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MAGNET SYSTEMS FOR THE TRANSPORT OF ION BEAMS WITH NO CHANGE OF PROPERTIES

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1. Introduction

Particle motion through a beam transport system may be described, to first order, by using the well known matrix formalism[1]. We review this method, only considering aspects relevant to this report.

We choose an orthogonal Cartesian co-ordinate system where

x denotes the horizontal displacement,

y denotes the vertical displacement, and

z denotes the displacement in the beam direction.

Beam handling elements (we consider only drift spaces and quadrupole magnets) are represented by 2 x 2 transformation matrices which operate on particle displacement-divergence vectors. Thus, for motion in the xz-plane we have

$$\begin{pmatrix} x \\ \frac{dx}{dz} \end{pmatrix} = M_i \begin{pmatrix} x_o \\ \frac{dx_o}{dz} \end{pmatrix},$$

where M_i is the i^{th} element transformation matrix.

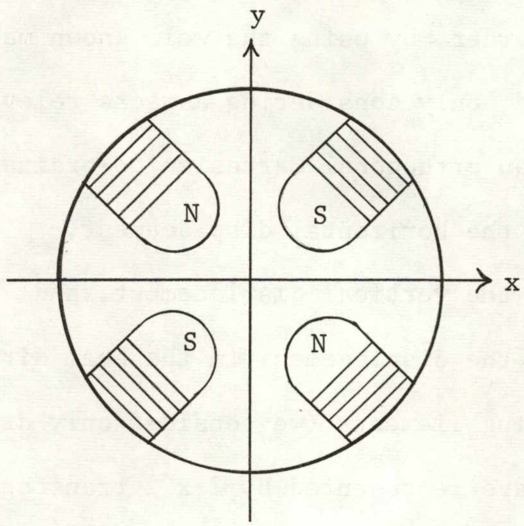
The zero subscripts denote input values, and the resultant vector on the left gives the particle displacement and slope at the exit of the element. A similar equation holds for motion in the yz-plane.

In a drift space the particles travel in straight lines and the transformation matrix is

$$M = \begin{pmatrix} 1 & D \\ 0 & 1 \end{pmatrix},$$

where D is the length of the field-free region. This matrix is the same in both the xz- and yz-planes.

A quadrupole magnet consists of four magnetic pole pieces mounted on a common yoke as shown below.



If the co-ordinate system is oriented as shown (z is out of page) then the magnetic field $\vec{B}$ arising from such a pole configuration can be described, to good approximation, by

$$B_x = gy ,$$

$$B_y = gx ,$$

and

$$B_z = 0 ,$$

where

$$g = \frac{\partial B_x}{\partial y} = \frac{\partial B_y}{\partial x}$$

is the field gradient which is assumed to be constant over the effective length L of the magnet and zero outside of it (hard edge model).

By substituting this field into the equations of motion

$$\frac{d}{dt}(m\vec{v}) = e\vec{v} \times \vec{B} \ ,$$

we get, to first order, the differential equations

$$\frac{d^2x}{dz^2} + k^2x = 0 \ ,$$

$$\frac{d^2y}{dz^2} - k^2y = 0 \ ,$$

with $$k^2 = \frac{eg}{P} = .003 \frac{g}{P} \ ,$$

and where the particle momentum P is to be expressed in Gev/c and the units of g are Gauss/cm.

It is apparent from the signs in the above two differential equations that the quadrupole is focussing in the xz-plane and defocussing (with the same strength) in the yz-plane. If the magnet polarities are reversed, these focussing effects are interchanged.

The solutions to the two equations of motion can be written in matrix form as

$$\begin{pmatrix} x \\ \frac{dx}{dz} \end{pmatrix} \equiv M_H \begin{pmatrix} x_o \\ \frac{dx_o}{dz} \end{pmatrix} = \begin{pmatrix} \cos KL & K^{-1}\sin KL \\ -K\sin KL & \cos KL \end{pmatrix} \begin{pmatrix} x_o \\ \frac{dx_o}{dz} \end{pmatrix} ,$$

$$\begin{pmatrix} y \\ \frac{dy}{dz} \end{pmatrix} \equiv M_V \begin{pmatrix} y_o \\ \frac{dy_o}{dz} \end{pmatrix} = \begin{pmatrix} \cosh KL & K^{-1}\sinh KL \\ K \sinh KL & \cosh KL \end{pmatrix} \begin{pmatrix} y_o \\ \frac{dy_o}{dz} \end{pmatrix} .$$

These transformation matrices M_H and M_V may be factored[2] into the product of three matrices describing two drift spaces separated by a thin lens; i.e.,

$$M_H = \begin{pmatrix} 1 & K^{-1}\tan\frac{KL}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -K\sin KL & 1 \end{pmatrix} \begin{pmatrix} 1 & K^{-1}\tan\frac{KL}{2} \\ 0 & 1 \end{pmatrix},$$

$$M_V = \begin{pmatrix} 1 & K^{-1}\tanh\frac{KL}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ K\sinh KL & 1 \end{pmatrix} \begin{pmatrix} 1 & K^{-1}\tanh\frac{KL}{2} \\ 0 & 1 \end{pmatrix}.$$

In the "thin lens approximation", we take

$$\sin KL \cong \sinh KL \cong KL ,$$

$$\tan\frac{KL}{2} \cong \tanh\frac{KL}{2} \cong \frac{KL}{2} .$$

Then both M_H and M_V can be written in the form

$$M_i = \begin{pmatrix} 1 & L/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix} \begin{pmatrix} 1 & L/2 \\ 0 & 1 \end{pmatrix},$$

where $f = 1/K^2L$ is the thin lens focal length which is positive or negative depending on whether the effect of the lens is focussing or defocussing.

The transformation matrix T representing a transport system is found by computing the product of the individual element matrices; i.e.,

$$T = \prod_{i=1}^{N} M_i ,$$

where N is the number of elements in the system.

We note that this product matrix always has unit determinant since this is true for the matrices representing drift spaces and quadrupoles.

It is of interest to design "identity sections" for which the system transformation matrix in both xz- and yz-planes satisfies

$$T = \pm I ,$$

where I is the unit matrix.

These systems change particle displacement-divergence vectors according to

$$\begin{pmatrix} x \\ \frac{dx}{dz} \end{pmatrix} = \pm \begin{pmatrix} x_o \\ \frac{dx_o}{dz} \end{pmatrix} ,$$

and thus any region in the input $(x_o, \frac{dx_o}{dz})$ phase space which is invariant under the operation of reflection through the origin is reproduced at the exit of an identity section. In particular, if the input beam is represented by a central ellipse in displacement-divergence phase space, then the output beam is represented by the same ellipse. An identity section is therefore useful for transporting such a beam effectively unchanged between two points on the beam line.

The initial investigations of these identity sections were carried out using the thin lens approximation. It is shown in Appendices I and II that there exist no thin lens doublet or triplet systems but in the case of four thin lens quadrupoles there exist systems with two degrees of freedom. The corresponding four quadrupole thick lens systems are discussed in Section 2 using assumptions based on the thin lens analysis.

The systems found have the following characteristics:

- the system transformation matrix is equal to minus the unit matrix
- the systems have a symmetry property; that is, they consist of two identical symmetric doublets in succession
- the total system length D is taken as a design parameter and can easily be adjusted
- a system of a given length has three degrees of freedom. These can be taken as:

  r, the ratio of the last drift length* to the first

  ℓ, the ratio of the central drift space* for the system to D

  d, the ratio of the quadrupole magnet effective length to D.

An expression is obtained which relates the quadrupole field gradient for the system to ℓ and d. This gradient is a minimum when the drift spaces between the magnets are of equal length.

Identity sections with 4n (n is a positive integer) quadrupoles can be formed by using n of the above systems in succession. If the r-, ℓ- and d- values are the same for each identity section in the series, then when n becomes infinite, we obtain an infinite periodic system where any segment with length equal to a multiple of D is an identity section. The infinite and four quadrupole systems are discussed in Section 2. The phase space acceptances for a number of four quadrupole identity sections were calculated numerically. It was found that for a system of a given length the acceptance is a minimum when the system field gradient is a minimum; i.e., when the drift spaces between magnets are equal in length. It was also found that a maximum acceptance occurred for particles of momentum greater than the design momentum for the system. The details of these computations are given in Section 3.

