Polarization measurements for $pp \rightarrow \pi^+D$

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We discuss how polarization experiments on $pp \rightarrow \pi D$ can provide information on the transition amplitudes relevant near the reaction threshold energy.
1. **INTRODUCTION**

The purpose of this report is to present to the experimental physicist a simple 'partial wave' analysis at low energy of the reaction

\[ pp \rightarrow \pi^+D. \]

This material has also been considered by other authors.\(^1\) I wish to simplify the notation and to illustrate how polarization measurements can be decisive in understanding this reaction. I am motivated to do so because it appears to me that details of this process are still poorly understood. If one consults the literature,\(^2\) one is struck by the strong time dependence of some data, and by the absence of other desirable data, many years after study of the reaction was initiated.

In Section 2 we shall enumerate the possible transitions which are relevant in the vicinity of threshold. It is then useful to express all possible spin-dependent transitions in terms of transitions between states of an 'L-S-J' basis; this we relegate to Appendix A. These resultant expressions are then employed in Section 3 to obtain the various spin-dependent cross-sections corresponding to feasible experiments.

In Appendix B, we discuss how the problem can be simplified by using the unitarity relation for the scattering amplitudes to infer some phases of transition amplitudes. This simplifies life by lowering the number of necessary polarization experiments (in order to solve for the magnitudes of all transition amplitudes) from 3 to 2.

In Section 4 we present numerical estimates for a set of experiments at 330 MeV, to illustrate which experiments might be most useful.

Finally, in Section 5 we comment on possible advantages to measurements on \( \pi D + pp \).
2. **LOW ENERGY TRANSITIONS**

Near threshold, we will consider only S and P waves in the $\pi$-D system. The restrictions of J conservation, parity conservation, and the Pauli exclusion principle applied to the pp system than leave us with only three transition amplitudes for pp $\rightarrow$ $^{1}D$. Defining the transition amplitudes analogous in normalization to the conventional elastic scattering partial wave amplitude

$$\frac{e^{i\delta}}{\sqrt{q_{\pi D} q_{pp}}} \sin \delta$$

we have the transitions

$$t_{pp}[S = 1, \ell = 1, J = 1] \rightarrow \pi D [S - wave, J = 1] \quad (1)$$

$\equiv a$

$$t_{pp}[S = 0, \ell = 2 = J] \rightarrow \pi D [P wave, J = 2]$$

$\equiv b$

$$t_{pp}[S = 0, \ell = 0 = J] \rightarrow \pi D [P wave, J = 0]$$

$\equiv c$

In Appendix A we expand the relevant scattering states in an L-S-J basis, and obtain scattering amplitudes in terms of $a$, $b$, $c$.

3. **CROSS-SECTIONS**

Using the transitions defined by eqn. (1) and the amplitudes of equations A(8), it is simple to write down expressions for various cross-sections. The results are now presented, followed by discussion.
\[
\frac{q_{\pi D}}{q_{pp}} \left( \frac{d\sigma}{d\Omega} \right)_{\text{unpolarized}} = \frac{1}{2} \left( \frac{3}{2} |a|^2 + \frac{c}{\sqrt{2}} + \frac{\sqrt{5}}{2} |b|^2 \right) + \frac{1}{2} \left( \frac{15}{4} |b|^2 - \frac{3\sqrt{5}}{\sqrt{2}} \text{Re} \, \alpha b \right) \cos^2 \theta.
\]

\[
\frac{d\sigma}{d\Omega} \left( \frac{d\sigma}{d\Omega} \right)_{\text{longitudinally polarized beam}} = \frac{d\sigma}{d\Omega} \left( \frac{d\sigma}{d\Omega} \right)_{\text{unpolarized}}. \tag{3}
\]

Eqn. (3) is just a consequence of parity conservation, which leads to all the equalities exhibited by the set of amplitudes in eqns. (8).

