TRIUMF

SIX LECTURES ON LATTICE FIELD THEORY

Michael Stone
Loomis Laboratory of Physics, University of Illinois

Lectures given at TRIUMF, May 24-June 6, 1983

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INTRODUCTION

These lectures are intended as an elementary introduction to some of the ideas of lattice quantum field theory[1] for an audience already familiar with field theory in the continuum.

There are, I think, two not wholly unrelated reasons why one should know something about field theories on a lattice: Firstly for numerical results in strongly interacting systems like QCD it seems to be the only procedure that works well enough to be implemented with some degree of control over the accuracy of the results. In the next few years practitioners of the art hope to be able to compute the hadron spectrum from essentially first principles (requiring only the quark masses and the QCD parameter as inputs). The results of this computation will either agree with what we see in nature or not. While I am sure that there will be much debate before a consensus is reached, first results in this direction seem promising.[2]

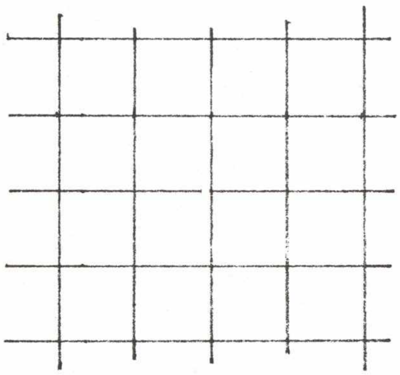
The second reason is more philosophical: When one first learns field theory in a traditional course, from Bjorken and Drell for example, most people, myself included, feel ill at ease with the subject because of the infinities. In any perturbative calculation there is a well-defined procedure to follow which, provided one does the sums correctly, will give an answer that makes physical sense and will be the same no matter who does it. As soon as one moves away from perturbation calculations towards general arguments, however, it is no longer clear what the rules of the game are and there exist situations in which wise men of good faith will disagree about the answer. With a lattice and view of the continuum limit being taken in the sense of critical phenomena (a view due essentially to K.G. Wilson), one has a well-defined game in which questions such as "is $(\lambda\phi^4)_4$ a free theory" can be posed precisely (if not yet answered decisively). This does away with the feeling of skating on thin ice and restores one's belief that we know <u>what</u> we are talking about even if we do not know all the answers.

While the second reason adduced above is the aim of "constructive field theory" which is a very formal subject, I do not intend to be formal in the organization of this material. I do, however, hope that there is some logic in the order that topics are presented.

LECTURE 1. SCALAR FIELDS - FREE AND INTERACTING

Throughout these lectures I am going to work in Euclidean space time [signature (+,+,+,+)]. Most of the time I shall be using a discrete lattice in all four directions with hypercubic symmetry (i.e. $\mathbb{Z}^4$). I shall occasionally allude to the Hamiltonian formalism which uses a continuous "time" variable ($\mathbb{R} \otimes \mathbb{Z}^3$) and of course mention the connection with the real world which has continuous time and space and has Minkowski signature (+,-,-,-). For the moment just visualize a four-dimensional square lattice. I am also going to assume that you are all familiar with relativistic quantum field theory in the continuum, preferably in a a functional integral approach.

In this first lecture we will go back to basics and study the free scalar field theory on $\mathbb{Z}^d$ with the particular intention of exploring the particle/field duality which is a fruitful way to think in lattice theories.



We put a real scalar variable $\phi(\vec{n})$ at each point $\vec{n} = (n_1 \ldots n_d)$ on $\mathbb{Z}^d$.

We look at the integral

$$Z = \int d[\phi] \exp\left\{- \sum_{\vec{n},i=1}^{d} a^d \left[\frac{1}{2} \frac{(\phi(\vec{n}+\vec{i}) - \phi(\vec{n}))^2}{a^2} + \frac{m_o^2}{2} \phi(\vec{n})\right]\right\}. \tag{1.1}$$

In this expression

$$d[\phi] = \prod_{\vec{n}} d\big(\phi(\vec{n})\big)$$

a = lattice spacing

$$\vec{i} = (0,\ldots \underset{\substack{\uparrow \\ i^{th} \text{ space}}}{1},0\ldots)$$

One should think of Z as the partition function of some statistical system of springs and masses. The exponent is clearly to be thought of as a lattice approximation to

$$L = \int d^dx \left(\frac{1}{2} (\partial\phi)^2 + \frac{m^2}{2} \phi^2\right) . \tag{1.2}$$

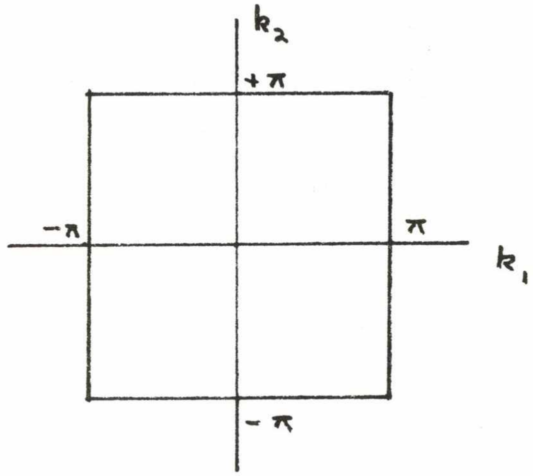
We are really interested in the Green's functions which can be thought of as the correlation functions of the $\phi(\vec{n})$ with respect to the probability distribution in (1.1)

$$\langle\phi(n_1)\ \dots\ \phi(n_i)\rangle = \frac{1}{Z}\int d[\phi]\ \phi(n_1)\ \dots\ \phi(n_i)\ \exp\left\{-\sum_{\vec{n},i=1}^{d}(\dots)\right\}. \tag{1.3}$$

We can evaluate these Green's functions by the simple but relatively unilluminating procedure of Fourier transforming $\phi(\vec{n})$. Let

$$\phi(\vec{n}) = \int_{-\pi}^{+\pi}\frac{d^dk}{(2\pi)^d}\, e^{i\vec{n}\cdot\vec{k}}\,\tilde{\phi}(\vec{k})\ . \tag{1.4}$$

Because of the discrete cubic lattice the $\vec{k}$ are restricted to lie in a Brillouin zone



The converse of this formula is

$$\tilde{\phi}(\vec{k}) = \sum_{n\varepsilon\mathbb{Z}^d} e^{-i\vec{n}\cdot\vec{k}}\,\phi(\vec{n}) \tag{1.5}$$

since

$$\int_{-\pi}^{+n}\frac{d^dk}{(2\pi)^d}\, e^{i\vec{k}\cdot(\vec{n}-\vec{n}')} = \delta_{n_1n_1'}\,\delta_{n_2n_2'}\,\dots\,\delta_{n_dn_d'}$$

$$\sum_{n} e^{i\vec{n}\cdot(\vec{k}-\vec{k}')} = \sum_{\vec{m}\varepsilon\mathbb{Z}^d}(2\pi)^d\,\delta^d(\vec{k}-\vec{k}'+2\pi\vec{m})\ . \tag{1.6}$$

In terms of $\tilde{\phi}(k)$ (and setting $a = 1$),

$$Z = \int d[\tilde{\phi}]\ \exp -\int\frac{d^dk}{(2\pi)^d}\,\frac{1}{2}\left[m_0^2 + 2\sum_{1}^{d}(1-\cos k_i)\right]|\tilde{\phi}(\vec{k})|^2 \tag{1.7}$$

and we can read off

$$\langle\tilde{\phi}(\vec{k}_1)\ \tilde{\phi}(\vec{k}_2)\rangle = \delta^d(\vec{k}_1-\vec{k}_2)\ \frac{(2\pi)^d}{m_o^2 + 2\sum_1^d (1-\cos k_i)} \tag{1.8}$$

or, in configuration space,

$$\langle\phi(\vec{n}_1)\ \phi(\vec{n}_2)\rangle = \int \frac{d^dk}{(2\pi)^d}\ \frac{e^{i\vec{k}\cdot(\vec{n}_1-\vec{n}_2)}}{m_o^2 + 2\sum_1^d (1-\cos k_i)}\ . \tag{1.9}$$

This is clearly the analog of the continuum propagator since $2(1-\cos k) = k^2 + O(k^4)$.

To get more insight into this expression let us reorganize the exponent in (1.1):

$$Z = \int d[\phi]\ \exp\left\{+\sum_{\vec{n},\vec{i}} \phi(\vec{n}+\vec{i})\phi(\vec{n}) - \sum_{n} (2d+m^2)\ \frac{\phi^2(\vec{n})}{2}\right\}. \tag{1.10}$$

Rescale

$$\phi \to \sqrt{2d+m^2}\ \phi \tag{1.11}$$

and absorb the Jacobean into the measure

$$Z = \int d\left[\frac{\phi}{\sqrt{2\pi}}\right]\ \exp\left\{+\frac{1}{(2d+m_o^2)}\sum_{\vec{n},i} \delta(\vec{n}+\vec{i})\phi(\vec{n}) - \sum_{\vec{n}} \frac{\phi^2(\vec{n})}{2}\right\}. \tag{1.12}$$

If m_o^2 is large it makes sense to expand out the first term in the exponent and obtain a typical "strong coupling" or "high temperature"[3] expansion:

$$Z = \quad d\left[\frac{\phi}{\sqrt{2\pi}}\right]\left(1 + \frac{1}{m_o^2+2d}\sum_{\vec{n},\vec{i}} \phi(\vec{n}+\vec{i})\phi(\vec{n})\ \ldots\right) \exp - \sum \frac{\phi^2(\vec{n})}{2}\ . \tag{1.13}$$

If we use the formulae

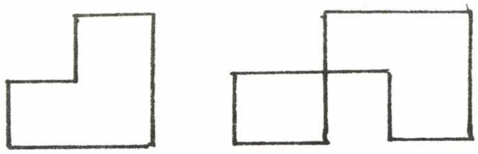
$$\int \frac{d\phi}{\sqrt{2\pi}}\ e^{-\phi^2/2} = 1$$

$$\int \frac{d\phi}{\sqrt{2\pi}}\ \phi^{2n+1}\ e^{-\phi^2/2} = 0$$

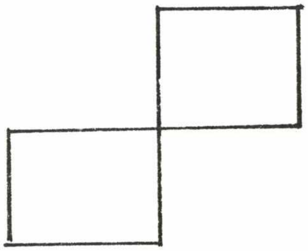
$$\int \frac{d\phi}{\sqrt{2\pi}}\ \phi^2\ e^{-\phi^2/2} = 1$$

$$\int \frac{d\phi}{\sqrt{2\pi}}\ \phi^4\ e^{-\phi^2/2} = 3 \quad \text{etc.}\ , \tag{1.14}$$

