Time-Variant Reliability Analysis using Polynomial Chaos Expansion

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ABSTRACT: An accurate structural reliability analysis throughout time needs to consider the random character of the material properties, geometry and loadings as well as their time variability. Such assessment can be done using time-variant reliability methods. However, progress in this field has been stagnant mainly because of the extremely high computational cost in stochastic analysis with time-dependent limit states and the error that would rise in nonlinear cases. A method based on the polynomial chaos (PC) expansion is proposed to address these issues. First, the time-variant reliability analysis is transformed into a time-invariant one by discretizing the time-dependent limit state over the structure lifetime into instantaneous limit states. Then, a principle component analysis (PCA) is performed followed by PC expansions of the retained components. The complex time-dependent limit state function is therefore substituted by a polynomial surrogate model on which Monte-Carlo simulation (MCS) can be performed at a lower computational cost. Three examples are dealt with to demonstrate the efficacy of the proposed method.

1. INTRODUCTION

Reliability is a key requirement in structural designs. It is the probability of efficiency of a system in which input parameters can have random and temporal character (Choi et al. 2007). The time dependency introduces complexity to the reliability analysis because the limit state of the system evolves in time.

In a classical time-variant reliability problem we study a mechanical system that can be described by a computational model (analytical or finite element) that depends on stochastic processes (i.e. loadings) and random variables combined with deterministic functions of time (i.e. degradation of material properties). The cumulative probability of failure of the system over a certain time interval $[t_i,t_f]$ is the probability that it exists at least one instant $t\in[t_i,t_f]$ on which failure occurs. This can be written as follows:

$$P_{f,c}\left(t_i,t_f\right)=\mathrm{Prob}\left(\exists t\in[t_i,t_f],G(\mathbf{X},\mathbf{Y}(t),t)\leq 0\right) \quad (1)$$

where $G(\mathbf{X},\mathbf{Y}(t),t)$ is the time-dependent limit state function (henceforth denoted by $G(t)$), $\mathbf{X}=[X_1,X_2,\ldots,X_n]$ is a vector of random variables and $\mathbf{Y}(t)=[Y_1(t),Y_2(t),\ldots,Y_m(t)]$ is a vector of stochastic processes. Theoretically Eq. (1) can be resolved using classical Monte-Carlo simulation (MCS), yet technically this is computationally prohibitive especially for systems with small probabilities of failure. To overcome this problem, two basic categories for time-variant reliability analysis have been developed in the literature: the extreme performance methods (Li et al. 2007, Wang and Wang 2012) and the out-crossing methods (Schrupp and Rackwitz 1988, Kuschel and Rackwitz 2000).

The aim of the extreme performance methods is to obtain the distribution of the extreme value of the limit state function over the structure lifetime. The probability of failure is thus estimated as the probability that the extreme value is greater than a critical threshold. Among these methods, the nested extreme response surface (NERS) approach has been recently proposed by Wang and Wang (2013) as an efficient simulation tool for time-variant reliability analysis. However, the distribution estimation is still computationally expensive requiring global optimization techniques (Jones et al. 1998).

The out-crossing methods evaluate the crossing rate of the likelihood that the limit state surface falls into the failure domain. Andrieu-Renaud et al. (2004) suggested the PHI2 approach that relies on FORM and parallel system analysis. Disadvantages of this approach are its limitation to linear limit states and its estimation of the upper bound of the cumulative probability of failure instead of the probability itself.

In this paper, a new method is proposed for time-variant reliability analysis. It consists on the use of PC expansions (Ghanem and Spanos 2003, Soize and Ghanem 2004) to approximate the time-dependent limit state function of the system with a polynomial surrogate model on which MCS can be easily carried out at a low computational cost.

The rest of the paper is organized as follows. In section 2, the proposed method is presented. Comparisons to existing time-variant reliability methods with analyzed results are made in section 3 and finally a concluding summary is given in Section 4.

