TRIUMF

STUDIES A PROPOS $\left\{ \begin{matrix} \mu^- p \rightarrow \nu n \gamma \\ \pi \rightarrow e \bar{\nu} \gamma \end{matrix} \right\}$ EXPERIMENTS

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Introduction

The purpose of this report is to examine the reactions

$$1)\quad \mu^- p \to \nu n \gamma$$

$$2)\quad \pi \to e\bar{\nu}\gamma$$

with an aim to clarifying the feasibility and most interesting kinematic domains for possible experiments.

The main interest in the μ^- radiative capture is the possibility of studying the 'induced pseudoscalar' weak interaction in a kinematic domain where it is relatively enhanced compared to other weak couplings. This will be seen in subsequent discussion to occur for the largest possible photon energies; our aim is to see to what extent the experimental sensitivity to this coupling varies with photon energy. We also shall explore the theoretical uncertainties in constructing a gauge invariant amplitude for this process: an ambiguity will be elucidated which seems to have escaped consideration in the literature.

For the radiative π decay, interest has generally focused on determining so-called 'structure dependent' vector and axial vector weak couplings which cannot be observed in the non-radiative decay. Here we wish to explore other possible 'aberrations of nature' to which experiment might be sensitive, particularly the presence of off-shell effects in the weak interaction and of tensor currents.

1. $\mu^- p \to \nu n \gamma$

1.1 Kinematics

We shall be concerned with the radiative decay rate for $\mu^- p \to \nu n\gamma$ at fixed photon (γ) and neutron (n) angles, as a function of photon energy k. This rate we denote by

$$\frac{d^5\Gamma}{d\Omega_\gamma d\Omega_n dk}\,; \tag{1.1}$$

it will be calculated for the initial μ having $\approx$ zero three momentum (in an atomic state).

The 4-momenta of the n,γ,ν and initial state are denoted by

muon	:	q	mass μ
neutron	:	p′	mass M (energy E′, momentum p′)
photon	:	k	
neutrino	:	K	
μ^- + p	:	P	

The angle between the neutron and photon will be called $\theta p'k$; its cosine and sine we denote by x and y, respectively.

By making use of the fact that (suppressing Lorentz indices)

$$K^2 = (P-p'-k)^2 \equiv 0 \tag{1.2}$$

we obtain, setting $W = M + \mu$,

$$2kp'x = W^2+M^2-2Wk - 2E'(W-k). \tag{1.3}$$

This can be solved for E′:

$$E'(k,x) = \frac{1}{2\left((W-k)^2 - k^2x^2\right)}\Big((W-k)(W^2+M^2-2kW)$$

$$\mp kx\left((W^2-M^2-2kW)^2 - 4M^2k^2y^2\right)^{\frac{1}{2}}\Big). \tag{1.4}$$

One can show that for $\theta'_{pk} \leqslant \pi/2$, only the - sign gives a valid solution. However, for $\theta'_{pk} > \pi/2$ *and* for sufficiently large k, both solutions are valid. This is an important point! (The reader can easily see that this is so by considering a special case, namely, $\theta_{p'k} = \pi$, k large, K small.)

Now that we have $E'(k,x)$ we trivially obtain K by energy conservation:

$$K = W - E' - k . \tag{1.5}$$

All momenta are now fully specified; relevant angles are easily obtained using momentum conservation.

To find the maximum k ($\equiv k_{max}$) at given x, we maximise k as given by Eq. (1.3). After some tedious algebra, we obtain the result

$$\theta_{p'k} \leqslant \pi/2 \qquad k_{max} = \mu/2$$

$$\theta_{p'k} > \pi/2 \qquad k_{max} = \frac{W^2-M^2}{2(W+yM)} . \tag{1.6}$$

For this latter value of k, the $(\ )^{\frac{1}{2}}$ in Eq. (1.4) vanishes, and both kinematic solutions are identical. In this case, we obtain

$$E'(k_{max},x) = \frac{M(W^2+M^2+2WMy)}{2WM + y(W^2+M^2)} \tag{1.7a}$$

$$p'(k_{max},x) = \frac{-xM(W^2-M^2)}{2WM + y(W^2+M^2)} . \tag{1.7b}$$

(Physically, for $\theta_{p'k} \leqslant \pi/2$, the photon energy is maximised when the neutron is at rest, with the γ and ν sharing the remaining energy = μ.)

Finally, it is useful to look at the relevant momentum transfers in this problem; we list the results for the various amplitudes for γ emission:

$$\mu \text{ emission} \quad Q^2 = (p' - p_{proton})^2 = 2M(M-E') < 0$$

$$\text{n,p emission} \quad Q^2 = (p' - p_{proton} + k)^2$$

$$\equiv (k - q)^2 = \mu(\mu-2k) \lesseqgtr 0. \tag{1.8}$$

Thus the hadron emission occurs with a relatively enhanced induced pseudoscalar coupling, since this coupling has a pole at $Q^2 = {}^{+}m_\pi^2$.

1.2 Phase space

If the initial state of $\mu^- p$ (or $p\mu p$ in liquid H_2) has wave function $\phi(\vec{r_\mu - r_p})$, then the decay rate is given by

$$\Gamma = \frac{|\phi(0)|^2}{(2\pi)^5\, 4\mu M} \int \frac{d^3k\, d^3K\, d^3p'}{8kKE'}\, \delta^4(p'+k+K-P) \left(\frac{1}{4}\sum_{\text{spins}} |T|^2\right) \tag{1.9}$$

and T is the usual Lorentz invariant transition amplitude.

