Identification Uncertainty of Close Modes in Operational Modal Analysis

Yi-Chen Zhu
Graduate Student, School of Engineering, University of Liverpool, Liverpool, UK

Siu-Kui Au
Professor, School of Engineering, University of Liverpool, Liverpool, UK

Steve Jones
Senior Lecturer, School of Engineering, University of Liverpool, Liverpool, UK

ABSTRACT: Operational modal analysis has attracted a lot of attention in both theory development and field applications for its high economy in implementation. It allows the modal properties (natural frequencies, damping ratios, mode shapes, etc.) to be identified based on 'output' vibration data only. In the absence of information about the input loading, the uncertainty associated with the identified modal parameters is a significant concern. Among the challenging situations encountered in practice, close modes (i.e., modes with similar frequencies) are significantly more difficult to identify than well-separated modes. The possible interaction of modes with similar frequencies complicates the identification model and in many cases reduces the identification precision, or even renders the situation unidentifiable. Using a Bayesian modal identification approach, this paper investigates the identification uncertainty of closely-spaced modes, which are identified using a multi-mode model with FFT (Fast Fourier Transform) data on the same frequency band. In this context, the identification uncertainty is investigated through the posterior covariance matrix, which can be computed for a given set of data. A series of numerical studies will be performed, where synthetic data in different specially designed situations are generated. Based on these data the modal properties in different situations are identified and their resulting posterior uncertainties are investigated. The effects of the proximity of close mode frequencies and mode shapes will be investigated. It is anticipated that this work will provide significant insights on the identification uncertainty in operational modal analysis for closely-spaced modes encountered in practice.

1. INTRODUCTION

Operational modal analysis aims at identifying the modal parameters (e.g. natural frequency, damping ratio and mode shapes) of the structure based on the measured response data (Ewins 2000; Hudson 1977). The ambient vibration test is gaining popularity as it can be economically conducted without knowing the actual loading but assumes the loading as statistically random (Brincker et al. 2001; Peeters and De Roeck 2001). However, the uncertainty of the identified modal parameters is significantly higher compared to forced or free vibration tests because of the absence of input loading information. The situation becomes even more complex when identifying closely-spaced modes (i.e. modes with similar frequencies), which are commonly encountered in tall buildings. It will be useful to know the identification accuracy beforehand based on the given test configurations and the prediction of the mode properties when identifying close modes. Compared to the identification uncertainty of well separated modes (Au 2014), it may be very difficult and tedious to develop the leading order behavior of the identification uncertainties (i.e. uncertainty law) for close modes mathematically at this stage. The objective of this paper is to assess the possible factors which have effects on the identification uncertainties of close modes based on numerical studies, where synthetic data under different situations is generated. A proximity index (PI) is first proposed and discussed in order to quantify the proximity of two modes. Using a Bayesian modal identification approach (Au 2012a; b; Yuen and Katafygiotis 2003), the influence of signal-to-noise ratio and the similarity of two mode shapes on the identification uncertainties of close modes are investigated.

2. BAYESIAN MODAL IDENTIFICATION THEORY

A fast Bayesian modal identification approach (Au 2012a; b; Yuen et al. 2002) is used in this paper to identify modal parameters and evaluate identification uncertainties. The method is briefly explained in this section. Let the measured acceleration data at n degrees of freedom (DOF) of a structure be $\{\hat{\ddot{\mathbf{x}}}_j \in R^n : j = 1,2...N\}$, where N is the number of samples per channel. It can be modelled as $\hat{\ddot{\mathbf{x}}}_j = \ddot{\mathbf{x}}_j(\boldsymbol{\theta}) + \boldsymbol{\varepsilon}_j$, which consists of the model response $\ddot{\mathbf{x}}_j(\boldsymbol{\theta})$ (depends on the set of modal parameters $\boldsymbol{\theta}$ to be identified) and prediction error $\boldsymbol{\varepsilon}_j$ (due to noise and modelling error). The scaled FFT of $\{\hat{\ddot{\mathbf{x}}}_j\}$ is defined as:

$$\mathcal{F}_k = \sqrt{\frac{2\Delta t}{N}} \sum_{j=1}^{N} \hat{\ddot{\mathbf{x}}}_j \exp\left[-2\pi\mathbf{i}\frac{(k-1)(j-1)}{N}\right] \quad (1)$$

where $\mathbf{i}^2 = -1$ and Δt is the sampling interval. Here, $\mathcal{F}_k$ corresponds to frequency $f_k = (k-1)/N\Delta t$ for k=1,..,N_q, where N_q=int[N/2]+1(int[.] denotes the integer part) is the index corresponding to the Nyquist frequency. The sample power spectral density (PSD) matrix can be calculated by multiplying the FFT by its conjugate transpose. The scaling factor is defined such that the PSD is one-sided with respect to frequency in Hz. In practice, only the $\mathcal{F}_k$ within the selected frequency band containing the modes of interested is used for identification. Let $\boldsymbol{\theta}$ denote the modal parameters to be identified:

$$\boldsymbol{\theta} = \{f, \zeta, S, S_e, \boldsymbol{\Phi}\} \quad (2)$$

where f and ζ are the natural frequency and damping ratio respectively; S and S_e are the PSD of modal excitation and prediction error respectively; $\boldsymbol{\Phi} = [\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, .., \boldsymbol{\varphi}_m] \in R^{n\times m}$ is the mode shape matrix with $\boldsymbol{\varphi}_i (i = 1,.., m)$ being the i-th mode shape confined to the measured DOFs. Let $\mathbf{Z}_k = [\mathrm{Re}\,\mathcal{F}_k; \mathrm{Im}\,\mathcal{F}_k] \in R^{2n}$, where $\mathrm{Re}\,\mathcal{F}_k$ and $\mathrm{Im}\,\mathcal{F}_k$ denote the real and imaginary part of $\mathcal{F}_k$. Let such collection be denoted by $\{\mathbf{Z}_k\}$. Using Bayes' theorem, the posterior PDF of $\boldsymbol{\theta}$ given the measured data is:

$$p(\boldsymbol{\theta}|\{\mathbf{Z}_k\}) = p(\{\mathbf{Z}_k\}|\boldsymbol{\theta}) \frac{p(\boldsymbol{\theta})}{p(\{\mathbf{Z}_k\})} \quad (3)$$

where $p(\boldsymbol{\theta})$ is the prior PDF that reflects one's knowledge about $\boldsymbol{\theta}$ in the absence of data; $p(\{\mathbf{Z}_k\})$ is a normalizing constant and it does not depend on $\boldsymbol{\theta}$. Assuming no prior information, the posterior PDF is directly proportional to the 'likelihood function' $p(\{\mathbf{Z}_k\}|\boldsymbol{\theta})$. For large N and small Δt , it can be shown that $\{\mathbf{Z}_k\}$ are asymptotically independent at different frequencies and $\mathbf{Z}_k$ is a zero mean Gaussian vector with covariance matrix given by:

$$\mathbf{C}_k = \frac{1}{2}\begin{bmatrix}\boldsymbol{\Phi} & \\ & \boldsymbol{\Phi}\end{bmatrix}\begin{bmatrix}\mathrm{Re}\,\mathbf{H}_k & -\mathrm{Im}\,\mathbf{H}_k \\ \mathrm{Im}\,\mathbf{H}_k & \mathrm{Re}\,\mathbf{H}_k\end{bmatrix}\begin{bmatrix}\boldsymbol{\Phi}^T & \\ & \boldsymbol{\Phi}^T\end{bmatrix} + \frac{S_e}{2}\mathbf{I}_{2n} \quad (4)$$

where $\mathbf{I}_{2n} \in R^{2n}$ denotes the identity matrix, $\mathbf{H}_k \in R^{m\times m}$ is the transfer matrix whose (i, j) entry given by:

$$\mathbf{H}_k(i, j) = S_{ij}[\beta_{ik}^2 - 1) + 2\mathbf{i}\zeta_i\beta_{ik}]^{-1} \times [\beta_{jk}^2 - 1) - 2\mathbf{i}\zeta_j\beta_{jk}]^{-1} \quad (5)$$

and $\beta_{ik} = f_i / f_k$; f_i is the natural frequency of the i-th mode; f_k is the frequency in the selected FFT frequency band. S_{ij} is the cross spectral density of the i-th and j-th modal excitation. The posterior PDF now becomes:

$$p(\boldsymbol{\theta}|\{\mathbf{Z}_k\}) \propto p(\{\mathbf{Z}_k\}|\boldsymbol{\theta}) = (2\pi)^{-nN_f} \times \prod_k (\det \mathbf{C}_k)^{-1/2} \exp\left[-\frac{1}{2}\mathbf{Z}_k^T \mathbf{C}_k^{-1} \mathbf{Z}_k\right] \quad (6)$$

For convenience in analysis and computation, the posterior PDF is expressed in terms of the negative log-likelihood function $L(\boldsymbol{\theta})$ so that:

$$p(\boldsymbol{\theta}|\{\mathbf{Z}_k\}) \propto \exp[-L(\boldsymbol{\theta})] \quad (7)$$

$$L(\boldsymbol{\theta}) = \frac{1}{2}\sum_k [\ln \det \mathbf{C}_k + \mathbf{Z}_k^T \mathbf{C}_k^{-1} \mathbf{Z}_k] \quad (8)$$

The MPV (most probable value) of the modal parameters are determined by maximizing $p(\boldsymbol{\theta}|\{\mathbf{Z}_k\})$ (which means minimizing $L(\boldsymbol{\theta})$). Comparing to non-Bayesian methods, one major advantage of the Bayesian identification method is that it can also provide a rigorous quantitative evaluation of the uncertainties of the identified modal parameters. The posterior covariance matrix can be approximated by the inverse of the Hessian of the $L(\boldsymbol{\theta})$ function. In this paper, the uncertainties of the identified modal parameters are discussed based on the 'coefficient of variation' (c.o.v.), which can be calculated as the ratio of the square root of variance to the MPV value. The variance is given by the corresponding diagonal element of the posterior covariance matrix.

3. SYNTHETIC DATA SIMULATION

Using the Bayesian approach explained in the previous section, the identification uncertainties of close modes under different conditions will be evaluated using synthetic data. The properties of the structure simulated will be explained. A proximity index will then be proposed in order to quantify the proximity of two close modes. The influence of the signal-to-noise (s/n) ratio and the modal assurance criterion (MAC) on the identification uncertainties will be discussed using the simulated data afterwards.

The synthetic data simulates the acceleration response of a 3DOF structure with two modes which have an average frequency of 1Hz and the same damping ratio of 1%. The mode shapes of these two modes are assumed to be $[1 \quad 2 \quad 3]^T / \sqrt{14}$ and $[1 \quad 3 \quad 1]^T / \sqrt{11}$ respectively. The structure is subjected to two i.i.d. (independent and identically distributed) white noise modal excitations with the same PSD of $10^{-8} g^2 / Hz$. The acceleration response of the structure is collected at a sampling rate of 100Hz for a duration of 1000s. The data is contaminated by i.i.d. channel noise with a PSD of $2.5 \times 10^{-8} g^2 / Hz$. Modal identification of these two modes is based on the selected frequency band of $[0.5 \quad 1.5] Hz$. Different situations are performed to investigate the effect of different attributes.

