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Approximating Sensitivity of Failure Probability in Reliability-Based Design Optimization

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ABSTRACT: This paper presents an efficient numerical method for approximating the parameter sensitivity of the failure probability with respect to design parameters. The method is computationally inexpensive and the obtained approximations are more accurate than the approximations based on first order reliability method (FORM). The method is particularly suitable for applications in reliability-based design optimization (RBDO), including reliability-based topology optimization (RBTO).

1. INTRODUCTION

Reliability analysis has been considered as a part of the standard design procedure to help engineers make safe and reliable designs subject to the inevitable randomness and uncertainties in nature. On the other hand, design optimization automates the design process and guides the decision makers to find efficient use of resources and maximize performance. Therefore, incorporating reliability-based design within the framework of design optimization becomes a natural goal for engineers and researchers. Reliability-based design optimization (RBDO) is often formulated as optimization with probabilistic constraint(s). There are a variety of approaches for solving such kind of problems in the literature using either gradient-free or gradient-based algorithms. For problems that can be described with differentiable functions, gradient-based routine is preferable due to its computational efficiency.

The challenge of using gradient-based optimization algorithm for RBDO is the evaluation of failure probability and its sensitivity with respect to design parameters, which affects the limit state function(s). Although numerical estimation of failure probability has been studied extensively, the literature on the numerical approximation of the parameter sensitivity is limited. Some analytical work can be found around the year of 1990. Hohenbichler and Rackwitz (1986) developed the parameter sensitivity of the estimated failure probability obtained by first order reliability method (FORM). The FORM-based expression is computationally efficient to evaluate, thus it is widely used in RBDO, but we should be careful when using this expression since the approximation can be quite inaccurate for cases where the limit state function is nonlinear. Breitung (1991) and Uryasev (1994) derived the analytical expression for the parameter sensitivity of the exact failure probability. However, such expression is in integral form and therefore, precise numerical evaluation of the integral is not likely to be computationally tractable. Approximations of the sensitivity is needed to perform RBDO.

There are two major formulations for RBDO in the literature: the Reliability Index Approach (RIA), which explicitly uses the gradient of probabilistic constraint(s) in the optimization; and the Performance Measure Approach (PMA), which constructs target performance function(s) as equivalent deterministic constraint(s) by the inverse reliability analysis, thus the gradient of probabilistic constraint(s) is involved implicitly (Tu et al. 2001; Cheng et al. 2006). Currently, the two approaches are mostly implemented in conjunction with FORM and FORM-based expression for the sensitivity of failure probability. Although FORM-based approximations for the sensitivity of the failure probability are not directly used in the PMA, the PMA essentially shares the same approximations of the sensitivity with RIA, if the probabilistic constraint(s) is(are) active when the optimization converges (Tu et al. 2001). Another type of gradient-based approach employs the sample average approximation (SAA) where the failure probability and its sensitivity calculation are both carried out based on Monte Carlo simulations (MCS) (Royset and Polak 2004). Although MCS-based can yield accurate estimations, they have very high computational costs.

The main purpose of this work is to provide an alternative method that is more accurate than a FORM-based approximation and require significantly less computational cost than MCS-based methods. The proposed method called Segmental Multi-point Linearization (SML) is developed to estimate the sensitivity of the failure probability with respect to design parameters in component reliability analysis. The method can be directly employed in the framework of RIA, enabling gradient-based algorithms to be used in RBDO.

2. FORM-BASED APPROACHES FOR RBDO, REVISITED

In this section, the two most used approaches of RBDO, namely FORM-based RIA and PMA are revisited via the Karush-Kuhn-Tucker (KKT) optimality conditions, in order to demonstrate the importance of the accuracy of sensitivity approximation. Consider a generic formulation of RBDO problems with one reliability component:

$$\begin{aligned} &\min_{\mathbf{x}} f(\mathbf{x}) \\ s.t. \quad &P_f = \int_{G(\mathbf{u},\mathbf{x})<0} \varphi_n(\mathbf{u}) \mathrm{d}\mathbf{u} \leqslant P_f^t \\ &\mathbf{h}(\mathbf{x}) \leqslant \mathbf{0} \end{aligned} \tag{1}$$

where P_f^t is the target failure probability; $\mathbf{x}$ is the vector of design variables; $\mathbf{u}$ is the vector of random variables that are transformed from the original distribution space to standard normal space by a probability preserving transformation; $G(\mathbf{u},\mathbf{x})$ is the limit state function in standard normal random space; $\varphi_n(\cdot)$ is the multi-variate standard normal PDF for n random variables; and $\mathbf{h}(\mathbf{x})$ is a set of deterministic constraints such as lower and upper bounds of $\mathbf{x}$. Equivalently, the constraint on failure probability can be expressed in terms of the generalized reliability index $\beta = \Phi^{-1}(1-P_f)$.

