Subset Simulation-based Random Finite Element Method for Slope Reliability Analysis and Risk Assessment

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ABSTRACT: Spatial variability is one of the most significant uncertainties in soil properties that affect the reliability of slope stability. It can be incorporated into slope reliability analysis and risk assessment through random finite element method (RFEM) in a rigorous manner. The great potential of RFEM in reliability analysis and risk assessment of soil slopes has been demonstrated in previous studies. Nevertheless, it often suffers from a common criticism of requiring extensive computational efforts and a lack of efficiency, particularly at small probability levels. This study develops an efficient RFEM that integrates RFEM with an advanced Monte Carlo Simulation method called "Subset Simulation (SS)". By this means, the computational efficiency of calculating the failure probability and risk is significantly improved. This enhances the applications of RFEM in slope reliability analysis and risk assessment at small probability levels. In addition, the proposed SS-based RFEM also provides insights into the relative contributions of slope failure risk at different probability levels to the overall risk. Finally, the proposed approach is illustrated through a soil slope example. It is shown that the slope failure probability and risk can be evaluated properly using SS-based RFEM.

1. INTRODUCTION

In the past few decades, several probabilistic analysis approaches have been developed for reliability analysis and risk assessment of slope stability in geotechnical reliability community. However, practicing engineers are still reluctant to adopt them in slope engineering practice. This dilemma can be attributed to, at least, two reasons, as observed by Griffiths and Fenton (2004): (1) the majority of probabilistic slope stability analysis methods in previous studies make use of traditional slope stability analysis techniques, i.e., limit equilibrium methods (LEMs), which usually need to assume the shape and location of slope failure surfaces prior to the calculation and may fail to locate the most critical failure mechanism in highly variable soils; (2) Only the spatial variability of soil strength parameters along the critical slip surface identified by LEMs can be taken into account when using LEMs in the probabilistic analysis of slope stability.

To address abovementioned problems, Griffiths and Fenton (2004) proposed a rigorous probabilistic analysis method called "random finite element method (RFEM)" for slope reliability analysis and risk assessment. RFEM consists of three major components: random field theory (Vanmarcke, 2010) to model inherent spatial variability of soil properties (Li et al. 2014, 2015), FEM to assess the safety factor (*FS*) of slope stability, and Monte Carlo Simulation (MCS) for uncertainty propagation and calculation of failure probability (P_f) and risk (R). Although the great potential of RFEM in reliability analysis and risk assessment of soil slopes has been demonstrated in previous studies (Griffiths and Fenton, 2004; Huang et al., 2013), it suffers from a common criticism of requiring extensive computational efforts and a lack of efficiency at small probability levels.

To enhance the applications of RFEM at small probability levels, this paper integrates RFEM with an advanced MCS method called "Subset Simulation (SS)" to develop an efficient RFEM, i.e., SS-based RFEM, for slope reliability analysis and risk assessment. SS is used for uncertainty propagation instead of direct MCS, which improves the computational efficiency of calculating P_f and R significantly. In addition, the proposed SS-based RFEM also provides insights into the relative contributions of slope failure risk at different probability levels to the overall risk. The paper starts with a brief description of SS, followed by development of the SS-based RFEM for slope reliability analysis and risk assessment. Then, the proposed approach is illustrated through a soil slope example.

2. SUBSET SIMULATION

SS is an advanced MCS method that uses conditional probability and Markov Chain Monte Carlo Simulation (MCMCS) method to efficiently compute small tail probability (Au and Wang, 2014). It expresses a rare failure event F with a small probability as a sequence of intermediate failure events $\{F_1, F_2, \ldots, F_m\}$ with larger conditional failure probabilities and employs specially designed Markov Chains to generate conditional failure samples of these intermediate failure events until the target failure domain is achieved. For example, consider the slope stability problem, whose failure probability P_f is defined as the probability of the performance function G (i.e., $FS-1$) smaller than a given threshold value g, i.e., $P_f = P(G < g)$. Let $g = g_m < g_{m-1} < \ldots < g_2 < g_1$ be an increasing sequence of intermediate threshold values. Then, the intermediate events $\{F_k, k = 1, 2, \ldots, m\}$ are defined as $F_k = \{G < g_k\}$, $k = 1, 2, \ldots, m$. By sequentially conditioning on the intermediate events $\{F_k, k = 1, 2, \ldots, m\}$, P_f can be written as (Au and Wang, 2014):

