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Adaptive Kriging Reliability-Based Design Optimization Of An Automotive Body Structure Under Crashworthiness Constraints

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ABSTRACT: The increasing use of surrogate models has widened the range of application of classical reliability-based design optimization (RBDO) techniques to industrial problems. In this paper, we consider such an approach to the lightweight design of an automotive body structure. Solving this problem while approximating the complex models (nonlinear, noisy and high-dimensional) with a single metamodel would require a very large and non affordable design of experiments (DOE). We thus investigate and propose a methodology of adaptive Kriging based RBDO where an initial DOE is iteratively updated so as to improve the Kriging models only in regions that actually matter. The nested reliability analysis is expressed in terms of quantiles assessment. Two stages of enrichment are performed. The first one seeks to gradually improve the accuracy of the metamodels where the probabilistic constraints are likely to be violated. The second one is embedded in an evolution strategy optimization scheme where, at each iteration, the accuracy of the quantile estimation is improved if necessary. The methodology is applied on an analytical and crashworthiness design problems showing good performance by enhancing accuracy and efficiency with respect to a traditional approach.

The computational cost of the latest high fidelity simulation codes make them unaffordable when it comes to structural design problems such as optimization or reliability analyses. Designers are now familiar with so-called *metamodelling* approaches where an easy-to-evaluate function is used as a proxy of the true model $\mathscr{M} : \boldsymbol{x} \in \mathbb{X} \subset \mathbb{R}^s \mapsto y = \mathscr{M}(\boldsymbol{x})$. The metamodel is fitted by learning over an initial *design of experiments* which is basically some pairs of known inputs-outputs of the code: $\mathscr{D} = \{(\boldsymbol{x}^{(i)}, y_i), i \in \{1,\ldots,n\}, \forall \boldsymbol{x}^{(i)} \in \mathbb{X} \subset \mathbb{R}^s, y_i = \mathscr{M}(\boldsymbol{x}^{(i)}) \in \mathbb{Y} \subset \mathbb{R}\}$. In this paper, we consider the application of such an approach to the lightweight design of an automotive body structure. The associated constraints have shown to be very noisy because of the chaotic nature of a vehicle frontal impact. To account for the various uncertainties associated with this problem, reliability-based design optimization (RBDO) is performed. However, performing the RBDO on surrogate models built on a single space-filling design does not yield accurate solutions nor is it efficient. This is because ultimately, the region of interest for optimization is very often only a small subset of the entire design space. Alternatively, an initial surrogate model can be iteratively updated to accurately approximate $\mathscr{M}$

only where necessary such as introduced in the framework of *expected global reliability analysis* (Bichon et al. (2008)).

We propose here a methodology for adaptive Kriging-based RBDO embedded in an evolution strategy where at each iteration, the accuracy of the surrogate model is checked and if necessary improved by local enrichment. The paper is organized as follows. First we set up the RBDO problem and briefly review Kriging. Then we introduce the techniques for adaptive design of experiments followed by the entire methodology. Finally, an analytical and a finite element-based examples are considered for application.

1. DESIGN OPTIMIZATION UNDER UNCERTAINTY

Uncertainties are ubiquitous in structural systems and play a key role in the robustness and reliability of optimized solutions. The reliability aspect is most often handled with RBDO. Such a technique seeks to balance some cost and a predefined level of reliability. Generally, the RBDO problem is stated in the following terms:

$$\boldsymbol{d}^* = \arg\min_{\boldsymbol{d}\in\mathbb{D}} \mathfrak{c}(\boldsymbol{d}) \quad \text{s.t.} \quad \begin{cases} \mathfrak{f}_j(\boldsymbol{d}) \leq 0, \{j=1,\ldots,n_s\} \\ \mathbb{P}(\mathfrak{g}_k(\boldsymbol{X}(\boldsymbol{d}),\boldsymbol{Z}) < 0) \leq P_{f_k}, \{k=1,\ldots,n_p\} \end{cases} \tag{1}$$

where a cost function $\mathfrak{c}$ is minimized with respect to design variables $\boldsymbol{d}$ while satisfying to a collection of n_s soft and n_p performance constraints respectively denoted by $\mathfrak{f}$ and $\mathfrak{g}$. The former simply bounds the design space and the latter splits it into safety and failure domains. To account for the uncertainties, the RBDO approach addresses the problem in terms of probability of failure being lower than a given threshold, herein $\boldsymbol{P}_f = \{P_{f_k}, k=1,\ldots,n_p\}$. In this respect, the random variables $\boldsymbol{X} \sim f_{\boldsymbol{X}|\boldsymbol{d}}$ and $\boldsymbol{Z} \sim f_{\boldsymbol{Z}}$ are introduced and respectively stand for the design and environment variables.

