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The Continuous Wavelet Transform as a Stochastic Process for Damage Detection

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ABSTRACT: This paper presents the formulation of a novel statistical model for the wavelet transform of the acceleration response of a structure based on Gaussian Process Theory. The model requires no prior knowledge of the structural properties and all the model parameters are learned directly from the measured data using Maximum Likelihood Estimation. The proposed model is applied to the data obtained from a series of shake table tests and the results are presented. The results, even at a proof-of-concept level, appear to correlate well with the ocurrence of damage, which is an indication of the validity of the underlying model. The results from the use of a simple metric for the detection of damage are presented as well.

1. Introduction

The application of Statistical Pattern Recognition (SPR) in the field of Structural Health Monitoring (SHM) has received significant attention by researchers over the past few decades, especially in the context of vibration analysis of structures. There has been considerable research in the application of various pattern recognition methods for damage detection (Farrar and Sohn (2000); Sohn et al. (2001); Sohn and Farrar (2001)) while a more formal presentation of the Statistical Pattern Recognition Paradigm can be found in Fugate et al. (2000) and Farrar and Worden (2007). In SPR, damage is detected through changes or outliers in statistical features that are obtained directly from the acquired data rather than by changes in estimates of structural properties. As a result, one of the advantages of SPR is that limited to no knowledge of the structural properties is required. This allows for tools and algorithms that are modular and eliminate the uncertainty around developing a structural model and estimating its parameters.

A mathematical model that is very widely used in SHM and especially under the Statistical Pattern Recognition Paradigm is the Continuous Wavelet Transform (CWT). Research on the application of the CWT for SHM includes the observation of changes in the wavelet coefficients under different loading conditions (Melhem and Kim (2003); Kim and Melhem (2004)), the extraction of features from the CWT (Sun and Tang (2002); Robertson et al. (2003); Noh et al. (2011)) and the combination of the CWT with other signal processing methods such as Empirical Mode Decomposition (EMD) (Li et al. (2007)). The literature on the application of wavelets in the field of SHM is so rich that has spurred the publication of several review papers. Comprehensive reviews on the intersection of the wavelet transform and SHM can be found in Peng and Chu (2004), Taha et al. (2006), or Staszewski and Robertson (2007).

This paper presents a novel statistical model of the wavelet coefficients at each time sample as a Gaussian Process (GP). The scope of the present paper is to present the mathematical formulation of the statistical model and provide proof-of-concept for the efficacy of the model for damage detection. In keeping with the Statistical Pattern Recognition Paradigm, all of the model parameters are estimated directly from the acquired data and no knowledge of the properties of the monitored structure is required, other than an undamaged baseline signal. Only the structural response is required in order to detect damage. The effect of the input excitation is accounted for through the model parameters.

The proposed model is applied to the data acquired from 13 sequential shake table tests on a reinforced concrete bridge pier conducted at the University of Nevada, Reno by Choi et al. (2007). This dataset provides an excellent testbed for the proposed model due to the large number of experimental runs, the progressive development of structural damage and the detailed documentation of the damage. An extensive presentation of the experimental setup, the testing protocol and the damage documentation can be found in Choi et al. (2007).

The present analysis is limited to Single Degree of Freedom (SDOF) systems under earthquake loading. The SDOF system was selected as a starting point due to its simplicity and the fact that it can still approximate well important types of structures such as bridge piers. However, the statistical model proposed is not necessarily limited to SDOF systems as the type of the structure does not affect the model, which only requires a response data stream as input.

The response of a structure to earthquake loading, as a non-stationary signal, is very well suited for wavelet analysis. The high intensity of the earthquake loading (compared to ambient vibration or wind loading) ensures that any potential occurrence of damage will be reflected in the recorded response while the relatively short duration of the earthquake isolates the effects of damage in the signal from environmental effects that can affect the response of the structure. As was the case with the type of structure, the proposed model need not be limited to this type of loading and can be used for other types of dynamic excitation.