$$P_f = P(F_m) = P(F_1)\prod_{k=2}^{m} P(F_k \mid F_{k-1}) \tag{1}$$

where $P(F_1) = P(G < g_1)$ and $P(F_k|F_{k-1}) = \{P(G < g_k|G < g_{k-1}), k = 2, 3, \ldots, m\}$. In implementations, $g_1, g_2, \ldots, g_{m-1}$ are generated adaptively using information from simulated samples so that the sample estimates of $P(F_1)$ and $\{P(F_k|F_{k-1}), k = 2, 3, \ldots, m-1\}$ always correspond to a common specified value of conditional probability p_0 (Au and Wang, 2014).

The efficient generation of conditional samples is pivotal to the success of SS, and it is made possible through the machinery of MCMCS. In MCMCS, a modified Metropolis algorithm (MMA) (Au and Wang, 2014) is used, which generates the candidate sample of a high dimensional random vector component by component to improve the acceptance ratio of candidate sample and avoid many repeated samples. For example, using MMA to generate the candidate sample of $\boldsymbol{X} = [X_1, X_2, \ldots, X_{Np}]$ contains N_p steps. In each step, the candidate sample of X_j ($j = 1, 2, \ldots, N_p$), is generated. After the candidate samples of all the components are obtained, they are collectively taken as the candidate sample of $\boldsymbol{X}$. If the $\boldsymbol{X}$'s candidate sample belongs to the intermediate failure event concerned, it is taken as the next state of $\boldsymbol{X}$ in Markov Chain. The MMA makes SS feasible in high dimensional problems, e.g., slope reliability analysis problems that consider geotechnical spatial variability using random fields. SS is integrated with RFEM to calculate P_f and R in the next two sections, respectively.

3. SS-BASED RFEM FOR RELIABILITY ANALYSIS OF SLOPE STABILITY

In SS-based RFEM, each random sample generated during SS is taken as the input in finite element analysis of slope stability to calculate its corresponding values of FS and G. The values of G are then used to gradually determine the respective intermediate threshold values g_1, g_2, …, g_m for intermediate failure events $\{F_k$, k = 1, 2, …, $m\}$. Note that the finite element analysis of slope stability can be performed in commercial software packages of finite element analysis, e.g., Abaqus. The commercial software package is repeatedly invoked to analyze the slope stability during SS. By this means, SS is deliberately decoupled from the deterministic finite element analysis of slope stability in a non-intrusive manner. This effectively removes the hurdle of reliability computational algorithm and allows geotechnical practitioners to focus on the deterministic slope stability analysis that they are familiar with.

SS procedures contain m+1 levels, including one direct MCS to generate N unconditional samples and m levels of MCMCS to simulate $(1-p_0)N$ conditional samples. Finally, $N+m(1-p_0)N$ samples are obtained from the m+1 levels of simulations. These samples fall into different subsets $\{\Omega_k, k = 0, 1, 2, \ldots, m\}$ defined by the m intermediate threshold values g_1, g_2, …, g_m that are determined adaptively during SS. According to the Total Probability Theorem, the failure probability is then written as:

$$P_f = \sum_{k=0}^{m} P(F \mid \Omega_k) P(\Omega_k) \tag{2}$$

where $\Omega_0 = \{G \geq g_1\}$; $\Omega_k = \{g_{k+1} \leq G < g_k\}$, k = 1, 2, …, $m-1$; $\Omega_m = \{G < g_m\}$; $P(\Omega_k)$ = the occurrence probability of Ω_k, and it is taken as $p_0^k - p_0^{k+1}$ for k = 0, 1, …, m−1, and p_0^m for $k = m$ (Wang and Cao, 2013); $P(F|\Omega_k)$ = the conditional failure probability given sampling in Ω_k, and it is estimated as the fraction of the failure samples in Ω_k. The failure samples are collected from samples generated by SS and are based on the performance failure criteria (i.e., $G < g$). Using the $N+m(1-p_0)N$ samples generated by SS and Eq. (2), the P_f is calculated accordingly.