The solution of Eq. (1) requires the assessment of the probability of failure for any given design during the optimization process. This is usually done by integrating the joint probability density function of $\boldsymbol{X}$ and $\boldsymbol{Z}$ over the failure domain, which can turn out to be a cumbersome task (Dubourg et al. (2011)).

Despite many techniques exist in the literature to solve this problem, we adopt a simpler yet efficient approach by replacing the probabilistic constraint with a quantile assessment. This is especially justified since the target probabilities of failure are relatively high in the problems we are addressing (actually around 5%).

Eq. (1) then rewrites:

$$\boldsymbol{d}^* = \arg\min_{\boldsymbol{d}\in\mathbb{D}} \mathfrak{c}(\boldsymbol{d}) \quad \text{s.t.} \begin{cases} \mathfrak{f}(\boldsymbol{d}) \leq \boldsymbol{0} \\ \mathfrak{g}_\alpha(\boldsymbol{X}(\boldsymbol{d}),\boldsymbol{Z}) \leq \boldsymbol{0} \end{cases} \tag{2}$$

where for any performance function $\mathfrak{g}_k$, the quantile $\mathfrak{g}_{k_\alpha}(\boldsymbol{X}(\boldsymbol{d}),\boldsymbol{Z}) \equiv \mathfrak{g}_{k_\alpha}(\boldsymbol{d})$ is defined such that:

$$\mathbb{P}(\mathfrak{g}_k(\boldsymbol{X}(\boldsymbol{d}),\boldsymbol{Z}) \leq \mathfrak{g}_{k_\alpha}(\boldsymbol{d})) = \alpha \tag{3}$$

The quantile estimation mostly resorts to Monte Carlo (MC) sampling. More specifically for a given design $\boldsymbol{d}^{(i)}$, a MC population is sampled following $f_{\boldsymbol{X}|\boldsymbol{d}}$ and $f_{\boldsymbol{Z}}$:

$$\mathfrak{C}_q^{(i)} = \left\{ (\boldsymbol{x}_j(\boldsymbol{d}^{(i)}), \boldsymbol{z}_j), j=1,\ldots,N_q \right\} \tag{4}$$

The model is then evaluated on these points and the results ranked in ascending order. The estimated quantile corresponds to the $\lfloor N_q\alpha \rfloor$-th term, where $\lfloor \bullet \rfloor$ denotes the floor function.

This simulation technique is nonetheless very-time consuming as it relies on multiple model evaluations. To make the RBDO affordable, a surrogate-based approach is adopted. In this paper, Kriging has been chosen as default surrogate and will be briefly reviewed in the next section.

2. KRIGING SURROGATE

Kriging or Gaussian process modeling relies on a major hypothesis which is to assume that the function to emulate is one realization of a stochastic process (Santner et al. (2003)) that reads:

$$\mathscr{M}(\boldsymbol{x}) = \sum_{j=1}^{p} \beta_j f_j(\boldsymbol{x}) + Z(\boldsymbol{x}) \tag{5}$$

where $\sum_{j=1}^{p} \beta_j f_j(\boldsymbol{x})$ is a linear combination of some basis functions which captures a global

trend, as conventionally assumed in universal Kriging, and $Z(\boldsymbol{x})$ is a zero-mean Gaussian Process with auto-covariance function $\mathrm{Cov}\,[Z(\boldsymbol{x}),Z(\boldsymbol{x'})] = \sigma^2 R(\boldsymbol{x},\boldsymbol{x'};\boldsymbol{\theta})$. In this setting, σ^2 is the variance, $R(\boldsymbol{x},\boldsymbol{x'};\boldsymbol{\theta})$ the auto-correlation function and $\boldsymbol{\theta}$ a vector gathering its hyperparameters.

Building up the model requires first to make some choices about the basis and auto-correlation functions, the most popular ones being respectively low order polynomials and parametric multi-variate stationary functions. The latter encodes the assumptions about the underlying process such as regularity.

