THE PROPERTIES OF ION ORBITS IN THE CENTRAL REGION OF A CYCLOTRON

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ABSTRACT

The behaviour of ion orbits in the magnetic and electric fields at the centre of a cyclotron is studied in detail. The objective is to optimize the phase acceptance and beam quality for a 500 MeV H\(^+\) isochronous cyclotron.

Since accurate electric fields are necessary for orbit calculations, a numerical method for calculating those fields is examined in detail. The method is suitable for complicated electrode shapes and converges rapidly, yielding potentials in three dimensions with average errors of less than 0.01%. The magnetic fields used in the orbit calculations are measured on model magnets.

The axial motions are examined using a thick lens approximation for the accelerating gaps. A method is demonstrated for calculating the axial acceptance of the cyclotron as a function of RF phase. This method is used to evaluate the merits of various central geometries and injection energies. This method is also used to examine the effects of flat-topping the RF voltage by adding some third harmonic to the fundamental waveform. It is found that addition of the optimum amount of third harmonic increases the phase acceptance by about 20 deg. Finally, the effects of field bumps on the axial motions are investigated.

To allow accurate radial motion calculations to high energy, an approximate formula is developed which yields accurate (<1%) values for the changes in orbit properties of an ion crossing a dee gap. The geometry of the orbit on the first turn is discussed in detail. The radial centring is studied by tracking ions from injection to 20 MeV, and a method is described for choosing the starting conditions of the beam so as to minimize the radial betatron amplitude over a desired phase range.

The problems associated with using a three-fold symmetric magnetic field with a two-fold symmetric electric field are also discussed. Besides the well-known gap-crossing resonance, a previously ignored phase-oscillation effect is found to be important for cyclotrons operating on a high harmonic of the ion rotation frequency.
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1. INTRODUCTION

1.1 Problems in the Cyclotron Central Region

The central region of a cyclotron requires special attention because the internal beam quality and phase acceptance are primarily determined during injection and the first few turns within the machine. During these initial turns, the beam has low energy and is therefore strongly influenced by the phase-dependent lens effects of the dee gaps. The objective of this work is to study the behaviour of ion orbits in the magnetic and electric fields at the cyclotron centre, and thereby to choose the beam injection conditions and magnet and electrode designs for optimum beam performance, i.e. a beam which is centred, has minimum spot size in both the radial and axial directions, and is in a phase interval which optimizes the acceleration process.

The usual studies of cyclotron central regions, for example Rose,\(^1\) and others,\(^2-4\) are concerned with machines with internal ion sources where the ion starts with zero energy and spends its first turn mainly within the electric field produced by the dee gap. With an external ion source, the problems are quite different; to solve them this study was undertaken.

Injection of ions into a cyclotron from an external source has been studied by Powell and Reece;\(^5\) however, the injection energy in their case was 11 keV, compared to a maximum energy gain of 50 keV per turn, whereas in this case the injection energy is 300 keV, compared to 400 keV per turn. Also, the electrode geometry is very different.

This report considers ion injection for a H\(^-\) cyclotron where the ions are extracted by electron stripping and the duty cycle is determined by the phase band the central region will transmit and not by the extraction system, as in some cyclotrons with resonant extraction schemes. Thus there is considerable emphasis on reducing phase-dependent effects in the central region.

The central region problems fall naturally into two groups, those concerning the axial motions and those concerning the radial motions.

The basic problem in the axial motion is that the focusing provided by the magnetic field becomes very small near the centre of the machine, while
the (phase-dependent) electric forces due to the dee gaps become very strong. It is well known\textsuperscript{1} that the electric forces are defocusing for half of the RF cycle. Since these electric forces will be larger than the (focusing) magnetic forces at low energy, a detailed study of the axial motions is required if a large range of RF phases is to be accepted. The situation is further complicated by the fact that space charge effects will also tend to expand the beam. Space charge effects will be most important at low energy and high current.

The basic problem in the radial motion is not lack of focusing but rather how to minimize the radial oscillation amplitudes of the ions. Since the ions are extracted when they reach a particular radius, a large spread in radial amplitudes means that ions from different turns may be present at the extraction radius, resulting in a large energy spread in the extracted beam. The initial motion of the ions in the cyclotron requires that the beam be injected off centre if it is to be centred at extraction; however, this effect is phase dependent, making it difficult to centre ions with a wide range of phases.

Since a knowledge of the electric fields involved is required for studies of both the axial and radial motions, Section 2 describes in detail a method for calculating these fields.

Section 3 considers the axial motions. A method is presented which allows calculation of the axial acceptance of the accelerator as a function of RF phase. This method is used to study various injection energies and the effects of adding third harmonic to the RF. Finally, the effects of field bumps, used to induce phase slip, are considered.

Section 4 considers the radial motions. The geometry of the first turn and how this is influenced by the accelerating electrodes is studied in detail. The radial centring is studied by tracking ions from injection out to 20 MeV. Finally, the effects of a finite beam size are considered.

Section 5 describes an approximation which allows the changes in orbit properties of an ion crossing a dee gap to be evaluated to high accuracy without numerical integration through the electric field. The accuracy of the method is given as a function of RF phase and incident ion energy. This approximation is used in the tracking of the radial motions in Section 4 between 5 and 20 MeV where this approximation is very accurate.
1.2 The TRIUMF Cyclotron

The studies described in this report were performed for the TRIUMF cyclotron, which because of its unique design has several special problems.

The TRIUMF cyclotron is a six-sector, azimuthally varying field (AVF), isochronous machine, designed to accelerate 100 μA of H⁻ ions to 500 MeV. The acceleration of H⁻ ions provides a convenient method of extraction by stripping two electrons from the H⁻ ions by passing the beam through a thin foil. This method gives an extraction efficiency of nearly 100% whereas conventional proton machines have not achieved efficiencies greater than 80% with a large duty cycle. Two other advantages of extraction by electron-stripping are variability of extraction energy by adjusting the foil position and simultaneous extraction of several beams at different energies. The disadvantages of this technique are that the lifetime of the H⁻ ions requires that the maximum magnetic field that the ions pass through must be low (5.7 kG at 500 MeV) to prevent disassociation of the H⁻ ions, and also there must be a vacuum < 7 x 10⁻⁸ Torr to prevent H⁻ stripping by residual gas molecules. The low magnetic field means that the radius of the machine is very large (500 MeV orbit radius of 311 in.), and the central magnetic field (3.0 kG) is five or six times lower than in conventional cyclotrons.

The accelerating voltage is provided by four resonant cavities which provide 0.4 MeV energy gain per turn. The low magnetic field means that the ion rotation frequency is low (4.53 MHz). To allow the cavity resonators to fit inside the vacuum tank, the RF is operated at the fifth harmonic of the ion frequency. The fact that the accelerating structures are cavity resonators means that some third harmonic of the ion frequency can be introduced into the cavity, squaring the RF waveform and giving significant improvements in orbit properties.

The arrangement of the TRIUMF central region is shown in Figs. 1.1 and 1.2. The centre post is required to support part of the weight of the upper magnet cores, the magnetic force between the magnet pole pieces and the atmospheric load. The H⁻ beam is produced in an external (Ehlers) ion source and accelerated to 300 keV before being transported to the cyclotron and bent into the median plane by the spiral electrostatic inflector. The beam leaves the centre post at the "injection gap", which provides an auxiliary 100 keV (the dee-to-ground potential) acceleration on the first
turn. The beam then spirals outward, gaining a maximum of 400 keV per
turn.

Several types of operating conditions must be considered. One of the
principal uses of the machine will be to produce mesons. In this case, the
current required is large, but the energy resolution is not important
(since the mesons are produced in a secondary target). Therefore, the
duty cycle may be maximized at the expense of energy resolution. It is
also planned to produce a high resolution proton beam. In this case, high
current is not required so a smaller duty cycle may be considered, giving
smaller radial oscillation amplitude and thus improving energy resolution.
It is also hoped that with the addition of third harmonic to the RF,
separated turn acceleration will be possible, i.e. spatial turn separation
will be maintained out to extraction so that the beam can be extracted
from one turn, giving very high energy resolution (hopefully, ±50 keV).
Again, the phase band accelerated would be quite narrow.

1.3 Equations of Motion

The force on a charged particle moving in electric and magnetic fields
is given by the sum of the Lorentz and electric forces

\[ \vec{F} = q(\vec{E} + \vec{v} \times \vec{B}). \]  

(1)

\( \vec{F} \) is the force on the particle which has charge \( q \), mass \( m \) and velocity \( \vec{v} \). The electric field is \( \vec{E} \) and the magnetic field is \( \vec{B} \).

We define a Cartesian co-ordinate system with the \( z \) axis upwards in
the axial direction (perpendicular to the plane of the orbits), the
\( x \) direction is along the centreline of the dee gap, and \( y \) is perpendicular
to the dee gap and the axial direction.

In a Cartesian system, Eq. (1) can be written

\[ F_x = q \left( \frac{\partial}{\partial x} E_x + \left( \frac{\partial}{\partial y} B_x - \frac{\partial}{\partial z} B_y \right) \right), \]  

(2)

\[ F_y = q \left( \frac{\partial}{\partial y} E_y + \left( \frac{\partial}{\partial z} B_y - \frac{\partial}{\partial x} B_z \right) \right), \]  

(3)

\[ F_z = q \left( \frac{\partial}{\partial z} E_z + \left( \frac{\partial}{\partial x} B_z - \frac{\partial}{\partial y} B_x \right) \right). \]  

(4)
The ion circulates in its orbit near the $x$-$y$ plane; hence the components of the velocity in this plane ($v_x$ and $v_y$) are much larger than $v_z$. Due to the symmetry of the magnet, the magnetic field in the median plane is in the axial direction only, i.e. $B_x = B_y = 0$. Errors in the construction of the magnet may cause the magnetic median surface to be different from the geometric median plane, giving non-zero values of $B_x$ and $B_y$ in the geometric median plane; however, these will be small, and we may write Eqs. (2) and (3) as

$$\frac{d}{dt}(mv_x) = q(E_x + v_y B_z)$$  \hspace{1cm} (5)$$

$$\frac{d}{dt}(mv_y) = q(E_y - v_x B_z).$$  \hspace{1cm} (6)

Eqs. (5) and (6) are relativistically correct, provided the changes in mass due to acceleration are not neglected. The relativistic mass is

$$m = \gamma m_0$$

where $m_0$ is the rest mass and $\gamma$ is the usual relativistic factor

$$\gamma = 1 + \frac{T}{m_0 c^2} = \left(1 - \beta^2\right)^{-\frac{1}{2}}$$

where $T$ is the kinetic energy of the ion, $c$ is the velocity of light and $\beta = v/c$.

The approximation used in deriving Eqs. (5) and (6), i.e. that terms in $v_x B_y$ and $v_y B_x$ are negligible, has removed coupling between motion in the median plane and motion in the axial direction, greatly simplifying the calculations. The solutions of Eqs. (5) and (6) [obtained by numerical integration through realistic electric and magnetic fields] are discussed in Section 4.

The axial motion is described by Eq. (4). The terms in $B_x$ and $B_y$ cannot be neglected in this case since they are multiplied by the (large) velocities $v_x$ and $v_y$. It is these terms which describe the axial magnetic focusing produced by flutter and spiral in the magnetic field when the ion is not in the median plane. The axial motion is discussed in Section 3.
2. ELECTRIC FIELD CALCULATIONS

2.1 Choice of Method

Accurate orbit calculations in the central region require a detailed knowledge of the electric and magnetic fields involved. The magnetic field can be obtained from measurements on model magnets. The electric field is produced by complicated electrode shapes (see Figs. 1.1 and 1.2) and hence cannot be calculated analytically. There are several methods which can be used to obtain the electric field in these circumstances:

1) Electroconductive analogies in which the potential is obtained by measuring the voltage in a conducting medium surrounding a model of the electrodes. This method yields potentials (in two or three dimensions) with errors of about 0.3%.  

2) Numerical solution of Laplace's equation. This method yields potentials with average errors of 0.1% or less, depending on the time available for computation. This method is described in detail below.

3) The induced current method in which a vibrating charged probe induces a current in the electrodes proportional to the component of the required field at the probe in the direction of vibration of the probe. This method gives field values with errors of 5.0% or less.

4) The magnetic analog in which the components of the magnetic field are a measure of the corresponding electric field components.

Methods 3 and 4 yield field values which can be used directly in orbit calculations while methods 1 and 2 give potentials which must be numerically differentiated to obtain the field components.

From this point of view, method 3 or 4 is more attractive. However, methods 1, 3 and 4 require a model of the electrode structure to be built. This means that changes in the electrodes require time-consuming and expensive changes in the model. In addition, these three methods involve mechanically-driven probes which are subject to alignment errors. Also, these methods use complicated electronic circuits which are subject to drift over long periods of time. For these reasons, the numerical solution
of Laplace's equation which avoids these difficulties is the most attractive choice. Solving Laplace's equation for a complicated boundary shape is a difficult computational problem; however, the availability of large, fast computers enables large problems to be solved in a reasonable amount of time.

2.2 Finite Difference Approximation

We wish to find the electrostatic potential \( \phi \) which is the solution of Laplace's equation, i.e.

\[
\nabla^2 \phi = 0
\]  

(7)

within the rectangular parallelepiped shown in Fig. 2.1. This volume is bounded by the planes \( x = 0, x = pH, y = 0, y = qh, z = 0, z = rh \). In the usual problem either the potential or its derivative is known on the surface of the volume (Dirichlet or Neumann boundary conditions, respectively) while the potential is unknown inside the volume. In the problems to be studied here every boundary plane has Dirichlet boundary conditions or is a plane of symmetry (described below). In addition, parts of the interior of the volume may have fixed potential values, i.e. the boundary conditions may extend inside the volume.

To solve Eq. (7) numerically we transform the differential equation to a difference equation and solve for the values of \( \phi \) at discrete nodes within the volume. Fig. 2.1 shows a rectangular grid with uniform spacing \( h \) in all three directions. The nodes occur at the intersections of the planes \( x = ih, y = jh \) and \( z = kh \) where \( i = 0,1,...,p \), \( j = 0,1,...,q \) and \( k = 0,1,...,r \). The number of nodes in the grid \( N \) is \((p+1)(q+1)(r+1)\).

To derive the finite difference approximation, we consider the potential \( \phi_{ijk} \) at some node \( i,j,k \). Expanding the potential in a Taylor series at the six nodes nearest to \( i, j, k \) we obtain

\[
\phi_{i\pm1,j,k} = \phi_{ijk} \pm h \left( \frac{\partial \phi}{\partial x} \right)_{ijk} + \frac{h^2}{2} \left( \frac{\partial^2 \phi}{\partial x^2} \right)_{ijk} + \frac{h^3}{6} \left( \frac{\partial^3 \phi}{\partial x^3} \right)_{ijk} + \frac{h^4}{24} \left( \frac{\partial^4 \phi}{\partial x^4} \right)_{ijk} + \ldots
\]

\[
\phi_{i,j\pm1,k} = \phi_{ijk} \pm h \left( \frac{\partial \phi}{\partial y} \right)_{ijk} + \frac{h^2}{2} \left( \frac{\partial^2 \phi}{\partial y^2} \right)_{ijk} + \frac{h^3}{6} \left( \frac{\partial^3 \phi}{\partial y^3} \right)_{ijk} + \frac{h^4}{24} \left( \frac{\partial^4 \phi}{\partial y^4} \right)_{ijk} + \ldots
\]

( cont'd. )
\[
\phi_{i,j,k+1} = \phi_{i,j,k} \pm h \left( \frac{\partial^4 \phi}{\partial z} \right)_{i,j,k} + \frac{h^2}{2} \left( \frac{\partial^2 \phi}{\partial z^2} \right)_{i,j,k} \pm \frac{h^3}{6} \left( \frac{\partial^3 \phi}{\partial z^3} \right)_{i,j,k} + \frac{h^4}{24} \left( \frac{\partial^4 \phi}{\partial z^4} \right)_{i,j,k} + \ldots
\]

Adding these, we obtain

\[
\phi_{i+1,j,k} + \phi_{i-1,j,k} + \phi_{i,j+1,k} + \phi_{i,j-1,k} + \phi_{i,j,k-1} + \phi_{i,j,k+1} = 6 \phi_{i,j,k} + \frac{h^2}{2} \nabla^2 \phi + O(h^4);
\]

using (7) and neglecting terms in \( h^4 \) and higher, we have,

\[
\phi_{i,j,k} = \frac{1}{6} \left( \phi_{i+1,j,k} + \phi_{i-1,j,k} + \phi_{i,j+1,k} + \phi_{i,j-1,k} + \phi_{i,j,k+1} + \phi_{i,j,k-1} \right)
\]

[interior points]

\[
= b_{i,j,k}.
\]

[boundary points]

In the right side of Eq. (8) we have abbreviated the notation by writing only those subscripts which are not equal to \( i, j \) or \( k \).

Eq. (8) describes a linear system of \( N \) equations which can be written

\[
Ad = b
\]

(9)

where \( A \) is an \( N \) by \( N \) matrix containing the coefficients of the system, \( d \) is a column vector containing the unknown potential values

\[
\begin{pmatrix}
\phi_{111} \\
\vdots \\
\phi_{pqr}
\end{pmatrix},
\]

and \( b \) is a column vector containing the potential values for those nodes which fall in the boundaries.

Now the solution of Eq. (7) is reduced to the solution of the linear system Eq. (9). It should be noted that the order of the system (9) is equal to the number of nodes in the mesh, which will be of the order of many thousands or millions.

Direct methods for solving linear systems such as Gaussian elimination or use of determinants have two disadvantages in the present case. Firstly, they require that the matrix \( A \) be stored. This is clearly unnecessary
since the elements of $A$ can be generated using Eq. (8). Secondly, they require about $N^3/3$ multiplications to solve a system of order $N$. To solve a system with $N = 10^6$ would take $10^{12}$ sec (many years) allowing 3 usec per multiplication. Such a system can be solved in about 2 hours using the iterative method described below.

Iterative methods offer two advantages over direct methods in this case. Firstly, they require only the current solution vector $x$ to be stored and secondly, they are much more efficient for solving large systems when the coefficient matrix $A$ contains many zero elements.

Many iterative methods for solving systems such as (9) have been developed and studied theoretically. An excellent review of the methods available is given by Forsythe and Wasow.\textsuperscript{12}

The method used here is based on a program developed by D. Nelson.\textsuperscript{13,14} Basically this program uses successive over-relaxation by points to solve the linear system.\textsuperscript{*} This method is applied in a manner which allows extremely large problems to be solved using a modest amount of computer memory. The theory of successive over-relaxation by points is reviewed in Appendix A. The important results are as follows:

We start with an initial approximation (usually zero) to the potential at each node $\phi_{ijk}^0$; then we obtain successive approximations using

$$\phi_{ijk}^{n+1} = \phi_{ijk}^n + \frac{\alpha(n+1)}{6} \left( \phi_{i-1,jk}^n + \phi_{i+1,jk}^n + \phi_{i,j-1,k}^n + \phi_{i,j,k-1}^n + \phi_{i,j,k+1}^n - 6\phi_{ijk}^n \right)$$

(10)

where the best value of the "over-relaxation factor" $\alpha$ for the ordinary successive over-relaxation method is given by

$$\alpha_b = \frac{2}{1 + \sin \theta} \approx 2 \left( 1 - \frac{1}{3} \sqrt{\frac{1}{q^2} + \frac{1}{r^2} + \frac{1}{n^2}} \right)$$

(11)

\textsuperscript{*} For this problem it appears that the Peaceman-Rachford method\textsuperscript{15} gives faster convergence.\textsuperscript{16} However, as has been pointed out by Young,\textsuperscript{16} it is difficult to devise an efficient storage scheme which allows the matrix $A$ to be accessed alternately by rows and columns. Any increase in convergence rate would probably be negated by increased time spent retrieving the data from the mass storage device.
where
\[
\cos \theta = \frac{1}{3} \left( \cos \frac{\pi}{p} + \cos \frac{\pi}{q} + \cos \frac{\pi}{r} \right) = 1 - \frac{\pi^2}{6} \left( \frac{1}{p^2} + \frac{1}{q^2} + \frac{1}{r^2} \right).
\]  

So solving the system consists of iterating over the nodes of the mesh in some order, replacing the value of \( \phi \) of each node by the values given by Eq. (10). The order we shall choose is, giving the \( i_ijk \) values of the point to be iterated,

\[
\begin{align*}
(0,0,0), (1,0,0) & \ldots (p,0,0) (0,1,0) \ldots (p,1,0) \ldots \ldots (p,q,0) \\
(0,0,1) & \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (p,q,1) \\
\vdots & \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (p,q,r)
\end{align*}
\]
or the reverse order.