4. SS-BASED RFEM FOR RISK ASSESSMENT OF SLOPE FAILURE

For a slope problem, there might exist multiple failure modes due to the spatial variability or stratification of geotechnical materials (e.g., Wang et al., 2011; Jiang et al., 2015). To account for the contributions of multiple failure modes to overall risk of slope failure, the risk of slope failure can be expressed as (Huang et al., 2013)

$$R = \sum_{i=1}^{N_f} P_i C_i = P_f \overline{C} \tag{3}$$

where $P_i = 1/N_T$ and C_i are the probability and consequence of the failure mode corresponding to the i-th failure sample during MCS, respectively; N_T = the total number of samples in MCS; N_f = the number of failure samples; P_f = N_f/N_T; $\overline{C} = \sum_{i=1}^{N_f} C_i / N_f$ = the average consequence of different failure modes. As pointed out by Huang et al. (2013), the consequence of slope failure depends on the sliding mass volume, which can, therefore, be taken as an “equivalent” quantity to quantify the consequence of slope failure in slope risk assessment. For this purpose, the consequence C_i (i.e., the sliding mass volume) for the i-th failure sample generated during MCS needs to be estimated. This can be achieved by the K-means clustering method (KMCM) (Huang et al., 2013). Based on KMCM, the finite-element nodes are classified into two categories (i.e., stable and unstable masses) according to node displacements obtained from the finite element analysis of slope stability. Then, the sliding mass volume can be estimated if the slope failure occurs.

Using the direct MCS-based RFEM, it is straightforward to evaluate $\overline{C}$. However, this is not the case for SS-based RFEM because the SS samples are conditional samples that fall into different subsets $\{\Omega_k, k = 0, 1, 2, \ldots, m\}$ and carry different weights. Thus, when using these conditional failure samples collected from SS to evaluate $\overline{C}$, a weighted summation is necessary, and $\overline{C}$ is written as:

$$\overline{C} = E(C \mid F) = \sum_{k=0}^{m} \overline{C}_k P(\Omega_k \mid F) \tag{4}$$

where $E(C|F)$ = the conditional expectation of the consequence given the occurrence of the slope failure; $P(\Omega_k|F)$ = the probability of the failure samples falling into Ω_k; $\overline{C}_k = \sum_{i=1}^{N_{f,k}} C_{k,i} / N_{f,k}$ = the average consequence of different failure modes corresponding to failure samples in Ω_k, in which $N_{f,k}$ = the number of failure samples generated in Ω_k during SS and $C_{k,i}$ is the consequence of the failure mode corresponding to the i-th failure sample generated in Ω_k. Note that $C_{k,i}$ is a key input in Eq. (4) to evaluate the average consequence $\overline{C}_k$ of slope failure occurring in Ω_k. For simplicity, $C_{k,i}$ is approximately taken as the sliding mass volume for the i-th failure sample generated in Ω_k in this study, which can be determined using KMCM as suggested by Huang et al. (2013).

In addition, $P(\Omega_k|F)$ in Eq. (4) can be calculated using the Bayes' Theorem as:

$$P(\Omega_k \mid F) = \frac{P(F \mid \Omega_k) P(\Omega_k)}{P_f} \tag{5}$$

where P_f, $P(F|\Omega_k)$, and $P(\Omega_k)$ are given in Eq. (2). Using Eqs. (4) and (5) gives

$$\overline{C} = \frac{\sum_{k=0}^{m} \overline{C}_k P(F \mid \Omega_k) P(\Omega_k)}{P_f} \tag{6}$$

Then, substituting Eq. (6) into Eq. (3) gives

$$R = \sum_{k=0}^{m} \overline{C}_k P(F \mid \Omega_k) P(\Omega_k) \tag{7}$$

Using Eq. (7), the risk R of slope failure is calculated using the conditional samples generated during SS. Herein, it is worthwhile to point out that although a simple and approximate way is adopted in this study to estimate the consequence of slope failure, the proposed approach is generally applicable for different methods to evaluate the slope failure consequence (e.g., $C_{k,i}$ and $\overline{C}_k$). Using a more accurate estimation of slope failure consequence, of course, leads to a more accurate estimation of slope failure risk.