The Kriging approximate for a new point $\boldsymbol{x}^{(0)}$ is provided by a realization of a Gaussian variable $G \sim \mathcal{N}(\mu_{\widehat{G}}, \sigma^2_{\widehat{G}})$:

$$\begin{cases} \mu_{\widehat{G}} = \boldsymbol{f}(\boldsymbol{x}^{(0)})^T \widehat{\boldsymbol{\beta}} + \boldsymbol{r}_0^T \boldsymbol{R}^{-1}(\boldsymbol{y} - \boldsymbol{F}\widehat{\boldsymbol{\beta}}) \\ \sigma^2_{\widehat{G}} = \widehat{\sigma^2}(1 - \boldsymbol{r}_0^T \boldsymbol{R}^{-1}\boldsymbol{r}_0 + \boldsymbol{u}^T(\boldsymbol{F}^T\boldsymbol{R}^{-1}\boldsymbol{F})^{-1}\boldsymbol{u}) \end{cases} \tag{6}$$

where $\widehat{\boldsymbol{\beta}} = (\boldsymbol{F}^T\boldsymbol{R}^{-1}\boldsymbol{F})^{-1}\boldsymbol{F}^T\boldsymbol{R}^{-1}\boldsymbol{y}$ and $\widehat{\sigma^2} = \frac{1}{n}(\boldsymbol{y} - \boldsymbol{F}\boldsymbol{\beta})^T\boldsymbol{R}^{-1}(\boldsymbol{y} - \boldsymbol{F}\boldsymbol{\beta})$ are the generalized least-square estimates of the Kriging parameters for a polynomial trend and $\boldsymbol{u} = \boldsymbol{F}^T\boldsymbol{R}^{-1}\boldsymbol{r_0} - \boldsymbol{f}(\boldsymbol{x}^{(0)})$. The following matricial notation have been introduced: $\boldsymbol{F} = \{F_{ij} = f_j(\boldsymbol{x}^{(i)}), i = 1,\dots,n, j = 1,\dots,p\}$, $\boldsymbol{R} = \{r_{ik} = R(\boldsymbol{x}^{(i)},\boldsymbol{x}^{(k)}), i = 1,\dots,n, k = 1,\dots,n\}$ and $\boldsymbol{r_0} = \{R(\boldsymbol{x}^{(0)},\boldsymbol{x}^{(i)}), i = 1,\dots,n\}$.

The parameters $\boldsymbol{\theta}$ are inferred from the data. This learning stage usually resorts to various techniques among which the widely used *maximum likelihood estimation*. It turns out to be an optimization problem which reads:

$$\widehat{\boldsymbol{\theta}} = \arg\min_{\boldsymbol{\theta}\in\mathbb{R}^{d_\theta}} \widehat{\sigma}^2(\boldsymbol{\theta})(\det \boldsymbol{R}(\boldsymbol{\theta}))^{\frac{1}{n}} \tag{7}$$

where d_θ is the number of parameters. This crucial optimization problem is solved here by an hybrid algorithm (genetic followed by BFGS) as proposed in the R package DiceKriging we are using (Roustant et al. (2012)).

Eq. (6) displays the mean and variance of the Kriging prediction. This variance provides Kriging with a local estimator of error mainly due to the sparsity of data. This feature is exploited in so-called *adaptive designs*.

3. ADAPTIVE DESIGNS

3.1. Learning function

Following the ideas developed in the framework of *efficient global optimization*, various approaches were introduced to adaptively update initial design of experiments. According to whether it is the objective or the constraint function that is surrogated, different families of learning functions are used. In our context of constraints handling with a perfectly known objective function, we focus on AK-MCS (Active Kriging - Monte Carlo Simulation) proposed by Echard et al. (2011). Many other learning functions exist and are to the authors experience *almost* equally efficient. However, AK-MCS has a very simple formulation and is easy to interpret.

The idea in AK-MCS is to sample a very large MC population of candidates for enrichment. The learning function will then select among these points the one which promises the highest expected gain of information. This gain of information is considered here with respect to either quantile estimation or contour approximation. Therefore it results in two slightly different formulations.

For quantile estimation, we follow the methodology proposed in Schöbi and Sudret (2014). Let us consider we are at iteration i of the optimization process. With the current design $\boldsymbol{d}^{(i)}$, the MC population $\mathfrak{C}_q^{(i)}$ considered for enrichment is sampled according to Eq. (4). The learning function then reads:

$$U_q(\boldsymbol{x},\boldsymbol{z}) = \frac{|\mu_{\widehat{G}}(\boldsymbol{x},\boldsymbol{z}) - \widehat{\mathfrak{g}}_\alpha(\boldsymbol{d}^{(i)})|}{\sigma_{\widehat{G}}(\boldsymbol{x},\boldsymbol{z})} \tag{8}$$

where $\widehat{\mathfrak{g}}_\alpha(\boldsymbol{d}^{(i)})$ is the estimate of the quantile computed through the Kriging model. The best next point is the one that minimizes U_q. It simply corresponds to points with estimates close to the current quantile or with high variance. By iteratively adding points in this fashion, the accuracy of the quantile estimate will be improved.

