

Simulation of Strongly Non-Gaussian Non-stationary Stochastic Processes utilizing Karhunen-Loeve Expansion

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ABSTRACT: The simulation of non-stationary and non-Gaussian stochastic processes is a challenging problem of considerable practical interest. Recently, Shields et al. have developed a class of conceptually simple and efficient methods for simulation of non-Gaussian processes using translation process theory (collectively referred to as the Iterative Translation Approximation Method - ITAM) that iteratively upgrades the underlying Gaussian power spectral density function for simulation using the spectral representation method. However, the currently existing ITAM method for generation of non-stationary and non-Gaussian processes requires additional approximations in the estimation of the evolutionary spectrum. An extension of the ITAM is proposed that utilizes the K-L expansion. The developed method iteratively upgrades the covariance function directly and, in so doing, avoids the complex and non-unique inverse problem of estimating the evolutionary spectrum from the non-stationary autocorrelation. The application of the method for a strongly non-Gaussian and non-stationary process with a prescribed target non-Gaussian correlation function is demonstrated.

1. INTRODUCTION

Probabilistic analysis of complex problems involving strong non-linearities and uncertainties represented by non-Gaussian stochastic processes/fields typically requires Monte Carlo simulation. Monte Carlo simulation, which is the most robust method available for these problems, requires accurate simulation of stochastic processes. Well established methods are readily available for the simulation of stationary Gaussian random processes. Unfortunately, these stationary and Gaussian processes are often not representative of reality. Instead, realizations of non-Gaussian and non-stationary stochastic process are necessary for accurate analysis. However, simulation of these processes present numerous challenges and consequently, the development of methods for their simulation has lagged considerably behind the stationary and Gaussian models.

The Karhunen-Loeve (K-L) expansion and the Spectral Representation Method (SRM) are the two most commonly employed methods for generation

of random processes in mechanics. The K-L expansion (Ghanem and Spanos (1991), Huang et al. (2001)) affords an optimally succinct representation of the stationary and non-stationary processes from a specified covariance function and its implementation is straightforward for Gaussian processes. Phoon et al. (2002) and Phoon et al. (2005) developed K-L-based methods for the simulation of non-Gaussian and non-stationary processes that iteratively update the probability density function's of the K-L random variables. Furthermore, Sakamoto and Ghanem (2002a,b) proposed a technique for simulating non-stationary and non-Gaussian stochastic processes that utilizes a polynomial chaos expansion (PCE) of the K-L random variables. Another class of algorithms for generating non-Gaussian processes couples the SRM, (Shinozuka and Jan (1972); Shinozuka and Deodatis (1991)) with translation process theory, (Grigoriu (1995)). Several developments along these lines have been proposed over the past 25 years includ-

ing works by Yamazaki and Shinozuka (1988), Deodatis and Micaletti (2001), and Bocchini and Deodatis (2008) among others. Most recently, Shields et al. (2011) and Shields and Deodatis (2013a) proposed the Iterative Translation Approximation Method (ITAM) for simulation of stationary non-Gaussian processes that upgrades the underlying Gaussian power spectral density function in a simple and efficient manner. This method has been extended to the simulation of non-stationary and non-Gaussian processes by Shields and Deodatis (2013b).

For strongly non-Gaussian and non-stationary processes, each of the aforementioned methods possesses notable drawbacks. K-L based simulation methods are subject to the Central Limit Theorem. Therefore, in the limit as the number of K-L random variables grows large the process tends toward Gaussian regardless of the distribution of the K-L variables themselves. Moreover, even for a small number of K-L variables, the process often cannot maintain the strongly non-Gaussian features that are desired. SRM-based methods, on the other hand, require significant approximations in order to produce non-stationary and non-Gaussian processes (Shields and Deodatis (2013b)). Most notably, no unique inverse exists to compute the evolutionary spectrum (ES) as defined by Priestley (1965) from the non-stationary autocorrelation function. Thus, approximations must be introduced that can cause significant errors for strongly non-Gaussian and non-stationary processes.

