d-Realizers and the Minimal Graph Without One

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We explore the work of Evans et al. [2], who in turn have built on Schnyder’s [1] definition of the dimension of a graph, and extended Schnyder woods to higher dimensions. Here we discuss d-realizers: sequences of \( d \) permutations on a set of vertices required to have empty intersection and \( d \) ‘suspension’ vertices. We will present a minimal graph having no d-realizer, and numerous graphs on 5, 6 and 7 vertices that do have one. Finally, we consider what possible characterization of graphs having a d-realizer could extend the triangulated-graph characterization found by Schnyder for 3 dimensional graphs.

1 Background

In Schnyder’s 1989 paper [1], he introduced a notion of dimension for graphs via a set of permutations he termed a \( d \)-dimensional representation. Working from a later paper by Evans et al. [2], we will examine how Shnyder’s results extend to dimensions higher than 3.

Both papers work with total orderings on the set of vertices \( V \), generally of some graph \( G \). A few basic definitions are in order. Given 2 total orderings, \( <_1, <_2 \), their intersection is the partial order containing all relationships \( <_1, <_2 \) agree on. To be precise: for \( x, y \in V \), \( x < y \) is in the intersection of \( <_1 \) and \( <_2 \) if and only if \( x <_1 y \) and \( x <_2 y \). One can easily see how this extends to the intersection of arbitrarily many orderings, with \( x < y \) in the intersection of \( <_1, \ldots, <_d \) if and only if \( x <_i y \) for all \( i \in \{1, \ldots, d\} \). An anti-chain, then, is defined to be any set of orders with empty (total) intersection. In Schnyder’s paper he
refers to this as the ‘vertex property’ of a sequence of orders.

Schnyder also introduces a second property:

**Definition 1.1.** Given some graph $G$ and sequence of orders $<_{1},...,<_{d}$, a pair of vertices $x,y \in V(G)$ is said to be a candidate pair if for all $z \in V(G)$, with $z \neq x,y$ there exists some $i \in \{1,..,d\}$ s.t. $z >_{i} x$ and $z >_{i} y$.

In [1] he refers to any graph satisfying this as having the edge property, for that sequence of orders.

We now have enough to define the aforementioned d-dimensional representations.

**Definition 1.2.** A d-dimensional representation on a set $V$ is an anti-chain $<_{1},...,<_{d}$ of total orders on $V$. This anti-chain is said to represent all graphs on $|V|$ vertices with an edge set such that the edge property holds, or equivalently, graphs with every edge formed by a candidate pair.

Note that by the above definition, if a graph has some d-dimensional representation then any subgraph, with the same set of vertices, will also.

Schnyder’s leap was in extending the definition of dimension from partially ordered sets, or posets, to graphs. He did so by constructing a poset from any given graph, but later gave an equivalent definition managing to skip the poset business entirely, and so this we present:

**Definition 1.3.** A graph $G$ with vertex set $V$ is said to have dimension $\leq d$ if and only if there exists an anti-chain $<_{1},...,<_{d}$ of total orders on $V$ under which every edge of $G$ is a candidate pair - that is if a d-dimensional representation of $G$ exists. If $d$ is the smallest number for which such a sequence exists, we say the dimension of $G$ equals $d$.

With this Schnyder obtained his famous characterization of planar graphs as those of dimension $\leq 3$. Other interesting results include showing all dimension 2 graphs were paths or subgraphs of paths, and that the only dimension 1 graph is a single node.

What we are most interested in now, however, is not simply all graphs represented by some anti-chain $<_{1},...,<_{d}$, but those induced by one:

**Definition 1.4.** A graph $G$ is induced by the anti-chain $<_{1},...,<_{d}$ if edge $(x,y)$ appears in $E(G)$ if and only if $x,y$ is a candidate pair with respect to $<_{1},...,<_{d}$.

Note that such a graph is unique to its sequence of orderings, containing the maximal number of edges possible for a graph with that representation.
Schnyder considered arbitrary d-dimensional representations, requiring only that their total orderings form an anti-chain and the edge property hold for the graph in question. For simplicity however, we will restrict ourselves to a special class of d-dimensional representation, which Schnyder defines as follows:

**Definition 1.5.** A d-dimensional representation \(<_1, \ldots, _d\) on a set \(V\) is **standard** if \(|V| \geq d\) and for all \(i \neq j\) the maximal element of \(_i\) is one of the \(d-1\) smallest elements of \(_j\).

Schnyder refers to the \(d\) maximal elements of a standard representation as the exterior vertices while [2] calls them the suspension vertices and an anti-chain with \(d\) suspension vertices 'suspended'. We will follow the latter convention as we are about to transition into the work by Evans et al. In that second paper, a cousin of d-dimensional representations is defined, the **d-realizer**:

**Definition 1.6.** A d-realizer is a suspended anti-chain \(<_1, \ldots, _d\) on the set \(V^+ = V \cup S\), where \(S\) is the set of suspensions. In our previous language this is simply a standard d-dimensional representation.

