Finite Element Reliability Analysis of Structures using the Dimensional Reduction Method

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ABSTRACT: Finite Element Reliability Analysis (FERA) has been used to evaluate the reliability of structures. Mean and variance of the structural response is often estimated with the use of approximate methods, while structural response distribution is approximated based on Monte Carlo simulation (MCS). In this paper, FERA is applied in an efficient manner with the use of a Multiplicative form of Dimensional Reduction Method (M-DRM), which can estimate accurately the statistical moments and the probability distribution of the structural response, e.g., drift of a structure. The proposed approach is combined with OpenSees FE software and illustrated through the nonlinear pushover and nonlinear dynamic analysis of a steel frame. MCS is also performed for comparison of the proposed method.

1. INTRODUCTION

Finite element analysis (FEA) has become a widely used tool for the numerical analysis of the structural response. But, uncertainties associated with input parameters such as material properties, geometry and loads may have to be accounted for in FEA. Finite element reliability analysis (FERA) can overcome this challenge by considering the input parameters as random variables, thus evaluating the reliability of large and practical multi degrees of freedom structures.

In a large multi degree of freedom problem, the following issues are usually encountered; (1) to minimize the number of function evaluations, especially when the deterministic evaluation of the model takes a long time, (2) to estimate accurately the probability distribution of the structural response function, especially in finite element analysis where the structural response function is in an implicit form, (3) to connect a general FEA software with a reliability platform, especially when knowledge on advanced programing languages is required.

At this time there is not any available commercial software in the market that includes in its interface both FEA and reliability. In order to apply FERA is required to link a general purpose FEA program, e.g., ABAQUS, with an existing reliability platform, e.g., NESSUS or ISIGHT, but the analyst has to purchase separately these reliability platforms. More information with respect to the connection between FEA software and structural reliability can be found in literature (Pellissetti and Schuëler 2006).

On the other hand, the analyst can take advantage of the free and open-source FE software OpenSees (McKenna et al. 2000), which already contains reliability capabilities (Der Kiureghian et al. 2006). OpenSees interface supports the use of Tcl programming language, thus in this paper Tcl programming is used for automating both MCS and M-DRM. The only disadvantages here is that writing source code for large scale structures is a tedious task which requires prior knowledge and enough experience in computer programming, which can be prohibited for researchers to apply FERA.

The objective of this paper is to present a Multiplicative form of Directional Reduction Method (M-DRM) for FERA of structures, which can overcome potential limitations as they were introduced in previous paragraphs. Monte Carlo simulation (MCS) is also applied for comparison and validation of the M-DRM results. Thus, M-DRM and MCS are both implemented on a structure selected from literature and analyzed with the use of the free, open-source code and object-oriented software framework OpenSees.

2. MUTIPLICATIVE DIMENSIONAL REDUCTION METHOD (M-DRM)

2.1. General Description

In the reliability analysis a system's response is frequently modeled as a function of numerous input variables. For instance, the lateral displacement of a structural frame can be described as a function of several input variables such as the strength of materials, the structural dimensions and the applied loads.

Mathematically this is denoted as

$$Y = h(\mathbf{x}) \tag{1}$$

where Y is a scalar output random variable and $(\mathbf{x})$ is a vector of input random variables $x_1, \dots, x_n$. If the structural failure is defined by a simple condition such as $Y > y_c$, the probability of failure (p_f) can be obtained from the cumulative distribution function (CDF) of the response, $F_Y(y)$, as

$$p_f = 1 - F_Y(y_c) \tag{2}$$

where y_c corresponds to a safety limit. Numerical approach using Monte Carlo simulation (MCS) provides an easy to implement alternative for reliability computation, but the computational cost can be prohibited for the reliability analysis, especially when it comes to complex and/or large scale structures. For that reason a multiplicative dimensional reduction method (M-DRM) is proposed for reliability analysis.