For contour approximation, the candidate MC population is defined in the joint space of design and environment variables: $\mathfrak{C}_c = \{(\boldsymbol{x}^{(j)},\boldsymbol{z}^{(j)}), j = 1,\dots,N\}$ and the learning function reads:

$$U_c(\boldsymbol{x},\boldsymbol{z}) = \frac{|\widehat{\mathfrak{g}}_\alpha(\boldsymbol{d})|}{\sigma_{\widehat{G}}(\boldsymbol{x},\boldsymbol{z})} \tag{9}$$

The same mechanisms as for U_q are in play here. Therefore samples with a high probability of constraint violation and/or high variance are likely to be added in the design. The computational cost of U_c is quite high as it implies $N \times N_q$ calls of the Kriging model. To reduce this computational burden, the quantile is estimated here with a very limited number of samples in a bootstrapping approach. More specifically in this paper, we go from the usual $N_q = 10^4$ to $N_q = 500$ samples with respectively 200 and 500 bootstrap replicates for the first and second applications. This setting offers a fair trade between accuracy and computational time.

3.2. Multi-constrained and multi-points enrichment

As stated above, we are concerned with multi-constrained optimization problems. Furthermore, the computational power of clustered PCs allows us to simultaneously launch many evaluations of the true model. In such a case, the adaptive scheme should allow for multi-points enrichment. Considering these two aspects, the enrichment methodology described above is slightly modified.

First, let us consider the presence of multiple constraints. Fauriat and Gayton (2014) proposed a composite criterion for system reliability problems. That is, the learning function is evaluated only with respect to the constraint that plays the largest role in the system failure. For Eqs. (8,9), this translates respectively to:

$$U_q^s(\boldsymbol{x},\boldsymbol{z}) = \frac{|\mu_{\widehat{G}_s}(\boldsymbol{x},\boldsymbol{z}) - \widehat{\mathfrak{g}}_{s_\alpha}(\boldsymbol{d}^{(i)})|}{\sigma_{\widehat{G}_s}(\boldsymbol{x},\boldsymbol{z})} \tag{10}$$

$$U_c^s(\boldsymbol{x},\boldsymbol{z}) = \frac{|\widehat{\mathfrak{g}}_{s_\alpha}(\boldsymbol{d})|}{\sigma_{\widehat{G}_s}(\boldsymbol{x},\boldsymbol{z})} \tag{11}$$

where s is the index of the performance function with the highest value at the evaluated point.

On the other hand, to allow for multiple points enrichment, we first define the following 95% margins of uncertainty:

$$\begin{aligned}\mathbb{M}_q = \cup_{k=1}^{n_p}\{(\boldsymbol{x},\boldsymbol{z}) \in \mathfrak{C}_q : \widehat{\mathfrak{g}}_{k_\alpha} - 2\sigma_{\widehat{G}_k} \leq \\ \mu_{\widehat{G}_k} \leq \widehat{\mathfrak{g}}_{k_\alpha} + 2\sigma_{\widehat{G}_k}\}\end{aligned} \tag{12}$$

$$\mathbb{M}_c = \cup_{k=1}^{n_p}\{(\boldsymbol{x},\boldsymbol{z}) \in \mathfrak{C}_c : -2\sigma_{\widehat{G}_k} \leq \widehat{\mathfrak{g}}_{k_\alpha} \leq 2\sigma_{\widehat{G}_k}\} \tag{13}$$

where $\mathbb{M}_q$ and $\mathbb{M}_c$ correspond respectively to the quantile estimation and contour approximation problems.

The learning function is evaluated on this subset of points. K clusters are identified by means of weighted K-means clustering with weight $\varphi(-U_c)$ (or $\varphi(-U_q)$), where φ is the standard Gaussian probability density function. By this weighting, regions where the criterion is high will be favored. The best next points to add are selected as the ones in $\mathbb{M}_c$ (or $\mathbb{M}_q$) which are the closest to the clusters centroids.

4. ADAPTIVE KRIGING BASED RBDO

The above-mentioned topics are now embedded in an optimization scheme so as to propose a methodology for adaptive Kriging-based RBDO.