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* measured from the center of the magnet

2. Thick Lens Identity Sections

a. Four Quadrupole Systems

It is demonstrated in Appendices I and II that there exist no doublet or triplet thin lens quadrupole identity sections but for the case of four thin lenses there exist a double infinity of solutions. Therefore the thick lens identity sections were investigated by studying systems with four thick quadrupole lenses.

We consider first a theorem taken from elementary matrix theory. This is the Hamilton-Cayley[5] theorem which states that a matrix satisfies its characteristic equation. For a 2 x 2 matrix M the characteristic equation is

$$\det\begin{pmatrix} M_{11}-\lambda & M_{22} \\ M_{21} & M_{22}-\lambda \end{pmatrix} = \lambda^2 - (\text{trace } M)\lambda + \det M = 0 \ ,$$

and hence we can write $$M^2 - (\text{trace } M)M + (\det M)I = 0 \ . \qquad 2.1$$

If M represents a system transformation matrix, then

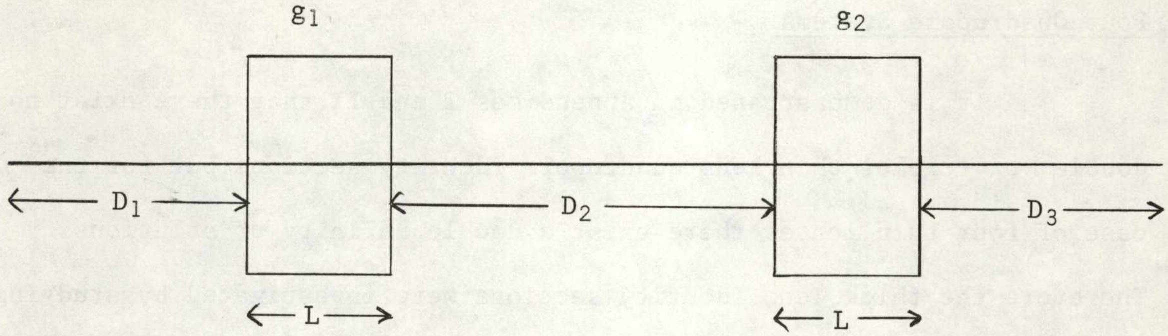
$$\det M = 1 \ ,$$

and from equation 2.1 $$M^2 = - I \ , \qquad 2.2$$

when we impose the condition $$\text{trace } M = 0 \ . \qquad 2.3$$

From equation 2.2 we conclude that if we can find a thick lens doublet satisfying equation 2.3, then a four quadrupole identity section can be formed by using two such doublets in succession.

To this end we consider the thick lens doublet shown below:



Here g_1 and g_2 are the quadrupole field gradients and L the quadrupole effective length. The drift lengths D_1, D_2, D_3 are measured between magnet edges. From our thin lens calculations (Appendix II), we suspect that the magnets must be of equal strength and alternately focussing and defocussing. Accordingly we assume

$$g_1 = -g_2 \equiv g \;.$$

The horizontal plane transformation matrix is then obtained from

$$H = \begin{pmatrix} 1 & D_3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \\ -K\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} 1 & D_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cosh\phi & \frac{\sinh\phi}{K} \\ K\sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} 1 & D_1 \\ 0 & 1 \end{pmatrix},$$

where, for brevity, $\phi = KL \;.$ 2.4

Multiplying out the above matrices we find that the diagonal components of H are given by

$$H_{11} = -D_2D_3K^2\sin\phi\sinh\phi + D_2K\cos\phi\sinh\phi + D_3K(\cos\phi\sinh\phi - \sin\phi\cosh\phi) + \cos\phi\cosh\phi + \sin\phi\sinh\phi \;, \qquad 2.5$$

and

$$H_{22} = -D_1D_2K^2\sin\phi\sinh\phi + D_1K(\cos\phi\sinh\phi - \sin\phi\cosh\phi) - D_2K\sin\phi\cosh\phi + \cos\phi\cosh\phi - \sin\phi\sinh\phi \;. \qquad 2.6$$

Now substituting equations 2.5 and 2.6 into equation 2.3 gives

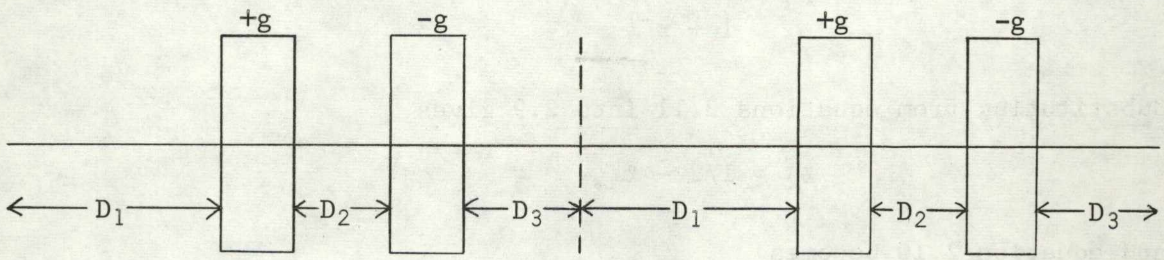
$$D_2(D_1 + D_3)K^2\tanh\phi\tan\phi - K(D_1 + D_2 + D_3)(\tanh\phi - \tan\phi) - 2 = 0 \cdot \qquad 2.7$$

Also, setting trace V = 0, where V is the vertical plane transformation matrix for this doublet, leads again to equation 2.7.

Since the doublet is to represent one half of the identity section the expression for the total system length D is

$$D = 2\,(\,D_1 + D_2 + D_3 + 2\,L)\;. \qquad 2.8$$

It is convenient to take D and L to be fixed parameters. Then we have two equations (2.7 and 2.8) in the four unknowns D_1, D_2, D_3, K. We choose to solve for D_2 and $D_1 + D_3$ in terms of K. Then the complete identity section, shown below, will be specified, except for the ratio of the first drift length to the last, when the K-value is given.



We solve for D_2 by rearranging equation 2.8

$$D_2 = \frac{D}{2} - 2L - (D_1 + D_3)\;, \qquad 2.9$$

and substituting this into equation 2.7 yields

$$D_1+D_3 = (\frac{D}{4} - L) \pm \left[(\frac{D}{4} - L)^2 - \frac{2\coth\phi\cot\phi}{K^2}\left\{1+K(\frac{D}{4} - L)(\tanh\phi - \tan\phi)\right\}\right]^{\frac{1}{2}}\;. \qquad 2.10$$

It is convenient to write these equations in terms of dimensionless parameters. We define

$$
\begin{aligned}
d &\equiv L/D \ , \\
\theta &\equiv KD \ , \\
\ell_1 &\equiv \frac{d}{2} + \frac{D_1}{D} \ , \\
\ell_2 &\equiv d + \frac{D_2}{D} \ , \\
\ell_3 &\equiv \frac{d}{2} + \frac{D_3}{D} \ , \\
\ell &\equiv d + \frac{D_1 + D_3}{D} = \ell_1 + \ell_3 \ , \\
r &\equiv \ell_3/\ell_1 \ .
\end{aligned}
\qquad 2.11
$$

From the last two defining equations we get

$$
\begin{aligned}
\ell_1 &= \frac{\ell}{1 + r} \ , \\
\ell_3 &= \frac{\ell}{1 + r^{-1}} \ .
\end{aligned}
\qquad 2.12
$$

Substituting from equations 2.11 into 2.9 gives

$$\ell_2 = 1/2 - \ell \ , \qquad 2.13$$

and equation 2.10 becomes

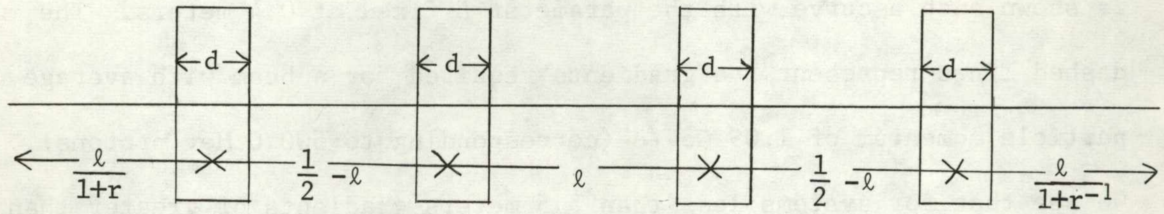
$$\ell = \frac{1}{4} \pm \left[\left(\frac{1}{4} - d\right)^2 - \frac{2\coth d\theta \cot d\theta}{\theta^2} \left\{ 1 + \theta\left(\frac{1}{4} - d\right)(\tanh d\theta - \tan d\theta) \right\} \right]^{\frac{1}{2}} . \qquad 2.14$$

Using this notation, the total system length is equal to unity. To scale to a system of length D we multiply all drift lengths by D and choose the system field gradient according to

$$g = \frac{PK^2}{.003} = \frac{P}{.003} \ \frac{\theta^2}{P^2} \ . \qquad 2.15$$

We note that a four quadrupole identity section of a given length is completely determined when the three dimensionless ratios r, ℓ, and d are specified.