Mathematically, the KKT optimality conditions of the optimization model as described in (1) would become:

(1) Stationary condition:
$\nabla_{\mathbf{x}} f + \lambda \nabla_{\mathbf{x}} P_f + \sum \gamma_i \nabla_{\mathbf{x}} h_i = 0$

(2) Primal feasibility:
$P_f - P_f^t \leqslant 0,\ h_i \leqslant 0\ \forall i$

(3) Dual feasibility:
$\lambda \geqslant 0,\ \gamma_i \geqslant 0\ \forall i$

(4) Complementary slackness:
$\lambda (P_f - P_f^t) = 0,\ \gamma_i h_i = 0\ \forall i$

where λ and γ_i's are the Lagrange multipliers. The KKT conditions are necessary for the solution to be optimal. In RBDO, for most cases both the value and sensitivity of the probabilistic constraint can not be evaluated exactly, thus the KKT conditions are only approximately satisfied at the optimum of a numerical solution. The more accurate the approximations are, the closer the solution is to the real optimum.

In the RIA formulation, the reliability constraint is considered directly. The sensitivity of failure probability with respect to design parameters is used to find the search direction in optimization.

The FORM-based expression by Hohenbichler and Rackwitz (1986) is given as:

$$\nabla_{\mathbf{x}} P_f \approx \nabla_{\mathbf{x}} P_{f,1} = -\frac{\varphi(\beta_1)}{\|\nabla_{\mathbf{u}} G^*\|} \nabla_{\mathbf{x}} G^* \tag{2}$$

where G^* denotes the limit state function evaluated at the design point $\mathbf{u}^*$ as defined in Eq. (3), $G(\mathbf{u}^*, \mathbf{x})$, with current design $\mathbf{x}$.

$$\mathbf{u}^* = \underset{\mathbf{u}}{\operatorname{argmin}} \{\|\mathbf{u}\| \mid G(\mathbf{u}, \mathbf{x}) = 0\} \tag{3}$$

Thus the KKT stationary condition becomes:

$$\nabla_{\mathbf{x}} f + \lambda^{RIA} \left[-\frac{\varphi(\beta)}{\|\nabla_{\mathbf{u}} G^*\|} \nabla_{\mathbf{x}} G^* \right] + \sum \gamma_i \nabla_{\mathbf{x}} h_i = 0 \tag{4}$$

This expression is an approximation of the KKT stationary condition, the error of which can be quite large when the limit state function G is nonlinear.

The FORM-based PMA formulation applies an inverse FORM reliability analysis. The approach defines a target performance function $G^t(\mathbf{x}) = G(\mathbf{x}, \mathbf{u}^t)$ and incorporates it as a deterministic constraint, where $\mathbf{u}^t$ is an estimation of the design point of the optimal design and is updated at each iteration as

$$\mathbf{u}^t = \underset{\mathbf{u}}{\operatorname{argmin}} \{G(\mathbf{u}, \mathbf{x}) \mid \|\mathbf{u}\| = \beta^t = \Phi^{-1}(1 - P_f^t)\} \tag{5}$$

The KKT stationary condition of the PMA is:

$$\nabla_{\mathbf{x}} f + \lambda^{PMA}(-\nabla_{\mathbf{x}} G^t) + \sum \gamma_i \nabla_{\mathbf{x}} h_i = 0 \tag{6}$$

Equations (4) and (6) become the same if the probabilistic constraint is active and the design point is unique, that is, $\mathbf{u}^t = \mathbf{u}^*$ and $\nabla_{\mathbf{x}} G^t = \nabla_{\mathbf{x}} G^*$ (Tu et al. 2001). Given that the probability of failure are both approximated by FORM, the KKT conditions of RIA and PMA become identical. Hence, although PMA tends to be more robust than RIA, it does not improve the numerical result of the optimization.