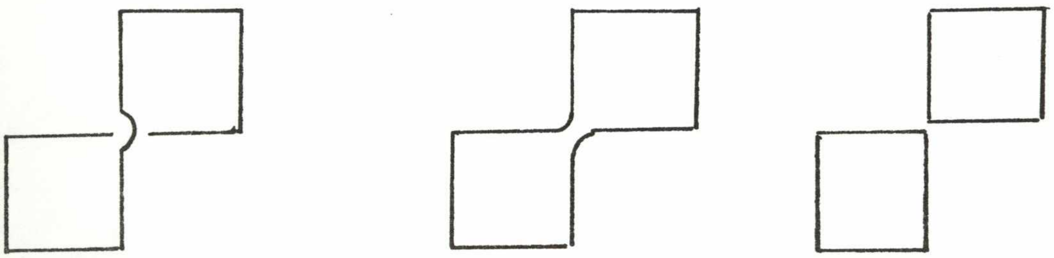
we can organize the expansion into diagrams which form closed loops



with a weight $1/(m_0^2+2d)$ for each link used. The factor of 3 in the ϕ^4 integral (1.14) means that a diagram such as this



has a weight of +3 which can be regarded as the sum of three diagrams



All the other n!, etc. in the expansion of the exponent in (1.14) lead to the plausible, and true, result that the expansion for Z is a gas (or solution) of closed, non-interacting, loops. If we were calculating Green's functions the $\phi(\vec{n})$ act as sources where the lines can start or stop, so we have a Feynman diagram-like expansion for them together with vacuum loops. We will interpret these lines and loops as the world lines of the particles created and destroyed by the $\phi(\vec{n})$ fields.

We will now sum these diagrams and recover the results we obtained from the Fourier transform method. [This is easy here but these methods can be used to solve non-trivial problems such as the d=2 Ising model.][4]

Random Walks

Let us temporarily forget about our field theory and just look at the theory of random walks on a hypercubic lattice. Let

$$\Gamma(\vec{n}_1,\vec{n}_2,t) \equiv \Gamma(0,\vec{n}_2-\vec{n}_1,t)$$
$$= \text{number of walks from } n_1 \text{ to } n_2 \text{ which are t steps long} \qquad (1.15)$$

Then it only takes a moment to see that

$$\Gamma(0,n,t) = \text{coeff. of } x_1^{n_1}\, x_2^{n_2} \ldots x^{n_d} \text{ in}$$

$$\left(x_1 + \frac{1}{x_1} + x_2 + \frac{1}{x_2} + \ldots + x_d + \frac{1}{x_d}\right)^t . \qquad (1.16)$$

If we put $x_i = e^{ik_i}$ we can extract this coefficient as a closed formula

$$\Gamma(0,\vec{n},\vec{t}) = \int \frac{d^dk}{(2\pi)^d}\, e^{-i\vec{k}\cdot\vec{n}}\, (2\cos k_1 + 2\cos k_2 + \ldots)^t . \qquad (1.17)$$

If we then define $G(0,n,\mu)$ by

$$G(0,\vec{n},\mu) = \sum_{t=0}^{\infty} e^{-t\mu}\, \Gamma(0,\vec{n},t) \qquad (1.18)$$

we see that G is essentially the propagator (1.7)

$$G(0,\vec{n},\mu) = \int \frac{d^dk}{(2\pi)^d}\, e^{-i\vec{k}\cdot\vec{n}}\, \frac{1}{1 - e^{-\mu} \sum_1^d 2\cos k_i}$$

$$= (m_0^2+2d) \int \frac{d^dk}{(2\pi)^d}\, e^{-i\vec{k}\cdot\vec{n}}\, \frac{1}{m_0^2 + 2\sum_1^d (1-\cos k_i)} \qquad (1.19)$$

provided $e^{-\mu} = (m_0^2 + 2d)^{-1}$. The factor in front of the integral reflects the scaling made in Eq. (1.11). We have thus proved that the sum over paths from n_1 to n_2 with the weight $(m_0^2 + 2d)^{-1}$ per link does indeed produce the propagator.

If we want to sum up the closed vacuum loops we have to be a little bit more careful to avoid overcounting. If we fix a point on a loop and sum over all loops through that point of length t we find:

$$e^{-\mu t}\, \Gamma(\vec{n},\vec{n},t) = \int \frac{d^dk}{(2\pi)^d}\, e^{-\mu t}\left(2 \sum_1^d \cos k_i\right)^t \qquad (1.20)$$

but if we want to sum on $\vec{n}$ to get all loops of length t we must divide by 2t. The factor t arises because all t vertices on the loop are equivalent and the factor of 2 occurs because the loop occurs twice with opposite orientation. (This factor would be absent if we were using complex ϕ fields as there would be arrows on the loop just as for charged continuum fields.)

The total sum over all configurations of one loop is therefore

$$N \sum_t \frac{1}{2t} \int \frac{d^dk}{(2\pi)^d} \left(\sum_1^d 2\cos k_i \; e^{-\mu} \right)^t$$

$$= - \frac{N}{2} \int \frac{d^dk}{(2\pi)^d} \, \ell n\left(1 - e^{-\mu} \sum_1^d 2\cos k_i\right)$$

$$= \text{const} - \frac{N}{2} \int \frac{d^dk}{(2\pi)^d} \, \ell n\left(m_o^2 + 2 \sum_1^d (1-\cos k_i)\right)$$

(N = # of sites in the lattice). (1.21)

If we sum over a gas of n loops one must divide by n! to avoid over-counting, so finally:

$$Z = \exp - \frac{N}{2} \int \frac{d^dk}{(2\pi)^d} \, \ell n\left(m_o^2 + \sum_1^d (1-\cos k_i)\right)$$

$$= \det^{-1/2} \left(\Delta^2 + m_o^2\right) \; . \tag{1.22}$$

We could have obtained this directly from (1.7) of course, but I think it is rather nice to see the determinant expression for the vacuum loops come directly from the grand canonical ensemble of a "loop gas" with fugacity $e^{-\mu}$ for each "monomer" or link.

I hope everyone is familiar with the continuum expressions

$$Z = \langle 0_{out} | 0_{in} \rangle = \exp\left\{- \frac{TV}{2} \int \frac{d^4k}{(2\pi)^4} \, \ell n(k^2+m^2)\right\}$$

$$= \exp\left(- E_o T\right) \tag{1.23}$$

$$E_o = \frac{V}{2} \int \frac{d^4k}{(2\pi)^4} \, \ell n(k^2+m^2) = \frac{V}{2} \int \frac{d^3k}{(2\pi)^3} \sqrt{\vec{k}^2+m^2} \; (+ \text{ m indep. term})$$

$$= \frac{1}{2} \, h\nu/\text{degree of freedom}) \times \text{number of states} \tag{1.24}$$

Putting in Interactions

We have seen that a free scalar field theory - i.e. one whose functional integral is purely Gaussian - could be interpreted as a gas of non-interacting world lines (a better physical model would be solution of polymers). If we introduce a non-Gaussian $\lambda\phi^4$ term ($\lambda > 0$) into the action

$$Z = \int d[\phi] \, \exp\left\{- \sum_{\vec{n},\vec{i}} \left[\frac{\phi(\vec{n}+\vec{i}) - \phi(n)^{\;2}}{2} + \frac{m_o^2\phi^2(\vec{n})}{2} + \frac{\lambda\phi^4(\vec{n})}{4!}\right]\right\}, \tag{1.25}$$

the $\lambda\phi^4$ term can be used in the large m expansion when lines cross. It has the effect (because of the minus sign) of reducing the contribution to the sum over loops of configurations where lines cross. In other words it leads to a short-range repulsion between loops and between one part of a single loop and another. This type of interaction is actually used to model the effects of short-range repulsion in polymer solutions.

If one were to take $\lambda < 0$ then the integral would diverge at large $\phi(\vec{n})$. This is reflected in the force between polymers being attractive. The force does not saturate and if one has n walks close together the cost in action per unit length is proportional to

$$S = \alpha n - \beta\lambda n^2$$

for some α,β (the factor n^2 is because there are $n(n-1)/2$ pairs of lines to interact). This is essentially the same as in the continuum where an n-particle bound state has mass:

$$mn - K\lambda n^2$$

and will become tachyonic if n is large enough. When this happens the the vacuum is filled with a tangled mess of world lines. A polymer precipitates out of solution if the interstrand forces become attractive.

If we return to the case $\lambda > 0$ but now take $m_0^2 < 0$ so that spontaneous symmetry-breaking occurs for small λ, then a similar picture holds; having m_0^2 negative means that:

$$e^{-\mu} > (2d)^{-1} \ . \tag{1.26}$$

At each step of a walk on a d dimensional hypercubic lattice one has a choice of 2d directions to go in. If (1.26) holds, the $e^{-\mu}$ factor is not enough to discourage long walks and the vacuum fills with a spaghetti of vacuum lines now limited in the density they achieve by the interparticle repulsion. This is the world line picture of a Bose condensate.

Gauge and Electromagnetic Interactions

In order to discuss charged particles we must, just as in continuum physics, allow $\phi(\vec{n})$ to be a complex field. This just has the effect of adding an arrow to the diagrams to distinguish the use of the now distinct hopping terms $\phi^*(\vec{n}+\vec{1})\phi(\vec{n})$ and $\phi(\vec{n}+\vec{1})\phi^*(\vec{n})$.

To introduce an electromagnetic interaction we modify the action by replacing the exponent with

$$S = -\sum_{\vec{1},\vec{n}} \left\{ |\phi(\vec{n}+\vec{1}) - e^{ieA(\vec{n}+\vec{1},\vec{n})}\phi(\vec{n})|^2 + m_0^2\phi^*\phi \right\} .$$

This means that the field at $\vec{n}+\vec{1}$ is compared with the field at $\vec{n}$ after it has been "parallelly transported" by multiplication with the phase factor on the link. For a non-Abelian gauge group the phase factor would be replaced by a matrix living in the representation of the group to which $\phi(n)$ belongs.

