Optimization of Non-linear Truss Considering Expected Consequences of Failure

Thaís G. Pedrosa
Graduate Student, Dept. of Structural Engineering, Univ. of São Paulo, São Carlos, SP, Brazil

André T. Beck
Assoc. Prof., Dept. of Structural Engineering, Univ. of São Paulo, São Carlos, SP, Brazil

ABSTRACT: Defying failure is the primary challenge of the structural engineer. It sounds paradoxical, but in order to achieve a successful design, the structural engineer must think about and account for all possible failure modes of a structure. This is not different in structural optimization. Hence, in structural optimization one also has to consider the expected costs of failure. In structural engineering design, economy and safety are apparently conflicting goals. However, when expected costs of failure are considered, it becomes evident that investments in safety are necessary in order to avoid paying for failure. The optimum point of compromise can be found by a risk optimization, where the objective function includes all costs over the life-cycle of the structure: construction, operation, inspection maintenance, disposal, and the expected costs of failure. The latter are an undeletable remainder of the failure modes that the structure needs to be designed against. This paper addresses the optimization of simple structural systems, considering the balance between competing failure modes such as yielding (squashing), buckling and snap-through. The study shows how different failure modes, associated to different costs of failure, lead to different optimal designs. A plane truss structure is studied as application example. The shape (nodal positions) and member size are considered as design variables. Results show that different optimal designs are obtained when the balance between competing failure modes is changed.

1. INTRODUCTION

Structural optimization involves seeking for the (optimal) shape and size of a structure, which allows it to fulfill its intended function with minimal resources use. It is today widely known that deterministic design optimization is not robust w.r.t. uncertainties in structural loads and material strength. In deterministic design optimization, uncertainties are taken into account in a subjective and indirect way, by means of partial safety factors specified in design codes.

As a consequence, deterministic optimal solutions may lead to reduced reliability levels (Beck & Gomes, 2012). As an example, consider a fully stressed design where all material points are designed against the limit: such a structure is more likely to collapse, since there are no alternative load paths.

Structural optimization should be robust with respect to load and resistance uncertainties. This perception has led to the development of different methodologies:

1. Robust optimization (Beyer & Sendhoff, 2006; Schueller & Jensen, 2009);
2. Fuzzy optimization or fuzzy programming (Zimmermann, 1992);
3. Reliability-based design optimization or RBDO (Cheng et al., 2006).

Typical objectives in robust optimization are maximizing mean system performance and minimizing the variance. In robust optimization, uncertainties are represented as random variables or stochastic processes. Otherwise, in fuzzy optimization, uncertainties are represented by fuzzy numbers. While robust optimization has a probabilistic approach, the fuzzy optimization presents a possibilistic approach. In general, the amount of information available determines the appropriate approach (Bastin, 2004).

RBDO seeks minimization of objective function involving material volume, manufacture cost or structural system performance. Uncertainty in loading and strength is addressed objectively and modeled probabilistically. Failure probabilities are normally used design constraints.

Consideration of uncertainties in structural design optimization is not a novelty. However, a literature review shows that in most classical RBDO articles (Du & Chen, 2004; Tu et al., 1999; Cheng et al., 2006; Youn & Choi, 2004; Agarwal et al., 2007; Yi et al., 2008; Aoues & Chateauneuf, 2010; Valdebenito & Schueller, 2010) failure consequences are ignored or discarded.

Objective functions in robust optimization and RBDO frequently involve costs, but such costs relate to material volume or manufacturing costs. A cost term often overlooked in classical RBDO problems is the expected failure cost. Expected failure cost is the product between failure costs by failure probabilities. This definition complies with the definition of risk: expected cost of failure can be understood as the risk associated to each failure mode of the structure. The total expected cost is the sum of expected failure costs for each possible failure mode, in addition to initial construction costs, maintenance and operation costs. The total expected cost of a structural system is directly affected by the risk offered to users, employees, the general public and the environment.

In this study, the risk optimization formulation is employed (Beck & Verzenhassi, 2008; Beck & Gomes, 2012; Beck et al., 2012). In this formulation, expected costs of failure are included in the objective function. The optimization problem becomes unconstrained.