In this work, an extension of the ITAM that utilizes the K-L expansion for simulation is proposed. The method developed in this study has the following advantages for simulation of non-stationary and non-Gaussian processes. First, estimation of the evolutionary spectrum is avoided; thus alleviating the associated approximations and allowing direct simulation from the covariance matrix. Second, generated realizations are produced directly from Gaussian K-L random variables with the resulting Gaussian process translated to match the target marginal non-Gaussian probability function exactly. Lastly, It is conceptually simple, straightforward to implement, and converges rapidly (usu-

ally in 10 iterations or less). The application of the method for a strongly non-Gaussian and non-stationary process with prescribed target non-Gaussian covariance is demonstrated.

2. KARHUNEN-LOEVE EXPANSION

Consider a random process $A(t, \theta)$ defined on the probability space (Ω, σ, P) over the domain $s, t \in [a, b]$ with zero mean and covariance function $C(s, t)$ possessing finite variance. The process can be expressed according to the K-L expansion as

$$A(t, \theta) = \sum_{i=1}^{\infty} \sqrt{\lambda_i} \zeta_i(\theta) f_i(t) \quad (1)$$

where λ_i and $f_i(x)$ are the eigenvalues and eigenfunctions of the covariance $C(s, t)$ determined by solving the Fredholm integral equation of the second kind given by

$$\int_a^b C(s, t) f_i(s) ds = \lambda_i f_i(t) \quad (2)$$

The parameters ζ_i are a set of uncorrelated random variables with zero mean $E[\zeta_i] = 0$ and correlation $E[\zeta_i \zeta_j] = \delta_{ij}$. The K-L expansion can be conveniently implemented for simulation by truncating a countable number, M , of its terms as

$$\tilde{A}(t, \theta) = \sum_{i=1}^M \sqrt{\lambda_i} \zeta_i(\theta) f_i(x) \quad (3)$$

For Gaussian processes, the random variables ζ_i follow the standard Gaussian distribution. However, the distribution of ζ_i for non-Gaussian processes can be difficult to determine and may require complex dependence structure in general (Grigoriu (2010)).

3. TRANSLATION PROCESS THEORY

Translation process theory (Grigoriu (1995)) represents a non-Gaussian process $Y(t)$ through the nonlinear transformation of a Gaussian process, $X(t)$, as follows:

$$Y(t) = g(X(t)) \quad (4)$$

where $g(\cdot)$ is a general nonlinear transformation. The so-called standard translation is defined such that $g(\cdot) = F^{-1} \cdot \Phi(\cdot)$ where $F(\cdot)$ and $\Phi(\cdot)$ are

the arbitrary non-Gaussian and standard Gaussian marginal cumulative distribution functions, respectively.

The extension of translation process theory for non-stationary processes was proposed by Ferrante et al. (2005) where the mapping in Eq. (4) is generalized to incorporate a time-dependent transformation as

$$Y(t) = g(X(t), t) = F^{-1} \cdot \{\Phi(X(t)), t\} \quad (5)$$

where $X(t)$ is a non-stationary Gaussian process and $Y(t)$ is a non-stationary and non-Gaussian process with time-varying marginal cumulative distribution function $F(\cdot, t)$.

Utilizing non-stationary translation process theory, the correlation $R(s, t)$ of the non-stationary and non-Gaussian process can be computed from the standard Gaussian correlation as

$$\begin{aligned} R(s, t) &= \mu(s)\mu(t) + \sqrt{C(s, s)C(t, t)}\xi(s, t) \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(u, s)g(v, t)\phi\{u, v; \rho(s, t)\} dudv \end{aligned} \quad (6)$$

where $\mu(t)$, $C(t, t)$, and $\xi(s, t)$ are the mean, variance, and correlation coefficient of the non-Gaussian process $Y(t)$ and $\phi(s, t)$ denotes the joint Gaussian PDF with correlation coefficient, $\rho(s, t)$.

A general challenge with translation process theory is that, given a prescribed covariance function and associated non-Gaussian marginal probability density function, it is not always possible to identify a corresponding underlying Gaussian correlation function. In other words, the inverse of Eq. (6) does not always exist. In such cases, the prescribed covariance function and marginal probability density function are said to be incompatible with the translation process model. However, given the convenience of the translation model and the lack of comparable methods for modeling non-Gaussian processes, it is often desirable to identify an underlying Gaussian process that, when mapped to the non-Gaussian distribution using Eq. (6), produces a non-Gaussian correlation function that is as close as possible to the prescribed non-Gaussian correlation function. This is the motivation for the

previously developed Iterative Translation Approximation Method (ITAM) described in the following section and its improvement in this work.

