On a Completion of Cohomological Functors Generalizing Tate Cohomology

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Abstract

Viewing group cohomology as a so-called cohomological functor, G. Mislin has generalized Tate cohomology from finite groups to all discrete groups by defining a completion for cohomological functors in his paper "Tate Cohomology for Arbitrary Groups via Satellites". We construct a Mislin completion for any cohomological functor whose domain category is an abelian category with enough projectives and whose codomain category is an abelian category in which all countable direct limits exist and are exact. This takes Tate cohomology to settings where it has never been introduced such as in condensed mathematics. Through the latter, one can define Tate cohomology for any topological group all whose points are closed. More specifically, we generalize four constructions of Mislin completions from the literature, prove that they yield isomorphic cohomological functors and provide explicit formulae for their connecting homomorphisms. For any morphism in the domain category we develop formulae for the induced morphism in each degree of the Mislin completion in terms of each construction. As their main feature, Mislin completions of Ext-functors detect finite projective dimension of objects in the domain category. Moreover, we establish a version of dimension shifting, an Eckmann-Shapiro result as well as Yoneda and external products.

Lay Summary

One of the simplest algebraic structures, a group is a set (an accumulation of items) together with a certain operation. For example: the integers with addition, the symmetries of an equilateral triangle. Homology and cohomology are algebraic tools that can discern different geometric objects by looking at their "holes". For example, one can discern a ball from a donut using them. They detect that a loop of string has a 1-dimensional hole in the middle, that a soap bubble has a 2-dimensional hole and that other objects have higher dimensional holes. There are counterparts for groups, so group homology and group cohomology are tools that can discern different groups. Tate cohomology unites both group homology and cohomology. We generalise Tate cohomology to settings where it has never been introduced. As its main feature, our generalisation detects whether an object has "holes" of arbitrarily high dimension.

Preface

This dissertation is original, unpublished, independent work by the author, Max Gheorghiu.

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Acknowledgements

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Chapter 1

Introduction

Because Tate cohomology is the main motivation of this work, we first outline *Tate cohomology* for a finite group G based on [1, p. 78–79]. For any module M over the group ring $\mathbb{Z}[G]$ and any integer $n \in \mathbb{Z}$ there is a Tate cohomology group $\widehat{H}^n(G, M)$. Note that $\widehat{H}^n(G, M) = H^n(G, M)$ for $n \geq 1$ and $\widehat{H}^n(G, M) = H_{-n-1}(G, M)$ for $n \leq -2$, whence Tate cohomology unites both group cohomology and group homology. According to [2, p. 136], Tate cohomology of G satisfies *dimension shifting* and possesses cup products. The former means that for any M there are $\mathbb{Z}[G]$-modules M_*, M^* such that $\widehat{H}^{n-1}(G, M_*) \cong \widehat{H}^n(G, M)$ and $\widehat{H}^{n+1}(G, M^*) \cong \widehat{H}^n(G, M)$ for every $n \in \mathbb{Z}$. Thus, if one desires to establish a property for every degree $\widehat{H}^n(G, -)$ of $\widehat{H}^\bullet(G, -)$, then one only needs to establish it for a carefully chosen degree n. If N is a $\mathbb{Z}[G]$-module and $m \in \mathbb{Z}$ another integer, then a *cup product* is a homomorphism of the form

$$\smile : \widehat{H}^n(G, M) \otimes \widehat{H}^m(G, N) \to \widehat{H}^{m+n}(G, M \otimes N)$$

where the tensor product $M \otimes N$ is a $\mathbb{Z}[G]$-module endowed with the diagonal action of G. Importantly for us, Tate cohomology for G *vanishes on projectives* according to [1, p. 79], meaning that $\widehat{H}^n(G, P) = 0$ for any projective $\mathbb{Z}[G]$-module P and $n \in \mathbb{Z}$. Then complete cohomology generalizes Tate cohomology for finite groups and was introduced for all discrete groups by G. Mislin, P. Vogel, D. J. Benson and J. F. Carlson using different constructions all yielding isomorphic cohomology groups [3, p. 196–197].

The bulk of this paper consists on the one hand in generalising these constructions to a previously unknown extent. On the the other hand, we develop for each construction explicit formulae for connecting homomorphisms and induced morphisms that have not been covered in the literature. In particular, since our generalisation takes Tate cohomology to condensed mathematics, one can construct Tate cohomology for any topological group in which every point is closed, meaning for any $T1$ *topological group*. More specifically, *condensed mathematics* is a novel theory developed by D.

Clausen and P. Scholze in 2018 [4]. In [5], P. Scholze writes that he wants "to make the strong claim that in the foundations of mathematics, one should replace topological spaces with condensed sets". Independently of whether this claim turns out to be true, it is a very promising tool for the study of continuous cohomology of topological groups and the study of representations of topological groups to topological abelian groups [4, p. 6]. It also serves as a foundation for a very general form of analytic geometry [6, p. 6]. More fundamentally, it provides a unified approach for studying topological groups, rings and modules [4, p. 6]. We end this work by showing that complete cohomology detects whether a group has finite cohomological dimension, by establishing a version of dimension shifting, an Eckmann-Shapiro result as well as Yoneda and external products.

Let us provide more detail on the nature of our constructions. As in the case of G. Mislin's work [7], the notion of a cohomological functor is essential. More specifically, if $\mathcal{C}$, $\mathcal{D}$ denote two abelian categories, then a family of additive functors $(T^n : \mathcal{C} \to \mathcal{D})_{n\in\mathbb{Z}}$ is a *cohomological functor* if there are so-called *connecting homomorphisms* $(\delta^n : T^n \to T^{n+1})_{n\in\mathbb{Z}}$ satisfying two (natural) axioms [3, p. 201–202]. In particular, if G is a discrete group, R a discrete ring and A a discrete R-module, then setting $H^n_R(G, -) = 0$ and $\mathrm{Ext}^n_R(A, -) = 0$ for $n < 0$ renders group cohomology and Ext-functors into cohomological functors [3, p. 201], [7, p. 295]. We refer the reader unfamiliar with abelian categories to the account in [8, p. 249–257], [9, Tag 00ZX] or [10, Section A.4]. G. Mislin defined *complete cohomology* $\widehat{H}^\bullet_R(G, -)$ as a specific completion of ordinary group cohomology $H^\bullet_R(G, -)$ in his paper [7]. Following the convention from [3, p. 197/202], we call such a completion of a cohomological functor a Mislin completion. Formally, a *Mislin completion* of $T^\bullet : \mathcal{C} \to \mathcal{D}$ is a cohomological functor $\widehat{T}^\bullet : \mathcal{C} \to \mathcal{D}$ together with a morphism $\Phi^\bullet : T^\bullet \to \widehat{T}^\bullet$ such that $\widehat{T}^n(P) = 0$ for any projective $P \in \mathrm{obj}(\mathcal{C})$ and $n \in \mathbb{Z}$ and such that any morphism $T^\bullet \to V^\bullet$ to a cohomological functor $V^\bullet : \mathcal{C} \to \mathcal{D}$ also vanishing on projectives factors uniquely through $\Phi^\bullet$ [7, Definition 2.1]. By its universal property, any Mislin completion is unique up to isomorphism [3, p. 202].

We generalize two constructions of Mislin completions due to G. Mislin [7] under the assumption that the domain category $\mathcal{C}$ has enough projective objects and that in the codomain category $\mathcal{D}$ all countable direct limits exist and are exact (Lemma 3.0.2 and Theorem 4.3.2). *Direct limits* are a specific kind of colimits [11, p. 14] and are vital to these constructions. In doing so, we demonstrate that there are countably many distinct constructions of Mislin completions (Lemma 4.4.5). If Ab denotes the *category of* *abelian groups*, then we generalize P. Vogel's [12] as well as D. J. Benson and J. F. Carlson's construction [13] mentioned above so that they give rise to the Mislin completion of (unenriched) Ext-functors $\mathrm{Ext}^{\bullet}_{\mathcal{C}}(A,-) : \mathcal{C} \to \mathrm{Ab}$ that we call *completed (unenrched) Ext-functors* (Theorem 5.3.2 and Corollary 6.4.4). Using the last construction, we prove for the canonical morphism $\Phi^{\bullet} : \mathrm{Ext}^{\bullet}_{\mathcal{C}}(A,-) \to \widehat{\mathrm{Ext}}^{\bullet}_{\mathcal{C}}(A,-)$ of the Mislin completion that its terms Φ^n fit into a long exact sequence relating three distinct cohomological functors (Lemma 7.5.2). Especially in the context of condensed mathematics, we also consider Hom-functors of the form $\mathrm{Hom}_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ called *enriched Hom-functors* where the category $\mathcal{D}$ does not need to be Ab and the corresponding *enriched Ext-functors* $\mathrm{Ext}^{\bullet}_{\mathcal{C}}(A,-) : \mathcal{C} \to \mathcal{D}$ together with their *completed enriched Ext-functors* $\widehat{\mathrm{Ext}}^{\bullet}_{\mathcal{C}}(A,-)$. For any morphism $f : M \to N$ in $\mathcal{C}$ we develop formulae for the induced morphism $\widehat{T}^n(f) : \widehat{T}^n(M) \to \widehat{T}^n(N)$ of the Mislin completion in terms of each construction (Proposition 3.0.4, Proposition 4.2.1, Definition 5.1.4, Definition 6.1.2). Using each construction, we provide explicit formulae for the connecting homomorphisms of $\widehat{T}^{\bullet}$. (Lemma 3.0.3, Definition 4.2.2, Lemma 5.3.1, Definition 6.2.2).

In order to present a key feature of Mislin completions, we remind the reader that group cohomology can be constructed as a specific Ext-functor. More specifically, if $\mathcal{C}_{R,G}$ is the respective category of module objects over the group ring (object) of G over R, one can define $H^n_R(G,-) := \mathrm{Ext}^n_{\mathcal{C}_{R,G}}(R,-)$ where G acts trivially on R. We say that an object has *finite projective dimension* if it admits a projective resolution of finite length [2, p. 152]. The group (object) G has *finite cohomological dimension* over R if the module object R has finite projective dimension in $\mathcal{C}_{R,G}$ [2, p. 184–185]. First we obtain

Lemma 1.0.1 *(= Lemma 7.1.1)*

1. *Assume that $T^{\bullet} : \mathcal{C} \to \mathcal{D}$ is a cohomological functor where $\mathcal{C}$ has enough projectives and in $\mathcal{D}$ all countable direct limits exist and are exact. If $M \in \mathrm{obj}(\mathcal{C})$ has finite projective dimension, then $\widehat{T}^n(M) = 0$ for every $n \in \mathbb{Z}$. In particular, if every object in $\mathcal{C}$ has finite projective dimension such as in a category of modules over a ring of finite global dimension, then $\widehat{T}^{\bullet} = 0$ for any cohomoological functor $T^{\bullet}$.*

2. *If one considers* **enriched** *Ext-functors $\mathrm{Ext}^n_{\mathcal{C}}(A,-) : \mathcal{C} \to \mathcal{D}$ with $A \in \mathrm{obj}(\mathcal{C})$ of finite projective dimension, then $\widehat{\mathrm{Ext}}^n_{\mathcal{C}}(A,-) = 0$ for every $n \in \mathbb{Z}$. In particular, complete cohomology $\widehat{H}^{\bullet}_R(G,M)$ vanishes if G has finite cohomological dimension or M has finite projective di-*

mension.

Then the main feature of completed Ext-functors is the following theorem which generalizes Lemma 4.2.4 from [3]:

Theorem 1.0.2 *(= Theorem 7.1.2) If* $\widehat{\mathrm{Ext}}^{\bullet}_{\mathcal{C}}(A,-) : \mathcal{C} \to \mathrm{Ab}$ *denote completed unenriched Ext-functors for* $A \in \mathrm{obj}(\mathcal{C})$*, then the following are equivalent.*

1. *The object* A *has finite projective dimension.*
2. $\widehat{\mathrm{Ext}}^{n}_{\mathcal{C}}(A,-) = \widehat{\mathrm{Ext}}^{n}_{\mathcal{C}}(-,A) = 0$ *for any* $n \in \mathbb{Z}$*.*
3. $\widehat{\mathrm{Ext}}^{0}_{\mathcal{C}}(A,A) = 0$*.*

In particular, the zeroeth complete cohomology group detects whether a group has finite cohomological dimension.

We provide a (partial) version of dimension shifting for Mislin completions.

Theorem 1.0.3 *(= Theorem 7.2.3) Let* $T^{\bullet} : \mathcal{C} \to \mathcal{D}$ *be a cohomological functor where* $\mathcal{C}$ *has enough projectives and in* $\mathcal{D}$ *all countable direct limits exist and are exact. Then for every* $M \in \mathrm{obj}(\mathcal{C})$ *there is* $M^{*} \in \mathrm{obj}(\mathcal{C})$ *such that* $\widehat{T}^{n+1}(M^{*}) \cong \widehat{T}^{n}(M)$ *for every* $n \in \mathbb{Z}$*. In addition, if there is a monomorphism* $f : M \to N$ *in* $\mathcal{C}$ *with* $\widehat{T}^{k}(N) = 0$ *for every* $k \in \mathbb{Z}$*, then* $\widehat{T}^{n-1}(\mathrm{Coker}(f)) \cong \widehat{T}^{n}(M)$*.*

The conditions of this theorem are satisfied for instance if $\mathcal{C}$ is the category of discrete R-modules for a discrete commutative ring R and G is a discrete group together with a finite index subgroup of finite cohomological dimension. Another instance is if $\mathcal{C}$ is the category of profinite S-modules for a commutative profinite ring S and if K is a profinite group with an open subgroup of finite cohomological dimension. (Example 7.2.4). Let us explain the terminology involved in the latter example. An inverse limit is a specific type of limit [14, p. 12]. A *profinite group* is a topological group that can be equivalently defined as an inverse system of discrete finite groups, as a closed subgroup of a cartesian product of finite discrete groups endowed with the product topology [14, Corollary 1.2.4] or as a Galois group endowed with the Krull topology [14, Theorem 3.3.2]. *Profinite spaces, rings, modules etc.* are defined analogously [11, p. 1]. The *completed group ring* $S[[K]]$ is a profinite ring that is a profinite version of a (discrete) group ring [11, p. 171]. We note that the category of profinite $S[[K]]$-modules has enough projectives [15, p. 353]. Lastly, we follow the convention in [16, p. 235] that the cohomology groups $H^{\bullet}_{S}(K,M)$ are $U(S)$-modules where $U(S)$ denotes the ring S without its topology. The reader is referred to [11] and [14] for more background on profinite groups and to [17] and [15] for sources specializing on cohomology of profinite groups.

Moving back to greater generality, we recall that any $M \in \mathrm{obj}(\mathcal{C}_{R,H})$ can be turned into an object in $\mathcal{C}_{R,G}$ by induction $\mathrm{Ind}_H^G(M)$ and coinduction $\mathrm{Coind}_H^G(M)$. Contrarily, any $M \in \mathrm{obj}(\mathcal{C}_{R,G})$ can be turned into an object in $\mathcal{C}_{R,H}$ by restriction $\mathrm{Res}_H^G(M)$. Under specific circumstances we can retrieve an Eckmann-Shapiro type result.

Lemma 1.0.4 *(= Lemma 7.2.1)*

1. *If both $\mathrm{Res}_H^G(-)$ and $\mathrm{Coind}_H^G(-)$ are exact and preserve projective objects, then $\widehat{\mathrm{Ext}}^n_{\mathcal{C}_{R,H}}(\mathrm{Res}_H^G(A),B) \cong \widehat{\mathrm{Ext}}^n_{\mathcal{C}_{R,G}}(A,\mathrm{Coind}_H^G(B))$ as unenriched completed Ext-functors for every $n \in \mathbb{Z}$, $A \in \mathrm{obj}(\mathcal{C}_{R,G})$ and $B \in \mathrm{obj}(\mathcal{C}_{R,H})$. In particular, in the case where $A = R$ we have $\widehat{H}^n_R(H,B) \cong \widehat{H}^n_R(G,\mathrm{Coind}_H^G(B))$.*
2. *If $\mathrm{Ind}_H^G(-)$ and $\mathrm{Res}_H^G(-)$ are exact functors and preserve projectives, then $\widehat{\mathrm{Ext}}^n_{\mathcal{C}_{R,G}}(\mathrm{Ind}_H^G(A),B) \cong \widehat{\mathrm{Ext}}^n_{\mathcal{C}_{R,G}}(A,\mathrm{Res}_H^G(B))$ unenriched completed Ext-functors for every $n \in \mathbb{Z}$, $A \in \mathrm{obj}(\mathcal{C}_{R,H})$ and $B \in \mathrm{obj}(\mathcal{C}_{R,G})$.*

Similar to before, the conditions of this lemma are satisfied by the category of discrete R-modules for a discrete commutative ring R and by a discrete group G together with a finite index subgroup. Or by the category of profinite S-modules for a commutative profinite ring S and by a profinite group K with an open subgroup (Example 7.2.2). Moving forward, we can establish two cohomology products for completed unenriched Ext-functors as in [13, p. 110].

Theorem 1.0.5 *(= Theorem 7.4.1) Let $\widehat{\mathrm{Ext}}^{\bullet}_{\mathcal{C}}(A,-) : \mathcal{C} \to \mathrm{Ab}$ denote unenriched completed Ext-functors and let $B, C, D \in \mathrm{obj}(\mathcal{C})$.*

1. *If $\otimes$ denotes the tensor product in Ab, then for every $m,n \in \mathbb{Z}$ Yoneda products*

$$\circ : \widehat{\mathrm{Ext}}^n_{\mathcal{C}}(B,C) \otimes \widehat{\mathrm{Ext}}^m_{\mathcal{C}}(A,B) \to \widehat{\mathrm{Ext}}^{m+n}_{\mathcal{C}}(A,C)$$

   can be defined.

2. *Let $A_\bullet$, $B_\bullet$, $C_\bullet$, $D_\bullet$ be projective resolutions and let $\otimes_{\mathcal{C}} : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ be a bi-additive functor such that $(h \otimes_{\mathcal{C}} i) \circ (f \otimes_{\mathcal{C}} g) = (h \circ f) \otimes_{\mathcal{C}} (i \circ g)$ for*

any morphisms f, g, h, i for which the compositions $h \circ f$ and $i \circ g$ are defined. If $k \in \mathbb{N}_0$ and $\widetilde{E}_k$ denotes any of $\widetilde{A}_k$, $\widetilde{B}_k$, $\widetilde{C}_k$ or $\widetilde{D}_k$, then we impose that $\widetilde{E}_k \otimes_{\mathcal{C}} - : \mathcal{C} \to \mathcal{C}$ as well as $- \otimes_{\mathcal{C}} \widetilde{E}_k$ is an exact functor and that $P \otimes_{\mathcal{C}} \widetilde{E}_k$ as well as $\widetilde{E}_k \otimes_{\mathcal{C}} P$ is projective for any projective $P \in \mathrm{obj}(\mathcal{C})$. Then external products

$$\vee : \widehat{\mathrm{Ext}}_{\mathcal{C}}^{m}(A, B) \otimes \widehat{\mathrm{Ext}}_{\mathcal{C}}^{n}(C, D) \to \widehat{\mathrm{Ext}}_{\mathcal{C}}^{m+n}(A \otimes_{\mathcal{C}} C, B \otimes_{\mathcal{C}} D)$$

can be defined.

The conditions of this theorem are fulfilled for example by the category of discrete $R[G]$-modules for a discrete group G and a principal ideal domain R. Moreover, the restriction of A, B, C, D to R-modules needs to be projective. Analogously, the conditions are satisfied by the category of profinite $S[[K]]$-modules for a profinite group K and profinite commutative ring S with a unique maximal open ideal where the restriction of A, B, C, D to R-modules needs to be projective (Example 7.4.3). In the latter case, there is a profinite tensor product $\otimes_S$ defined for profinite $S[[K]]$-modules [11, p. 177/191].

Moreover, complete cohomology generalizes *Tate-Farrell cohomology.* More specifically, F. T. Farrell generalized Tate cohomology of finite groups to discrete groups having a finite index subgroup of finite cohomological dimension in his paper [18] by using so-called complete resolutions. A *complete resolution* of an object A in $\mathcal{C}$ is a particular acyclic chain complex $(\overline{A}_n)_{n \in \mathbb{Z}}$ of projective objects in $\mathcal{C}$ that agrees with a projective resolution of M in sufficiently high degree [19, Definition 1.1]. Following [18, p. 158], we define the *Tate-Farrell Ext-functor* $\overline{\mathrm{Ext}}_{\mathcal{C}}^{\bullet}(A, B)$ as the cohomology of the cochain complex $\mathrm{Hom}_{\mathcal{C}}(\overline{A}_{\bullet}, B)$ where we the Hom-functor is unenriched. Then we show

Lemma 1.0.6 *(= Lemma 7.5.6) The Tate-Farrell Ext-functors $\overline{\mathrm{Ext}}_{\mathcal{C}}^{\bullet}(A, -)$ are isomorphic to the completed unenriched Ext-functors $\widehat{\mathrm{Ext}}_{\mathcal{C}}^{\bullet}(A, -)$ as cohomological functors.*

Using F. T. Farrell's work, P. Symonds deduces in [20, p. 34] that Tate-Farrell cohomology exists for every profinite group having an open subgroup of finite cohomological dimension. In particular, complete cohomology for profinite groups generalizes P. Symonds Tate-Farrell cohomology taking coefficients in profinite modules (Example 7.5.7).

Let us mention a few examples. In [21], P. Symonds computes the Tate-Farrell cohomology of the Morava stabilizer group S_{p-1} with coefficients in the moduli space E_{p-1} for odd primes p. However, there are groups for which their complete cohomology cannot be calculated by a complete resolution. For instance, a free abelian group of countable rank $\bigoplus_{n\in\mathbb{N}}\mathbb{Z}$, $GL_n(K)$ for K a subfield of the algebraic closure of $\mathbb{Q}$, the Thompson group F and the free product $\ast_{n\in\mathbb{N}}\mathbb{Z}^n$ do not admit complete resolutions [19, Example 5.3], [22, p. 119–120]. By [7, p. 297–298], the cohomology of the first three examples is isomorphic to their complete cohomology while all cohomology groups of the last group are distinct from its complete cohomology groups [23, p. 432]. The zeroeth complete cohomology group of this last example is calculated in [22, Corollary 2.4 and Theorem A]. In [24], F. Dembegioti calculates the zeroeth complete cohomology group of a class of discrete polycyclic groups, but also explains why one cannot determine a general formula calculating the zeroeth complete cohomology groups for all polycyclic groups. Lastly, if p is a prime number, then *pro-p groups* are a class of profinite groups for which there is a rich theory of their group cohomology. See for instance [17, Chapter I] or [15]. One example of a pro-p group that is also a pro-p ring are the *p-adic integers* $\mathbb{Z}_p$ which are studied in [14, Section 1.5]. These examples give rise to the following questions.

Question 1.0.7 *As in [11, Example 3.3.8(c)], let $\prod_{\mathbb{N}}\mathbb{Z}_p$ denote the free abelian pro-p group over $\mathbb{N}$. Is its cohomology $H^\bullet_{\mathbb{Z}_p}(\prod_{\mathbb{N}}\mathbb{Z}_p,-)$ isomorphic to its complete cohomology $\widehat{H}^\bullet_{\mathbb{Z}_p}(\prod_{\mathbb{N}}\mathbb{Z}_p,-)$ as a cohomological functor?*

Question 1.0.8 *By [25, p. 137–140], write $G=\coprod_{n\in\mathbb{N}\cup\{\infty\}}G_n$ for the following free pro-p product over the one-point compactification of the integers $\mathbb{N}\cup\{\infty\}$. Set $G_n=\mathbb{Z}_p^n$ for $n\in\mathbb{N}$ and $G_\infty=\{1\}$. Can one compute the zeroeth complete cohomology $\widehat{H}^0_{\mathbb{Z}_p}(G,A)$ for an $\mathbb{Z}_p[[G]]$-module A?*

Let us elaborate on the implementation of complete cohomology into condensed mathematics. If Pro denotes the category of profinite spaces, then, informally, a *condensed set* can be thought as a contravariant functor of the form $X:\mathrm{Pro}\to\mathrm{Set}$ satisfying certain compatibility conditions [4, p. 7]. Given any $T1$ topological space Y, one can turn it into a condensed set

$$\underline{Y}:\mathrm{Pro}^{\mathrm{op}}\to\mathrm{Set}, S\mapsto\{f:S\to Y\text{ continuous}\}$$

called the *condensate* of T [4, p. 15/16 and Proposition 1.7]. *Condensed groups* (resp. rings, modules etc.) are defined analogously [4, p. 7] and condensates of $T1$ topological groups are condensed groups (resp. rings, modules etc) [4, p. 8]. Because we define condensed sets and modules by using sheaves, they possess excellent category theoretic properties such as the existence of all small limits and colimits [4, p. 11/15] as well as enough projectives [6, p. 16]. Moreover, all products, direct sums and direct limits of condensed modules are exact [4, Theorem 2.2]. J. Anschütz and A.-C. Le Bras have defined condensed group cohomology over the condensed group ring $\mathcal{R}[\mathcal{G}]$ in [26, p. 5] since condensed abelian groups form an abelian category contrary to topological abelian groups [4, p. 6/11]. We note that for condensed $\mathcal{R}$-modules over a condensed ring $\mathcal{R}$ there is not only an unenriched Hom-functor $\mathrm{Hom}_{\mathrm{Cond}(\mathcal{R})}(-,-)$ mapping to abelian groups, but also an *internal Hom-functor* $\underline{\mathrm{Hom}}_{\mathrm{Cond}(\mathcal{R})}(-,-)$ mapping to condensed abelian groups [4, p. 13].

Theorem 1.0.9 *(= Theorem 7.6.1) Let $\mathcal{R}$ be a condensed ring and A a condensed $\mathcal{R}$-module. Then there are completed condensed unenriched Ext-functors $\widehat{\mathrm{Ext}}^{\bullet}_{\mathrm{Cond}(Mod(\mathcal{R}))}(A,-)$ and completed condensed internal Ext-functors of the form $\widehat{\underline{\mathrm{Ext}}}^{\bullet}_{\mathrm{Cond}(Mod(\mathcal{R}))}(A,-)$. If $\mathcal{G}$ is a condensed group, then complete condensed unenriched group cohomology $\widehat{H}^{\bullet}_{\mathcal{R}}(\mathcal{G},-)$ and complete condensed internal group cohomology $\widehat{\underline{H}}^{\bullet}_{\mathcal{R}}(\mathcal{G},-)$ can be defined.*

Corollary 1.0.10 *One can define group cohomology and thus complete cohomology for any $T1$ topological group G by taking its condensate $\underline{G}$.*

Specifically for profinite groups, it is more natural to consider so-called solid modules. In the same manner as a profinite group ring can be seen as a completion of an ordinary group ring [14, Proposition 7.1.2], one can introduce a notion of completion for condensed modules over a condensed ring $\mathcal{R}$. We call such completed condensed modules *solid* [4, p. 32/44] and $\mathcal{R}$ *analytic* if it admits a choice of solid modules [4, Definition 7.4].

Lemma 1.0.11 *(= Lemma 7.6.3) Let $\mathcal{A}$ be an analytic ring and M a solid $\mathcal{A}$-module. Then there are completed solid unenriched Ext-functors $\widehat{\mathrm{Ext}}^{\bullet}_{\mathrm{Solid}(\mathcal{A})}(M,-)$.*

For a profinite group G and for analytic rings $\mathcal{R}$ of a specific form, J. Anschütz and A.-C. Le Bras define in [26, p. 2/5] *solid group cohomology* $H^{\bullet}_{\mathrm{Solid}(\mathcal{R})}(\underline{G},-)$ of the condensate of G taking coefficients in solid $\mathcal{R}[\underline{G}]$-modules. They prove that the condensates $\underline{R}$ of some profinite rings R are analytic rings of this form [26, p. 2]. As any condensate $\underline{M}$ of a profinite or discrete R-module M is a solid $\underline{R}$-module for such rings [26, p. 4], one has $H^{\bullet}_{\mathrm{Solid}(\underline{R})}(\underline{G},\underline{M}) \cong H^{\bullet}_{R}(G,M)$ [26, Lemma 2.1]. We prove that there exists complete solid cohomology $\widehat{H}^{\bullet}_{\mathrm{Solid}(\mathcal{R})}(\underline{G},-)$ mapping to abelian groups and complete solid internal group cohomology $\widehat{\underline{H}}^{\bullet}_{\mathrm{Solid}(\mathcal{R})}(\underline{G},-)$ mapping to solid $\underline{\mathbb{Z}}$-modules whenever the analytic ring $\mathcal{R}$ is of the required form (Lemma 7.6.5). In particular, the following question arises.

Question 1.0.12 *When is the complete cohomology of a profinite group $\widehat{H}^{\bullet}_{R}(G,-)$ isomorphic to the complete solid cohomology of its condensate $\widehat{H}^{\bullet}_{\mathrm{Solid}(\underline{R})}(\underline{G},-)$ as a cohomological functor?*

We refer the reader to M. Land's notes [27] and P. Scholze's notes [4] for more detail on condensed mathematics. For results on condensed and solid group cohomology the reader is referred to J. Anschütz and A.-C. Le Bras' preprint [26].

Let us detail how our work compares to the literature. Recently, S. Guo and L. Liang have generalised in [28] the above constructions by G. Mislin [7], by D. J. Benson and J. F. Carlson [13] and by P. Vogel [12] for unenriched relative Ext-functors over any abelian category $\mathcal{C}$ with a special precovering subcategory $\mathcal{W}$ and only proved that the resulting completed relative Ext-functors $\widehat{\mathrm{Ext}}^{n}_{\mathcal{WC}}(A,-) : \mathcal{C} \to \mathrm{Ab}$ are naturally isomorphic for every $n \in \mathbb{Z}$. Although they prove that their completed Ext-functors $\widehat{\mathrm{Ext}}^{n}_{\mathcal{WC}}(A,-)$ form a cohomological functor, we do not know whether they also form a Mislin completion of the Ext-functors $\mathrm{Ext}^{\bullet}_{\mathcal{WC}}(A,-)$ (Question 6.4.8). On the other hand, A. Beligiannis and I. Reiten establish specific Mislin completions in their book on torsion theories [29]. In the elegant account from their Section IX.2, they consider an abelian category $\mathcal{C}$ containing a particular pair of subcategories $(\mathcal{X},\mathcal{Y})$, define a completed unenriched relative Ext-functor $\widehat{\mathrm{Ext}}^{\bullet}_{(\mathcal{X},\mathcal{Y})}(-,-)$ using a version of D. J. Benson and J. F. Carlson's construction from [13] and prove that $\mathrm{Ext}^{\bullet}_{\mathcal{X}\cap\mathcal{Y}}(A,-) \to \widehat{\mathrm{Ext}}^{\bullet}_{(\mathcal{X},\mathcal{Y})}(A,-)$ forms a Mislin completion for any $A \in \mathrm{obj}(\mathcal{C})$. Lastly, J. Hu et al. use in [30] a version of P. Vogel's construction from [12] to establish completed unenriched Ext-functors $\widehat{\mathrm{Ext}}^{\bullet}_{\mathcal{C}}(-,-)$ for an extriangulated category $\mathcal{C}$ where extriangulated categories are a generalisation of exact categories and triangulated categories. Among their applications, they provide a criterion for the validity of the Wakamatsu Tilting Conjecture. As a common application of completed unenriched Ext-functors, all above mentioned authors consider various notions of homological dimension for which they characterise when a homological dimension of an object in $\mathcal{C}$ is finite.

Apart from providing different applications, our contribution mainly consists in establishing a Mislin completion whenever we are given any cohomological functor between two particular abelian categories. For instance, we can define completed internal condensed Ext-functors which cannot be established through the previous frameworks (Remark 7.6.2). Moreover, in none of the above sources, connecting homomorphisms or induced morphisms of Mislin completions are constructed explicitly.

For an overview of various other generalisations of Tate cohomology and their applications, the reader is referred to M. Paganin's survey paper [31]. To mention two developments that are not contained in it, a version of completed Ext-functors for complexes of modules over a ring is defined in [32] and a generalisation of Tate cohomology in the language of spectra can be found in [33]. B. E. A. Nucinkis observed first in [34] that all construction can be performed dually by assuming that there are enough injectives instead of enough projectives. More specifically, if $\mathcal{C}$ is an abelian category with enough injectives and $\mathcal{D}$ is an abelian category in which all countable direct limits exist and are exact, then one can define for a cohomological functor $T^\bullet : \mathcal{C} \to \mathcal{D}$ a "*Nucinkis completion*" $T^\bullet \to \widetilde{T}^\bullet$ such that $\widetilde{T}^\bullet$ vanishes on injectives and is universal with respect to this property. For Ext-functors, this completion takes the form $\mathrm{Ext}^\bullet_{\mathcal{C}}(-, A) \to \widetilde{\mathrm{Ext}}^\bullet_{\mathcal{C}}(-, A)$. S. Guo and L. Liang perform all their constructions of completed unenriched relative Ext-functors in a dual manner. A. Beligiannis and I. Reiten provide in [29, Theorem IX.4.4] a criterion for when $\widehat{\mathrm{Ext}}^\bullet_{\mathcal{C}}(-, -)$ is isomorphic to $\widetilde{\mathrm{Ext}}^\bullet_{\mathcal{C}}(-, -)$ as well as S. Hu et al. in [35, Theorem 4.4].

After outlining four constructions of Mislin completions in Chapter 2, we prove in the subsequent four chapters that they result in isomorphic cohomological functors and provide explicit formulae of their connecting homomorphisms and induced morphisms. More specifically, in Chapter 3 we demonstrate that one construction in [7] by G. Mislin gives rise to Mislin completions. In Chapter 4 we prove that another construction of a cohomological functor from [7] is isomorphic to the former. Thereby we deduce that there are countably many distinct constructions of Mislin completions. Then in Chapter 5 we show that P. Vogel's construction in [12] gives rise to a cohomological functor that is isomorphic to the one from the latter construction by G. Mislin. In Chapter 6 we prove that the cohomological functor resulting from D. J. Benson and J. F. Carlson's construction in [13] is also isomorphic to the one from the latter construction by G. Mislin. Lastly, in Chapter 7, we establish various properties of completed Ext-functors and more generally of Mislin completions such as detection of finite projective dimension, dimension shifting, an Eckmann-Shapiro result and Yoneda and external products.

We also indicate how our completed Ext-functors generalize Tate-Farrell cohomology and how one can introduce completed Ext-functors in condensed mathematics.

1.1 Notation and terminology

We adopt the convention that the natural numbers $\mathbb{N}$ include 1, but do not include 0. We write $\mathbb{N}_0$ for $\mathbb{N} \cup \{0\}$. Moreover, we adopt B. Poonen's convention from [36] that a ring is an abelian group together with a totally associative product, meaning a binary associative relation admitting an identity element. In particular, ring homomorphisms are understood to map identity elements to identity elements. Lastly, we use the symbol $\varinjlim$ only to denote direct limits. If $\mathcal{D}$ is a category, then $\varinjlim_{\mathcal{D}}$ denotes a direct limit in this category and if I is a directed set, then $\varinjlim_{i \in I}$ denotes a direct limit indexed over I. We write $\varinjlim_{k \in \mathbb{N}}(M_k, \mu_k)$ if $\mu_k : M_k \to M_{k+1}$ are the morphisms giving rise to the direct limit.

Chapter 2

Outline of constructions

To showcase the relevant constructions of Mislin completions, let us axiomatically define cohomological functors. For two abelian categories $\mathcal{C}$, $\mathcal{D}$ we call a family of additive functors $(T^n : \mathcal{C} \to \mathcal{D})_{n\in\mathbb{Z}}$ a cohomological functor if it satisfies the following two axioms [3, p. 201–202].

Axiom 2.0.1 *For every short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{C}$ there is a natural connecting homomorphism $\delta^n : T^n(C) \to T^{n+1}(A)$ for any integer $n \in \mathbb{Z}$.*

Being natural means in this context that for every commuting diagram in $\mathcal{C}$

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, "g"] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

with exact rows there is a commuting diagram

$$\begin{tikzcd} T^n(A) \arrow[r, "\delta^n"] \arrow[d, "T^n(g)"] & T^{n+1}(A) \arrow[d, "T^{n+1}(f)"] \\ T^n(A') \arrow[r, "\delta^n"] & T^{n+1}(C') \end{tikzcd}$$

in $\mathcal{D}$.

Axiom 2.0.2 *For every short exact sequence $0 \to A \xrightarrow{\iota} B \xrightarrow{\pi} C \to 0$ in $\mathcal{C}$ there is a long exact sequence*

$$\dots \xrightarrow{T^{n-1}(\pi)} T^{n-1}(C) \xrightarrow{\delta^{n-1}} T^n(A) \xrightarrow{T^n(\iota)} T^n(B) \xrightarrow{T^n(\pi)} T^n(C) \xrightarrow{\delta^n} \dots$$

Assume that $(T^\bullet, \delta^\bullet)$ and $(U^\bullet, \varepsilon^\bullet)$ are cohomological functors from $\mathcal{C}$ to $\mathcal{D}$. Then a family of natural transformations $(\nu^n : T^n \to U^n)_{n\in\mathbb{Z}}$ is a morphism of cohomological functors if it satisfies the axiom [3, p. 202]

Axiom 2.0.3 *For every short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{C}$ and $n \in \mathbb{Z}$, the following square commutes:*

$$\begin{tikzcd} T^n(C) \arrow[r, "\delta^n"] \arrow[d, "\nu^n"] & T^{n+1}(A) \arrow[d, "\nu^{n+1}"] \\ U^n(C) \arrow[r, "\varepsilon^n"] & U^{n+1}(A) \end{tikzcd}$$

Let us generalize G. Mislin's Definition 2.1 from [7].

Definition 2.0.4 (Mislin completion) *Let $(T^\bullet, \delta^\bullet)$ be a cohomological functor from $\mathcal{C}$ to $\mathcal{D}$. Then its Mislin completion is a cohomological functor $(\widehat{T}^\bullet, \widehat{\delta}^\bullet)$ from $\mathcal{C}$ to $\mathcal{D}$ together with a morphism $\nu^\bullet : T^\bullet \to \widehat{T}^\bullet$ satisfying the following two conditions:*
1. *For every $n \in \mathbb{Z}$ and every projective $P \in \mathrm{obj}(\mathcal{C})$ we have $\widehat{T}^n(P) = 0$.*
2. *If $(U^\bullet, \varepsilon^\bullet)$ is any cohomological functor vanishing on projectives, then each morphism $T^\bullet \to U^\bullet$ factors uniquely as $T^\bullet \xrightarrow{\nu^\bullet} \widehat{T}^\bullet \to U^\bullet$.*

Hence, if $(U^\bullet, \varepsilon^\bullet)$ is another Mislin completion of $(T^\bullet, \delta^\bullet)$, then there is an isomorphism $\mu^\bullet : \widehat{T}^\bullet \to U^\bullet$, meaning that $\mu^n(M) : \widehat{T}^n(M) \to U^n(M)$ is an isomorphism for any $n \in \mathbb{Z}$ and $M \in \mathrm{obj}(\mathcal{C})$ [3, p. 202]. This allows us to state G. Mislin's definition from [7, p. 297] in greater generality.

Definition 2.0.5 (Axiomatic, Mislin) *For any $A \in \mathrm{obj}(\mathcal{C})$ we extend the (enriched or unenriched) Ext-functors to cohomological functor by setting $\mathrm{Ext}^n_{\mathcal{C}}(A, -) = 0$ for $n < 0$ and define completed Ext-functors as the Mislin completion $(\widehat{\mathrm{Ext}}^\bullet_{\mathcal{C}}(A, -), \widehat{\delta}^\bullet)$. Analogously, if G is a group object in $\mathcal{C}$ and R a ring object, then we extend group cohomology to a cohomological functor $(H^\bullet_R(G, -), \delta^\bullet)$ by imposing $H^n_R(G, -) = 0$ for $n < 0$ and define complete cohomology as the Mislin completion $(\widehat{H}^\bullet_R(G, -), \widehat{\delta}^\bullet)$.*

To ensure that complete cohomology exists, we introduce *left satellite functors* to present a construction due to G. Mislin. As we assume that $\mathcal{C}$ has enough projective objects, there is for any $M \in \mathrm{obj}(\mathcal{C})$ a short exact sequence $0 \to K \to P \to M \to 0$ in $\mathcal{C}$ with P projective. For a cohomological functor $(T^\bullet, \delta^\bullet)$, we define the zeroeth left satellite functor of T^n as $S^0T^n := T^n$, the first left satellite functor as

$$S^{-1}T^n(M) := \mathrm{Ker}\big(T^n(K) \to T^n(P)\big)$$

and the k^{th} left satellite functor as $S^{-k}T^n := S^{-1}(S^{-k+1}T^n)$ for $k \geq 2$ [37, p. 36]. It is shown in [37, Section III.1] that left satellite functors do not depend on the choice of short exact sequence. Since they have been defined as kernels, it follows from Axiom 2.0.2 that $\delta^n : T^n \to T^{n+1}$ induces a morphism $\underline{\delta}^n : T^n(M) \to S^{-1}T^{n+1}(M)$ and therefore a morphism $S^{-k}\underline{\delta}^{n+k} : S^{-k}T^{n+k}(M) \to S^{-k-1}T^{n+k+1}(M)$ for any $k \in \mathbb{N}$ [3, p. 207–208]. Here, our assumption hits in that all countable direct limits exist in the codomain category $\mathcal{D}$ of $T^\bullet$.

Definition 2.0.6 *A partially ordered set $(I, \leq)$ is a directed set if for every $i, j \in I$ there is $k \in I$ such that $i, j \leq k$ [11, p. 1]. According to [11, p. 14], a diagram $\{D_i\}_{i\in I}$ in $\mathcal{D}$ indexed over a directed set is called a direct system in $\mathcal{D}$. More formally, I can be turned into a category whose objects are its elements and there is a unique morphism $i \to j$ whenever $i \leq j \in I$. Then a direct system is a covariant functor $I \to \mathcal{D}, i \mapsto D_i$. A direct limit $\varinjlim_{i\in I} D_i$ in $\mathcal{D}$ is a colimit of a direct system $\{D_i\}_{i\in I}$. Direct limits in $\mathcal{D}$ are called exact if for every direct system $\{0 \to A_i \to B_i \to C_i \to 0\}_{i\in I}$ of short exact sequence also*

$$0 \to \varinjlim_{i\in I} A_i \to \varinjlim_{i\in I} B_i \to \varinjlim_{i\in I} C_i \to 0$$

is a short exact sequence [9, Tag 079A].

Hence, we can extend G. Mislin's construction from [7, p. 293].

Definition 2.0.7 (Via satellite functors, Mislin) *The Mislin completion of a cohomological functor $(T^\bullet, \delta^\bullet)$ can be defined as*

$$\widehat{T}^n(M) := \varinjlim_{k\in\mathbb{N}_0} (S^{-k}T^{n+k}(M), S^{-k}\underline{\delta}^{n+k})$$

for any $M \in \mathrm{obj}(\mathcal{C})$ and $n \in \mathbb{Z}$. Accordingly,

$$\widehat{H}^n_R(G, M) := \varinjlim_{k\in\mathbb{N}_0} (S^{-k}H^{n+k}_R(G, M), S^{-k}\underline{\delta}^{n+k})$$

is a definition of complete cohomology.

In order that the above forms a cohomological functor, one needs to impose that all direct limits in $\mathcal{D}$ are exact. The reader might be aware that one can define cohomology of discrete groups and cohomology of profinite groups taking discrete coefficients by using that the respective category $\mathcal{C}$ has enough injectives [2, p. 61], [17, p. 9]. One could assume instead that

$\mathcal{C}$ has enough injectives and in the category $\mathcal{D}$ all countable inverse limits exist and are exact. One could then perform a construction via right satellite functors dual to the above. This completion of group cohomology would have a universal property as a Mislin completion, except that it would vanish on injective objects instead of projective ones. However, cohomology of discrete groups as well as of profinite groups taking discrete coefficients already vanishes on injectives [2, p. 61], [17, p. 9]. Thus, such a completion would not yield Tate cohomology whence we assume that $\mathcal{C}$ has enough projectives instead of enough injectives.

Notation 2.0.8 *For the rest of the thesis, $\mathcal{C}$ always denotes an abelian category with enough projectives and $\mathcal{D}$ an abelian category in which all countable direct limits exist and are exact.*

Let us go over to what we term the resolution construction that occurs in [38, Lemma B.3] and can be retrieved from page 299 in G. Mislin's paper [7]. If $(M_n)_{n\in\mathbb{N}_0}$ is a projective resolution of $M \in \operatorname{obj}(\mathcal{C})$, let us define $\widetilde{M}_0 := M$ and $\widetilde{M}_k := \operatorname{Ker}(M_{k-1} \to \widetilde{M}_{k-1})$ for $k \in \mathbb{N}$. This is called the k^{th} *syzygy* of $M_\bullet$ in the Gorenstein context [31, p. 89]. The choice of our notation is meant to reflect that our syzygies do not necessarily arise from a specific choice of projective resolution as in [3] and [7]. Since for every $k \in \mathbb{N}_0$ we have the short exact sequence $0 \to \widetilde{M}_{k+1} \to M_k \to \widetilde{M}_k \to 0$, there is a connecting homomorphism $\delta^{n+k} : T^{n+k}(\widetilde{M}_k) \to T^{n+k+1}(\widetilde{M}_{k+1})$ for every $n \in \mathbb{Z}$. Then the following definition makes it more apparent why Mislin completions vanish on projective objects.

Definition 2.0.9 (Resolutions, Mislin) *The Mislin completion of a cohomological functor $(T^\bullet, \delta^\bullet)$ can be defined as*

$$\widehat{T}^n(M) := \varinjlim_{k\in\mathbb{N}_0} (T^{n+k}(\widetilde{M}_k), \delta^{n+k})$$

for any $n \in \mathbb{Z}$ and $M \in \operatorname{obj}(\mathcal{C})$. Accordingly,

$$\widehat{H}^n_R(G, M) := \varinjlim_{k\in\mathbb{N}_0} (H^{n+k}_R(G, \widetilde{M}_k), \delta^{n+k})$$

is a definition of complete cohomology.

The next two constructions only give rise to completed unenriched Ext-functors. Let $A_\bullet$, $B_\bullet$ are projective resolutions of $A, B \in \operatorname{obj}(\mathcal{C})$ and let $\widetilde{f}_{n+k} : \widetilde{A}_{n+k} \to \widetilde{B}_k$ be a morphism for $n \in \mathbb{Z}$ and $k \in \mathbb{N}_0$ such that $n+k \geq 0$. Then we can write the commuting diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{A}_{n+k+1} & \longrightarrow & A_{n+k} & \longrightarrow & \widetilde{A}_{n+k} & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \widetilde{f}_{k+1}} & & \Big\downarrow{\scriptstyle f_{k+1}} & & \Big\downarrow{\scriptstyle \widetilde{f}_k} & & \\ 0 & \longrightarrow & \widetilde{B}_{k+1} & \longrightarrow & B_{k+1} & \longrightarrow & \widetilde{B}_k & \longrightarrow & 0 \end{array}$$

whose terms arise as follows. Since the bottom row is exact and the term A_{n+k} projective, there is a lift f_k of $\widetilde{f}_k$ making the right-hand square commute. Because $\widetilde{B}_{k+1} \to B_{k+1}$ is a kernel, there is a morphism $\widetilde{f}_{k+1}$ making the left-hand side commute. If $\mathrm{Hom}_{\mathcal{C}}(-,-)$ denotes the (unenriched) Hom-functor in $\mathcal{C}$, then $\mathrm{Hom}_{\mathcal{C}}(\widetilde{A}_{n+k}, \widetilde{B}_k)$ is an abelian group by virtue of $\mathcal{C}$ being an abelian category. We define $\mathcal{P}_{\mathcal{C}}(\widetilde{A}_{n+k}, \widetilde{B}_k)$ to be the subgroup of $\mathrm{Hom}_{\mathcal{C}}(\widetilde{A}_{n+k}, \widetilde{B}_k)$ consisting of all morphisms factoring through a projective and write the quotient as $[\widetilde{A}_{n+k}, \widetilde{B}_k]_{\mathcal{C}} := \mathrm{Hom}_{\mathcal{C}}(\widetilde{A}_{n+k}, \widetilde{B}_k)/\mathcal{P}_{\mathcal{C}}(\widetilde{A}_{n+k}, \widetilde{B}_k)$ [3, p. 203]. As in the case of modules over a ring covered by [3, p. 204], we prove that

$$\begin{aligned} t_{\widetilde{A}_{n+k},\widetilde{B}_k} &: [\widetilde{A}_{n+k}, \widetilde{B}_k]_{\mathcal{C}} \to [\widetilde{A}_{n+k+1}, \widetilde{B}_{k+1}]_{\mathcal{C}}, \\ &\widetilde{f}_k + \mathcal{P}_{\mathcal{C}}(\widetilde{A}_{n+k}, \widetilde{B}_k) \mapsto \widetilde{f}_{k+1} + \mathcal{P}_{\mathcal{C}}(\widetilde{A}_{n+k+1}, \widetilde{B}_{k+1}) \end{aligned}$$

is a well defined homomorphism. Through this we generalize the following definition from D. J. Benson and J. F. Carlson's paper [13, p. 109].

Definition 2.0.10 (naive construction, Benson & Carlson) *We can define the n^{th} completed unenriched Ext-functor as*

$$\widehat{\mathrm{Ext}}^n_{\mathcal{C}}(A, B) := \varinjlim_{k\in\mathbb{N}_0, n+k\geq 0} ([\widetilde{A}_{n+k}, \widetilde{B}_k]_{\mathcal{C}}, t_{\widetilde{A}_{n+k},\widetilde{B}_k})$$

for $n \in \mathbb{Z}$. If $R_\bullet$ is a projective $R[G]$-resolution of $R \in \mathrm{obj}(\mathcal{C}_{R,G})$, we can define complete unenriched cohomology as

$$\widehat{H}^n_R(G, B) := \varinjlim_{k\in\mathbb{N}_0, n+k\geq 0} ([\widetilde{R}_{n+k}, \widetilde{B}_k]_{\mathcal{C}}, t_{\widetilde{R}_{n+k},\widetilde{B}_k}) \,.$$

Lastly, we present what we call the hypercohomology construction of complete cohomology. We define the chain complex $(A'_n)_{n\in\mathbb{Z}}$ by $A'_n = A_n$ for $n \geq 0$ and $A'_n = 0$ for $n < 0$ and similarly $(B'_n)_{n\in\mathbb{Z}}$ [3, p. 209]. Define the *hypercohomology complex* $(\mathrm{Hyp}_{\mathcal{C}}(A'_\bullet, B'_\bullet)_n, d^n)_{n\in\mathbb{Z}}$ by having n-cochains $\mathrm{Hyp}_{\mathcal{C}}(A'_\bullet, B'_\bullet)_n = \prod_{k\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(A'_{k+n}, B'_k)$. To ease notation in the following, we view abelian groups as $\mathbb{Z}$-modules. If we denote by $a_n : A'_n \to A'_{n-1}$

and $b_n : B'_n \to B'_{n-1}$ the differentials induced from the respective projective resolution, we define for $n \in \mathbb{Z}$ the differential

$$\begin{aligned} d^n : \mathrm{Hyp}_{\mathcal{C}}(A'_\bullet, B'_\bullet)_n &\to \mathrm{Hyp}_{\mathcal{C}}(A'_\bullet, B'_\bullet)_{n+1} \\ (\varphi_{n+k})_{k\in\mathbb{Z}} &\mapsto (b_{k+1} \circ \varphi_{n+k+1} - (-1)^n \varphi_{n+k} \circ a_{n+k+1})_{k\in\mathbb{Z}} \end{aligned}$$

Let us define the *total complex* $\mathrm{Tot}_{\mathcal{C}}(A'_\bullet, B'_\bullet)_{n\in\mathbb{Z}}$ to be the subcomplex of $\mathrm{Hyp}_{\mathcal{C}}(A'_\bullet, B'_\bullet)_{n\in\mathbb{Z}}$ given by $\mathrm{Tot}_{\mathcal{C}}(A'_\bullet, B'_\bullet)_n = \bigoplus_{k\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(A'_{k+n}, B'_k)$. Now we define the *Vogel complex* $\mathrm{Vog}_{\mathcal{C}}(A'_\bullet, B'_\bullet)_{n\in\mathbb{Z}}$ to be the quotient complex $\mathrm{Hyp}_{\mathcal{C}}(A'_\bullet, B'_\bullet)_n / \mathrm{Tot}_{\mathcal{C}}(A'_\bullet, B'_\bullet)_n$ [3, p. 209]. By this, we generalize Definition 1.2 from F. Goichot's paper [12] where he attributes it to P. Vogel on page 39.