Thus we obtain

$$\left.\frac{d^5\Gamma}{d\Omega k\; d\Omega p'\; dk}\right|_{\text{unpol.}} = \frac{|\phi(0)|^2}{(2\pi)^5\; 32\mu M} \frac{k(p')^2}{X} \left(\frac{1}{4}\sum_{\text{spins}} |T|^2\right) \tag{1.10}$$

where

$$X = KE'\left(\frac{p'}{E'} + \frac{p'+kx}{K}\right) \tag{1.11a}$$

$$\equiv KE'\left(\frac{p'}{E'} - \cos\theta_{Kp'}\right) \tag{1.11b}$$

$$\equiv Kp' + E'(p'+kx) \tag{1.11c}$$

$$\equiv p'(W-k) + E'kx\;. \tag{1.11d}$$

Straightforward algebra then reveals that (except for $x = -1$) when $\theta_{p'k} > \pi/2$

$$X\left(k_{max}, x\right) = 0\;. \tag{1.12a}$$

This singularity of the k-spectrum no longer remains if we perform an averaging over x as does any finite experimental detector. We examine this now in more detail, by considering $X\left(k_{max}^{(x_o)}, x\right)$ for x near x_o. Clearly, as $x \to -1$, $k_{max}^{(x)}$ increases; this has a corollary for $\theta_{p'k} \geq \pi/2$:

$$\text{Phase space for } k_{max}^{(x_o)} \text{ vanishes for } x > x_o. \tag{1.12b}$$

To proceed further, we consider

$$\left.\frac{\partial X}{\partial x}\right|_k = \frac{k^2 x M^2}{X}\;. \tag{1.13}$$

This result is obtained by using kinematics Eq.(1.3) for p', phase space Eq.(1.11d) for X, and simply differentiating. It follows that at constant k,

$$X(x,k) = kM\sqrt{x^2+a^2}, \text{ a constant.} \tag{1.14}$$

Thus for $k = k_{max}^{(x_o)}$

$$X\left(k_{max}^{(x_o)},x\right) = k_{max}^{x_o}\, M\sqrt{x^2-x_o^2}\,. \tag{1.15}$$

We now consider a detector which measures (with equal weight) a range of x denoted by 2Δ, centred at x_o. The counting rate (for 100% efficiency) at energy $k_{max}^{(x_o)}$ is then given by

$$R = \frac{|\phi(0)|^2\, k_{max}^{(x_o)}}{(2\pi)^5\, 32\mu M}\cdot 2\pi \int_{x_o}^{x_o-\Delta} \frac{(p')^2\, |T|^2\, dx}{k_{max}^{(x)}\, M\sqrt{x^2-x_o^2}}$$

$$\approx \frac{|\phi|^2\, (p')^2\, |T|^2_{k_{max},x_o}}{(2\pi)^5\, 32\mu M^2\sqrt{2x_o}}\cdot 2\pi \int_{x_o}^{x_o-\Delta} \frac{dx}{\sqrt{x-x_o}}\,. \tag{1.16}$$

The integral is simply $2\sqrt{\Delta}$; we used Eq.(1.12b) to eliminate half the range of integration. The 'averaged $d\Gamma$' is then given by

$$\overline{\left(\frac{d^5\Gamma}{d\Omega\, d\Omega\, dk}\right)} = \frac{R}{2\pi\cdot 2\Delta} = \frac{|\phi(0)|^2 (p')^2\, |T|^2_{k_{max}}}{(2\pi)^5\, 32\mu M^2\, \sqrt{2\Delta x_o}}\,. \tag{1.17}$$

which is finite for finite angular resolution.

1.3 Matrix elements

For capture from an atomic state $\phi\left(r_{\mu p}\right)$, we have a matrix element

$$\overline{T} \sim \phi(0)\left\langle n\nu k|T|\ \mu(\vec{q}=0),\ p(\vec{p}=0)\right\rangle. \tag{1.18}$$

We now need to look at various contributions to the momentum space amplitude above. First we write the np weak current:

$$\left(J_{np}^{weak}(Q)\right)_{\mu} = \overline{U}_n \, \hat{O}_\mu \, U_p \tag{1.19}$$

$$\hat{O}_\mu(Q) = \gamma_\mu F_v^E(Q^2) - \frac{(\gamma_\mu \not{Q} - \not{Q}\gamma_\mu)}{4M} F_v^M(Q^2) + \gamma_\mu \gamma_5 g_A(Q^2) + \gamma_5 Q_\mu g_p(Q^2) \tag{1.20}$$

where

a) $1+\gamma_5$ is the left-handed helicity projection operator

b) $F_v^{E,M}$ are the isovector nucleon charge and magnetic moment form factors with $F_v^E(0) = 1 \quad F_v^M(0) = 3.7$

c) $g_A(0) = 1.2$

d) $g_p(Q^2)$ is the induced pseudoscalar coupling constant, expected to be dominated by the π pole. We take

$$g_p(Q^2) = \frac{2Mg_A(0)}{m_\pi^2 - Q^2} \times r. \tag{1.21}$$

e) $Q = p' - p$

and where $r = 1$ according to theory.

The amplitude for μ external emission is then (Fig. 1.1):

$$T_1 = \overline{U}(\nu)\,\gamma_\mu(1+\gamma_5)(\not{q} - \not{k} + \mu)\not{\varepsilon}\,U(\mu)\Big/2q\cdot k$$

$$\times \left(J_{np}^{weak}(p'-p)\right)_\mu \times \left(\sqrt{4\pi\alpha}\,\frac{G}{\sqrt{2}}\right) \tag{1.22}$$

where

$$\varepsilon = \gamma \text{ polarization 4-vector}$$

$$\alpha = \frac{1}{137}$$

$$G = 10^{-5}/M^2 .$$

The amplitudes for n,p emission are then (Figs. 1.2, 1.3):

$$T_2(n) = \sqrt{4\pi\alpha}\,\frac{G}{\sqrt{2}}\left[L_\mu\,\overline{U}_n\,\frac{\not{\varepsilon}\not{k}}{2M}\,\frac{\not{p}'+\not{k}+m}{2k\cdot p'}\,\hat{O}_\mu(p'+k-p)U_p\right.$$

$$\left.\times F_n^M\left((p'+k-p)^2\right)\right] =$$

$$T_2(p) = \sqrt{4\pi\alpha}\,\frac{G}{\sqrt{2}}\left[L_\mu\,\overline{U}_n\,\hat{O}_\mu(p'+k-p)\,\frac{\not{p}-\not{k}+m}{-2k\cdot p}\left(\not{\varepsilon} + \frac{\not{\varepsilon}\not{k}}{2M}F_p^M\right)U_p\right] \tag{1.23}$$

where

$$L_\mu = \overline{U}(\nu)\ \gamma_\mu(1+\gamma_5)\ U(\mu), \tag{1.24}$$

and $F^M_{n,p}$ are the anomalous moment form factors of n,p, with

$$F_n(0) = -1.91$$

$$F_p(0) = +1.79.$$

Note that

$$F^M_V(Q^2) \equiv F^M_p(Q^2) - F^M_n(Q^2).$$

We shall assume that $F^{E,M}_V$, $F^M_{n,p}$, g_p are constant in the range of momentum transfers occurring in this reaction. If we write