A typical system is shown below:



In Fig. 1 we have plotted θ as a function of ℓ for values of d equal to .01, .02, .03 and .04. As would be expected, we see that if we use longer magnets (larger d-values) the values of θ become smaller, i.e. we can use weaker magnet gradients.

b. Minimum Gradient System

It is apparent that each curve shown in Fig. 1 goes through a minimum at $\ell = 1/4$. This means that for a system (with fixed D, d) the minimum gradient occurs when the drift spaces between the magnets are equal in length since from equation 2.12, $\ell = 1/4$ implies also that $\ell_2 = 1/4$. These minimum gradient (maximum focal length) systems were also predicted by the thin lens analysis of Appendix II. For these systems, we can find a relationship between the total system length and the quadrupole constant K. We set the quantity under the square root sign in equation 2.10 equal to zero and obtain

$$\left(\frac{D}{4} - L\right)^2 = \frac{2\coth\phi\cot\phi}{K^2}\left\{1 + K\left(\frac{D}{4} - L\right)(\tanh\phi - \tan\phi)\right\},$$

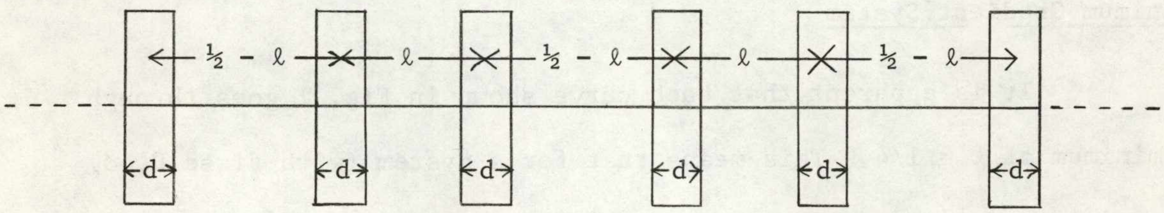
which can be put in the form

$$D = \frac{1}{K}\left(L + (\cot\phi - \coth\phi) + \sqrt{\coth^2\phi + \cot\phi^2}\right). \qquad 2.15$$

We can get a feeling for the strength of magnets required for typical identity sections by plotting K as a function of D. In Figure 2 is shown such a curve with the parameter L fixed at 0.4 meters. The dashed lines represent the gradients required for a beam with average particle momentum of 1.09 Gev/c (corresponding to 500.0 Mev protons). We see that for systems less than 5.5 meters gradients of greater than 1000 Gauss/cm. will be required.

c. Infinite Periodic System

We can use the theory just developed to form a periodic system consisting of an infinite number of symmetric thick lens doublets as shown below.



Here, d and ℓ are the dimensionless parameters defined previously and the magnet field gradients are obtained from equation 2.15 where θ satisfies equation 2.14. Any segment of length 1 unit is an identity section independent of where the start of the unit length section is taken (since the curves in Fig. 1 are symmetric about $\ell = 1/4$). In fact any section of length m (m is a positive integer) has a transformation matrix T given by

$$T = M^{2m} = (- 1)^{m} I , \qquad 2.14$$

where M denotes the matrix representing each doublet.

It is known that the condition for overall focussing in a periodic system of quadrupoles is

$$|\text{trace } M| < 2 \,, \qquad 2.15$$

where M is the matrix for one period.

From the derivation given for the identity sections (recall equation 2.3) we note that the infinite system under discussion satisfies this criterion.*

d. Recovery of Thin Lens Formulae as L Approaches Zero

As $L \to 0$ the thick lens relationships established in this section reduce to those found in Appendix I using the thin lens approximation. Thus equation 2.9 becomes

$$D_2 \simeq \frac{D}{2} - (D_1 + D_3) \,, \qquad 2.16$$

and taking $\tanh\phi \simeq \tan\phi \simeq KL$,

in equation 2.10 we get

$$D_1 + D_3 \simeq \frac{D}{4} \pm \left[\left(\frac{D}{4}\right)^2 - \frac{2}{K^4L^2}\right]^{\frac{1}{2}} .$$

Squaring to remove the radical gives

$$\left(D_1 + D_3 - \frac{D}{4}\right)^2 \simeq \left(\frac{D}{4}\right)^2 - \frac{2}{K^4L^2} \,,$$

or

$$f \simeq \frac{1}{K^2L} \simeq \left[\frac{(D_1 + D_3)}{2}\left\{\frac{D}{2} - (D_1 + D_3)\right\}\right]^{\frac{1}{2}} \,, \qquad 2.17$$

where f is the thin lens focal length.

In addition, the second and fourth drift lengths of the identity section are equal, due to the assumption of symmetry ($M^2 = -I$). Thus equations 2.16 and 2.17 are equivalent to the thin lens formulae found in Appendix II.

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* Evidently, from equation 2.14, the trace of the periodic system oscillates between -2 and +2 with a period of length 2 units.

3. Acceptance Calculations for Thick Lens Identity Sections

The acceptance of a beam transport system is defined as that area in displacement-divergence phase space which contains the initial conditions of all particles that will pass through the system. The UVic IBM 360/44 computer was used to calculate the acceptance, assuming a four inch diameter beam tube, for the four quadrupole thick lens identity sections described in Section 2. The computations were done using the following parameter values:

$D = 7.0, 10.0, 13.0, 16.0, 19.0$ meters,

$L = 0.4$ meters (all magnet effective lengths were equal),

$r = 1.0$ (first and last drift lengths were equal).

The results for the xz-plane are shown in Fig. 3.* We see that in each case the acceptance is a slowly varying function of the independent variable ℓ, with a minimum at $\ell = 1/4$. These curves are a bit deceptive in that the actual acceptance area shapes change radically with changing ℓ-values. This effect is seen in Fig. 4. The acceptance areas for both horizontal and vertical planes for a 10.0 meter identity section with $\ell = .05, .25,$ and $.45$ are shown. The acceptance areas in the two planes are related to each other via a reflection through the ordinate or abscissa.

* The results for the yz-plane were the same. This can be predicted theoretically by noting that, for systems symmetrical about the midpoint, a particle traversing the system in one plane experiences the same focussing as one travelling in reverse order in the other plane.

It can be seen from Fig. 4 that large acceptance areas with small divergences are obtained by using ℓ-values close to 1/2. Acceptance shapes for identity sections of length other than 10.0 meters were investigated and similar results were obtained.

From Figs. 3 and 4 it is seen that there is no single value of ℓ which should be chosen in all design situations. If the phase space area representing the input beam is small (much less than the acceptance with $\ell = 1/4$) then the minimum gradient system is the most economical. When the input beam has a large phase space area (due to multiple scattering from a target, say), then one may have to increase or decrease the ℓ-value in order to accommodate the beam using reasonable quadrupole apertures.

For the 10.0 meter identity section with $\ell = .25$ a plot* was also made of acceptance as a function of particle momentum. It turns out that the maximum acceptance does not occur at the design momentum (1.09 Gev/c) but at a higher value. This indicates that identity sections would probably not be useful in designing a system (e.g. a muon channel) where maximum acceptance is required.

* See Figure 5.

ACKNOWLEDGMENTS

The author wishes to thank Dr. R.M. Pearce for many interesting discussions regarding this report and in particular for pointing out the relationship of this work to that of Wolfe. The author also wishes to thank Dr. D. Lobb and Dr. L.P. Robertson for their valuable comments during the progress of the work. The author also wishes to thank Mr. B. Gray for checking some of the algebra for the thin lens analysis.

The acceptance calculations for Fig. 3 and Fig. 4 were done on a computer program written by B. Harrison under the direction of Dr. Lobb. The calculations for Fig. 5 were done on the computer program "IPSO FACTO".