It is shown in Appendix A that the convergence of the method is determined by the largest eigenvalue of the matrix \( A \). If the best value of \( \alpha \), i.e., \( \alpha_b \), is used, this eigenvalue is

\[
\lambda_m = \frac{1 - \sin \beta}{1 + \sin \beta} = 1 - 2 \pi \sqrt{\frac{1}{3} \left( \frac{1}{p^2} + \frac{1}{q^2} + \frac{1}{r^2} \right)} = \alpha_b - 1.
\]  

Values of \( \lambda_m \) and \( \alpha_b \) for the problems which are discussed in this section are given in Table I.

The number of iterations required to reduce the error by a factor \( f \) is approximately

\[
n = \log f / \log \lambda_m.
\]  

2.3 Computational Details

The program as described by Nelson\textsuperscript{14} used an iteration subroutine coded in FORTRAN. This was rewritten in assembler language giving a factor of twelve increase in speed. In addition, the new iteration routine allows the iteration to be done in alternating directions. Details of these changes are given in internal report TRI-1-71-1.

The advantage in iterating in alternating directions is that it ensures that the effect of the boundary conditions is quickly propagated through the volume. If, for example, uni-directional iteration was used
**TABLE I**

Largest eigenvalue ($\lambda_m$) and best over-relaxation factor ($\alpha_b$) for various size relaxation problems

<table>
<thead>
<tr>
<th>Problem size</th>
<th>Total number of mesh points</th>
<th>$\lambda_m$</th>
<th>$\alpha_b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$32 \times 32 \times 16$</td>
<td>16,384</td>
<td>0.7569</td>
<td>1.7569</td>
</tr>
<tr>
<td>$64 \times 64 \times 32$</td>
<td>131,072</td>
<td>0.8841</td>
<td>1.8841</td>
</tr>
<tr>
<td>$128 \times 32 \times 8$</td>
<td>32,768</td>
<td>0.6245</td>
<td>1.6245</td>
</tr>
<tr>
<td>$256 \times 64 \times 16$</td>
<td>262,144</td>
<td>0.7946</td>
<td>1.7946</td>
</tr>
<tr>
<td>$512 \times 128 \times 32$</td>
<td>2,097,152</td>
<td>0.8895</td>
<td>1.8895</td>
</tr>
</tbody>
</table>
going from small $i,j,k$ to large $i,j,k$, and all boundaries were zero except the plane with the largest $k$ value, many iterations would be required before the effect of the boundary at large $k$ would be felt at small $k$. Alternating the direction of iteration avoids this difficulty.

The values of the $\phi_{i,j,k}$'s are stored on a mass storage device (tape, disc or drum). Subsets of this total "volume" are transferred to core storage, iterated over and returned to the mass storage device. To increase efficiency (by decreasing the number of data swaps) several iterations are done over each subset of the total volume while it is in core storage. This causes the convergence rate to be very slow; however, the program has a novel feature, described below, which allows good starting values to be found, hence reducing the number of iterations required. The iterations over the subsets of the total volume must be done carefully, to avoid discontinuities where the edges of these subsets occur. Consider the volume shown in Fig. 2.1 broken into blocks, each block containing 16 x 16 x 8 points; then there are $b_1 = (p+1)/16$ blocks along the $x$ co-ordinate, $b_2 = (q+1)/16$ blocks along the $y$ co-ordinate and $b_3 = (r+1)/8$ blocks along the $z$ co-ordinate. The data area in core storage in which the iterations are done (the physical work area) contains a $2 \times 2 \times 2$ block subset of the total problem. The iteration is done as follows:

The physical work area is loaded starting at block co-ordinates $(1,1,1)$ and then iterated. During this iteration all potentials on the boundaries of the physical work area are held fixed except boundaries which are symmetry planes of the total volume. The next load origin is $(2,1,1)$, and this iteration is repeated. Since two blocks along each co-ordinate are iterated each time while the increment between iterations is one block, discontinuities in the data should be reduced. The sequence of load points for the iteration is either

$$(1,1,1), (2,1,1) \ldots \ldots (b_1,1,1),(1,2,1),(2,1,1) \ldots \ldots (b_1,b_2,1)$$

$$(1,1,2), (2,1,2) \ldots \ldots (b_1,1,2),(1,2,2),(2,1,2) \ldots \ldots (b_1,b_2,2)$$

or the reverse one (alternating direction iteration over the blocks).
It should be noted that in one sweep over the data using this procedure \(8(b_1-1)(b_2-1)(b_3-1)\) blocks are iterated. On the average, this is

\[
\frac{8(b_1-1)(b_2-1)(b_3-1)}{b_1 b_2 b_3}
\]  

iterations over each block.

The novel feature mentioned above which allows good starting values to be found operates as follows. After the boundary values have been assigned but before any iterations have been done, the mesh size is doubled reducing the problem to one with an eighth as many data points as the original problem. This process is repeated until the problem size is close to the size of the physical work area (32 x 32 x 16 points). This "reduced" problem is solved iteratively and expanded back to the original size problem. During the expansion process, the value assigned to each unknown point is the value for the nearest known node with smaller or equal \(i\), \(j\), and \(k\) values, i.e. if \(\phi_{ijk}\) is known, the program sets (omitting subscripts which are \(i\), \(j\), or \(k\))

\[
\phi_{i+1} = \phi_{j+1} = \phi_{i+1j+1} = \phi_{k+1} = \phi_{i+1k+1} = \phi_{j+1k+1} = \phi_{i+1j+1k+1} = \phi.
\]

This procedure provides good starting values for the final iteration.

The boundary values are assigned either by calling a user-supplied subroutine which returns the value of the potential at each point, or by the method given in internal report TRI-1-71-1 or by a combination of both.

In many situations, the boundary values at an edge of the problem are not known, but this edge is a plane of symmetry. In this case, the program calculates the potentials on the symmetry plane using the fact that the potentials outside it are the same as those inside. For example, if the \(i = 0\) plane were a plane of symmetry, then on this plane Eq. (10) would be

\[
\phi_{0jk}^{n+1} = \phi_{0jk}^n + \frac{1}{6} \left( \phi_{1jk}^n + \phi_{1jk}^n + \phi_{0j-1k}^n + \phi_{0j+1k}^n + \phi_{0jk-1}^n + \phi_{0jk+1}^n \right)
\]

When estimating the convergence rate for a problem which contains planes of symmetry, it is important to remember that the errors are not
zero at the plane of symmetry (as they would be if the plane were a boundary plane). Thus the errors and convergence rates will be those appropriate for the "effective size" of the problem, which is the size the problem would be if the symmetry properties were not utilized. Thus, if a problem contains one plane of symmetry, the effective size is twice the actual size, in general; if there are $n$ symmetry planes, the effective size is $2^n$ times the actual size.

2.4 Convergence Tests

To test the convergence and accuracy of the method, a problem for which the analytic solution was known was solved using the relaxation method. The problem is the one used by D. Nelson as a test case; it consists of a $64 \times 64 \times 32$ point "box" with boundary values of zero on all sides except the $k = 32$ surface where the potential is

$$V = \sin \left( \frac{2\pi l}{64} \right) \sin \left( \frac{2\pi j}{64} \right).$$

The sequence of operations carried out in solving this problem is given in Table II.

The first question which must be answered is how many iterations are required on the reduced problem. To answer this, several runs were done. For each run $n$ sweeps were done with $a = 1.5$, $n$ with $a = 1.3$ and $n$ with $a = 1.0$ (a total of $3n$ sweeps). The value of $a$ used was reduced from 1.5 (close to the best value) to 1.0 to ensure that the difference equations are solved as exactly as possible when the iterations are finished. The problem was then expanded to full size, and two iterations were done on the full volume. All iterations were done with alternating directions. The results are summarized in Table III. In all cases the error is very small. Since iterating over the small volume is relatively fast, there is no large penalty paid for over-estimating the number of iterations required, and $n = 100$ was chosen. For this case ($n = 100$), the average change per iteration before expanding was $< 10^{-6}$, i.e. the reduced problem had been solved exactly to the precision of the arithmetic used. Thus the error of 0.25% after expanding is due to the expansion process.

Now the problem was expanded back to full size $64 \times 64 \times 32$ points, and the convergence of this problem was investigated. Since we are doing
TABLE II

Sequence of operations used to solve a 64x64x32 node relaxation problem

<table>
<thead>
<tr>
<th>Step</th>
<th>Operation</th>
<th>Problem size after this step</th>
<th>Number of iterations</th>
<th>Average change per iteration after this step</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>set boundary conditions</td>
<td>64 x 64 x 32</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>reduce to coarse grid</td>
<td>32 x 32 x 16</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>3</td>
<td>iterate (α = 1.5)</td>
<td>32 x 32 x 16</td>
<td>100</td>
<td>0.9 x 10^{-3}</td>
</tr>
<tr>
<td>4</td>
<td>iterate (α = 1.3)</td>
<td>32 x 32 x 16</td>
<td>100</td>
<td>0.6 x 10^{-5}</td>
</tr>
<tr>
<td>5</td>
<td>iterate (α = 1.0)</td>
<td>32 x 32 x 16</td>
<td>100</td>
<td>0.5 x 10^{-6}</td>
</tr>
<tr>
<td>6</td>
<td>expand to fine grid</td>
<td>64 x 64 x 32</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>7</td>
<td>iterate (α = 1.5)</td>
<td>64 x 64 x 32</td>
<td>see Fig. 2.3</td>
<td>see Fig. 2.3</td>
</tr>
</tbody>
</table>

TABLE III

Average error after various numbers of iterations over the reduced problem

<table>
<thead>
<tr>
<th>Case</th>
<th>n</th>
<th>Average error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>25</td>
<td>0.40</td>
</tr>
<tr>
<td>2</td>
<td>50</td>
<td>0.30</td>
</tr>
<tr>
<td>3</td>
<td>75</td>
<td>0.31</td>
</tr>
<tr>
<td>4</td>
<td>100</td>
<td>0.25</td>
</tr>
<tr>
<td>5</td>
<td>200</td>
<td>0.28</td>
</tr>
</tbody>
</table>
the iterations over subsets of the problem each containing $32 \times 32 \times 16$
points, the best value of $\alpha$ is, from Eq. (11), $\alpha_b = 1.75$. Two other
values of $\alpha$ were used, 1.87 because this is $\alpha_b$ for a problem containing
$64 \times 64 \times 32$ points and 1.50 for reasons discussed below. Fig. 2.2 shows
the average error as a function of the number of iterations for these
values of $\alpha$. The discrepancy in the error after two iterations over the
fine grid between Table II and Fig. 2.2 is due to different iteration
directions where iterating over the fine grid. As expected, $\alpha = 1.75$ pro-
duces the fastest convergence, but the convergence is satisfactory in all
three cases. Eq. (A.23) predicts that the number of iterations required
to reduce the error by a factor $f$ is $n = \frac{\log f}{\log \lambda}$. The value of $\lambda_m$ appropriate to $\alpha = 1.75$ is 0.75; hence the number of iterations required to reduce
the error by a factor 10 is

$$\frac{-1}{\log_{10}(0.75)} = 8.0.$$  

As can be seen from Fig. 2.2, about 40 sweeps over the data are required to
achieve the same reduction ($\alpha = 1.75$). Since the problem contains
$4 \times 4 \times 4$ blocks of data, by Eq. (15), each sweep corresponds to

$$\frac{8(4-1)(4-1)(4-1)}{(4)(4)(4)} = 3.375.$$  

iterations. Hence the 40 sweeps correspond to 135 iterations, indicating
that the convergence is about sixteen times slower than the theoretically
expected rate for ordinary successive over-relaxation. This slow conver-
genre rate is probably due to the way in which the iterations are done,
i.e. many iterations over a small subset of the total volume. However,
with the good starting values provided by the reducing and expanding
procedure, the convergence of the problem is acceptable.

For practical problems it has been found by the author and by
D. Nelson\textsuperscript{13} that $\alpha = 1.5$ gives the best results. This is probably due to
the fact that, in practical cases, fixed points occur within the volume.
This means that the "wavelength" of the errors will be smaller than that
assumed in Eq. (A.8), leading to smaller values of $\alpha_b$. Since $\alpha = 1.5$
seemed to be best for "real" problems and since $\alpha = 1.5$ still gives accept-
convergence for the test problem, only this value was studied further.
Fig. 2.3 shows the average error and the average change per iteration for the $\alpha = 1.5$ case. The bars on the points giving the average error indicate the error at which the number of points vs error curve (Fig. 2.4) has fallen to half its peak value. As we would expect, since we are using $\alpha > 1$, the change per iteration is always larger than the error. Of course, there may be a few large local errors which do not produce a large average error.

Fig. 2.4 shows the distribution of errors for various numbers of iterations. The graph is actually a histogram; the vertical lines give the (approximately equal) intervals in which the numbers of points are counted. Several points are worth noting. Even after many iterations, about 0.04% of the points have errors larger than 5.0%, despite the average error being less than 0.04%. It appears that this situation will not change significantly even if many more iterations are done. It seems that the large errors must be removed before the smaller ones are affected. This is shown more clearly in Fig. 2.5 where the number of points with a given error is plotted as a function of the number of errors. It can be seen that the number of points with small errors remains relatively constant until the number of large errors has been reduced.

2.5 The Practical Problem

Problems which are useful in practice usually contain many more points than the case discussed in Section 2.4. The same reducing and expanding procedure is followed, so that the starting values for the iterations on the large problems are quite good. However, since the number of points is larger, the convergence will be slower [as predicted by Eq. (A.22)], and each iteration will take longer.

The practical case discussed here is a 128 in. by 32 in. by 8 in. section from the centre of the TRIUMF cyclotron. The 8 in. dimension is in the axial direction and extends from the cyclotron median plane to the vacuum tank. The 128 in. dimension is along the dee gap, and the 32 in. dimension is perpendicular to the dee gap. The geometry in the median plane and the electric equipotentials calculated using this method are shown in Fig. 1.1. The geometry in the axial direction is shown in Fig. 1.2. It was felt that a 0.25 in. grid size adequately defined the boundaries;
hence the problem contained $512 \times 128 \times 32 = 2,097,152$ data points.

The sequence of operations used in solving this problem is given in Table IV. At the end of step 6, the change per iteration at each point is less than $10^{-7}$, so after expansion to $256 \times 64 \times 16$ points, we would expect the average error to be about 0.40% as it was in the test case. The errors are of course unknown, but the average change per iteration at the beginning of step 8 was about 0.1%. The reason for this value being smaller than the value for the test case is probably that there are more fixed points in the real case. After 75 iterations over the $256 \times 64 \times 16$ problem, the average change per iteration is less than $10^{-6}$. The iterations on the full-size problem (step 12) are very costly since we now have over two million data points; however, very few iterations are required. Step 12 consisted of four iterations over the full volume to smooth out any bumps left by the expanding process. The average change per iteration at the end of step 12 was less than 0.01%. Local errors will be larger than this, of course. In the test problem the largest errors were more than 100 times as large as the average error but only for 0.04% of the points; hence in this case we can expect local errors of the order of 1 or 2% at a very small number of points. However, the convergence of the problem is very satisfactory. As can be seen in Fig. 1.1, the equipotentials have no unexpected kinks and fit the boundary conditions extremely well.
TABLE IV

Sequence of operations used to solve a 512x128x32 node relaxation problem

<table>
<thead>
<tr>
<th>Step</th>
<th>Operation</th>
<th>Problem size after this step</th>
<th>Number of iterations</th>
<th>Average change per iteration after this step</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>set boundary conditions</td>
<td>512 x 128 x 32</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>reduce to coarse grid</td>
<td>256 x 64 x 16</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>3</td>
<td>reduce to coarser grid</td>
<td>128 x 32 x 8</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>4</td>
<td>iterate ((\alpha = 1.5))</td>
<td>128 x 32 x 8</td>
<td>100</td>
<td>(0.3 \times 10^{-3})</td>
</tr>
<tr>
<td>5</td>
<td>iterate ((\alpha = 1.3))</td>
<td>128 x 32 x 8</td>
<td>100</td>
<td>(&lt;10^{-7})</td>
</tr>
<tr>
<td>6</td>
<td>iterate ((\alpha = 1.0))</td>
<td>128 x 32 x 8</td>
<td>100</td>
<td>(&lt;10^{-7})</td>
</tr>
<tr>
<td>7</td>
<td>expand</td>
<td>256 x 64 x 16</td>
<td>-</td>
<td>?</td>
</tr>
<tr>
<td>8</td>
<td>iterate ((\alpha = 1.5))</td>
<td>256 x 64 x 16</td>
<td>25</td>
<td>(0.2 \times 10^{-4})</td>
</tr>
<tr>
<td>9</td>
<td>iterate ((\alpha = 1.3))</td>
<td>256 x 64 x 16</td>
<td>25</td>
<td>(0.2 \times 10^{-6})</td>
</tr>
<tr>
<td>10</td>
<td>iterate ((\alpha = 1.0))</td>
<td>256 x 64 x 16</td>
<td>25</td>
<td>(0.1 \times 10^{-6})</td>
</tr>
<tr>
<td>11</td>
<td>expand to full size</td>
<td>512 x 128 x 32</td>
<td>-</td>
<td>?</td>
</tr>
<tr>
<td>12</td>
<td>iterate ((\alpha = 1.5))</td>
<td>512 x 128 x 32</td>
<td>4</td>
<td>(&lt;10^{-4})</td>
</tr>
</tbody>
</table>
3. AXIAL MOTIONS

3.1 Introduction

The axial motions of ions within a cyclotron are influenced by three effects: magnetic forces due to slope, flutter and spiral of the magnetic field, electric forces due to the lens effect of the dee gap, and space charge forces due to the electric field produced by the beam. The magnetic force is small and focusing at the centre of the machine. The electric force is very strong and phase dependent (focusing for some phases and defocusing for others). The space charge force is weak and always defocusing. It will be shown that the axial motions during the first few turns are controlled almost entirely by the electric forces.

Since one of the main design objectives of TRIUMF is to accelerate ions over a wide interval of the RF waveform (i.e. ions with large differences in their initial RF phase) careful design is required to prevent loss of those ions which start at unfavourable phases. The improvements which can be achieved by 'squaring' the radio-frequency waveform (by adding a small fraction of third harmonic to the fundamental) will be demonstrated.

Obviously, the axial displacement of the beam must not exceed the aperture of the dees but a more stringent limit on the amplitude of the axial oscillation is set by the fact that passage of the beam through regions where the forces are not linear causes distortion of the beam emittance. This causes a decrease in the "effective density" of the beam within the elliptical contour enclosing the beam's phase space. Recent work by Han\textsuperscript{17} indicates that about 60\% of the dee aperture is linear to within 5\%.\textsuperscript{*} The axial motions must be adjusted so that as wide a range of phases as possible is transmitted. In addition, the beam must be matched to the magnetic field so that the amplitude of the axial oscillations is minimized.

3.2 Magnetic Field

The axial restoring force ($F_z$) exerted on an ion by the magnetic field can be expressed in terms of the axial oscillation frequency ($\nu_z$) which an

\* C. Han integrated the equations of motion numerically through fields for three gap geometries. For a gap height of 1.6 in. and gap widths of 3.0 in., 6.5 in., and 7.4 in., the deviations from linearity were less than 5\% over 60\% of the gap height.
ion would have in the absence of other forces in the axial direction. The oscillation frequency is related to the force by

$$F_z = -m \omega_0^2 \left( v_z \right)^2,$$

where $\omega_0$ is the ion rotation frequency. This is a linear approximation valid only for $z \ll g$, the magnet pole gap height. In a sector-focused cyclotron the oscillation is given approximately by

$$\left( v_z \right)_m^2 = -\mu' + F^2 (1 + 2 \tan^2 \theta).$$

where $\mu'$ describes the radial variation in the magnetic field. The azimuthal variation of the magnetic field is described by the flutter function $F^2$ where

$$F^2 = \frac{(B - \overline{B})^2}{\overline{B}^2}.$$ 

$\overline{B}$ is the mean field at a given radius and the angular brackets denote a mean at one radius. Near the centre of the cyclotron the magnetic field is given to a good approximation by $B = \overline{B} \left[ 1 + f \cos \theta_m \right]$ where $f$ is the amplitude of the sixth harmonic component ($= \overline{B}_6 / B$) and $\theta_m$ is the azimuth angle of the peak field. If the azimuthal variation of the field is related to one harmonic only, the flutter function $F^2$ is related to the amplitude of the harmonic $f$ by $F^2 = \frac{f^2}{2}$. The angle $\epsilon$ is the so-called "spiral angle" defined by $\tan \epsilon = r \theta_m / dr$. For an isochronous field, $\mu'$ is positive and, near the centre of the cyclotron, the spiral angle $\epsilon$ is zero, hence the focusing provided by the magnetic field is due only to the "flutter" in the field. If the net effect of the magnetic field is to be focusing, the term due to the flutter must be larger than the defocusing term due to the field slope. Unfortunately, it is difficult to obtain large flutter in the central region of a cyclotron magnet because the vertical magnet gap is much larger than the horizontal distance between pole pieces. A typical plot of $(v_z^2)$ as a function of radius is shown in Fig. 3.1. We plot $(v_z^2)$ rather than $(\mu')$ since $(v_z^2)$ is proportional to the force exerted on the ion. Some cyclotrons use a magnetic "cone" in the central region to increase focusing. This consists of a central "bump" on the isochronous value of the field. This means that $\mu'$ is negative,
hence the magnetic focusing is increased. In addition, phase slip is introduced due to the fact that the field is not isochronous. The ions must be started at positive (late) phases so that they have slipped into phase by the time they have reached the isochronous field region. The positive phase histories are advantageous from the point of view of electric focusing.