3.1. Proximity Index

To quantify how close two modes are, a proximity index (PI) is proposed and discussed in this section. A well-separated mode can be defined as one where there is a frequency band around its natural frequency without significant contribution from other modes. Conversely, if the resonance bands of two modes significantly overlap, then they are considered close. The dynamic amplification factor between the PSD of the acceleration response and the modal excitation is (Au 2011):

$$D_k = \left[(\beta_k^2 - 1)^2 + (2\zeta\beta_k)^2\right]^{-1} \quad (9)$$

where $\beta_k = f / f_k$. For modes with the same damping ratio, the shape of the modes at high frequencies will have wider spread over the frequency band. On the other hand, modes with small damping will have a sharper peak in the frequency domain compared to ones with large damping. For simplicity, considering two modes with the same damping ratio, a proximity index (PI) is proposed in this case as:

$$PI = \frac{\Delta f}{\bar{f} \times \zeta} \quad (10)$$

where Δf is the difference of the natural frequencies between two modes; $\bar{f}$ is the average frequency and ζ is the damping ratio. For well-separated modes, the frequency band for modal identification is commonly symmetrically centered about the resonance peak of the mode with bandwidth of $2\kappa\zeta f$. Here, κ is defined as the bandwidth factor. For example, the half-power band is $f(1\pm\zeta)$, κ in this case equals to 1. Typically, the band with $\kappa = 6$ may account for 90% of the contributions of one mode (Au 2014). In real application, the bandwidth factor is chosen depending on the trade-off between the amount of information used for identification and the modelling error included within the band. Considering two modes with the same damping ratio and the same κ used for frequency band selection, PI is equal to 2κ when these two frequency bands is connected with each other. This means two modes can be considered as close modes when their PI is smaller than 2κ given certain κ used for frequency band selection.

In this paper, the uncertainty of two modes with PI ranging from 0.2 to 10 will be assessed. An example using synthetic data is presented to demonstrate how close two modes are in frequency domain when PI equals 0.2 and 10.

Figure 1(a,b) shows that when the PI of two modes is 10, these two modes can be distinguished visually in the frequency domain but they already have a small overlapped area. In this case, it is possible to select the frequency bands separately for these two modes or select them in one frequency band and consider them as two close modes. This mainly depends on how the bandwidth factor κ is selected. Figure 1(c,d) shows that when PI is 0.2, the two modes are extremely close to each other and they can only be inferred from the singular value spectrum. This means they can only be selected in one frequency band.

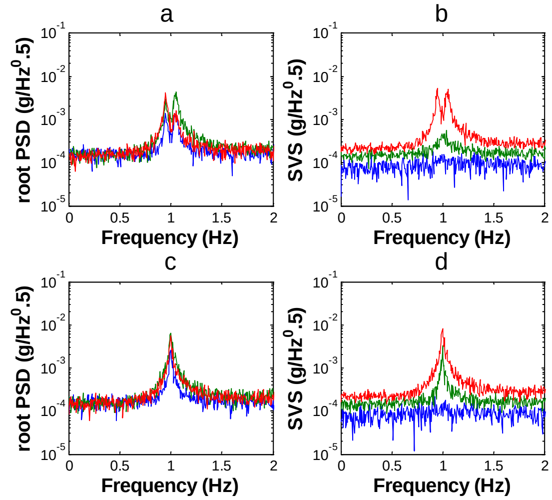

Figure 1: root PSD and Singular Value Spectrum (SVS) for PI=10(a,b) and PI=0.2(c,d).

Now the PI value of two close modes will be evaluated using an application example. The demonstrated structure is a tall building in Hong Kong, which is 320m tall and 50m by 50m in plan. The building has two close modes shown in Table 1 which were identified by Au et al (2012) using the fast Bayesian FFT method.

