Towards the Construction of the Heterotic Moduli

by

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Abstract

We study the moduli space of the heterotic system, which holds significant importance in physics. Fixing the complex structure, we explore the moduli space by considering two different yet “dual” deformation paths starting from a Kähler solution. They correspond to deformations along the Bott-Chern cohomology class and the Aeppli cohomology class respectively. Together with the deformation of the gauge bundle, we prove the existence of heterotic solutions along these two paths using the implicit function theorem. Hence, we construct local coordinates in the neighborhood of a Kähler solution along the submanifold of fixed complex structure on the full heterotic moduli. This is an initial step in constructing the full metric of heterotic moduli.

Lay Summary

The heterotic system comprises four equations: two Hermitian-Yang-Mills equations, the balanced condition equation, and the anomaly cancellation equation. Our objective is to construct a family of solutions to the heterotic system, starting from a Kähler solution. Each point in the heterotic moduli corresponds to a solution to this system, thus forming a path originating from a Kähler point.

We adopt parallel treatments of two distinct yet 'dual' deformation paths. These paths correspond to different initial assumptions satisfying two separate partitions of the four equations of the heterotic system. Specifically, they are associated with the Bott-Chern cohomology class and the Aeppli cohomology class. Using the implicit function theorem, we establish that these classes can serve as parameters on the moduli. Consequently, local coordinates are constructed around a Kähler solution within the heterotic moduli.

Preface

This thesis is based on a joint work with Sébastien Picard.

A version of Part II has been published as a preprint arXiv:2401.05331 [S. Picard and P-L. Wu. Balanced and Aeppli Parameters for the Heterotic Moduli. 2024]

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Dedicated to my parents for their love and support.

Chapter 1

Introduction

1.1 Motivation

From a physical point of view, the heterotic system [47, 60] is a set of supersymmetry conditions on the geometry of the compactification of the heterotic string theory [45, 46]. It can be regarded as a natural extension in the Candelas-Horowitz-Strominger-Witten model [12] where the superstring compactification is a Calabi-Yau 3-fold. Studies of the moduli space of the Calabi-Yau 3-folds initiated by [16] lead to some remarkable results in mirror symmetry. Therefore, it is motivated to study its natural extension - the moduli space of the heterotic system, and the heterotic moduli in short.

By heterotic moduli, we mean a parameter space of the heterotic system, meaning that each point in the heterotic moduli corresponds to a specific solution to the heterotic system and hence corresponds to a particular internal manifold. Therefore, the geometry of the heterotic moduli encodes the information on the deformation of the internal geometry. To study this parameter space, we construct a family of solutions to the heterotic system. A natural starting point is the deformation of the Calabi-Yau 3-fold into the non-Kähler one nearby. Namely, we wish to probe this extended moduli space near a Calabi-Yau 3-fold solution, i.e. a neighborhood of a Kähler point on the heterotic moduli.

1.2 Survey

We provide a brief and incomplete survey of the studies on the heterotic system and its moduli space. From a mathematical point of view, the heterotic system consists of two main ingredients: a Hermitian manifold X and a complex vector bundle E over X. It can be seen as the geometric interaction between these two main objects.

Due to this, the heterotic system is related to various fields of pure mathematics. It has been reformulated in terms of generalized geometry, initiated by observations in [1, 22, 33], and the moment map picture [6, 39, 40]. The parabolic version of the heterotic system, known as the anomaly flow, is understood as a supersymmetric Ricci flow with α' corrections by [4, 29, 55–57]. Special solutions to the heterotic system are constructed in [20, 27, 28, 30–34, 42, 54].

The framework for understanding variations of families of solutions to the heterotic system is initiated by [1, 22, 38], with many follow-up works. In particular, the geometry of the total space is investigated in [14, 15], and the moduli space is given a Kähler structure as a consequence of supersymmetry. These works assume the existence of local coordinates on the heterotic moduli and explore the induced geometric structure of the total space.

We prove the existence of local coordinates in the case of a fixed complex structure. In our setup, we assume the existence of a background Kähler solution and probe its nearby solutions. Li and Yau [50] constructed the first family of solutions to the heterotic system by the inverse function theorem and was later generalized by Andreas and Garcia-Fernandez [2]. [19] gives a new proof where the inverse function theorem was applied to a fixed balanced class. We then use the implicit function theorem to vary the balanced class parameter and give a parallel treatment to a "dual" case in the Aeppli class.

1.3 Outline

This thesis is divided into two parts. The first part reviews the historical construction of the heterotic string theory and the heterotic system. The second part deals with the construction of the submanifold of the heterotic moduli space along the orbit of a fixed complex structure.

In Part I, we give some background in string theory and motivate the heterotic system. In Chapter 2, we review the construction of the heterotic string theory starting from the bosonic string theory in the perspective of the non-linear σ-model and then introduce the supersymmetry to obtain the superstring theory. The heterotic string theory arises from the heterosis of the bosonic string theory and the superstring theory with appropriate toroidal compactifications. The equations of motion of the low-effective field theory from the heterotic string give rise to the heterotic system upon flux compactification. In Chapter 3, we identify the mathematical ingredients of the heterotic system.

In Part II, we establish two natural deformation ansatzes and their related cohomology classes in Chapter 4. We then set up the desired map $\mathcal{F}$ for the implicit function theorem for each ansatz in Chapter 5. We provide the calculation of the linearization $D\mathcal{F}$ and verify the isomorphism for the application of the implicit function theorem in Chapter 6. This leads to the conclusion of two existence theorems of local coordinates on the heterotic moduli given below.

1.4 Main Results

Let X be a complex manifold of dimension 3 with holomorphic volume form Ω and hermitian metric ω. Let $E \to X$ be a complex vector bundle with connection A. Let $\alpha' > 0$. We will construct families of solutions to the following system:

$$\begin{aligned} &d(|\Omega|_\omega \omega^2) = 0, \quad i\partial\bar{\partial}\omega = \alpha'(\operatorname{Tr} F_A \wedge F_A - \operatorname{Tr} R \wedge R) \\ &F_A^{2,0} = F_A^{0,2} = 0, \quad F_A \wedge \omega^2 = 0. \end{aligned} \tag{1.1}$$

We make the following assumptions on the background data: [BG]

- X admits a Kähler Ricci-flat metric ω_{CY} [63].
- $c_1(X) = c_1(E) = 0$ and $c_2(X) = c_2(E)$.
- $E \to (X, \omega_{\mathrm{CY}})$ admits the structure of a holomorphic stable bundle.

Two natural deformation ansatzes can be constructed by two different choices of partitioning the four equations. Consequently, the ansatzes for the deformation of the metric ω are associated with the following two cohomology classes:

- Bott-Chern cohomology

$$H^{2,2}_{\text{B-C}}(X, \mathbb{C}) = \frac{\operatorname{Ker}(d) \ \cap \ \Lambda^{2,2}(X, \mathbb{R})}{\operatorname{Im}(\partial\bar{\partial}) \ \cap \ \Lambda^{1,1}(X, \mathbb{R})}.$$

- Aeppli cohomology

$$H^{1,1}_{\mathrm{A}}(X, \mathbb{C}) = \frac{\operatorname{Ker}(\partial\bar{\partial}) \ \cap \ \Lambda^{1,1}(X, \mathbb{R})}{(\operatorname{Im}(\partial) \cap \Lambda^{0,1}(X, \mathbb{R})) \oplus (\operatorname{Im}(\bar{\partial}) \cap \Lambda^{1,0}(X, \mathbb{R}))}.$$

And the deformation for the gauge connection A are associated with the Dolbeault cohomology $H^{0,1}(X, \mathrm{End}E)$ on the holomorphic vector bundle E,

$$H^{0,1}(X, \mathrm{End}E) = \frac{\operatorname{Ker}(\bar{\mathcal{D}}) \ \cap \ \Lambda^{0,1}(\mathrm{End}E)}{\operatorname{Im}(\bar{\mathcal{D}}) \ \cap \ \Gamma(\mathrm{End}E)}.$$

Our main results are the following two theorems showing the existence of local coordinates on the sub-manifold of the heterotic moduli where the complex structure remains invariant.

Theorem 1 (Local existence under Bott-Chern deformation)**.** *Suppose the background assumptions [BG] are satisfied. There exists a small parameter* $\alpha' > 0$ *such that the heterotic system admits solutions* $(\tilde{\omega}, \tilde{A})$ *nearby a Kähler structure* $(\omega_{\mathrm{CY}}, A_{\mathrm{DUY}})$ *along deformation paths parameterized by*

$$[\mathfrak{b}] \in H^{2,2}_{B\text{-}C}(X, \mathbb{R}), \quad [\alpha_1] \in H^{0,1}(X, \operatorname{End} E).$$

This equips the heterotic moduli with local coordinates $([\mathfrak{b}], [\alpha_1])$ *in a small* ϵ*-neighborhood of a Kähler solution.*

Theorem 2 (Local existence under Aeppli deformation)**.** *Suppose the background assumptions* $[BG]$ *are satisfied. There exists a small parameter* $\alpha' > 0$ *such that the heterotic system* (1.1) *with an additional spurious gauge field admits solutions* $(\tilde{\omega}, \tilde{A}, \tilde{\theta})$ *nearby a Kähler structure* $(\omega_{\mathrm{CY}}, A_{\mathrm{DUY}}, \Gamma_{\mathrm{CY}})$ *along deformation path parameterized by*

$$[\mathfrak{a}] \in H^{1,1}_{A}(X, \mathbb{R}), \quad [\alpha_1] \in H^{0,1}(X, \operatorname{End} E), \quad [\alpha_2] \in H^{0,1}(X, \operatorname{End} T^{1,0}X).$$

Thus the heterotic moduli admits local coordinates $([\mathfrak{a}], [\alpha_1], [\alpha_2])$ *in a small* ϵ*-neighborhood of a Kähler solution.*

We discuss the implications and possible future works in Chapter 7.

Part I

Background

Chapter 2

Physics Origin

In this chapter, we first give a brief account of the physical origin of the heterotic string theory. The heterotic string is a hybrid of bosonic string theory and superstring theory. We therefore need to recall both and then present a conceptual path to the construction of the heterotic string theory. Our discussion is restricted to the classical level for simplicity, and we will only provide brief remarks on its quantum behavior. This chapter is mainly a historical treatment and many details are omitted. The main references for this chapter are the textbooks [8, 23, 43].

In general, string theory studies extended objects, like strings and membranes, which are higher-dimensional analogs of the 0-dimensional point particle studied in quantum mechanics. When a 1-dimensional string moves through spacetime, it sweeps out a 2-dimensional worldsheet Σ. Its action can be formulated in terms of the non-linear σ-model.

2.1 Non-linear σ-model

The non-linear σ-model[1] describes a scalar field X taking on values in a nonlinear manifold called Σ, which is the submanifold swept out by the extended physical objects. We first motivate it in the point particle case where Σ is the worldline and then extend the discussion to the string case where Σ is the worldsheet.

2.1.1 Point bosonic particle

The free particle of mass m moves in the Minkowski spacetime M of dimension $D = d + 1$ with metric $\eta_{\mu\nu} = \mathrm{diag}(- + \cdots +)$. The principle of least action implies we can take the length of its worldline to be the action,

$$S[x] = m \int_{X_0}^{X_1} dX = m \int_{\tau_0}^{\tau_1} \left(\frac{dX^\mu}{d\tau} \frac{dX^\nu}{d\tau} g_{\mu\nu} \right)^{\frac{1}{2}} d\tau, \tag{2.1}$$

[1]Historically, the name σ-model comes from a field in their model corresponding to a spinless meson called σ.

where τ is an arbitrary parametrization along the worldline, and X^μ is the scalar field that maps from the spacetime manifold M to the worldline Σ. This action has several drawbacks, it contains the square root and does not describe massless particles. To circumvent these drawbacks, we can introduce an auxiliary worldline metric $h(\tau)$, and consider the action,

$$S[x,h] = -\frac{1}{2}\int_{\tau_0}^{\tau_1}\left(h^{-1}\dot{X}^2 - hm^2\right)d\tau, \tag{2.2}$$

where we denote $\dot{X} = \frac{dX}{d\tau}$. A quick calculation of the Euler-Lagrange equations for X and h shows the equivalence to the previous action. Take $m = 0$, we obtain the action for a free massless point particle,

$$S[x,h] = -\frac{1}{2}\int_{\tau_0}^{\tau_1}\left(h^{-1}\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}\eta_{\mu\nu}\right)d\tau, \tag{2.3}$$

Following the same intuition, we can extend this action to the higher dimensional case.

2.1.2 Worldsheet action

A string is a one-dimensional object. When it moves in the $(d+1)$-dimensional spacetime manifold M with metric $g_{\mu\nu}$ and coordinate x^μ, $\mu = 0, \cdots, d$. It sweeps out a two-dimensional worksheet which is the target space Σ with coordinate σ^α, $\alpha = 0, 1$. How the worldsheet Σ is embedded in M is encoded in the coordinates $X^\mu(\tau,\sigma)$, $\mu = 0, \cdots, d$. In other words, X are the maps $\Sigma \to M$.

Analogous to the point particle case, we take the area of the worldsheet to be our action,

$$S = -T\int_\Sigma dA = -T\int_\Sigma (\det G_{\alpha\beta})^{\frac{1}{2}} d^2\sigma, \tag{2.4}$$

where $G_{\alpha\beta}$ is the induced metric on Σ inherited from M,

$$G_{\alpha\beta} = \frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X^\nu}{\partial\sigma^\beta}g_{\mu\nu} \equiv \langle\partial_\alpha X, \partial_\beta X\rangle_{(M,g)}. \tag{2.5}$$

We have also introduced the tension T which is of unit $[\text{Area}]^{-1}$ to make S dimensionless. This action was first considered by Nambu and Goto, hence called Nambu-Goto action, but it suffers from the problem of the square root. Therefore, we can again introduce an auxiliary worldsheet metric $h_{\alpha\beta}$ instead of using the induced metric $G_{\alpha\beta}$, and write the analog of (2.3),

$$S[X,h] = -\frac{T}{2}\int \left(h^{\alpha\beta}\frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X^\nu}{\partial\sigma^\beta}g_{\mu\nu}\right)\mathrm{dvol}_\Sigma, \tag{2.6}$$

where the volume form on Σ is given by

$$d\mathrm{vol}_\Sigma = \left(\sqrt{\det h_{\alpha\beta}}\right) d\tau\wedge d\sigma) \equiv \sqrt{h}\ d^2\sigma. \tag{2.7}$$

A quick calculation can show that the Euler-Lagrange equation of $h^{\alpha\beta}$ is algebraic and recovers to the Nambu-Goto action when substituted back into the action.

A closer look at the integrand leads to a nice differential geometric formulation. This action is just the inner product of differential forms out of this scalar field[2] X, where the volume form is conveniently encoded in the Hoghe star $\star$ with respect to the worldsheet metric h,

$$S = -\frac{T}{2}\int_\Sigma dX\wedge\star dX \equiv -\frac{T}{2}\langle\langle dX, dX\rangle\rangle^{(\Sigma,h)}_{(M,g)}. \tag{2.8}$$

This action is called the Polyakov action. This is our starting point of bosonic string theory.

For notation, we usually denote $\partial_\alpha X^\mu = \frac{\partial X^\mu}{\partial\sigma^\alpha}$, and in particular $\dot{X}^\mu = \partial_\tau X^\mu$ and $X'^\nu = \partial_\sigma X^\mu$.

Due to historical reasons, the Polyakov action and Nambu-Goto action were proposed to explain the enormous proliferation of strongly interacting hadrons whose resonances continue to exist to a rather high spin and its mass square m^2 is roughly linear to the spin J, $m^2 = \frac{J}{\alpha'}$. This linear slope α' is known as the Regge slope. The tension T is related to the Regge slope by $T = \frac{1}{2\pi\alpha'}$, and we usually work in the unit $T = 1$ for simplicity and restore it when necessary. It is natural to expect that the limit $\alpha' \to 0$, corresponds to a string with infinite tension, and hence the string should collapse to a point particle with some higher-order string corrections. We will see this in the later discussion of low-energy effective field theory.

Symmetry

In Minkowski spacetime ($g_{\mu\nu} = \eta_{\mu\nu}$), the Polyakov action enjoys several symmetries.

[2]One can think of this scalar field as a section $X : M \to \Sigma$ of the fiber bundle over M with Σ as its fiber.

First, we have the usual spacetime Poincaré invariance as a global symmetry, whose infinitesimal transformation is given by,

$$\delta X^\mu = a^\mu{}_\nu X^\nu + b^\mu, \quad a_{\mu\nu} = -a_{\nu\mu}, \tag{2.9}$$
$$\delta h_{\alpha\beta} = 0. \tag{2.10}$$

Noether's theorem then concludes the conserved charge associated with this global symmetry is our usual energy-momentum tensor $T_{\alpha\beta}$.

Second, as seen from the geometric definition of Polaykov action, it is invariant under the reparametrization of the worldsheet as a local symmetry. Its infinitesimal transformation is given by,

$$\begin{aligned} \delta\sigma^\alpha &= \xi^\alpha, & (2.11)\\ \delta X^\mu &= \xi^\alpha \partial_\alpha X^\mu, & (2.12)\\ \delta h_{\alpha\beta} &= (\xi^\gamma \partial_\gamma h_{\alpha\beta} - \partial_\alpha \xi^\gamma h_{\gamma\beta} - \partial_\beta \xi^\gamma h_{\alpha\gamma}) & (2.13)\\ &= (\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha), & (2.14) \end{aligned}$$

where ξ^α is a constant.

Apart from the above two, we have an additional local symmetry called the Weyl scaling, which is a conformal scaling of the worldsheet metric $h_{\alpha\beta}$. Its transformation is given by,

$$\begin{aligned} h_{\alpha\beta} &\mapsto e^\Lambda h_{\alpha\beta}, & (2.15)\\ \delta h_{\alpha\beta} &= \Lambda h^{\alpha\beta}, & (2.16)\\ \delta X^\mu &= 0, & (2.17) \end{aligned}$$

where Λ is a constant.

A direct consequence of the Weyl invariance is that the energy-momentum tensor is traceless, $h^{\alpha\beta}T_{\alpha\beta} = 0$. This is a crucial ingredient to obtain a consistent theory, which leads to the well-known Virasoro algebra of the Fourier component of the energy-stress tensor, and consequently the determination of spacetime dimension: $D = 26$ for bosonic strings and $D = 10$ for superstring in flat Minkowski spacetime.

In fact, since we require $h_{\alpha\beta}$ to be auxiliary, we must be able to fix $h_{\alpha\beta}$ using the above two local symmetries. If we have instead considered a generic n-dimensional object sweeping out a $(n+1)$-dimensional worldvolume, under the Weyl scaling, we would have

$$\sqrt{h}h^{\alpha\beta} \mapsto \Lambda^{\frac{1}{2}(n+1)-1}\sqrt{h}h^{\alpha\beta}, \tag{2.18}$$

invariant only when $n = 1$. And a simple count of the degree of freedom shows that only in $n = 1$ case, we can fully fix the $\frac{1}{2}(n+1)(n+2) = 3$ components of $h_{\alpha\beta}$. This shows we can not extend the Polyakov action to higher-dimensional objects like membranes in a consistent way. Hence, we have only string theory, not a membrane theory or higher-dimensional analog[3] with the desired property of fixing the worldsheet metric h above.

Equations of motion

We now work in the flat Minkowski spacetime, $g_{\mu\nu} = \eta_{\mu\nu}$. The variation of S with respect to the inverse metric $h^{\alpha\beta}$ gives the vanishing energy-momentum tensor,

$$0 = T_{\alpha\beta} = -\frac{T}{2}\frac{1}{\sqrt{h}}\frac{\delta S}{\delta h^{\alpha\beta}} = \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2} h_{\alpha\beta} h_{\alpha\beta} h^{\gamma\delta} \partial_\gamma X^\mu \partial_\delta X_\mu, \quad (2.19)$$

which is manifestly traceless as a consequence of the Weyl scaling symmetry. The vanishing of $T_{\alpha\beta}$ leads to the condition on the induced metric $G_{\alpha\beta} = \frac{1}{2} h_{\alpha\beta} \mathrm{Tr}(G_{\alpha\beta})$, when imposed reduce the Polyakov action to Nambu-Goto action.

The variation of the S with respect to X^μ is the usual two-dimensional wave equation,

$$0 = \Box X^\mu = \frac{1}{\sqrt{h}} \partial_\alpha \Big(\sqrt{h} h^{\alpha\beta} \partial_\beta X^\mu \Big). \quad (2.20)$$

This simplifies after we fix $h_{\alpha\beta} = \eta_{\alpha\beta}$ in conformal gauge using our local symmetries[4],

$$-\ddot{X}^\mu + X''^\mu = (-\partial_\tau^2 + \partial_\sigma^2) X^\mu = 0. \quad (2.21)$$

Mode expansion

To solve for X, we need to impose appropriate boundary conditions. Since there are two topological types of a string: open or closed, we have two suitable boundary conditions of X^μ on Σ. Set $\sigma \in [0, \pi]$ for convenience.

For closed strings, we require the fixed-time periodic boundary condition,

$$X^\mu(\tau, \sigma) = X^\mu(\sigma + \pi). \quad (2.22)$$

[3] The membrane arises in string theory in a different context, via the endpoints of the open string satisfying Dirichlet boundary conditions along some p directions, hence called *Dp*-branes.

[4] It is worth noting that this conformal gauge-fixing is a local statement. In general, one can not fix the worldsheet metric h to be flat globally on Σ.

For open strings, we require the worldsheet spatial derivative $\frac{\partial X^\mu}{\partial \sigma}$ must vanish, representing that the momentum vanishes at the endpoint of the open string,

$$\frac{\partial X^\mu}{\partial \sigma}(\tau, 0) = \frac{\partial X^\mu}{\partial \sigma}(\tau, \pi) = 0. \tag{2.23}$$

Now, imposing the appropriate boundary condition, we can solve for X^μ for the wave equation. We omit the calculation and cite results from [43] below.

For open strings,

$$X^\mu(\tau, \sigma) = x^\mu + l^2 p^\mu \tau + il \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos n\sigma, \tag{2.24}$$

where the α_n^μ's are the Fourier components to be interpreted as the oscillator coordinates and satisfy the usual Poison brackets,

$$[\alpha_m^\mu, \alpha_n^\nu]_{\text{PB}} = im\delta_{m+n}\eta^{\mu\nu}. \tag{2.25}$$

We will not provide further details for open string here, as it is not relevant in heterotic string theory.

For closed strings, the periodic boundary condition implies that we can further split it into right-movers and left-movers,

$$X^\mu(\tau, \sigma) = X_{\text{R}}^\mu(\tau - \sigma) + X_{\text{L}}^\mu(\tau + \sigma), \tag{2.26}$$

$$X_{\text{R}}^\mu(\tau - \sigma) = \frac{x^\mu}{2} + \frac{l^2 p^\mu}{2}(\tau - \sigma) + \frac{i}{2} l \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-2in(\tau - \sigma)}, \tag{2.27}$$

$$X_{\text{L}}^\mu(\tau + \sigma) = \frac{x^\mu}{2} + \frac{l^2 p^\mu}{2}(\tau + \sigma) + \frac{i}{2} l \sum_{n \neq 0} \frac{1}{n} \tilde{\alpha}_n^\mu e^{-2in(\tau + \sigma)}. \tag{2.28}$$

The right-movers and left-movers are decoupled, and non-interact with each other. Hence we have two sets of oscillator coordinates α_n^μ and $\tilde{\alpha}_n^\mu$ with same commutation relations. The non-interaction between left- and right-movers is reflected in the vanishing of the Poison bracket between α_m's and $\tilde{\alpha}_n$'s,

$$[\alpha_m^\mu, \tilde{\alpha}_n^\nu]_{\text{PB}} = 0. \tag{2.29}$$

Such decoupling indicates the possibility of a hybridization taking right-movers and left-movers from different theories. In particular, the heterotic string is given by the hybridization of the right-movers of $D = 26$ bosonic string and the left-movers of $D = 10$ superstring.

The quantization follows by promoting the Poison bracket of α_n^μ's to the Dirac commutators via the substitution,

$$[\cdot,\cdot]_{\mathrm{PB}} \mapsto -\frac{i}{\hbar}[\cdot,\cdot], \tag{2.30}$$

and we usually work in unit $\hbar = 1$. After quantization, the oscillator coordinate can be rewritten in terms of ladder operators a_m^μ with suitable normalization $\alpha_m^\mu = \sqrt{m}a_m^\mu$. Those ladder operators give the full spectrum of the bosonic string by acting on the vacuum state.

However, the spectrum we obtained here is off-shell, since we have not imposed the first equation of motion that $T_{\alpha\beta} = 0$. Without explicit discussion, we comment that the Fourier component of the energy-momentum tensor after quantization satisfies the Virasoro algebra, and the vanishing of energy-momentum tensor helps to determine the correct spacetime dimension $D = 26$ such that there are no negative-norm states (called the ghosts)[5]. We point the reader to Table 3.1 and Table 3.2 in Chapter 3 of [8] for the explicit table of the bosonic string spectrum.

So far, we have constructed a bosonic string theory. Yet, it has several drawbacks. First, it has a tachyonic ground state with a negative mass and is hence unstable. Second, we have not included any fermionic states. These would be the two main motivations for constructing a superstring theory where we introduce an additional internal degree of freedom propagating along the string, i.e. the fermionic partner of the bosonic field. Then the two drawbacks can be resolved in superstring theory by introducing the supersymmetry that relates the fermionic fields to the bosonic fields.

2.2 Supersymmetry

In this section, we provide some motivations and general principles of supersymmetry.

2.2.1 Why supersymmetry?

The supersymmetry is a symmetry that relates bosons to fermions. Despite the absence of experimental evidence for its realization in Nature, there are several motivations for studying such a rather bizarre symmetry.

[5]This is a well-known argument that the renormalization of vacuum Casimir effect of the string by the Riemann zeta function regularization $\zeta(-1) = -\frac{1}{12}$. This will determine the central charge of the Virasoro algebra and consequently the spacetime dimension $D = 26$ for bosonic string theory and $D = 10$ for superstring theory. This was one of many appealing features of the string theory in early days.

The first is related to the S-matrix. The S-matrix encodes the information of the scattering process between asymptotic states. In a generic quantum field theory, we expect some very reasonable and physical conditions for the scattering process and hence for the S-matrix, like locality, causality, positivity of energy, and finiteness of the number of particles. Then the Coleman-Mandula theorem states that the only possible continuous symmetries for such an S-matrix are a direct product of Poincaré and some internal (gauge) symmetry group. This puts a restriction on the symmetry of the generic quantum theory. However, supersymmetry is one of the loopholes of this theorem. In particular, there is no restriction for adding some fermionic generators. Yet one must add them in a specific way to promote Poincaré to super-Poincaré. The Haag-Lopuszanski-Sohnius is the supersymmetric extension of the Coleman-Mandula theorem, and it states the only possible continuous symmetries are a direct product of super-Poincaré and some internal (gauge) symmetry group.

The second comes from a rather meta-theoretical perspective for the unification. The Standard Model provides a unifying framework for electromagnetic, strong, and weak interactions. The difficulty in unifying gravity lies in the non-renormalizability of general relativity. This implies that general relativity is really an effective theory. Supersymmetry provides some alleviation and the pure supergravity theory is renormalizable up to one and two loops. Another reason for supersymmetry comes from a phenomenological point of view. The three gauge couplings of the Standard model meet exactly just by considering the minimal supersymmetric extension.

The third and most compelling reason for supersymmetry arises in string theory. A consistent string theory automatically unified the gravity and the Standard Model provided correct compactifications of extra dimensions. For that, the consistent string theories require supersymmetry and critical dimension. In this sense, if string theory turns out to be correct, these two conditions would be the most striking predictions. If any one of these two is experimentally disproved, then we know the string theory is not realized in Nature. Unfortunately, we have no experimental evidence for any of these two predictions. Hence, all investigations and discussions of supersymmetry and string theory are purely theoretical.