Since

$$-\sum_{\vec{1},\vec{n}} |\phi(\vec{n}+\vec{1}) - e^{ieA(\vec{n}+\vec{1},\vec{n})}\,\phi(\vec{n})|^2 + m_0^2\phi\,\phi^* =$$

$$-\sum (2d + m^2)\phi^*\phi + \phi(\vec{n}+\vec{1})\,e^{ieA(\vec{n}+\vec{1},\vec{n})}\,\phi(\vec{n})$$

$$+\,\phi(\vec{n}+\vec{1})\,e^{-ieA(\vec{n}+\vec{1},\vec{n})}\,\phi^*(\vec{n})$$

the hopping terms are

$$\phi^*(\vec{n}+\vec{1})\,e^{ieA(\vec{n}+\vec{1},\vec{n})}\,\phi(n),\ \phi(\vec{n}+\vec{1})\,e^{-ieA(\vec{n}+\vec{1},\vec{n})}\,\phi^*(\vec{n})\ ,$$

which give rise to the diagrams:

$$\underset{\vec{n}+\vec{1}}{\overset{\phi}{\bullet}} \longrightarrow \underset{\vec{n}}{\overset{\phi^*}{\bullet}} \qquad\qquad \underset{\vec{n}+\vec{1}}{\overset{\phi^*}{\bullet}} \longrightarrow \underset{\vec{n}}{\overset{\phi}{\bullet}}$$

with weights for a world line of t steps of

$$\frac{1}{(2d+m_0^2)}\prod_{\text{links}} e^{ieA(\vec{n}+\vec{1},\vec{n})}\ .$$

For closed loops we find that each loop has a "Wilson loop" factor

$$W = \prod_{\circlearrowright} e^{ieA(\vec{n}+\vec{1},\vec{n})}$$

which is reponsible for the interactions. If we recollect that for a continuum

$$\langle W\rangle = \int d[A]\ \exp\left\{-\int\left(\frac{F^2}{4} + iJ_\mu\ A_\mu\right)d^4x\right\}$$

$$J_\mu = e\oint dt\ \frac{dx^\mu}{dt}\ \delta^4\big(x-x(t)\big)$$

$$\langle W\rangle = \exp -\frac{1}{2}\int J_\mu(x)\ G_{\mu\nu}(x-y)J_\nu(y)\ d^4x$$

we see that after integrating over gauge fields the effect is, for Abelian fields, of providing a Biot-Savart-like force between the world lines which carry a current proportional to their charge. The factor of "i" in the phase factor has the important effect of changing the sign of the interaction so that parallel conductors repel one another and anti-parallel world lines attract.

LECTURE 2. CONTINUUM LIMIT, RESTORATION OF ROTATIONAL SYMMETRY AND ASYMPTOTIC FREEDOM

To discuss the topics in the title of this lecture I am going to use a simple solvable model which illustrates them all in a very transparent way. This is the two-dimensional O(N) symmetric non-linear σ model in the large N limit.

The partition function is the same as the free scalar theory except for a constraint on ϕ:

$$Z = \int d[\phi]\ \delta(\phi^2-1)\ \exp\left\{ -\frac{1}{g^2}\sum_{\vec{n},\vec{1}} \frac{1}{2}\ [\phi^\alpha(\vec{n}+\vec{1}) - \phi^\alpha(\vec{n})] \right\}\ ; \tag{2.1}$$

here $\vec{\phi} = (\phi^\alpha)$ is an N component vector in some internal space. We can solve this model in the $N \to \infty$ limit just as we can in the continuum. We introduce a "Lagrange" multiplier $\lambda(\vec{n})$ to enforce the constraint

$$Z = \int d[\lambda] \int d[\phi]\ \exp - \left\{ \frac{1}{g^2}\sum_{\vec{n},\vec{1}} \frac{1}{2}\ [\phi(\vec{n}+\vec{1}) - \phi(\vec{n})]^2 + \sum i\lambda(\vec{n})[\phi^2-1] \right\}\ . \tag{2.2}$$

If we put $g^2 N = \alpha$ = const, $\lambda = N\tilde{\lambda}$ we can use the method of steepest descent. If we assume λ is a constant we can perform the ϕ integrals to get

$$\int d\lambda\ \exp - N\left[\frac{1}{2}\int_{-\pi}^{+\pi} \ell n\left(\sum_i^d (2-2\cos k_i) - 2i\tilde{\lambda}\alpha\right) \frac{d^2k}{(2\pi)^d} + i\tilde{\lambda}\right] \tag{2.3}$$

so the λ saddle point equation is

$$-\frac{2i\alpha}{2}\int_{-\pi}^{+\pi} \frac{d^dk}{(2\pi)^2}\ \frac{1}{\Sigma(2-2\cos k_i) - 2i\lambda\alpha} + i = 0$$

or if we put $-2i\lambda = m^2$

$$\alpha \int_{-\pi}^{+\pi} \frac{d^dk}{(2\pi)^2}\ \frac{1}{\Sigma(2-2\cos k_i) + m^2} = 1\ . \tag{2.4}$$

Using this method to compute the correlation functions we find that

$$\langle\phi^\alpha(\vec{n}_1)\ \phi^\beta(\vec{n}_2)\rangle = \delta_{\alpha\beta}\ \frac{\alpha}{N}\ \Delta(\vec{n}_1,\vec{n}_2,m^2)$$

where
$$\Delta(\vec{n}_1,\vec{n}_2,m^2) = \int_{-\pi}^{+\pi} \frac{d^dk}{(2\pi)^d}\ \frac{e^{i\vec{k}\cdot(\vec{n}_1-\vec{n}_2)}}{2\,\Sigma\,(1-\cos k_i) + m^2} \tag{2.5}$$

and (2.4) can be written as

$$<\sum_{1}^{N} \phi^2 - 1> = 0 \ . \qquad (2.6)$$

In two dimensions the integral on the LHS of (2.4) is infrared divergent as $m^2 \to 0$, so for all α we can find a solution with $m^2 > 0$. In more than two dimensions $\Delta(0,m^2)$ is bounded above by the finite number $\Delta(0,0)$ so for $\alpha < \alpha_c = [\Delta(0,0)]^{-1}$ there is no solution of (2.4) for real m and we must set $m^2 = 0$. There is a phase transition at α_c.

In two dimensions, after some searching in Gradshteyn and Ryzhik, we discover that

$$\Delta(0,m^2) = \frac{1}{2\pi} \frac{1}{(1+m^2/2)} K\left(\frac{1}{(1+m^2/2)}\right) \qquad (2.7)$$

where K is a complete elliptic integral. As $m \to 0$

$$\Delta(0,m^2) = \frac{1}{4\pi} \ln \frac{32}{m^2} + O(m^2) \qquad (2.8)$$

so we can solve for the connection between α and m^2 when m^2 is small:

$$\frac{1}{\alpha} = \frac{1}{4\pi} \ln \frac{32}{m^2} \ . \qquad (2.9)$$

This means that m^2 is small when g^2 is small.

When g^2 and m^2 are small most of the contribution to the propagator comes from small k where the integrand looks like $(k^2+m^2)^{-1}$ which is rotationally invariant. Let us look in detail how this comes about:

$$\Delta(\vec{x},m^2) = \int_{-\pi}^{+\pi} \frac{d^2k}{(2\pi)^2} \frac{e^{i\vec{k}\cdot\vec{x}}}{m^2 + \sum_{1}^{2} (2-2\cos k)} \ . \qquad (2.10)$$

Suppose $\vec{x}$ is becoming large in a particular direction specified by a unit vector $\vec{e}$

$$\vec{x} = |r|\vec{e} \qquad e^2 = 1 \ .$$

We expect Δ to fall off exponentially:

$$\Delta(r\vec{e},m^2) \approx e^{-\kappa(\vec{e})|r|} \ . \qquad (2.11)$$

The problem is to compute $\kappa(\vec{e})$, the inverse correlation length in the direction $\vec{e}$. We expect it to be anisotropic at large m^2 but to become isotropic as m^2 becomes small.

Define

$$f(\xi) = \int_{0}^{\infty} dr \ e^{-i\xi r} \Delta(r\vec{e},m^2) \ . \qquad (2.12)$$

This should be analytic in the lower half ξ plane and the asymptotic behaviour at large $|r|$ of $\Delta(r\vec{e},m^2)$ will be given by the nearest singularity to the real ξ axis (which will be on the positive imaginary axis if Δ does not oscillate)

$$\left[\text{recall} \int_{-\infty}^{+\infty} e^{i\xi r} \frac{1}{\xi - i\xi_0} \frac{d\xi}{2\pi i} = e^{-\xi_0 r}\right] .$$

Now

$$f(\xi) = \pi \int \frac{d^2k}{(2\pi)^2} \frac{\delta(k \cdot e - \xi)}{m^2 + \sum_1^2 (2 - 2\cos k_i)}$$

$$= \pi \int \frac{d^2k}{(2\pi)^2} \frac{\delta(k \cdot e - \xi)}{D(k)} . \tag{2.13}$$

For small m the singularity in $f(\xi)$ is expected to be caused by zeros of $D(k)$ pinching the contour of integration. [Just as in the continuum:

$$f(i\xi) = \int dk \frac{1}{k^2 + m^2 - \xi^2} \tag{2.14}$$

where the integrand has poles at $\pm i\sqrt{m^2 - \xi^2}$ which pinch if $\xi = m$. This makes the continuum propagator fall off as e^{-mr} as indeed we know it does.]