Definition 2.0.11 (Hypercohomology, Vogel) *For $n \in \mathbb{Z}$ we can define the n^{th} completed unenriched Ext-functor as*

$$\widehat{\mathrm{Ext}}^n_{\mathcal{C}}(A, B) := H^n((\mathrm{Vog}_{\mathcal{C}}(A'_\bullet, B'_\bullet)_k, d^k)_{k\in\mathbb{Z}}) .$$

We can thus define complete unenriched cohomology as

$$\widehat{H}^n_R(G, M) := H^n((\mathrm{Vog}_R(R'_\bullet, B'_\bullet)_k, d^k)_{k\in\mathbb{Z}}) .$$

Let us remark why it is unlikely that the previous two constructions could yield Mislin completions of more general enriched Ext-functors. For the naive construction, one aims to find a morphism

$$\mathrm{Hom}_{\mathcal{C}}(\widetilde{A}_{n+k}, \widetilde{B}_k) \to \mathrm{Hom}_{\mathcal{C}}(\widetilde{A}_{n+k+1}, \widetilde{B}_{k+1}) .$$

If one tries to lift along the morphisms induced by $A_{n+k} \to \widetilde{A}_{n+k}$ and $B_{n+k} \to \widetilde{B}_{n+k}$, one requires that $\mathrm{Hom}_{\mathcal{C}}(A_{n+k}, -)$ preserves epimorphisms. For unenriched Hom-functors, this role is exactly played by projective objects [10, Lemma 2.2.3]. For the hypercohomology construction we require that the coproduct $\mathrm{Tot}_{\mathcal{C}}(A'_\bullet, B'_\bullet)$ maps into the product $\mathrm{Hyp}_{\mathcal{C}}(A'_\bullet, B'_\bullet)$. Moreover, as we shall see in Section 6.1, the cohomology groups of the Vogel complex $\mathrm{Vog}_{\mathcal{C}}(A'_\bullet, B'_\bullet)$ correspond exactly to so-called almost chain maps modulo almost chain homotopy. However, this implies that the objects $\mathrm{Hom}_{\mathcal{C}}(A_p, B_q) \in \mathrm{obj}(\mathcal{D})$ describe morphisms of the form $A_p \to B_q$ in $\mathcal{C}$.

Chapter 3

The satellite functor construction

Satellite functors are treated in the third chapter of H. Cartan and S. Eilenberg's book on homological algebra [37]. By using morphisms of cohomological functors, G. Mislin constructs Mislin completions through satellite functors in the proof of Theorem 2.2 from [7] while P. H. Kropholler provides more explicit descriptions of the terms of such a Mislin completion on pages 206–208 of [3]. In this chapter we generalize G. Mislin and P. H. Kropholler's work that is performed only using modules over a ring. As we also aim to determine explicit formulae for the induced morphisms and for the connecting homomorphisms of the resulting Mislin completions, we provide full details on the satellite functor construction. In turn, these details are needed later to manufacture the induced morphisms and connecting homomorphisms of all other constructions of Mislin completions.

For the reader's convenience, we summarize relevant results on left satellite functors from the third chapter of [37]. They may be working explicitly in a category of modules over a ring, but A. Grothendieck has generalised their satellite functors in his Tôhoku paper [39, p. 140–143] and thus, their arguments pertain to our abelian categories. In the first section, they show that left satellite functors are well defined and additive functors. More specifically, let $f : C' \to C$ be a morphism in $\mathcal{C}$. As we assume that the abelian category $\mathcal{C}$ has enough projectives, one can take short exact sequences

$$0 \to M' \to P' \to C' \to 0 \quad \text{and} \quad 0 \to M \to P \to C \to 0$$

with P' and P projective. Using this and the fact that every monomorphism is normal in an abelian category, we deduce the existence of morphisms

$\overline{f}^* : P' \to P$ and $\overline{f}' : M' \to M$ rendering the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & P' & \longrightarrow & C' & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \overline{f}^*} & & \big\downarrow{\scriptstyle \overline{f}} & & \big\downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & M & \longrightarrow & P & \longrightarrow & C & \longrightarrow & 0 \end{array} \tag{3.0.1}$$

commutative. To ease future notation, let $T^\bullet : \mathcal{C} \to \mathcal{D}$ be a cohomological functor with connecting homomorphisms denote by $\delta^\bullet$. The morphism $T^n(\overline{f}^*) : T^n(M') \to T^n(M)$ induces the corresponding morphism between the kernels $S^{-1}T^n(f) : S^{-1}T^n(C') \to S^{-1}T^n(C)$. Pictorially,

$$\begin{array}{ccc} S^{-1}T^n(C') & \xrightarrow{\varepsilon^{-1}} & T^n(M') \\ \big\downarrow{\scriptstyle S^{-1}T^n(f)} & & \big\downarrow{\scriptstyle T^n(\overline{f}^*)} \\ S^{-1}T^n(C) & \xrightarrow{\varepsilon^{-1}} & T^n(M) \end{array}$$

where the morphisms ε^{-1} are the canonical monomorphisms from the kernels. Being independent of the choices of short exact sequences, the morphism $S^{-1}T^n(f)$ is well defined. In the same manner we define the morphisms $S^{-k}T^n(f) : S^{-k}T^n(C') \to S^{-k}T^n(C)$ for higher satellite functors. Thereafter, in the second section, they construct for every exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{C}$ and any $k \in \mathbb{N}_0$ a *connecting homomorphism* $\varepsilon^{-k-1} : S^{-k-1}T^n(C) \to S^{-k}T^n(A)$ in the following manner. If B is projective, then ε^{-k-1} is the monomorphism from the kernel $S^{-k-1}T^n(C)$. Otherwise let $0 \to M \to P \to C \to 0$ be a short exact sequence with P projective. As we have seen above, there are morphisms $h : P \to B$ and $h^* : M \to A$ extending the identity morphism $\mathrm{id}_C : C \to C$ making the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & P & \longrightarrow & C & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle h^*} & & \big\downarrow{\scriptstyle h} & & \big\downarrow{\scriptstyle \mathrm{id}_C} & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \end{array} \tag{3.0.2}$$

commute. Then $\varepsilon^{-k-1} : S^{-k-1}T^n(C) \to S^{-k}T^n(M) \xrightarrow{S^{-k}T^n(h^*)} S^{-k}T^n(A)$ is a connecting homomorphism where the first morphism is the monomorphism of the kernel $S^{-n-1}T^n(C)$. Moreover they show that the every connecting homomorphism is a natural transformation. Extending into the third section, they use these to construct the long exact sequence

$$\ldots \to S^{-k-1}T^n(C) \xrightarrow{\varepsilon^{-k-1}} S^{-k}T^n(A) \to S^{-k}T^n(B) \to S^{-k}T^n(C) \xrightarrow{\varepsilon^{-k}} \ldots$$

where the unlabelled morphisms are induced from the underlying short exact sequence. In the fifth section, they provide the following crucial result corresponding to Proposition 5.2 on page 46:

Lemma 3.0.1 *Let $T^\bullet$ and $U^\bullet$ be cohomological functors and assume that $\Phi^0 : T^0 \to U^0$ is a natural transformation. If for every short exact sequence $0 \to K \to P \to M \to 0$ with P projective and every $n \in \mathbb{N}$ the sequence involving the connecting homomorphism $0 \to U^{-n-1}(M) \to U^{-n}(K)$ is exact, then Φ^0 extends uniquely to a (partial) morphism of cohomological functors $(\Phi^n : T^n \to U^n)_{n\leq 0}$ only defined for $n \leq 0$. In particular, if also $T^\bullet$ has the property that the sequence $0 \to T^{-n-1}(M) \to T^{-n}(K)$ is exact for $n \in \mathbb{N}$ and Φ^0 is an equivalence, then the extension $\Phi^\bullet$ is an isomorphism.*

G. Mislin uses this lemma to perform his satellite functor construction on pages 295–296 in [7] which we generalize in the following. For any $m \in \mathbb{N}_0$ we can define the cohomological functor

$$T^k\langle m\rangle(M) := \begin{cases} S^{k-m}T^m(M) & \text{if } k < m \\ T^k(M) & \text{if } k \geq m \end{cases} \tag{3.0.3}$$

where the connecting homomorphisms are given by $\delta^k\langle m\rangle = \delta^k : T^k \to T^{k+1}$ for $k \geq m$ and $\delta^k\langle m\rangle = \varepsilon^{k-m} : S^{k-m}T^m \to S^{k-m+1}T^m$ for $k < m$. Consider the partial morphism $(\Phi^k_m := \mathrm{id} : T^k \to T^k\langle m\rangle)_{m\leq k}$ of cohomological functors defined only in degrees $m \leq k$. According to the previous lemma we can extend $\Phi^\bullet_m$ uniquely to a morphism of cohomological functors defined in all degrees. Using the same arguments we have for $m \leq n$ a unique morphism $\Phi^\bullet_{m,n} : T^\bullet\langle m\rangle \to T^\bullet\langle n\rangle$ that is equal to the identity morphism in any degree $n \leq k \in \mathbb{Z}$. Using the uniqueness of all these morphisms, we conclude for $m \leq n \leq o \in \mathbb{N}_0$ that $\Phi^\bullet_n = \Phi^\bullet_{m,n} \circ \Phi^\bullet_m$ and $\Phi^\bullet_{m,o} = \Phi^\bullet_{n,o} \circ \Phi^\bullet_{m,n}$. This yields a morphism to a direct system $T^\bullet \to (T^\bullet\langle m\rangle, \Phi^\bullet_{m,n})_{m\leq n\in\mathbb{N}_0}$. This is where countable direct limits $\varinjlim_{\mathcal{D}}$ in the codomain category $\mathcal{D}$ come into play. For every $n \in \mathbb{Z}$ we define the n^{th} term of the satellite functor construction as

$$\widehat{T}^n(M) := \varinjlim_{k\in\mathbb{N}_0} T^n\langle k\rangle(M)\,.$$

Unravelling the definitions and noting that $\{k \in \mathbb{N}_0 \mid k \geq n\}$ is cofinal in $\mathbb{N}_0$, we obtain

$$\begin{aligned} \widehat{T}^n(M) &= \varinjlim_{k\in\mathbb{N}_0, k\geq n} \left(S^{n-k}T^k(M), \Phi^n_{k,k+1}(M)\right) \\ &= \varinjlim_{k\in\mathbb{N}_0} \left(S^{-k}T^{n+k}(M), \Phi^n_{n+k,n+k+1}(M)\right). \end{aligned} \tag{3.0.4}$$

As $T^\bullet$ is a cohomological functor and left satellite functors are additive, each functor $S^{-k}T^{n+k}$ is additive. Since $\varinjlim_{\mathcal{D}}$ is additive, $\widehat{T}^n$ is an additive functor. Every short exact sequence $0 \to A \to B \to C \to 0$ in the category $\mathcal{C}$ yields a long exact sequence in the cohomological functors $T^\bullet$ and $T^\bullet\langle m\rangle$ for any $m \in \mathbb{N}_0$ by Axiom 2.0.2. Hence we obtain a long exact sequence in $\widehat{T}^\bullet = \varinjlim_{\mathcal{D}, m\in\mathbb{N}_0} T^\bullet\langle m\rangle$ and in particular, a connecting homomorphism $\widehat{\delta}^\bullet$ only if the direct limit functor $\varinjlim_{\mathcal{D}, m\in\mathbb{N}_0}$ is exact.

More specifically, to demonstrate that $(\widehat{T}^\bullet, \widehat{\delta}^\bullet)$ is a cohomological functor it suffices to show that $\widehat{\delta}^\bullet$ is a natural transformation and how a long exact sequence in $\widehat{T}^\bullet$ arises from a short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ in $\mathcal{C}$. As for every $m \leq n \in \mathbb{N}_0$ we have a morphism of cohomological functors $\Phi^\bullet_{m,n}$, the following diagram commutes for every $k \in \mathbb{Z}$:

$$\begin{tikzcd}
T^k\langle m\rangle(C) \arrow[r, "\delta^k\langle m\rangle"] \arrow[d, "\Phi^k_{m,n}(C)"] & T^{k+1}\langle m\rangle(A) \arrow[d, "\Phi^{k+1}_{m,n}(A)"] \\
T^k\langle n\rangle(C) \arrow[r, "\delta^k\langle n\rangle"] & T^{k+1}\langle n\rangle(A)
\end{tikzcd}$$

As $\Phi^k_{m,n}(-)$ (resp. $\Phi^{k+1}_{m,n}(-)$) is a functor and $\delta^k\langle m\rangle$ (resp. $\delta^{k+1}\langle n\rangle$) is natural, the connecting homomorphism

$$\widehat{\delta}^k = \varinjlim_{\mathcal{D}, n\in\mathbb{N}_0} \delta^k\langle n\rangle : \widehat{T}^k(C) \to \widehat{T}^{k+1}(A)$$

is natural by the above square. Using again that $\Phi^\bullet_{m,n}$ is a morphism of cohomological functors, also the following diagram commutes:

$$\begin{tikzcd}
... \arrow[r] & T^{k-1}\langle m\rangle(C) \arrow[r, "\delta^{k-1}\langle m\rangle"] \arrow[d, "\Phi^{k-1}_{m,n}(C)"] & T^k\langle m\rangle(A) \arrow[r, "T^k\langle m\rangle(f)"] \arrow[d, "\Phi^k_{m,n}(A)"] & T^k\langle m\rangle(B) \arrow[r, "T^k\langle m\rangle(g)"] \arrow[d, "\Phi^k_{m,n}(B)"] & T^k\langle m\rangle(C) \arrow[r, "\delta^k\langle m\rangle"] \arrow[d, "\Phi^k_{m,n}(C)"] & ... \\
... \arrow[r] & T^{k-1}\langle n\rangle(C) \arrow[r, "\delta^{k-1}\langle n\rangle"] & T^k\langle n\rangle(A) \arrow[r, "T^k\langle n\rangle(f)"] & T^k\langle n\rangle(B) \arrow[r, "T^k\langle n\rangle(g)"] & T^k\langle n\rangle(C) \arrow[r, "\delta^k\langle n\rangle"] & ...
\end{tikzcd}$$

Since the above constitutes a direct system of long exact sequences and countable direct limits $\varinjlim_{\mathcal{D}}$ are exact, we obtain the corresponding long exact sequence in $(\widehat{T}^\bullet, \widehat{\delta}^\bullet)$

$$\ldots \longrightarrow \widehat{T}^{k-1}(C) \xrightarrow{\widehat{\delta}^{k-1}} \widehat{T}^k(A) \xrightarrow{\widehat{T}^k(f)} \widehat{T}^k(B) \xrightarrow{\widehat{T}^k(g)} \widehat{T}^k(C) \xrightarrow{\widehat{\delta}^k} \ldots$$

Lemma 3.0.2 *The cohomological functor $(\widehat{T}^\bullet, \widehat{\delta}^\bullet)$ constructed via satellite functors is a Mislin completion of $(T^\bullet, \delta^\bullet)$.*

Proof As for G. Mislin's satellite functor construction, the proof is an adaptation of material found on page 296 of his paper [7]. Let P be projective projective object in $\mathcal{C}$. If we choose the short exact sequence $0 \to \big(0 \to P \to P\big) \to 0$, then we see that $S^{-1}T^n(P) = 0$ and thus $S^{-k}T^{n+k}(P) = 0$ for any $k \geq 1$. In particular, $(\widehat{T}^\bullet, \delta^\bullet)$ vanishes on projectives. Let us define a canonical morphism $\Phi^\bullet : (T^\bullet, \delta^\bullet) \to (\widehat{T}^\bullet, \widehat{\delta}^\bullet)$ such that any morphism $\Psi^\bullet : (T^\bullet, \delta^\bullet) \to (U^\bullet, \zeta^\bullet)$ to a cohomological functor vanishing on projectives factors uniquely as

$$\Psi^\bullet : (T^\bullet, \delta^\bullet) \xrightarrow{\Phi^\bullet} (\widehat{T}^\bullet, \widehat{\delta}^\bullet) \to (U^\bullet, \zeta^\bullet)\,.$$

By Lemma 3.0.1 there is a morphism $\Phi^\bullet_m : (T^\bullet, \delta^\bullet) \to (T^\bullet\langle m\rangle, \delta^\bullet\langle m\rangle)$ for any $m \in \mathbb{N}_0$ and denote by $\widehat{\Phi}^\bullet_m : (T^\bullet\langle m\rangle, \delta^\bullet\langle m\rangle) \to (\widehat{T}^\bullet, \widehat{\delta}^\bullet)$ the morphism to the direct limit. Defining $\Phi^\bullet := \widehat{\Phi}^\bullet_m \circ \Phi^\bullet_m : (T^\bullet, \delta^\bullet) \to (\widehat{T}^\bullet, \widehat{\delta}^\bullet)$ is then independent of $m \in \mathbb{N}_0$ since

$$\widehat{\Phi}^\bullet_m \circ \Phi^\bullet_m = (\widehat{\Phi}^\bullet_n \circ \Phi^\bullet_{m,n}) \circ \Phi^\bullet_m = \widehat{\Phi}^\bullet_n \circ (\Phi^\bullet_{m,n} \circ \Phi^\bullet_m) = \widehat{\Phi}^\bullet_n \circ \Phi^\bullet_n$$

for any $m \leq n$. Define a morphism $\Psi^\bullet_m : (T^\bullet\langle m\rangle, \delta^\bullet\langle m\rangle) \to (U^\bullet, \zeta^\bullet)$ by setting $\Psi^n_m = \Psi^n$ for any $m \leq n$. Using Lemma 3.0.1 we extend $\Psi^\bullet_m$ uniquely to a morphism of cohomological functors. Given that $\Phi^n_m = \mathrm{id}_{\mathcal{D}}$ for every $m \leq n \in \mathbb{Z}$, we have $\Psi^n = \Psi^n_m \circ \Phi^n_m$ and Lemma 3.0.1 implies that

$$\Psi^\bullet = \Psi^\bullet_m \circ \Phi^\bullet_m : (T^\bullet, \delta^\bullet) \to (T^\bullet\langle m\rangle, \delta^\bullet\langle m\rangle) \to (U^\bullet, \zeta^\bullet)\,.$$

Deduce analogously $\Psi^\bullet_m = \Psi^\bullet_n \circ \Phi^\bullet_{m,n}$ for any $m \leq n \in \mathbb{N}_0$. We apply the universal property of direct limits in order to obtain the unique morphism $\widehat{\Psi}^\bullet : (\widehat{T}^\bullet, \widehat{\delta}^\bullet) \to (U^\bullet, \zeta^\bullet)$ such that $\Psi^\bullet_m = \widehat{\Psi}^\bullet \circ \widehat{\Phi}^\bullet_m$. Therefore

$$\Psi^\bullet = \Psi^\bullet_m \circ \Phi^\bullet_m = \widehat{\Psi}^\bullet \circ \widehat{\Phi}^\bullet_m \circ \Phi^\bullet_m = \widehat{\Psi}^\bullet \circ \widehat{\Phi}^\bullet : (T^\bullet, \delta^\bullet) \to (\widehat{T}^\bullet, \widehat{\delta}^\bullet) \to (U^\bullet, \zeta^\bullet)\,.$$

We want to show that this factorization is unique. So assume that there is another factorization $\Psi^\bullet = X^\bullet \circ \widehat{\Phi}^\bullet$. Thus

$$\widehat{\Psi}^\bullet \circ \widehat{\Phi}^\bullet_m \circ \Phi^\bullet_m = \widehat{\Psi}^\bullet \circ \widehat{\Phi}^\bullet = X^\bullet \circ \widehat{\Phi}^\bullet = X^\bullet \circ \widehat{\Phi}^\bullet_m \circ \Phi^\bullet_m\,.$$

As $\Phi^n_m = \mathrm{id}_{\mathcal{D}}$ for $m \leq n \in \mathbb{N}_0$, $\widehat{\Psi}^n \circ \widehat{\Phi}^n_m = X^n \circ \widehat{\Phi}^n_m$ and

$$\Psi_m = \widehat{\Psi}^\bullet \circ \widehat{\Phi}^\bullet_m = X^\bullet \circ \widehat{\Phi}^\bullet_m : (T^\bullet\langle m\rangle, \delta^\bullet\langle m\rangle) \to (U^\bullet, \zeta^\bullet)$$

by Lemma 3.0.1. Since $\widehat{\Psi}^\bullet$ is the unique morphism such that $\Psi^\bullet_m = \widehat{\Psi}^\bullet \circ \widehat{\Phi}^\bullet_m$ for any $m \in \mathbb{N}_0$, we conclude that $X^\bullet = \widehat{\Psi}^\bullet$. $\square$

Let us provide an explicit formula for the connecting homomorphisms of the satellite functor construction. For this, we require H. Cartan and S. Eilenberg's extension of the concept of left satellite functors to natural transformations from [37, Corollary 5.3] that we present in the following. As before, their considerations for modules over a ring pass to abelian categories. Let $\varphi : F \to G$ be a natural transformation between two half-exact functors $F, G : \mathcal{C} \to \mathcal{D}$, meaning that for any short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{C}$ the induced sequences in $\mathcal{D}$

$$F(A) \to F(B) \to F(C) \text{ and } G(A) \to G(B) \to G(C)$$

are exact [3, p. 205]. For $C \in \mathrm{obj}(\mathcal{C})$, let $0 \to M \to P \to C \to 0$ be a short exact sequence with P projective. This induces the commutative diagram

$$\begin{tikzcd} F(M) \arrow[r] \arrow[d, "\varphi(M)"] & F(P) \arrow[r] \arrow[d, "\varphi(P)"] & F(C) \arrow[d, "\varphi(C)"] \\ G(M) \arrow[r] & G(P) \arrow[r] & G(C) \end{tikzcd}$$

In the same manner as we have defined $S^{-1}T^n(f) : S^{-1}T^n(C') \to S^{-1}T^n(C)$ between first left satellite functors, write $S^{-1}\varphi(C) : S^{-1}F(C) \to S^{-1}G(C)$ for the morphism induced from $\varphi(M) : F(M) \to G(M)$ between the corresponding kernels. Pictorially

$$\begin{tikzcd} S^{-1}F(C) \arrow[r, "\varepsilon^{-1}"] \arrow[d, dashed, "S^{-1}\varphi(C)"] & F(M) \arrow[d, "\varphi(M)"] \\ S^{-1}G(C) \arrow[r, "\varepsilon^{-1}"] & G(M) \end{tikzcd}$$

Write $S^0\varphi := \varphi$ and

$$S^{-k}\varphi := S^{-1}(S^{-k+1}\varphi) : S^{-k}F \to S^{-k}G\,,$$

for any $k \in \mathbb{N}$. We term this the k^{th} *left satellite transformation* of φ. Now remember that any connecting homomorphism $\delta^n : T^n \to T^{n+1}$ is a natural transformation. We have seen in Section 2 that it factors through $\underline{\delta}^n : T^n(M) \to S^{-1}T^{n+1}(M)$ for any $M \in \mathrm{obj}(\mathcal{C})$. Then P. H. Kropholler's account [3, p. 206–208] suggests the following result.

Lemma 3.0.3 *For any $n \in \mathbb{Z}$, $\underline{\delta}^n : T^n \to S^{-1}T^{n+1}$ is a natural transformation. In case that $0 \to A \to B \to C \to 0$ is a short exact sequence in $\mathcal{C}$, we take the connecting homomorphism $\varepsilon^{-k} : S^{-k}T^{n+k}(C) \to S^{-k+1}T^{n+k}(A)$*

for $k \in \mathbb{N}_0$. Then the n^{th} connecting homomorphism of the satellite functor construction is given by

$$\widehat{\delta}^n = \varinjlim_{k \in \mathbb{N}} \varepsilon^{-k} : \widehat{T}^n(C) \to \widehat{T}^{n+1}(A)$$

where the terms ε^{-k} are connected over $S^{-k}\underline{\delta}^{n+k} : S^{-k}T^{n+k} \to S^{-k-1}T^{n+k+1}$.

Proof The proof is based on determining all terms of each morphism of cohomological functors $\Phi^\bullet_{k,k+1} : T^\bullet\langle k\rangle \to T^\bullet\langle k+1\rangle$ that give rise to the satellite functor construction. We already know that $\Phi^n_{k,k+1} = \mathrm{id}_{\mathcal{D}}$ for $k < n$. To describe $\Phi^k_{k,k+1}$, consider a short exact sequence $0 \to M \to P \to C \to 0$ with P projective. We note that $\Phi^\bullet_{k,k+1}$ is meant to commute with the connecting homomorphisms of $T^\bullet\langle k\rangle$ and $T^\bullet\langle k+1\rangle$, in particular with the respective k^{th} connecting homomorphisms

$$\delta^k\langle k\rangle = \delta^k : T^k(C) \to T^{k+1}(A) \text{ and } \delta^k\langle k+1\rangle = \varepsilon^{-1} : S^{-1}T^{k+1}(C) \to T^{k+1}(A)\,.$$

Since $(T^\bullet, \delta^\bullet)$ is a cohomological functor, we obtain from Diagram 3.0.2 the commutative diagram

$$\begin{tikzcd} \dots \arrow[r] & T^k(P) \arrow[r] \arrow[d, "T^k(h)"] & T^k(C) \arrow[r, "\delta^k"] \arrow[d, "\mathrm{id}_{T^k(C)}"] & T^{k+1}(M) \arrow[r] \arrow[d, "T^{k+1}(h^*)"] & T^{k+1}(P) \arrow[r] \arrow[d, "T^{k+1}(h)"] & \dots \\ \dots \arrow[r] & T^k(B) \arrow[r] & T^k(C) \arrow[r, "\delta^k"] & T^{k+1}(A) \arrow[r] & T^{k+1}(B) \arrow[r] & \dots \end{tikzcd} \tag{3.0.5}$$

Recalling that $S^{-1}T^{k+1}(C)$ is a kernel and $\varepsilon^{-1} : S^{-1}T^{k+1}(C) \to T^{k+1}(M)$ its canonical monomorphism, we can write the middle square as

$$\begin{tikzcd} T^k(C) \arrow[r, "\underline{\delta}^k"] \arrow[d, "\mathrm{id}_{T^k(C)}"] & S^{-1}T^{k+1}(C) \arrow[r, "\varepsilon^{-1}"] & T^{k+1}(M) \arrow[d, "T^{k+1}(h^*)"] \\ T^k(C) \arrow[rr, "\delta^n"] & & T^{k+1}(A) \end{tikzcd}$$

By definition,

$$\delta^k\langle k+1\rangle = \varepsilon^{-1} : S^{-1}T^{k+1}(C) \xrightarrow{\varepsilon^{-1}} T^{k+1}(M) \xrightarrow{T^{k+1}(h^*)} T^{k+1}(A)\,.$$

Tilting the above diagram, we obtain the desired commutative diagram

$$\begin{tikzcd} T^k(C) \arrow[r, "\delta^k"] \arrow[d, "\underline{\delta}^k"] & T^{k+1}(A) \arrow[d, "\mathrm{id}_{T^{k+1}(A)}"] \\ S^{-1}T^{k+1}(C) \arrow[r, "\varepsilon^{-1}"] & T^{k+1}(A) \end{tikzcd} \tag{3.0.6}$$

We conclude that $\underline{\delta}^k = \Phi^k_{k,k+1} : T^k \to S^{-1}T^{k+1}$ is a natural transformation by the proof of Lemma 3.0.1 found in [37, p. 46–47]. By the construction of left satellite transformations, we infer that

$$\Phi^n_{k,k+1} = S^{n-k}\underline{\delta}^k : S^{n-k}T^k \to S^{n-k+1}T^{k+1}$$

for any $n < k$. In particular, the square

$$\begin{array}{ccc} S^{n-k}T^k(C) & \xrightarrow{\varepsilon^{n-k}} & S^{-k+1}T^n(A) \\ \downarrow{\scriptstyle S^{n-k}\underline{\delta}^k} & & \downarrow{\scriptstyle S^{n-k+1}\underline{\delta}^k} \\ S^{n-k-1}T^{k+1}(C) & \xrightarrow{\varepsilon^{n-k-1}} & S^{n-k}T^{k+1}(A) \end{array} \tag{3.0.7}$$

commutes. In the direct limit, these squares give rise to the connecting homomorphism $\widehat{\delta}^n$ and the formula in the lemma follows from reindexing as in Equation 3.0.4. □

Lastly, we provide an explicit formula for the induced morphisms in the satellite functor construction.

Proposition 3.0.4 *Let $f : A \to B$ be a morphism in $\mathcal{C}$. Then for any $n \in \mathbb{Z}$ the commuting square*

$$\begin{array}{ccc} S^{-k}T^{n+k}(A) & \xrightarrow{S^{-k}T^{n+k}(f)} & S^{-k}T^{n+k}(B) \\ \downarrow{\scriptstyle S^{-k}\underline{\delta}^{n+k}} & & \downarrow{\scriptstyle S^{-k}\underline{\delta}^{n+k}} \\ S^{-k-1}T^{n+k+1}(A) & \xrightarrow{S^{-k-1}T^{n+k+1}(f)} & S^{-k-1}T^{n+k+1}(B) \end{array}$$

give rise to $\widehat{T}^n(f) = \varinjlim_{\mathcal{D},k\in\mathbb{N}_0} S^{-k}T^{n+k}(f) : \widehat{T}(A) \to \widehat{T}(B)$.

Proof This follows from Equation 3.0.4 and the description of the cohomological functors $\Phi^\bullet_{n+k,n+k+1}$ found in the proof of Lemma 3.0.3. □

Chapter 4

The resolution and the satellite functor construction

The main goal of this chapter is to prove that the resolution construction gives rise to Mislin completions of cohomological functors. First, we show in Section 4.1 that the cohomology groups resulting from the satellite functor construction are isomorphic to those from the resolution construction. In Section 4.2, we provide explicit formulae for the induced morphisms and connecting homomorphisms of the resolution construction. Then, in Section 4.3, we prove that the isomorphisms between the cohomology groups extend to isomorphisms of cohomological functors. This sets the ground for Section 4.4 where we demonstrate that there are countably many distinct constructions of Mislin completions.

4.1 Isomorphism of cohomology groups

To set up notation for the resolution construction, let $M \in \mathrm{obj}(\mathcal{C})$, $(M_n)_{n\in\mathbb{N}_0}$ be a projective resolution of M and $\widetilde{M}_k$ be the k^{th} syzygy of $M_\bullet$ for $k \in \mathbb{N}_0$. Keeping our notation from last chapter, we denote for $n \in \mathbb{Z}$ the n^{th} term of the satellite functor construction by $\widehat{T}^n(M)$. If the connecting homomorphisms $\delta^{n+k} : T^{n+k}(\widetilde{M}_k) \to T^{n+k+1}(\widetilde{M}_{k+1})$ are taken as in Section 2, we write $T^n_{Res}(M) := \varinjlim_{k\in\mathbb{N}_0} \left(T^{n+k}(\widetilde{M}_k), \delta^{n+k}\right)$ for the n^{th} term of the resolution construction. However it depends on a choice of projective resolution a priori. This issue is resolved by demonstrating

Lemma 4.1.1 *There is an isomorphism $\omega_n(M) : \widehat{T}^n(M) \to T^n_{Res}(M)$ in $\mathcal{D}$ for any $n \in \mathbb{Z}$.*

A proof of this lemma can be extracted from [7, p. 299] for modules over rings and can be found in [28, p. 20] for unenriched relative Ext-functors. In the following, we present an analogous proof for the cohomological functor

$T^\bullet : \mathcal{C} \to \mathcal{D}$. The manner in which we conduct it allows us in Section 4.3 to extend the resulting isomorphism to an isomorphism of cohomological functors.

Proof The proof consists of a two step endeavor where we introduce the relevant notation throughout its course. In the first step, we construct an isomorphism

$$\eta_n(M) : \varinjlim_{k \in \mathbb{N}} (S^{-1}T^{n+k}(\widetilde{M}_{k-1}), \Sigma^{-1}\delta^{n+k}) \to T^n_{Res}(M)$$

and extend it to an isomorphism $\omega_n(M) : \widehat{T}^n(M) \to T^n_{Res}(M)$ in the second step.

Step 1 Defining $\eta_n(M) : \varinjlim_{k \in \mathbb{N}} (S^{-1}T^{n+k}(\widetilde{M}_{k-1}), \Sigma^{-1}\delta^{n+k}) \to T^n_{Res}(M)$

By Axiom 2.0.2, the short exact sequence $0 \to \widetilde{M}_{k+1} \xrightarrow{\iota_{k+1}} M_k \xrightarrow{\pi_k} \widetilde{M}_k \to 0$ induces a long exact sequence in the cohomological functor $(T^\bullet, \delta^\bullet)$ one of whose parts is

$$\begin{aligned} T^{n+k}(\widetilde{M}_k) \xrightarrow{\delta^{n+k}} T^{n+k+1}(\widetilde{M}_{k+1}) &\xrightarrow{T^{n+k+1}(\iota_{k+1})} T^{n+k+1}(M_k) \\ &\xrightarrow{T^{n+k+1}(\pi_k)} T^{n+k+1}(\widetilde{M}_k)\,. \end{aligned}$$

Because M_k is projective, it follows from the exactness of the above and from the definition of left satellite functors that

$$\mathrm{Im}(\delta^{n+k}) = \mathrm{Ker}(T^{n+k+1}(\iota_{k+1})) = S^{-1}T^{n+k+1}(\widetilde{M}_k)\,.$$

We can think of $\underline{\delta}^{n+k} : T^{n+k}(\widetilde{M}_k) \to S^{-1}T^{n+k+1}(\widetilde{M}_k)$ as being obtained by restricting the codomain of δ^{n+k} onto the subobject $S^{-1}T^{n+k+1}(\widetilde{M}_k)$ of $T^{n+k+1}(\widetilde{M}_{k+1})$. Because $\varepsilon^{-1} : S^{-1}T^{n+k}(\widetilde{M}_{k-1}) \to T^{n+k}(\widetilde{M}_k)$ is the canonical monomorphism from a kernel, we can interpret

$$\Sigma^{-1}\delta^{n+k} := \underline{\delta}^{n+k} \circ \varepsilon^{-1} : \quad S^{-1}T^{n+k}(\widetilde{M}_{k-1}) \to T^{n+k}(\widetilde{M}_k) \to S^{-1}T^{n+k+1}(\widetilde{M}_k)$$

as arising by restricting the domain of δ^{n+k} onto $S^{-1}T^{n+k}(\widetilde{M}_{k-1})$ and its codomain onto $S^{-1}T^{n+k+1}(\widetilde{M}_k)$. Using this and Diagram 3.0.6 we obtain the commuting diagram

$$\begin{tikzcd} S^{-1}T^{n+k}(\widetilde{M}_{k-1}) \arrow[r, "\varepsilon^{-1}"] \arrow[d, "\Sigma^{-1}\delta^{n+k}"] & T^{n+k}(\widetilde{M}_k) \arrow[d, "\delta^{n+k}"] \arrow[dl, "\underline{\delta}^{n+k}"] \\ S^{-1}T^{n+k+1}(\widetilde{M}_k) \arrow[r, "\varepsilon^{-1}"] & T^{n+k+1}(\widetilde{M}_{k+1}) \end{tikzcd} \tag{4.1.1}$$

Since the above squares give rise to a direct system of morphisms, we define

$$\eta_n(M) = \varinjlim \varepsilon^{-1} : \varinjlim_{k\in\mathbb{N}} (S^{-1}T^{n+k}(\widetilde{M}_{k-1}), \Sigma^{-1}\delta^{n+k}) \to T^n_{Res}(M)\,.$$

If the cokernel of the monomorphism $\varepsilon^{-1} : S^{-1}T^{n+k}(\widetilde{M}_{k-1}) \to T^{n+k}(\widetilde{M}_k)$ is denoted by N_{n+k}, then it follows from its universal property that we can extend Diagram 4.1.1 by a morphism $m_{n+k} : N_{n+k} \to N_{n+k+1}$ in a commutative manner as

$$\begin{tikzcd}
0 \arrow[r] & S^{-1}T^{n+k}(\widetilde{M}_{k-1}) \arrow[r, "\varepsilon^{-1}"] \arrow[d, "\Sigma^{-1}\delta^{n+k}"] & T^{n+k}(\widetilde{M}_k) \arrow[r, "q_{n+k}"] \arrow[d, "\delta^{n+k}"] \arrow[dl, "\underline{\delta}^{n+k}"'] & N_{n+k} \arrow[r] \arrow[d, "m_{n+k}"] & 0 \\
0 \arrow[r] & S^{-1}T^{n+k+1}(\widetilde{M}_k) \arrow[r, "\varepsilon^{-1}"] & T^{n+k+1}(\widetilde{M}_{k+1}) \arrow[r, "q_{n+k+1}"] & N_{n+k+1} \arrow[r] & 0
\end{tikzcd} \tag{4.1.2}$$

However, note that

$$m_{n+k} \circ q_{n+k} = q_{n+k+1} \circ \varepsilon^{-1} \circ \underline{\delta}^{n+k} : T^{n+k}(\widetilde{M}_k) \to N_{n+k+1}$$

is the zero morphism. Since q_{n+k} is surjective, we deduce that $m_{n+k} = 0$. Given that these diagrams form a direct system of short exact sequences and countable direct limits in $\mathcal{D}$ are exact, we obtain a short exact sequence

$$0 \to \varinjlim_{k\in\mathbb{N}} (S^{-1}T^{n+k}(\widetilde{M}_{n+k}), \Sigma^{-1}\delta^{n+k}) \xrightarrow{\eta_n(M)} T^n_{Res}(M) \to 0 \to 0\,.$$

Hence, $\eta_n(M)$ is an isomorphism.

Step 2 Defining an isomorphism $\omega_n(M) : \widehat{T}^n(M) \to T^n_{Res}(M)$
The way we construct this isomorphism is by connecting the terms of the direct system $(S^{-k}T^{n+k}(M), S^{-k}\underline{\delta}^{n+k})_{k\in\mathbb{N}_0}$ to the ones of $(T^{n+k}(\widetilde{M}_k), \delta^{n+k})_{k\in\mathbb{N}_0}$. This requires a larger commutative diagram for which we introduce all the necessary morphisms. By the previous step, we can extend Diagram 3.0.7 to the commutative diagram

$$\begin{tikzcd}
S^{-1}T^{n+k+2}(\widetilde{M}_{k+1}) \arrow[r, "\varepsilon^{-1}"] \arrow[d, "S^{-1}\underline{\delta}^{n+k+2}"] \arrow[dr, "\Sigma^{-1}\delta^{n+k+2}"] & T^{n+k+2}(\widetilde{M}_{k+2}) \arrow[d, "\underline{\delta}^{n+k+2}"] \\
S^{-2}T^{n+k+3}(\widetilde{M}_{k+1}) \arrow[r, "\varepsilon^{-2}"] & S^{-1}T^{n+k+3}(\widetilde{M}_{k+2})
\end{tikzcd}$$

Since $\varepsilon^{-2} : S^{-2}T^{n+k+3}(\widetilde{M}_{k+1}) \to S^{-1}T^{n+k+3}(\widetilde{M}_{k+2})$ is the monomorphism of a kernel, we can think of

$$S^{-1}\underline{\delta}^{n+k+2} : S^{-1}T^{n+k+2}(\widetilde{M}_{k+1}) \to S^{-2}T^{n+k+3}(\widetilde{M}_{k+1})$$

as being obtained by restricting the codomain of $\Sigma^{-1}\delta^{n+k+2}$ onto the subobject $S^{-2}T^{n+k+3}(\widetilde{M}_{k+1})$ of $S^{-1}T^{n+k+3}(\widetilde{M}_{k+2})$. Furthermore, we can interpret the morphism

$$\begin{aligned}\Sigma^{-2}\delta^{n+k+2} &:= S^{-1}\underline{\delta}^{n+k+2} \circ \varepsilon^{-2} : \\ &S^{-2}T^{n+k+2}(\widetilde{M}_k) \to S^{-1}T^{n+k+2}(\widetilde{M}_{k+1}) \to S^{-2}T^{n+k+3}(\widetilde{M}_{k+1})\end{aligned}$$

as arising from restricting the domain of $\Sigma^{-1}\delta^{n+k+2}$ onto $S^{-2}T^{n+k+2}(\widetilde{M}_k)$ and its codomain onto $S^{-2}T^{n+k+3}(\widetilde{M}_{k+1})$. As for left satellite functors, we may define $\Sigma^0\delta^{n+k} := \delta^{n+k}$ and

$$\begin{aligned}\Sigma^{-l}\delta^{n+k+l} &:= S^{-l+1}\underline{\delta}^{n+k+l} \circ \varepsilon^{-l} : \\ &S^{-l}T^{n+k+l}(\widetilde{M}_k) \to S^{-l+1}T^{n+k+l}(\widetilde{M}_{k+1}) \to S^{-l}T^{n+k+l+1}(\widetilde{M}_{k+1})\end{aligned} \tag{4.1.3}$$

for any $l \in \mathbb{N}$. We can think of $\Sigma^{-l}\delta^{n+k+l}$ as being obtained by restricting the domain of $\Sigma^{-l+1}\delta^{n+k+l-1}$ onto $S^{-l}T^{n+k+l}(\widetilde{M}_k)$ and its codomain onto $S^{-l}T^{n+k+l+1}(\widetilde{M}_{k+1})$. We can again extend Diagram 3.0.7 to another commutative diagram of the form

$$\begin{tikzcd} S^{-l}T^{n+k+l}(\widetilde{M}_k) \arrow[r, "\varepsilon^{-l}"] \arrow[d, "S^{-l}\underline{\delta}^{n+k+l}"'] \arrow[dr, "\Sigma^{-l}\delta^{n+k+l}"] & S^{-l+1}T^{n+k+l}(\widetilde{M}_{k+1}) \arrow[d, "S^{-l+1}\underline{\delta}^{n+k+l}"] \\ S^{-l-1}T^{n+k+l+1}(\widetilde{M}_k) \arrow[r, "\varepsilon^{-l-1}"] & S^{-l}T^{n+k+l+1}(\widetilde{M}_{k+1}) \end{tikzcd} \tag{4.1.4}$$

By [37, Section III.2], for every $2 \leq l \in \mathbb{N}$ and $k \in \{0, \dots, l-1\}$ the short exact sequence $0 \to \widetilde{M}_{k+1} \to M_k \to \widetilde{M}_k \to 0$ gives rise to a long exact sequence one of whose parts is

$$S^{-l}T^{n+k+l}(M_k) \to S^{-l}T^{n+k+l}(\widetilde{M}_k) \xrightarrow{\varepsilon^{-l}} S^{-l+1}T^{n+k+l}(\widetilde{M}_{k+1}) \to S^{-l+1}T^{n+k+l}(M_k)\,.$$

By construction, left satellite functors vanish on projective objects such as M_k. Hence the morphism $\varepsilon^{-l} : S^{-l}T^{n+k+l}(\widetilde{M}_k) \to S^{-l+1}T^{n+k+l}(\widetilde{M}_{k+1})$ is an isomorphism. Define the isomorphisms

$$\begin{aligned}\widetilde{o}^k := \varepsilon^{-3} \circ \ldots \circ \varepsilon^{-k+1} \circ \varepsilon^{-k} :\ & S^{-k}T^{n+k}(M) \to S^{-2}T^{n+k}(\widetilde{M}_{k-2}) \\ \text{and} \quad o^k := \varepsilon^{-2} \circ \widetilde{o}^k :\ & S^{-k}T^{n+k}(M) \to S^{-1}T^{n+k}(\widetilde{M}_{k-1})\,.\end{aligned} \tag{4.1.5}$$

We can form the following diagram with all morphisms we have introduced thus far:

$$
\begin{tikzcd}
& & & T^{n}(M) \arrow[dl, "\underline{\delta}^{n}"'] \arrow[d, "\delta^{n}"] \\
& & S^{-1}T^{n+1}(M) \arrow[r, "\varepsilon^{-1}"] \arrow[dl, "S^{-1}\underline{\delta}^{n+1}"'] \arrow[d, "\Sigma^{-1}\delta^{n+1}"] & T^{n+1}(\widetilde{M}_1) \arrow[dl, "\underline{\delta}^{n+1}"'] \arrow[d, "\delta^{n+1}"] \\
& S^{-2}T^{n+2}(M) \arrow[r, "\varepsilon^{-2}"] \arrow[dl, "S^{-2}\underline{\delta}^{n+2}"'] \arrow[d, "\Sigma^{-2}\delta^{n+2}"] & S^{-1}T^{n+2}(\widetilde{M}_1) \arrow[r, "\varepsilon^{-1}"] \arrow[dl, "S^{-1}\underline{\delta}^{n+2}"'] \arrow[d, "\Sigma^{-1}\delta^{n+2}"] & T^{n+2}(\widetilde{M}_2) \arrow[dl, "\underline{\delta}^{n+2}"'] \arrow[d, "\delta^{n+2}"] \\
S^{-3}T^{n+3}(M) \arrow[r, "\widetilde{o}^{3}"] \arrow[d, "S^{-3}\underline{\delta}^{n+3}"] & S^{-2}T^{n+3}(\widetilde{M}_1) \arrow[r, "\varepsilon^{-2}"] \arrow[d, "\Sigma^{-2}\delta^{n+3}"] & S^{-1}T^{n+3}(\widetilde{M}_2) \arrow[r, "\varepsilon^{-1}"] \arrow[dl, "S^{-1}\underline{\delta}^{n+3}"'] \arrow[d, "\Sigma^{-1}\delta^{n+3}"] & T^{n+3}(\widetilde{M}_3) \arrow[dl, "\underline{\delta}^{n+3}"'] \arrow[d, "\delta^{n+3}"] \\
S^{-4}T^{n+4}(M) \arrow[r, "\widetilde{o}^{4}"] \arrow[d] & S^{-2}T^{n+4}(\widetilde{M}_2) \arrow[r, "\varepsilon^{-2}"] \arrow[d] & S^{-1}T^{n+4}(\widetilde{M}_3) \arrow[r, "\varepsilon^{-1}"] \arrow[d] & T^{n+4}(\widetilde{M}_4) \arrow[d] \\
\vdots \arrow[r] \arrow[d] & \vdots \arrow[r] \arrow[d, dashed] & \vdots \arrow[r] \arrow[d] & \vdots \arrow[d] \\
S^{-k}T^{n+k}(M) \arrow[r, "\widetilde{o}^{k}"] \arrow[rr, bend left, "o^{k}"] \arrow[d, "S^{-k}\underline{\delta}^{n+k}"] & S^{-2}T^{n+k}(\widetilde{M}_{k-2}) \arrow[r, "\varepsilon^{-2}"] \arrow[d, "\Sigma^{-2}\delta^{n+k}"] & S^{-1}T^{n+k}(\widetilde{M}_{k-1}) \arrow[r, "\varepsilon^{-1}"] \arrow[dl, "S^{-1}\underline{\delta}^{n+k}"'] \arrow[d, "\Sigma^{-1}\delta^{n+k}"] & T^{n+k}(\widetilde{M}_k) \arrow[dl, "\underline{\delta}^{n+k}"'] \arrow[d, "\delta^{n+k}"] \\
S^{-k-1}T^{n+k+1}(M) \arrow[r, "\widetilde{o}^{k+1}"] \arrow[d] & S^{-2}T^{n+k+1}(\widetilde{M}_{k-1}) \arrow[r, "\varepsilon^{-2}"] \arrow[d] & S^{-1}T^{n+k+1}(\widetilde{M}_k) \arrow[r, "\varepsilon^{-1}"] \arrow[d] & T^{n+k+1}(\widetilde{M}_{k+1}) \arrow[d] \\
\vdots & \vdots & \vdots & \vdots
\end{tikzcd}
\tag{4.1.6}
$$

We need to argue why this diagram commutes before we can demonstrate how it yields the desired isomorphism. The triangles at the very right side containing the morphisms $\delta^{n+k} : T^{n+k}(\widetilde{M}_k) \to T^{n+k+1}(\widetilde{M}_{k+1})$ commute by Diagram 3.0.6. All other triangles commute due to Diagram 4.1.4. Finally, due to Diagram 3.0.7 and 4.1.4 the following diagram commutes for any $k \geq 3$.

$$
\begin{tikzcd}
S^{-k}T^{n+k}(M) \arrow[r, "\varepsilon^{-k}"] \arrow[d, "S^{-k}\underline{\delta}^{n+k}"] \arrow[ddrrrr, bend left, "o^k"] & S^{-k+1}T^{n+k}(\widetilde{M}_1) \arrow[r, "\varepsilon^{-k+1}"] \arrow[d, "S^{-k+1}\underline{\delta}^{n+k}"] & \ldots & & \\
S^{-k-1}T^{n+k+1}(M) \arrow[r, "\varepsilon^{-k-1}"] \arrow[ddrrrr, bend right, "\widetilde{o}^{k+1}"'] & S^{-k}T^{n+k+1}(\widetilde{M}_1) \arrow[r, "\varepsilon^{-k}"] & \ldots & & \\
& & \ldots \arrow[r, "\varepsilon^{-3}"] & S^{-2}T^{n+k}(\widetilde{M}_{k-2}) \arrow[r, "\varepsilon^{-2}"] \arrow[d, "S^{-2}\underline{\delta}^{n+k}"] \arrow[dr, "\Sigma^{-2}\delta^{n+k}"] & S^{-1}T^{n+k}(\widetilde{M}_{k-1}) \arrow[d, "S^{-1}\underline{\delta}^{n+k}"] \\
& & \ldots \arrow[r, "\varepsilon^{-4}"] & S^{-3}T^{n+k+1}(\widetilde{M}_{k-2}) \arrow[r, "\varepsilon^{-3}"] & S^{-2}T^{n+k+1}(\widetilde{M}_{k-1})
\end{tikzcd}
\tag{4.1.7}
$$

We conclude from it that $\Sigma^{-2}\delta^{n+k} \circ \widetilde{o}^k = \widetilde{o}^{k-1} \circ S^{-k}\underline{\delta}^{n+k}$. In particular, Diagram 4.1.6 commutes as all its stand-alone squares also do so.

