The Primordial Inflation Polarization Explorer (PIPER)

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ABSTRACT

The Primordial Inflation Polarization Explorer (PIPER) is a balloon-borne instrument designed to search for the faint signature of inflation in the polarized component of the cosmic microwave background (CMB). Each flight will be configured for a single frequency, but in order to aid in the removal of the polarized foreground signal due to Galactic dust, the filters will be changed between flights. In this way, the CMB polarization at a total of four different frequencies (200, 270, 350, and 600 GHz) will be measured on large angular scales. PIPER consists of a pair of cryogenic telescopes, one for measuring each of Stokes Q and U in the instrument frame. Each telescope receives both linear orthogonal polarizations in two 32 × 40 element planar arrays that utilize Transition-Edge Sensors (TES). The first element in each telescope is a variable-delay polarization modulator (VPM) that fully modulates the linear Stokes parameter to which the telescope is sensitive. There are several advantages to this architecture. First, by modulating at the front of the optics, instrumental polarization is unmodulated and is therefore cleanly separated from source polarization. Second, by implementing this system with the appropriate symmetry, systematic effects can be further mitigated. In the PIPER design, many of the systematics are manifest in the unmeasured linear Stokes parameter for each telescope and thus can be separated from the desired signal. Finally, the modulation cycle never mixes the Q and U linear Stokes parameters, and thus residuals in the modulation do not twist the observed polarization vector. This is advantageous because measuring the angle of linear polarization is critical for separating the inflationary signal from other polarized components.

Keywords: Cosmic Microwave Background, Polariometry

1. INTRODUCTION

The measurement of the polarization of the Cosmic Microwave Background offers a tool with which to probe the earliest epoch of the Universe’s history. Recent observations have significantly advanced our understanding of the history of the Universe and have hinted at new physics. Specifically, the flatness of the Universe, its uniformity, and the presence of a nearly scale-invariant spectrum of density perturbations in the early Universe have

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supported the idea that the Universe underwent a brief period of exponential expansion, known as “inflation,” early in its history.

Inflation is expected to produce a stochastic background of gravitational waves (tensor perturbations) whose amplitude depends only on the energy scale at which inflation occurred. The CMB temperature anisotropy is sensitive to a combination of scalar and tensor perturbations, but the CMB polarization anisotropy has a component that depends only on the gravitational wave background. Thus the polarization signal provides a way to directly measure the energy scale of inflation. This is a potential probe of physics some 12 orders of magnitude higher in energy than is achievable using terrestrial accelerators.

The CMB anisotropy is weakly polarized due to anisotropic Thompson scattering at the epochs of recombination and reionization. The polarization pattern that is induced by density perturbations is “curl-free”, due to the symmetry of the scalar perturbations. This curl-free pattern is referred to as “E-mode” polarization. The pattern induced by gravitational waves does not possess scalar symmetry, thus these perturbations can give rise to an additional divergence-free component, which is termed “B-mode” polarization. A detection of B-mode polarization thus provides a direct measure of the gravitational wave amplitude imparted by inflation. To date, the E-mode signal has been measured by several groups and is consistent with the predictions of anisotropic Thompson scattering of the primary temperature anisotropy. The B-mode signal has yet to be detected.

The inflationary gravitational wave amplitude is parameterized by \( r \), the “tensor-to-scalar” ratio. Models for inflation connect this parameter to the spectral tilt of the density perturbations, \( n_s \). Recent measurements have found that \( n_s \) is measurably less than 1. For such a value, the simplest and arguably most compelling inflation models predict \( r > 0.01 \). Therefore, polarization measurements at this sensitivity provide a means to test specific inflationary models. The energy scale associated with such scales is consistent with that predicted for GUT scale physics.

The Primordial Inflation Polarization Explorer (PIPER) is a balloon-borne experiment designed to search for the B-mode polarization signal in the Cosmic Microwave background as a test of inflation. PIPER will use a pair of liquid helium-cooled telescopes to observe a large fraction of the sky during turn-around flights from both the Northern and Southern Hemispheres. Each flight will be configured for a single frequency, and the frequency response of the instrument will be changed between campaigns.

In Section 2, we summarize the predicted instrument performance. In Section 3 we briefly describe the optics. In Section 4, the polarization modulation for PIPER is described. The detectors are discussed in Section 5, and a discussion of systematics is included in Section 6.