4. ITERATIVE TRANSLATION APPROXIMATION METHOD

The ITAM, originally developed by Shields et al. (2011) and extended for non-Gaussian and non-stationary processes by Shields and Deodatis (2013b), utilizes a simple and efficient iterative method to upgrade the underlying Gaussian power spectral density (stationary) or evolutionary spectrum (non-stationary). In the non-stationary case the ITAM upgrades the underlying Gaussian evolutionary spectrum as

$$S_G^{(i+1)}(\omega, t) = \left[\frac{S_N^T(\omega, t)}{S_N^{(i)}(\omega, t)} \right]^\beta S_G^{(i)}(\omega, t) \quad (7)$$

where $S_G^{(i)}(\omega, t)$ and $S_N^{(i)}(\omega, t)$ are the underlying Gaussian and the computed non-Gaussian ES at iteration i , respectively and β is parameter utilized to optimize the convergence.

The challenge associated with this method is that it requires estimation of the evolutionary spectrum from the non-stationary correlation function. However, in general the evolutionary spectrum is not uniquely defined for a given non-stationary autocorrelation (Priestley (1965); Benowitz et al. (2014)). Thus, the ITAM proposed by Shields and Deodatis (2013b) requires an estimation that may cause significant errors in the case of strongly non-Gaussian and non-stationary processes. Recently, Benowitz et al. (2014) has investigated the uniqueness of this inversion and the results indicate that it may be possible to obtain a unique evolutionary spectrum under specific conditions, but the computational cost required to estimate this evolutionary spectrum with reasonable accuracy is extremely high.

5. PROPOSED ITAM WITH K-L EXPANSION

A new ITAM utilizing the K-L expansion for simulation is proposed in this work. This new ITAM iteratively upgrades the underlying non-stationary autocorrelation function (ACF) directly (removing

approximations associated with evolutionary spectrum estimation). The following procedure describes the proposed methodology for finding the compatible underlying Gaussian ACF.

1. Initialize underlying Gaussian ACF, $R_G^{(0)}(s, t)$.
2. Upgrade the underlying Gaussian ACF.
3. Find the nearest positive semi-definite ACF to the upgraded ACF.
4. Compute the non-Gaussian ACF at iteration i , $R_N^{(i)}(s, t)$, by non-stationary translation process theory.
5. Check the difference between the computed and target non-Gaussian ACF. If convergence, proceed to the next step. Otherwise, iterate back to step 2.
6. Simulate the processes using the K-L expansion and translate process theory.

When an incompatible pair of marginal non-Gaussian non-stationary PDF and ACF is prescribed, the initial underlying Gaussian ACF can be defined arbitrarily - as long as it is a properly defined autocorrelation function (i.e. is symmetric and positive semi-definite). In practice, it is useful to reduce calculation time by setting the initial underlying Gaussian ACF equal to the target non-Gaussian ACF.

The proposed upgrading method in step 2 is similar to that of the SRM-based ITAM given in Eq. (7). Specifically, the underlying Gaussian normalized ACF, $\rho(s, t)$ is upgraded as

$$\rho^{(i+1)}(s, t) = \left[\frac{\xi^T(s, t)}{\xi^{(i)}(s, t)} \right] \rho^{(i)}(s, t) \quad (8)$$

where $\xi^{(i)}(s, t)$ is the computed non-Gaussian ACF at iteration i and $\xi^T(s, t)$ is the prescribed target standardized non-Gaussian ACF. However, there are a few noteworthy differences with the SRM-based ITAM. Most notably, the evolutionary spectrum is a strictly positive quantity while the ACF is not. This has two significant consequences. First, the exponent β is removed in the proposed method because iterations using a non-integer exponent will produce imaginary numbers for negative ratios in the parentheses. Second, the resulting ACF from the iterations in Eq. (8) is not necessary positive semi-definite and is therefore not a valid autocor-

relation function. To correct this, it is necessary to identify the nearest positive semi-definite ACF (step 3). This is achieved using the method proposed by Qi and Sun (2006) who use a quadratically convergent newton method whose quadratic convergence has been proven.