We see in Diagram 4.1.6 that the right most column corresponds to the direct system $(T^{n+k}(\widetilde{M}_k), \delta^{n+k})_{k\in\mathbb{N}_0}$ while the top diagonal together with the left most column forms the direct system $(S^{-k}T^{n+k}(M), S^{-k}\underline{\delta}^{n+k})_{k\in\mathbb{N}_0}$. The right most squares in the diagram recover the isomorphism

$$\eta_n(M) = \varinjlim \varepsilon^{-1} : \varinjlim_{k\in\mathbb{N}} (S^{-1}T^{n+k}(\widetilde{M}_{k-1}), \Sigma^{-1}\delta^{n+k}) \to T^n_{Res}(M)\,.$$

For $k \geq 3$, all the remaining squares yield together squares of the form

$$
\begin{tikzcd}
S^{-k}T^{n+k}(M) \arrow[r, "o^k"] \arrow[d, "S^{-k}\underline{\delta}^{n+k}"] & S^{-1}T^{n+k}(\widetilde{M}_{k-1}) \arrow[d, "\Sigma^{-1}\delta^{n+k}"] \\
S^{-k-1}T^{n+k+1}(M) \arrow[r, "o^{k+1}"] & S^{-1}T^{n+k+1}(\widetilde{M}_k)
\end{tikzcd}
$$

As the morphisms o^k are isomorphisms and countable direct limits in $\mathcal{D}$ are exact, this gives rise to the isomorphism

$$\varinjlim o^k : \widehat{T}^n(M) \to \varinjlim_{k\in\mathbb{N}} (S^{-1}T^{n+k}(\widetilde{M}_{k-1}), \Sigma^{-1}\delta^{n+k})\,.$$

By the universal property of direct limits, $\varinjlim o^k \circ \varinjlim \varepsilon^{-1} = \varinjlim o^k \circ \varepsilon^{-1}$. Because $\eta_n(M) = \varinjlim \varepsilon^{-1}$ is already an isomorphism,

$$\omega_n(M) = \varinjlim o^k \circ \varepsilon^{-1} : \widehat{T}^n(M) \to T^n_{Res}(M)$$

is our desired isomorphism. Since the morphisms $o^k \circ \varepsilon^{-1}$ span the entire width of Diagram 4.1.6, every single morphism was needed to construct this isomorphism. ☐

4.2 Induced morphisms and connecting homomorphisms

The first step to turn $\omega_n(M) : \widehat{T}^n(M) \to T^n_{Res}(M)$ into an isomorphism of cohomological functor is to show that it is natural. Let $f : M \to N$ be a morphism in $\mathcal{C}$ and $M_\bullet$, $N_\bullet$ be projective resolutions of M, N. Set $\widetilde{f}_0 := f : \widetilde{M}_0 \to \widetilde{N}_0$ and assume that $\widetilde{f}_{k+1} : \widetilde{M}_k \to \widetilde{N}_k$ has been defined for $k \geq 0$. Let us denote by $\pi_k : M_k \to \widetilde{M}_k$ the epimorphism from the projective M_k onto $\widetilde{M}_k$ and by $\iota_{k+1} : \widetilde{M}_{k+1} \to M_k$ the monomorphism from the $(k+1)^{\mathrm{st}}$ syzygy $\widetilde{M}_{k+1}$ into M_k. We adopt an analogous convention for $\widetilde{N}_{k+1}$, N_k and $\widetilde{N}_k$. Then we can lift $\widetilde{f}_k$ to fit in the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \widetilde{M}_{k+1} \arrow[r, "\iota_{k+1}"] \arrow[d, "\widetilde{f}_{k+1}"] & M_k \arrow[r, "\pi_k"] \arrow[d, "f_k"] & \widetilde{M}_k \arrow[r] \arrow[d, "\widetilde{f}_k"] & 0 \\ 0 \arrow[r] & \widetilde{N}_{k+1} \arrow[r, "\iota_{k+1}"] & N_k \arrow[r, "\pi_k"] & \widetilde{N}_k \arrow[r] & 0 \end{tikzcd} \tag{4.2.1}$$

since M_0 is a projective object and $\iota_{k+1} : \widetilde{N}_{k+1} \to N_k$ can be taken to be the kernel of π_k.

Proposition 4.2.1 *Let $f : M \to N$ and $(\widetilde{f}_k : \widetilde{M}_k \to \widetilde{N}_k)_{k\in\mathbb{N}_0}$ be a sequence of morphism as arising from Diagram 4.2.1. As $(T^\bullet, \delta^\bullet)$ is a cohomological functor, we denote the morphism in the direct limit of the commuting squares*

$$\begin{tikzcd} T^{n+k}(\widetilde{M}_k) \arrow[r, "T^{n+k}(\widetilde{f}_k)"] \arrow[d, "\delta^{n+k}"] & T^{n+k}(\widetilde{N}_k) \arrow[d, "\delta^{n+k}"] \\ T^{n+k+1}(\widetilde{M}_{k+1}) \arrow[r, "T^{n+k+1}(\widetilde{f}_{k+1})"] & T^{n+k+1}(\widetilde{N}_{k+1}) \end{tikzcd}$$

by $T^n_{Res}(f) := \varinjlim_{k\in\mathbb{N}_0} T^{n+k}(\widetilde{f}_k)$. Then the square

$$\begin{tikzcd} \widehat{T}^n(M) \arrow[r, "\omega_n(M)"] \arrow[d, "\widehat{T}^n(f)"] & T^n_{Res}(M) \arrow[d, "T^n_{Res}(f)"] \\ \widehat{T}^n(N) \arrow[r, "\omega_n(N)"] & T^n_{Res}(N) \end{tikzcd}$$

commutes. In particular, every term $T^n_{Res}(-)$ forms an additive functor and the homomorphisms $\omega_n : \widehat{T}^n \to T^n_{Res}$ a natural isomorphism.

Proof Recall that $\widehat{T}^n(f) = \varinjlim_{k\in\mathbb{N}_0} S^k T^{n+k}(f)$ by Proposition 3.0.4. Let $k \in \mathbb{N}_0$, $l \in \mathbb{N}$ and write $\Lambda = n + k + l$. Consider the diagram

$$\begin{tikzcd}
 & S^{-l}T^{\Lambda}(\widetilde{M}_k) \arrow[rr, "\varepsilon^{-l}"] \arrow[dl, "S^{-l}T^{\Lambda}(\widetilde{f}_k)"] \arrow[dd, "S^{-l}\underline{\delta}^{\Lambda}"] & & S^{-l+1}T^{\Lambda}(\widetilde{M}_{k+1}) \arrow[dl, "S^{-l+1}T^{\Lambda}(\widetilde{f}_{k+1})"] \arrow[dd, "S^{-l+1}\underline{\delta}^{\Lambda}"] \\
S^{-l}T^{\Lambda}(\widetilde{N}_k) \arrow[rr, "\varepsilon^{-l}"] \arrow[dd, "S^{-l}\underline{\delta}^{\Lambda}"] & & S^{-l+1}T^{\Lambda}(\widetilde{N}_{k+1}) \arrow[dd, "S^{-l+1}\underline{\delta}^{\Lambda}"] & \\
 & S^{-l-1}T^{\Lambda+1}(\widetilde{M}_k) \arrow[rr, "\varepsilon^{-l-1}"] \arrow[dl, "S^{-l-1}T^{\Lambda+1}(\widetilde{f}_k)"] & & S^{-l}T^{\Lambda+1}(\widetilde{M}_{k+1}) \arrow[dl, "S^{-l+1}T^{\Lambda+1}(\widetilde{f}_{k+1})"] \\
S^{-l-1}T^{\Lambda+1}(\widetilde{N}_k) \arrow[rr, "\varepsilon^{-l-1}"] & & S^{-l}T^{\Lambda+1}(\widetilde{N}_{k+1}) &
\end{tikzcd}$$

The front and back side commute by Diagram 4.1.4. The left hand and right hand side commute since $S^{-l}\underline{\delta}^{\Lambda}$ and $S^{-l+1}\underline{\delta}^{\Lambda}$ are natural transformations by Lemma 3.0.3. Lastly, the top and bottom side commute by Equation 3.0.3. Remember that we have defined the morphisms $o^k = \varepsilon^{-2} \circ \varepsilon^{-3} \circ \ldots \circ \varepsilon^{-k}$ in the proof of Lemma 4.1.1. Because δ^{n+k} is also a natural transformation, we conclude from Diagram 4.1.6 and the above cube that the diagram

$$\begin{tikzcd}
 & S^{-k}T^{n+k}(M) \arrow[rr, "\varepsilon^{-1}\circ o^k"] \arrow[dl, "S^{-k}T^{n+k}(f)"] \arrow[dd, "S^{-k}\underline{\delta}^{n+k}"] & & T^{n+k}(\widetilde{M}_k) \arrow[dl, "T^{n+k}(\widetilde{f}_k)"] \arrow[dd, "\delta^{n+k}"] \\
S^{-k}T^{n+k}(N) \arrow[rr, "\varepsilon^{-1}\circ o^k"] \arrow[dd, "S^{-k}\underline{\delta}^{n+k}"] & & T^{n+k}(\widetilde{N}_k) \arrow[dd, "\delta^{n+k}"] & \\
 & S^{-k-1}T^{n+k+1}(M) \arrow[rr, "\varepsilon^{-1}\circ o^{k+1}"] \arrow[dl, "S^{-k-1}T^{n+k+1}(f)"] & & T^{n+k+1}(\widetilde{M}_{k+1}) \arrow[dl, "T^{n+k+1}(\widetilde{f}_{k+1})"] \\
S^{-k-1}T^{n+k+1}(N) \arrow[rr, "\varepsilon^{-1}\circ o^{k+1}"] & & T^{n+k+1}(\widetilde{N}_{k+1}) &
\end{tikzcd}$$

is commutative. Referring again to the proof of Lemma 4.1.1, we recall that we have defined $\omega_n = \varinjlim_{k\in\mathbb{N}} \varepsilon^{-1} \circ o^k : \widehat{T}^n \to T^n_{Res}$. Therefore, the bottom square of the proposition commutes and $T^n_{Res}(f)$ is well defined in the sense that it does not depend on the choice of sequence $(\widetilde{f}_k : \widetilde{M}_k \to \widetilde{N}_k)_{k\in\mathbb{N}_0}$. Because every term $\widehat{T}^n$ is an additive functor, any T^n_{Res} is so too. Or more directly, for any morphism $g : M \to N$ and every corresponding sequence $(\widetilde{g}_k : \widetilde{M}_k \to \widetilde{N}_k)_{k\in\mathbb{N}_0}$ we see that

$$\begin{aligned} T^n_{Res}(f+g) &= \varinjlim_{k\in\mathbb{N}_0} T^{n+k}(\widetilde{f}_k + \widetilde{g}_k) \\ &= \varinjlim_{k\in\mathbb{N}_0} \big(T^{n+k}(\widetilde{f}_k) + T^{n+k}(\widetilde{g}_k)\big) = T^n_{Res}(f) + T^n_{Res}(g)\,. \qquad \square \end{aligned}$$

In order to turn $T^\bullet_{Res}$ into a cohomological functor, we need to construct connecting homomorphisms. For this let $0 \to \widetilde{A}_k \to \widetilde{B}_k \to \widetilde{C}_k \to 0$ be a short exact sequence in $\mathcal{C}$. By the Horseshoe Lemma [10, p. 37], there is a commutative diagram

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & \widetilde{A}_{k+1} \arrow[r, "\widetilde{f}_{k+1}"] \arrow[d] & \widetilde{B}_{k+1} \arrow[r, "\widetilde{g}_{k+1}"] \arrow[d] & \widetilde{C}_{k+1} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A_k \arrow[r, "f_k"] \arrow[d] & B_k \arrow[r, "g_k"] \arrow[d] & C_k \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \widetilde{A}_k \arrow[r, "\widetilde{f}_k"] \arrow[d] & \widetilde{B}_k \arrow[r, "\widetilde{g}_k"] \arrow[d] & \widetilde{C}_k \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd} \tag{4.2.2}$$

where A_k, B_k and C_k are projective and all rows and columns are exact. Since Proposition III.4.1 from H. Cartan and S. Eilenberg's book [37] can be generalized to abelian categories, the square

$$\begin{tikzcd} T^{n+k}(\widetilde{C}_k) \arrow[r, "\delta^{n+k}"] \arrow[d, "\delta^{n+k}"] & T^{n+k+1}(\widetilde{A}_k) \arrow[d, "\delta^{n+k+1}"] \\ T^{n+k+1}(\widetilde{C}_{k+1}) \arrow[r, "\delta^{n+k+1}"] & T^{n+k+2}(\widetilde{A}_{k+1}) \end{tikzcd} \tag{4.2.3}$$

anticommutes for any $n \in \mathbb{Z}$ and $k \in \mathbb{N}_0$, meaning that the composite morphisms in the abelian group $\mathrm{Hom}_{\mathcal{D}}(T^{n+k}(\widetilde{C}_k), T^{n+k+2}(\widetilde{A}_{k+1}))$ have opposite signs.

Definition 4.2.2 *For a short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{C}$ we define the (proposed) connecting homomorphism of $T^\bullet_{Res}$ to be*

$$\widetilde{\delta}^n := \varinjlim_{k\in\mathbb{N}_0} (-1)^k \delta^{n+k} : T^n_{Res}(C) \to T^{n+1}_{Res}(A)$$

where the morphisms $(-1)^k\delta^{n+k}$ are connected over the morphisms δ^{n+k} and δ^{n+k+1} from Diagram 4.2.3.

4.3 Isomorphisms of cohomological functors

The key step in establishing that $(T^\bullet_{Res}, \widetilde{\delta}^\bullet)$ is a cohomological functor and proving that $\omega_\bullet : \widehat{T}^\bullet \to T^\bullet_{Res}$ extends to an isomorphism of cohomological functors is

Lemma 4.3.1 *Denote by $\widehat{\delta}^\bullet$ the connecting homomorphism from the satellite functor construction $\widehat{T}^\bullet$. Then for every $n \in \mathbb{Z}$ and every short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{C}$ the diagram*

$$\begin{array}{ccc} \widehat{T}^n(C) & \xrightarrow{\omega_n} & T^n_{Res}(C) \\ \downarrow{\scriptstyle\widehat{\delta}^n} & & \downarrow{\scriptstyle\widetilde{\delta}^n} \\ \widehat{T}^{n+1}(A) & \xrightarrow{\omega_{n+1}} & T^{n+1}_{Res}(A) \end{array}$$

commutes.

Proof The proof is based on showing that the diagram on the next page is commutative. To be precise, it is an infinite prism extending downwards indefinitely. The front side is a copy of Diagram 4.1.6 as well as the back side up to its commuting right most column. The right hand side of the diagram gives rise to $\widetilde{\delta}^n : T^n_{Res}(C) \to T^{n+1}_{Res}(A)$ in the direct limit and the top left hand side to $\widehat{\delta}^n : \widehat{T}^n(C) \to \widehat{T}^{n+1}(A)$. In particular, all squares and triangles from these sides commute. Using the cohomological functors defined in Equation 3.0.3 we see that the horizontal squares commute by [37, Proposition III.4.1]. To investigate the vertical squares running from back to front, we can insert commuting triangles of the form

$$\begin{tikzcd} & S^{-l}T^{n+k}(\widetilde{M}_{k-l}) \arrow[dl, "S^{-l}\underline{\delta}^{n+k}"'] \arrow[d, "\Sigma^{-l}\delta^{n+k}"] \\ S^{-l-1}T^{n+k+1}(\widetilde{M}_{k-l}) \arrow[r, "\varepsilon^{-l-1}"] & S^{-l}T^{n+k+1}(\widetilde{M}_{k-l+1}) \end{tikzcd}$$

into the front and back side by Diagram 4.1.6.

$$
\begin{tikzcd}
 & & & & & & & & T^{n+1}(A) \arrow[dd, "\delta^{n+1}"] \arrow[ddll, bend right, "\mathrm{id}"'] & \\
 & & & & & & & & & T^n(C) \arrow[ul, "\delta^n"'] \arrow[dd, "\delta^n"] \arrow[ddll, bend right, "\underline{\delta}^n"'] \\
 & & & & & & T^{n+1}(A) \arrow[rr, "\delta^{n+1}"] \arrow[dd, "\delta^{n+1}"] \arrow[ddll, bend right, "\underline{\delta}^{n+1}"'] & & T^{n+2}(\widetilde{A}_1) \arrow[dd, "\delta^{n+2}"] & \\
 & & & & & & & S^{-1}T^{n+1}(C) \arrow[ul, "\varepsilon^{-1}"'] \arrow[rr, "\varepsilon^{-1}"] \arrow[dd, "\Sigma^{-1}\delta^{n+1}"] \arrow[ddll, bend right, "S^{-1}\underline{\delta}^{n+1}"'] & & T^{n+1}(\widetilde{C}_1) \arrow[ul, "-\delta^{n+1}"'] \arrow[dd, "\delta^{n+1}"] \\
 & & & & S^{-1}T^{n+2}(A) \arrow[rr, "\varepsilon^{-1}"] \arrow[dd, "\Sigma^{-1}\delta^{n+2}"] \arrow[ddll, bend right, "S^{-1}\underline{\delta}^{n+1}"'] & & T^{n+2}(\widetilde{A}_1) \arrow[rr, "\delta^{n+2}"] \arrow[dd, "\delta^{n+2}"] & & T^{n+3}(\widetilde{A}_2) \arrow[dd, "\delta^{n+3}"] & \\
 & & & & & S^{-2}T^{n+2}(C) \arrow[ul, "\varepsilon^{-2}"'] \arrow[rr, "\varepsilon^{-2}"] \arrow[dd, "\Sigma^{-2}\delta^{n+2}"] \arrow[ddll, bend right, "S^{-2}\underline{\delta}^{n+2}"'] & & S^{-1}T^{n+2}(\widetilde{C}_1) \arrow[ul, "-\varepsilon^{-1}"'] \arrow[rr, "\varepsilon^{-1}"] \arrow[dd, "\Sigma^{-1}\delta^{n+2}"] & & T^{n+2}(\widetilde{C}_2) \arrow[ul, "\delta^{n+2}"'] \arrow[dd, "\delta^{n+2}"] \\
 & & S^{-2}T^{n+3}(A) \arrow[rr, "\varepsilon^{-2}"] \arrow[dd, "\Sigma^{-2}\delta^{n+3}"] \arrow[ddll, bend right, "S^{-2}\underline{\delta}^{n+3}"'] & & S^{-1}T^{n+3}(\widetilde{A}_1) \arrow[rr, "\varepsilon^{-1}"] \arrow[dd, "\Sigma^{-1}\delta^{n+3}"] & & T^{n+3}(\widetilde{A}_2) \arrow[rr, "\delta^{n+3}"] \arrow[dd, "\delta^{n+3}"] & & T^{n+4}(\widetilde{A}_3) \arrow[dd, "\delta^{n+4}"] & \\
 & & & S^{-3}T^{n+3}(C) \arrow[ul, "\varepsilon^{-3}"'] \arrow[rr, "\varepsilon^{-3}=\tilde{\delta}^3"] \arrow[dd, "\Sigma^{-3}\delta^{n+3}"] \arrow[ddll, bend right, "S^{-2}\underline{\delta}^{n+3}"'] & & S^{-2}T^{n+3}(\widetilde{C}_1) \arrow[ul, "-\varepsilon^{-2}"'] \arrow[rr, "\varepsilon^{-2}"] \arrow[dd, "\Sigma^{-2}\delta^{n+3}"] & & S^{-1}T^{n+3}(\widetilde{C}_2) \arrow[ul, "\varepsilon^{-1}"'] \arrow[rr, "\varepsilon^{-1}"] \arrow[dd, "\Sigma^{-1}\delta^{n+3}"] & & T^{n+3}(\widetilde{C}_3) \arrow[ul, "-\delta^{n+3}"'] \arrow[dd, "\delta^{n+3}"] \\
\cdots & & \cdots & & \cdots & & \cdots & & \cdots & \\
 & \cdots & & \cdots & & \cdots & & \cdots & & \cdots
\end{tikzcd}
\tag{4.3.1}
$$

If we write $N := n + k$ and $K := k - l$, then these triangles give rise to prisms of the form

$$\begin{tikzcd}
 & & & S^{-l+1}T^{N}(\widetilde{A}_K) \arrow[dd, "\Sigma^{-l+1}\delta^N"] \arrow[dlll, bend right, "S^{-l+1}\underline{\delta}^N"'] & \\
 & & & & S^{-l}T^{N}(\widetilde{C}_K) \arrow[ul, "(-1)^K\varepsilon^{-l}"'] \arrow[dd, "\Sigma^{-l}\delta^N"] \arrow[ddlll, bend right, "S^{-l}\underline{\delta}^N"'] \\
S^{-l}T^{N+1}(\widetilde{A}_K) \arrow[rrr, "\varepsilon^{-l}"] & & & S^{-l+1}T^{N+1}(\widetilde{A}_{K+1}) & \\
 & S^{-l-1}T^{N+1}(\widetilde{C}_K) \arrow[ul, "(-1)^K\varepsilon^{-l-1}"] \arrow[rrr, "\varepsilon^{-l-1}"] & & & S^{-l}T^{N+1}(\widetilde{C}_{K+1}) \arrow[ul, "(-1)^{K+1}\varepsilon^{-l}"]
\end{tikzcd}$$

where their right hand square correspond to the vertical squares in Diagram 4.3.1. These squares commute because the bottom square of the latter prism does so and the top left square by Diagram 3.0.7. In particular, Diagram 4.3.1 commutes.

Denote by $\chi = \varinjlim \delta^{n+k+1} : T^{n+1}_{Res}(A) \to T^{n+1}_{Res}(A)$ the morphism resulting from the right most column of the back side of Diagram 4.3.1 and write $\Psi^k_{n+1} : T^{n+k+1}(\widetilde{A}_k) \to T^{n+1}_{Res}(A)$ for the morphism to the direct limit. Since every square in this column can be written as

$$\begin{tikzcd}
T^{n+k+1}(\widetilde{A}_k) \arrow[r, "\delta^{n+k+1}"] \arrow[d, "\delta^{n+k+1}"] & T^{n+k+2}(\widetilde{A}_{k+1}) \arrow[d, "\delta^{n+k+2}"] \\
T^{n+k+2}(\widetilde{A}_{k+1}) \arrow[r, "\delta^{n+k+2}"] \arrow[ur, "\mathrm{id}"] & T^{n+k+3}(\widetilde{A}_{k+2})
\end{tikzcd} \tag{4.3.2}$$

we deduce that the diagram

$$\begin{tikzcd}
 & T^{n+k+2}(\widetilde{A}_{k+1}) \arrow[r, "\Psi^{k+1}_{n+1}"] \arrow[d, "\delta^{n+k+2}"] & T^{n+1}_{Res}(A) \\
T^{n+k+2}(\widetilde{A}_{k+1}) \arrow[ur, "\mathrm{id}"] \arrow[r, "\delta^{n+k+2}"] & T^{n+k+3}(\widetilde{A}_{k+2}) \arrow[ur, bend right, "\Psi^{k+2}_{n+1}"] &
\end{tikzcd} \tag{4.3.3}$$

commutes. Observing that that the direct system $(T^{n+k+1}(\widetilde{A}_k), \delta^{n+k+1})_{k\in\mathbb{N}}$ is cofinal in $(T^{n+k+1}(\widetilde{A}_k), \delta^{n+k+1})_{k\in\mathbb{N}_0}$, we conclude from the previous two diagrams that $\mathrm{id} = \chi : T^{n+1}_{Res}(A) \to T^{n+1}_{Res}(A)$.

Let us consider the left most edge in the front side of Diagram 4.3.1 yielding $\widehat{T}^n(C)$. Going over the front side yields $\omega_n : \widehat{T}^n(C) \to T^n_{Res}(C)$. Through the right hand side we obtain $\widetilde{\delta}^n : T^n_{Res}(C) \to T^{n+1}_{Res}(A)$. Thus, going around these sides yields $\widetilde{\delta}^n \circ \omega_n$. Instead, if we go over the left hand side first, we obtain the connecting homomorphism $\widehat{\delta}^n : \widehat{T}^n(C) \to \widehat{T}^{n+1}(A)$. Going subsequently over the back side yields first $\omega_{n+1} : \widehat{T}^{n+1}(A) \to T^{n+1}_{Res}(A)$ and then $\chi : T^{n+1}_{Res}(A) \to T^{n+1}_{Res}(A)$. Hence, going around those sides, we obtain $\chi \circ \omega_{n+1} \circ \widehat{\delta}^n$. Because Diagram 4.3.1 commutes, we conclude that $\widetilde{\delta}^n \circ \omega_n = \chi \circ \omega_{n+1} \circ \widehat{\delta}^n$. □

Theorem 4.3.2 *The resolution construction $(T^\bullet_{Res}, \widetilde{\delta}^\bullet)$ forms a cohomological functor and $\omega_\bullet : (\widehat{T}^\bullet, \widehat{\delta}^\bullet) \to (T^\bullet_{Res}, \widetilde{\delta}^\bullet)$ is an isomorphism of cohomological functors.*

Proof First, we show that the proposed connecting homomorphisms $\widetilde{\delta}^\bullet$ are natural. For this let $n \in \mathbb{Z}$ and let

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\overline{\varphi}^*"] & B \arrow[r, "g"] \arrow[d, "\overline{\varphi}"] & C \arrow[r] \arrow[d, "\varphi"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r] & 0
\end{tikzcd}$$

be a commutative diagram in $\mathcal{C}$ with exact rows.Consider the diagram

$$\begin{tikzcd}
\widehat{T}^{n+1}(C) \arrow[rr, "\omega_{n+1}"] \arrow[dd, "\widehat{T}^{n+1}(\overline{\varphi}^*)"] & & T^{n+1}_{Res}(C) \arrow[dd, "T^{n+1}_{Res}(\overline{\varphi}^*)"] & \\
& \widehat{T}^n(A) \arrow[ul, "\widehat{\delta}^n"] \arrow[rr, "\omega_n"] \arrow[dd, "\widehat{T}^n(\varphi)"] & & T^n_{Res}(A) \arrow[ul, "\widetilde{\delta}^n"'] \arrow[dd, "T^n_{Res}(G,\varphi)"] \\
\widehat{T}^{n+1}(C') \arrow[rr, "\omega_{n+1}"] & & T^{n+1}_{Res}(C') & \\
& \widehat{T}^n(A') \arrow[ul, "\widehat{\delta}^n"] \arrow[rr, "\omega_n"] & & T^n_{Res}(A') \arrow[ul, "\widetilde{\delta}^n"']
\end{tikzcd}$$

By Proposition 4.2.1, the isomorphism ω_n (resp. ω_{n+1}) is natural and thus the front and back squares of the cube commute. According to Lemma 4.3.1, the top and bottom squares of the cube commute. As $(\widehat{T}^\bullet, \widehat{\delta}^\bullet)$ is a cohomological functor, the left hand side commutes. Due to this and the fact that $\omega_n : \widehat{T}^n(A) \to T^n_{Res}(A)$ is an isomorphism, we infer that also the right hand side commutes. Hence $(T^\bullet_{Res}, \widetilde{\delta}^\bullet)$ satisfies Axiom 2.0.1 of a cohomological functor. Using this together with the fact $\omega_n : \widehat{T}^n \to T^n_{Res}$ is a natural transformation for every $n \in \mathbb{Z}$, we see that the diagram

$$\begin{tikzcd}
\dots \arrow[r] & T^n_{Res}(A) \arrow[r, "T^n_{Res}(f)"] \arrow[d, "\omega_n"] & T^n_{Res}(B) \arrow[r, "T^n_{Res}(g)"] \arrow[d, "\omega_n"] & T^n_{Res}(C) \arrow[r, "\widetilde{\delta}^n"] \arrow[d, "\omega_n"] & T^{n+1}_{Res}(A) \arrow[r] \arrow[d, "\omega_{n+1}"] & \dots \\
\dots \arrow[r] & \widehat{T}^n(A) \arrow[r, "\widehat{T}^n(G,f)"] & \widehat{T}^n(B) \arrow[r, "\widehat{T}^n(G,g)"] & \widehat{T}^n(C) \arrow[r, "\widehat{\delta}^n"] & \widehat{T}^{n+1}(A) \arrow[r] & \dots
\end{tikzcd}$$

commutes. Using again that every $\omega_n : \widehat{T}^n \to T^n_{Res}$ is an isomorphism, we infer by the Five Lemma [37, Proposition I.1.1] that every image and kernel in the bottom row is isomorphic to the corresponding image or kernel in the top row. Because $(\widehat{T}^\bullet, \widehat{\delta}^\bullet)$ is a cohomological functor, the bottom row forms a long exact sequence and thus the top row does so too. In particular, $(T^\bullet_{Res}, \widetilde{\delta}^\bullet)$ satisfies Axiom 2.0.2 and is a cohomological functor. From this we conclude that $\omega_\bullet : \widehat{T}^\bullet \to T^\bullet_{Res}$ is an isomorphism of cohomological functors.$\square$

4.4 Countably many constructions of Mislin completions

Based on the material of this chapter thus far, we can demonstrate that there are countably many distinct constructions of Mislin completions.

Lemma 4.4.1 *Let $a = (a_k)_{k\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ be a sequence taking values in 0 and 1. Define $(P(a)_k)_{k\in\mathbb{N}_0}$ by $P(a)_0 := 0$ and $P(a)_k := \sum_{i=1}^k a_i$ and $(D(a)_k)_{k\in\mathbb{N}_0}$ by $D(a)_0 := 0$ and $D(a)_k := \sum_{i=1}^k (1-a_i)$. For any $M \in \mathrm{obj}(\mathcal{C})$ and $n \in \mathbb{Z}$ form the direct system $\left(S^{-P(a)_k} T^{n+k}(\widetilde{M}_{D(a)_k}), \delta_a^{n+k}\right)_{k\in\mathbb{N}_0}$ where*

$$\delta_a^{n+k} := \Sigma^{-P(a)_k} \delta^{n+k} : S^{-P(a)_k} T^{n+k}(\widetilde{M}_{D(a)_k}) \to S^{-P(a)_k} T^{n+k+1}(\widetilde{M}_{D(a)_k+1})$$

if $P(a)_{k+1} = P(a)_k$ and

$$\delta_a^{n+k} := S^{-P(a)_k} \underline{\delta}^{n+k} : S^{-P(a)_k} T^{n+k}(\widetilde{M}_{D(a)_k}) \to S^{-P(a)_k-1} T^{n+k+1}(\widetilde{M}_{D(a)_k})$$

if $P(a)_{k+1} = P(a)_k + 1$. Here the morphisms $\Sigma^{-P(a)_k}\delta^{n+k}$ are taken as in Equation 4.1.3. Write

$$T^n_a(M) := \varinjlim_{\mathcal{D}, k\in\mathbb{N}_0} S^{-P(a)_k} T^{n+k}(\widetilde{M}_{D(a)_k}) .$$

Then there exists an isomorphism $\omega_{a,n} : T^n_a(M) \to T^n_{Res}(M)$.

Proof For any $k \in \mathbb{N}_0$ we define the morphism

$$\widehat{o}_a^k := \begin{cases} \varepsilon^{-1}\circ\ldots\circ\varepsilon^{-P(a)_k} : S^{-P(a)_k}T^{n+k}(\widetilde{M}_{D(a)_k}) \to T^{n+k}(\widetilde{M}_k) & \text{if } P(a)_k \geq 1 \\ \mathrm{id} : T^{n+k}(\widetilde{M}_k) \to T^{n+k}(\widetilde{M}_k) & \text{if } P(a)_k = 0 \end{cases}$$

similarly to the morphism o^k from Equation 4.1.5. By Diagram 4.1.6 we infer that the square

$$\begin{tikzcd} S^{-P(a)_k}T^{n+k}(\widetilde{M}_{D(a)_k}) \arrow[r, "\widehat{o}_a^k"] \arrow[d, "\delta_a^{n+k}"] & T^{n+k}(\widetilde{M}_k) \arrow[d, "\delta^{n+k}"] \\ S^{-P(a)_{k+1}}T^{n+k+1}(\widetilde{M}_{D(a)_{k+1}}) \arrow[r, "\widehat{o}_a^{k+1}"] & T^{n+k+1}(\widetilde{M}_{k+1}) \end{tikzcd}$$

is commutative. We conclude as in the proof of Lemma 4.1.1 that the morphism $\omega_{a,n} := \varinjlim_{k\in\mathbb{N}_0} \widehat{o}_a^k$ is an isomorphism. $\square$

We can reiterate the proof of Proposition 4.2.1 to demonstrate

Proposition 4.4.2 *Let $f : M \to N$ be a morphism in $\mathcal{C}$ and assume that $(\widetilde{f}_k : \widetilde{M}_k \to \widetilde{N}_k)_{k\in\mathbb{N}_0}$ is a sequence of morphism as in Diagram 4.2.1. Then the squares*

$$\begin{tikzcd} S^{-P(a)_k}T^{n+k}(\widetilde{M}_{D(a)_k}) \arrow[r, "S^{-P(a)_k}T^{n+k}(\widetilde{f}_{D(a)_k})"] \arrow[d, "\delta_a^{n+k}"] & S^{-P(a)_k}T^{n+k}(\widetilde{N}_{D(a)_k}) \arrow[d, "\delta_a^{n+k}"] \\ S^{-P(a)_{k+1}}T^{n+k+1}(\widetilde{M}_{D(a)_{k+1}}) \arrow[r, "S^{-P(a)_{k+1}}T^{n+k+1}(\widetilde{f}_{D(a)_{k+1}})"] & S^{P(a)_{k+1}}T^{n+k+1}(\widetilde{N}_{D(a)_{k+1}}) \end{tikzcd}$$

commute. If we define $T_a^n(f) := \varinjlim_{k\in\mathbb{N}_0} S^{-P(a)_k}T^{n+k}(\widetilde{f}_{D(a)_k})$, then the square

$$\begin{tikzcd} T_a^n(M) \arrow[r, "\omega_{a,n}(M)"] \arrow[d, "T_a^n(f)"] & T_{Res}^n(M) \arrow[d, "T_{Res}^n(f)"] \\ T_a^n(N) \arrow[r, "\omega_{a,n}(N)"] & T_{Res}^n(N) \end{tikzcd}$$

is also commutative. In particular, every term $T_a^n(-)$ forms an additive functor and the homomorphisms $\omega_{a,n} : \widehat{T}^n \to T_{Res}^n$ a natural isomorphism.

Lemma 4.4.3 *If $0 \to A \to B \to C \to 0$ is a short exact sequence, define* *for any $k \in \mathbb{N}_0$ the morphism*

$$\delta_{a,k}^{n+k} := \varepsilon^{-P(a)_k - 1} \circ S^{-P(a)_k} \underline{\delta}^{n+k} : \\ S^{-P(a)_k} T^{n+k}(\widetilde{C}_{D(a)_k}) \to S^{-P(a)_k - 1} T^{n+k+1}(\widetilde{C}_{D(a)_k}) \\ \to S^{-P(a)_k} T^{(n+1)+k}(\widetilde{A}_{D(a)_k})$$

Then the squares

$$\begin{array}{ccc} S^{-P(a)_k} T^{n+k}(\widetilde{C}_{D(a)_k}) & \xrightarrow{\widehat{\sigma}_a^k} & T^{n+k}(\widetilde{C}_k) \\ \downarrow \delta_a^{n+k} & & \downarrow \delta^{n+k} \\ S^{-P(a)_k} T^{(n+1)+k}(\widetilde{A}_{D(a)_k}) & \xrightarrow{\widehat{\sigma}_a^{k+1}} & T^{(n+1)+k}(\widetilde{A}_k) \end{array} \tag{4.4.1}$$

commute whence we define $\widetilde{\delta}_a^n := \varinjlim_{k \in \mathbb{N}_0} \delta_{a,k}^{n+k} : T_a^n(C) \to T_a^{n+1}(A)$. Therefore, the square

$$\begin{array}{ccc} T_a^n(C) & \xrightarrow{\omega_{a,n}} & T_{Res}^n(C) \\ \downarrow \widetilde{\delta}_a^n & & \downarrow \widetilde{\delta}^n \\ T_a^{n+1}(A) & \xrightarrow{\omega_{a,n+1}} & T_{Res}^{n+1}(A) \end{array} \tag{4.4.2}$$

commutates, $(T_a^\bullet, \widetilde{\delta}_a^\bullet)$ is a cohomological functor and $\omega_{a,\bullet} : (T_a^\bullet, \widetilde{\delta}_a^\bullet) \to (T_{Res}^\bullet, \widetilde{\delta}^\bullet)$ an isomorphism of cohomological functors.

Proof Looking at Diagram 4.4.1, the term $S^{-P(a)_k} T^{n+k}(\widetilde{C}_{D(a)_k})$ lies in the front side of Diagram 4.3.1. We reach the very right hand side of the latter diagram through the morphism $\widehat{\sigma}_a^k : S^{-P(a)_k} T^{n+k}(\widetilde{C}_{D(a)_k}) \to T^{n+k}(\widetilde{C}_k)$. The morphism $\delta^{n+k} : T^{n+k}(\widetilde{C}_k) \to T^{(n+1)+k}(\widetilde{A}_k)$ leads us to the back side and we move by $\mathrm{id} : T^{(n+1)+k}(\widetilde{A}_k) \to T^{(n+1)+k}(\widetilde{A}_k)$ from Diagram 4.3.2 across a square in the very right hand column of the back side. On the other had, the morphism $\delta_{a,k}^{n+k} : S^{-P(a)_k} T^{n+k}(\widetilde{C}_{D(a)_k}) \to S^{-P(a)_k} T^{(n+1)+k}(\widetilde{A}_{D(a)_k})$ takes us first down the front side of Diagram 4.3.1 and then over to the back side. Lastly, we reach the left hand side of the right most column of the back side by the morphism $\widehat{\sigma}_a^{k+1} : S^{-P(a)_k} T^{(n+1)+k}(\widetilde{A}_{D(a)_k}) \to T^{(n+1)+k}(\widetilde{A}_k)$. Since Diagram 4.3.1 commutes, we conclude that

$$\mathrm{id} \circ \delta^{n+k} \circ \widehat{\sigma}_a^k = \widehat{\sigma}_a^{k+1} \circ \delta_{a,k}^{n+k} : \\ S^{-P(a)_k} T^{n+k}(\widetilde{C}_{D(a)_k}) \to S^{-P(a)_k} T^{(n+1)+k}(\widetilde{A}_{E_k}) . \tag{4.4.3}$$

Therefore, Diagram 4.4.1 commutes. Using again that Diagram 4.3.1 is commutative, we observe that the squares of Diagram 4.4.1 form a direct system in whose direct limit we obtain Diagram 4.4.2. The remainder of the lemma is proved identically to Theorem 4.3.2. □

Corollary 4.4.4 *Let us denote by $e \in \{0,1\}^{\mathbb{N}}$ the sequence all whose terms are set to 0 and by $f \in \{0,1\}^{\mathbb{N}}$ the sequence all whose terms are set to 1. Then $(\widehat{T}^\bullet, \widehat{\delta}^\bullet) = (T_f^\bullet, \widetilde{\delta}_f^\bullet)$ and $(T_{Res}^\bullet, \widetilde{\delta}^\bullet) = (T_e^\bullet, \widetilde{\delta}_e^\bullet)$.*

Proof By definition, $\widehat{T}^n = T_f^n$ and $T_{Res}^n = T_e^n$ for every $n \in \mathbb{Z}$. In the same manner as we have concluded by Diagram 4.3.2 and 4.3.3 that the morphism χ in the proof of Lemma 4.3.1 agrees with the identity, we see that

$$\widetilde{\delta}_f^n = \varinjlim_{k\in\mathbb{N}_0} \delta_{f,k}^{n+k} = \varinjlim_{k\in\mathbb{N}} \varepsilon^{-k-1} \circ S^{-k}\underline{\delta}^{n+k} = \varinjlim_{k\in\mathbb{N}} \varepsilon^{-k} = \widehat{\delta}^n$$

where

$$\varepsilon^{-k-1} \circ S^{-k}\underline{\delta}^{n+k} : S^{-k}T^{n+k} \to S^{-k}T^{(n+1)+k}$$

and

$$\varepsilon^{-k} : S^{-k}T^{n+k} \to S^{-k+1}T^{n+k}\,.$$

By Equation 4.4.3, we observe that $\widetilde{\delta}_e^n = \widetilde{\delta}^n$. □

Lemma 4.4.5 *All constructions of Mislin completions thus far are based on direct limits. Two constructions are said to agree if their underlying direct limits always agree. Otherwise, call the constructions are distinct. Then there are countably many distinct constructions of Mislin completions of the form $(T_a^\bullet, \widetilde{\delta}_a^\bullet)$.*

Proof Remember that

$$T_a^n(M) = \varinjlim_{k\in\mathbb{N}_0} S^{-P(a)_k}T^{n+k}(\widetilde{M}_{D(a)_k})$$

for any $a \in \{0,1\}^{\mathbb{N}}$ and $M \in \mathrm{obj}(\mathcal{C})$. By [40, Proposition 8.2] and the fact that Diagram 4.1.6 commutes, we infer that $(T_a^\bullet, \widehat{\delta}_a^\bullet) = (T_b^\bullet, \widehat{\delta}_b^\bullet)$ for another sequence $b \in \{0,1\}^{\mathbb{N}}$ if for every $k \in \mathbb{N}$ there exists $K \geq k$ such that $P(a)_K = P(b)_K$. On the other hand, if there is $L \in \mathbb{N}$ such that $P(a)_k \neq P(b)_k$ for every $k \geq L$, then the constructions giving rise to $(T_a^\bullet, \widehat{\delta}_a^\bullet)$ and $(T_b^\bullet, \widehat{\delta}_b^\bullet)$ are distinct. Define the binary relation '$\sim$' on $\{0,1\}^{\mathbb{N}}$ by setting $a \sim b$ if for every $k \in \mathbb{N}$ there is $K \geq k$ such that $P(a)_K = P(b)_K$. If $\sim'$ denotes the transitive closure of $\sim$, then the proof reduces to showing that there are countably many elements in $\{0,1\}^{\mathbb{N}}/\sim'$. Hence, define for any $m \in \mathbb{N}$ the sequences $e^m, f^m \in \{0,1\}^{\mathbb{N}}$ by

$$e_k^m := \begin{cases} 1 & \text{if } k < m \\ 0 & \text{if } k \geq m \end{cases} \qquad \text{and} \qquad f_k^m := \begin{cases} 0 & \text{if } k < m \\ 1 & \text{if } k \geq m \end{cases}$$

Comparing with Corollary 4.4.4, we see that $e = e_0$ and $f = f_0$. For any $g \in \{0,1\}^{\mathbb{N}}$ taking finitely many times either the value 0 or the value 1 there is $m \in \mathbb{N}$ such that $g \sim e_m$ or $g \sim f_m$. For any distinct $m, p \in \mathbb{N}$ we note that $e_m \nsim' e_p$, $e_m \nsim' f_p$ and $f_m \nsim' f_p$. To complete the proof, we demonstrate that for any $a, b \in \{0,1\}^{\mathbb{N}}$ taking both the values 0 and 1 infinitely often there is another sequence c such that $a \sim c$ and $c \sim b$. If $a \nsim b$, then assume that there is $K \in \mathbb{N}$ such that $P(a)_k < P(b)_k$ for every $k \geq K$. We construct c inductively as follows. For the base case, set $c_k := a_k$ for any $1 \leq k \leq K$ and let $c_l := 1$ for the subsequent $K \leq l$. Since $P(c)$ grows faster than $P(b)$, there is $l_1 \in \mathbb{N}$ such that $P(c)_{l_1} = P(b)_{l_1}$. Assume for $i \in \mathbb{N}$ that we have found $k_i < l_i \in \mathbb{N}$ such that $P(a)_{k_i} = P(c)_{k_i}$ and $P(c)_{l_i} = P(b)_{l_i}$. Setting $c_k := 0$ for $k \geq l_i$ yields $k_{i+1} \in \mathbb{N}$ and defining $c_l := 1$ for $l \geq k_{i+1}$ yields $l_{i+1} \in \mathbb{N}$. Then $a \sim c \sim b$ as desired. □

Notation 4.4.6 *In the rest of the thesis, we write $(\widehat{T}^\bullet, \widehat{\delta}^\bullet)$ for the Mislin completion of a cohomological functor $(T^\bullet, \delta^\bullet)$ independently from which construction it arises. When required, we specify the construction or use a special notation for it all of which we keep consistent throughout.*

Chapter 5

The naive and the resolution construction

In this chapter we prove that the naive construction gives rise to Mislin completions of unenriched Ext-functors. For this, we rigorosly perform this construction in Section 5.1. Thereafter, we show in Section 5.2 that its cohomology groups are naturally isomorphic to the ones of the resolution construction. Lastly, in Section 5.3, we construct a connecting homomorphism for the naive construction and extend the isomorphism between the cohomology groups to an isomorphism of cohomological functors.

5.1 Performing the naive construction

To start, we consider the unenriched Hom-functors

$$\mathrm{Hom}_{\mathcal{C}}(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Ab}$$

which are additive in both variables. As in Definition 2.0.5, we turn the corresponding unenriched Ext-functors into cohomological functors by setting $\mathrm{Ext}^n_{\mathcal{C}}(A,-) = 0$ for $n < 0$ and any $A \in \mathrm{obj}(\mathcal{C})$. Both A. Beligiannis and I. Reiten in [29, p. 37–38/163–165] as well as S. Guo and L. Liang in [28, p. 14–15] have established relative homological versions of the naïve construction in great generality. However, in order to construct external products in Theorem 7.4.1, we require an (absolute homological) generalisation faithful to D. J. Benson and J. F. Carlson's original construction found in [13, p. 109], which is why our account differs from the previous two. First, we define a bifunctor $[-,-]_{\mathcal{C}}$ that we have already encountered in Section 2. Let $\mathcal{P}_{\mathcal{C}}(A,B) \subseteq \mathrm{Hom}_{\mathcal{C}}(M,N)$ be the subset of all morphisms factoring through a projective object. Since the zero object $0 \in \mathcal{C}$ is projective, the zero morphism is in $\mathcal{P}_{\mathcal{C}}(A,B)$. Let $\varphi : M \to P \to N$ and $\psi : M \to Q \to N$ be morphisms factoring though projectives. If $\Delta_A : A \to A \oplus A$ is the diagonal map and $\nabla_B : B \oplus B \to B$ the codiagonal, then $\varphi + \psi = \nabla_B \circ (\varphi \oplus \psi) \circ \Delta_A$ by [8, p. 253]. Factoring the map $\varphi \oplus \psi : A \oplus A \to B \oplus B$ further, we obtain

$$\varphi + \psi : A \xrightarrow{\Delta_A} A \oplus A \to P \oplus Q \to B \oplus B \xrightarrow{\nabla_B} B\,.$$

As $P \oplus Q$ is a projective object [8, p. 250–251], $\varphi + \psi \in \mathcal{P}_{\mathcal{C}}(M, N)$ and $\mathcal{P}_{\mathcal{C}}(A, B)$ is a subgroup of $\mathrm{Hom}_{\mathcal{C}}(A, B)$. Define

$$[A, B]_{\mathcal{C}} := \mathrm{Hom}_{\mathcal{C}}(A, B)/\mathcal{P}_{\mathcal{C}}(A, B)\,.$$

The following proposition is not only relevant to the construction, but also later when discussing Yoneda and external products in Section 7.4.

Proposition 5.1.1 *The bifunctor*

$$\circ : \mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A, C)$$

descends to a bifunctor $\circ : [B, C]_{\mathcal{C}} \times [A, B]_{\mathcal{C}} \to [A, C]_{\mathcal{C}}$ *that is associative and additive in both variables.*

Proof First we compose the composition bifunctor with the projection homomorphism $\mathrm{Hom}_{\mathcal{C}}(A, C) \to [A, C]_{\mathcal{C}}$. If we quotient out the subgroup $\mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathcal{P}_{\mathcal{C}}(A, B)$ in the domain, this yields

$$\mathrm{Hom}_{\mathcal{C}}(B, C) \times [M, N]_{\mathcal{C}} \to [A, C]_{\mathcal{C}}\,.$$

Lastly, we quotient out $\mathcal{P}_{\mathcal{C}}(B, C) \times [A, B]_{\mathcal{C}}$ in the domain. Pictorially, this can be summarized in the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(B,C)\times\mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[r] \arrow[d, "-\circ-"] & \mathrm{Hom}_{\mathcal{C}}(B,C)\times[A,B]_{\mathcal{C}} \arrow[r] \arrow[dr] & {[B,C]_{\mathcal{C}}\times[A,B]_{\mathcal{C}}} \arrow[d, "-\circ-"] \\
\mathrm{Hom}_{\mathcal{C}}(A,C) \arrow[rr] & & {[A,C]_{\mathcal{C}}}
\end{tikzcd} \tag{5.1.1}$$

Since the original composition functor is associative as well as additive in both variables and the new one is obtained by subsequent quotients, also the resulting composition functor is associative as well as additive in both variables. $\square$

If $(A_n)_{n\in\mathbb{N}_0}$, $(B_n)_{n\in\mathbb{N}_0}$ are projective resolutions of A, B, then the first step of the naive construction is given by the following generalization of results found in [3, p. 204]

Proposition 5.1.2 *There exists a homomorphism* $t_{A,B} : [A,B]_{\mathcal{C}} \to [\widetilde{A}_1, \widetilde{B}_1]_{\mathcal{C}}$ *with the property that* $t_{B,C}(-) \circ t_{A,B}(-) = t_{A,C}(- \circ -)$.