2.2. Integer Moments of Response using M-DRM

In literature, the high dimensional model representation (HDMR) method (Rabitz and Aliş 1999; Li et al. 2001) uses an additive form to approximate a scalar function as

$$h(\mathbf{x}) \approx \sum_{i=1}^{n} h_i(x_i) - (n-1)h_0 \tag{3}$$

where n is the number of random variables, $h_i(x_i)$ is a unidimensional cut function defined as

$$h_i(x_i) = h(c_1, \dots, c_{i-1}, x_i, c_{i+1}, \dots, c_n) \tag{4}$$

and h_0 represents the system response which is evaluated at the cut point (Li et al. 2001) and is calculated when all input random variables are set equal to their mean values $(c_1, c_2, \dots, c_n)$ as

$$h_0 = h(c_1, c_2, \dots, c_n) = a\ constant \tag{5}$$

Alternate to the additive form, a multiplicative DRM method is developed to simplify the evaluation of both integer and fractional moments of the response. In this approach, the structural response is approximated as

$$h(\mathbf{x}) \approx h_0^{(1-n)} \times \prod_{i=1}^{n} h_i(x_i) \tag{6}$$

Using the above representation, a k^{th} product moment of the response function (Y) can be approximated as

$$E\{[h(\mathbf{x})]^k\} \approx E\left\{\left[h_0^{(1-n)} \times \prod_{i=1}^{n} h_i(x_i)\right]^k\right\} \tag{7}$$

where $E[\ ]$ denotes the mathematical expectation operation, which for $k = 1$ is the mean value. Considering all random variables as independent, Eq. (7) can be written as

$$E\{[h(\mathbf{x})]^k\} \approx h_0^{k(1-n)} \prod_{i=1}^{n} E\left[\left(h_i(x_i)\right)^k\right] \tag{8}$$

Then is defined the mean and mean square of a k^{th} cut function as $\rho_k = E_k[h_k(X_k)]$ and $\theta_k = E[\left(h_k(X_k)\right)^2]$, respectively, and using Eq. (8) the mean of the response is approximated as

$$\mu_Y \approx h_0^{(1-n)} \times \prod_{k=1}^{n} \rho_k \qquad (9)$$

and the mean square is approximated as

$$\mu_{2Y} \approx h_0^{(2-2n)} \times \prod_{k=1}^{n} \theta_k \qquad (10)$$

The variance is expressed as a difference between the mean square and the square of the first product moment (Ang and Tang 2007). Thus, using Eqs. (9) and (10), the variance of the response can be estimated as

$$V_Y \approx (\mu_Y)^2 \times \left[\left(\prod_{k=1}^{n} \frac{\theta_k}{\rho_k^2}\right) - 1\right] \qquad (11)$$

The evaluation of any k^{th} product moment of response requires one dimensional integration of all the cut functions using the scheme of Gauss quadrature, for which more details can be found in Zhang and Pandey (2013). For example, a k^{th} moment of an i^{th} cut function can be approximated as a weighted sum as

$$E\{[h_i(x_i)]^k\} \approx \sum_{j=1}^{L} w_j \left[h_i(x_j)\right]^k \qquad (12)$$

where L is the number of evaluation points of the Gauss quadrature, x_j and w_j are the coordinates and weights, respectively, of the Gauss quadrature points $(j = 1, \dots, L)$, h_i $(i = 1,2, \dots, n)$ is the response of the system when i^{th} cut function is set at j^{th} Gauss quadrature point and n is the number of random variables.

2.3. Fractional Moments of Response using M-DRM

After calculating the first two integer moments of the response, i.e., mean and variance respectively, the problem that arises is the estimation of response probability density function (PDF). Here, is used the maximum entropy (MaxEnt) principle (Jaynes, 1957) with fractional moment constraints, i.e., $E[X^\alpha] = M_X^\alpha$ where α not an integer, where the fractional moments (M_X^α) are derived from M-DRM method. The fractional moment of a positive random variable x is defined as (Inverardi and Tagliani 2003)

$$E[X^\alpha] = M_X^\alpha = -\int_X x^\alpha f_X(x)\, dx \qquad (13)$$

where α is a real number. Fractional moments have the important property of being capable of characterizing completely the probability distribution of a positive random variable (Pandey and Zhang 2012). Using the M-DRM as defined in Eq. (8), an α^{th} Moment calculation of the response can be approximated as

$$M_X^\alpha \approx \left[{h_0}^{(1-n)}\right]^\alpha \times E\{[h_1(x_1)]^\alpha\} \times \dots \times E\{[h_n(x_n)]^\alpha\} \qquad (14)$$

where h_0 represents the system response as shown in Eq. (5) and each expected value $E[\ \ ]$ is calculated as shown in Eq. (12).