2. TIME-VARIANT RELIABILITY ANALYSIS USING POLYNOMIAL CHAOS EXPANSION

Sparse polynomial chaos expansion (SPCE) was originally developed by Blatman and Sudret (2010) for scalar random response, thereafter it was combined to a principal component analysis (PCA) for a vector-valued response (Blatman and Sudret 2011). This approach was widely used for time-invariant reliability analysis and has proven its efficiency in tackling high dimensional problems. The idea behind our proposed methodology, henceforth denoted by "TV-PCA-SPCE", is to take advantage of the high efficiency of the PC expansion in time-invariant reliability analysis and exploit it in time-dependent problems. In what follows, the method is detailed.

2.1. Discretization of the time-dependent limit state function

As a random process, $G(t)$ becomes a simple function of time for a given sample of random inputs (i.e. a trajectory). For the corresponding sample, failure occurs if at least one t exists between t_i and t_f on which $G(t)$ is negative (Eq. (1)). In fact, limit states are seldom described with simple models so instead of studying the continuous response, we discretize it over $[t_i,t_f]$ at a certain number of nodes N_t with a discretization step Δt chosen so as to ensure that at most one out-crossing may occur per step. Thus, for each input simulation, $G(t)$ is computed uniquely at instants t_j, $j=1,...,N_t$ and all $G(t_j)$'s are grouped in one response matrix **G** of size $N_s \times N_t$, with N_s the total number of simulations. Every column $G(t_j)$ of the response matrix represents the system response at a specific time t_j. Thereby the time-variant reliability analysis is transformed into a time-invariant one (i.e. the time-dependent response is substituted with a multiple component one).

2.2. Polynomial chaos approximation of the time-dependent response

Each column (i.e. instantaneous limit state) of the response matrix can be seen as an independent component of the response, hence the similarity to the vector-valued response in time-invariant reliability analysis proposed by Blatman and Sudret (2011). Therefore we propose to approximate each random variable $G(t_j)$ using SPCE. However, when N_t increases it is no longer feasible to approximate the response components one by one using SPCE. To overcome this problem, a PCA should be carried out first allowing us to represent the entire response (i.e. all components $G(t_j)$, $j=1,...,N_t$) in a reduced space with a reduced number N_λ ($N_\lambda<N_t$) of non-physical components with minimum deflection of the reality.

The PCA is carried out on the covariance matrix **C** of **G** defined by its general term $C_{i,j}$ with $i,j=1,...,N_t$:

$$C_{ij} = \frac{1}{N_s - 1}\sum_{k=1}^{N_s}\left(G(t_i)^{(k)} - \bar{g}_i\right)\left(G(t_j)^{(k)} - \bar{g}_j\right) \quad (2)$$

where $G(t_j)^{(k)}$ is the k^{th} value of $G(t_j)$ and $\bar{g}_j$ its mean value given by:

$$\bar{g}_j = \frac{1}{N_s}\sum_{k=1}^{N_s} G(t_j)^{(k)} \quad (3)$$

Then N_λ principal components (i.e. eigenvalues λ_i, $i=1,..., N_\lambda$) of **C** are retained so as to ensure that the error ε induced by this N_λ-truncation is smaller than a preset target error ε^{tgt}.

$$\varepsilon = \frac{\sum_{i=N_\lambda+1}^{N_t} \lambda_i}{\mathrm{trace}(\mathbf{C})} \quad (4)$$

Therefore the time-dependent response can be approximated with the following equation:

$$\hat{\mathbf{G}}^{(N_\lambda)} = \overline{\mathbf{G}} + \sum_{i=1}^{N_\lambda} \boldsymbol{b}_i \boldsymbol{w}_i^t \quad (5)$$

where $\hat{\mathbf{G}}^{(N\lambda)}$ is the approximated response on the basis of N_λ principle components, $\bar{\mathbf{G}}$ is the matrix of mean values of size N_s x N_t containing N_s times the vector $\bar{\mathbf{g}}=\{\bar{g}_1,..., \bar{g}_{Nt}\}$ and $\boldsymbol{b}_i$ is a non-physical vector of size N_s obtained by projection of **G** onto the eigenvector $\boldsymbol{w}_i$ of **C**:

$$\boldsymbol{b}_i = \left(\mathbf{G} - \overline{\mathbf{G}}\right)\boldsymbol{w}_i \quad (6)$$

The $\boldsymbol{b}_i$'s are approximated using SPCE. So we have to do as many expansions as the number of retained eigenvalues.