Proof Consider the following version of Diagram 4.2.1:

$$\begin{tikzcd} 0 \arrow[r] & \widetilde{A}_1 \arrow[r, "\iota_A"] \arrow[d, "\widetilde{f}_1"] & A_0 \arrow[r, "\pi_A"] \arrow[d, "f_0"] & A \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & \widetilde{B}_1 \arrow[r, "\iota_B"] & B_0 \arrow[r, "\pi_B"] & B \arrow[r] & 0 \end{tikzcd} \tag{5.1.2}$$

If $f_0' : A_0 \to B_0$ and $\widetilde{f}_1' : \widetilde{A}_1 \to \widetilde{B}_1$ are different lifts, then the diagram

$$\begin{tikzcd} 0 \arrow[r] & \widetilde{A}_1 \arrow[r, "\iota_A"] \arrow[d, "\widetilde{f}_1 - \widetilde{f}_1'"] & A_0 \arrow[r, "\pi_A"] \arrow[d, "f_0 - f_0'"] & A \arrow[r] \arrow[d, "0"] & 0 \\ 0 \arrow[r] & \widetilde{B}_1 \arrow[r, "\iota_B"] & B_0 \arrow[r, "\pi_B"] & B \arrow[r] & 0 \end{tikzcd}$$

commutes. Because ι_B is the kernel of π_B, there is a morphism $e : A_0 \to \widetilde{B}_1$ making the bottom right triangle of

$$\begin{tikzcd} 0 \arrow[r] & \widetilde{A}_1 \arrow[r, "\iota_A"] \arrow[d, "\widetilde{f}_1 - \widetilde{f}_1'"] & A_0 \arrow[d, "f_0 - f_0'"] \arrow[dl, "e"] \\ 0 \arrow[r] & \widetilde{B}_1 \arrow[r, "\iota_B"] & B_0 \end{tikzcd}$$

commute. As the above square also commutes and ι_B is a monomorphism, we observe that $\widetilde{f}_1 - \widetilde{f}_1' = e \circ \iota_A$. Using that A_0 is projective, $\widetilde{f}_1$ and $\widetilde{f}_1'$ agree in $[\widetilde{A}_1, \widetilde{B}_1]_{\mathcal{C}}$. Therefore, there is a well define map

$$s_{A,B} : \mathrm{Hom}_{\mathcal{C}}(A,B) \to [\widetilde{A}_1, \widetilde{B}_1]_{\mathcal{C}}, f \mapsto \widetilde{f}_1 + \mathcal{P}_{\mathcal{C}}(\widetilde{A}_1, \widetilde{B}_1)\,.$$

We infer by construction that

$$s_{A,B}(f+g) = \widetilde{f}_1 + \widetilde{g}_1 + \mathcal{P}_{\mathcal{C}}(\widetilde{A}_1, \widetilde{B}_1)$$

and $s_{A,B}$ is a homomorphism. If $h \in \mathrm{Hom}_{\mathcal{C}}(B,C)$, then one can concatenate the corresponding diagrams of the above form in order to obtain that $s_{B,C}(h) \circ s_{A,B}(f) = s_{A,C}(h \circ f)$. In particular, if $f \in \mathcal{P}_{\mathcal{C}}(A,B)$, then one can choose a projective resolution of the respective projective such that $s_{A,B}(f) = 0$. Therefore, we conclude that $s_{A,B}$ descends to the desired homomorphism $t_{A,B} : [A,B]_{\mathcal{C}} \to [\widetilde{A}_1, \widetilde{B}_1]_{\mathcal{C}}$. Using again the observation that $s_{B,C}(-) \circ s_{A,B}(-) = s_{A,C}(- \circ -)$, we can factor the respective maps through quotient homomorphisms as in the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(B,C)\times[A,B]_{\mathcal{C}} \arrow[r] \arrow[rr, bend left, "s_{B,C}\times t_{A,B}"] & {[B,C]_{\mathcal{C}}\times[A,B]_{\mathcal{C}}} \arrow[r, "t_{B,C}\times t_{A,B}"] & {[\widetilde{B}_1,\widetilde{C}_1]\times[\widetilde{A}_1,\widetilde{B}_1]} \arrow[dd, "-\circ-"] \\
\mathrm{Hom}_{\mathcal{C}}(B,C)\times\mathrm{Hom}_{\mathcal{C}}(A,B) \arrow[u] \arrow[urr, "s_{B,C}\times s_{A,B}"] \arrow[d, "-\circ-"] & & \\
\mathrm{Hom}_{\mathcal{C}}(A,C) \arrow[r] \arrow[rr, bend right, "s_{A,C}"'] & {[A,C]_{\mathcal{C}}} \arrow[r, "t_{A,C}"] & {[\widetilde{A}_1,\widetilde{C}_1]}
\end{tikzcd}$$

Note that Diagram 5.1.1 can be fitted into the left hand side of the rectangle that is embedded within the above diagram. Since the quotient homomorphisms from the proof of Proposition 5.1.1 give rise to the surjective map

$$\mathrm{Hom}_{\mathcal{C}}(B,C)\times\mathrm{Hom}_{\mathcal{C}}(A,B)\to\mathrm{Hom}_{\mathcal{C}}(B,C)\times[A,B]_{\mathcal{C}}\to[B,C]_{\mathcal{C}}\times[A,B]_{\mathcal{C}}\,,$$

found in the top left corner of the above diagram, we conclude that the square

$$\begin{tikzcd}
{[B,C]_{\mathcal{C}}\times[A,B]_{\mathcal{C}}} \arrow[r, "t_{B,C}\times t_{A,B}"] \arrow[d, "-\circ-"] & {[\widetilde{B}_1,\widetilde{C}_1]\times[\widetilde{A}_1,\widetilde{B}_1]} \arrow[d, "-\circ-"] \\
{[A,C]_{\mathcal{C}}} \arrow[r, "t_{A,C}"] & {[\widetilde{A}_1,\widetilde{C}_1]}
\end{tikzcd}$$

commutes, meaning that $t_{B,C}(-)\circ t_{A,B}(-)=t_{A,C}(-\circ-)$. $\square$

Definition 5.1.3 *Let $n\in\mathbb{Z}$. There is for any $k\in\mathbb{N}_0$ with $n+k\geq 0$ a homomorphism $t_{\widetilde{A}_{n+k},\widetilde{B}_{n+k}}:[\widetilde{A}_{n+k},\widetilde{B}_k]_{\mathcal{C}}\to[\widetilde{A}_{n+k+1},\widetilde{B}_{k+1}]_{\mathcal{C}}$. We define the n^{th} term of the naive construction to be*

$$BC^n_{\mathcal{C}}(A,B):=\varinjlim_{k\in\mathbb{N}_0,n+k\geq 0}\left([\widetilde{A}_{n+k},\widetilde{B}_k]_{\mathcal{C}},t_{\widetilde{A}_{n+k},\widetilde{B}_k}\right).$$

In order to define induced homomorphisms for $BC^n_{\mathcal{C}}(-,-)$, let A, B, C be objects in $\mathcal{C}$ and consider corresponding projective resolutions $A_\bullet$, $B_\bullet$, $C_\bullet$. For a morphism $f:B\to C$ in $\mathcal{C}$ let us construct a homomorphism

$BC^n_{\mathcal{C}}(A,f)$ using these projective resolutions. Lifting f as in the proof of Proposition 5.1.2, we obtain a sequence of morphisms $(\widetilde{f}_k : \widetilde{B}_k \to \widetilde{C}_k)_{k\in\mathbb{N}_0}$ where

$$t_{\widetilde{B}_k,\widetilde{C}_k}\big(\widetilde{f}_k + \mathcal{P}_{\mathcal{C}}(\widetilde{B}_k,\widetilde{C}_k)\big) = \widetilde{f}_{k+1} + \mathcal{P}_{\mathcal{C}}(\widetilde{B}_{k+1},\widetilde{C}_{k+1})\,.$$

Since $f = \widetilde{f}_0$, the element $(\widetilde{f}_k + \mathcal{P}_{\mathcal{C}}(\widetilde{B}_k,\widetilde{C}_k))_{k\in\mathbb{N}_0}$ of $([\widetilde{B}_k,\widetilde{C}_k]_{\mathcal{C}})_{k\in\mathbb{N}_0}$ is independent of the lifts of f. Again by Proposition 5.1.2,

$$t_{\widetilde{A}_{n+k},\widetilde{C}_k}(-\circ-) = t_{\widetilde{B}_k,\widetilde{C}_k}(-)\circ t_{\widetilde{A}_{n+k},\widetilde{B}_k}(-)$$

and thus the square of homomorphisms of abelian groups

$$\begin{tikzcd}
{[\widetilde{A}_{n+k},\widetilde{B}_k]_{\mathcal{C}}} \arrow[r, "\big(\widetilde{f}_k+\mathcal{P}_{\mathcal{C}}(\widetilde{B}_k,\widetilde{C}_k)\big)\circ-"] \arrow[d, "t_{\widetilde{A}_{n+k},\widetilde{B}_k}"] & {[\widetilde{A}_{n+k},\widetilde{C}_k]_{\mathcal{C}}} \arrow[d, "t_{\widetilde{A}_{n+k},\widetilde{C}_k}"] \\
{[\widetilde{A}_{n+k+1},\widetilde{B}_{k+1}]_{\mathcal{C}}} \arrow[r, "\big(\widetilde{f}_{k+1}+\mathcal{P}_{\mathcal{C}}(\widetilde{B}_{k+1},\widetilde{C}_{k+1})\big)\circ-"] & {[\widetilde{A}_{n+k+1},\widetilde{C}_{k+1}]_{\mathcal{C}}}
\end{tikzcd} \tag{5.1.3}$$

commutes. Analogously, if $g : C \to A$ is another morphism, then we can construct an element $(\widetilde{g}_k + \mathcal{P}_{\mathcal{C}}(\widetilde{C}_k,\widetilde{A}_k))_{k\in\mathbb{N}_0}$ of $([\widetilde{C}_k,\widetilde{A}_k]_{\mathcal{C}})_{k\in\mathbb{N}_0}$ that is independent of the lifts of g.

Definition 5.1.4 *Taking the notation from before, we define*

$$BC^n_{\mathcal{C}}(A,f) := \varinjlim_{k\in\mathbb{N}_0,n+k\geq 0}\big(\big(\widetilde{f}_k+\mathcal{P}_{\mathcal{C}}(\widetilde{B}_k,\widetilde{C}_k)\big)\circ-\big) : BC^n_{\mathcal{C}}(A,B)\to BC^n_{\mathcal{C}}(A,C)$$

and

$$BC^n_{\mathcal{C}}(g,B) := \varinjlim_{k\in\mathbb{N}_0,n+k\geq 0}\big(-\circ\big(\widetilde{g}_{n+k}+\mathcal{P}_{\mathcal{C}}(\widetilde{C}_{n+k},\widetilde{A}_{n+k})\big)\big) : \\ BC^n_{\mathcal{C}}(A,B)\to BC^n_{\mathcal{C}}(C,B)\,.$$

Proposition 5.1.5 *For every $n\in\mathbb{Z}$, $BC^n_{\mathcal{C}}(-,-) : \mathcal{C}^{\mathrm{op}}\times\mathcal{C}\to \mathrm{Ab}$ forms a well defined bifunctor that is additive in both variables.*

Proof We demonstrate that $BC_{\mathcal{C}}(A,B)$ is independent of choices of projective resolutions. Hence let $A_\bullet$ and $A'_\bullet$ be projective resolutions of A. As in Definition 5.1.4, take $(\widetilde{\iota}_k + \mathcal{P}_{\mathcal{C}}(\widetilde{A}_k,\widetilde{A}'_k))_{k\in\mathbb{N}_0}$ and $(\widetilde{\iota}'_k + \mathcal{P}_{\mathcal{C}}(\widetilde{A}'_k,\widetilde{A}_k))_{k\in\mathbb{N}_0}$ lifting $\mathrm{id}_A : A\to A$. Given that

$$(\widetilde{\iota}'_k\circ\widetilde{\iota}_k + \mathcal{P}_{\mathcal{C}}(\widetilde{A}_k,\widetilde{A}_k))_{k\in\mathbb{N}_0} \text{ and } (\widetilde{\iota}_k\circ\widetilde{\iota}'_k + \mathcal{P}_{\mathcal{C}}(\widetilde{A}'_k,\widetilde{A}'_k))_{k\in\mathbb{N}_0}$$

are also lifts of id_A, we conclude by Proposition 5.1.2 that

$$\iota'_k \circ \iota_k + \mathcal{P}_{\mathcal{C}}(\widetilde{A}_k, \widetilde{A}_k) = \mathrm{id}_{\widetilde{A}_k} + \mathcal{P}_{\mathcal{C}}(\widetilde{A}_k, \widetilde{A}_k) \text{ and}$$
$$\iota_k \circ \iota'_k + \mathcal{P}_{\mathcal{C}}(\widetilde{A}'_k, \widetilde{A}'_k) = \mathrm{id}_{\widetilde{A}'_k} + \mathcal{P}_{\mathcal{C}}(\widetilde{A}'_k, \widetilde{A}'_k)$$

for any $k \in \mathbb{N}_0$. As there is an analogous statement for any two projective resolutions of B, $BC^n_{\mathcal{C}}(A, B)$ is unique up to isomorphism. Knowing that composition of morphisms is associative by Proposition 5.1.1, we see that

$$BC_{\mathcal{C}}(A, f'' \circ f) = BC_{\mathcal{C}}(A, f'') \circ BC_{\mathcal{C}}(A, f) \text{ and}$$
$$BC_{\mathcal{C}}(g \circ g'', B) = BC_{\mathcal{C}}(g, B) \circ BC_{\mathcal{C}}(g'', B)$$

for any $f'' : C \to D$ and $g'' : D \to C$. Thus, $BC^n_{\mathcal{C}}(-,-)$ a well defined bifunctor. Because composition of morphisms is bi-additive also by Proposition 5.1.1 and Diagram 5.1.3 commutes, we infer that

$$BC^n_{\mathcal{C}}(A, f + f') = BC^n_{\mathcal{C}}(A, f) + BC^n_{\mathcal{C}}(A, f') \text{ and}$$
$$BC^n_{\mathcal{C}}(g + g', B) = BC^n_{\mathcal{C}}(g, B) + BC^n_{\mathcal{C}}(g', B)$$

for any $f, f' : B \to C$ and $g, g' : C \to A$. In particular, $BC^n_{\mathcal{C}}(-,-)$ is additive in both variables. $\square$

5.2 Isomorphism of cohomology groups

Let us construct a homomorphism from completed Ext-functors $\widehat{\mathrm{Ext}}^n_{\mathcal{C}}(A, B)$ arising from the resolution construction to the terms $BC^n_{\mathcal{C}}(A, B)$ of the naive construction. The idea of the construction and of the proof that it yields an isomorphism can be retrieved from page 299 of G. Mislin's paper [7] and are generalized to our setting in the following. Again, this differs from S. Guo and L. Liang's account containing a similar construction and proof on pages 18–19 of their paper [28]. If $\partial_{n+1} : A_{n+1} \to A_n$ denotes a morphism of a projective resolution $A_\bullet$ of A, then

$$\mathrm{Ker}(\mathrm{Hom}_{\mathcal{C}}(\partial_{n+1}, B)) = \{f \in \mathrm{Hom}_{\mathcal{C}}(A_n, B) \mid f \circ \partial_{n+1} = 0\}\,.$$

The cokernel of ∂_{n+1} can be realized by the morphism $\pi_n : A_n \to \widetilde{A}_n$ occurring in the factorization $\partial_n : A_n \xrightarrow{\pi_n} \widetilde{A}_n \xrightarrow{\iota_n} A_{n-1}$. Thus, for every element $f \in \mathrm{Ker}(\mathrm{Hom}_{\mathcal{C}}(\partial_{n+1}, B))$ there is a unique morphism $f' : \widetilde{A}_n \to B$ such that $f = f' \circ \pi_n$. Therefore, there is a bijection

$$\alpha_n(B) : \mathrm{Ker}(\mathrm{Hom}_{\mathcal{C}}(\partial_{n+1}, B)) \to \mathrm{Hom}_{\mathcal{C}}(\widetilde{A}_n, B), \quad f \mapsto f'\,.$$

As $(\alpha_n(B)(f)+\alpha_n(B)(g))\circ\pi_n = f+g$, $\alpha_n(B)$ is an isomorphisms of abelian groups. Pictorially, one can summarize the content of this isomorphism as

$$\begin{tikzcd} A_n \arrow[r, "\pi_n"] \arrow[dr, "f"'] & \widetilde{A}_n \arrow[d, "\alpha_n(B)(f)"] \\ & B \end{tikzcd} \tag{5.2.1}$$

Before we proceed we need to observe that in the case $n = 0$ this yields an isomorphism

$$\alpha_0(B) : \mathrm{Ext}^0_{\mathcal{C}}(A,B) \to \mathrm{Hom}_{\mathcal{C}}(A,B)$$

since $\mathrm{Ker}(\mathrm{Hom}_{\mathcal{C}}(\partial_0,B)) = \mathrm{Ext}^0_{\mathcal{C}}(A,B)$. If $n \geq 1$, assume that f is an element in $\mathrm{Im}(\mathrm{Hom}_{\mathcal{C}}(\partial_n,B)) \leq \mathrm{Ker}(\mathrm{Hom}_{\mathcal{C}}(\partial_{n+1},B))$. Being of the form

$$f : A_n \xrightarrow{\partial_n} A_{n-1} \to B\,,$$

it follows from the above factorization of ∂_n that

$$\alpha_n(B)(f) : \widetilde{A}_n \xrightarrow{\iota_n} A_{n-1} \to B\,.$$

As A_{n-1} is a projective object, $\alpha_n(B)(f) \in \mathcal{P}_R(\widetilde{A}_n,B)$. Thus $\alpha_n(B)$ descends to a homomorphism of abelian groups

$$\begin{aligned} \beta_n(B)\colon\ \mathrm{Ext}^n_{\mathcal{C}}(A,B) &\to [\widetilde{A}_n,B]_{\mathcal{C}} \\ f+\mathrm{Im}(\mathrm{Hom}_{\mathcal{C}}(\partial_n,B)) &\mapsto \alpha_n(B)(f)+\mathcal{P}_R(\widetilde{A}_n,B)\,. \end{aligned}$$

Due to the above, we remark that $\beta^n(B)$ is only defined for $n \geq 1$ while we already have the isomorphism $\alpha_0(B)$ in the case $n = 0$. The last ingredient to construct our desired isomorphism between the cohomology groups is

Proposition 5.2.1 ([40, p. 261]) *In the category* Ab *of abelian groups, the direct limit $\{M_i,\psi_{i,j}\}_{i\leq j\in I}$ can be given as $\bigoplus_{i\in I} M_i/\sim$ where $m_i \in M_i \sim m_j \in M_j$ if there is $k \in I$ such that $\psi_{i,k}(m_i) = \psi_{j,k}(m_j) \in M_k$. In particular, if $\{m_i \in M_i\}_{i\in I}$ is a collection of nonzero elements such that $\psi_{i,j}(m_i) = m_j$ for any $i \leq j$, then the element $(m_i)_{i\in I} \in \varinjlim_{i\in I} M_i$ is nonzero.*

Hence we can generalize G. Mislin's Theorem 4.1 and its proof from [7].

Lemma 5.2.2 *For every $n \in \mathbb{Z}$ and $k \in \mathbb{N}_0$ with $n+k \geq 1$ the square*

$$\begin{tikzcd} \mathrm{Ext}^{n+k}_{\mathcal{C}}(A,\widetilde{B}_k) \arrow[r, "\beta_{n+k}(\widetilde{B}_k)"] \arrow[d, "\delta^{n+k}"] & {[\widetilde{A}_{n+k},\widetilde{B}_k]_{\mathcal{C}}} \arrow[d, "t_{\widetilde{A}_{n+k},\widetilde{B}_k}"] \\ \mathrm{Ext}^{n+k+1}_{\mathcal{C}}(A,\widetilde{B}_{k+1}) \arrow[r, "\beta_{n+k+1}(\widetilde{B}_{k+1})"] & {[\widetilde{A}_{n+k+1},\widetilde{B}_{k+1}]_{\mathcal{C}}} \end{tikzcd} \tag{5.2.2}$$

commutes. The induced homomorphism

$$\beta^n(B) := \varinjlim_{k\in\mathbb{N}_0, n+k\geq 1} \beta_{n+k}(\widetilde{B}_k) : \widehat{\operatorname{Ext}}{}^n_{\mathcal{C}}(A,B) \to BC^n_{\mathcal{C}}(A,B)$$

is an isomorphism.

Proof Note that we impose $n+k \geq 1$ in Diagram 5.2.2 and in the definition of $\beta^n(B)$ so that the morphisms $\beta_{n+k}(B)$ are well defined. To prove that the diagram is commutative, let $f + \operatorname{Im}(\operatorname{Hom}_{\mathcal{C}}(\partial_{n+k}, \widetilde{A}_k)$ be an element in $\operatorname{Ext}^{n+k}_{\mathcal{C}}(A, \widetilde{B}_k)$. It suffices to show that the diagram

$$\begin{tikzcd}
A_{n+k+1} \arrow[r, "\pi_{n+k+1}"] \arrow[dr, "F"'] & \widetilde{A}_{n+k+1} \arrow[r, "\iota_{n+k+1}"] \arrow[d, "\overline{\alpha_{n+k}(\widetilde{B}_k)(f)}^*"] & A_{n+k} \arrow[r, "\pi_{n+k}"] \arrow[d, "\overline{\alpha_{n+k}(\widetilde{B}_k)(f)}"] \arrow[dr, bend left, "f"] & \widetilde{A}_{n+k} \arrow[d, "\alpha_{n+k}(\widetilde{B}_k)(f)"] \\
& \widetilde{B}_{k+1} \arrow[r, "\iota_{k+1}"] & B_k \arrow[r, "\pi_k"] & \widetilde{B}_k
\end{tikzcd} \tag{5.2.3}$$

commutes where we explain in the following how to construct it. The right most triangle corresponds to Diagram 5.2.1. We can lift $\alpha_{n+k}(\widetilde{B}_k)(f)$ to $\overline{\alpha_{n+k}(\widetilde{B}_k)(f)}$ and $\overline{\alpha_{n+k}(\widetilde{B}_k)(f)}^*$ as we have done in Diagram 5.1.2. In particular, the morphism $\overline{\alpha_{n+k}(\widetilde{B}_k)(f)}^*$ is a representative of

$$t_{\widetilde{A}_{n+k},\widetilde{B}_k} \circ \beta_{n+k}(\widetilde{B}_k)(f + \operatorname{Im}(\operatorname{Hom}_{\mathcal{C}}(\partial_{n+k}, \widetilde{B}_k)))\,.$$

Writing F for $\overline{\alpha_{n+k}(\widetilde{B}_k)(f)}^* \circ \pi_{n+k+1}$, let us deduce that

$$\delta^{n+k}\big(f + \operatorname{Im}(\operatorname{Hom}_{\mathcal{C}}(\partial_{n+k}, \widetilde{B}_k))\big) = F + \operatorname{Im}(\operatorname{Hom}_{\mathcal{C}}(\partial_{n+k+1}, \widetilde{B}_{k+1}))$$

in $\operatorname{Ext}^{n+k+1}_{\mathcal{C}}(A, \widetilde{B}_{k+1})$. Following the construction of the connecting homomorphism δ^{n+k} by means of the Snake Lemma as in [10, p. 11–12], we see that $\overline{\alpha_{n+k}(\widetilde{B}_k)(f)}$ is a lift of f along π_k. As the boundary map ∂_{n+k+1} equals $\iota_{n+k+1} \circ \pi_{n+k+1} : A_{n+k+1} \to A_{n+k}$ and $\overline{\alpha_{n+k}(\widetilde{B}_k)(f)} \circ \partial_{n+k+1}$ equals $\iota_{k+1} \circ F$, F is a representative of $\delta^{n+k}(f + \operatorname{Im}(\operatorname{Hom}_{\mathcal{C}}(\partial_{n+k}, \widetilde{B}_k)))$. Thus, $\overline{\alpha_{n+k}(\widetilde{B}_k)(f)}^*$ is a representative of $\beta_{n+k+1}(\widetilde{B}_{k+1}) \circ \delta^{n+k}(f + \operatorname{Im}(\operatorname{Hom}_{\mathcal{C}}(\partial_{n+k}, \widetilde{B}_k)))$ and Diagram 5.2.2 commutes.

To show that $\beta^n(B)$ is an isomorphism, let us first prove that it is surjective. For this, let $g + \mathcal{P}_R(\widetilde{A}_{n+k}, \widetilde{B}_k) \in [\widetilde{A}_{n+k}, \widetilde{B}_k]_{\mathcal{C}}$. Due to the diagram

$$\begin{tikzcd}
A_{n+k} \arrow[r, "\pi_{n+k}"] \arrow[dr, bend right, "g\circ\pi_{n+k}"] & \widetilde{A}_{n+k} \arrow[d, "g"] \\
& \widetilde{B}_k
\end{tikzcd}$$

and Diagram 5.2.1, we see that $\alpha_{n+k}(\widetilde{B}_k)(g \circ \pi_{n+k}) = g$ and thus

$$\beta_{n+k}(\widetilde{B}_k)(g \circ \pi_{n+k} + \mathrm{Im}(\mathrm{Hom}_{\mathcal{C}}(\partial_{n+k}, \widetilde{B}_k))) = g + \mathcal{P}_R(\widetilde{A}_{n+k}, \widetilde{B}_k)\,.$$

Since $\beta_{n+k}(\widetilde{B}_k)$ is surjective and direct limits in the category of abelian groups are exact, $\beta^n(B) = \varinjlim_{k\in\mathbb{N}_0, n+k\geq 1} \beta_{n+k}(\widetilde{B}_k)$ is surjective.

In order to demonstrate that $\beta^n(B)$ is injective, let $\varphi \in \widehat{\mathrm{Ext}}^n_{\mathcal{C}}(A,B)$ be an element such that $\beta^n(B)(\varphi) = 0$. By the resolution construction, $\widehat{\mathrm{Ext}}^n_{\mathcal{C}}(A,B)$ can be defined as the direct limit $\varinjlim_{k\in\mathbb{N}_0} \mathrm{Ext}^{n+k}_{\mathcal{C}}(A, \widetilde{B}_k)$ in the category Ab. By Proposition 5.2.1, there exists $k \in \mathbb{N}_0$ and $h \in \mathrm{Ker}\big(\mathrm{Hom}_{\mathcal{C}}(\partial_{n+k+1}, \widetilde{B}_k))\big)$ such that the element $h + \mathrm{Im}(\mathrm{Hom}_{\mathcal{C}}(\partial_{n+k}, \widetilde{B}_k))$ of $\mathrm{Ext}^{n+k}_{\mathcal{C}}(A, \widetilde{B}_k)$ is mapped to $\varphi \in \widehat{\mathrm{Ext}}^n_{\mathcal{C}}(A,B)$ by the homomorphism to the direct limit. Still using Proposition 5.2.1, we may assume $\beta_{n+k}(\widetilde{B}_k)\big(h + \mathrm{Im}(\mathrm{Hom}_{\mathcal{C}}(\partial_{n+k}, \widetilde{B}_k))\big) = 0$. This implies that the morphism $\alpha_{n+k}(\widetilde{B}_k)(h) : \widetilde{A}_{n+k} \to \widetilde{B}_k$ factors through a projective. Because the morphism π_k at the bottom of Diagram 5.2.3 is an epimorphism, it follows from the lifting property of the previous projective object that there is a morphism l making the diagram

$$\begin{tikzcd} & \widetilde{A}_{n+k} \arrow[dl, "l"'] \arrow[d, "\alpha_{n+k}(\widetilde{B}_k)(h)"] \\ B_k \arrow[r, "\pi_k"] & \widetilde{B}_k \end{tikzcd}$$

commute. From this diagram and Diagram 5.2.3 we infer that

$$\pi_k \circ (l \circ \pi_{n+k}) = \alpha_{n+k}(\widetilde{B}_k)(h) \circ \pi_{n+k} = \pi_k \circ \overline{\alpha_{n+k}(\widetilde{B}_k)(h)}\,.$$

In particular, $\pi_k \circ (\overline{\alpha_{n+k}(\widetilde{B}_k)(h)} - l \circ \pi_{n+k}) = 0$. Note that the kernel of π_k is given by the morphism $\iota_{k+1} : \widetilde{B}_{k+1} \to B_k$ whence there is a L such that

$$\begin{tikzcd} & A_{n+k} \arrow[dl, "L"'] \arrow[d, "\overline{\alpha_{n+k}(\widetilde{B}_k)(h)} - l\circ\pi_{n+k}"] \\ \widetilde{B}_{k+1} \arrow[r, "\iota_{k+1}"] & B_k \end{tikzcd}$$

commutes. Using the morphism $\iota_{n+k+1} : \widetilde{A}_{n+k+1} \to A_{n+k}$ from Diagram 5.2.3, we see that

$$\iota_{k+1} \circ L \circ \iota_{n+k+1} = (\overline{\alpha_{n+k}(\widetilde{B}_k)(h)} - l \circ \pi_{n+k}) \circ \iota_{n+k+1}\,.$$

As $\pi_{n+k} \circ \iota_{n+k+1} = 0$, we deduce from Diagram 5.2.3 that

$$\iota_{k+1} \circ L \circ \iota_{n+k+1} = \overline{\alpha_{n+k}(\widetilde{B}_k)(h)} \circ \iota_{n+k+1} = \iota_{k+1} \circ \overline{\alpha_{n+k}(\widetilde{B}_k)(h)}^{*} .$$

Because ι_{n+k+1} is a monomorphism, $L \circ \iota_{n+k+1} = \overline{\alpha_{n+k}(\widetilde{B}_k)(f)}^{*}$. This conclusion together with the previous diagram and Diagram 5.2.3 can be pictorially summarized as

$$\begin{tikzcd}
A_{n+k+1} \arrow[r, "\pi_{n+k+1}"] \arrow[dr, "H"'] & \widetilde{A}_{n+k+1} \arrow[r, "\iota_{n+k+1}"] \arrow[d, "\overline{\alpha_{n+k}(\widetilde{B}_k)(h)}^{*}"] & \widetilde{A}_{n+k} \arrow[dl, bend left, "L"] \arrow[d, "\overline{\alpha_{n+k}(\widetilde{B}_k)(h)}"] \\
& \widetilde{B}_{k+1} \arrow[r, "\iota_{k+1}"] & B_k
\end{tikzcd}$$

Recalling that $\pi_{n+k+1} \circ \iota_{n+k+1} = \partial_{n+k+1}$, we deduce that

$$H = L \circ \partial_{n+k+1} \in \mathrm{Hom}_{\mathcal{C}}(\partial_{n+k+1}, \widetilde{B}_{k+1}) .$$

We infer that $\delta^{n+k}(h + \mathrm{Im}(\mathrm{Hom}_{\mathcal{C}}(\partial_{n+k}, \widetilde{B}_k))) = 0$ since the above is a representative in $\mathrm{Im}(\delta^{n+k})$. According to the resolution construction, the latter as well as $h + \mathrm{Im}(\mathrm{Hom}_{\mathcal{C}}(\partial_{n+k}, \widetilde{B}_k))$ are both mapped to $\varphi \in \widehat{\mathrm{Ext}}^n_{\mathcal{C}}(A, B)$. Therefore $\varphi = 0$, $\mathrm{Ker}(\beta^n(B)) = 0$ and $\beta^n(B)$ is injective. $\square$

Proposition 5.2.3 *For every $n \in \mathbb{Z}$ the isomorphism $\beta^n(-)$ is natural. That is, for every $f \in \mathrm{Hom}_{\mathcal{C}}(B, C)$ the square*

$$\begin{tikzcd}
\widehat{\mathrm{Ext}}^n_{\mathcal{C}}(A, B) \arrow[r, "\beta^n(B)"] \arrow[d, "\widehat{\mathrm{Ext}}^n_{\mathcal{C}}(A, f)"] & BC^n_{\mathcal{C}}(A, B) \arrow[d, "BC^n_{\mathcal{C}}(A, f)"] \\
\widehat{\mathrm{Ext}}^n_{\mathcal{C}}(A, C) \arrow[r, "\beta^n(C)"] & BC^n_{\mathcal{C}}(A, C)
\end{tikzcd}$$

commutes.

Proof By construction of $\widehat{\mathrm{Ext}}^n_{\mathcal{C}}(A, f)$ through the resolution construction from Proposition 4.2.1 and of $BC^n_{\mathcal{C}}(A, f)$ from Definition 5.1.4, it suffices to consider the cube

$$\begin{tikzcd}
& \mathrm{Ext}^{n+k}_{\mathcal{C}}(A,\widetilde{B}_k) \arrow[rr, "\beta_{n+k}(\widetilde{B}_k)"] \arrow[dl, "\mathrm{Ext}^{n+k}_{\mathcal{C}}(A,\widetilde{f}_k)"'] \arrow[dd, "\delta^{n+k}" near end] & & {[\widetilde{A}_{n+k},\widetilde{B}_k]_{\mathcal{C}}} \arrow[dl, "{[\widetilde{A}_{n+k},\widetilde{f}_k]_{\mathcal{C}}}"] \arrow[dd, "t_{\widetilde{A}_{n+k},\widetilde{B}_k}"] \\
\mathrm{Ext}^{n+k}_{\mathcal{C}}(A,\widetilde{C}_k) \arrow[rr, "\beta_{n+k}(\widetilde{C}_k)" near end] \arrow[dd, "\delta^{n+k}"] & & {[\widetilde{A}_{n+k},\widetilde{C}_k]_{\mathcal{C}}} \arrow[dd, "t_{\widetilde{A}_{n+k},\widetilde{C}_k}" near end] & \\
& \mathrm{Ext}^{n+k+1}_{\mathcal{C}}(A,\widetilde{B}_{k+1}) \arrow[rr, "\beta_{n+k+1}(\widetilde{B}_{k+1})" near end] \arrow[dl, "\mathrm{Ext}^{n+k+1}_{\mathcal{C}}(A,\widetilde{f}_{k+1})"] & & {[\widetilde{A}_{n+k+1},\widetilde{B}_{k+1}]_{\mathcal{C}}} \arrow[dl, "{[\widetilde{A}_{n+k+1},\widetilde{f}_{k+1}]_{\mathcal{C}}}"] \\
\mathrm{Ext}^{n+k+1}_{\mathcal{C}}(A,\widetilde{C}_{k+1}) \arrow[rr, "\beta_{n+k+1}(\widetilde{C}_{k+1})"] & & {[\widetilde{A}_{n+k+1},\widetilde{C}_{k+1}]_{\mathcal{C}}} &
\end{tikzcd}$$

for any $k \in \mathbb{N}_0$ with $n + k \geq 1$. The right hand side commutes due to the choice of the sequence of morphisms $(\widetilde{f}_k : \widetilde{B}_k \to \widetilde{C}_k)_{k\in\mathbb{N}_0}$ and the left hand side since connecting homomorphisms are natural. The front and back side correspond to Diagram 5.2.2. Regarding its top and bottom side, let $g \in \mathrm{Ker}(\mathrm{Hom}_{\mathcal{C}}(\partial_{n+k+1}, \widetilde{B}_k))$. Then

$$\begin{aligned}
\big([\widetilde{A}_{n+k}, \widetilde{f}_k]_{\mathcal{C}} \circ \beta_{n+k}(\widetilde{B}_k)\big)&\big(g + \mathrm{Im}(\mathrm{Hom}_{\mathcal{C}}(\partial_{n+k}, \widetilde{B}_k))\big) \\
&= [\widetilde{A}_{n+k}, \widetilde{f}_k]_{\mathcal{C}}\big(\alpha_{n+k}(g) + \mathcal{P}_{\mathcal{C}}(\widetilde{A}_{n+k}, \widetilde{B}_k)\big) \\
&= \widetilde{f}_k \circ \alpha_{n+k}(g) + \mathcal{P}_{\mathcal{C}}(\widetilde{A}_{n+k}, \widetilde{C}_k)\,.
\end{aligned}$$

Note that

$$\widetilde{f}_k \circ \alpha_{n+k}(\widetilde{B}_k)(g) \circ \pi_{n+k} = \widetilde{f}_k \circ g = \alpha_{n+k}(\widetilde{C}_k)(\widetilde{f}_k \circ g) \circ \pi_{n+k}\,.$$

By definition of the morphism $\alpha_{n+k}(\widetilde{C}_k) : \mathrm{Ext}^{n+k}_{\mathcal{C}}(A, \widetilde{C}_k) \to [\widetilde{A}_{n+k}, \widetilde{C}_k]_{\mathcal{C}}$ pictorially given in Diagram 5.2.1, we deduce that

$$\widetilde{f}_k \circ \alpha_{n+k}(\widetilde{B}_k)(g) = \alpha_{n+k}(\widetilde{C}_k)(\widetilde{f}_k \circ g)\,.$$

In particular,

$$\begin{aligned}
\big([\widetilde{A}_{n+k}, \widetilde{f}_k]_{\mathcal{C}} \circ \beta_{n+k}(\widetilde{B}_k)\big)&\big(g + \mathrm{Im}(\mathrm{Hom}_{\mathcal{C}}(\partial_{n+k}, \widetilde{B}_k))\big) \\
&= \alpha_{n+k}(\widetilde{C}_k)(\widetilde{f}_k \circ g) + \mathcal{P}_{\mathcal{C}}(\widetilde{A}_{n+k}, \widetilde{C}_k) \\
&= \beta_{n+k}(\widetilde{C}_k)\big(\widetilde{f}_k \circ g + \mathrm{Im}(\mathrm{Hom}_{\mathcal{C}}(\partial_{n+k}, \widetilde{C}_k))\big) \\
&= \big(\beta_{n+k}(\widetilde{C}_k) \circ \mathrm{Ext}^{n+k}_{\mathcal{C}}(A, \widetilde{f}_k)\big)\big(g + \mathrm{Im}(\mathrm{Hom}_{\mathcal{C}}(\partial_{n+k}, \widetilde{B}_k))\big)\,.
\end{aligned}$$

Thus, the top and bottom side of the above cube also commute and form a direct system of commuting squares. The statement of the proposition follows by passing to the direct limit. $\square$

5.3 Connecting homomorphisms of the naive construction

To render the naive construction into a cohomological functor, connecting homomorphisms are needed. It is formed in a similar manner to the morphisms $\varepsilon^{-m} : S^{-m}T \to S^{-m+1}T$ between satellite functors that we have established at the start of Chapter 3. Namely, let $0 \to B \xrightarrow{f} C \xrightarrow{g} D \to 0$ be a short exact sequence in $\mathcal{C}$. By Diagram 4.2.2, we can lift the morphisms f and g such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{B}_{k+1} \arrow[r, "\widetilde{f}_{k+1}"] \arrow[d] & \widetilde{C}_{k+1} \arrow[r, "\widetilde{g}_{k+1}"] \arrow[d] & \widetilde{D}_{k+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B_k \arrow[r, "f_k"] \arrow[d] & C_k \arrow[r, "g_k"] \arrow[d] & D_k \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \widetilde{B}_k \arrow[r, "\widetilde{f}_k"] & \widetilde{C}_k \arrow[r, "\widetilde{g}_k"] & \widetilde{D}_k \arrow[r] & 0
\end{tikzcd}$$

is commutative and has short exact sequences as rows for every $k \in \mathbb{N}_0$. Given that the domain of the connecting homomorphism is supposed to be $BC^n_R(A, D) = \varinjlim_{k\in\mathbb{N}_0, n+k\geq 0}[\widetilde{A}_{n+k}, \widetilde{D}_k]_{\mathcal{C}}$, let $l \in \mathrm{Hom}_{\mathcal{C}}(\widetilde{A}_{n+k}, \widetilde{D}_k)$. According to Diagram 5.1.2, we can lift l such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{A}_{n+k+1} \arrow[r] \arrow[d, "\bar{l}^*"] & A_{n+k} \arrow[r] \arrow[d, "\bar{l}"] & \widetilde{A}_{n+k} \arrow[r] \arrow[d, "l"] & 0 \\
0 \arrow[r] & \widetilde{D}_{k+1} \arrow[r] & D_k \arrow[r] & \widetilde{D}_k \arrow[r] & 0
\end{tikzcd}$$

commutes and $t_{\widetilde{A}_{n+k}, \widetilde{D}_k}(l + \mathcal{P}_{\mathcal{C}}(\widetilde{A}_{n+k}, \widetilde{D}_k)) = \bar{l}^* + \mathcal{P}_{\mathcal{C}}(\widetilde{A}_{n+k+1}, \widetilde{D}_{k+1})$. Using Diagram 3.0.2, we can extend it in a commutative manner to

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{A}_{n+k+1} \arrow[r] \arrow[d, "\bar{l}^*"] & A_{n+k} \arrow[r] \arrow[d, "\bar{l}"] & \widetilde{A}_{n+k} \arrow[r] \arrow[d, "l"] & 0 \\
0 \arrow[r] & \widetilde{D}_{k+1} \arrow[r] \arrow[d, "h^*"] & D_k \arrow[r] \arrow[d, "h"] & \widetilde{D}_k \arrow[r] \arrow[d, "\mathrm{id}_{\widetilde{D}_k}"] & 0 \\
0 \arrow[r] & \widetilde{B}_k \arrow[r, "\widetilde{f}_k"] & \widetilde{C}_k \arrow[r, "\widetilde{g}_k"] & \widetilde{D}_k \arrow[r] & 0
\end{tikzcd}$$

We define the map

$$\tau_{n+k,k} := [\widetilde{A}_{n+k+1}, h^*]_{\mathcal{C}} \circ t_{\widetilde{A}_{n+k},\widetilde{D}_k} : \quad [\widetilde{A}_{n+k}, \widetilde{D}_k]_{\mathcal{C}} \to [\widetilde{A}_{n+k+1}, \widetilde{B}_k]_{\mathcal{C}}$$
$$l + \mathcal{P}_{\mathcal{C}}(\widetilde{A}_{n+k}, \widetilde{D}_k) \mapsto h^* \circ \bar{l}^* + \mathcal{P}_{\mathcal{C}}(\widetilde{A}_{n+k+1}, \widetilde{B}_k)$$

Following the proof of Proposition 5.1.2, it is well defined in the sense that it does not depend on the particular lift $h^* : \widetilde{D}_{k+1} \to \widetilde{B}_k$ of $\mathrm{id}_{\widetilde{D}_k}$. By Proposition 5.1.1 and 5.1.2, $\tau_{n+k,k}$ is a homomorphism.

Lemma 5.3.1 *Both the squares*

$$\begin{array}{ccc} \mathrm{Ext}^{n+k}_{\mathcal{C}}(A,\widetilde{D}_k) & \xrightarrow{\beta_{n+k}(\widetilde{D}_k)} & [\widetilde{A}_{n+k}, \widetilde{D}_k]_{\mathcal{C}} \\ \big\downarrow{\scriptstyle \delta^{n+k}} & & \big\downarrow{\scriptstyle \tau_{n+k,k}} \\ \mathrm{Ext}^{n+k+1}_{\mathcal{C}}(A,\widetilde{B}_k) & \xrightarrow{\beta_{n+k+1}(\widetilde{B}_k)} & [\widetilde{A}_{n+k+1}, \widetilde{B}_k]_{\mathcal{C}} \end{array} \quad \textit{and} \quad \begin{array}{ccc} [\widetilde{A}_{n+k}, \widetilde{D}_k]_{\mathcal{C}} & \xrightarrow{(-1)^k\tau_{n+k,k}} & [\widetilde{A}_{n+k+1}, \widetilde{B}_k]_{\mathcal{C}} \\ \big\downarrow{\scriptstyle t_{\widetilde{A}_{n+k},\widetilde{D}_k}} & & \big\downarrow{\scriptstyle t_{\widetilde{A}_{n+k+1},\widetilde{B}_k}} \\ [\widetilde{A}_{n+k+1}, \widetilde{D}_{k+1}]_{\mathcal{C}} & \xrightarrow{(-1)^{k+1}\tau_{n+k+1,k+1}} & [\widetilde{A}_{n+k+2}, \widetilde{B}_{k+1}]_{\mathcal{C}} \end{array}$$

commute for every $k \in \mathbb{N}_0$ with $n + k \geq 1$. In particular, the induced homomorphism

$$\tau^n := \varinjlim_{k\in\mathbb{N}_0, n+k\geq 1} (-1)^k \tau_{n+k,k} : BC^n_{\mathcal{C}}(A, D) \to BC^{n+1}_{\mathcal{C}}(A, B)$$

fits into the commuting square

$$\begin{array}{ccc} \widehat{\mathrm{Ext}}^n_{\mathcal{C}}(A, D) & \xrightarrow{\beta^n(D)} & BC^n_{\mathcal{C}}(A, D) \\ \big\downarrow{\scriptstyle \widehat{\delta}^n} & & \big\downarrow{\scriptstyle \tau^n} \\ \widehat{\mathrm{Ext}}^{n+1}_{\mathcal{C}}(A, B) & \xrightarrow{\beta^{n+1}(B)} & BC^{n+1}_{\mathcal{C}}(A, B) \end{array}$$

Therefore, $\tau^n : BC^n_{\mathcal{C}}(A, -) \to BC^{n+1}_{\mathcal{C}}(A, -)$ represents the (proposed) connecting homomorphism of $BC^\bullet_{\mathcal{C}}(A, -)$.

Proof Taking a morphism $h^* : \widetilde{D}_{k+1} \to \widetilde{B}_k$ lifting $\mathrm{id}_{\widetilde{D}_k} : \widetilde{D}_k \to \widetilde{D}_k$ as before, we apply the Ext-functor $\mathrm{Ext}^\bullet_{\mathcal{C}}(A, -)$ to obtain the commutative square

$$\begin{array}{ccc} \mathrm{Ext}^{n+k}_{\mathcal{C}}(A, \widetilde{D}_k) & \xrightarrow{\delta^{n+k}} & \mathrm{Ext}^{n+k+1}_{\mathcal{C}}(A, \widetilde{D}_{k+1}) \\ \big\downarrow{\scriptstyle \mathrm{id}} & & \big\downarrow{\scriptstyle \mathrm{Ext}^{n+k+1}_{\mathcal{C}}(A, h^*)} \\ \mathrm{Ext}^{n+k}_{\mathcal{C}}(A, \widetilde{D}_k) & \xrightarrow{\delta^{n+k}} & \mathrm{Ext}^n_{\mathcal{C}}(A, \widetilde{B}_k) \end{array}$$

Using this and the definition of the morphism $\tau_{n+k,k}$, we can factorize the top left square in the statement of the Lemma as

$$
\begin{tikzcd}
\mathrm{Ext}^{n+k}_{\mathcal{C}}(A, \widetilde{D}_k) \arrow[r, "\beta_{n+k}(\widetilde{D}_k)"] \arrow[d, "\delta^{n+k}"] & {[\widetilde{A}_{n+k}, \widetilde{D}_k]_{\mathcal{C}}} \arrow[d, "t_{\widetilde{A}_{n+k}, \widetilde{D}_k}"] \\
\mathrm{Ext}^{n+k+1}_{\mathcal{C}}(A, \widetilde{D}_{k+1}) \arrow[r, "\beta_{n+k+1}(\widetilde{D}_{k+1})"] \arrow[d, "\mathrm{Ext}^{n+k+1}_{\mathcal{C}}(A, h^*)"] & {[\widetilde{A}_{n+k+1}, \widetilde{D}_{k+1}]_{\mathcal{C}}} \arrow[d, "{[\widetilde{A}_{n+k+1}, h^*]_{\mathcal{C}}}"] \\
\mathrm{Ext}^{n+k+1}_{\mathcal{C}}(A, \widetilde{B}_k) \arrow[r, "\beta_{n+k+1}(\widetilde{B}_k)"] & {[\widetilde{A}_{n+k+1}, \widetilde{B}_k]_{\mathcal{C}}}
\end{tikzcd}
$$

It commutes by Lemma 5.2.2 and Proposition 5.2.3. As for the top right square in the statement of the lemma, we can fit it as the right hand side of the cube

$$
\begin{tikzcd}
& \mathrm{Ext}^{n+k}_{\mathcal{C}}(A, \widetilde{D}_k) \arrow[rr, "\beta_{n+k}(\widetilde{D}_k)"] \arrow[dl, "(-1)^k \delta^{n+k}"'] \arrow[dd, "\delta^{n+k}"] & & {[\widetilde{A}_{n+k}, \widetilde{D}_k]_{\mathcal{C}}} \arrow[dl, "(-1)^k \tau_{n+k,k}"'] \arrow[dd, "t_{\widetilde{A}_{n+k}, \widetilde{D}_k}"] \\
\mathrm{Ext}^{n+k+1}_{\mathcal{C}}(A, \widetilde{B}_k) \arrow[rr, "\beta_{n+k+1}(\widetilde{B}_k)"] \arrow[dd, "\delta^{n+k+1}"] & & {[\widetilde{A}_{n+k+1}, \widetilde{B}_k]_{\mathcal{C}}} \arrow[dd, "t_{\widetilde{A}_{n+k+1}, \widetilde{B}_k}"] & \\
& \mathrm{Ext}^{n+k+1}_{\mathcal{C}}(A, \widetilde{D}_{k+1}) \arrow[rr, "\beta_{n+k+1}(\widetilde{D}_{k+1})"] \arrow[dl, "(-1)^{k+1} \delta^{n+k+1}"'] & & {[\widetilde{A}_{n+k+1}, \widetilde{D}_{k+1}]_{\mathcal{C}}} \arrow[dl, "(-1)^{k+1} \tau_{n+k+1,k+1}"'] \\
\mathrm{Ext}^{n+k+2}_{\mathcal{C}}(A, \widetilde{B}_{k+1}) \arrow[rr, "\beta_{n+k+2}(\widetilde{B}_{k+1})"] & & {[\widetilde{A}_{n+k+1}, \widetilde{B}_{k+1}]_{\mathcal{C}}} &
\end{tikzcd}
$$

We have just seen that the top and bottom side commutes. The left hand side corresponds to Diagram 4.2.3 while the front and back side commute by Lemma 5.2.2. The right hand commutes as desired since $\beta_{n+k+1}(\widetilde{B}_k)$ is an epimorphism by the proof of Lemma 5.2.2. Lastly, the top and bottom sides of these cubes form a direct system of commuting squares in whose direct limit we retrieve the bottom square from the statement of the lemma. $\square$

Theorem 5.3.2 *The naive construction $(BC^\bullet_{\mathcal{C}}(A,-), \tau^\bullet)$ forms a cohomological functor and $\beta^\bullet : (\widehat{\mathrm{Ext}}^\bullet_{\mathcal{C}}(A,-), \widehat{\delta}^\bullet) \to (BC^\bullet_{\mathcal{C}}(A,-), \tau^\bullet)$ is an isomorphism of cohomological functors.*

Proof We already know that $(\widehat{\mathrm{Ext}}^\bullet_{\mathcal{C}}(A,-), \widehat{\delta}^\bullet)$ is a cohomological functor by Lemma 3.0.2, that $\beta^\bullet$ is a natural transformation by Proposition 5.2.3 and that every β^n is an isomorphism by Lemma 5.2.2. Hence, we can reiterate the proof of Theorem 4.3.2 where we invoke Lemma 5.3.1 instead of Lemma 4.3.1

whenever required. □

Remark 5.3.3 *On page 299 of [7], G. Mislin constructs for every* $n \in \mathbb{Z}$ *an isomorphism* $\varinjlim_{k\in\mathbb{N}_0} S^{-k}\mathrm{Ext}^{n+k}(A,-) \to BC^n(A,-)$ *in the case of modules over a ring. Then he asserts that one can turn* $BC^\bullet(A,-)$ *into a Mislin completion by transferring the structure of* $\widehat{\mathrm{Ext}}^\bullet(A,-)$ *and claims that homomorphisms of the form* $[\widetilde{A}_{n+k}, \widetilde{D}_k] \to [\widetilde{A}_{n+k+1}, \widetilde{B}_k]$ *induce the connecting homomorphism* $\tau^n : BC^n(A,D) \to BC^{n+1}(A,B)$*. However, he does not explain how to construct the latter homomorphisms and in particular, does not indicate that there alternating signs involved in the direct system giving rise to* τ^n *as can be seen in Lemma 5.3.1.*

Chapter 6

The hypercohomology and the resolution construction

We demonstrate in this chapter that the hypercohomology construction gives rise to the Mislin completion of unenriched Ext-functors in the same way as the naive construction. For this, we relate in Section 6.1 the hypercohomology complex with chain maps and the Vogel complex with so-called almost chain maps. It is known that unenriched Ext-groups can be retrieved as cohomology groups of the former complex. In Section 6.2 we establish an explicit isomorphism between the latter two and use use it to construct connecting homomorphisms through (almost) chain maps. This sets the base for Section 6.3 where we construct a natural isomorphism from the cohomology groups of the resolution construction to the ones of the hypercohomolgy construction. In Section 6.4, we extend it to an isomorphism of cohomological functors. Moreover, we provide an explicit isomorphism of cohomological functors from the Mislin completion resulting from the hypercohomology construction to the one resulting from the naive construction. Lastly, although S. Guo and L. Liang have turned the cohomology groups of the Vogel complex into a cohomological functor in [28, Proposition 4.8], we do not know whether their construction forms a Mislin completion (Question 6.4.8). Similarly, J. Hu et al. introduce the Vogel complex in [30, p. 7] in a far more general setting than we or S. Guo and L. Liang consider, but we also do not know whether it yields a Mislin completion.