2. RISK OPTIMIZATION

Consider $\boldsymbol{X}$ and $\boldsymbol{d}$ two parameter vectors of a structural system. Vector $\boldsymbol{X}$ represents all random variables of the problem such as dimensions, resistance properties of materials and structural members and loads. Some of these parameters are by nature random variables; others cannot be deterministically defined due to several uncertainty sources. Vector $\boldsymbol{d}$ contains the system design variables or optimization variables, such as partial safety factors, dimensions, nodal coordinates and even some (deterministic) parameters of the probability distribution models of random vector $\boldsymbol{X}$.

The existence of uncertainties implies risks, that is, the possibility of undesired structural responses. The boundary between desired and undesired structural responses is formulated in limit state equations $g(\boldsymbol{X}, \boldsymbol{d}) = 0$ such that:

$$\Omega_f(\boldsymbol{d}) = \{x | g(\boldsymbol{X}, \boldsymbol{d}) \leq 0\} \tag{1}$$

$$\Omega_s(\boldsymbol{d}) = \{x | g(\boldsymbol{X}, \boldsymbol{d}) > 0\} \tag{2}$$

where $\Omega_f(\boldsymbol{d})$ and $\Omega_s(\boldsymbol{d})$ are the failure and survival domains, respectively.

Each limit state function describes a possible failure mode in terms of serviceability or ultimate capacity. The probability of an undesirable response, or failure probability, is given by:

$$P_f(\boldsymbol{d}) = P\left[\boldsymbol{X} \in \Omega_f(\boldsymbol{d})\right] = \int_{\Omega_f(\boldsymbol{d})} f_{\boldsymbol{X}}(\boldsymbol{x}) d\boldsymbol{x} \tag{3}$$

where $f_{\boldsymbol{X}}(\boldsymbol{x})$ represents the joint probability density function of random vector $\boldsymbol{X}$ and $P[.]$ represents probability.

The failure probability for each limit state function, as well as for system failure, is calculated using techniques of structural reliability, such as FORM (First Order Reliability Method), SORM (Second Order Reliability Method) or Monte Carlo simulation (Melchers, 1999; Ang & Tang, 2007).

2.1. Quantifying failure consequences

The total expected cost (C_{te}) of a structural system in a risk optimization problem is given by the sum of the following costs:

1. $C_{initial}$: initial or construction cost;
2. $C_{operation}$: operation cost;
3. $C_{maint.}$: Cost of inspections, maintenance, repair and replacement;

4. $C_{discard}$: cost of discard;
5. C_{ef}: expected cost of failure.

The expected cost of failure is the product between failure cost by the respective failure probability (Beck & Gomes, 2012):

$$C_{ef}(\boldsymbol{d}) = C_f(\boldsymbol{d}) P_f(\boldsymbol{d}) \tag{4}$$

Failure costs include repair or replacement costs for damaged components, complete system rebuild, compensation costs due to accidental failure, and others. The failure probability is evaluated by structural reliability theory, following Eq. (3). To each failure mode there is an associated expected cost of failure term, hence:

$$C_{te}(\boldsymbol{d}) = C_{initial}(\boldsymbol{d}) + C_{operation}(\boldsymbol{d}) + C_{maint.}(\boldsymbol{d}) + C_{discard}(\boldsymbol{d}) + \sum_{k=1}^{fail.\ modes} C_f(\boldsymbol{d}) P_f(\boldsymbol{d}) \tag{5}$$

The optimal amount of resources to invest in safety of the built structure, and in inspection and maintenance policies, is the one that leads to the minimum expected total cost. This amount can be found by solving the following risk optimization problem:

$$\boldsymbol{d}^* = arg\,min\,[C_{te}(\boldsymbol{d}) : \boldsymbol{d} \in S] \tag{6}$$

where $S = \{\boldsymbol{d}_{min} \leq \boldsymbol{d} \leq \boldsymbol{d}_{max}\}$ is the feasible design space and $\boldsymbol{d}_{min}$ and $\boldsymbol{d}_{max}$ are the lower and upper limits of the design variables. In this formulation, the reliability constraints are incorporated in the objective function.