### 2. INSTRUMENT SUMMARY

The PIPER payload is shown in Figure 1A. PIPER consists of two telescopes that are cooled to 1.5 K via liquid helium. The telescopes are mirror images of each other and are aligned such that one measures Stokes \( Q \) and the other Stokes \( U \) simultaneously and on the same location on the sky. The polarization modulation is accomplished via variable-delay polarization modulators (VPMs). In each telescope, a modulator is placed at the front of the optical system so as to encode the modulation onto the signal before instrumental polarization is introduced by the optics.

Both telescopes are located in a 3000 L bucket dewar that is filled with liquid helium before launch. Once at float altitude, the low atmospheric pressure reduces the temperature of the liquid to 1.5 K. The fore-optics, which are located in the same volume as the liquid cryogen, are cooled using a combination of evaporating liquid and superfluid helium pumps. There are no windows at the exit of the dewar; frost is prevented from collecting on the optics by the outflow of evaporating helium. These techniques were first demonstrated by the ARCADE experiment. The reimaging optics, filters, and detectors are located inside a vacuum vessel. An adiabatic demagnetization refrigerator (ADR) cools the detectors to \( \sim 100 \) mK.

Cooling the optics to 1.5 mK lowers the background photon noise on the detectors significantly as compared to the case with warm fore-optics, as shown in Figure 1B. The decreased noise in combination with the ability to simultaneously detect large numbers of modes from the CMB enables PIPER to reach low statistical limits on \( r \) in a series of turn-around flights.
Figure 1. (A) A schematic diagram of the PIPER payload is shown. The fore-optics are cooled to 1.5 K via superfluid pumps and the evaporating cryogen. The reimaging optics, filters, and detectors are located inside a vacuum vessel. Polarization modulation is accomplished via variable-delay polarization modulators (VPMs). (B) The calculated background photon noise contributions for PIPER are compared to that of an equivalent system with warm mirrors. For PIPER, the dominant photon noise contribution is from the residual atmosphere at float altitude. Approximate PIPER bands are shown (shaded bands).

By flying each frequency of PIPER in both the Southern and Northern Hemisphere, PIPER is able to cover a large fraction of the sky. This enables PIPER to be sensitive to the “reionization bump” of the B-mode spectrum that is present at large angular scales (low l). This signal is expected to be free of the gravitation lensing foreground signal that is likely to affect polarimetric observations at finer angular scales. Figure 2 shows both the sky coverage and the statistical sensitivity of PIPER. These calculations are based on 2 flights at each of 4 different frequencies. Table 1 gives a summary of various specified and predicted instrument properties for the four bands of PIPER.

<table>
<thead>
<tr>
<th>Property</th>
<th>Band 1 (GHz)</th>
<th>Band 2 (GHz)</th>
<th>Band 3 (GHz)</th>
<th>Band 4 (GHz)</th>
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<tr>
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<td>0.30</td>
<td>0.30</td>
<td>0.15</td>
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<td>0.70</td>
<td>0.50</td>
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<td>Bolometer (Phonon) NEP (W Hz⁻⁰.⁵)</td>
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<td>3.8 × 10⁻¹⁸</td>
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<td>Total NEP (W Hz⁻⁰.⁵)</td>
<td>4.7 × 10⁻¹⁸</td>
<td>5.9 × 10⁻¹⁸</td>
<td>5.1 × 10⁻¹⁸</td>
<td>7.1 × 10⁻¹⁸</td>
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<tr>
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<td>147</td>
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<td>877</td>
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</table>

3. OPTICS

The optics for PIPER are described in detail in a companion paper, so we will only briefly summarize them here. Figure 3 shows the basic layout for a single telescope. Each telescope consists of three distinct sections. The first section is the VPM which will be discussed in Section 4. The second section is the fore-optics. This
section (along with the VPM) is cooled to 1.5 K by the evaporating helium and a series of superfluid pumps and consists of a set of two powered mirrors and a fold mirror that reimage the primary pupil (located at the VPM) to a secondary pupil located lower in the optical system.

Finally, the third section is a set of reimaging optics that is located internal to a vacuum vessel. This includes a Lyot stop that is located at the secondary pupil formed by the fore-optics to control the illumination of the VPM. A silicon lens slows the beam before it passes through a analyzer grid which splits the polarization into two orthogonal linear components. Each polarization passes through a final lens which converts the beam to f/1.6 before illuminating each detector array. Band-defining filtering is integrated into baffles in the vicinity of the Lyot stop. Short wavelength rejection is provided by a cold quartz pressure window that is indium sealed to the vacuum vessel.