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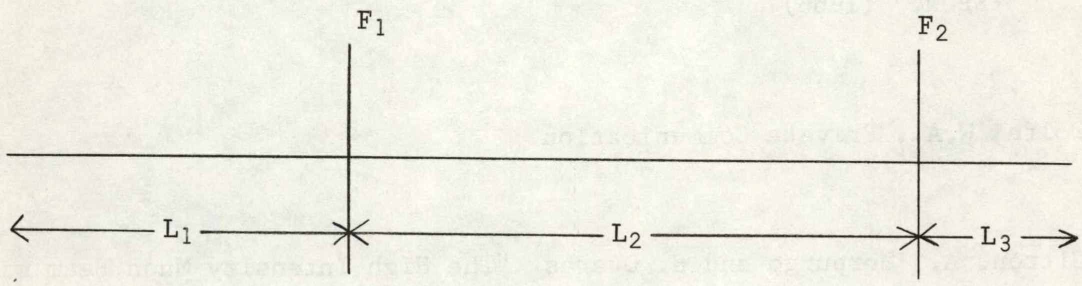
Appendix I.

Non-Existence of Doublet or Triplet Thin Lens Identity Sections

a. Doublet

We consider the thin lens quadrupole doublet shown below.



Here, F_1 and F_2 are the focal lengths and L_1, L_2, L_3 are the drift lengths for the system. The transformation matrix components for this system (for the xz-plane) are obtained from

$$T = \begin{pmatrix} 1 & L_3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1/F_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & L_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1/F_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & L_1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \begin{array}{l} 1+L_2/F_1+L_3/F_1+L_3/F_2 \\ \qquad +L_2L_3/(F_1F_2) \end{array} & \begin{array}{l} L_1+L_2+L_3+L_1L_2/F_1+L_1L_3/F_2+L_2L_3/F_2 \\ \qquad +L_1L_3/F_1+L_1L_2L_3/(F_1F_2) \end{array} \\ \\ 1/F_1+1/F_2+L_2/(F_1F_2) & 1+L_1/F_1+L_1/F_2+L_2/F_2+L_1L_2/(F_1F_2) \end{pmatrix} . \qquad \text{I-1}$$

The transformation matrix components for the yz-plane are the same except that

$$F_1 \rightarrow -F_1 ,$$

$$F_2 \rightarrow -F_2 .$$

If the system is to be an identity section, the off diagonal components of the matrix T must be zero. Applying this condition to component T_{21} in both planes gives

$$1/F_1 + 1/F_2 + L_2/(F_1F_2) = 0 \; , \tag{I-2}$$

$$- 1/F_1 - 1/F_2 + L_2/(F_1F_2) = 0 \; . \tag{I-3}$$

Adding equations I-2 and I-3 yields

$$L_2/(F_1F_2) = 0 \; . \tag{I-4}$$

For finite F_1 and F_2 this equation has only the solution

$$L_2 = 0 \; , \tag{I-5}$$

and putting this in equation I-2 we get

$$F_1 = - F_2 \; . \tag{I-6}$$

Substituting equations I-5 and I-6 into I-1 shows that the transformation matrix for both the xz- and yz-planes becomes

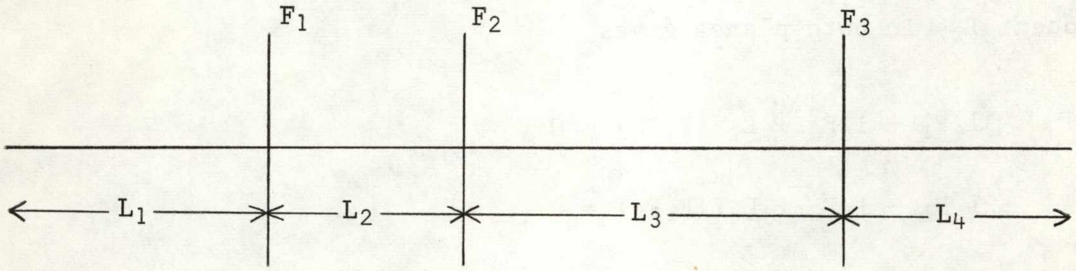
$$\begin{pmatrix} 1 & L_1 + L_3 \\ 0 & 1 \end{pmatrix} ,$$

and we conclude that an identity section occurs only in the trivial case that

$$L_1 = L_3 = 0 \; .$$

b. Triplet

We consider the thin lens quadrupole triplet shown below.



Here, L_1 , L_2 , L_3 , L_4 are the drift lengths and F_1 , F_2 , F_3 are the focal lengths for the system. To make the calculations easier we write the system transformation matrix as

$$T = SM , \qquad \text{I-7}$$

where M represents a doublet and S represents the last thin lens and drift space. For the xz-plane the components of M are given by equation I-1 and for S we have

$$S = \begin{pmatrix} 1 & L_4 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1/F_3 & 1 \end{pmatrix} = \begin{pmatrix} 1 + L_4/F_3 & L_4 \\ 1/F_3 & 1 \end{pmatrix} . \qquad \text{I-8}$$

The transformation matrices for the yz-plane are the same except that

$$F_1 \rightarrow - F_1 ,$$

$$F_2 \rightarrow - F_2 ,$$

$$F_3 \rightarrow - F_3 .$$

The condition for the system to be an identity section is

$$T = SM = \pm I ,$$

or

$$M = \pm S^{-1} = \begin{pmatrix} \pm S_{22} & \mp S_{12} \\ \mp S_{21} & \pm S_{11} \end{pmatrix} ,$$

where we have used

$$\det S = 1 .$$

In particular, we have

$$M_{21} = \mp S_{21} \quad , \tag{I-9}$$

which for the horizontal plane is equivalent to (from equations I-8 and I-1)

$$1/F_1 + 1/F_2 + L_2/(F_1F_2) = \mp 1/F_3 \quad , \tag{I-10}$$

and similarily for the vertical plane

$$- 1/F_1 - 1/F_2 + L_2/(F_1F_2) = \pm 1/F_3 \quad . \tag{I-11}$$

Adding equations I-10 and I-11 leads to

$$L_2/(F_1F_2) = 0 \quad ,$$

and for finite F_1 and F_2 we have

$$L_2 = 0 \quad . \tag{I-12}$$

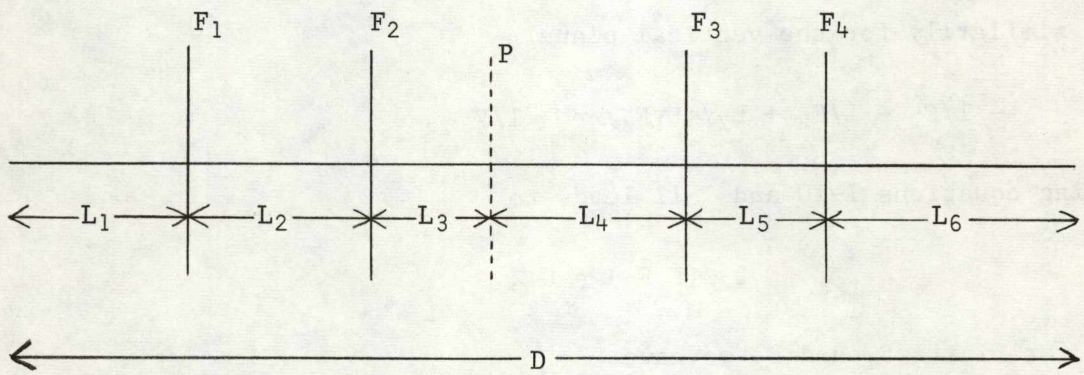
Since L_2 is zero, the first and second thin lenses are adjacent and we are effectively back to the doublet case which has already been treated. Hence there are no thin lens triplet identity sections.

Appendix II.

Thin Lens Identity Sections

a. Four Thin Lens Quadrupole Systems

We consider the four thin lens quadrupole system shown below.



Here F_1 , F_2 , F_3 , F_4 are the focal lengths and L_1 , L_2 , L_3 , L_4 , L_5 , L_6 are the drift lengths for the system. D is the total system length. We note that this system has been divided at some arbitrary point P between the two middle thin lenses so that we have effectively two doublets. This division does not restrict the system in any way and will simplify the subsequent calculations.

If we label the transformation matrices for the two doublets by M and D then the condition for an identity section is

$$T = RM = \pm I ,$$

or $$M = \pm R^{-1} = \begin{pmatrix} \pm R_{22} & \mp R_{12} \\ \mp R_{21} & \pm R_{11} \end{pmatrix},$$

where we have used $\det R = 1$.