Many algorithms, which are developed based on RIA and PMA incorporate SORM, MCS or other reliability methods to heuristically improve the approximation of P_f (i.e. the primal feasibility condition) (Royset et al. 2006; Nguyen et al. 2011), but little attention has been paid to the accuracy of the sensitivity which can be more influential in the search of the optimal solution. Furthermore, the error in the sensitivity is cumulative because it determines the search direction at each iteration of gradient-based optimization schemes.

3. THE METHOD OF SEGMENTAL MULTI-POINT LINEARIZATION

It can be found in Breitung (1991) and Uryasev (1994) that the analytical expression of the sensitivity of failure probability with respect to design parameters has the following form:

$$\nabla_{\mathbf{x}} P_f = -\int_S \frac{\varphi_n(\mathbf{u})}{\|\nabla_{\mathbf{u}} G\|} \nabla_{\mathbf{x}} G \mathrm{d}S \tag{7}$$

where S denotes the limit state surface. In most cases, the surface integral in Eq. (7) is not numerically tractable to compute since it is a multi-dimensional surface integral. *We propose an efficient numerical method for approximation of this surface integral. The method is developed based on local linearization of the limit state surface.*

If the surface S is composed of a set of pieces of hyperplanes described by linear functions, Eq. (7) can be analytically simplified to probability evaluation problems. The idea of the proposed method is to fit the limit state surface with plane segments in a piecewise manner. Then we can perform the integration on each hyperplane segment without much effort and compute their summation as the approximation. Denote the limit state surface as S and the piecewise linear fitting as $\bar{S}$ where each of the plane segment is denoted as $\bar{S}_i$. For each segment $\bar{S}_i$, which is assumed to be described by a linear function $\bar{G}_i$, the gradients $\nabla_{\mathbf{u}} \bar{G}_i$ and $\nabla_{\mathbf{x}} \bar{G}_i$ will be constant vectors on the corresponding surface. Thus, they can be taken out of the integral, which leads to the expression:

$$\nabla_{\mathbf{x}} P_{f,i} = -\frac{1}{\|\nabla_{\mathbf{u}} \bar{G}_i\|} \nabla_{\mathbf{x}} \bar{G}_i \int_{\bar{S}_i} \varphi_n(\mathbf{u}) \mathrm{d}\bar{S}_i \tag{8}$$

Because all fitting segments are pieces of hyperplanes, each hyperplane segment then has piecewise linear boundaries, thus its geometry appears as polygons. To further simplify Eq. (8), we can then rotate the coordinates of the standard normal space such that the positive direction of the first axis is along the opposite direction of the normal direction of the plane. As the function of the hyperplane is linear, the normal direction of the hyperplane is in the same direction of $\nabla_{\mathbf{u}}\bar{G}$ as shown in Fig. 1. Due to the rotational symmetry of the standard normal space, we can rewrite Eq. (8) by separating coordinate u'_1 from the integral, i.e.:

$$\nabla_{\mathbf{x}} P_{f,i} = -\frac{\varphi(b_i)}{\|\nabla_{\mathbf{u}}\bar{G}_i\|}\nabla_{\mathbf{x}}\bar{G}_i \int_{\bar{S}_i} \varphi_{n-1}(\hat{\mathbf{u}}')\mathrm{d}\hat{\mathbf{u}}' \qquad (9)$$

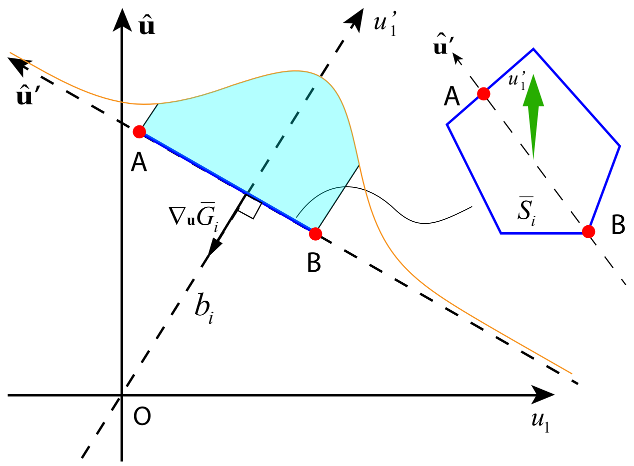


Figure 1: Illustration of the calculation of Eq. (7) on a hyperplane segment.

