Searching for Solitons in the Magnetosphere of a Magnetar

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Abstract

We describe a non-pertrubative method of searching for solitary waves travelling through the magnetosphere of a magnetar. The wave will be supported by the combined effects of nonlinearity and dispersion caused by the presence of a quantum electrodynamic (QED) vacuum in a strongly magnetized field and by a strongly magnetized plasma of the magnetosphere. Using this method, we have found a soliton in the form of an infinite current sheet. The method and the results of this paper could be conducive to research studying the emission of strongly magnetized stars.

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Chapter 1

Introduction

1.1 Pulsar Emission

Mitra and Rankin observed the emission from Pulsar B1857-26 [4]. Each time the pulsar did a sweep along the observer's line of sight, a pulse emission was recorded. In a data set of 200 pulses, where the pulses were aligned by the emission's central beam intensity, they noticed that substructures to the left and the right of the main intensity emission would "disappear" in one pulse and then "reappear" in another (see Figure 1.1). This was called the nulling effect. Mitra and Rankin then presented these pulses in a carousel configuration (see Figure 1.2). By assuming that that the pulsar beam slowly rotated around its beam axis, each beam sweep across the Earth's line of sight, provided data along a slice of the pulsar beam cone. By artificially placing the pulse intensities to where they may have originiated across the pulsar beam, Mitra and Rankin constructed a picture of a slice of the pulsar beam, cut perpendicular to its beam axis - carousel configuration. The high intensity, central structure of the emission corresponded to the area near the main axis of the pulsar beam, while the intensity substructures corresponded to a ring of 20 smaller beamlets surrounding the central beam.

It is unclear why these beamlets are so regularly spaced apart from one another or why they exist. We suggest that perhaps these beamlets are in actuality all one and the same. That is, a single, solitary electromagnetic wave is rotating around the pulsar beam axis. As the wave travels, its presence can alter the local electric and magnetic field, and apply a force on the local ions. By affecting the locations of the ions, the wave alters the locations of where radiation is emitted. If there is a soliton rotating around the pulsar beam axis, as it travels, it will go in and out of our line of sight. Hence if an intensity substructure is a soliton, the nulling effect would be the soliton leaving our line of sight.

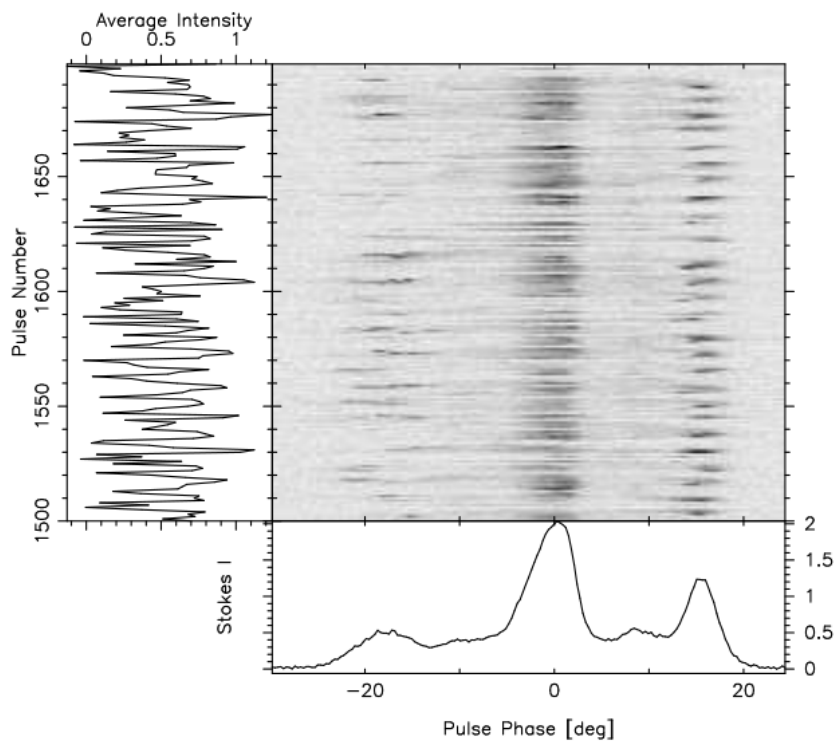

Figure 1.1: The intensity emissions of 200 pulses. Each horizontal line in the square corresponds to a pulse. The pulses are lined up by their central beam intensity. Notice that the intensity substructures on either side of the central intensity appears in some pulses, while disappearing in others.[4]

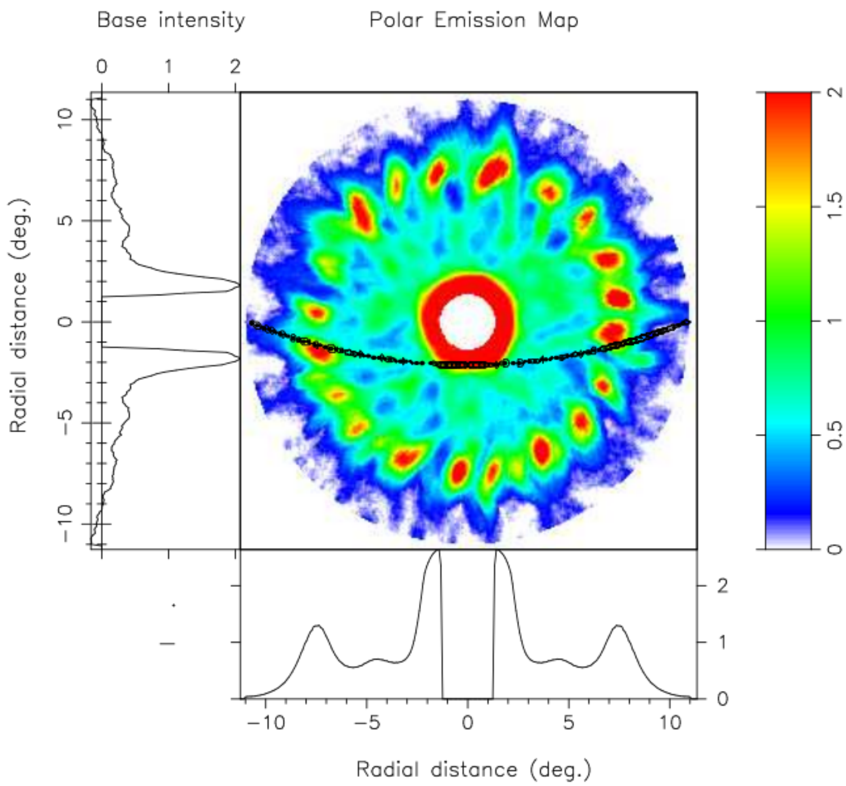

Figure 1.2: Intensity emission from Pulsar B1857-26 presented in carousel configuration. It is the reconstructed intensity emission of a slice of the pulsar beam, cut perpendicular to the beam axis. Black line indicates the position of one pulse. A series of pulses are needed to form this carousel configuration. [4]

1.2 Method for Finding Nonlinear Wave Solutions in the Magnetosphere

In a study presented by Mazur and Heyl, they were interested in stabilizing electromagnetic waves in the magnetosphere. They were able to find stabilized pulse trains in the magnetosphere by integrating over 15 coupled nonlinear ordinary differential equations (ODE) [3]. We are also interested in stabilizing waves in the magnetosphere, in particular we are interested in solitons. We will present a computationally simpler method of finding stabilized waves by integrating over three nonlinear ODEs.

1.3 Project Overview

In this study, we are searching for mathematical solutions that could indicate the possible existence of solitons in the magnetosphere. Our strategy will be to imagine an already formed wave travelling through this medium and then try to figure out how to best maintain the shape of this travelling wave. We decided that we could stabilize this wave using nonlinearity and dispersion effects. That is, the effect of the strongly magnetized plasma and the effect of the QED vacuum in a strongly magnetized field, respectively. As we will further discuss in Sections 2.2.2 and 2.2.3, these effects when taken individually, would cause a wave to change shape. However, when these effects are put together, there may be cases where the effects can counter one another and end up supporting a wave - this is what we are looking for.

Our study will make an ansantz that the travelling wave is a plane wave with the parameter $S = x - vt$. The variable x refers to position, as v and t refer to velocity and time, respectively. We will assume that our solitary wave is propagating in a homogenous medium, where we will neglect the effects of gravity and the magnetic field forces on the plasma medium. In addition, we will describe our travelling electromagnetic wave in terms of the magnetic vector potential and the electric potential in order to simplify computations. We are doing this in an effort to explore the possibility of a single, solitary electromagnetic wave surviving the magnetosphere.