$$T = \sum_{i=1}^{3} T_i = \sqrt{4\pi\alpha}\ \frac{G}{\sqrt{2}}\,\varepsilon\cdot J, \tag{1.25}$$

then the divergence of this current is explicitly

$$\begin{aligned} k\cdot J &= L_\mu \left\{\left(J^{weak}_{np}(p'-p)\right)_\mu - \left(J^{weak}_{np}(p'+k-p)\right)_\mu\right\} \\ &= L_\mu \left\{\overline{U}(n)\left(\frac{F^M_V}{2M}\left(k_\mu-\gamma_\mu \not{k}\right) + g_p\left((p'-p)^2\right)\gamma_5\left(p'-p\right)_\mu\right.\right. \\ &\qquad \left.\left. - g_p\left((p'+k-p)^2\right)\gamma_5\left(p'+k-p\right)_\mu\right)U(p)\right\}. \end{aligned} \tag{1.26}$$

We now consider the construction of a 'counter-current' to be added to J, such that its divergence cancels the divergence of Eq.(1.26). The total current-conserving amplitude desired will then be

$$T = \left(\varepsilon\cdot J + \varepsilon\cdot J^{counter}\right) \times \left(\sqrt{4\pi\alpha}\,\frac{G}{\sqrt{2}}\right). \tag{1.27}$$

First, the weak magnetism counter-current gives a counter term

$$\left(\sqrt{4\pi\alpha}\ \frac{G}{\sqrt{2}}\right)\frac{F^M_V}{2M}\,L_\mu\ \overline{U}_n\left(\gamma_\mu\not{\varepsilon}-\varepsilon_\mu\right)U_p \tag{1.28}$$

unambiguously (for F^M_V = constant). However, for the induced pseudoscalar counter term there is an important ambiguity not previously noted in the literature: we may take this counter term either as

$$\overline{U}_n\gamma_5 U_p\left(\frac{g_p\left((Q+k)^2\right) - g_p(Q^2)}{k\cdot Q}\,\varepsilon\cdot Q(Q+k)_\mu + g_p(Q^2)\varepsilon_\mu\right)L_\mu \qquad (1.29a)$$

$$Q = p' - p$$

$$\overline{U}_n\gamma_5 U_p\left(\frac{g_p\left((Q+k)^2\right) - g_p(Q^2)}{k\cdot Q}\,\varepsilon\cdot Q(Q_\mu) + g_p\left((Q+k)^2\right)\varepsilon_\mu\right)L_\mu . \qquad (1.29b)$$

Both choices have the same divergence. The first is the conventional choice, resulting from treating the induced pseudoscalar term as due to emission from an internal π plus a contact term at the π-lepton vertex. The second choice corresponds to interpreting the derivative coupling at the π-lepton vertex as a derivative of the hadron fields, with the resulting contact term at the hadron vertex. The first approach is clearly implied by a microscope Lagrangian field theory. We leave this question open for now and explore it numerically later. We do not present trace expressions for

$$\sum_{\text{spins}} |T|^2$$

as we prefer to simply let the computer do our Dirac matrix algebra in constructing each independent spin amplitude.

1.4 Presentation of results

We shall present below our calculated $d^5\sigma/d\Omega d\Omega dk$ rather than integrations w.r.t. variables Ω_N or k, in the event that it might be experimentally useful to detect the final neutron in coincidence with the γ in order to reduce experimental background noise. To avoid difficulties in the visual presentation, the k_{max} point in the spectrum (but *only* this point) is calculated for an angular resolution of

$$2\Delta = 2\sin\theta\,\Delta\theta \text{ with } \Delta\theta = 0.05 \text{ rad} \qquad (1.30)$$

according to the prescription of Eq. (1.18).

We shall separately present results for the cases in which the initial state is either

a) single, preferred in the μ^-p atom because of the hyperfine interaction;

b) statistically averaged singlet, triplet, appropriate to a μ^-

bound to a nucleus described as a fermi gas of uncorrelated nucleons; or

c) doublet p-μ-p molecule.

For case d) we use the molecular ground state

$$|0\rangle = |\text{space; antisymm. in } \vec{r}p_1, \vec{r}p_2\rangle \otimes |\text{spin; symm. in } \vec{s}_1, \vec{s}_2\rangle \quad (1.31)$$

where

$$|\text{spin; doublet, } J_z\text{=m}\rangle = \frac{1}{\sqrt{6}} \Big[|\text{proton } 1 = \text{m } (\mu^- p_2)\text{singlet}\rangle$$

$$+ (\mu^- p_1)\text{singlet} \times \text{proton } 2 = \text{m}\rangle \Big] .$$

Thus

$$= \frac{1}{\sqrt{6}} |\mu^- \uparrow ;\ (\uparrow\downarrow + \downarrow\uparrow)_{\text{protons}} - 2\mu^- \downarrow (\uparrow\uparrow)_{\text{protons}} .$$

If we do not observe the spectator proton for case c) [and this is almost always the case since it emerges with almost no KE], then the capture rate is the incoherent sum of the capture rate for each proton. We use previous estimates for

$$\int d^3 r_2\ |4(\overrightarrow{r_\mu - r_{p_1}} = 0; \vec{r}_{p_2})|^2 = 0.505 \times (\mu^- \text{p atom case}). \quad (1.32)$$

The following questions seem *a priori* interesting to us:

1) Are the results sensitive to how we construct the counter term, i.e. to the choice of (1.29a) or (1.29b).

2) For what kinematic region is the spectrum maximally sensitive to the induced-pseudoscalar coupling constant.

On a more trivial kinematic level one also needs to keep in mind the relation between k_{max} and $\theta_{p'k}$; since one prefers to look at high energy γ to avoid background confusion with $\mu \to e\bar{\nu}\nu\ \gamma$. From Eq. (1.6) we see that

$$k_{max} = 80 \text{ MeV when } \theta_{p'k} = 164 \text{ deg.}$$

We shall therefore present our most detailed results only for $\theta_{p'k} \geqslant 164$ deg.