4. POLARIZATION MODULATION

Each telescope of PIPER employs a Variable-delay Polarization Modulator (VPM) to modulate the polarization signal. In each case, the wires of the VPMs are rotated in opposite directions relative to the dewar centerline. This enables one telescope to detect Stokes $Q$ and the other to detect Stokes $U$. The VPM is placed at the front of the optical system so that the modulation can be encoded on the astronomical polarization signal before the optics can partially polarize the unpolarized background. Though it is always important to minimize, characterize, and remove instrumental polarization, this is especially true for CMB measurements where the polarization signal must be measured to 1 part in $>10^8$ of the background power on the detectors.

4.1 VPM Principle of Operation

The basic architecture for the VPM is shown in the left panel of Figure 4 and has been described in previous work.\textsuperscript{16-18} The VPM consists of a wire grid polarizer placed parallel to and in front of a moving mirror. The linearly polarized component of the incident radiation that has its electric field parallel to the wires of the grid will be reflected off of the grid. The perpendicular component will pass through and be reflected off of the mirror. The two polarizations will be recombined with a phase introduced between them. In the coordinate system in which Stokes $U$ is parallel to the wires of the polarizer, the polarization transfer function for a monochromatic source is

$$Q_{\text{out}} = Q_{\text{in}} \cos \phi + V_{\text{in}} \sin \phi$$

(1)
Figure 3. (LEFT) The ray trace analysis for a single PIPER telescope is shown. The reflective optics are exposed to the superfluid helium environment and are cooled to 1.5 K via the evaporating liquid and superfluid pumps. (RIGHT) The reimaging optics are located inside the vacuum vessel.

where $Q$ is the Stokes parameter that is rotated by $45^\circ$ from Stokes $U$, and $V$ is the Stokes parameter for circular polarization. VPMs have been demonstrated in the Hertz/VPM submillimeter polarimeter. In the PIPER implementation, the output polarization state is measured by splitting the polarization using an analyzer grid oriented at an angle of $45^\circ$ with respect to the VPM wires. Each polarization is directed to an independent detector array.

4.2 Implementation

The PIPER VPMs will each have a diameter of 39 cm and are located at the primary (entrance) pupil of each telescope in the design. There are two advantages to placing the VPMs at pupils. First, as the grid-mirror separation is modulated, the two beams will be shifted parallel to one another. By placing the VPM at a pupil, this so-called “beam walkoff” does not shift the two beams at the detector. Second, placement of the VPM at the pupil allows the VPM to be constructed with the smallest diameter possible consistent with the desired edge illumination.

Because the VPMs will be cooled to 1.5 K, care must be taken to ensure proper operation at cryogenic temperatures. The current design of the PIPER VPM is shown in the right side of Figure 4. In previously-constructed VPMs, gold coated tungsten was used for the grid wires, the flexures were made of titanium, and the frame and mirror were aluminum. However, because of CTE issues, a single material is desirable. For PIPER, the VPM will be constructed entirely of stainless steel. The grid wires and mirrors will be coated with gold to increase reflectivity.

The VPMs will be actuated by a warm drive system. At the warm end, a stepper motor is coupled to the shaft via a cam that mechanically limits the throw of the mirror to maximize the polarization modulation efficiency for a given band. Two shafts extend through a tube into the cold volume. They are actuated 180$^\circ$ out of phase such that one is used to drive the mirror and the other is used to drive a symmetric pair of counterweights to compensate the motion. The motion at the cold end of the shaft is all regulated by frictionless flexures so as to avoid the use of cold bearings. Connections to the PIPER optics frame and between the grid and mirror are done kinematically using flexures such that the thermal contraction can be predicted.

The grid-mirror separation will be monitored by using three capacitance sensors spaced at 120$^\circ$ angles around the circumference of each VPM. This will allow us to monitor the grid mirror separation and the parallelism
simultaneously. The capacitance sensors have been tested cryogenically and are accurate to within a fraction of a micron over throws up to 2 mm. The nominal throw for the long wavelength of PIPER is \( \sim 0.5 \) mm.

Figure 4. (LEFT) The VPM uses a wire grid positioned in front of and parallel to a moving mirror. The phase delay introduced between the two orthogonal polarizations is controlled by modulating the grid-mirror separation. (RIGHT) A VPM CAD model for PIPER is shown. The PIPER VPM will be actuated by a warm motor connected via a cam to a shaft. A rotary flexure couples the motion of the shaft to a flexure that maintains the parallelism between the grid and mirror over the stroke of the modulation.