Chapter 2

Theory

2.1 Soliton

It is a single, solitary wave that has a finite extent in space and time. The wave has a start and an end, with an amplitude that approaches zero far from the peak of the wave. As it travels through a medium, the soliton will maintain its shape.

2.1.1 Nonperturbative Method

We are making an ansatz that our soliton is a plane wave with a parameter $S = v - xt$. This parameter choice will help to simplify computations (see Section 3.1) and make the solutions that we derive, nonperturbative.

Since the electromagnetic wave is travelling through a nonlinear medium (see 2.2.3), it is key that we use a nonperturbative method of representing our solution. That is, we will be solving our solutions directly, rather than using perturbative methods of linearly approximating the solution. Perturbative methods, such as using Fourier Theory to approximate a wave, will not provide a generalized solution to nonlinear problems. They will only work for limited cases as we will need to solve these nonlinear equations computationally and can only integrate over a finite number of Fourier modes.

2.2 Magnetosphere

A magnetar is a pulsar with an exceptionally strong magnetic field on the scale of 10^{11} Teslas [5]. Due to the quickly changing magnetic fields near the spinning magnetar, a strong electric field is induced on to the surface of the star. This electric field works against gravity, and pulls electrons up from along the poles, forming the magnetosphere.

2.2.1 Phase and Group Velocity

If we imagine a set of harmonic waves, each travelling at different velocities, these velocities will be referred to as phase velocities, v_p. If we add up the amplitudes of these waves while they are travelling, they form a wave packet that will travel at its own charactersitic velocity. This is the group velocity, v_g. This is not surprising as the position at which the wave crests coincide is itself a location that alters with time [6].

$$v_p = \frac{\omega}{k} \tag{2.1}$$

$$v_g = \frac{\partial \omega}{\partial k} \tag{2.2}$$

Note that group velocity cannot exceed the speed of light, while phase velocity can. This is because phase velocity cannot carry information, while group velocity can.

2.2.2 Dispersion

As will later be seen in the Section 3.2, the equations will use the plasma frequency, ω_p. If some of the plasma gets displaced, the Coulomb force will cause the particles in the plasma to oscillate at this frequency ω_p. Since this value is set based on the medium, then from the definition of the phase velocity (see Equation 2.1), waves of different spatial frequencies will travel at different speeds. If we imagine that our solitary wave is made up of a combination of different sinusoidal waves, it is easy to imagine that the wave will disperse as different spatial frequency components propagate at different speeds.

2.2.3 Nonlinearity

Nonlinearity occurs when the dipoles of the medium no longer respond the alternating electric field of our wave in a linear fashion. This nonlinearity is generated by the radiative corrections of QED. In this case, the polarization has a nonlinear relationship with field.

The significance is more obvious by seeing the relationship,

$$\mathbf{D} = \mathbf{E} + \mathbf{P}(\mathbf{E}, \mathbf{B}) \tag{2.3}$$

$$\mathbf{H} = \mathbf{B} - \mathbf{M}(\mathbf{E}, \mathbf{B}) \tag{2.4}$$

In the magnetosphere, the medium that causes this nonlinearity is the QED vacuum in the presence of a strong magnetic field. The magnetic field causes the QED vacuum to become magnetized and polarized[1]. This effect can become so powerful that it can alter the shape of a wave as it is travelling through this magnetized vacuum. More specifically, this medium adds nonlinearity to the wave such that different amplitudes of the wave will travel at different speeds, much like a surfer's wave where the top of the wave overtakes the base the wave. This nonlinearity will cause the wave to steepen and shock.

In our equations, we will account for this nonlinearity in our vacuum dielectric and inverse magnetic permeability tensors, ϵ_{ij} and μ_{ij}^{-1} respectively. These tensors were derived from the following electromagnetic Lagrangian relationships [1].

$$\mathbf{D} = \mathbf{E} + \mathbf{P} = \frac{\partial \mathcal{L}}{\partial \mathbf{E}} \tag{2.5}$$

$$\mathbf{H} = \mathbf{B} - \mathbf{M} = -\frac{\partial \mathcal{L}}{\partial \mathbf{B}} \tag{2.6}$$

Note that rather than using the full, generalized equation, we are using the weak field approximation, where the field is less than $B_k = 4.413 \times 10^{13}$ G. The reason for this is that the generalized equation will lead to complicated tensors that will bring difficulties in the computation. So in this study we will test our computational method with the weak field approximation. Futher studies can then apply this computational method to the generalized tensors.

Chapter 3

Mathematical Detail

Here we will be rearranging variables in Maxwell's equations into terms of magnetic vector potential and electric potential. The purpose of this is to minimize the number of variables that we need to keep track of in order to simplify our compuations later on. After redefining the variables in Maxwell's equations into values of potentials, we can manipulate the equations and solve for the values of magnetic vector potential and electric potential. From these values, we can then figure out the other variables in Maxwell's equations.

We will begin by presenting the mathematical benefit of using $S = x - vt$ as its properties will be applied to our derivation.

3.1 Usefulness of $S = x - vt$ parameter

Our choice in using the parameter $S = x - vt$ allows us to solve ordinary differential equations (ODE) instead of partial differential equations (PDE). To demonstrate,

$$\frac{\partial f(S)}{\partial x} = \frac{\mathrm{d}f}{\mathrm{d}S}\frac{\mathrm{d}S}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}x} \tag{3.1}$$

$$\frac{\partial f(S)}{\partial t} = \frac{\mathrm{d}f}{\mathrm{d}S}\frac{\mathrm{d}S}{\mathrm{d}t} = -v\frac{\mathrm{d}f}{\mathrm{d}x} \tag{3.2}$$

Here we have transformed the PDEs that had partial derivatives with respect to x and to t into equations of ODE. Note that the derivatives with respect to y and to z will go to zero. We will frequently use these parameter S properties as we manipulate Maxwell's equations in the following section 3.2.

3.2 Storing Variables in Terms of Magnetic Vector Potential and Electric Potential

Since we are interested in an electromagnetic wave, we begin our derivations from Maxwell's equations in matter, in Gaussian units. We are using Gaussian units so that the units for electric and magnetic fields will be the same.

Recall that the variables are in terms of the parameter $S = x - vt$, and as such, the partial derivatives with respect to x and t are outlined in equations 3.1 and 3.2. Also note that in our notation, primes indicate derivatives with respect to S (or as we now have shown in equation 3.1, with respect to x, as well).

$$\nabla \cdot \mathbf{B} = 0 \tag{3.3}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \tag{3.4}$$

$$\nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{J_f} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} \tag{3.5}$$

$$\nabla \cdot \mathbf{D} = \mathbf{4}\pi\rho_{\mathbf{f}} \tag{3.6}$$

We will now proceed to describe the Maxwell variables in terms of potentials. Taking the first Maxwell equation, 3.3, we make use of the magnetic field divergence being zero.

$$\begin{aligned} \mathbf{B} &= \nabla \times \mathbf{A} + \mathbf{B_o} \\ \mathbf{B} &= -A_z'\mathbf{y} + A_y'\mathbf{z} + \mathbf{B_o} \end{aligned} \tag{3.7}$$

Taking the second Maxwell's Equation 3.4 and applying 3.7,

$$\begin{aligned} \nabla \times \mathbf{E} &= -\frac{1}{c}\frac{\partial}{\partial t}(\nabla \times \mathbf{A}) \\ \nabla \times \mathbf{E} &= \nabla \times (-\frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}) \\ 0 &= \nabla \times (\mathbf{E} + \frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}) \end{aligned} \tag{3.8}$$

Since the curl of $\mathbf{E} + \frac{1}{c}\frac{\delta}{\delta t}\mathbf{A}$ is zero, then we can set the term to equal the negative gradient of a scalar field ϕ. We will choose a scalar field such that $A_x = 0$ by Coloumb Gauge theory.

$$\begin{aligned} -\nabla\phi &= \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial S}\frac{\partial S}{\partial t} \\ \mathbf{E} &= \frac{v}{c}\frac{\partial \mathbf{A}}{\partial x} - \nabla\phi \\ \mathbf{E} &= \frac{v}{c}\mathbf{A}' - \phi'\mathbf{x} \end{aligned} \tag{3.9}$$

For the third Maxwell Equation 3.5,

$$\begin{aligned} 0 &= \nabla \times \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{D}}{\partial S}\frac{\partial S}{\partial t} - \frac{4\pi}{c}\mathbf{J} \\ 0 &= -H_z'\mathbf{y} + H_y'\mathbf{z} + \frac{v}{c}\mathbf{D}' - \frac{4\pi}{c}\mathbf{J} \\ \frac{4\pi}{c}\mathbf{J} &= -H_z'\mathbf{y} + H_y'\mathbf{z} + \frac{v}{c}\mathbf{D}' \end{aligned} \tag{3.10}$$

Focusing on the x, y and z components of equation 3.10 individually, we find that

$$\frac{4\pi}{c}J_x = \frac{v}{c}D_x' \tag{3.11}$$

$$\frac{4\pi}{c}J_y = -H_z' + \frac{v}{c}D_y' \tag{3.12}$$

$$\frac{4\pi}{c}J_z = H_y' + \frac{v}{c}D_z' \tag{3.13}$$

As will be shown in equation 3.16, there is a relationship between the current density **J** and the electric field. Utilizing equation 3.9, we can then define the components of the current density in terms of magnetic potential..