The condition for a pinch singularity at $i\xi_0$ is, if $\vec{K} = i\vec{k}$,

$$D(\vec{K}) = 0 \tag{2.15a}$$

$$\frac{\partial D}{\partial \vec{K}} = 0 \quad \text{on } \vec{K} \cdot \vec{e} = \xi_0 . \tag{2.15b}$$

One can impose the constraint in (2.15b) by means of a Lagrange multiplier

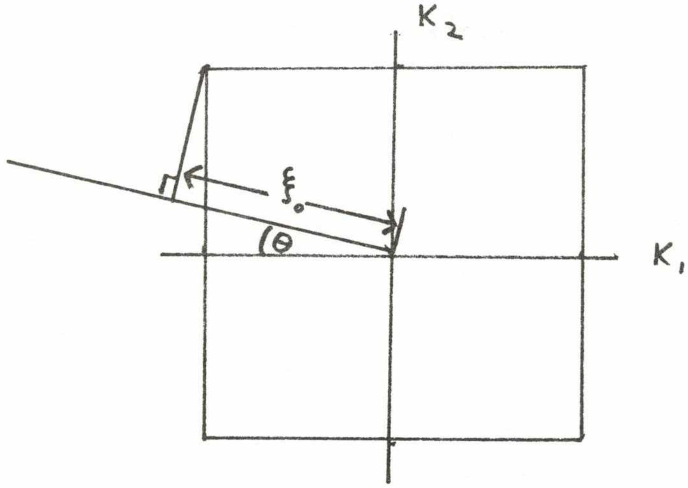
$$\frac{\partial}{\partial \vec{K}} \left\{ D(\vec{K}) - \lambda \vec{K} \cdot \vec{e} \right\} = 0 . \tag{2.15c}$$

These equations have a simple geometric interpretation. The set of points $\vec{K} \cdot \vec{e} = \xi_0$ is a straight line perpendicular to $\vec{e}$ and at a distance ξ_0 from the origin. Equations (2.15) say that if there is a pinch singularity at $i\xi_0$ then this line must be tangent to the curve $D(K) = 0$. A more direct way to obtain this condition is to note that the points of intersection of $\vec{K} \cdot \vec{e} = \xi_0$ with $D(K) = 0$ are the locations of the poles of the integrand in the complex $\vec{k}$ plane. Clearly they can pinch only if they are coincident and the line is tangent.

In our case:

$$D(K) = 4 + m^2 - 2(\cosh K_1 + \cosh K_2) . \tag{2.16}$$

For large m^2, $D(\vec{K}) = 0$ is essentially a square with sides at $\pm\cosh^{-1}(2+m^2)/2$.



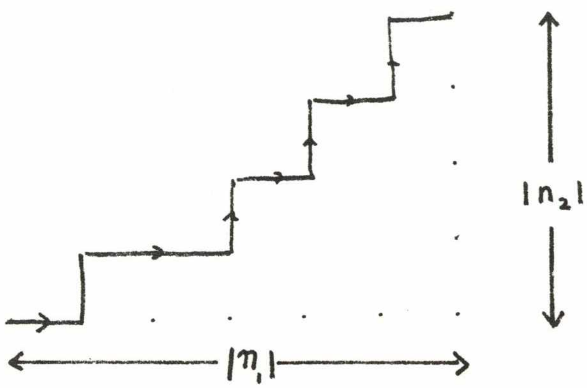
A little geometry shows that

$$\xi_0 \simeq (\cos\theta + \sin\theta)\ \cosh^{-1}(2+m^2)/2$$

$$\simeq (\cos\theta + \sin\theta)\ \ell n(2+m^2) \qquad (2.17)$$

$$\therefore \Delta(r\vec{e},m) \sim e^{-r\xi_0} = \frac{1}{(2+m^2)^{|n_1|+|n_2|}} \qquad (2.18)$$

which means that at large m^2 the propagator is dominated by the shortest route between 0 and $\vec{n}$.



As soon as m^2 is away from $m^2 = \infty$ the corners of the curve round off and as m^2 becomes small, the curve $D(\vec{K}) = 0$ becomes circular and the correlation length becomes isotropic.

It is easy to see that on axis

$$\kappa = \cosh^{-1}(1 + m^2/2)$$

while at 45° to an axis

$$\kappa = \sqrt{2}\ \cosh^{-1}(1 + m^2/4) \ .$$

Both expressions are equal to m when m^2 is small.

We can now explain what is meant by the "continuum limit" of this lattice model: As we let the coupling constant g^2 become smaller two things happen

(i) The correlation length measured in units of lattice spacing grows and approaches infinity as g^2 approaches zero.

(ii) The correlation functions become isotropic.

We can regard (i) and (ii) as meaning that if we take g^2 to zero and the lattice spacing a to zero in such a way that the correlation length measured in physical units (fermi) stays the same, then the anisotropy due to the lattice disappears and we have a continuum theory. The inverse of the correlation length is the mass f the particle created by the ϕ field and is given by

$$M^2 = \frac{m^2}{a^2} = 32 \frac{1}{a^2} \exp - \left(\frac{4\pi}{g^2 N}\right) . \qquad (2.19)$$

The mass M^2 satisfies a renormaliztion group equation

$$\left(a \frac{\partial}{\partial a} + \beta(g^2) \frac{\partial}{\partial g^2}\right) M^2(a,g^2) = 0 \qquad (2.20)$$

where

$$\beta(g^2) = + g^4 \frac{N}{2n} . \qquad (2.21)$$

It is common in lattice theories and statistical mechanics to define the β function this way (i.e. by varying a length instead of a mass) and we have a change of sign compared to the particle physics convention. Thus the plus sign in Eq. (2.21) means that this theory is asymptotically free.[6]

Finally, a brief word on Λ parameters. (I will touch on them later.) Suppose, instead of a lattice cut-off, we solved a continuum version of this model using a Pauli-Villars cut-off. Equation (2.4) would become

$$g^2 N \int_{-\infty}^{+\infty} \frac{d^2k}{(2\pi)^2} \left[\frac{1}{k^2+m^2} - \frac{1}{k^2+\mu^2}\right] = 1$$

so

$$M^2 = \mu^2 e^{-4\pi/g^2 N} .$$

The quantities called the Λ parameters are defined by

$$\Lambda^2_{PV} = \mu^2 e^{-4\pi/g^2 N}$$

$$\Lambda^2_{lattice} = a^{-2} e^{-4\pi/g^2 N}$$

and since the physical quantity M^2 must be independent of the regularization scheme we must have[7]

$$32 \Lambda^2_{lattice} = \Lambda^2_{PV} .$$

LECTURE 3. FRACTALS, SCALING AND RENORMALIZATION

We saw last time that to obtain a continuum limit of a lattice theory we need to find a "critical point" where the correlation length tends to infinity. I do not wish to discuss the formal theory of critical phenomena and field theory here as there are excellent discussions in the literature. What I do want to do is to be more intuitive and try to bridge the generation and culture gap that has grown up between those who learned renormalization in the traditional way and those who grew up on the work of Wilson. We all do the same sums but the language is often very different. I can paraphrase (parody?) the essentials of the two world views as follows:

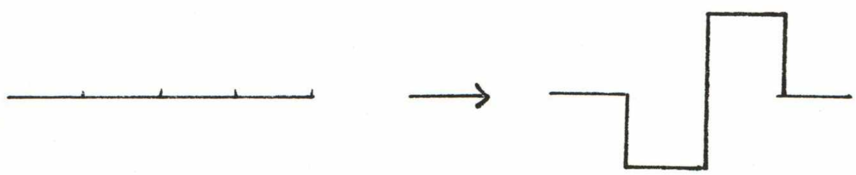
1) "A field theory is a set of Feynman rules together with a prescription (called renormalization) for getting rid of unwanted ultraviolet divergences."

2) "A field theory is the large distance behaviour of a system near a critical point. Renormalization is the language in which to describe how infrared divergences miraculously make this behaviour independent of the details of the short distance interactions."

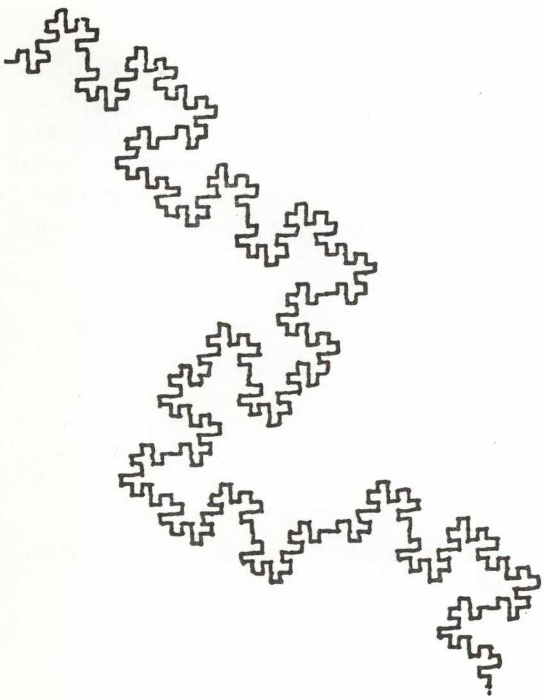
Perhaps not coincidentally most people who prefer world view #1 spend their working day in the Fourier transform of Minkowski space while those who favour world view #2 live and work in Euclidean space. Let me begin the discussion by asking

<u>How Long is the Coastline of England?[8]</u>

To be more mathematical let us ask how long is the curve generated recursively from a square by successively replacing each line element by another one as follows:



So we eventually obtain a figure with a "self similarity" property:



The length of the limiting set is clearly infinite but if we measure with a fixed resolution we get a finite number that depends on the resolving power. If we increase the resolving power by a factor of four we find that the measured length has increased eight times. Such a set is called a fractal and the way in which the length scales with the resolving power is related to the Hausdorff dimension of the set. The Hausdorff dimension is defined by trying to cover the set with little discs of radius ε. If the number of discs required to do this increases as

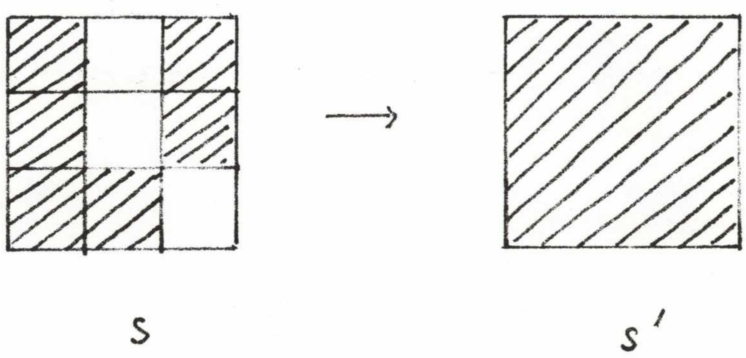
$$\text{Number} = \left(\frac{\varepsilon}{\mu}\right)^{-\alpha} \tag{3.1}$$

then α is the Hausdorff dimension. For a straight line $\alpha = 1$, for a plane figure $\alpha = 2$, etc. In our case if we reduce ε by 1/4 we need eight times as many discs so $\alpha = 3/2$. The total length we estimate is:

$$\text{Length} = \varepsilon\left(\frac{\varepsilon}{\mu}\right)^{-\alpha} \tag{3.2}$$

Notice the way that an additional length scale (renormalization point?) has crept into the length to soak up the extra "anomalous dimension". This behaviour is very similar to the behaviour of Green's functions in a massless field theory. A propagator canonically varies as p^{-2} but when interactions are present it may vary as $p^{-2-\eta}$ (e.g. the massless Thirring model).