In the absence of "squaring" of the RF, it will be shown that the phase acceptance has a sharp cutoff at -5 deg, i.e., only phases more positive than this can be accepted. A small field bump could be used to shift these positive phases into isochronism so that the range of phases which is accelerated is centred about 0 deg. It is shown in Section 3.8 that this field bump does not contribute appreciably to the focusing.

A field bump will not be required when addition of third harmonic to the RF shifts the lower limit of the phase acceptance from -5 deg to -25 deg. Field bumps are undesirable for three reasons.

Firstly, the ions start off phase and since the error in centring depends on the cosine of the largest phase angle, the centring errors are increased.

Secondly, since the ions start off phase the energy gain is reduced. Because of the relatively high injection energy, this makes the clearance between the centre post and the beam small on the first turn.

Thirdly, the field bump will cause the beam to pass through the \( v = 1 \) resonance (when \( \mu' = 0 \)) possibly leading to an increase in the radial oscillation amplitude.

Another way to increase the focusing is to increase the flutter. This may be achieved by cutting three of the pole pieces at a radius of 30 in., giving a three-sector geometry in the central region. This produced a \( \nu_r \) of about 0.2 from \( r = 10 \) in. outwards; however, the large flutter with three-fold symmetry caused undesirable effects in the radial behaviour of the beam (see Section 4) and had to be abandoned.

Tests have also shown that a set of "floating" pole pieces 1.66 in. above and below the median plane between 12.5 in. and 30.0 in. radius, with six-fold symmetry, can also provide \( (\nu_r)_{m} = 0.2 \) in the central region. However, it is virtually impossible to mount pole pieces in such a position without disturbing the alignment of the resonator hot arms.
At this time it seems that the best magnetic focusing that can be achieved is that shown by the solid curve in Fig. 3.1.

3.3 Space Charge Forces

The ions in the beam produce an electric field which exerts a force on each ion in the beam. This is the space charge effect. This effect can be analyzed by considering the force on an ion on the surface of a bunch due to the other ions in the bunch and the force due to the ions in other bunches. Reiser\textsuperscript{19} has analyzed this problem; we find for the case of TRIUMF (400 kV voltage gain per turn and low magnetic field) that at low energy, more than 90\% of the space charge force on an ion is due to the field produced by the other ions in the bunch. This force is directed radially outward from the centre of the bunch and can be written\textsuperscript{19}

\[ F = \frac{q I G}{4\varepsilon_0 \Delta \phi v z_m} \]

where \( z_m \) and \( \Delta \phi \) are the maximum height and the length of the bunch in degrees of RF, respectively, \( v \) is the velocity of the ions, \( I \) is the average current, \( q \) is the charge on the ion, \( \varepsilon_0 \) is the permittivity of free space, and \( G \) is a factor which depends on the height-to-width ratio of the beam bunch.

The vertical oscillation frequency produced by this force is

\[ \left( \nu_B^2 \right)_{\phi} = -\frac{q}{4\varepsilon_0 m \omega^2} \frac{G I}{z_m^2} \frac{1}{\Delta \phi} \frac{1}{v}. \] (17)

In the case of TRIUMF, the source will produce about 2 mA, hence without bunching we can expect \( I/\Delta \phi = 5 \mu A/\text{deg} \). Since the axial focusing is four or five times weaker than the radial focusing, we can expect beam width-to-height ratios of the order of 0.5. For this value, the geometrical factor \( G \) is 4.8. Fig. 3.2 shows how \( (\nu_B^2)_{\phi} \) varies with energy for various values of \( I/\Delta \phi \) and \( z_m \).

A graph showing the variation of \( G \) with the gap and height of the dee can be found in the paper by Reiser.\textsuperscript{19}

This force is roughly the same order of magnitude as the magnetic force and can be accounted for by using an "effective" magnetic \( \nu_B^2 \) which
is the difference between the actual magnetic $v_z$ and the space charge $v_z$ for the beam intensity under consideration.

3.4 Electric Lens Effects

The importance of electric focusing effects in cyclotrons was recognized soon after the invention of the cyclotron. Rose\(^1\) developed approximate expressions for the lens effects of cyclotron dee gaps using the symmetry properties of the electric fields and a description of the field derived by Kottler.\(^20\) These studies indicated that the lens properties arose from two effects. As can be seen in Fig. 3.3, the first part of the gap is focusing, while the second part is defocusing. These would cancel exactly, except that

(i) the field is changing, and

(ii) the ion is accelerated.

The deflection due to the field variation arises because the ion sees a different electric field in the second half of the gap than in the first. Since the first half of the gap is focusing and the second half is defocusing, there is a differential focusing effect. The change in $z' = \frac{dZ}{dx}$ due to this "field variation" effect is, to first order in $qV_0/E_0$,

$$\left(\Delta z'\right)_{fV} = -\frac{qV_0}{E_0} \frac{1}{r} z \sin \phi_0$$

where $V_0$ is the dee voltage and $E_0$, $r$ and $\phi_0$ are the energy, radius and RF phase of the ion at the gap centre. This effect is linear in $z$ and is focusing when the field is falling (positive phases) but defocusing when the field is rising (negative phases).

The second effect is due to the ion spending less time in the second half of the gap, hence the defocusing force in the second half of the gap produces less deflection than an equal force would in the first half. This effect is always focusing and is given by

$$\left(\Delta z'\right)_{eO} = -g \left(\frac{qV_0}{E_0}\right)^2 z \cos^2 \phi$$

where $g$ is a numerical factor depending on the geometry.

There is also a collimation term due to the fact that the forward momentum $p_x$ increases while the transverse momentum $p_z$ remains constant;
however, this term disappears when the change in \( p_z \) rather than \( z' \) is considered.

In addition, Rose predicts a change in axial position

\[
\Delta z = \left( 1 - \frac{qV_0 \cos \phi}{E_0} \right) z_0 .
\]  

(20)

Rose's analysis was extended by Cohen\(^3\) who used an electric field developed by Murray and Ratner.\(^21\) This more detailed analysis showed that the expressions developed by Rose are the first two terms in a series in \( \sqrt{1/E_0} \). More recently, the analysis has been further extended by Reiser.\(^22\) This most recent analysis includes the effect of the dee liner, i.e., \( c \) is not \( \infty \) (see Fig. 3.3) as was assumed in the previous analyses. Reiser's expression for the deflection is

\[
\Delta z' = -z_0 \left( \frac{N(qV_0)}{\mu E_0} \right) \sin \phi_c + \frac{2F(a,b,c)}{\pi b} \left( \frac{qV_0}{E_0} \right)^2 \cos^2 \phi_c - \frac{qV_0}{E_0} \cos \phi_c z_0' \]  

(21)

where \( z_0 \) is the axial displacement of the ion when it enters the lens, \( N \) is the harmonic ratio of the RF frequency to the ion frequency, and \( F(a,b,c) \) is a dimensionless function which depends on the geometry (\( a, b \) and \( c \) are described by Fig. 3.3).

The linearity in \( z \) of the above expressions for the deflection and displacement permits an enormous simplification in the axial motion calculations. However, this formula is based on the assumptions that the transit time of the particle across the gap is small and that the energy gain across the gap is much smaller than the incident ion energy. Since the electric forces become strong just where these approximations are likely to become invalid (i.e., at low energy), it is important to investigate the validity of this formula. For the case of TRIUMF, the RF operates at the fifth harmonic of the ion frequency (\( N=5 \)) and the transit times are of the order of 60 to 70 deg of RF on the first turn, so the small transit time approximation is not valid; however, the relatively high injection energy (300 keV) means that the approximation that the energy gain is small compared to the incident energy is reasonably valid after a few accelerations. Recently, Han\(^23\) has published a compilation of the focusing effects of cyclotron-like gaps for geometries applicable to TRIUMF. These results
were obtained by numerically integrating the equations of motion through electric fields calculated using the relaxation method described in Section 2. The data given in Han's Tables 6-1 to 6-7 provide a relevant source of numerical results to compare with the theory. To allow comparison of the electric forces to the magnetic and space charge forces, we can approximate the focusing effects at the dee gaps by an equivalent which would give the same deflection over a half-turn.

If \( \Delta z' \ll z_0/\pi r \),

\[
\nu^2_{z^2} = -\frac{F_0}{\omega^2} = -\frac{d^2 z}{\omega^2 dt^2} = -\frac{d^2 z}{d\theta^2} = -\frac{1}{\pi} \Delta \left( \frac{dz}{d\theta} \right) = -\frac{r}{\pi} \Delta z'.
\]  

(21)

For the numerical results given by Han \(^{23}\)

\[
\left( \frac{\nu^2_{z^2}}{z^2} \right)_e = \frac{r}{\pi} \left( \frac{1}{F_2} \right)
\]

where \( F_2 \) is the forward focal power of the lens.

Fig. 3.4 compares values of \( \left( \frac{\nu^2_{z^2}}{z^2} \right)_e \) obtained by exact numerical integration with those obtained from Eq. (21) for various phases and energies. The agreement is better for negative phases than for positive ones. In all cases the analytic description given by Eq. (21) overestimates the strength of the electric forces. It should be noted that for TRIUMF the injection energy is 300 keV, and the ion energy after the first main gap crossing is about 600 keV, so the approximation is valid to within 15% in the first turn. Hence we can use the expressions given in Eqs. (20) and (21) and obtain a reasonable estimate of the axial motions.

It should be noted that the electric forces are much larger than the magnetic and space charge forces. In addition, the electric forces are defocusing for, roughly speaking, negative phases. This causes a sharp cut-off in the phase acceptance near 0 deg. This cut-off can be shifted to more negative phases by providing addition (for example, magnetic) focusing. To shift this cut-off to -30 deg at 500 keV, magnetic focusing equivalent to a \( (\nu^2_{z^2})^2 \) of 0.3 would be required to overcome the defocusing effects of the electric field.

This sharp cut-off for negative phases is due to the field variation effect. The deflection due to field variation is proportional to \( \sin \phi \).
hence rapidly becomes large for negative phases. The focusing due to the energy change is proportional to $\cos^2 \phi$ and is multiplied by a smaller coefficient than the field variation term. The relative magnitude of these two effects is shown in Fig. 3.4. The maximum contribution to $(v_z)^2$ from the energy change term is given by the curve for $\phi = 0$. Hence the net effect of the electric field closely follows the field variation effect and is defocusing for negative phases.

3.5 Calculation of Cyclotron Acceptance

Since the linear description of the electric lens effect is reasonably accurate, and the magnetic and space charge focusing can be described in a linear manner, we can track the axial motions of the ions using the matrix method for tracking beams as suggested by Penner. If the axial focusing frequency due to the combined effects of the magnetic field and space charge is $v_z$, then in a region where $v_z$ is constant, the axial motion will be given by

$$z(s) = z_0 \cos v_z \theta + \frac{r s_0}{v_z} \sin v_z \theta$$

$$z'(s) = -\frac{v_z s_0}{r} \sin v_z \theta + z_0' \cos v_z \theta$$

where $z_0$ and $z_0'$ are the initial (axial) displacement and slope, respectively, $\theta = s/r$ is the azimuthal angle subtended by the ion, $s$ is the path length, and $r$ is the radius of curvature of the ion.

We are dealing with low energies so we can use a non-relativistic energy-momentum relationship

$$z^* = \frac{p_z}{k \sqrt{E}}$$

where $p_z$ is the axial momentum, $E$ is the kinetic energy and $k = \sqrt{2m_0}$. It is convenient to measure momenta as $\beta \gamma r_\infty$ where $\beta$ and $\gamma$ are the usual relativistic factors and $r_\infty$ is the "cyclotron radius" ($= m_0 c/qB_\infty$). This momentum is numerically equal to the radius of curvature the ion would have in the central magnetic field ($B_\infty$). For the TRIUMF central region $B_\infty = 3.0$ kG, $\beta \gamma r_\infty (\text{in.}) = 18.94 \sqrt{E} (\text{MeV})$. 
Using Eq. (23), we can write the magnetic transfer matrix

\[
\begin{pmatrix}
  z \\
  \frac{p_B}{k}
\end{pmatrix} = \begin{pmatrix}
  \cos \nu_z \theta & \frac{r}{v_B \sqrt{E}} \sin \nu_z \theta \\
  -\frac{v_B \sqrt{E}}{r} \sin \nu_z \theta & \cos \nu_z \theta
\end{pmatrix} \begin{pmatrix}
  z_0 \\
  \frac{p_B}{k}
\end{pmatrix} = T \begin{pmatrix}
  z_0 \\
  \frac{p_B}{k}
\end{pmatrix}.
\]

(24)

The expressions given in Eq. (20) and (21) can also be written in this form. If we call the transfer matrix for the dee gap

\[
T_e = \begin{pmatrix}
  a & b \\
  c & d
\end{pmatrix}
\]

then

\[
a = 1 - \frac{qV_0}{E_c} \cos \phi
\]

(25)

\[
c = -\sqrt{E_f} \left \{ \frac{N(qV_0)}{q} \sin \phi + \frac{2P(a, b, c)}{\pi b} \left ( \frac{qV_0}{E_c} \right )^2 \cos^2 \phi \right \}
\]

(26)

where $E_c$ and $E_f$ are the ion energies at the centre and end of the dee gap, respectively.

The expressions derived by Rose and Cohen do not include any dependence of the final position on the initial divergence, i.e. $b$ is assumed to be zero. The numerical results given by Han indicate $d = 1$, and since Liouville’s Theorem required $ad - cb = 1$, we choose

\[
d = 1
\]

(27)

\[
b = \frac{1}{a} (ad - 1)
\]

(28)

The fact that $b$ is non-zero means that there is a displacement term which depends on the initial slope ($z_1^\prime$). The existence of this term is confirmed in the numerical results given by Han; however, it is a small effect.

Now that the transfer matrices for the various parts of the trajectory are known, the complete trajectory can be calculated by the usual matrix multiplication method:
A method for analysing optic systems, much more powerful than trajectory tracking, has been developed by Steffen. This method allows the tracking of elliptical beam phase space areas through the system. It is usual to consider elliptical beam emittances, since ellipses can be specified by only three parameters, and make a good approximation to the actual phase space shape, which would presumably be polygonal. We will use Steffen's notation to describe the phase space ellipses. If the ellipse is described by the equation

\[ \gamma z^2 + 2 \alpha \frac{p_z}{k} + \beta \left( \frac{p_z}{k} \right)^2 = \varepsilon \]

then, as derived by Steffen, the maximum displacement and momentum are

\[ z_{max} = \sqrt{\varepsilon \beta} \]

\[ \frac{p_z}{k}_{max} = \sqrt{\varepsilon \gamma} \]

If we define the transfer matrix which transforms the vector

\[ \begin{pmatrix} z \\ \frac{p_z}{k} \end{pmatrix} \]
then Steffen shows that the ellipse parameters are transformed according to

\[
\begin{pmatrix}
\frac{p_{z_2}}{k} \\
\frac{p_{z_1}}{k}
\end{pmatrix} \rightarrow
\begin{pmatrix}
a \\
as
\end{pmatrix} \begin{pmatrix}
z_2 \\
z_1
\end{pmatrix}
\]

This allows tracking of the beam ellipse through the system by multiplication of 3x3 matrices.

Since the electric and space charge forces decrease with increasing energy, the only important focusing force outside the central region is the magnetic field. At this point (where electric forces have become negligible) the beam must be matched to the magnetic field; that is, the amplitude of the axial oscillations must be minimized and a beam of uniform envelope obtained. For a constant \( \nu_B \) the phase space ellipse which minimizes \( z_{max} \) is given by

\[
\begin{align*}
\alpha_m &= 0 \\
\gamma_m &= \frac{\nu_B \sqrt{E}}{r} = \text{constant} \cdot \nu_B \\
\beta_m &= \frac{1}{\gamma_m}.
\end{align*}
\]

Once the central region geometry has been decided, the transfer matrix from injection to the radius where electric forces are negligible can be calculated (\( T \)). Now if this matrix is inverted (\( T^I \)), the phase space ellipse required at injection to provide a beam matched to the magnetic field is

\[
\begin{pmatrix}
\beta_i \\
\alpha_i \\
\gamma_i
\end{pmatrix} = T^I \begin{pmatrix}
\beta_m \\
\alpha_m \\
\gamma_m
\end{pmatrix}.
\]
Unfortunately, due to the fact that the electric forces are phase dependent, $\mathcal{T}^{I}$ will be different for every initial RF phase. This means that the required initial phase space shape will be phase dependent; however, the initial phase space shape cannot easily be varied with phase. The best that can be done is to choose the initial ellipse shape for one RF phase and accept the fact that for other phases only those ions whose points in phase space fall within the chosen ellipse shape will be accelerated with $z < z_{\text{ideal envelope}}$. This provides a method for calculating the phase acceptance of an accelerator. The ellipse shape required for one phase is chosen as the one to be provided; then the overlap of the ellipses for other phases with the chosen ellipse gives the acceptance of the accelerator for each phase.

### 3.6 Phase Space Acceptance for Various TRIUMF Central Geometries and Injection Energies

Early in the design of TRIUMF it was necessary to fix the injection energy and injection geometry. The original suggestion was that the injection energy should be 150 keV; however, it was soon realized that the strong and phase-dependent electric forces would allow only a poor duty factor for this injection energy. Raising the injection energy would alleviate this problem and also reduce the spread in orbit centre points due to different energy gains for different RF phases, thus improving the radial beam quality. However, a higher injection energy makes bunching, chopping and the design of the spiral electrostatic inflector more difficult. Thus we must investigate the axial motions to determine how the phase acceptance varies with injection energy and make a compromise between increased phase acceptance and the difficulties mentioned above.

Various initial orbit geometries had been suggested, ranging from the one injection gap case shown in Fig. 1.1 to the multi-gap case shown in Fig. 3.5.

At the time of these studies it was hoped that a three-sector magnetic field could be used in the central region. Model tests showed that this three-sector geometry produced a magnetic $\nu_2$ of about 0.2, and so the axial motion studies were done with this value. The three-sector magnetic geometry was later replaced by a six-sector geometry for reasons which are
explained in Section 4. The six-sector geometry produces much smaller values of $v_z$ at small radius (see Fig. 3.1); however, tests with smaller $v_z$ values show that the conclusions reached here are still valid even with much reduced values of $v_z$.

Most of the geometries studied had posts defining the first two gaps. It was estimated that these posts would reduce the electric forces by a factor of 4, and this is included in the calculations.\(^\star\)

For each geometry the transfer matrices and the ellipse shapes required at injection were calculated. Fig. 3.6 shows the ellipses required for one geometry. The cyclotron acceptance was calculated as discussed above. For comparison purposes, the overlap of the ellipses with the ellipse for 20 deg is used, since this is approximately in the middle of the acceptance interval and gives as good matching with other phases as is possible.

Fig. 3.7 shows the overlap with the ellipse for a phase of +20 deg for injection energies of 150, 306 and 472 keV with one 100 keV gap in the first turn. The sharp cutoff at about -15 deg is due to the defocusing action of the electric field. Fig. 3.8 shows the overlap with the ellipse for a phase of +20 deg for injection energies of 120, 286 and 454 keV with three 100 keV gaps in the first turn. This geometry gives no significant improvement in the phase acceptance, and the multi-gap geometry worsens the radial centre point spread and complicates the resonator design; hence multi-gap geometries were abandoned.

The data for the one-gap geometries is summarized in Fig. 3.9. The average acceptance (averaged from -30 deg to +60 deg) seemed to flatten out above 300 keV, and this seemed to be the highest reasonable energy from the point of view of bunching and inflection, so it was decided to raise the injection energy from 150 to 300 keV.

Now after it was decided that a three-sector magnetic geometry could not be used, the code was rewritten to accept a magnetic $v_z$ varying with radius, and the calculations for 300 keV injection energy were repeated using the much smaller $v_z$ measured on the six-sector magnet model (Fig. 3.1).

\(^\star\) The study by Han\(^{23}\) indicates that the presence of posts in the dee gap actually reduces the focusing forces by a factor of six or seven.
Fig. 3.10 shows the acceptance as a function of phase for this case for various choices for the initial ellipse. If the 20 deg ellipse is chosen as the axial phase space shape of the beam at injection, over 90% of the beam would be accepted at phases greater than 10 deg while the fraction of the beam accepted would be 64% at 0 deg, 30% at -10 deg and zero for phases less than -10 deg. For this choice of initial ellipse shape the amplitude of the beam envelope does not exceed 0.55 in. for phases between -10 deg and 60 deg.