The benefit here is that we do not need to specify the fractions α_i $(i = 1,2, \dots, m)$ a priory, as they will be determined during the entropy optimization procedure (Inverardi and Tagliani 2003). The true entropy $(H[f])$ of a continuous positive random variable x is defined in terms of its PDF $\left(f_X(x)\right)$ as

$$H[f] = -\int_X f_X(x)\, ln[f_X(x)]dx \qquad (15)$$

The Lagrangian function associated with the MaxEnt problem is given as

$$\begin{aligned} &\mathcal{L}[\lambda, \alpha; f_X(x)] \\ &= -\int_X f_X(x)\, ln[f_X(x)]dx \\ &- (\lambda_0 - 1)\left[\int_X f_X(x)\, dx - 1\right] \\ &- \sum_{i=1}^{m} \lambda_i \left[\int_X x^{\alpha_i} f_X(x) dx - M_X^{\alpha_i}\right] \end{aligned} \qquad (16)$$

where $\lambda = [\lambda_0, \lambda_1, \dots, \lambda_m]^T$ are the Lagrange multipliers and $\alpha = [\alpha_0, \alpha_1, \dots, \alpha_m]^T$ are the fractions associated with the fractional moments. For optimal solution the following key condition is applied as

$$\frac{\partial \mathcal{L}[\lambda, \alpha; f_X(x)]}{\partial f_X(x)} = 0 \tag{17}$$

which leads to the estimated PDF $\left(\hat{f}_X(x)\right)$ of the true PDF $(f_X(x))$ as

$$\hat{f}_X(x) = \exp\left(-\sum_{i=0}^{m} \lambda_i x^{\alpha_i}\right) \tag{18}$$

For $i = 0$, $\alpha_0 = 0$ and λ_0 is derived as

$$\lambda_0 = \log\left[\int_X \exp\left(-\sum_{i=1}^{m} \lambda_i x^{\alpha_i}\right) dx\right] \tag{19}$$

based on the normalization condition that the integration of the PDF must be equal to one.

Then MaxEnt optimization procedure with constraints in term of fractional moments is applied, contrary to the traditional MaxEnt method which uses integer moments (Ramírez and Carta 2006). The reason is that as the order of the integer moment increases, the estimation error increases too (Pandey and Zhang 2012).

In order to implement the idea of MaxEnt optimization with fractional moments, an alternate formulation is proposed based on the minimization of the Kullback-Leibler (K−L) divergence, also called cross-entropy, between the true PDF and the estimated PDF as

$$\begin{aligned} K[\,f, \hat{f}\,] &= \int_X f_X(x)\, ln\left[f_X(x) \,/\, \hat{f}_X(x)\right] dx \\ &= \int_X f_X(x)\, ln[f_X(x)] dx - \int_X f_X(x)\, ln\left[\hat{f}_X(x)\right] dx \end{aligned} \tag{20}$$

Taking into account Eq. (13) and substituting $H[f]$ from Eq. (15) and $\hat{f}_X(x)$ from Eq. (18) into Eq. (20), the K−L divergence $\left(K[\,f, \hat{f}\,]\right)$ can be rewritten as

$$K[\,f, \hat{f}\,] = -H[f] + \lambda_0 + \sum_{i=1}^{m} \lambda_i M_X^{\alpha_i} \tag{21}$$

where $K[\,f, \hat{f}\,]$ is the divergence measure of the distance between the true PDF and the estimated PDF. Here it can be noticed that the entropy $(H[f])$ of the true PDF does not depend on λ and α, so the K−L minimization implies the minimization of the following function

$$K[\,f, \hat{f}\,] + H[f] = \lambda_0 + \sum_{i=1}^{m} \lambda_i M_X^{\alpha_i} \tag{22}$$

Therefore, the MaxEnt parameters, i.e., the Lagrange multipliers (λ_i) and the fractional exponents (α_i), can be obtained by applying the following optimization

$$\begin{cases} \boldsymbol{Find}: \{\alpha_i\}_{i=1}^{m} \quad \{\lambda_i\}_{i=1}^{m} \\ \boldsymbol{Minimize}: I(\lambda, \alpha) = \lambda_0 + \sum_{i=1}^{m} \lambda_i M_X^{\alpha_i} \end{cases} \tag{23}$$

which is implemented in MATLAB by using the simplex search method (Lagarias et al. 1998).