Let us see if we can relate such an anomalous dimension to some sort of self similarity in the field theory. Consider an Ising model near T_c. We define a "renormalization group transformation" by letting blocks of nine spins vote[9]:



In an Ising model the original spins S have a probability distribution given by the usual Ising Hamiltonian

$$P(S_i) = Z^{-1} \exp K \sum S_i S_j \ . \tag{3.3}$$

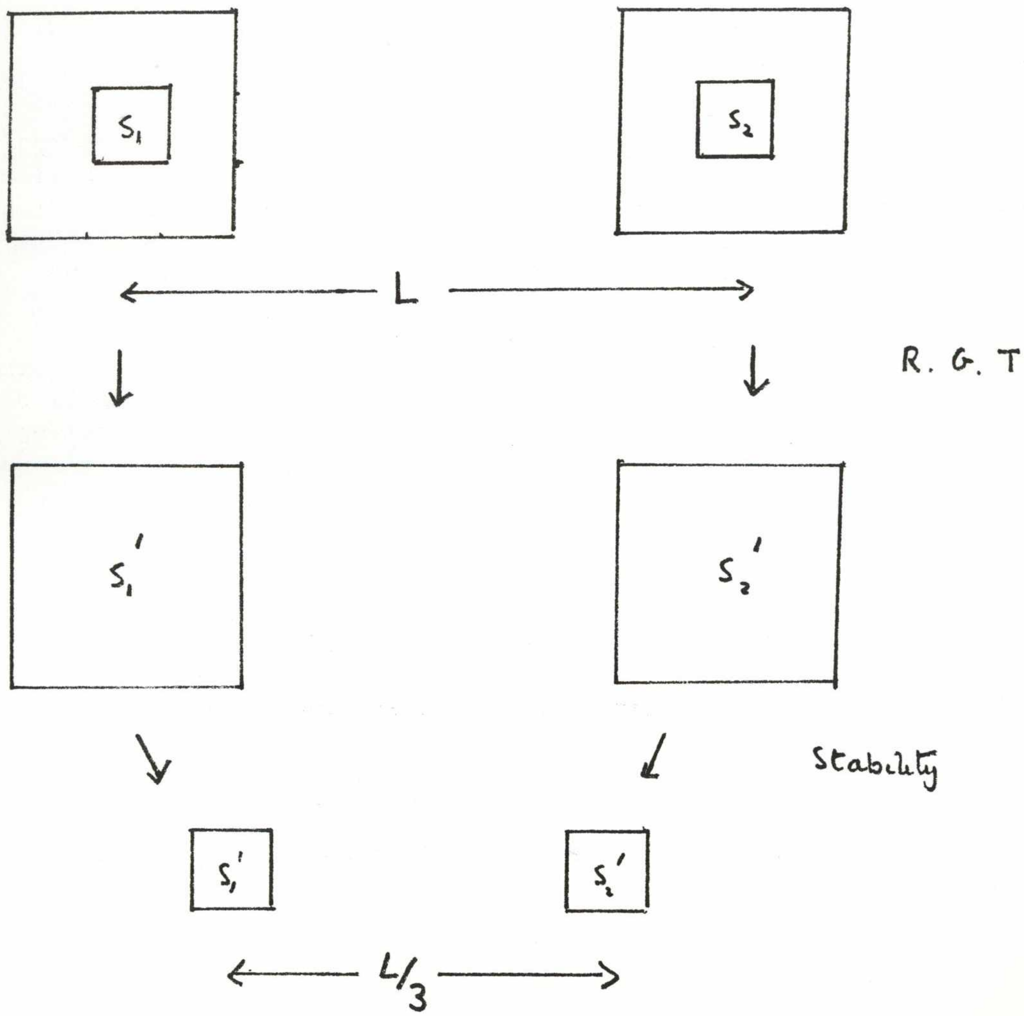
From this we can find the distribution P(S') for the new spins. If we imagine this used to describe spins on a lattice 1/3 the size we can compare it with the original P:

$$P(S) \xrightarrow[\text{(vote, reduce by 1/3)}]{} P'(S') \ . \tag{3.4}$$

In general P'(S') is not exactly of Ising form; it will contain interactions and all sorts of junk. A very important feature of the critical point is, however, that if we keep iterating

$$P(S) \to P'(S') \to P''(S'') \to \ldots$$

eventually we reach a fixed distribution which does not change under this process. If we are not at a critical point this does not happen. (This is very similar to the central limit theorem of probability - keep adding random variables from the same distribution and eventually the distribution of the sum is Gaussian.) This <u>stability</u> of the distribution is the analog of self similarity for the fractal. It essentially says that inside islands of up-spin there are smaller islands of down-spin and so on ad infinitum. Let us use this to obtain a power law for the spin-spin correlation functions:



A formal statement of this stability is as follows: Let $P(S_1' = \mu_1', S_2' = \mu_2'; L)$ be the probability for two blocks of nine spins on the original lattice, separated by a distance L, to vote for the values μ_1', μ_2'. Let $P(S_1 = \mu_1', S_2 = \mu_2', L/3)$ be the probability of individual spins at distance L/3 to have the values μ_1', μ_2'. Then at the fixed (stable point)

$$P(S_1' = \mu_1', S_2' = \mu_2'; L) = P(S_1 = \mu_1', S_2 = \mu_2', L/3) . \qquad (3.5)$$

The spin-spin correlation function is

$$\langle S_1 S_2 \rangle_L = \sum_{\mu_1, \mu_2 = \pm 1} \mu_1 \mu_2 \, P(S_1 = \mu_1, S_2 = \mu_2; L) . \qquad (3.6)$$

Now $P(S_1 = \mu_1, S_2 = \mu_2; L) = \sum_{\mu_1' \mu_2} P(S_1 = \mu_1 | S' = \mu_1') P(S_2 = \mu_2 | S_2 = \mu_2')$

$$\times P(S_1' = \mu_1', S_2' = \mu_2'; L) \qquad (3.7)$$

$$= \sum_{\mu_1' \mu_2'} P(S_1 = \mu_1 | S' = \mu_1') P(S_2 = \mu_2 | S_2 = \mu_2')$$

$$\times P(S_1 = \mu_1', S_2 = \mu_2'; L/3) . \qquad (3.8)$$

If the interaction is short range then $P(S_1 = \mu_1 | S_1' = \mu_1')$ only depends on the value of the Block spin. So

$$\sum_{\mu_1} \mu_1 P(S_1 = \mu_1 | S_1' = \mu_1') = f(\mu_1') \qquad (3.9)$$

is a function of μ' only. From symmetry considerations

$$\sum_{\mu_1} \mu_1 \, P(S_1 = \mu_1 | S_1' = \mu_1') = Z^{1/2} \mu_1' . \qquad (3.10)$$

Thus we find from (3.10) and (3.6), (3.8) that

$$\langle S_1 S_2 \rangle = Z \, \langle S_1 S_2 \rangle_{L/3}$$

so

$$\langle S_1 S_2 \rangle_{L_0 3^n} = Z^n \langle S_1 S_2 \rangle_{L_0}$$

or

$$\langle S \; S \rangle \propto \frac{1}{|L|^\eta}$$

$$\eta = -\ln Z / \ln(3) .$$

(Recall that anomalous dimensions in field theory are defined as

$$\gamma = \frac{d(\ln Z)}{d(\ln \mu)} . \qquad)$$

We can rewrite this by introducing a lattice spacing a and a renormalization point R_0

$$\langle S\ S\rangle = \left(\frac{L_0}{L}\right)^{\eta}$$

$$= \left(\frac{aL_0}{aL}\right)^{\eta} = \left(\frac{R_0}{R}\right)^{\eta} \cdot \left(\frac{aL_0}{R_0}\right)^{\eta}$$

$$= \left(\frac{R_0}{R}\right)^{\eta} \exp \eta\ \ell n \frac{aL_0}{R_0}$$

$$\therefore \langle S\ S\rangle = \langle S\ S\rangle_{\text{renormalized}} \times Z(a,R_0)$$

As $a \to 0$ we have logarithmic divergences in the wave function renormalization factor.

This is at T_c and corresponds to a massless theory. If we want a massive theory we would work close to, but not quite at, T_c and let T approach T_c as we take a to zero.

The idea that there is a fixed point in the probability distribution has a number of consequences for continuum field theories which are worth appreciating:

1) There are many different lattice approximations to any given continuum theory. To give rise to the same theory they just have to be in the domain of attraction of that fixed point (so-called universality class).

2) Theories that are in the same universality class differ by what are called "irrelevant" interactions. These irrelevant interactions cannot affect the continuum theory. In the conventional language of field theory these irrelevant interactions were called "non-renormalizable".

3) Operator ordering: We all know that in quantum mechanics it matters in which order quantum operators appear in the Hamiltonian - but we never seem to worry about this in continuum field theory. This is essentially due to point 1). The operator ordering ambiguities come about in the functional integral formalism because of different ways of discretizing the functional integral. As the dimension of space time increases more interactions become non-renormalizable or irrelevant and the size of the universality classes increases. This means that most operator-ordering problems disappear as we take the continuum limit.

LECTURE 4. GAUGE FIELDS, MONTE-CARLO AND STRING TENSION

We briefly mentioned in the first lecture how matter fields interact with background gauge fields via "parallel transport". We now need to discuss the form of the gauge field action.