Finally, in step 5 the relative difference between the computed and target non-Gaussian non-stationary ACF as

$$\varepsilon^{(i)} = 100 \sqrt{\frac{\sum_{n=0}^{N-1} \sum_{m=0}^{N-1} [\xi^{(i)}(s_n, t_m) - \xi^T(s_n, t_m)]^2}{\sum_{n=0}^{N-1} \sum_{m=0}^{N-1} [\xi^T(s_n, t_m)]^2}} \quad (9)$$

When the value of the relative difference stabilizes to a constant value, the iterations are stopped and the K-L expansion is used to generate the Gaussian process from the converged underlying Gaussian ACF using Eq. (3). Then, the generated Gaussian sample function is translated to the non-Gaussian distribution using Eq. (5).

6. NUMERICAL EXAMPLE

To demonstrate the capabilities of the improved ITAM, consider the simulation of the strongly non-Gaussian and non-stationary process with target non-stationary covariance in Figure 1 is given by

$$C(s, t) = \min(s, t) \cdot \cos[4\pi(s - t)] \quad (10)$$

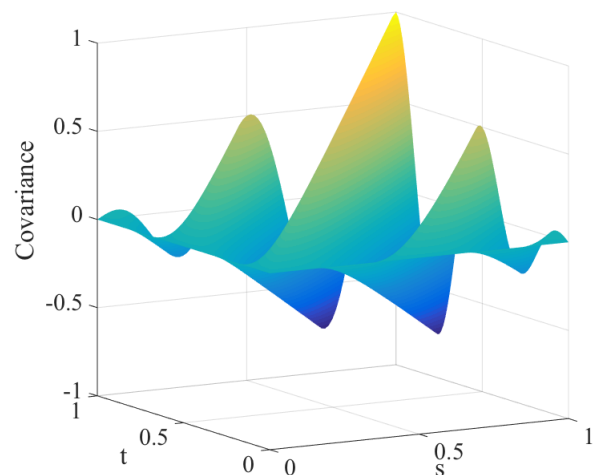


Figure 1: Target non-stationary covariance.

The process possesses a non-stationary "U-shaped" beta marginal distribution with cumulative distribution function given by

$$F(y; p, q) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} \int_0^u z^{p-1} (1-z)^{q-1} dz \quad (11)$$

where $u = \frac{y-y_{\min}}{y_{\max}-y_{\min}}$ with upper and lower bounds $y_{\min} = \mu_b(t) + \sigma_b(t) \sqrt{\frac{p(p+q+1)}{q}}$ and $y_{\max} = \mu_b(t) - \sigma_b(t) \sqrt{\frac{q(p+q+1)}{p}}$, and $\Gamma(\cdot)$ is the gamma function with parameters $p = 0.342$ and $q = 0.528$. The mean $\mu_b(t) = 0$ and the variance $\sigma_b^2(t) = C(t, t) = t$. The shape of this "U-shaped" beta PDF is shown in Figure 2 along with the standard normal pdf.

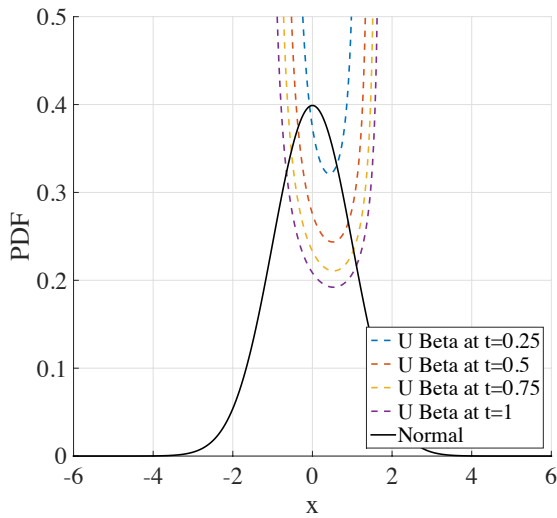
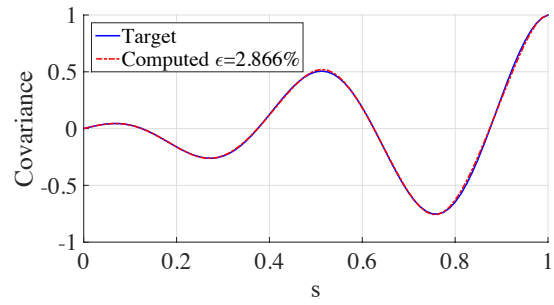
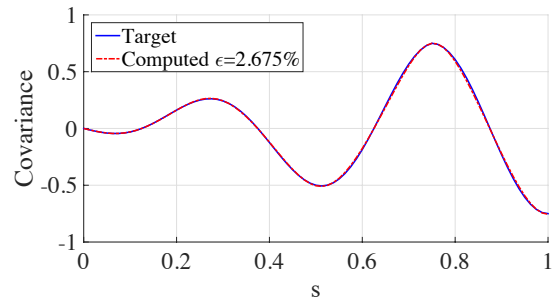