In Fig. 1.2 we indicate the various momentum transfers as a function of photon energy, at fixed $\theta_{p'k}$. In Fig. 1.3a, b, c we present the actual photon spectra for $\theta_{p'k} = 3$ rad. Note the numerical sensitivity to the counter-term construction. The singlet and $(p\text{-}\mu\text{-}p)_{1/2}$ results are considerably smaller than the triplet case [incidentally, $d^5\sigma$ for $(p\text{-}\mu\text{-}p)_{1/2}$ can be shown to equal $1/4\ d^5\sigma)_{\text{triplet}} + 3/4\ d^5\sigma)_{\text{singlet}}$] and, as experience has proven to us, therefore sensitive to any details of the amplitude. We show calculations in which the constant r which specifies the strength of the induced pseudoscalar coupling g_p is either 1 (as theoretically predicted) or 2. Also, for the $(p\text{-}\mu\text{-}p)_{1/2}$ case, we increased the strength of g_p by taking

$$g_p = \frac{2Mg_A}{\mu_\pi^2 - G^2} + \left(\frac{2Mg_A}{\mu_\pi^2} \equiv C\right). \tag{1.33}$$

This gives a result quite similar to merely doubling r, as shown in Fig. 1.3c. The main point to notice is: the $(p\text{-}\mu\text{-}p)_{1/2}$ and singlet rates are indeed increasingly sensitive to g_p as k increases.

Our calculation differs from the earlier one by Opat in that we have chosen to look at a differential rate for *fixed* gamma-neutron opening angle. This choice was motivated by the possible desirability of detecting the neutron as well as the gamma, in order to decrease background noise experimentally. A typical anticipated counting rate is presented below; we assume

a) $(d^5\sigma)$ 'standard' $= 10^{-3}/(\text{sec} - 10\ \text{MeV} - \text{sr}^2)$.

Thus, fraction of stopped μ's radiatively decaying (per sr^2 - 10 MeV) equals

$$\frac{10^{-3}}{1/(2.2 \times 10^{-6})} = 2.2 \times 10^{-9}.$$

Next we assume

b) 10 γ-detectors, each 10×10 cm at 2/3 m from the target, i.e.

$$(\Delta\Omega)_\gamma = 9/40\ \text{sr}$$

c) Neutron detector (large enough to encompass much of the cone of neutrons associated with high energy γ's) has $(\Delta\Omega)_n = 0.1$ sr

d) 100% efficient detectors

e) 10 MeV 'bin' for γ's

f) 10^7 stopped μ's per second

This gives R = 43 events/day.

Thus, for 70-90 MeV γ's (i.e. 20 MeV bin), according to Fig. 1.3c for $r(g_p) = 1$, $R \approx 40$ events/day.

$\mu^- p \to \nu n \gamma$ Bibliography

The most recent $\mu^- p \to \nu n \gamma$ calculation that the author knows of is

Geoffrey I. Opat, Phys. Rev. 134, B428 (1964).

The present work overlaps very much with this calculation but hopefully presents some material more explicitly useful to experimental design estimates.

Acknowledgements

I thank Dr. B.M.K. Nefkens (UCLA), J-M Poutissou and M. Hasinoff (UBC) for arousing my interest in these radiative processes; I also thank Dr. D. Smith (UBC) for assistance in checking the ($\mu^- p \to \ldots$) calculations.

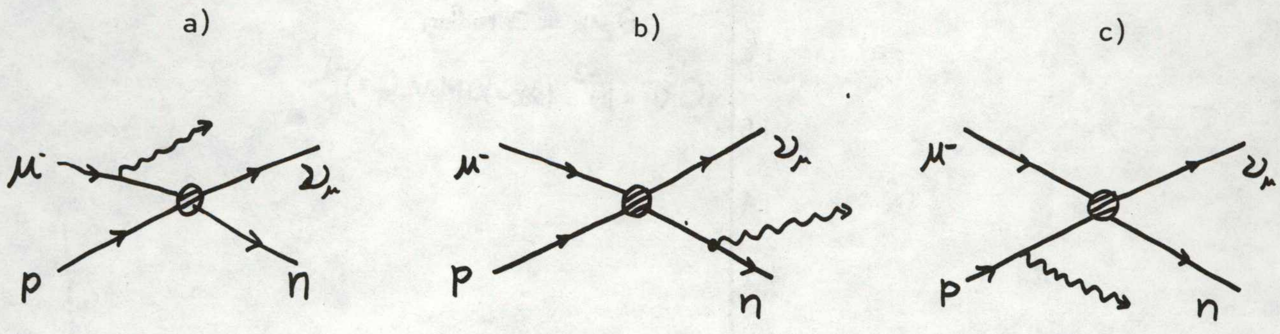


Fig. 1.1. External emission diagrams.

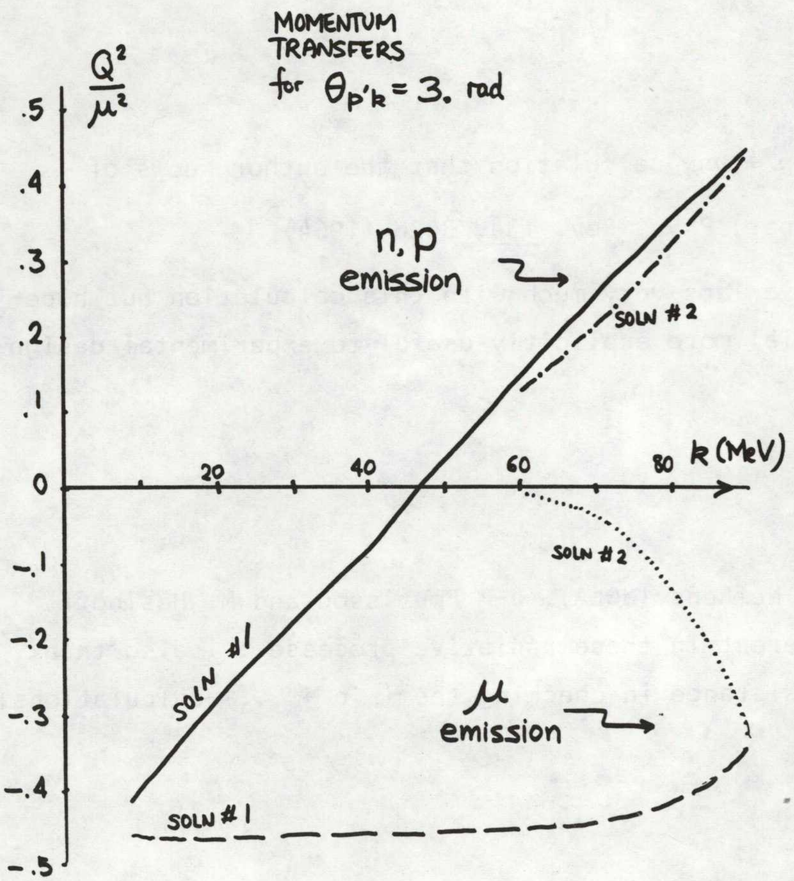


Fig. 1.2

Momentum transfers vs photon energy.