2.2.2 Supersymmetry in string theory

Before we construct the superstring, it is worth having a bird-eye view of the supersymmetry that appears in superstring theory.

In general, there are two points of view of the supersymmetry in superstirng theories: one is the worldsheet supersymmetry indicated by the equal number of bosonic and fermionic states at every mass level, and another is the spacetime supersymmetry that arises from 10-dimensional supersymmetry multiplets organized by the bosonic and fermionic states in the 10-dimensional effective supergravity that describes the massless spectrum of its corresponding string theory. Hence, there are two natural and equivalent approaches to a supersymmetric string theory depending on which type of supersymmetries you choose to implement explicitly.

One is to implement N supersymmetry on the worldsheet Σ explicitly, meaning that we introduce N fermionic partner $\psi^{A\mu}$ of the worldsheet bosonic coordinate X^μ related to each other by the worldsheet supersymmetry, where $A = 1, \cdots, N$ is the indices of N supersymmetries, in other words, we are promoting Σ to a super-manifold $\tilde{\Sigma}^{2|2N}$ and its embedding in M is encoded in N super-pairs $(X^\mu, \psi^{1\mu}), \cdots, (X^\mu, \psi^{N\mu})$. This approach was originally developed by Pierre Ramond, André Neveu, and John Schwarz (RNS) in 1971 [53, 59]. The action principle with $N = 1$ supersymmetry gives rise to a consistent $D = 10$ string theory by truncating the spectrum proposed by Gliozzi-Scherk-Olive (GSO)[41]. In this approach, the spacetime supersymmetry of the superstring theory is hidden and revealed by combining the Ramond sector and the Neveu-Schwarz sector. The difficulty of understanding the spacetime supersymmetry is the motivation of the alternative approach.

The other is to implement N supersymmetries on the spacetime manifold M explicitly, meaning that we introduce N extra fermionic coordinate θ^A, $A = 1, \cdots, N$, i.e. promoting M to a super-manifold $\tilde{M}^{D|N}$ with coordinate (x^μ, θ^A). Note the difference to the previous approach that θ has no spacetime indices μ. This approach is called Green-Schwarz (GS) formalism. In GS formalism, the GSO condition is automatically manifested to obtain the physical spectrum. The worldsheet supersymmetry is then revealed by the analysis of the spectrum that there are equal numbers of bosonic and fermionic states at every mass level.

2.3 Worldsheet supersymmetry

In this section, we look at the first approach - RNS formalism.

For $N = 1$, we introduce a fermionic partner $\psi^\mu(\tau, \sigma)$ which is a D-plet Majorana fermion. By adding a Dirac action to the Polyakov action, we consider the action in conformal gauge $h_{\alpha\beta} = \eta_{\alpha\beta}$,

$$S = -\frac{1}{2}\int \left(\partial_\alpha X^\mu \partial^\alpha X_\mu - i\bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu\right) d^2\sigma, \tag{2.31}$$

where ρ^α are the 2-dimensional Dirac matrices. The barred $\bar{\psi}$ is the Dirac conjugate of ψ,

$$\bar{\psi} = \psi^\dagger \rho^0 = \psi^\intercal \rho^0, \tag{2.32}$$

where the last equality holds only if ψ is the real Majorana spinor. In 2-dimension, the choice of Majorana spinor leads to the most interesting theory.

We choose a convenient pure imaginary basis for ρ since we are working with real Majorana ψ^μ,

$$\rho^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{2.33}$$

A simple computation verifies that ρ^α indeed satisfies the Clifford algebra

$$\{\rho^\alpha, \rho^\beta\} = -2\eta^{\alpha\beta}. \tag{2.34}$$

In this basis, ψ has components,

$$\psi^\mu = \begin{pmatrix} \psi^\mu_- \\ \psi^\mu_+ \end{pmatrix}, \tag{2.35}$$

and they have the chiralites[6],

$$\bar{\rho}\psi_\pm = \mp\psi_\pm, \quad \bar{\rho} = \rho^0\rho^1. \tag{2.36}$$

This action has global worldsheet supersymmetry whose infinitesimal transformation is given below,

$$\delta X^\mu = \bar{\epsilon}\psi^\mu, \tag{2.37}$$

$$\delta\psi^\mu = -i\rho^\alpha \partial_\alpha X^\mu \epsilon, \tag{2.38}$$

where ϵ is a constant infinitesimal Majorana spinor. Noether's theorem implies that there exists a conserved worldsheet super-current[7] for this global worldsheet symmetry,

$$J_\alpha = \frac{1}{2}\rho^\beta \rho_\alpha \psi^\mu \partial_\beta X^\mu, \tag{2.39}$$

[6] $\pm$ in $\psi_\pm$ implies the relation to right- and left-mover as shown in (2.45) and (2.44)

[7] The normalization is chosen for later convenience

which is the super-analog of the energy-momentum tensor, given by (applying Noether's theorem to worldsheet translation)

$$T_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu + \frac{i}{2}\bar{\psi}^\mu \rho_{(\alpha}\partial_{\beta)}\psi_\mu - (\text{trace}). \tag{2.40}$$

$T_{\alpha\beta}$, like in bosonic theory, are manifestly traceless, and the super-curent J_α is also traceless in the sense of $\rho^\alpha J_\alpha$, which is a consequence of the 2-dimensional identity $\rho^\alpha \rho^\beta \rho_\alpha = 0$. After quantization, the traceless of $T_{\alpha\beta}$ and J_α will be related to the super analog of the Virasoro algebra and help to determine the critical dimension $D = 10$.

Equations of motion

The equations of motion with respect to the bosonic X^μ is the usual wave equations,

$$\partial_\alpha \partial^\alpha X^\mu = 0, \tag{2.41}$$

The equation of motion with respect to the fermionic ψ^μ is the Dirac equations,

$$\rho^\alpha \partial_\alpha \psi^\mu = 0. \tag{2.42}$$

In lightcone coordinates $\sigma^\pm = \tau \pm \sigma$, and $\partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$, we can rewrite the fermionic action in terms of the component $\psi_\pm$,

$$S_f = \frac{i}{2}\int d\sigma^2 (\psi_- \partial_+ \psi_- + \psi_+ \partial_- \psi_+). \tag{2.43}$$

This implies the decoupling of the positive- and negative-chirality. The Dirac equations and the wave equations read,

$$\partial_+ \psi_-^\mu = \partial_+ \partial_- X^\mu = 0, \tag{2.44}$$

$$\partial_- \psi_+^\mu = \partial_- \partial_+ X^\mu = 0. \tag{2.45}$$

It then becomes more transparent why there can be a symmetry between the bosons and fermions, namely, $\partial_\pm X^\mu$ behaves like $\psi_\pm^\mu$.

Mode expansion

The mode expansion for X^μ is the same. We therefore focus on the mode expansion of ψ^μ. Like in the bosonic theory, a superstring can be open or closed.

For open strings, upon the variation of the fermionic action, we require the vanishing of the boundary terms, namely,

$$\psi_+\delta\psi_+ - \psi_-\delta\psi_- = 0 \quad \text{at } \sigma = 0, \pi. \tag{2.46}$$

This implies,

$$\psi_+ = \pm\psi_- \quad \text{at } \sigma = 0, \pi. \tag{2.47}$$

Without loss of generality, we can fix the sign of $\psi_+ = \psi_-$ at $\sigma = 0$, and leave two possible boundary conditions at $\sigma = \pi$. The one with the positive sign is called the Ramond (R) boundary condition, and the other with the negative sign is called the Neveu-Schwarz (NS) boundary condition.

Then the mode expansion for $\psi^\mu_\pm$ reads,

$$\text{(R)} \quad \psi^\mu_+(\tau,\pi) = +\psi^\mu_-(\tau,\pi), \quad \psi^\mu_\pm(\tau,\sigma) = \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} d^\mu_n e^{-in(\tau\pm\sigma)}. \tag{2.48}$$

$$\text{(NS)} \quad \psi^\mu_+(\tau,\pi) = -\psi^\mu_-(\tau,\pi), \quad \psi^\mu_\pm(\tau,\sigma) = \frac{1}{\sqrt{2}}\sum_{r\in\mathbb{Z}+\frac{1}{2}} b^\mu_r e^{-in(\tau\pm\sigma)}. \tag{2.49}$$

For closed strings, the boundary term vanishes when the boundary conditions are periodicity (R) or anti-periodicity (NS) for each component of ψ separately,

$$\psi^\mu_\pm(\tau,\sigma) = \pm\psi^\mu_\pm(\tau,\sigma+\pi). \tag{2.50}$$

Thus we can have for the right-movers

$$\text{(R)}_\text{R} \quad \psi^\mu_- = \sum_{n\in\mathbb{Z}} d^\mu_n e^{-2in(\tau-\sigma)}, \tag{2.51}$$

or

$$\text{(NS)}_\text{R} \quad \psi^\mu_- = \sum_{n\in\mathbb{Z}+\frac{1}{2}} b^\mu_r e^{-2ir(\tau-\sigma)}. \tag{2.52}$$

And for left-movers

$$\text{(R)}_\text{L} \quad \psi^\mu_+ = \sum_{n\in\mathbb{Z}} \tilde{d}^\mu_n e^{-2in(\tau+\sigma)}, \tag{2.53}$$

or

$$\text{(NS)}_\text{L} \quad \psi^\mu_+ = \sum_{n\in\mathbb{Z}+\frac{1}{2}} \tilde{b}^\mu_r e^{-2ir(\tau+\sigma)}. \tag{2.54}$$

Therefore, we have four distinct closed-string sectors corresponding to the 4 pairings: $(\mathrm{R})_{\mathrm{L}} - (\mathrm{R})_{\mathrm{R}}$, $(\mathrm{NS})_{\mathrm{L}} - (\mathrm{NS})_{\mathrm{R}}$, describing bosonic states[8], and $(\mathrm{R})_{\mathrm{L}} - (\mathrm{NS})_{\mathrm{R}}$, $(\mathrm{NS})_{\mathrm{L}} - (\mathrm{R})_{\mathrm{R}}$ describing fermionic states.

Upon quantization, the commutation of the mode operators read,

$$\{d_m^\mu, d_n^\nu\} = \delta_{m+m}\eta^{\mu\nu}, \tag{2.55}$$

$$\{b_r^\mu, b_s^\nu\} = \delta_{r+s}\eta^{\mu\nu}. \tag{2.56}$$

We now can study the spectrum of the superstring by acting those operators on the vacuum states. We remark that the theory we have here is still not a consistent quantum theory, the physical spectrum arises from a very specific truncation of the spectrum by GSO projection which we will not go into the details. For interested readers, we point to Table 8.1 and Table 8.2 in [8] for the spectrum of open and closed superstring spectrum.

One of the remarkable consequences of the GSO projection is that it eliminates the tachyonic state and also it gives a 10-dimensional supersymmetry theory as the Ramond sectors and the Neveu-Schwarz sectors combine into a 10-dimensional supersymmetric multiplet. This would be the motivation for the other approach to implement supersymmetry in string theory discussed in the next section.

So far, we have constructed a fairly simple prototype superstring action. Yet, our prototype action suffers from the choice of fixing $h_{\alpha\beta}$, which means that it is not covariant. To proceed, we need to know how to introduce spinors on a curved worldsheet, namely, need to formulate a 2-dimensional supergravity theory. For that, we point the reader to Chapter 4 in [43].

2.4 Spacetime supersymmetry

In this section, we look at the GS formulation, where the spacetime supersymmetry is explicitly implemented.

The spacetime supersymmetry is realized by generalizing Minkowski space with its bosonic coordinates $\{x^\mu\}$ to a superspace with $\{x^\mu, \theta^A\}$, where θ^A is the fermionic coordinates, meaning it anticommutes. If there are to be N supersymmetries, we introduce N anticommuting spinor coordinates θ^{Aa}, $A = 1, \cdots, N$. For a general Dirac spinor, the spinor indices $a = 1, \cdots, 2^{\frac{D}{2}}$, and it will be suppressed in most formulas.

[8]Note the difference in purely bosonic string, in superstring theory, besides α oscillators, the bosonic states contain those from involves d, b oscillators.

Then the supersymmetry is realized in the superspace by a ϵ-translation in fermionic coordinates θ and consequent transformation on bosonic coordinate x,

$$\delta_\epsilon \theta^{Aa} = \epsilon^{Aa}, \tag{2.57}$$

$$\delta_\epsilon x^\mu = i\bar{\epsilon}^A \Gamma^\mu \theta^A, \tag{2.58}$$

where ϵ^A is a constant (τ-independent) spinor of same type as the corresponding θ^A coordinates, and Γ are the D-dimensional Dirac gamma matrices satisfying the Clifford algebra.

We can view the supersymmetry more geometrically by defining the following generator,

$$Q_a^A = \frac{\partial}{\partial \bar{\theta}^{Aa}} + i(\Gamma^\mu \theta^A)_a \partial_\mu, \tag{2.59}$$

such that the supersymmetry transformation is generated by,

$$\delta_\epsilon \theta^A = [\bar{\epsilon}^A Q^A, \theta^A] = \epsilon^A, \tag{2.60}$$

$$\delta_\epsilon x^\mu = [\bar{\epsilon}^A Q^A, \theta^A] = i\bar{\epsilon}^A \Gamma^\mu \theta^A. \tag{2.61}$$

Simple calculation verifies the general property of the supersymmetry that the commutator of two supersymmetry transformations gives a spatial translation,

$$[\bar{\epsilon}_1 Q, \bar{\epsilon}_2 Q] = -2i\bar{\epsilon}_1 \Gamma^\mu \epsilon_2 \partial_\mu. \tag{2.62}$$

2.4.1 Point super-particle

We start with the case of the point super-particle. Consider first for a worldline action for a massless bosonic point particle,

$$S[x,h] = \frac{1}{2}\int h^{-1}\big(\dot{x}^2\big) d\tau. \tag{2.63}$$

Notice that we can already form two supersymmetric invariants $p^\mu := \dot{x}^\mu - i\bar{\theta}^A \Gamma^\mu \theta^A$ and $\dot{\theta}^{Aa}$, where the dot denotes the τ-derivative. Then using our first supersymmetric invariant p^μ, we can immediately generalize to a worldline action for a supersymmetric point particle,

$$S[x,\theta,h] = \frac{1}{2}\int h^{-1} p^2 d\tau = \frac{1}{2}\int h^{-1}\big(\dot{x}^\mu - i\bar{\theta}^A \Gamma^\mu \theta^A\big)^2 d\tau \tag{2.64}$$

It is manifestly Lorentz invariant and global supersymmetric.

Now we have a nontrivial local fermionic symmetry called κ-symmetry of the form,

$$\delta_\kappa \theta^A = i(\Gamma \cdot p)\kappa^A, \tag{2.65}$$

$$\delta_\kappa x^\mu = i\bar{\theta}^A \Gamma^\mu \big(\delta_\kappa \theta^A\big), \tag{2.66}$$

$$\delta_\kappa h = 4h\dot{\bar{\theta}}^A \kappa^A, \tag{2.67}$$

where $\kappa^A(\tau)$ denotes N Grassmannian spinor parameters that is dependent on τ.

There is one more local bosonic reparametrization symmetry for (2.64),

$$\delta_\lambda \theta^A = \lambda \dot{\theta}^A \tag{2.68}$$

$$\delta_\lambda x^\mu = i\bar{\theta}^A \Gamma^\mu \big(\delta_\lambda \theta^A\big), \tag{2.69}$$

where $\lambda(\tau)$ is a scalar parameter dependent on τ.

2.4.2 Supersymmetric non-linear σ-model

We now generalize the above discussion to string $d = 1$ case. A natural extension of p^μ to worldsheet is,

$$\Pi^\mu_\alpha := \partial_\alpha X^\mu - i\bar{\theta}^A \Gamma^\mu \partial_\alpha \theta^A. \tag{2.70}$$

Then, the prototype for a supersymmetric superstring action follows by analogy to (2.64), (for simplicity, we work in unit $2\alpha' = 1$)

$$S_1[X, \theta, h] = -\frac{1}{2\pi} \int_\Sigma \sqrt{h} \Big(h^{\alpha\beta} \Pi^\mu_\alpha \Pi^\nu_\beta \eta_{\mu\nu} \Big) d^2\sigma. \tag{2.71}$$

Symmetries

By construction, S_1 possesses local reparametrization invariance and N global supersymmetries. However, the local κ-supersymmetry of the supersymmetric particle action is lost in this naïve generalization.

To restore the local κ-symmetry, we need to add an extra term. Also, we can not take arbitrary N. We need to take $N \leq 2$, meaning that there are at most two supersymmetries[9]. For the general $N = 2$ case, this extra

[9] This is because, by the Weinberg-Witten theorem, theories with more than 32 supersymmetry generators automatically have massless fields with spin (or helicity) greater than 2, which is deemed ill and not renormalizable. Hence the maximum number of supersymmetry generators possible is 32. Since in $D = 10$ dimension, the size of the spinor is $2^{10/2-1} = 16$, Therefore $N = 2$ is sufficient to host all 32 supersymmetry generators. In fact, maximal 32 symmetric generators indicate that $D = 11$ is the maximal dimension for a renormalizable supersymmetric theory. Note, this is purely a physical argument, the supersymmetric algebra can be defined in arbitrary dimensions.

action term[10] reads,

$$\begin{aligned} S_2[X,\theta] &= \frac{1}{\pi}\int_\Sigma \varepsilon^{\alpha\beta}\big[-i\partial_\alpha X^\mu\big(\bar{\theta}^1\Gamma^\nu\partial_\beta\theta^1 - \bar{\theta}^2\Gamma^\nu\partial_\beta\theta^2\big)\big]\eta_{\mu\nu}\, d^2\sigma \\ &+ \frac{1}{\pi}\int_\Sigma \varepsilon^{\alpha\beta}\big[\big(\bar{\theta}^1\Gamma^\mu\partial_\alpha\theta^1\big)\big(\bar{\theta}^2\Gamma^\nu p_\beta\theta^2\big)\big]\eta_{\mu\nu}\, d^2\sigma. \qquad (2.72) \\ &= \frac{1}{\pi}\int_\Sigma -id\sigma^\mu\wedge\big(\bar{\theta}^1\Gamma_\mu d\theta^1 - \bar{\theta}^2\Gamma_\mu d\theta^2\big) \\ &+ \frac{1}{\pi}\int_\Sigma (\bar{\theta}^1\Gamma^\mu d\theta^1)\wedge(\bar{\theta}^2\Gamma_\mu d\theta^2). \qquad (2.73) \end{aligned}$$

And the $N=1$ case and $N=0$ case can be obtained by setting appropriate θ^A to zero.

Subjugate S_2 to δ_ϵ transformation will yield a term[11] of form,

$$2\bar{\epsilon}\gamma_\mu\phi_{[1}\bar{\phi}_2\gamma^\mu\phi_{3]}, \qquad (2.74)$$

where $(\phi_1,\phi_2,\phi_3) = (\theta,\theta',\dot{\theta})$. S_2 is supersymmetric only when this term vanishes, which puts a restriction on the spacetime dimension to the following four cases:

1. $D=3$ and θ is Majorana.
2. $D=4$ and θ is Majorana or Weyl.
3. $D=6$ and θ is Weyl.
4. $D=10$ and θ is Majorana-Weyl.

Quantum consideration will single out $D=10$ case.

To proceed, it is convenient to define the projection[12] to self-dual and anti-self-dual tensor on the worldsheet Σ,

$$P_\pm^{\alpha\beta} = \frac{1}{2}\bigg(h^{\alpha\beta} \pm \frac{1}{\sqrt{h}}\varepsilon^{\alpha\beta}\bigg). \qquad (2.75)$$

It can be verified that the full action S_1+S_2 now restores the κ-symmetry of the form,

$$\delta_\kappa\theta^A = 2i(\Gamma\cdot\Pi_a)\kappa^{Aa}, \qquad (2.76)$$

[10] This action can be recast in the form of a Wess-Zumino-Witten action.

[11] Such term also appears in supersymmetric Yang-Mills (SYM) theories. Hence following the same argument, we only have SYM theories in the 4 allowed dimensions.

[12] Where the normalization factor is chosen for later convenience.

$$\delta_\kappa X^\mu = i\bar{\theta}^A \Gamma^\mu \big(\delta_\kappa \theta^A\big), \tag{2.77}$$

provided that κ^A parameters are restricted to be anti-self-dual for $A = 1$ that $\kappa^{1\alpha} = P_-^{\alpha\beta}\kappa_\beta^1$ and self-dual for $A = 2$ that $\kappa^{2\alpha} = P_+^{\alpha\beta}\kappa_\beta^2$. It turns out that $A = 1$ describes right-moving modes and symmetries, and $A = 2$ describes left-moving modes and symmetries.

In addition to the local reparametrization and κ-symmetries, there is a local bosonic symmetry for $S_1 + S_2$,

$$\delta_\lambda \theta^1 = \sqrt{h} P_-^{\alpha\beta} \partial_\beta \theta^1 \lambda, \tag{2.78}$$

$$\delta_\lambda \theta^2 = \sqrt{h} P_+^{\alpha\beta} \partial_\beta \theta^2 \lambda, \tag{2.79}$$

$$\delta_\lambda X^\mu = i\theta^A \Gamma^\mu \big(\delta_\lambda \theta^A\big), \tag{2.80}$$

$$\delta_\lambda(\sqrt{h} h^{\alpha\beta}) = 0. \tag{2.81}$$

In summary, our supersymmetric action $S_1 + S_2$ is invariant under the spacetime and worldsheet Lorentz transformation, worldsheet reparametrization, Weyl rescaling, δ_ϵ supersymmetric transformations globally, and the δ_κ, δ_λ, and δ_{λ_a} transformations locally.

Type I v.s. Type II

In the critical dimension $D = 10$, θ are Majorana-Weyl spinors, hence we have the freedom to choose the chirality of θ^1 and θ^2. This choice leads to three main possibilities of superstring theory:

1. Type I: a superstring based on open superstring, i.e. the coordinates $\{X^\mu, \theta^A\}$ satisfy the similar boundary conditions as (2.23). Such a boundary condition will reduce the spacetime supersymmetry to $N = 1$, hence we call it of type "1". We will not discuss it in further detail.

2. Type II: in closed superstring theory, we have two choices for the chiralities of θ^A:

   (a) Type IIA: θ^1 and θ^2 have opposite chiralities, we say it has $N = (1, 1)$ supersymmetry.

   (b) Type IIB: θ^1 and θ^2 have same chiralities, we say it has $N = (2, 0)$ supersymmetry.

We now proceed to understand the equations of motion of this action $S_1 + S_2$.

Equations of motion

Up to now, we have constructed a $N \leq 2$ supersymmetric string action of the form,

$$S_{\rm s} = S_1[X, \theta^A, h] + S_2[X, \theta^A]. \tag{2.82}$$

The equation of motion for supersymmetric string action $S_1 + S_2$ can be obtained by variation $\delta_{h^{\alpha\beta}} S_s = \delta_{\theta^A} S_s = \delta_{X^\mu} S_s = 0$.

$$0 = T_{\alpha\beta} = \frac{-2}{\sqrt{h}} \delta_{h^{\alpha\beta}} S_s = (\Pi_\alpha \cdot \Pi_\beta) - \frac{1}{2} h_{\alpha\beta} h^{\gamma\delta} (\Pi_\gamma \cdot \Pi_\delta), \tag{2.83}$$

$$0 = (\Gamma \cdot \Pi_\alpha) P_-^{\alpha\beta} \big(\partial_\beta \theta^1\big), \tag{2.84}$$

$$0 = (\Gamma \cdot \Pi_\alpha) P_+^{\alpha\beta} \big(\partial_\beta \theta^2\big), \tag{2.85}$$

$$0 = \partial_\alpha \Big[\sqrt{h} \Big(h^{\alpha\beta} \partial_\beta X^\mu - 2i P_-^{\alpha\beta} \bar{\theta}^1 \Gamma^\mu \partial_\beta \theta^1 - 2i P_+^{\alpha\beta} \bar{\theta}^2 \Gamma^\mu \partial_\beta \theta^2 \Big) \Big]. \tag{2.86}$$

These equations of motion are highly non-linear, and it does not have a covariant quantization as θ's do not have a kinetic term. Due to this difficulty, we work in the lightcone gauge.

Lightcone gauge

We now consider free superstring in the critical dimension $D = 10$, and then θ^A are Majorana-Weyl. In lightcone gauge, we can use the local reparametrization, Weyl invariance, and local κ-symmetry to set the worldsheet metric to be flat $h_{ab} = \eta_{ab}$. We still have some residual symmetry to make the lightcone gauge choice as follows,

$$\Gamma^+ \theta^1 = \Gamma^+ \theta^2 = 0, \quad \Gamma^\pm = \frac{1}{\sqrt{2}} (\Gamma^0 \pm \Gamma^9), \quad \big(\Gamma^\pm\big)^2 = 0. \tag{2.87}$$

We can also use the residual conformal invariance to impose the following condition on the right-moving bosonic lightcone coordinate,

$$X^+(\tau, \sigma) = x^+ + p^+ \tau, \tag{2.88}$$

It is worth counting the physical degrees of freedom in lightcone gauge, we note that a general 10-dimensional Dirac spinor is of size $2^{10/2} = 32$, and the Majorana-Weyl condition making it real and eliminates half together reduces to 16, then the lightcone condition reduces the count by another factor of 2. Hence we have 8 real components remaining. We use the symbol S for the 8 surviving components of θ in the lightcone gauge.

This forms an 8-dimensional representation of Spin(8). There are three 8-dimensional irreducible representations, $(8)_{\mathrm{v}}$ the fundamental vector representations (labeled by i, j, k), and two other spinor representations $(8)_{\mathrm{s}}$ (labeled by a, b, c) and 8_{c} (labeled by $\dot{a}, \dot{b}, \dot{c}$).

Using the symbol S for the 8 surviving component, we identify,

$$\sqrt{p^+}\theta^1 \to S^{1a} \text{ or } S^{1\dot{a}}, \tag{2.89}$$

$$\sqrt{p^-}\theta^2 \to S^{2a} \text{ or } S^{2\dot{a}}. \tag{2.90}$$

where the choice is determined by the chirality of the corresponding θ^A. In convention, we choose S^1 to belong $(8)_{\mathrm{s}}$ without loss of generality, then for type I and IIB, S^2 belongs to $(8)_{\mathrm{s}}$ since they are not chiral theories, and for type IIA, S^2 belongs to $(8)_{\mathrm{c}}$.

Mode expansion

The equation of motions reduces to the following,

$$\left(-\partial_\tau^2 + \partial_\sigma^2\right)X^i = 0, \tag{2.91}$$

$$(\partial_\tau + \partial_\sigma)S^{1a} = 0, \tag{2.92}$$

$$(\partial_\tau - \partial_\sigma)S^{2a} = 0. \tag{2.93}$$

Solve the S's for periodic boundary condition, we have the fermionic mode expansion for closed strings,

$$S^{1a}(\tau, \sigma) = \sum S^a_n e^{-2in(\tau-\sigma)}, \tag{2.94}$$

$$S^{2a}(\tau, \sigma) = \sum \tilde{S}^a_n e^{-2in(\tau+\sigma)}. \tag{2.95}$$

Then the Fourier components S's and $\tilde{S}$'s can be turned into fermionic operators upon quantization, and they act on the ground state to give the spectrum. We will not provide further details.

Lightcone action

We note that the equations of motions (2.91),(2.92) and (2.93) can be obtained from the following lightcone action, (with suitable rescaling of S^a)

$$S_{\mathrm{lc}} = -\frac{1}{2}\int_\Sigma \Big(\eta^{\alpha\beta}\partial_\alpha X^i \partial_\beta X_i + i\bar{S}^a\Gamma^-(\partial_\tau + \partial_\sigma)S^a\Big)d^2\sigma, \tag{2.96}$$

where $i = 1, \cdots, 8$ the transverse directions to the lightcone $X^\pm$. This lightcone action is equivalent to our previous worldsheet action (2.31), by bosonizing the fermions ψ^i and then re-fermionizing them. We point the reader to (5.2.22)-(5.2.24) in [43] for explicit redefinition.