Henceforth, the value of the time-dependent limit state can be obtained for any input simulation and at any discretized time using Eq. (5).

2.3. Evolution of the cumulative probability of failure with time

In reliability analysis, MCS is the best way to compute the probability of failure (Melchers 1987), however its application to the underlying computer code is most often very time consuming. An alternative is to apply MCS on the polynomial surrogate model (Eq. (5)) whose evaluation is much easier and less computationally expensive.

The cumulative probability of failure calculated with MCS over a time interval $[t_i,t]$ is given by:

$$P_{f,c}(t_i,t) = \frac{N_{fail}(t_i,t)}{N_s} \quad (7)$$

It is a simple ratio between the number of failing trajectories $N_{fail}(t_i,t)$ over $[t_i,t]$ and the total number of simulations N_s. A trajectory is called failing over $[t_i,t]$ if at least one out-crossing from the safe zone into the failure zone occurs. Thereby, the real cumulative probability of failure is estimated and not an upper-bound of it.

To proceed with this method, N_s samples of input parameters are first simulated. For each sample, the values of the instantaneous limit states are evaluated with Eq. (5) starting with t_i, $t_i+\Delta t$, $t_i+2\Delta t$,... Note that we do not have to evaluate the limit state at all instants since we are interested in detecting the first out-crossing (i.e. the time t_{cr} on which the limit state function is negative for the first time). By determining t_{cr}, one can say that there is failure over any time interval $[t_i,t]$ when $t \geq t_{cr}$ otherwise (i.e. if $t \leq t_{cr}$) the system is safe.

3. APPLICATION EXAMPLES

In this section three examples are dealt with to validate TV-PCA-SPCE. The first example illustrates the reliability analysis of a corroded beam under time-dependent random loading and the two others are mathematical problems. To show the accuracy improvement of the approach, its results are compared with those of two recent time-variant reliability methods: PHI2 and NERS. The MCS is used as a reference because no exact solutions are available in the three cases.

3.1. First example: Reliability of a corroded beam under time-dependent random loading

The beam problem studied in Andrieu-Renaud et al. (2004) is recalled with some modifications. The structure shown in Figure 1 consists of a steel bending beam of length L=5m and a rectangular section (b_0=0.2m, h_0=0.04m) submitted to its own weight $W=\rho_{st}b_0h_0$ (N/m), with ρ_{st}=78.5kN/m^3 the steel mass density, as well as a pinpoint load F applied at mid span. Random input parameters are given in Table 1.

The load is a Gaussian stochastic process with an exponential square autocorrelation coefficient function and a correlation length ℓ =1 year.

$$\rho_F(t_1,t_2)=\exp\left(-\left(\frac{t_2-t_1}{\ell}\right)^2\right) \tag{8}$$

The maximum bending moment occurs at mid span and is given by:

$$M_{\max}(t)=\frac{F(t)L}{4}+\frac{\rho_{st}b_0h_0L^2}{8} \tag{9}$$

Let us denote by f_y the steel yield stress and assume that the four faces of the steel beam are isotropically attacked by corrosion that is linear in time between t_i=0 and t_f=25 years with a corrosion coefficient c=0.05mm/year. The ultimate bending moment is given by:

Table 1: Corroded beam-random variables and parameters

	Distribution	Mean	COV
Variable			
f_y	*Lognormal*	*240 MPa*	*10%*
b_0	*Lognormal*	*0.2 m*	*5%*
h_0	*Lognormal*	*0.04 m*	*10%*
Process			
F	*Gaussian*	*3500 N*	*20%*