3.7 Effects of Third Harmonic in the RF on Axial Motions

The unique features of the TRIUMF resonators allow the addition of higher odd harmonics to the fundamental mode.\(^{27}\) If the third harmonic of the fundamental is added, the resonators operate as a \(3\lambda/4\) cavity as well as a \(\lambda/4\) cavity. We now have an RF voltage given by

\[
V = V_0 \left( \cos \phi - \varepsilon \cos (3\phi + \delta) \right)
\]

(29)

where \(\varepsilon = \frac{\text{amplitude of third harmonic}}{\text{amplitude of fundamental}}\) and \(\delta\) is the phase of the third harmonic with respect to the fundamental. Small positive values of \(\varepsilon\) are required to square or "flat-top" the fundamental. A fraction \(\varepsilon = 1/9\) produces perfect flat-topping at 0 deg, while more third harmonic than this produces a slight dip in the total voltage at 0 deg (see Fig. 3.11).

Now we must modify the formulae describing the lens effects of the gaps (Eqs. 25 to 28) to reflect the fact that the accelerating voltage is given by Eq. (29) instead of a pure cosine waveform. The field variation term given in Eq. (18) is essentially proportional to \(dV/d\phi\), i.e. to the rate at which the field is changing, and the energy change term given in Eq. (19) depends on the square of the energy gain. Hence with the RF voltage given by Eq. (29) we have

\[
\sigma = \sqrt{E_f} \left\{ \frac{N}{r} \left[ \frac{qV_o}{E_o} \right] \left[ \sin \phi - 3\varepsilon \sin (3\phi + \delta) \right] + \frac{2r(3\delta - 2\phi)}{\pi b} \left( \frac{qV_o}{E_o} \right)^2 \left[ \cos \phi - \varepsilon \cos (3\phi + \delta) \right]^2 \right\}
\]

(30)

and

\[
\alpha = 1 - \left( \frac{qV_o^2}{E_o} \right) \left[ \cos \phi - \varepsilon \cos (3\phi + \delta) \right].
\]

(31)

[cf. Eqs. (25) and (26) for no third harmonic.]
As before,

\[ d = 1 \]
\[ b = \frac{1}{a}(v - 1). \]

The negative limit on the axial phase acceptance is determined by the most positive phase for which the total force acting on the ion is defocusing (see Section 3.8). The electric force is defocusing due to the field variation effect when the field is increasing, since the (always focusing) energy gain effect is much smaller than the field variation effect. Hence we want to choose \( \varepsilon \) and \( \delta \) so that the negative of the slope of the voltage (due to the field variation effect)

\[
-\frac{d}{d\phi} \left( \frac{V}{V_0} \right) = +\sin \phi - 3\varepsilon \sin(3\phi + \delta)
\]

remains positive over as wide a range as possible. Fig. 3.12 shows \(-\frac{d}{d\phi} \left( \frac{V}{V_0} \right)\) for various values of \( \varepsilon \) and \( \delta \). The widest interval where \(-\frac{d}{d\phi} \left( \frac{V}{V_0} \right)\) remains positive is produced by \( \delta = -10 \pm 2 \text{ deg} \) and \( \varepsilon = 0.15 \pm 0.01 \); however, when \( \delta \neq 0 \), the presence of the beam causes coupling between the first and third harmonics in the resonators, so that the third harmonic becomes detuned increasing, by a large amount, the power required to maintain the third harmonic voltage. These coupling effects are not yet fully understood, so we will consider two cases, \( \delta = 0 \) and \( \delta \neq 0 \). When \( \delta = 0 \) the limit on \( \varepsilon \) is set by the value which causes \(-\frac{d}{d\phi} \left( \frac{V}{V_0} \right)\) to become negative; \( \varepsilon = 0.17 \), for example, produces a "hole" in the phase acceptance for \( 5 \text{ deg} < \phi < 25 \text{ deg} \) (see Fig. 3.13) due to the fact that \(-\frac{d}{d\phi} \left( \frac{V}{V_0} \right)\) is negative there. With \( \delta \neq 0 \), the maximum value of \( \varepsilon \) which can be tolerated is \( \varepsilon = 0.12 \pm 0.01 \). The phase acceptance produced by this value is shown in Fig. 3.14. If we allow non-zero values of \( \delta \), the best choice is \( \delta = -10 \text{ deg}, \varepsilon = 0.15 \). The phase acceptance for this case is shown in Fig. 3.15.

The optimum values of \( \varepsilon \) and \( \delta \) will depend to some extent on the final details of the magnetic field, the current being accelerated and on the RF system, since the amount of power required to keep the peak voltage at 100 keV increases as \( \varepsilon \) increases; however, the use of third harmonic in the RF appears to shift the cut-off in acceptance due to electric defocusing from -5 deg to -25 deg.
These conclusions are based purely on approximate analytic formulae and should be confirmed by numerical orbit tracking, i.e. by integrating the equations of motion numerically through three-dimensional electric and magnetic fields.

3.8 Effects of Field Bumps

By a field bump we mean here an increase in the magnetic field above the isochronous value. The usual procedure in cyclotrons is to make the bump largest at the centre of the cyclotron and decrease with radius. This produces additional axial magnetic focusing due to the negative field gradient [see Eq. (16)]. In addition, the bump causes the phase of the ions to change, since the magnetic field is no longer isochronous. The change in the sine of the phase angle is given by Smith and Garren

\[ \Delta (\sin \phi) = \frac{2 \pi N q B_c}{m_0 \Delta E} \int \delta B r dr = \frac{1}{8858 \text{ G} \cdot \text{in}^2} \int \delta B r dr. \]  

(33)

\( \delta B \) is the field bump, \( B_c \) is the central magnetic field, \( \Delta E \) is the energy gain per turn, \( N \) is the harmonic number, and \( q \) and \( m_0 \) are the ion charge and mass, respectively, and the constant is appropriate to the TRIUMF cyclotron. If \( \delta B > 0 \), the ions "catch up", i.e. positive phase (late) ions (favourable for electric focusing) are brought into phase as the energy increases and the electric focusing becomes less important than the magnetic flutter focusing. In a conventional cyclotron with the RF not operating at a high harmonic of the ion frequency, a carefully chosen field bump can be of great help in overcoming the electric forces, since the two effects mentioned above both help to increase the useful phase acceptance.

For the case of TRIUMF, the operation of the RF at the fifth harmonic of the ion frequency means that the electric focusing is very strong, and the two advantages mentioned above are reduced. For example, a bump of 25 G at 10 in. diminishing to zero at 25 in. gives the required phase shift of about 30 deg and an equivalent \( v_{\Phi}^2 \) due to the field gradient of 0.01. As can be seen from Fig. 3.4, however, this is much smaller than the force produced by the electric fields, and hence would have only a small effect on the phase acceptance. Larger field bumps cannot be used because (i) they produce more phase shift, causing ions to be shifted to a phase where the electric forces are defocusing, and (ii) ions starting at large positive
phases will not gain enough energy to clear the centre post on the first turn. A field bump can, however, be used to shift the range of phases which is accepted (-5 deg to +25 deg) to a range which is centred about 0 deg. This must be done carefully so that none of the useful phase range is shifted to a phase where the electric field is defocusing before the magnetic focusing is strong enough to make the total focusing positive.

Fig. 3.16 shows the total effective $v^2$ produced by the magnetic and electric fields. It can be seen that the sharp cut-off in phase acceptance at -5 deg demonstrated in Fig. 3.10 is caused by the fact that ions with phases more negative than -5 deg experience a force which is defocusing at about 1.5 MeV. Since Fig. 3.16 shows at what energy the total focusing becomes positive as a function of phase, we can calculate how the phase of the ions should be "programmed" so that the ion is brought into isochronism as soon as possible but not subjected to defocusing forces. Fig. 3.17 shows the phase at which the focusing becomes positive as a function of energy. An ideal magnetic field would produce no phase gain out to 1.5 MeV; then it would cause the phase of the ion whose phase was -5 deg at 1.5 MeV to become more negative, as shown in Fig. 3.17. The amount of phase gain desired is determined by the phase range to be accepted. If we aim to accept all ions with phases between -5 deg and +45 deg, the amount of phase gain is given by $\Delta \sin \phi$

where, after the phase gain has taken place,

$$\left( \sin(-5^\circ) + \Delta \sin \phi \right) = -\left( \sin(45^\circ) + \Delta \sin \phi \right).$$

This gives $\Delta \sin \phi = -0.31$, and the final range of phases is $\pm 23.4$ deg. The phase history shown in Fig. 3.17 is produced by the $\Delta \sin \phi$ shown in Fig. 3.18. The field bump required to produce this variation in $\Delta \sin \phi$ is shown in Fig. 3.19. A bump with such a sharp cut-off cannot be produced in practice; however, a bump with the same $\int \delta B \cdot dr$ as the one shown in Fig. 3.19, and which shifts the phases no faster than the bump shown in Fig. 3.19, could be used. It should be noted that the positive slope of this bump will decrease $\left(\frac{v_B}{m} \right)^2$ by about 0.005. Another disadvantage is that the ions will spend 5 or 6 turns far from the optimum phase. This may cause a large spread in centre points to develop unless the radial starting conditions are carefully chosen. This problem is considered in Section 4.

Of course, with third harmonic in the RF, the negative phase limit due to electric focusing is -25 deg and a field bump is not required.
4. RADIAL MOTIONS

4.1 Introduction

The central region of the cyclotron must be designed with two general objectives in mind. Firstly, the geometry of the electrodes must be arranged so that ions with the desired range of phases can be accelerated without hitting the electrodes. Secondly, the central region must produce a beam which is centred to within the desired tolerances.

We have shown in Section 1 that the motion in the median plane and the axial motion are independent to a good approximation. This section will discuss motion in the median plane only.

Fig. 1.1 is a section through the median plane of TRIUMF. The beam is injected down the axis of the cyclotron, then bent into the median plane by the spiral electrostatic inflector. The problem of the inflector is discussed elsewhere. We will assume that at the exit of the inflector we have a mono-energetic beam whose shape in phase space is a free parameter. The fact that the RF operates at the fifth harmonic of the ion frequency allows the "injection gap" to provide an extra 100 keV acceleration on the first turn. This eases the geometrical problems somewhat but causes the co-ordinate of the orbit centre point, perpendicular to the gap, to vary with phase. After reaching the first main gap, the ions are accelerated and spiral outward as in an ordinary two-dee cyclotron. The main geometrical constraint is clearance of the centre post on the first turn. Ions more than 45 deg from peak phase will hit the dee and be lost; however, centring requirements limit the acceptable phases to a range smaller than this.

The use of fifth harmonic acceleration means that transit time effects are large. This reduces the energy gain at low energy. To alleviate this situation, the dee gap is tapered in both the horizontal and axial directions (see Figs. 1.1 and 1.2) so that the electric fields are compressed and the energy gains are increased. In addition, the injection gap and first main gap are defined by posts which compress the electric field further and decrease axial focusing effects, as described in Section 3. These refinements make the geometry quite complicated and necessitate numerical tracking of the orbits at least out to the radius where the taper ends (30 in. or about 3 MeV). The orbit tracking was done using a slightly
modified version of the computer program PINWHEEL. The magnetic fields were obtained from measurements on a 1:20 model magnet, and the electric fields were calculated using the methods described in Section 2.

4.2 Basic Design

The first ion orbit in the cyclotron is shown schematically in Fig. 4.1. After leaving the inflector the ions travel, under the influence of the magnetic field only and with centre of curvature \( S \), until they reach the injection gap. At the injection gap the ions are accelerated, and hence the centre of curvature changes. In addition, if the centre line of the injection gap is at an angle to the beam, the ions are deflected. Since the energy gains and deflections are phase dependent, the centre points and radii of curvature will be different for different phases. The ions now travel, again under the influence of the magnetic field only, to the first main gap where they are again accelerated. Due to the phase-dependent effects at the injection gap, the radius and RF phase at which the first main gap is crossed will depend on the initial RF phase and so will the centre points. In designing the central geometry it is desirable to choose the position and orientation of the injection gap so that as wide a phase interval as possible clears the centre post on the first turn and is close enough to being centred to be useful.

As far as the placement of the injection gap is concerned, the quantities of interest are the radius, RF phase, energy and angle at which ions with various initial phases cross the first main gap.

To get a first order description of the effects we can use the approximation that the energy gains are instantaneous and give the ions \( 93.0 \cos \phi \) keV at the injection gap and \( 174.5 \cos \phi \) keV at the first main gap. These values are based on the results of numerically integrating ion trajectories through the gaps. We will also use a non-relativistic expression for the radius of curvature of the ion which, for a 3 kG magnetic field, is

\[
\rho(\text{in.}) = 0.60 \sqrt{E(\text{keV})} = 18.94 \sqrt{E(\text{MeV})}. \tag{34}
\]

Since in most cases we will be interested in differences between ions with different phases, we will label the ion whose centre of curvature is \((x_\phi,0)\) [i.e. its centre of curvature is on the centreline of the dee gap]
by the subscript 1. We label an ion at some other phase by the subscript 2. Quantities referring to the injection gap are further labelled with the superscript \( ig \), while those referring to the first main gap have superscript \( mg \).

The geometry of the orbit near the injection gap and first main gap is shown in Fig. 4.1. If the ion reaches the injection gap making an angle \( \beta \) to the gap axis, then for a peak dee voltage \( V_o \) an ion with charge \( q \) experiences a force \(-qV \cos \phi \dot{q} g \cos \beta \) along the orbit and a force \(-qV \cos \phi \dot{q} g \sin \beta \) perpendicular to the orbit. The injection gap causes a straightening or "collimating" deflection, given by

\[
A = \tan^{-1} \left( \frac{\Delta p \sin \beta}{p + \Delta p \cos \beta} \right) = \frac{\Delta E}{2E} \cos \phi \dot{q} g \sin \beta
\]

(35)

where \( p \) and \( \Delta p \) are the initial momentum and momentum gain, respectively. The approximation is valid if \( \Delta p \ll p \). Because of radial centring considerations, two quantities of interest are the radius and angle at which the ions cross the first main gap, or the differences in these quantities for different phases. The length \( a \) is given by

\[
a^2 = r_1^2 + r_2^2 = 2r_1 r_2 \cos (A_2 - A_1)
\]

(36)

and the angle \( E \) by

\[
E = \sin^{-1} \left( \frac{\Delta p}{p} (A_2 - A_1) \right).
\]

(37)

The radius difference at the first main gap (\( \delta r \)) is given by

\[
\delta r = a \cos (E-\psi) - r_1 + \sqrt{r_2^2 + a^2 \sin^2(E-\psi)}.
\]

(38)

The angle at which the ions cross the centreline of the gap is given by

\[
C = \sin^{-1} \left( \frac{a}{\Delta p} \sin(E-\psi) \right).
\]

(39)

The RF phase at which the ion reaches the main gap is given by

\[
\phi^{mg} = \phi^{ig} + 5(\psi + A_2 - A_1 - C).
\]

(40)

The co-ordinate of the centre point perpendicular to the dee gap is given by

\[
y_0 = -a \sin(E-\psi).
\]

(41)
It appears from Eqs. (35) and (38) that if $\beta \neq 0$, the radius and angle at which the ion crosses the main gap can be varied with phase. This would be useful because it provides a method of improving the "match" between the orbit starting conditions provided by the injection gap and those required for centred orbits. However, because of the posts, the injection gap acts as a lens (similar in properties to the lenses studied in Section 3). Han\textsuperscript{23} has studied the properties of a lens similar to the injection gap and found that it is convergent for positive phases and divergent for negative phases. The deflections due to the lens effects are larger than those produced by placing the injection gap at an angle to the beam. This effect has been confirmed by numerical orbit tracks through electric fields with the injection gap at various angles. Since radial centring considerations require $r^m_g$ for both positive and negative phases to be less than $r^m_g$ for zero phase, slanting the injection gap to the ion path does not improve the centring and will not be considered further.

Now with $\beta = 0$, Eqs. (35) to (41) are much simplified and will be stated again:

$$A_1 = A_2 = 0$$

$$a = r_1 - r_2 = \frac{r_0}{2} \frac{\Delta E'g}{E_0} (\cos \phi_1 - \cos \phi_2) \quad (42)$$

$$E = 0$$

$$\delta r = a \cos \psi - r_1 + \sqrt{r_2^2 + a^2 \sin^2 \psi} \quad (43)$$

$$\sin C = \frac{r_2 - r_1 \sin \psi}{r_2} = -\frac{a}{r_2} \sin \psi \quad (44)$$

$$y_c = (r_1 - r_2) \sin \psi = -r_2 \sin C \quad (45)$$

$$= \frac{r_0}{2} \frac{\Delta E'g}{E_0} \sin \psi (\cos \phi_1 - \cos \phi_2)$$

$$\psi^m_g = \phi^m_g + 5 \psi - 5C. \quad (46)$$

The subscript $0$ refers to quantities before the injection gap is reached.

In the central region of TRIUMF the radius of curvature is given by Eq. (34). The quantities $\Delta E'g$ and $E_0$ are 93 and 300 keV, respectively; hence the constant appearing in Eq. (45) is $\frac{r_0}{2} \frac{\Delta E'g}{E_0} = 1.61$ in.
The first order choice for the available parameters is $\psi = 36$ deg and $\phi_1 = 0$ deg, i.e. the ion with zero phase at the injection gap has $y_c = 0$ after the injection gap. If we use this geometry, then the centre points predicted by Eq. (45) are as shown by the solid curve in Fig. 4.2. The energy gain calculated as $93 \cos i g + 174.5 \cos \psi m g$, with $\psi m g$ given by Eq. (46), is shown by the solid curve in Fig. 4.3. As expected, the centre points for all ions which gain less energy than the ion with zero phase lie above the centreline of the dee (positive values of $y_c$). The asymmetry in the energy gain is due to the fact that negative phases are favoured by this arrangement, which delays all non-zero phases. Consider two ions which reach the injection gap with phases $i g = \pm 30$ deg; the energy gains will be identical here but the ions will arrive at the main gap at $\pm 30$ deg - $5C$, as predicted by Eq. (46). Since $\phi$ is about $-0.55$ deg for this case, the ion with phase $-30$ deg at the injection gap will reach the main gap at $-27.25$ deg and the ion with phase $+30$ deg at the injection gap will reach the main gap at $+32.25$ deg. The phase at the main gap as a function of phase at the injection gap for this case is shown by the solid line in Fig. 4.4.

The different values of $y_c$ for various phases are inherent in the use of the injection gap at an angle to the main gap. This centring is undesirable because it leads to a phase oscillation. An orbit with radius of curvature $r$ and off centre by an amount $y_c$ must turn through an angle of $\pi + 2y_c/r$ between dee gap crossings. This means that the ion will arrive at one gap early by $10y_c/r$ deg of RF phase and late at the next gap by the same amount. Fig. 4.5 shows the magnitude of this phase oscillation as a function of $y_c$ for various values of $r$. Fig. 4.2 indicates that we can expect values of $y_c$ of about 0.10 in. for an ion with phase of $+30$ deg. This leads to a phase oscillation amplitude of 4 deg at a radius of 14 in. (the radius of the first turn). The existence and order of magnitude of these phase oscillations are confirmed by the phase histories shown in Fig. 4.6. These phase histories are from a numerically integrated orbit. The $+30$ deg ion has an oscillation amplitude of about 4.5 deg at a radius of 14 in. (the radius of the first turn). The existence and order of magnitude of these phase oscillations are confirmed by the phase histories shown in Fig. 4.6. These phase histories are from a numerically integrated orbit. The $+30$ deg ion has an oscillation amplitude of about 4.5 deg, in good agreement with the expected value. The asymmetry between positive and negative phases in Fig. 4.6 is probably due to the zero phase ion not being exactly centred. The phase oscillation damps out as the energy (and hence $r$) increases. The magnitude of the centring errors (and hence the phase oscillations) can be reduced by centring the spread of $y_c$'s about the centreline of the dee.
instead of having all the \( y_\alpha \) of one sign, as was assumed for the solid curve in Fig. 4.2. This is achieved by moving the injection gap closer to the main gap without changing its orientation. To centre the spread of \( y_\alpha \)'s for a phase interval of \( \pm \Delta \phi \), the phase of the ion whose \( y_\alpha \) value is zero is

\[
\phi_1^{ig} = \cos^{-1}\left(\frac{1 + \cos \Delta \phi}{2}\right);
\]

hence

\[
\sin \frac{\phi_1^{ig}}{2} = \frac{1}{\sqrt{2}} \sin \frac{\Delta \phi}{2}.
\]

(47)

So, if we wanted to centre \( \frac{\sqrt{2}}{\sin \frac{\Delta \phi}{2}} \) for a phase range of \( \pm 45 \) deg, we would choose \( \phi_1^{ig} = 31.4 \) deg. The produces the \( y_\alpha \) and \( \Delta E \) values shown by the dashed lines in Figs. 4.2 and 4.3. In order to make \( y_\alpha \) zero for a phase of 31.4 deg, the injection gap must be shifted 0.12 in. (see Fig. 4.2) closer to the main gap. The maximum phase oscillation is now reduced to about \( \pm 5 \) deg rather than \( \pm 10 \) deg. The phase change between injection gap and main gap [given by Eq. (46)] is now less than 180 deg; hence we are shifting the ions towards negative phases for ions with \( |\phi^{ig}| < 31.4 \) deg where they tend to be defocused in the axial direction on subsequent turns. In fact it is useful to reduce the angle \( \psi \) (i.e. rotate the injection gap towards the main gap about the centre point for \( \phi_1 \)). This reduces the energy spread for positive phases; for example \( \psi = 32 \) deg produces the energy gain curve given by the dotted line in Fig. 4.3. The maximum is shifted towards positive phases because reducing \( \psi \) reduces \( \psi^{mg} \) [see Eq. (46)]; hence positive phases gain more energy. The reduced energy spread alleviates the problems of centring and clearing the centre post on the first turn. However, as can be seen from Eq. (46) and the dotted curve on Fig. 4.4, reducing \( \psi \) to 32 deg causes a large phase shift (about 23 deg) towards negative phases. The small shift towards negative phases required to centre the spread of \( y_\alpha \)'s is tolerable, since it produces a large improvement in the centring; however, reducing \( \psi \) to, say, 32 deg produces an unacceptably large shift towards negative phases. Hence \( \psi \) must be chosen so that the ion with phase \( \phi_1^{ig} \) [see Eq. (47)] has \( y_\alpha = 0 \) after passing through the injection gap. The radius of the injection gap is fixed because the injection energy is fixed; hence the injection gap position is determined.