6.1 Defining Ext-functors through chain maps

There is a reformulation of the hypercohomology construction occurring in D. J. Benson and J. F. Carlson's paper [13, p. 109] in terms of almost chain maps. For this, we introduce the following notation. If $(M_k, \mu_k)_{k\in\mathbb{Z}}$ is a chain complex in the category $\mathcal{C}$ with boundary maps $\mu_k : M_k \to M_{k-1}$ and $n \in \mathbb{Z}$, then we define $(M[n]_k, \mu[n]_k)_{k\in\mathbb{Z}}$ to be $M[n]_k := M_{k+n}$ and

$\mu[n]_k := (-1)^n \mu_{k+n}$ [41, p. 154], [10, p. 9–10]. Recall that we have constructed the hypercohomology complex $\mathrm{Hyp}_{\mathcal{C}}(M_\bullet, N_\bullet)_\bullet$ for a chain complexes $(N_k, \nu_k)_{k\in\mathbb{Z}}$ before Definition 2.0.11. According to [10, p. 62–63], an n-cocycle of $\mathrm{Hyp}_{\mathcal{C}}(M_\bullet, N_\bullet)_\bullet$ is exactly a chain map of the form $M[n]_\bullet \to N_\bullet$ where n-coboundaries are nullhomotopic chain maps of this form. In the same way, we observe that an n-cocycle of the Vogel complex $\mathrm{Vog}_{\mathcal{C}}(M_\bullet, N_\bullet)_\bullet$ is a collection of morphisms $(f_{k+n} : M[n]_k \to N_k)_{k\in\mathbb{Z}}$ such that all but finitely many of the squares

$$\begin{array}{ccc} M_{k+n} & \xrightarrow{f_{k+n}} & N_k \\ \Big\downarrow{\scriptstyle (-1)^n \mu_{n+k}} & & \Big\downarrow{\scriptstyle \nu_k} \\ M_{k-1+n} & \xrightarrow{f_{k-1+n}} & N_{k-1} \end{array}$$

commute. We follow the convention in [13, p. 109] by calling this an *almost chain map* of degree n. We call $f_{\bullet+n} : M[n]_\bullet \to N_\bullet$ *nullhomotopic* if there is a collection of morphisms $(e_{k+n} : M[n]_k \to N_{k+1})_{k\in\mathbb{Z}}$ such that $f_{k+n} = e_{k-1+n} \circ \mu[n]_k + \mu[n]_{k+1} \circ e_{k+n}$ for all but finitely many $k \in \mathbb{Z}$. Then we observe that an n-cboundary of $\mathrm{Vog}_{\mathcal{C}}(M_\bullet, N_\bullet)_\bullet$ is exactly such a nullhomotopic almost chain map. Denote by $\mathrm{Hom}_{\mathrm{Ch}(\mathcal{C})}(M[n]_\bullet, N_\bullet)$ the set of chain maps $M[n]_\bullet \to N_\bullet$ and by $\mathrm{Null}_{\mathrm{Ch}(\mathcal{C})}(M[n]_\bullet, N_\bullet)$ the subset of nullhomotopic maps. Analogously, write $\widehat{\mathrm{Hom}}_{\mathrm{Ch}(\mathcal{C})}(M[n]_\bullet, N_\bullet)$ for the *set of almost chain maps* $M[n]_\bullet \to N_\bullet$ and by $\widehat{\mathrm{Null}}_{\mathrm{Ch}(\mathcal{C})}(M[n]_\bullet, N_\bullet)$ the *subset of nullhomotopic almost chain maps*. Above, we have just seen that

$$\begin{aligned} H^n(\mathrm{Hyp}_{\mathcal{C}}(M_\bullet, N_\bullet)_\bullet) &= \mathrm{Hom}_{\mathrm{Ch}(\mathcal{C})}(M[n]_\bullet, N_\bullet)/\mathrm{Null}_{\mathrm{Ch}(\mathcal{C})}(M[n]_\bullet, N_\bullet) \text{ and} \\ H^n(\mathrm{Vog}_{\mathcal{C}}(M_\bullet, N_\bullet)_\bullet) &= \widehat{\mathrm{Hom}}_{\mathrm{Ch}(\mathcal{C})}(M[n]_\bullet, N_\bullet)/\widehat{\mathrm{Null}}_{\mathrm{Ch}(\mathcal{C})}(M[n]_\bullet, N_\bullet)\,. \end{aligned}$$

If $M_\bullet$ is a projective resolution of an object $M \in \mathrm{obj}(\mathcal{C})$ and $N_\bullet$ a projective resolution of $N \in \mathrm{obj}(\mathcal{C})$, then the former equals the unenriched Ext-group $\mathrm{Ext}^n_{\mathcal{C}}(M, N)$ according to [41, p. 166] and [10, Section 10.7]. The proper setting to illustrate this are derived categories. The idea and basic properties of the *derived category* $\mathcal{D}(\mathcal{C})$ of $\mathcal{C}$ are summarized on pages 143–144 in S. I. Gelfand and Yu. I. Manin's book on homological algebra [41]. Morally, objects of $\mathcal{D}(\mathcal{C})$ are chain complexes $(M_k)_{n\in\mathbb{Z}}$ in $\mathcal{C}$ and morphisms are chain maps where any chain map $(f_k : M_k \to N_k)_{k\in\mathbb{Z}}$ is taken to be an isomorphism if for every $n \in \mathbb{Z}$ the induced morphism in homology $H_n(f_\bullet) : H_n(M_\bullet) \to H_n(N_\bullet)$ is an isomorphism. Every $A \in \mathrm{obj}(\mathcal{C})$ can be turned into an object $\iota(A)_\bullet \in \mathrm{obj}(\mathcal{D}(\mathcal{C}))$ where $\iota(A)_0 = A$, $\iota(A)_k = 0$ for $k \neq 0$ and all boundary maps are zero. If $(A_k)_{k\in\mathbb{N}_0}$ is a projective resolution of A, then we can extend it to a chain complex $(A_k)_{k\in\mathbb{Z}}$ by setting $A_k = 0$ for $k < 0$ and all other boundary maps to be zero. By means of the augmentation map $A_0 \to A$ we see that $\iota(A)_\bullet$ is isomorphic to $A_\bullet$ in $\mathcal{D}(\mathcal{C})$. As we index our projective resolutions to have positive degrees, our notation in the following agrees with the one found in [10] and disagrees with the one in [41]. According to [41, p. 166] and [10, p. 399], one can define unenriched Ext-groups as $\mathrm{Ext}^n_{\mathcal{C}}(A,B) := \mathrm{Hom}_{\mathcal{D}(\mathcal{C})}(\iota(A)[n]_\bullet, \iota(B)_\bullet)$. Note that $\mathrm{Ext}^n_{\mathcal{C}}(A,B) \cong \mathrm{Hom}_{\mathcal{D}(\mathcal{C})}(A[n]_\bullet, \iota(B)_\bullet)$. Since homotopic chain maps are identified in $\mathcal{D}(\mathcal{C})$ [41, p. 159], this agrees with the definition of $\mathrm{Ext}^n_{\mathcal{C}}(-,B)$ as the n^{th} right derived functors of the unenriched Hom-functor $\mathrm{Hom}_{\mathcal{C}}(-,B)$. We refer the reader to Sections 2.4 and 2.5 of [10] for a thorough account on derived categories. In order to distinguish the two emerging notions of Ext-functors, we introduce the following notation.

Notation 6.1.1 *From this point on, if $(A_k)_{k\in\mathbb{Z}}$ is a chain complex in $\mathcal{C}$, then it is assumed that $(A_k)_{k\in\mathbb{N}_0}$ is a projective resolution of an object A in $\mathcal{C}$ and $A_k = 0$ for $k < 0$. We write*

$$\mathcal{E}xt^n_{\mathcal{C}}(A,B) := \mathrm{Hom}_{\mathrm{Ch}(\mathcal{C})}(A[n]_\bullet, B_\bullet)/\mathrm{Null}_{\mathrm{Ch}(\mathcal{C})}(A[n]_\bullet, B_\bullet)$$

whenever we consider the n^{th} Ext-group as arising from chain maps modulo chain homotopy. Analogously, we write

$$\widehat{\mathcal{E}xt}^n_{\mathcal{C}}(A,B) := \widehat{\mathrm{Hom}}_{\mathrm{Ch}(\mathcal{C})}(A[n]_\bullet, B_\bullet)/\widehat{\mathrm{Null}}_{\mathrm{Ch}(\mathcal{C})}(A[n]_\bullet, B_\bullet)\,.$$

These agree with the n^{th} cohomology group of the hypercohomology complex and of the Vogel complex respectively. If $a_k : A_k \to A_{k-1}$ denote the boundary maps of $A_\bullet$, then we continue to write

$$\mathrm{Ext}^n_{\mathcal{C}}(A,B) = \mathrm{Ker}(\mathrm{Hom}_{\mathcal{C}}(a_{n+1},B))/\mathrm{Im}(\mathrm{Hom}_{\mathcal{C}}(a_n,B))$$

for the n^{th} Ext-group defined as the n^{th} derived functor of $\mathrm{Hom}_{\mathcal{C}}(-,B)$*.*

For morphisms $f : B \to C$ and $g : C \to A$ in $\mathcal{C}$, let us construct $\mathcal{E}xt^n_{\mathcal{C}}(A,f)$ and $\mathcal{E}xt^n_{\mathcal{C}}(g,B)$ as well as $\widehat{\mathcal{E}xt}^n_{\mathcal{C}}(A,f)$ and $\widehat{\mathcal{E}xt}^n_{\mathcal{C}}(g,B)$. Consider projective resolutions $A_\bullet$, $B_\bullet$, $C_\bullet$ of A, B, C. By the Comparison Theorem [10, Theorem 2.2.6] there is a chain map $f_\bullet : B_\bullet \to C_\bullet$ that is unique up to chain homotopy such that the diagram

$$\begin{tikzcd}
\dots \arrow[r] & B_2 \arrow[r] \arrow[d, "f_2"] & B_1 \arrow[r] \arrow[d, "f_1"] & B_0 \arrow[r] \arrow[d, "f_0"] & B \arrow[d, "f"] \\
\dots \arrow[r] & C_2 \arrow[r] & C_1 \arrow[r] & C_0 \arrow[r] & C
\end{tikzcd}$$

commutes. There is an analogous chain map $g_\bullet : C_\bullet \to A_\bullet$ that is unique up to chain homotopy. Define $g[n]_\bullet : C[n]_\bullet \to A[n]_\bullet$ by $g[n]_k := g_{k+n}$ for every $k \in \mathbb{Z}$ [10, p. 10]. Since the category of chain complexes $\mathrm{Ch}(\mathcal{C})$ with chain maps as morphisms forms an abelian category [10, Theorem 1.2.3], there are homomorphisms of abelian groups

$$\mathrm{Hom}_{\mathrm{Ch}(\mathcal{C})}(A[n]_\bullet, f_\bullet) : \mathrm{Hom}_{\mathcal{C}}(A[n]_\bullet, B_\bullet) \to \mathrm{Hom}_{\mathcal{C}}(A[n]_\bullet, C_\bullet),\ \varphi_\bullet \mapsto f_\bullet \circ \varphi_\bullet \text{ and}$$
$$\mathrm{Hom}_{\mathrm{Ch}(\mathcal{C})}(g[n]_\bullet, B_\bullet) : \mathrm{Hom}_{\mathcal{C}}(A[n]_\bullet, B_\bullet) \to \mathrm{Hom}_{\mathcal{C}}(C[n]_\bullet, B_\bullet),\ \varphi_\bullet \mapsto \varphi_\bullet \circ g[n]_\bullet$$

It follows by definition and [10, p. 5] that any $\widehat{\mathrm{Hom}}_{\mathrm{Ch}(\mathcal{C})}(A[n]_\bullet, B_\bullet)$ is an abelian group and that the maps

$$\widehat{\mathrm{Hom}}_{\mathrm{Ch}(\mathcal{C})}(A[n]_\bullet, f_\bullet) : \widehat{\mathrm{Hom}}_{\mathcal{C}}(A[n]_\bullet, B_\bullet) \to \widehat{\mathrm{Hom}}_{\mathcal{C}}(A[n]_\bullet, C_\bullet),\ \varphi_\bullet \mapsto f_\bullet \circ \varphi_\bullet \text{ and}$$
$$\widehat{\mathrm{Hom}}_{\mathrm{Ch}(\mathcal{C})}(g[n]_\bullet, B_\bullet) : \widehat{\mathrm{Hom}}_{\mathcal{C}}(A[n]_\bullet, B_\bullet) \to \widehat{\mathrm{Hom}}_{\mathcal{C}}(C[n]_\bullet, B_\bullet),\ \varphi_\bullet \mapsto \varphi_\bullet \circ g[n]_\bullet$$

are also homomorphisms of abelian groups.

Definition 6.1.2 *The above descend to homomorphisms*

$$\begin{aligned} \mathcal{E}xt^n_{\mathcal{C}}(A, f) :\ & \mathcal{E}xt^n_{\mathcal{C}}(A, B) \to \mathcal{E}xt^n_{\mathcal{C}}(A, C) \\ \varphi_\bullet + \mathrm{Null}_{\mathrm{Ch}(\mathcal{C})}(A[n]_\bullet, B_\bullet) &\mapsto f_\bullet \circ \varphi_\bullet + \mathrm{Null}_{\mathrm{Ch}(\mathcal{C})}(A[n]_\bullet, C_\bullet) \end{aligned}$$

and

$$\begin{aligned} \mathcal{E}xt^n_{\mathcal{C}}(g, B) :\ & \mathcal{E}xt^n_{\mathcal{C}}(A, B) \to \mathcal{E}xt^n_{\mathcal{C}}(C, B) \\ \varphi_\bullet + \mathrm{Null}_{\mathrm{Ch}(\mathcal{C})}(A[n]_\bullet, B_\bullet) &\mapsto \varphi_\bullet \circ g[n]_\bullet + \mathrm{Null}_{\mathrm{Ch}(\mathcal{C})}(C[n]_\bullet, B_\bullet)\,. \end{aligned}$$

The homomorphisms

$$\widehat{\mathcal{E}xt}^n_{\mathcal{C}}(A, f) : \widehat{\mathcal{E}xt}^n_{\mathcal{C}}(A, B) \to \widehat{\mathcal{E}xt}^n_{\mathcal{C}}(A, C) \text{ and}$$
$$\widehat{\mathcal{E}xt}^n_{\mathcal{C}}(g, B) : \widehat{\mathcal{E}xt}^n_{\mathcal{C}}(A, B) \to \widehat{\mathcal{E}xt}^n_{\mathcal{C}}(C, B)$$

are defined analogously.

Although F. Goichot only works with modules over a ring, we generalize his remark from [12, p. 41] that $\widehat{\mathcal{E}xt}^n_{\mathcal{C}}(A, B)$ does not depend on choices of projective resolutions of A and B.

Proposition 6.1.3 *For any $n \in \mathbb{Z}$, both*

$$\mathcal{E}xt^n_{\mathcal{C}}(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Ab} \text{ and } \widehat{\mathcal{E}xt}^n_{\mathcal{C}}(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Ab}$$

form well defined functors that are additive in both variables with respect to the induced morphisms from Definition 6.1.2.

Proof Let $B_\bullet$, $B'_\bullet$ be two projective resolutions of B. According to [10, Theorem 2.2.6] there are chain maps $\iota_\bullet : B_\bullet \to B'_\bullet$ and $\iota'_\bullet : B'_\bullet \to B_\bullet$ lifting $\mathrm{id}_B : B \to B$ that are unique up to chain homotopy. In particular, we see that

$$\begin{aligned}\iota'_\bullet \circ \iota_\bullet + \mathrm{Null}_{\mathrm{Ch}(\mathcal{C})}(B_\bullet, B_\bullet) &= \mathrm{id} + \mathrm{Null}_{\mathrm{Ch}(\mathcal{C})}(B_\bullet, B_\bullet) \text{ and}\\ \iota_\bullet \circ \iota'_\bullet + \mathrm{Null}_{\mathrm{Ch}(\mathcal{C})}(B'_\bullet, B'_\bullet) &= \mathrm{id} + \mathrm{Null}_{\mathrm{Ch}(\mathcal{C})}(B'_\bullet, B'_\bullet)\,.\end{aligned}$$

This constitutes a direct proof that the abelian groups $\mathcal{E}xt_{\mathcal{C}}(A,B)$ and $\widehat{\mathcal{E}xt}_{\mathcal{C}}(A,B)$ do not depend on a choice of projective resolution of B. We may conclude analogously that they do not depend on the choice of projective resolution of A. If $f : A \to B$ and $g : C \to A$ are morphisms in $\mathcal{C}$, then this also proves that $\mathcal{E}xt_{\mathcal{C}}(A,f)$ and $\widehat{\mathcal{E}xt}_{\mathcal{C}}(A,f)$ as well as $\mathcal{E}xt_{\mathcal{C}}(g,B)$ and $\widehat{\mathcal{E}xt}_{\mathcal{C}}(g,B)$ are well defined homomorphisms. We conclude from [10, p. 5] that $\mathcal{E}xt^n_{\mathcal{C}}(-,-)$ and $\widehat{\mathcal{E}xt}^n_{\mathcal{C}}(-,-)$ are additive in both variables. □

6.2 Connecting homomorphisms through chain maps

We construct an explicit isomorphism from the Ext-group $\mathrm{Ext}^n_{\mathcal{C}}(A,B)$ to $\mathcal{E}xt^n_{\mathcal{C}}(A,B)$ which allows us to determine connecting homomorphisms for $\mathcal{E}xt^\bullet_{\mathcal{C}}(A,-)$ and $\widehat{\mathcal{E}xt}^\bullet_{\mathcal{C}}(A,-)$ in terms of (almost) chain maps. The construction of this isomorphism is of similar nature to pages 63–64 of C. A. Weibel's book [10] where he compares the chain complex $\mathrm{Hom}_{\mathcal{C}}(A_\bullet, B)$ to $\mathrm{Hyp}_{\mathcal{C}}(A_\bullet, B_\bullet)_{\bullet,\bullet}$ seen as a double complex. However, as our techniques differ and we have not found this isomorphism in the literature otherwise, we provide full details.

Given that Ext-groups vanish in negative degrees, let $n \in \mathbb{N}_0$. Let $(A_\bullet, a_\bullet)$ be a projective resolution of A, $(B_\bullet, b_\bullet)$ be a projective resolution of B and assume that $\varphi \in \mathrm{Ker}(\mathrm{Hom}_{\mathcal{C}}(a_{k+1}, B))$. By Diagram 5.2.1 and the Comparison Theorem [10, Theorem 2.2.6], there exists $(\varphi[n]_k : A[n]_k \to B_k)_{k \in \mathbb{N}_0}$ that is unique up to chain homotopy making the diagram

$$\begin{tikzcd} \dots \arrow[r] & A_{n+2} \arrow[r, "(-1)^n a_{n+2}"] \arrow[d, "\varphi[n]_2"] & A_{n+1} \arrow[r, "(-1)^n a_{n+1}"] \arrow[d, "\varphi[n]_1"] & A_n \arrow[r, "\pi_n"] \arrow[d, "\varphi[n]_0"] \arrow[dr, "\varphi"] & \widetilde{A}_n \arrow[d, "\alpha_n(B)(\varphi)"] \\ \dots \arrow[r] & B_2 \arrow[r, "b_2"] & B_1 \arrow[r, "b_1"] & B_0 \arrow[r, "b"] & B \end{tikzcd} \tag{6.2.1}$$

commute. There is an extension $(\varphi[n]_k : A[n]_k \to B_k)_{k\in\mathbb{Z}}$ since $B_k = 0$ for $k < 0$. Hence the map

$$\begin{aligned}\zeta'_n : \operatorname{Ker}(\operatorname{Hom}_{\mathcal{C}}(a_{n+1}, B)) &\to \mathcal{E}xt^n_{\mathcal{C}}(A, B)\\ \varphi &\mapsto \varphi[n]_\bullet + \operatorname{Null}_{\operatorname{Ch}(\mathcal{C})}(A[n]_\bullet, B_\bullet)\end{aligned}$$

is a well defined homomorphism of abelian groups.

Assume that $n \geq 1$ and $\psi \in \operatorname{Im}(\operatorname{Hom}_{\mathcal{C}}(a_n, B))$. Then there is a morphism $\psi : A_{n-1} \to B$ such that $\varphi = \psi \circ a_n$. Because A_{n-1} is projective and $b : B_0 \to B$ an epimorphism, there $e_{n-1} : A_{n-1} \to B_0$ such that $\psi = b \circ e_{n-1}$. Knowing that $a_n \circ a_{n+1} = 0$, we see that the diagram

$$\begin{tikzcd} A_{n+1} \arrow[r, "(-1)^n a_{n+1}"] \arrow[d, "0"] & A_n \arrow[r, "\pi_n"] \arrow[d, "e_{n-1}\circ a_n"] \arrow[dr, "\varphi"] & \widetilde{A}_n \arrow[d, "\alpha_n(B)(\varphi)"] \\ B_1 \arrow[r, "b_1"] & B_0 \arrow[r, "b"] & B \end{tikzcd}$$

is commutative. Set $\varphi[n]_\bullet \in \operatorname{Hom}_{\operatorname{Ch}(\mathcal{C})}(A[n]_\bullet, B_\bullet)$ to be zero in every degree except for $k = 0$ where $\varphi[n]_0 = e_{n-1} \circ a_n$. Similarly, set $e[n]_\bullet : A[n]_\bullet \to B_{\bullet+1}$ to be also zero in every degree except for $k = -1$ where

$$e[n]_{-1} = (-1)^n e_{n-1} : A_{n-1} \to B_0\,.$$

Then $e[n]_\bullet$ is a chain homotopy between $\varphi[n]_\bullet$ and the zero chain map. Therefore, ζ'_n descends to a homomorphism

$$\begin{aligned}\zeta_n :\ \operatorname{Ext}^n_{\mathcal{C}}(A, B) &\to \mathcal{E}xt^n_{\mathcal{C}}(A, B)\\ \varphi + \operatorname{Im}(\operatorname{Hom}_{\mathcal{C}}(a_n, B)) &\mapsto \varphi[n]_\bullet + \operatorname{Null}_{\operatorname{Ch}(\mathcal{C})}(A[n]_\bullet, B_\bullet)\,.\end{aligned}$$

As we have assumed $n \geq 1$ for this, we define $\zeta_0 := \zeta'_0$ where we exploit the fact that $\operatorname{Ker}(\operatorname{Hom}_{\mathcal{C}}(a_1, B)) = \operatorname{Ext}^0_{\mathcal{C}}(A, B)$. Given that unenriched Ext-functors of negative degree vanish, we set $\zeta_n := 0$ for $n < 0$.

Proposition 6.2.1 *For every $n \in \mathbb{Z}$, $\zeta_n : \operatorname{Ext}^n(A, B) \to \mathcal{E}xt^n_{\mathcal{C}}(A, B)$ is a isomorphism that is natural in both variables A and B.*

Proof Since ζ_n is a natural isomorphism for $n < 0$, let us assume that $n \in \mathbb{N}_0$ for the rest of the proof. To prove that ζ_n is surjective, take an element $\psi_{\bullet+n} \in \operatorname{Hom}_{\mathcal{C}}(A[n]_\bullet, B_\bullet)$. As $B_k =$ for $k < 0$, $b_1 \circ b = 0$ and the diagram

$$\begin{tikzcd}
A_{n+1} \arrow[r, "(-1)^n a_{n+1}"] \arrow[d, "\psi_{n+1}"] & A_n \arrow[d, "\psi_n"] \arrow[dr, "b\circ\psi_n"] & \\
B_1 \arrow[r, "b_1"] & B_0 \arrow[r, "b"] & B
\end{tikzcd}$$

commutes, we infer that $b \circ \psi_n \in \operatorname{Ker}(\operatorname{Hom}_{\mathcal{C}}(a_{n+1}, B))$ and that

$$\zeta_n(b \circ \psi_0 + \operatorname{Im}(\operatorname{Hom}_{\mathcal{C}}(a_n, B))) = \psi_{\bullet+n} + \operatorname{Null}_{\operatorname{Ch}(\mathcal{C})}(A[n]_\bullet, B_\bullet)\,.$$

Let us show that ζ_n is injective. Assume first that $n \geq 1$. If there is an element $\varphi \in \operatorname{Ker}(\operatorname{Hom}_{\mathcal{C}}(a_{n+1}, B))$ is such that $\zeta_n(\varphi+\operatorname{Im}(\operatorname{Hom}_{\mathcal{C}}(a_n, B))) = 0$, then there are $\varphi_n : A_n \to B_0$, $e_n : A_n \to B_1$ and $e_{n-1} : A_{n-1} \to B_0$ such that

$$\varphi = b \circ \varphi_n = b \circ \big(e_{n-1} \circ (-1)^n a_n + b_1 \circ e_n\big) = (-1)^n (b \circ e_n) \circ a_n\,.$$

In particular, $\varphi \in \operatorname{Im}(\operatorname{Hom}_{\mathcal{C}}(a_n, B))$ and ζ_n is injective. Suppose that $n = 0$ and that $\varphi \in \operatorname{Ker}(\operatorname{Hom}_{\mathcal{C}}(a_1, B))$ is such that $\zeta_0(\varphi) = 0$. Because $A_{-1} = 0$ and $a_0 = 0$, we conclude that $\varphi = 0$ and that ζ_0 is also injective.

Let us demonstrate that ζ_n is natural in both variables. Let $g : C \to A$ and $f : B \to D$ be a morphism in $\mathcal{C}$ and $\varphi \in \operatorname{Ker}(\operatorname{Hom}_{\mathcal{C}}(a_{n+1}, B))$. Taking lifts $f_\bullet : B_\bullet \to D_\bullet$ and $g[n]_\bullet : C[n]_\bullet \to A[n]_\bullet$ as needed in Definition 6.1.2, we see that the diagram

$$\begin{tikzcd}
\dots \arrow[r] & C_{n+2} \arrow[r, "(-1)^n c_{n+2}"] \arrow[d, "g[n]_2"] & C_{n+1} \arrow[r, "(-1)^n c_{n+1}"] \arrow[d, "g[n]_1"] & C_n \arrow[d, "g[n]_0"] & \\
\dots \arrow[r] & A_{n+2} \arrow[r, "(-1)^n a_{n+2}"] \arrow[d, "\varphi[n]_2"] & A_{n+1} \arrow[r, "(-1)^n a_{n+1}"] \arrow[d, "\varphi[n]_1"] & A_n \arrow[d, "\varphi[n]_0"] \arrow[dr, "\varphi"] & \\
\dots \arrow[r] & B_2 \arrow[r, "b_2"] \arrow[d, "f_2"] & B_1 \arrow[r, "b_1"] \arrow[d, "f_1"] & B_0 \arrow[r, "b"] \arrow[d, "f_0"] & B \arrow[d, "f"] \\
\dots \arrow[r] & D_2 \arrow[r, "d_2"] & D_1 \arrow[r, "d_1"] & D_0 \arrow[r, "d"] & C
\end{tikzcd}$$

commutes. This shows that

$$\begin{aligned}
&\zeta_n\Big(\operatorname{Ext}^n_{\mathcal{C}}(A, f)\big(\varphi+\operatorname{Im}(\operatorname{Hom}_{\mathcal{C}}(a_{n+1}, B))\big)\Big) \\
&\qquad = \mathcal{E}xt^n_{\mathcal{C}}(A, f)\Big(\zeta_n\big(\varphi + \operatorname{Im}(\operatorname{Hom}_{\mathcal{C}}(a_{n+1}, B))\big)\Big)
\end{aligned}$$

and that

$$\zeta_n\Big(\mathrm{Ext}^n_{\mathcal{C}}(g,B)\big(\varphi+\mathrm{Im}(\mathrm{Hom}_{\mathcal{C}}(a_{n+1},B))\big)\Big) \\ = \mathcal{E}xt^n_{\mathcal{C}}(g,B)\Big(\zeta_n\big(\varphi+\mathrm{Im}(\mathrm{Hom}_{\mathcal{C}}(a_{n+1},B))\big)\Big)\,.$$

In short,

$$\zeta_n \circ \mathrm{Ext}^n_{\mathcal{C}}(A,-) = \mathcal{E}xt^n_{\mathcal{C}}(A,-)\circ\zeta_n \text{ and } \zeta_n \circ \mathrm{Ext}^n_{\mathcal{C}}(-,B) = \mathcal{E}xt^n_{\mathcal{C}}(-,B)\circ\zeta_n\,,$$

which prove naturality. $\square$

Our construction of the desired connecting homomorphisms for $\mathcal{E}xt^\bullet_{\mathcal{C}}(A,-)$ and $\widehat{\mathcal{E}xt}^\bullet_{\mathcal{C}}(A,-)$ is similar to the definitions found in [42, p. 2202]. We consider a projective resolution $(D_\bullet, d_\bullet)$ of D. For any $k \in \mathbb{N}_0$ define the chain complex $(D^{+1}_\bullet, d^{+1}_\bullet)$ by setting $(D^{+1}_m, d^{+1}_m)_{m\in\mathbb{N}} := (D_{m+1}, d_{m+1})_{m\in\mathbb{N}}$, $D^{+1}_0 := D_1$ and $D^{+1}_m := 0$ for $m < 0$. Observe that $D^{+1}_\bullet$ is a projective resolution of $\widetilde{D}_1$. We define a chain map $\pi^n_\bullet : D_\bullet \to D^{+1}[-1]_\bullet$ by setting

$$\pi^n_m := \begin{cases} (-1)^{m-1+n}\mathrm{id}_{D_m} : D_m \to D_m & \text{if } m \geq 1 \\ 0 & \text{if } m \leq 0 \end{cases}$$

Pictorially

$$\begin{tikzcd}
\dots \arrow[r] & D_4 \arrow[r, "d_4"] \arrow[d, "(-1)^{n+3}\mathrm{id}"] & D_3 \arrow[r, "d_3"] \arrow[d, "(-1)^{n+2}\mathrm{id}"] & D_2 \arrow[r, "d_2"] \arrow[d, "(-1)^{n+1}\mathrm{id}"] & D_1 \arrow[r, "d_1"] \arrow[d, "(-1)^{n}\mathrm{id}"] & D_0 \arrow[d] \\
\dots \arrow[r] & D_4 \arrow[r, "-d_4"] & D_3 \arrow[r, "-d_3"] & D_2 \arrow[r, "-d_2"] & D_1 \arrow[r] & 0
\end{tikzcd} \tag{6.2.2}$$

As is noted in [41, p. 166], we can equally well express the $(n+1)^{\text{st}}$ Ext-group as

$$\mathcal{E}xt^{n+1}_{\mathcal{C}}(A,D) = \mathrm{Hom}_{\mathrm{Ch}(\mathcal{C})}(A[n]_\bullet, D[-1]_\bullet)/\mathrm{Null}_{\mathrm{Ch}(\mathcal{C})}(A[n]_\bullet, D[-1]_\bullet)$$

with an analogous statement true for $\widehat{\mathcal{E}xt}^{n+1}_{\mathcal{C}}(A,D)$. The following definition is similar to the connecting homomorphisms $\varepsilon^{-m} : S^{-m}T \to S^{-m+1}T$ established at the start of Chapter 3 and with the morphisms used in the construction of the connecting homomorphisms of the naive construction found in Lemma 5.3.1.

Definition 6.2.2 *We set*

$$\begin{aligned}\Delta^n : \mathcal{E}xt^n_{\mathcal{C}}(A,D) &\to \mathcal{E}xt^{n+1}_{\mathcal{C}}(A,\widetilde{D}_1)\\ \varphi_{\bullet+n} + \mathrm{Null}_{\mathrm{Ch}(\mathcal{C})}(A[n]_\bullet, D_\bullet) &\mapsto \pi^n_\bullet \circ \varphi_{\bullet+n} + \mathrm{Null}_{\mathrm{Ch}(\mathcal{C})}(A[n]_\bullet, D^{+1}[-1]_\bullet)\end{aligned}$$

to be the (proposed) connecting homomorphism associated too the short exact sequence $0 \to \widetilde{D}_1 \to D_0 \to D \to 0$*. Analogously, we define the (proposed) connecting homomorphism* $\widehat{\Delta}^n : \widehat{\mathcal{E}xt}^n_{\mathcal{C}}(A,D) \to \widehat{\mathcal{E}xt}^{n+1}_{\mathcal{C}}(A,\widetilde{D}_1)$.

If $0 \to B \to C \to D \to 0$ *is another short exact sequence in* $\mathcal{C}$*, then consider the lift* $h^* : \widetilde{D}_1 \to B$ *of* $\mathrm{id}_D : D \to D$ *from Diagram 3.0.2 and a chain map* $h^*_{\bullet+1} : D^{+1}_\bullet \to B_\bullet$ *as in [10, Theorem 2.2.6]. We define the (proposed) connecting homomorphism for this short exact sequence as*

$$\begin{aligned}\Delta^n : \mathcal{E}xt^n_{\mathcal{C}}(A,D) &\to \mathcal{E}xt^{n+1}_{\mathcal{C}}(A,B)\\ \varphi_{\bullet+n} + \mathrm{Null}_{\mathrm{Ch}(\mathcal{C})}(A[n]_\bullet, D_\bullet) &\mapsto h^*[-1]_{\bullet+1} \circ \pi^n_\bullet \circ \varphi_{\bullet+n}\\ &\qquad + \mathrm{Null}_{\mathrm{Ch}(\mathcal{C})}(A[n]_\bullet, B[-1]_\bullet)\end{aligned}$$

where the definition of $\widehat{\Delta}^n : \widehat{\mathcal{E}xt}^n_{\mathcal{C}}(A,D) \to \widehat{\mathcal{E}xt}^{n+1}_{\mathcal{C}}(A,B)$ *is analogous.*

Because the maps in the above definition are defined by compositions of (almost) chain maps, they are homomorphisms of abelian groups. However, they are not well defined a priori because they depend on a choice of morphism $h^* : \widetilde{D}_1 \to B$.

Lemma 6.2.3 *The family* $(\mathcal{E}xt^\bullet_{\mathcal{C}}(A,-), \Delta^\bullet)$ *forms a cohomological functor. Furthermore,* $\zeta_\bullet : (\mathrm{Ext}^\bullet_{\mathcal{C}}(A,-), \delta^\bullet) \to (\mathcal{E}xt^\bullet_{\mathcal{C}}(A,-), \Delta^\bullet)$ *is an isomorphism of cohomological functors.*

Proof It suffices to prove that $\zeta_n : \mathrm{Ext}^n_{\mathcal{C}}(A,-) \to \mathcal{E}xt^\bullet_{\mathcal{C}}(A,-)$ commutes with the (proposed) connecting homomorphisms. Let $0 \to B \to C \to D \to 0$ be a short exact sequence in $\mathcal{C}$ and let $h^* : \widetilde{D}_1 \to B$ be a lift of id_D as before. We have seen at the start of the proof of Lemma 5.3.1 that the associated connecting homomorphism factors through the one associated to $0 \to \widetilde{D}_1 \to D_0 \to D \to 0$ as

$$\delta^n : \mathrm{Ext}^n_{\mathcal{C}}(A,D) \xrightarrow{\delta^n} \mathrm{Ext}^{n+1}_{\mathcal{C}}(A,\widetilde{D}_1) \xrightarrow{\mathrm{Ext}^{n+1}_{\mathcal{C}}(A,h^*)} \mathrm{Ext}^{n+1}_{\mathcal{C}}(A,B)\,.$$

Thus, the left hand triangle in the diagram

$$\begin{tikzcd}
\mathrm{Ext}^n_{\mathcal{C}}(A,D) \arrow[rrr, "\zeta_n"] \arrow[dd, "\delta^n"] \arrow[dr, "\delta^n"] & & & \mathcal{E}xt^n_{\mathcal{C}}(A,D) \arrow[dl, "\Delta^n"'] \arrow[dd, "\Delta^n"] \\
& \mathrm{Ext}^{n+1}_{\mathcal{C}}(A,\widetilde{D}_1) \arrow[r, "\zeta_{n+1}"] \arrow[dl, "\mathrm{Ext}^{n+1}_{\mathcal{C}}(A,h^*)"] & \mathcal{E}xt^{n+1}_{\mathcal{C}}(A,\widetilde{D}_1) \arrow[dr, "\mathcal{E}xt^{n+1}_{\mathcal{C}}(A,h^*)"] & \\
\mathrm{Ext}^{n+1}_{\mathcal{C}}(A,B) \arrow[rrr, "\zeta_{n+1}"] & & & \mathcal{E}xt^{n+1}_{\mathcal{C}}(A,B)
\end{tikzcd} \tag{6.2.3}$$

commutes. By Definition 6.1.2 and 6.2.2, the right hand triangle triangle also commutes. According to Proposition 6.2.1, the bottom trapezium is commutative. The top trapzium can be seen as a generalization of Proposition 1.1 from [42] where we use a different method to prove that it commutes. Let $\varphi \in \mathrm{Ker}(\mathrm{Hom}_{\mathcal{C}}(a_n, D))$. We explain how one can combine Diagram 5.2.3 and 6.2.1 as

$$\begin{tikzcd}
\cdots \arrow[r] & A_{n+2} \arrow[r, "(-1)^n a_{n+2}"] \arrow[d, "{\varphi[n]_2}"] & A_{n+1} \arrow[r, "\pi_{n+1}"] \arrow[rr, bend left, "a_{n+1}"] \arrow[d, "{\varphi[n]_1}"] \arrow[dr, "\Phi"] & \widetilde{A}_{n+1} \arrow[r, "\iota_{n+1}"] \arrow[d] & A_n \arrow[r, "\pi_n"] \arrow[d, "{\varphi[n]_0}"] \arrow[dr, "\varphi"] & \widetilde{A}_n \arrow[d, "\alpha_{n+k}(B)(\varphi)"] \\
\cdots \arrow[r] & D_2 \arrow[r, "d_2"] & D_1 \arrow[r, "\pi_1"] \arrow[rr, bend right, "d_1"'] & \widetilde{D}_1 \arrow[r, "\iota_1"] & D_0 \arrow[r, "\pi_0"] & D
\end{tikzcd} \tag{6.2.4}$$

The right most morphisms up to $\Phi : A_{n+1} \to \widetilde{B}_1$ are taken from Diagram 5.2.3 while $\varphi : A_n \to D$ and $\varphi[n]_\bullet : A[n]_\bullet \to D_\bullet$ are the morphisms as they occur in Diagram 6.2.1. In particular,

$$\delta^n\big(\varphi + \mathrm{Im}(\mathrm{Hom}_{\mathcal{C}}(a_n, D))\big) = \Phi + \big(\varphi + \mathrm{Im}(\mathrm{Hom}_{\mathcal{C}}(a_n, \widetilde{D}_1))\,.$$

However, note that $\iota_{n+1} \circ \pi_{n+1} = a_{n+1}$ might not agree with the morphism $(-1)^n a_{n+1} : A_{n+1} \to A_n$. This together with the construction of Diagram 5.2.3 and the proof of the Comparison Theorem from [10, p. 36] implies that $\Phi = (-1)^n \pi_1 \circ \varphi[n]_1$. Therefore, we can choose the representative $\Phi[n+1]_\bullet : A[n+1] \to \widetilde{D}^{+1}_\bullet$ of a lift of Φ as in Diagram 6.2.1 which is defined as

$$\Phi[n+1]_m := \begin{cases} (-1)^{n+m}\varphi[n]_{m+1} : A_{m+n+1} \to D_{m+1} & \text{if } m \geq 0 \\ 0 & \text{if } m < 0 \end{cases}$$

The alternating signs are due to the fact that the boundary maps of $A[n]_\bullet$ and $A[n+1]$ have opposite signs. We conclude by Diagram 6.2.2, Definition 6.2.2 and Diagram 6.2.4 that

$$\zeta_{n+1}\Big(\delta^n\big(\varphi + \mathrm{Im}(\mathrm{Hom}_{\mathcal{C}}(a_n, D))\big)\Big) = \Delta^n\Big(\zeta^n\big(\varphi + \mathrm{Im}(\mathrm{Hom}_{\mathcal{C}}(a_n, D))\big)\Big).$$

In particular, the top trapezium and with it all of Diagram 6.2.3 commutes. As for Theorem 5.3.2, we can now reiterate the proof of Theorem 4.3.2 with Diagram 6.2.3, the fact that $(\mathrm{Ext}^\bullet_{\mathcal{C}}(A,-), \delta^\bullet)$ is a cohomological functor and that $\zeta_\bullet : \mathrm{Ext}^\bullet_{\mathcal{C}}(A,-) \to \mathcal{E}xt^\bullet_{\mathcal{C}}(A,-)$ is a natural isomorphism by Proposition 6.2.1. □

6.3 Isomorphism of cohomology groups

In this section we construct for every $n \in \mathbb{Z}$ a natural isomorphism from the completed unenriched Ext-functor $\mathrm{Ext}^n_{Res,\mathcal{C}}(A,-)$ defined through the resolution construction to $\widehat{\mathcal{E}xt}^n_{\mathcal{C}}(A,-)$. In order to do so, we first reformulate the former completed Ext-functor.

Definition 6.3.1 *For any $B \in \mathrm{obj}(\mathcal{C})$, $f : B \to C$ and $n \in \mathbb{Z}$ define*

$$\mathcal{E}xt^n_{Res,\mathcal{C}}(A,B) := \varinjlim_{k\in\mathbb{N}_0} (\mathcal{E}xt^{n+k}_{\mathcal{C}}(A, \widetilde{B}_k), \Delta^{n+k})$$

and

$$\mathcal{E}xt^n_{Res,\mathcal{C}}(A,f) := \varinjlim_{k\in\mathbb{N}_0} \mathcal{E}xt^{n+k}_{\mathcal{C}}(A, \widetilde{f}_k) : \mathcal{E}xt^n_{\mathcal{C}}(A,B) \to \mathcal{E}xt^n_{\mathcal{C}}(A,C)$$

where $\widetilde{f}_k : \widetilde{B}_k \to \widetilde{C}_k$ are taken as in Proposition 4.2.1. For any short exact sequence $0 \to B \to C \to D \to 0$ we set

$$\widetilde{\Delta}^n := \varinjlim_{k\in\mathbb{N}_0} (-1)^k \Delta^{n+k} : \mathcal{E}xt^n_{Res,\mathcal{C}}(A,D) \to \mathcal{E}xt^{n+1}_{Res,\mathcal{C}}(A,B)$$

where $\Delta^{n+k} : \mathcal{E}xt^{n+k}_{\mathcal{C}}(A, \widetilde{D}_k) \to \mathcal{E}xt^{n+k+1}_{\mathcal{C}}(A, \widetilde{B}_k)$ are taken as in Diagram 4.2.3. Using the isomorphisms $\zeta_{n+k} : \mathrm{Ext}^{n+k}_{\mathcal{C}}(A,-) \to \mathcal{E}xt^{n+k}_{\mathcal{C}}(A,-)$ from Lemma 6.2.3, we can extend the natural isomorphisms

$$\zeta^n := \varinjlim_{k\in\mathbb{N}_0} \zeta_{n+k} : \mathrm{Ext}^n_{Res,\mathcal{C}}(A,-) \to \mathcal{E}xt^n_{Res,\mathcal{C}}(A,-)$$

to $\zeta^\bullet : (\mathrm{Ext}^\bullet_{Res,\mathcal{C}}(A,-), \widetilde{\delta}^\bullet) \to (\mathcal{E}xt^\bullet_{Res,\mathcal{C}}(A,-), \widetilde{\Delta}^\bullet)$ which is an isomorphism of cohomological functors.

We only consider the cofinal system $(\mathcal{E}xt_{\mathcal{C}}^{n+2k}(A, \widetilde{B}_{2k}), \Delta^{n+2k+1} \circ \Delta^{n+2k})_{k \in \mathbb{N}_0}$ and specifically form $\mathcal{E}xt^n_{Res,\mathcal{C}}(A, B)$ as its direct limit in order to ease notation when constructing a homomorphism from any cohomology group $\mathcal{E}xt^n_{Res,\mathcal{C}}(A, B)$ to $\widehat{\mathcal{E}xt}^n_{\mathcal{C}}(A, B)$. Nonetheless, we require

Notation 6.3.2 *If $(B_\bullet, b_\bullet)$ is a projective resolution of B and $k \in \mathbb{N}_0$, then we define the projective resolution $(B^{+k}_\bullet, b^{+k}_\bullet)_{m \in \mathbb{Z}}$ of $\widetilde{B}_k$ by declaring for $m \in \mathbb{Z}$ that*

$$(B^{+k}_m, b^{+k}_{m+1}) := \begin{cases} (B_{m+k},\, b_{m+1+k} : B_{m+1+k} \to B_{m+k}) & \textit{if } m \geq 0 \\ 0 & \textit{if } m < 0 \end{cases}$$

Lemma 6.3.3 *There is an isomorphism $\vartheta^n(B) : \mathcal{E}xt^n_{Res,\mathcal{C}}(A, B) \to \widehat{\mathcal{E}xt}^n_{\mathcal{C}}(A, B)$ for any $n \in \mathbb{Z}$.*

Proof For any $k \in \mathbb{N}_0$ we construct a map

$$\begin{gathered} \vartheta^{2k}_n(B) : \mathcal{E}xt^{n+2k}_{\mathcal{C}}(A, \widetilde{B}_{2k}) \to \widehat{\mathcal{E}xt}^n_{\mathcal{C}}(A, B) \\ \varphi_{\bullet+n+2k} + \mathrm{Null}_{\mathrm{Ch}(\mathcal{C})}(A[n+2k]_\bullet, B^{+2k}_\bullet) \mapsto \Phi_{\bullet+n} + \widehat{\mathrm{Null}}_{\mathrm{Ch}(\mathcal{C})}(A[n]_\bullet, B_\bullet) \end{gathered}$$

as follows. For a representative $\varphi_{\bullet+n+2k} \in \mathrm{Hom}_{\mathrm{Ch}(\mathcal{C})}(A[n+2k]_\bullet, B^{+2k}_\bullet)$ of an element in $\mathcal{E}xt^{n+2k}_{\mathcal{C}}(A, \widetilde{B}_{2k})$ we define $\Phi_{\bullet+n} : A[n]_\bullet \to B_\bullet$ by

$$\Phi_{m+n} := \begin{cases} \varphi_{m+n} : A_{m+n} \to B_m & \text{if } m \geq 2k \\ 0 & \text{if } m < 2k \end{cases}$$

As the boundary maps of $A[n]$ and $A[n+2k]$ agree modulo a shift of $2k$ in their indices, all squares of the form

$$\begin{tikzcd} A_{m+n} \arrow[r, "\Phi_{m+n}"] \arrow[d, "(-1)^n a_{m+n}"] & B_m \arrow[d, "b_m"] \\ A_{m-1+n} \arrow[r, "\Phi_{n+m-1}"] & B_{m-1} \end{tikzcd}$$

commute for $m \in \mathbb{Z} \setminus \{2k\}$. We conclude that $\Phi_{\bullet+n} \in \widehat{\mathrm{Hom}}_{\mathrm{Ch}(\mathcal{C})}(A[n]_\bullet, B_\bullet)$. If the chain map $\varphi_{\bullet+n+2k}$ is nullhomotopic, then $\Phi_{\bullet+n}$ is genuinely nullhomotopic in any degree $m \in \mathbb{Z} \setminus \{2k-1\}$, hence nullhomotopic as an almost chain map. Thus, ϑ^{2k}_n is a well defined map. It follows by construction that it is also a homomorphism of abelian groups.

We argue that the triangle

$$\begin{tikzcd}
\mathcal{E}xt^{n+2k}_{\mathcal{C}}(A, \widetilde{B}_{2k}) \arrow[d, "\Delta^{n+2k+1}\circ\Delta^{n+2k}"'] \arrow[dr, "\vartheta^{2k}_n(B)"] & \\
& \widehat{\mathcal{E}xt}^n_{\mathcal{C}}(A, B) \\
\mathcal{E}xt^{n+2k+2}_{\mathcal{C}}(A, \widetilde{B}_{2k+2)}) \arrow[ur, "\vartheta^{2k+2)}_n(B)"] &
\end{tikzcd} \tag{6.3.1}$$

commutes. By Definition 6.2.2, Δ^{n+2k} and Δ^{n+2k+2} are constructed using the chain maps

$$\pi^n_\bullet : B^{+2k}_\bullet \to B^{+2k+1}[-1]_\bullet \text{ and } \pi^{n+1}[-1]_\bullet : B^{+2k+1}[-1]_\bullet \to B^{+2k+2}[-2]_\bullet$$

which are pictorially rendered in Diagram 6.2.2. Then $\pi^{n+1}[-1]_\bullet \circ \pi^n_\bullet$ is given by

$$\pi^{n+1}[-1]_m \circ \pi^n_m = \begin{cases} \mathrm{id}_{B_m} : B_m \to B_m & \text{if } m \geq 2 \\ 0 & \text{if } m \leq 1 \end{cases}$$

since $(-1)^{(m-1)-1+(n+1)} \cdot (-1)^{m-1+n} = 1$. Hence, the m^{th} degree of an almost chain map representative of the image of an element $\varphi_{\bullet+n+2k} + \mathrm{Null}_{\mathrm{Ch}(\mathcal{C})}(A[n+2k], B^{+2k}_\bullet)$ in $\mathcal{E}xt^{n+2k}_{\mathcal{C}}(A; \widetilde{B}_{2k})$ under $\Delta^{n+2k+1} \circ \Delta^{n+2k}$ is

$$\begin{cases} \varphi_{m+n+2k+2} : A_{m+n+2k+2} \to B_{m+2k+2} & \text{if } m \geq 0 \\ 0 & \text{if } m < 0 \end{cases} \tag{6.3.2}$$

We infer by definition of $\vartheta^{2k}_n(B)$ and $\vartheta^{2k+2}_n(B)$ that

$$\vartheta^{2k}_n(B)\big(\varphi_{\bullet+n+2k} + \mathrm{Null}_{\mathrm{Ch}(\mathcal{C})}(A[n+2k], B^{+2k}_\bullet)\big) = \\ \big(\vartheta^{2k+2}_n \circ \Delta^{n+2k+1} \circ \Delta^{n+2k}\big)\big(\varphi_{\bullet+n+2k} + \mathrm{Null}_{\mathrm{Ch}(\mathcal{C})}(A[n+2k], B^{+2k}_\bullet)\ \big)$$

The resulting homomorphism in the direct limit of the above commuting triangles is

$$\vartheta^n(B) := \varinjlim_{k\in\mathbb{N}_0} \vartheta^{2k}_n(B) : \ \mathcal{E}xt^n_{Res,\mathcal{C}}(A, B) \to \widehat{\mathcal{E}xt}^n_{\mathcal{C}}(A, B)\,.$$

To show that it is surjective, let $\Phi_{\bullet+n}+\widehat{\mathrm{Null}}_{\mathrm{Ch}(\mathcal{C})}(A[n]_\bullet, B_\bullet)$ be in $\widehat{\mathcal{E}xt}^n_{\mathcal{C}}(A,B)$. Being an almost chain map, there is $k \in \mathbb{N}_0$ such that $(\Phi_{m+n} : A_{m+n} \to B_m)_{m\geq 2k}$ is a chain map. Defining $\varphi_{\bullet+n+2k} : A[n+2k]_\bullet \to B^{+2k}_\bullet$ by setting

$$\varphi_{m+n+2k} := \begin{cases} \Phi_{m+n+2k} : A_{m+n+2k} \to B_{m+2k} & \text{if } m \geq 0 \\ 0 & \text{if } m < 0 \end{cases}$$

gives an element in $\mathcal{E}xt^{n+2k}_{\mathcal{C}}(A, \widetilde{B}_{2k})$ that $\vartheta^{2k}_n(B)$ maps to

$$\Phi_{\bullet+n} + \widehat{\mathrm{Null}}_{\mathrm{Ch}(\mathcal{C})}(A[n]_\bullet, B_\bullet)\,.$$

To prove that $\vartheta^n(B)$ is injective, let $x \in \varinjlim \mathcal{E}xt^{n+2k}_{\mathcal{C}}(A, \widetilde{B}_{2k})$ be an element with $\vartheta^n(B)(x) = 0$. According to Proposition 5.2.1, there exists $k \in \mathbb{N}_0$ and an element

$$\psi_{\bullet+n+2k} \in \mathrm{Hom}_{\mathrm{Ch}(\mathcal{C})}(A[n+2k]_\bullet, B^{+2k}_\bullet)$$

mapping to x in the direct limit that lies in the kernel of the homomorphism $\vartheta^{2k}_n(B)$. If we construct an almost chain map representative

$$\Psi_{\bullet+n} : A[n]_\bullet \to B_\bullet$$

from $\psi_{\bullet+n+k}$ as above, then $\Psi_{\bullet+n} \in \widehat{\mathrm{Null}}_{\mathrm{Ch}(\mathcal{C})}(A[n]_\bullet, B_\bullet)$. Therefore, there is $K \geq k$ such that $(\Psi_m)_{m\geq 2K}$ is nullhomotopic as a chain map and not just as an almost chain map. The chain map $\overline{\psi}_{\bullet+n+2K} : A[n+2K]_\bullet \to B^{+2K}_\bullet$ that we define by

$$\overline{\psi}_{m+n+2K} := \begin{cases} \Psi_{m+n+2K} = \psi_{m+n+2K} : A_{m+n+2K} \to B_{m+2K} & \text{if } m \geq 0 \\ 0 & \text{if } m < 0 \end{cases}$$

lies in $\mathrm{Null}_{\mathrm{Ch}(\mathcal{C})}(A[n+2K]_\bullet, B^{+2K}_\bullet)$. By Equation 6.3.2, $\overline{\psi}_{\bullet+n+2K}$ is a representative of the image of $\psi_{\bullet+n+2k} + \mathrm{Null}_{\mathrm{Ch}(\mathcal{C})}(A[n+2k]_\bullet, B^{+2k}_\bullet)$ under $\Delta^{n+2K-1} \circ \ldots \circ \Delta^{n+2k}$, which also maps to the element x in the direct limit. As $x = 0$, this implies that $\vartheta^n(B)$ is injective and thus an isomorphism. □

Proposition 6.3.4 *For every $n \in \mathbb{Z}$ the isomorphism $\vartheta^n(-)$ is natural. That is, for every morphism $f : B \to C$ in $\mathcal{C}$ the square*

$$\begin{array}{ccc} \mathcal{E}xt^n_{Res,\mathcal{C}}(A,B) & \xrightarrow{\vartheta^n(B)} & \widehat{\mathcal{E}xt}^n_{\mathcal{C}}(A,B) \\ \Big\downarrow{\scriptstyle \mathcal{E}xt^n_{Res,\mathcal{C}}(A,f)} & & \Big\downarrow{\scriptstyle \widehat{\mathcal{E}xt}^n_{\mathcal{C}}(A,f)} \\ \mathcal{E}xt^n_{Res,\mathcal{C}}(A,C) & \xrightarrow{\vartheta^n(C)} & \widehat{\mathcal{E}xt}^n_{\mathcal{C}}(A,C) \end{array}$$

commutes.