In the final Maxwell Equation 3.6, we apply equation 3.11 and find

$$\begin{aligned} \rho &= \frac{D_x'}{4\pi} \\ \rho &= \frac{1}{4\pi}\frac{4\pi J_x}{v} \\ J_x &= v\rho \end{aligned} \tag{3.14}$$

In order to close the system, we will also apply Lorentz Force Equation. First, we start with the change in current density with respect to time and use the relationship $\mathbf{J} = \rho\mathbf{v}$. The variables $\mathbf{J}, \rho$, and v represent volume current density, charge density, and velocity respectively.

$$\begin{aligned} \frac{\partial \mathbf{J}}{\partial t} &= \frac{\partial \mathbf{J}}{\partial S}\frac{\partial S}{\partial t} \\ \frac{\partial(\rho\mathbf{v})}{\partial t} &= -v\frac{\partial \mathbf{J}}{\partial S} \\ \rho\frac{\partial \mathbf{v}}{\partial t} &= -v\mathbf{J}' \end{aligned} \tag{3.15}$$

Now we will use the Lorentz Force Equation, $\mathbf{F} = q(\mathbf{E} - \mathbf{v} \times \mathbf{B})$. By assuming that the velocity of the charges in the plasma are close to zero, $\mathbf{v} \times \mathbf{B} \to 0$, we find that $m\mathbf{a} = q\mathbf{E}$. Returning to equation 3.15,

$$\begin{aligned} -v\mathbf{J}' &= \rho\left(\frac{q\mathbf{E}}{m}\right) \\ &= nq\left(\frac{q\mathbf{E}}{m}\right) \\ &= \frac{n}{m}q^2\mathbf{E} \\ &= \frac{\omega_p^2}{4\pi}\mathbf{E} \end{aligned} \tag{3.16}$$

Above, we used definition of the charge density, $\rho = nq$, where q and n are the charge and the particle density, respectively. We then applied the relationship $\frac{nq^2}{m} = \frac{\omega_p^2}{4\pi}$, where ω_p is the plasma frequency.

Here we take the x-component of the equation 3.16 and plug in the electric field from 3.9. We make use of the fact that we have set $A_x = 0$, and then we apply equation 3.11.

$$
\begin{aligned}
-vJ'_x &= \frac{\omega_p^2}{4\pi}\left(\frac{v}{c}A'_x - \phi'\right) \\
-vJ'_x &= -\frac{\omega_p^2}{4\pi}\phi' \\
vJ_x &= \frac{\omega_p^2}{4\pi}\phi \\
\frac{v^2}{4\pi}D'_x &= \frac{\omega_p^2}{4\pi}\phi \\
v^2 D'_x &= \omega_p^2\phi
\end{aligned} \tag{3.17}
$$

By applying the electric field relationship in equation 3.9 onto equation 3.16, we get $-v\mathbf{J'} = \frac{\omega_p^2}{4\pi}(\frac{v}{c}\mathbf{A'} - \phi'\mathbf{x})$. From this, we can find the x, y, and z components of $\mathbf{J}$. In addition, if we include the relationships 3.12, and 3.13, we find

$$-H'_z + \frac{v}{c}D'_y = \frac{4\pi}{c}J_y = -\frac{\omega_p^2}{c^2}A_y \tag{3.18}$$

$$H'_y + \frac{v}{c}D'_z = \frac{4\pi}{c}J_z = -\frac{\omega_p^2}{c^2}A_z \tag{3.19}$$

Finally, we will use the definition of the fields.

$$D_i = \epsilon_{ij}E_j \tag{3.20}$$

$$H_i = \mu_{ij}^{-1}B_j \tag{3.21}$$

For this study, we will apply the weak field limit of ϵ_{ij} and μ_{ij}^{-1} (recovered from Klein and Nigam's paper [2]). A more generalized field can be found in a paper by Heyl and Hernquist [1].

$$\epsilon_{ij} = \delta_{ij} + \frac{1}{45\pi}\frac{\alpha}{B_k^2}[2(E^2 - B^2]\delta_{ij} + 7B_iB_j] \tag{3.22}$$

$$\mu_{ij}^{-1} = \delta_{ij} + \frac{1}{45\pi}\frac{\alpha}{B_k^2}[2(E^2 - B^2)\delta_{ij} - 7E_iE_j] \tag{3.23}$$

We have now redefined all of the necessary variables into terms of the magnetic vector and electric potentials. More specifically, into values of magnetic potential components A_y and A_z, and the electric potential ϕ.

3.3 Setting Up Equations for Computation

In this section we will be expanding our previous equations (from 3.2) and will solve for A_y, A_z, and ϕ numerically with the Runge Kutta 4 method. A copy of this numerical solver is attached A.

As we are dealing with many different terms, Maple will be used to do the algebraic manipulations. Also note that we will be making the equations dimensionless by setting identities for the length and the field strength.

3.3.1 Writing Equations Explicitly in A_y, A_z, and ϕ

Here we will rewrite the components of **H**' and **D**' into terms of A_y, A_z, and ϕ. This is necessary so that on both sides of the equation, for the equations 3.18, 3.19, and 3.17, will be written as values of the potentials. Later, these rewritten versions of 3.18, 3.19, and 3.17 will be used to numerically solve for A_y, A_z, and ϕ.

From the field definitions in 3.20 and 3.21, we see that the components of **H**' and **D**' are defined by the fields **B** and **E**, and the matrices ϵ_{ij} and μ_{ij}^{-1}. Recall from 3.7 and 3.8 that $\mathbf{B} = \langle B_{0,x}, B_{0,y} - A_z', B_{0,z} + A_y' \rangle$ and $\mathbf{E} = \langle -\phi', \frac{v}{c} A_y', \frac{v}{c} A_z' \rangle$. The extended form of the matrices ϵ_{ij} and μ_{ij}^{-1} are shown below (taken from [2]).

$$\epsilon_{ij} = \begin{bmatrix} 1 + \frac{\alpha}{45\pi B_k^2}\left(2\left(E^2 - B^2\right) + 7B_x^2\right) & \frac{\alpha}{45\pi B_k^2}\left(7B_xB_y\right) & \frac{\alpha}{45\pi B_k^2}\left(7B_xB_z\right) \\ \frac{\alpha}{45\pi B_k^2}\left(7B_xB_y\right) & 1 + \frac{\alpha}{45\pi B_k^2}\left(2\left(E^2 - B^2\right) + 7B_y^2\right) & \frac{\alpha}{45\pi B_k^2}\left(7B_yB_z\right) \\ \frac{\alpha}{45\pi B_k^2}\left(7B_xB_z\right) & \frac{\alpha}{45\pi B_k^2}\left(7B_yB_z\right) & 1 + \frac{\alpha}{45\pi B_k^2}\left(2\left(E^2 - B^2\right) + 7B_z^2\right) \end{bmatrix} \tag{3.24}$$

$$\mu_{ij}^{-1} = \begin{bmatrix} 1+\frac{\alpha}{45\pi B_k^2}\left(2\left(E^2-B^2\right)-7E_x^2\right) & -\frac{\alpha}{45\pi B_k^2}\left(7E_xE_y\right) & -\frac{\alpha}{45\pi B_k^2}\left(7E_xE_z\right) \\ -\frac{\alpha}{45\pi B_k^2}\left(7E_xE_y\right) & 1+\frac{\alpha}{45\pi B_k^2}\left(2\left(E^2-B^2\right)-7E_y^2\right) & -\frac{\alpha}{45\pi B_k^2}\left(7E_yE_z\right) \\ -\frac{\alpha}{45\pi B_k^2}\left(7E_xE_z\right) & -\frac{\alpha}{45\pi B_k^2}\left(7E_yE_z\right) & 1+\frac{\alpha}{45\pi B_k^2}\left(2\left(E^2-B^2\right)-7E_z^2\right) \end{bmatrix} \tag{3.25}$$

The extended form of the matrices are still not explicitly written in terms of the potentials. Here we will resolve that by taking the products of **E** and **B** relevant to the matrices and finding their values in terms of potentials.