In terms of matrix components we get

$$M_{11} = \pm R_{22} \,, \qquad \text{II-1}$$

$$M_{12} = \mp R_{12} \,, \qquad \text{II-2}$$

$$M_{21} = \mp R_{21} \,, \qquad \text{II-3}$$

$$M_{22} = \pm R_{11} \,, \qquad \text{II-4}$$

where the upper sign is for $T = + I$.

Using the matrix components for a doublet (given in equation I-1) we get for the xz-plane the following four equations:

- from equation II-1

$$1+L_2/F_1+L_3/F_2+L_3/F_1+L_2L_3/(F_1F_2) = \pm(1+L_4/F_3+L_4/F_4+L_5/F_4+L_4L_5/(F_3F_4)) \,, \qquad \text{II-5}$$

- from equation II-2

$$L_1+L_2+L_3+L_1L_2/F_1+L_1L_3/F_2+L_2L_3/F_2 +L_1L_3/F_1+L_1L_2L_3/(F_1F_2) = \mp(L_4+L_5+L_6+L_4L_5/F_3+L_4L_6/F_4+L_5L_6/F_4 +L_4L_6/F_3+L_4L_5L_6/(F_3F_4)) \,, \qquad \text{II-6}$$

- from equation II-3

$$1/F_1+1/F_2+L_2/(F_1F_2) = \mp(1/F_3+1/F_4+L_5/(F_3F_4)) \,, \qquad \text{II-7}$$

- from equation II-4

$$1+L_1/F_1+L_1/F_2+L_1L_2/(F_1F_2) = \pm(1+L_5/F_3+L_6/F_4+L_6/F_3+L_5L_6/(F_3F_4)) \,. \qquad \text{II-8}$$

Similarly, for the xz-plane we have

$$1-L_2/F_1-L_3/F_2-L_3/F_1+L_2L_3/(F_1F_2) = \pm(1-L_4/F_3-L_4/F_4-L_5/F_4+L_4L_5/(F_3F_4)) \,, \qquad \text{II-9}$$

$$L_1+L_2+L_3-L_1L_2/F_1-L_1L_3/F_2 -L_2L_3/F_2-L_1L_3/F_1+L_1L_2L_3/(F_1F_2) = \mp(L_4+L_5+L_6-L_4L_5/F_3-L_4L_6/F_4-L_5L_6/F_4 -L_4L_6/F_3+L_4L_5L_6/(F_3F_4)) \,, \qquad \text{II-10}$$

$$-1/F_1-1/F_2+L_2/(F_1F_2) = \mp(-1/F_3-1/F_4+L_5/(F_3F_4)) , \qquad \text{II-11}$$

$$1-L_1/F_1-L_1/F_2-L_2/F_2+L_1L_2/(F_1F_2) = \pm(1-L_5/F_3-L_6/F_4-L_6/F_3+L_5L_6/(F_3F_4)) . \qquad \text{II-12}$$

It is useful to consider the total system length as a parameter. We write

$$L_1 + L_2 + L_3 + L_4 + L_5 + L_6 = D . \qquad \text{II-13}$$

Since $\det M = \det R = 1$ only six of the eight equations II-5 to II-12 are independent. We have in total seven independent equations in the nine unknowns L_1, L_2, $L_3 + L_4$, L_5, L_6, F_1, F_2, F_3 and F_4. In the above, the central drift space for the system (given by $L_3 + L_4$) is counted as a single unknown since its division into lengths L_3 and L_4 was made only for convenience and has no physical significance.

To solve this set of equations, we manipulate as follows:

equation II-7 + II-11 gives

$$F_3F_4/F_1F_2 = \mp L_5/L_2 , \qquad \text{II-14}$$

equation II-7 - II-11 gives

$$1/F_1 + 1/F_2 = \mp(1/F_3 + 1/F_4) , \qquad \text{II-15}$$

equation II-5 + II-9 gives

$$L_2L_3/(F_1F_2) = (-1 \pm 1 \pm L_4L_5/(F_3F_4)) . \qquad \text{II-16}$$

If we choose the upper sign we get

$$F_3F_4/(F_1F_2) = L_4L_5/L_2L_3 = (L_5/L_2)\, L_4/L_3 ,$$

and from equation II-14

$$L_4/L_3 = -1 .$$

This result indicates that there are no feasible solutions since the middle drift space of the system, $L_3 + L_4$, is zero and we are back to the three lens case. Hence there are no four lens identity sections with $T = + I$.

We will therefore consider only the case where $T = - I$.

Then in equation II-16 we use the lower sign to get

$$L_2L_3/(F_1F_2) = - 2 - L_4L_5/(F_3F_4) \quad . \qquad \text{II-17}$$

From equation II-8 + II-12

$$L_1L_2/(F_1F_2) = - 2 - L_5L_6/(F_3F_4) \quad , \qquad \text{II-18}$$

and from equation II-18 - II-17

$$F_3F_4/(F_1F_2) = \frac{L_5}{L_2}(L_4 - L_6)/(L_1 - L_3) \quad . \qquad \text{II-19}$$

Substituting equation II-14 into II-19 gives

$$L_4 + L_3 = L_6 + L_1 \quad . \qquad \text{II-20}$$

We have also equation II-5 - II-9

$$(L_2 + L_3)/F_1 + L_3/F_2 = - L_4/F_3 - (L_4 + L_5)/F_4 \quad , \qquad \text{II-21}$$

and equation II-8 - II-12

$$L_1/F_1 + (L_1 + L_2)/F_2 = - (L_5 + L_6)/F_3 - L_6/F_4 \quad . \qquad \text{II-22}$$

Then equation II-21 + II-22 combined with II-13 yields

$$(1/F_1 + 1/F_2) = (1/F_3 + 1/F_4)(1 - D/(L_1+L_2+L_3)) \, . \qquad \text{II-23}$$

Taking equation II-23 - II-15 gives

$$0 = (1/F_3 + 1/F_4)(D/(L_1+L_2+L_3)) \quad .$$

We ignore the trivial solution $D = 0$ and get

$$1/F_3 + 1/F_4 = 0 \quad ,$$

and it follows from the above equation and II-15 that

$$F_3 = - F_4 \quad ,$$
$$F_1 = - F_2 \quad . \qquad \text{II-24}$$

We define $$F \equiv F_1 = - F_2 ,$$ II-25
and $$G \equiv F_3 = - F_4 .$$

Now from equation II-14 $$G/F = \sqrt{L_5/L_2} .$$ II-26

Substituting equation II-25 in II-21 we find

$$G/F = L_5/L_2 ,$$

and from equation II-26 we have

$$\sqrt{L_5/L_2} = L_5/L_2 ,$$

with solutions $$L_5/L_2 = 0, 1 .$$

We discard the zero solution as this leads us back to the three lens case.

We are left with $$L_5 = L_2 ,$$ II-27

and from equation II-26 we get $F = G$. II-28

It is apparent from equations II-25 that the four focal lengths for the system are equal in absolute value and alternate in sign.

We can determine L_2, L_5 in terms of D and $L_3 + L_4$.

From equations II-13 and II-27 we obtain

$$L_2 + L_5 = 2\,L_2 = 2\,L_5 = D - (L_1 + L_3 + L_4 + L_6) ,$$

and using equation II-20

$$L_2 = L_5 = D/2 - (L_3 + L_4) .$$ II-29

Substitution of equations II-27, II-28, II-25 and II-29 into equation II-16 gives the system focal length also in terms of D and $L_3 + L_4$.