To gain more insights into the R estimated by SS, Eq. (7) is further written as:

$$R = \sum_{k=0}^{m} R_k P(\Omega_k) \tag{8}$$

where $R_k = P(F \mid \Omega_k)\overline{C}_k$ is an analogue of Eq. (3) and represents the conditional risk of slope failure when sampling in Ω_k. In the light of Eq. (8), the overall risk of slope failure can be considered as a weighed aggregation of the slope failure risk in different sampling spaces (i.e., $\{\Omega_k, k = 0, 1, 2, \ldots, m\}$) that are progressively determined during SS and have different occurrence probabilities. The contribution of slope failure risk (COR) in each sampling space to the overall slope failure risk is calculated as:

$$COR_k = 100 R_k P(\Omega_k) / R \tag{9}$$

where COR_k = contribution of the risk of slope failure occurring in Ω_k to the overall risk R in percent (%). Since the occurrence probabilities of Ω_k, $k = 0, 1, 2, \ldots, m$, are different, COR_k quantifies the relative contribution of the slope failure risk at different probability levels to the overall risk.

5. ILLUSTRATIVE EXAMPLE

For illustration, this section applies the proposed SS-based RFEM to evaluate P_f and R of a soil slope example, which has been analyzed by Cho (2007). As shown in Fig. 1, the slope has a height of 10 m and a slope angle of 26.6°, and it is comprised of two soil layers. Table 1 summarizes the statistics of soil properties (including unit weight γ_1 and γ_2, cohesion c_1 and c_2, and friction angle ϕ_1 and ϕ_2) in the two soil layers, which are consistent with those adopted by Cho (2007). All the five uncertain parameters are lognormally distributed. In addition, to enable the finite element analysis of slope stability in this study, the information on the Young's modulus (i.e., E_1 and E_2) and Poisson's ratio (i.e., v_1 and v_2) are assumed to be 100MPa and 0.3 (Griffiths and Lane, 1999), respectively.

In this study, a finite element model of the slope is created using a commercial software

Table 1. Summary of statistics and values of soil parameters (after Cho (2007))

Soil Parameters		Mean	COV
Upper Layer	γ_1(kN/m^3)	18	0.05
	c_1(kPa)	38.31	0.4
	ϕ_1(°)	0	/
	E_1(MPa)	100	/
	v_1	0.3	/
Lower Layer	γ_2(kN/m^3)	18	0.05
	c_2(kPa)	23.94	0.2
	ϕ_2(°)	12	0.1
	E_2(MPa)	100	/
	v_2	0.3	/

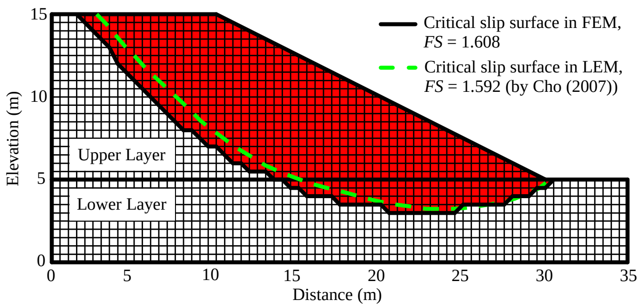

Fig. 1. An illustrative slope example

package Abaqus as shown in Fig. 1. Based on the mean values of soil properties shown in Table 1, the finite element analysis of slope stability is performed using an elastic-perfectly plastic constitute model with a Mohr-Coulomb failure criterion in Abaqus. Using the shear strength reduction technique, the resulting *FS* is 1.608, and it is slightly different from the *FS* value (i.e., 1.592) of the deterministic critical slip surface reported by Cho (2007). Such a slight difference is attributed to different slope stability analysis methods used in this study and Cho (2007), that is, FEM and Spencer's LEM, respectively.