Note that from eqn. (2)

\[
\sigma_{\text{TOT}} \equiv \frac{\pi q_{\pi D}}{q_{pp}} (3|a|^2 + 5|b|^2 + |c|^2). \tag{4}
\]

If we have a transversely polarized proton beam, polarized 100% in the x direction, then using

\[
|J = \frac{1}{2}, J_x = \frac{1}{2}\rangle = \frac{1}{\sqrt{2}} \left( |J, J_z = \frac{1}{2}\rangle + i |J, J_z = -\frac{1}{2}\rangle \right) \tag{5}
\]

we obtain (see Appendix A)

\[
\sum_F \left| \langle F | \hat{t}^{A+iD} \rangle \right|^2 + \left| \langle F | \hat{t}^{C+iB} \rangle \right|^2
\]

\[
\equiv \frac{q_{pp}}{q_{\pi D}} \left( \frac{d\sigma}{d\Omega} \right)_{x\text{-polarized beam of protons}}
\]
If we can polarize both the target and the beam, then we obtain information in a very nicely "separated" manner.

For beam longitudinally polarized, and target polarized the same as the beam, we obtain

\[
\frac{q_{pp}}{q_{\pi D}} \, d\sigma \bigg|_{\text{beam longitudinally polarized, target same polarization}} = \frac{3}{2} |a|^2.
\]

This is particularly nice because it immediately tells us the part of the total cross-section due to production of S-wave pions. This is of interest, for example, in making comparison of data with soft-pion theories\(^3\) for the reaction, which essentially make predictions for S-wave pions only.

With the information of eqn. (6) (which effectively includes the unpolarized beam information) and eqn. (7), we still cannot obtain all the magnitudes \(|a|, |b|, |c|\), as well as two relative phases. Thus three distinct polarization experiments are required for a complete "measurement" of amplitudes (and relative phases) at a given energy.

A third experiment could be to measure the final deuteron polarization (with unpolarized protons). This measurement gives us the polarization \(\hat{P}_D\) transverse to the reaction plane:

\[
\frac{q_{pp}}{q_{\pi D}} \, d\sigma \bigg|_{\text{unpolarized}} \times \hat{P}_D
\]
which now permits us to solve for all relative phases and magnitudes of amplitudes. In this case, explicit solution (algebraically) is very clumsy, but enough data is available to give a unique (numerical) solution.

(However, note the comments in Appendix B.)

Finally, having outlined a reasonable set of experiments, I will conclude this section with expressions for the case of only partially polarized beam or target.

**Partially Polarized transverse beam:** If the transverse polarization is $\alpha$, then multiply the last two terms of equation (6) by $\alpha$.

**Partially polarized (longitudinal) beam and partially polarized target:** Let $\alpha \equiv$ beam longitudinal polarization and $\beta$ the target polarization ($||$ the beam polarization, say).

Then

$$
\frac{d\sigma}{d\Omega} = \left(1 - \alpha \beta \right) \frac{d\sigma}{d\Omega}_{\text{unpolarized}} + \frac{3}{2} \alpha \beta \cdot |a|^2 \cdot \frac{q_{\pi D}}{q_{pp}}.
$$

Thus even with partial polarization, this double polarization experiment enables us to pick out the ($S$-wave-pion) amplitude $a$. Unfortunately, the product $\alpha \beta$ under typical experimental conditions (which can be realized at the present time) would be of the order of 0.2, so that rather good data statistics are required.
NUMERICAL EXAMPLES

To see what order of magnitude one expects for possible measurements, we shall consider experiments at 330 MeV proton laboratory kinetic energy; a typical estimate of P wave production\(^2\) is that \(r = \sigma_\text{TOT}^P/\sigma_\text{TOT} = 1/3\). For lack of any information at present, we take \(|b| = |c|\) and use the phases for \(b\) and \(c\) estimated via unitarity in Appendix B, namely

\[+10^\circ = \text{Phase of } b = -\text{Phase of } c.\]

As pointed out in Appendix B, we cannot trust simple unitarity approximations for the amplitude \(a\). Nevertheless, since no data is available, we use the guess

\[
\text{Phase } a \equiv \delta_{J=S=1}^{J=S=1} = -30^\circ \text{ at } 330 \text{ MeV.} \quad (10)
\]

From eqn. (4), we have \(|b| = |c| = \sqrt{r/(1-r)} \cdot |a|\).