4.3 Problems with Three-Sector Magnetic Fields

As demonstrated in Section 3, the lower limit on the phase acceptance
is set by axial focusing requirements. The acceptable range of phases can be increased if a phase-independent focusing force can be found to counteract the defocusing effects of the electric field. The only phase-independent source of axial focusing is the magnetic field; hence efforts were made to increase \( \langle v_s \rangle_m \) near the centre of the machine. Increasing \( \langle v_s \rangle_m \) requires that the "flutter" of the magnetic field be increased. Unfortunately, the central geometry of TRIUMF makes this very difficult because the magnet gap is large and there are six sectors, making the spacing between the magnet sectors small at small radius. One way of increasing the flutter is to transform the field from a six-sector geometry to a three-sector geometry in the central region. This is done by cutting off alternate magnet pole pieces at \( r = 40 \, \text{in.} \) and adding to the remaining pole pieces steel wedges (see Fig. 1.2) extending to the centre of the cyclotron. This produces a field which is dominated by the third harmonic rather than the sixth. This "three-sector geometry" produced a considerable improvement in \( \langle v_s \rangle_m \), as is shown by the dashed line in Fig. 3.1. With the three-sector geometry, \( \langle v_s \rangle_m \) is greater than 0.1 for \( r > 10 \, \text{in.} \). However, the large third harmonic caused undesirable effects in the radial orbit behaviour.

There are two effects caused by the three-sector geometry, an increase in phase oscillation amplitude and the gap-crossing resonance. Which of these effects is most important depends on the orientation of the electric field to the magnetic field. We define this orientation by the angle \( \delta \) shown in Fig. 4.7.

The phase oscillation effect results because, if \( \delta \neq 0 \), the orbit covers 2 valleys and 1 hill on one half-turn and 1 valley and 2 hills on the next half-turn. Hence, the lengths of the orbit on successive half-turns are different, as can be seen in Fig. 4.7. If the \( n^{\text{th}} \) harmonic dominates the variation in the magnetic field, the orbit equation may be written in the approximate form (e.g. Walkinshaw and King\(^{31}\))

\[
 r = r_0 \left[ 1 + \frac{1}{n^2-1} \frac{B_n}{B} \cos n\delta \right]
\]

where \( r_0 \) is the radius of the (circular) orbit if the field had no azimuthal variation, \( B_n \) is the amplitude of the \( n^{\text{th}} \) harmonic in the field and \( B \) is.
the average field. For the present case with \( n = 3 \),

\[
  r = r_0 (1 + a \cos 3\theta)
\]

where

\[
  a = \frac{1}{8} \frac{B_3}{B}.
\]

Now we wish to calculate the path length \( s \) between dee gaps. Using Eq. (49), we have

\[
  \frac{ds}{d\theta} = \left( \frac{dr}{d\theta} \right)^2 + r^2)^{1/2}
\]

\[
  = r_0 \left( 1 + a \cos 3\theta \right).
\]

The approximation which has been made is that \( B_3 \ll 8 B \). Hence, over one half-turn we have

\[
  s_1 = \int_{\theta=\delta}^{\theta=\delta+\pi} (1 + a \cos 3\theta) d\theta = \pi + \frac{1}{12} \frac{B_3}{B} \sin 3\delta
\]

(50)

and over the following half-turn

\[
  s_1 = \int_{\theta=\delta}^{\theta=\delta+\pi} (1 + a \cos 3\theta) d\theta = \pi - \frac{1}{12} \frac{B_3}{B} \sin 3\delta
\]

(51)

So, between successive gap crossings the phase oscillates by

\[
  \frac{5}{12} \frac{B_3}{B} \sin 3\delta
\]

(since the RF operates on the fifth harmonic of the ion frequency).

The variation of this phase change as a function of \( \delta \) is shown in Fig. 4.8 for a third harmonic amplitude \( B_3 \) which will produce \( v_z = 0.2 \).

Phase histories for a numerical orbit track corresponding to the worst case \( (\delta = 30 \text{ deg}) \) are shown in Fig. 4.9. The amplitude of the phase oscillation is about 8.5 deg for the zero phase ion (for which the phase oscillation would be zero without the three-sector magnetic field). This is in reasonable agreement with the theory. Since any phase oscillation such as this will decrease the duty factor, \( \delta \) must be small, i.e. the centreline of the dee gap should be close to the line running from a hill top at \( \delta = 0 \) to the opposite valley bottom at \( \delta = 180 \text{ deg} \). To keep the amplitude of the
phase oscillation less than 5 deg, we must have $\delta < 16$ deg. The effect of this phase oscillation is important here because the RF operates at the fifth harmonic of the ion frequency. It has been dismissed as unimportant for three-sector cyclotrons operating with $N = 1$.

This phase oscillation effect can be eliminated by placing the dee gap along a hill-valley centreline ($\delta = 0$ in Fig. 4.7). Unfortunately, this orientation maximizes another undesirable effect, the gap-crossing resonance. This is essentially a shift in the orbit centre points along the dee gap caused by a larger magnetic field at one dee gap than the other. This effect has been discussed in detail by Gordon, but we can make an estimate of the effects as follows. Referring again to Fig. 4.7, we can refer to the dee gap on the right side by the subscript 1 and on the left side by the subscript 2; then the magnetic fields at the gap are

$$B_1 = \bar{B} + B_3 \cos \delta,$$
$$B_2 = \bar{B} - B_3 \cos \delta.$$  \hspace{1cm} (52)

Since these effects are important at low energy, the radius of curvature can be approximated by

$$\rho(in.) = \frac{56.92}{B(kG)} \sqrt{E(\text{MeV})}.$$  \hspace{1cm} (53)

If the increase in energy at the dee gap is $\Delta E \text{ MeV}$, the change in radius of curvature at the gap, assuming the ions always cross normally, is

$$\Delta \rho(E) = \frac{56.92}{B} \left( \sqrt{E + \Delta E} - \sqrt{E} \right).$$  \hspace{1cm} (54)

The radial position of the ion does not change appreciably as the gap is crossed, so the change in radius of curvature is reflected in a change in the position of the centre of curvature. As the ion alternately crosses gaps 1 and 2, the centre of curvature oscillates back and forth approximately along the centreline of the dee. If the magnetic fields are different at the two gaps, there is a net drift of the centre of curvature towards the higher field, given by

$$\delta \rho = 56.92 \left\{ \sum_{i=1 \text{ odd}}^{k} \frac{1}{B_1} \left( \sqrt{E_{i+1}} - \sqrt{E_{i}} \right) - \sum_{i=2 \text{ even}}^{k+1} \frac{1}{B_2} \left( \sqrt{E_{i+1}} - \sqrt{E_{i}} \right) \right\}.$$  \hspace{1cm} (54)
where \( E_{i+1} = E_i + \Delta E_i \); \( i \) is the half-turn number and \( \Delta E_i \) is the energy gained at the \( i \)th dee gap.

Using Eq. (52) and the fact that \( B^3/\mathcal{B} \ll 1 \), this can be expressed as

\[
\delta \rho = \frac{56.92}{B} \sum_{i=1}^{k} \left[ \frac{\Delta E_i}{2\sqrt{E_i}} \left[ (-1)^{i+1} - \frac{B_3}{B} \cos 3\delta \right] \right].
\]  

(55)

The first term in the square bracket is the displacement of the orbit centre from the cyclotron centre. This term oscillates, hence its sum depends on the differences of the \( \Delta E_i \)'s. The second term in the square bracket is the centre point drift due to the third harmonic component in the magnetic field. This term always has the same sign, hence will accumulate rapidly if \( B_3 \) is large. \( B_3 \) varies widely with radius (see Fig. 4.10); hence the sum depends on the magnetic field used. Numerically summing the series for the values of \( B_3 \) shown in Fig. 4.10, and using \( \Delta E_i = 0.2 \text{ MeV} \) at all gaps, produced a centring error of 0.3 in. Numerical tracking of ions through the measured magnetic field using the computer code GOBLIN gave a centring error of about 0.5 in. for this field. Eq. (55) shows that the centre point drift due to \( B_3 \) is proportional to \( \cos 3\delta \) and hence could be eliminated to this approximation by choosing \( \delta = 30 \text{ deg} \). This means that the dee gap runs along a hill-valley interface, but this is unfortunately the situation which produces the large phase oscillations discussed above.

The drift in centre point could be reduced by putting a first harmonic in the magnetic field. The first harmonic causes the centre point to drift and could be arranged to cancel out the drift due to the gap-crossing resonance, as has been described by Gordon\(^{33}\) and van Kranenburg \textit{et al.}\(^{34}\). However, producing a first harmonic varying accurately enough with radius would be extremely difficult and necessitate special coils or shimming of the magnet. In addition, the compensation is exact for only one RF phase.

In summary, aligning the dee gap along the centreline of a hill (or valley) produces a phase oscillation of about 10 deg. Aligning the dee gap along a hill-valley interface excites the gap-crossing resonance causing a centring error of about 0.5 in., which can be only partially cancelled by a first harmonic in the magnetic field. Orientations between the two described above do not bring the phase oscillation and the centre point
drift within acceptable limits, and hence the three-sector magnetic field has not been adopted.

4.4 Radial Centring

The central region of a cyclotron must produce a beam which is centred at extraction. By centred we mean that the oscillations of the orbit centre point approach the geometric centre of the machine as the energy increases. In TRIUMF the large energy gain per turn and low magnetic field produce large oscillations of the centre point at low energy. The centre point at injection must be off centre by about half the radius gain per half-turn (see, e.g., Gordon33) if the orbit centre point is to approach the centre of the machine as the energy becomes large. This centre point displacement required because of the acceleration can be derived in a manner similar to the derivation of Eq. (55). If we assume circular orbits, then the change in centre of curvature at one gap is

\[ x_{c_i} - x_{c_{i+1}} = \rho_i - \rho_{i+1} \]

and at the next gap

\[ x_{c_{i+2}} - x_{c_{i+1}} = \rho_{i+1} - \rho_{i+2}. \]

Hence over one turn the change in centre point is

\[ x_{c_{i+2}} - x_{c_i} = -\rho_{i+2} + 2\rho_{i+1} - \rho_i. \]

If the energy gain per gap crossing (\(\Delta E\)) is \(\ll\) the energy \(E\) and the change in radius per gap crossing (\(\Delta r\)) is \(\ll\) the radius \(r\), we can approximate Eq. (56) by a differential equation

\[ \frac{dx_c}{dE} = \frac{\Delta E}{2} \frac{d^2\rho}{dE^2}. \]

Now if the centre point \(x_c\) at infinite energy is zero, i.e. the beam is centred, integrating Eq. (57) once yields

\[ x_c(E) = \frac{\Delta E}{2} \frac{d\rho}{dE} = \frac{r_0}{2} \frac{\Delta E}{m_0 c^2 \beta \gamma^3}. \]
where $\rho$ and $\delta$ are the usual relativistic factors and $r_\infty$ is the cyclotron radius $= m_0 c^2 / q B_0$ ($\approx 410$ in. for TRIUMF). The right-hand side of Eq. (58) is just one-half the radius gain per half-turn at energy $E$. The above estimate provides a good starting point for finding central orbits, especially at high energy. However, at low energy where the geometry is complicated by the presence of the injection gap, we must resort to numerical orbit tracks to optimize the centring.

The determination of what constitutes a centred orbit is complicated by several factors. The azimuthal variation of the magnetic field causes scalloping of the orbit; hence the instantaneous centre point depends on the azimuthal angle. The average orbit radius and maximum scalloping are shown in Fig. 4.11 as a function of energy.

The quantities we will mainly be concerned with in this section are $r$, the radius of the ion from the geometric centre of the cyclotron, and $p_\rho$, the radial momentum. The momentum will be written in the form

$$p = \beta \gamma r_\infty.$$ 

In these units, the momentum of the ion is represented by its radius of curvature in the central magnetic field ($B_0$). The radial momentum is that component of the momentum which is directed in the radial direction, i.e.

$$p_\rho = \frac{dr}{ds} = \frac{\dot{r}}{\gamma v} = p \sin \chi$$

where $\tan \chi = dr / d\theta$. In the central region, the flutter in the magnetic field is small, hence the orbit scalloping is small, and we can roughly approximate $B$ by $B_0$ and $p_\theta$ by $p$; then the component of the centre point perpendicular to the dee gap ($y_\theta$) equals $p_\rho$ at $\theta = \delta$, i.e. the dee gap.

The essential features of the central orbits of TRIUMF are shown in Fig. 4.12. The magnetic field has six-fold symmetry. The centreline of the dee gap is 5.5 deg from the centreline of a valley. The azimuthal angle $\theta$ is measured from the centreline of the dee gap as shown.

One way to remove the complicating effects of orbit scalloping, and to determine how close the orbit is to an ideal centred orbit, is to calculate, at some azimuthal angle, the difference between the radius and radial
momentum of the actual accelerated orbit (a.o.) and an equilibrium orbit (e.o.) at the same energy. An e.o. is a fixed energy orbit which closes upon itself, has average centre of curvature at the centre of the machine, and is stable for small displacements in radius and momentum. The e.o.'s are calculated by the program CYCLOPS.* Now for any energy at one azimuth, we know the radius and radial momentum \((r_{eo} \text{ and } p_{r eo})\) of the equilibrium orbit. Hence, when tracking an a.o., we can calculate the differences in radius and momentum between the e.o. and the a.o. at this azimuth. If an orbit is to be centred at the final energy, the differences between the e.o. and the a.o. during acceleration (i.e. \(\Delta r = r_{ao} - r_{eo} \) and \(\Delta p_r = p_{r ao} - p_{r eo}\)) are due to centre point displacements along the dee gap only (changes in \(x_C\)) due to acceleration. Hence, as the energy increases and the changes in \(x_C\) decrease, the values of \(\Delta r\) and \(\Delta p_r\) (due only to a non-zero value of \(x_C\)) will decrease, and the a.o. will approach the e.o. The locus of the point \((\Delta r, \Delta p_r)\) in phase space on successive turns (at one azimuth angle) as the acceleration proceeds forms an "accelerated phase plot". The gross features of the accelerated phase plot depend on the amount the instantaneous centre point of the a.o. differs from the instantaneous centre point of the e.o. \((\Delta x_C \text{ and } \Delta y_C \text{ in the } x- \text{ and } y-\text{directions, respectively})\) and on the azimuthal angle at which the accelerated phase plot is calculated.

Suppose at some angle \(\theta_0\) the a.o. has energy \(E\), radius \(r_{ao}\) and radial momentum \(p_{r ao}\). We interpolate in a table of equilibrium orbit radii and radial momenta values for azimuth \(\theta_0\) to obtain \(r_{eo}\) and \(p_{r eo}\), which are the radius and radial momentum of the equilibrium orbit at energy \(E\). Now if we neglect the variation in the magnetic field along \(\theta_0\) between \(r_{eo}\) and \(r_{ao}\), then the radii of curvature are the same, i.e. \(r_{eo} = r_{ao} = r\), and we have the situation shown in Fig. 4.13. The angle \(x\) will be small since \(p_{r}\) is much less than \(p\) and \(\Delta p_r\) will also be much less than \(p\), hence we can approximate the arc \(\rho(\Delta p/p)\) by a straight line and the centre point components are related to the differences in radius and radial momenta by

\[
\begin{pmatrix}
\Delta y_C \\
\Delta x_C
\end{pmatrix} = \begin{pmatrix}
\cos(\theta_0 - x) & \sin(\theta_0 - x) \\
-\sin(\theta_0 - x) & \cos(\theta_0 - x)
\end{pmatrix} \begin{pmatrix}
\rho \Delta p_r \\
\Delta r
\end{pmatrix}.
\]

\(\text{(60)}\)

* CYCLOPS was kindly made available to TRIUMF by Dr. M. Gordon of Michigan State University.
Thus the accelerated phase plot removes the "motions" in the orbit centre point due to scalloping of the orbit and allows the actual errors in centring to be determined. Fig. 4.14 shows $\Delta x_c$ calculated using Eq. (60) [using $\Delta \rho$ and $\Delta \rho_r$ values from a numerical track of a centred orbit] compared to the values of $\Delta x_c$ predicted by Eq. (58). $|x_c|$ is plotted rather than $x_c$ to allow comparison of the curves for $\theta_o = 54.5$ deg and $\theta_o = 234.5$ deg. The values of $x_c$ are all negative for $\theta_o = 54.5$ deg and all positive for $\theta_o = 234.5$ deg. The agreement is fairly good; however, the only way to do the final optimization of the centring seems to be to work backwards from the centred orbit. That is, we start an ion on a centred orbit at high energy and numerically track it backwards into the centre of the machine. If we do this for several RF phases, we will know what the starting conditions should be if ions with various phases are to be centred. Using a typical magnetic field (01-03-06-70), ions with various starting phases were tracked backwards into the centre of the machine.

The procedure which is used for tracking orbits is as follows. For energies less than 5 MeV the program PINWHEEL is used. This solves the relativistic equations of motion using measured magnetic fields and electric fields calculated by the method described in Section 2. For energies greater than 5 MeV the program GOBLIN is used. This program solves the relativistic equations of motion using measured magnetic fields but approximating the effects of the electric fields by the "impulse" approximation described in Section 5. The transition is made at 5 MeV because above this energy there is no significant radial variation in the field across the dee gap, while for energies below 5 MeV there is such a variation because of the tapering of the electrodes.

The ions were started at 20 MeV at the centreline of a valley ($\theta = -5.5$ deg), with $\Delta \rho_r = 0$ and with $\Delta r$ equal to one-half the turn separation per half-turn, as indicated by Eq. (58). The accelerated phase plots at $\theta_o = 54.5$ deg and $\theta_o = -126.5$ deg (see Fig. 4.12), i.e. at the centreline of a valley, are shown in Figs. 4.15, 4.16 and 4.17 for ions with starting phases of -30 deg, 0 and +30 deg, respectively. For the ideal case where $y_c$ is always zero and $x_c$ becomes zero at high energy, then Eq. (60) shows that $\Delta \rho_r$ and $\Delta r$ values will always lie on the straight line passing through $\Delta r = 0$ and $\Delta \rho_r = 0$ and at an angle of $\pi - \theta_o$ to the $\Delta r = 0$ axis. This is
the straight line shown in Figs. 4.15, 4.16 and 4.17. Using Eq. (60) and
the data shown in Fig. 4.16, the values of \( x_c \) shown in Fig. 4.14 were calcu­
lated. Extrapolation of this curve down to 0.4 MeV (the energy of the beam
between the injection gap and the first main gap) indicates that the beam
should be off centre by about 1.32 ± 0.05 in. at this energy. Since the
radius of curvature of the beam is 11.88 in., this means that the radius at
which the first main gap should be crossed is 13.20 ± 0.05 in. Accelerated
phase plots for three different choices of radius at the first main gap
crossing are shown in Fig. 4.18. The arrow on Fig. 4.18 gives twice the
radial oscillation amplitude. The curve for \( r = 13.20 \) in. clearly leads to
the smallest amplitude radial oscillation. To determine what happens to
other phases an ion was tracked backwards from \( r = 13.20 \) in. at the first
main gap through the injection gap, into the centre post, providing initial
conditions for outward tracks. Using these initial conditions, trajectories
were followed outwards for various phases, producing the phase plot shown
in Fig. 4.19. The -25 deg ion gives a radial oscillation amplitude of about
0.5 in., while the +25 deg ion gives an amplitude of about 0.8 in. These
oscillations are much too large, as they would lead to a very large energy
spread at extraction. In order to achieve more than a very narrow phase
band, the starting conditions must be adjusted to favour ions which start
with phases other than zero. Since the difference in the accelerating con­
ditions which causes the large oscillation amplitudes to develop is essen­
tially the energy gain, which varies roughly as \( \cos \phi \), it is reasonable to
centre an ion whose phase corresponds to the average cosine in the phase
band to be accelerated.