2. Statistical Model Formulation

Let $a(t)$ be the acceleration response of the system, where t denotes time. The Continuous Wavelet Transform (CWT) of the signal $a(t)$ will be denoted as $Wa(u,s)$ and is defined as

$$Wa(u,s) = \int_{-\infty}^{\infty} a(t) \frac{1}{\sqrt{s}} \psi^* \left(\frac{t-u}{s} \right) dt \tag{1}$$

where u refers to shift (a measure of time), s refers to wavelet scale (a measure of frequency) and the $(\cdot)^*$ operator is the complex conjugate. Let $y_t(s)$ be the wavelet coefficients at shift t which will be referred to as wavelet "slice" at time t.

$$y_t(s) = Wa(t,s) \tag{2}$$

Let us define a random process of wavelet scale, $\Psi(s)$ that represents the fundamental shape of the wavelet slices and only depends on the damage state of the structure. The realizations of $\Psi(s)$. for each time t are denoted by $\Psi_t(s)$.

Assumption 1. The realizations of $\Psi(s)$ can be written as:

$$\Psi_t(s) = \overline{\Psi}(s) + \varepsilon_t(s) \tag{3}$$

where $\overline{\Psi}(s)$ is an unobservable function of wavelet scale that only depends on the damage state of the structure.

Assumption 2. The error terms $\varepsilon_t(s)$ are realizations of a zero-mean Gaussian Process (GP) with covariance function $k_\varepsilon(s,s')$.

While a different statistical model or distribution could potentially be used, the assumption of a Gaussian Process is made for simplicity and computational efficiency. Testing the validity of this assumption is part of the authors' current work and will be presented in a future publication.

Define the functional $\mathscr{F}$ for any function f such that:

$$\mathscr{F}(f;a,b,c)(s) = a \cdot f(b \cdot s + c) \tag{4}$$

Assumption 3. Each wavelet slice, $y_t(s)$ can be expressed as:

$$\begin{aligned} y_t(s) &= \mathscr{F}(\Psi; a_t, b_t, c_t)(s) + \Delta y(s) \\ &= a_t \Psi(b_t \cdot s + c_t) + \Delta y(s) \end{aligned} \tag{5}$$

where a_t, b_t and c_t are, generally unobservable, scalar parameters that represent the effect of the input motion on the system's response and $\Delta y(s)$ is a function of scale that represents effects not captured by the first term.

Null Hypothesis: while the structure is undamaged, the realizations of $\Psi(s)$ are drawn from the same distribution. When damage occurs, the behavior of the structure changes and, thus, the shape of $\overline{\Psi}(s)$ is assumed to change. Since $\Psi(s)$ is unobservable, that change in shape is manifested as a change in the distribution of $\Psi(s)$.

Combining Equations 3 and 5, we can obtain the following expression for the wavelet coefficients at time t:

$$\begin{aligned} y_t(s) &= a_t \overline{\Psi}(b_t \cdot s + c_t) + a_t \varepsilon_t(b_t \cdot s + c_t) + \Delta y(s) \\ &= \mathscr{F}\left(\overline{\Psi}; a_t, b_t, c_t\right)(s) + \mathscr{F}(\varepsilon_t; a_t, b_t, c_t)(s) \\ &\quad + \Delta y(s) \end{aligned} \tag{6}$$

Without loss of generality, we can define the slice at time t_0 as a reference slice. The reference slice will serve as a baseline for the model and, thus, would have to correspond to the undamaged state of the structure. In the present analysis, the reference slice is selected manually so that it is relatively smooth an its general shape is representative of the shape of the majority of the rest of the slices. Equation 5 can be written for the reference slice:

$$\begin{aligned} y_0(s) &= \mathscr{F}(\Psi; a_0, b_0, c_0)(s) + \Delta y(s) \\ &= a_0 \Psi(b_0 \cdot s + c_0) + \Delta y(s) \\ &= a_0 \overline{\Psi}(b_0 \cdot s + c_0) + a_0 \varepsilon_0(b_0 \cdot s + c_0) \\ &\quad + \Delta y(s) \end{aligned} \tag{7}$$

Solving Equation 6 for $\overline{\Psi}(s)$ and substituting in Equation 7, we obtain:

$$\begin{aligned} y_0(s) =& \frac{a_0}{a_t} y_t\left(\frac{b_0}{b_t}s + \frac{c_0 - c_t}{b_t}\right) \\ & - a_0 \varepsilon_t(b_0 s + c_0) - \frac{a_0}{a_t}\Delta y\left(\frac{b_0}{b_t}s + \frac{c_0 - c_t}{b_t}\right) \\ & + a_0 \varepsilon_0(b_0 s + c_0) + \Delta y(s) \end{aligned} \tag{8}$$