The above formulation allows one to solve the safety-economy tradeoff in the design and operation of a given structure. Moreover, it allows one to find the optimal point of balance between competing failure modes, which is particularly important when significantly different failure costs are associated to different failure modes (Gomes & Beck, 2013). The risk optimization formulation allows finding the appropriate safety margins w.r.t. each failure mode.

3. THE PROBLEM

The problem addressed in this paper is a simple two-bar truss as illustrated in Figure 1. Geometrical and material non-linearities are considered. The solution is formulated in analytical form.

In the initial configuration, the two bars form an angle θ_0 with the imaginary horizontal line connecting the supports. Vertical force P at node B causes the vertical displacement u. The problem is symmetric w.r.t. the vertical line passing through B and B'. The half-span is b and height is h. The solution is parameterized w.r.t. $h/2b$.

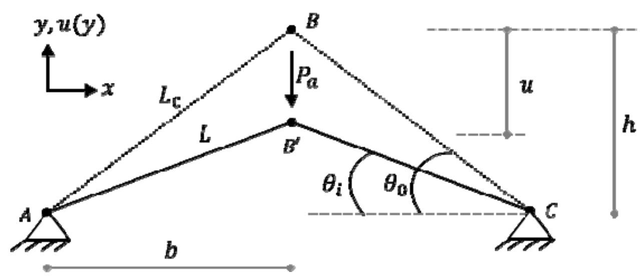

Figure 1: Simple two-bar truss.

By equilibrium condition, the relation between applied force and normal force in the bars is given by:

$$P(u) = N \sin\theta_i = \frac{N(h+u)}{L} \tag{7}$$

where θ_i is the bar angle in displaced position.

A non-linear elastic material is considered, with Green deformation:

$$E(u) = E_0\left(1 + \eta\varepsilon_G(u)\right) \tag{8}$$

$$\varepsilon_G(u) = \frac{1}{2}\left(\frac{2hu+u^2}{{L_0}^2}\right) \tag{9}$$

By means of Hooke's Law, the normal force is given as product of material elasticity constant by deformation:

$$N(u,h,\eta) = E_0 S_0 \left[\frac{1}{2}\left(\frac{2hu+u^2}{{L_0}^2}\right) + \frac{\eta}{4}\left(\frac{2hu+u^2}{{L_0}^2}\right)^2\right] \tag{10}$$

Equilibrium paths for the structure, with geometrical and material nonlinearities, are shown in Figure 2, for $h/2b = 0.09$ and different material constants (η). Effect of pure geometrical nonlinearity can be observed for $\eta \to 0$. Note that with large deflections, as physical nonlinearity increases (larger η), more instability regions are observable, and limit loads are reduced.

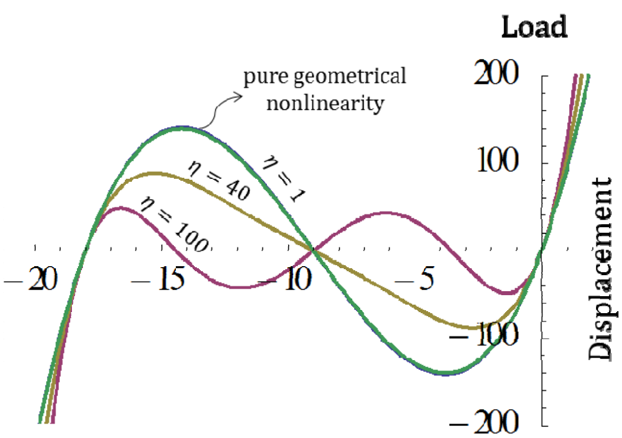

Figure 2: Equilibrium paths for different elastic nonlinearity constants for initial configuration ($h/2b =$ 0.09).