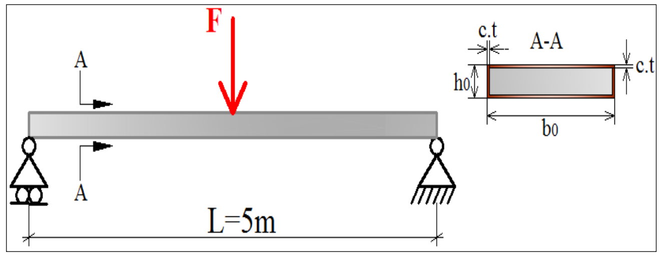

Figure 1: Corroded beam under random loading.

$$M_u(t)=\frac{(b_0-2ct)(h_0-2ct)^2}{4}f_y \tag{10}$$

The beam fails whenever $M_{max}(t)$ is greater than $M_u(t)$. Thus, the limit state function can be written as follows:

$$G(\mathbf{X},\mathbf{Y}(t),t)=\frac{(b_0-2ct)(h_0-2ct)^2}{4}f_y-\left(\frac{F(t)L}{4}+\frac{\rho_{st}b_0h_0L^2}{8}\right) \tag{11}$$

In the present work, the Expansion Optimal Linear Estimation (EOLE) method is used for the process discretization (Li and Der Kiureghian 1993) on 5 intervals of five years where only q=7 terms are retained from the decomposition that corresponds to a discretization error of 0.7%.

The reference method in this example is MCS that is based on 1,000,000 simulations. For each simulation the limit state function is evaluated at 500 equidistant time nodes and this by considering a discretization step of 0.05 year. Such a step provides a compromise between results accuracy (increases for small values of Δt) and method efficiency (decreases for small values of Δt). The cumulative probability of failure is then computed using Eq. (7).

To apply TV-PCA-SPCE, the same discretization scheme as for MCS is used but with only N_s=200 simulations of input parameters that serve to build up the polynomial surrogate model. Then all instantaneous limit state functions are grouped in a response matrix and a PCA is carried out afterwards on the covariance matrix. A target precision ε^{tgt}=0.05% yields to N_λ=8 principle components that are approximated each by a SPCE, hence the importance of the PCA that has reduced the number of components from 500 to 8 (i.e. optimization up to 98%). The 8 PC's are then combined together in one polynomial model (Eq. (5)) that describes the entire time-dependent response. For the computation of the cumulative probability of failure, MCS is applied on the polynomial surrogate model with the same number of simulation as for the classical MCS.

PHI2 is as well applied to this example. Results of the three methods are graphically presented in Figure 2. Table 2 gives the different values of the cumulative probability of failure over the studied time interval.

Figure 2 shows that TV-PCA-SPCE gives a very fine estimation of the time evolution of the cumulative probability of failure with a reduced number of evaluations of the limit state function (Eq. (11)) with respect to MCS (100,000 evaluations are needed for TV-PCA-SPCE and 500,000,000 for MCS). On the other hand, PHI2 requires only 18,720 evaluations of the limit state function except that it provides less accurate (conservative) results. This can be related to the nonlinearity of the limit state function and the fact that PHI2 does not estimate the real probability of failure but an upper bound of it.

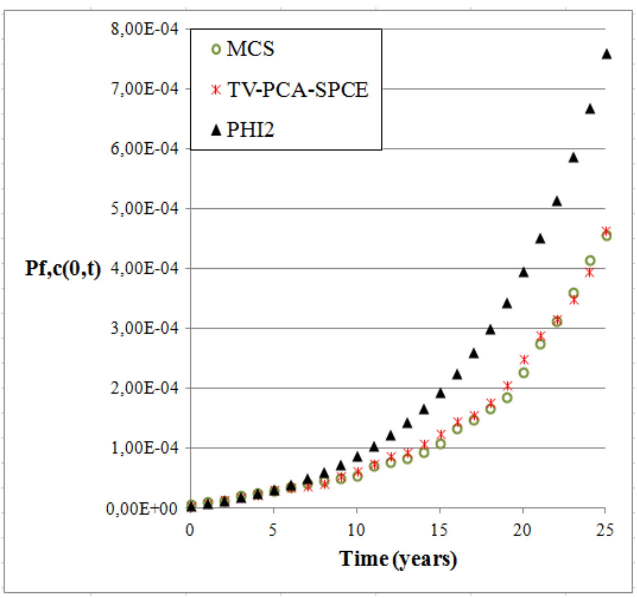

Figure 2: Corroded beam-evolution with time of the probability of failure.