The following characteristics then ensue for polarized beam and/or target experiments.

The transverse polarization angular distribution has the form (eqn. (6))

\[
\frac{d\sigma}{d\Omega} \propto \left( \frac{3}{2} \frac{1-r}{r} + \frac{13}{8} \right) - \frac{3}{8} \cos^2 \theta + \alpha \sqrt{\frac{1-r}{r}} (-1.2 \sin \theta \cos \phi) \quad (11a)
\]

where S-P wave interference is responsible for the last terms.

For \(r \sim \frac{1}{3}\)

\[
\frac{d\sigma}{d\Omega} \sim 1 - 0.08 \cos^2 \theta + \alpha (0.37 \sin \theta \cos \phi) \quad (11b)
\]
Thus, with a 50% polarized beam, this experiment might probably still provide useful information if high precision is feasible. Unfortunately, the $\cos^2 \theta$ terms is probably too small to be measured with useful precision, so only two quantities can be actually obtained rather than the theoretical three independent quantities which this experiment could provide.

Next we look at the "double-polarization" experiment described by eqn. (9). The only useful measured quantity here is the fractional change in $\sigma_{\text{TOT}}$ (compared with $\sigma$ unpolarized) which is independent of assumptions about $b$, $c$. We thus find

$$\delta = \frac{\Delta \sigma_{\text{TOTAL}}}{\sigma_{\text{TOTAL}}} = \alpha \beta (1 - 2r)$$

where

$$\frac{\sigma_{\text{TOT}} \text{P wave}}{\sigma_{\text{TOT}}} = r.$$ 

(12)

Thus, $r = \frac{1}{10}$ and $\frac{1}{3}$ respectively, and $\alpha \beta = 0.2$, gives us $\delta = 0.16$ and $\delta = 0.067$.

Therefore, rather high precision experiments will be necessary with presently feasible low polarizations ($\alpha \beta$).

Using $r = \frac{1}{3}$, we next estimate the final state Deuteron polarization from eqn. (8). For $\theta = \pi/2$, $\phi = 0$, we find

$$\hat{P} \sim 5\%$$

which is rather discouraging to the experimentalist.

In summary, for the near threshold region it appears that one might be able (with feasible double polarization experiments) to obtain a good estimate of $r$. Because the anisotropic P-wave term in the angular distribution is so small, it is unlikely to be
measured with a precision that contributes to the evaluation of the P-wave amplitudes—in effect, a certain amount of information is, for practical purposes, inaccessible. The same comment applies to final state Deuteron polarization. At higher energies these pessimistic comments are no longer applicable, but then the unitarity approximations of Appendix B, which give us \( \phi_b, \phi_c \) become suspect. (One might be able to trust our approximations up to about 370 MeV (Lab proton K.E.)).

Finally, if I were asked what single measurement would be most interesting, my reply would be a determination of \( r \) via a double polarization experiment at low energies (where \( r \) is the order of \( 1/10 \), e.g. around 310-315 MeV).

5. \( \pi D + pp \)

There are some advantages to studying this inverse reaction, due to statistical spin factors in various cross-sections. In particular, the final transverse proton polarization is given by

\[
P = \frac{\frac{4}{3} \sqrt{3} \sin \theta \cos \phi \Im a^*(\frac{c}{\sqrt{2}} + \frac{\sqrt{5}}{2} b)}{\frac{1}{3} \left( \frac{\partial q_{pp}}{\partial \Omega} \right)_{pp + \pi D}}.
\]  

(13)

For \( r = \frac{1}{3} \), and the assumptions of the previous section, at \( \theta = \pi/2, \phi = 0 \), we estimate \( \hat{P} = 75\% \). It appears, therefore, that this measurement can be more useful than the analogous \( pp + \pi d \) measurement described by eqns. (6) and (11b).
APPENDIX A

We first express scattering states, as a direct product of spin states $|S, S_z\rangle$ and plane wave states. For the initial pp state, let particle 1 move in the +z direction.