2.4. Computational Efficiency using M-DRM

M-DRM combined with the rules of Gaussian quadrature requires a magnitude of nL trials for the evaluation of the structural response. Considering the Gauss-type integration scheme, the total number of functional evaluations can be assessed as $MDRM_{trials} = 1 + (nL)$, where 1 represents the case when all random variables are fixed to their mean values, i.e., cut point, n is the number of the random variables and L is the order of the Gaussian quadrature. Further details regarding Gaussian quadrature can be found in Zhang and Pandey (2013).

3. NUMERICAL EXAMPLE

3.1. General Description

In order to illustrate the applicability and to examine the efficiency of the proposed M-DRM, FERA is applied on a steel frame (Figure 1) taken from Haukaas and Scott (2006). Each frame member is discretized in 8 displacement-based finite elements. Gravity load of 50 kN and 100 kN is applied at the external and internal connections, respectively. In additional to the gravity loads, the frame is subjected; (1) to static pushover analysis and (2) to dynamic analysis. For the pushover analysis, lateral loads of 400 kN, 267 kN and 133 kN are applied on nodes 13, 9 and 5, respectively.

For the dynamic analysis, the Imperial Valley earthquake ground motion is used, taken from the PEER Strong Motion Database (http://peer.berkeley.edu/smcat/). The Magnitude of the earthquake was 6.53, with a PGA (g) equals to 0.143 at 10.84 sec, as it was recorded from Station USGS 931 El Centro Array #12 (1979/10/15, 23:16). Pushover analysis gives a total reaction force at the supports equals to 800 kN, and the accelerogram of the earthquake was scaled so as to produce the same reaction force at the time of the PGA. Both analyses performed by using the open source FE software OpenSees, where Tcl programming is also used in order to automatically update the random variables in each trial for both M-DRM and MCS. This idea of the parameter updating functionality is further described in Scott and Haukaas (2008).

The material properties of the steel frame are considered as independent random variables, forming 21 members x 3 parameters = 63 random variables in total (Table 1). The objective here is to estimate the mean, variance and distribution of the response, i.e., maximum lateral displacement of node 13 (u_{13}).

Table 1: Statistical properties of random variables.

Parameter	Distribution	Mean	COV
E (MPa)	Lognormal	200,000	5.0%
f_y (MPa)	Lognormal	300	10.0%
b	Lognormal	0.02	10.0%

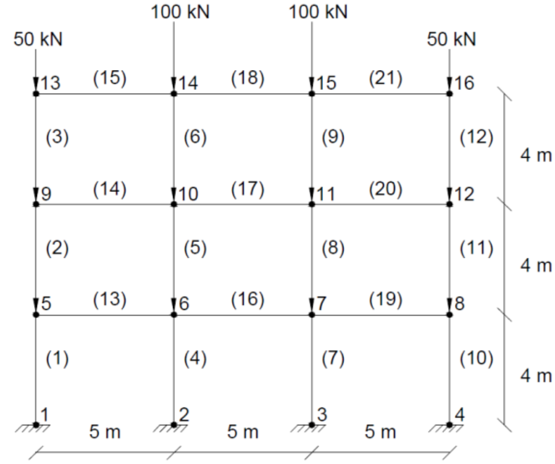

Figure 1: Steel frame showing node numbers, member numbers (in parenthesis) and gravity loads.

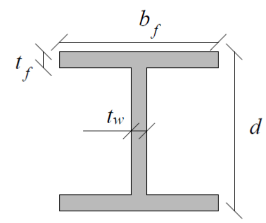

Figure 2: Steel cross section (b_f =d=250 mm, t_f =t_w=20 mm).