3.1. Variables and limit state functions

The bars are assumed hollow circular, with external diameter d and thickness t. The vector of design variables is given by initial configuration and cross-section area: $\boldsymbol{d} = \{b, h, d, t\}$.

Three random variables are considered: applied load P_a, deformation modulus E_0 and yielding stress σ_y. The random variables follow Gaussian distributions with parameters $P_a \sim \mathrm{N}(50,5)$ kN, $E_0 \sim \mathrm{LN}(20500,1025)$ MPa and $\sigma_y \sim \mathrm{LN}(50,2.5)$ MPa. Hence, random vector $\boldsymbol{X}$ is: $\boldsymbol{X} = \{P_a, E_0, \sigma_y\}$.

Three failures modes are considered:

1. Snap-through or limit point instability;
2. Buckling or bifurcation instability;
3. Yielding or squashing;

To formulate a solution, the final displacement, which delimits the equilibrium path extension, is first defined. This point is calculated by iterative Newton Raphson or by looking for a root of $P(u) - P_a$. The result is a displacement u_f given as a function of $(P_a, E_0, \boldsymbol{d}, \eta)$.

Snap-through failure is defined by the limit point, which indicates the beginning of an unstable equilibrium. This point u_{lim} is obtained by nullifying the tangent stiffness matrix. The limit displacement is function of (h, b, η). The limit state function for snap-through failure is written as:

$$g_1(\boldsymbol{X}, \boldsymbol{d}, \eta) = P_{lim}(\boldsymbol{X}, \boldsymbol{d}, \eta) - P_a \qquad (11)$$

where P_{lim} is the load corresponding to limit displacement u_{lim}, and P_a is the applied load.

The second failure mode is bifurcation instability or buckling. The Euler relation (Eq. 12) that describes buckling is given by:

$$N_{cr}(u) = \frac{\pi^2 E\, I}{L_0^2} = \frac{\pi^2\, E_0 I\,(1+\eta\varepsilon_G(u))}{L_0^2} \qquad (12)$$

where $N_{cr}(u)$ is the critical normal force. For bi-articulated bars, buckling length is equal to bar length.

The limit state equation for buckling is:

$$g_2(\boldsymbol{X}, \boldsymbol{d}, \eta) = N_{cr}(u_i) - N(u_i) \qquad (13)$$

where $N(u_i)$ is the acting normal force, given in Eq. (10).

During the load-displacement equilibrium path for the structure, Eq. (13) presents three local minima. The points of minima (u_i in Eq. (13)) are found by taking the derivative of Eq. (13) w.r.t. displacement u and equating it to zero. Failure due to buckling occurs at one of these points of minima or at the final displacement (u_f). The point of minima where buckling failure occurs changes according to the initial configuration ($h/2b$).

The third failure mode is compressive squashing or tensile yielding of the cross-section. The limit state function in Eq. (14) is given in terms of the yielding tension σ_y, compared for both tensile and compression normal forces:

$$g_3(\boldsymbol{X}, \boldsymbol{d}, \eta) = Min \begin{bmatrix} (\sigma_y . S_0 - \mathrm{Max}_u[|N_{comp}(u_j)|]) \\ (\sigma_y . S_0 - \mathrm{Max}_u[N_{tensile}(u_j)]) \end{bmatrix} \qquad (14)$$

where $N_{comp}(u_j)$ and $N_{tensile}(u_j)$ are evaluated from the load-displacement curve, by taking the derivative of normal force w.r.t. displacements ($dN(u)/du$), and by considering the final displacement.

These displacements (u_j in Eq. 14) are illustrated in Figure 3, which shows normal force versus displacement curves. One observes in Figure 3 the existence of two points of maximum compression, one point of maximum tensile force and the final displacement (black dot). The correct evaluation of the maximum normal (compression or tensile) force is done by a displacement analysis, which identifies the displacements that belong to the equilibrium path, according to final displacement (u_f). Figure 3 also shows the displacements u_i in Eq. (13). The difference between u_i and u_j is shown by the variation of minimum point in $N(u)$ and $N_{cr}(u) - N(u)$ curve.