Proof Let $(\widetilde{f}_k : \widetilde{M}_k \to \widetilde{N}_k)_{k \in \mathbb{N}_0}$ be a sequence of morphism as arising from Diagram 4.2.1. Setting $K := 2k$ and $N := n + 2k$, it suffices by Definition 6.3.1 to consider the diagram

$$\begin{tikzcd}
 & \mathcal{E}xt^N_{\mathcal{C}}(A, \widetilde{C}_K) \arrow[drr, "\vartheta^K_n(C)"] \arrow[dd, "\Delta^{N+1} \circ \Delta^N"] & & \\
\mathcal{E}xt^N_{\mathcal{C}}(A, \widetilde{B}_K) \arrow[ur, "\mathcal{E}xt^N_{\mathcal{C}}(A, \widetilde{f}_K)"] \arrow[drr, "\vartheta^K_n(B)"] \arrow[dd, "\Delta^{N+1} \circ \Delta^N"] & & & \widehat{\mathcal{E}xt}^n_{\mathcal{C}}(A, C) \\
 & \mathcal{E}xt^{N+2}_{\mathcal{C}}(A, \widetilde{C}_{K+2}) \arrow[urr, "\vartheta^{K+2}_n(C)"] & \widehat{\mathcal{E}xt}^n_{\mathcal{C}}(A, B) \arrow[ur, "\widehat{\mathcal{E}xt}^n_{\mathcal{C}}(A, f)"] & \\
\mathcal{E}xt^{N+2}_{\mathcal{C}}(A, \widetilde{B}_{K+2}) \arrow[ur, "\mathcal{E}xt^{N+2}_{\mathcal{C}}(A, \widetilde{f}_{K+2})"] \arrow[urr, "\vartheta^{K+2}_n(B)"'] & & &
\end{tikzcd} \tag{6.3.3}$$

The left hand side of the prism commutes because Δ^{n+2k} and Δ^{n+2k+1} are connecting homomorphisms. The triangles in the front and the back commute by Diagram 6.3.1. In Definition 6.1.2, we have constructed $\widehat{\mathcal{E}xt}^n_{\mathcal{C}}(A, f)$ by lifting $f : B \to C$ to a chain map $f_\bullet : B_\bullet \to C_\bullet$ through the Comparison Theorem [10, Theorem 2.2.6] and by composing any almost chain map representative of an element in $\widehat{\mathcal{E}xt}^n_{\mathcal{C}}(A, B)$ with $f_\bullet$. Note that we can lift $\widetilde{f}_{2k} : \widetilde{B}_{2k} \to \widetilde{C}_{2k}$ to a chain map $f^{+2k}_\bullet : B^{+2k}_\bullet \to C^{+2k}_\bullet$ given by

$$f^{+2k}_m := \begin{cases} f_{m+2k} : B_{m+2k} \to C_{m+2k} & \text{if } m \geq 0 \\ 0 & \text{if } m < 0 \end{cases}$$

In particular, the square

$$\begin{tikzcd}
\mathcal{E}xt^{n+2k}_{\mathcal{C}}(A, \widetilde{B}_{2k}) \arrow[r, "\mathcal{E}xt^{n+2k}_{\mathcal{C}}(A, \widetilde{f}_{2k})"] \arrow[d, "\vartheta^{2k}_n(B)"] & \mathcal{E}xt^{n+2k}_{\mathcal{C}}(A, \widetilde{C}_{2k}) \arrow[d, "\vartheta^{2k}_n(C)"] \\
\widehat{\mathcal{E}xt}^n_{\mathcal{C}}(A, B) \arrow[r, "\widehat{\mathcal{E}xt}^n_{\mathcal{C}}(A, f)"] & \widehat{\mathcal{E}xt}^n_{\mathcal{C}}(A, C)
\end{tikzcd}$$

commutes and with it Diagram 6.3.3. The top right and bottom right squares of the latter diagram form a direct system in whose direct limit we obtain the commuting square in the statement of the proposition. □

Corollary 6.3.5 *For every $n \in \mathbb{Z}$ we define the natural isomorphism*

$$\sigma^n := \vartheta^n \circ \zeta^n : \widehat{\operatorname{Ext}}^n_{\mathcal{C}}(A, -) \to \mathcal{E}xt^n_{Res,\mathcal{C}}(A, -) \to \widehat{\mathcal{E}xt}^n_{\mathcal{C}}(A, -)$$

where ζ^n is taken from Definition 6.3.1.

6.4 Isomorphism of cohomological functors

In order to prove that the isomorphisms $\vartheta^n : \mathcal{E}xt^n_{Res,\mathcal{C}}(A, -) \to \widehat{\mathcal{E}xt}^n_{\mathcal{C}}(A, -)$ extend to an isomorphism of cohomological functors, we need the following technical result.

Proposition 6.4.1 *Let $0 \to B \xrightarrow{f} C \xrightarrow{g} D \to 0$ be a short exact sequence in $\mathcal{C}$. If $h : \widetilde{D}_1 \to B$ denotes a lift of $\mathrm{id}_D : D \to D$ as in Diagram 3.0.2 and $D_\bullet$ a projective resolution of D, consider a lift of h^* to a chain map $h_{\bullet+1} : D^{+1}_\bullet \to B_\bullet$ given by [10, Theorem 2.2.6]. Denote by $\widetilde{h}^*_{k+1} : \widetilde{D}_{k+1} \to \widetilde{B}_k$ the induced morphism between the corresponding syzygies where $\widetilde{h}^*_1 = h^*$. Then there is a projective resolution $\underline{D}'_\bullet$ of D such that for any $k \in \mathbb{N}_0$, $\widetilde{D}_k = \underline{\widetilde{D}}_k$ and there is a commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & \widetilde{D}_{k+1} \arrow[r, "\iota_{k+1}"] \arrow[d, "\widetilde{h}^*_{k+1}"] & \underline{D}_k \arrow[r, "\pi_k"] \arrow[d, "h_k"] & \widetilde{D}_k \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & \widetilde{B}_k \arrow[r, "\widetilde{f}_k"] & \widetilde{C}_k \arrow[r, "\widetilde{g}_k"] & \widetilde{D}_k \arrow[r] & 0 \end{tikzcd} \tag{6.4.1}$$

where the morphisms ι_{k+1}, π_k stem from $\underline{D}_\bullet$. In particular, any $\widetilde{h}^*_{k+1}$ is a lift of $\mathrm{id}_{\widetilde{D}_k}$ as in Diagram 3.0.2.

Let us explain the context of this result. In Definition 6.2.2, we have constructed a connecting homomorphism $\widehat{\Delta}^n : \widehat{\mathcal{E}xt}^n_{\mathcal{C}}(A, D) \to \widehat{\mathcal{E}xt}^{n+1}_{\mathcal{C}}(A, \widetilde{D}_1)$ first associated to the short exact sequence $0 \to \widetilde{D}_1 \to D_0 \to D \to 0$. Then we defined

$$\widehat{\mathcal{E}xt}^n_{\mathcal{C}}(A, D) \xrightarrow{\widehat{\Delta}^n} \widehat{\mathcal{E}xt}^{n+1}_{\mathcal{C}}(A, \widetilde{D}_1) \xrightarrow{\widehat{\mathcal{E}xt}^{n+1}_{\mathcal{C}}(A, h^*)} \widehat{\mathcal{E}xt}^{n+1}_{\mathcal{C}}(A, B)$$

to be the connecting homomorphism associated to $0 \to B \to C \to D \to 0$. Later, we shall compare this to

$$\mathcal{E}xt_{\mathcal{C}}^{n+2k}(A, \widetilde{D}_{2k}) \xrightarrow{\Delta^n} \mathcal{E}xt_{\mathcal{C}}^{n+1+2k}(A, \widetilde{D}_{2k+1}) \xrightarrow{\mathcal{E}xt_{\mathcal{C}}^{n+1+2k}(A, \widetilde{h}^*_{2k+1})} \mathcal{E}xt_{\mathcal{C}}^{n+1+2k}(A, \widetilde{B}_{2k})$$

which we would like to be the connecting homomorphism associated to the short exact sequence $0 \to \widetilde{B}_{2k} \to \widetilde{C}_{2k} \to \widetilde{D}_{2k} \to 0$. This is where Diagram 6.4.1 comes into play.

Proof (Proposition 6.4.1) Proceed by induction over $k \in \mathbb{N}_0$. If $k = 0$, then we take $\underline{D}_0 := D_0$ where Diagram 3.0.2 yields the desired diagram for this case. Assume that $\underline{D}_0, \ldots, \underline{D}_k$ have been defined and that there is a diagram of the required form for a number $k \geq 0$. To construct $\underline{D}_{k+1}$ and establish Diagram 6.4.1 with $h_{k+1} : \underline{D}_{k+1} \to \widetilde{C}_k$ and $\widetilde{h}^*_{k+2} : \widetilde{D}_{k+2} \to \widetilde{B}_{k+1}$, we consider two instance of Diagram 4.2.2 from the Horseshoe Lemma. For this, we follow the construction found in [10, p. 37–38] that yields the proof of this so-called lemma. We explain how to obtain the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \widetilde{D}_{k+2} \arrow[r, "\iota''_{k+2}"] \arrow[d, "\iota'_{k+2}"] & \operatorname{Ker}(\varepsilon) \arrow[r, "\pi''_{k+1}"] \arrow[d] & \widetilde{D}_{k+1} \arrow[r] \arrow[d, "\iota_{k+1}"] & 0 \\
0 \arrow[r] & D_{k+1} \arrow[r, shift left, "i_1^1"] \arrow[d, "\pi'_{k+1}"] \arrow[dr, "\iota_{k+1}\circ\pi'_{k+1}"] & D_{k+1} \oplus \underline{D}_k \arrow[l, shift left, "p_1^1"] \arrow[r, shift left, "p_2^1"] \arrow[d, "\epsilon"] & \underline{D}_k \arrow[l, shift left, "i_2^1"] \arrow[r] \arrow[d, "\pi_k"] \arrow[dl, "\mathrm{id}"] & 0 \\
0 \arrow[r] & \widetilde{D}_{k+1} \arrow[r, "\iota_{k+1}"] \arrow[d] & \underline{D}_k \arrow[r, "\pi_k"] \arrow[d] & \widetilde{D}_k \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{6.4.2}$$

The morphisms π_k, ι_{k+1} stem from the projective resolution $(\underline{D}_m)_{m=1}^k$ and the morphisms π'_{k+1}, ι'_{k+2} from $D_\bullet$. In particular, the triangle in the bottom left corner and the one in the bottom right corner commute. The morphisms of the middle row are the ones of a direct product as they can be found in [8, p. 250]. Moreover, $\varepsilon = (\iota_{k+1} \circ \pi'_{k+1}) \circ p_1^1 + \mathrm{id} \circ p_2^1$ by [8, p. 250] and [10, p. 37]. Since ι'_{k+2} and ι_{k+1} can be taken as kernels of π'_{k+1} and π_k, the morphisms ι'_{k+2} and π''_{k+1} are induced by the morphisms i_1^1 and p_2^1. All rows and columns are exact. Because $\underline{D}_k$ is projective, the middle column is a split exact sequence by [9, Tag 010G]. This together with $D_{k+1} \oplus \underline{D}_k$ being projective implies that $\operatorname{Ker}(\varepsilon)$ is also projective. Similarly, we construct with the morphisms $\widetilde{f}_k$ and $\widetilde{g}_k$ from Diagram 6.4.1 the diagram

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \widetilde{B}_{k+1} \arrow[r, "\widetilde{f}_{k+1}"] \arrow[d] & \widetilde{C}_{k+1} \arrow[r, "\widetilde{g}_{k+1}"] \arrow[d] & \widetilde{D}_{k+1} \arrow[r] \arrow[d, "\iota_{k+1}"] & 0 \\
0 \arrow[r] & B_k \arrow[r, shift left, "i_1^2 = f_k"] \arrow[d, "\breve{\pi}_k"] \arrow[dr, "\widetilde{f}_k \circ \breve{\pi}_k"] & B_k \oplus \underline{D}_k \arrow[l, shift left, "p_1^2"] \arrow[r, shift left, "p_2^2 = g_k"] \arrow[d, "\eta"] & \underline{D}_k \arrow[l, shift left, "i_2^2"] \arrow[r] \arrow[d, "\pi_k"] \arrow[dl, "h_k"'] & 0 \\
0 \arrow[r] & \widetilde{B}_k \arrow[r, "\widetilde{f}_k"] \arrow[d] & C_k \arrow[r, "\widetilde{g}_k"] \arrow[d] & \widetilde{D}_k \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
\tag{6.4.3}
$$

Note that $\eta = (\widetilde{f}_k \circ \breve{\pi}_k) \circ p_1^2 + h_k \circ p_2^2$ and that the triangle at the bottom right corner commutes by Diagram 6.4.1. Now consider the diagram

$$
\begin{tikzcd}
 & D_{k+1} \arrow[rr, "i_1^1"] \arrow[dl, "h^*_{k+1}"'] \arrow[dd, "\pi'_{k+1}"] \arrow[ddrr, bend right, "\iota_{k+1} \circ \pi'_{k+1}"] & & D_{k+1} \oplus \underline{D}_k \arrow[rr, "p_2^1"] \arrow[dl, "h^*_{k+1} \oplus \mathrm{id}"'] \arrow[dd, "\epsilon"] & & \underline{D}_k \arrow[dl, "\mathrm{id}"] \arrow[dd, "\pi_k"] \arrow[ddll, bend left, "\mathrm{id}"] \\
B_k \arrow[rr, "i_1^2"] \arrow[dd, "\breve{\pi}_k"] \arrow[ddrr, bend right, "\widetilde{f}_k \circ \breve{\pi}_k"] & & B_k \oplus \underline{D}_k \arrow[rr, "p_2^2"] \arrow[dd, "\eta"] & & \underline{D}_k \arrow[dd, "\pi_k"] \arrow[ddll, bend left, "h_k"] & \\
 & \widetilde{D}_{k+1} \arrow[rr, "\iota_{k+1}"] \arrow[dl, "\widetilde{h}^*_{k+1}"'] & & \underline{D}_k \arrow[rr, "\pi_k"] \arrow[dl, "h_k"'] & & \widetilde{D}_k \arrow[dl, "\mathrm{id}"] \\
\widetilde{B}_k \arrow[rr, "\widetilde{f}_k"] & & C_k \arrow[rr, "\widetilde{g}_k"] & & \widetilde{D}_k &
\end{tikzcd}
\tag{6.4.4}
$$

The bottom side stems from Diagram 6.4.1, the back side from the lower half of Diagram 6.4.2 and the front side from the lower half of Diagram 6.4.3. The left hand side arises from the chain map $h^*_\bullet : D^{+1}_\bullet \to B_\bullet$ while the right hand side commutes trivially. The morphism $h^*_{k+1} \oplus \mathrm{id} : D_{k+1} \oplus \underline{D}_k \to B_k \oplus \underline{D}_k$ is constructed as in [8, p. 251] and renders the top side commutative. In particular, all squares on the outside of Diagram 6.4.4 commute. The slanted squares in the interior give rise to the commuting diagram

$$\begin{tikzcd} D_{k+1} \arrow[r, "\iota_{k+1}\circ\pi'_{k+1}"] \arrow[d, "h^*_{k+1}"] & \underline{D}_k \arrow[d, "h_k"] & \underline{D}_k \arrow[l, "\mathrm{id}"'] \arrow[d, "\mathrm{id}"] \\ B_k \arrow[r, "\widetilde{f}_k\circ\breve{\pi}_k"] & C_k & \underline{D}_k \arrow[l, "h_k"'] \end{tikzcd}$$

We can use it together with Diagram 6.4.2, Diagram 6.4.3 and the content of [8, p. 251] to prove that that the middle square in the interior of Diagram 6.4.4 also commutes. Namely,

$$\begin{aligned} \eta \circ (h^*_{k+1} \oplus \mathrm{id}) &= \big((\widetilde{f}_k \circ \breve{\pi}_k) \circ p_1^2 + h_k \circ p_2^2\big) \circ (h^*_{k+1} \oplus \mathrm{id}) \\ &= (\widetilde{f}_k \circ \breve{\pi}_k) \circ p_1^2 \circ (h^*_{k+1} \oplus \mathrm{id}) + h_k \circ p_2^2 \circ (h^*_{k+1} \oplus \mathrm{id}) \\ &= (\widetilde{f}_k \circ \breve{\pi}_k) \circ h^*_{k+1} \circ p_1^1 + h_k \circ \mathrm{id} \circ p_2^1 \\ &= h_k \circ (\iota_{k+1} \circ \pi'_{k+1}) \circ p_1^1 + h_k \circ \mathrm{id} \circ p_2^1 \\ &= h_k \circ \big((\iota_{k+1} \circ \pi'_{k+1}) \circ p_1^1 + \mathrm{id} \circ p_2^1\big) \\ &= h_k \circ \varepsilon\,. \end{aligned}$$

Therefore, all squares in Diagram 6.4.4 commute. Its top side connected by epimorphisms to its bottom side according to Diagram 6.4.2 and 6.4.3. Still using the latter two diagrams, we can take kernels to obtain the commuting diagram

$$\begin{tikzcd} 0 \arrow[r] & \widetilde{D}_{k+2} \arrow[r, "\iota''_{k+2}"] \arrow[d, "\widetilde{h}^*_{k+2}"] & \mathrm{Ker}(\varepsilon) \arrow[r, "\pi''_{k+1}"] \arrow[d, "H"] & \widetilde{D}_{k+1} \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & \widetilde{B}_{k+1} \arrow[r, "\widetilde{f}_{k+1}"] & \widetilde{C}_{k+1} \arrow[r, "\widetilde{g}_{k+1}"] & \widetilde{D}_{k+1} \arrow[r] & 0 \end{tikzcd}$$

where the rows are short exact sequences. Having seen that $\mathrm{Ker}(\varepsilon)$ is projective, we complete the inductive step by setting $\underline{D}_{k+1} := \mathrm{Ker}(\varepsilon)$, $\iota_{k+2} := \iota''_{k+2}$, $\pi_{k+1} := \pi''_{k+1}$ and $h_{k+1} := H$. □

Theorem 6.4.2 *For any short exact sequence $0 \to B \to C \to D \to 0$ in $\mathcal{C}$ the square*

$$\begin{tikzcd} \mathcal{E}xt^n_{Res,\mathcal{C}}(A,D) \arrow[r, "\vartheta^n"] \arrow[d, "\widetilde{\Delta}^n"] & \widehat{\mathcal{E}xt}^n_{\mathcal{C}}(A,D) \arrow[d, "\widehat{\Delta}^n"] \\ \mathcal{E}xt^{n+1}_{Res,\mathcal{C}}(A,B) \arrow[r, "\vartheta^{n+1}"] & \widehat{\mathcal{E}xt}^{n+1}_{\mathcal{C}}(A,B) \end{tikzcd}$$

commutes for every $n \in \mathbb{Z}$. In particular, $(\widehat{\mathcal{E}xt}^{\bullet}_{\mathcal{C}}(A,-), \widehat{\Delta}^{\bullet})$ forms a cohomological functor and $\vartheta^{\bullet} : \mathcal{E}xt^{\bullet}_{Res,\mathcal{C}}(A,-) \to \widehat{\mathcal{E}xt}^{\bullet}_{\mathcal{C}}(A,-)$ an isomorphism of cohomological functors.

Proof We write in the following $N := n+2k$, $N' := n+1+2k$ and $K := 2k$. Analogously to the proof of Proposition 6.3.4, it suffices by Definition 6.3.1 to show that the diagram

$$\begin{tikzcd}
 & \mathcal{E}xt^{N'}_{\mathcal{C}}(A, \widetilde{B}_K) \arrow[drr, "\vartheta^K_{n+1}(B)"] \arrow[dd, "\Delta^{N'+1} \circ \Delta^{N'}"] & & \\
\mathcal{E}xt^{N}_{\mathcal{C}}(A, \widetilde{D}_K) \arrow[ur, "\Delta^K"] \arrow[drr, "\vartheta^K_n(D)"] \arrow[dd, "\Delta^{N+1} \circ \Delta^{N}"] & & & \widehat{\mathcal{E}xt}^{n+1}_{\mathcal{C}}(A,B) \\
 & \mathcal{E}xt^{N'+2}_{\mathcal{C}}(A, \widetilde{B}_{2k+2}) \arrow[urr, "\vartheta^{K+2}_{n+1}(B)"] & \widehat{\mathcal{E}xt}^{n}_{\mathcal{C}}(A,D) \arrow[ur, "\widehat{\Delta}^n"] & \\
\mathcal{E}xt^{N+2}_{\mathcal{C}}(A, \widetilde{D}_{K+2} \arrow[ur, "\Delta^{N+2}"] \arrow[urr, "\vartheta^{K+2}_n(D)"] & & &
\end{tikzcd} \tag{6.4.5}$$

is commutative in order to conclude that the square in the statement of the theorem commutes. One can reformulate this prism in form of the diagram

$$
\begin{tikzcd}
\mathcal{E}xt^{N}_{\mathcal{C}}(A,\widetilde{D}_K) \arrow[r, "\Delta^N"] \arrow[d, "\Delta^N"] \arrow[rr, bend right, "\Delta^N"] \arrow[ddd, bend left, "\vartheta^K_n(D)"] & \mathcal{E}xt^{N+1}_{\mathcal{C}}(A,\widetilde{D}_{K+1}) \arrow[r, "\mathcal{E}xt^{N+1}_{\mathcal{C}}(A,\widetilde{h}^*_{K+1})"] \arrow[d, "\Delta^{N+1}"] \arrow[ddd, bend left, "\vartheta^K_{n+1}(\widetilde{D}_1)"] & \mathcal{E}xt^{N+1}_{\mathcal{C}}(A,\widetilde{B}_K) \arrow[d, "\Delta^{N+1}"] \arrow[ddd, bend left, "\vartheta^K_{n+1}(B)"] \\
\mathcal{E}xt^{N+1}_{\mathcal{C}}(A,\widetilde{D}_{K+1}) \arrow[r, "-\Delta^{N+1}"] \arrow[d, "\Delta^{N+1}"] \arrow[rr, bend right, "-\Delta^{N+1}"] & \mathcal{E}xt^{N+2}_{\mathcal{C}}(A,\widetilde{D}_{K+2}) \arrow[r, "\mathcal{E}xt^{N+2}_{\mathcal{C}}(A,\widetilde{h}^*_{K+2})"] \arrow[d, "\Delta^{N+2}"] & \mathcal{E}xt^{N+2}_{\mathcal{C}}(A,\widetilde{B}_{K+1}) \arrow[d, "\Delta^{N+2}"] \\
\mathcal{E}xt^{N+2}_{\mathcal{C}}(A,\widetilde{D}_{K+2}) \arrow[r, "\Delta^{N+2}"] \arrow[d, "\vartheta^{K+2}_n(D)"] \arrow[rr, bend right, "\Delta^{N+2}"] & \mathcal{E}xt^{N+3}_{\mathcal{C}}(A,\widetilde{D}_{K+3}) \arrow[r, "\mathcal{E}xt^{N+2}_{\mathcal{C}}(A,\widetilde{h}^*_{K+3})"] \arrow[d, "\vartheta^{K+2}_{n+1}(\widetilde{D}_1)"] & \mathcal{E}xt^{N+3}_{\mathcal{C}}(A,\widetilde{B}_{K+2}) \arrow[d, "\vartheta^{K+2}_{n+1}(B)"] \\
\widehat{\mathcal{E}xt}^{n}_{\mathcal{C}}(A,D) \arrow[r, "\widehat{\Delta}^n"] \arrow[rr, bend right, "\widehat{\Delta}^n"] & \widehat{\mathcal{E}xt}^{n+1}_{\mathcal{C}}(A,\widetilde{D}_1) \arrow[r, "\widehat{\mathcal{E}xt}^{n+1}_{\mathcal{C}}(A,h^*)"] & \widehat{\mathcal{E}xt}^{n+1}_{\mathcal{C}}(A,B)
\end{tikzcd}
\tag{6.4.6}
$$

The left hand side of Diagram 6.4.5 corresponds to the top four squares of Diagram 6.4.6 while the bottom right hand side of the former corresponds to the bottom two squares of the latter. In Diagram 6.4.6, the top and bottom row together with the morphisms connecting them directly retrieve the top right side of Diagram 6.4.5.

Let us explain how to construct Diagram 6.4.6 by making use of Proposition 6.4.1 and its proof. We take the morphism $h^* : \widetilde{D}_1 \to B$ as in that proposition, meaning that any term $\widetilde{h}^*_{2k+1}$ originates from a chain map $h^*_{\bullet+1} : D^{+1}_{\bullet} \to B_{\bullet}$. By Definition 6.2.2, the connecting homomorphism $\widehat{\Delta}^n : \widehat{\mathcal{E}xt}^{n}_{\mathcal{C}}(A,D) \to \widehat{\mathcal{E}xt}^{n+1}_{\mathcal{C}}(A,B)$ associated to the short exact sequence $0 \to B \to C \to D \to 0$ is rendered in the very bottom row. The top left square arises from Diagram 4.2.3 and 6.4.2. More specifically, its top connecting homomorphism is associated to the short exact sequence at the bottom of Diagram 6.4.2, its right hand connecting homomorphism to the short exact sequence at the left hand side of the latter diagram, its bottom one to the top short exact sequence and its left hand one to the right hand short exact sequence. Because the connecting homomorphism at the right hand side of the top right square is associated to the short exact sequences at the left hand side of Diagram 6.4.3, the top two squares commute. By

Proposition 6.4.1, the homomorphism

$$\mathcal{E}xt_{\mathcal{C}}^{n+2k}(A,\widetilde{D}_{2k}) \xrightarrow{\Delta^{n+2k}} \mathcal{E}xt_{\mathcal{C}}^{n+2k+1}(A,\widetilde{D}_{2k+1}) \xrightarrow{\mathcal{E}xt_{\mathcal{C}}^{n+2k+1}(A,\widetilde{h}^*_{2k+1})} \mathcal{E}xt_{\mathcal{C}}^{n+2k+1}(A,\widetilde{B}_{2k})$$

at the very top of the diagram is the connecting homomorphism Δ^{n+2k} associated to the short exact sequence $0 \to \widetilde{B}_k \to \widetilde{C}_k \to \widetilde{D}_k \to 0$. This does not only imply that the entire top third of the diagram commutes, but also that the middle third does so too. Due to Proposition 6.3.4 the entire right hand side of the diagram commutes. Regarding the left hand side, since we are in the same situation as in the proof of the latter proposition, we are left to show that the diagram

$$\begin{tikzcd}
\mathcal{E}xt_{\mathcal{C}}^{n+2k}(A,\widetilde{D}_{2k}) \arrow[r, "\Delta^{n+2k}"] \arrow[d, "\vartheta_n^{2k}(D)"] & \mathcal{E}xt_{\mathcal{C}}^{n+1+2k}(A,\widetilde{D}_{2k+1}) \arrow[d, "\vartheta_{n+1}^{2k}(\widetilde{D}_1)"] \\
\widehat{\mathcal{E}xt}_{\mathcal{C}}^{n}(A,D) \arrow[r, "\widehat{\Delta}^n"] & \widehat{\mathcal{E}xt}_{\mathcal{C}}^{n+1}(A,\widetilde{D}_1)
\end{tikzcd}$$

commutes. If $\varphi_{\bullet+n+2k} + \mathrm{Hom}_{\mathrm{Ch}(\mathcal{C})}(A[n+2k], D_{\bullet}^{+2k})$ is a an element in $\mathcal{E}xt_{\mathcal{C}}^{n+2k}(A,\widetilde{D}_{2k})$. Then

$$\begin{cases} (-1)^{m-1+n}\varphi_{m+n} : A_{m+n} \to D_m & \text{if } m \geq 2k+1 \\ 0 & \text{if } m \leq 2k \end{cases}$$

is an almost chain map representative of the image under the homomorphism $\widehat{\Delta}^n \circ \vartheta_n^{2k}(D)$ while

$$\begin{cases} (-1)^{m-2k-1+n}\varphi_{m+n} : A_{m+n} \to D_m & \text{if } m \geq 2k+1 \\ 0 & \text{if } m \leq 2k \end{cases}$$

is a representative of the image under $\vartheta_{n+1}^{2k} \circ \Delta^{n+2k}$. Thus, Diagram 6.4.5 and 6.4.6 commute as well as the square in the statement of the theorem.

By Definition 6.3.1, $(\mathcal{E}xt^{\bullet}_{Res,\mathcal{C}}(A,-),\widetilde{\Delta}^{\bullet})$ forms a cohomological functor and by Lemma 6.3.3 and Proposition 6.3.4, ϑ^n is a natural isomorphism for any $n \in \mathbb{Z}$. As for Theorem 5.3.2 and Lemma 6.2.3, we can reiterate the proof of Theorem 4.3.2 to demonstrate that $(\widehat{\mathcal{E}xt}^{\bullet}_{\mathcal{C}}(A,-),\widehat{\Delta}^{\bullet})$ is a cohomological functor and that $\vartheta^{\bullet} : \mathcal{E}xt^{\bullet}_{Res,\mathcal{C}}(A,-) \to \widehat{\mathcal{E}xt}^{\bullet}_{\mathcal{C}}(A,-)$ is an isomorphism of cohomological functors. $\square$

Remark 6.4.3 *Given any short exact sequence* $0 \to B \to C \to D \to 0$ *in* $\mathcal{C}$*, we conclude by means of Theorem 6.4.2 that every associated connecting homomorphism* $\widehat{\Delta}^n : \widehat{\mathcal{E}xt}^n_{\mathcal{C}}(A, D) \to \widehat{\mathcal{E}xt}^{n+1}_{\mathcal{C}}(A, B)$ *is well defined in the sense that it does not depend on a choice of a morphism* $h^* : \widetilde{D}_1 \to B$ *as it occurs in Definition 6.2.2. We have not been able to prove this directly.*

Corollary 6.4.4 *We can extend the natural isomorphisms* σ^n *from Corollary 6.3.5 to isomorphism of cohomological functors*

$$\sigma^\bullet = \vartheta^\bullet \circ \zeta^\bullet : \left(\widehat{\mathrm{Ext}}^n_{\mathcal{C}}(A, -), \widehat{\delta}^\bullet\right) \to \left(\mathcal{E}xt^n_{Res,\mathcal{C}}(A, -), \widetilde{\Delta}^\bullet\right) \to \left(\widehat{\mathcal{E}xt}^n_{\mathcal{C}}(A, -), \widehat{\Delta}^\bullet\right).$$

In [28, p. 21–22], S. Guo and L. Liang establish for every $n \in \mathbb{Z}$ an isomorphism $\widehat{\mathcal{E}xt}^n_{\mathcal{C}}(A, B) \to BC^n_{\mathcal{C}}(A, B)$ from the cohomology groups of the hypercohomology construction to the ones of the naive construction. As it is used in the construction of Yoneda and external products in Section 7.4, we present it below and prove that it is an isomorphism of cohomological functors.

Definition 6.4.5 *([28, p. 21–22]) For any* $A, B \in \mathrm{obj}(\mathcal{C})$ *and* $n \in \mathbb{Z}$ *let the map*

$$\rho^n : \widehat{\mathcal{E}xt}^n_{\mathcal{C}}(A, B) \to BC^n_{\mathcal{C}}(A, B)$$

be given as follows. If $x = \varphi_{\bullet+n} + \widehat{\mathrm{Null}}_{\mathrm{Ch}(\mathcal{C})}(A[n]_\bullet, B_\bullet)$ *is an element in* $\widehat{\mathcal{E}xt}^n_{\mathcal{C}}(A, B)$*, then there exists* $K \in \mathbb{N}_0$ *such that* $(\varphi_{m+n} : A_{m+n} \to B_m)_{m \geq 2K}$ *forms a chain map. Denote by* $\widetilde{\varphi}_{2K+n} : \widetilde{A}_{2K+n} \to \widetilde{B}_{2K}$ *the morphism induced between the corresponding syzygies. We set* $\rho^n(x)$ *to be the element in* $BC^n_{\mathcal{C}}(A, B)$ *to which the element* $\widetilde{\varphi}_{2K} + \mathcal{P}_{\mathcal{C}}(\widetilde{A}_{2K+n}, \widetilde{B}_{2K})$ *in* $[\widetilde{A}_{2K+n}, B_{2K}]_{\mathcal{C}}$ *is mapped by the corresponding homomorphism going to the direct limit.*

Lemma 6.4.6 *The maps* ρ^n *form an isomorphism of cohomological functors such that*

$$\rho^\bullet = \beta^\bullet \circ (\sigma^\bullet)^{-1} : (\widehat{\mathcal{E}xt}^\bullet_{\mathcal{C}}(A, -), \widehat{\Delta}^\bullet) \to (\mathrm{Ext}^n_{Res,\mathcal{C}}(A, -), \widetilde{\delta}^\bullet) \to (BC^\bullet_{\mathcal{C}}(A, -), \tau^\bullet)$$

where $\beta^\bullet$ *is taken from Theorem 5.3.2.*

Proof Since $\beta^\bullet$ and $\sigma^\bullet$ are already isomorphisms of cohomological functors, it suffices to prove that $\rho^n = \beta^n \circ (\sigma^n)^{-1}$ for every $n \in \mathbb{Z}$. Assume that $\varphi_{\bullet+n} + \widehat{\mathrm{Null}}_{\mathrm{Ch}(\mathcal{C})}(A[n]_\bullet, B_\bullet)$ be an element in $\widehat{\mathcal{E}xt}^n_{\mathcal{C}}(A, B)$. If $K \in \mathbb{N}_0$ is such that $(\varphi_{m+n})_{m\geq 2K}$ is chain map, then for any $k \geq K$ we define the chain map $\varphi^{\{2k\}}_{\bullet+n+2k} : A[n+2k]_\bullet \to B^{+2k}_\bullet$ by setting

$$\varphi^{\{2k\}}_{m+n+2k} := \begin{cases} \varphi_{m+n+2k} : A_{m+n+2k} \to B_{m+2k} & \text{if } m \geq 0 \\ 0 & \text{if } m < 0 \end{cases}$$

By Corollary 6.3.5, $(\sigma^n)^{-1} = (\zeta^n)^{-1} \circ (\vartheta^n)^{-1}$. As $\mathrm{Ext}^n_{Res,\mathcal{C}}(A,B)$ is formed as a direct limit of abelian groups, we introduce the following notation. Let $(A_i)_{i\in\mathbb{N}}$ be a direct system of abelian groups with homomorphisms given by $a_{i,j} : A_i \to A_j$ for $i \leq j$. We write $(x_k)_{k\geq K}$ if $x_k \in A_k$ and $a_{i,j}(x_i) = x_j$ for any $j \geq i \geq K$. This results in an element in $\varinjlim_{i\in\mathbb{N}} A_i$ that we identify with $(x_k)_{k\geq K}$. Then we see that

$$(\vartheta^n)^{-1}\big(\varphi_{\bullet+n}+\widehat{\mathrm{Null}}_{\mathrm{Ch}(\mathcal{C})}(A[n]_\bullet, B_\bullet)\big) = \big(\varphi^{\{2k\}}_{\bullet+n+2k}+\mathrm{Null}_{\mathrm{Ch}(\mathcal{C})}(A[n+2k]_\bullet, B^{+2k}_\bullet)\big)_{k\geq K}.$$

Denote by $a_m : A_m \to A_{m-1}$ the boundary maps in the projective resolution $A_\bullet$ and by $\pi_m : B_m \to \widetilde{B}_m$ the morphisms to the m^{th} syzygy. According to the construction of ζ_m in the proof of Lemma 6.2.3 and of σ^n in Definition 6.3.1, we infer that

$$(\sigma^n)^{-1}\big(\varphi_{\bullet+n}+\widehat{\mathrm{Null}}_{\mathrm{Ch}(\mathcal{C})}(A[n]_\bullet, B_\bullet)\big) = \Big(\pi_{n+2k}\circ\varphi_{n+2k}+\mathrm{Im}\big(\mathrm{Hom}_{\mathcal{C}}(a_{n+2k}, \widetilde{B}_{2k})\big)\Big)_{k\geq K}.$$

We deduce from the construction of β^n in Lemma 5.2.2 that

$$\beta^n\circ(\sigma^n)^{-1}\big(\varphi_{\bullet+n}+\widehat{\mathrm{Null}}_{\mathrm{Ch}(\mathcal{C})}(A[n]_\bullet, B_\bullet)\big) = \big(\alpha_{n+2k}(\widetilde{B}_{2k})(\varphi_{n+2k})+[\widetilde{A}_{2k+n}, \widetilde{B}_{2k}]_{\mathcal{C}}\big)_{k\geq K}$$

where $\alpha_{n+2k}(\widetilde{B}_{2k})$ denotes the homomorphism from Diagram 5.2.1. Because

$$\beta^n \circ (\sigma^n)^{-1}\big(\varphi_{\bullet+n} + \widehat{\mathrm{Null}}_{\mathrm{Ch}(\mathcal{C})}(A[n]_\bullet, B_\bullet)\big) = \big(\widetilde{\varphi}_{n+2k} + [\widetilde{A}_{2k+n}, \widetilde{B}_{2k}]_{\mathcal{C}}\big)_{k\geq K}$$

by Diagram 6.2.1, we conclude that $\rho^n = \beta^n \circ (\sigma^n)^{-1}$. $\square$

We close our account on constructions of Mislin completions by posing a question. In Notation 6.1.1 we have remarked that any completed Ext-group $\widehat{\mathcal{E}xt}^n_{\mathcal{C}}(A,B)$ can be defined as the n^{th} cohomology group of the Vogel complex $\mathrm{Vog}^n_{\mathcal{C}}(A_\bullet, B_\bullet)_\bullet$ whence one obtains

Lemma 6.4.7 *([28, Proposition 4.8]) If $0 \to B \to C \to D \to 0$ is a short exact sequence in $\mathcal{C}$, one can lift it by the Horseshoe Lemma [10, p. 37] to a short exact sequence $0 \to B_\bullet \to C_\bullet \to D_\bullet \to 0$ of chain complexes.*

Then, by the Snake Lemma [10, p. 11–12], there are associated connecting homomorphisms

$$\overline{\delta}^n : \widehat{\mathcal{E}xt}^n_{\mathcal{C}}(A, D) \to \widehat{\mathcal{E}xt}^{n+1}_{\mathcal{C}}(A, B) \text{ and } \overline{\delta}^n : \widehat{\mathcal{E}xt}^n_{\mathcal{C}}(B, A) \to \widehat{\mathcal{E}xt}^{n+1}_{\mathcal{C}}(D, A)\,.$$

In particular, $\big(\widehat{E}xt^\bullet_{\mathcal{C}}(A, -), \overline{\delta}^\bullet\big)$ *and* $\big(\widehat{E}xt^\bullet_{\mathcal{C}}(-, A), \overline{\delta}^\bullet\big)$ *form cohomological functors.*

Question 6.4.8 *Does the connecting homomorphism* $\widehat{\Delta}^\bullet$ *constructed in Definition 6.2.2 agree with* $\overline{\delta}^\bullet$*? More specifically, is* $\big(\widehat{\mathcal{E}xt}^\bullet_{\mathcal{C}}(A, -), \overline{\delta}^\bullet\big)$ *a Mislin completion of the unenriched Ext-functors* $\big(\mathrm{Ext}^\bullet_{\mathcal{C}}(A, -), \delta^\bullet\big)$*?*

Chapter 7

Properties of Mislin completions

We establish properties of Mislin completions and in particular, of completed Ext-functors in this chapter. In Section 7.1, we prove the most important feature of Mislin completions, namely that unenriched completed Ext-functors detect when an object in the domain category has finite projective dimension. Thereafter, in Section 7.2, we demonstrate a (partial) version of dimension shifting and prove an Eckmann-Shapiro type lemma. We also provide examples for which these results hold. In Section 7.4, we construct both Yoneda products and external products for completed unenriched Ext-functors under certain conditions and provide examples which meet these conditions. This requires requires a few auxiliary results that are demonstrated in Section 7.3. Section 7.5 contains a result on canonical morphisms from unerniched Ext-functors to their Mislin completions and establishes that one can construct completed unenriched Ext-functors using so-called complete resolutions. Lastly, we introduce Tate cohomology in condensed mathematics in Section 7.6.

7.1 Detecting finite projective dimension

Recall from Chapter 1 that an object in a category is said to have finite projective dimension if it admit a projective resolution of finite length. Remember that group cohomology as well as complete cohomology can be defined as a specific (completed) Ext-functor. Then we re-establish Lemma 4.2.3 from [3] in greater generality.

Lemma 7.1.1 *1. Assume that $T^\bullet : \mathcal{C} \to \mathcal{D}$ is a cohomological functor where $\mathcal{C}$ has enough projectives and in $\mathcal{D}$ all countable direct limits exist and are exact. If $M \in \mathrm{obj}(\mathcal{C})$ has finite projective dimension, then $\widehat{T}^n(M) = 0$ for every $n \in \mathbb{Z}$. In particular, if every object in $\mathcal{C}$*

has finite projective dimension such as in a category of modules over a ring of finite global dimension, then $\widehat{T}^\bullet = 0$ for any cohomoological functor $T^\bullet$.

2. *If one considers* ***enriched*** *Ext-functors* $\mathrm{Ext}^n_{\mathcal{C}}(A, -) : \mathcal{C} \to \mathcal{D}$ *with* $A \in \mathrm{obj}(\mathcal{C})$ *of finite projective dimension, then* $\widehat{\mathrm{Ext}}^n_{\mathcal{C}}(A, -) = 0$ *for every* $n \in \mathbb{Z}$*. In particular, complete cohomology* $\widehat{H}^\bullet_R(G, M)$ *vanishes if* G *has finite cohomological dimension or* M *has finite projective dimension.*

Proof If M has finite projective dimension, then there is $m \in \mathbb{N}_0$ such that $\widetilde{M}_k = 0$ for any $k \geq m$. In particular, $T^{n+k}(\widetilde{B}_k) = 0$ for any $k \geq m$ and $\widehat{T}^n(B) = 0$ according to the resolution construction. On the other hand, if A has finite projective dimension, then there $m' \in \mathbb{N}_0$ such that $\mathrm{Ext}^{n+k}_{\mathcal{C}}(A, -) = 0$ for any $n + k \geq m'$. We conclude that $\widehat{\mathrm{Ext}}^n_{\mathcal{C}}(A, -) = 0$ as before. □

This allows us to re-establish the following version of a theorem that appears in the literature as [29, Proposition IX.1.3], [28, Theorem 4.11] and [30, Theorem 3.10].

Theorem 7.1.2 *If* $\widehat{\mathrm{Ext}}^\bullet_{\mathcal{C}}(A, -) : \mathcal{C} \to \mathrm{Ab}$ *denote completed unenriched Ext-functors for* $A \in \mathrm{obj}(\mathcal{C})$*, then the following are equivalent.*

1. *The object A has finite projective dimension.*
2. $\widehat{\mathrm{Ext}}^n_{\mathcal{C}}(A, -) = \widehat{\mathrm{Ext}}^n_{\mathcal{C}}(-, A) = 0$ *for any* $n \in \mathbb{Z}$.
3. $\widehat{\mathrm{Ext}}^0_{\mathcal{C}}(A, A) = 0$.

In particular, the zeroeth complete cohomology group detects whether a group has finite cohomological dimension.

Given the torsion-theoretic framework in which A. Beligiannis and I. Reiten work, this theorem follows from their definitions in [29]. On the other hand, S. Guo and L. Liang in [28] as well as J. Hu et al. in [30] prove this theorem using the hypercohomology construction. For completeness, we generalise a proof by the naïve construction found in [3, p. 205].

Proof According to Lemma 7.1.1, it suffices to assume that $\widehat{\mathrm{Ext}}^0_{\mathcal{C}}(A, A) = 0$. By the construction of direct limits of abelian groups from Proposition 5.2.1, there is $k \in \mathbb{N}_0$ such that $\mathrm{id}_{\widetilde{A}_k} + \mathcal{P}_{\mathcal{C}}(\widetilde{A}_k, \widetilde{A}_k) = 0$ in $[\widetilde{A}_k, \widetilde{A}_k]_{\mathcal{C}}$. Because unenriched Hom-functors are used, we conclude that $\mathrm{id}_{\widetilde{A}_k}$ factors through a projective. In particular, $\widetilde{A}_k$ is projective and A has finite projective dimension. □

7.2 Dimension shifting and an Eckmann-Shapiro Lemma

One of the properties for Tate cohomology of finite groups that we have seen at the start of Chapter 1 is dimension shifting. As we only recover a partial version, we need an Eckmann-Shapiro type result in order to provide examples where dimension shifting holds in full generality. For this, we formally define *induction*, *coinduction* and *restriction* as can be found in [2, p. 62–63 and p. 67]. Denote by $\mathcal{C}_{R,G}$ the subcategory of $\mathcal{C}$ of module objects over the group ring (object) $R[G]$. Assume that the category $\mathcal{C}$ contains a tensor product $-\otimes - : \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ that is right-adjoint to an internal Hom-functor of the form $\underline{\mathrm{Hom}}_{\mathcal{C}}(-,-):\mathcal{C}^{\mathrm{op}}\times\mathcal{C}\to\mathcal{C}$. Let H be a subgroup object of G. If we regard the group ring object $R[G]$ as an $(R[G],R[H])$-bimodule object, then we define restriction and coinduction as

$$\mathrm{Res}_H^G(-) := -\otimes_{R[G]} R[G] :\ \mathcal{C}_{R,G}\to\mathcal{C}_{R,H} \text{ and}$$
$$\mathrm{Coind}_H^G(-) := \underline{\mathrm{Hom}}_{\mathcal{C}_{R,H}}(R[G],-) :\ \mathcal{C}_{R,H}\to\mathcal{C}_{R,G}\,.$$

These form adjoint functors in the sense that

$$\mathrm{Hom}_{\mathcal{C}_{R,H}}(\mathrm{Res}_H^G(A),B)\cong \mathrm{Hom}_{\mathcal{C}_{R,G}}(A,\mathrm{Coind}_H^G(B))$$

for every $A\in\mathcal{C}_{R,G}$ and $B\in\mathcal{C}_{R,H}$ where $\mathrm{Hom}_{\mathcal{C}}(-,-):\mathcal{C}^{\mathrm{op}}\times\mathcal{C}\to\mathrm{Ab}$ denotes the unenriched Hom-functor. Instead, if we regard $R[G]$ as an $(R[H],R[G])$-bimodule, then we define induction and restriction as $\mathrm{Ind}_H^G(-):=-\otimes_{R[H]} R[G]:\mathcal{C}_{R,H}\to\mathcal{C}_{R,G}$ and $\mathrm{Res}_H^G(-):=\underline{\mathrm{Hom}}_{\mathcal{C}_{R,G}}(R[G],-):\mathcal{C}_{R,G}\to\mathcal{C}_{R,H}$. This redefinition of restriction allows us to conclude that $(\mathrm{Ind}_H^G(-),\mathrm{Res}_H^G(-))$ is a pair of adjoint functors. We say that a functor $F:\mathcal{C}\to\mathcal{D}$ preserves projectives if for every projective $P\in\mathrm{obj}(\mathcal{C})$ also $F(P)\in\mathrm{obj}(\mathcal{D})$ is projective.

Lemma 7.2.1 (Eckmann-Shapiro) *1. If both* $\mathrm{Res}_H^G(-)$ *and* $\mathrm{Coind}_H^G(-)$ *are exact and preserve projectives, then*

$$\widehat{\mathrm{Ext}}^n_{\mathcal{C}_{R,H}}(\mathrm{Res}_H^G(A),B)\cong \widehat{\mathrm{Ext}}^n_{\mathcal{C}_{R,G}}(A,\mathrm{Coind}_H^G(B))$$

as unenriched completed Ext-functors for every $n\in\mathbb{Z}$, $A\in\mathrm{obj}(\mathcal{C}_{R,G})$ *and* $B\in\mathrm{obj}(\mathcal{C}_{R,H})$. *In particular,* $\widehat{H}^n_R(H,B)\cong\widehat{H}^n_R(G,\mathrm{Coind}_H^G(B))$ *if* $A=R$.

2. *If* $\mathrm{Ind}_H^G(-)$ *and* $\mathrm{Res}_H^G(-)$ *are exact and preserve projectives, then*

$$\widehat{\mathrm{Ext}}^n_{\mathcal{C}_{R,G}}(\mathrm{Ind}_H^G(A), B) \cong \widehat{\mathrm{Ext}}^n_{\mathcal{C}_{R,G}}(A, \mathrm{Res}_H^G(B))$$

as unenriched completed Ext-functors for every $n \in \mathbb{Z}$, $A \in \mathrm{obj}(\mathcal{C}_{R,H})$ *and* $B \in \mathrm{obj}(\mathcal{C}_{R,G})$.

Proof As the other one is analogous, we only prove the assertion for the pair of adjoint functors $(\mathrm{Res}_H^G(-), \mathrm{Coind}_H^G(-))$. For this we use the resolution construction. If $f : A \to C$ is a morphism in $\mathcal{C}_{R,H}$, then it follows from the naturality of the adjunction and the Five Lemma [37, Proposition I.1.1] that

$$\mathrm{Ker}\big(\mathrm{Hom}_{\mathcal{C}_{R,H}}(\mathrm{Res}_H^G(f), B)\big) \cong \mathrm{Ker}\big(\mathrm{Hom}_{\mathcal{C}_{R,G}}(f, \mathrm{Coind}_H^G(B)\big) \text{ and}$$
$$\mathrm{Coker}\big(\mathrm{Hom}_{\mathcal{C}_{R,H}}(\mathrm{Res}_H^G(f), B)\big) \cong \mathrm{Coker}\big(\mathrm{Hom}_{\mathcal{C}_{R,G}}(f, \mathrm{Coind}_H^G(B)\big)\,.$$

To ease notation, write $\mathrm{Kr}(\mathrm{Rs}(f), B)$ for $\mathrm{Ker}\big(\mathrm{Hom}_{\mathcal{C}_{R,H}}(\mathrm{Res}_H^G(f), B)\big)$ and $\mathrm{Kr}(f, \mathrm{Cd}(B))$ for $\mathrm{Ker}\big(\mathrm{Hom}_{\mathcal{C}_{R,H}}(f, \mathrm{Coind}_H^G(B)\big)$. We denote the corresponding cokernels by $\mathrm{Cr}(\mathrm{Rs}(f), B)$, $\mathrm{Cr}(f, \mathrm{Cd}(B))$. If $(A_l, a_l)_{l\in\mathbb{N}_0}$ is a projective resolution of A, then let $m, k \in \mathbb{N}_0$ with $m = n + k$ and consider the short exact sequence $0 \to \widetilde{B}_{k+1} \to B_k \to \widetilde{B}_k \to 0$. This gives rise to the diagram on the next page.