We start by applying a few iterations of enrichment for contour approximation. This is to roughly locate and enrich regions of the space where the constraints are likely to be violated. This allows us to start from a very scarce initial design of experiments. We stop this enrichment procedure when the size of the 95% margin of uncertainty is considerably reduced. Particularly, we consider the following criterion: the ratio of Card($\mathbb{M}_c$) between the current and the initial iteration being lower than a given threshold, say 0.15. The residual epistemic uncertainty will be reduced through enrichment for quantile estimation during the optimization.

Let us now consider we are solving the RBDO problem in an iterative way:

$$\boldsymbol{d}^{(i+1)} = \boldsymbol{d}^{(i)} + \boldsymbol{v}^{(i)} \tag{14}$$

where $\boldsymbol{v}^{(i)}$ is a step in the search space promising an improvement of the objective function.

Since the true functions have been replaced by Kriging approximations, a sufficient level of accuracy of the metamodels should be ensured before proceeding to the updating scheme in any given iteration. We consider that the following relationship should hold:

$$\frac{|\widehat{\mathfrak{g}}_{k_\alpha}^{+} - \widehat{\mathfrak{g}}_{k_\alpha}^{-}|}{\mu_{\widehat{G}_k}^{max} - \mu_{\widehat{G}_k}^{min}} \leq \varepsilon_{g_k}, \quad \forall k = \{1,\ldots,n_p\} \tag{15}$$

where $\widehat{\mathfrak{g}}_{k_\alpha}^{\pm}$ are the quantiles estimated with the functions $\mu_{\widehat{G}_k} \pm 2\sigma_{\widehat{G}_k}$, $\mu_{\widehat{G}_k}^{max}$ and $\mu_{\widehat{G}_k}^{min}$ are respectively the maximum and minimum values of $\mu_{\widehat{G}_k}$ evaluated on $\mathfrak{C}_q^{(i)}$ and ε_{g_k} a predefined threshold.

If Eq. (15) holds for any small values of ε_{g_k}, then the metamodels are deemed accurate enough to be trusted as surrogates of the true functions and one can proceed to the updating scheme. However, if this relationship does not hold, instances of enrichment should be performed so as to improve the accuracy of the quantile estimates.

For this strategy to make sense, the optimization algorithm should handle only one point per iteration. Otherwise the accuracy of the quantiles would have to be estimated on all the points generated in an iteration. This would make the strategy cumbersome. The condition of one point per iteration is generally well fulfilled by gradient-based approaches. However and globally speaking, local search algorithms are not adapted to the multimodal problems we are intending to solve. In this paper, we use the *covariance matrix adaptation - evolution strategy* (CMA-ES), more specifically the (1+1)-CMA-ES for constrained optimization proposed by Arnold and Hansen (2012). The overall strategy is summarized in the algorithm in Figure 1.

1. Intialize $\mathscr{D}$, $\boldsymbol{d}^{(0)}$
2. Enrich for contour approximation and update $\mathscr{D}$ - Eq.(9)
3. Start optimization, $i = 0$
4. Build MC population $\mathfrak{C}_q^{(i)}$ - Eq. (4)
5. Compute $\widehat{\mathfrak{g}}_\alpha$, $\widehat{\mathfrak{g}}_\alpha^+$, $\widehat{\mathfrak{g}}_\alpha^-$, $\boldsymbol{\mu}_{\widehat{G}}^{max}$ and $\boldsymbol{\mu}_{\widehat{G}}^{min}$
6. Check for accuracy - Eq. (15)
7. While not accurate enough, enrich for quantile estimation - Eqs. (4,8)
8. Do one iteration of CMA-ES, $i \leftarrow i+1$
9. While no convergence of CMA-ES, go to 4.

Figure 1: Methodology for adaptive Kriging based-RBDO.

4.1. A short introduction to (1+1)-CMA-ES for constrained optimization

In a nutshell, CMA-ES is an evolution strategy which relies on multivariate normal distributions to iteratively sample solutions in the descent direction of the objective function. Considering Eq. (14), $\boldsymbol{\nu}_i$, the so-called *mutation term*, is a realization of a zero-mean Gaussian process of auto-covariance $\boldsymbol{C}^{(i)}$. The entire strategy relies on an appropriate update of $\boldsymbol{C}^{(i)}$ so as to iteratively increase the probability of sampling offsprings which promise the largest fitness progress.

In a practical point of view, the updating scheme in the constrained (1+1)-CMA-ES reads:

$$\boldsymbol{d}^{(i+1)} = \boldsymbol{d}^{(i)} + \sigma^{(i)} \boldsymbol{A}^{(i)} \boldsymbol{z}_c \tag{16}$$

where $\boldsymbol{z}_c$ is a realization of a standard Gaussian random variable, $\sigma^{(i)}$ an optimal step size and $\boldsymbol{A}^{(i)}$ the Cholesky decomposition of $\boldsymbol{C}^{(i)}$, introduced as a mean to sample from $\mathscr{N}(\boldsymbol{0}, \boldsymbol{C}^{(i)})$.