The gauge field degrees of freedom live on the links of the hypercubic lattice and are elements of the gauge group. For an SU(2) group, for example,

$$U_\mu(\vec{n}) = e^{iB_\mu(\bar{n})} \tag{4.1}$$

$$B_\mu(\vec{n}) = \frac{1}{2}\, ag\, \tau_i\, A^i_\mu(\vec{n}) = \frac{1}{2}\, ag\, \tau\cdot A_\mu(\vec{n}) \ . \tag{4.2}$$

When we need links in a backward direction we associate them with $U^{-1}_\mu(\vec{n})$:

$$U_{-\mu}(\vec{n}+\vec{\mu}) = U^{-1}_\mu(\vec{n}) \ . \tag{4.3}$$

The most commonly used action is the Wilson action[1] which is made up out of a sum over plaquettes (μ,ν):

$$S = -\frac{1}{ag^2}\sum_{n,\mu,\nu} \mathrm{Tr}\left\{U_\mu(\vec{n})U_\nu(\vec{n}+\vec{\mu})U_{-\mu}(\vec{n}+\vec{\mu}+\vec{\nu})U_{-\nu}(n+\nu)\right\} \ . \tag{4.4}$$

It is less obvious than in the scalar field case what motivates this action. The basic idea is to parallel transport around the four sides of a plaquette and to look how far the resulting group element is away from the identity (measured in the intrinsic Riemannian geometry on the group manifold). For slowly varying fields we have

$$\mathrm{Tr}\left\{U_\mu(\vec{n})\ U_\nu(\vec{n}+\vec{\mu})\ U_{-\mu}(\vec{n}+\vec{\mu}+\vec{\nu})\ U_{-\nu}(\vec{n}+\vec{\nu})\right\}$$

$$\approx e^{iB_\mu}\ e^{iB_\nu+a\partial_\mu B_\nu}\ e^{-i(B_\mu+a\partial_\nu B_\mu)}\ e^{-iB_\nu}$$

$$= e^{i(B_\mu+B_\nu+a\partial_\mu B_\nu)-1/2[B_\mu,B_\nu]}\ e^{-i(B_\mu+B_\nu+a\partial_\nu B_\mu)-1/2[B_\mu,B_\nu]}$$

$$(\text{using } e^x\ e^y = e^{x+y-1/2[x,y]+\dots}\ )$$

$$\approx e^{ia(\partial_\mu B_\nu-\partial_\nu B_\mu-[B_\mu B_\nu])}$$

$$= e^{ia^2 gF_{\mu\nu}} \tag{4.5}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - g[A_\mu,A_\nu] \ . \tag{4.6}$$

Now

$$\mathrm{Tr}\{U\ U\ U\ U\} = \mathrm{Tr}\left(1 + i\ a\ gF_{\mu\nu} - \frac{a^4g^2}{2}\ (F_{\mu\nu})^2\ \dots\right)$$

$$= -a^4g^2\left(F^a_{\mu\nu}\cdot F^a_{\mu\nu}\right)/4 \ . \tag{4.7}$$

So $$\sum_n \sum_{\mu\nu} \frac{1}{2g^2} \mathrm{Tr}(U\ U\ U\ U) \simeq -\int d^4x \frac{1}{4} F_{\mu\nu}F_{\mu\nu} , \tag{4.8}$$

which is the usual continuum action and is proportional to the square of the distance of the group elements from the identity.

The simplest quantity to study with some physical significance in a pure gauge theory is the vacuum expectation value of a Wilson loop or product of the U's around a closed curve. If we take a rectangular loop of sides T,R then the expectation value will give the vacuum-vacuum amplitude in the presence of a quark-antiquark pair from which we can read off the extra energy the pair have above the vacuum:

$$\langle \Pi U \rangle = e^{-TV(R)} \tag{4.9}$$

V(R) = potential energy of the quark-antiquark pair at distance R. If there is a linear confining potential between the particles then

$$V(R) = \sigma R + \mathrm{const} , \tag{4.10}$$

where σ is the "string tension", i.e.

$$\langle \Pi U \rangle = e^{-\sigma RT + c(R+T) + \mathrm{const}} \tag{4.11}$$

or the logarithm of the Wilson loop has an area term - the string tension, a perimeter term which is the quark self energy and possibly a constant term. One can determine the string tension by Monte-Carlo procedures which I will now attempt to describe. We want to evaluate the integral

$$\frac{\int d[U]\ \Pi\ U\ e^{-\beta S(U)}}{\int d[U]\ e^{-\beta S(U)}} = \langle \Pi U \rangle \tag{4.12}$$

by a stochastic method. One might try choosing sets of U's at random and weighting the result by $e^{-\beta S}$. This is very inefficient and a better procedure is to let the action term play a role in the choice of the U's.[10] We will choose a sequence of configurations U by a Markov process governed by a master equation for the probabilities $P_n(U)$ at step n:

$$P_{n+1}(U) = \sum_{U' \neq U} P(U' \to U) P_n(U') + P(U \to U) P_n(U) . \tag{4.13}$$

Write

$$P(U \to U) = 1 - \sum_{U' \neq U} P(U \to U')$$

so that

$$P_{n+1}(U) = P_n(U) + \sum_{U \neq U'} \Big(P(U' \to U) P_n(U') - P(U \to U') P_n(U) \Big) . \tag{4.14}$$

We want to choose the transition probabilities $P(U \to U')$ so that as n increases the P_n converge to a stationary distribution:

$$P(U) = e^{-\beta S(U)} \Big/ \sum_U e^{-\beta S(U)} . \tag{4.15}$$

We can arrange this by making each term in the sum on U in (4.14) vanish (the principle of "detailed balance"), i.e.

$$\frac{P(U'\to U)}{P(U\to U')} = \frac{P(U)}{P(U')} = e^{-\beta\left(S(U)-S(U')\right)} \;. \qquad (4.16)$$

There are two popular methods for satisfying (4.16):

1) The Metropolis algorithm[11]

One starts by generating a table of about 50 matrices chosen randomly from the gauge group. No particular distribution for these matrices is needed but one hopes that between them they will generate the whole group. (Just in case they do not, one occasionally generates a new set.) Then one proceeds, one link at a time, to try replacing the U on that link by U multiplied by one of the fifty matrices. If this new U' leads to a configuration with lower action, then one goes ahead and replaces U by it. If this U' leads to a larger action S(U') one may still accept it but with a probability

$$P = e^{-\beta\left(S(U')-S(U)\right)} \;.$$

If one does not accept the change you just leave the original U in place and move on to the next link.

2) The heat bath[12]

This is similar to the Metropolis algorithm in that one proceeds one link at a time but now one chooses a new U' with no reference to the old value on the link. One chooses U' with a probability proportional to:

$$e^{-\beta S(U')} \;.$$

It is easy to check that both these procedures satisfy the detailed balance principle.

Suppose now that we have iterated our Markov chain enough times that P_n has settled down to its asymptotic distribution. We use this to estimate the Wilson loops by evaluating them for a sequence of configurations and taking the time average

$$\overline{W} = \frac{1}{N}\sum_{1}^{N} W(U_n) \;. \qquad (4.18)$$

As $N \to \infty$ this should become equal to the expectation value of W with respect to the distribution P (this is the content of the ergodic theorem). One can easily see that $\overline{W}$ is an unbiased estimator for $\langle W\rangle$:

$$\langle \overline{W}\rangle = \frac{1}{N}\sum_{1}^{N} \langle W(U_n)\rangle = \langle W\rangle \qquad (4.19)$$

since the distribution for all the U_n is equal to P. Provided that the $W(U_n)$ are only correlated over a finite number of Markov steps:

$$\langle W(U_n)W(U_{n'})\rangle - \langle W\rangle^2 \propto e^{-|n-n'|/\tau_o} \tag{4.20}$$

one can see that the variance of $\overline{W}$ tends to zero:

$$\langle(\overline{W} - \langle\overline{W}\rangle)^2\rangle = \frac{1}{N^2}\sum_{n,n'}\left\{\langle W(U_n)W(U_{n'})\rangle - \langle W\rangle^2\right\}$$

$$\simeq \tau_o/N \ . \tag{4.21}$$

In this case we see that, almost certainly, as $N \to \infty$ the time average of the $W(U_n)$ is equal to the ensemble average (4.12). We also find that the errors decrease as $N^{-1/2}$ provided we take independent samples, i.e. at intervals greater than τ_o. Unfortunately near a critical point τ_o becomes large and actually at a critical point the correlations decay algebraically making the errors decrease more slowly than $N^{-1/2}$.

To estimate the string tension it is convenient to take combinations of Wilson loops in which the constant term and the perimeter term cancel out. For example:

$$X(I,J) = -\ell n\left(\frac{W(I,J)\ W(I-1,J-1)}{W(I,J-1)\ W(I-1,J)}\right)$$

$$= \sigma \tag{4.22}$$

provided W is of the form (4.11). Using this combination estimates of the string tension have been made for values of the couplings sufficiently small that σ scales as any quantity of dimensions $(\text{mass})^2$ should:

$$\sigma = c\ \Lambda_L^2 \tag{4.23}$$

where

$$\Lambda_L^2 = \frac{1}{a^2}(\beta_o g^2)^{-\beta_1/\beta_o^2}\ e^{-1/\beta_o g^2} \tag{4.24}$$

$$\beta_o = \frac{11}{3}\left(\frac{N}{16\pi^2}\right) \qquad \beta_1 = \frac{34}{3}\left(\frac{N}{16\pi^2}\right)^2 \tag{4.25}$$

for SU(N). By fitting (4.23), (4.24) to the Monte-Carlo data one can estimate c and find[13]:

$$\sqrt{\sigma} = (79 \pm 12)\ \Lambda_L \qquad SU(2)$$

$$\sqrt{\sigma} = (220 \pm 66)\ \Lambda_L \qquad SU(3)\ . \tag{4.26}$$

As this stands it is not much use but fortunately one can calculate the connection between the Λ_L parameter and the Λ parameters for continuum schemes just as we did in Lecture 2[7]:

$$\Lambda^{mom}/\Lambda_L = 83.5 \text{ in } SU(3)\ . \tag{4.27}$$

Now one also believes that

$$\sqrt{\sigma} \sim 450 \text{ MeV} \tag{4.28}$$

from either potential models of quarkonia or from string models where there is a connection between σ and the Regge slope α'

$$\frac{\alpha'}{2\pi} = \sigma \ . \tag{4.29}$$

From (4.26), (4.27), (4.28), we find an estimate for Λ^{mom}

$$\Lambda^{mom} = 180 \text{ MeV} \tag{4.30}$$

which is roughly consistent with what is seen in deep inelastic scaling violation.

LECTURE 5. FERMIONS

We now come to the problem of incorporating fermions. Putting fermions on a lattice is a little tricky and there are a number of problems that have not yet been solved in a really satisfactory manner.