The homomorphisms from the front to the back are isomorphisms arising from the above adjunction. By naturality of this adjunction, the front and back squares as well as the vertical squares running from front to back commute. Using the universal property of kernels and cokernels together with the naturality of the adjunction, we can deduce that also the horizontal squares running from front to back commute. Since we assume that restriction and coinduction are exact and preserve projectives, all rows are exact. Thus, the front and back side give rise to the connecting homomorphisms of the respective Ext-functors. In particular, this diagram gives rise to the commuting square

$$\begin{array}{ccc}
\mathrm{Ext}^m_{\mathcal{C}_{R,H}}(\mathrm{Res}_H^G(A), \widetilde{B}_k) & \xrightarrow{\cong} & \mathrm{Ext}^m_{\mathcal{C}_{R,G}}(A, \mathrm{Coind}_H^G(\widetilde{B}_k)) \\
\Big\downarrow{\scriptstyle \delta^m} & & \Big\downarrow{\scriptstyle \delta^m} \\
\mathrm{Ext}^{m+1}_{\mathcal{C}_{R,H}}(\mathrm{Res}_H^G(A), \widetilde{B}_{k+1}) & \xrightarrow{\cong} & \mathrm{Ext}^{m+1}_{\mathcal{C}_{R,H}}(A, \mathrm{Coind}_H^G(\widetilde{B}_{k+1}))
\end{array}$$

$$
\begin{tikzcd}[column sep=small]
& & & & & \mathrm{Ext}^m_{\mathcal{C}_{R,G}}(A, \mathrm{Cd}(\widetilde{B}_k)) \arrow[dd] \\
& & & & \mathrm{Ext}^m_{\mathcal{C}_{R,H}}(\mathrm{Rs}(A), \widetilde{B}_k) \arrow[ur] \arrow[dd] & \\
& \mathrm{Cr}(a_m, \mathrm{Cd}(\widetilde{B}_{k+1})) \arrow[rr] \arrow[dd] & & \mathrm{Cr}(a_m, \mathrm{Cd}(B_k)) \arrow[rr] \arrow[dd] & & \mathrm{Cr}(a_m, \mathrm{Cd}(\widetilde{B}_k)) \arrow[dd] \\
\mathrm{Cr}(\mathrm{Rs}(a_m), \widetilde{B}_{k+1}) \arrow[ur] \arrow[rr] \arrow[dd] & & \mathrm{Cr}(\mathrm{Rs}(a_m), B_k) \arrow[ur] \arrow[rr] \arrow[dd] & & \mathrm{Cr}(\mathrm{Rs}(a_m), \widetilde{B}_k) \arrow[ur] \arrow[dd] & \\
& \mathrm{Kr}(a_{m+1}, \mathrm{Cd}(\widetilde{B}_{k+1})) \arrow[rr] \arrow[dd] & & \mathrm{Kr}(a_{m+1}, \mathrm{Cd}(B_k)) \arrow[rr] & & \mathrm{Kr}(a_{m+1}, \mathrm{Cd}(\widetilde{B}_k)) \\
\mathrm{Kr}(\mathrm{Rs}(a_{m+1}), \widetilde{B}_{k+1}) \arrow[ur] \arrow[rr] \arrow[dd] & & \mathrm{Kr}(\mathrm{Rs}(a_{m+1}), B_k) \arrow[ur] \arrow[rr] & & \mathrm{Kr}(\mathrm{Rs}(a_{m+1}), \widetilde{B}_k) \arrow[ur] & \\
& \mathrm{Ext}^{m+1}_{\mathcal{C}_{R,G}}(A, \mathrm{Cd}(\widetilde{B}_{k+1})) & & & & \\
\mathrm{Ext}^{m+1}_{\mathcal{C}_{R,H}}(\mathrm{Rs}(A), \widetilde{B}_{k+1}) \arrow[ur] & & & & &
\end{tikzcd}
$$

These squares form a direct system in whose direct limit we obtain the desired isomorphism. □

For the following example, recall that we have defined profinite groups and their associated completed group rings in Chapter 1.

Example 7.2.2 *The conditions of Lemma 7.2.1 are satisfied in the following two instances.*

1. *Let $\mathcal{C}$ be the category of discrete R-modules for a discrete commutative ring R and let G be a discrete group together with a finite index subgroup H. The group rings $R[G]$ and $R[H]$ are taken to be discrete.*
2. *Let $\mathcal{C}$ be the category of profinite S-modules for a profinite commutative ring S and let K be a profinite group together with an open subgroup L. The group rings $S[[K]]$ and $S[[L]]$ are taken to be profinite.*

Proof Let us first clarify some points regarding the profinite setting. Since the subgroup L is open in K, it is of finite index [14, Lemma 0.3.1]. Although there is no internal Hom-functor for profinite modules in general, coinduction and restriction can be nevertheless defined because L is open subgroup of the profinite group K. Namely, for any $S[[L]]$-module M and any $S[[K]]$-module N, endowing $\mathrm{Coind}_L^K(M) = \mathrm{Hom}_{S[[L]]}(S[[K]], M)$ and $\mathrm{Res}_L^K(N) = \mathrm{Hom}_{S[[K]]}(S[[K]], N)$ with the compact-open topology turns them into profinite modules [15, p. 369–371]. As there is a tensor product for profinite modules [11, p. 177/191], induction and restriction can be defined in this case. By [43, Lemma 7.8] restriction is left adjoint to coinduction and induction left adjoint to restriction in the profinite case. As both the discrete and profinite case can be treated analogously from this point, we write $R[G]$ for either the discrete or profinite group ring of G over R and denote by H the finite index (open) subgroup. It follows from the description $\mathrm{Res}_H^G(-) = \mathrm{Hom}_{R[G]}(R[G], -)$ that restriction is exact. By [2, Proposition I.3.1], [11, Proposition 5.4.2] and [11, Proposition 5.7.1] the $R[H]$-module $R[G]$ is projective. We thus infer by [10, Lemma 2.2.3] that coinduction is exact and by [11, Proposition 5.5.3] and [10, p. 68] that induction is exact. According to the proof of [43, Corollary 7.9], induction preserves projectives. Because the (open) subgroup H is of finite index in G, $\mathrm{Ind}_H^G(M) \cong \mathrm{Coind}_H^G(M)$ for every $R[H]$-module M by [2, Proposition III.5.9] and [15, p. 371]. Due to this isomorphism and [10, Lemma 2.2.3], coinduction preserves projectives. □

Theorem 7.2.3 (Dimension shifting) *If $T^\bullet : \mathcal{C} \to \mathcal{D}$ is a cohomological functor, then for any $M \in \mathrm{obj}(\mathcal{C})$ there is $M^* \in \mathrm{obj}(\mathcal{C})$ such that $\widehat{T}^{n+1}(M^*) \cong \widehat{T}^n(M)$ for every $n \in \mathbb{Z}$. In addition, if there is a monomorphism $f : M \to N$ in $\mathcal{C}$ with $\widehat{T}^k(N) = 0$ for every $k \in \mathbb{Z}$, then we observe that $\widehat{T}^{n-1}(\mathrm{Coker}(f)) \cong \widehat{T}^n(M)$.*

Proof We consider the short exact sequence $0 \to \widetilde{M_1} \to M_0 \to M \to 0$ and set $M^* := \widetilde{M_1}$. Then the first assertion follows from Axiom 2.0.2 and Definition 2.0.4. The second assertion is deduced analogously. □

Example 7.2.4 *The conditions of Theorem 7.2.3 are satisfied in the following two instances.*

1. *Let $\mathcal{C}$ be the category of discrete R-modules for a discrete commutative ring R and let G be a discrete group together with a finite index subgroup H of finite cohomological dimension. The group rings $R[G]$ and $R[H]$ are taken to be discrete.*
2. *Let $\mathcal{C}$ be the category of profinite S-modules for a profinite commutative ring S and let K be a profinite group together with an open subgroup L of finite cohomological dimension. The group rings $S[[K]]$ and $S[[L]]$ are taken to be profinite.*

Proof As the discrete and profinite case can be treated largely analogously, we denote by $R[G]$ either the discrete or profinite group ring and by H the finite index (open) subgroup of finite cohomological dimension. Then by [2, Proposition III.5.9] and [15, p. 371] we deduce that there is a (continuous) injective $R[H]$-module homomorphism

$$\mathrm{const} : M \to \mathrm{Hom}_{R[H]}(R[G], \mathrm{Res}^G_H(M)) = \mathrm{Coind}^G_H(\mathrm{Res}^G_H(M))$$

defined by

$$\mathrm{const}(m) : R[G] \to \mathrm{Res}^G_H(M),\ x \mapsto \begin{cases} x \cdot m & \text{if } x \in R[H] \\ 0 & \text{if } x \notin R[H] \end{cases}$$

for every $m \in M$. Specifically in the profinite case, since const is a continuous R-module homomorphism that is invariant under the G-action, it is invariant under the ordinary group ring of G over R. Because the latter is dense in the profinite group ring $R[G]$ [11, Lemma 5.3.5], const is a continuous $R[G]$-module homomorphisms. Returning to full generality, we conclude by Lemma 7.1.1 and Example 7.2.2 that

$$\widehat{H}^n_R\big(G, \mathrm{Coind}_H^G(\mathrm{Res}_H^G(M))\big) \cong \widehat{H}^n_R(H, \mathrm{Res}_H^G(M)) = 0$$

for every $n \in \mathbb{Z}$. Specifically for discrete groups of finite virtual cohomological dimension this can be proved in a different manner. As mentioned in Chapter 1, Tate-Farrell cohomology can be defined by taking a complete resolution in this case [18, p. 157–158]. By [10, Lemma 2.3.4], $\widehat{H}^\bullet_R(G, -)$ vanishes on injectives where the category of discrete $R[G]$-modules has enough injectives [2, p. 61]. However, this argumentation does not pass through to profinite groups because the category of profinite $R[G]$-modules does not have enough injectives in general [15, p. 353]. $\square$

7.3 Preliminaries to Yoneda and external products

In order to generalize external product from page 110 of D. J. Benson and J. F. Carlson's paper [13] using the naive construction, we need a few auxiliary results.

Proposition 7.3.1 *Let $E, F, P, Q \in \mathrm{obj}(\mathcal{C})$ with P and Q projective. Then*

$$\xi_{P,Q} : [E,F]_{\mathcal{C}} \to [E\oplus P, F\oplus Q]_{\mathcal{C}},\ f + \mathcal{P}_{\mathcal{C}}(E,F) \mapsto f\oplus 0 + \mathcal{P}_{\mathcal{C}}(E\oplus P, F\oplus Q)$$

is an isomorphism.

Proof Following [8, p. 250], we write $i_1 : E \to E\oplus P$, $i_1 : P \to E\oplus P$ for the (canonical) monomorphisms and $p_1 : E\oplus P \to E$, $p_2 : E\oplus P \to P$ for the (canonical) epimorphisms of the direct product $E\oplus P$. We also see that $\mathrm{id}_{E\oplus P} = i_1 \circ p_1 + i_2 \circ p_2$ and that $i_2 \circ p_2 : E\oplus P \to P \to E\oplus P$ factors through a projective. Hence,

$$\begin{aligned} i_1 \circ p_1 + \mathcal{P}_{\mathcal{C}}(E\oplus P, E\oplus P) &= \mathrm{id}_{E\oplus P} + \mathcal{P}_{\mathcal{C}}(E\oplus P, E\oplus P) \text{ and} \\ p_1 \circ i_1 + \mathcal{P}_{\mathcal{C}}(E,E) &= \mathrm{id}_E + \mathcal{P}_{\mathcal{C}}(E,E)\,. \end{aligned}$$

In particular, i_1 and p_1 become isomorphisms when using the functors $[-,-]_{\mathcal{C}}$ instead of the unenriched Hom-functors $\mathrm{Hom}_{\mathcal{C}}(-,-)$. Denote by $i : F \to F\oplus Q$ the canonical monomorphism as before. If $f : E \to F$ is any morphism and $0 : P \to Q$ the zero morphism, then

$$i \circ f \circ p_1 = f \oplus 0 : E\oplus P \to F\oplus Q$$

by [8, p. 251]. On the other, since the composition functor in $[-,-]_{\mathcal{C}}$ is associative by Proposition 5.1.1,

$$\xi_{P,Q} : [E,F]_{\mathcal{C}} \to [E\oplus P, F\oplus Q]_{\mathcal{C}},\ f+\mathcal{P}_{\mathcal{C}}(E,F) \mapsto i\circ f\circ p_1+\mathcal{P}_{\mathcal{C}}(E\oplus P, F\oplus Q)$$

is an isomorphism. □

This allows us to related tensor products of syzygies with syzygies of a tensor product. We require that as external products involve taking tensor products.

Proposition 7.3.2 *Let $A, B, C, D \in \mathrm{obj}(\mathcal{C})$ and take $A_\bullet$, $B_\bullet$, $C_\bullet$, $D_\bullet$ to be corresponding projective resolutions. Assume that $\otimes_{\mathcal{C}} : \mathcal{C}\times\mathcal{C} \to \mathcal{C}$ is a bi-additive functor such that $(h\otimes_{\mathcal{C}} i)\circ(f\otimes_{\mathcal{C}} g) = (h\circ f)\otimes_{\mathcal{C}}(i\circ g)$ for any morphisms f, g, h, i for which the compositions $h\circ f$ and $i\circ g$ are defined. If $k\in\mathbb{N}_0$ and $\widetilde{E}_k$ denotes any of $\widetilde{A}_k$, $\widetilde{B}_k$, $\widetilde{C}_k$ or $\widetilde{D}_k$, impose that $\widetilde{E}_k\otimes_{\mathcal{C}} - : \mathcal{C}\to\mathcal{C}$ as well as $-\otimes_{\mathcal{C}}\widetilde{E}_k$ is an exact functor and that $P\otimes_{\mathcal{C}}\widetilde{E}_k$ as well as $\widetilde{E}_k\otimes_{\mathcal{C}} P$ is projective for any projective $P\in\mathrm{obj}(\mathcal{C})$. Then there is an isomorphism*

$$\xi^{k,l}_{m,n} : [\widetilde{A}_K\otimes_{\mathcal{C}}\widetilde{C}_{l+n}, \widetilde{B}_k\otimes_{\mathcal{C}}\widetilde{D}_l]_{\mathcal{C}} \to [(\widetilde{A\otimes_{\mathcal{C}} C})_{K+l+n}, (\widetilde{B\otimes_{\mathcal{C}} D})_{k+l}]_{\mathcal{C}}$$

for any $m,n\in\mathbb{Z}$ and $k,l\in\mathbb{N}_0$ with $K, l+n\geq 0$.

Proof We prove by induction on k and l that there are projective objects $P^{k,l}, Q^{k,l}$ in $\mathcal{C}$ such that

$$(\widetilde{B}_k\otimes_{\mathcal{C}}\widetilde{D}_l)\oplus P^{k,l} \cong (\widetilde{B\otimes_{\mathcal{C}} D})_{k+l}\oplus Q^{k,l}\,. \tag{7.3.1}$$

For $k=l=0$ we set $P^{0,0}=Q^{0,0}=0$. Regarding the inductive step, note that the sequences

$$0\to(\widetilde{B\otimes_{\mathcal{C}} D})_{k+l+1}\xrightarrow{\iota_1}(B\otimes_{\mathcal{C}} D)_{k+l}\xrightarrow{\pi^1_1}(\widetilde{B\otimes_{\mathcal{C}} D})_{k+l}\to 0 \text{ and}$$
$$0\to\widetilde{B}_{k+1}\otimes_{\mathcal{C}}\widetilde{D}_l\xrightarrow{\iota_2}B_k\otimes_{\mathcal{C}}\widetilde{D}_l\xrightarrow{\pi^1_2}\widetilde{B}_k\otimes_{\mathcal{C}}\widetilde{D}_l\to 0\,.$$

are exact because we have assumed that $-\otimes_{\mathcal{C}}\widetilde{D}_l$ is an exact functor. Moreover, $B_k\otimes_{\mathcal{C}}\widetilde{D}_l$ is projective by assumption. It follows from [8, p. 250–251] that the sequences

$$0\to(\widetilde{B\otimes_{\mathcal{C}} D})_{k+l+1}\oplus Q^{k,l}\xrightarrow{\iota_1\oplus\mathrm{id}_{Q^{k,l}}}(B\otimes_{\mathcal{C}} D)_{k+l}\oplus Q^{k,l}\xrightarrow{\pi^1_1\oplus\mathrm{id}_{Q^{k,l}}}(\widetilde{B\otimes_{\mathcal{C}} D})_{k+l}\oplus Q^{k,l}\to 0 \text{ and}$$
$$0\to(\widetilde{B}_{k+1}\otimes_{\mathcal{C}}\widetilde{D}_l)\oplus P^{k,l}\xrightarrow{\iota_2\oplus\mathrm{id}_{P^{k,l}}}(B_k\otimes_{\mathcal{C}}\widetilde{D}_l)\oplus P^{k,l}\xrightarrow{\pi^1_2\oplus\mathrm{id}_{P^{k,l}}}(\widetilde{B}_k\otimes_{\mathcal{C}}\widetilde{D}_l)\oplus P^{k,l}\to 0\,.$$

remain exact. This puts us into the position to apply Schanuel's Lemma. Let us write $T^{k,l} := (\widetilde{B}_k \otimes_{\mathcal{C}} \widetilde{D}_l) \oplus P^{k,l}$. By [44, Lemma 11.28] and [45, p. 15/34] we can form with the above short exact sequences the commuting diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & (\widetilde{B\otimes_{\mathcal{C}} D})_{k+l+1}\oplus Q^{k,l} \arrow[r, "\mathrm{id}"] \arrow[d] & (\widetilde{B\otimes_{\mathcal{C}} D})_{k+l+1}\oplus Q^{k,l} \arrow[d] & \\
0 \arrow[r] & (\widetilde{B}_{k+1}\otimes_{\mathcal{C}}\widetilde{D}_l)\oplus P^{k,l} \arrow[r, "i^{k+1,l}"] \arrow[d, "\mathrm{id}"] & T^{k+1,l} \arrow[r] \arrow[d] \arrow[u, dashed, bend left, "p^{k+1,l}"] & (B\otimes_{\mathcal{C}} D)_{k+l}\oplus Q^{k,l} \arrow[r] \arrow[d, "\pi_2^1\oplus\mathrm{id}_{Q^{k,l}}"] & 0 \\
0 \arrow[r] & (\widetilde{B}_{k+1}\otimes_{\mathcal{C}}\widetilde{D}_l)\oplus P^{k,l} \arrow[r] & (B_k\otimes_{\mathcal{C}}\widetilde{D}_l)\oplus P^{k,l} \arrow[r, "\pi_1^1\oplus\mathrm{id}_{P^{k,l}}"] \arrow[d] & T^{k,l} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd} \tag{7.3.2}$$

In the bottom right square, $T^{k+1,l}$ is the pullback of $\pi_1 \oplus \mathrm{id}_{Q^{k,l}}$ and $\pi_2 \oplus \mathrm{id}_{P^{k,l}}$. Since $(B_k \otimes_{\mathcal{C}} \widetilde{D}_l) \oplus P^{k,l}$ and $(B \otimes_{\mathcal{C}} D)_{k+l} \oplus Q^{k,l}$ are projective, the row and the column containing the object $T^{k+1,l}$ are split exact. Accordingly, $i^{k+1,l}$ and $p^{k+1,l}$ denote morphisms in these split exact sequences. Thus, if we define

$$\begin{aligned} P^{k+1,l} &:= P^{k,l} \oplus (B \otimes_{\mathcal{C}} D)_{k+l} \oplus Q^{k,l} \text{ and} \\ Q^{k+1,l} &:= Q^{k,l} \oplus (B_k \otimes_{\mathcal{C}} \widetilde{D}_l) \oplus P^{k,l}\,, \end{aligned} \tag{7.3.3}$$

then

$$(\widetilde{B}_{k+1} \otimes_{\mathcal{C}} \widetilde{D}_l) \oplus P^{k+1,l} \cong T^{k+1,l} \cong (\widetilde{B \otimes_{\mathcal{C}} D})_{k+l+1} \oplus Q^{k+1,l}\,. \tag{7.3.4}$$

Given that constructing $P^{k,l+1}$, $Q^{k,l+1}$ and $T^{k,l+1}$ is analogous, this completes the induction. One can perform all arguments analogously for $\widetilde{A}_K \otimes_{\mathcal{C}} \widetilde{C}_{l+n}$ and $(\widetilde{A \otimes_{\mathcal{C}} C})_{K+l+n}$. Namely, if we set $K := K$ and $L := l + n$, then

Diagram 7.3.2 becomes

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & (\widetilde{A\otimes_{\mathcal{C}}C})_{K+L+1}\oplus Q^{k,l}_{m,n} \arrow[r, "\mathrm{id}"] \arrow[d, "i^{k+1,l}_{m,n}"] & (\widetilde{A\otimes_{\mathcal{C}}C})_{K+L+1}\oplus Q^{k,l}_{m,n} \arrow[d] & \\
0 \arrow[r] & (\widetilde{A}_{K+1}\otimes_{\mathcal{C}}\widetilde{C}_L)\oplus P^{k,l}_{m,n} \arrow[r, shift left] \arrow[d, "\mathrm{id}"] & T^{k+1,l}_{m,n} \arrow[r] \arrow[l, dashed, shift left, "p^{k+1,l}_{m,n}"] \arrow[d] & (A\otimes_{\mathcal{C}}C)_{K+L}\oplus Q^{k,l}_{m,n} \arrow[r] \arrow[d, "\pi_2^2\oplus\mathrm{id}_{Q^{k,l}_{m,n}}"] & 0 \\
0 \arrow[r] & (\widetilde{A}_{K+1}\otimes_{\mathcal{C}}\widetilde{C}_L)\oplus P^{k,l}_{m,n} \arrow[r] & (A_K\otimes_{\mathcal{C}}\widetilde{C}_L)\oplus P^{k,l}_{m,n} \arrow[r, "\pi_1^2\oplus\mathrm{id}_{P^{k,l}_{m,n}}"] \arrow[d] & T^{k,l}_{m,n} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd} \tag{7.3.5}$$

and Equation 7.3.4 becomes

$$(\widetilde{A}_K \otimes_{\mathcal{C}} \widetilde{C}_L) \oplus P^{k,l}_{m,n} \cong T^{k+1,l}_{m,n} \cong (\widetilde{A \otimes_{\mathcal{C}} C})_{K+L} \oplus Q^{k,l}_{m,n}\,. \tag{7.3.6}$$

If $(k,l) \neq (0,0) \neq (K,L)$, then the morphisms $i^{k,l}$, $p^{k,l}$, $i^{kl}_{m,n}$ and $p^{k,l}_{m,n}$ from Diagram 7.3.2 and 7.3.5 become isomorphisms when using $[-,-]_{\mathcal{C}}$ by the proof of Proposition 7.3.1. Due to the base case of the previous induction, we can define $i^{0,0}, p^{0,0} := \mathrm{id}_{B\otimes_{\mathcal{C}}D}$ for $(k,l) = (0,0)$ and $i^{k,l}_{-k,-l}, p^{k,l}_{-k,-l} := \mathrm{id}_{A\otimes_{\mathcal{C}}C}$ for $(K,L) = (0,0)$. For respective $r, s \in \{0,1\}$ with

$$k-r, K-r, l-s, L-s \geq 0 \text{ and } r+s \leq 1$$

this yields an isomorphism

$$\begin{aligned}
\upsilon^{k,l}_{m,n} :\ & [(\widetilde{A}_K \otimes_{\mathcal{C}} \widetilde{C}_L) \oplus P^{k-r,l-s}_{m,n}, (\widetilde{B}_k \otimes_{\mathcal{C}} \widetilde{D}_l) \oplus P^{k-r,l-s}] \\
&\quad \to [(\widetilde{A \otimes_{\mathcal{C}} C})_{K+L} \oplus Q^{k-r,l-s}_{m,n}, (\widetilde{B \otimes_{\mathcal{C}} D})_{k+l} \oplus Q^{k-r,l-s}] \\
f + \mathcal{P}_{\mathcal{C}}\big(& (\widetilde{A}_K \otimes_{\mathcal{C}} \widetilde{C}_L) \oplus P^{k-r,l-s}_{m,n}, (\widetilde{B}_k \otimes_{\mathcal{C}} \widetilde{D}_l) \oplus P^{k-r,l-s}\big) \\
&\quad \mapsto i^{k,l} \circ p^{k,l} \circ f \circ p^{k,l}_{m,n} \circ i^{k,l}_{m,n} \\
&\qquad\qquad + \mathcal{P}_{\mathcal{C}}\big((\widetilde{A \otimes_{\mathcal{C}} C})_{K+L} \oplus Q^{k-r,l-s}_{m,n}, (\widetilde{B \otimes_{\mathcal{C}} D})_{k+l} \oplus Q^{k-r,l-s}\big)
\end{aligned}$$

The desired isomorphism is given by

$$\begin{aligned}
\xi^{k,l}_{m,n} :=\ & \xi^{-1}_{P^{k-r,l-s},Q^{k-r,l-s}} \circ \upsilon^{k-r,l-s}_{m,n} \circ \xi_{P^{k-r,l-s}_{m,n},Q^{k-r,l-s}_{m,n}} : \\
& [\widetilde{A}_K \otimes_{\mathcal{C}} \widetilde{C}_L, \widetilde{B}_k \otimes_{\mathcal{C}} \widetilde{D}_l]_{\mathcal{C}} \to [(\widetilde{A \otimes_{\mathcal{C}} C})_{K+L}, (\widetilde{B \otimes_{\mathcal{C}} D})_{k+l}]_{\mathcal{C}}
\end{aligned}$$

where the isomorphisms $\xi_{P^{k-r,l-s},Q^{k-r,l-s}}$ and $\xi_{P^{k-r,l-s}_{m,n},Q^{k-r,l-s}_{m,n}}$ are taken as in Proposition 7.3.1. □

Since the naive construction involves taking direct limits, we require the following result that we have not found in the literature in this manner.

Proposition 7.3.3 *Let $(M_i, m_i)_{i\in\mathbb{N}}$, $(N_i, n_i)_{i\in\mathbb{N}}$ be direct systems of abelian groups and denote by $\otimes$ the tensor product of abelian groups. Then*

$$\varinjlim_{i\in\mathbb{N}}(M_i \otimes N_i, m_i \otimes n_i) \cong (\varinjlim_{j\in\mathbb{N}} M_j) \otimes (\varinjlim_{k\in\mathbb{N}} N_k)\,.$$

Proof We demonstrate that $\varinjlim_{i\in\mathbb{N}}(M_i \otimes N_i, m_i \otimes n_i)$ satisfies the universal property of the tensor product $(\varinjlim_{j\in\mathbb{N}} M_j) \otimes (\varinjlim_{k\in\mathbb{N}} N_k)$ that can be found in [9, Tag 00CV] for instance. Denote by $M_i \times N_i$ the cartesian product and observe that the squares

$$\begin{array}{ccc} M_i \times N_i & \xrightarrow{q_i} & M_i \otimes N_i \\ \downarrow{\scriptstyle m_i \times n_i} & & \downarrow{\scriptstyle m_i \otimes n_i} \\ M_{i+1} \times N_{i+1} & \xrightarrow{q_{i+1}} & M_{i+1} \otimes N_{i+1} \end{array} \tag{7.3.7}$$

commute. Although $M_i \times N_i$ and $M_{i+1} \times N_{i+1}$ are abelian groups and $m_i \otimes n_i$ a homomorphism, we regard the former as sets and the latter as a function. In particular, Diagram 7.3.7 gives rise to a direct system of functions in whose direct limit we obtain

$$q := \varinjlim_{i\in\mathbb{N}} q_i : \varinjlim_{\mathrm{Set}, i\in\mathbb{N}} (M_i \times N_i, m_i \times n_i) \to \varinjlim_{\mathrm{Set}, i\in\mathbb{N}} (M_i \otimes N_i, m_i \otimes n_i) \tag{7.3.8}$$

where the latter is taken to be a direct limit of sets. By [9, Tag 002W], the former direct limit can be given by

$$\varinjlim_{\mathrm{Set}, i\in\mathbb{N}} (M_i \times N_i, m_i \times n_i) \cong (\varinjlim_{\mathrm{Set}, j\in\mathbb{N}} M_j) \times (\varinjlim_{\mathrm{Set}, k\in\mathbb{N}} N_k)\,. \tag{7.3.9}$$

In particular, if $m_{i,\infty} : M_i \to (\varinjlim_{\mathrm{Set}, j\in\mathbb{N}} M_j)$ and $n_{i,\infty} : N_i \to (\varinjlim_{\mathrm{Set}, k\in\mathbb{N}} N_k)$ denote functions to the respective direct limit, then the function

$$m_{i,\infty} \times n_{i,\infty} : M_i \times N_i \to (\varinjlim_{\mathrm{Set}, j\in\mathbb{N}} M_j) \times (\varinjlim_{\mathrm{Set}, k\in\mathbb{N}} N_k) \tag{7.3.10}$$

represents the function to the direct limit $\varinjlim_{\mathrm{Set}, i\in\mathbb{N}}(M_i \times N_i, m_i \times n_i)$. According to [9, Tag 04AX], if $\sqcup$ denotes the disjoint union of sets, then the latter direct limit in Equation 7.3.8 can be constructed as

$$\varinjlim_{\mathrm{Set}, i\in\mathbb{N}} (M_i \otimes N_i, m_i \otimes n_i) = \bigsqcup_{n\in\mathbb{N}} M_n \otimes N_n / \sim \tag{7.3.11}$$

where $x_i \in M_i \otimes N_i \sim y_j \in M_j \otimes N_j$ if there is $i, j \leq k$ such that x_i and y_j are mapped to the same element in $M_k \otimes N_k$. It follows from the proof of [11, Proposition 1.2.1] and [9, Tag 09WR] that it can be endowed with the structure of an abelian group such that it forms the colimit $\varinjlim_{i\in\mathbb{N}}(M_i \otimes N_i, m_i \otimes n_i)$ of abelian groups and not just of sets. Same holds true for $\varinjlim_{i\in\mathbb{N}} M_i$ and $\varinjlim_{i\in\mathbb{N}} N_i$ from Equation 7.3.9. Hence, the function q from Equation 7.3.8 can be written as

$$q : (\varinjlim_{j\in\mathbb{N}} M_j) \times (\varinjlim_{k\in\mathbb{N}} N_k) \to \varinjlim_{i\in\mathbb{N}}(M_i \otimes N_i, m_i \otimes n_i)\,.$$

To prove that q satisfies the above mentioned universal property, consider a bilinear function $a : (\varinjlim_{j\in\mathbb{N}} M_j) \times (\varinjlim_{k\in\mathbb{N}} N_k) \to A$ for an abelian group A. Composing with the homomorphisms $m_{i,\infty}$ and $n_{i,\infty}$ occurring in Equation 7.3.10 yields a bilinear function

$$M_i \times N_i \xrightarrow{m_{i,\infty}\times n_{i,\infty}} (\varinjlim_{j\in\mathbb{N}} M_j) \times (\varinjlim_{k\in\mathbb{N}} N_k) \xrightarrow{a} A\,.$$

By the universal property of the tensor product, there exists a unique homomorphism $b_i : M_i \otimes N_i \to A$ such that the square

$$\begin{tikzcd} M_i \times N_i \arrow[r, "q_i"] \arrow[d, "m_{i,\infty}\times n_{i,\infty}"] & M_i \otimes N_i \arrow[d, "b_i"] \\ (\varinjlim_{j\in\mathbb{N}} M_j) \times (\varinjlim_{k\in\mathbb{N}} N_k) \arrow[r, "a"] & A \end{tikzcd} \tag{7.3.12}$$

commutes. By this and Diagram 7.3.7 we infer that the triangle

$$\begin{tikzcd} M_i \otimes N_i \arrow[d, "m_i\otimes n_i"] \arrow[dr, bend left, "b_i"] & \\ M_{i+1} \otimes N_{i+1} \arrow[r, "b_{i+1}"] & A \end{tikzcd} \tag{7.3.13}$$

is commutative whence there is a homomorphism

$$b := \varinjlim_{i\in\mathbb{N}} b_i : \varinjlim_{i\in\mathbb{N}}(M_i \otimes N_i, m_i \otimes n_i) \to A$$

By Diagram 7.3.7, 7.3.12 and 7.3.13 we obtain the factorization

$$a : \varinjlim_{\mathrm{Set}, i\in\mathbb{N}} (M_i \times N_i, m_i \times n_i) \xrightarrow{q} \varinjlim_{\mathrm{Set}, i\in\mathbb{N}} (M_i \otimes N_i, m_i \otimes n_i) \xrightarrow{b} A\,.$$

Because b is unique in this factorization due to Equation 7.3.11 and the uniqueness of the homomorphisms b_i from Diagram 7.3.12 and 7.3.13, the direct limit $\varinjlim_{i\in\mathbb{N}}(M_i \otimes N_i, m_i \otimes n_i)$ satisfies the universal property of the tensor product tensor product $(\varinjlim_{j\in\mathbb{N}} M_j) \otimes (\varinjlim_{k\in\mathbb{N}} N_k)$. $\square$

7.4 Existence of Yoneda and external products

At least for Yoneda products recall that for every $n \in \mathbb{Z}$ the completed unenriched Ext-functor $\widehat{\mathrm{Ext}}^n_{\mathcal{C}}(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Ab}$ forms a bifunctor that is additive in both variables by Proposition 5.1.5 and Proposition 6.1.3.

Theorem 7.4.1 *Let* $\widehat{\mathrm{Ext}}^\bullet_{\mathcal{C}}(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Ab}$ *denote unenriched completed Ext-functors and let* $A, B, C, D \in \mathrm{obj}(\mathcal{C})$.

1. *If* $\otimes$ *denotes the tensor product in* Ab, *then for every* $m, n \in \mathbb{Z}$ *Yoneda products*

$$\circ : \widehat{\mathrm{Ext}}^n_{\mathcal{C}}(B, C) \otimes \widehat{\mathrm{Ext}}^m_{\mathcal{C}}(A, B) \to \widehat{\mathrm{Ext}}^{m+n}_{\mathcal{C}}(A, C)$$

   can be equivalently defined by the hypercohomology construction as composition of almost chain maps or by the naive construction as a direct limit of the composition functors of the functors $[-,-]_{\mathcal{C}}$ *from Proposition 5.1.1.*

2. *Let* $A_\bullet$, $B_\bullet$, $C_\bullet$, $D_\bullet$ *be projective resolutions and let* $\otimes_{\mathcal{C}} : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ *be a bi-additive functor satisfying the condition in Proposition 7.3.2. Then external products*

$$\vee : \widehat{\mathrm{Ext}}^m_{\mathcal{C}}(A, B) \otimes \widehat{\mathrm{Ext}}^n_{\mathcal{C}}(C, D) \to \widehat{\mathrm{Ext}}^{m+n}_{\mathcal{C}}(A \otimes_{\mathcal{C}} C, B \otimes_{\mathcal{C}} D)$$

   can be defined through the naive construction.

Remark 7.4.2 *External products cannot be defined through the hypercohomology construction in general. Namely, if we identify unerniched Ext-functors with chain maps modulo chain homotopy as in Proposition 6.2.1, then external products descend from the tensor product of chain maps by [8, p. 220–222 and p. 228–229]. Even for modules over a ring, D. J. Benson and J. F. Carlson note in [13, p. 110] that the tensor product of two almost chain maps is not necessarily an almost chain map.*

Proof (1) Let us construct Yoneda products in the two ways we have indicated. In Notation 6.1.1 we have seen that elements in $\widehat{\mathrm{Ext}}^m_{\mathcal{C}}(A,B)$ and $\widehat{\mathrm{Ext}}^n_{\mathcal{C}}(B,C)$ are exactly almost chain maps modulo almost chain homotopy. Then we can define the Yoneda product as for unenriched Ext-functors, which is covered in [41, p. 166] for instance. Namely, if the almost chain map $f[n]_{\bullet+m} : A[m+n]_\bullet \to B[n]_\bullet$ is a representative an element in $\widehat{\mathrm{Ext}}^m_{\mathcal{C}}(A,B)$ and $g_{\bullet+n} : B[n]_\bullet \to C_\bullet$ a representative of an element in $\widehat{\mathrm{Ext}}^n_{\mathcal{C}}(B,C)$, then their composition $g_{\bullet+n} \circ f[n]_{\bullet+m} : A[m+n]_\bullet \to C_\bullet$ is again an almost chain map. If $f'_{\bullet+m}$ is chain homotopic to $f_{\bullet+m}$ and $g'_{\bullet+n}$ chain homotopic to $g_{\bullet+n}$, then $g_{\bullet+n} \circ f[n]_{\bullet+m}$ is chain homotopic to $g_{\bullet+n} \circ f'[n]_{\bullet+m}$ which is in turn chain homotopic to $g'_{\bullet+n} \circ f'[n]_{\bullet+m}$. Hence, $g_{\bullet+n} \circ f[n]_{\bullet+m} + \widehat{\mathrm{Null}}_{\mathrm{Ch}(\mathcal{C}}(A[m+n]_\bullet, C_\bullet)$ is a well defined element in $\widehat{\mathrm{Ext}}^{m+n}_{\mathcal{C}}(A,C)$. This operation constitutes the first Yoneda product $\widehat{\mathrm{Ext}}^n_{\mathcal{C}}(B,C) \times \widehat{\mathrm{Ext}}^m_{\mathcal{C}}(A,B) \to \widehat{\mathrm{Ext}}^{m+n}_{\mathcal{C}}(A,C)$. Moving over to he naive construction, we have seen in Proposition 5.1.1 that for any $k \in \mathbb{N}_0$ with $n+m+k, n+k \geq 0$ the composition functor of morphisms descends to a bifunctor

$$\circ : [\widetilde{B}_{n+k}, \widetilde{C}_k]_{\mathcal{C}} \times [\widetilde{A}_{n+m+k}, \widetilde{B}_{n+k}]_{\mathcal{C}} \to [\widetilde{A}_{n+m+k}, \widetilde{C}_k]_{\mathcal{C}}$$

In particular, if we take $f'_{n+m+k} + \mathcal{P}_{\mathcal{C}}(\widetilde{A}_{n+m+k}, \widetilde{B}_{n+k})$ in $[\widetilde{A}_{n+m+k}, \widetilde{B}_{n+k}]_{\mathcal{C}}$ and an element $g'_{n+k} + \mathcal{P}_{\mathcal{C}}(\widetilde{B}_{n+k}, \widetilde{C}_k)$ in $[\widetilde{B}_{n+k}, \widetilde{C}_k]_{\mathcal{C}}$, then they yield a well defined element $g'_{n+k} \circ f'_{n+m+k} + \mathcal{P}_{\mathcal{C}}(\widetilde{A}_{n+m+k}, \widetilde{C}_k)$ in $[\widetilde{A}_{n+m+k}, \widetilde{C}_k]_{\mathcal{C}}$. By Proposition 5.1.2 and Definition 5.1.3 we can pass to the direct limit to obtain $\widehat{\mathrm{Ext}}^n_{\mathcal{C}}(B,C) \times \widehat{\mathrm{Ext}}^m_{\mathcal{C}}(A,B) \to \widehat{\mathrm{Ext}}^{m+n}_{\mathcal{C}}(A,C)$, the second Yoneda product. By construction, both Yoneda products are bi-additive and thus, they can be written in the form they are stated in the theorem. Applying the isomorphism $\rho^{n+m} : \widehat{\mathcal{E}xt}^{n+m}_{\mathcal{C}}(A,C) \to BC^{n+m}_{\mathcal{C}}(A,C)$ from Definition 6.4.5, we deduce from the proof of Lemma 6.4.6 that

$$\begin{aligned}\rho^{n+m}\big(g_{\bullet+n} \circ f[n]_{\bullet+m} + &\widehat{\mathrm{Null}}_{\mathrm{Ch}(\mathcal{C})}(A[n+m]_\bullet, C_\bullet)\big) \\ &= \big(\widetilde{g}_{n+2k} \circ \widetilde{f}_{n+m+2k} + \mathcal{P}_{\mathcal{C}}(\widetilde{A}_{n+m+2k}, \widetilde{C}_{2k})\big)_{k \geq K}\end{aligned}$$

where $K \in \mathbb{N}_0$ is chosen such that both $(f_{n+m+k})_{k\geq 2K}$ and $(g_{n+k})_{k \geq 2K}$ are chain maps. This demonstrate that the two Yoneda products agree.

(2) In order to establish external products, let $k, l \in \mathbb{N}_0$ with $K := k+m \geq 0$ and $L := l+n \geq 0$. We first need to determine that the bi-additive tensor product $\otimes_{\mathcal{C}}$ descends to a well defined homomorphism

$$\otimes'_{[-,-]_{\mathcal{C}}} : [\widetilde{A}_K, \widetilde{B}_k]_{\mathcal{C}} \otimes [\widetilde{C}_L, \widetilde{D}_l]_{\mathcal{C}} \to [\widetilde{A}_K \otimes_{\mathcal{C}} \widetilde{C}_L, \widetilde{B}_k \otimes_{\mathcal{C}} \widetilde{D}_l]_{\mathcal{C}} \tag{7.4.1}$$
$$\big(\widetilde{f}_K + \mathcal{P}_{\mathcal{C}}(\widetilde{A}_K, \widetilde{B}_k)\big) \otimes \big(\widetilde{h}_L + \mathcal{P}_{\mathcal{C}}(\widetilde{C}_L, \widetilde{D}_l)\big) \mapsto \widetilde{f}_K \otimes_{\mathcal{C}} \widetilde{h}_L + \mathcal{P}_{\mathcal{C}}(\widetilde{A}_K \otimes_{\mathcal{C}} \widetilde{C}_L, \widetilde{B}_k \otimes_{\mathcal{C}} \widetilde{D}_l)$$

Assume that $\widetilde{f}_K$ factors through a projective. Since tensoring a projective with either $\widetilde{C}_L$ or $\widetilde{D}_l$ yields another projective object, $\widetilde{f}_K \otimes_{\mathcal{C}} \widetilde{h}_L$ factors through a projective. It follows that $\widetilde{f}_K \otimes_{\mathcal{C}} \widetilde{h}_L + \mathcal{P}_{\mathcal{C}}(\widetilde{A}_K \otimes_{\mathcal{C}} \widetilde{C}_L, \widetilde{B}_k \otimes_{\mathcal{C}} \widetilde{D}_l)$ does not depend on the choice of representative $\widetilde{f}_K$. Analogously, it also does not depend on the choice of representative $\widetilde{h}_L$. Since we have imposed that the tensor product $\otimes_{\mathcal{C}}$ is additive in both variables, $\otimes'_{[-,-]_{\mathcal{C}}}$ is indeed well defined homomorphism. According to Proposition 7.3.2 we can thus define

$$\otimes_{[-,-]_{\mathcal{C}}} := \xi^{k,l}_{m,n} \circ \otimes_{[-,-]_{\mathcal{C}}} : \tag{7.4.2}$$
$$[\widetilde{A}_K, \widetilde{B}_k]_{\mathcal{C}} \otimes [\widetilde{C}_L, \widetilde{D}_l]_{\mathcal{C}} \to [\widetilde{A}_K \otimes_{\mathcal{C}} \widetilde{C}_L, \widetilde{B}_k \otimes_{\mathcal{C}} \widetilde{D}_l] \to [(\widetilde{A \otimes_{\mathcal{C}} C})_{K+L}, (\widetilde{B \otimes_{\mathcal{C}} D})_{k+l}]_{\mathcal{C}}\,.$$

It remains to show that $\otimes_{[-,-]_{\mathcal{C}}}$ descends to an external product

$$\widehat{\operatorname{Ext}}^m_{\mathcal{C}}(A, B) \otimes \widehat{\operatorname{Ext}}^n_{\mathcal{C}}(C, D) \to \widehat{\operatorname{Ext}}^{m+n}_{\mathcal{C}}(A \otimes_{\mathcal{C}} C, B \otimes_{\mathcal{C}} D)$$

in the direct limit. The following version of Diagram 4.2.1

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{A}_{K+1} \arrow[r] \arrow[d, "\widetilde{f}_{K+1}"] & A_K \arrow[r] \arrow[d, "f_K"] & \widetilde{A}_K \arrow[r] \arrow[d, "\widetilde{f}_K"] & 0 \\
0 \arrow[r] & \widetilde{B}_{k+1} \arrow[r] & B_k \arrow[r] & \widetilde{B}_k \arrow[r] & 0
\end{tikzcd}$$

can be used to perform one step in constructing $\widehat{\operatorname{Ext}}^m_{\mathcal{C}}(A, B)$ according to the naive construction as in Definition 5.1.3. Because tensoring with $\widetilde{C}_L$ and $\widetilde{D}_l$ is exact, we can extend the above in a commutative manner to

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{A}_{K+1} \otimes_{\mathcal{C}} \widetilde{C}_L \arrow[r] \arrow[d, "\widetilde{g}^L_{K+1} := \widetilde{f}_{K+1} \otimes_{\mathcal{C}} \widetilde{h}_L"] & A_K \otimes_{\mathcal{C}} \widetilde{C}_L \arrow[r] \arrow[d, "g^L_K := f_K \otimes_{\mathcal{C}} \widetilde{h}_L"] & \widetilde{A}_K \otimes_{\mathcal{C}} \widetilde{C}_L \arrow[r] \arrow[d, "\widetilde{g}^L_K := \widetilde{f}_K \otimes_{\mathcal{C}} \widetilde{h}_L"] & 0 \\
0 \arrow[r] & \widetilde{B}_{k+1} \otimes_{\mathcal{C}} \widetilde{D}_l \arrow[r] & B_k \otimes_{\mathcal{C}} \widetilde{D}_l \arrow[r] & \widetilde{B}_k \otimes_{\mathcal{C}} \widetilde{D}_l \arrow[r] & 0
\end{tikzcd} \tag{7.4.3}$$

such that the rows are exact. As $A_K \otimes_{\mathcal{C}} \widetilde{C}_L$ and $B_k \otimes_{\mathcal{C}} \widetilde{D}_l$ are projective, we conclude that the diagram

$$\begin{tikzcd}
{[\widetilde{A}_K, \widetilde{B}_k]_{\mathcal{C}} \otimes [\widetilde{C}_L, \widetilde{D}_l]_{\mathcal{C}}} \arrow[r, "\otimes'_{[-,-]_{\mathcal{C}}}"] \arrow[d, "t_{\widetilde{A}_K, \widetilde{B}_k} \otimes \mathrm{id}_{\widetilde{C}_L, \widetilde{D}_l}"] & {[\widetilde{A}_K \otimes_{\mathcal{C}} \widetilde{C}_L, \widetilde{B}_k \otimes_{\mathcal{C}} \widetilde{D}_l]_{\mathcal{C}}} \arrow[d, "t_{\widetilde{A}_K \otimes_{\mathcal{C}} \widetilde{C}_L, \widetilde{B}_k \otimes_{\mathcal{C}} \widetilde{D}_l}"] \\
{[\widetilde{A}_{K+1}, \widetilde{B}_{k+1}]_{\mathcal{C}} \otimes [\widetilde{C}_L, \widetilde{D}_l]_{\mathcal{C}}} \arrow[r, "\otimes'_{[-,-]_{\mathcal{C}}}"] & {[\widetilde{A}_{K+1} \otimes_{\mathcal{C}} \widetilde{C}_L, \widetilde{B}_{k+1} \otimes_{\mathcal{C}} \widetilde{D}_l]_{\mathcal{C}}}
\end{tikzcd} \tag{7.4.4}$$

commutes where we have used notation from Proposition 5.1.2. To expand the homomorphisms $\otimes'_{[-,-]_{\mathcal{C}}}$ in the rows of the above square to $\otimes_{[-,-]_{\mathcal{C}}}$, we construct a diagram whose front portion is found on the next page. Its back portion is attached to its right-hand half and is represented by the cube

$$\begin{tikzcd}[column sep=small]
& (\widetilde{A\otimes_{\mathcal{C}} C})_{K+L+1} \oplus Q^{k,l}_{m,n} \arrow[rr, "\mathrm{id}"] \arrow[dl, "i^{k+1,l}_{m,n}"'] \arrow[dd, "\widetilde{g}_{K+L+1}"] & & (\widetilde{A\otimes_{\mathcal{C}} C})_{K+L+1} \oplus Q^{k,l}_{m,n} \arrow[dl] \arrow[dd, "\widetilde{g}_{K+L+1}"] \\
T^{k+1,l}_{m,n} \arrow[rr] \arrow[dd, "G^L_K"] & & (A\otimes_{\mathcal{C}} C)_{K+L} \oplus Q^{k,l}_{m,n} \arrow[dd, "g_{K+L}"] & \\
& (\widetilde{B\otimes_{\mathcal{C}} D})_{k+l+1} \oplus Q^{k,l} \arrow[rr, "\mathrm{id}"] \arrow[dl] & & (\widetilde{B\otimes_{\mathcal{C}} D})_{k+l+1} \oplus Q^{k,l} \arrow[dl] \\
T^{k+1,l} \arrow[rr] \arrow[ur, dashed, bend left, "p^{k+1,l}"] & & (B\otimes_{\mathcal{C}} D)_{k+l} \oplus Q^{k,l} &
\end{tikzcd}$$

The bottom side of the entire diagram is given by Diagram 7.3.2, the top side by Diagram 7.3.5 and the front hand side by Diagram 7.4.3. Because $(A \otimes_{\mathcal{C}} C)_{K+L} \oplus Q^{k,l}_{m,n}$ is projective and $\pi^1_2 \oplus \mathrm{id}_{Q^{k,l}}$ an epimorphism, the morphism g_{K+L} on the right hand side is a lift of $\widetilde{g}^L_K : T^{k,l}_{m,n} \to T^{k+l}$. If we take also the morphism g^L_K from the front side, then it sits together with g_{K+L} in the front right hand cube from which we obtain the commuting square

$$\begin{tikzcd}
T^{k,l}_{m,n} \arrow[r] \arrow[d] & (A \otimes_{\mathcal{C}} C)_{K+L} \oplus Q^{k,l}_{m,n} \arrow[d, "\pi^1_2 \oplus \mathrm{id}_{Q^{k,l}}"] \\
(B_k \otimes_{\mathcal{C}} \widetilde{D}_l) \oplus P^{k,l} \arrow[r, "\pi^1_1 \otimes \mathrm{id}_{P^{k,l}}"] & T^{k,l}
\end{tikzcd}$$

$$
\begin{tikzcd}
 & (\widetilde{A}_{K+1}\otimes_{\mathcal{C}}\widetilde{C}_L)\oplus P^{k,l}_{m,n} \arrow[rr] \arrow[dl, "\mathrm{id}"'] \arrow[dd, "\widetilde{g}^L_{K+1}"] & & T^{k+1,l}_{m,n} \arrow[ll, dashed, bend left=10, "p^{k+1,l}_{m,n}"] \arrow[rr] \arrow[dl] \arrow[dd, "G^L_K"] & & (A\otimes_{\mathcal{C}}C)_{K+L}\oplus Q^{k,l}_{m,n} \arrow[dl, "\pi^2_2\oplus\mathrm{id}_{Q^{k,l}_{m,n}}"] \arrow[dd, "g_{K+L}"] \\
(\widetilde{A}_{K+1}\otimes_{\mathcal{C}}\widetilde{C}_L)\oplus P^{k,l}_{m,n} \arrow[rr] \arrow[dd, "\widetilde{g}^L_{K+1}"] & & (A_K\otimes_{\mathcal{C}}\widetilde{C}_L)\oplus P^{k,l}_{m,n} \arrow[rr, "\pi^2_1\oplus\mathrm{id}_{P^{k,l}_{m,n}}"] \arrow[dd, "g^L_K"] & & T^{k,l}_{m,n} \arrow[dd, "\widetilde{g}^L_K"] & \\
 & (\widetilde{B}_{k+1}\otimes_{\mathcal{C}}\widetilde{D}_l)\oplus P^{k,l} \arrow[rr, "i^{k+1,l}"] \arrow[dl, "\mathrm{id}"'] & & T^{k+1,l} \arrow[rr] \arrow[dl] & & (B\otimes_{\mathcal{C}}D)_{k+l}\oplus Q^{k,l} \arrow[dl, "\pi^1_2\oplus\mathrm{id}_{Q^{k,l}}"] \\
(\widetilde{B}_{k+1}\otimes_{\mathcal{C}}\widetilde{D}_l)\oplus P^{k,l} \arrow[rr] & & (B_k\otimes_{\mathcal{C}}\widetilde{D}_l)\oplus P^{k,l} \arrow[rr, "\pi^1_1\oplus\mathrm{id}_{P^{k,l}}"] & & T^{k,l} &
\end{tikzcd}
\tag{7.4.5}
$$

As $T^{k+1,l}$ in the bottom front right hand square is a pullback of $\pi_1^1 \otimes \mathrm{id}_{P^{k,l}}$ and $\pi_2^1 \oplus \mathrm{id}_{Q^{k,l}}$, this yields the morphism $G_K^L : T_{m,n}^{k,l} \to T^{k+1,l}$ rendering the front right hand square commutative. The morphisms in the front left hand cube going to the front right hand cube are kernel from morphisms in the latter cube. Thus, all morphisms in the very left hand square arise by taking kernels where the morphisms in the very back square arise analogously.