Figure 2: Strongly non-Gaussian and non-stationary "U-shaped" beta marginal PDF shown with the standard normal distribution

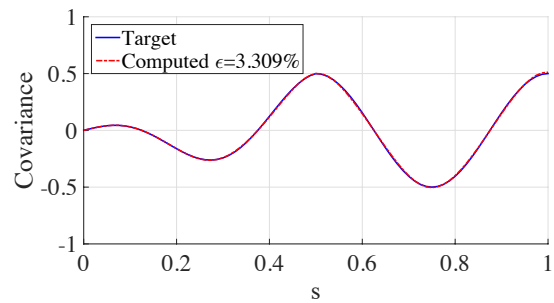
The converged non-Gaussian covariance at several time instants are shown in Figure 3. Despite the strongly non-Gaussian and non-stationary nature of the target process, the computed non-Gaussian and non-stationary covariance converges to the target with high accuracy for all values of t . The converged results are calculated in only 5 iterations. Across the entire time domain, the maximum relative difference is 5.58% with the converged results maintaining the shape and magnitude of the target covariance very well.



(a) $t = 1$



(b) $t = 0.75$



(c) $t = 0.5$



(d) $t = 0.25$

Figure 3: Comparison of the computed non-Gaussian and non-stationary covariance with the target at different times

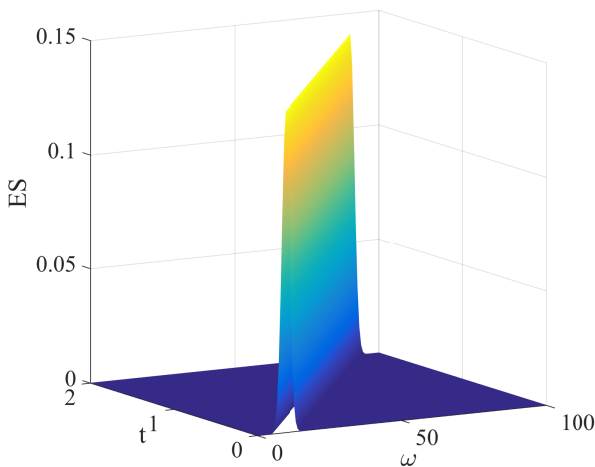
7. COMPARISON WITH THE ORIGINAL SRM-ITAM

The performance of the SRM-based ITAM and the proposed ITAM is demonstrated using the strongly

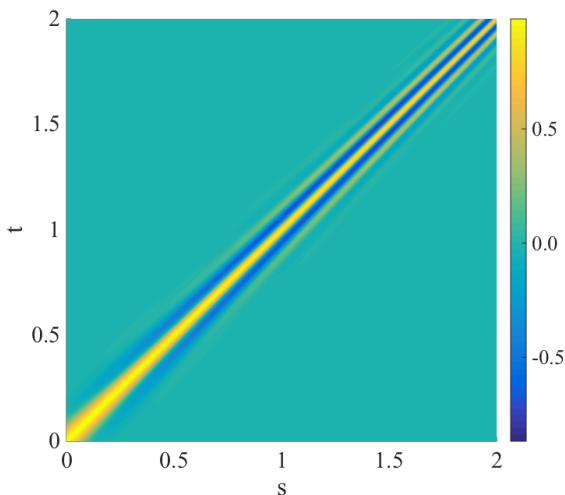
non-stationary target ES given as

$$S(t, \omega) = \frac{1}{4\sqrt{\pi}} \exp \left[- \left(\frac{\omega - \omega_0(t)}{2} \right)^2 \right] \quad (12)$$

where $\omega_0(t) = 10 + 40t$. Plots of the target ES and the corresponding covariance are presented in Figure 4. The prescribed PDF is the "U-shaped" beta PDF in Eq. (11) with constant mean $\mu_b(t) = 0$ and variance $\sigma_b^2(t) = 1$.