And now, to explore the graph corresponding to a d-realizer. We begin with a sibling to the candidate pair property, also introduced by [2]:

**Definition 1.7.** Given some graph \(G\) and anti-chain \(<_1, \ldots, _d\), a pair of vertices \(x, y \in V(G)\) has the **1-of-d-property** if there exists unique \(i \in \{1, \ldots, d\}\) s.t. \(_i \prec x \prec y\) and \(_j \succ x \succ y\) for all \(j \neq i\).

Evans et al. then defines the graph of a d-realizer \(<_1, \ldots, _d\) as \(G_R = (V^+, E^+)\) where \(V^+ = V \cup S\) again and \(E^+ = E_R \cup E_S\). \(E_S\) here is simply all edges of the clique on the suspension vertices, and \(E_R\) composed of all edges \((x, y)\) who’s vertices \(x, y\) satisfy both the candidate pair and 1-of-d properties.

This sounds very similar to our previous definition of the graph induced by a d-representation, the only differences being the added 1-of-d criterion, and the handling of the clique edges.

To resolve the latter of these difference we suggest an additional requirement on the d-realizer\(^1\). Let \(v_1, \ldots, v_d\) be the suspension vertices, already \(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_d\) are required to appear in the smallest \(d-1\) spots of \(_i\) in the realizer. Perhaps regrettably, we will call a d-realizer standard if those \(d-1\) vertices are ordered \(v_1 < \ldots v_{i-1} < v_{i+1} < \ldots < v_d\) in

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\(^1\)The author admits this additional requirement was suggested in [2] in their proof of Theorem 1 of the section on box-representations.
<i>as well. The reader can check this guarantees all edges of the clique on the suspension vertices now satisfy the candidate pair and 1-of-d properties. And so for a standard d-realizer, an equivalent definition of the corresponding graph \( G_R \) is \( G_R = (V^+, E_R) \), since \( E_S \subseteq E_R \). Note that since the ordering of the suspension vertices within the first \( d - 1 \) elements normally has no effect on the graph \( G_R \), exactly the same graphs have a standard d-realizer as those with a non-standard realizer.

And so, given a standard d-realizer, the corresponding graph \( G_R \) is almost now almost the same as the one induced - save for the added 1-of-d condition. In the case \( d = 3 \) however, the 1-of-d condition becomes automatic: given any pair \( x, y \in V(G) \), \( x <_1 y, x <_2 y \) and \( x <_3 y \) would imply \( x < y \) in the intersection and so our supposed d-realizer is not even an anti-chain. And so there must be \( i, j \in \{1, 2, 3\} \) with \( x <_i y \) and \( x >_j y \); then whatever order the remaining \( <_k \) choses, there will be an odd order out.

And so we have in the \( d = 3 \) case, that the set of graphs having a d-realizer is exactly the set with a standard d-dimensional representation. In [1] Schnyder proves these are precisely the maximal planar graphs. And indeed, [2] notes in the abstract that all graphs with a 3 dimensional realizer are maximal planar graphs.

Observe also that the 1-of-d criteria appears even more easily in dimensions \(< 3 \). And so one can say for \( d \leq 3 \), a graph has a d-realizer if and only if it has a d-dimensional representation. Thus the question of which graphs have a d-realizer was solved for \( d \leq 3 \) when Schnyder published in 1989. We are now interested in higher dimensions, in particular we will focus on the case \( d = 4 \).

2 The Quest for the Smallest Counter-example

In general, one would like to know when a graph does and does not have a d-realizer for some dimension \( d \). To this end we searched for a smallest graph, in vertex number, that is without a d-realizer and yet with sufficient edges to in theory have one.

In this section many of our proofs will rely on properties found in the resulting Schnyder wood of a d-realizer, and so we introduce these now. The Schnyder wood was originally defined only for 3-dimensional representations - requiring the 1-of-d property that one gets for free in \( d = 3 \). With [2]'s definition of the graph \( G_R \) of a d-realizer, and in particular their 1-of-d requirement, the Schnyder wood construction was extended to arbitrary dimensions.

\[ ^2 \text{I am not sure if they note this is a characterization.} \]
Definition 2.1. Given a d-realizer $<_{1},...,<_{d}$ and associated graph $G_{R}$ we let $T_{i}$ be the set of directed edges $(x,y)$ where $x,y$ form a candidate pair, and $y >_{i} x$ while $y <_{j} x$ for all $j \neq i$. Thus $T_{i} \cap T_{j} = \emptyset$ for $i \neq j$ and $\cup_{i} T_{i} = G_{R}$.