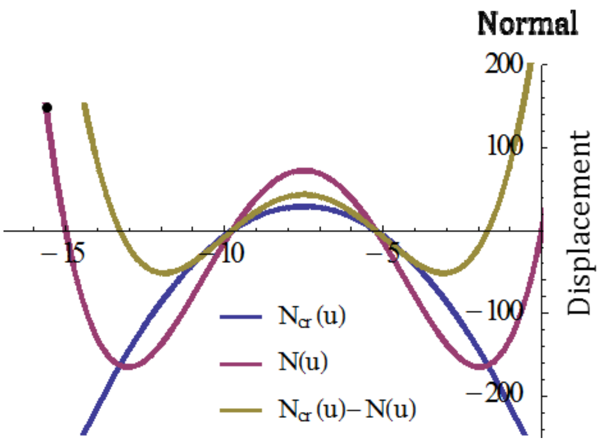

Figure 3: Comparison between acting and critical normal force.

3.2. Initial and Failure Costs

The initial cost is given by material volume ($V = 2\,L_0 S_0\ cm^3$), multiplied by specific mass ρ (g/cm^3) and unit mass price ($1000\ \$/kg$):

$$C_{initial}(\boldsymbol{d}) = 2000\,\rho\,\mathrm{c}_{unit}\,L_0\,S_0 = C_{ref} L_0\,S_0 \quad (15)$$

where C_{ref} is a reference value for costs, L_0 is given in *cm* and the area in *cm*². The initial cost is, of course, function of the design variables.

It is well known that different failure modes are associated to different costs. Fragile or sudden failure, which occurs without warning, has more serious consequences than ductile failure. Service failures can lead to temporary suspension of use, which generally represents smaller costs than collapse (ultimate failure).

The simple two-bar structure considered herein could have many different applications; hence the costs of failure are immaterial. In order to avoid limiting the study to one possible application, the costs of failure are given generically in proportion to the initial cost.

Total expected cost can be written as:

$$C_{te}(\boldsymbol{d}) = C_{inicial}(1 + A\,P_{f\,snap} + B\,P_{f buckling} + C\,P_{f tension}) \quad (16)$$

where A, B, C, are failure cost factors that depend on the failure mode. Different failure cost scenarios are considered in the sequence.

4. RESULTS

In a first analysis, failure probabilities are evaluated by changing initial truss height, keeping other design variables fixed at $d = 3.34\ vcm, t = 3.4\ mm, b = 50\ cm$ and $\eta = 100$. This study helps to understand the different failure mode behavior when the angle θ_0 is changed.

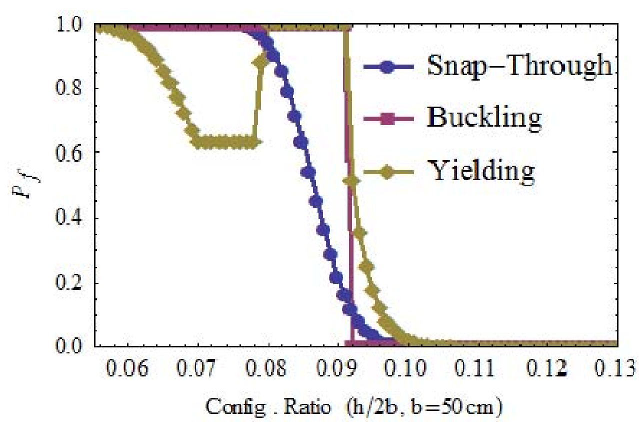

Figure 4: Failure probabilities for each failure mode.

In Figure 4, one observes that the transition to snap-through failure is continuous, whereas transition to buckling failure is discrete. The yielding failure can be explained with aid of Figure 5, where each load-displacement curve corresponds to a configuration ratio in Figure 4. For large $h/2b$ ($h/2b > 0.95$), normal load is compressive and failure may occur due to squashing (compressive yielding). For $h/2b < 0.08$, snap-trough failure is highly likely, and in this case there is a reversal in the normal load. Because of this reversal, there is a probability that the bars fail also due to tensile yielding.