Since in the absence of third harmonic in the RF we are restricted
essentially to positive phases, we will choose starting conditions so that
various phases are centred and observe how this affects the magnitude of the
radial oscillations. Fig. 4.20 shows accelerated phase plots for ions with
the same phase range as in Fig. 4.19 but with starting conditions chosen to
give centred orbits for starting a phase of +17 deg. Phase plots such as
shown in Fig. 4.20 were calculated using starting conditions to give centred
orbits for initial phases of +15 deg, +17 deg, +19 deg and +21 deg. The
results are summarized in Fig. 4.21. For a phase range of -5 deg to +25 deg
the amplitudes of the radial oscillations are minimized if an ion with
initial phase of 15 deg to 17 deg is centred. If a small amplitude of
oscillation were desired (and a small phase width could be tolerated), one would choose the case where the 0 deg ion was centred.

To first order the energy resolution obtainable in the extracted beam is related to the radial oscillation amplitude by the energy gain per turn. For the case of TRIUMF, the maximum energy gain per turn (400 keV) produces a 0.064 in. increase in radius at 500 MeV. When operating with a wide phase spread, the beam will be spread out fairly uniformly with radius, so that a ±0.064 in. oscillation will worsen the energy resolution by ±400 keV (or alternatively ±0.1 in. will produce ±600 keV).

Eq. (58) shows that the \( x_Q \) required to allow for centre point motions due to acceleration is proportional to \( \Delta E \) and hence is also proportional to \( \cos \phi \) since \( \Delta E = qV_0 \cos \phi \). Therefore, if the zero phase ion is centred at high energy and has centre point \( x_Q^{E_1} \) at injection and \( x_Q^{E_2} \) at some other energy, an ion with some other phase (\( \phi \)) will have a centring error \( (1 - \cos \phi)x_Q^{E_1} \) at injection. If this centring error did not alter the behaviour of the centre point motions with energy, we would expect this initial centring error to produce a centring error \( (1 - \cos \phi)(x_Q^{E_1} - x_Q^{E_2}) \) at energy \( E_2 \). The dashed line in Fig. 4.21 shows this centring error as a function of phase at injection. Fig. 4.21 shows that the oscillation amplitude is much larger than this, so some mechanism is causing this centring error to produce a large amplitude radial oscillation. One such mechanism is described in Section 4.5.

Of course, the ion beam will contain particles with various displacements and divergences from the central ray, and we must investigate how the beam as a whole is centred. This is discussed in Section 4.5.

In Section 3.8 it was shown that a field bump could be used to shift the acceptable range of phases so that the accepted phase interval is centred about zero degrees. Fig. 4.22 shows the phase histories for four ions in a field which has the bump described in Fig. 3.19 added to it. As expected, the initial phase interval of 0 deg to +50 deg is shifted to about -21 to +25 deg. The dashed line shows the theoretically expected phase shift for the ideal bump. Note that the phase change is never faster than ideal, so no ions which are initially focused are shifted to defocusing phases. Accelerated phase plots for ions with various starting phases in the magnetic field with the bump added are shown in Fig. 4.23. The starting
conditions are adjusted to favour the $+17$ deg ion (as in Fig. 4.20 without the field bump). The oscillation amplitudes for phases of $0$ deg, $+15$ deg and $+30$ deg are $0.13$, $0.12$ and $0.42$ in., respectively, while without the bump they are $0.20$, $0.14$ and $0.46$ in. Thus the effect of the bump is to slightly decrease the oscillation amplitudes in this case. We would expect (from Fig. 4.21) that large positive phases would have very large oscillation amplitudes, and this is confirmed in Fig. 4.23, which shows that the ion that starts at $+45$ deg is unacceptable. This undesirable behaviour for large positive phases is not significantly improved if we arrange the starting conditions to favour the $+21$ deg ion. Note that the radial centring requirement effectively sets a positive phase limit of about $+25$ deg, so that the field bump used (designed for a phase interval of $-5$ deg to $+45$ deg, see Section 3.8) is too large. However, the effects of the bump on the radial motion are small.

4.5 Effects of Finite Beam Emittance

Now we will consider how the centring varies over a beam with a realistic size. The expected emittance of the TRIUMF ion source is $0.50 \, \text{in. mrad}$ (at $300$ keV). We will assume that this is not significantly increased by the transport system up to the point of injection into the cyclotron dees. To minimize the amplitude of the radial oscillations we want to choose the initial ellipse shape to match the radial focusing, as described in Section 3 for axial focusing. Since the lens effects of the dee gaps are small, we first try matching to the magnetic focusing, for which $\nu_r = 1.0$ in the central region. To see how the emittance is transformed as the beam is accelerated, four particles were tracked, starting on the edge of the emittance ellipse. Figs. 4.24(a), 4.25(a) and 4.26(a) show accelerated phase plots for these four points for initial phases of $0$ deg, $+15$ deg and $+25$ deg, respectively. As can be seen from these figures, the ellipse is "stretched" as the acceleration proceeds, producing a large amplitude radial oscillation. This is due to an effect explained by Mackenzie. Briefly, the effect is important in this case because of the low field and large energy gain per turn causing the initial orbits to be far from the equilibrium orbits. Consider the trajectories in phase space of $+\phi$ and $-\phi$. An initial displacement from the origin ($\Delta r \neq 0$ or $\Delta \theta \neq 0$) will cause precession through an angle of approximately (if $\nu_2$ is close
to $\pi (v_p - 1)$ during a half-turn in the magnetic field. Since $\Delta p \neq 0$ or $\Delta p_n \neq 0$ means that the beam is not centred, the ions arrive at the next dee gap later or earlier than they left the previous gap (as described in Section 4.2). Hence the energy gain is not the same for the $+\phi$ ion as for the $-\phi$ one. This means that on the next half-turn one ion will be closer to its e.o. than the other to its e.o., and while they both precess through the same angle, the $-\phi$ ion will precess so as to reduce its displacement from the origin in phase space, while the displacement of the $+\phi$ ion increases if $v_p > 1$. The effect is reversed if $v_p < 1$. These displacements in phase space cause "stretching" of the emittance ellipse, producing a large amplitude radial oscillation. This effect is important when the ion energy is small and when $v_p$ is different from one, so that the precession is large. Numerical orbit tracks have shown that the effect is unimportant above 10 MeV.

The amplitude of these oscillations can be reduced by choosing a different initial ellipse shape. If, for example, we choose an ellipse which is reduced by a factor of two in the $\Delta p$ direction but increased by a factor of two in the $\Delta p_n$ direction from the ellipse that is matched to $v_p = 1$, we obtain the phase plots shown in Figs. 4.24(b), 4.25(b) and 4.26(b) for the same three initial phases as used previously. These phase plots show that the oscillation amplitude is reduced to 0.25 in. over the phase range 0 deg to +25 deg. This represents an effective increase by a factor of almost four in the oscillation amplitude due to the phase-dependent acceleration.
5. RADIAL LENS EFFECTS OF CYCLOTRON DEE GAPS

5.1 Introduction

The calculation of radial motions in a cyclotron at low energies requires a detailed knowledge of the electric field produced by the dees. The calculation or measurement of this field is a difficult problem (see Section 2), and the numerical integration of the equations of motion through the field is a slow procedure. To integrate the equations of motion from injection to extraction would require a prohibitively large amount of computer time. It is therefore useful to have an approximate method of calculating the radial motions. One way of doing this is to represent the radial motion as half-turns in a purely magnetic field separated by accelerating impulses induced by the electric fields at the dee gap. The magnetic field is approximated by an isochronous field with \( v_r \) (the radial oscillation frequency) constant over each half-turn and determined by interpolation in the values computed by the equilibrium orbit code for the real field. The effects of the dee gaps are approximated by instantaneous changes in the energy \( \delta E \), RF phase \( \delta t \), radial position \( \delta x \) and angle to the gap \( \delta \xi \) when the ion reaches the azimuthal angle of the centre line of the dee gap. Thus in a two-dee cyclotron such as TRIUMF the ion will pass through a 180 deg long magnetic field region (with \( v_r \) constant), then have its energy, radial position, RF phase and angle to the dee gap instantaneously changed as it crosses the gap, then pass through another magnetic field region and dee gap, etc. This section investigates various approximations which give the quantities describing the dee gap \( \delta E, \delta t, \delta x \) and \( \delta \xi \). The results from the approximations are compared to numerical orbit tracks through a real electric field.

Since we have \( v_r \) constant over each half-turn, we can approximate the radial motion in the magnetic field by a sinusoidal oscillation about the equilibrium orbit. There will also be a significant oscillation at the principal flutter frequency, but this will produce no change over 180 deg in a six-sector machine.

We will call the amplitude of this oscillation \( \Delta r \) and the slope \( \frac{\Delta p_r}{p} \) where \( p \) and \( p_r \) are the total and radial momenta, respectively. The transformation of these quantities are given by an equation equivalent to
where \( r \) is the radius of curvature and \( \theta \) is the azimuthal angle in the magnetic field.

Now we need a description of the changes in momentum and position produced by the dee gaps. Fig. 5.1 shows a typical dee gap. The radial motion of the ion is confined close to the median plane \( z = 0 \). Fig. 5.2 shows a plot of the instantaneous electric potential in the median plane. This figure suggests that a first approximation to the effects of the dee gap can be obtained by assuming that the gradient of the electric field is constant over some region and zero elsewhere.

### 5.2 Constant Gradient Approximation with No Magnetic Field

We assume the gap is as shown in Fig. 5.3, uniform in the \( x \)-direction, with a gap width of \( \ell \) and a total voltage across the gap of \( V_0 \). At time \( t = 0 \), the ion is at \( x = x_0, y = \ell/2 \) with velocity \( \dot{x} = \dot{x}_0, \dot{y} = \dot{y}_0 \), \((\dot{x} \equiv dx/dt, \dot{y} \equiv dy/dt)\). The phase of the accelerating voltage at \( t = 0 \) is \( \phi_0 \) and its frequency is \( \omega \). The equations of motion are

\[
\ddot{x} = 0
\]

\[
\ddot{y} = \frac{qV_0}{m\ell} \cos(\omega t + \phi_0)
\]

where \( k = \frac{qV_0}{m\ell} \), \( q \) and \( m \) being the charge and mass of the ion, respectively. Integrating Eq. (62) gives

\[
\dot{x} = \dot{x}_0
\]

\[
x = x_0 + \dot{x}_0 t.
\]

Integrating Eq. (63) gives

\[
\dot{y} = \dot{y}_0 + \frac{k}{\omega} \left[ \sin(\omega t + \phi_0) - \sin(\phi_0) \right],
\]
In practical cases, the electrodes which produce the field are located above and below the median plane, and so the width of the field there is larger than the physical gap between the electrodes (as demonstrated in Fig. 5.1). We therefore treat \( \lambda \) as a free parameter to obtain the best agreement with numerically integrated orbits. The time required for the ion to cross the gap \( \tau \) (the transit time) is also as yet unknown. Within the validity of the approximation, the width of the electric field will depend only on the geometry, while the transit time will depend on the electric field and the velocity of the ion; hence we choose \( \lambda \) so that the constant gradient approximation gives the same energy gain as the numerical results for one case, i.e. one incident energy and RF phase. Now, using this value of \( \lambda \), we calculate the transit time \( \tau \) which is the value of \( \tau \) that solves Eq. (67) when \( y - y_0 = -\lambda \), i.e.

\[
\lambda + \dot{y}_0 \tau + \frac{k}{\omega} \left[ \frac{1}{\omega} \left( \cos(\phi_0) - \cos(\omega t + \phi_0) \right) - \tau \sin(\phi_0) \right] = 0. \tag{68}
\]

This can be solved by any standard numerical technique, for example the Newton-Raphson method.\(^{36}\) Once \( \tau \) is known, the changes in \( x, y, \dot{x} \) and \( \dot{y} \) across the gap can be calculated. We will call the above approximation the iterative approximation, since it requires an iterative solution of Eq. (68) to find the transit time.

Now, within the validity of the approximation, the value of \( \lambda \) found to be best in one case should also give the best results for other incident energies and phases. To select an appropriate value of \( \lambda \), the best method seems to be to compare the results of numerical integrations to the results predicted by the constant gradient approximation at high energy, where we expect the approximation to be most valid.

Since the iterative solution of Eq. (68) may be time consuming, one is tempted to look for simpler approximations. If the transit time is small enough so that we can approximate \( \sin(\omega t) \) by \( \omega t \) and \( \cos(\omega t) \) by 1, then we obtain the "linear" approximation

\[
\dot{y} = \dot{y}_0 + k t \cos \phi_0 \tag{69}
\]
\[
y = y_0 + \dot{y}_0 t \tag{70}
\]
and the transit time is
\[ \tau = \frac{k}{y_0}. \] (71)

In this case the assumption of small transit time is equivalent to assuming that the velocity of the ion is constant across the gap.

A more exact approximation is obtained if we keep terms up to \((\omega t)^2\) in the expansions of \(\sin \omega t\) and \(\cos \omega t\); then we obtain the "quadratic" approximation which is
\[ \dot{y} = \dot{y}_0 + k \left( t \cos \phi_0 - \frac{\omega^2 t^2}{2} \sin \phi_0 \right) \] (72)
and
\[ y = y_0 + \dot{y}_0 t + \frac{k t^2}{2} \cos \phi_o. \] (73)

The transit time is obtained from Eq. (73) when \(y = \frac{y_0}{2}\) and is
\[ \tau = \frac{-\dot{y}_0 - \sqrt{\dot{y}_0^2 + 2k \frac{t}{\cos \phi_0}}} {k \cos \phi_0}. \] (74)

This approximation is equivalent to assuming that the ion velocity across the gap is the average of the initial and final velocities.

A still better approximation can be obtained by retaining one more term in the expansion of \(\sin \omega t\); then we obtain the "cubic" approximation
\[ \dot{y} = \dot{y}_0 + k \left( t \cos \phi_0 - \frac{\omega^2 t^2}{2} \sin \phi_0 - \frac{\omega^4 t^3}{6} \cos \phi_0 \right). \] (75)

The last term in Eq. (75), which was neglected in Eq. (72), is usually as large as the second last term in Eq. (75). In the cubic approximation we still calculate the transit time using Eq. (74).

The validity of these approximations was tested by comparing the changes in \(x, \dot{x}, y\) and \(\dot{y}\) to those given by numerical integration through a real electric field. The numerical calculations solve the exact relativistic equations of motion. The various constant gradient approximations assume that the mass is constant across the gap; however, the mass used is the relativistic mass appropriate to the initial ion energy. The electric field used was that for the gap shown in Fig. 5.1, i.e. a total gap height of
4.0 in. and a total gap width of 6.0 in. In the TRIUMF cyclotron the field produced by a gap of these dimensions is reached at a radius of 40 in. (about 5 MeV). The value of \( \lambda \) was selected so that the iterative approximation gave the same energy gain as the numerical integration for \( \phi_0 = 0 \) deg and \( E_0 = 100 \) MeV. The value selected was 8.97 in., considerably larger than the physical gap width of 6.0 in. The gradient used for the approximation is shown by the dashed line in Fig. 5.2.

In this case, there is no force in the \( x \)-direction; hence \( P_x \) remains constant and displacements in the \( x \)-direction are just \( \dot{x}t \). The change in \( P_y \) causes the energy of the ion to increase. We will express the energy gain by the so-called gap factor \( G \),

\[
G = \frac{\Delta E}{qV_o \cos \phi_c} \cdot 100\% \tag{76}
\]

where \( \Delta E \) is the actual energy gained by the ion and \( \phi_c \) is the RF phase at which the ion crosses the centre of the gap.

Fig. 5.4 compares the energy gain obtained by numerical orbit tracking in the real field for \( \phi_c = 0 \) deg with the energy gains predicted by the various constant gradient approximations. The small transit time approximations give very much less accurate results than the approximation based on exact computation of the transit time, and hence they will not be considered further.

Values of \( G \) for various phases and energies from the iterative constant gradient approximation and from numerical integration are given in Table V. Fig. 5.5 shows the differences between the values calculated by numerical integration and those from the iterative constant gradient approximation. In all cases, over a phase range of \(-45\) deg to \(+45\) deg and an energy range of \(1\) to \(100\) MeV, the differences are less than \(0.5\)%.

The errors displayed in Fig. 5.5 are inherent in the constant gradient approximation and not a result of an inappropriate choice of \( \lambda \), since changing \( \lambda \) merely displaces the family of error curves.

The other quantity of interest is the transit time. In all cases, the transit time was within \(0.1\)% of the expected time of \( \lambda/v_a \) where \( v_a \) is the average of the initial and final velocities.
### Gap factors given by numerical integration and by the constant gradient approximation (no magnetic field)

<table>
<thead>
<tr>
<th>Energy (MeV)</th>
<th>IA</th>
<th>N</th>
<th>IA</th>
<th>N</th>
<th>IA</th>
<th>N</th>
<th>IA</th>
<th>N</th>
<th>IA</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>78.599</td>
<td>78.319</td>
<td>78.921</td>
<td>78.840</td>
<td>79.242</td>
<td>79.302</td>
<td>79.512</td>
<td>79.683</td>
<td>79.697</td>
<td>79.963</td>
</tr>
<tr>
<td>2</td>
<td>88.718</td>
<td>88.726</td>
<td>88.775</td>
<td>88.888</td>
<td>88.847</td>
<td>89.034</td>
<td>88.905</td>
<td>89.157</td>
<td>88.935</td>
<td>89.249</td>
</tr>
<tr>
<td>5</td>
<td>95.351</td>
<td>95.349</td>
<td>95.318</td>
<td>95.381</td>
<td>95.304</td>
<td>95.412</td>
<td>95.291</td>
<td>95.440</td>
<td>95.272</td>
<td>95.462</td>
</tr>
<tr>
<td>20</td>
<td>98.833</td>
<td>98.790</td>
<td>98.802</td>
<td>98.791</td>
<td>98.780</td>
<td>98.794</td>
<td>98.762</td>
<td>98.796</td>
<td>98.743</td>
<td>98.800</td>
</tr>
</tbody>
</table>

N = numerical results from iterative approximation
IA = results from iterative approximation
Thus in this case the effect of the gap can be approximated by instantaneous changes at the centre line of the gap as follows: \( \delta E \) is the energy change appropriate to the velocity change given by Eq. (66), \( \delta t \) and \( \delta x \) are zero and \( \delta \xi \) is the change in angle due to the energy gain. This change in angle results because \( p_y \) changes while \( p_x \) remains constant and is

\[
\delta \xi = \tan^{-1} \left( \frac{p_x}{p_y + \Delta p} \right) - \tan^{-1} \left( \frac{p_x}{p_y} \right)
\]

(77)

where \( \Delta p \) is the increase in momentum given by Eq. (66), and \( p_x \) and \( p_y \) are the initial momentum components.

5.3 Sine Gradient Approximation

To improve the results given by the constant gradient approximation, we should use an electric field which more closely approximates the real field. Fig. 5.2 suggests that a better approximation than a linearly varying potential (constant gradient) might be a potential of the form

\[
\frac{V}{V_0} = \cos (\omega t + \phi_0) \cos \left( \frac{\pi}{L} (y - \frac{L}{2}) \right)
\]

(78)

where \( L \) is a wavelength which describes the electric field. If we define the average velocity \( v_a = \frac{\ell}{\tau} \) (gap width \( \ell \neq L \), see below), the equation of motion is, in so far as the actual velocity can be approximated by the average velocity,

\[
\ddot{y} = k \pi \cos(\omega t + \phi_0) \sin \frac{\pi v_a t}{L}.
\]

(79)

Integrating Eq. (79) [assuming \( v_a \) is constant] gives

\[
\dot{y} - \dot{y}_0 = \frac{k \pi}{2} \left[ \cos \phi_0 - \cos \left( \frac{t(\omega + \pi v_a/L) + \phi_0}{\omega + \pi v_a/L} \right) \right]
\]

(80)

and

\[
y - y_0 = \dot{y}_0 t + \frac{k \pi}{2} \left[ \sin \phi_0 - \sin \left( \frac{t(\omega + \pi v_a/L) + \phi_0}{\omega + \pi v_a/L} \right) \right]
\]

(81)

\[
+ \sin \left( \frac{t(\omega - \pi v_a/L) + \phi_0}{\omega - \pi v_a/L} \right) - \sin \phi_0 \frac{t \cos \phi_0}{\omega - \pi v_a/L}.
\]
It was found that the results were improved if we allowed the gap width ($\lambda$) to be less than the wavelength which describes the electric field ($L$). Thus we use only part of one cycle of the sine function to describe the electric field gradient. Of course, if $\lambda < L$, $V_0$ must be increased to

$$\frac{V_0}{\cos \left( \frac{\pi \lambda}{L} \left( \frac{L}{2} - \frac{L}{2} \right) \right)}$$

to maintain the same total voltage across the gap.