We will often refer to edges of $T_{i}$ as the edges of ‘colour $i$’. Evans et al. show that these $T_{i}$ carry over our favourite behaviours from the 3-dimensional Schnyder Wood case, in particular:

Proposition 2.2. Given d-realizer $<_{1},...,<_{d}$, for each $i \in \{1,...,d\}$ the directed graph $T_{i}$ forms an in-arborescence with root, $s_{i}$, being the suspension vertex associated with $<_{i}$.

Proposition 2.3. The graph $G_{R}$ of a d-realizer on $n + d$ vertices will have precisely $dn + \binom{n}{d}$ edges.

Proposition 2.4. Again, suppose we have some d-realizer, $<_{1},...,<_{d}$ for graph $G$. Then for any $j \in \{1,...,d\}$ the directed graph $\sum_{i \neq j} T_{i} + (T_{j})^{-1}$ is a acyclic.

2.0.1 A counter-example on 7 vertices

We now introduce, in figure 2.1, a graph $G^{*}$ on 7 vertices found to have no d-realizer, and yet with enough edges to support one. To be precise, the claim is that no d-realizer, $<_{1},...,<_{d}$, has this graph as its corresponding $G_{R}$. First we handle $d < 4$: as explained previously, these graphs were shown by Schnyder to be planar graphs. Since $G^{*}$ contains $K_{3,3}$, for example see figure 2.1 (b), it cannot have a d-realizer for $d \leq 3$. Meanwhile, counting edges will allow us to rule out all $d > 4$. Using proposition 2.3, we see that $d = 5$ implies $n = 2$ and so requires $10 + 10 = 20$ edges. Similarly $d = 6$ corresponds to 16 edges, $d = 7$ to 21. Since none of these numbers are 18, which the diligent reader may check is our edge count, $d \neq 5, 6$ or 7. Larger $d$ of course, simply require more suspension vertices than our graph can support and so can be ruled out immediately.

This leaves us with the case $d = 4$. At first glance the proof seems tedious, as we must allow any 4-clique of the graph to try forming the suspension vertices. Looking closely however, $G^{*}$ is highly symmetric. See the complement in figure 2.2 (a) for a clear picture. In short, there is a single vertex of degree 6, and 6 of degree 5. The 5 degree vertices have been paired up by the 3 missing edges, and so any 4-clique can contain only one vertex of each pair. Since that allows for only 3 such vertices, the 6-degree vertex must always be chosen - along with precisely 3 non-neighbouring (in the complement) degree-5 vertices. The key observation is then that all 4-cliques $H \subset G^{*}$ result in the same remainder $G \setminus H$, up to an isomorphism.

And so we need only consider the removal of an arbitrary 4-clique, see figure 2.2 (b) for the remainder on such a removal. There, the black triangle contains the non-suspension vertices, the other four having formed the 4-clique. As shown in the blue edges, there
Figure 2.1: A graph, $G^*$, on 7 vertices with no d-realizer

will be a single suspension vertex, originally the degree 6 vertex, with all non-suspension vertices already in it’s tree. Meanwhile it is clear that the red, orange and green suspension vertices (those vertices touching only red, orange or green edges in the figure) each require an additional edge to span. Thus the black triangle must contain 3 different edge colours. But then it is doomed - if the edges are arranged cyclically flipping the blue tree presents us with a cycle in $T^{-1}_{\text{blue}} + T_{\text{red}} + T_{\text{orange}} + T_{\text{green}}$, while an acyclic 3-cycle can always be made cyclic with the flip of a single edge. And so in either case, the trees would contradict proposition 2.4 and therefore no 4-realizer can exist.

3 How About Some Graphs that Do Work?

The search for a counter example involved, unsurprisingly perhaps, the discovery of many actual examples. In particular, the author believes they have found all graphs on 7 vertices with a 4-realizer - though if 3 graphs seems like a ridiculously small number the reader is advised to take this with a grain of salt.

We will start by proving our previously described counter example is indeed on the minimum number of vertices possible, and then describe in more detail our findings on $|V| = 7$.

Note that a graph on 4 or fewer vertices is planar, and so has a 3,2 or 1-realizer. Thus the first case to consider is that of $n = 5$. Here the only graph with sufficient edges is $K_5$
The complement of $G^*$

(a) The complement of $G^*$ on removing any $K_4$

(b) The remainder of $G^*$ itself, and the reader can check the following provides a valid 4-realizer for this graph (see figure 3.1):

\[
\begin{align*}
<1: & \ v_2, v_3, v_4, v_1, v_5 \\
<2: & \ v_2, v_3, v_5, v_1, v_4 \\
<3: & \ v_2, v_4, v_5, v_1, v_3 \\
<4: & \ v_3, v_4, v_5, v_1, v_2
\end{align*}
\]