From dot product of 3.8

$$E^2 = \phi'^2 + \left(\frac{v}{c}\right)^2 A_y'^2 + \left(\frac{v}{c}\right)^2 A_z'^2 \tag{3.26}$$

From dot product of 3.7

$$B^2 = B_{0,x}^2 + \left(B_{0,y} - A_z'\right)^2 + \left(B_{0,z} + A_y'\right)^2 \tag{3.27}$$

Subtracting Equation 3.27 from 3.26

$$E^2 - B^2 = \phi'^2 + \left[\left(\frac{v}{c}\right)^2 - 1\right] A_y'^2 + \left[\left(\frac{v}{c}\right)^2 - 1\right] A_z'^2 - B_{0,x}^2 - B_{0,y}^2 - B_{0,z}^2 + 2B_{0,y}A_z' + 2B_{0,z}A_y' \tag{3.28}$$

Taking the derivative of 3.28 with respect to x

$$\left[E^2 - B^2\right]' = 2\phi'\phi'' + 2\left[\left(\frac{v}{c}\right)^2 - 1\right] A_y'A_y'' + 2\left[\left(\frac{v}{c}\right)^2 - 1\right] A_z'A_z'' + 2B_{0,y}A_z'' + 2B_{0,z}A_y'' \tag{3.29}$$

Now that $\mathbf{B}, \mathbf{E}, \epsilon_{ij}$, and μ_{ij}^{-1} are explicitly defined by A_y, A_z, and ϕ, we can find the components of **D** and **H** using relationships 3.20 and 3.21.

Note that the equations below are split into three rows in order to increase readability. Each row in the D_i equation represents a cell in the matrix product $\epsilon_{ij}E_j$. Likewise, each H_i equation is split into three rows so that each row corresponds to a cell $\mu_{ij}^{-1}B_j$.

$$\begin{aligned}D'_x =& -\phi'' - \frac{\alpha}{45\pi B_k^2}\left(2\left[E^2-B^2\right]'\phi' + 2\left[E^2-B^2\right]\phi'' + 7B_{0,x}^2\phi''\right) \\ &+ \frac{7\alpha}{45\pi B_k^2}\frac{v}{c}B_{0,x}\left[-A''_z A'_y + (B_{0,y}-A'_z)A''_y\right] \\ &+ \frac{7\alpha}{45\pi B_k^2}\frac{v}{c}B_{0,x}\left[-A''_y A'_z + (B_{0,z}-A'_y)A''_z\right]\end{aligned} \tag{3.30}$$

$$\begin{aligned}D'_y =& \frac{7\alpha}{45\pi B_k^2}\left[B_{0,x}A''_z\phi' + \left(B_{0,x}A'_z - B_{0,x}B_{0,y}\right)\phi''\right] \\ &+ \frac{v}{c}A''_y + \frac{\alpha}{45\pi B_k^2}\frac{v}{c}\left[2\left[E^2-B^2\right]'A'_y + 2\left[E^2-B^2\right]A''_y - 14\left(B_{0,y}-A'_z\right)A''_z A'_y + 7\left(B_{0,y}-A'_z\right)^2 A''_y\right] \\ &+ \frac{7\alpha}{45\pi B_k^2}\frac{v}{c}\left[\left(B_{0,y}A''_z - 2A'_z A''_z\right)\left(B_{0,z}+A'_y\right) + \left(B_{0,y}A'_z - A'^2_z\right)A''_y\right]\end{aligned} \tag{3.31}$$

$$\begin{aligned}D'_z =& -\frac{7\alpha}{45\pi B_k^2}B_{0,x}\left[\phi''(B_{0,z}+A'_y) + \phi' A''_y\right] \\ &+ \frac{7\alpha}{45\pi B_k^2}\frac{v}{c}\left[-A''_z(B_{0,z}A'_y + A'^2_y) + (B_{0,y}-A'_z)(B_{0,z}A''_y + 2A'_y A''_y)\right] \\ &+ \frac{v}{c}A''_z + \frac{\alpha}{45\pi B_k^2}\frac{v}{c}\left[2[E^2-B^2]A''_z + 2[E^2-B^2]'A'_z + 14(B_{0,z}+A'_y)A''_y A'_z + 7(B_{0,z}+A'_y)^2 A''_z\right]\end{aligned} \tag{3.32}$$

$$\begin{aligned}H'_x =& \frac{\alpha}{45\pi B_k^2}\left(2[E^2-B^2]' - 14\left(\frac{v}{c}\right)^2\phi'\phi''\right)B_{0,x} \\ &+ \frac{7\alpha}{45\pi B_k^2}\frac{v}{c}\left[\phi''(B_{0,y}A'_y - A'_y A'_z) + \phi'(B_{0,y}A''_y - A''_y A'_z - A'_y A''_z)\right] \\ &+ \frac{7\alpha}{45\pi B_k^2}\frac{v}{c}\left[\phi''(B_{0,z}A'_z - A'_y A'_z) + \phi'(B_{0,z}A''_z - A''_y A'_z - A'_y A''_z)\right]\end{aligned} \tag{3.33}$$

$$\begin{aligned}H'_y =& \frac{7\alpha}{45\pi B_k^2}\frac{v}{c}\left[\phi'' A'_y + \phi' A''_y\right]B_{0,x} \\ &- A''_z + \frac{\alpha}{45\pi B_k^2}\left[2[E^2-B^2]'(B_{0,y}-A'_z) - 2[E^2-B^2]A''_z - 14\left(\frac{v}{c}\right)^2 A'_y A''_y(B_{0,y}-A'_z) + 7\left(\frac{v}{c}\right)^2 A'^2_y A''_z\right] \\ &- \frac{7\alpha}{45\pi B_k^2}\left(\frac{v}{c}\right)^2\left[A''_z(B_{0,z}A'_y + A'^2_y) + A'_z(B_{0,z}A''_y + 2A'_y A''_y)\right]\end{aligned} \tag{3.34}$$

$$
\begin{aligned}
H_z' =& \frac{7\alpha}{45\pi B_k^2}\frac{v}{c}\left[\phi'' A_z' + \phi' A_z''\right] B_{0,x} \\
&- \frac{7\alpha}{45\pi B_k^2}\left(\frac{v}{c}\right)^2\left[A_y'' A_z' B_{0,y} + A_y' A_z'' B_{0,y} - A_y'' A_z'^2 - 2A_y' A_z' A_z''\right] \\
&+ A_y'' + \frac{\alpha}{45\pi B_k^2}\left[\left(2\left[E^2 - B^2\right] - 7\left(\frac{v}{c}\right)^2 A_z'^2\right) A_y'' + \left(2\left[E^2 - B^2\right]' - 14\left(\frac{v}{c}\right)^2 A_z' A_z''\right)(B_{0,z} + A_y')\right]
\end{aligned} \tag{3.35}
$$

Now that **H** and **D** are explicitly defined by the magnetic and electric potentials, we can write the equations of 3.18, 3.19, and 3.17 entirely in terms of potentials. In this way we can manipulate the equations so that we can solve for A_y, A_z, and ϕ, using nonlinear ODEs.

As our equations are nonlinear and have many different terms, finding a solution can be difficult. So instead, we will be turning on our variables one at a time. That way, we can slowly but steadily build our way up to the full solution.

3.3.2 Turning on A_y

In this section, we are only concerned with A_y, its derivatives and the constants in the equation. Our goal here is to find a general solution to our nonlinear A_y equation. With this general solution, we will try to find a case where a soliton might exist. Plugging in 3.35 and 3.31 into equation 3.18, we find that

$$
\begin{aligned}
-\left(\frac{\omega_p}{c}\right)^2 A_y =& - A_y'' - \frac{\alpha}{45\pi B_k^2}[2(E^2 - B^2)A_y'' + 2(E^2 - B^2)' A_y'] \\
&+ \left(\frac{v}{c}\right)^2 A_y'' + \frac{\alpha}{45\pi B_k^2}\left(\frac{v}{c}\right)^2[2(E^2 - B^2)' A_y' + 2(E^2 - B^2)A_y'']
\end{aligned} \tag{3.36}
$$

If we only focus on the A_y variables in 3.28 and 3.29, we find $E^2 - B^2 = \left[\left(\frac{v}{c}\right)^2 - 1\right] A_y'^2$ and $2\left(E^2 - B^2\right)' = 2\left[\left(\frac{v}{c}\right)^2 - 1\right] A_y' A_y''$. Therefore, equation 3.36 becomes

$$
\begin{aligned}
-\left(\frac{\omega_p}{c}\right)^2 A_y =& \left[\left(\frac{v}{c}\right)^2 - 1\right] A_y'' + \frac{6\alpha}{45\pi B_k^2}\left[\left(\frac{v}{c}\right)^2 - 1\right]^2 (A_y')^2 A_y'' \\
-\left(\frac{\omega_p}{c}\right)^2 \left|\left(\frac{v}{c}\right)^2 - 1\right|^{-1} A_y =& A_y'' + \frac{6\alpha}{45\pi B_k^2}\left|\left(\frac{v}{c}\right)^2 - 1\right| (A_y')^2 A_y''
\end{aligned} \tag{3.37}
$$

We can simplify the equation by making it dimensionless. We will set the term $\frac{2\alpha}{45\pi B_k^2} 3\left|\left(\frac{v}{c}\right)^2 - 1\right|$, which is a measurement of field strength, to 1. Likewise, we will set $\left(\frac{\omega_p}{c}\right)^2 \left|\left(\frac{v}{c}\right)^2 - 1\right|^{-1}$, which is in units of length^{-2}, also to 1.