a)

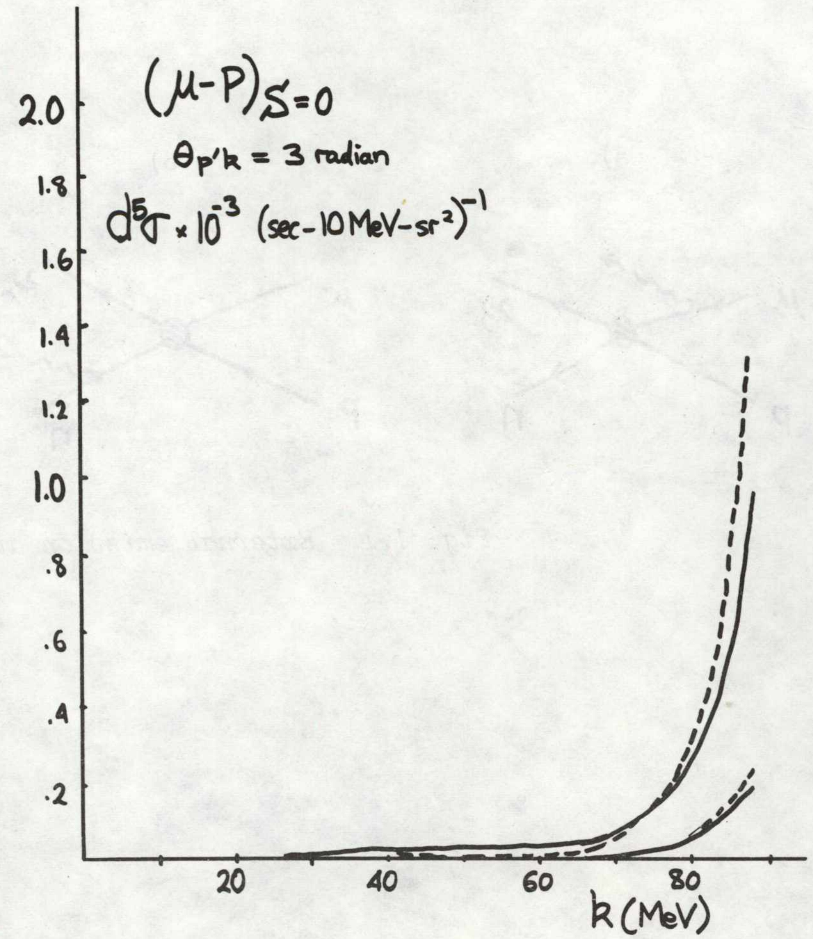
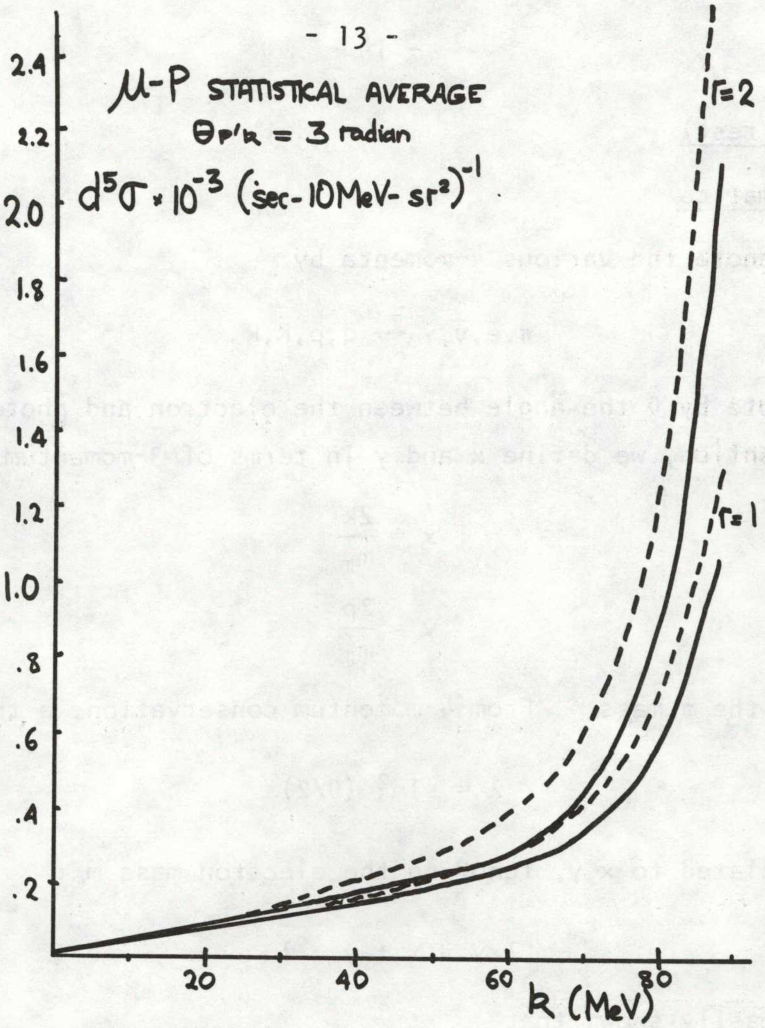