To incorporate the effects of spatial variability of soil properties into reliability analysis and risk assessment, the inherent spatial variability of γ_1, γ_2, c_1, c_2, and ϕ_2 are taken into account in this example. For each parameter, the random field is simulated using the covariance

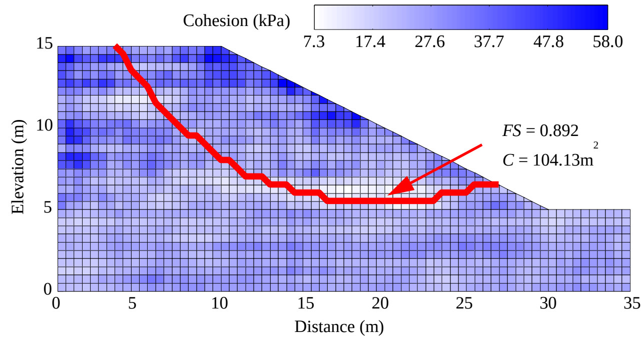

Fig. 2. A typical realization of random fields and its corresponding results of slope stability analysis

matrix decomposition method, in which 2-D single exponential correlation function is adopted and the horizontal and vertical scale of fluctuation are taken as 15m and 3m, respectively. Each realization of the random field is mapped onto the finite-element mesh shown in Fig. 1. Then, the finite element analysis of slope stability is performed to calculate the values of *FS* and *G* (i.e., *FS*−1) and the node displacements. The node displacements are subsequently used to determine the critical slip surface by KMCM. Based on the critical slip surface, the slope failure consequence is estimated as the sliding mass volume if the slope failure occurs (i.e., $G < g$). For example, Fig. 2 shows a typical realization of the random fields of c_1 and c_2 and its corresponding results of slope stability analysis obtained from the finite element analysis, including the *FS* (i.e., 0.892), the critical slip surface identified by KMCM (see the bold solid line), and the consequence (i.e., sliding mass volume in this study) of about 104.13m^3/m (or 104.13m^2). Note that 2-D slope stability analysis is performed in this study. Strictly speaking, the identified sliding mass is an "area" but not a "volume" in this example, and it, therefore, has a unit of "m^3/m" (or "m^2" for simplicity), rather than "m^3". However, the term "sliding mass volume" is still used herein for easy understanding and communication.

Based on the deterministic finite element model and random field model of soil parameters described above, a SS run with m = 3, p_0 = 0.1, and N = 500 is performed in this study. This

Table 2. Results of reliability analysis and risk assessment in the slope example

Simulation level k	0	1	2	3
$P(\Omega_k)$	0.9	0.09	0.009	0.001
$P(F\|\Omega_k)$	0	0	23/450	500/500
$\overline{C}_k\,(m^2)$	0	0	110.6	111.92
$COR_k\,(\%)$	0	0	31	69
$P_f\,(\%)$	0.146			
$\overline{C}\,(m^2)$	111.64			
$R(m^2)$	0.163			

leads to a total of $500+3\times(1-0.1)\times 500 = 1850$ random samples.

5.1. Results of Reliability Analysis and Risk Assessment

Table 2 summarizes the procedures of evaluation of P_f and R in this example. Among the 1850 samples from SS, 523 samples are identified as failure samples for $g = 0$ (i.e., $FS < 1$). These 523 failure samples include 500 samples in simulation level 3 and 23 samples in simulation level 2, and no failure sample occurs in simulation levels 0 and 1. Using Eq. (2), P_f is estimated as 0.146% (corresponding COV(P_f) is about 0.3). Such a value is favorably comparable with that (i.e., 0.12%) reported by Cho (2007). In addition, $\overline{C}$ and R are also estimated as 111.64 m^2 and 0.163 m^2 by Eqs. (6) and (7), respectively. Although the individual value of the estimated R seems meaningless, the estimated R is meaningful in a relative scale and provides a useful tool to quantify the relative contributions of slope failure risk at different probability levels to the overall risk.