Note that we are currently taking the field strength and length^{-2} terms as absolute values. In order to account for all cases of the equation 3.37, we need to consider when these terms are postive and when they are negative. The sign of these terms will depend on whether the phase velocity v is greater or less than the speed of light in a vacuum, c. The case when $v = c$ will make the equation a trivial problem as the field strength and length^{-2} terms will be zero.

For now, we will just focus on the case when the phase velocity is greater than c. In particular, that the terms are set to 1.

$$
\begin{aligned}
-A_y &= A_y'' + A_y'^2 A_y'' \\
-A_y &= A_y''(1 + A_y'^2) \\
A_y'' &= -\frac{A_y}{1 + A_y'^2}
\end{aligned}
\tag{3.38}
$$

In this case, with just one variable A_y and its derivatives, we can solve for A_y analytically. We shall do so right now.

Let

$$
\begin{aligned}
x &= A_y, \quad \frac{\mathrm{d}x}{\mathrm{d}t} = A_y' = y \\
y &= A_y', \quad \frac{\mathrm{d}y}{\mathrm{d}t} = A_y'' = y'
\end{aligned}
\tag{3.39}
$$

Then we can rewrite equation 3.38 as the expression

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x} \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{x}{1+y^2}\frac{1}{y} \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{x}{y+y^3} \qquad (3.40) \\
\int \left(y+y^3\right)\mathrm{d}y &= -\int x\mathrm{d}x \\
\frac{y^2}{2}+\frac{y^4}{4} &= -\frac{x^2}{2} \qquad (3.41)
\end{aligned}$$

Removing the substitions from 3.39, we find

$$(A'_y)^2 + \frac{(A'_y)^4}{2} = -A_y^2 \qquad (3.42)$$

From the equation 3.40, we can use the information to make a phase diagram with A'_y and A_y for the case when the phase velocity is greater than c.

$$\frac{\mathrm{d}A'_y}{\mathrm{d}A_y} = -\frac{A_y}{A'_y + (A'_y)^3} \qquad (3.43)$$

In the case of when the phase velocity is less than c, the field strength and the length^{-2} terms become negative. As such, we can test this case by setting these terms as -1. We find that

$$\frac{\mathrm{d}A'_y}{\mathrm{d}A_y} = \frac{A_y}{A'_y - (A'_y)^3} \qquad (3.44)$$

$$\begin{aligned}
(A'_y)^2 - \frac{(A'_y)^4}{2} &= A_y^2 \\
\pm\left[(A'_y)^2 - \frac{(A'_y)^4}{2}\right]^{\frac{1}{2}} &= A_y \qquad (3.45)
\end{aligned}$$

Due to the symmetry between A_y and A_z, this method is also applicable to A_z.

3.3.3 Turning on ϕ

Similar to section 3.3.2, we will only look at ϕ and its derivatives; the other variables will be set to zero. We will solve for ϕ using equation 3.17 and 3.30. Putting these equations together, we find

$$-\frac{\omega_p^2}{v^2}\phi = \phi'' + \frac{6\alpha}{45\pi B_k^2}\phi'^2\phi'' \tag{3.46}$$

If we set the constants $-\frac{\omega_p^2}{v^2}$ and $\frac{6\alpha}{45\pi B_k^2}$ to one, we see that this equation will be identical to the expression in 3.38. Hence the solution found in section 3.3.2 will also be applicable to ϕ.

3.3.4 Turning on A_y and A_z

Here we are allowing mixing between the variables A_y and A_z, and its derivatives. Using equations 3.18 and 3.19, and applying 3.34, 3.35, 3.31, and 3.32, we find

$$-\left(\frac{w_p}{c}\right)^2 A_y = \left[\left(\frac{v}{c}\right)^2 - 1\right] A_y'' + M_1 A_y'' A_z'^2 + 2M_1 A_y' A_z' A_z'' + 3M_1 A_y'^2 A_y'' \tag{3.47}$$

$$-\left(\frac{w_p}{c}\right)^2 A_z = \left(\left(\frac{v}{c}\right)^2 - 1\right) A_z'' + M_1 A_y'^2 A_z'' + 2M_1 A_y' A_y'' A_z' + 3M_1 A_z'^2 A_z'' \tag{3.48}$$

where M_1 is

$$M_1 = -\frac{4\alpha v^2 c^2 - 2\alpha c^4 - 2v^4\alpha}{45c^4\pi B_k^2} \tag{3.49}$$

We see that if we switch the y and z subscripts of one of the equations, then the equations will become identical. As such, we see that the equations are perfectly symmetic.

Here we take the equation 3.47 and divide both sides by $\|\left(\frac{v}{c}\right)^2 - 1\|$, being careful with the absolute values. Recall in Section 3.3.2 that we set the term $\left(\frac{w_p}{c}\right)^2 \|\left(\frac{v}{c}\right)^2 - 1\|^{-1} = 1$, here we will do it again. Below, we will pull $MA_y''A_z'^2 + 2MA_y'A_z'A_z''$ together as $[MA_y'A_z'^2]'$ and factor the numerator of M_1.

$$-A_y = \frac{v^2 - c^2}{\|v^2 - c^2\|} A_y'' + 3M_2 A_y'^2 A_y'' + [M_2 A_y' A_z'^2]' \tag{3.50}$$

$$M_2 = \frac{2\alpha}{45c^2\pi B_k^2} \frac{\left(v^2 - c^2\right)^2}{\|v^2 - c^2\|} \tag{3.51}$$

Since A_y and A_z are symmetric in this section, the equation for $-A_z$ is just 3.50 with the y and z subscripts switched.

Chapter 4

Results

Using the equations from Chapter 3 and plugging them into our nonlinear ODE solver (see A), we were able to produce phase diagrams of our equations. From these diagrams we can infer the characteristics of a wave and determine whether a soliton can be found with particular solution.

Note that based on making the equations dimensionless (see Section 3.3.2), we realize that we need to consider both the case when the phase velocity, v, is less than c and the case when it is greater than c. As well, recall in Chapter 3, we were turning the variables on one at a time. So in the same fashion, our results are grouped by the variables turned on in the equation and by the phase velocity.

4.1 Turning on A_y with Phase Velocity Greater Than C

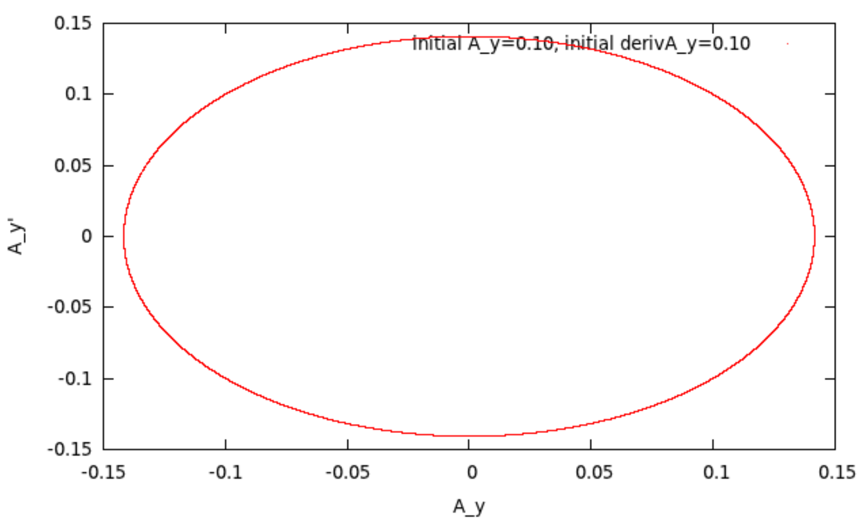

Figure 4.1: Phase diagram of A_y with a $v_p > c$ and a small input. Notice how circular it is when the input is small.