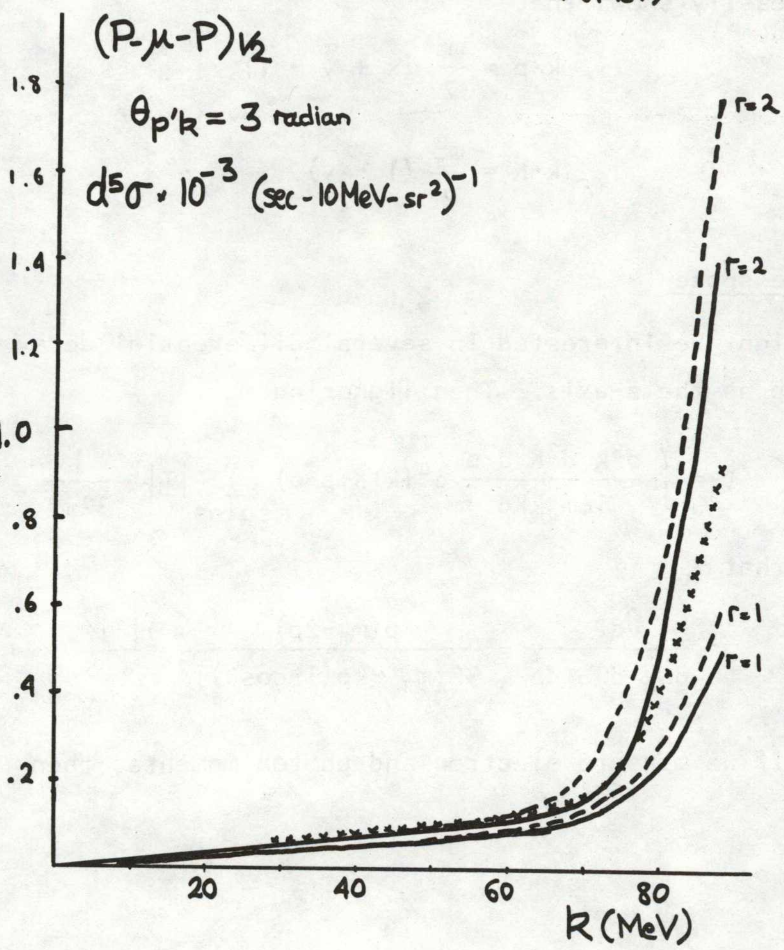
Fig. 1.2

Photon spectra. In these figures solid lines refer to the conventional counter term (Eq. 1.29a) while dotted lines refer to the alternative (Eq. 1.29b). In c) the crosses are for the conventional counter term, but with g_P as described in Eq. 1.33.

b)


μ-P STATISTICAL AVERAGE
θp'k = 3 radian
d⁵σ × 10⁻³ (sec-10MeV-sr²)⁻¹
2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
.8
.6
.4
.2
20
40
60
80
k (MeV)
r=2
r=1

c)


(P-μ-P)1/2
θp'k = 3 radian
d⁵σ × 10⁻³ (sec-10MeV-sr²)⁻¹
1.8
1.6
1.4
1.2
1.0
.8
.6
.4
.2
20
40
60
80
k (MeV)
r=2
r=2
r=1
r=1

2. $\pi \to e\bar{\nu}\gamma$ (at rest)

2.1 Kinematics

We denote the various 4-momenta by

$$\pi, e, \bar{\nu}, \gamma \leftrightarrow q, p, K, k.$$

We also denote by θ the angle between the electron and photon momenta. Following convention, we define x and y in terms of 3-momentum magnitude:

$$x = \frac{2k}{m_\pi} \tag{2.1a}$$

$$y = \frac{2p}{m_\pi} \tag{2.1b}$$

where m_π is the π mass. From 4-momentum conservation, a third variable

$$\lambda = \sin^2 (\theta/2) \tag{2.1c}$$

is easily related to x,y, ignoring the electron mass m_e:

$$\lambda xy = x + y - 1 . \tag{2.2}$$

It is then easily shown that

$$k \cdot p = \frac{m_\pi^2}{2} (x + y - 1), \tag{2.3a}$$

$$k \cdot K = \frac{m_\pi^2}{2} (1 - y). \tag{2.3b}$$

2.2 Phase space

We might be interested in several differential decay rates. We shall always use p as the z-axis. Then (ignoring m_e):

$$\Gamma = \int \frac{d^3k\ d^3K\ d^3p}{16 m_\pi k K p}\ \delta^4(k+K+p-q) \sum_{\text{spins}} |T|^2 \frac{1}{32\pi^5} . \tag{2.4}$$

It follows that

1) $$\frac{d\Gamma}{d\Omega_e\ d\Omega_\gamma\ dp} = \frac{p(m_\pi - 2p)}{32\big(m_\pi - p(1-\cos\theta)\big)^2} \frac{\sum|T|^2}{32\pi^5} \tag{2.5}$$

2) If we measure electron and photon momenta, then

$$\frac{d\Gamma}{dx\,dy} = \frac{\pi^2 m_\pi}{8} \sum |T|^2 \frac{1}{32\pi^5} . \tag{2.6}$$

The fact that the weight multiplying $|T|^2$ is constant is well known to users of Dalitz plots; our x,y variables are just the energy variables used in these plots.

Recall our discussion of double kinematic solutions in radiative μ decay ($\mu^- p \to \nu n \gamma$):

The double-valued solutions to kinematics for fixed θ and k only apply for $k > (m_\pi^2 - m_e^2)/2m_\pi$ here. Thus only for k within $m_e^2/2m_\pi$ of k_{max} do we need worry about these; in practice this is of no consequence evidently.

2.3 Matrix elements

We start with the $\pi e \nu$ interaction Lagrangian in momentum space

$$\mathcal{L}_{weak} = F\, \overline{U}(e) \not{q} (1+\gamma_5)\, V(\overline{\nu}) , \tag{2.7}$$

where our convention gives $(1+\gamma_5) \equiv \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ in 2×2 notation, as the projection operator for right handed $\overline{\nu}$.