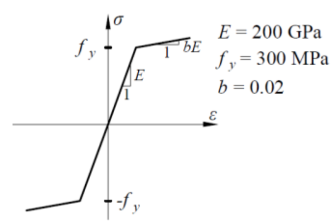

Figure 3: Steel material model.

3.2. Mean and Variance of Response

Based on M-DRM and using the five point Gauss-Hermite integration scheme, the response of the steel frame is a product of 63×5+1 = 316 cut functions, since the problem involves 63 random variables. Thus, an input grid is created (Table 2) and FEA is performed. For instance, for the first five trials the modulus of elasticity (E) of member 17 changes, while the rest of 62 random variables remain fixed to their mean values. This input grid is used for both pushover and dynamic analysis, and the response u_{13} obtained for each one is shown on Table 3.

Then, the mean of each cut function is calculated as $\rho_i = \sum_{j=1}^{L} w_j u_{13ij}$ $(i = 1,2, \dots, n)$ where u_{13ij} is the lateral displacement of thirteenth node when i^{th} cut function is set at j^{th} quadrature point, and w_j is the Gauss weights of the five order rule ($L = 5$) of Gauss-Hermite quadrature. Similarly the mean square of cut functions is calculated as $\theta_i = \sum_{j=1}^{L} w_j \left(u_{3ij}\right)^2$. The overall response mean and variance is then approximated by Eq. (9) and Eq. (11), respectively, for both pushover (Table 4) and dynamic analysis (Table 5).

Table 2: Input grid for M-DRM.

Input RV	Trial	E (N/mm^2)	…	f_y (N/mm^2)
E (member 17)	1	173,176	…	300
	2	186,667	…	300
	3	199,750	…	300
	4	213,750	…	300
	5	230,402	…	300
…	…	…	…	…
f_y (member 16)	311	200,000	…	224
	312	200,000	…	260
	313	200,000	…	298
	314	200,000	…	341
	315	200,000	…	396
Fixed Mean Values	316	200,000	…	300

Table 3: Output response based on input grid for M-DRM.

Trial	u_{13} (mm)	
	Pushover	Dynamic
1	226.33	145.94
2	225.59	145.42
3	224.94	144.95
4	224.315	144.47
5	223.64	143.93
…	…	…
311	247.64	144.94
312	233.56	144.94
313	225.11	144.94
314	223.51	144.94
315	223.50	144.94
316	224.93	144.94

Table 4: Output response statistics: Pushover Analysis.

Pushover Analysis	Max lateral displacement (u_{13})		
	MCS (10^5 Trials)	M-DRM (316 Trials)	Relative Error (%)
Mean (mm)	237.32	238.47	0.48
Stdev (mm)	22.36	21.62	3.33
COV	0.0942	0.0906	3.80

Table 5: Output response statistics: Dynamic Analysis.

Dynamic Analysis	Max lateral displacement (u_{13})		
	MCS (10^4 Trials)	M-DRM (316 Trials)	Relative Error (%)
Mean (mm)	149.06	146.21	1.92
Stdev (mm)	10.05	13.19	31.21
COV	0.0674	0.0902	33.77

Note: Stdev = Standard Deviation; COV = Coefficient of Variation; Relative Error = $|MCS - MDRM|/MCS$.

The numerical results obtained from M-DRM and MCS are compared. M-DRM method requires 316 trials, whereas simulation results are based on 10^5 and 10^4 trials for pushover and dynamic analysis, respectively. For pushover analysis, mean and standard deviation estimations based on M-DRM have a very small error compared to MCS results (Table 4). For dynamic analysis, M-DRM estimates have a larger error, but still are in a good agreement with the MCS results (Table 5).

3.3. Probability Distribution of Response

The distribution of maximum lateral displacement (u_{13}) is estimated based on the maximum entropy (MaxEnt) principle. The MaxEnt algorithm provides the Lagrange multipliers (λ_i) and the fractional exponents (α_i) ($i = 1,2,\ldots,m$) defining the probability distribution in Eq. (18). Usually, three fractional moments ($m = 3$) are sufficient for the analysis, as entropy converges rapidly (Table 6).