where b_i is the distance from the origin to the plane, and $\hat{\mathbf{u}} = [u'_2, u'_3, \ldots, u'_n]^{\mathrm{T}}$. The surface integral now is simplified to the volume integral of probability. Assuming we have a proper piecewise linear fitting of the limit state surface where each piece of plane segment is representative of a portion of the limit state surface, we should be able to construct an approximation of $\nabla_{\mathbf{x}} P_f$ based on Eq. (9) that has the following form of weighted sum:

$$\nabla_{\mathbf{x}} P_f \approx \sum_{i=1}^{p} W_i \nabla_{\mathbf{x}}\bar{G}_i \qquad (10)$$

where

$$W_i = -\frac{\varphi(b_i)}{\|\nabla_{\mathbf{u}}\bar{G}_i\|}\int_{\bar{S}_i} \varphi_{n-1}(\hat{\mathbf{u}}')\mathrm{d}\hat{\mathbf{u}}'. \qquad (11)$$

In particular, if we use only one hyperplane to fit the limit state surface which is defined by taking a tangent expansion at the design point, then from Eqs. (10) and (9) we can obtain Eq. (2), indicating that FORM-based approximation is a special case of the proposed method. It is important to notice that, due to the exponential decay of the probability density in the standard normal space, we only need to focus on the region that is close to the origin where the probability density is high. According to Eq. (7), the integrand becomes too small to make an impact on the overall integration when it is evaluated far from the origin. In addition, as we increase the number of fitting plane segments, the accuracy of the approximation can be improved.

A proper fitting scheme is essential to the approximation. In general, each plane segment can be completely defined by a fitting point, the normal direction of the plane, and the boundary of the segment. A straightforward thought is to let $\bar{G}_i$ be the first order expansion of the limit state function G at the corresponding fitting point $\mathbf{u}_i$, which ensures first order accuracy. Therefore, $\nabla_{\mathbf{x}}\bar{G}_i = \nabla_{\mathbf{x}} G(\mathbf{u}_i, \mathbf{x})$ and $\nabla_{\mathbf{u}}\bar{G}_i = \nabla_{\mathbf{u}} G(\mathbf{u}_i, \mathbf{x})$, where $\mathbf{x}$ is the current design. This leads to a tangent fitting scheme. If the locally most central points are selected to be the fitting points, the tangent fitting has the same approximation of the limit state surface as the multi-point FORM (Ditlevsen and Madsen 2007), but here the approximation of the limit state surface is primarily used to calculate the sensitivity of failure probability rather than the failure probability itself.

However, in high dimensional random space, the tangent fitting scheme makes it quite difficult to track the boundaries of the planes segments. Moreover, it can be affected a lot by possible perturbed local information of the limit state function. To overcome these challenges, we can project the gradient of the limit state function at a fitting point in the random space to a prescribed direction and define the hyperplane segment by the projected gradient. Since the design variables $\mathbf{x}$ are not in the random space, the value of $\nabla_{\mathbf{x}}\bar{G}_i$ will remain as the first order expansion of $G(\mathbf{u},\mathbf{x})$. That is, we take $\nabla_{\mathbf{u}}\bar{G}_i = (\mathbf{n}^{\mathrm{T}}\nabla_{\mathbf{u}}G(\mathbf{u}_i,\mathbf{x}))\mathbf{n}$ where $\mathbf{n}$ is the prescribed normal direction for the plane segment, and still keep $\nabla_{\mathbf{x}}\bar{G}_i = \nabla_{\mathbf{x}}G(\mathbf{u}_i,\mathbf{x})$. Hence, this compromise would only affect the computation of the weights. However, if the angle between the gradient vector $\nabla_{\mathbf{u}}G(\mathbf{u}_i,\mathbf{x}))$ and normal direction $\mathbf{n}$ is large, the computed weight can be erroneous. For example, in an extreme case, if the angle becomes 90°, the corresponding weight will be infinitely large. There are many alternative ways to specify the normals, and different choice of the normals leads to different fitting schemes.

An orthogonal fitting (OF) scheme is developed for a simple and quick construct of the approximation. The basic idea is to fit the limit state surface with plane segments that have normals along an orthogonal basis of the space. The general procedure is described as follows:

(1) Select a reference point on the limit state surface;

(2) Rotate the coordinates such that the reference point lies on the positive part of the first axis of the new coordinates;

(3) Search for intersection points of the new axes and the limit state surface within radius $r = k_1 b_1$ from the origin in both positive and negative directions, where k_1 is a user defined parameter and b_1 is distance from the reference point to the origin;

(4) Define plane segment i by its fitting point with the normal $\mathbf{n}_i$ being the direction of the axis on which the fitting point lies;

(5) For the half axis $\pm\mathbf{e}'_j$, that has no intersection point within the search region, a plane segment with normal along $\mathbf{e}'_1$ direction is fitted at the off-axis point $\mathbf{u}_j$ (denote as $\mathbf{u}_{j+n}$ for $-\mathbf{e}'_j$) with coordinate $\pm k_2 b_1 \mathbf{e}'_j + b_j \mathbf{e}'_1$.