Table 1: Identified modal parameters of the tall building

Mode	Frequency (Hz)	Damping Ratio (%)
1	0.175	0.9
2	0.176	1.7

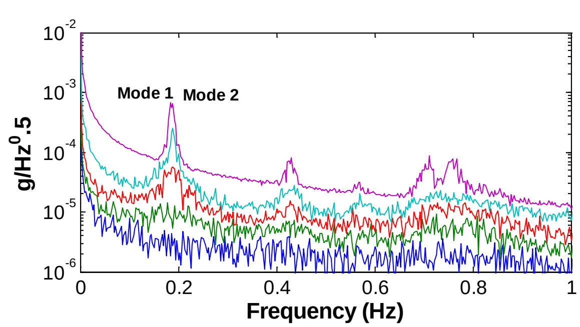

Figure 2: Singular Value Spectrum of the tall building

The singular value spectrum in Figure 2 shows that these two modes are extremely close to each other. The PI of these two close modes can be calculated as 0.438 based on the identified frequency results.

From the two examples above, it can be seen that the proximity index provides a good reference for describing the proximity of two close modes and the PI range researched in this paper (from 0.2 to 10) also has a practical meaning in practice.

3.2. Signal-to-noise Ratio

The signal-to-noise (s/n) ratio is defined as the ratio of the PSD of the modal response to the PSD of the prediction error at the resonance peak for each mode (Au 2011):

$$\gamma = \frac{S}{4S_e\zeta^2} \tag{11}$$

where the PSD of the modal response S is the intensity of the modal excitation at the natural frequency; the PSD of prediction error S_e consists of the channel noise and the modelling error in the selected frequency band and ζ is the damping ratio of the mode. The s/n ratio reflects the uncertainty level in the collected data, which depends on the quality of measurement equipment used (e.g. sensor, cable) and modelling error in the excitations.

In this section, the identification uncertainty of the frequency and damping ratio under different s/n ratios is investigated using synthetic data. Considering the structure discussed in Section 3, the PSD of the channel noise is modified based on the required s/n ratio. The natural frequencies of two modes approach each other from 0.95Hz and 1.05Hz to 0.995Hz and 1.005Hz so that the corresponding PI ranges from 0.2 to 10.

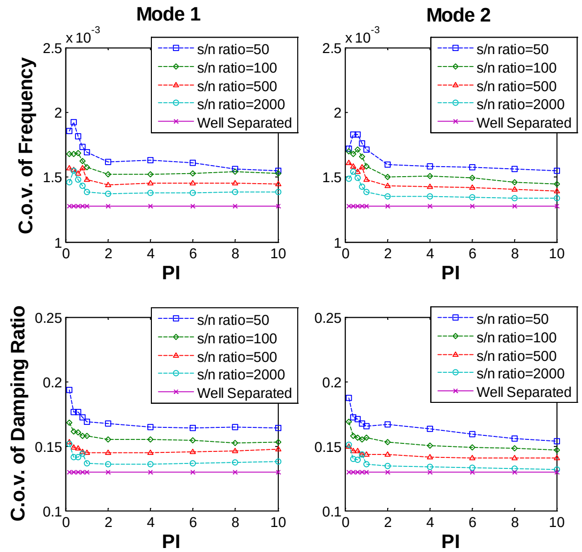

Figure 3: C.o.v. of identified frequency and damping ratio against PI (solid line—uncertainty law for well separated modes)

Figure 3 shows that although there is some slight increase in c.o.v. when PI becomes small, the posterior c.o.v. of the identified frequency and damping ratio is insensitive to the PI value overall. Compared to the uncertainty law for well separated modes, the c.o.v. of frequency and damping ratio are still acceptable. The uncertainty of identified modal parameters decreases with the s/n ratio. This is further explained in Figure 4, which plots the posterior c.o.v. of the frequency and damping ratio against different s/n ratio when PI equals 4 as an example.

When the s/n ratio is low, a small increase in s/n ratio can effectively reduce the posterior uncertainty. This effect diminishes when the modal s/n ratio is high. Note that the two modes have the same order of identification uncertainties, and so the following figures will only plot the identification uncertainties of mode 1 to simplify discussion.