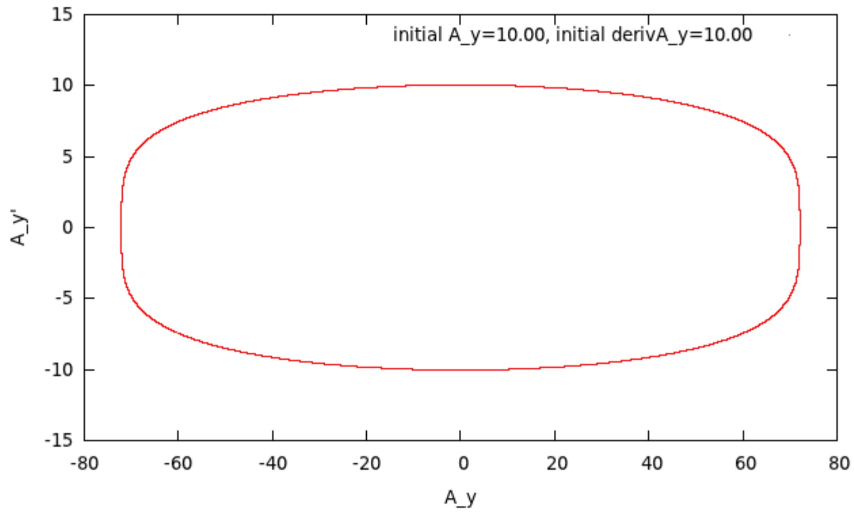

Figure 4.2: Phase diagram of A_y with a $v_p > c$ and a large input. Notice that when input is large, it is an ellipse and it is more squared out.

When the phase velocity is greater than c, we observe that the phase diagrams of A'_y versus A_y are stable and continuous (see figures 4.1 and 4.2). This indicates that although the wave is supported by nonlinearity and dispersion in the magnetosphere, the wave is periodic. Since a soliton is a single, solitary wave in space, it is not periodic. Therefore, this type of solution will not reveal a soliton.

What is interesting about these phase diagrams is that we can see the affects of nonlinearity. Previously, we were accounting for the nonlinearity in the magnetosphere through the vacuum dielectric and inverse magnetic permeability tensors, ϵ_{ij} and μ_{ij}^{-1}. As shown in the equations 3.22 and 3.23, the nonlinearity of these tensors are dependent on the electric and magnetic fields. Also, recall that the electric and magnetic fields are dependent on A'_y. Thus, the larger the initial value of A'_y, the stronger the nonlinearity.

In the figure 4.1, a small initial A'_y value was put in and as such, the affects of nonlinearity is muted. The phase diagram is very round, which is similar to the phase diagrams of a linear system, such as a linear harmonic oscillator.

In the figure 4.2, a large initial A'_y value was put in, making the affects of nonlinearity apparent. This is seen by the squaring of the phase diagram.

4.2 Turning on A_y with Phase Velocity Less Than C

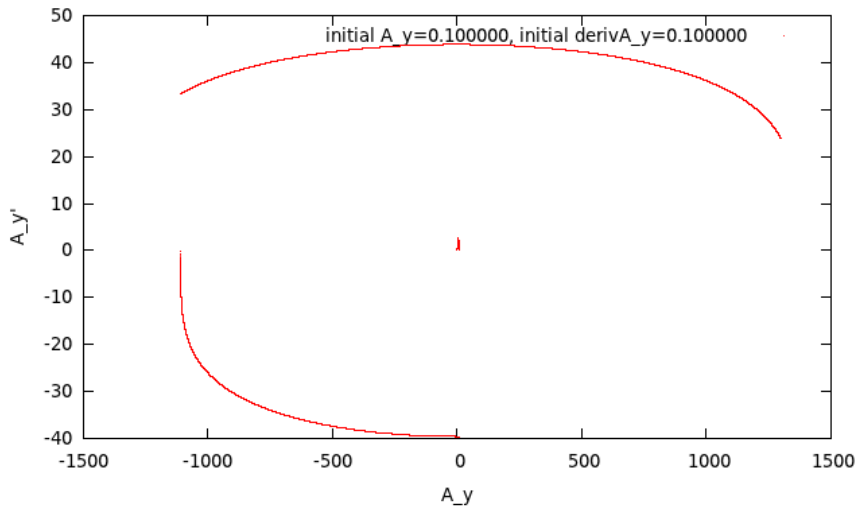

Figure 4.3: Phase diagram of A_y with a $v_p < c$ and a small input.

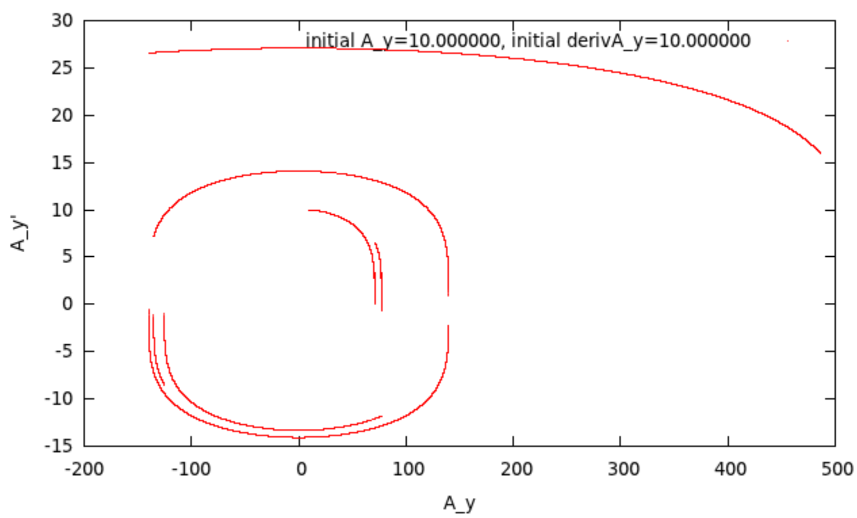

Figure 4.4: Phase diagram of A_y with a $v_p < c$ and a large input.

As discussed in the previous section, phase diagrams can help to infer characteristics of the wave. Previously when the phase velocity was greater than c, the phase diagram was stable and continuous. Here, we see that the phase diagram is discontinous and unstable, indicating that the wave is aperiodic. Hence there is possibility of finding a soliton in this type of equation.

Notice that the discontinuity of the phase diagram always occurs at $A_y' = \pm 1$. From equation

3.44, we see that this is because at $A'_y = \pm 1$, we are dividing zero and the slope becomes infinite. These discontinuous phase diagrams 4.3 and 4.4 are not very useful because they are inaccurate. The reason is that we are numerically solving for these values using Runge Kutta 4 (RK 4), a method that depends on the slope. When the slope heads towards infinity, the values that RK 4 solves for afterwards will be thrown off.

Hence, in order to get a useful phase diagram, we will not use RK 4 here. Instead, we will use the analytically solved equation 3.45. We will plot the phase diagram, where A_y is a function of A'_y. That is, we will be plugging in values for A'_y and then solving for A_y. In this way, we can plot out our phase diagram, Figure 4.5.

As seen in Figure 4.5, the smaller A'_y inputs have a rounder shape than the larger A'_y inputs. The larger values seem to be more squared off. Recall from the previous section that this is due to the larger A'_y, the stronger the nonlinearity effects.

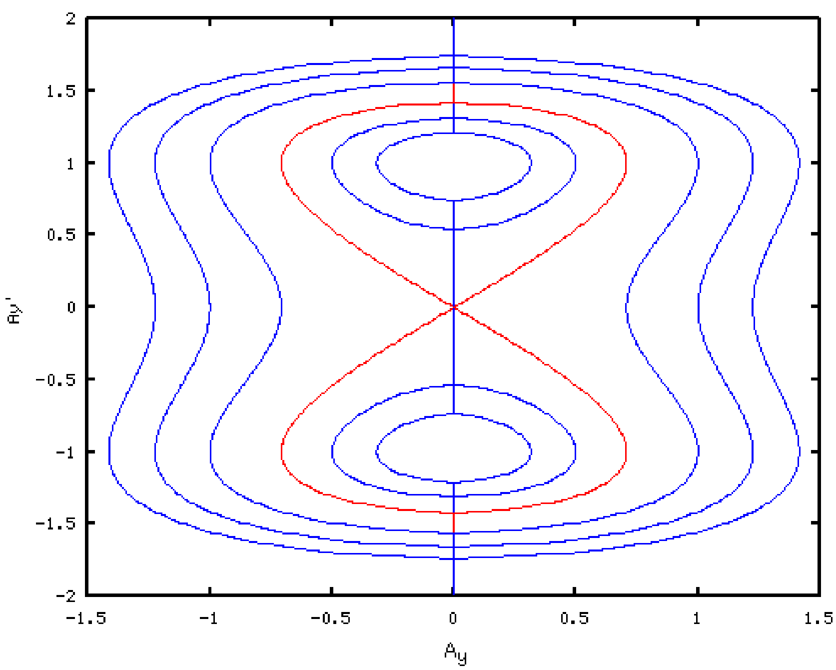

Figure 4.5: Soliton is outlined in red.