We express pp spin states in terms of total spin:

$$
|\frac{1}{2} \frac{1}{2} \rangle |\frac{1}{2} \frac{1}{2} \rangle = |1,1\rangle
$$

$$
|\frac{1}{2} + \frac{1}{2} \rangle |\frac{1}{2} - \frac{1}{2} \rangle = \frac{1}{\sqrt{2}} |1,0\rangle + \frac{1}{\sqrt{2}} |00\rangle \tag{A1}
$$

$$
|\frac{1}{2} - \frac{1}{2} \rangle |\frac{1}{2} \frac{1}{2} \rangle = \frac{1}{\sqrt{2}} |1,0\rangle - \frac{1}{\sqrt{2}} |00\rangle
$$

$$
|\frac{1}{2} - \frac{1}{2} \rangle |\frac{1}{2} - \frac{1}{2} \rangle = |1,-1\rangle
$$

Therefore $|S, S_z\rangle e^{ikz} = |S, S_z\rangle \sum_{\ell} i^\ell \sqrt{(2\ell+1)4\pi} |\ell,0\rangle \tag{A2}$

(where $|\ell,0\rangle$ is normalized angular momentum state,

$$
\langle \ell,0|\ell',0\rangle = \delta_{\ell\ell'} \equiv \sum_{\ell J} i^\ell \sqrt{(2\ell+1)4\pi c(S\ell J; S_z 0)} |J\ell S; S_z\rangle
$$

Thus, omitting terms in the expansion which do not contribute because of the Pauli exclusion principle,

$$
A = \left|\frac{1}{2} \frac{1}{2} \right| \left|\frac{1}{2} \frac{1}{2} \right| e^{ikz} = \sqrt{4\pi} \sqrt{\frac{3}{2}} |J=1, \ell=1, S=1\rangle + ...
$$

$$
B = \left|\frac{1}{2} \frac{1}{2} \right| \left|\frac{1}{2} \frac{1}{2} \right| e^{ikz} = -\sqrt{4\pi} \sqrt{\frac{3}{2}} |J=1, S=1, \ell=1\rangle + ...
$$
\[ C \equiv \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} e^{ikz} \equiv + \sqrt{4 \pi} \begin{pmatrix} \frac{1}{\sqrt{2}} |J=0; S=0, \varepsilon=0 \rangle \\ + \frac{\sqrt{5}}{2} |J=2, S=0, \varepsilon=2 \rangle \end{pmatrix} \]

\[ D \equiv \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} e^{ikz} \equiv + \sqrt{4 \pi} \begin{pmatrix} \frac{1}{\sqrt{2}} |J=0; S=0, \varepsilon=0 \rangle \\ + \frac{\sqrt{5}}{2} |J=2, S=0, \varepsilon=2 \rangle \end{pmatrix}. \]

For the final \( \pi D \) state, we simply use

\[ e^{ik \cdot r} = 4 \pi \sum_{LM} Y_{LM}^{*} (\Omega, k) \times |L, M\rangle \]

where \( \langle L, M|L^{-1}M^{-1} \rangle = \delta_{LL} \cdot \delta_{MM} \).

Thus we obtain, expanding as in A(3),

\[ A = |1, +1 \rangle e^{i k \cdot r} = 4 \pi Y_{oo}^{*} (\Omega, k) |J=1, M=1 \rangle \]

\[ + 4 \pi \sum_{M} Y_{1M}^{*} (\Omega, k) C(112, 1M) |J=2, L=1; J_z=M+1 \rangle \]

\[ + \frac{4 \pi Y_{1-1}^{*}}{\sqrt{3}} |J=0; L=1 \rangle \]

\[ B = |1, -1 \rangle e^{i k \cdot r} = 4 \pi Y_{oo}^{*} |J=1, L=0, J_z=-1 \rangle \]

\[ + \frac{4 \pi Y_{11}^{*}}{\sqrt{3}} |J=0, L=1, J_z=0 \rangle \]

\[ + 4 \pi \sum_{M} Y_{1M}^{*} C(112; -1, M) |J=2, J_z=1-1 \rangle \]

\[ L=1 \]