In lightcone gauge, the supersymmetry (2.57)and (2.58) with parameter ϵ need not obey the lightcone condition $\Gamma^+\theta \sim \Gamma^+ S = 0$. Therefore, there are two types of supersymmetry transformations characterized by $\Gamma^+\epsilon_+ = 0$ and $\Gamma^-\epsilon_+ = 0$ of positive and negative chirality on the transverse component $\epsilon_\pm$. Hence it split into two sets of 8 supersymmetries, the one of ϵ_+ are particularly simple,

$$\delta_{\epsilon_+} X^i = 0 \tag{2.97}$$

$$\delta_{\epsilon_+} S^a = -2i\sqrt{p^+}\epsilon_+, \tag{2.98}$$

and the other set is given by,

$$\delta_\epsilon X^i = \frac{1}{\sqrt{p^+}}\bar{\epsilon}^a \Gamma^i S^a, \tag{2.99}$$

$$\delta_\epsilon S^a = \frac{i}{\sqrt{p^+}}\Gamma_-\Gamma_\mu(\partial_\tau - \partial_\sigma)X^\mu \epsilon^a, \tag{2.100}$$

where ϵ is Majorana-Weyl lightcone spinor.

Up to now, we briefly introduce the purely bosonic and the supersymmetric string theory. For the closed string, we have seen the decoupling between the right- and left-movers. This provides a new possibility of the hybridization of purely bosonic and super-string theory, namely the heterotic string theory.

2.5 Heterotic strings

The heterotic string theory was introduced by Gross-Harvey-Martinec-Rohm[13] [45, 46]. Its vacuum configuration was given by Candelas-Horowitz-Strominger-Witten[12] where the internal manifold is Kähler Ricci-flat, i.e. a Calabi-Yau 3-fold. The general supersymmetric configuration in the heterotic string was subsequently worked out by Strominger [60] and Hull [47], where the internal manifold is no longer a Calabi-Yau 3-fold.

2.5.1 Why heterotic?

The most natural motivation comes from the consideration of the gauge group. The Green-Schwarz anomaly cancellation [44] showed that the Lorentz

[13] The so-called "Princeton string quartet"

and Yang-Mills chiral anomalies cancel if and only if the gauge group of the theory is $\mathrm{Spin}(32)/\mathbb{Z}_2$ or $E_8 \times E_8$. Yet neither Type I nor Type II can incorporate $E_8 \times E_8$. Hence, the motivation to construct a consistent string theory that realizes the $E_8 \times E_8$ gauge group leads to the heterotic string theory.

The heterotic string is constructed by a chiral hybrid of the purely bosonic closed string and the closed superstring. As we have seen from free closed string (2.26), the field can be split into right- and left-movers. This is even true with string interactions. Therefore, the idea is to take the hybrid of the right-moving sector of the 10-dimensional fermionic closed superstring and the left-moving sector of the 26-dimensional closed bosonic string. The mismatching 16-dimensional bosonic dimensions must then be compactified on a flat torus for consistency, which leads to the realization[14] of the gauge group $G = E_8 \times E_8$ or $G = \mathrm{Spin}(32)/\mathbb{Z}_2$.

2.5.2 Heterosis

In the critical dimension $D = 10$, to construct the heterotic superstring, we take the right-moving sector of the fermionic superstring $(X^i(\tau-\sigma), S^a(\tau-\sigma))$, where $X^i(\tau-\sigma) = X^i_{\mathrm{R}}$ whose mode expansion is given in (2.27) and $S^a = S^{1a}(\tau,\sigma)$ whose mode expansion is given in (2.94). take the left-moving sector of the 26-dimensional bosonic string $X^\mu(\tau+\sigma)$ whose mode expansion is given by (2.28), consists of the 8 transverse coordinates X^i_L with index $i = 1, \cdots, 8$, and X^I_L with index $I = 10, \cdots, 25$.

The dynamics of the free heterotic string are then governed by the light-cone action, ($h^{ab} = \eta^{ab}$, and recover $T = \frac{1}{2\pi\alpha'}$)

$$\begin{aligned} S_{\text{het-lc}} = &-\frac{1}{4\pi\alpha'}\int_\Sigma \Big(\eta^{\alpha\beta}\partial_\alpha X^i \partial_\beta X_i + i\bar{S}^a \Gamma^-(\partial_\tau + \partial_\sigma) S^a\Big) d^2\sigma \\ &- \frac{1}{4\pi\alpha'}\int_\Sigma \Big(\eta^{\alpha\beta}\partial_\alpha X^I \partial_\beta X_I\Big) d^2\sigma, \end{aligned} \tag{2.101}$$

together with constraints ϕ^I that X^I consists left-moving modes alone,

$$\Phi^I = (\partial_\tau - \partial_\sigma) X^I = 0, I = 10, \cdots, 25. \tag{2.102}$$

Then $S_{\text{het-lc}}$ admits the same supersymmetries given in (2.99) and (2.100) where now the transformation parameter ϵ is Majorana-Weyl right-moving

[14] The toroidal compactification on T^{16} leads to a set of massless vector bosons $(496) = (16) \oplus (480)$, where (16) as the ordinary Kaluza-Klein gauge bosons and (480) are massless solitons. They will fill out the adjoint representation of $G = E_8 \times E_8$ or $G = \mathrm{Spin}(32)/\mathbb{Z}_2$

lightcone spinor. It relates the right-moving bosons and fermions, while leaving the left-moving bosonic X_L^i unchanged.

2.5.3 Toroidal compactification T^{16}

Since the critical dimension $D = 10$, the mismatched 16 extra bosonic coordinates X^I must then be interpreted as parametrizing an internal compact space T. For consistency reasons, T is a 16-dimensional torus that is flat and compact, so that the result theory does not contain interactions with inevitable anomalies.

Recall the mode expansion of X^I in (2.28), read (in unit $l = 1$)

$$X_L^I = x_L^I + p_L^I(\tau + \sigma) + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^I e^{-2in(\tau+\sigma)}, \tag{2.103}$$

where we rewrite the indices $I = 1, \cdots, 16$ for convenience. The compactification of X^I on 16-dimension torus T^{16} means that each X^I is identified with that of addition to some period vectors. We start with 16-dimensional Euclidean space $\mathbb{R}^{16}$, and introduce the period vector e_i^I $i = 1, \cdots, 16$.

To describe such a torus T^{16}, we can define a lattice Γ consisting points of the form,

$$\sum_{i=1}^{16} n_i e_i^I, \quad n_i \in \mathbb{Z}. \tag{2.104}$$

Then a point x^I on the torus $T^{16} = \mathbb{R}^{16}/\pi\Gamma$ (where the factor π is chosen for simplicity) is identified to a point x^I in $\mathbb{R}^{16}$ subject to the equivalence relation,

$$x^I \sim x^I + \pi\sum_{i=1}^{16} n_i e_i^I = x^I + 2\pi L^I. \tag{2.105}$$

The momentum space hence is characterized by the dual lattice $\tilde{\Gamma}$. This is because from the canonical commutation relation,

$$[x^I, p^I] = i\delta^{IJ}, \quad x^I = x_L^I + x_R^I, \quad p^I = p_L^I + p_R^I, \tag{2.106}$$

we have,

$$[x_L^I, p_L^I] = \frac{i}{2}\delta^{IJ}. \tag{2.107}$$

It implies that,

$$p_L^I \sim -\frac{i}{2}\frac{\partial}{\partial x_L^I}. \tag{2.108}$$

Then the quantization condition for the allowed momenta K^I is given by the requirement that $\exp(2iK \cdot x)$ be single valued and such K^I's forms the dual lattice $\tilde{\Gamma}$ with basis e_i^{*I} obeying,

$$\sum_{I=1}^{16} e_i^{*I} e_j^I = \delta_{ij}. \tag{2.109}$$

Hence, the allowed momenta K^I must be of form,

$$K^I = \sum_{i=1}^{16} m_i e_i^{*I}, \quad m_i \in \mathbb{Z} \tag{2.110}$$

Then the zero-frequency mode of the Virasoro condition requires level matching that $K^I = 2L^I$, meaning,

$$\sum_{i=1}^{16} m_i e_i^{*I} = \sum_{i=1}^{16} n_i e_i^I. \tag{2.111}$$

This is equivalent to say Γ is an even and self-dual lattice, which is a rather strong constraint on the basis vector e_i^I.

In fact, even and self-dual lattices are extremely rare. They only exist in $8n$ dimensions. In our 16-dimensional case, we have two such lattices available: 1) the direct product of two root lattices Γ_8 of E_8, $\Gamma_8 \times \Gamma_8$. This corresponds to introducing gauge group $E_8 \times E_8$ for the resultant 10-dimensional theory. 2) Γ_{16} which is related to the root lattices of $SO(32)$ plus additional points corresponding to the spinor weights of $\mathrm{Spin}(32)/\mathbb{Z}_2$. This corresponds to introducing gauge group $\mathrm{Spin}(32)/\mathbb{Z}_2$ for the resultant 10-dimensional theory. We will not dive into the detail here, Chapter 6 of [43] serves to be a good reference.

Up to now, we have constructed the free heterotic string theory with the gauge group $E_8 \times E_8$ or $\mathrm{Spin}(32)/\mathbb{Z}_2$. This is a promising starting point for a realistic model for particle physics.

2.5.4 Heterotic strings in background fields

To proceed, we need to generalize the free heterotic string in curved spacetime and introduce background fields [9]. We start with the bosonic case first and then move on to the supersymmetric case.

Bosonic strings in background fields

We first look at how we can generalize the purely bosonic strings to include background fields. Take our Polyakov action of form,

$$S_1[X,h,g] = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\sqrt{h}\Big(h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu g_{\mu\nu}\Big), \tag{2.112}$$

where the $g_{\mu\nu}(X^\rho)$ is viewed as a background gravitational field which is a symmetric spcetime tensor field, with the help of the worldsheet symmetric tensor $h_{\alpha\beta}$. This perspective motivates the inclusion of an anti-symmetric spacetime tensor field $B_{\mu\nu}(X^\rho)$ with the help of the worldsheet anti-symmetric tensor $\varepsilon_{\alpha\beta}/\sqrt{h}$ (where the normalization is chosen for simpler expression). Namely, we can consider a term of form,

$$S_2[X,B] = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\Big(\varepsilon^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu B_{\mu\nu}\Big). \tag{2.113}$$

Additionally, we can add a dilaton field Φ in a term as a weighted total worldsheet scalar curvature,

$$S_3[X,\Phi] = -\frac{1}{4\pi}\int_\Sigma d^2\sigma\sqrt{h}\Phi R^{(2)}. \tag{2.114}$$

If $\Phi = 1$, this term by the Gauss-Bonnet theorem, S_3 is the Euler characteristics $\chi(\Sigma)$ of the worldsheet Σ, which has important physical consequences[15].

$S_1 + S_2 + S_3$ describes bosonic string in general background fields.

Bosonic low energy effective actions

We want to maintain the Weyl scaling symmetry (2.15) even at the quantum level, as it dictates whether a string theory is consistent or not. The Weyl scaling can be regarded as a change of scale μ. We can regard the background fields $g_{\mu\nu}$, $B_{\mu\nu}$ and Φ as function-valued couplings and probe their dependence on the Weyl scaling transformation. In the renormalization group description, how couplings depend on a scale μ is called the β-function. Schematically, β-function for a generic coupling $\lambda(\mu)$ is of form,

$$\beta(\lambda) \sim \mu\frac{\partial\lambda(\mu)}{\partial\mu}. \tag{2.115}$$

[15]Since the perturbative string theory expands according to the genus of the worldsheet, the constant mode of the dilaton $\langle\Phi\rangle$ determines the string coupling constant. Usually, it is given by the asymptotic value of the dilaton,$g_s = e^{\Phi_0}$ where $\Phi_0 = \lim_{X\to\infty}\Phi(X)$.

The quantum theory remains Weyl invariant only if the coupling does not have μ-dependence, namely,

$$\beta(\lambda) = 0. \tag{2.116}$$

In our case of $S_1 + S_2 + S_3$, the β-function arises from probing the dependence of $g_{\mu\nu}$, $B_{\mu\nu}$ and Φ on the Weyl scaling. The Weyl invariance dictates the traceless of the worldsheet energy-momentum tensor $T_{\alpha\beta}$. At the quantum level, we are looking for the contributions to the expected value of the trace of $T_{\alpha\beta}$ under a Weyl scaling transformation, namely a scale change,

$$\begin{aligned}\langle T^{\alpha}{}_{\alpha} \rangle = &- \frac{1}{2\alpha'} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \beta_{\mu\nu}(g) \\ &- \frac{i}{2\alpha'} \varepsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\mu \beta_{\mu\nu}(B) \\ &- \frac{1}{2} \beta(\Phi) R^{(2)}.\end{aligned} \tag{2.117}$$

And the Weyl invariance condition translates to the condition that $\beta_{\mu\nu} g = \beta_{\mu\nu}(B) = \beta(\Phi) = 0$. It is impossible to fully display the calculation of the quantum expected value of the trace $\langle T^{\alpha}{}_{\alpha} \rangle$, as it relies on the one-loop and two-loop calculations of the perturbative expansion (in powers of α') of the σ-model action $S_1 + S_2 + S_3$. We point the interested reader to [9, 11] for the full calculations. The quantum calculation up to the first order of α' yields the following results, cited from [43],

$$0 = \beta_{\mu\nu}(g) = \alpha' \left(R_{\mu\nu} + \frac{1}{4} H_\mu{}^{\lambda\rho} H_{\nu\lambda\rho} - 2 D_\mu D_\nu \Phi \right) + \mathcal{O}(\alpha'^2), \tag{2.118}$$

$$0 = \beta_{\mu\nu}(B) = \alpha' \left(D_\lambda H^\lambda{}_{\mu\nu} - 2 (D_\lambda \Phi) H^\lambda{}_{\mu\nu} \right) + \mathcal{O}(\alpha'^2), \tag{2.119}$$

$$0 = \beta(\Phi) = \alpha' \left(4 (D_\mu \Phi)^2 - 4 D_\mu D^\mu \Phi + R + \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} \right) + \mathcal{O}(\alpha'^2), \tag{2.120}$$

where H is the field strength of B-field, $H = dB$, namely,

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}, \tag{2.121}$$

and D_μ is the covariant derivative on M.

The key insight comes that all the β-functions up to first-order of α' can be organized as the equations of motion of the following gravity action [10],

$$S_{\text{eff}} = -\frac{1}{2\kappa^2} \int d^D x \sqrt{g} e^{-2\Phi} (R - 4 D_\mu \Phi D^\mu \Phi + \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho}),$$

$$= -\frac{1}{2\kappa^2}\int_M e^{-2\Phi}(\star R - 4|D\Phi|^2 + \frac{1}{2}|H|^2), \tag{2.122}$$

where we write $|A|^2 = A \wedge \star A$ for any form A for simplicity and a more transparent geometric interpretation. Therefore, in the limit, $\alpha' \to 0$, all higher order contribution to the β-function vanishes and S_{eff} describes the massless modes of the bosonic theory. In this sense, it is the low-energy effective action of $S_1 + S_2 + S_3$.

Heterotic string in background fields

We can generalize our discussion to the case of heterotic string. Without really discussing the details, we can write down the lightcone[16] action for a heterotic string in background fields (see [60]),

$$\begin{aligned} S = &-\frac{1}{4\pi\alpha'}\int d^2\sigma\big(\partial_\alpha X^i \partial^\alpha X^j g_{ij} + i\bar{S}\Gamma^- D_+ S\big) \\ &-\frac{1}{4\pi\alpha'}\int d^2\sigma\Big(\varepsilon^{\alpha\beta}\partial_\alpha X^i \partial_\beta X^j B_{ij}\Big) \\ &-\frac{1}{4\pi}\int d^2\sigma\Big(\Phi R^{(2)}\Big) \\ &-\frac{1}{4\pi\alpha'}\int d^2\sigma\Big(\psi^a D_- \psi^a + F^A_{ij}\bar{S}\Gamma^-\Gamma^{ij} S T^A_{ab}\psi^a\psi^b\Big), \end{aligned} \tag{2.123}$$

where $i, j = 1, \cdots, 8$ run over the transverse space of lightcone in 10-dimensional M, S is the right-moving lightcone ($\Gamma^+ S = 0$) Majorana-Weyl spinor as in (2.101). However, we now have the complication of the background gauge field representing $E_8 \times E_8$ or Spin(32)/$\mathbb{Z}_2$ from the toroidal compactification. This is encoded in field strength F^A_{ij} of the gauge field A_i, the generator T^A_{ij} and ψ^i left-moving worldsheet spinors in the representation of the gauge group. The covariant derivative $D_\pm$ now includes torsion and acts as $D_-\psi = \partial_-\psi + (\partial_- X^i)A_i\psi$ and $D_+ S = \partial_+ S + \partial_+ X^i \frac{1}{4}(\omega_i - H_i)S$, where ω is the spin connection, and $H = dB$.

In flat space, this S is invariant under the supersymmetry transformations (2.97), (2.98) and (2.99),(2.100) To require then unbroken during compactification would then yield geometric conditions on the internal space to be discussed later.

[16] A covariant action with background fields can be formulated in the language of superspace, see [9].

Heterotic low-energy effective theory

Since supersymmetry is preserved in the massless spectrum that is described by the low-energy effective theory, we can focus on the bosonic part of the action and obtain the low-energy effective bosonic theory and then use the supersymmetry to recover the fermionic part. This is why supersymmetry is quite powerful.

The complication arises in the introduction of the background gauge field A from the gauge group $E_8 \times E_8$ or $\mathrm{Spin}(32)/\mathbb{Z}_2$. This will introduce a Yang-Mills term and one to make sure the contribution to the dependence of the Weyl scaling from this gauge field cancels with that of from gravitation g, namely the Green-Schwarz anomaly cancellation [44]. Without going into the detail, it requires we modify the field strength H of B-field to add Chern-Simons 3-form from the gauge field A to be viewed as a connection on the principal bundle and the usual Levi-Civita connection D on TM that gives the covariant derivative D, with appropriate normalization,

$$H = dB + \alpha'(\mathrm{CS}_3(A) - \mathrm{CS}_3(D)), \tag{2.124}$$

such that the theory remains Weyl invariant at the quantum level (i.e. anomaly-free). From this, the Bianchi identity for H reads,

$$dH = \alpha'(\mathrm{Tr} F_A \wedge F_A - \mathrm{Tr} R \wedge R). \tag{2.125}$$

Account for this and follow the same line of argument using β-function which requires a lot of calculation, we obtain the bosonic part of the low-energy effective supergravity theory for the heterotic string up to the first order α' [7, 23], with suitable normalization,

$$S_{\text{eff}} = \int e^{-2\Phi}\left(\star R - 4|D\Phi|^2 + \frac{1}{2}|H|^2 + \alpha'(\mathrm{Tr}|F_A|^2 - \mathrm{Tr}|R|^2)\right) + \mathcal{O}(\alpha'^3). \tag{2.126}$$

We have the gravitino ψ_μ as the super-partner for the graviton $g_{\mu\nu}$, the dilatino λ for the dilaton Φ, and the gaugino χ for the gauge field A.

The supersymmetry variation of those fermionic fields is then recovered via the bosonic fields[17], with appropriate normalization, and read as [7, 60],

$$\delta\psi_\mu = D^+_\mu \epsilon = \left(D_\mu + \frac{1}{2}H_\mu\right)\epsilon + \mathcal{O}(\alpha') = 0, \tag{2.127}$$

$$\delta\lambda = \left(\Gamma^\mu D_\mu \Phi + \frac{1}{3}H\right)\epsilon + \mathcal{O}(\alpha') = 0, \tag{2.128}$$

[17] This is not a trivial process, and was derived from [18].

$$\delta\chi = -\frac{1}{2}F_{\mu\nu}\Gamma^{\mu\nu}\epsilon + \mathcal{O}(\alpha') = 0, \tag{2.129}$$

where $H_\mu = H_{\mu\nu\rho}\Gamma^{\nu\rho}$, $H = H_{\mu\nu\rho}\Gamma^{\mu\nu\rho}$ and $\Gamma^{\mu\nu\cdots\rho} = \Gamma^{[\mu}\Gamma^{\nu}\cdots\Gamma^{\rho]}$ the anti-symmetric product of gamma metrics. Supersymmetry invariance then says that all those field variations vanish.

2.6 Heterotic system

2.6.1 Heterotic compactification

To relate the resultant 10-dimension heterotic theory to our 4-dimensional reality. We need to perform another compactification of the 6 extra dimensions. Namely, we consider the 10-dimensional spacetime M_{10} as a direct product $M_4 \times X_6$, where M_4 is the 4-dimensional Minkowski spacetime manifold and X_6 is the 6-dimensional internal manifold. We now use M, N, P to denote indices on M_{10}, m, n, p to denote indices on X_6 with coordinates y, and μ, ν, ρ to denote indices on M_4 with coordinate x. Namely, we consider superstring compactification with the product metric of form,

$$g_{MN}(x,y) = e^{2D(y)}\begin{pmatrix} g_{mn}(y) & 0 \\ 0 & g_{\mu\nu}(x) \end{pmatrix}, \tag{2.130}$$

where we had introduced a wrap factor $D(y)$, which can be shown to equate to the dilaton field Φ by supersymmetry.

However, X_6 is not arbitrary. We would like a resultant theory on a maximally symmetric M_4 with unbroken supersymmetry, which puts a constraint on the geometry of X_6. Namely, a 10-dimensional spinor ϵ_{10} decomposes as a direct product $\epsilon_4 \otimes \epsilon_6$, and we desire the supersymmetry conditions (2.127), (2.128) and (2.129) holds for 6-dimensional ϵ_6 and corresponding fields.

In the case of $H = 0$, Candelas-Horowitz-Strominger-Witten [12] showed that the in X_6 has to be Kähler Ricci-flat, namely a Calabi-Yau 3-fold, and admits $SU(3)$ holonomy group. In the case of $H \neq 0$, Strominger [60] and Hull [47] showed that the geometry of X_6 is dictated by a set of equations called the heterotic system. The main surprise is that with $H \neq 0$, X_6 is still a complex manifold. Therefore, from a geometric point of view, the geometry of X_6 is in some sense the minimal extension of a Kähler manifold to a non-Kähler manifold, hence the reason why the heterotic system is an active research topic.

2.6.2 Supersymmetry conditions

We now follow Strominger [60] and give a brief discussion of the heterotic system. Picard's lecture note [58] provides more details of the calculation.

If $H = 0$,

The supersymmetry conditions (2.127) and (2.128) imply the existence of positive and negative chirality[18] 6-dimensional spinors $\eta_\pm$ (with appropriate normalization $\eta_\pm^\dagger \eta_\pm = 1$) that is H-covariantly constant, meaning

$$D^\pm \eta_\pm = \left(D \pm \frac{1}{2} g^{-1} H \right) \eta_\pm = 0, \tag{2.131}$$

where this connection D^+ is called the Strominger-Bismut connection, and D^- is called the Hull connection.

Then we can construct a tensor J from the positive chirality spinor η_+ by,

$$J_m{}^n = i\eta_+^\dagger \gamma_m{}^n \eta_+. \tag{2.132}$$

By the Fierz rearrangement identity $\gamma_m \gamma^n = \gamma_m{}^n - \delta_m{}^n$, we have

$$J_m{}^n J_n{}^p = -\delta_m{}^p, \tag{2.133}$$

meaning J is an almost complex structure on X_6. Then it is natural to investigate the integrability of J, which is given by the Nijenhaus tensor,

$$N_{mn}{}^p = J_m{}^q J_{[n}{}^p{}_{,q]} - J_n{}^q J_{[m}{}^p{}_{,q]}. \tag{2.134}$$

where $,n$ means taking covariant derivative D_n. The J is H-covariantly constant meaning

$$D_m^+ J_n{}^p = D_m J_n{}^p - H_{sm}{}^p J_n{}^s - H^s{}_{mn} J_s{}^p = 0. \tag{2.135}$$

Using this fact and the dilatino equation (2.128), and some gamma matrices identity (see [13]), we can calculate,

$$\begin{aligned} N_{mnp} &= (H_{mnp} - 3J_{[m}{}^q J_n{}^r H_{p]qr}), \qquad (2.136) \\ &= -\frac{1}{3}\eta_+^\dagger \{H, \gamma_{mnp} + 3i\gamma_{[m} J_{np]}\} \eta_+, \end{aligned}$$

[18]The chirality is reflected by their eigenvalues under the chiral operator γ^7,

$$\gamma^7 \eta_\pm = i^3 \gamma^1 \gamma^2 \gamma^3 \gamma^4 \gamma^5 \gamma^6 \eta_\pm = \pm \eta_\pm,$$

where γ^m are 6-dimensional gamma matrices.

$$= -\eta_+^\dagger \left[\gamma^m D_m \Phi, i\gamma_{[m} J_{np]}\right] \eta_+,$$
$$= 0, \tag{2.137}$$

namely, J is indeed integrable. Then by the Newlander-Nirenberg theorem (see Thm. 3), the almost complex structure J with vanishing Nijenhuis tensor gives holomorphic coordinate charts and hence defines a complex structure. Therefore, we can write the complex structure J in a local holomorphic chart with α, β now denoting the complex indices, as,

$$J = J_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta = i g_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta. \tag{2.138}$$

Multiplying the H-covariantly constant condition on J (2.135) by $J_q{}^m$ and using (2.136), we can obtain,

$$H = \frac{i}{2}(\bar{\partial} - \partial)J, \tag{2.139}$$

where $\partial, \bar{\partial}$ now denotes the Dolbeault holomorphic and anti-holomorphic derivatives. One can see that if $H = 0$, then $dJ = 0$, and X_6 is indeed Kähler.

Now take the exterior derivative, then recall the Bianchi identity of H given by (2.125), we have the anomaly cancellation condition,

$$dH = i\partial\bar{\partial}J = \alpha'(\mathrm{Tr}F^2 - \mathrm{Tr}R^2). \tag{2.140}$$

This condition implies a topological condition on X_6 and the principal bundle (denoted later as E) that encodes the gauge group, namely, via integration, the second Chern class $c_2(X_6) = c_2(E)$.

Moreover, the supersymmetry condition implies the existence of a non-vanishing holomorphic 3-form, given by,

$$\Omega = e^{8\Phi} \eta_-^\intercal \gamma_{123} \eta_- dz^1 \wedge dz^2 \wedge dz^3, \quad \bar{p}\Omega = 0. \tag{2.141}$$

Then using Φ and H, we can rewrite the dilatino equation (2.128) as a equation for J,

$$d^\dagger J + i(\bar{\partial} - \partial) \ln |\Omega| = 0. \tag{2.142}$$

It is worth noting that the existence of Ω gives a topological restriction on that $h^{3,0} = 1$ and the vanishing of the first Chern class $c_1(X) = 0$.