We will begin by looking at the "naive" fermion action:

$$S = \sum_n \left\{ \bar{\psi}(\vec{n})\psi(n) + K\,\bar{\psi}(n) \sum_\mu \gamma_\mu \left\{ \psi(n+\mu) - \psi(n-\mu) \right\} \right\} . \tag{5.1}$$

Comparing (5.1) with the usual continuum action

$$\int d^4x (\bar{\psi}\gamma^\mu \partial_\mu \psi + m\bar{\psi}\psi) \tag{5.2}$$

shows that we need to take

$$\psi_{\text{continuum}} = \sqrt{2K}\, \psi_{\text{lattice}} \tag{5.3}$$

$$m = \frac{1}{2K} . \tag{5.4}$$

I am using γ matrices which are Hermitian and satisfy

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu} . \tag{5.5}$$

We want to use this action in a functional integral. In order to get the correct Feynman rules we need to make the $\underline{\psi}$'s into Grassman variables and use the Berezin integral. So the ψ's and $\bar{\psi}$'s anticommute

$$\{\psi(\vec{n})\psi(n')\} = \{\bar{\psi}(n), \bar{\psi}(n')\} = \{\bar{\psi}(n), \psi(n')\} = 0 \tag{5.6}$$

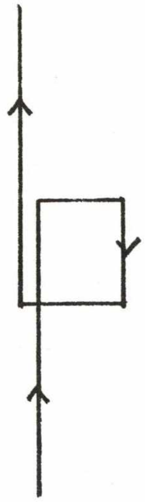
and

$$\int d(\psi) = 0 \qquad \int d(\psi)\psi = 1, \quad \text{etc.} \tag{5.7}$$

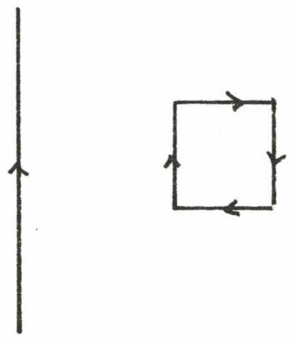
With these definitions one can see why I chose the coefficient of the $\bar{\psi}\psi$ term to be one. We can either use the $\bar{\psi}\psi$ term at each site, to satisfy the requirements for a non-zero answer, or a pair of hopping terms. In this way, just as in lecture one, we quickly see, at least as long as we do not try to use the same link twice, that we get a path expansion for the propagator:

$$\langle \psi(n')\bar{\psi}(n) \rangle = \sum_{\text{path}} K^{|L|} \prod \gamma_\mu \quad . \tag{5.8}$$

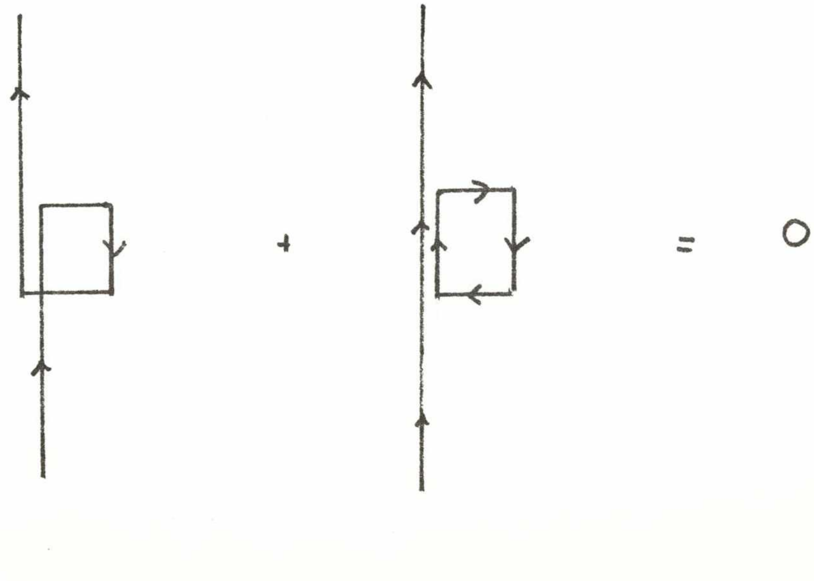
The γ_μ here is shorthand for $\pm\gamma_\mu$ depending on whether we traverse a link in the positive or negative direction. At first one may think, however, that the sum should be over self-avoiding paths or else we get in trouble with the $\psi^2 = \bar{\psi}^2 = 0$ conditions. The problem is actually illusory as there is a conspiracy between graphs contributing to the propagator and closed loop graphs contributing to the vacuum diagrams which allows us to sum over unconstrained paths in Eq. (5.8). To see how this comes about let us look at a simple example. Consider the path in the figure below:



The part of the diagram where the two lines share the same link is forbidden in the expansion, but consider also the diagram where the propagator goes straight through and there is a vacuum loop:



Again we cannot put the loop in contact with the propagator line but the closed vacuum loop has a minus sign while the figure with the coil in the propagator has a plus sign. Thus adding both forbidden diagrams changes nothing:



That this conspiracy works in general is best seen from the identity for the diagonalized action

$$Z = \int d[\bar{\psi}]d[\psi] \ \exp\left(\sum \lambda_i \bar{\psi}_i \psi_i = \Pi \lambda_i\right) \tag{5.9}$$

$$\langle \bar{\psi}_k \psi_k \rangle = \frac{1}{Z} \int d[\bar{\psi}]d[\psi]\bar{\psi}_k \psi_k \ \exp\left(\sum \lambda_i \bar{\psi}_i \psi_i\right)$$

$$= \frac{1}{\Pi \lambda_i} \left(\prod_{i \neq k} \lambda_i\right) = \frac{1}{\lambda_k} \ . \tag{5.10}$$

One can see that there is co-operation between the vacuum terms and the other integral, which leads to the propagator being the inverse of the matrix in the exponent. We can now sum Eq. (5.8) with a trick similar to the one used on the scalar fields in lecture one:

$$\sum_{\text{path}} K^{|L|}\left(\Pi \gamma_\mu\right) = \sum_{|L|} \int \frac{d^d k}{(2\pi)^d} e^{-i\vec{k}\cdot\vec{n}} \left(\sum_\mu \gamma_\mu 2i \ \sin k_\mu\right)^{|L|}$$

$$= \int \frac{d^d k}{(2\pi)^d} e^{-i\vec{k}\cdot\vec{n}} \frac{1}{1 - \sum_\mu \gamma_\mu 2i \ \sin k_\mu}$$

$$= \int \frac{d^d k}{(2\pi)^d} e^{-ik\cdot n} \frac{1 + {}_\mu \gamma_\mu 2i \ \sin k_\mu}{1 + \sum_1^4 4K^2 \ \sin^2 k_\mu} \ . \tag{5.11}$$

This looks like the continuum propagator for small k:

$$G = \int \frac{d^d k}{(2\pi)^d} e^{-ik\cdot n} \frac{(m + i\not{k})}{m^2 + k^2} \tag{5.12}$$

but there is a problem. If we were to compute, say, $\bar{\psi}\psi$:

$$\langle \psi\psi \rangle = \int \frac{d^d k}{(2\pi)^d} \frac{1}{1 + 4\not{K}^2 \sum_1^4 \sin^2 k_\mu} \tag{5.13}$$

Because $\sin\pi = 0$ we get similar contributions from many other places in the domain of integration. We actually get 2^d times the correct answer (16 time in 4 dimensions). This is the notorious "fermion doubling problem" - we thought we had only one species of fermion in our sums but we turn out to have 16.

There are several ways that have been proposed to circumvent this multiplication of degrees of freedom:

1. Wilson Fermions[1]

We replace the $\pm\gamma_\mu$ factors in the hopping terms by $1 \pm \gamma_\mu$. Then

$$G = \int_{-\pi}^{+\pi} \frac{d^d k}{(2\pi)^2} e^{-ik\cdot n} \frac{\left(1 + 2ik \sum \gamma_\mu \ \sin k_\mu + 2k \sum \cos k_\mu\right)}{\left(1 - 2\sum_\mu \ \cos k_\mu\right)^2 + 4k^2 \sum_1^4 \ \sin k_\mu} \ . \tag{5.14}$$

2. Staggered or "Kogut-Susskind" fermions[14]

These are a little more arcane. The best route to understanding them is due to Kawamoto and Smit.[15] We replace ψ by ψ' where

$$\psi(n_1 \cdots n_4) = \gamma_4^{n_4}\,\gamma_3^{n_3}\,\gamma_2^{n_2}\,\gamma_1^{n_1}\,\psi'(n_1 \cdots n_4)\ .$$

$$\bar\psi(n_1 \cdots n_4) = \bar\psi'(n_1 \cdots n_4)\gamma_1^{n_1}\,\gamma_2^{n_2}\,\gamma_3^{n_3}\,\gamma_4^{n_4}$$

so (5.1) becomes

$$S = \sum_n \left\{\bar\psi'(n)\psi'(n) + (-1)^{\phi(\mu,n)}\,\bar\psi'(\vec n)\{\psi(\vec n+\vec\mu) - \psi(\vec n-\vec\mu)\}\right\}\ .$$

The phases $(-1)^{\phi(\mu,n)}$ arise from commuting the γ's through one another, e.g.

$$\begin{aligned}\bar\psi(n)\gamma_1\ \{\psi(n+1) - \psi(n-1)\} &= \bar\psi'(n)\ \gamma_1^{n_1}\,\gamma_2^{n_2}\,\gamma_3^{n_3}\,\gamma_4^{n_4}(\gamma_1)\{\gamma_4^{n_4}\,\gamma_3^{n_3}\,\gamma_2^{n_2}\,\gamma_1^{n_1+1}\ \psi(\vec n+\vec 1) \\ &\quad - \gamma_4^{n_4}\,\gamma_3^{n_3}\,\gamma_2^{n_2}\,\gamma_1^{n_1+1}\ \psi'(n-1)\} \\ &= (-1)^{n_2+n_3+n_4}\ \bar\psi'(n)\{\psi'(n+1) - \psi'(n-1)\}\ .\end{aligned}$$

Similarly

$$\phi(2,\vec n) = n_3 + n_4$$

$$\phi(3,\vec n) = n_4$$

$$\phi(4,\vec n) = 0$$

The only remnant of the γ algebra is now the fact that

$$\prod_{\text{plaquette}} (-1)^{\gamma} = -1$$

which arises from $\gamma^\mu\gamma^\nu\gamma^\mu\gamma^\nu = -1$ if $\mu \neq \nu$. The action is now diagonal in the spin label and if we just keep one of the spins we reduce the number of fermi species by one quarter leaving, in four dimensions, four flavours of fermions. These are the Kogut-Susskind fermions.