\[ C = |1, 0 \rangle e^{i k \cdot r} = 4 \pi Y_{oo}^{*} |J=1, L=0, J_z=0 \rangle \]

\[ - \frac{4 \pi Y_{10}^{*}}{\sqrt{3}} |J=0, L=1, J_z=0 \rangle \]
Now, as in section 2, defining

\[ 4\pi \langle J=1, L=0 \mid \pi D \mid J=1, \ell=1, S=1 \mid pp \rangle = a \]
\[ 4\pi \langle J=2, L=1 \mid \pi D \mid J=2, \ell=2, S=0 \mid pp \rangle = b \]
\[ 4\pi \langle J=0, L=1 \mid \pi D \mid J=0, \ell=2, S=0 \mid pp \rangle = c \]

we obtain from A(4), A(6), A(7) the following amplitudes

\[ 0 = f_{B A} = f_{A B} = f_{A C} = f_{B C} \]

and

\[ -f_{B B} = f_{A A} = \sqrt{4} \sqrt{3/2} Y_{10} a \]
\[ -f_{D A} = f_{C A} = \sqrt{4\pi/3} Y_{11} \left( \frac{c}{\sqrt{2}} + \frac{\sqrt{5}}{2} b \right) \]
\[ -f_{D B} = f_{C B} = \sqrt{4\pi/3} Y_{11} \left( \frac{c}{\sqrt{2}} + \frac{\sqrt{5}}{2} b \right) \]
\[ -f_{D C} = f_{C C} = \sqrt{4\pi/3} Y_{10} \left( -\frac{c}{\sqrt{2}} + \frac{\sqrt{5}}{2} b \right). \]

Since we have used an "\( S_z, J_z \)" basis rather than helicity states, we have to repeat this whole process for the inverse reaction \( \pi D \to pp \)--there are no trivial "reciprocity" relations to help us. The technique is now obvious, so we present only results. Again, \( A, B, C, D, \alpha, \beta, \gamma, \) label spin states (quantization axis = \( \pi \) momentum direction) as before:

\[ f_{\alpha A} = \frac{3}{2} a^* \cos \theta = -f_{\alpha B} \]
\[ f_{\alpha B} = 0 = f_{\beta A} \]
\[ f_{ac} = \frac{1}{2} \sin \theta e^{-i\phi} \left( \sqrt{\frac{3}{2}} a^* - 3 \sqrt{\frac{5}{2}} b^* \cos \theta \right) \]
\[ = e^{-2i\phi} f_{\phi D} \]
\[ f_{ad} = \frac{1}{2} \sin \theta e^{-i\phi} \left( \sqrt{\frac{3}{2}} a^* + 3 \sqrt{\frac{5}{2}} b^* \cos \theta \right) \]
\[ = e^{-2i\phi} f_{\phi C} \]
\[ f_{\mathcal{E} A} = \frac{\sqrt{3}}{2} a^* e^{i\phi} \sin \theta = e^{2i\phi} f_{\mathcal{E} B} \]
\[ f_{\mathcal{E} C} = \frac{\sqrt{5}}{2} b^* (3 \cos^2 \theta - 1) - \frac{1}{\sqrt{2}} c^* = -f_{\phi D}. \]
\[ A(9) \]
APPENDIX B: UNITARITY AND PHASES

Equations A(8) (and A(9)) are now the starting point for all manipulations to obtain cross-sections, etc. Note that our $f$'s are defined analogously to the elastic amplitude

$$ e^{i\delta} \sin \delta k $$

Thus, here

$$ f = \frac{e^{i\delta} \sin \delta}{\sqrt{q_{\pi D} q_{pp}}} $$

Thus, for S-waves at $\pi D$ threshold, $f \approx$ constant. The cross-section is given by

$$ \frac{d\sigma}{d\Omega}_{pp + \pi D} = \frac{q_{\pi D}}{q_{pp}} |f|^2. $$

For states of a given $J$, parity, $f$ satisfies the unitary relation, neglecting 3 body intermediate states