The gluino equation (2.129) can be rewritten in the form of the hermitian Yang-Mills equation,

$$J^{\alpha\bar{\beta}} F_{\alpha\bar{\beta}} = i\eta_+^\dagger F_{mn} \gamma^{mn} \eta_+ = 0 \tag{2.143}$$

$$F_{\bar{\alpha}\bar{\beta}} = \frac{1}{8}\eta_+^\dagger\{\gamma_{\bar{\alpha}\bar{\beta}}, F_{mn}\gamma^{mn}\}\eta_+ = 0 = F_{\alpha\beta}. \tag{2.144}$$

Therefore, Strominger [60] concludes the necessary and sufficient conditions for spacetime supersymmetry on X_6, and in summary,

1. The internal manifold must be complex, by the existence of integrable almost complex structure given by,

$$J_m{}^n := i\eta_+^\dagger\Gamma_m{}^n\eta_+,$$
$$N_{mnp} = H_{mnp} - 3J_{[m}{}^q J_n{}^r H_{p]qr} = 0.$$

2. The fundamental form $J = ig_{a\bar{b}}dz^\alpha \wedge d\bar{z}^\beta$ must obey,

$$-i\partial\bar{\partial}J = \alpha'(\text{Tr}F \wedge F - \text{Tr}R \wedge R),$$
$$d^\dagger J = i(\partial - \bar{\partial})\ln||\Omega||.$$

3. the Yang-Mills field strength F must satisfy,

$$J^{a\bar{b}}F_{a\bar{b}} = 0,$$
$$F_{ab} = F_{\bar{a}\bar{b}} = 0.$$

This is the heterotic system. Hull [47] later gives a more detailed treatment of spacetime supersymmetry in the GS formalism and the equivalence to that of worldsheet supersymmetry in RNS formalism.

For the proceeding chapter, we will reformulate the heterotic system in the language of complex geometry and vector bundles, and recast them in a simpler form for our later analysis of the heterotic system.

Before that, it is worth emphasizing the significance of the heterotic system. It gives the necessary condition for the unbroken supersymmetry in the heterotic compactification. When $\alpha' = 0$ or $H = 0$, we return to case of Candelas-Horowitz-Strominger-Witten [12] that X_6 is Kähler Ricci-flat and hence a Calabi-Yau 3-fold. While maintaining natural topological constraints $c_1(X_6) = 0$, $c_2(X_6) = c_2(E)$, the heterotic system provides a minimal extension to the non-Kähler scenarios for X_6 of many interests.

Chapter 3

Mathematical Ingredients

The heterotic system, motivated in Chapter 1, consists of two main mathematical ingredients: the base manifold (X, ω) which is a hermitian manifold representing the internal space X_6 for the heterotic compactification, and the gauge bundle which is a holomorphic vector bundle (E, A) where its curvature is invariant under gauge transformation. The heterotic system is viewed as the system of equations that dictates their interactions.

In this chapter, we establish the conventions of these two main ingredients and collect some necessary facts for our construction of heterotic moduli.

3.1 Base manifold (X, ω)

Let X be a compact complex manifold of complex dimension $n = \dim_{\mathbb{C}} X$. In particular, we are interested in $n = 3$. X then admits a nowhere vanishing holomorphic 3-form $\Omega \in \Lambda^{n,0}(X, \mathbb{C})$. Let $\omega \in \Lambda^{1,1}(X, \mathbb{R})$ be a hermitian form[19] on X.

Given a holomorphic coordinates of X, denoted as $\{z^\alpha\}_{\alpha=1}^n$, we can express the hermitian form ω in terms of holomorphic coordinates locally as:

$$\omega = i g_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta \tag{3.1}$$

where $g_{\alpha\bar{\beta}}$ is hermitian, meaning $\overline{g_{\alpha\bar{\beta}}} = g_{\beta\bar{\alpha}}$, and we denote its inverse $g^{\bar{\alpha}\beta}$ such that $g_{\alpha\bar{\gamma}} g^{\bar{\gamma}\beta} = \delta_\alpha{}^\beta$. We therefore also call ω hermitian metric.

The nowhere vanishing holomorphic 3-form Ω can be expressed in terms of holomorphic coordinates locally as:

$$\Omega = f dz^1 \wedge \cdots \wedge dz^n, \tag{3.2}$$

where $f \in \mathcal{O}_X^*$ a non-vanishing holomorphic function. We also define the

[19] To be related to the J in Chapter 1.

norm of Ω with respect to ω, to be denoted as $|\Omega|_\omega$, by the following relation,

$$i\Omega \wedge \bar{\Omega} = |\Omega|^2_\omega \frac{\omega^n}{n!}. \tag{3.3}$$

Its local expression[20] can be calculated to be,

$$|\Omega|^2_\omega = f\bar{f}(\det g_{\bar{\beta}\alpha})^{-1}. \tag{3.4}$$

Given (X, ω), we can look at the tangent bundle $T^{1,0}(X)$, and form the Chern connection associated with the hermitian form ω, to be denoted as ∇^{C}. Its action on any section $V \in \Gamma(T^{1,0}(X))$ is defined by,

$$\nabla^{\mathrm{C}}_\alpha V^\beta = \partial_\alpha V^\beta + g^{\bar{\gamma}\beta}\partial_\alpha g_{\delta\bar{\gamma}} V^\delta, \tag{3.5}$$

$$\nabla^{\mathrm{C}}_{\bar{\alpha}} V^\beta = \bar{\partial}_\alpha V^\beta, \tag{3.6}$$

namely, its connection forms has vanishing component in $\bar{\partial}$-direction and we denote the component in ∂-direction as Γ, locally given by,

$$\Gamma^\beta{}_{\alpha\delta} = g^{\bar{\gamma}\beta}\partial_\alpha g_{\delta\bar{\gamma}}. \tag{3.7}$$

It is easy to see that taking conjugates induced the Chern connection on the conjugate tangent bundle $(T^{(1,0)}(X))^*$.

Then the curvature of ∇^{C} is called the Chern curvature, to be denoted as R and locally given by,

$$R_{\alpha\bar{\beta}}{}^\mu{}_\nu = -\bar{\partial}_\beta \Gamma^\mu{}_{\alpha\nu} = -\bar{\partial}_\beta\big(g^{\bar{\gamma}\mu}\partial_\alpha g_\nu \bar{\gamma}\big), \tag{3.8}$$

$$(R_\omega)^\mu{}_\nu = R_{\alpha\bar{\beta}}{}^\mu{}_\nu dz^\alpha \wedge d\bar{z}^\beta, \tag{3.9}$$

and R_ω is the endomorphism-valued curvature form.

If the hermitian form ω is closed,

$$d\omega = 0, \quad \omega \in [\omega], \quad [\omega] \in H^{1,1}_{\mathrm{dR}}(X, \mathbb{R}) \tag{3.10}$$

then we call it the Kähler form and X is a Kähler manifold. Then Yau's theorem [63] states that for a Kähler manifold X, there exists a unique Kähler Ricci-flat metric $\omega_{\mathrm{CY}} \in [\omega]$.

ω is called conformally balanced if

$$d\big(|\Omega|_\omega \omega^2\big) = 0. \tag{3.11}$$

Kähler Ricci-flat metrics are automatically conformally balanced as $|\Omega|_\omega$ is constant and $d\omega^2 = 0$. In particular, Li and Yau [50] showed that Strominger's original condition (2.142) can be recast into this conformally balanced condition (3.11).

[20]This local expression will be useful in the calculation of variation of metric along the deformation path later in Lemma. 6.1.1.

3.2 Gauge bundle (E, A)

Let $E \to X$ be a smooth complex vector bundle over X of rank r, equipped with a hermitian metric h.

Let $D = d + A$ be the covariant derivative of a unitary connection A on E, namely, $Dh = 0$. We will typically work in a smooth local unitary frame $\{e_\alpha\}$, so that in this frame the bundle metric,

$$h(e_\alpha, e_\beta) = \delta_{\alpha\beta} \tag{3.12}$$

is the identity and A is unitary

$$A = -A^\dagger. \tag{3.13}$$

Then by type decomposition, we denote

$$A = -\mathcal{A}^\dagger + \mathcal{A}, \tag{3.14}$$

where $\mathcal{A}$ is of type $(0, 1)$ and $-\mathcal{A}^\dagger$ is of type $(1, 0)$. Then the covariant derivative can also be type decomposed

$$D = \mathcal{D} + \bar{\mathcal{D}}, \tag{3.15}$$

$$\mathcal{D} = \partial - \mathcal{A}^\dagger, \tag{3.16}$$

$$\bar{\mathcal{D}} = \bar{\partial} + \mathcal{A}. \tag{3.17}$$

The properties of the covariant derivative are recorded in Appendix C.

The curvature F_A of the unitary connection A is given by

$$F_A = dA + A \wedge A. \tag{3.18}$$

Suppose F_A is only of type $(1, 1)$, meaning its $(0, 2)$ type component vanishes and so does component of type $(2, 0)$ by unitarity of A,

$$F_A^{0,2} = F_A^{2,0} = 0, \tag{3.19}$$

then by a well-known integrability result (see Theorem 2.1.53 in [26]), E admits trivializations by holomorphic frames so that $E \to X$ is a holomorphic vector bundle. In such a holomorphic frame $\{f_\alpha\}$,

$$A^{0,1} = 0, \quad \bar{\mathcal{D}} = \bar{\partial}. \tag{3.20}$$

Then The metric appears as

$$h_{\alpha\bar{\beta}} = h(f_\alpha, f_\beta) \tag{3.21}$$

and metric compatibility implies

$$A(h) = h^{-1}\partial h \tag{3.22}$$

The curvature is then given by,

$$F_A = \bar{\partial}(h^{-1}\partial h). \tag{3.23}$$

Now, given a holomorphic vector bundle $(E, A(h)) \to (X, \omega)$ over a Kähler manifold, the notion of stability is given as follows. We define the ω-degree $\deg(E)$ of E by

$$\deg(E) = \int_X c_1(E) \wedge \omega^2, \tag{3.24}$$

and the ω-slope $\mu(E)$ of E by

$$\mu(E) = \frac{\deg(E)}{\mathrm{rank}(E)}. \tag{3.25}$$

Then E is called stable if for any coherent subsheaf $\mathcal{F} \subset \mathcal{O}(E)$ satisfying $0 < \mathrm{rank}(F) < \mathrm{rank}(E)$, we have $\mu(\mathcal{F}) < \mu(E)$. The bundle E is said to be poly-stable if E is the direct sum of stable vector bundles with the same ω-slope. The Donaldson-Uhlenbeck-Yau theorem [25, 61] states that for a compact Kähler manifold (X, ω) and holomorphic vector bundle $E \to X$, the bundle E is poly-stable if and only if E admits a Hermitian-Yang-Mills metric h, meaning that its curvature

$$F_h = \bar{\partial}(h^{-1}\partial h) \tag{3.26}$$

solves

$$\Lambda_\omega F_h = \mu \mathbf{1}_E. \tag{3.27}$$

In our setup, we will take as initial data a Kähler Calabi-Yau threefold X and a holomorphic vector bundle $E \to X$ which is stable with respect to ω_{CY} and satisfies

$$c_1(E) = 0, \qquad c_2(E) = c_2(X). \tag{3.28}$$

It is well-known that a stable bundle E admits no global holomorphic endomorphism other than homotheties, i.e. it does not admit any holomorphic endomorphism other than multiplies of the identity. We will use this later on to rule out elements of the kernel of a linearized operator in Lemma.

6.1.3. We apply the Donaldson-Uhlenbeck-Yau theorem and take the bundle metric h to be Hermitian-Yang-Mills, so that

$$F_h \wedge \omega_{\mathrm{CY}}^2 = 0. \tag{3.29}$$

As discussed before, the Chern connection of h is $D = d + A$ with $Dh = 0$ and $A = h^{-1}\partial h$ in a holomorphic frame. If we work instead in a smooth local unitary frame $\{e_\alpha\}$, then we are back in setup (3.14). Such frame transformation is always possible as shown in Appendix.B. Then, the assumption $c_1(E) = 0$ gives

$$\mathrm{Tr}(iF_A) = d\gamma \tag{3.30}$$

for some $\gamma \in \Lambda^1(X)$.

3.3 Heterotic system $[(X,\omega),(E,A)]$

Now let X be a complex 3-fold, Ω a holomorphic volume form, and $(E,h) \to X$ a smooth complex vector bundle with metric h. With the data (X,Ω), $(E,A(h))$ considered fixed, the heterotic system [60] can be translated into a system for a pair (ω, A) satisfying the following three conditions:

1. The hermitian metric ω is conformally balanced:

$$d(|\Omega|_\omega \omega^2) = 0. \tag{3.31}$$

2. The unitary connection A solves the Hermitian-Yang-Mills (HYM) equation:

$$F_A^{0,2} = 0 = F_A^{2,0}, \tag{3.32}$$
$$F_A \wedge \omega^2 = 0. \tag{3.33}$$

3. The anomaly cancellation relation holds:

$$i\partial\bar{\partial}\omega = \alpha'(\mathrm{Tr}F_A \wedge F_A - \mathrm{Tr}R_\omega \wedge R_\omega). \tag{3.34}$$

These equations of heterotic systems have been historically split into two terms:

1. D-terms:

$$d(|\Omega|_\omega \omega^2) = 0,$$
$$F_A \wedge \omega^2 = 0.$$

2. F-terms:

$$\begin{aligned} i\partial\bar{\partial}\omega &= \alpha'(\mathrm{Tr}F_A \wedge F_A - \mathrm{Tr}R_\omega \wedge R_\omega), \\ F_A^{0,2} &= 0 = F_A^{2,0} \end{aligned}$$

Each relates to different supersymmetric couplings in the 4-dimensional effective supergravity theory.

Again to emphasize the subtlety, we choose the Chern connection and its corresponding Chern curvature R_ω. In the physics literature, one usually takes the Hull connection [47] and views (3.34) not as an equality but as an expansion where the remaining terms are of order $O(\alpha'^2)$. Nevertheless, since our setup involves a perturbation of a Kähler background, our constructed solutions solve the equations of heterotic string theory to appropriate order in α'. See Appendix A for justification for taking the Chern connection.

Part II

Heterotic Moduli

Chapter 4

Deformation Ansatzes

In this chapter, we motivate two natural deformation ansatzes of (ω, A) fixing the complex structure, and set up the corresponding implicit function theorem for both cases in the proceeding chapter.

To understand the moduli space of the heterotic system, we can start with a Kähler Calabi-Yau solution

$$d\omega_{\mathrm{C}Y} = 0 \tag{4.1}$$

which satisfies the heterotic system at $\alpha' = 0$. We therefore start from a pair of a Kähler Ricci-flat metric $\omega = \omega_{\mathrm{C}Y}$ and a hermitian Yang-Mills connection A, so that (ω, A) solves the heterotic system at $\alpha' = 0$:

$$d(|\Omega|_\omega \omega^2) = 0, \tag{4.2}$$

$$F_A^{0,2} = 0, \tag{4.3}$$

$$F_A \wedge \omega^2 = 0, \tag{4.4}$$

$$i\partial\bar{\partial}\omega = 0. \tag{4.5}$$

Then we can probe nearby local solutions $(\tilde{\omega}, \tilde{A})$ solving the heterotic system to the next order $\mathcal{O}(\alpha')$,

$$d(|\Omega|_{\tilde{\omega}} \tilde{\omega}^2) = 0, \tag{4.6}$$

$$\tilde{F}_{\tilde{A}}^{0,2} = 0, \tag{4.7}$$

$$\tilde{F}_{\tilde{A}} \wedge \tilde{\omega}^2 = 0, \tag{4.8}$$

$$i\partial\bar{\partial}\tilde{\omega} = \alpha' \Big(\mathrm{Tr} \tilde{F}_{\tilde{A}} \wedge \tilde{F}_{\tilde{A}} - \mathrm{Tr} \tilde{R}_{\tilde{\omega}} \wedge \tilde{R}_{\tilde{\omega}} \Big). \tag{4.9}$$

We have the freedom to fix the deformation ansatz to satisfy some of these four equations and then solve the remaining. Different partitions lead to different deformation ansatzes. The two most natural ansatz are as follows:

1. Bott-Chern case: fix the deformation ansatz $(\tilde{\omega}, \tilde{A})$ to satisfy

$$d(|\Omega|_{\tilde{\omega}} \tilde{\omega}^2) = 0, \tag{4.10}$$

$$\tilde{F}^{0,2}_{\tilde{A}} = 0. \tag{4.11}$$

And then we solve for the remaining two,

$$i\partial\bar{\partial}\tilde{\omega} = \alpha'(\mathrm{Tr}\tilde{F}_{\tilde{A}} \wedge \tilde{F}_{\tilde{A}} - \mathrm{Tr}\tilde{R}_{\tilde{\omega}} \wedge \tilde{R}_{\tilde{\omega}}), \tag{4.12}$$

$$\tilde{F}_{\tilde{A}} \wedge \tilde{\omega}^2 = 0. \tag{4.13}$$

This approach corresponds to deforming the metric along the Bott-Chern cohomology class.

2. Aeppli case: fix the deformation ansatz $(\tilde{\omega}, \tilde{A})$ to satisfy

$$i\partial\bar{\partial}\tilde{\omega} = \alpha'(\mathrm{Tr}\tilde{F}_{\tilde{A}} \wedge \tilde{F}_{\tilde{A}} - \mathrm{Tr}\tilde{R}_{\tilde{\omega}} \wedge \tilde{R}_{\tilde{\omega}}), \tag{4.14}$$

$$\tilde{F}^{0,2}_{\tilde{A}} = 0. \tag{4.15}$$

And we solve for

$$d(|\Omega|_{\tilde{\omega}}\tilde{\omega}^2) = 0, \tag{4.16}$$

$$\tilde{F}_{\tilde{A}} \wedge \tilde{\omega}^2 = 0. \tag{4.17}$$

This approach corresponds to deforming the metric along the Aeppli cohomology class.

It is worth noting that there are isomorphisms between the de-Rham cohomology, Bott-Chern cohomology, and Aeppli cohomology on Kähler background [24] (See [3] for a good survey.). We can regard these two deformation ansatzes as "dual" to each other on the initial Kähler solution. We hence provide a parallel treatment of these two approaches towards constructing local coordinates on the heterotic moduli space.

4.1 Deformation of complex structure

We fix the complex structure J along the deformation, hence fixing ∂ and Ω. Otherwise, we need more effort to construct a natural deformation ansatz because the deformation of complex structure couples to both deformations of metric and gauge connection.

Therefore, technically speaking, we are probing a gauge orbit of the full heterotic moduli where the complex structure is invariant. We leave the further extension to the full moduli to future investigations and provide some directions in the final discussion.

4.2 Metric deformation $\tilde{\omega}$

There are two natural partitions of the heterotic system, hence we have two parallel deformation ansatzes.

4.2.1 Bott-Chern case

We would like our ansatz of $\tilde{\omega}$ to satisfy the balanced condition,

$$d(|\Omega|_{\tilde{\omega}}\tilde{\omega}^2) = 0. \tag{4.18}$$

For this reason, we choose to deform the metric ω along the Bott-Chern cohomology class $H^{2,2}_{\text{B-C}}(X, \mathbb{C})$. Recall the Bott-Chern cohomology is defined by,

$$H^{2,2}_{\text{B-C}}(X, \mathbb{C}) = \frac{\text{Ker}(d) \ \cap \ \Lambda^{2,2}(X, \mathbb{R})}{\text{Im}(\partial\bar{\partial}) \ \cap \ \Lambda^{1,1}(X, \mathbb{R})}. \tag{4.19}$$

From a conformally balanced metric $\tilde{\omega}$, we can produce a balanced class whose elements satisfy the balanced condition,

$$\underline{b}(\tilde{\omega}) = [|\Omega|_{\tilde{\omega}}\tilde{\omega}^2] \in H^{2,2}_{\text{B-C}}(X, \mathbb{R}). \tag{4.20}$$

With $\omega = \omega_{\text{CY}}$ as our initial reference metric, we hence take the deformation of metric $\tilde{\omega}$ to be of the form,

$$\underline{b}(\tilde{\omega}) = [|\Omega|_{\omega_{\text{CY}}}\omega^2_{\text{CY}}] + [\mathfrak{b}], \tag{4.21}$$

more precisely,

$$|\Omega|_{\tilde{\omega}}\tilde{\omega}^2 = |\Omega|_{\omega}\omega^2 + \mathfrak{b} + \Theta, \tag{4.22}$$

where

$$(\mathfrak{b} + \Theta) \in \Lambda^{2,2}(X, \mathbb{R}) : \ d\mathfrak{b} = 0, \ \Theta = i\partial\bar{\partial}\gamma \in i\partial\bar{\partial}\Lambda^{1,1}(X, \mathbb{R}). \tag{4.23}$$

For $(\mathfrak{b}+\Theta)$ small enough, taking the square root defines our deformed metric $\tilde{\omega}$. The precise formula for $\tilde{\omega}$ can be calculated and is given in [19]. Now, the deformed metric satisfies the balanced condition automatically,

$$d(|\Omega|_{\tilde{\omega}}\tilde{\omega}^2) = d(|\Omega|_{\omega}\omega^2) + d\mathfrak{b} + d(i\partial\bar{\partial}\gamma) = 0. \tag{4.24}$$

To solve the heterotic system, we will substitute this ansatz into the remaining hermitian Yang-Mills equation and the anomaly cancellation equation.

4.2.2 Aeppli case

We now demand our ansatz of $\tilde{\omega}$ solves the anomaly cancellation relation,

$$i\partial\bar{\partial}\tilde{\omega} = \alpha'(\text{Tr}\tilde{F}_{\tilde{A}} \wedge \tilde{F}_{\tilde{A}} - \text{Tr}\tilde{R}_{\tilde{\omega}} \wedge \tilde{R}_{\tilde{\omega}}). \tag{4.25}$$

We therefore choose to deform the metric ω along the Aeppli cohomology class $H^{1,1}_{\text{A}}(X, \mathbb{C})$. Recall the Aeppli cohomology is defined by,

$$H^{1,1}_{\text{A}}(X, \mathbb{C}) = \frac{\text{Ker}(\partial\bar{\partial}) \ \cap \ \Lambda^{1,1}(X, \mathbb{R})}{(\text{Im}(\partial) \cap \Lambda^{0,1}(X, \mathbb{R})) \oplus (\text{Im}(\bar{\partial}) \cap \Lambda^{1,0}(X, \mathbb{R}))}. \tag{4.26}$$

There is one caveat: due to the complexity of $\tilde{R}_{\tilde{\omega}}$, we introduce a spurious degree of freedom, namely an extra gauge field θ on $T^{1,0}(X)$, to compute $\text{Tr}R \wedge R$ with a modified anomaly cancellation relation,

$$i\partial\bar{\partial}\tilde{\omega} = \alpha'\Big(\text{Tr}\tilde{F}_{\tilde{A}} \wedge \tilde{F}_{\tilde{A}} - \text{Tr}\tilde{R}_{\tilde{\theta}} \wedge \tilde{R}_{\tilde{\theta}}\Big), \tag{4.27}$$

and requiring this spurious gauge field $\tilde{\theta}$ to satisfy the hermitian Yang-Mills equation,

$$\tilde{R}_{\tilde{\theta}} \wedge \omega^2 = 0, \tag{4.28}$$

$$\tilde{R}^{0,2}_{\tilde{\theta}} = 0. \tag{4.29}$$

Such an approach with the introduction of the gauge field θ was considered by previous works [2, 22, 36]. We provide an explanation why this setup in the Kähler large radius limit also solves the physical equations in Appendix. A.

The topological condition on Chern classes $c_1(E) = c_1(X) = 0$ and $c_2(E) = c_2(X)$, together with the $\partial\bar{\partial}$-lemma, implies the existence of $\beta \in \Lambda^{1,1}(X, \mathbb{R})$ such that,

$$\text{Tr}F_A \wedge F_A - \text{Tr}R_\theta \wedge R_\theta = i\partial\bar{\partial}\beta. \tag{4.30}$$

Here (ω, A, θ) are the background reference fields. To ensure β is real, we take the real part if necessary since $i\partial\bar{\partial}\text{Im}\beta = 0$.

It has been notice in [37], a solution $(\tilde{\omega}, \tilde{A}, \tilde{\theta})$ to the anomaly cancellation relation create an Aeppli class,

$$\underline{a}(\tilde{\omega}, \tilde{A}, \tilde{\theta}) = \Big[\tilde{\omega} - \alpha'(R_2[\tilde{A}, A] - R_2[\tilde{\theta}, \theta] + \beta)\Big] \in H^{1,1}_{\text{A}}(X, \mathbb{R}). \tag{4.31}$$

Here the R_2's are the Bott-Chern-Simons secondary characteristics defined respectively by,

$$i\partial\bar{\partial}R_2[\tilde{A},A] = \mathrm{Tr}\tilde{F}_{\tilde{A}} \wedge \tilde{F}_{\tilde{A}} - \mathrm{Tr}F_A \wedge F_A, \tag{4.32}$$

$$i\partial\bar{\partial}R_2[\tilde{\theta},\theta] = \mathrm{Tr}\tilde{R}_{\tilde{\theta}} \wedge \tilde{R}_{\tilde{\theta}} - \mathrm{Tr}R_\theta \wedge R_\theta. \tag{4.33}$$

The explicit construction is given in [25]. In our construction of $R_2[\tilde{A},A]$ will be slightly different since A, $\tilde{A}$ give different holomorphic structure on E. Before we provide a detailed construction of R_2, we first write down our Aeppli ansatz below.

We start from a reference (ω, A, θ) satisfying the heterotic system with $\alpha' = 0$, where $\omega = \omega_{\mathrm{CY}}$, A is hermitian Yang-Mills, and θ is the Chern connection of ω. We deform the Aeppli class by

$$\underline{a}(\tilde{\omega},\tilde{A},\tilde{\theta}) = [\omega_{\mathrm{CY}}] + [\mathfrak{a}], \tag{4.34}$$

more precisely,

$$\tilde{\omega} = \omega + \mathfrak{a} + b + \alpha'\Big(R_2[\tilde{A},A] - R_2[\tilde{\theta},\theta] + \beta\Big), \tag{4.35}$$

where

$$(\mathfrak{a}+b) \in \Lambda^{1,1}(X,\mathbb{R}): \ \partial\bar{\partial}\mathfrak{a} = \partial\bar{\partial}b = 0, \ b \in (\mathrm{Im}(\partial) + \mathrm{Im}(\bar{\partial})) \cap \Lambda^{1,1}(X,\mathbb{R}). \tag{4.36}$$

Then this ansatz automatically solves the anomaly cancellation relation. This ansatz was first introduced in [37] concerning the positive metric on a Bott-Chern algebroid.

To solve the heterotic system, we will substitute this ansatz into the remaining hermitian Yang-Mills equation and the balanced equation.

Construction of R_2

We now define the Bott-Chern-Simons secondary characteristics $R_2[\tilde{A},A]$ aforementioned in our ansatz. First, by Chern-Weil theory, there exists γ such that,

$$\mathrm{Tr}\tilde{F}_{\tilde{A}} \wedge \tilde{F}_{\tilde{A}} - \mathrm{Tr}F_A \wedge F_A = d\theta. \tag{4.37}$$

Applying the $\partial\bar{\partial}$-lemma, we may write $d\theta = i\partial\bar{\partial}\chi$ for some χ. However, χ has undetermined dependence on the unknown connection $\tilde{A}$, therefore such construction is *ad hoc*. We are looking for a canonical construction, which is provided by the usage of the Kodaira-Spencer operator [49],

$$E = \partial\bar{\partial}\bar{\partial}^\dagger\partial^\dagger + \bar{\partial}^\dagger\partial^\dagger\partial\bar{\partial} + \bar{\partial}^\dagger\partial\partial^\dagger\bar{\partial} + \partial^\dagger\bar{\partial}\bar{\partial}^\dagger\partial + \bar{\partial}^\dagger\bar{\partial} + \partial^\dagger\partial, \tag{4.38}$$

where the adjoints † is with respect to g_{CY}. E is a 4-th order self-adjoint elliptic operator, and then by the elliptic theory, there exists a unique solution $\gamma \in \mathrm{Ker}(E)^\perp$ solving,

$$E(\gamma) = \mathrm{Tr}\tilde{F}_{\tilde{A}} \wedge \tilde{F}_{\tilde{A}} - \mathrm{Tr}F_A \wedge F_A, \tag{4.39}$$

providing that right-hand side is perpendicular to $\mathrm{Ker}(E)$.