5. DETECTORS

PIPER utilizes the Backshort-Under-Grid (BUG) architecture\(^{20}\) for its bolometer arrays. These detectors are described in greater detail in a companion paper,\(^{21}\) but a brief description is included here for completeness. Each array contains 32 \( \times \) 40 planar absorbing elements each thermally coupled to a transition-edge sensor (TES). Four such arrays are included in PIPER, each used to detect a single linear polarization in each of the two telescopes.

A BUG array (a 32\( \times \)40 test device is shown in Fig. 5) is a two-dimensional structure that is sized to mate to the two-dimensional multiplexers developed by NIST for SCUBA-2.\(^{22}\) Prototype BUG arrays have been demonstrated in the GISMO instrument on the IRAM telescope.\(^{23}\) The BUG array consists of two pieces. The first is the grid of suspended membranes that make up the sensing part of the array. The electrical signals for the TES bolometers are routed to the back of the grid structure using “wrap-around vias,” or electrical paths that are deposited around each wall. A second grid that consists of an array of reflecting elements is nested into the back of the first grid in order to provide a backshort for each detector. In the case of PIPER, the backshort distance was chosen to maximize detector absorption over the three lowest frequencies which are those that contain the CMB signal. Because the pixels are close to the size of the wavelength for the PIPER bands, few modes exist within the detector volume. Because of this, we used full electromagnetic models to calculate the desired detector-backshort spacing in order to accurately account for the boundary conditions.

6. SYSTEMATIC ERROR CONTROL

Because B-mode measurement involves the detection of a small polarized signal in the presence of much larger unpolarized sources, systematic error control is critical. PIPER is designed to avoid such systematics where possible and to mitigate those it cannot avoid.

6.1 \( T \rightarrow B \)

The mixing of unpolarized flux into a false polarized signal, commonly referred to as “instrumental polarization,” is especially to control given that the unpolarized flux is likely to be \( > 10^8 \) larger than the B-mode signal. Scattering, off-axis reflections, and transmission through dielectrics can all induce polarization on an initially unpolarized signal. PIPER’s VPMs are placed at the front of the optical system so that the polarization modulation is encoded on the CMB before any of these effects can occur within the instrument.
It is conceivable to get some polarization at the VPM due to the small asymmetry between the boundary conditions for the two polarization components; however, the modulation architecture mitigates this in the following way. This class of systematic is manifest as a differential decrease in the linear polarization basis that defines the Stokes $U$-basis. The analyzer grid is oriented at a $45^\circ$ angle with respect to this base, and so detection is done in the Stokes $Q$-basis. In practice, a true sky polarization will induce a signal with opposite signs in the two oppositely-polarized detector arrays within a single telescope. The systematic signal will have the same sign in the two detectors. Mitigation of this class of systematic is dependent upon gain stability between the two detectors.

6.2 \( \Delta T \rightarrow B \)

Asymmetry between the vertical and horizontal polarization sensitivity of an instrument can cause unpolarized map features (such as the primary CMB anisotropy or foregrounds) to manifest themselves as a polarization. The front-end VPM mitigates this effect by switching the beam’s polarization sensitivity without altering the beam shape. In addition, the primary modulation is done in polarization rather than scanning, and so this provides a nearly diagonal covariance matrix.

6.3 \( E \rightarrow B \)

Mixing Stokes $Q$ and $U$ lead to $E$, $B$ mixing. Because, the VPM modulates the polarization from a single linear Stokes parameter to circular polarization, there is no $Q$-$U$ mixing inherent in the modulation cycle. To illustrate this, it is useful to compare the VPM to the half-wave plate (HWP) that modulates between $Q$ and $U$. For example, in the presence of cross-polarization, the spatial distribution of polarization within the beam can cause some degree of $Q$-$U$ mixing. The left side of Figure 6 shows a simulated patch of sky in Stokes $Q$, $U$, and $V$. The contours show the co- and cross-polar beam patterns, assuming a -25 dB peak cross-polarization. On the right hand side of this figure, raw signals for a HWP and VPM are shown along with the error signals. For an E-mode RMS amplitude of 1.3 $\mu$K, we find that the HWP generates a modulated error of 147 nK. The VPM generates an offset of 3 nK. The details for the cross-polar dependent mixing are given in the Appendix.

6.4 Foregrounds

PIPER’s four frequencies are all in the spectral regime in which dust is the dominant foreground with polarized synchrotron being negligible. The multiple frequencies will allow the characterization of the dust foreground for removal. Ultimately, it is likely that the limiting factor in CMB polarization measurements will be determined by the degree of complexity of the foreground signal.