It is $$F = \left[\left\{\frac{D}{2} - (L_3 + L_4)\right\}\frac{(L_3 + L_4)}{2}\right]^{\frac{1}{2}} .$$ II-30

We have now found the general solution for an identity section of length D. If we define the dimensionless quantities

$$
\begin{aligned}
\ell &= (L_3 + L_4)/D \ , \\
\ell_i &= L_i/D \ , \qquad i = 1, 2, 5, 6 \\
r &= L_6/L_1 \ , \\
f &= F/D \ ,
\end{aligned}
\qquad \text{II-31}
$$

then we can summarize the results obtained so far as follows:

- from equation II-30 $\quad f = \left[(\tfrac{1}{2} - \ell)\,\frac{\ell}{2}\right]^{\frac{1}{2}} , \quad 0 < \ell < \tfrac{1}{2}$ $\qquad$ II-32

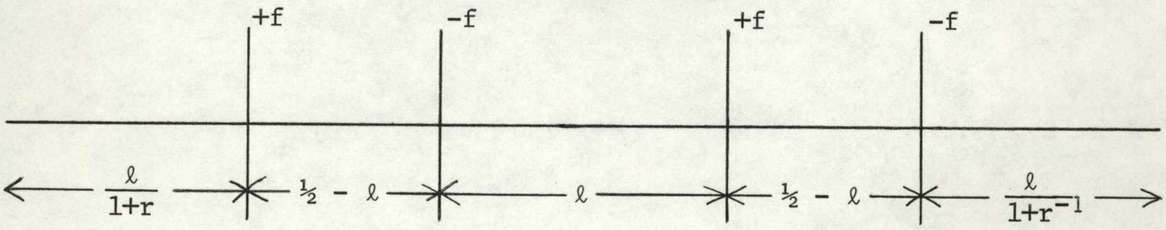
- from equation II-29 $\quad \ell_2 = \ell_5 = \tfrac{1}{2} - \ell \ ,$ $\qquad$ II-33

- from equation II-20 $\quad \ell_1 + \ell_6 = \ell \ ,$ $\qquad$ II-34

with

$$
\begin{aligned}
\ell_1 &= \frac{\ell}{1 + r} \ , \\
\ell_2 &= \frac{\ell}{1 + r^{-1}} \ .
\end{aligned}
\qquad 0 < r \qquad \text{II-35}
$$

Using this notation, the total system length is equal to unity and to scale to a system of length D we multiply all focal lengths and drift lengths by D. We note that an identity section of a given length is completely determined when the two dimensionless ratios ℓ and r are specified. The system is as shown below.



We see that this system has a fundamental symmetry. It can be regarded as two identical symmetric doublets of length one-half a unit; i.e., if the doublet matrix is denoted by M, then the identity section transformation matrix is M^2.

b. Maximum Focal Length System

We can fix the degree of freedom specified by ℓ by requiring that the system focal lengths be a maximum.

We have $$\frac{df}{d\ell} = \frac{d}{d\ell}\left[(\tfrac{1}{2} - \ell)\frac{\ell}{2}\right]^{\frac{1}{2}} = 0 ,$$

which yields $$\ell = 1/4 . \qquad \text{II-36}$$

From equations II-32 to II-35 the rest of the system is then specified by

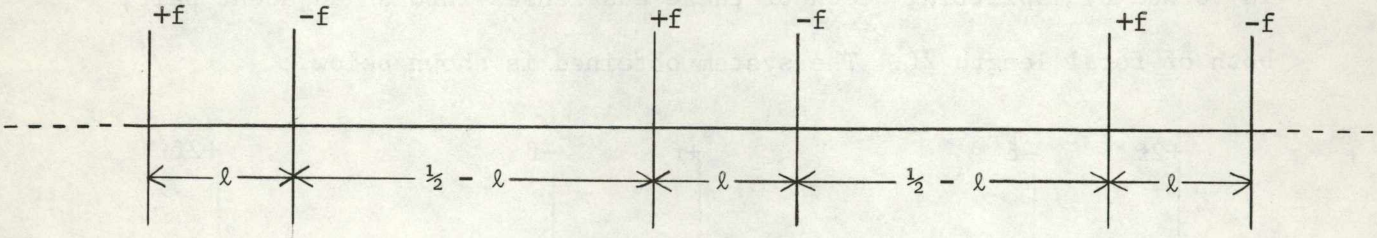
$$\ell_2 = \ell_5 = 1/4 ,$$

$$\ell_1 + \ell_6 = 1/4 ,$$

and $$f = \frac{1}{4\sqrt{2}} .$$

c. Infinite Periodic System

We now apply the results of the previous section to the periodic system consisting of an infinite number of thin lens quadrupole doublets all of length one-half unit. This system is shown below.



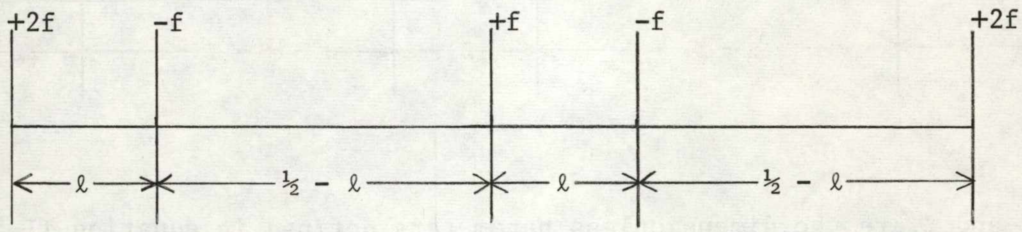
Here, f and ℓ are the dimensionless parameters defined in equation II-31. If we restrict f and ℓ by the relation

$$f = \left[\left(\tfrac{1}{2} - \ell\right)\frac{\ell}{2}\right]^{\frac{1}{2}} , \qquad 0 < \ell < \tfrac{1}{2} \qquad \text{II-36}$$

then from equations II-32 to II-34 any section of unit length is an identity section independent of where the start of the unit section occurs. In fact, any section of length n units (n is a positive integer) has the transformation matrix $(-1)^n I$.

d. Five Thin Lens Quadrupole System

A particular section of this periodic system has been discussed by Wolfe[3]. In this case, the end points of a unit length section lie on the "centers" of two thin lenses and a five quadrupole identity section is formed by "splitting" each of these end lenses into an adjacent pair, both of focal length 2f. The system obtained is shown below.