To verify this, we first note that integration by parts shows that the kernel of E is given by,

$$\mathrm{Ker}(E) = \{\varphi \in \Lambda^{2,2}(X) : \ d\varphi = 0, \ \bar{\partial}^\dagger \partial^\dagger \varphi = 0\}. \tag{4.40}$$

Then, it is easy to see that indeed,

$$i\partial\bar{\partial}\chi \perp \mathrm{Ker}(E). \tag{4.41}$$

Hence, we may solve,

$$E(\gamma) = i\partial\bar{\partial}\chi, \tag{4.42}$$

which gives a solution to (4.39).

Then, by an integration by parts argument (see (4.3) in [35]), we can show that (4.42) implies,

$$d\gamma = 0. \tag{4.43}$$

Therefore,

$$\partial\bar{\partial}\bar{\partial}^\dagger\partial^\dagger\gamma = \mathrm{Tr}\tilde{F}_{\tilde{A}} \wedge \tilde{F}_{\tilde{A}} - \mathrm{Tr}F_A \wedge F_A, \tag{4.44}$$

and we let

$$R_2[\tilde{A}, A] = i\bar{\partial}^\dagger\partial^\dagger E^{-1}\Big(\mathrm{Tr}\tilde{F}_{\tilde{A}} \wedge \tilde{F}_{\tilde{A}} - \mathrm{Tr}F_A \wedge F_A\Big). \tag{4.45}$$

We take the real part if necessary so that $R_2[\tilde{A}, A]$ is a real $(1,1)$-form.

4.3 Gauge deformation $\tilde{A}$

We now focus on constructing a suitable deformation of the gauge connection $\tilde{A}$. Our partition of the heterotic system takes advantage of the aforementioned Donaldson-Uhlenbeck-Yau theorem, which allows us to describe the holomorphic vector bundle in terms of unitary connection $\tilde{A}$. This provides a complementary approach to previous work using the bundle hermitian metric h [19].

We start with a reference unitary connection A and its associated hermitian metric h on E. A admits a type decomposition in an orthonormal frame, $A = -\mathcal{A}^\dagger + \mathcal{A}$. The associated covariant derivative $D = d + A$ also decomposes into types $D = \mathcal{D} + \bar{\mathcal{D}}$. It is assumed that for the reference connection satisfies,

$$\bar{\mathcal{D}}^2 = 0. \tag{4.46}$$

This condition leads to the Dolbeault cohomology $H^{0,1}(X, \mathrm{End}E)$ on the holomorphic vector bundle E,

$$H^{0,1}(X, \mathrm{End}E) = \frac{\mathrm{Ker}(\bar{\mathcal{D}}) \ \cap \ \Lambda^{0,1}(\mathrm{End}E)}{\mathrm{Im}(\bar{\mathcal{D}}) \ \cap \ \Gamma(\mathrm{End}E)}. \tag{4.47}$$

By our ansatz, $\tilde{A}$ need to solve $\tilde{F}^{0,2}_{\tilde{A}} = 0$. We can then pick an equivalent class element $[\alpha_1]$ of the Dolbeault cohomology $H^{0,1}(X, \mathrm{End}E)$ as our parameter such that this ansatz is automatically satisfied. Therefore, our deformation for the gauge connection is as follows,

$$\mathcal{A} \mapsto \mathcal{A} + [\alpha_1], \quad [\alpha_1] \in H^{0,1}(X, \mathrm{End}E). \tag{4.48}$$

In other words, in an orthonormal frame, our deformed $\tilde{\mathcal{A}}$ reads,

$$\tilde{\mathcal{A}} = \mathcal{A} + \alpha_1 + \bar{\mathcal{D}}u_1, \quad u_1 \in \Gamma(\mathrm{End}E), \tag{4.49}$$

$$\tilde{A} = -\tilde{\mathcal{A}}^\dagger + \tilde{\mathcal{A}}. \tag{4.50}$$

We can then calculate the deformed curvature $\tilde{F}_{\tilde{A}}$ along this deformation path. Then the $(0,2)$-part of the deformed curvature $\tilde{F}$ indeed solves the ansatz automatically,

$$\tilde{F}^{0,2}_{\tilde{A}} = F^{0,2}_A + \bar{\mathcal{D}}\alpha_1 + \bar{\mathcal{D}}^2 u_1 = 0. \tag{4.51}$$

The calculation of the deformed $\tilde{F}_{\tilde{A}}$ is detailed at the end of this section, For now, we continue the discussion for the Aeppli case.

In the Aeppli case, we have introduced an addition spurious gauge field θ. Its deformation follows similarly as $\tilde{\mathcal{A}}$. Namely, from a reference connection $\theta \in T^{1,0}X$ and a reference metric g, we work in an orthonormal frame and let

$$\tilde{\theta}^{0,1} = \theta^{0,1} + \alpha_2 + \bar{\mathcal{D}}u_2, \quad [\alpha_2] \in H^{0,1}(\mathrm{End}T^{1,0}X), \quad u_2 \in \Gamma(\mathrm{End}T^{1,0}X), \tag{4.52}$$

$$\tilde{\theta}^{1,0} = -(\tilde{\theta}^{0,1})^\dagger, \quad \tilde{\theta} = \tilde{\theta}^{1,0} + \tilde{\theta}^{0,1}. \tag{4.53}$$

For the rest of the thesis, we take the reference metric h solving hermitian Yang-Mills equations on a holomorphic stable bundle $(E, h) \to (X, \omega_{\mathrm{CY}})$ and the associated reference gauge connection $A(h)$ is its Chern connection. On $T^{1,0}X$, the reference metric g is the Calabi-Yau metric and the reference connection θ is its Chern connection.

4.3.1 Deformed curvature $\tilde{F}_{\tilde{A}}$

Lemma 4.3.1 (Deformation of F_A)**.** *The deformed curvature $\tilde{F}$ of the deformed connection $\tilde{\mathcal{A}}_i$ taking value of either $\tilde{\mathcal{A}}$ for $i = 1$ or $\tilde{\theta}^{0,1}$ for $i = 2$, along the deformation path,*

$$\tilde{\mathcal{A}}_i = \mathcal{A}_i + \alpha_i + \bar{\mathcal{D}} u_i, \tag{4.54}$$

(4.48) *or* (4.52) *respectively is given by,*

$$\tilde{F}_{\tilde{A}_i} = F_{A_i} + D(-\alpha_i^\dagger + \alpha_i) + (\mathcal{D}\bar{\mathcal{D}} - \bar{\mathcal{D}}\mathcal{D})u_i. \tag{4.55}$$

Proof. For simplicity, we omit the subscript i of α and u in this proof, until the later discussion in the Aeppli case where distinguishing is necessary.

Along deformation path (4.54), by the conjugation relation (C.2), we have the $(1,0)$-type part as,

$$-\tilde{\mathcal{A}}^\dagger = -\mathcal{A}^\dagger - \alpha^\dagger - \mathcal{D}u. \tag{4.56}$$

The deformed curvature can then be calculated as below, for $\tilde{\mathcal{A}} = \mathcal{A} + \beta$, where $\beta = \alpha + \bar{\mathcal{D}}u \in \Lambda^{0,1}(\mathrm{End}E)$. The covariant derivative of β hence reads,

$$\mathcal{D}\beta = \partial\beta - \mathcal{A}^\dagger \wedge \beta + (-1)\beta \wedge \mathcal{A}^\dagger, \tag{4.57}$$

$$\bar{\mathcal{D}}\beta = \bar{\partial}\beta + \mathcal{A} \wedge \beta - (-1)\beta \wedge \mathcal{A}. \tag{4.58}$$

Then we have,

$$\begin{aligned}
\tilde{F}_{\tilde{A}} &= F_A + (\partial + \bar{\partial})(-\beta^\dagger + \beta) \\
&\qquad + (-\beta^\dagger + \beta) \wedge (-\mathcal{A}^\dagger + \mathcal{A}) + (-\mathcal{A}^\dagger + \mathcal{A}) \wedge (-\beta^\dagger + \beta) \\
&= F_A - \partial\beta^\dagger + \beta^\dagger \wedge \mathcal{A}^\dagger + \mathcal{A}^\dagger \wedge \beta^\dagger - \bar{\partial}\beta^\dagger - \beta^\dagger \wedge \mathcal{A} - \mathcal{A} \wedge \beta^\dagger \\
&\qquad + \partial\beta - \beta \wedge \mathcal{A}^\dagger - \mathcal{A}^\dagger \wedge \beta + \bar{\partial}\beta + \beta \wedge \mathcal{A} + \mathcal{A} \wedge \beta \\
&= F_A + (\mathcal{D} + \bar{\mathcal{D}})(-\beta^\dagger + \beta).
\end{aligned} \tag{4.59}$$

Hence, substituting $\beta = \alpha + \bar{\mathcal{D}}u$, we obtain,

$$\tilde{F}_{\tilde{A}} = F_{\mathcal{A}} + (\mathcal{D} + \bar{\mathcal{D}})(-\alpha^\dagger + \alpha) + (-\mathcal{D}\mathcal{D} + \mathcal{D}\bar{\mathcal{D}} - \bar{\mathcal{D}}\mathcal{D} + \bar{\mathcal{D}}\bar{\mathcal{D}})u. \tag{4.60}$$

We proceed to show the following three identities,

1. $\mathcal{D}\mathcal{D}u = [F_A^{2,0}, u] = 0$ which is a consequence of hermitian Yang-Mills equation (3.32),

$$\begin{aligned}
\mathcal{D}\mathcal{D}u &= \mathcal{D}\Big(\partial u - \mathcal{A}^\dagger \wedge u + u \wedge \mathcal{A}^\dagger\Big) \\
&= \partial\partial u + (-\partial\mathcal{A}^\dagger + \mathcal{A}^\dagger \wedge \mathcal{A}^\dagger) \wedge u + u \wedge (-\partial\mathcal{A}^\dagger + \mathcal{A}^\dagger \wedge \mathcal{A}^\dagger) \\
&= [F_A^{2,0}, u] = 0.
\end{aligned} \tag{4.61}$$

2. $\bar{\mathcal{D}}\bar{\mathcal{D}}u = [F_A^{0,2}, u] = 0$, which follows immediately by taking the conjugate of the above,

$$\bar{\mathcal{D}}\bar{\mathcal{D}}u = \overline{\mathcal{D}\mathcal{D}u} = \overline{[F_A^{2,0}, u]} = [F_A^{0,2}, u] = 0. \tag{4.62}$$

3. We can follow a similar calculation and use the following identity,

$$\bar{\partial}\partial u = \bar{\partial}_\mu \partial_\nu u dz^\nu \wedge d\bar{z}^\mu = -\partial_\nu \bar{\partial}_\mu u d\bar{z}^\mu \wedge dz^\nu, \tag{4.63}$$

to write,

$$(\mathcal{D}\bar{\mathcal{D}} - \bar{\mathcal{D}}\mathcal{D})u = (2\mathcal{D}\bar{\mathcal{D}}u - [F_A^{1,1}, u]), \tag{4.64}$$

where,

$$F_A^{1,1} = (\partial\mathcal{A} - \mathcal{A} \wedge \mathcal{A}^\dagger) - (\bar{\partial}\mathcal{A}^\dagger + \mathcal{A}^\dagger \wedge \mathcal{A}). \tag{4.65}$$

We will be using this identity in the calculation of linearization operator L_2 later in Section. 6.1.3.

Then, we read the deformed curvature $\tilde{F}_{\tilde{A}}$ as (4.55) since $\mathcal{D}\mathcal{D}$ and $\bar{\mathcal{D}}\bar{\mathcal{D}}$ terms vanish. □

It is easy to see that our deformed curvature remains to be anti-self-adjoint,

$$\tilde{F}_{\tilde{A}}^\dagger = -\tilde{F}_{\tilde{A}}, \tag{4.66}$$

which follows from the anti-self-adjoint identities below,

$$\Big(D(-\alpha^\dagger + \alpha)\Big)^\dagger = -D(-\alpha^\dagger + \alpha), \tag{4.67}$$

$$\big((\mathcal{D}\bar{\mathcal{D}} - \bar{\mathcal{D}}\mathcal{D})u\big)^\dagger = -(\mathcal{D}\bar{\mathcal{D}} - \bar{\mathcal{D}}\mathcal{D})u. \tag{4.68}$$

They are direct consequences of the conjugation identities (C.1) and (C.2).

Chapter 5

Setup for the Implicit Function Theorem

We have determined our deformation ansatzes, we now seek to solve the remaining equations of the heterotic system respectively. This chapter details the setup for the implicit function theorem on the heterotic moduli.

To do this, we separate coordinates as,

$$\mathrm{X} = (\alpha', [\text{cohomology class}]), \quad \mathrm{Y} = (\text{parameter within equivalence class}), \tag{5.1}$$

and construct the map $\mathcal{F}$ such that,

$$\mathcal{F}(\mathrm{X}, \mathrm{Y}) = [\text{remaining heterotic equations}], \tag{5.2}$$

and look for the zeros of the map $\mathcal{F}(\mathrm{X}, \mathrm{Y})$. Then if the zero locus exists, we will have constructed heterotic solutions, and X will be the desired coordinate of the parameter space.

Since we start the deformation from a reference metric ω which is Kähler, then,

$$\mathcal{F}(0, 0) = 0. \tag{5.3}$$

We then compute the linearization $D_{\mathrm{Y}}\mathcal{F}\big|_0$ at $\alpha' = 0$ and show it is an isomorphism. Then by implicit function theorem D.1.1, there exists $\epsilon > 0$, such that for any $\alpha' < \epsilon$, there exists solution Y(X) such that $\mathcal{F}(\mathrm{X}, \mathrm{Y}(\mathrm{X})) = 0$. In other words, for each α', there is a neighborhood of the starting Kähler solution parameterized by X, which produced solutions $(\tilde{\omega}, \tilde{A})$ to the heterotic system.

In the proceeding chapter, we will establish the validity of such application of the implicit function theorem on our two deformation paths corresponding to the Bott-Chern case and the Aeppli case, in other words, to show that the respective $D_{\mathrm{Y}}\mathcal{F}\big|_0$'s are indeed isomorphism.

We now proceed to formulate the explicit setups for both cases.

5.1 Bott-Chern case

The deformation ansatz reads by (4.22) and (4.48),

$$|\Omega|_{\tilde{\omega}}\tilde{\omega}^2 = |\Omega|_{\omega}\omega^2 + \mathfrak{b} + \Theta, \tag{5.4}$$

$$\tilde{\mathcal{A}} = \mathcal{A} + \alpha_1 + \bar{\mathcal{D}}u_1, \tag{5.5}$$

where $\mathfrak{b}$ and α_1 represent cohomology classes. To set this up as a Banach space, we use the Hodge theorem to identify cohomology with the space of harmonic representatives. Let,

$$\mathbb{H}^{p,p} = \{\Theta \in \Lambda^{p,p}(X,\mathbb{R}) : \ \Delta_{\omega_{\mathrm{CY}}}\Theta = 0\}, \tag{5.6}$$

$$\mathbb{H}^{0,1}_{\bar{\mathcal{D}}} = \{\alpha \in \Lambda^{0,1}(\mathrm{End}E) : \ \Delta_{\bar{\mathcal{D}}}\,\alpha = 0\}. \tag{5.7}$$

These are well-known to be finite-dimensional vector spaces.

Hence, the coordinates X and Y read,

$$\mathrm{X} = (\alpha', \mathfrak{b}, \alpha_1) \in \mathbb{R} \times \mathbb{H}^{2,2} \times \mathbb{H}^{0,1}_{\bar{\mathcal{D}}}, \tag{5.8}$$

$$\mathrm{Y} = (\Theta, u_1) \in \partial\bar{\partial}\Lambda^{1,1}(X,\mathbb{R}) \times \Gamma(\mathrm{End}_0 E), \tag{5.9}$$

where $\mathrm{End}_0 E$ consists of endomorphisms with travel zero. This is needed since shifts $u \mapsto u + C\mathrm{id}_E$ by constant multiples of the identity are not seen by the ansatz.

As our ansatz is automatically conformally balanced by construction, the map $\mathcal{F}_{\text{B-C}}$ includes the remaining equations in the heterotic system and reads,

$$\mathcal{F}_{\text{B-C}} = \begin{bmatrix} i\partial\bar{\partial}\tilde{\omega} - \alpha'(\mathrm{Tr}\tilde{F}_{\tilde{A}} \wedge \tilde{F}_{\tilde{A}} - \mathrm{Tr}\tilde{R}_{\tilde{\omega}} \wedge \tilde{R}_{\tilde{\omega}}) \\ |\Omega|_{\tilde{\omega}}\tilde{\omega}^2 \wedge i\tilde{F}_{\tilde{A}} \end{bmatrix}. \tag{5.10}$$

Solutions to the system are then cut out by the zero locus $\mathcal{F}_{\text{B-C}}(\mathrm{X},\mathrm{Y}) = 0$.

Our initial data is a Kähler Ricci-flat metric $\omega = \omega_{\mathrm{CY}}$ and Donaldson-Uhlenbeck-Yau connection $A = A_{\mathrm{D}UY}$, and so $\mathcal{F}_{\text{B-C}}(0,0) = 0$.

To apply the implicit function theorem on Banach spaces, we take $k \leq 2$, $0 < \gamma < 1$, and set the domain and codomain to be,

$$\mathcal{F}_{\text{B-C}} : \mathbb{R} \times \mathbb{H}^{2,2} \times \mathbb{H}^{0,1}_{\bar{\mathcal{D}}} \times C^{k+2,\gamma}(\partial\bar{\partial}\Lambda^{1,1}) \times C^{k+2,\gamma}(\mathrm{E}nd_0\,E) \longrightarrow C^{k,\gamma}(\partial\bar{\partial}\Lambda^{1,1}) \times C^{k,\gamma}(V(E)). \tag{5.11}$$

We have further constrained the image of $\mathcal{F}_{\text{B-C}}$ to the following subspace,

$$V(E) = \{v \in \Lambda^6(X, \mathrm{End}E) : \ \int_X \mathrm{Tr}v = 0, \ v^\dagger = v\}. \tag{5.12}$$

This will be needed when showing that the linearized operator is subjective later on. We also note that the spaces involved are all Banach spaces. For example, $C^{k,\gamma}(\partial\bar{\partial}\Lambda^{1,1})$ is a Banach space. To see this, it is well-known that $C^{k,\gamma}(d\Lambda^{1,1})$ is a Banach space (see [51]), and then applying $\partial\bar{\partial}$-lemma makes $C^{k,\gamma}(\partial\bar{\partial}\Lambda^{1,1})$ a closed subspace, hence Banach.

It is easy to check that the image of $\mathcal{F}_{\text{B-C}}$ is indeed contained in $\partial\bar{\partial}\Lambda^{1,1}\times V(E)$. The first row is contained in the image of $\partial\bar{\partial}$, since we assume $c_2(E) = c_2(X)$ and we may apply the $\partial\bar{\partial}$-lemma. For containment of the second row in $V(E)$, it suffices to show that the following two conditions hold,

1. Vanishing integration of trace:

$$\int_X \text{Tr}(|\Omega|_{\tilde{\omega}}\tilde{\omega}^2 \wedge i\tilde{F}_{\tilde{A}}) = 0.$$

   This uses that $c_1(E) = 0$ so that $\mathrm{T}r\,\tilde{F} = d\gamma$, and the ansatz is such that $|\Omega|_{\tilde{\omega}}\tilde{\omega}^2$ is closed.

2. Self-adjoint:

$$(|\Omega|_{\tilde{\omega}}\tilde{\omega}^2 \wedge i\tilde{F}_{\tilde{A}})^\dagger = |\Omega|_{\tilde{\omega}}\tilde{\omega}^2 \wedge i\tilde{F}_{\tilde{A}}.$$

   This follows from the fact that the deformed curvature $\tilde{F}_{\tilde{A}}$ remains anti-self-adjoint.

Now we remain to check the invertibility of $D_{\mathrm{Y}}\mathcal{F}_{\text{B-C}}$ at $(\mathrm{X}, \mathrm{XY}) = (0, 0)$,

$$D_{\mathrm{Y}}\mathcal{F}_{\text{B-C}}\Big|_{(0,0)} : C^{k+2,\gamma}(\partial\bar{\partial}\Lambda^{1,1}) \times C^{k+2,\gamma}(\mathrm{E}nd_0\, E) \longrightarrow C^{k,\gamma}(\partial\bar{\partial}\Lambda^{1,1}) \times C^{k,\gamma}(V(E))$$

$$D_{\mathrm{Y}}\mathcal{F}_{\text{B-C}}\Big|_{(0,0)}(\dot{\Theta}, \dot{u}_1) = \begin{bmatrix} L_1^{\text{B-C}} & 0 \\ C & L_2 \end{bmatrix} \begin{bmatrix} \dot{\Theta} \\ \dot{u}_1 \end{bmatrix}, \tag{5.13}$$

and prove that $D_{\mathrm{Y}}\mathcal{F}_{\text{B-C}}\big|_{(0,0)}$ is an isomorphism.

5.2 Aeppli case

Our deformation ansatz constructed in previous chapter reads by (4.35), (4.48) and (4.52),

$$\tilde{\omega} = \omega + \mathfrak{a} + b + \alpha'\Big(R_2[\tilde{A}, A] - R_2[\tilde{\theta}, \theta] + \beta\Big), \tag{5.14}$$

$$\tilde{\mathcal{A}} = \mathcal{A} + \alpha_1 + \bar{\mathcal{D}}u_1, \tag{5.15}$$
$$\tilde{\theta}^{0,1} = \theta^{0,1} + \alpha_2 + \bar{\mathcal{D}}u_2, \tag{5.16}$$

where $\mathfrak{a}$, α_1 and α_2 represent cohomology classes. The coordinates X and Y read,

$$\mathrm{X} = (\alpha', \mathfrak{a}, \alpha_1, \alpha_2) \in \mathbb{R} \times \mathbb{H}^{1,1} \times \mathbb{H}^{0,1}_{\bar{\mathcal{D}}}(\mathrm{End}E) \times \mathbb{H}^{0,1}_{\bar{\partial}}(\mathrm{End}T^{1,0}X), \tag{5.17}$$
$$\mathrm{Y} = (b, u_1, u_2) \in \big(\mathrm{Im}(d) \cap \Lambda^{1,1}(X, \mathbb{R})\big) \times \Gamma(\mathrm{E}nd_0 E) \times \Gamma(\mathrm{End}_0 T^{1,0}X). \tag{5.18}$$

We will be using de Rham cohomology to represent harmonic forms $\mathbb{H}^{1,1}$ rather than Aeppli cohomology $H^{1,1}_{\mathrm{A}}$. This is equivalent, since at $(\mathrm{X}, \mathrm{Y}) = (0, 0)$, the metric ω is Kähler. By a non-trivial result that for a Kähler manifold X, there are isomorphisms between Bott-Chern, Aeppli, and de Rham cohomology (see Lemma 5.15 in [24], and [3]).

$$H^{2,2}_{\text{B-C}}(X, \mathbb{R}) \cong H^{2,2}(X, \mathbb{R}) \cong H^{1,1}(X, \mathbb{R}) \cong H^{1,1}_{\mathrm{A}}(X, \mathbb{R}). \tag{5.19}$$

In any case, since $d\mathfrak{a} = 0$ and $b \in \mathrm{Im}(d)$, it is still the case that,

$$\underline{a}(\tilde{\omega}, \tilde{A}, \tilde{\theta}) = [\omega]_{\mathrm{A}} + [\mathfrak{a}]_{\mathrm{A}}, \tag{5.20}$$

which is the main property of this ansatz.

We take the map $\mathcal{F}_A$ to be the remaining equations in the heterotic system with slight modifications,

$$\mathcal{F}_{\mathrm{A}} = \begin{bmatrix} \star_\omega d\big(|\Omega|_{\tilde{\omega}} \tilde{\omega}^2\big) \\ |\Omega|_{\tilde{\omega}} \tilde{\omega}^2 \wedge i\tilde{F}_{\tilde{A}} - d_1 \tilde{\omega}^3 \otimes \mathrm{id} \\ |\Omega|_{\tilde{\omega}} \tilde{\omega}^2 \wedge i\tilde{R}_{\tilde{\theta}} - d_2 \tilde{\omega}^3 \otimes \mathrm{id} \end{bmatrix}, \tag{5.21}$$

where,

$$d_1 = \frac{1}{\mathrm{r}k\, E} \frac{\int_X |\Omega|_{\tilde{\omega}} \tilde{\omega}^2 \wedge \mathrm{T}r\, i\tilde{F}_{\tilde{A}}}{\int_X |\Omega|_{\tilde{\omega}} \tilde{\omega}^3} \tag{5.22}$$
$$d_2 = \frac{1}{3} \frac{\int_X |\Omega|_{\tilde{\omega}} \tilde{\omega}^2 \wedge \mathrm{T}r\, i\tilde{R}_{\tilde{\theta}}}{\int_X |\Omega|_{\tilde{\omega}} \tilde{\omega}^3}. \tag{5.23}$$

The domain and codomain are

$$\begin{aligned} \mathcal{F}_{\mathrm{A}} : \mathbb{R} \times \mathbb{H}^{1,1} \times \mathbb{H}^{0,1}_{\bar{\mathcal{D}}}(\mathrm{End}E) \times \mathbb{H}^{0,1}_{\bar{\partial}}(\mathrm{End}T^{1,0}X) \\ \times C^{k+2,\gamma}\big(\mathrm{Im}(d) \cap \Lambda^{1,1}(X, \mathbb{R})\big) \times C^{k+2,\gamma}(\mathrm{E}nd_0\, E) \times C^{k+2,\gamma}(\mathrm{E}nd_0\, T^{1,0}X) \end{aligned}$$

$$\longrightarrow C^{k+1,\gamma}(d^\dagger_\omega \Lambda^2) \times C^{k,\gamma}(V(E)) \times C^{k,\gamma}(V(T^{1,0}X)). \tag{5.24}$$

Here $d^\dagger_\omega = - \star_\omega d \star_\omega$ is with respect to the background Calabi-Yau metric $\omega = \omega_{\mathrm{CY}}$, and $C^{k,\gamma}(d^\dagger \Lambda^r)$ is well-known to be a Banach space (see [51]). The inclusion of d_1 and d_2 terms is to that the images of the last two rows of $\mathcal{F}$ are contained in $V(E)$ and $V(T^{1,0}(X))$ respectively as before in (5.12), meaning that the integral of the trace is equal to zero.

The set $F_{\mathrm{A}}(\mathrm{X}, \mathrm{Y}) = 0$ is our solution space. This is because if,

$$d|\Omega|_{\tilde{\omega}} \tilde{\omega}^2 = 0, \quad |\Omega|_{\tilde{\omega}} \tilde{\omega}^2 \wedge i \tilde{F}_{\tilde{A}} = d_1 \tilde{\omega}^3 \otimes \mathrm{id}, \tag{5.25}$$

and then upon taking the trace and integrating we see that,

$$\int_X |\Omega|_{\tilde{\omega}} \tilde{\omega}^2 \wedge i \mathrm{Tr} \tilde{F}_{\tilde{A}} = \int_X |\Omega|_{\tilde{\omega}} \tilde{\omega}^2 \wedge i \mathrm{Tr} F_A = 0, \tag{5.26}$$

since $c_1(E) = 0$ implies $\mathrm{Tr} F_A = d\gamma$. Therefore $d_1 = 0$ and similarly $d_2 = 0$ on shell. We will give justification on why $\mathcal{F}_{\mathrm{A}}$ is a differentiable map in 5.2.1.