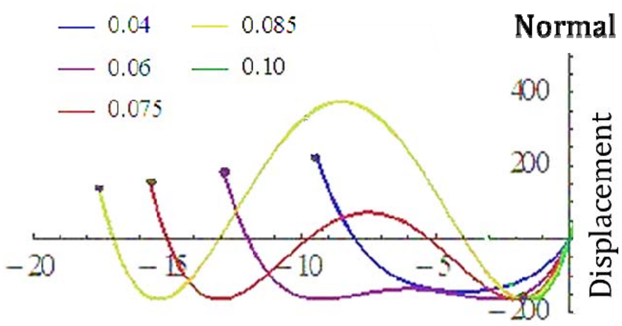

Figure 5: Normal force versus displacements for five initial configurations (h/2b).

Figure 6 shows results for a parametric analysis, where different costs of failure are considered. The objective function (C_{te}) is placed: in terms of truss height (above), diameter (center) and thickness (below). For each plot, the "remaining" design variables are fixed at: $d = 3.34\ cm, t = 3.4\ mm, b = 50\ cm$ and $h = 10.7\ cm$, which is the optimal height for the corresponding configurations. Total expected cost for this "reference" configuration, with $A = B = C = 1$, is $C_{te}/C_{ref} = 0.1639$. Reliability indexes for this configuration are: $\beta_{snap} = 4.03$, $\beta_{buckling} = \infty$ and $\beta_{yield} = 3.29$.

In Figure 6, results are also shown for other cost configurations, were each cost term increased at once to ten units (10). Since buckling is not relevant for this configuration, one observes that changing the cost of failure for buckling does not change the objective function nor optimal values of design variables. However, increasing the cost of failure for snap-through and yielding affects the objective function, increasing optimal values of design variables h, d and t. The change is not dramatic, but it shows that a different equilibrium point is obtained between the competing failure modes. Observed changes in design variables are of 4.67% for h, and 7.74% for d and 9.68% for t.

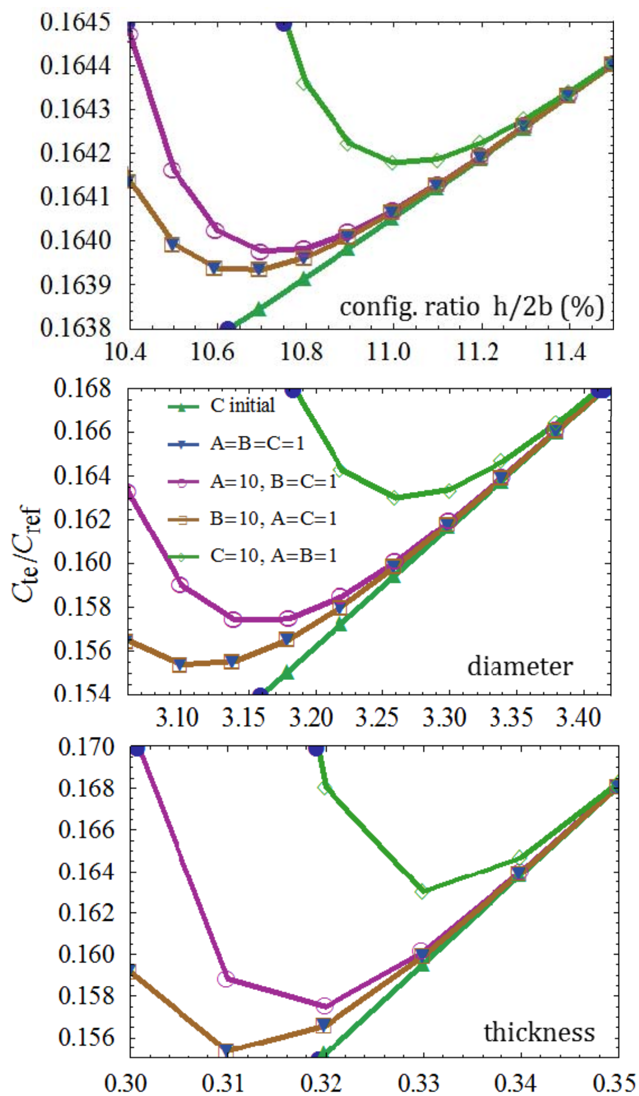

Figure 6: Total expected cost (C_{te}) for four failure cost scenarios, in terms of design variables h, d and t.