Wolfe considered the case of maximum focal length and obtained the value

$$\ell = 1/4 \quad .$$

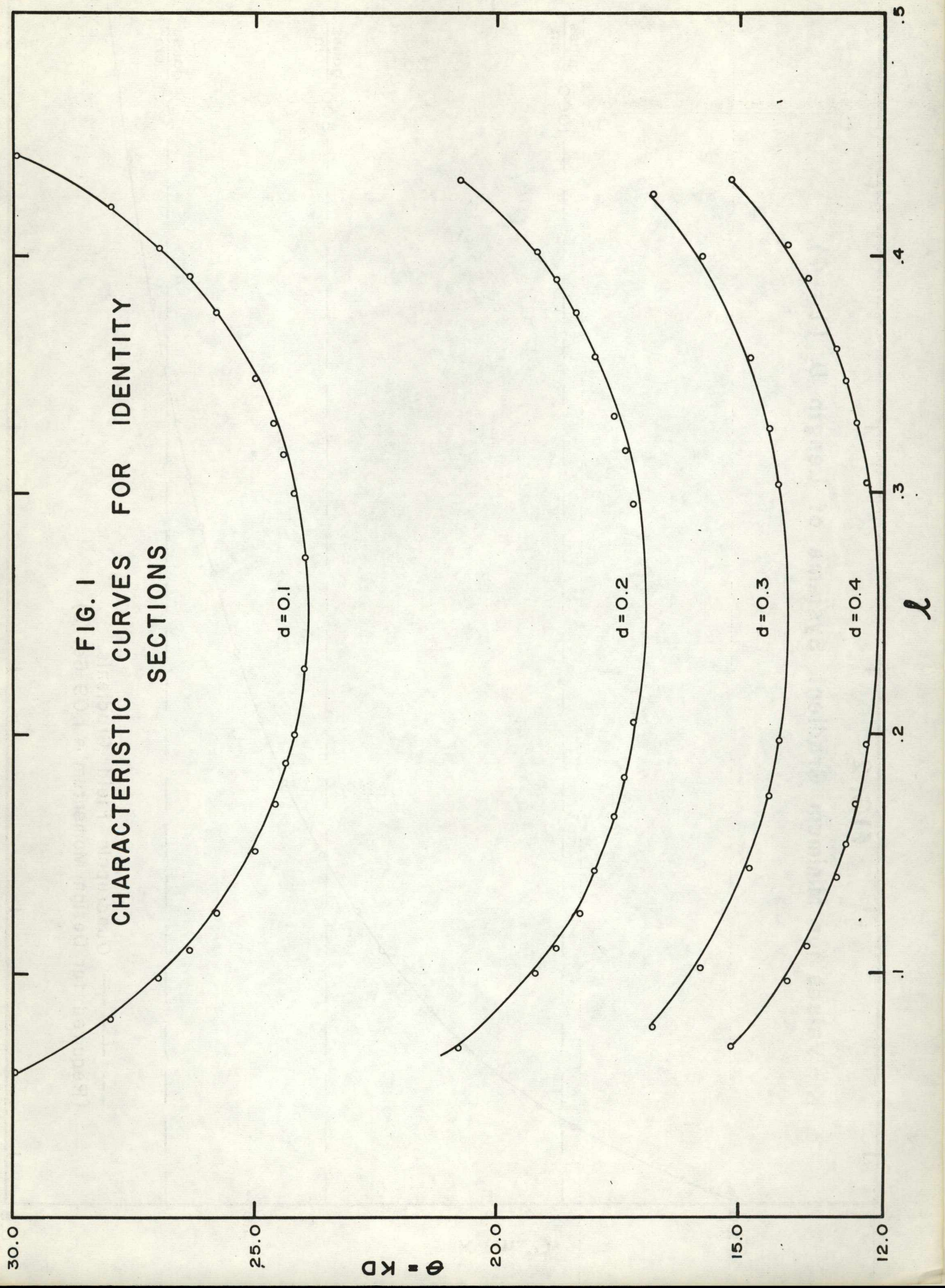


FIG. I CHARACTERISTIC CURVES FOR IDENTITY SECTIONS

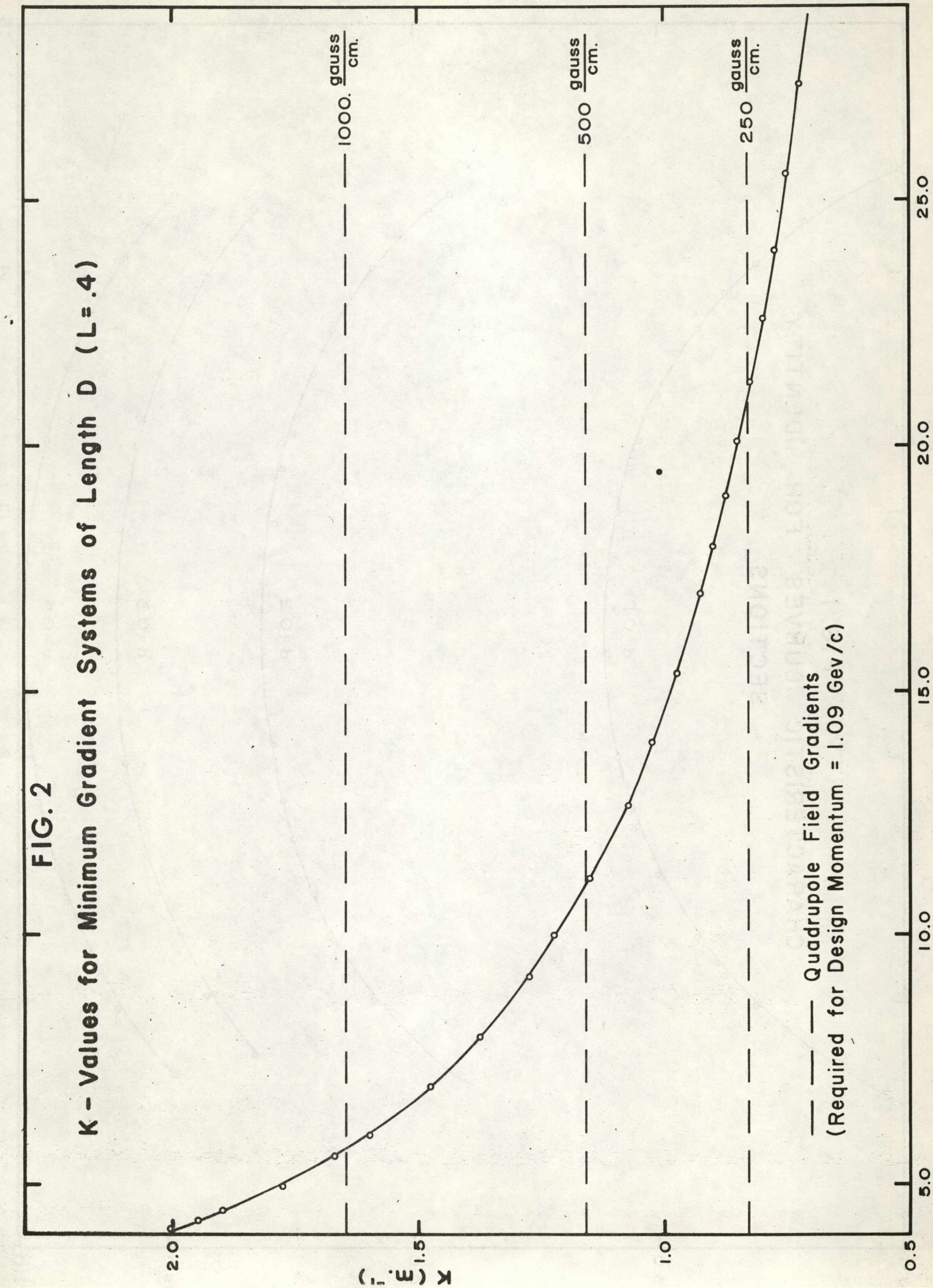
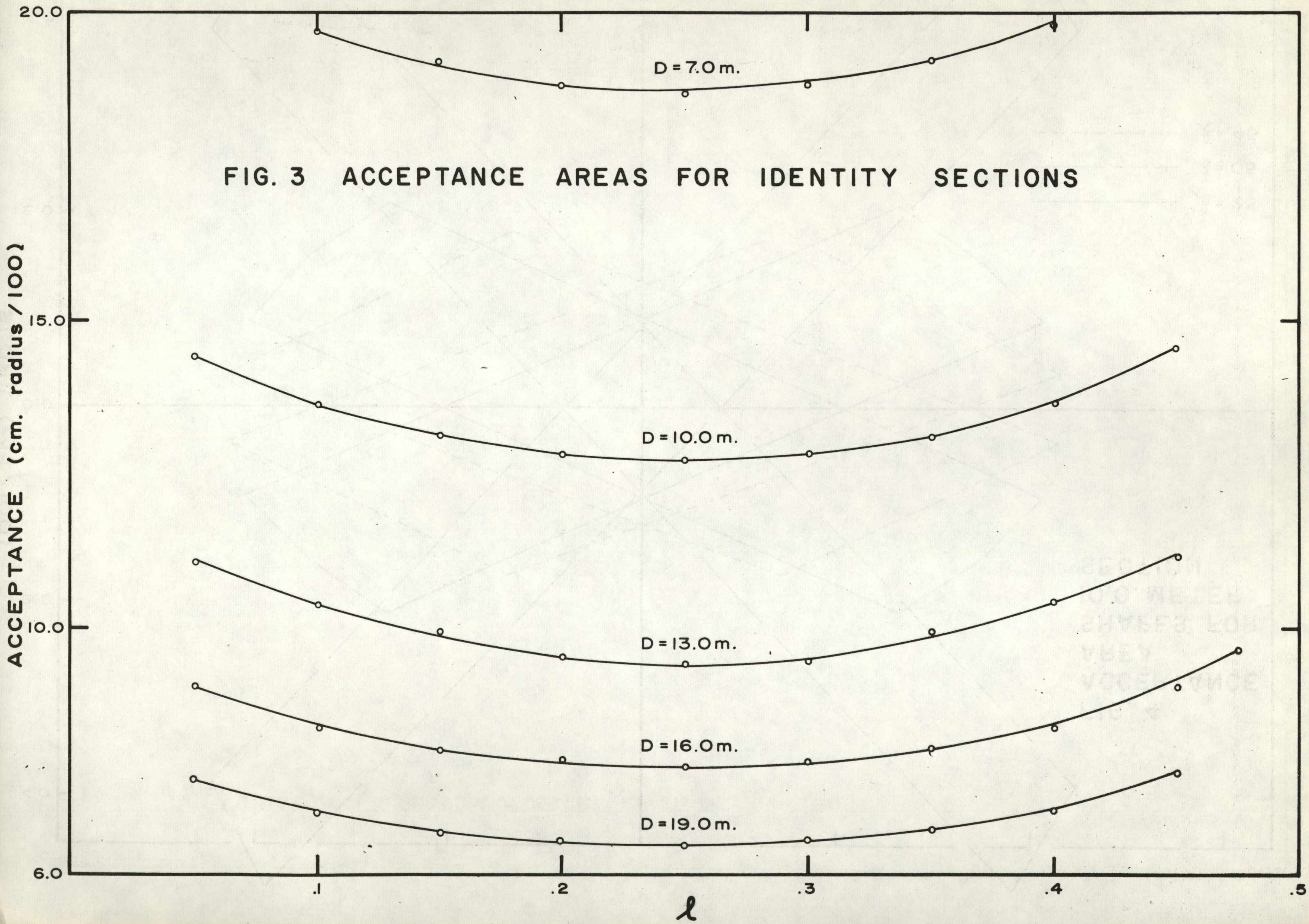