The adaptation of the covariance is directly carried out on $\boldsymbol{A}$ to avoid the costly successive Cholesky decompositions. Besides, directions of unfeasible region in the vicinity of the parent are identified and variance of the distribution in these directions are appropriately decreased. Hence, this allows the constraint handling. Many heuristics are additionally introduced by the authors to fine-tune the efficiency of the algorithm. The reader is invited to refer to Arnold and Hansen (2012) for further details.

5. APPLICATIONS

In this section, we apply the methodology to two examples which share the following settings.

The anisotropic Matérn $5/2$ auto-correlation function is chosen for the Kriging models. They are built on the so-called *augmented space* as defined in Dubourg et al. (2011). This allows us to build a single Kriging model for all the nested reliability analyses (quantile estimation). In other words, we have $\boldsymbol{x} \in \prod_{j=1}^{s_d} \left[d_j^- - 5\sigma_j ; d_j^+ + 5\sigma_j \right]$ where d_j^- and d_j^+ are respectively the minimum and maximum admissible values of the j-th component of $\boldsymbol{d}$ and σ_j corresponds to its associated standard deviation. For the case where there is no random-

ness in $\boldsymbol{d}$, $\sigma_j = 0$ for any $j = \{1,\dots,s_d\}$. An $\mathscr{L}_2$-discrepancy optimized Latin Hypercube is used to sample in the hypercube. On the other hand, the environment variables are uniformly sampled in an hypersphere of radius r_0: $\mathbb{Z} = \{\boldsymbol{z} \in \mathbb{R}^{s_e} | \|\boldsymbol{z}\|_2 \leq r_0\}$ where $\|\bullet\|_2$ denotes the $\mathscr{L}_2$-norm in $\mathbb{R}^{s_e}$. For the following applications r_0 is set respectively to 5 and 3).

Considering CMA-ES, the initial step size $\sigma^{(0)}$ is set equal to $1/3$ of the length of the widest search direction in the design space. Besides, the threshold for quantile accuracy is set loose in the early iterations of CMA-ES for the sake of efficiency. This is because CMA-ES will likely be exploring in these iterations and it might not be necessary to have a very accurate estimate of the quantile when we are far away from the potential optimum. Gradually decreasing values of the threshold are therefore set in a simulated-annealing fashion with four levels.

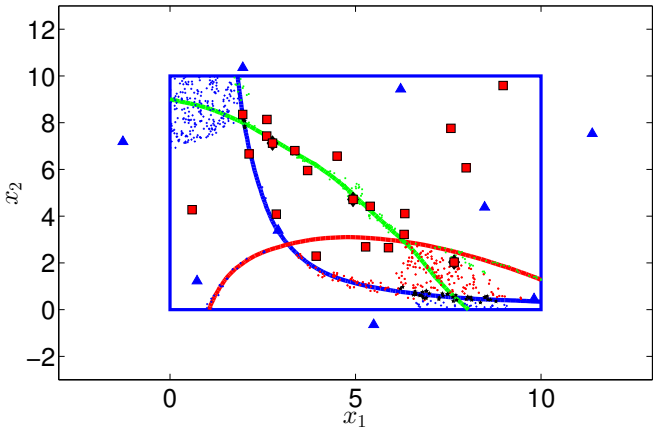
5.1. The modified Choi problem

Consider the modified Choi problem which writes (Lee and Jung (2008)):

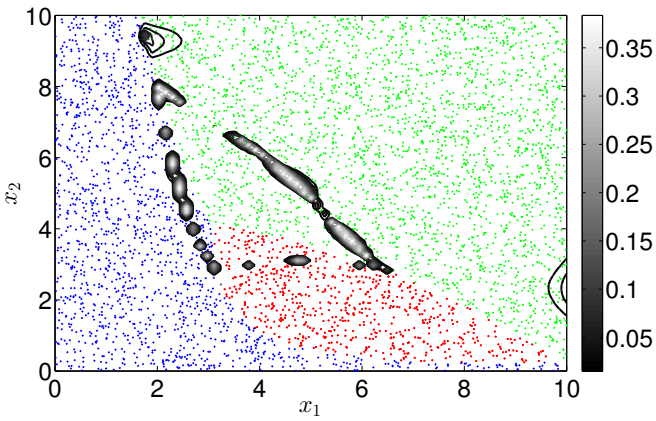
$$\boldsymbol{d}^* = \arg \min_{\boldsymbol{d}\in[0,10]^2} 10 - d_1 + d_2 \quad \text{s.t.}$$
$$\begin{cases} \mathfrak{g}_1(\boldsymbol{x}) = \frac{x_1^2 x_2}{20} - 1 \\ \mathfrak{g}_2(\boldsymbol{x}) = \frac{(x_1+x_2-5)^2}{30} + \frac{(x_1-x_2-12)^2}{120} - 1 \\ \mathfrak{g}_3(\boldsymbol{x}) = \frac{80}{(x_1^2+8x_2+5)} - 1 \end{cases} \tag{17}$$