As indicated by Eq. (8), the risk of slope failure is de-aggregated into different sampling spaces Ω_0, Ω_1, Ω_2, and Ω_3. The overall risk of slope failure is mainly attributed to the slope failure occurring in Ω_3 (69%) in this example. To enable a desired accuracy of the estimated R, a large number of failure samples shall be generated in Ω_3. This can be achieved through SS with relative ease. As shown in Table 2, all the 500 samples generated in Ω_3 by SS are failure samples. On the other hand, if using direct MCS to generate such a number of samples in Ω_3, a total of 500/0.001 = 500,000 direct MCS samples are required on average, because the $P(\Omega_3) = 0.001$. Compared with direct MCS, the proposed SS-based RFEM not only provides more insights (e.g., contributions of slope failure risk in different sampling spaces or at different probability levels) into the overall risk R, but also improves significantly the computational efficiency of generating the samples of interest (e.g., samples in Ω_3) for slope risk assessment.

5.2. Validation of Reliability Analysis and Risk Assessment Results

To validate the reliability analysis and risk assessment results obtained from the SS-based RFEM, the direct MCS run with 10,000 samples is performed to calculate the P_f and R of the slope example.

Fig. 3 shows the variation of the failure probability $P_f = P(G < g)$ as a function of g (i.e., the cumulative distribution function (CDF) of the performance function G) obtained from SS and direct MCS by a solid line and a dashed line, respectively. The solid line generally plots closely to the dashed line. For $g = 0$, the P_f estimated from direct MCS is 0.14%, which agrees well with the value (i.e., 0.146%) of P_f obtained from SS. Such observations indicate that the P_f is calculated properly using the SS-based RFEM proposed in this study. Note that only 1850 random samples are generated in the SS-based RFEM, and hence a total 1850 of finite element analyses of slope stability are performed, which are much less than the number (i.e., 10,000) of finite element analyses of slope stability required in the original RFEM with direct MCS. In addition, as shown in Fig. 3, the CDF of G obtained from direct MCS becomes erratic as the probability level deceases from

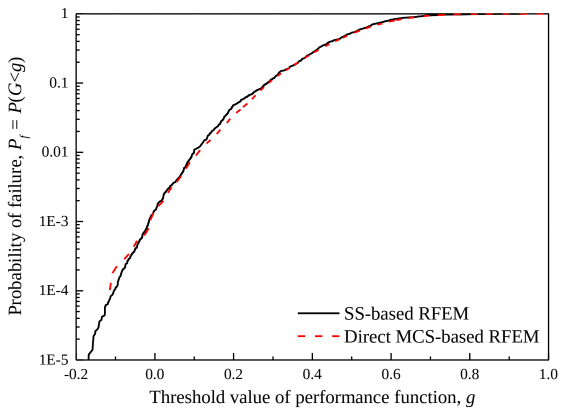

Fig. 3. Cumulative distribution function of the performance function of slope stability