7. SUMMARY

PIPER is a balloon-borne polarimeter for searching for the polarized signature from inflation. It observes with two cryogenic co-pointed telescopes, each of which has a front end VPM for polarization modulation and detects both orthogonal linear polarization components in order to both maximize sensitivity and to monitor and remove systematic effects. PIPER will fly the first of four frequency configurations in 2013.
REFERENCES


APPENDIX: CROSS-POLAR COUPLING IN HWPS AND VPMS

The power absorbed by a polarization-sensitive detector can be expressed in the following way:\textsuperscript{24,25}

\[ \langle p(t) \rangle = \int_0^\infty \phi(\omega) d\omega \int_S d\hat{\Omega}_1 \int_S d\hat{\Omega}_2 \overline{G}(\hat{\Omega}_1, \hat{\Omega}_2, \omega) \cdot \overline{E}(\hat{\Omega}_1, \hat{\Omega}_2, \omega) \] \hspace{1cm} (2)

Here, the \( \cdots \) refers to complete tensor contraction (the trace of the matrix product), and \( \phi(\omega) \) describes the passband of the system. In addition, \( \overline{G}(\hat{\Omega}_1, \hat{\Omega}_2, \omega) = \langle G^\ast(\hat{\Omega}_1, \omega) G(\hat{\Omega}_2, \omega) \rangle \) is the detector response expressed as a dyadic, and \( \overline{E}(\hat{\Omega}_1, \hat{\Omega}_2, \omega) = \langle E^\ast(\hat{\Omega}_1, \omega) E(\hat{\Omega}_2, \omega) \rangle \) is the cross-spectral dyadic of the source. Here, the beam pattern \( G(\hat{\Omega}, \omega) \) is the system response or gain in the direction \( \hat{\Omega} \) at frequency \( \omega \), and \( E(\hat{\Omega}, \omega) \) is the electric field of the source in the direction \( \hat{\Omega} \) at frequency \( \omega \). The angle brackets represent a time average.

This expression is extremely general since it decomposes both the beam and the sky in terms of their second-order statistical properties. This also makes for a convenient representation for a polarization analysis. In our case, the sky is partially coherent at each point (partially-polarized), but there are no correlations from point-to-point. Thus, we only have to consider the terms of the dyadic in which \( \hat{\Omega} = \hat{\Omega}_1 = \hat{\Omega}_2 \), reducing the expression for the cross-spectral dyadic of the sky to

\[ \overline{E}(\hat{\Omega}, \omega) = I(\hat{\Omega}, \omega) \overline{\sigma}_0 + Q(\hat{\Omega}, \omega) \overline{\sigma}_1 + U(\hat{\Omega}, \omega) \overline{\sigma}_2 + V(\hat{\Omega}, \omega) \overline{\sigma}_3. \] \hspace{1cm} (3)

Here, \( \overline{\sigma}_0 \) is the identity matrix, \( \overline{\sigma}_i \) are the Pauli matrices, and \( (I, Q, U, V) \) are the Stokes parameters.

In contrast to the dyadic representing the sky, the dyadic representing the detector response can be highly correlated. For example, the beam pattern of a single-mode feedhorn is highly correlated over the entirety of a beam. Multi-mode systems can be represented by incoherent sums of coherent modes and will be only partially correlated.

The gain can be expressed in the co- and cross-polarization basis.

\[ G(\theta, \phi) = C(\theta, \phi) \hat{e}_C + R(\theta, \phi) \hat{e}_R \] \hspace{1cm} (4)

where \( R(\theta, \phi) \) and \( C(\theta, \phi) \) are the co- and cross-polarization beam patterns, respectively.\textsuperscript{26}

Therefore, the correlation dyadic is

\[ G_{ij}(\theta, \phi) = \hat{e}_i \cdot \overline{G}(\hat{\Omega}) \cdot \hat{e}_j = \begin{pmatrix} |C(\theta, \phi)|^2 & C^\ast(\theta, \phi) R(\theta, \phi) \\ C(\theta, \phi) R^\ast(\theta, \phi) & |R(\theta, \phi)|^2 \end{pmatrix}. \] \hspace{1cm} (5)

The contracted dyadics in the integrand of Equation 2 can be written analytically as

\[ \int_S d\hat{\Omega}_1 \int_S d\hat{\Omega}_2 \overline{G}(\hat{\Omega}_1, \hat{\Omega}_2, \omega) \cdot \overline{E}(\hat{\Omega}_1, \hat{\Omega}_2, \omega) = \int_S d(\cos \theta) d\phi \{ I(\theta, \phi) [C^2 + R^2] + Q(\theta, \phi) [C^2 - R^2] + U(\theta, \phi) [2R(R^\ast C)] + V(\theta, \phi) [23(R^\ast C)] \}. \] \hspace{1cm} (6)
Here we have suppressed the explicit dependence of the Stokes parameters on frequency and the co- and cross-polarization dependence on θ and φ.