To solve the RBDO problem, we introduce the random design variables $X_i \sim \mathscr{N}(d_i, 0.6^2)$ and a target probability of failure of 5.4% (corresponding to a reliability index of 2). The reference solution is considered as the one found with the analytical functions while using the quantile assessment for the reliability analysis and reads $\boldsymbol{d}^*_{ref} = \{5.7060, 3.4836\}$.

Let us first illustrate the methodology with some diagnostic plots. Figure 2 refers to the enrichment for contour approximation. The thick blue, red and green lines respectively represent the contours $\widehat{\mathfrak{g}}_{k\alpha} = 0, k = \{1,2,3\}$. In Figure 2a, the blue triangles are the initial DOE and the red squares the points added for enrichment. Additionally, the small marks show $\mathbb{M}_c$, each color corresponding to a specific constraint (blue '+', red '*' and green 'x' respectively for $\mathfrak{g}_{1\alpha}$, $\mathfrak{g}_{2\alpha}$ and $\mathfrak{g}_{3\alpha}$). One can see how this margin of uncertainty is reduced at the last iteration as the constraints are more accurately approximated. A non-negligible level of uncertainty remains in the design space and will be reduced in the optimization process only where necessary. Figure 2b shows contours of the learning function $\varphi(-U_c^s)$. The marks with different colors highlight which constraint is considered for the computation of U_c^s.



(a) Kriging models and added points



(b) Contour of the learning function U_c^s

Figure 2: Kriging models and enrichment for contour approximation.

After this enrichment for contour approximation, the optimization is performed with CMA-ES. Figure 3 shows the history of the sampled points during optimization. Points for which an enrichment has been done are circled in cyan with this rule: the larger the radius, the more points were added in

the corresponding $\mathfrak{C}_q^{(i)}$. The red triangles are those which do not improve the current best point during CMA-ES and the blue squares those which improve it but are not feasible. The green circles are the successive admissible improved solutions sampled during CMA-ES. The final solution is shown as a black diamond.

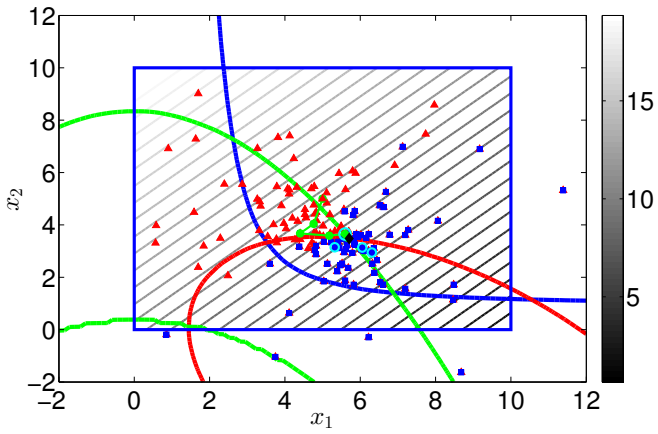


Figure 3: Illustration of CMA-ES with enrichment.

To assess the solutions accuracy, we repeat the optimization five times. The results are gathered in Table 1 where $MRE = |\boldsymbol{d}^* - \boldsymbol{d}^*_{ref}|/\boldsymbol{d}^*_{ref}$ is the mean relative error. Two cases are considered. The first one gives fairly accurate solutions with an average design of size $\bar{n} = 30.4$. The quantile accuracy criterion was set to 0.05 in this case. This threshold is decreased to 0.01 for the case #2. We now have a very accurate solution but at the cost of functions evaluations which average to 33.4. Note that, with the same number of points but without enrichment, the optimization does not even converge. This shows the great improvement brought by the adaptive procedure.

Table 1: Summary of the results for replicate optimizations.