It follows that the sum of electron emission, pion emission and contact term due to the π-momentum dependence in Eq.(2.7) gives, after using the Dirac equation to simplify the diagrams of Fig. 2.1,

$$(\sqrt{4\pi})^{-1}\, T = \overline{U}(p) \left\{ \underbrace{\left(\not{\varepsilon} + m_e \frac{\varepsilon \cdot p}{k \cdot p} + m_e \frac{\not{\varepsilon}\not{k}}{2k \cdot p} \right)}_{\text{electron emission}} \underbrace{- m_e \frac{q \cdot \varepsilon}{q \cdot k}}_{\pi \text{ emission}} \underbrace{- \not{\varepsilon}}_{\text{contact}} \right\} (1+\gamma_5)\, V(K)\, F$$

$$= \overline{U} \left(\frac{\varepsilon \cdot p}{k \cdot p} + \frac{\not{\varepsilon}\not{k}}{2k \cdot p} \right) \left(1+\gamma_5 \right) V\, F m_e \quad \text{in lab frame.} \tag{2.8}$$

Note that this amplitude $\sim m_e$, in agreement with the present experimental observation. Any attempt to delete internal emission (contact) would give an amplitude of order m_π instead of m_e, in gross disagreement with experiment.

The vector internal emission of Fig. 2.2 contributes a gauge invariant amplitude

$$\left(\sqrt{4\pi\alpha}\right)^{-1} T_{vector} = \overline{U}\,\gamma_\mu\left(1+\gamma_5\right)V\,\frac{g_{\mu\nu} - (P_\mu P_\nu)/M^2}{P^2 - M^2}\,\varepsilon_{\nu\alpha\sigma\tau}\,\varepsilon_\alpha k_\sigma q_\tau\,\overline{g}_V \quad (2.9)$$

where $P = q - k$ and we have pretended that a 1^- boson contributes. For very large boson mass M we write this as

$$\left(\sqrt{4\pi\alpha}\right)^{-1} T_{vector} \sim g_V\,\overline{U}\not{A}\,(1+\gamma_5)V$$

from which
$$A_\mu = \varepsilon_{\mu\alpha\sigma\tau}\,\varepsilon_\alpha k_\sigma q_\tau. \quad (2.10)$$

By using the CVC hypothesis the $\pi\gamma\nu$ coupling is related to $\pi^0\gamma\gamma$, and one estimates that

$$g_V = \left(\frac{1}{\alpha m_\pi}\sqrt{\Gamma\pi^0/\pi m_\pi}\right)G$$

$$2m_\pi^6\,g_V^2 \approx 21 \times (8F^2 m_e^2).$$

Similarly, an axial internal emission amplitude can be considered as due to an axial vector boson as in Fig. 2.3. This contributes

$$\left(\sqrt{4\pi\alpha}\right)^{-1} T_{axial} = \overline{g}_A\,\overline{U}\gamma_\mu\left(1+\gamma_5\right)V\,\varepsilon_\alpha\left(\frac{g_{\alpha\mu} - (Q_\alpha Q_\mu)/M^2}{Q^2 - M^2}\right) \quad (2.11)$$

plus a counter term for gauge invariance; the net result for large M is

$$\left(\sqrt{4\pi\alpha}\right)^{-1} T_{axial} = \overline{g}_A\,\overline{U}\,\left(\not{\varepsilon}k\cdot q - \not{k}\varepsilon\cdot q\right)\left(1+\gamma_5\right)V \quad (2.12)$$

with $\overline{g}_A$ model dependent, *a priori* unknown, and of order (m_e).

Without changing the accepted currents we can still speculate about possible *off-shell* weak interactions, which we write as

$$\mathcal{L}_{\substack{off\\axial}} = F\,\frac{bm_e}{m_\pi^2}\,\overline{U}\,(\not{p}-m_e)q\,\gamma_5\,V \quad (2.13a)$$

$$\mathcal{L}_{\substack{off\\scalar}} = F\,\frac{dm_e}{m_\pi^2}\,\overline{U}\,(\not{p}-m_e)\,V. \quad (2.13b)$$

These contribute (together with associated counter terms) to radiative decay

$$\left(\sqrt{4\pi\alpha}\right)^{-1} T'_{axial} = F\,\frac{bm_e}{m_\pi^2}\,\overline{U}\,\not{\varepsilon}\not{k}\,\gamma_5\,V \quad (2.14a)$$

$$T'_{scalar} = 0 \text{ due to current conservation.} \quad (2.14b)$$

Finally, we speculate about an on-shell tensor weak interaction, which cannot be manifested in non-radiative π decay: it contributes a radiative amplitude

$$\left(\sqrt{4\pi\alpha}\right)^{-1} T_{\text{tensor}} = \frac{Fcm_e}{2m_\pi^2} \overline{U} \left(\not{\epsilon}\not{K}-\not{K}\not{\epsilon}\right)\left(1+\gamma_5\right) V$$

$$= \frac{Fcm_e}{m_\pi^2} \overline{U} \not{\epsilon}\not{K}\left(1+\gamma_5\right) V. \qquad (2.15)$$

2.4 Summary

Next we tabulate the rates ensuing from these matrix elements and their likely significant interferences only. In Table I we show

$$\left(\frac{\pi^2 m_\pi m_e^2 F^2 \cdot 4\pi\alpha}{32\pi^5}\right)^{-1} \sum_{\text{spins}} \frac{d\Gamma}{dx\, dy} . \qquad (2.16)$$

Note that our expression $\left(m_\pi^3 g_A\right)/\left(2m_e F\right)$ is usually called γ, the ratio of axial to vector (internal) couplings. The expressions in Table II are obtained from amplitudes (1.8), (1.10), (1.12), (1.14a), (1.15) by standard trace algebra.

Note that

$$\Gamma_{\pi^+ e\nu} = \frac{m_\pi m_e^2 F^2}{4\pi}$$

so that

$$\left.\frac{d\Gamma}{dx\, dy}\right)_{\text{external}} = \frac{\alpha}{2\pi} \Gamma_{\pi^+\to e\nu} \times \begin{array}{l}\text{Table I}\\ \text{entry}\end{array}$$

i.e. in Eq. (1.16),

$$\left(\;\right) = \frac{\alpha}{2\pi} \Gamma_{\pi^+\to e\nu} \cdot$$

Table I

	Nature of Term	
1)	External	$\frac{(1-y)}{(x+y-1)x^2}\left(1+(1-x)^2\right)$
2)	Off-shell axial	$\frac{1}{2}b^2\ (1-y)(x+y-1)$
3)	Interference of 1,2	$b(1-y)/x$
4)	Vector (internal) (r ~ 21 from CVC)	$r(1-x)\left((x+y-1)^2+(1-y)^2\right)$
5)	1,4 Interference	0
6)	Axial (internal)	$\left(1-x\right)\left((x+y-1)^2+(1-y)^2\right)\left(\frac{g_A m_\pi^3}{2m_e F}\right)^2$
7)	1,6 Interference	$-\left(\frac{m_\pi^3 g_A}{2Fm_e}\right)\left(\frac{(1-y)x}{(x+y-1)}\right)\left(\frac{(x+y-1)(1-x)}{x^2}+\frac{1}{2}\right)\cdot\left(\frac{2m_e}{m_\pi}\right)$
8)	4,6 interference	$2\left(\frac{m_\pi^3 g_A}{2m_e F}\right)\sqrt{r}\cdot x(1-x)(x-2+2y)$
9)	Tensor	$(1/2)c^2\ (x+y-1)(1-y)$: looks like 2
10)	1,9 interference	$c(1-y)x$: looks like 3