Define the following:

$$\tilde{a}_t = \frac{a_0}{a_t} \tag{9a}$$

$$\tilde{b}_t = \frac{b_0}{b_t} \tag{9b}$$

$$\tilde{c}_t = \frac{c_0 - c_t}{b_t} \tag{9c}$$

$$\tilde{\varepsilon}_t(s) = a_0 \varepsilon_0(b_0 s + c_0) - a_0 \varepsilon_t(b_0 s + c_0) \tag{9d}$$

$$\widetilde{\Delta y}(s) = \Delta y(s) - \tilde{a}_t \Delta y\left(\tilde{b}_t s + \tilde{c}_t\right) \tag{9e}$$

Equation 8 then becomes:

$$y_0(s) = \tilde{a}_t y_t\left(\tilde{b}_t s + \tilde{c}_t\right) + \tilde{\varepsilon}_t(s) + \widetilde{\Delta y}(s) \tag{10}$$

The scalar parameters $\tilde{a}_t$, $\tilde{b}_t$ and $\tilde{c}_t$ can be estimated from the data once the reference slice, $y_0(s)$ has been selected.

In order to obtain an estimate for the transformed error term, $\tilde{\varepsilon}_t(s)$, the wavelet slices can be transformed as follows:

$$y_t'(s) = \tilde{a}_t y_t\left(\tilde{b}_t s + \tilde{c}_t\right) \tag{11}$$

As a result, and since the transformed error term, $\tilde{\varepsilon}_t(s)$ is zero-mean, an estimate for the transformed, unmodeled effects term, $\widetilde{\Delta y}(s)$ can be obtained by:

$$\widehat{\Delta y}(s) = \sum_{t=1}^{t=N} y_0(s) - y_t'(s) \tag{12}$$

n estimate for the transformed noise terms, $\tilde{\varepsilon}_t(s)$ can be obtained as:

$$\hat{\varepsilon}_t(s) = y_0(s) - y_t'(s) - \widehat{\Delta y}(s) \tag{13}$$

In essence, this transformation "fits" each wavelet slice to the reference one. While the initial parameters, **a**, **b** and **c**, are not recovered, all the transformed slices refer to the same baseline (the reference slice) in terms of signal energy and bandwidth and, thus, the influence of the amplitude and frequency content of the input motion is removed. The transformed noise estimates, as well as the parameters calculated from Equation 9, can then be used to test whether the slices are indeed drawn from the same distribution. The case where not all slices are drawn from the undamaged distribution implies that damage has occurred in the structure.

3. Estimation of the model parameters

The parameters that need to be estimated are: $\tilde{\mathbf{a}}$, $\tilde{\mathbf{b}}$ and $\tilde{\mathbf{c}}$, the bias term $\widetilde{\Delta y}$ and the transformed noise covariance matrix, $\Sigma_{\tilde{\varepsilon}}$. The vectors $\tilde{\mathbf{a}}$, $\tilde{\mathbf{b}}$ and $\tilde{\mathbf{c}}$ have size $N \times 1$, where N is the number of time samples, the bias term is an $M \times 1$ vector and the transformed noise covariance matrix is an $M \times M$ matrix, where M is the number of scales at which the wavelet transform is calculated.

The estimation of the model parameters requires the estimation of $3 \cdot N + M + M \cdot (M+1)/2$ values. For a typical acceleration response record and a reasonable amount of wavelet scales, the simultaneous estimation of the parameters is computationally very expensive. For that reason, a recursive algorithm is used in the present analysis:

Step 1 Initialize Δy and $\Sigma_{\tilde{\varepsilon}}$ to an uninformed prior. In the present analysis, $\Delta y^{(0)} = 0$ and $\Sigma_{\tilde{\varepsilon}}^{(0)} = I$, where I is the $M \times M$ identity matrix.

Step 2 For each $t = 1 \ldots N$, calculate the parameters $\hat{a}_t^{(k+1)}$, $\hat{b}_t^{(k+1)}$, $\hat{c}_t^{(k+1)}$ using Maximum Likelihood Estimation.

Step 3 Calculate the transformed slices from Equation 11.

Step 4 Calculate the bias term from Equation 12.