Diagram 7.4.3 and 7.4.5 together with the construction of the isomorphism $\xi_{m,n}^{k,l}$ from Proposition 7.3.2 imply that we can extend Diagram 7.4.4 in a commutative manner to

$$\begin{tikzcd}
{[\widetilde{A}_K, \widetilde{B}_k]_{\mathcal{C}} \otimes [\widetilde{C}_L, \widetilde{D}_l]_{\mathcal{C}}} \arrow[r, "\otimes'_{[-,-]_{\mathcal{C}}}"] \arrow[d, "t_{\widetilde{A}_K, \widetilde{B}_k} \otimes \mathrm{id}_{\widetilde{C}_L, \widetilde{D}_l}"] \arrow[rr, bend left, "\otimes_{[-,-]_{\mathcal{C}}}"] & {[\widetilde{A}_K \otimes_{\mathcal{C}} \widetilde{C}_L, \widetilde{B}_k \otimes_{\mathcal{C}} \widetilde{D}_l]_{\mathcal{C}}} \arrow[r, "\xi_{m,n}^{k,l}"] \arrow[d, "t_{\widetilde{A}_K \otimes_{\mathcal{C}} \widetilde{C}_L, \widetilde{B}_k \otimes_{\mathcal{C}} \widetilde{D}_l}"] & {[(\widetilde{A \otimes_{\mathcal{C}} C})_{K+L}, (\widetilde{B \otimes_{\mathcal{C}} D})_{k+l}]_{\mathcal{C}}} \arrow[d, "t_{(\widetilde{A \otimes_{\mathcal{C}} C})_{K+L}, (\widetilde{B \otimes_{\mathcal{C}} D})_{k+l}}"] \\
{[\widetilde{A}_{K+1}, \widetilde{B}_{k+1}]_{\mathcal{C}} \otimes [\widetilde{C}_L, \widetilde{D}_l]_{\mathcal{C}}} \arrow[r, "\otimes'_{[-,-]_{\mathcal{C}}}"] \arrow[rr, bend right, "\otimes_{[-,-]_{\mathcal{C}}}"'] & {[\widetilde{A}_{K+1} \otimes_{\mathcal{C}} \widetilde{C}_L, \widetilde{B}_{k+1} \otimes_{\mathcal{C}} \widetilde{D}_l]_{\mathcal{C}}} \arrow[r, "\xi_{m,n}^{k+1,l}"] & {[(\widetilde{A \otimes_{\mathcal{C}} C})_{K+L+1}, (\widetilde{B \otimes_{\mathcal{C}} D})_{k+l+1}]_{\mathcal{C}}}
\end{tikzcd} \tag{7.4.6}$$

Instead, we could have started with a different version of Diagram 4.2.1 leading one step in constructing $\widehat{\mathrm{Ext}}_{\mathcal{C}}^n(C, D)$ and obtained the following version of Diagram 7.4.3

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{A}_K \otimes_{\mathcal{C}} \widetilde{C}_{L+1} \arrow[r] \arrow[d, "\widetilde{f}_K \otimes_{\mathcal{C}} \widetilde{h}_{L+1}"] & \widetilde{A}_K \otimes_{\mathcal{C}} C_L \arrow[r] \arrow[d, "\widetilde{f}_K \otimes_{\mathcal{C}} h_L"] & \widetilde{A}_K \otimes_{\mathcal{C}} \widetilde{C}_L \arrow[r] \arrow[d, "\widetilde{f}_K \otimes_{\mathcal{C}} \widetilde{h}_L"] & 0 \\
0 \arrow[r] & \widetilde{B}_k \otimes_{\mathcal{C}} \widetilde{D}_{l+1} \arrow[r] & \widetilde{B}_k \otimes_{\mathcal{C}} D_l \arrow[r] & \widetilde{B}_k \otimes_{\mathcal{C}} \widetilde{D}_l \arrow[r] & 0
\end{tikzcd}$$

We conclude analogously that there is also a commutative diagram of the form

$$\begin{tikzcd}
{[\widetilde{A}_K, \widetilde{B}_k]_{\mathcal{C}} \otimes [\widetilde{C}_L, \widetilde{D}_l]_{\mathcal{C}}} \arrow[r, "\otimes'_{[-,-]_{\mathcal{C}}}"] \arrow[d, "\mathrm{id}_{\widetilde{A}_K, \widetilde{B}_k} \otimes t_{\widetilde{C}_L, \widetilde{D}_l}"] \arrow[rr, bend left, "\otimes_{[-,-]_{\mathcal{C}}}"] & {[\widetilde{A}_K \otimes_{\mathcal{C}} \widetilde{C}_L, \widetilde{B}_k \otimes_{\mathcal{C}} \widetilde{D}_l]_{\mathcal{C}}} \arrow[r, "\xi_{m,n}^{k,l}"] \arrow[d, "t_{\widetilde{A}_K \otimes_{\mathcal{C}} \widetilde{C}_L, \widetilde{B}_k \otimes_{\mathcal{C}} \widetilde{D}_l}"] & {[(\widetilde{A \otimes_{\mathcal{C}} C})_{K+L}, (\widetilde{B \otimes_{\mathcal{C}} D})_{k+l}]_{\mathcal{C}}} \arrow[d, "t_{\widetilde{A \otimes_{\mathcal{C}} C})_{K+L}, (\widetilde{B \otimes_{\mathcal{C}} D})_{k+l}}"] \\
{[\widetilde{A}_K, \widetilde{B}_k]_{\mathcal{C}} \otimes [\widetilde{C}_{L+1}, \widetilde{D}_{l+1}]_{\mathcal{C}}} \arrow[r, "\otimes'_{[-,-]_{\mathcal{C}}}"] \arrow[rr, bend right, "\otimes_{[-,-]_{\mathcal{C}}}"'] & {[\widetilde{A}_K \otimes_{\mathcal{C}} \widetilde{C}_{L+1}, \widetilde{B}_k \otimes_{\mathcal{C}} \widetilde{D}_{l+1}]_{\mathcal{C}}} \arrow[r, "\xi_{m,n}^{k,l+1}"] & {[(\widetilde{A \otimes_{\mathcal{C}} C})_{K+L+1}, (\widetilde{B \otimes_{\mathcal{C}} D})_{k+l+1}]_{\mathcal{C}}}
\end{tikzcd} \tag{7.4.7}$$

Diagram 7.4.6 and 7.4.7 give rise to a direct system of commuting squares

$$\begin{array}{ccc} [\widetilde{A}_K, \widetilde{B}_k]_{\mathcal{C}} \otimes [\widetilde{C}_L, \widetilde{D}_l]_{\mathcal{C}} & \xrightarrow{\otimes_{[-,-]_{\mathcal{C}}}} & [(\widetilde{A \otimes_{\mathcal{C}} C})_{K+L}, (\widetilde{B \otimes_{\mathcal{C}} D})_{k+l}]_{\mathcal{C}} \\ \Big\downarrow t_{\widetilde{A}_K, \widetilde{B}_k} \otimes t_{\widetilde{C}_L, \widetilde{D}_l} & & \Big\downarrow t_{*,*} \circ t_{*,*} \\ [\widetilde{A}_{K+1}, \widetilde{B}_{k+1}]_{\mathcal{C}} \otimes [\widetilde{C}_{L+1}, \widetilde{D}_{l+1}]_{\mathcal{C}} & \xrightarrow{\otimes_{[-,-]_{\mathcal{C}}}} & [(\widetilde{A \otimes_{\mathcal{C}} C})_{K+L+2}, (\widetilde{B \otimes_{\mathcal{C}} D})_{k+l+2}]_{\mathcal{C}} \end{array}$$

whose direct limit is the desired external product

$$\vee : \widehat{\operatorname{Ext}}^m_{\mathcal{C}}(A, B) \otimes \widehat{\operatorname{Ext}}^n_{\mathcal{C}}(C, D) \to \widehat{\operatorname{Ext}}^{m+n}_{\mathcal{C}}(A \otimes_{\mathcal{C}} C, B \otimes_{\mathcal{C}} D)$$

by Proposition 7.3.3. □

Example 7.4.3 *All conditions of Theorem 7.4.1 are satisfied in the following two instances.*

1. *Let $\mathcal{C}$ be the category of discrete $R[G]$-modules for a discrete group G and a principal ideal domain R. The restriction of A, B, C, D to R-modules needs to be projective.*
2. *Let $\mathcal{C}$ be the category of profinite $S[[H]]$-modules for a profinite group H and profinite commutative ring S with a unique maximal open ideal. Then the restriction of A, B, C, D to R-modules needs to be projective.*

Remark 7.4.4 *The p-adic integers $\mathbb{Z}_p$ are an example of a profinite commutative ring with the unique maximal ideal $p\mathbb{Z}_p$ [14, p. 27]. The restriction of any p-torsion-free profinite $\mathbb{Z}_p[[G]]$-module to a $\mathbb{Z}_p$-ideal is projective [15, Corollary 2.1.2].*

Proof Since Yoneda products exist, we only need to verify the conditions required for the existence of external products. There is a tensor product $\otimes_R$ for discrete $R[G]$-modules and one $\widehat{\otimes}_S$ for profinite $S[[H]]$-modules satisfying the conditions of Proposition 7.3.2 [11, Section 5.5 and 5.8]. Hence denote by E any of A, B, C or D. By [2, p. 10–11] and [14, Lemma 9.8.2], there is a resolution $E_\bullet$ of E such that E_k is a free $R[G]$-module (resp. $S[[H]]$-module) for every $k \in \mathbb{N}_0$. Then E_k is free as an R-module (resp. S-module) where the result for discrete modules follows by construction and for profinite ones by [11, Corollary 5.7.2]. In particular, E_k is a projective R-module where one invokes [11, Proposition 5.4.2] for the profinite case. Because E is projective as an R-module (resp. S-module), the short exact sequence $0 \to \widetilde{E}_1 \to E_0 \to E \to 0$ is split and hence $\widetilde{E}_1$ projective as an R-module (resp. S-module). Inductively, we conclude for every $k \in \mathbb{N}_0$ that $\widetilde{E}_k$ is projective as an R-module (resp. S-module). According to [2, p. 29] and [11, Proposition 5.5.3], $\widetilde{E}_k$ is flat and according to [46, Theorem 9.8] and [14, Proposition 7.5.1], it is also free as an R-module. It then follows from the proof of [15, Proposition 3.3.2] that tensoring $\widetilde{E}_k$ with a projective $R[G]$-module (resp. $S[[H]]$-module) gives rise to another projective module. $\square$

7.5 Canonical morphisms and Tate-Farrell Ext-functors

In this section we establish two properties of completed unenriched Ext-functors $\widehat{\mathrm{Ext}}^{\bullet}_{\mathcal{C}}(A,-)$. On the one hand, a Mislin completion of a cohomological functor $T^\bullet$ comes with a canonical morphism $\Phi^\bullet : T^\bullet \to \widehat{T}^\bullet$ of cohomological functors according to Definition 2.0.4. For $T^\bullet = \mathrm{Ext}^{\bullet}_{\mathcal{C}}(A,-)$, we prove that the terms Φ^n fit into a long exact sequence relating three distinct cohomological functors. On the other hand, we prove that one can compute $\widehat{\mathrm{Ext}}^{\bullet}_{\mathcal{C}}(A,-)$ using complete resolutions whenever such resolutions exist. In order to prove the former result, we generalize a remark from [3, p. 210] to

Proposition 7.5.1 *For unenriched Ext-functors* $\mathrm{Ext}^{\bullet}_{\mathcal{C}}(A,-) : \mathcal{C} \to \mathrm{Ab}$ *the quotient map* $\mathrm{Hyp}_{\mathcal{C}}(A_\bullet, B_\bullet)_\bullet \to \mathrm{Vog}_{\mathcal{C}}(A_\bullet, B_\bullet)_\bullet$ *of chain complexes from the hypercohomology construction induces the canonical morphism of cohomological functors* $\Phi^\bullet : \mathrm{Ext}^{\bullet}_{\mathcal{C}}(A,B) \to \widehat{\mathrm{Ext}}^{\bullet}_{\mathcal{C}}(A,B)$ *from the definition of a Mislin completion.*

Proof Denote by $\Phi^\bullet : \mathrm{Ext}^{\bullet}_{\mathcal{C}}(A,-) \to \widehat{\mathrm{Ext}}^{\bullet}_{\mathcal{C}}(A,-)$ the canonical morphism of cohomological functors established through the satellite functor construction in Lemma 3.0.2. If $\omega_\bullet : \widehat{\mathrm{Ext}}^{\bullet}_{\mathcal{C}}(A,-) \to \mathrm{Ext}^{\bullet}_{Res\mathcal{C}}(A,-)$ is the isomorphisms of cohomological functors to the resolution construction from Theorem 4.3.2, then $\omega_\bullet \circ \Phi^\bullet : \mathrm{Ext}^{\bullet}_{\mathcal{C}}(A,-) \to \mathrm{Ext}^{\bullet}_{Res,\mathcal{C}}(A,-)$ is also a canonical morphism to the Mislin completion. By Diagram 4.1.6 in the proof of Lemma 4.1.1, each term $\omega_n \circ \Phi^n : \mathrm{Ext}^{n}_{\mathcal{C}}(A,B) \to \mathrm{Ext}^{n}_{Res,\mathcal{C}}(A,B)$ is a homomorphism to the direct limit occurring in the resolution construction as in Definition 2.0.9. If $\mathcal{E}xt^{\bullet}_{\mathcal{C}}(A,-)$ and $\mathcal{E}xt^{\bullet}_{Res,\mathcal{C}}(A,-)$ are taken as in Definition 6.3.1, denote by $\Psi^\bullet : \mathcal{E}xt^{\bullet}_{\mathcal{C}}(A,-) \to \mathcal{E}xt^{\bullet}_{Res,\mathcal{C}}(A,-)$ an analogous morphism to the direct limit. Taking the isomorphisms of cohomological functors $\zeta_\bullet$ and $\zeta^\bullet$ also from Definition 6.3.1, we see that the diagram

$$\begin{CD}
\mathrm{Ext}^{\bullet}_{\mathcal{C}}(A,-) @>{\zeta_{\bullet}}>> \mathcal{E}xt^{\bullet}_{\mathcal{C}}(A,-)\\
@V{\omega_{\bullet}\circ\Phi^{\bullet}}VV @VV{\Psi^{\bullet}}V\\
\mathrm{Ext}^{Res,\bullet}_{\mathcal{C}}(A,-) @>{\zeta^{\bullet}}>> \mathcal{E}xt^{\bullet}_{Res,\mathcal{C}}(A,-)
\end{CD}$$

commutes. Since $\mathcal{E}xt^{\bullet}_{\mathcal{C}}(A,-)$ is a different description of the Ext-functors $\mathrm{Ext}^{\bullet}_{\mathcal{C}}(A,-)$ according to Notation 6.1.1, $\Psi^{\bullet}$ also represents a canonical morphism to the Mislin completion. If $\vartheta^0_{\bullet}$ denote homomorphisms from the proof of Lemma 6.3.3 and $\vartheta^{\bullet}$ the isomorphism of a cohomological functors from Theorem 6.4.2, then the diagram

$$\begin{tikzcd}
\mathcal{E}xt^{\bullet}_{\mathcal{C}}(A,-) \arrow[d, "\Psi^{\bullet}"] \arrow[dr, "\vartheta^0_{\bullet}"] & \\
\mathcal{E}xt^{\bullet}_{Res,\mathcal{C}}(A,-) \arrow[r, "\vartheta^{\bullet}"] & \widehat{\mathcal{E}xt}^{\bullet}_{\mathcal{C}}(A,-)
\end{tikzcd}$$

is commutative. As before, we infer that $\vartheta^0_{\bullet}$ is a canonical morphism to the Mislin completion $\widehat{\mathcal{E}xt}^{\bullet}_{\mathcal{C}}(A,-)$. Note that we can restrict the quotient map of chain complexes

$$(\mathrm{Hyp}_{\mathcal{C}}(A_{\bullet},B_{\bullet})_n, d^n)_{n\in\mathbb{Z}} \to (\mathrm{Vog}_{\mathcal{C}}(A_{\bullet},B_{\bullet})_n, \overline{d}^n)_{n\in\mathbb{Z}}$$

to a homomorphism $\mathrm{Ker}(d^n) \to \mathrm{Ker}(\overline{d}^n)$ for any $n \in \mathbb{Z}$. Again by Notation 6.1.1, the latter is equivalent to

$$\mathrm{Hom}_{\mathrm{Ch}(\mathcal{C})}(A[n]_{\bullet},B_{\bullet}) \to \widehat{\mathrm{Hom}}_{\mathrm{Ch}(\mathcal{C})}(A[n]_{\bullet},B_{\bullet})\,.$$

By definition, this further descends to $\vartheta^0_n : \mathcal{E}xt^n_{\mathcal{C}}(A,B) \to \widehat{\mathcal{E}xt}^n_{\mathcal{C}}(A,B)$ as desired. □

From this proposition we deduce

Lemma 7.5.2 *The short exact sequence of chain complexes*

$$0 \to \mathrm{Tot}_{\mathcal{C}}(A_{\bullet},B_{\bullet})_{\bullet} \to \mathrm{Hyp}_{\mathcal{C}}(A_{\bullet},B_{\bullet})_{\bullet} \to \mathrm{Vog}_{\mathcal{C}}(A_{\bullet},B_{\bullet})_{\bullet} \to 0$$

from the hypercohomology construction induces the long exact sequence

$$\begin{aligned}\ldots \to H^n(\mathrm{Tot}_{\mathcal{C}}(A_{\bullet},B_{\bullet})_{\bullet}) \to \mathrm{Ext}^n_{\mathcal{C}}(A,B) &\xrightarrow{\Phi^n} \widehat{\mathrm{Ext}}^n_{\mathcal{C}}(A,B)\\ &\to H^{n+1}(\mathrm{Tot}_{\mathcal{C}}(A_{\bullet},B_{\bullet})_{\bullet}) \to \ldots\end{aligned}$$

where $\Phi^\bullet : \operatorname{Ext}^\bullet_{\mathcal{C}}(A,B) \to \widehat{\operatorname{Ext}}^\bullet_{\mathcal{C}}(A,B)$ *is the canonical morphism from the definition of a Mislin completion. In particular, the terms* Φ^n *fit into a long exact sequence relating three distinct cohomological functors.*

Remark 7.5.3 *This lemma is similar to Proposition 4.6 in S. Guo and L. Liang's paper [28]. Both results contain the same long exact sequence where our contribution lies in determining that it involves the terms of the canonical morphism to the Mislin completion.*

The following generalisation of Lemma 2.3 in [7] provides a criterion determining when a cohomology group T^n of a cohomological functor agrees the corresponding cohomology group $\widehat{T}^n$ of its Mislin completion

Proposition 7.5.4 *Let* $T^\bullet : \mathcal{C} \to \mathcal{D}$ *be a cohomological functor. Then for* $n \in \mathbb{Z}$ *the following are equivalent.*

1. *For any* $k \geq n$ *the functor* T^k *vanishes on projective objects.*
2. *For any* $k \geq n$ *the functor* T^k *is naturally isomorphic to* $\widehat{T}^k$.

Proof As the second assertion implies the first, assume that T^k vanishes on projective objects for any $k \geq n$. Let $T^\bullet \to U^\bullet$ be a morphism of cohomological functors where $U^\bullet$ vanishes on projective objects. Then we have seen in the proof of Lemma 3.0.2 that this morphism factors uniquely as $T^\bullet \to T^\bullet\langle n\rangle \to U^\bullet$. Because $T^\bullet\langle n\rangle$ vanishes on projectives by our assumptions, we infer that it is a Mislin completion. In particular, we observe for any $k \geq n$ that $T^k = T^k\langle n\rangle \cong \widetilde{T}^k$. □

Let us present a criterion for when a cohomological functor vanishing on projectives is a Mislin completion. The following is a generalisation of Lemma 2.4 in [7].

Proposition 7.5.5 *Let* $T^\bullet, V^\bullet : \mathcal{C} \to \mathcal{D}$ *be cohomological functors where* $V^\bullet$ *vanishes on projective objects. Assume that* $\Phi^\bullet : T^\bullet \to V^\bullet$ *is a morphism such that there is* $n \in \mathbb{Z}$ *with the property that* Φ^k *is an isomorphism for any* $k \geq n$*. Then* $V^\bullet$ *together with* Φ *forms a Mislin completion of* $T^\bullet$.

Proof Note that T^k vanishes on projectives for any $k \geq n$. It follows from the proof of Proposition 7.5.4 that there is a unique factorization

$$\Phi^\bullet : T^\bullet \to T^\bullet\langle n\rangle \xrightarrow{\Psi^\bullet} V^\bullet$$

where $T^\bullet\langle n\rangle$ is the Mislin completion of $T^\bullet$. According to the proof of Lemma 3.0.2, $\Psi^k = \Phi^k$ is an isomorphism for any $k \geq n$. Therefore, $\Psi^\bullet$ is an isomorphism of cohomological functors by Lemma 3.0.1 and $V^\bullet$ is a Mislin completion. □

Moving forward, a complete resolution of $A \in \mathrm{obj}(\mathcal{C})$ is an acyclic chain complex $(\overline{A}_n)_{n\in\mathbb{Z}}$ of projectives such that there is $n \in \mathbb{N}_0$ for which $(\overline{A}_k)_{k\geq n}$ agrees with a projective resolution of A and such that for any projective $P \in \mathrm{obj}(\mathcal{C})$ the cochain complex $\mathrm{Hom}_{\mathcal{C}}(\overline{A}_\bullet, P)$ is acyclic where $\mathrm{Hom}_{\mathcal{C}}(-,-)$ denotes the unenriched Hom-functor [19, Definition 1.1]. In accordance with [18, p. 158], we define $\overline{\mathrm{Ext}}^\bullet_{\mathcal{C}}(A,B) := H^\bullet(\mathrm{Hom}_{\mathcal{C}}(\overline{A}_\bullet, B))$. By [19, Lemma 2.4], any two complete resolutions are chain homotopic and thus, Tate-Farrell Ext-functors do not depend on the choice of a complete resolution. In [28, Proposition 4.15], S. Guo and L. Liang demonstrate that the Tate-Farrell Ext-functor $\overline{\mathrm{Ext}}^n_{\mathcal{C}}(A,-)$ is naturally isomorphic to $\widehat{\mathrm{Ext}}^n_{\mathcal{C}}(A,-)$ for every $n \in \mathbb{Z}$. Then we obtain the following version of Theorem 4.6 from [30]

Lemma 7.5.6 *The Tate-Farrell Ext-functors* $\overline{\mathrm{Ext}}^\bullet_{\mathcal{C}}(A,-)$ *are isomorphic to the completed unenriched Ext-functors* $\widehat{\mathrm{Ext}}^\bullet_{\mathcal{C}}(A,-)$ *as cohomological functors.*

Apart from the proof found in [30], one can demonstrate this by using the proof of Theorem 1.2 in [19] together with Proposition 7.5.5.

Example 7.5.7 *P. Symonds constructs in [20, p. 34] Tate-Farrell cohomology for a profinite group with an open subgroup of finite cohomological dimension taking coefficients in profinite modules. Thus, Complete cohomology generalizes his Tate-Farrell cohomology.*

7.6 Complete cohomology in condensed mathematics

As we implement complete cohomology in condensed mathematics, we provide hereby more detail on the latter. First, we define condensed sets following D. Clausen and P. Scholze's account [6, Lecture I and II]. After giving a definition that is sheaf-theoretic in nature, we provide a more down-to-earth definition. A finite collection of (continuous) maps $\{f_i : S_i \to S\}_{i=1}^n$ in the category Pro of profinite spaces is called a *covering* if the induced map

$\bigsqcup_{i=1}^n S_i \to S$ is surjective. Although this turns Pro into a *Grothendieck site*, it is not well suited for sheaf theoretic purposes as it is not essentially small. A cardinal number κ is called a *strong limit cardinal* if for every $\lambda < \kappa$ we also have $2^\lambda < \kappa$. Then we define the site of *κ-small profinite spaces* Pro_κ for an uncountable strong limit cardinal κ as the full subcategory of Pro consisting of all profinite spaces with less than κ clopen subsets and restrict the Grothendieck topology to it. Note that the Grothendieck site Pro_κ is essentially small. Then a *κ-condensed set* is a sheaf of sets on Pro_κ. In more down-to-earth terms, a (contravariant) functor

$$X : \mathrm{Pro}_\kappa^{\mathrm{op}} \to \mathrm{Set}$$

is a κ-condensed set if the following two conditions are satisfied.

1. For every finite collection $\{S_i\}_{i=1}^n$ of objects in the category Pro_κ there is a bijection $X(\bigsqcup_{i=1}^n S_i) \to \prod_{i=1}^n X(S_i)$.
2. Let $f : T \to S$ be a (continuous) surjection in Pro_κ and denote by $p_1, p_2 : T \times_S T \to T$ the projections from the fibre product. Then the function $X(f) : X(S) \to X(T)$ is injective with image

$$\mathrm{Im}(X(f)) = \{x \in X(T) \mid X(p_1)(x) = X(p_2)(x) \in X(T \times_S T)\}\,.$$

One can think of X as encoding a topological space and of $X(S)$ as the continuous maps "$S \to X$" from a κ-profinite space. If $\kappa < \kappa'$ is another strong limit cardinal, then there is a fully faithful functor

$$\mathrm{Cond}_\kappa(\mathrm{Set}) \to \mathrm{Cond}_{\kappa'}(\mathrm{Set})$$

due to our choice of the cardinal numbers. This means that one can extend any κ-condensed set to a κ'-condensed set. If K denotes the class of uncountable strong limit cardinals, then the *category of condensed sets* is defined as the colimit category

$$\mathrm{Cond}(\mathrm{Set}) := \varinjlim_{\kappa \in K} \mathrm{Cond}_\kappa(\mathrm{Set})\,.$$

Hence, a condensed set is an equivalence class of κ-condensed sets. According to [4, p. 15/16 and Proposition 1.7] we can establish the following relation with topological spaces. If $T1$-Top denotes the category of $T1$ topological spaces, then there is a well-defined faithful functor $T1$-Top $\to$ Cond(Set) mapping any Y to its so-called condensate

$$\underline{Y} : \mathrm{Pro}_\kappa^{\mathrm{op}} \to \mathrm{Set},\ S \mapsto \{f : S \to Y \text{ continuous}\}$$

where κ is chosen to be a sufficiently large strong limit cardinal. This functor is fully faithful when restricted to $T1$ compactly generated topological spaces. It follows from the proofs of Proposition 2.9 and Proposition 2.15 in [4] that the functor does not depend on κ. Namely, for every strong limit cardinal $\kappa < \kappa'$ and κ'-profinite space $S = \varprojlim_{i \in I} S_i$ written as a κ-inverse limit of κ-profinite spaces S_i, the function $\varinjlim_{i \in I} \underline{Y}(S_i) \to \underline{Y}(S)$ is a bijection. Although Cond(Set) does not form the category of sheaves over any site [4, Remark 2.12], it nevertheless inherits good sheaf-theoretic properties from the categories $\mathrm{Cond}_\kappa(\mathrm{Set})$ [6, p. 13].

Condensed groups (resp. rings, modules etc.) are defined analogously [4, p. 7] and condensates of $T1$ topological groups are condensed groups (resp. rings, modules etc) [4, p. 8]. In the category of condensed abelian groups Cond(Ab) all limits and colimits exist where arbitrary products, arbitrary direct sums and direct limits are exact [4, Theorem 1.10]. According to [4, p. 12], Cond(Ab) has enough projectives. Following the source, these assertions also hold for the category of condensed $\mathcal{R}$-modules $\mathrm{Cond}(Mod(\mathcal{R}))$ where $\mathcal{R}$ is a condensed ring. If S is a projective object in Pro, then one can form the *free condensed $\mathcal{R}$-module* $\mathcal{R}[\underline{S}]$ as the sheafification of the presheaf of $\mathcal{R}$-modules

$$\mathrm{Pro}_\kappa^{\mathrm{op}} \to \mathrm{Ab}, T \mapsto \mathcal{R}(T)[\underline{S}(T)]$$

where $\mathcal{R}(T)[\underline{S}(T)]$ denotes the free $\mathcal{R}(T)$-module over the set $\underline{S}(T)$ and κ is a sufficiently large strong limit cardinal [4, p. 12]. Since the Grothendieck site Pro_κ is essentially small, sheafification exists and again by the proof of [4, Proposition 2.9], the construction of $\mathcal{R}[\underline{S}]$ is independent of κ. Moreover, the condensed $\mathcal{R}$-modules $\mathcal{R}[\underline{S}]$ are compact projective generators, meaning that for every condensed $\mathcal{R}$-module A there is a collection of projective profinite spaces S_i such that there is an epimorphism $\bigoplus_{i \in I} \mathcal{R}[\underline{S_i}] \to A$ [4, p. 12]. For condensed $\mathcal{R}$-modules A, B one does not only have the "usual" Hom-set $\mathrm{Hom}_{\mathrm{Cond}(Mod(\mathcal{R}))}(A, B) \in \mathrm{obj}(\mathrm{Ab})$, but also an internal Hom-set which is constructed in [4, p. 13]. Namely, for any $C \in \mathrm{obj}(\mathrm{Cond}(Mod(\mathcal{R})))$ one can form the tensor product $C \otimes_{\mathcal{R}} A \in \mathrm{Cond}(Mod(\mathcal{R}))$ as the sheafification of

$$\mathrm{Pro}_\kappa^{\mathrm{op}} \to \mathrm{Ab}, T \mapsto C(T) \otimes_{\mathcal{R}(T)} A(T)\,.$$

Then define the internal Hom-set $\underline{\mathrm{Hom}}_{\mathrm{Cond}(Mod(\mathcal{R}))}(A, B) \in \mathrm{obj}(\mathrm{Cond}(\mathrm{Ab}))$ by

$$\mathrm{Pro}_\kappa^{\mathrm{op}} \to \mathrm{Ab}, T \mapsto \mathrm{Hom}_{\mathrm{Cond}(Mod(\mathcal{R}))}(A \otimes_{\mathcal{R}} \mathcal{R}[\underline{T}], B)\,.$$

By definition, it satisfies the adjunction

$$\mathrm{Hom}_{\mathrm{Cond}(Mod(\mathcal{R}))}(C, \underline{\mathrm{Hom}}_{\mathrm{Cond}(Mod(\mathcal{R}))}(A, B)) \cong \mathrm{Hom}_{\mathrm{Cond}(Mod(\mathcal{R}))}(C \otimes_{\mathcal{R}} A, B)\,.$$

Because the category $\mathrm{Cond}(Mod(\mathcal{R}))$ has enough projectives, we can define the unenriched Ext-functors

$$\mathrm{Ext}^{\bullet}_{\mathrm{Cond}(Mod(\mathcal{R}))}(A,-) : \mathrm{Cond}(Mod(\mathcal{R})) \to \mathrm{Ab}$$

and the internal Ext-functors

$$\underline{\mathrm{Ext}}^{\bullet}_{\mathrm{Cond}(Mod(\mathcal{R}))}(A,-) : \mathrm{Cond}(Mod(\mathcal{R})) \to \mathrm{Cond}(\mathrm{Ab})$$

to be the derived functors of

$$\mathrm{Hom}_{\mathrm{Cond}(Mod(\mathcal{R}))}(A,-) \text{ and } \underline{\mathrm{Hom}}_{\mathrm{Cond}(Mod(\mathcal{R}))}(A,-)\,.$$

This can be used to define cohomology of a condensed group $\mathcal{G}$. Following [26, p. 2–3] one can form the condensed group ring $\mathcal{R}[\mathcal{G}]$ such that the category $\mathrm{Cond}(Mod(\mathcal{R}[\mathcal{G}]))$ is equivalent to the subcategory of $\mathrm{Cond}(\mathcal{R})$ of all $\mathcal{R}$-modules M with a $\mathcal{G}$-action $\mathcal{G}\times M \to M$. By [26, p. 5/8], one can define condensed unenriched group cohomology $H^{\bullet}_{\mathcal{R}}(\mathcal{G},-)$ and the condensed internal group cohomology $\underline{H}^{\bullet}_{\mathcal{R}}(\mathcal{G},-)$ as the Ext-functors $\mathrm{Ext}^{\bullet}_{\mathrm{Cond}(Mod(\mathcal{R}[\mathcal{G}]))}(\mathcal{R},-)$ and $\underline{\mathrm{Ext}}^{\bullet}_{\mathrm{Cond}(Mod(\mathcal{R}[\mathcal{G}]))}(\mathcal{R},-)$. By the above, we conclude

Theorem 7.6.1 *For $\mathcal{R}$ a condensed ring and A a condensed $\mathcal{R}$-module there are completed condensed unenriched Ext-functors $\widehat{\mathrm{Ext}}^{\bullet}_{\mathrm{Cond}(Mod(\mathcal{R}))}(A,-)$ and completed condensed internal Ext-functors $\widehat{\underline{\mathrm{Ext}}}^{\bullet}_{\mathrm{Cond}(Mod(\mathcal{R}))}(A,-)$. If $\mathcal{G}$ is a condensed group, then complete condensed unenriched group cohomology $\widehat{H}^{\bullet}_{\mathcal{R}}(\mathcal{G},-)$ and complete condensed internal group cohomology $\widehat{\underline{H}}^{\bullet}_{\mathcal{R}}(\mathcal{G},-)$ can be defined.*

Remark 7.6.2 *Because both $\widehat{\underline{\mathrm{Ext}}}^{\bullet}_{\mathrm{Cond}(Mod(\mathcal{R}))}(A,-)$ and $\widehat{\underline{H}}^{\bullet}_{\mathcal{R}}(\mathcal{G},-)$ have as codomain category* $\mathrm{Cond}(\mathrm{Ab})$*, they are an example of completed Ext-functors that are not covered under the previous frameworks such as the one by A. Beligiannis and I. Reiten in [29], the one by S. Guo and L. Liang's in [28] and the one by J. Hu et al. in [30].*

Having discussed condensed modules, let us consider solid modules. We have already mentioned in Chapter 1 that one can think of these as being a form of completed condensed modules where a condensed ring $\mathcal{A}$ is called analytic if it admits a choice of solid modules. More formally, let ProjPro denote the full subcategory of Pro consisting of projective objects. Then an analytic ring $\mathcal{A}$ comes with a particular functor

$$c_{\mathcal{A}} : \mathrm{ProjPro} \to \mathrm{Cond}(Mod(\mathcal{A})),\ S \mapsto \mathcal{A}[S]_{\blacksquare}\,.$$

and a condensed $\mathcal{A}$-module homomorphism $C_{\mathcal{A}}(S) : \mathcal{A}[\underline{S}] \to \mathcal{A}[\underline{S}]_{\blacksquare}$ for every projective profinite space S [4, p. 44]. One can think of $c_{\mathcal{A}}$ as assigning to every projective profinite space S a canonical choice of free solid $\mathcal{A}$-module $\mathcal{A}[S]_{\blacksquare}$. According to [4, Proposition 7.5], analytic rings have the following properties. The full subcategory Solid($\mathcal{A}$) of Cond($Mod(\mathcal{A})$) with objects M such that

$$\mathrm{Hom}_{\mathrm{Cond}(Mod(\mathcal{A}))}(C_{\mathcal{A}}(S), M) : \mathrm{Hom}_{\mathrm{Cond}(Mod(\mathcal{A}))}(\mathcal{A}[S]_{\blacksquare}, M) \to M(S)$$

is an isomorphism for all $S \in \mathrm{ProjPro}$ is an abelian category stable under limits, colimits and extensions. The objects $\mathcal{A}[S]_{\blacksquare}$ are compact projective generators in Solid($\mathcal{A}$). Moreover, there is an extension of the morphisms $C_{\mathcal{A}}(S) : \mathcal{A}[\underline{S}] \to \mathcal{A}[\underline{S}]_{\blacksquare}$ to a functor $\mathrm{Cond}(Mod(\mathcal{A})) \to \mathrm{Solid}(\mathcal{A})$ left adjoint to the inclusion of categories that we call the *solidification functor*. Lastly, if $\mathcal{A}$ is commutative, then there exists a unique symmetric monoidal tensor product on Solid($\mathcal{A}$) rendering the solidification functor symmetric monoidal. In particular, since the category Solid($\mathcal{A}$) has enough projectives, we can define the Ext functors

$$\mathrm{Ext}^{\bullet}_{\mathrm{Solid}(\mathcal{A})}(M, -) : \mathrm{Solid}(\mathcal{A}) \to \mathrm{Ab}$$

as the derived functors of $\mathrm{Hom}_{\mathrm{Solid}(\mathcal{A})}(M, -)$.

Lemma 7.6.3 *Let $\mathcal{A}$ be an analytic ring and M a solid $\mathcal{A}$-module. Then there are completed solid Ext-functors $\widehat{\mathrm{Ext}}^{\bullet}_{\mathrm{Solid}(\mathcal{A})}(M, -)$.*

Example 7.6.4 ([26, p. 2]) *Let $S = \varprojlim_{i \in I}$ be a profinite space and define the condensed $\underline{\mathbb{Z}}$-module $\mathbb{Z}[S]^{\blacksquare} := \varprojlim_{i \in I} \mathbb{Z}[\underline{S_i}]$. Then the condensed ring $\underline{\mathbb{Z}}$ together with the assignment $S \mapsto \mathbb{Z}[S]^{\blacksquare}$ for every projective profinite space S forms an analytic ring. By extension, we denote the solidification functor by $(-)^{\blacksquare} : \mathrm{Cond}(\mathrm{Ab}) \to \mathrm{Solid}(\underline{\mathbb{Z}})$. If Λ denotes one of the profinite rings $\mathbb{Z}_p$, $\mathbb{Z}/p\mathbb{Z}$ or $\mathbb{Z}/p^n\mathbb{Z}$, then also $(\underline{\Lambda}, S \mapsto (\underline{\Lambda}[\underline{S}])^{\blacksquare})$ is an analytic ring.*

Lemma 7.6.5 *Let G be a profinite group and $(L, S \mapsto L[S]^{\blacksquare})$ an analytic ring where $(-)^{\blacksquare}$ denotes the solidification functor of the analytic ring $\underline{\mathbb{Z}}$ from Example 7.6.4. Then the group ring $(L[\underline{G}], S \mapsto L[\underline{G}][S]^{\blacksquare})$ is also an analytic ring and one can thus define complete solid cohomology*

$$\widehat{H}^{\bullet}_{\mathrm{Solid}(L)}(\underline{G}, -) : \mathrm{Solid}(L[\underline{G}]) \to \mathrm{Ab}$$

and complete solid internal cohomology

$$\underline{\widehat{H}}^{\bullet}_{\mathrm{Solid}(L)}(\underline{G}, -) : \mathrm{Solid}(L[\underline{G}]) \to \mathrm{Solid}(\underline{\mathbb{Z}})\,.$$

Proof It follows from Corollary 5.5 and Corollary 6.1 of [4] that $L[S]^{\blacksquare}$ is a projective solid L-module for every profinite (but not necessarily projective) space S. Then, by [26, Lemma 1.3], the group ring $(L[\underline{G}], S \mapsto L[\underline{G}][S]^{\blacksquare})$ is analytic. It follows from [26, p. 3–4] that the category of solid $L[\underline{G}]$ is equivalent to the category of solid L-modules together with an action $\underline{G} \times M \to M$ in the category of solid L-modules. In particular, there is an unenriched Hom-functor $\mathrm{Hom}_{\mathrm{Solid}(L[\underline{G}])}(L, -)$ from which we can form the solid group cohomology $H^{\bullet}_{\mathrm{Solid}(L)}(\underline{G}, -)$ [26, p. 5] and thus the corresponding complete solid cohomology. On the other hand, the internal Hom-functor of condensed $L[\underline{G}]$-modules restricted onto the full subcategory $\mathrm{Solid}(L[\underline{G}])$

$$\mathrm{Solid}(L[\underline{G}]) \to \mathrm{Cond}(\mathrm{Ab}), M \to \underline{\mathrm{Hom}}_{\mathrm{Cond}(L[\underline{G}])}(L, M)$$

maps into solid $\underline{\mathbb{Z}}$-modules according to [26, p. 8]. Therefore, we can also form the complete solid internal cohomology. $\square$

Chapter 8

Conclusion

8.1 Contribution

In Chapter 3, 4, 5 and 6 we have generalised constructions of cohomological functors by G. Mislin [7], by D. J. Benson and J. F. Carlson [13] and by P. Vogel [12] so that they give rise to Mislin completions. Not only do we prove that all constructions yield isomorphic cohomological functors, but also provide explicit formulae for connecting homomorphisms and induced morphism for each construction. By doing so, we deduce that there are countably many distinct constructions of Mislin completions. To our knowledge, this has not been done in the literature before. In particular, since our generalisation takes Tate cohomology to condensed mathematics, one can construct Tate cohomology for any $T1$ topological group.

In Chapter 7, we establish some properties of these general Mislin completions. By recovering a version of Theorem 4.11 from [28], we deduce that completed unenriched Ext-functors detect finite projective dimension of objects in the domain category. We prove a partial version of dimension shifting and under specific conditions an Eckmann-Shapiro result. We construct Yoneda products for completed unenriched Ext-functors in full generality and external products if certain conditions are met. In addition, we demonstrate that the terms of the canonical morphism of unenriched Ext-functors to their Mislin completion fit in a long exact sequence relating three different cohomological functors and that completed unenriched Ext-functors generalise Tate-Farrell Ext-functors. For the above properties of complete cohomology we provide examples. Due to the author's background these are discrete groups or profinite groups.

8.2 Limitations

Although S. Guo and L. Liang have also generalised the above constructions by G. Mislin, D. J. Benson, J. F. Carlson and P. Vogel for completed unenriched Ext-functors, we do not know whether their resulting cohomological functor forms a Mislin completion of unenriched Ext-functors (Question 6.4.8). This is due to the fact that they construct connecting homomorphisms for completed unenriched Ext-functors differently than we do.

We also do not know to which extent one can further generalise the properties of completed Ext-functors and more generally of Mislin completions presented in Chapter 7. Although we do not think that dimension shifting holds in full generality, we wonder how one can establish further criteria under which it does hold. We are curious under what other conditions one can obtain an Eckmann-Shapiro result and external products for completed unenriched Ext-functors.

In order to generalise the above properties, it would be beneficial to know more examples of completed unenriched Ext-functors and more generally of Mislin completions. We would like to know whether complete cohomology of a profinite groups agrees with the complete solid cohomology of its condensate (Question 1.0.12). As there exist already computations of complete cohomology of discrete groups, we wonder whether some of them can be taken over to profinite groups (Question 1.0.7 and Question 1.0.8).

Bibliography

[1] A. Adem and R. J. Milgram. *Cohomology of Finite Groups*, volume 309 of *Grundlehren der mathematischen Wissenschaften / A Series of Comprehensive Studies in Mathematics*. Springer-Verlag Berlin Heidelberg New York, 2nd edition, 2004.

[2] K. S. Brown. *Cohomology of Groups*, volume 87 of *Graduate Texts in Mathematics*. Springer-Verlag New York, Inc., 1982.

[3] P. H. Kropholler. Hierarchical Decompositions, Generalized Tate Cohomology, and Groups of Type $(\mathrm{FP})_\infty$. In *Combinatorial and Geometric Group Theory, Edinburgh 1993*, volume 204 of *London Mathematical Society Lecture Note Series*, pages 190–216. Cambridge University Press, January 1995.

[4] P. Scholze. Lectures on Condensed Mathematics. `https://www.math.uni-bonn.de/people/scholze/Condensed.pdf`, May 2019. This website has been consulted on 21 February 2023.

[5] P. Scholze. Liquid Tensor Experiment. `https://xenaproject.wordpress.com/2020/12/05/liquid-tensor-experiment/`, December 2020. This website has been consulted on 25 September 2023.

[6] D. Clausen and P. Scholze. Condensed Mathematics and Complex Geometry. `https://people.mpim-bonn.mpg.de/scholze/Complex.pdf`, April 2022. This website has been consulted on 8 April 2023.

[7] G. Mislin. Tate Cohomology for Arbitrary Groups via Satellites. *Topology and its Applications*, 56:293–300, 1994.

[8] S. Mac Lane. *Homology*. Classics in Mathematics. Springer-Verlag Berlin Heidelberg, 1995.

[9] The Stacks project authors. The Stacks Project. `https://stacks.math.columbia.edu`, 2024.

[10] C. A. Weibel. *An Introduction to Homological Algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 1994.

[11] L. Ribes and P. Zalesskii. *Profinite Groups*, volume 40 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics*. Springer-Verlag Berlin Heidelberg, 2nd edition, 2010.

[12] F. Goichot. Homologie de Tate-Vogel Équivariante. *Journal of Pure and Applied Algebra*, 82(1):39–64, 1992.

[13] D. J. Benson and J. F. Carlson. Products in Negative Cohomology. *Journal of Pure and Applied Algebra*, 82(2):107–129, 1992.

[14] J. S. Wilson. *Profinite Groups*, volume 19 of *London Mathematical Society Monographs New Series*. Oxford University Press Inc., New York, 1998.

[15] P. Symonds and T. Weigel. Cohomology of p-adic Analytic Groups. In *New Horizons in Pro-p Groups*, volume 184 of *Progress in Mathematics*, pages 347–408. Birkhäuser, Boston, 2000.

[16] G. C. Cook. On Profinite Groups of Type FP_∞. *Advances in Mathematics*, 294:216–255, 2016.

[17] J.-P. Serre. *Cohomologie Galoisienne*, volume 5 of *Lecture Notes in Mathematics*. Springer-Verlag Berlin Heidelberg, 5th edition, 1994.

[18] F. T. Farrell. An Extension of Tate Cohomology to a Class of Infinite Groups. *Journal of Pure and Applied Algebra*, 10(2):153–161, 1977.

[19] J. Cornick and P. H. Kropholler. On Complete Resolutions. *Topology and its Applications*, 78(3):235—250, 1997.

[20] P. Symonds. Permutation Complexes for Profinite Groups. *Commentarii Mathematici Helvetici*, 82(1):1–37, 2007.

[21] P. Symonds. The Tate-Farrell Cohomology of the Morava Stabilizer Group s_{p-1} with Coefficients in e_{p-1}. In *Homotopy Theory: Relations with Algebraic Geometry, Group Cohomology, and Algebraic K-Theory*, volume 346 of *Contemporary Mathematics*, pages 485–402. American Mathematical Society, 2004.

[22] F. Dembegioti. On the Zeroeth Complete Cohomology. *Journal of Pure and Applied Algebra*, 203(1–3):119–132, 2005.

[23] J. H. Jo. Complete Homology and Related Dimensions of Groups. *Journal of Group Theory*, 12(3):431–448, 2009.

[24] F. Dembegioti. On the Zeroeth Complete Cohomology of Certain Polycyclic Groups. *Communications in Algebra*, 36(5):1927–1941, 2008.

[25] L. Ribes. *Profinite Graphs and Groups*, volume 66 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics*. Springer Nature, 2017.

[26] J. Anschütz and A.-C. Le Bras. Solid Group Cohomology. `https://www.math.uni-bonn.de/people/ja/homology_profinite.pdf`, September 2020. This website has been consulted on 15 July 2023.

[27] M. Land. Condensed Mathematics. `https://www.markus-land.de/teaching/`, December 2022. This website has been consulted on 22 September 2023.

[28] S. Guo and L. Liang. Stable Functors and Cohomology Theory in Abelian Categories. *arXiv e-prints*, page arXiv:2006.13562v3, October 2023.

[29] A. Beligiannis and I. Reiten. *Homological and Homotopical Aspects of Torsion Theories*, volume 188 of *Memoirs of the American Mathematical Society*. American Mathematical Society, 2007.

[30] J. Hu, D. Zhang, T. Zhao, and P. Zhou. Complete Cohomology for Extriangulated Categories. *Algebra Colloquium*, 28(4):701–720, 2021.

[31] M. Paganin. On Some Generalizations of Tate Cohomology: an Overview. *Pro Mathematica*, 29(58):63–115, 2016.

[32] J. Asadollahi and S. Salarian. Cohomology Theories for Complexes. *Journal of Pure and Applied Algebra*, 210(3):771–787, 2007.

[33] J. R. Klein. Axioms for Generalized Farrell–Tate Cohomology. *Journal of Pure and Applied Algebra*, 172(2–3):225–238, 2002.

[34] B. E. A. Nucinkis. Complete Cohomology for Arbitrary Rings Uusing Injectives. *Journal of Pure and Applied Algebra*, 131(3):297–318, 1998.

[35] J. Hu, D. Zhang, T. Zhao, and P. Zhou. Balance of Complete Cohomology in Extriangulated Categories. *Electronic Research Archive*, 29(5):3341–3359, 2021.

[36] B. Poonen. Why All Rings should have a 1. `https://math.mit.edu/~poonen/papers/ring.pdf`, June 2018. This website has been consulted on 25 October 2023.

[37] H. Cartan and S. Eilenberg. *Homological Algebra*, volume 19 of *Princeton Mathematical Series*. Princeton University Press, 1956.

[38] O. Celikbas, L. W. Christensen, L. Liang, and G. Piepmeyer. Complete Homology over Associative Rings. *Israel Journal of Mathematics*, 221(1):1–24, 2017.

[39] A. Grothendieck. Sur Quelques Points d'Algèbre Homologique. *Tohoku Mathematical Journal*, 9(2):119–221, 1957.

[40] M. S. Osborne. *Basic Homological Algebra*, volume 196 of *Graduate Texts in Mathematics*. Springer-Verlag New York, Inc., 2000.

[41] S. I. Gelfand and Yu. I. Manin. *Methods of Homological Algebra*. Springer Monographs in Mathematics. Springer-Verlag Berlin Heidelberg, 2nd edition, 2003.

[42] I. Emmanouil and P. Manousaki. On the Stable Homology of Modules. *Journal of Pure and Applied Algebra*, 221(9):2198–2219, 2017.

[43] M. Boggi and G. C. Cook. Continuous Cohomology and Homology of Profinite Groups. *Documenta Mathematica*, 21:1269–1312, 2016.

[44] C. Faith. *Algebra: Rings, Modules and Categories I*, volume 190 of *Grundlehren der mathematischen Wissenschaften / A Series of Comprehensive Studies in Mathematics*. Springer-Verlag Berlin Heidelberg New York, 1973.

[45] B. Mitchell. *Theory of Categories*. Academic Press, 1965.

[46] J. J. Rotman. *Advanced Modern Algebra*. Prentice Hall, 2002.