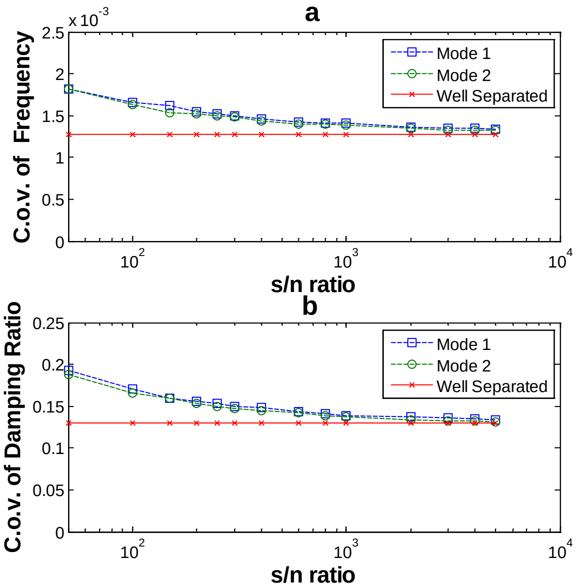

Figure 4: C.o.v. of identified frequency (a) and damping ratio (b) against s/n ratio when PI=4 (solid line--uncertainty law for well separated modes)

3.3. Modal Assurance Criterion

The modal assurance criterion (MAC) is used to measure the degree of linearity between two modes shapes which is defined as:

$$MAC = \frac{\left|\boldsymbol{\varphi}_i^T \boldsymbol{\varphi}_j\right|}{\left\|\boldsymbol{\varphi}_i\right\|\left\|\boldsymbol{\varphi}_j\right\|} \tag{12}$$

where $\boldsymbol{\varphi}_i$ and $\boldsymbol{\varphi}_j$ is the mode shapes of two modes. The MAC takes on values from zero (clearly linearly independent) to one (linearly dependent). In this paper, it is used to describe the similarity of the mode shapes for two modes. Based on the simulated structure, the mode shapes of the two modes are set as $[1 \quad 2 \quad x]^T/\sqrt{5+x^2}$ and $[-2 \quad 1 \quad x]^T/\sqrt{5+x^2}$ respectively, where the parameter x is adjusted to have MAC values ranging from 0.1 to 0.98. The identification uncertainty results simulated by synthetic data are shown in Figure 5.

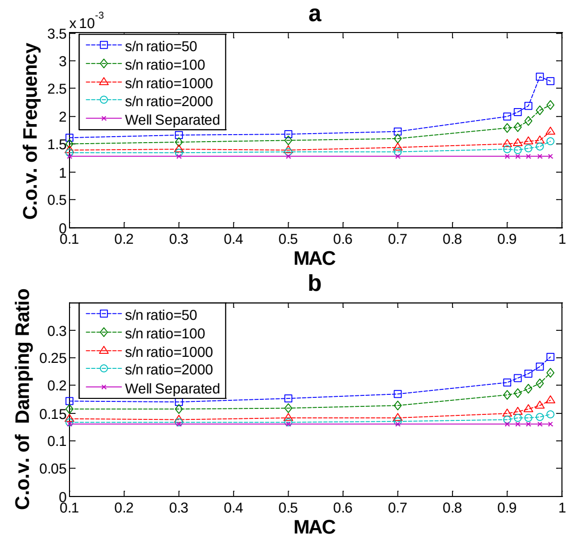

Figure 5: C.o.v. of identified frequency (a) and damping ratio (b) for mode 1 against MAC (solid line--uncertainty law for well separated modes)

This shows that for clearly linearly independent mode shapes (MAC less than 0.9), the identification uncertainties are insensitive to the MAC. For two mode shapes with high similarity (MAC larger than 0.9), there will be an increase in the identification uncertainties. However, high s/n ratios can effectively restrain the increase of identification uncertainties. The influence of the s/n ratio on the identification uncertainty with different MACs is further investigated as shown in Figure 6.