The values for k_1 and k_2 are based on heuristic rules. The parameter k_1 determines the size of the search region for the intersection point, and it is suggested to have the value such that

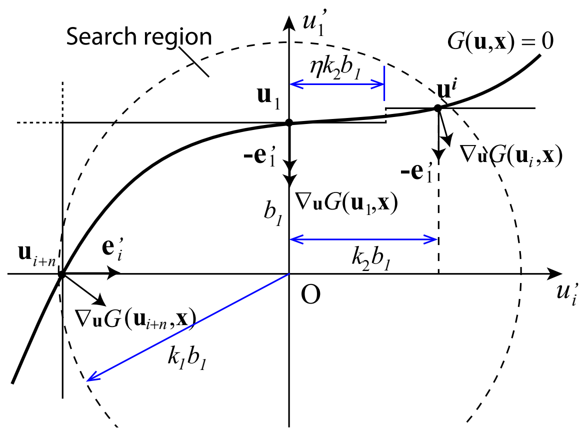


Figure 2: Illustration of orthogonal fitting SML.

$\varphi(k_1 b_1)/\varphi(b_1) = \varepsilon$ where ε is a small value (e.g. $\varepsilon = 0.1$) to ensure that the search region is large enough while the intersection fitting points are also close to the origin. On the other hand, k_2 is to ensure that the off-axis fitting points stay not too far from the origin. It is generally good to set k_2 be the minimum of 1 and $3/b_1$, which is the same rule as in the point-fitting SORM proposed by Der Kiureghian et al. (1987) except that the reference point is not necessarily the design point. In addition, as shown in Fig. 2, a partition coefficient η is used to determine the boundaries of the plane segments determined by the reference point and off-axis fitting points. In practice, η can take the value between 0.5 and 1.0.

The choice of the reference point also influences the accuracy of the SML approximation obtained by this fitting scheme. Intuitively, the reference point should be close to the origin. The design point is a good candidate for the reference point, however, in some particular cases, other choices of the reference point would actually make the approximation more accurate than using the design point as the reference point.

In addition, it is worth noticing that the method also provides an approximation of P_f associatively. Based on the existing approximation of the limit state surface, computing the approximate failure probability is a very light task. The output is observed to be generally better than FORM. Thus, in an implementation of RBDO, one can get the approximation of the gradient of failure probability and the failure probability itself using the segmental multi-point linearization method. One may also couple other reliability methods, for example SORM, point-fitting SORM and MCS with the SML method for approximation of the failure probability, and only use SML for approximating the gradient of failure probability.

4. NUMERICAL EXAMPLES

4.1. Preliminary Investigation

This example is to show the accuracy of the estimation obtained by the proposed method. We consider a typical quadratic limit state function defined with original random variables $\mathbf{v}$:

$$g(\mathbf{v},\mathbf{x}) = x_3 - v_3 - x_2 v_2^2 - x_1 v_1^2 \qquad (12)$$

where x_1, x_2 and x_3 are the design parameters. We assume that the random variables have the same marginal standard normal distribution. In addition, v_2 and v_3 are correlated with a correlation coefficient of 0.2 while the other random variables are taken to be statistically independent. We will compare the estimations of sensitivity by (1) the proposed method, (2) FORM-based expression (i.e. Eq. 2) and (3) the MCS-based approximation proposed by Royset and Polak (2004). We use the maximum number of simulations as suggested in Royset and Polak (2004), which equals 25000. We can expect that the MCS-based approximations are very close to the actual values. In addition, the corresponding approximations of the reliability index by the three methods are compared as well. The design point is selected to be the reference point in the SML method using the suggested OF scheme.