Before that, we finish the discussion of the setup. With F_{A} defined, X is the coordinate on moduli space, and we will compute the linearization of $\mathcal{F}$ with respect to Y, denoted $D_{\mathrm{Y}} \mathcal{F}_{\mathrm{A}}$ at $\alpha' = 0$:

$$D_{\mathrm{Y}} \mathcal{F}_{\mathrm{A}} \Big|_{(0,0,0)} :$$

$$C^{k+2,\gamma}\big(\mathrm{Im}(d) \cap \Lambda^{1,1}(X, \mathbb{R})\big) \times C^{k+2,\gamma}(\mathrm{E}nd_0\, E) \times C^{k+2,\gamma}(\mathrm{E}nd_0\, T^{1,0} X)$$

$$\longrightarrow C^{k+1,\gamma}(d^\dagger_\omega \Lambda^2) \times C^{k,\gamma}(V(E)) \times C^{k,\gamma}(V(T^{1,0}X))$$

$$D_{\mathrm{Y}} \mathcal{F}_{\mathrm{A}} \Big|_{(0,0,0)} (\dot{b}, \dot{u}_1, \dot{u}_2) = \begin{bmatrix} L_1^{\mathrm{A}} & 0 & 0 \\ C_1 & L_2 & 0 \\ C_2 & 0 & L_3 \end{bmatrix} \begin{bmatrix} \dot{b} \\ \dot{u}_1 \\ \dot{u}_2 \end{bmatrix}, \tag{5.27}$$

We will prove the invertibility of this operator and obtain the local existence of heterotic solutions $\mathcal{F}_{\mathrm{A}}(\mathrm{X}, \mathrm{Y}(\mathrm{X})) = 0$ near

$$\mathcal{F}_{\mathrm{A}}(0,0) = 0 \tag{5.28}$$

by the implicit function theorem. The point $(0,0)$ is in the zero locus of $\mathcal{F}_{\mathrm{A}}$ since our initial data is a Kähler Ricci-flat metric $\omega = \omega_{\mathrm{CY}}$, a Donaldson-Uhlenbeck-Yau connection $A = A_{\mathrm{DUY}}$, and the Chern connection of the Kähler Ricci-flat metric $\theta = \Gamma^{\mathrm{C}}_{g_{\mathrm{CY}}}$.

5.2.1 Differentiability of $\mathcal{F}_{\mathbf{A}}$

We now comment on why $\mathcal{F}_{\mathrm{A}}$ is a differentiable map. This is a standard argument as everything is explicit, except for differences

$$R_2[\tilde{A}+h, A] - R_2[\tilde{A}, A].$$

First, we note that for connections $A, \tilde{A} \in C^{k+1,\gamma}$ and $h \in C^{k+1,\gamma}(\Lambda^1(\mathrm{E}nd\, E))$, then $R[\tilde{A}+h, A] \in C^{k+2,\gamma}$. This is from the expression (4.45) and elliptic regularity

$$E^{-1} : C^{k,\gamma} \to C^{k+4,\gamma}$$

for the 4th order elliptic operator E. Next, we set $L_{\tilde{A}}h = \frac{d}{dt}\big|_{t=0}(\mathrm{T}r\, F_{\tilde{A}+th} \wedge F_{\tilde{A}+th})$, and explain why $R_2[\tilde{A}, A]$ is differentiable in the unknown $\tilde{A}$, namely

$$\lim_{h\to 0} \frac{\|R_2[\tilde{A}+h, A] - R_2[\tilde{A}, A] - \bar{\partial}^\dagger\partial^\dagger E^{-1}(L_{\tilde{A}}h)\|_{C^{k+2,\gamma}}}{\|h\|_{C^{k+1,\gamma}}} = 0. \tag{5.29}$$

For this, we use (4.45) to obtain

$$\begin{aligned}&\|R_2[\tilde{A}+h, A] - R_2[\tilde{A}, A] - \bar{\partial}^\dagger\partial^\dagger E^{-1}(L_{\tilde{A}}h)\|_{C^{k+2,\gamma}}\\ &= \|\bar{\partial}^\dagger\partial^\dagger E^{-1}(\mathrm{T}r\, F_{\tilde{A}+h} \wedge F_{\tilde{A}+h} - \mathrm{T}r\, F_{\tilde{A}} \wedge F_{\tilde{A}} - L_{\tilde{A}}h)\|_{C^{k+2,\gamma}}.\end{aligned}$$

To obtain bounds on E^{-1}, we use the standard elliptic estimate

$$\|\gamma\|_{C^{4+k,\gamma}} \leq C\|E(\gamma)\|_{C^{k,\gamma}}, \quad \gamma \in (\ker E)^\perp.$$

Thus

$$\begin{aligned}&\|R_2[\tilde{A}+h, A] - R_2[\tilde{A}, A] - \bar{\partial}^\dagger\partial^\dagger E^{-1}(L_{\tilde{A}}h)\|_{C^{k+2,\gamma}}\\ &\leq C\|(\mathrm{T}r\, F_{\tilde{A}+h} \wedge F_{\tilde{A}+h} - \mathrm{T}r\, F_{\tilde{A}} \wedge F_{\tilde{A}} - L_{\tilde{A}}h)\|_{C^{k,\gamma}}\\ &\leq C\left\|\int_0^1 \frac{d}{dt}(\mathrm{T}r\, F_{\tilde{A}+th} \wedge F_{\tilde{A}+th})dt - \frac{d}{dt}\bigg|_{t=0}(\mathrm{T}r\, F_{\tilde{A}+th} \wedge F_{\tilde{A}+th})\right\|_{C^{k,\gamma}}.\end{aligned}$$

Since

$$\frac{d}{dt}\mathrm{T}r\,(F_t)^2 = 2\mathrm{T}r\, F_t d_{A_t}h, \quad A_t = \tilde{A}+th,$$

we have

$$\begin{aligned}&\|R_2[\tilde{A}+h, A] - R_2[\tilde{A}, A] - \bar{\partial}^\dagger\partial^\dagger E^{-1}(L_{\tilde{A}}h)\|_{C^{k+2,\gamma}}\\ &\leq C\left\|\int_0^1 (\mathrm{T}rF_t d_{A_t} - \mathrm{T}rF_{\tilde{A}}d_{\tilde{A}})h\, dt\right\|_{C^{k,\gamma}}\end{aligned}$$

$$= C\left\| \int_0^1 \int_0^1 \frac{d}{ds}(\mathrm{Tr}\, F_s d_{A_s}) h ds\, dt \right\|_{C^{k,\gamma}}$$

where $A_s = \tilde{A} + tsh$. It follows from here that

$$\|R_2[\tilde{A} + h, A] - R_2[\tilde{A}, A] - \bar{\partial}^\dagger \partial^\dagger E^{-1}(L_{\tilde{A}} h)\|_{C^{k+2,\gamma}} \leq C\|h\|^2_{C^{k+1,\gamma}},$$

where the constant C depends on $\|\tilde{A}\|_{C^{k+1,\gamma}}$. The limit (5.29) follows, and so does differentiability of $\tilde{\omega}(\mathfrak{a}, b, \tilde{A}, \tilde{\theta})$ in the argument $\tilde{A}$ (and similarly for $\tilde{\theta}$).

Chapter 6

Calculations of $D\mathcal{F}$

We now calculate $D\mathcal{F}$ and proceed to show the invertibility of $\mathcal{D}_{\mathrm{Y}}\mathcal{F}$ for both cases and establish the validity of the application of the implicit function theorem to conclude the local existence of the heterotic solution near a Kähler solution.

6.1 Calculation of $D_{\mathbf{Y}}\mathcal{F}_{\textbf{B-C}}$

Consider a deformation path $(\Theta(s), u_1(t))$ along $\mathrm{X} = 0$ with $(\Theta(0), u_1(0)) = (0, 0)$ since the starting point is a Kähler solution. Then our deformation path on moduli by (4.22) and (4.48) now reads

$$|\Omega|_{\tilde{\omega}(s)}\tilde{\omega}^2(s) = |\Omega|_\omega \omega^2 + \Theta(s)$$
$$\tilde{\mathcal{A}}(t) = \mathcal{A} + \bar{\mathcal{D}}u_1(t),$$

and we will compute the partial derivatives $\partial_s|_0$, $\partial_t|_0$ of $\mathcal{F}_{\text{B-C}}$ along this path.

6.1.1 Invertibility of $D_{\mathbf{Y}}\mathcal{F}_{\textbf{B-C}}\big|_{(0,0)}$ at $\alpha' = 0$

For the implicit function theorem to hold, we need $D_{\mathrm{Y}}\mathcal{F}_{\text{B-C}}$ to be an isomorphism at $(\mathrm{X}, \mathrm{Y}) = (0, 0)$, hence $D_{\mathrm{Y}}\mathcal{F}_{\text{B-C}}$ should have an inverse $(D_{\mathrm{Y}}\mathcal{F}_{\text{B-C}})^{-1}$. From (5.13), it must be

$$(D_{\mathrm{Y}}\mathcal{F}_{\text{B-C}})^{-1} = \begin{bmatrix} (L_1^{\text{B-C}})^{-1} & 0 \\ -(L_1^{\text{B-C}})^{-1} C L_2^{-1} & L_2^{-1} \end{bmatrix}. \tag{6.1}$$

Therefore, we only need to establish the invertibility of $L_1^{\text{B-C}}$ and L_2 respectively.

6.1.2 $L_1^{\textbf{B-C}}$

To calculate $L_1^{\text{B-C}}$, we need to understand the variation of the hermitian metric $\tilde{\omega}$ along the Bott-Chern deformation path (4.22). This is provided by the following lemma.

Notation: In this subsection, we write $\delta_{s=0}$ as δ and $\delta\Theta = \dot{\Theta}$ for simplicity.

Lemma 6.1.1 (Variation of $\tilde{\omega}$ under Bott-Chern deformation)**.** *On the Bott-Chern deformation path* (4.22) *given by* $|\Omega|_{\tilde{\omega}}\tilde{\omega}^2 = |\Omega|_\omega \omega^2 + \Theta$*, then*

$$\delta\tilde{\omega} = \frac{1}{2|\Omega|_\omega}\Lambda_\omega \dot{\Theta}. \tag{6.2}$$

This formula is known and can be found in [19] and its calculation can be completed by various methods. We provide here a proof which uses the Lefschetz operator commutator relations.

Proof. First, we find the variation of $|\Omega|_{\tilde{\omega}}$ at $\alpha' = 0$, using the local expression (3.4),

$$\begin{aligned}
\delta|\Omega|^2_{\tilde{\omega}} &= \Omega\bar{\Omega}\delta(\det \tilde{g}_{\alpha\bar{\beta}})^{-1} \\
2|\Omega|_\omega \delta|\Omega|_{\tilde{\omega}} &= -\Omega\bar{\Omega}(\det g_{\alpha\bar{\beta}})^{-2}\delta(\det \tilde{g}_{\alpha\bar{\beta}}) \\
&= -\Omega\bar{\Omega}(\det g_{\alpha\bar{\beta}})^{-1} g^{\alpha\bar{\beta}}\delta\tilde{g}_{\alpha\bar{\beta}} \\
&= -|\Omega|^2_\omega g^{\alpha\bar{\beta}}\delta\tilde{g}_{\alpha\bar{\beta}} \\
&= -|\Omega|^2_\omega(\Lambda_\omega \delta\tilde{\omega}),
\end{aligned}$$

hence, rearranging gives,

$$\delta|\Omega|_{\tilde{\omega}} = -\frac{1}{2}|\Omega|_\omega(\Lambda_\omega\delta\tilde{\omega}). \tag{6.3}$$

Then the variation of $(|\Omega|_{\tilde{\omega}}\tilde{\omega}^2)$ follows,

$$\begin{aligned}
\delta(|\Omega|_{\tilde{\omega}}\tilde{\omega}^2) &= \delta(|\Omega|_{\tilde{\omega}})\omega^2 + |\Omega|_\omega\delta(\tilde{\omega}^2) \\
\delta\Theta &= |\Omega|_\omega\left(-\frac{1}{2}(\Lambda_\omega\delta\tilde{\omega})\omega^2 + 2\omega\wedge\delta\tilde{\omega}\right) \\
\dot{\Theta} &= |\Omega|_\omega\left(-\frac{1}{2}(\Lambda_\omega\delta\tilde{\omega})L_\omega\omega + 2L_\omega\delta\tilde{\omega}\right) \qquad (6.4)
\end{aligned}$$

where we have rewrite ω^2 and $\omega\wedge\delta\tilde{\omega}$ using Lefschetz operator L_ω via (D.23). Contracting both sides with ω,

$$\Lambda_\omega\dot{\Theta} = |\Omega|_\omega\left(-\frac{1}{2}(\Lambda_\omega\delta\tilde{\omega})\Lambda_\omega L_\omega\omega + 2\Lambda_\omega L_\omega\delta\tilde{\omega}\right) \tag{6.5}$$

Now using the Lefschetz identities (D.30), we write,

$$\Lambda_\omega \dot{\Theta} = |\Omega|_\omega \left(-\frac{1}{2}(\Lambda_\omega \delta\tilde{\omega})(L_\omega \Lambda_\omega - H)\omega + 2(L_\omega\Lambda_\omega - H)\delta\tilde{\omega} \right) \tag{6.6}$$

now invoke our linear algebra results: (D.38) and (D.27), we write,

$$\Lambda_\omega \dot{\Theta} = |\Omega|_\omega \left(\frac{1}{2}(-2n + 2 + 4)(\Lambda_\omega \delta\tilde{\omega})\omega - 2(2-n)\delta\tilde{\omega} \right) = 2|\Omega|_\omega \delta\tilde{\omega} \tag{6.7}$$

taking $n = 3$ and rearranging, we arrive at the desired variation of $\tilde{\omega}$ as in (6.2). □

Then $L_1^{\text{B-C}}$ follows from (6.2) immediately,

$$L_1^{\text{B-C}}\dot{\Theta} = i\partial\bar{\partial}\delta\tilde{\omega} = \frac{i}{2|\Omega|_\omega}\partial\bar{\partial}(\Lambda_\omega \dot{\Theta}) = -\frac{1}{2|\Omega|_\omega}\partial\partial_\omega^\dagger \dot{\Theta} = -\frac{1}{2|\Omega|_\omega}\Delta_\omega \dot{\Theta}, \tag{6.8}$$

where we have used the Kähler identity (D.73) in Proposition. D.5.1 in the third equation, and can also replace $\partial\partial_\omega^\dagger$ with the Laplacian $\Delta_\omega \equiv \partial\partial_\omega^\dagger + \partial_\omega^\dagger\partial$ as both Θ and $\dot{\Theta}$ are $\partial\bar{\partial}$-exact in our ansatz and $\partial^2 = 0$.

We now proceed to establish the invertibility of $L_1^{\text{B-C}}$, which follows from the Hodge theory and $\partial\bar{\partial}$-lemma. By the Fredholm alternative, it suffices to consider the kernel of Δ_ω restricted to exact forms, which is $\{0\}$ as desired. We detail it in the following lemma.

Lemma 6.1.2 (Invertibility of $L_1^{\text{B-C}}$)**.** *The Bott-Chern linearization operator $L_1^{B\text{-}C}$ defined as below,*

$$\begin{aligned} &L_1^{B\text{-}C} : C^{k+2,\gamma}(\partial\bar{\partial}\Lambda^{1,1}(X,\mathbb{R})) \to C^{k,\gamma}(\partial\bar{\partial}\Lambda^{1,1}(X,\mathbb{R})), \\ &L_1^{B\text{-}C} = -\frac{1}{2|\Omega|_\omega}\Delta_\omega, \end{aligned} \tag{6.9}$$

is invertible.

Proof. For an element $\theta = i\partial\bar{\partial}\beta \in \text{Ker}(\Delta_\omega)$, we can deduce from,

$$(\Delta_\omega \theta, \theta) = 0, \tag{6.10}$$

that

$$(\partial_\omega^\dagger \theta, \partial_\omega^\dagger \theta) + (\partial\theta, \partial\theta) = 0, \tag{6.11}$$

by integration by part. Hence,

$$\partial_\omega^\dagger \theta = \partial\theta = 0. \tag{6.12}$$

Then,

$$(\bar{\partial}\beta, \partial_\omega^\dagger\theta) = (\theta, \theta) = 0, \tag{6.13}$$

hence,

$$\mathrm{Ker}(L_1^{\text{B-C}}) = \{0\}. \tag{6.14}$$

Since now, by Fredholm alternative and that Δ_ω is self-adjoint,

$$\mathrm{Im}(L_1^{\text{B-C}}) = \Big(\mathrm{Ker}\Big(\big(Ł_1^{\text{B-C}}\big)^\dagger\Big)\Big)^\perp = (\{0\})^\perp, \tag{6.15}$$

is the full space. Then the elliptic regularity provides an inverse,

$$\big(L_1^{\text{B-C}}\big)^{-1} : C^{k,\gamma}(\partial\bar{\partial}\Lambda^{1,1}(X,\mathbb{R})) \to C^{k+2,\gamma}(\partial\bar{\partial}\Lambda^{1,1}(X,\mathbb{R})). \tag{6.16}$$

Hence $L_1^{\text{B-C}}$ is invertible. □

6.1.3 L_2

To show the invertibility of L_2, we need to understand the deformed curvature $\tilde{F}_{\tilde{A}}$ on the deformation path (4.48), which is calculated by (4.55) in the Lemma 4.3.1.

We parameterize $u(t)$ and write $\delta_{t=0}$ as δ and $\delta u = \dot{u}$ for simplicity. Then the variation of $\tilde{F}_{\tilde{A}}$ along the deformation path (4.48) reads,

$$\delta\tilde{F}_{\tilde{A}} = (\mathcal{D}\bar{\mathcal{D}} - \bar{\mathcal{D}}\mathcal{D})\dot{u}. \tag{6.17}$$

The linearization of the Hermitian-Yang-Mills equation then reads,

$$\begin{aligned}
L_2\dot{u} &= \delta\Big(|\Omega|_\omega\omega^2 \wedge i\tilde{F}_{\tilde{A}}\Big) \\
&= \delta\Big(i\Lambda_\omega\tilde{F}_{\tilde{A}} \otimes |\Omega|_\omega\frac{\omega^3}{3!}\Big) \\
&= i\Lambda_\omega\delta\tilde{F}_{\tilde{A}} \otimes |\Omega|_\omega\frac{\omega^3}{3!} \\
&= i\Lambda_\omega\big(\mathcal{D}\bar{\mathcal{D}} - \bar{D}\mathcal{D}\big)\dot{u} \otimes |\Omega|_\omega\frac{\omega^3}{3!}, \\
&= 2g^{\bar{\nu}\mu}\mathcal{D}_\mu\bar{\mathcal{D}}_\nu\dot{u} \otimes |\Omega|_\omega\frac{\omega^3}{3!}.
\end{aligned} \tag{6.18}$$

We have used the identity (4.64) and the initial data F_A satisfies Hermitian-Yang-Mills $i\Lambda_\omega F_A = 0$.

Now, we collect the invertibility of L_2 in the following lemma.

Lemma 6.1.3 (Invertibility of L_2)**.** *The Hermitian-Yang-Mills linearization operator L_2 defined as below,*

$$\begin{aligned} &L_2 : C^{k+2,\gamma}(End_0 E) \to C^{k,\gamma}(V(E)), \\ &L_2 = 2g^{\bar{\nu}\mu}\mathcal{D}_\mu\bar{\mathcal{D}}_\nu(\cdot) \otimes |\Omega|_\omega \frac{\omega^3}{3!}, \end{aligned} \tag{6.19}$$

is invertible.

Before we provide the proof, we need to investigate the kernel of the operator $\Lambda_\omega(\mathcal{D}\bar{\mathcal{D}} - \bar{\mathcal{D}}\mathcal{D})$, and collect the result in the following lemma.

Lemma 6.1.4 (Kernel of $\Lambda_\omega(\mathcal{D}\bar{\mathcal{D}}-\bar{\mathcal{D}}\mathcal{D})$)**.** *Let $E \to X$ be a holomorphic vector bundle, and let (ω, Ω) be a conformally balanced structure, $d(|\Omega|_\omega\omega^2) = 0$. Then if $\gamma \in \ker(\Lambda_\omega(\mathcal{D}\bar{\mathcal{D}} - \bar{\mathcal{D}}\mathcal{D}))$, then γ is holomorphic.*

Proof. Recall the inner product of the sections of End$E$ is defined as,

$$\langle\gamma_1, \gamma_2\rangle = \int_X \mathrm{Tr}(\gamma_1\gamma_2^\dagger)|\Omega|_\omega \frac{\omega^3}{3!}, \quad \forall \gamma_1, \gamma_2 \in \Gamma(\mathrm{End}E). \tag{6.20}$$

Take $\gamma \in \Gamma(\mathrm{End}_0 E)$ and with $\gamma \in \mathrm{Ker}\big(\Lambda_\omega(\mathcal{D}\bar{\mathcal{D}} - \bar{\mathcal{D}}\mathcal{D})\big)$, using the identity (4.64), and the Hermitian-Yang-Mills equation $i\Lambda_\omega F_A = 0$, we have

$$\begin{aligned} 0 &= \langle i\Lambda_\omega(\mathcal{D}\bar{\mathcal{D}} - \bar{\mathcal{D}}\mathcal{D})\gamma, \gamma\rangle \\ &= \int_X \mathrm{Tr}(i\Lambda_\omega(2\mathcal{D}_\mu\mathcal{D}_{\bar{\nu}}\gamma - [F_{\mu\bar{\nu}}, \gamma])dx^{\mu\bar{\nu}}) \cdot \gamma^\dagger|\Omega|_\omega \frac{\omega^3}{3!} \\ &= 2\int_X i\Lambda_\omega \mathrm{Tr}(\mathcal{D}_\mu\bar{\mathcal{D}}_{\bar{\nu}}\gamma \cdot \gamma^\dagger)dx^{\mu\bar{\nu}}|\Omega|_\omega \frac{\omega^3}{3!} =: 2I_1. \end{aligned}$$

For the evaluation of I_1, integration by part yields

$$\begin{aligned} I_1 = &\int_X i\Lambda_\omega \mathrm{Tr}(\mathcal{D}_\mu\bar{\mathcal{D}}_\nu(\gamma\gamma^\dagger))dx^{\mu\bar{\nu}}|\Omega|_\omega \frac{\omega^3}{3!} \\ &- \int_X i\Lambda_\omega \mathrm{Tr}(\bar{\mathcal{D}}_{\bar{\nu}}\gamma \cdot \mathcal{D}_\mu\gamma^\dagger)dx^{\mu\bar{\nu}}|\Omega|_\omega \frac{\omega^3}{3!} \\ &- \int_X i\Lambda_\omega \mathrm{Tr}(\mathcal{D}_\mu\gamma \cdot \bar{\mathcal{D}}_{\bar{\nu}}\gamma^\dagger)dx^{\mu\bar{\nu}}|\Omega|_\omega \frac{\omega^3}{3!} \\ &- \int_X i\Lambda_\omega \mathrm{Tr}(\gamma \cdot \mathcal{D}_\mu\bar{\mathcal{D}}_\nu(\gamma^\dagger))dx^{\mu\bar{\nu}}|\Omega|_\omega \frac{\omega^3}{3!}. \end{aligned}$$

Then since γ is self-adjoint, the above yields

$$\langle\bar{\mathcal{D}}\gamma,\bar{\mathcal{D}}\gamma\rangle = -I_1 + \frac{1}{2}\int_X i\Lambda_\omega \mathrm{Tr}(\mathcal{D}_\mu\bar{\mathcal{D}}_\nu(\gamma\gamma^\dagger))dx^{\mu\bar{\nu}}|\Omega|_\omega\frac{\omega^3}{3!}.$$

Setting $I_1 = 0$ and commuting $\mathcal{D}$ with the trace, we obtain

$$\begin{aligned}\langle\bar{\mathcal{D}}\gamma,\bar{\mathcal{D}}\gamma\rangle &= \frac{1}{2}\int_X i\Lambda_\omega\partial\bar{\partial}\mathrm{Tr}(\gamma\gamma^\dagger)|\Omega|_\omega\frac{\omega^3}{3!}\\ &= \frac{3}{2}\int_X i\partial\bar{\partial}\mathrm{Tr}(\gamma\gamma^\dagger)\wedge|\Omega|_\omega\frac{\omega^2}{3!} =: \frac{3}{2\cdot 3!}I_2,\end{aligned}$$

where the last equality holds as for any function f, we have the identity

$$(\Lambda_\omega i\partial\bar{\partial}f)\omega^n = ni\partial\bar{\partial}f\wedge\omega^{n-1}.$$

Integrating by parts again, we have

$$I_2 = \int_X id(\bar{\partial}\mathrm{Tr}(\gamma\gamma^\dagger)\wedge|\Omega|_\omega\omega^2) + \int_X i\bar{\partial}\mathrm{Tr}(\gamma\gamma^\dagger)\wedge d(|\Omega|_\omega\omega^2) = 0,$$

where the first term vanishes by Stoke's theorem and compactness of base manifold X, and the other vanishes due to the balanced condition $d(|\Omega|_\omega\omega^2) = 0$. Therefore, $\bar{\mathcal{D}}\gamma = 0$, or in other words, γ is holomorphic. □

We now give the proof for Lemma. 6.1.3.

Proof for Lemma. 6.1.3. Take $\gamma\in\Gamma(\mathrm{End}_0E)$ and with $\gamma\in\mathrm{Ker}\big(\Lambda_\omega(\mathcal{D}\bar{\mathcal{D}}-\bar{\mathcal{D}}\mathcal{D})\big)$, integrate by parts and using the balanced condition, we can deduce from

$$\langle\Lambda_\omega(\mathcal{D}\bar{\mathcal{D}}-\bar{\mathcal{D}}\mathcal{D})\gamma\gamma\rangle = 0, \tag{6.21}$$

by Lemma. 6.1.4 that γ is holomorphic.

$E\to X$ is assumed to be stable. The stability implies E is simple, hence any holomorphic endomorphism section γ of End$E$ is proportional to the identity,

$$\gamma = \lambda\cdot\mathrm{id}_E, \tag{6.22}$$

for some constant λ. Together with the constraint of vanishing trace as $\gamma\in\Gamma(\mathrm{End}_0E)$, λ must be zero, and hence our kernel of L_2 must be zero and L_2 is injective as expected.

Now we continue to show that L_2 is subjective. We use again the Fredholm alternative,

$$\mathrm{Im}(L_2) = \Big(\mathrm{Ker}(L_2^\dagger)\Big)^\perp, \tag{6.23}$$

where the adjoint $\dagger$ is with respect to the L^2-inner product (6.20). If $\gamma \in \ker(L_2^\dagger)$, then (6.21) holds. By Lemma. 6.1.4, γ is holomorphic section of E. It then follows above that $\gamma = \lambda \cdot \mathrm{id}_E$.

Since $\gamma \in V(E)$, then traceless condition yields $\lambda = 0$ and hence $\gamma = 0$. Therefore,

$$\ker L_2^2 = \{0\}. \tag{6.24}$$

and L_2 is subjective indeed.

Elliptic regularity then gives the existence of inverse L_2^{-1},

$$L_2^{-1} : C^{k,\gamma}(V(E)) \to C^{k+2,\gamma}(\mathrm{End}_0 E). \tag{6.25}$$

Hence, L_2 is invertible. □

We are now in a position to conclude the invertibility of $D_{\mathrm{Y}}\mathcal{F}_{\text{B-C}}$, and hence it is indeed an isomorphism. Then the application of the implicit function theorem for Banach spaces concludes Theorem 1.

6.2 Calculation of $D_{\mathbf{Y}}\mathcal{F}_{\mathbf{A}}$

Consider a deformation path $(\Theta(s), u_1(t_1), u_2(t_2)) \in \mathrm{Y}$ along $\mathrm{X} = 0$ with starting point the Kähler solution $(\Theta(0), u_1(0), u_2(0)) = (0, 0, 0)$. Then, Then our deformation path on moduli by (4.35), (4.48) and (4.52) now reads,

$$\tilde{\omega}(s) = \omega + b(s) + \alpha'\Big(R_2[\tilde{A}, A] - R_2[\tilde{\theta}, \theta] + \beta\Big), \tag{6.26}$$

$$\tilde{\mathcal{A}}(t_1) = \mathcal{A} + \bar{\mathcal{D}}u_1(t_1) \tag{6.27}$$

$$\tilde{\theta}^{0,1}(t_2) = \theta^{0,1} + \bar{\mathcal{D}}u_2(t_2), \tag{6.28}$$

and we will compute the partial derivatives $\partial_s\big|_0$, $\partial_{t_1}\big|_0$ and $\partial_{t_2}\big|_0$ of $\mathcal{F}_{\mathrm{A}}$ along this path.