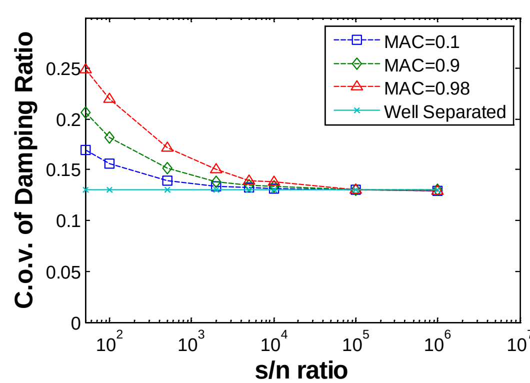

Figure 6: C.o.v. of identified damping ratio for mode 1 against s/n ratio (solid line--uncertainty law for well separated modes)

It can be seen from Figure 6 that the c.o.v. of the damping ratio under different MAC values all converges to the uncertainty law for well separated modes when the s/n ratio is high. At the same time, it shows that the MAC of the close modes to be identified does not affect the uncertainty law for the close modes identification.

4. CONCLUSIONS

This paper has investigated the identification uncertainty of close modes modal properties using the fast Bayesian ambient modal identification method. A proximity index (PI) has been proposed which provides a quantitative way to reflect the proximity of two close modes. Synthetic data simulations demonstrated that with high s/n ratio and clearly linearly independent modes, the posterior c.o.v. of frequency and damping ratio is insensitive to the proximity of two modes. In this case, the posterior uncertainty for close modes is similar to that of well-separated modes. Increasing the s/n ratio can decrease the posterior uncertainties of both frequency and damping ratio. However, the s/n ratio just needs to be high enough for identification. Further increase has little effect on improving the accuracy of the identified modal parameters. For high s/n ratios, the identification uncertainty under different MAC values converges to the uncertainty law of well separated modes, which means the similarity of the mode shapes do not have an effect on the uncertainty law for close modes in this case. This paper provides significant insights on the uncertainty of closely-spaced modes, which can help estimate the identification accuracy beforehand based on the test configurations.

5. REFERENCES

Au, S.-K. (2011). “Fast Bayesian FFT method for ambient modal identification with separated modes.” *Journal of Engineering Mechanics*.

Au, S.-K. (2012a). “Fast Bayesian ambient modal identification in the frequency domain, Part I: Posterior most probable value.” *Mechanical Systems and Signal Processing*, Elsevier, 26, 60–75.

Au, S.-K. (2012b). “Fast Bayesian ambient modal identification in the frequency domain, Part II: Posterior uncertainty.” *Mechanical Systems and Signal Processing*, Elsevier, 26, 76–90.

Au, S.-K. (2014). “Uncertainty law in ambient modal identification---Part I: Theory.” Elsevier, 48(1-2), 15–33.

Au, S.-K., Zhang, F.-L., and To, P. (2012). “Field observations on modal properties of two tall buildings under strong wind.” *Journal of Wind Engineering and Industrial Aerodynamics*, Elsevier, 101, 12–23.

Brincker, R., Zhang, L., and Andersen, P. (2001). “Modal identification of output-only systems using frequency domain decomposition.” *Smart Materials and Structures*, 10(3), 441–445.

Ewins, D. J. (2000). *Modal testing : theory, practice, and application*. Mechanical engineering research studies: Engineering dynamics series 10, Baldock : Research Studies Press, 2000.

Hudson, D. E. (1977). “Dynamic Tests of Full-Scale Structures.” *Journal of the Engineering Mechanics Division*, ASCE, 103(6), 1141–1157.

Peeters, B., and De Roeck, G. (2001). “Stochastic System Identification for Operational Modal Analysis: A Review.” *Journal of Dynamic Systems, Measurement, and Control*, 123(4), 659.

Yuen, K., and Katafygiotis, L. (2003). “Bayesian fast Fourier transform approach for modal updating using ambient data.” *Advances in Structural Engineering*, 6(2), 81–95.

Yuen, K.-V., Katafygiotis, L. S., and Beck, J. L. (2002). “Spectral density estimation of stochastic vector processes.” *Probabilistic Engineering Mechanics*, 17(3), 265–272.