Step 5 Calculate the error terms from Equation 13.

Step 6 Estimate the covariance matrix from the error terms.

Step 7 Repeat Steps 2 through 6 until the bias term and covariance matrix converge.

A more extensive presentation of this algorithm, presentation of alternative methods and discussion on the estimation of the model parameters will be provided in a future paper.

It should be noted that this algorithm need only be applied to a reference signal where the structure is a priori assumed to be undamaged. Once the covariance matrix and bias terms are estimated, they essentially describe the structure's undamaged behavior since, as mentioned previously, a change in the damage state of the structure will be reflected in these two parameters. Then, the learned covariance matrix and bias term can be directly applied to a signal where the structure's damage state is unknown.

The outlined algorithm requires the estimation and inversion of the covariance matrix of a generally high-dimensional random variable. The estimation of the covariance matrix is a well-studied problem and several parametric and non-parametric estimation methods exist in the literature (e.g. Anderson (1973); Andrews (1991); Chen et al. (2010)). The sample covariance matrix was found to be numerically unstable, especially after a few iterations of the presented algorithm and did not converge. To overcome this problem, a parametric covariance function was fit to the data and its parameters were estimated by maximizing the marginal log-likelihood of the data. In the results presented in subsequent sections, a Matern covariance function was used and the fitting was performed using the Gaussian Processes for Machine Learning (GPML) toolbox, which is based on Rasmussen and Williams (2006).

4. Experimental validation

4.1. Dataset description

In order to evaluate the validity of the proposed model and its underlying assumptions, it is applied to the experimental data obtained from a series of shake table tests conducted at the University of Nevada, Reno by Choi et al. (2007). This dataset, as mentioned in a previous section, presents an excellent benchmark for this model due to the large number of experimental runs and the progression of damage and documentation thereof. The specimen used was a 3 : 10 scale model of a reinforced concrete bridge pier. The acceleration response that was used in the analysis was measured at the top of the pier. The testing sequence consisted of 13 earthquake runs, each with increasing intensity. The first two runs are damage-free, damage first occurs during Run 3 and progresses with each subsequent run. A comprehensive presentation of the experimental set up and damage description can be found in Choi et al. (2007).

4.2. *Parameter distributions*

The transformed slices, residuals (error terms) and model parameters for the 13 experimental runs were calculated. The reference slice was selected manually from the first experimental run where the structure was known to be damage-free. Figure 1 shows heatmaps of the transformed residuals for six different experimental runs. For clarity, only Runs 1, 2, 3, 6, 9 and 12 are shown. As mentioned previously, the first two runs are undamaged, Run 3 marks the first occurrence of damage and Runs 4 through 13 exhibit progressively increasing damage. This is consistent with what can be visually observed in Figure 1. The residuals in the first two runs are approximately zero (the mean of the error term process) for almost all time samples while clusters of slices with significantly different shape appear from Run 3 onwards with increasing duration and intensity.

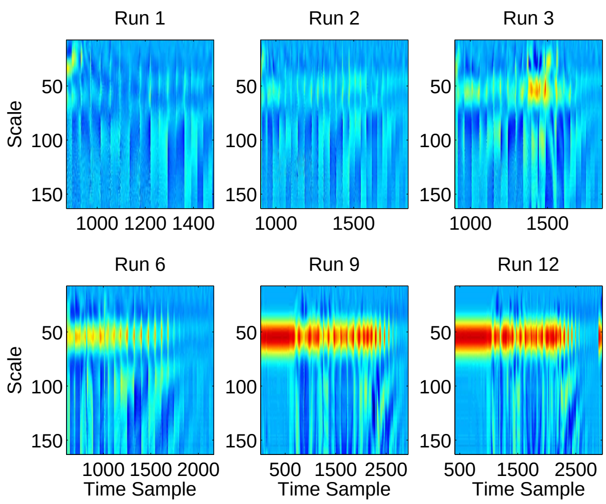

Figure 1: Transformed residuals for different experimental runs

Figures 2 through 4 show histograms of the amplitude, stretch and shift parameters, respectively, for the same experimental runs shown in Figure 1. It can be observed that the distributions of the parameters are generally stable in the two undamaged cases (Runs 1 and 2) and change when damage occurs in the structure, even at low levels of damage (Run 3). This shift in distribution for the model parameters clearly demonstrates a sensitivity of the proposed model to damage in the structure.