Table 2: Corroded beam-cumulative probability of failure over [0,25 years]

Method	$P_{f,c}(0,25)$
MCS	4.54×10^{-4}
PHI2	7.58×10^{-4}
TV-PCA-SPCE	4.64×10^{-4}

3.2. Second example: A mathematical problem

In this paragraph TV-PCA-SPCE is compared to NERS. Let us consider a system described by the following time-dependent limit state function (Wang and Wang 2013):

$$G(\mathbf{X},t) = -20 + X_1^2 X_2 - 5X_1 t + (X_2 + 1)t^2 \quad (12)$$

where t represents the time variable, varying within [0,5]. X_1 and X_2 are normally distributed random variables with mean μ=3.5 and standard deviation σ=0.3. To apply MCS 100,000 trajectories are simulated and evaluated at 100 nodes equidistantly distributed within the interval [0,5].

To apply TV-PCA-SPCE, N_s=20 samples of input parameters are simulated. For ε^{tgt}=0.05%, only 2 eigenvalues are retained (i.e. 2 components instead of 100 and thus an optimization of 98%).

Results of these methods are compared to those given by PHI2 and NERS with MCS (Table 3). The results show that PHI2 has a high estimation error while TV-PCA-SPCE and NERS associated to MCS are both very accurate. However, for the same accuracy TV-PCA-SPCE is computationally less expensive. Therefore, in this case TV-PCA-SPCE is the best substitute of the classical MCS.

3.3. Third example: A mathematical problem of strongly nonlinear limit state function

Let us consider a mathematical example in which the limit state is hyper-paraboloid (i.e. strongly nonlinear). Such an example is proposed in Lemaire (2005) for time-independent limit state function:

$$G(\mathbf{X}) = 3 + X_1 - \frac{1}{6}\sum_{i=2}^{10} X_i^2 \quad (13)$$

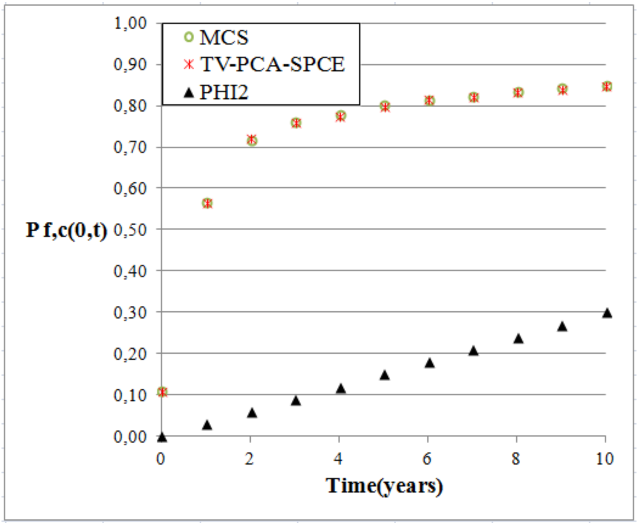

Figure 3: Strongly nonlinear limit state function-evolution with time of the probability of failure.

T able 3: Mathematical problem- probability of failure over [0,5] and the number N_{func} of function calls.

Method	*$P_{f,c}(0,5)$*	*Error*	*N_{func}*
MCS	*0.1848*	*---*	*10,000,000*
NERS+MCS	*0.1842*	*0.11%*	*>100,000*
PHI2	*0.1596*	*13.64%*	*8,860*
TV-PCA-SPCE	*0.1842*	*0.11%*	*2,000*

where X_i,i=1,…,10 are independent standard Gaussian random variables.

To integrate the time dependency, let us replace X_1 with $Y(t)$ a stationary Gaussian random process. The autocorrelation coefficient function is of exponential square type (Eq. (8)) with a correlation length ℓ =1 month and the interval of study is [0,10 years]. The time dependent limit state function becomes as follows:

$$G(\mathbf{X},\mathbf{Y}(t)) = 3 + Y(t) - \frac{1}{6}\sum_{i=2}^{10} X_i^2 \qquad (14)$$

MCS is applied with 100,000 simulations and Eq. (14) is evaluated at 500 time nodes evenly distributed within [0, 10 years].