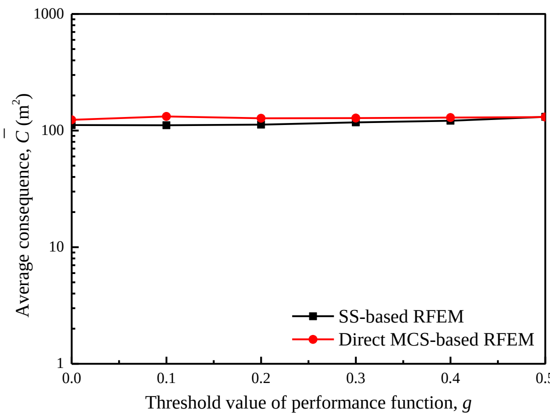

Fig. 4. Average consequence at different safety levels

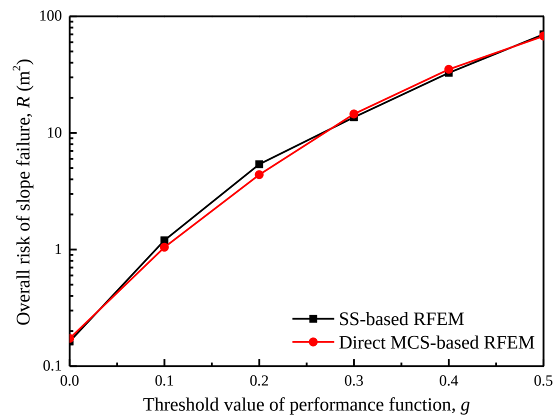

Fig. 5. Overall risk of slope failure at different safety levels

0.001 to 0.0001, and no information on the CDF is obtained from direct MCS with 10,000 samples at the probability level less than 0.0001. In contrast, the CDF of G obtained from SS remains consistent as the probability level decreases from 0.001 to 0.00001. Integrating SS with RFEM improves, significantly, the computational efficiency and resolution at small probability levels and reduces the computational efforts. This enhances the applications of RFEM in the slope reliability analysis problems, particularly at small probability levels.

Fig. 4 shows the estimated $\bar{C}$ for different values (i.e., 0, 0.1, 0.2, 0.3, 0.4, and 0.5) of g obtained from RFEM with SS and direct MCS by the lines with squares and circles, respectively. The line with squares plots closely to the line with circles. The $\bar{C}$ values estimated from the SS-based RFEM are in good agreement with those obtained from the original RFEM with direct MCS. In addition, it is also noted that the estimated $\bar{C}$ almost remains unchanged as g varies from 0 to 0.5 in this example. Fig. 5 shows the variation of R as a function of g obtained from RFEM with SS and direct MCS by the lines with squares and circles, respectively. Similar to Figs. 3 and 4, the line with squares plots closely to the line with circles. The R values estimated from the SS-based RFEM agree well with those obtained from the original RFEM with direct MCS. This further validates the risk assessment results obtained from the SS-based RFEM.

As shown in Fig. 5, R increases significantly as g increases. Note that the g value indicates the desired safety level of the slope. As g increases, the desired safety level of the slope increases. As expected, for a given slope problem, R increases as the desired safety level increases. Such an increase in R, which is a product of P_f and $\bar{C}$, is mainly attributed to the increase in P_f as g increases (see Fig. 3) since $\bar{C}$ remains almost unchanged in this example as g varies (see Fig. 4). The proposed approach properly depicts the slope failure risk at different safety levels using a single run of SS.

6. SUMMARY AND CONCLUSIONS

This paper developed an efficient random finite element method (RFEM) for slope reliability analysis and risk assessment. The proposed approach integrates RFEM with an advanced MCS method called “Subset Simulation (SS)” to evaluate the failure probability (P_f) of slope stability and the slope failure risk (R) in spatially variable soils. The proposed SS-based RFEM expresses the overall risk of slope failure as a weighed aggregation of slope failure risk in different sampling spaces at different probability levels that are progressively determined during SS. It quantifies the relative contributions of the slope failure risk at different probability levels to the overall risk. In addition, SS is deliberately decoupled from the deterministic finite element analysis of slope stability in the proposed approach, so that the slope reliability analysis and risk assessment using RFEM can proceed as an extension of deterministic finite element analysis of slope stability in a non-intrusive manner.

The SS-based RFEM proposed in this study can be viewed as a new development of the original RFEM that makes use of direct MCS to evaluate P_f and R. Equations were derived for evaluating P_f and R using the random samples generated by SS. These equations were illustrated using a soil slope example. The results obtained from the SS-based RFEM are validated against those obtained using the direct MCS-based RFEM. It was shown that the P_f and R at different safety levels (i.e., different g values) are calculated properly using the proposed approach. Compared with the direct MCS-based RFEM, integrating SS with RFEM improves, significantly, the computational efficiency of generating failure samples for evaluating P_f and R. This enhances the applications of RFEM in slope reliability analysis and risk assessment, particularly at small probability levels, which are of great interest in slope engineering practice.

7. ACKNOWLEDGMENTS

This work was supported by the National Science Fund for Distinguished Young Scholars (Project No. 51225903), the National Natural Science Foundation of China (Project No. 51329901, 51409196), the Natural Science Foundation of Hubei Province of China (Project No. 2014CFA001). The financial support is gratefully acknowledged.

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