To maximize sensitivity to b (or c) type terms (i.e. terms 2, 3, 9, 10) it is preferable to look at large X, which also minimizes the large internal vector term 4. Small y then maximizes sensitivity to these terms (6). In Table II we indicate the evaluation of external, vector internal and b terms of Table I, for judiciously selected (x,y). We omit evaluation of axial terms for now in view of the current experimental indications that these are small (Stetz *et al.*).

Table II

X = 0.9	y = 0.9	0.7	0.5	0.3
External	0.156	0.625	1.55	4.4
Vector internal	1.36	0.94	0.84	1.1
b term (b=1)	0.11	0.33	0.55	0.8
b^2 term (b=1)	0.04	0.09	0.2	0.07

X = 0.8	y = 0.9	0.7	0.5	0.3
External	0.23	1.0	2.7	11.4
Vector internal	2.1	1.47	1.43	2.1
b term (b=1)	0.22	0.375	0.65	0.875
b^2 term (b=1)	0.035	0.075	0.075	0.075

We see that large X, $y \sim 0.5$ maximizes sensitivity to terms such as off-shell axial, or (on-shell) tensor.

$\Pi e\nu\gamma$ Bibliography (with thanks to J-M Poutissou, University of British Columbia)

The following papers of 'Group 1' are in error in the calculation of the CVC estimate for internal vector emission, being too small by $\sqrt{2}$ in this amplitude; 'Group 2' is correct in this respect.

Group 1:

1. V.G. Vaks and B.L. Ioffe, Nuovo Cimento X, 342 (1958)
2. V.F. Muller, Zeitschrift fur Physik 173, 438 (1963)
3. F. Scheck and A. Wullschleger, Nucl. Phys. B67, 504 (1973)

Group 2:

1. S. Bludman and J. Young, Phys. Rev. 118, 602 (1960)
2. D.E. Neville, Phys. Rev. 124, 2037 (1961)

Recent results:

A.W. Stetz *et al.*, Phys. Rev. Letters 33, 1455 (1974)

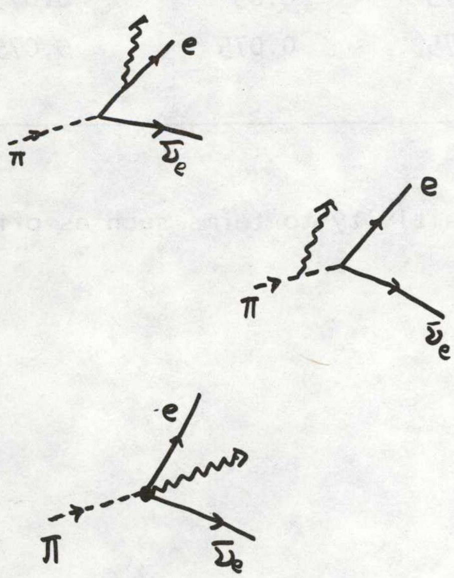


Fig. 2.1

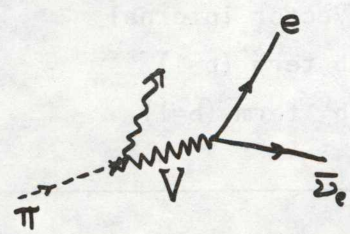


Fig. 2.2. Internal 'vector'

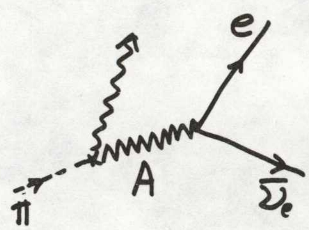


Fig. 2.3. Internal 'axial'

3. Conclusions

$\mu^- p \rightarrow \nu n \gamma$

1) The 'counter current' necessary to maintain current conservation is not uniquely prescribed; numerically this ambiguity is of the order of 20% at large photon energies.

2) Ignoring the above feature, one sees that photon spectra are most sensitive to g_P at large k (>70 MeV, say) so that an experiment concentrating on this kinematic domain would be most welcome.

3) Estimated counting rates indicate that this experiment is only marginally feasible with even the full anticipated flux of stopped μ's at TRIUMF. Further work to be reported elsewhere indicates a much larger cross-section for $\mu\ {}^3He \rightarrow {}^3H \nu \gamma$, which appears to therefore be a better candidate for experimental work.

$\pi \rightarrow e \bar{\nu} \gamma$

To enhance contributions of tensor currents (or off-shell axial interaction) one should look for high-energy photons and medium-energy electrons. This contrasts slightly with the preferred domain for sensitivity to internal axial couplings, namely high-energy electrons and photons.

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[illegible] Conclusions

K_L [illegible]

1) The induced current necessary to maintain current conservation is not uniquely prescribed; numerically this ambiguity is of the order of 10% at large photon energies.

2) Ignoring the above ambiguity, one finds that photon spectra are most sensitive to g_A at large k ([illegible] MeV). [illegible] an experiment concentrating on this kinematic domain [illegible] be [illegible].

3) Estimated counting rates indicate that this experiment is only marginally feasible with even the full anticipated flux of [illegible] at TRIUMF. Further work to be reported elsewhere indicates a much larger cross section for [illegible], which appears to therefore be a better candidate for experimental work.

[illegible]

[illegible] enhance contributions of tensor currents (or [illegible] exist) [illegible] one should look for high energy photons and medium energy electrons. This contrasts [illegible] with the preferred domain for sensitivity to [illegible] axial coupling, namely high energy electrons and photons.