LECTURE 6. CHIRAL SYMMETRY-BREAKING

The Kogut-Susskind fermion action

$$S = \sum_n \left\{ \bar{\psi}(n)\psi(n) + \bar{\psi}(n) \sum_\mu (-1)^{\phi(n,\mu)} \left\{ \psi(\vec{n}+\vec{\mu}) - \psi(\vec{n}-\vec{\mu}) \right\} \right\} \quad (6.1)$$

has a number of symmetries. Two of them are continuous:

(a) "Vector" symmetry

$$\begin{aligned} \psi(n) &\to e^{i\theta}\, \psi(n) \\ \bar{\psi}(n) &\to e^{-i\theta}\, \bar{\psi}(n) \ . \end{aligned} \quad (6.2)$$

(b) "Axial" symmetry (only good if m = 0)

$$\left.\begin{aligned} \psi(n) &\to e^{i\theta}\, \psi(n) \\ \bar{\psi}(n) &\to e^{i\theta}\, \bar{\psi}(n) \end{aligned}\right\} \quad \vec{n} \text{ odd (i.e. } \sum n_i = \text{odd)}$$

$$\left.\begin{aligned} \psi(n) &\to e^{-i\theta}\, \psi(n) \\ \bar{\psi}(n) &\to e^{-i\theta}\, \bar{\psi}(n) \end{aligned}\right\} \quad \vec{n} \text{ even} \ . \quad (6.3)$$

The first symmetry simply leads to fermion number conservation (i.e. continuity of world lines) while the second is only good when $K \to \infty$ and the world-line picture is no use. This second symmetry can be broken spontaneously and leads to the existence of massless Goldstone bosons - "pions".

To discuss these symmetries it is easiest to go back to the naive fermion language where the "axial" symmetry takes the form

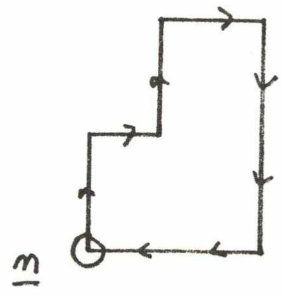
$$\begin{aligned} \psi &\to e^{i\theta\gamma_5}\, \psi \\ \bar{\psi} &\to \bar{\psi}\, e^{i\theta\gamma_5} \ . \end{aligned} \quad (6.4)$$

(It should be said that this formula is a trifle deceptive because the actual symmetry, in terms of the flavours that are really present, is <u>not</u> the U(1) axial symmetry. It is a U(1) subgroup of the axial symmetry but in a combination with a flavour matrix which is anomaly free.) From this one can derive a Ward identity by the usual methods. I shall, following the general spirit of the lectures, prove it by a diagrammatic argument.

The Ward identity is

$$\langle\bar{\psi}(n)\psi(n)\rangle = \sum_{n'} \langle\bar{\psi}(n')\gamma_5\psi(n')\ \bar{\psi}(n)\gamma_5\psi(n)\rangle \ . \quad (6.5)$$

To prove it consider a graph contributing to $\langle\bar{\psi}\psi\rangle$:



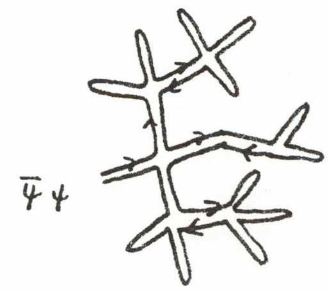
Insert a γ_5 matrix at $\vec{n}$ and another at $\vec{n}'$ on the loop and slide n' round the loop. Since γ_5 anticommutes with the γ_μ this gives rise to an alternation in sign. On adding all the terms they cancel except for one (where the two γ_5's are coincident at $\vec{n}$. In this way we get the LHS of (6.5).) The terms which are being added can also be regarded as contributions to the RHS of (6.5). By summing over all loops through n we get the whole of the (6.5).

Rescaling the fields by $\sqrt{2K}$, as in the last lecture, to make contact with the continuum field normalization gives:

$$\langle\bar{\psi}\psi\rangle = m\langle\bar{\psi}\gamma^5\psi,\ \bar{\psi}\gamma^5\psi\rangle_{p=0}\ . \tag{6.6}$$

If $\langle\bar{\psi}\psi\rangle \neq 0$ and we take $m \to 0$ we must have a divergence in the zero momentum Green's function which will be caused by a zero mass pion.

There is a nice argument due to Brout and Englert[16] which enables one to see how $\bar{\psi}\psi$ becomes nonzero at strong coupling. When $g^2 \to 0$ the quarks and antiquarks have to pair up on each link so the diagrammatic expansion for $\bar{\psi}\psi$ is a sum of rooted trees:



We can sum these recursively

$$(NC)^{-1}\ \langle\bar{\psi}\psi\rangle = \frac{1}{1 - x\langle\bar{\psi}\psi\rangle(NC)^{-1}}\ . \tag{6.7}$$

The factors of N, C are the number of fermi components and the number of colours, respectively. The x is the factor for each link of the tree. Since each link has a $+\gamma_\mu$ and a $-\gamma_\mu$ and can point in any of 2d directions we see that

$$x = -2dK^2\ . \tag{6.8}$$

Thus:

$$\langle\bar{\psi}\psi\rangle = \frac{NC}{4dK^2}\left\{\sqrt{1+8K^2} - 1\right\} . \tag{6.9}$$

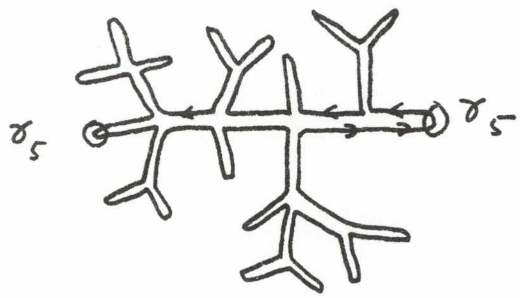
Again rescaling by $\sqrt{2K}$ and putting $m = (2K)^{-1}$ we can rewrite this as

$$\langle\bar{\psi}\psi\rangle = \frac{NC}{d}\left\{\sqrt{m^2+2d} - m\right\} \tag{6.10}$$

and as $m \to 0$ we find that (6.10) goes to a finite limit of

$$\langle\bar{\psi}\psi\rangle_{n=0} = NC\sqrt{\frac{2}{d}} . \tag{6.11}$$

One can also see the existence of the zero mass pion in this limit. Diagrammatically the pion looks like



We can sum this by noticing that the effect of the γ^5's at the ends cancels the (-1) for all the links in the "backbone" of the graph. Thus:

$$\langle\bar{\psi}\gamma_5\psi\ \bar{\psi}\gamma_5\psi\rangle = NC\left(\frac{\langle\bar{\psi}\psi\rangle}{NC}\right)^2 \sum_{\text{path}} \left[K^2\left(\frac{\langle\bar{\psi}\psi\rangle}{NC}\right)^2\right]^{|L|}$$

$$= NC\left(\frac{\langle\bar{\psi}\psi\rangle}{NC}\right)^2 \int \frac{d^dk}{(2\pi)}\, e^{ik\cdot n}\, \frac{1}{1 - K^2\left(\frac{\langle\bar{\psi}\psi\rangle}{NC}\right)^2 \sum_1^d (2\cos k_i)} . \tag{6.12}$$

When $m \to 0$ (or equivalently $K \to 0$) this goes on to

$$G_\pi(0,n) \propto \int \frac{d^dk}{(2\pi)^d}\, e^{ik\cdot n}\, \frac{1}{1 - 1/2d \sum_1^d 2\cos k_i} \tag{6.13}$$

which is the propagator for a massless particle.

A number of groups[17] have performed Monte-Carlo simulations of chiral symmetry-breaking in the "quenched approximation" where the fermion determinant is ignored. In this case the chiral symmetry-breaking order parameter $\langle\bar{\psi}\psi\rangle$ is obtained from the inverse of the Dirac operator in the background gauge field:

$$\langle\bar{\psi}\psi\rangle = \mathrm{Tr}\ G(x\ x) \tag{6.14}$$

$$= \mathrm{Tr}\ \langle\frac{1}{\not{\partial} + m}\rangle\ . \tag{6.15}$$

In (6.15) the angular brackets can be read as averaging over gauge configurations produced with the Wilson action. If the eigenvalues of the Euclidean Dirac operator are $i\lambda_n$ we can rewrite (6.15) as

$$\langle\bar{\psi}\psi\rangle = \frac{1}{V}\sum_n \frac{1}{i\lambda_n + m}\ . \tag{6.16}$$

As V becomes large the poles in (6.16) merge to form a branch cut and we can regard the large V limit of (6.16) as a dispersion relation

$$\langle\bar{\psi}\psi\rangle = \int d\lambda\ \frac{\rho(\lambda)}{i\lambda + m}\ , \tag{6.17}$$

where $\rho(\lambda)$ is the density of states with eigenvalue λ. The symmetry breaking now comes about because the integral is discontinuous across the branch cut (i.e. as m changes from positive to negative) and we find

$$\langle\bar{\psi}\psi\rangle_{m=0} = \pi\rho(0)\ . \tag{6.18}$$

This branch cut is in evidence in Eq. (6.10) because of the square root sign. We can interpret Eq. (6.10) in the light of these remarks as a calculation of the strong coupling density of eigenvalues of the Dirac operator. We find

$$\rho(\lambda) = \frac{NC}{d\pi}\sqrt{2d-\lambda^2}\ . \tag{6.19}$$

This can be compared with the free theory where $\lambda = |k|$ and

$$\rho(\lambda)d\lambda = \frac{d^4k}{(2\pi)^4} = |k|^3\ \frac{dk}{(2\pi)^4} \tag{6.20}$$

so $\rho(\lambda) \propto \lambda^3$ for small λ. In the quenched strong coupling calculation the symmetry-breaking comes about because the eigenvalues slump towards $\lambda = 0$ to make up the semicircle distribution which seems characteristic of many random matrix problems.

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