Case	Case #1		Case #2	
$\bar{n}$	30.4		33.4	
$\boldsymbol{d}^*$	d_1	d_2	d_1	d_2
Mean	5.7249	3.4792	5.7047	3.4832
MRE	3.310^{-3}	1.310^{-3}	$2.1\,10^{-4}$	$1.3\,10^{-4}$

5.2. *The sidemember sub-system*

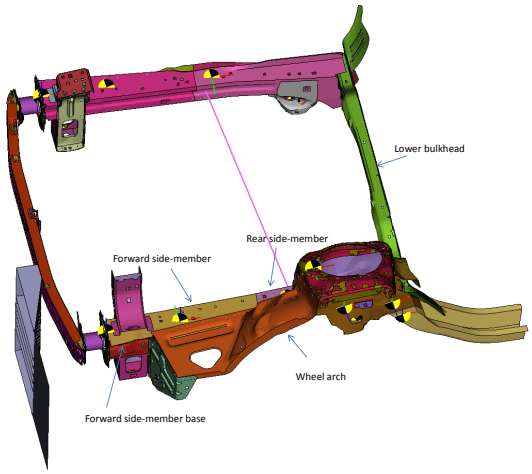


Figure 4: The sidemember subsystem

This application is about the lightweight design of a so-called *sidemember subsystem* under frontal impact. As its name suggests, it is a subsystem of an automotive front end. Globally speaking, it is a collection of parts which impact the crash behavior of the car. The associated finite element model is time-consuming and most importantly prone to noise. This noise is due to the high sensitivity of crash simulations to the initial conditions. To account for the uncertainties in the initial conditions, a probabilistic model is set up considering some information from crash certification procedures. As a result, the initial speed of the car V (in km/h) and the lateral position of the barrier P (in mm) are considered random with the following uniform and normal distributions:

$$\boldsymbol{Z} = \{V, P\} : V \sim \mathscr{U}(34, 35), P \sim \mathscr{N}(0, 2) \quad (18)$$

Thicknesses of five parts are considered as design variables. Their nominal values are $\boldsymbol{d} = \{1.95, 1.95, 2.44, 1.97, 0.87\}$ corresponding to a weight of 9.67 kg. Two constraints are considered, namely the maximum wall force (y_1) and sidemember compression (y_2). Their maximum admissible values are respectively 170 kN and 525 mm. It should be stressed as this point that all numerical values in this application are different from those of an entire car model.

To solve this problem, we start with an initial design of 64 points in the 7-dimensional augmented space. The first stage of enrichment results in 78 additional points (8 iterations of 10 points each among which two failed). It allows us to globally reduce the margin of uncertainty of the models with respect to the probabilistic constraints. The optimization converged in 650 iterations, among which only 14 were selected for enrichment summing up to 48 more points (3 to 5 points per enrichment according to whether the sampled point improves the current best solution or not, some simulations failed). The final size of the DOE is then 190. The overall procedure leads to an optimum estimated at $\boldsymbol{d}^* = \{2.1601, 1.6991, 2.0278, 1.5981, 0.6002\}$ corresponding to a weight of 8.48 kg (12.30% of weight saving).

To validate this optimum, we simulate the associated probabilistic constraint with the true finite element model. 200 points are generated for the quantile estimation. Results are gathered in the Table 2 below. They show that the found optimum is actually feasible as expected. As a comparison, a previous work without adaptive design resulted in an optimum which was not feasible with respect to the true model, despite a DOE of size 285.

Table 2: Results of simulation with respect to the true finite element model. Computed on 200 *points with* 500 *bootstrap replicates.*

Model	$\widehat{\mathfrak{g}}_{1\alpha}$	$\mathfrak{g}_{1\alpha}$	$\widehat{\mathfrak{g}}_{2\alpha}$	$\mathfrak{g}_{2\alpha}$
Bootstrap mean	156.3	163.1	513.6	518.4

6. CONCLUSION

The proposed methodology for adaptive Kriging RBDO provides an updating procedure which is first global then local. The first stage simply seeks to reduce the overall Kriging epistemic uncertainty. The second one, embedded in an optimization scheme, focuses on improving the probabilistic constraint evaluation expressed in the nested reliability problem. A special care is given to efficiency by saving a higher computational budget to iterations where (1+1)-CMA-ES is exploiting (in opposition to early iterations of space exploration). The two applications have shown improvement over a traditional approach both in terms of solutions feasibility and number of model evaluations. In an industrial context however, this splitting of the enrichment into many iterations might be an issue in situations where the overall project lead time matters.

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