Figure 7 shows the objective function for other combinations of failure costs, in terms external diameter. The variation in total expected costs for these scenarios is presented in Table 1.

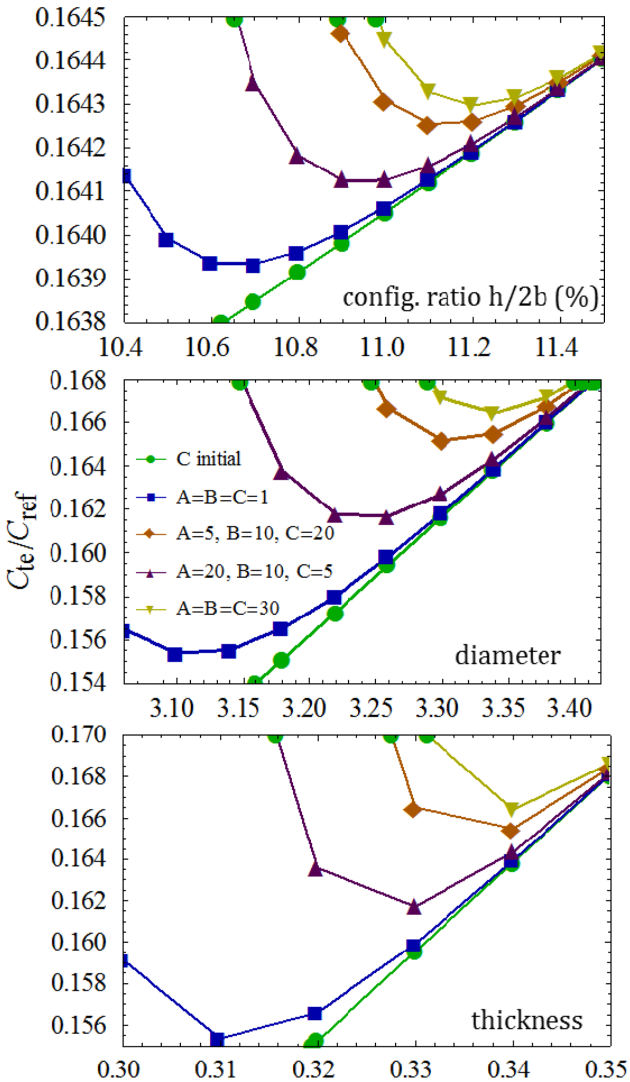

Figure 7: Total expected cost (C_{te}) for different failure cost scenarios, in terms of design variables $h/2b, d$ and t.

Table 1: Percentage change in design parameters for different cost configurations.

A	B	C	h/2b	%	d	%	t	%
1	1	1	0.107	-	3.10	-	0.31	-
10	1	1	0.107	0	3.14	1.29	0.31	0
1	10	1	0.107	0	3.10	0	0.32	3.23
1	1	10	0.110	2.80	3.26	5.16	0.33	6.45
5	10	20	0.111	3.74	3.30	6.45	0.34	9.68
20	10	5	0.109	1.87	3.26	5.16	0.33	6.45
30	30	30	0.112	4.67	3.34	7.74	0.34	9.68

5. CONCLUSIONS

This paper addressed the optimal design of a non-linear truss structure, considering the effects of expected costs of failure. Physical and geometrical non-linearities were considered. Failure by yielding, buckling and snap-through were considered. Problem selection and its solutions targeted the competition between failure modes, when costs of failure are different. The differences between the optimal designs found herein were subtle, but they demonstrated how the competition between failure modes can affect optimal structural design.

ACKNOWLEDGMENTS

Financial support by Brazilian Agencies CAPES and FAPESP is greatly acknowledged.

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