Table 6: Entropy.

Fractional Moments	Entropy	
	Pushover	Dynamic
m=1	-0.852	-0.012
m=2	-2.435	-2.915
m=3	-2.434	-2.915
m=4	-2.434	-2.914

Table 7: MaxEnt distribution parameters for 3 fractional moments.

m=3	Pushover			Dynamic		
i	λ_i	α_i	$M_X^{\alpha_i}$	λ_i	α_i	$M_X^{\alpha_i}$
0	35.51	-	-	365	-	-
1	39.39	0.72	0.35	-4E-05	2.53	0.01
2	-33.6	-0.4	1.79	1125	1.42	0.07
3	0.03	-3.95	311	-695	0.23	0.64

PDF of lateral displacement at node 13 obtained from M-DRM and MCS (simulations) are in fairly close agreement (Figure 4). Three fractional moments can accurately model the distribution, as shown from the probability of exceedance (POE) curves (Figure 5, Figure 6).

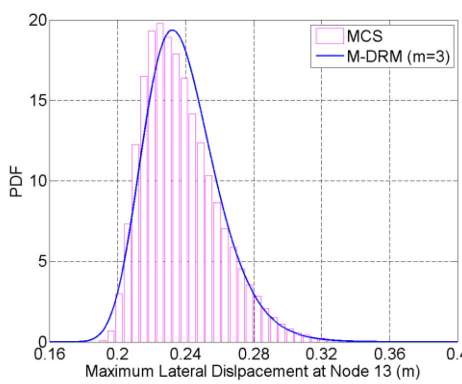

Figure 4: Probability distribution of the maximum lateral displacement at node 13: Pushover Analysis.

For instance, if the maximum allowable lateral displacement of node 13 is 0.36m (3% of the frame height), the probability of exceeding this limit (or probability of failure) is estimated by M-DRM as 9.18×10^{-5} and by MCS as 8.99×10^{-5} (Figure 5). This result confirms high accuracy of M-DRM achieved by a relatively small number of structural analyses (316 trials) contrary to MCS (10^5 trials).

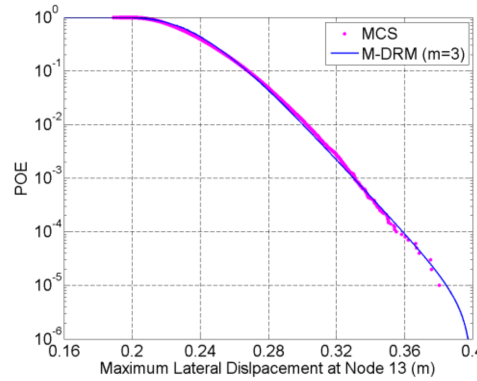

Figure 5: Probability of Exceedance of the maximum lateral displacement at node 13: Pushover Analysis.

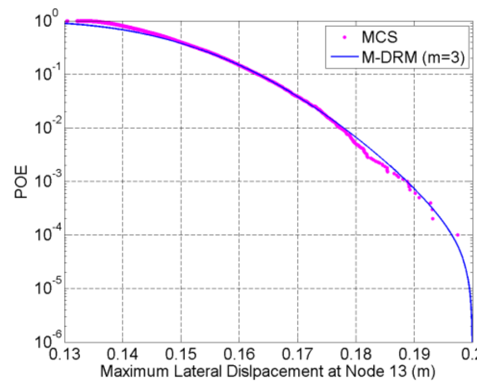

Figure 6: Probability of Exceedance of the maximum lateral displacement at node 13: Dynamic Analysis.

3.4. Computational Time

M-DRM provides an enormous saving of computational time. Using a personal computer with Intel i7-3770 3rd Generation Processor and 16GB of RAM, for the pushover analysis MCS with 100,000 FE simulations takes 5.84 hours and for the dynamic analysis MCS with 10,000 FE simulations takes 14.02 hours. M-DRM approximation based on 316 FE analyses, for the pushover analysis takes 0.76 minutes and MaxEnt optimization requires 0.34 minutes, and for the dynamic analysis takes 32.46 minutes and MaxEnt optimization requires 1.03 minutes. Thus, total time taken by M-DRM is merely 0.31% and 3.99% of the time taken by MCS for the pushover and dynamic analysis, respectively. Note that as the complexity of the problem increases, the computational cost is relatively reduced.