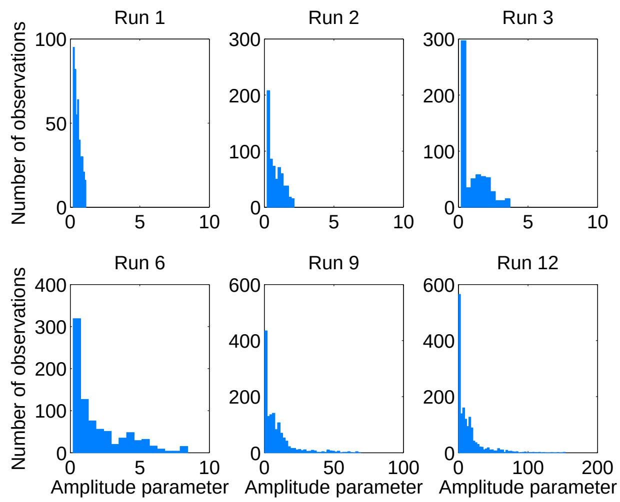

Figure 2: Histograms of the absolute value of the amplitude parameter

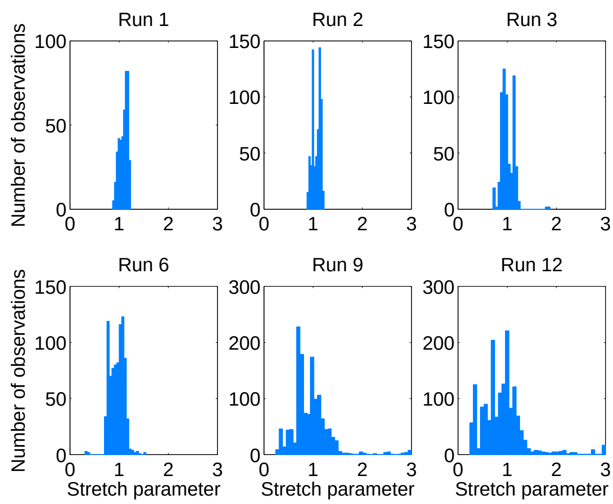

Figure 3: Histograms of the stretch parameter

5. Application on Damage Detection

Based on visual inspection of Figure 1, the occurrence of damage can be correlated with the presence of clusters of outlying slices. For that rea-

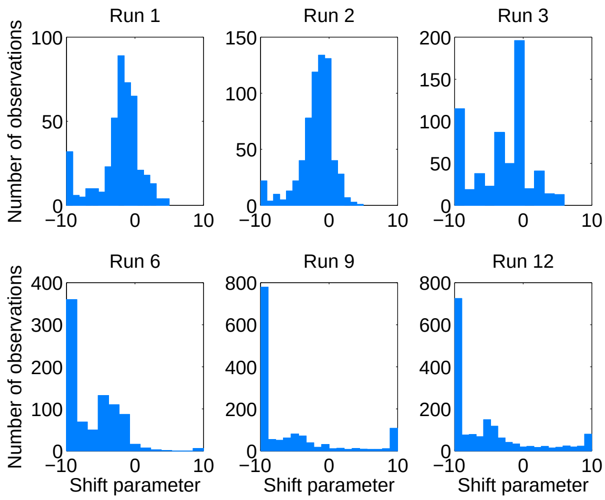

Figure 4: Histograms of the shift parameter

son, it is reasonable to investigate the temporal progression and statistical distribution of the Squared Mahalanobis Distance (SMD), which, for a random variable X, is defined as:

$$d_M^2 = (X - \mu_X)^T \Sigma_X^{-1} (X - \mu_X) \qquad (14)$$

and is a commonly used metric for the identification of multivariate outliers. Figure 5 shows time-series plots for the SMD during the experimental runs shown in previous plots, while Figure 6 shows the corresponding histograms of the SMD.