4.2.1 Soliton

As seen in Figure 4.5, the outline in red is a soliton. We know this because there is an abrupt change at coordinates (0.7, 1), (0.7, -1), (-0.7, 1), and (-0.7, -1) in Figure 4.5. This can be observed if we start from the origin of the figure and travel outwards along the red outline towards coordinate (0.7, 1). Along this path, A'_y and A_y are both positive, which makes sense. However, continue travelling along this red outline, we see that while the derivative of A_y, that is A'_y, is positive, the A_y is heading towards the negative direction, which does not make sense. The only sensible solution is that upon reaching the coordinate (0.7, 1), the solution jumps down towards (0.7, -1) and continues inwards towards the origin.

Recall from equation 3.7, that the magnetic field is proportional to A'_y, hence if there is a jump in A'_y, then there is also a jump in the magnetic field. Based on the path that we just described, we see that A'_y is steadily increasing before it makes an abrupt change towards the negative direction. In the same way, we would expect the the magnetic field over space to continuously increase before it suddenly switches into the other direction (see Figure 4.7). Based on the magnetic field shown in Figure 4.7, we conclude that the soliton must be an infinite current sheet that sits right where the magnetic field switches direction.

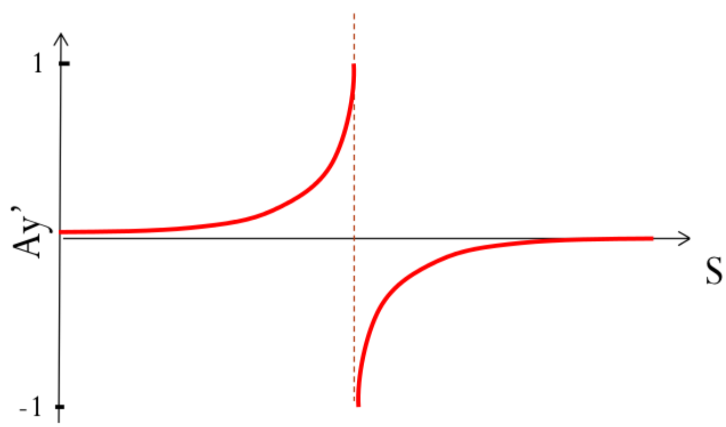

Figure 4.6: The discontinuity of A'_y.

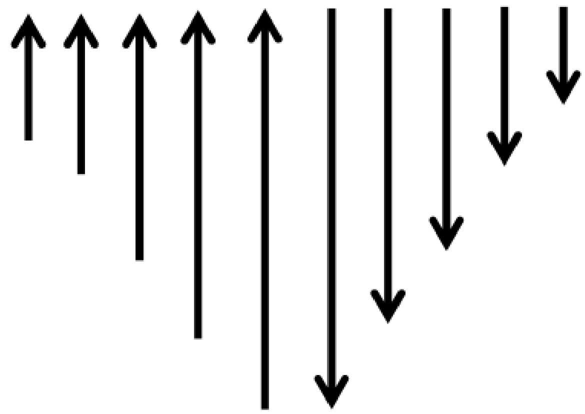

Figure 4.7: Discontinuity of the magnetic field.

4.3 Turning on A_y and A_z Values, Circumpolar

Here we are applying the equations when both A_y and A_z are turned on. We will only look at the case when the phase velocity is greater than c. (The case when the phase velocity is less than c does not seem to produce a coherent solution.)

In the figures 4.8 and 4.9, we are observing circularly polarized waves. The numerical difference between the two figures is that figure 4.8 started with a smaller initial input than figure 4.9. Similar to the previous sections, the solution with the larger initial input was more affected by nonlinearity. Here we see that the phase diagram that is experiencing more nonlinearity has an angular precession in a A_z versus A_y plot. This precession indicates that there is energy shifting between the axes. The upper and lower boundary of the phase diagram “radius” is maintained, suggesting some sort of energy conservation.

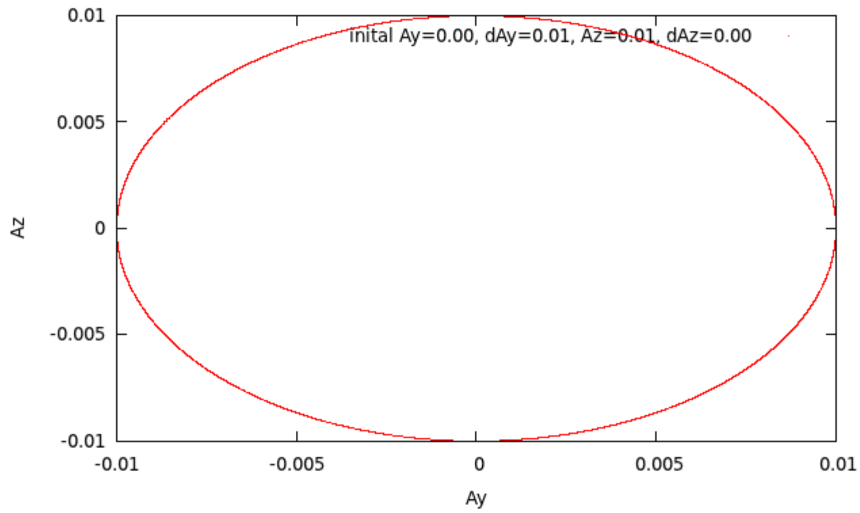

Figure 4.8: Circularly polarized wave, where $v_p > c$. Small initial input.

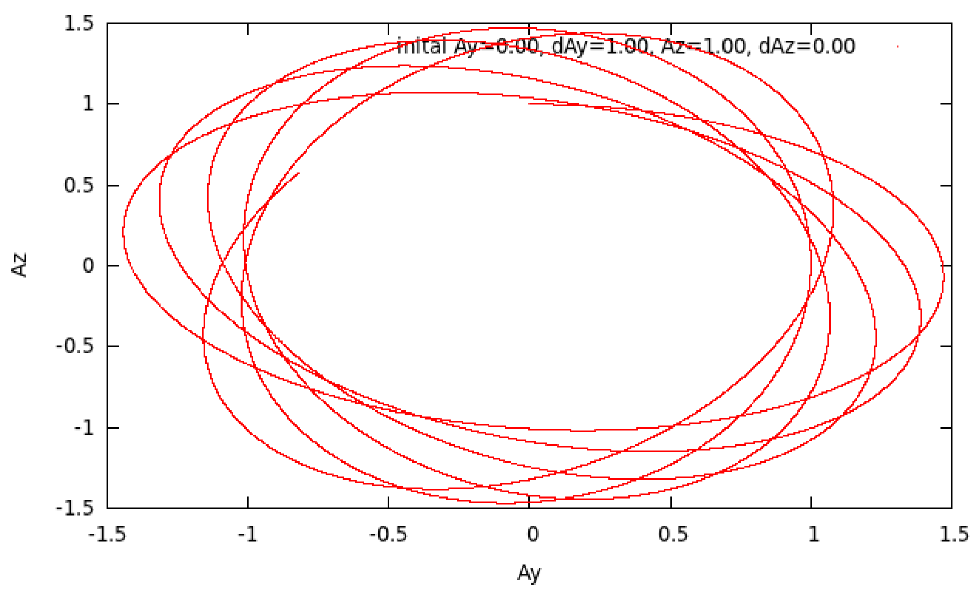

Figure 4.9: Circularly polarized wave, where $v_p > c$. Large initial input

Chapter 5

Conclusion

We discussed about a nonperturbative method of finding stabilized waves travelling in the magnetosphere of a magnetar. A nonpertubative method was necessary as the medium the wave was propagating through was nonlinear. This method was achieved by making an ansantz that the wave was a plane wave with the parameter $S = x - vt$. In addition, our method of storing values in terms of magnetic vector potential, electric potential and plasma density has simplified the process of numerically solving for stabilized waves.