$$ \text{Im} \ f^J_{a+b} = \sum_n q_n f^J_{b+n} f^J_{a+n}. $$

Thus

$$ \text{Im} \ c = q_{pp} C^* f^{J=0, S=0}_{pp} $$

$$ + q_{\pi D} f^{*J=0, L=1}_{\pi D + \pi D} C $$

$$ + q_{\pi D} f^{*J=2, L=2}_{pp + \pi D} \left\{ \begin{array}{l} f^{*J=2, L=3, S=0}_{\pi D + \pi D} f^{J=2, L=2, S=0}_{pp + \pi D} \end{array} \right\} $$

$$ + q_{\pi D} f^{*J=2, L=2, S=0}_{pp + \pi D} f^{J=2, L=2, S=0}_{pp + \pi D} $$

$$ + q_{\pi D} f^{*J=2, L=3, S=0}_{pp + \pi D} f^{J=2, L=2, S=0}_{pp + \pi D} $$

$$ + q_{\pi D} f^{*J=2, L=2, S=0}_{pp + \pi D} f^{J=2, L=2, S=0}_{pp + \pi D} $$
\[ \text{Im } a = q_{pp} A^* f_{pp}^{S=1} \]
\[ = \rho_{\pi D}^* \left( f_{\pi D}^{S=1} (J=1, L=0; J=1, L=2) f_{pp}^{S=1} (J=1, L=1; J=1, L=2) + f_{\pi D}^{S=1} (J=1, L=0) \right) \]

We observe that near threshold, we expect (see A(9))

\[ C \sim q_{\pi D} \]
\[ f_{\pi D}^{(L=1)} \sim q_{\pi D}^2. \]

Therefore in the eqn. for \text{Im } c, the 2nd term on the right hand side is of order \( q_{\pi D}^3 \times c \). This argument applies equally to the equation for \text{Im } b. Thus, close to the \( \pi D \) threshold, we should be safe in ignoring these terms. As a result, it should be a very good approximation that, from eqns. B(4),

\[ \text{phase } c = \delta_{J=0}^1 (pp) \]
\[ \text{phase } b = \delta_{J=2}^1 (pp) \]

and also should be a good approximation to ignore 3-body states.
Knowing these phases, one only needs a measurement of deuteron polarization together with the unpolarized angular distribution in order to "solve" for \(|a|\), \(|b|\), \(|c|\). To see this, note that from eqn. 8 (deuteron polarization) if one knows the difference in phase between \(b\) and \(c\), then the measurement tells us \(|b| \times |c|\) which couples with eqn. (2) to provide \(|b|\) and \(|c|\) separately.

At 330 MeV\(^5\) (Lab Kinetic energy)

\[
\delta_{j=0}^1 \approx -10^\circ
\]

and

\[
\delta_{j=2}^1 \approx +10^\circ.
\]

Thus the relative phase of \(b\) and \(c\) is 20\(^\circ\), or \(\sin^{-1}(0.35)\) which is perhaps large enough so that the resultant deuteron polarization could be feasibly measured.

Note that in the unitarity relation for \(a\), the 2nd term on the right side is only suppressed by a factor of \(q_{\pi D}^2\) instead of \(q_{\pi D}^3\), so I would feel a priori cautious about neglecting this term.\(^4\)
REFERENCES

2. For the most recent survey, see C. M. Rose, Phys. Rev. 154, B 1304 (1967).
4. Consider the 3 body contribution to Im c: it must involve positive parity npπ+ (or ppπ0) states and therefore P-wave states. This contribution is then of form

\[ \int \text{phase space} \times f(J=0 \rightarrow \text{(one P-wave)}) \]

\[ (\text{small}) \quad pp \quad np\pi^+ \]

\[ \times f^P\left( \text{P-wave} \quad np\pi^+ \rightarrow \pi D, \text{P-wave} \right) \]

which is small because of all the P-wave factors. Similarly the 3-body contributions to Im b should be negligible because J=2 states are involved.

The 3-body contribution to Im a may be substantial, however, since this is a transition to S-wave π-D state and the S-wave π^+-n-p state. The latter is not small; in fact, total cross-sections for pp→π^+d and pp→npπ^+ are comparable.