Since $v_\lambda$ and the time required to cross the gap are both unknown, we must solve Eq. (81) for the transit time $\tau$ when $y - y_0 = -\lambda$. We define $v_\lambda \equiv \lambda/\tau$ so the transit time is given by

$$\lambda + y_0 \tau + \frac{k\pi}{2} \left( \frac{\sin \phi_0 - \sin (\omega \tau + \pi \lambda/L + \phi_0)}{(\omega + \pi \lambda/L \tau)^2} + \frac{\tau \cos \phi_0}{\omega + \pi \lambda/L \tau} \right) + \frac{\sin (\omega \tau - \pi \lambda/L + \phi_0) - \sin \phi_0}{(\omega - \pi \lambda/L \tau)^2} - \frac{\tau \cos \phi_0}{\omega - \pi \lambda/L \tau} = 0. \tag{82}$$

This approximation was compared to the results of numerical integration through the real field as described above for the constant gradient approximation. Over a phase interval of -45 deg to +45 deg and an energy interval of 1 to 100 MeV, the errors for the sine gradient approximation were several times larger than for the constant gradient approximation; hence the sine gradient approach was not pursued further.

5.4 Constant Gradient Approximation with Third Harmonic in the Electric Field

Since it is planned to "flat-top" the RF voltage by adding a small fraction of third harmonic to the fundamental, it is useful to derive the constant gradient approximation for the case where the RF voltage is given by

$$\frac{V}{V_0} = \cos(\omega t + \phi_0) - \epsilon \cos(3\omega t + \phi_0 + \delta). \tag{83}$$

$\epsilon$ is the fraction of third harmonic and $\delta$ is the phase of the third harmonic with respect to the fundamental. The equation of motion in the $y$-direction
is now
\[ \ddot{y} = k \left( \cos(\omega t + \phi_o) - \epsilon \cos(3\omega t + \phi_o + \delta) \right) \] (84)
which when integrated gives
\[ \dot{y} - y_o = \frac{k}{\omega} \left[ \sin(\omega t + \phi_o) - \sin\phi_o - \frac{\epsilon}{\delta} \left( \sin(3\omega t + \phi_o + \delta) - \sin(\phi_o + \delta) \right) \right] \]
and
\[ y - y_o = \dot{y_o} t + \frac{k}{\omega} \left[ \frac{1}{\omega} \left( \cos(\omega t + \phi_o) - \cos\phi_o \right) - t \sin\phi_o \right. \]
\[ \left. + \frac{\epsilon}{\delta} \left( \frac{1}{\delta \omega} \cos(3\omega t + \phi_o + \delta) - \cos(\phi_o + \delta) \right) - t \sin(\phi_o + \delta) \right] \] (85)

The validity of Eqs. (85) and (86) have not been checked by comparison with numerical integration. However, it is reasonable to expect them to be at least as accurate as (66) and (67) when the RF voltage is not changing more rapidly than it does for the fundamental only. Fortunately, the phase region of interest is precisely where the waveform given by (83) is flat, so that Eqs. (85) and (86) should be of as much utility as (66) and (67).

5.5 Constant Gradient Approximation with Magnetic Field

We now consider the case where the ion being accelerated sees a magnetic field \( B \) perpendicular to its plane of motion and an electric field with constant gradient. The equations of motion are
\[ \ddot{x} = \frac{q}{m} B \dot{y} \] (87)
and
\[ \ddot{y} = k \cos(\omega t + \phi_o) - \frac{q}{m} B \dot{x} \] (88)
where as before \( k = q V_o / m \).

If we assume that the magnetic field is isochronous, then the ion rotation frequency is constant and equal to \( qB/m \). Integrating Eq. (87) once and using \( \omega = NqB/m \) (the RF frequency), we obtain
\[ \dot{x} - \dot{x}_o = \frac{\omega}{N}(y - y_o) \] (89)
and

$$\ddot{y} = k \cos(\omega t + \phi) - \frac{\omega}{N} \left( \dot{x}_0 + \frac{\omega}{N} (y - y_0) \right).$$  \hspace{1cm} (90)$$

The solutions to Eqs. (89) and (90) are, for \( N \neq 1 \),

$$x = x_0 - \frac{N}{\omega} \left( \dot{y}_0 + \frac{k}{\omega} \frac{N^2}{1-N^2} \sin\phi_0 \right) \left( \cos\frac{\omega t}{N} - 1 \right)$$

$$+ \frac{N}{\omega} \left( \dot{x}_0 - \frac{k}{\omega} \frac{N}{1-N^2} \cos\phi_0 \right) \sin\frac{\omega t}{N}$$

$$+ \frac{k}{\omega^2} \frac{N}{1-N^2} \left( \sin(\omega t + \phi) - \sin\phi_0 \right),$$

$$\dot{x} = \left( \dot{y}_0 + \frac{k}{\omega} \frac{N^2}{1-N^2} \sin\phi_0 \right) \sin\frac{\omega t}{N}$$

$$+ \left( \dot{x}_0 - \frac{k}{\omega} \frac{N}{1-N^2} \cos\phi_0 \right) \cos\frac{\omega t}{N} + \frac{k}{\omega^2} \frac{N^2}{1-N^2} \cos(\omega t + \phi),$$

$$y = y_0 - \frac{N^2}{\omega} \ddot{x}_0 + \frac{N}{\omega} \left( \dot{y}_0 + \frac{k}{\omega} \frac{N^2}{1-N^2} \sin\phi_0 \right) \sin\frac{\omega t}{N}$$

$$+ \frac{N}{\omega} \left( \dot{x}_0 - \frac{k}{\omega} \frac{N}{1-N^2} \cos\phi_0 \right) \cos\frac{\omega t}{N} + \frac{k}{\omega^2} \frac{N^2}{1-N^2} \cos(\omega t + \phi),$$

and

$$\dot{y} = \left( \dot{y}_0 + \frac{k}{\omega} \frac{N^2}{1-N^2} \sin\phi_0 \right) \cos\frac{\omega t}{N}$$

$$- \left( \dot{x}_0 - \frac{k}{\omega} \frac{N}{1-N^2} \cos\phi_0 \right) \sin\frac{\omega t}{N} - \frac{k}{\omega^2} \frac{N^2}{1-N^2} \sin(\omega t + \phi_0).$$

The transit time \( \tau \) is given by

$$\tau = \frac{N}{\omega} \left( -\dot{x}_0 + \left( \dot{y}_0 + \frac{k}{\omega} \frac{N^2}{1-N^2} \sin\phi_0 \right) \sin\frac{\omega t}{N} \right)$$

$$+ \left( \dot{x}_0 - \frac{k}{\omega} \frac{N}{1-N^2} \cos\phi_0 \right) \cos\frac{\omega t}{N} + \frac{k}{\omega^2} \frac{N^2}{1-N^2} \cos(\omega t + \phi).$$
Note that Eq. (95) takes into account the increase in path length due to the magnetic field.

Cohen, Comiti and Reiser have derived equations similar to Eqs. (93) and (94), but they give no method for calculating the transit time nor any indication of the accuracy of their approximations.

For the case without a magnetic field, displacements in the $x$-direction depend on $p_x$ only, since the electric field produces no component of force in the $x$-direction. Similarly, when a magnetic field is present the change in $p_x$ should be that due to the magnetic field only, as if the electric field were not present. That this is true is verified by the work of Comiti and by the results of numerical integration in the present case.

The energy gains predicted by Eq. (94) and those obtained from numerical integration are given in Table VI. The value of $l$ (the gap width) used was the one chosen for the zero magnetic field case, i.e. 8.97 in. The differences in the gap factor $G$ are shown in Fig. 5.6. The errors in this case are about ten times larger than for the case where no magnetic field was present. This is possibly due to the fact that the curvature of the ion path causes the ion to spend more time near the edges of the field where the constant gradient approximation is least accurate. However, over the region of interest ($-30 \text{ deg} < \phi < 30 \text{ deg}$ and $E > 5 \text{ MeV}$), the errors are still less than 1%. Fig. 5.7 shows the phase variation of the error in $G$. A more accurate description of the energy gain could be obtained by fitting some function to the curves shown in Fig. 5.7 and using this as a correction to the energy gain predicted by Eq. (94).

The errors in timing caused by assuming that the change in energy occurs discontinuously at the centre of the gap are about 1.5 deg (RF) at -45 deg and 1 MeV, 0.2 deg at 0 deg and 1 MeV, decreasing rapidly with energy (<0.01 deg at 100 MeV).

In the present case (with a magnetic field), there is an apparent displacement of the ion due to the change of radius of curvature. The numerical integration of the ion orbit is done over a distance $d$ in the $y$-direction. If the radius of curvature before the gap is $\rho_1$, and after the gap is $\rho_2$, the displacement we would expect from $y = \frac{d}{2}$ to $y = -\frac{d}{2}$, if the change in radius of curvature occurs discontinuously at the centre of
### TABLE VI

Gap factors given by numerical integration and by the constant gradient approximation (isochronous magnetic field)

<table>
<thead>
<tr>
<th>Energy (MeV)</th>
<th>-45</th>
<th>-30</th>
<th>-15</th>
<th>0</th>
<th>+15</th>
<th>+30</th>
<th>+45</th>
</tr>
</thead>
<tbody>
<tr>
<td>N</td>
<td>IA</td>
<td>N</td>
<td>IA</td>
<td>N</td>
<td>IA</td>
<td>N</td>
<td>IA</td>
</tr>
<tr>
<td>1</td>
<td>75.487</td>
<td>78.126</td>
<td>77.009</td>
<td>78.654</td>
<td>78.199</td>
<td>79.124</td>
<td>79.207</td>
</tr>
<tr>
<td>2</td>
<td>86.375</td>
<td>88.681</td>
<td>87.395</td>
<td>88.844</td>
<td>88.166</td>
<td>88.991</td>
<td>88.824</td>
</tr>
<tr>
<td>5</td>
<td>93.836</td>
<td>95.345</td>
<td>94.459</td>
<td>95.378</td>
<td>94.919</td>
<td>95.409</td>
<td>95.318</td>
</tr>
<tr>
<td>10</td>
<td>96.590</td>
<td>97.634</td>
<td>97.028</td>
<td>97.643</td>
<td>97.352</td>
<td>97.652</td>
<td>97.631</td>
</tr>
<tr>
<td>20</td>
<td>98.080</td>
<td>98.791</td>
<td>98.390</td>
<td>98.793</td>
<td>98.618</td>
<td>98.795</td>
<td>98.816</td>
</tr>
</tbody>
</table>

N = numerically integrated results

IA = results from iterative approximation
the gap with no displacement along the gap, is

\[ \Delta x = \rho_2 \cos \left( \sin^{-1} \left( \frac{d}{2\rho_2} \right) \right) - (\rho_2 - \rho_1) - \rho_1 \cos \left( \sin^{-1} \left( \frac{d}{2\rho_1} \right) \right) \]

\[ = \frac{d^2}{8} \left( \frac{1}{\rho_1} - \frac{1}{\rho_2} \right). \]  

(96)

In Fig. 5.8 the values of \( \Delta x \) found from Eq. (96) are compared with the numerically-tracked orbits. The agreement is excellent. This means that \( \delta x \), the displacement of the orbit at the dee gap, is effectively zero, and the energy gain results in a displacement of the centre of curvature.

Thus, in this case as in the case where no magnetic field was present, we approximate the effect of the gap by instantaneous changes at the centre-line of the gap as follows: \( \delta E \) is the energy change corresponding to the velocity change given by Eq. (94), \( \delta t \) and \( \delta x \) are given zero values, and \( \delta \xi \) is the "collimation" given by Eq. (77).

5.6 Constant Gradient Approximation with Magnetic Field and Third Harmonic in the Electric Field

Using the RF voltage given by Eq. (83) and including the effects of an isochronous magnetic field, the equations of motion are

\[ \dot{x} - \dot{x}_0 = \frac{\omega}{N} (y - y_0) \]  

(97)

and

\[ \ddot{y} + \frac{\omega^2}{N^2} y + \frac{\omega}{N} \left( \dot{x}_0 - \frac{\omega}{N} y_0 \right) = k \left( \cos(\omega t + \phi_0) - \varepsilon \cos(3\omega t + \phi_0 + \delta) \right). \]  

(98)

For \( N \neq 1 \), the solutions to Eqs. (97) and (98) are

\[ \dot{x} = \left[ \dot{x}_0 + \frac{k}{\omega} N^2 \left( \sin \phi_0 \frac{1}{1-N^2} - \frac{3\varepsilon}{1-9N^2} \sin(\phi_0 + \delta) \right) \right] \sin \left( \frac{\omega t}{N} \right) \]  

\[ + \left[ \dot{x}_0 - \frac{k}{\omega} N \left( \cos \phi_0 \frac{1}{1-N^2} - \frac{\varepsilon}{1-9N^2} \cos(\phi_0 + \delta) \right) \right] \cos \left( \frac{\omega t}{N} \right) \]  

\[ + \frac{k}{\omega} N \cos(\omega t + \phi_0) - \frac{\varepsilon k}{\omega} \frac{N}{1-9N^2} \cos(3\omega t + \phi_0 + \delta), \]  

(99)
These equations have not been compared to the results of numerical integrations, but the differences should be comparable to those quoted in the previous section over the region of interest.
6. SUMMARY AND CONCLUSIONS

The motion of the ions at the centre of a cyclotron has been studied with particular reference to the TRIUMF cyclotron. The object was to investigate the factors determining the phase acceptance and beam quality of the cyclotron, and to consider how the design might be adjusted to optimize these quantities.

The calculation of both axial and radial motion requires knowledge of the electric and magnetic fields. The magnetic fields used were measured on model magnets. The electric fields were calculated by numerically solving Laplace's equation using the relaxation method. The convergence and accuracy of this method was investigated in detail. Numerically solving a problem for which the solution could be found analytically showed that the numerical solution contained average errors less than 0.01%. The method uses a novel feature to obtain accurate starting values for the iteration, and the solution time for a very large problem \((2 \times 10^6\) data points) is about 3 hours on an IBM 360/67.

The axial motions were studied using the thick lens description of the dee gaps developed by Rose, Cohen and Reiser. A method was developed for calculating the axial acceptance of the accelerator as a function of RF phase. It was found that the axial acceptance exhibits a sharp cut-off at about -5 deg, i.e. ions with phases more negative than -5 deg cannot be accelerated. This effect results because, for negative phases, the field is rising, and the field variation effect causes the dee gaps to defocus the ions. This effect is more important for TRIUMF than for other cyclotrons because the RF operates at the fifth harmonic of the ion frequency, causing the electric forces to be much stronger. The negative phase limit can be shifted to more negative values by flat-topping the RF waveform. This flattopping can be produced by adding some third harmonic of the RF frequency to the fundamental waveform. It is shown that addition of 12% of third harmonic in phase with the fundamental shifts the cut-off due to the field variation effect to about -15 deg. This situation can be further improved by adding 15% of third harmonic shifted 10 deg from the fundamental. For this case the negative phase limit is shifted to about -25 deg.

The effect of field bumps is investigated. For the case of TRIUMF a radially decreasing field bump at the cyclotron centre cannot produce enough
axial focusing to overcome the strong electric forces. However, a carefully
designed field bump can be used to shift the phases of the ions. It is shown
how to design a field bump to shift those phases initially favoured by elec­
tric focusing (positive phases) into phase with the peak of the RF voltage
when electric focusing is less important. This is done without shifting the
ions to phases where they are defocused by the electric field.

The radial motions of the ions in the first turn were studied to find
the best position of the injection gap. A position close to 36 deg back
along the orbit from the main gap was found to provide the best centring and
phase histories.

To allow economical orbit tracking out to high energies, an analytic
description of the changes in radial orbit properties on crossing a dee gap
was developed. The results of this approximation differ from the exact changes
(found by numerical integration) by less than 1% for energies above 5 MeV.

The beam centring was studied by tracking ions through realistic electric
and magnetic fields (to 5 MeV), then to 20 MeV by integrating through the
magnetic field and using the approximation mentioned above. The results of
these orbit tracks showed that the transformation of the radial beam ellipse
is quite phase dependent. This may be reduced by reducing phase-dependent
effects at the dee gaps. The energy resolution of the beam is investigated
by tracking ions to 20 MeV. The radial oscillations present at 20 MeV are
reduced by a factor of about 1.5 during acceleration to 500 MeV due to adia­
batic compression. The finite emittance of the beam also worsens the energy
resolution by ±300 keV.

If the ion with initial RF phase of 0 deg is centred, large radial
oscillations develop for ions with other initial RF phases. For example, if
we require an energy resolution of ±600 keV, then half of this can be
allowed to the coherent radial oscillations, meaning that the oscillation
amplitude allowed is 0.05 in. at 500 MeV or 0.07 in. at 20 MeV. This allows
a phase acceptance of 16 deg. For an energy resolution of ±1200 keV the
phase acceptance is 26 deg. For the case where large duty cycle is required,
the largest phase acceptance is obtained if an ion in the centre of the phase
interval is centred (rather than the ion with 0 deg initial phase). For
±1200 keV energy resolution, for example, the phase acceptance can be
increased to -17 deg to +26 deg by centring the ion with initial phase of 17 deg.

In summary, the axial motions place a positive limit >60 deg on the phase acceptance. The negative limit is -5 deg without third harmonic, -15 deg with 12% of third harmonic in phase with the fundamental, and -25 deg with 15% of third harmonic shifted by 10 deg from the fundamental. The radial motions allow a phase acceptance of -8 deg to +6 deg for an energy resolution of ±600 keV or -17 deg to +26 deg for an energy resolution of ±1200 keV, in both cases with RF fundamental only.
ACKNOWLEDGEMENTS

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APPENDIX A
THEORY OF SUCCESSIVE OVER-RELAXATION

We wish to solve a system of equations described by

\[ \Phi_{ijk} = \frac{1}{\Theta} \left( \Phi_{i-1,jk} + \Phi_{i+1,jk} + \Phi_{i,j-1,k} + \Phi_{i,j+1,k} + \Phi_{i,jk-1} + \Phi_{i,jk+1} \right) \]

[interior nodes]

\[ = b_{ijk} \]

[boundary nodes]

\[ i = 0, 1, 2 \ldots \ p \]
\[ j = 0, 1, 2 \ldots \ q \]
\[ k = 0, 1, 2 \ldots \ r \]

The system contains \( N \) equations where \( N = (p + 1)(q + 1)(r + 1) \).

We will start with some initial approximation for each unknown value of \( \Phi \) denoted \( \Phi^{0}_{ijk} \). We will sequentially modify each of these values in the order

\[ \Phi_{111} \Phi_{211} \ldots \Phi_{p11} \Phi_{121} \Phi_{221} \ldots \Phi_{pq1} \Phi_{1p2} \Phi_{2p2} \ldots \]
\[ \ldots \Phi_{pq2} \ldots \ldots \ldots \Phi_{pqr} \]

or in the opposite order.

The successive over-relaxation method can be described by the iterative sequence

\[ \phi^{n+1}_{ijk} = \phi^{n}_{ijk} + \alpha \left[ \phi^{n+1}_{i-1,jk} + \phi^{n}_{i+1,jk} + \phi^{n+1}_{i,j-1,k} + \phi^{n}_{i,j+1,k} + \phi^{n+1}_{i,jk-1} + \phi^{n}_{i,jk+1} \right] \]

[interior nodes]

\[ - \Theta \phi^{n}_{ijk} \]

[boundary nodes]

\[ = b_{ijk}. \]

\( \phi^{n}_{ijk} \) is the \( n^{th} \) estimate of the value of the potential at the node \( i,j,k \) and \( \alpha \) is a constant.
Now we define the error at the node $ijk$ at the $n$th iteration as

$$\varepsilon_{ijk}^n = \phi_{ijk}^n - \phi_{ijk}$$  \hspace{1cm} (A.2)

where $\phi_{ijk}$ is the correct value at this node; then

$$\varepsilon_{ijk}^{n+1} + \phi_{ijk} = \varepsilon_{ijk}^n + \phi_{ijk} + \frac{\alpha}{6} \left[ \varepsilon_{i-1,jk}^{n+1} + \phi_{i-1,jk} + \varepsilon_{i+1,jk}^{n+1} + \phi_{i+1,jk} \\ + \varepsilon_{i,j-1k}^{n+1} + \phi_{i,j-1k} + \varepsilon_{i,j+1k}^{n+1} + \phi_{i,j+1k} \\ + \varepsilon_{i,jk+1}^{n+1} + \phi_{i,jk+1} - 6\varepsilon_{ijk}^{n} - 6\phi_{ijk} \right].$$

According to Eq. (8), the terms in $\phi$ in the square bracket cancel, leaving

$$\varepsilon_{ijk}^{n+1} = \varepsilon_{ijk}^n + \frac{\alpha}{6} \left[ \varepsilon_{i-1,jk}^{n+1} + \varepsilon_{i+1,jk}^{n+1} + \varepsilon_{i,j-1k}^{n+1} + \varepsilon_{i,j+1k}^{n+1} + \varepsilon_{i,jk+1}^{n+1} \\
- 6\varepsilon_{ijk}^{n} \right]$$  \hspace{1cm} [interior points]

or

$$\varepsilon_{ijk}^{n+1} - \frac{\alpha}{6} \left[ \varepsilon_{i-1,jk}^{n+1} + \varepsilon_{i+1,jk}^{n+1} + \varepsilon_{i,j-1k}^{n+1} + \varepsilon_{i,j+1k}^{n+1} \right] = \varepsilon_{ijk}^n + \frac{\alpha}{6} \left[ \varepsilon_{i+1,jk}^{n} + \varepsilon_{i,j+1k}^{n} + \varepsilon_{i,jk+1}^{n} \right].$$

Clearly this method of iteration leads to a linear dependence of the $\varepsilon_{ijk}^{n+1}$ on the $\varepsilon_{ijk}^n$, so we may write

$$\varepsilon_{ijk}^{n+1} = K\varepsilon_{ijk}^n$$  \hspace{1cm} (A.4)

and

$$\varepsilon_{ijk}^{n+1} = K\varepsilon_{ijk}^n = K^2\varepsilon_{ijk}^{n-1} = \ldots = K^n\varepsilon_{ijk}^0$$

where $\varepsilon_{ijk}^{n+1}$ and $\varepsilon_{ijk}^n$ are $N$-vectors whose elements are the $N$ errors after $n+1$ and $n$ iterations, respectively. $K$ is an $N \times N$ matrix which depends on $\alpha$, $p$, $q$ and $r$, but not on $n$.