For TV-PCA-SPCE, 150 simulations are needed. A PCA is then carried out and N_λ =16 eigenvalues are retained for ε^{tgt}=0.05% (an optimization of 97%). PHI2 is also applied to this example and results of the three methods are graphically presented in Figure 3.

The results clearly show that PHI2 is not suitable for highly nonlinear limit state functions while TV-PCA-SPCE still procures very accurate results with a reduced computational cost.

4. CONCLUSION

In this paper we propose a new method (TV-PCA-SPCE) for time-variant reliability analysis that efficiently estimates the evolution of the probability of failure of a mechanical system.

It consists on an extension of the time-invariant reliability method developed by Blatman and Sudret (2011) onto time-variant reliability analysis. The aim is to substitute the complex time-dependent limit state function with a polynomial surrogate model on which MCS can be easily performed.

Examples treated in this paper demonstrate efficacy of TV-PCA-SPCE that has shown good accuracy and suitability for even highly nonlinear limit state functions at an affordable computational cost.

5. REFERENCES

Andrieu-Renaud, C., Sudret, B., and Lemaire, L. (2004). "The PHI2 method: a way to compute time-variant reliability." *Reliability Engineering and System Safety*, 84, 75-86.

Blatman, G., and Sudret, B. (2010). "An adaptive algorithm to build up sparse polynomial chaos expansions for stochastic finite element analysis." *Probabilistic Engineering Mechanics*, 25, 183-197.

Blatman, G., and Sudret, B. (2011). "Principal component analysis and Least angle regression in spectral stochastic finite element analysis." *In M. Faber (Ed), Proc. 11th Int. Conf. on Applications of Stat. and Prob. in Civil Engineering (ICASP11)*, Zurich, Switzerland.

Choi, S.K., Grandhi R.V., and Canfield, R.A. (2007). "*Reliability-based structural design*." New York: Springer.

Ghanem, R., and Spanos, P. (2003). "*Stochastic finite elements: a spectral approach.*" Courier Dover Publications.

Jones, D., Schonlau, M., and Welch, W. (1998). "Efficient global optimization of expensive black-box functions." *Journal of Global optimization*, 134(12), 121007-14.

Kuschel, N., and Rackwitz, R. (2000). "Optimal Design Under Time-Variant Reliability Constraints." *Structural Safety,* 22(2), 113-127.

Lemaire, M., Chateauneuf, A. and Mitteau, J.C. (2005). "*Fiabilité des structures: couplage mécano-fiabiliste statique*." Hermès sciences publications.

Li, C., and Der Kiureghian, A. (1993). "Optimal discretization of random fields."*Journal of Engineering Mechanics*, 119(6), 1136-1154.

Li, J., Chen, J., and Fan, W. (2007). "The Equivalent Extreme-Value Event and Evaluation of the Structural System Reliability." *Structural Safety,* 29(2), 112-131.

Melchers, R.E. (1987). "*Structural reliability analysis and prediction.*" Chichester: Ellis Horwood.

Schrupp, K., and Rackwitz, R. (1988). "Out-crossing Rates of Marked Poisson Cluster Processes in Structural Reliability." *Applied Mathematical Modelling,* 12(5), 482-490.

Soize, C., and Ghanem, R. (2004). "Physical systems with random uncertainties: chaos representations with arbitrary probability measure." *SIAM Journal on Scientific Computing,* 26(2), 395-410.

Wang, Z., and Wang, P. (2012). "A Nested Extreme Response Surface Approach for Time-Dependent Reliability-Based Design Optimization." *Journal of Mechanical Design,* 134(12), 121007 (14 pages).

Wang, Z., and Wang, P. (2013). "A new approach for reliability analysis with time-variant performance characteristics." *Reliability Engineering and System Safety,* 115, 70-81.