(a) Target non-stationary evolutionary spectrum



(b) Target non-stationary covariance

Figure 4: Target non-stationary evolutionary spectrum and covariance

Numerical results based on the two different ITAMs are presented on Table 1. As the original

Table 1: Relative difference of the computed non-Gaussian non-stationary covariance between the SRM-ITAM and KL-ITAM

Time	Relative difference (%)	
	SRM-ITAM	KL-TAM
$s = 0$	17.81	1.086
$s = 1$	12.11	5.317

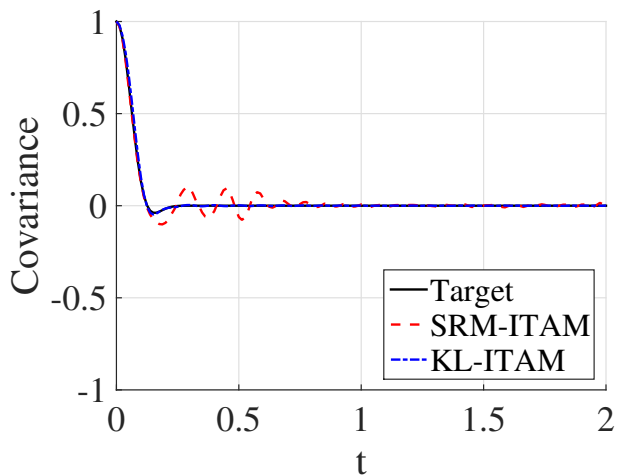
ITAM with SRM upgrades the underlying Gaussian ES, which is difficult to estimate for strongly non-stationary cases and requires computation of a pseudo-ES (Shields and Deodatis (2013a)), it has larger relative difference throughout time. However, the presented KL-ITAM, which upgrades the underlying correlation function directly, approximates the target covariance better than the previous method. The relative difference of the KL-based ITAM does not exceed 6%, the other difference is almost 15% over the time. These differences will be more significant for stronger non-Gaussian and non-stationary targets.

8. CONCLUSIONS

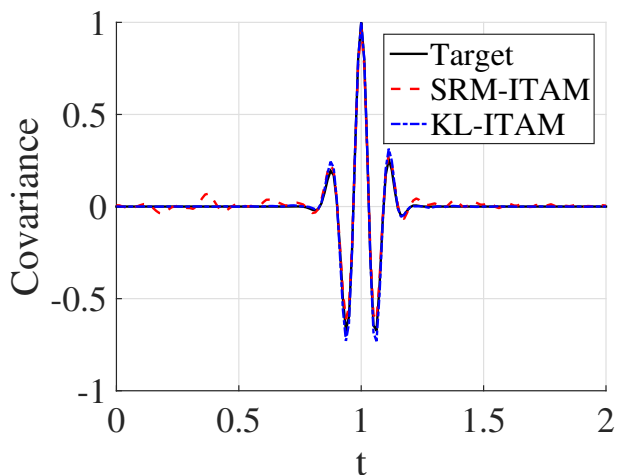
An improvement to the Iterative Translation Approximation Method for generation of strongly non-Gaussian and non-stationary process is proposed which utilizes direct upgrading on the non-stationary autocorrelation function and simulation with the K-L expansion. The method improves upon existing K-L based simulation methods by pairing them with translation process theory in order to match marginal non-Gaussian probability density functions exactly. It also improves upon the existing SRM-based ITAM simulation methods by removing the need to estimate the evolutionary spectrum which can produce significant errors. The method is shown to be highly accurate, straightforward to implement, and converge rapidly - even for processes that are very difficult to simulate.

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(a) $s = 0$



(b) $s = 1$

Figure 5: Comparison of the computed non-Gaussian and non-stationary covariance with the target between SRM-ITAM and KL-ITAM

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