FIG. 2

K - Values for Minimum Gradient Systems of Length D (L=.4)

FIG. 3 ACCEPTANCE AREAS FOR IDENTITY SECTIONS



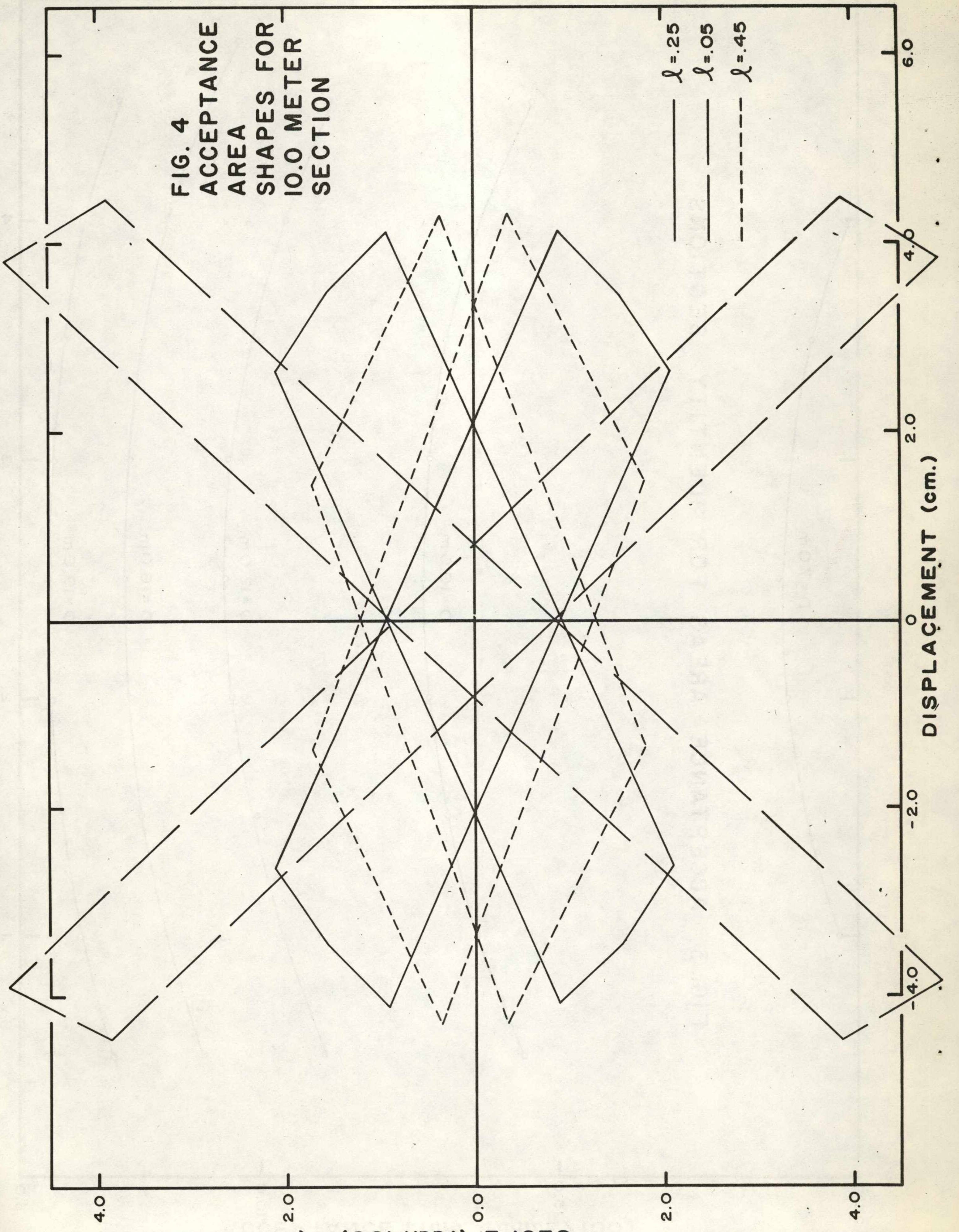


FIG. 4
ACCEPTANCE
AREA
SHAPES FOR
10.0 METER
SECTION
ℓ =.25
ℓ =.05
ℓ =.45
6.0
4.0
2.0
0
-2.0
-4.0
DISPLACEMENT (cm.)
4.0
2.0
0.0
2.0
4.0

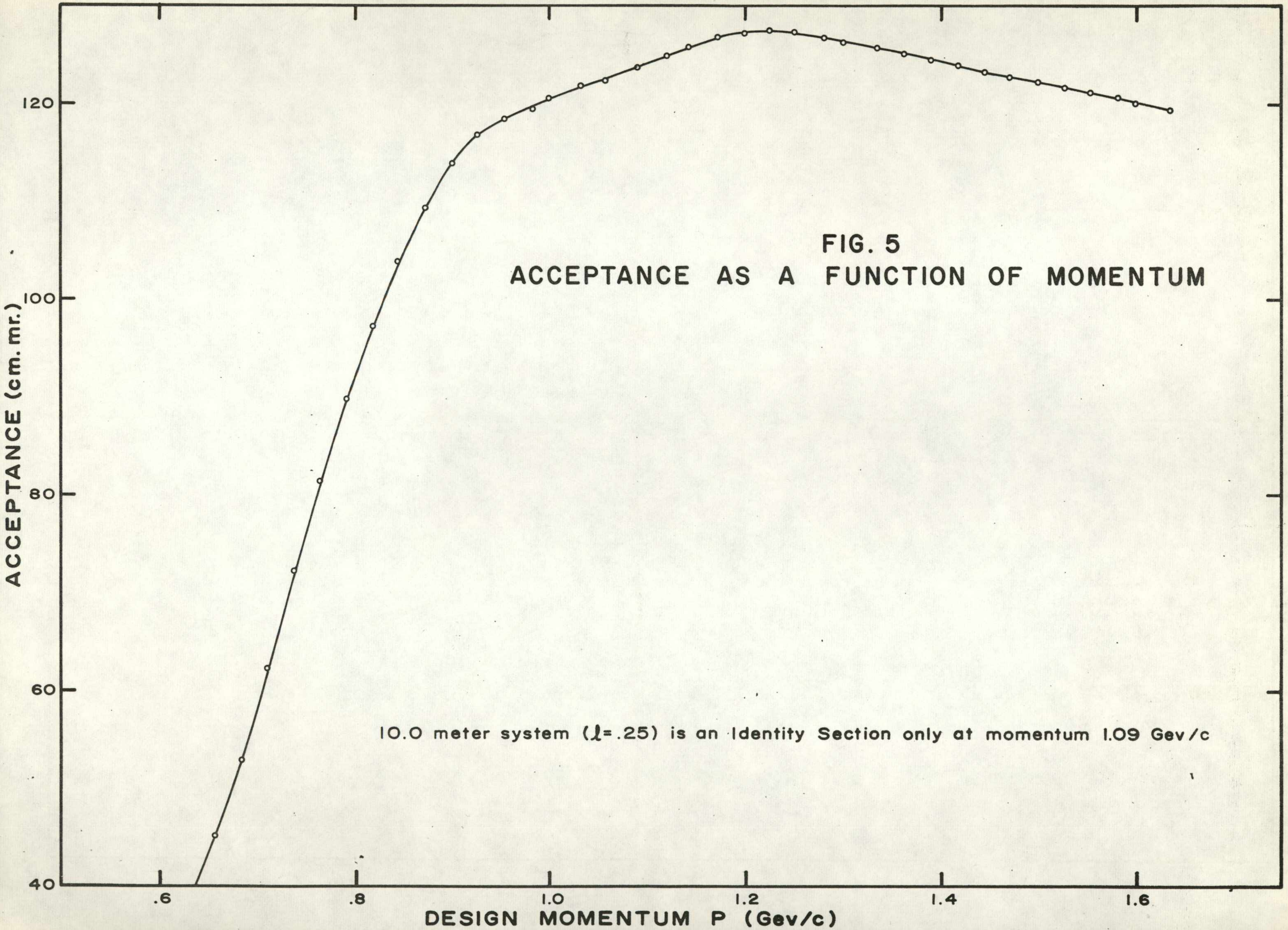


FIG. 5
ACCEPTANCE AS A FUNCTION OF MOMENTUM