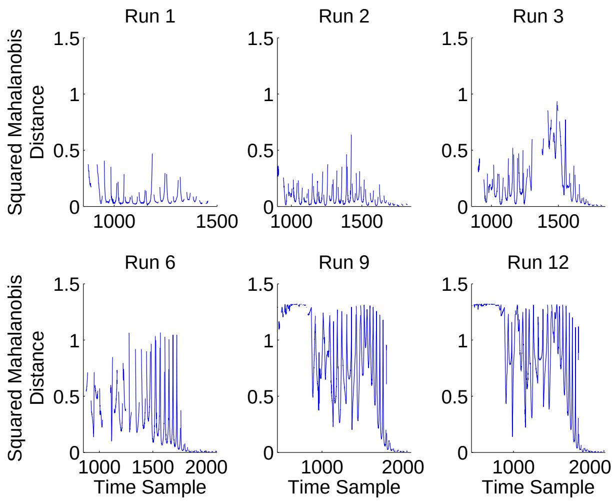

Figure 5: Time-series of the Squared Mahalanobis Distance

Visually, a differentiation between runs without damage (Runs 1 and 2) and runs with damage (Runs

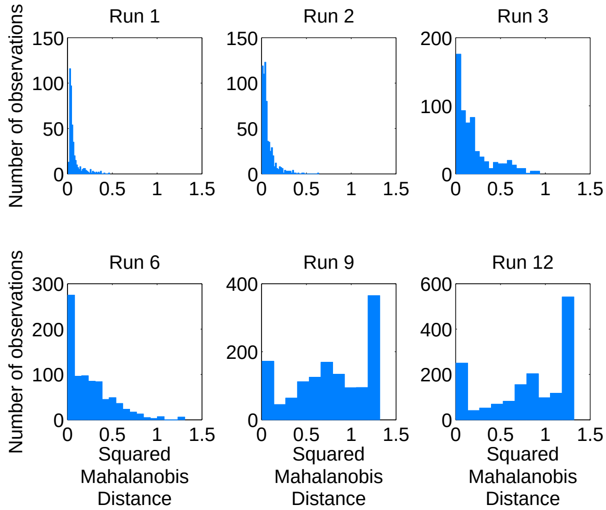

Figure 6: Histograms of the Squared Mahalanobis Distance

3 through 13) can be observed in both Figure 5 and 6. Despite the presence of a high-frequency component, the lower envelope of the plots in Figure 5 gives a good indication of the time intervals where outliers occur, which can be used for estimating the extent of damage.
While this is, by no means, a complete damage detection scheme, it presents an important validation of the underlying model and its assumptions. The SMD, while a useful metric, collapses a multivariate random variable into just one number and does not take into account additional information that the proposed model provides in the form of the amplitude, stretch and scale parameters. The information from the residuals and the model parameters will be combined to develop a robust damage detection algorithm. The details of this algorithm will be presented by the authors in a subsequent paper (Balafas et al. (2015)).

6. Conclusions

This paper presents the formulation of a novel statistical model for the wavelet transform of the acceleration response of a structure. In the proposed model, the wavelet coefficients at each moment in time are considered transformed realizations of a fundamental Gaussian Process that is only dependent upon the presence or not of damage in the structure. The model only considers the structural response; the input excitation is not required and its effects on the response signal are taken into account by the model using an amplitude parameter and two, stretch and shift, parameters in the wavelet scale domain. The model also requires no prior knowledge of the structural properties and all the model parameters are learned directly from the measured data. As such, it eliminates the need for complex Finite Element Models and the uncertainty involved with material or geometric properties of the structure. The parameters of the model are estimated using Maximum Likelihood Estimation in combination with a recursive algorithm due to their large number of values to be estimated. In calculating the model parameters, it is required to estimate the covariance matrix of a multivariate sample. In order to avoid a poorly conditioned matrix, the covariance matrix is estimated through training of a Gaussian Process. The proposed model is applied to the data obtained from a series of shake table tests and the results are presented. The distributions of the model parameters are stable while the structure remains undamaged, and shift with the occurrence and progression of damage. Furthermore, based on visual observation of the model residuals, there is potential for temporal localization of damage. This would provide information on the number and duration of time intervals where damage occurs, information that is potentially important for the assessment of the extent of damage. The results of the presented analysis, even at a proof-of-concept level, are very encouraging, which is an indication of the validity of the underlying model. This paper shows the results from the use of a simple metric for the detection of damage, also with encouraging results. The derivation of a damage detection algorithm that would make use of all the model parameters and learn the parameter distributions in order to detect and classify damage through hypothesis testing is part of the authors' work and will be presented in a future paper.

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