Meanwhile on 6 vertices with $d = 4$ we have $n = 2$ and so by proposition 2.3, require $8 + 6 = 14$ edges. But $K_6$ only has $\binom{6}{2} = 15$ edges to begin with, and so there is only one graph to consider (see figure 3.3). Given this, we suggest the following 4-realizer:

\[
\begin{align*}
<1: & \ v_3, v_4, v_5, v_2, v_1, v_6 \\
<2: & \ v_3, v_4, v_6, v_2, v_1, v_5 \\
<3: & \ v_3, v_5, v_6, v_1, v_2, v_4 \\
<4: & \ v_4, v_5, v_6, v_1, v_2, v_3
\end{align*}
\]

As assumed in figure 3.3 (b), here $v_3, v_4, v_5, v_6$ form our clique. The reader can check the above forms a suspended anti-chain, and that an edge appears in 3.3 (a) if and only if it satisfies the candidate pair and 1-af-d properties in our 4-realizer. Since all graphs with 14 edges on 6 vertices are isomorphic (under a relabelling) to 3.3 (a), this shows all
such graphs indeed have a d-realizer.

Thus 7 is truly the smallest number of vertices it is possible to find a counter example on. We close this section with a figure displaying, supposedly, the only 3 graphs on 7 vertices who have some 4-realizer (see figure 3.3). Generating such graphs was an attempt by the author to confirm their supposed 7-vertex counter example indeed had no 4-realizer. The observant reader may be alarmed by their similarity to our counter-example, but on checking the degree sequence we find they are not actually isomorphic.

These graphs were generated by computing all possible (ordered) quadruples of permutations of 1-2-3, keeping only those with empty intersection, and then appending suspension vertices. The graphs were then ‘induced’ according to our two edge requirements and drawn in Mathematica. Mathematica was also used to remove isomorphic graphs - whittling down an original 906 to merely 3. The c++ code used can be found here: https://github.com/krstnmnlsn/d-realizers\textsuperscript{3}.

4 Thoughts on a Characterization

In $d = 3$, as explained above, we get that the set of graphs with a 3-realizer is exactly the set of maximal graphs, and so the author wondered naturally if the characterization of those graphs that have a d-realizer somehow generalized this quality.

\textsuperscript{3}though hack job would be an understatement in describing it and caution is advised
(a) The graph $K_6$ minus a single edge
(b) With an example 4-clique dropped

Figure 3.2: The only graph possible on 6 vertices is $K_6$ short one edge. For clarity (b) omits one possible choice of 4-clique, using the suspension vertices $v_1, v_2, v_3, v_4$

Figure 3.3: The 3 graphs found on 7 vertices to have a 4-realizer
Of course, without a lot of background graph-theoretic knowledge there wasn’t much hope, but wikipedia was trawled and the author now has quite a few questions. We started at the definition of maximality using planarity: where a graph is maximal if a single additional edge will always make it non-planar. But this seems a bit ugly, since there are several triangular graphs that don’t fit this definition ($C_3$ and $K_4$) - and our use of maximal certainly includes these graphs.

One interesting characterization found was that the triangulated graphs are precisely those forming the skeletons of the simplicial polyhedra, or those containing only triangular faces. We wonder then if the graphs with d-realizers form the 1-skeletons of some class of possibly simplicial d-polytopes. Unfortunately d-polytopes, even 4-polytopes or polychorons, don’t seem to be terribly well understood yet so the author was unable to find many examples or even a sanity check.

The only actual simplical polychoron we were able to find was the 4-simplex, apparently the 4-dimensional analog of a tetrahedron. The 1-skeleton of this is $K_5$, which we have shown previously to indeed have a 4-realizer.

One other fact of interest is a result found on wikipedia, that the 1-skeleton of a k-dimensional convex polytope is a k-connected graph. Perhaps this could be used to quickly find a counter-example to our above hypothesis, though Mathematica has so far confirmed the graphs previously shown to have a 4-realizer also are 4-vertex connected.

5 Unrelated Remarks

I came across 2 minor typos in [2], first in the definition of candidate pair I believe $z$ should be greater than $x$ and $y$ in some order $\pi_i$, not less. Probably it would be okay to flip absolutely everything, but currently it is inconsistent with the proof of proposition 1. The only other thing is in the first line of the proof of prop. 1 $v \in V$ should be $x \in V$, or else all the later $x$’s flipped to $v$’s.

Finally, this paper would not have been possible without Paul Liu’s life-saving sanity check abilities and Mathematica knowledge. I’d also like to thank Will Evans for the very cool project and for having the patience to accepting this paper so late.

4 and really just are curious, and not trying to suggest it is so, having no intuition or knowledge at all
5 d spanning trees should give one d-edge-connectedness, but not necessarily d-vertex-connectedness I believe
6 “By Friday” effectively means, by the time reasonable people wake up on Saturday, right?
References