6.2.1 Invertibility of $D_{\mathbf{Y}}\mathcal{F}_{\mathbf{A}}\big|_{(0,0,0)}$ at $\alpha' = 0$

Similarly to the Bott-Chern case, we need $D_{\mathrm{Y}}\mathcal{F}_{\mathrm{A}}$ to be an isomorphism at $(\mathrm{X}, \mathrm{Y}) = (0, 0)$ and show $D_{\mathrm{Y}}\mathcal{F}$ has an inverse $(D_{\mathrm{Y}}\mathcal{F}_{\mathrm{A}})^{-1}$. From (5.27), we have,

$$(D_{\mathrm{Y}}\mathcal{F}_{\mathrm{A}})^{-1} = \begin{bmatrix} (L_1^{\mathrm{A}})^{-1} & 0 & 0 \\ -(L_1^{\mathrm{A}})^{-1}C_1 L_2^{-1} & L_2^{-1} & 0 \\ -(L_1^{\mathrm{A}})^{-1}C_2 L_3^{-1} & 0 & L_3^{-1} \end{bmatrix}.$$

Therefore, we only need to establish the invertibility of L_1^{A}, L_2, and L_3 respectively. Again, by matching the domain and range of L_1^{A}, L_2, and L_3 respectively, we can apply the Fredholm alternative and use that $\mathrm{Im}(L_i) = \mathrm{Ker}(L_i^\dagger)^\perp$, $i = 1, 2, 3$ respectively.

6.2.2 $L_1^{\mathbf{A}}$

To calculate L_1^{A}, we need to understand the deformation of the metric on the Aeppli deformation path (4.35). In particular, we can write $\tilde{\omega}$ in the local expression,

$$\tilde{g}_{\alpha\bar{\beta}} = g_{\alpha\bar{\beta}} - i\mathfrak{a}_{\alpha\bar{\beta}} - ib_{\alpha\bar{\beta}} - i\mathcal{O}(\alpha'), \tag{6.29}$$

where we denote $\mathfrak{a} = \mathfrak{a}_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta$ and $b = b_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta$.

We parameterize $b(s)$ and write $\delta_{s=0}$ as δ and $\delta\tilde{\omega} = \delta b = \dot{b}$ for simplicity. Hence, locally $\delta g_{\alpha\bar{\beta}} = -i\dot{b}_{\alpha\bar{\beta}}$.

By previous calculation (6.4), we have at $s = 0$,

$$\begin{aligned} \delta\big(\star_\omega d\big(|\Omega|_{\tilde{\omega}}\tilde{\omega}^2\big)\big) &= \star_\omega d\big(\delta\big(|\Omega|_{\tilde{\omega}}\tilde{\omega}^2\big)\big) \\ &= \star_\omega d\big(|\Omega|_\omega\big(-\tfrac{1}{2}(\Lambda_\omega\delta\tilde{\omega})\omega^2 + 2\omega\wedge\delta\tilde{\omega}\big)\big). \end{aligned} \tag{6.30}$$

We now invoke the following lemma about the reflection relation.

Lemma 6.2.1 (The reflection relation)**.** *For any* 2*-form* $\gamma \in \Lambda^2(X, \mathbb{C})$*, then,*

$$\frac{1}{2}(\Lambda_\omega\gamma)\omega^2 = \star_\omega\gamma + \omega\wedge\gamma. \tag{6.31}$$

This relation is a consequence of a general behavior of the interplay of the Lefschetz operator L_ω and Hodge $\star_\omega$-operator on a hermitian manifold.

Proof. This relation follows from the Lefschetz decomposition (see Proposition. D.2.3 that any 2-form γ can be decomposed by (D.33),

$$\gamma = \alpha + \frac{1}{3}(\Lambda_\omega\gamma)\omega,$$

where α is a primitive 2-form, meaning that $\Lambda_\omega\alpha = 0$. Then use the "mysterious but extremely useful" Proposition. D.2.4. In particular, we have for a primitive $(1, 1)$-form α, by (D.37)

$$\star_\omega\alpha = -\frac{1}{(n-2)!}\omega^{n-2}\wedge\alpha = -\omega\wedge\alpha,$$

and for the hermitian $(1,1)$-form ω, by (D.36)

$$\star_\omega \omega = \frac{\omega^{n-1}}{(n-1)!} = \frac{1}{2}\omega^2$$

Hence, we have,

$$\begin{aligned}\star_\omega \gamma &= \star_\omega \alpha + \frac{1}{3}(\Lambda_\omega \gamma) \star_\omega \omega \\ &= -\omega \wedge \alpha + \frac{1}{3}(\Lambda_\omega \gamma)\omega^2 \\ &= -\omega \wedge \left(\gamma - \frac{1}{3}(\Lambda_\omega \gamma)\omega\right) + \frac{1}{3!}(\Lambda_\omega \gamma)\omega^2 \\ &= -\omega \wedge \gamma + \frac{1}{2}(\Lambda_\omega \gamma)\omega^2.\end{aligned}$$

By rearranging, we conclude the desired reflection relation. □

We can take $\gamma = \delta\tilde{\omega} = \dot{b}$, and we have,

$$\begin{aligned}\delta\big(\star_\omega d\big(|\Omega|_{\tilde{\omega}}\tilde{\omega}^2\big)\big) &= \star_\omega d\Big(-|\Omega|_\omega(\star_\omega \dot{b} + \omega \wedge \dot{b}) + 2|\Omega|_\omega \omega \wedge \dot{b}\Big) \\ &= -\star_\omega d \star_\omega \left(|\Omega|_\omega \dot{b}\right) + \star_\omega d(|\Omega|_\omega \omega \wedge \dot{b}) \\ &= d^\dagger_\omega\left(|\Omega|_\omega \dot{b}\right) + \star_\omega(|\Omega|_\omega \omega \wedge d\dot{b}) \\ &= |\Omega|_\omega (d + d^\dagger_\omega)\dot{b},\end{aligned}$$

where in the second last equation, we note that $b(s), \dot{b} \in \mathrm{Im}(d) \cap \Lambda^{1,1}(X,\mathbb{R})$, we have $db = d\dot{b} = 0$, and the second term drops, and that $|\Omega|_\omega$ is a constant at the reference Calabi-Yau metric $\omega = \omega_{\mathrm{CY}}$. We hence obtain $L_1^{\mathrm{A}} = |\Omega|_\omega(d + d^\dagger)$.

We collect the invertibility of L_1^{A} in the following lemma mirroring $L_1^{\text{B-C}}$.

Lemma 6.2.2 (Invertibility of L_1^{A})**.** *The Aeppli linearization operator L_1^A defined as below,*

$$\begin{aligned}&L_1^A : C^{k+2,\gamma}(Im(d) \cap \Lambda^{1,1}(X,\mathbb{R})) \to C^{k,\gamma}(d^\dagger \Lambda^2(X,\mathbb{C})) \\ &L_1^A = |\Omega|_\omega (d + d^\dagger_\omega),\end{aligned} \tag{6.32}$$

is invertible.

Proof. The operator $d + d^\dagger_\omega$ is self-adjoint and elliptic. If $\gamma \in \mathrm{Ker}(L_1^{\mathrm{A}})$, then $\gamma = d\beta$ and consider,

$$0 = (L_1^{\mathrm{A}}\gamma, \beta) = |\Omega|_\omega(d^\dagger\gamma, \beta) = |\Omega|_\omega(\gamma, d\beta) = |\Omega|_\omega|\gamma|^2,$$

hence $\mathrm{Ker}(L_1^{\mathrm{A}}) = \{0\}$ and L_1^{A} is injective.

If $\gamma \in \ker(L_1^{\mathrm{A}})^\dagger$, then $\gamma = d^\dagger\beta$ and

$$0 = (L_1^{\mathrm{A}}\gamma, \beta) = |\Omega|_\omega(d\gamma, \beta) = |\Omega|_\omega(\gamma, d^\dagger\beta) = |\Omega|_\omega|\gamma|^2.$$

Therefore $\ker(L_1^{\mathrm{A}}) = \{0\}$ and by Fredholm alternative,

$$\mathrm{Im}(L_1^{\mathrm{A}}) = \ker((L_1^{\mathrm{A}})^\dagger)^\perp = \{0\}^\perp,$$

is the full space, hence L_1^{A} is subjective.

Then by elliptic regularity,

$$(L_1^{\mathrm{A}})^{-1} : C^{k+1,\gamma}(d^\dagger\Lambda^2) \to C^{k+2,\gamma}(\mathrm{Im}\, d \cap \Lambda^{1,1}),$$

and we have constructed an inverse, hence L_1^{A} is indeed invertible. □

6.2.3 L_2 and L_3

The L_2 and L_3 are the same compared to the Bott-Chern case, hence we can conclude their invertibility respectively.

Therefore, the implicit function theorem for Banach spaces can be applied to conclude the existence of solutions of heterotic systems near the small ϵ-neighborhood of the reference Kähler solution, i.e. Theorem 2.

Chapter 7

Discussion

We have established the existence of local coordinates near a Kähler solution with two different deformation paths. Several future directions for research arises naturally:

1. Throughout our paper, we have fixed the complex structure on the complex manifold X. Therefore, technically speaking, the constructed local coordinates are not true local coordinates of the moduli space but provide local coordinates to a subspace of the moduli instead. In particular, they are local coordinates of the submanifold where the complex structure is invariant. Nevertheless, it is sufficient to show that the full heterotic moduli is smooth as this submanifold of invariant complex structure as a submanifold is smooth. Hence, a further extension to the full moduli is needed for future investigations.

2. The isomorphism between $H^{2,2}_{\text{B-C}}$ and $H^{1,1}_{\text{A}}$ suggests that the constructed local coordinates can be related by some coordinate transformation in the tangent space of the moduli space. However, as we need to introduce a spurious gauge field θ in the Aeppli case, such coordinate transformation without this introduction of θ is unclear, despite the hint of its existence. One can always match the Bott-Chern case with the Aeppli case by introducing θ *ad hoc*, but its physical origin is unclear.

3. We have established local coordinates of the heterotic moduli near a Kähler solution taking advantage of many nice properties of Kähler manifold. Can we establish the existence of local coordinates on a general heterotic solution? Away from the Kähler background, our two deformation paths parameterized by the Bott-Chern cohomology class and the Aeppli cohomology class are no longer isomorphic, hence this discrepancy suggests that there is not a trivial coordinates transformation between the constructed local coordinates on a general heterotic solution. Yet the $\partial\bar{\partial}$-lemma yields the dimension matching between Aeppli and Bott-Chern cohomology suggesting the existence of such a

coordinate transformation.

4. We have shown the existence of local coordinates which allows us to probe near a Kähler point. This can be thought of as a short-time existence theorem of PDEs from the heterotic system. Yet, we do not have a long-time existence theorem that tells us the global structure of the heterotic moduli. However, [14] provides the physics argument that the heterotic moduli should be Kähler by the supersymmetry condition with the assumption of the existence of coordinates implicitly. One main result is that the heterotic moduli is governed by a Kähler potential which is a direct generalization from the $\alpha' = 0$ case. Our work is then a step of explicitly constructing such coordinates on the heterotic moduli. We expect to explicitly verify this after incorporating the deformation of the complex structure.

5. While our setup involves varying the metric and gauge field (g, A), other investigations of the moduli e.g. [14, 39] introduces an additional field called the B-field and vary (g, B, A). One motivation for adding the B-field is to give the moduli space the structure of a complex manifold [15, 40].

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Appendix A

Justification for the Chern connection

The equations of heterotic supergravity are derived by an expansion in a small parameter denoted α'. In this model, the fields are a metric tensor g, the curvature F of a connection on a vector bundle E, a scalar function Φ, and a 3-form H. The 3-form is encoded into the geometry by producing a connection $D^{\rm H}$ as in (2.131) which has Christoffel symbols

$$\Gamma^{\rm H}{}_i{}^k{}_j = \Gamma^{\rm L-C}{}_i{}^k{}_j + \frac{1}{2} H_i{}^k{}_j.$$

The low-energy effective action to first order in α' is given by (2.126). The principle of least action states that the supersymmetry invariance should correspond to stationary points of the action.

Special critical points of the action involve extra structure on the manifold. As we briefly discussed in 2.6.2, certain non-Kähler structures in complex geometry were identified by Strominger [60] and Hull [47] as satisfying the equations of supersymmetry. In this setup, X is a complex manifold with holomorphic volume form Ω and hermitian metric $\omega = i g_{\mu\bar\nu} dz^\mu \wedge d\bar z^\nu$, and the associated fields are given by

$$H = i(\partial - \bar\partial)\omega, \quad \Phi = -\frac{1}{2} \log |\Omega|_\omega.$$

The supersymmetry constraints are

$$d(|\Omega|_\omega \omega^2) = O(\alpha'^2), \quad F^{0,2} = O(\alpha'), \quad F \wedge \omega^2 = O(\alpha') \tag{A.1}$$

and the heterotic Bianchi identity is

$$i\partial\bar\partial\omega = \alpha'({\rm Tr}\, F \wedge F - {\rm Tr}\, R^{\rm H} \wedge R^{\rm H}) + O(\alpha'^2). \tag{A.2}$$

The consistency of these equations with conformal invariance of the sigma model and anomaly cancellation was established by Hull [47]. It is well-known in the string theory literature (see [17] and more recent work e.g. [5, 21]) that solutions to (A.1), (A.2) are critical points of the action S.

From the perspective of string theory, these constraints should not be taken as literal equality, as they are not valid to higher order in α'. That said, as a mathematical problem we may look for solutions where the right-hand side of the supersymmetry constraints (A.1) is exactly zero. However, the constraint (A.2) is problematic as a literal equality without the higher order $O(\alpha'^2)$ contributions, since $\mathrm{Tr}\, R^{\mathrm{H}} \wedge R^{\mathrm{H}}$ is generally not a $(2,2)$ form. Therefore, we can safely use the Chern connection for the analysis.

The solutions constructed in this paper solve the mathematical equations (1.1) near a Kähler background. The equations (1.1) are almost the same as the truncated physical system, except that $\mathrm{Tr}\, R \wedge R$ is computed with respect to the Chern connection instead of the Hull connection D^{H}. We now explain why in the setting of this paper, this is not a problem as these solutions do satisfy the physical system to this order in α'. This is because applying the implicit function theorem over a Kähler background gives a smooth path $g_{\alpha'}$ in the parameter α' with $g|_{\alpha'=0} = g_{\mathrm{CY}}$. In other words,

$$g_{\alpha'} = g_{\mathrm{CY}} + O(\alpha'). \tag{A.3}$$

It follows that these solutions satisfy the estimate $H = O(\alpha')$ and so

$$R^{\mathrm{H}} - R^{\mathrm{Ch}} = \mathcal{O}(H, \nabla H) = O(\alpha').$$

Thus (A.2) is satisfied and solutions to the heterotic Bianchi identity with Chern connection are not distinguishable from solutions with Hull connection at this order in α'.

We also considered another setup in this paper where the heterotic Bianchi identity is taken with extra connection θ on $T^{1,0}X$.

$$i\partial\bar{\partial}\omega = \alpha'(\mathrm{Tr}\, F \wedge F - \mathrm{Tr}\, R_\theta \wedge R_\theta). \tag{A.4}$$

Here we applied the implicit function theorem over a Kähler background to obtain a path $\theta_{\alpha'}$ such that $\theta_{\alpha'} \to \Gamma^{\mathrm{LC}}_{g_{\mathrm{CY}}}$ as $\alpha' \to 0$. It follows that

$$R_\theta - R^{\mathrm{H}}_{g_{\mathrm{CY}}} = O(\alpha').$$

Combining this with the fact that the metric g is Calabi-Yau at zeroth order (A.3), it follows that (A.2) is satisfied.

In summary, the mathematical equations solved by Li and Yau [50] and Andreas and Garcia-Fernandez [2] (see [19] for follow-up work) by inverse function theorem near a Kähler Calabi-Yau solution solve the physical system. The present work deforms these solutions by the implicit function theorem, and so both setups considered in this paper satisfy the physical system. From the perspective of string theory, exact solutions to the truncated equations are not needed as solving the physical equations by large radius expansion about a Kähler Calabi-Yau is well-known in the string theory literature by works of Witten-Witten [62] and Melnikov-Minasian-Sethi [52].

We make one further comment on the difference between the setup of the physical system in string theory and the setup considered in the current paper. A fundamental equation in our approach is the 4-form Bianchi identity (A.4), while in string theory the anomaly cancellation relation is typically written as a 3-form equation

$$H = dB + \alpha'(CS[A] - CS[\theta])$$

where $CS[A]$, $CS[\theta]$ are Chern-Simons local 3-forms and B is a local 2-form which transforms on overlaps in such a way that H is a well-defined global 3-form. In string theory, variations of the 3-form field strength H are related to variations of the 2-form B field [1, 14]. The approach to the moduli problem in the present work is to vary the 3-form H directly by variations of the metric ω via $H = i(\partial - \bar{\partial})\omega$.

Appendix B

Frame transformation

Throughout our paper, we write the reference Chern connection A in a fixing orthonormal frame such that $A = -\mathcal{A}^\dagger + \mathcal{A}$ where $A^{0,1} = \mathcal{A}$. Here we justify the existence of such a frame and how to obtain it from the usual holomorphic frame.

Say given a Chern connection A, which naturally admits a simple expression in the holomorphic frame $\{f_\alpha\}$, where there is no component $A_f^{0,1} = 0$ along the anti-holomorphic direction and $A_f^{1,0} = A_f$ is purely $(1,0)$-form. Then the covariant derivative D reads,

$$\begin{aligned} D_f &= d + A_f = D_f^{1,0} + D_f^{0,1}, \\ D_f^{1,0} &= \partial + A_f^{1,0} = \partial + A_f, \\ D_f^{0,1} &= \bar{\partial} + A_f^{0,1} = \bar{\partial}. \end{aligned} \tag{B.1}$$

where the subscript f indicates that it is written in holomorphic frame $\{f_\alpha\}$. We can write A_f using the Hermitian metric h associated to the Chern connection as below,

$$\mathcal{A}_f = h^{-1}\partial h. \tag{B.2}$$

Now, we desire an orthonormal frame $\{e_a\}$ such that the Chern connection is unitary $A = -\mathcal{A}^\dagger + \mathcal{A}$ where $\mathcal{A} = A_e^{0,1}$. The subscript e indicates that it is written in the orthonormal frame $\{e_a\}$. Then the covariant derivative reads,

$$\begin{aligned} D_e &= d + A_e = \mathcal{D} + \bar{\mathcal{D}}, \\ \mathcal{D} &= D_e^{1,0} = \partial - \mathcal{A}^\dagger \\ \bar{\mathcal{D}} &= D_e^{0,1} = \bar{\partial} + \mathcal{A}. \end{aligned} \tag{B.3}$$

Hence, we want to find a frame transformation B from the holomorphic frame to the desired orthonormal frame,

$$e_a = f_\alpha B^\alpha{}_a. \tag{B.4}$$

Under this frame transformation, given A_f, we obtain A_e written in the orthonormal frame as,

$$\begin{aligned} A_e &= B^{-1}A_fB + B^{-1}dB = A_e^{1,0} + A_e^{0,1}, \\ A_e^{1,0} &= B^{-1}A_fB + B^{-1}\partial B, \\ A_e^{0,1} &= B^{-1}\bar{\partial}B. \end{aligned} \tag{B.5}$$

Now solving the unitary condition $-\left(A_e^{0,1}\right)^\dagger = A_e^{1,0}$, we get

$$\begin{aligned} A_f &= -B\Big(B^{-1}\partial B + \partial B^\dagger (B^\dagger)^{-1}\Big)B^{-1} \\ &= -\partial B B^{-1} - B\partial B^\dagger \left(B^\dagger\right)^{-1} B^{-1} \\ &= B\partial B^{-1} + BB^\dagger \partial\left(B^\dagger\right)^{-1} B^{-1} \\ &= \left(BB^\dagger\right)\partial\left(BB^\dagger\right)^{-1}. \end{aligned} \tag{B.6}$$

Since $A_f = h^{-1}\partial h$, we can therefore obtain the solution for such desired frame transformation as the solution of the following equation

$$h = \left(BB^\dagger\right)^{-1}. \tag{B.7}$$

Appendix C

Covariant derivatives D of a unitary connection A

For our reference unitary connection $A = -\mathcal{A}^\dagger + \mathcal{A}$ on E, we denote upon type decomposition that $D = d + A = \mathcal{D} + \bar{\mathcal{D}}$. Their actions on any section $s \in \Gamma(E)$ reads,

$$\mathcal{D}s = \partial s - \mathcal{A}^\dagger \wedge s,$$
$$\bar{\mathcal{D}}s = \bar{\partial}s + \mathcal{A} \wedge s.$$

Their actions on any section $\gamma \in \Gamma(\mathrm{End}E)$ reads,

$$\mathcal{D}\gamma = \partial\gamma - \mathcal{A}^\dagger \wedge \gamma + \gamma \wedge \mathcal{A}^\dagger,$$
$$\bar{\mathcal{D}}\gamma = \bar{\partial}\gamma + \mathcal{A} \wedge \gamma - \gamma \wedge \mathcal{A}.$$

From above, one can easily see the following conjugation relations,

$$(\mathcal{D}\gamma)^\dagger = \Big(\partial\gamma - \mathcal{A}^\dagger \wedge \gamma + \gamma \wedge \mathcal{A}^\dagger\Big)^\dagger = \bar{\partial}\gamma^\dagger + \mathcal{A} \wedge \gamma^\dagger - \gamma^\dagger \wedge \mathcal{A} = \bar{\mathcal{D}}\gamma^\dagger, \tag{C.1}$$

$$\big(\bar{\mathcal{D}}\gamma\big)^\dagger = \big(\bar{\partial}\gamma + \mathcal{A} \wedge \gamma - \gamma \wedge \mathcal{A}\big)^\dagger = \partial\gamma^\dagger - \mathcal{A}^\dagger \wedge \gamma^\dagger + \gamma^\dagger \wedge \mathcal{A}^\dagger = \mathcal{D}\gamma^\dagger. \tag{C.2}$$

Also, one can easily see the following trace relations, using the cyclic property of trace,

$$\mathrm{Tr}(\mathcal{D}\gamma) = \mathrm{Tr}\Big(\partial\gamma - \mathcal{A}^\dagger \wedge \gamma + \gamma \wedge \mathcal{A}^\dagger\Big) = \partial(\mathrm{Tr}\gamma), \tag{C.3}$$

$$\mathrm{Tr}\big(\bar{\mathcal{D}}\gamma\big) = \mathrm{Tr}\big(\bar{\partial}\gamma + \mathcal{A} \wedge \gamma - \gamma \wedge \mathcal{A}\big) = \bar{\partial}(\mathrm{Tr}\gamma). \tag{C.4}$$

For general $\lambda \in \Lambda^{p,q}(\mathrm{End}E)$, we have such trace identities,

$$\begin{aligned}\mathrm{Tr}(\mathcal{D}\lambda) &= \mathrm{Tr}\Big(\partial\lambda - \mathcal{A}^\dagger \wedge \lambda + (-1)^{p+1}\lambda \wedge \mathcal{A}^\dagger\Big) \\ &= \partial(\mathrm{Tr}\lambda) - \mathrm{Tr}\Big(\mathcal{A}^\dagger \wedge \lambda\Big) + (-1)^{2(p+1)}\mathrm{Tr}\Big(\mathcal{A}^\dagger \wedge \lambda\Big) \\ &= \partial(\mathrm{Tr}\lambda),\end{aligned} \tag{C.5}$$

$$\begin{aligned}\mathrm{Tr}\big(\bar{\mathcal{D}}\lambda\big) &= \mathrm{Tr}\big(\bar{\partial}\lambda + \mathcal{A}\wedge\lambda - (-1)^{q+1}\lambda\wedge\mathcal{A}\big)\\ &= \bar{\partial}(\mathrm{Tr}\lambda) + \mathrm{Tr}(\mathcal{A}\wedge\lambda) - (-1)^{2(q+1)}\mathrm{Tr}(\lambda\wedge\mathcal{A})\\ &= \bar{\partial}(\mathrm{Tr}\lambda).\end{aligned} \tag{C.6}$$

Hence we can easily show the following trace identities:

1. $\mathrm{Tr}D(-\alpha^{\dagger}+\alpha) = d\mathrm{Tr}(-\alpha^{\dagger}+\alpha)$, which follows from trace identities (C.3) and (C.4).
2. $\mathrm{Tr}\mathcal{D}\bar{\mathcal{D}}u = \partial\bar{\partial}\mathrm{Tr}u$ and $\mathrm{Tr}\bar{\mathcal{D}}\mathcal{D}u = \bar{\partial}\partial\mathrm{Tr}u$, which follows from application of trace identities (C.5) and (C.6).

Appendix D

Complex Geometry

Here we review some necessary background and key facts about complex geometry, and establish the notations and conventions. The main reference is [48]

D.1 Holomorphic functions

We start by recalling the definition of holomorphicity in the theory of the holomorphic function of a single variable.

Definition 1 (Holomorphicity of $f : \mathbb{C} \supset U \to \mathbb{C}$). *Let $U \subset \mathbb{C}$ be an open subset, a function $f : U \to \mathbb{C}$ is called holomorphic if for any point $z_0 \in U$, there exists $\epsilon > 0$,such that the ball $B_\epsilon(z_0) \subset U$ upon which f admits a convergent power series expansion,*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad \forall z \in B_\epsilon(z_0). \tag{D.1}$$

This can be immediately extended to $\mathbb{C}^m$ case.

Definition 2 (Holomorphicity of $f : \mathbb{C}^m \supset U \to \mathbb{C}$). *Let $U \subset \mathbb{C}^m$ be an open subset, a function $f : U \to \mathbb{C}^n$ is holomorphic if all coordinate functions $f_1, \cdots, f_n$ are holomorphic functions $U \to \mathbb{C}$.*

Then the complex Jacobian of such a holomorphic f at point $z \in U$ is the matrix,

$$J(f)(z) = \left[\frac{\partial f_i}{\partial z_j}\right]_{ij}, \quad 1 \leq i \leq n, \quad 1 \leq j \leq m. \tag{D.2}$$

Then the implicit function theorem is given in analogy to the real case.

Proposition D.1.1 (Implicity function theorem). *Let $U \in \mathbb{C}^m$ be an open subset and let $f : U \to \mathbb{C}^n$ be a holomorphic map, where $m \geq n$. Suppose $z_0 \in U$ is a point such that $\det(J(f)(z_0)) \neq 0$. Then there exists open subsets $U_1 \subset \mathbb{C}^{m-n}$, $U_2 \subset \mathbb{C}^n$ and a holomorphic map $g : U_1 \to U_2$ such that $U_1 \times U_2 \subset U$ and $f(z) = f(z_0)$ if and only if $g(z_{n+1}, \cdots, z_m) = (z_1, \cdots, z_n)$.*

D.2 Almost complex and hermitian structure

In general, given a finite-dimensional real vector space V, we can equip it with additional structures, such as scalar products and almost complex structures. They induce linear operators on the exterior algebra such as the Hodge $\star$-operator, the Lefschetz operator L_ω, the contraction operator Λ_ω, and so on. The interplay between those linear operators is important in our later discussion.

Definition 3 (Almost complex structure J on V)**.** *An endomorphism $J : V \to V$ with $J^2 = -id$ is called an almost complex structure on V.*

In fact, any almost complex structure J on V induces a natural orientation on V.