Because the sensitivity of failure probability with respect to design parameters is essentially a gradient vector, the vector direction and vector length (L2-norm) of the approximate sensitivity are the two factors that determine its accuracy. However, in most optimization problems, the direction of the gradient vector is of most interest, because the search direction at each iteration in an optimization process is determined by the direction of the gradient and most modern gradient-based optimization algorithms employ techniques to adaptively find the proper step length (e.g., line search) (Ascher and Greif 2011). We define an angle γ that measures the relative angle between the approximations of the gradient and the actual gradient. In the example, the MCS-based approximation is taken as the actual gradient, hence the angle γ for MCS-based method is assumed to be 0. The magnitude of the angle γ provides the information about the error in the vector direction. Thus, γ is small when the direction of an approximate gradient is accurate. Given a nonlinear limit state function g with $x_3 = 3.0$ and $x_2 = 0.15$, at different values of x_1, Fig. 3 shows the measures of γ for the approximations by FORM-based expression and the proposed method. We can observe that the SML method yield smaller error in vector direction of the approximate gradient than the commonly-used FORM-based approximation, i.e. Eq. (2). In addition, Fig. 4 shows the approximations of reliability index β by the three methods. Overall, although the accuracy of the approximations by the proposed method is not as high as MCS-based method, it is much better than traditional FORM/FORM-based approximation. In terms of the computational cost, the SML method is just a little bit larger than the evaluation of FORM/FORM-based method, but much less than MCS.

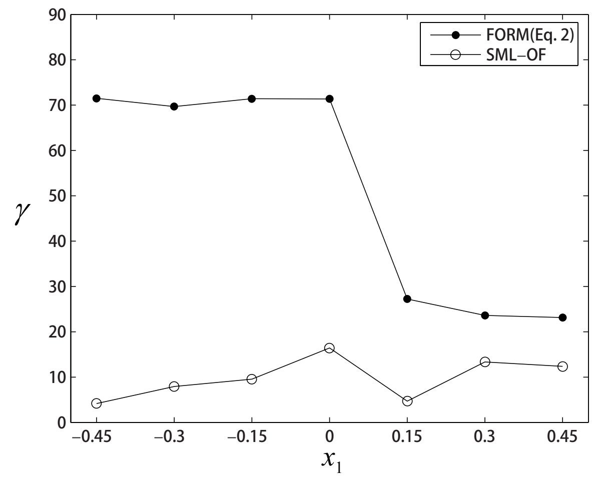


Figure 3: The angle γ between the approximate gradients and the actual gradient of $\nabla_{\mathbf{x}} P_f$.

4.2. Application to RBTO

Among various RBDO problems, the reliability-based topology optimization is one of the most challenging task. We conduct the RBTO on a

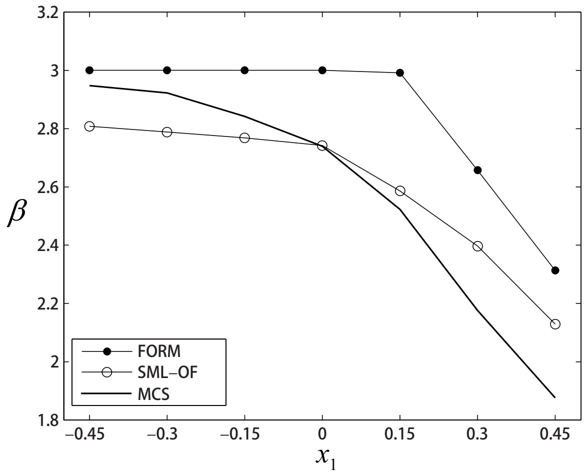


Figure 4: Approximations of reliability index β.

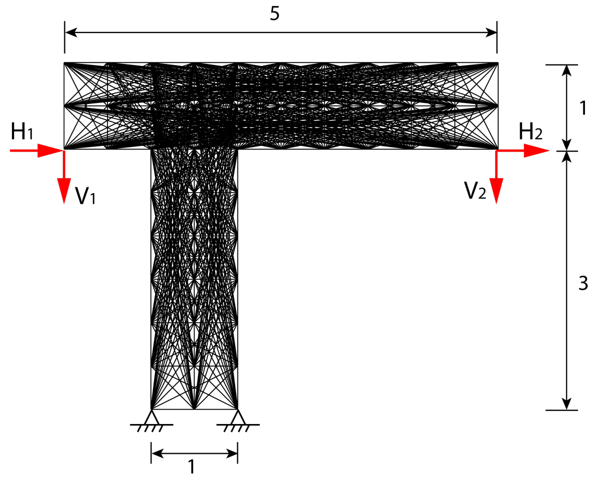


Figure 5: Design domain, initial ground structure and boundary conditions.