4. CONCLUSIONS

In this paper is presented a Multiplicative form of Dimensional Reduction Method (M-DRM), which can be used efficiently for finite element reliability analysis (FERA) of structures. Based on Gauss quadrature scheme, an input grid is created and response moments (mean and variance) are calculated. Then the MaxEnt algorithm is applied to compute the distribution parameters of the response. Here, M-DRM is implemented in OpenSees FE software with the aid of Tcl programming language

The proposed method provides a robust and computationally viable method for full probabilistic analysis of practical problems, as requires relatively small FE simulations based on a specified input grid. The main benefit of M-DRM is large computationally economy with a high accuracy comparing to simulations, as illustrated in this paper. Nonlinear pushover and nonlinear dynamic analysis of a steel frame with 63 random variables illustrates this point very well, as M-DRM with 316 FE simulations provides close results to 100,000 and 10,000 simulations for the pushover and dynamic analysis, respectively, while M-DRM is merely a fraction (0.31%, 3.99%) of that of the Monte Carlo simulations.

5. AKNOWLEDGEMENTS

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6. REFERENCES

Ang, H-S A., and Tang, H. W. (2007). *Probability Concepts in Engineering: Emphasis on Applications in Civil & Environmental Engineering*, John Wiley and Sons Inc, Hoboken, NJ.

Der Kiureghian, A., Haukaas, T., and Fujimura, K. (2006). “Structural Reliability Software at the University of California, Berkeley”, *J. Structural Safety*, 28(1-2), 44–67.

Haukaas, T., and Scott, M. H. (2006). “Shape Sensitivities in the Reliability analysis of Nonlinear Frame Structures”, *J. Probabilistic Engineering Mechanics*, 21(2), 133–147.

Inverardi, P., and Tagliani, A., (2003). “Maximum Entropy Density Estimation from Fractional Moments”, *J. Communication in Statistics − Theory and Methods*, 32(2), 327–345.

Jaynes, E. (1957). “Information Theory and Statistical Mechanics”, *Physical Review*, 106(4), 620–630.

Lagarias, J., Reeds, J., Wright, M., and Wright, P. (1998). “Convergence Properties of the Nelder-Mead Simplex Method in Low Dimensions”, *SIAM J. Optimization*, 9(1), 112–147.

Li, G., Rosenthal, C., and Rabitz, H. (2001). “High Dimensional Model Representations”, *J. Physical Chemistry*, American Chemical Society, 105(33), 7765–7777.

McKenna, F., Fenves G. L,. and Scott, M. H. (2000). *Open System for Earthquake Engineering Simulation*, Univ. of California–Berkeley, CA, <http://opensees.berkeley.edu>.

Pandey, M.D., and Zhang, X. (2012). “System Reliability Analysis of the Robotic Manipulator with Random Joint Clearances”, *J. Mechanism and Machine Theory*, 58, 137–152.

Pellissetti, F. M., and Schuëler, I. G. (2006). “On General Purpose Software in Structural Reliability–An Overview”, *J. Structural Safety*, 28(1-2), 3–16.

Rabitz, H., and Aliş, Ö. (1999). “General Foundations of High-Dimensional Model Representations”, *J. Mathematical Chemistry*, Springer, 25(2-3), 197–233.

Ramírez, P., and Carta, J.A. (2006). “The use of Wind Probability Distribution derived from the Maximum entropy Principle in the analysis of wind energy. A case study”, *J. Energy Conversion and Management*, 47(15−16), 2564–2577.

Scott, M. H., and Haukaas, T. (2008). “Software Framework for Parameter Updating and Finite-Element Response Sensitivity Analysis”, *J. Computing in Civil Engineering*, 22(5), 281–291.

Zhang, X., and Pandey, M.D. (2013). “Structural Reliability Analysis based on the concepts of Entropy, Fractional Moment and Dimensional Reduction Method”, *J. Structural Safety*, 43, 28–40.