For an individual error $\varepsilon_{ijk}$, Eq. (A.4) can be written

$$\varepsilon_{ijk}^{n+1} = \sum_{i'j'k'} K_{i'j'k'} \varepsilon_{i'j'k'}^n$$  \hspace{1cm} (A.5)
Now we denote the \( N \) eigenvalues of \( K \) by \( \lambda_k \) (\( k = 1, 2, \ldots, N \)) and the corresponding eigenvectors by \( \beta_k \)

\[
K \beta_k = \lambda_k \beta_k .
\]  
(A.6)

Since the eigenvectors form an orthogonal set, we can express the error vectors as a sum over the eigenvectors

\[
e^n = \sum_k \sigma_k^n \beta_k .
\]  
(A.7)

from Eq. (A.4)

\[
e^{n+1} = \sum_k \sigma_k^n K \beta_k = \sum_k \sigma_k^n \lambda_k \beta_k
\]  
(A.8)

hence

\[
\sigma_k^{n+1} = \lambda_k \sigma_k^n = \ldots = \lambda_k^{n+1} \sigma_k^0 .
\]

Now to evaluate the eigenvalues, we substitute Eqs. (A.7) and (A.8) into (A.3), giving

\[
\begin{aligned}
\left[ \sum_l \sigma_k^{n+1} \beta_l \right]_{ijk} &= \left[ \sum_l \sigma_k^n \beta_l \right]_{ijk} + \frac{\alpha}{\delta} \left[ \sum_l \sigma_k^{n+1} \beta_l \right]_{i-1jk} + \left[ \sum_l \sigma_k^n \beta_l \right]_{i+1jk} \\
&+ \left[ \sum_l \sigma_k^n \beta_l \right]_{ij-1k} + \left[ \sum_l \sigma_k^n \beta_l \right]_{ij+1k} + \left[ \sum_l \sigma_k^{n+1} \beta_l \right]_{ijk-1} \\
&+ \left[ \sum_l \sigma_k^n \beta_l \right]_{ijk+1} - \delta \left[ \sum_l \sigma_k^n \beta_l \right]_{ijk} .
\end{aligned}
\]

Using the second part of Eq. (A.8), this becomes

\[
\sum_l \sigma_k^n \left[ \lambda_k - 1 + \alpha \right] \beta_l_{ijk} - \frac{\alpha}{\delta} \lambda_k \left[ \beta_l \right]_{i-1jk} + \left[ \beta_l \right]_{i+1jk} + \lambda_k \left[ \beta_l \right]_{i-1k} \\
+ \left[ \beta_l \right]_{i+j+1k} + \lambda_k \left[ \beta_l \right]_{i+j-1k} + \left[ \beta_l \right]_{i+jk+1} = 0 .
\]
But this must be true for any error $\varepsilon^n$, i.e. for any set of $\varepsilon^n$; hence

$$\left(\lambda_l - 1 + a\right) \beta_{l,jk} = \frac{a}{\theta}\left(\lambda_l \beta_{l,jk-1} + \beta_{l,jk+1} + \lambda_l \beta_{l,jk-1} + \beta_{l,jk+1}\right)$$

$$l = 1, 2 \ldots N$$

and

$$\beta_{l,0,jk} = \beta_{l,i0,jk} = \beta_{l,ij0} = \beta_{l,ip,jk} = \beta_{l,ijq} = \beta_{l,ijn} = 0.$$  \hspace{1cm} (A.10)

To evaluate the eigenvectors and eigenvalues we will follow the procedure first given by Frankel.\textsuperscript{39} A more general treatment has been given by Young.\textsuperscript{40} The elements $\beta_{l,jk}$ of the eigenfunctions are evidently\textsuperscript{41}

$$\beta_{l,jk} = A^i \sin \frac{\pi sj}{p} B^j \sin \frac{\pi tj}{q} C^k \sin \frac{\pi uk}{r}$$  \hspace{1cm} (A.11)

where $s = 1, 2, \ldots p - 1$

$\quad t = 1, 2, \ldots q - 1$

$\quad u = 1, 2, \ldots r - 1$.

Substituting Eq. (A.11) into Eq. (A.9) gives

$$(\lambda - 1 + a)A^i \sin \frac{\pi sj}{p} B^j \sin \frac{\pi tj}{q} C^k \sin \frac{\pi uk}{r} =$$

$$\frac{a}{\theta}\left(\lambda a^{i-1} \sin \frac{\pi (i-1)}{p} + A^i + 1 \sin \frac{\pi (i+1)}{p}\right)$$

$$+ A^i \sin \frac{\pi sj}{p} C^k \sin \frac{\pi uk}{r} \left(\lambda B^j - 1 \sin \frac{\pi (j-1)}{q} + B^j + 1 \sin \frac{\pi (j+1)}{q}\right)$$

$$+ A^i \sin \frac{\pi sj}{p} B^j \sin \frac{\pi tj}{q} \left(\lambda C^k - 1 \sin \frac{\pi (k-1)}{r} + C^k + 1 \sin \frac{\pi (k+1)}{r}\right).$$

For Eq. (A.12) to be satisfied for all values of $i$, $j$ and $k$, we must have

$$\lambda = A^2, \lambda = B^2, \lambda = C^2.$$
Since multiplication of $A$ (or $B$ or $C$) by $-1$ is equivalent to replacing $s$ by $p - s$ (or $t$ by $q - t$ or $u$ by $r - u$), we may take $A = +B = +C$.

Then Eq. (A.12) becomes

$$(A^2 - 1 + a)A^i \sin \frac{\pi s_i}{p} A^j \sin \frac{\pi t_j}{q} A^k \sin \frac{\pi u_k}{r} =$$

$$\frac{\alpha}{3} \left[ A^i \sin \frac{\pi t_i}{q} A^j \sin \frac{\pi u_k}{r} 2A^{i+1} \sin \frac{\pi s_i}{p} \cos \frac{\pi s}{p} + A^i \sin \frac{\pi u_k}{r} 2A^{i+1} \sin \frac{\pi t_j}{q} \cos \frac{\pi t}{q} + A^i \sin \frac{\pi t_j}{q} 2A^{i+1} \sin \frac{\pi u_k}{r} \cos \frac{\pi u}{r} \right]$$

so

$$(A^2 - 1 + a) = \frac{\alpha}{3} A \left( \cos \frac{\pi s}{p} + \cos \frac{\pi t}{q} + \cos \frac{\pi u}{r} \right)$$

$$A^2 - Aav + (a-1) = 0 \quad (A.13)$$

$$A = \frac{1}{2} \left[ av \pm \sqrt{a^2v^2 - 4(a-1)} \right] \quad (A.14)$$

where $v = \frac{1}{3} \left( \cos \frac{\pi s}{p} + \cos \frac{\pi t}{q} + \cos \frac{\pi u}{r} \right) = \cos \theta$.

Now in order to investigate the convergence rate we note that we have expressed the error vectors as linear combinations of the eigenvectors, and Eq. (A.8) can be written

$$\epsilon^n = \sum_l \alpha_l^n \beta_l = \sum_l \lambda_l^n \sigma_l^0 \beta_l. \quad (A.15)$$

We will call the $\lambda_{l_m}$ with the largest absolute value $l_m$. We can then write Eq. (A.15) as

$$\frac{\epsilon^n}{\lambda_m} = c_m^0 \beta_m + \sum_{l \neq m} \left( \frac{\lambda_l}{\lambda_m} \right)^n \sigma_l^0 \beta_l.$$
but since \( \lambda_m > \lambda, \left( \frac{\lambda}{\lambda_m} \right)^n \) goes to zero as \( n \) becomes large and we obtain, for large \( n \),

\[
\varepsilon^n = c_m^o \beta \lambda_m^n. \quad (A.16)
\]

So, to achieve the maximum convergence rate, we want \( \lambda_m (= A^2, \text{etc.}) \) to be as small as possible. (\( \lambda_m \) must be less than one if the process is to converge.) Returning to our equation for \( A \), then if we consider

\[
a^2 \nu^2 < 4(a - 1), \quad (A.17)
\]

the roots of Eq. (A.14) are complex conjugates with magnitude \(|A|^2 = a - 1\); however, if \( a^2 \nu^2 > 4(a - 1) \), the roots will be real and unequal. Since the product of the roots is \((a - 1)\), one of the roots must have a magnitude greater than \( \sqrt{a - 1} \). Hence, the minimum \( \lambda_m = \max \left( |A_1|^2, |A_2|^2 \right) \) occurs for \( a \) in the range \( a^2 \nu^2 < 4(a - 1) \). Since the magnitude of \( \lambda_m \) in this range is \((a - 1)\), the value of \( a \) (called \( a_b \)) giving the smallest \( \lambda_m \) is the smaller of the two roots of

\[
a_b^2 \nu^2 = 4(a_b - 1),
\]

i.e.

\[
a_b = \frac{2}{1 + \sin \theta} = \frac{2 - 2\sqrt{1-\nu^2}}{\nu^2} = 1 + \left( \frac{\nu}{1 + \sqrt{1-\nu^2}} \right)^2 \quad (A.18)
\]

and

\[
\lambda_m = A^2 = \left( \frac{a_b \nu}{2} \right)^2 = \frac{1 - \sin \theta}{1 + \sin \theta} = a_b - 1 = \frac{2}{\nu^2}(1 - \sqrt{1-\nu^2}) - 1.
\]

Since we are calculating the minimum of the maximum values of \( A \), we must take the worst case, i.e. the largest value of \( v \), which is

\[
v = \frac{1}{3} \left( \cos \frac{\pi}{p} + \cos \frac{\pi}{q} + \cos \frac{\pi}{r} \right). \quad (A.19)
\]

In practical problems, \( p, q \) and \( r \) are \( >> 1 \), and we can obtain approximate expressions for \( a_b \) and \( \lambda_m \).
\[ v = 1 - \frac{\pi^2}{\theta} \left( \frac{1}{p^2} + \frac{1}{q^2} + \frac{1}{r^2} \right), \quad \sin \theta = \pi \sqrt{\frac{1}{p^2} + \frac{1}{q^2} + \frac{1}{r^2}}, \quad (A.20) \]

\[ \alpha_b = 2 - 2\pi \sqrt{\frac{1}{3} \left( \frac{1}{p^2} + \frac{1}{q^2} + \frac{1}{r^2} \right)}, \quad (A.21) \]

\[ \lambda_m = 1 - 2\pi \sqrt{\frac{1}{3} \left( \frac{1}{p^2} + \frac{1}{q^2} + \frac{1}{r^2} \right)} = \alpha_b - 1. \quad (A.22) \]

Now that we have found the value of \( \alpha \) which gives fastest convergence, the question of interest is how fast does it converge. Referring to Eq. (A.8), each iteration reduces each error by at least a factor \( \lambda_m \); hence \( n \) iterations reduce the error by at least a factor \( (\lambda_m)^n \). Hence, to reduce the errors by a factor \( f \), the number of iterations required is

\[ n = \frac{\log f}{\log \lambda_m}. \quad (A.23) \]
Central region of the TRIUMF cyclotron - median plane

Fig. 1.1 Central region of the TRIUMF cyclotron - median plane
Fig. 1.2 Central region of the TRIUMF cyclotron - section through centreline of hill #3
Fig. 2.1 Relaxation Mesh Organization. Total number of nodes is \((p+1)(q+1)(r+1)\).
Error bars indicate error at which number of points has fallen to half the peak value.

\( \alpha = 1.5 \)

Fig. 2.2 Average error and average change per iteration vs number of sweeps over large volume.
Fig. 2.3 Average error vs number of sweeps over large volume for various values of $\alpha$. 

- $\alpha = 1.50$
- $\alpha = 1.87$
- $\alpha = 1.75$
Fig. 2.4 Number of nodes with a given error vs size of error for various number of sweeps over large volume
Fig. 2.5 Number of nodes with a given error vs number of sweeps over large volume for various size errors
Fig. 3.1 Magnetic axial focusing frequency ($\nu_z$) vs energy for three- and six-sector magnetic geometries

- Field 1-30-06-70 (six sector)
- Field 1-14-05-70 (three sector)
Fig. 3.2 Equivalent axial focusing frequency produced by space charge forces vs energy for various beam currents and axial beam heights
Fig. 3.3 Cross-section of dees near accelerating region showing electric equipotentials and (schematically) an ion trajectory.
Fig. 3.4 Comparison between equivalent electric axial focusing frequencies predicted by the thin lens approximation and determined by numerical integration.
Possible TRIUMF central geometry with three accelerating gaps in the first half-turn.
Fig. 3.6 Axial emittance ellipses required at injection for various RF phases

\[ \sqrt{2} = 0.2 \]
emittance \( \sim 2.0 \text{ in. mrad} \)
injection \( 0.5 \pi \text{ in. mrad} \) (worsened by \( k \))
Fig. 3.7 Axial acceptance vs RF phase for various injection energies (one accelerating gap in the first half turn)
Fig. 3.8 Axial acceptance vs RF phase for various injection energies
(three accelerating gaps in the first half-turn)
Fig. 3.9 Average axial acceptance (averaged from -30 deg to +60 deg) vs injection energy (one accelerating gap in the first half-turn)
Fig. 3.10 Axial acceptance vs RF phase for various choices of the initial emittance ellipse
RF voltage waveforms with various amounts of third harmonic and phase shift between fundamental and third harmonic.

\[ \phi + \delta \cos 3 - (\phi) \cos \frac{\theta A}{A} = \delta \]
Fig. 3.12  Slope of RF voltage waveform with various amounts of third harmonic and phase shift between fundamental and third harmonic
Fig. 3.13 Axial acceptance vs RF phase for various choices of the initial emittance ellipse, $\varepsilon = 0.17, \delta = 0$
Fig. 3.14 Axial acceptance vs RF phase for various choices of the initial emittance ellipse, $\epsilon = 0.12, \phi = 0$,

- overlap with $\phi = 0$ deg ellipse
- $\phi = +20$ deg
- $\phi = +40$ deg

Axial emittance ellipse overlap (%) vs RF phase (°) degrees
Fig. 3.15 Axial acceptance vs RF phase for various choices of the initial emittance ellipse, $\varepsilon = 0.15$, $\delta = -10$ deg
Fig. 3.16  Total (Magnetic and electric) equivalent axial focusing frequency vs energy for various RF phases
Fig. 3.17 Transition phase (of total axial focusing from negative to positive) vs energy
Fig. 3.18  Change in sine of RF phase required to keep ion at transition phase vs radius
Fig. 3.19 Magnetic field bump required to keep ion at transition phase vs radius
Fig. 4.1 Geometry of injection gap and first main gap for two RF phases
Fig. 4.2 $\gamma_c$ vs RF phase at injection gap for various injection gap positions

Fig. 4.3 Energy gain in injection gap and first main gap vs RF phase at injection gap for various injection gap positions
Fig. 4.4 RF phase at first main gap vs RF phase at injection gap for various injection gap positions
Fig. 4.5 Phase oscillation amplitude vs centring error at various radii

- $r = 14$ in.
- $r = 20$ in.
- $r = 30$ in.
- $r = 50$ in.
Fig. 4.6 RF phase vs half-turn number for various initial phases with no flutter in the magnetic field.
Fig. 4.7 Geometry of an orbit in a three-sector magnetic field
Fig. 4.8 RF phase difference on succeeding half-turns as a function of orientation of the dee gap ($\delta$)
Fig. 4.9 RF phase vs half-turn number for various initial phases with a three-sector magnetic field ($\delta = 30$ deg)
Fig. 4.10 Ratio of third harmonic amplitude in magnetic field to average field vs radius for (three-sector) field 1-14-5-70
Fig. 4.11 Average orbit radius and maximum orbit scalloping vs radius for (six-sector) field 1-30-06-70.
Fig. 4.12 Geometry of an orbit in a six-sector magnetic field
Fig. 4.13 Geometry of the difference between an equilibrium orbit and an accelerated orbit

\[ C_{eo} = \text{instantaneous centre point of e.o.} \]

\[ C_{ao} = \text{"" "" "" a.o.} \]
Fig. 4.14 Centre-point displacement along the dee gap vs energy showing values from numerical orbit tracks and from an analytic approximation.
Fig. 4.15 Accelerated phase plot inwards from 5 MeV, $\phi = -30^\circ$
Fig. 4.16 Accelerated phase plot inwards from 5 MeV, $\phi = 0$ deg

$\phi = 0$

$\theta_0 = 54.5^\circ$

$\theta_0 = 234.5^\circ$

0.79 MeV

0.61 MeV

5.34 MeV

5.53 MeV

$\Delta r$ (in.) -0.4

-0.6

-0.8

(\text{in.})$

$\Delta \gamma$

0.8

0.6

0.4
Fig. 4.17 Accelerated phase plot inwards from 5 MeV, $\phi = +30^\circ$ deg
Fig. 4.18 Accelerated phase plot outwards for various radii at first main dee gap, \( \phi = 0 \) deg

starting energy = 0.48 MeV
final energy = 20 MeV
Fig. 4.19 Accelerated phase plot outwards from inflector exit for various phases; ion with $\phi = 0$ is centred

starting energy = 0.3 MeV
final energy = 20 MeV
Fig. 4.20 Accelerated phase plot outwards from inflector exit for various phases; ion with $\phi = +17$ deg is centred

Starting energy = 0.3 MeV
Final energy = 20 MeV

$\theta_0 = 54.5^\circ$
Fig. 4.21 Betatron oscillation amplitude vs RF phase for various starting conditions
Fig. 4.22 Phase histories of ions with various starting phases in a magnetic field with a field bump
Fig. 4.23 Accelerated phase plot outwards from inflector exit for various phases using the magnetic field with the field bump; ion with $\phi = 17$ deg is centred.
Fig. 4.24 Accelerated phase plots with $\phi = 0^\circ$ for four points on the edge of the emittance ellipse.
a) matched to $v = 1$, and b) chosen to reduce the radial oscillation amplitude over the phase range $-5^\circ$ to $+25^\circ$.
Fig. 4.25 Accelerated phase plots with $\phi = +15$ deg for four points on the edge of the emittance ellipse a) matched to $v_r = 1$, and b) chosen to reduce the radial oscillation amplitude over the phase range -5 deg to +25 deg.
Fig. 4.26 Accelerated phase plots with $\phi = +25$ deg for four points on the edge of the emittance ellipse a) matched to $v_r = 1$, and b) chosen to reduce the radial oscillation amplitude over the phase range $-5$ deg to $+25$ deg.
Fig. 5.1 Cross-section of a deep gap showing electric equipotentials.
Fig. 5.2 Electric potential vs distance from dee gap centre showing actual values and constant gradient approximation
Fig. 5.3 Geometry of an ion crossing a dee gap
Fig. 5.4: Gap factors vs. energy for $\phi_c = 0$ deg.
Fig. 5.5 Differences between gap factors obtained from numerical integration and those obtained from the constant gradient approximation as a function of energy, no magnetic field.
Fig. 5.6 Differences between gap factors obtained from numerical integration and those obtained from the constant gradient approximation as a function of energy, with an isochronous magnetic field.
Fig. 5.7 Differences between gap factors obtained from numerical integration and those obtained from the constant gradient approximation as a function of RF phase, with an isochronous magnetic field.
Fig. 5.8 Apparent displacement due to change in radius of curvature of the ion path while crossing the dee gap

Numerical results
Impulse approximation (equation 5.36)