While travelling through the magnetosphere, the wave will maintain its shape through the combined effects of nonlinearity and dispersion caused by the QED vacuum in a strongly magnetized field and by the strongly magnetized plasma, respectively. We were interested in cases where the wave steepening effect from the nonlinearity will balance with the effect from dispersion, allowing us to oberve a soliton. We were able to find a soliton in the form of an infinite current sheet.

In this study we only looked at several variables and used a weak field approximation. In the future, we hope to extend our search by adding more variables and using a more general form of the field equations.

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Appendix A

Appendix

A sample `C` program is that solves nonlinear ODEs and plots out a 2D plot. When initialized, it will ask the user for initial input values for the equation. It will also ask the user what two variables the user wishes to plot.

```
/*
* Nonlinear ODE solver and plotter.
*/

#include <stdio.h>
#include <math.h>
#include <stdlib.h>
#include <assert.h>

void initialize(double *, double *, double *, double *, double *, double *, double *,
    double *,double *, double *, int *, int *);
void derivatives(double, double *, double *, double *, double *, double, double,
    double, double);
void rungeKutta4(double *, double *, double *, double *, int, double, double, double
    *, double *, double, double, double, double, void (*derivatives)(double, double *,
    double *, double *, double *, double,double, double, double));

void output( FILE *, double, double, double *, double *, double);

int main(){
       FILE *outputFile;
```

```
FILE *pipe;

//***Declaring Variables***
double initialY, initialDy, finalt, initialZ,initialDz,C, By,Gamma,a;
double h;
double plotScale;
int xAxis, yAxis;

double y[2], dydt[2], yOut[2];
double z[2], dzdt[2], zOut[2];
double t;

const char *plotLabel[7];
outputFile = fopen("Data.dat", "w");
pipe = popen("gnuplot -persist", "w");

//***Read User Input From Screen And Feed To Variables ***
initialize(&initialY, &initialDy, &initialZ, &initialDz,&By, &Gamma, &a,
    &finalt, &h, &plotScale,&xAxis, &yAxis);

y[0] = initialY;
y[1] = initialDy; // dx/dt = y
z[0] = initialZ;
z[1] = initialDz;

C=1; // (v^2-c^2)/|v^2-c^2|
t=0.;

//****Computing and Writing Data Values to "outputFile"****
output(outputFile, h, t, y, z, plotScale);

while(t<=finalt){
```

```
        t+=h;
        derivatives(t,y,dydt,z,dzdt,C,By,Gamma,a);
        rungeKutta4(y,z,dydt,dzdt,2,t,h,yOut,zOut,C,By,Gamma,a,derivatives);
        y[0]=yOut[0];
        y[1]=yOut[1];
        z[0]=zOut[0];
        z[1]=zOut[1];
        output(outputFile,h,t,y,z,plotScale);
}
fclose(outputFile);

//******************PLOTTING SECTION ********************
//***Plot Labels***
plotLabel[0]="h";
plotLabel[1]="t";
plotLabel[2]="plotScale*(y[1]+z[1])";
plotLabel[3]="y[0]";
plotLabel[4]="y[1]";
plotLabel[5]="z[0]";
plotLabel[6]="z[1]";



//***Actual Plotting Using An Imbedded Gnuplot***
fprintf(pipe, "plot 'Data.dat' using %i:%i title 'inital Ay=%4.2lf, dAy=%4.2lf,
    Az=%4.2lf, dAz=%4.2lf' with
    dots\n",xAxis,yAxis,initialY,initialDy,initialZ,initialDz);
fprintf(pipe, "set xlabel '%20s'\n",plotLabel[xAxis-1]);
fprintf(pipe, "set ylabel '%20s'\n",plotLabel[yAxis-1]);
fprintf(pipe, "replot\n");
close(pipe);

return 0;
```

```
}


//****************METHOD CODE *******************
void initialize(double *initialY, double *initialDy, double *initialZ, double
    *initialDz, double *By, double *Gamma, double *a, double *finalt, double *h,
    double *plotScale, int *xAxis, int *yAxis){
       double numOfPeriod;

       printf("\n\nAy, Az and By. \n");
       printf("We have set Bk=1, c=1, w=1 \n");

       //***Prompting User For Input Values and Plotting Axes***
       printf("Please enter the initial ay, ay’, az, az’,By, Gamma,alpha, final t,
           step size, and plot scaler\n");
       scanf("%lf %lf %lf %lf %lf %lf %lf %lf %lf %lf", initialY, initialDy, initialZ,
           initialDz, By, Gamma,a, finalt, h, plotScale);
       printf("PLOTTING: Enter x and y axes of plot. Choose from the following:\n 1.
           h\n 2. t\n 3. plotScale*(y[1]+z[1])\n 4. y[0]\n 5. y[1]\n 6. z[0]\n 7.
           z[1]\n");
       scanf("%i %i", xAxis, yAxis);

}

//******Taking Derivatives*********
// Probably most important section of this script. The dydt[1] and dzdt[1] are where
    the
// second order derivatives (of either A_y, A_z, \phi) that we calculated in Chapter 3
    are placed.
void derivatives(double t, double *y, double *dydt, double *z, double *dzdt, double C,
    double By, double Gamma,double a){
       double w=1;
```

```
    double Bk=1;
    double c=1;
    dydt[0] = y[1]; // slope of position (aka velocity)
    dydt[1] = **PUT SECOND DERIVATIVE EQUATION HERE**
                //This is where we put the equations that we found in Chapter 3
                    Mathematical Details.
            //So if y = A_y, then you would put the equation for A_y'' here.

    dzdt[0] = z[1]; // slope of position (aka velocity)
    dzdt[1] = **PUT SECOND DERIVATIVE EQUATION HERE**
                //This is where we put the equations that we found in Chapter 3
                    Mathematical Details.
            //So if z = A_z, then you would put the equation for A_z'' here.
}

//****************RK 4******************/
void rungeKutta4(double *y, double *z, double *dydx, double *dzdx, int n, double x,
    double h, double *yOut, double *zOut, double C, double By, double Gamma, double
    a,void (*derivatives)(double, double *, double *, double *, double *, double,
    double, double,double)){
    int i;
    double xh, hh, h6;
    double *y_otherSlope, *y_newSlope, *y_next;
    double *z_otherSlope, *z_newSlope, *z_next;

    y_otherSlope = (double *) malloc(n*sizeof(double));
    y_newSlope = (double *) malloc(n*sizeof(double));
    y_next = (double *) malloc(n*sizeof(double));
    z_otherSlope = (double *) malloc(n*sizeof(double));
    z_newSlope = (double *) malloc(n*sizeof(double));
    z_next = (double *) malloc(n*sizeof(double));
    hh = h*0.5;
```

```
h6 = h/6.;
xh = x+hh;
for(i=0; i<n; i++){
        y_next[i] = y[i]+hh*dydx[i];
        z_next[i] = z[i]+hh*dzdx[i];
}
(*derivatives)(xh,y_next,y_newSlope,z_next,z_newSlope,C,By,Gamma,a);
for(i=0; i<n; i++){
        y_next[i] = y[i]+hh*y_newSlope[i]; //y_newSlope has been updated in
            derivatives(). So y_next will change
        z_next[i] = z[i]+hh*z_newSlope[i]; //y_newSlope has been updated in
            derivatives(). So y_next will change
}
(*derivatives)(xh,y_next,y_otherSlope,z_next,z_otherSlope,C,By,Gamma,a);
for(i=0; i<n; i++){
        y_next[i] = y[i]+h*y_otherSlope[i];
        z_next[i] = z[i]+h*z_otherSlope[i];
        y_otherSlope[i] += y_newSlope[i]; //save a malloc call. ie. other slope
            is now otherslope + y_newSlope
        z_otherSlope[i] += z_newSlope[i];
}
(*derivatives)(x+h,y_next,y_newSlope,z_next,z_newSlope,C,By,Gamma,a);
for(i=0; i<n; i++){
        yOut[i] = y[i]+h6*(dydx[i]+y_newSlope[i]+2.0*y_otherSlope[i]);
        zOut[i] = z[i]+h6*(dzdx[i]+z_newSlope[i]+2.0*z_otherSlope[i]);
}
free(y_otherSlope);
free(y_newSlope);
free(y_next);
free(z_otherSlope);
free(z_newSlope);
free(z_next);
```

```
}

void output(FILE *outputFile, double h, double t, double *y, double *z, double
    plotScale) {
        fprintf(outputFile, "%lf %lf %lf %lf %lf %lf %lf\n", h, t,
            plotScale*(y[1]+z[1]), y[0], y[1], z[0], z[1]);
}
```