The action of a modulator can be represented by transforming the beam correlation dyadic using a representative Jones matrix. In general, this Jones matrix is frequency-dependent. It is unitary at each frequency for an ideal modulator; non-ideal effects can be introduced by relaxing the unitary condition.

The total power from the detector can then be represented by

\[
\langle p(t) \rangle = \int_0^\infty \phi(\omega) d\omega \int_S d\Omega_1 \int_S d\Omega_2 \bar{J}_{\text{mod}}^{\dagger} (\Omega_1, \Omega_2, \omega) \bar{J}_{\text{mod}} \cdot \bar{E}(\Omega_1, \Omega_2, \omega)
\]

(7)

Note that the Jones matrix representing the modulator will depend on frequency, the internal modulation parameters (such as phase or rotation angle), and possibly \( \hat{\Omega} \). Equation 7 provides a method for coding the polarization-modulated response of a detector to an arbitrary sky. As an example, we will look at the cases for ideal single-frequency half-wave plate (HWP) and a variable-delay polarization modulator (VPM).

The Jones matrix for an ideal HWP is (see e.g. \(^{16}\))

\[
\bar{J}_{\text{HWP}}(\alpha) = i \left( -\cos 2\alpha & -\sin 2\alpha \\
-\sin 2\alpha & \cos 2\alpha \right)
\]

(8)

where \( \alpha \) is the rotation angle of the waveplate.

Using Equations 8, 3, and 5, we get the following expression for the integrand

\[
\int_S d\Omega_1 \int_S d\Omega_2 \bar{J}_{\text{HWP}}^{\dagger} (\Omega_1, \Omega_2, \omega) \bar{J}_{\text{HWP}} \cdot \bar{E}(\Omega_1, \Omega_2, \omega) =
\]

\[
\int_S d(\cos \theta)d\phi \left\{ I(\theta, \phi)(C^2 + R^2) + Q(\theta, \phi)((C^2 - R^2) \cos 4\alpha + 2\Re(C^*R) \sin 4\alpha) + U(\theta, \phi)((C^2 - R^2) \sin 4\alpha + 2\Re(C^*R) \cos 4\alpha) + V(\theta, \phi)[23(R^*C)] \right\}
\]

(9)

The Jones matrix for an ideal VPM oriented at 45° with respect to the polarization axis of the detector is \(^{16}\)

\[
\bar{J}_{\text{VPM}}(\delta) = \left( \begin{array}{cc} \cos \delta/2 & -i \sin \delta/2 \\ i \sin \delta/2 & -\cos \delta/2 \end{array} \right).
\]

(10)

where \( \delta \) is the phase delay between the two orthogonal linear polarizations.

Using Equations 10, 3, and 5, we get the following expression for the integrand

\[
\int_S d\Omega_1 \int_S d\Omega_2 \bar{J}_{\text{VPM}}^{\dagger} (\Omega_1, \Omega_2, \omega) \bar{J}_{\text{VPM}} \cdot \bar{E}(\Omega_1, \Omega_2, \omega) =
\]

\[
\int_S d(\cos \theta)d\phi \left\{ I(\theta, \phi)(C^2 + R^2) + Q(\theta, \phi)[(C^2 - R^2) \cos \delta + 2\Re(C^*R) \sin \delta] + U(\theta, \phi)[2\Re(C^*R)] + V(\theta, \phi)[(C^2 - R^2) \sin \delta - 2\Re(C^*C) \cos \delta] \right\}
\]

(11)

For the HWP, the term proportional to the product of the cross- and co-polarization causes leakage between \( Q \) and \( U \). For the VPM, the leakage terms are between \( Q \) and \( V \) rather than \( Q \) and \( U \). For a VPM measuring \( Q \), \( U \) is unmodulated. The VPM does mix \( Q \) and \( V \), as expected from its path on the Poincaré sphere. However, it is expected that \( V \) is close to zero for the astrophysical case in question.