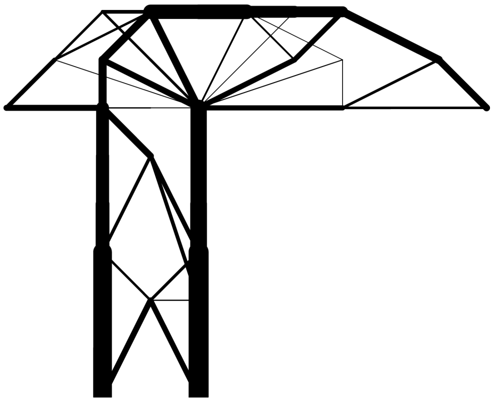
Table 1: Statistics of Random Variables

Variable	Distribution	μ	σ	C.C.
V_1	Normal	-5	1	0.2
V_2	Normal	-3	2	
H_1	Normal	0	3	0.3
H_2	Normal	0	3	
E	Lognormal	100	10	0.0

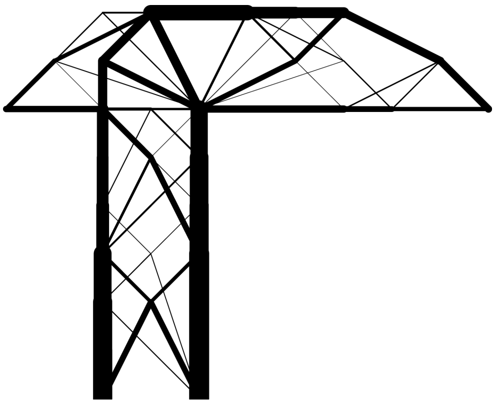
ground structure based elastic formulation where we are trying to find the structure with minimal volume by sizing a large number of potential members (Christensen and Klarbring 2008; Zegard and Paulino 2014). The final topology can be used to indicate the optimal structural layout. The limit state function appears as a threshold on the total compliance, which is reciprocal to the overall stiffness of the structure. Due to the nature of topology optimization, the structural layout indicated by the solution can be highly affected by the accuracy of sensitivity approximation. On the other hand, the evaluation of the gradient of compliance, which is necessary for the computation of the gradient of the failure probability, is quite computationally expensive since it requires FE analysis, thus MCS-based approximation is not applicable.

Consider a layout design of a crane in 2D. The design domain (ground structure) and boundary conditions of the problem is shown in Fig. 5. The statistics of the random variables are shown in Table. 1. The correlations coefficients (C.C.) between loads are included to make the design condition more practical. Uncertainty in material property (i.e. Young's Modulus) is also considered and modeled as a random variable with lognormal distribution since negative value of E is not physically admissible.

The example is performed using the approximations of P_f and its sensitivity by FORM-based method and the proposed SML method separately. Again, the design point is the reference point in SML. The target reliability index is 3.0 for both cases. The obtained topologies are shown in Fig. 6a and Fig. 6b. A simple cutoff strategy is adopted to obtain the members shown in the final topology (Zegard and Paulino 2014). The local stability and equilibrium in the plotted layouts may not be satisfied, which is inherent from this kind of topology optimization approach. A crude MCS is performed with c.o.v = 5% to check the actual reliability of the structure. We can observe that not only the optimal volumes and actual reliability indices are different for the two cases, but the structural layout is also different. The conclusion is that the RBTO based on the approximations by SML method yields a structural layout that possesses more efficient use of material. In topology optimization, the final topology usually highly depends on the values of sensitivities. Thus comparing to traditional FORM-based approach, the SML-based approach is more likely to produce result that is closer to the exact solution of a RBTO problem since the sensitivity approximation is more accurate.



(a)



(b)

Figure 6: Optimal topology by RBTO using (a) FORM-based method and (b) SML method. (a) Volume $= 650$, $\beta_{MCS} = 2.48$; *(b) Volume* $= 696$, $\beta_{MCS} = 2.91$. $Cutoff = 0.01$.

5. CONCLUSIONS

The proposed method can be used as a general tool for reliability analysis, and it is suitable for a variety of RBDO problems especially when the accuracy of the parameter sensitivity is essential for convergence to an optimal solution (e.g., RBTO). The benefit of low computational cost enables the method to be applied to practical engineering problems. The theory behind the method is general and other fitting schemes can be developed to better approximate the sensitivity of the failure probability with respect to design parameters.

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