Lemma D.2.1 ($\pm i$-eigenspace decomposition of $V_{\mathbb{C}}$)**.** *Let $V_{\mathbb{C}} = V \otimes \mathbb{C}$ be the complexification of the real vector space V via the map $v \mapsto v \otimes 1$. Then the $\mathbb{C}$-linear extension $J : V_{\mathbb{C}} \to V_{\mathbb{C}}$ induces the decomposition of $V_{\mathbb{C}}$ into $\pm i$-eigenspace of J,*

$$V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}, \tag{D.3}$$

$$V^{1,0} = \{v \in V_{\mathbb{C}} : \ J(v) = i \cdot v\}, \tag{D.4}$$

$$V^{0,1} = \{v \in V_{\mathbb{C}} : \ J(v) = -i \cdot v\}. \tag{D.5}$$

And the complex conjugation on $V_{\mathbb{C}}$, which is defined by $\overline{(v \otimes \lambda)} = v \otimes \bar{\lambda}$, induces an $\mathbb{R}$-linear isomorphism $V^{1,0} \cong V^{0,1}$.

This also applies to the dual space $V^* = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$.

Lemma D.2.2 ($\pm i$-eigenspace decomposition of $V^*_{\mathbb{C}}$)**.** *The almost complex structure J induced on V^* is given by $J(f)(v) = f(J(v))$ forall $f \in V^*$. And the induced decomposition on $(V_{\mathbb{C}})^* = Hom_{\mathbb{R}}(V, \mathbb{C}) = (V^*)_{\mathbb{C}}$,*

$$(V_{\mathbb{C}})^* = (V^*)^{1,0} \oplus (V^*)^{0,1}), \tag{D.6}$$

$$(V^*)^{1,0} = \{f \in Hom_{\mathbb{R}}(V, \mathbb{C}) : \ f(J(v)) = if(v)\} = (V^{1,0})^*, \tag{D.7}$$

$$(V^*)^{0,1} = \{f \in Hom_{\mathbb{R}}(V, \mathbb{C}) : \ f(J(v)) = -if(v)\} = (V^{0,1})^*. \tag{D.8}$$

If V is of dimension d, the natural decomposition of its exterior algebra is of the form,

$$\bigwedge^* V = \bigoplus_{k=0}^{d} \bigwedge^k V. \tag{D.9}$$

In analogy, the exterior algebra of the complexified $V_{\mathbb{C}}$ also admits a natural decomposition,

$$\bigwedge^* V_{\mathbb{C}} = \bigoplus_{k=0}^{d} \bigwedge^k V_{\mathbb{C}}. \tag{D.10}$$

If $d = 2n$ is even. By the $\pm i$-eigenspace decomposition of $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$, each eigenspace is then of dimension n. It is then natural to consider the bidegree (p, q) element formed out of the exterior algebra of each eigenspace and their space.

Definition 4. *We define*

$$\bigwedge^{p,q} V = \bigwedge^p V^{1,0} \otimes_{\mathbb{C}} \bigwedge^q V^{0,1}. \tag{D.11}$$

Then it has the following properties.

Proposition D.2.1. *For a real vector space V equipped with an almost complex structure J,*

1. *$\bigwedge^{p,q} V$ is in a canonical way a subspace of $\bigwedge^{p+q} V_{\mathbb{C}}$.*
2. *The exterior algebra of $V_{\mathbb{C}}$ admits a bidegree decomposition:*

$$\bigwedge^k V_{\mathbb{C}} = \bigoplus_{p+q=k} \bigwedge^{p,q} V. \tag{D.12}$$

3. *Complex conjugation on $\bigwedge^* V_{\mathbb{C}}$ defines a $\mathbb{C}$-antilinear isomorphism $\bigwedge^{p,q} V \cong \bigwedge^{q,p} V$.*
4. *The exterior product if of bidegree $(0, 0)$, $\bigwedge^{p,q} V \times \bigwedge^{r,s} V \to \bigwedge^{p+r,q+s} V$, $(\alpha, \beta) \mapsto \alpha \wedge \beta$.*

Since we will be interested in the degree and bidegree elements, we can define the corresponding projections.

Definition 5. *The natural projections to the degree and bidegree elements are given by,*

$$\Pi^k : \bigwedge^* V_{\mathbb{C}} \to \bigwedge^k V_{\mathbb{C}}, \tag{D.13}$$

$$\Pi^{p,q} : \bigwedge^* V_{\mathbb{C}} \to \bigwedge^{p,q} V_{\mathbb{C}}. \tag{D.14}$$

It is also convenient to define an endomorphism $\mathbf{J}$ *on* $\bigwedge^* V_{\mathbb{C}}$*,*

$$\mathbf{J} = \sum_{p,q} i^{p-q} \cdot \Pi^{p,q}, \tag{D.15}$$

which is the multiplicative extension of the almost complex structure J *on* $V_{\mathbb{C}}$*.*

We now equip V with a scalar product $\langle \cdot, \cdot \rangle$ which is a positive symmetric bilinear form. Hence, now V is a finite-dimensional euclidean vector space. We now want to understand the interplay between the scalar product $\langle \cdot, \cdot \rangle$ and the almost complex structure J.

Definition 6. *An almost complex structure* J *on* V *is compatible with the the scaler product* $\langle \cdot, \cdot \rangle$ *if*

$$\langle J(v), J(w) \rangle = \langle v, w \rangle, \quad \forall v, w \in V. \tag{D.16}$$

This defines the fundamental form of the following.

Definition 7. *The fundamental form associated to* $(V, \langle \cdot, \cdot \rangle, J)$ *is of form,*

$$\omega = -\langle \cdot, J(\cdot) \rangle = \langle J(\cdot), \cdot \rangle. \tag{D.17}$$

And ω *is real and of type* $(1,1)$*,* $\omega \in \bigwedge^2 V^* \cap \bigwedge^{1,1} V^*$*.*

It is worth noting that any two of the three structures $\{\langle \cdot, \cdot \rangle, J, \omega\}$ determines the rest.

We can encode the scalar product $\langle \cdot, \cdot \rangle$ and the fundamental form ω by a natural hermitian form.

Lemma D.2.3 (Hermitian form $(\cdot, \cdot)$)**.** *Given* $(V, \langle \cdot, \cdot \rangle, J)$*, the form,*

$$(\cdot, \cdot) = \langle \cdot, \cdot \rangle - i \cdot \omega, \tag{D.18}$$

is a positive hermitian form on (V, J)*. And the extension of the scalar product* $\langle \cdot, \cdot \rangle$ *to a positive definite hermitian form* $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ *on* $V_{\mathbb{C}}$ *is given by,*

$$\langle v \otimes \lambda, w \otimes \mu \rangle_{\mathbb{C}} = (\lambda \bar{\mu}) \cdot \langle v, w \rangle, \quad \forall v, w \in \mathbb{C}, \ \lambda, \mu \in \mathbb{C}. \tag{D.19}$$

Concretely, let $\{z_1, \cdots, z_n\}$ be a $\mathbb{C}$-basis of $V^{1,0}$, then $\{\bar{z}\}_i$ is a basis for $V^{0,1}$. Then the fundamental form ω can be locally expressed as,

$$\omega = \frac{i}{2} \sum_{i,j=1}^{n} h_{ij} z^i \wedge \bar{z}^j, \tag{D.20}$$

where $\frac{1}{2}h_{ij}$ is an hermitian matrix.

We are now in the position of defining some interesting linear operators on the exterior algebra of the dual space $V_{\mathbb{C}}^*$. We start with the Hodge $\star$-operator.

Definition 8. *Given $(V, \langle\cdot,\cdot\rangle, J)$. Let $vol_\omega \in \bigwedge^d V$ the orientation of V of norm 1, then the Hodge $\star_\omega$-operator is a linear operator $\star_\omega : \bigwedge^k V \to \bigwedge^{d-k} V$, defined by a non-degenerating pairing $\bigwedge^k V \times \bigwedge^{d-k} V \to \bigwedge^d V = vol \cdot \mathbb{R}$, i.e.*

$$\alpha \wedge \star_\omega \beta = \langle \alpha, \beta \rangle \cdot vol, \quad \forall \alpha, \beta \in \bigwedge^* V. \tag{D.21}$$

Definition 9. *Given $(V, \langle\cdot,\cdot\rangle, J)$, the Lefschetz operator $L_\omega : \bigwedge^k V^* \to \bigwedge^{k+2} V^*$ with respect to the fundamental form ω is defined by,*

$$L_\omega : \bigwedge^* V_{\mathbb{C}}^* \to \bigwedge^* V_{\mathbb{C}}^*, \tag{D.22}$$

$$\alpha \mapsto \omega \wedge \alpha. \tag{D.23}$$

The Lefschetz operator L_ω comes with a dual operator Λ_ω.

Definition 10. *The dual Lefschetz operator Λ_ω, also called the contraction operator with respect to ω is a operator $\Lambda_\omega : \bigwedge^* V^* \to \bigwedge^* V^*$ adjoint to L_ω with respect to $\langle\cdot,\cdot\rangle$, i.e.*

$$\langle \Lambda_\omega \alpha, \beta \rangle = \langle \alpha, L_\omega \beta \rangle, \quad \forall \alpha, \beta \in \bigwedge^* V^* \tag{D.24}$$

It has again $\mathbb{C}$-linear extension denoted also as $\Lambda_\omega : \bigwedge^* V_{\mathbb{C}}^* \to \bigwedge^* V_{\mathbb{C}}^*$.

By definition, we have,

$$\Lambda_\omega = \star_\omega^{-1} \circ L_\omega \circ \star_\omega. \tag{D.25}$$

We now define a counting operator.

Definition 11. *Let $H : \bigwedge^* V \to \bigwedge^* V$ be the counting operator defined by, for $dim_{\mathbb{R}} V = d = 2n$,*

$$H = \sum_{k=0}^{2n} (k - n) \cdot \Pi^k. \tag{D.26}$$

Hence, we have for our fundamental form ω,

$$H\omega = (2 - n)\omega. \tag{D.27}$$

In fact, L_ω, Λ_ω and H defines a natural $\mathfrak{sl}_2$-representation on $\bigwedge^* V^*$, hence the commutation relations are given by the following.

Proposition D.2.2. *Given $(V, \langle\cdot,\cdot\rangle, J)$, we have,*

$$[H, L_\omega] = 2L_\omega, \tag{D.28}$$
$$[H, \Lambda_\omega] = -2\Lambda_\omega, \tag{D.29}$$
$$[L_\omega, \Lambda_\omega] = H. \tag{D.30}$$

These results will be useful in the calculation of the variation of ω in Lemma. 6.1.1 later on.

There are some elements of the exterior algebra that vanish under contraction.

Definition 12. *Given $(V, \langle\cdot,\cdot\rangle, J)$, let L_ω, Λ_ω and H be as above. An element $\alpha \in \bigwedge^k V^*$ is called primitive if $\Lambda_\omega \alpha = 0$. The linear subspace of all primitive elements $\alpha \in \bigwedge^k V^*$ is denoted by $P^k \subset \bigwedge^k V^*$. This extend to the $\mathbb{C}$-linear extension to $\bigwedge^k V^*_{\mathbb{C}}$. If $k > n$, $P^k = 0$.*

Then, we have the important Lefschetz decomposition for the exterior algebra.

Proposition D.2.3 (Lefschetz decomposition for the exterior algebra)**.** *Given $(V, \langle\cdot,\cdot\rangle, J)$, and the associated Lefschetz operators L_ω and Λ_ω defined as above. There exists a direct sum decomposition of the form,*

$$\bigwedge^k V^* = \bigoplus_{i\geq 0} L^i_\omega (P^{k-2i}). \tag{D.31}$$

The Lefschetz decomposition is orthogonal with respect to $\langle\cdot,\cdot\rangle$.

In particular, in degree 2, we have,

$$\bigwedge^2 V^* = \omega\mathbb{R} \oplus P^2. \tag{D.32}$$

This means that,

$$\gamma = \alpha + \frac{1}{3}(\Lambda_\omega \gamma)\omega, \quad \alpha \in P^2. \tag{D.33}$$

The interplay between the Lefschetz operator L_ω and the Hodge $\star_\omega$-operator is given by the following "mysterious yet extremely useful" proposition.

Proposition D.2.4 (The reflection relation of the primitives)**.** *For all $\alpha \in P^k$, we have,*

$$\star_\omega L^j_\omega \alpha = (-1)^{\frac{k(k+1)}{2}} \frac{k!}{(n-k-j)!} \cdot L^{n-k-j}_\omega \mathbf{J}(\alpha). \tag{D.34}$$

In particular, we have the following special cases:

1. $k = j = 0$ and $\alpha = 1$, we obtain,

$$\star_\omega 1 = \frac{1}{n!} L_\omega^n 1 = \frac{\omega^n}{n!} = \mathrm{vol}_\omega. \tag{D.35}$$

2. $k = 0$, $j = 1$ and $\alpha = 0$, we obtain,

$$\star_\omega \omega = \frac{1}{(n-1)!}\omega^{n-1}. \tag{D.36}$$

3. For $\alpha \in P^2 \cap \bigwedge^{1,1} V^*$,

$$\star_\omega \alpha = -\frac{1}{(n-2)!}\omega^{n-2} \wedge \alpha. \tag{D.37}$$

Those will be useful in our later proof of Lemma. 6.2.1. Note also by (D.36) and (D.25), we have,

$$\Lambda_\omega \omega = \star_\omega^{-1} \circ L_\omega \circ \star_\omega \omega = n. \tag{D.38}$$

D.3 Differential Forms

We now apply the result of the previous section to the tangent bundle TM, and the k-form bundle $\bigwedge^k (TM)^*$ of a real manifold M. Locally we take $U \subset \mathbb{C}^n$ as $2n$-dimensional real manifold, with coordinates $\{x^i, y^i\}_{i=1}^n$. Then at point $x \in U$, the canonical basis for the tangent space T_xU is given by $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\}_{i=1}^n$ and the canonical basis for the dual tangent space $(T_xU)^*$ is given by $\{dx^i, dy^i\}_{i=1}^n$. Both admit the natural almost complex structures defined by,

$$J : T_xU \to T_xU, \qquad \frac{\partial}{\partial x^i} \mapsto \frac{\partial}{\partial y^i}, \quad \frac{\partial}{\partial y^i} \mapsto -\frac{\partial}{\partial x^i}. \tag{D.39}$$

$$J : (T_xU)^* \to (T_xU)^*, \qquad dx^i \mapsto -dy^i, \quad dy^i \mapsto dx^i. \tag{D.40}$$

Then our results in the previous section give the $\pm i$-eigenspace decomposition of the complexifed tangent bundle $T_{\mathbb{C}}U = TU \otimes \mathbb{C}$ and dual tangent bundle $T^*_{\mathbb{C}}U = (TU)^* \otimes \mathbb{C}$.

Proposition D.3.1 ($\pm i$-eigenspace decomposition of $T_{\mathbb{C}}U$ and $T^*_{\mathbb{C}}U$)**.** *$T_{\mathbb{C}}U$ and $T^*_{\mathbb{C}}U$ admit $\pm i$-eigenspace decompositions as follows,*

$$T_{\mathbb{C}}U = T^{1,0}U \oplus T^{0,1}U, \tag{D.41}$$

$$T^*_{\mathbb{C}}U = (T^*U)^{1,0} \oplus (T^*U)^{0,1}. \tag{D.42}$$

Locally, the vector bundle $T^{1,0}U$ and $T^{0,1}U$ is trivialized by sections $\frac{\partial}{\partial z^i} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} - i\frac{\partial}{\partial x^i}\right)$ and $\frac{\partial}{\partial \bar{z}^i} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} + i\frac{\partial}{\partial x^i}\right)$, $i = 1, \cdots, n$. The vector bundle $(T^*U)^{1,0}$ and $(T^*U)^{0,1}$ are trivializaed by sections $dz^i = dx^i + idy^i$ and $d\bar{z}^i = dx^i - idy^i$.

The $\mathbb{C}$-linear extension of the differential df of a holomorphic map f respects the above decomposition.

Definition 13. *Let $U \subset \mathbb{C}^n$ be an open subset. Over U, we define that complex vector bundles,*

$$\bigwedge^{p,q} U = \bigwedge^{p} (T^*U)^{1,0} \otimes \bigwedge^{q} (T^*U)^{0,1}. \tag{D.43}$$

We note the space of the sections of $\bigwedge_{\mathbb{C}}^k U = \bigwedge^k T_{\mathbb{C}}^* U$ as $\mathcal{A}_{\mathbb{C}}^k(U)$, and the space of sections of $\bigwedge^{p,q} U$ as $\mathcal{A}^{p,q}(U)$.

The bundles of k-forms admit bidegree decompositions.

Proposition D.3.2 (Bidegree decomposition of k-form bundle)**.** *There are natural decompositions,*

$$\bigwedge_{\mathbb{C}}^{k} U = \bigoplus_{p+q=k} \bigwedge^{p,q} U, \tag{D.44}$$

$$\mathcal{A}_{\mathbb{C}}^k(U) = \bigoplus_{p+q=k} \mathcal{A}^{p,q}(U). \tag{D.45}$$

We denote the natural projection operator,

$$\Pi^{p,q} : \bigwedge_{\mathbb{C}}^{k} U \to \bigwedge^{p,q} U, \tag{D.46}$$

$$\Pi^{p,q} : \mathcal{A}_{\mathbb{C}}^k(U) \to \mathcal{A}^{p,q}(U). \tag{D.47}$$

We also have the decomposition of the exterior differential d.

Definition 14. *Let $d : \mathcal{A}_{\mathbb{C}}^k(U) \to \mathcal{A}_{\mathbb{C}}^{k+1}(U)$ be the $\mathbb{C}$-linear extension of the usual exterior differential. Via projection we define,*

$$\partial : \mathcal{A}^{p,q}(U) \to \mathcal{A}^{p+1,q}(U), \quad \partial = \Pi^{p+1,q} \circ d, \tag{D.48}$$

$$\bar{\partial} : \mathcal{A}^{p,q}(U) \to \mathcal{A}^{p,q+1}(U), \quad \bar{\partial} = \Pi^{p,q+1} \circ d. \tag{D.49}$$

They share the usual properties of the exterior differential d.

Proposition D.3.3. *For the differential operator ∂ and $\bar{\partial}$, we have,* *1.* $d = \partial + \bar{\partial}$.

2. $\partial^2 = \bar{\partial}^2 = 0$ *and* $\partial\bar{\partial} = -\bar{\partial}\partial$.

3. ∂ *and* $\bar{\partial}$ *satisfies the Leibniz rule, for any* $\alpha \in \mathcal{A}^{p,q}(U)$ *and* $\beta \in \mathcal{A}^{r,s}(U)$,

$$d(\alpha \wedge \beta) = d(\alpha) \wedge \beta + (-1)^{p+q} \alpha \wedge d(\beta), \tag{D.50}$$

$$\partial(\alpha \wedge \beta) = \partial(\alpha) \wedge \beta + (-1)^{p+q} \alpha \wedge \partial(\beta), \tag{D.51}$$

$$\bar{\partial}(\alpha \wedge \beta) = \bar{\partial}(\alpha) \wedge \beta + (-1)^{p+q} \alpha \wedge \bar{\partial}(\beta). \tag{D.52}$$

Now if we equip U with a Riemannian metric g, which effectively introduces a scalar product g_x on $T_x U$ at any point $x \in U$. If g_x is compatible with the almost complex structure J for any $x \in U$, i.e. $g_x(\cdot, \cdot) = g_x(J(\cdot), J(\cdot))$, then we say g is compatible with J. Then the fundamental form $\omega \in \mathcal{A}^{1,1}(U) \cap \mathcal{A}^2(U)$ is defined by,

$$\omega = g(J(\cdot), \cdot), \tag{D.53}$$

and the associated hermitian form reads,

$$h = g - i\omega. \tag{D.54}$$

Locally the fundamental form can be written as,

$$\omega = \frac{i}{2} \sum_{i=1}^{n} h_{ij} dz^i \wedge d\bar{z}^j, \tag{D.55}$$

where $\frac{1}{2} h_{ij}$ is positive definite hermitian matrix.

D.4 Complex manifolds

In short, the complex manifold is the holomorphic analogue of differential manifolds.

Definition 15. *A holomorphic atlas on a differentiable manifold is an atlas* $\{(U_i, \varphi_i)\}$ *of the form*

We now extend the local theory to the global theory and determine the complex manifold by an integrable almost complex structure on its tangent space. We first extend the notion of an almost complex structure to a differentiable manifold X.

Definition 16. *An almost complex manifold is a differentiable manifold X together with a real tangent bundle endomorphism,*

$$J : TX \to TX, \quad J^2 = -id. \tag{D.56}$$

If an almost complex structure exists, then the real dimension of X is even.

Let (X, J) be such an almost complex manifold endowed with J, its complexifcation of tangent bundle $T_{\mathbb{C}}X = TX \otimes \mathbb{C}$ admits $\pm i$-eigenspace decomposition.

Proposition D.4.1 ($\pm i$-eigenspace decomposition of $T_{\mathbb{C}}X$). *Let (X, J) be above, then there exists a direct sum decomposition,*

$$T_{\mathbb{C}}X = T^{1,0}(X) \oplus T^{0,1}X, \tag{D.57}$$

We call the bundles $T^{1,0}X$ ($T^{0,1}(X)$ resp.) the holomorphic (anti-holomorphic resp.) tangent bundle of X.

Similar bidegree (or type) decomposition happens in dual bundles.

Definition 17. *Given (X, J), we define the complex vector bundles,*

$$T_{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X, \tag{D.58}$$

$$T^*_{\mathbb{C}}X = (T^*X)^{1,0} \oplus_{\mathbb{C}} (T^*X)^{0,1}. \tag{D.59}$$

Their sheaves of sections are denoted by $\mathcal{A}^k_{X,\mathbb{C}}$ and $\mathcal{A}^{p,q}_X$ respectively. The elements in $\mathcal{A}^{p,q}_X$, i.e. global sections, are called forms of type (p, q). We denote the natural projection as,

$$\Pi^k : \mathcal{A}^*(X) \to \mathcal{A}^k(X), \tag{D.60}$$

$$\Pi^{p,q} : \mathcal{A}^*(X) \to \mathcal{A}^{p,q}(X). \tag{D.61}$$

Then, we have the type decomposition for the k-form on X.

Proposition D.4.2. *Given (X, J), there exists a natural direct sum decomposition,*

$$\bigwedge\nolimits^k_{\mathbb{C}} X = \bigoplus_{p+q=k} \bigwedge\nolimits^{p,q} X, \tag{D.62}$$

$$\mathcal{A}^k_{X,\mathbb{C}} = \bigoplus_{p+q=k} \mathcal{A}^{p,q}_X, \tag{D.63}$$

and we have,

$$\overline{\bigwedge\nolimits^{p,q}} X = \bigwedge\nolimits^{q,q} X, \quad \overline{\mathcal{A}^{p,q}_X} = \mathcal{A}^{q,p}_X. \tag{D.64}$$

We also have the decomposition of the exterior differential d.

Definition 18. *For X almost complex manifold, Let $d : \mathcal{A}^k_{X,\mathbb{C}} \to \mathcal{A}^{k+1}_{X,\mathbb{C}}$ be the $\mathbb{C}$-linear extension of the usual exterior differential. Via projection we define,*

$$\partial : \mathcal{A}^{p,q}_X \to \mathcal{A}^{p+1,q}_X, \quad \partial = \Pi^{p+1,q} \circ d, \tag{D.65}$$

$$\bar{\partial} : \mathcal{A}^{p,q}_X \to \mathcal{A}^{p,q+1}_X, \quad \bar{\partial} = \Pi^{p,q+1} \circ d. \tag{D.66}$$

They also satisfy the usual properties of d and the Leibniz rule.

To induce a complex structure from an almost complex structure, we turn to the following proposition.

Proposition D.4.3. *Let X be an almost complex manifold, then the following are equivalent,*

1. *$d\alpha = \partial\alpha + \bar{\partial}\alpha$ for all $\alpha \in \mathcal{A}^*(X)$.*
2. *On $\mathcal{A}^{1,0}(X)$, $\Pi^{0,2} \circ d = 0$.*

Both conditions hold true if X is a complex manifold.

Definition 19. *An almost complex structure J on X is called integrable if the above conditions hold.*

There is another characterization of an integrable almost complex structure via the Lie bracket.

Proposition D.4.4. *An almost complex structure J is integrable if and only if the Lie bracket of vector fields preserves $T^{0,1}_X$, i.e. $[T^{0,1}_X, T^{0,1}_X] \subset T^{0,1}_X$.*

As a consequence, if J is an integrable almost complex structure, then $\partial^2 = \bar{\partial}^2 = 0$ and $\partial\bar{\partial} = -\bar{\partial}\partial$. Conversely, if $\bar{\partial}^2 = 0$, then J is integrable.

Then by the highly non-trivial theorem, the differentiable manifolds endowed with an integrable almost complex structure describe the same geometrical objects as the complex manifolds.

Theorem 3 (Newlander-Nierenberg)**.** *Any integrable almost complex structure is induced by a complex structure.*

D.5 Kähler manifold

The complex manifold endows with a complex structure J. If the complex manifold X also endows with an hermitian structure g, then we say X is an hermitian structure. (g, J) will induce a real $(1,1)$-form $\omega = g(J\cdot, \cdot)$ as in (D.17) called the fundamental form.

Our local theory extends by analogy, where we can split the exterior derivative $d = \partial + \bar{\partial}$ as in Proposition. D.3.3, and define the Lefschetz operator L_ω as in (D.22), the Hodge $\star_\omega$-operator as in (D.21) and the contraction operator Λ_ω as in (D.25). We retain the Lefschetz decomposition as in Proposition. D.2.3 and the linear algebra works if we pass from linear operator to differential operator.

On even dimensional complex manifold X, the adjoint of exterior derivative d is defined as,

$$d^\dagger = - \star \circ d \circ \star, \tag{D.67}$$

and analogously for its type decomposition $\partial^\dagger, \bar{p}^\dagger$. Then the integrability of J will yield the following lemma.

Lemma D.5.1. *If (X, g) is aN hermitian manifold, then $d^\dagger = \partial^\dagger + \bar{\partial}^\dagger$ and $(\partial^\dagger)^2 = (\bar{\partial}^\dagger)^2 = 0$.*

The Laplacian operator and its type decomposition are given by,

$$\Delta = d^\dagger d + dd^\dagger, \quad \Delta_\partial = \partial^\dagger \partial + \partial\partial^\dagger, \quad \Delta_{\bar{\partial}} = \bar{\partial}^\dagger \bar{\partial} + \bar{\partial}\bar{\partial}^\dagger. \tag{D.68}$$

Definition 20. *A Kähler structure is an hermitian structure g for which the associated fundamental form ω is closed, $d\omega = 0$. If X endows a Kähler structure, then it is called a Kähler manifold and ω is called the Kähler form.*

The Kähler condition and the local linear algebra yield the following key result.

Proposition D.5.1 (Kähler identities)**.** *Let X be a complex manifold endowed with a Kähler metric g, then the following identities holds,*

$$[\bar{p}, L] = [\partial, L] = 0, \tag{D.69}$$

$$[\bar{p}^\dagger, \Lambda] = [\partial^\dagger, \Lambda] = 0, \tag{D.70}$$

$$[\bar{\partial}^\dagger, L] = i\partial, \tag{D.71}$$

$$[\partial^\dagger, L] = -i\bar{\partial} \tag{D.72}$$

$$[\Lambda, \bar{\partial}] = -i\partial^\dagger, \tag{D.73}$$

$$[\Lambda, \partial] = i\bar{\partial}^\dagger, \tag{D.74}$$

$$\Delta_\partial = \Delta_{\bar{\partial}} = \frac{1}{2}\Delta, \tag{D.75}$$

and Δ *commutes with* $\star$, ∂, $\bar{\partial}$, $\partial^\dagger$, $\bar{\partial}^\dagger$, L *and* Λ.