The Schur-Positivity of Generalized Nets

by

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Abstract

A graph is Schur-positive if its chromatic symmetric function expands nonnegatively in the Schur basis. All claw-free graphs are conjectured to be Schur-positive. We introduce a combinatorial object corresponding to a graph G, called a special rim hook G-tabloid, which is a variation on the special rim hook tabloid. These objects can be employed to compute any Schur coefficient of the chromatic symmetric function of a graph. Special rim hook tabloids have previously been used to prove the non-Schur-positivity of some graphs. We construct sign-reversing maps on these special rim hook G-tabloids to prove that a family of claw-free graphs called generalized nets are Schur-positive. Thus, we demonstrate a new method for proving the Schur-positivity of graphs, which has the potential to be applied to make further progress toward the aforementioned conjecture.

Lay Summary

In the mathematical field of graph theory, graphs are structures composed of vertices (dots) and edges (lines) connecting these vertices. We can associate a mathematical function with a graph, known as the chromatic symmetric function, which encodes information about how we can color the vertices of the graph in a particular way.

One way to represent this chromatic symmetric function is by expressing it in terms of other well-known functions called Schur functions. If we can write this representation using only positive numbers, the graph is "Schur-positive."

A conjecture from the 1990s suggests that graphs lacking a specific subgraph configuration called the "claw" should all be Schur-positive. In this thesis, we provide a mathematical proof that a family of graphs named "generalized nets" satisfy this conjecture. In other words, we show that these generalized nets possess the Schur-positivity property, furthering our understanding of how certain graphs can be characterized mathematically.

Preface

This thesis is original, unpublished, independent work by the author, E. Shelburne.

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List of Symbols

C_n	cycle graph on n vertices
$\mathcal{C}$	proper coloring of a graph
χ_G	chromatic polynomial of a graph G
$f(C, D)$	$[s_{(2^C,1^D)}]X_{GN_{C+D,C}}$ or 0 if the coefficient is not properly defined
$GN_{n,m}$	generalized net with n body vertices and m pendants
$hd(T)$	head of an SRH G-tabloid T
$inc(P)$	incomparability graph of a poset P
K_n	complete graph on n vertices
κ_T	content of an SRH tabloid T
$\ell(\kappa)$, $\lvert\kappa\rvert$	length, size of a composition κ (respectively)
$\ell(\lambda)$, $\lvert\lambda\rvert$	length, size of a partition λ (respectively)
$\lambda \setminus 1^t$	partition λ with t parts equal to 1 removed
$\Lambda(\kappa)$	partition obtained from composition κ
P_n	path graph on n vertices
$\mathbb{P}$	set of positive integers
$\mathbb{Q}$	set of rational numbers
$\mathbb{Q}[[\mathbf{x}]]$	algebra of formal power series over $\mathbb{Q}$ in commuting variables $\mathbf{x} = \{x_1, x_2, x_3, \ldots\}$
s_λ	Schur function associated to partition λ

$[s_\lambda]f(\mathbf{x})$	Schur coefficient of $f(\mathbf{x})$ corresponding to partition λ
$\mathrm{sgn}(T)$	sign of an SRH tabloid T
$\mathrm{Sym}(\mathbf{x})$	algebra of symmetric functions over $\mathbb{Q}$
$\mathrm{tl}(T)$	tail of an SRH G-tabloid T
$\mathcal{T}_\lambda$	set of SRH tabloids of shape equal to partition λ
$\mathcal{T}_{\lambda,G}$	set of SRH G-tabloids of shape equal to partition λ
$\mathcal{T}^S_{\lambda,G}$	set of SRH G-tabloids of shape equal to partition λ such that some $\nu \in S$ is in the left-most position of the bottom row
$\mathcal{T}^\nu_{\lambda,G}$	set of SRH G-tabloids of shape equal to partition λ such that ν is in the left-most position of the bottom row
$V(G)$	set of vertices of a graph G
$\mathrm{wt}(Q)$	weight of a semistandard tableaux Q
X_G	chromatic symmetric function of a graph G
$\xi(\lambda, G)$	$[s_\lambda]X_G$ or 0 if the coefficient is not properly defined

Acknowledgements

I am very grateful to my advisor, Steph van Willigenburg, for her continued support, mentorship, and mathematical insight. This project would not have been possible without her.

I would also like to acknowledge that this research has taken place on University of British Columbia's Vancouver campus, which is located on the traditional, ancestral, unceded, and stolen territory of the Musqueam people.

Chapter 1

Introduction

The chromatic polynomial is a function associated to a graph which takes an integer k as input and outputs the number of colorings of G in k colors that are "proper," that is, satisfy the property that no two adjacent vertices are colored identically. This combinatorial function was introduced in 1912 by George Birkhoff in [3]. The chromatic polynomial became the object of many different graph theoretical studies, questions, and conjectures.

In 1995, Richard Stanley introduced the chromatic symmetric function in [18]. The chromatic symmetric function extends the idea of counting proper colorings of graphs to the algebra of symmetric functions. Symmetric functions are formal power series in countably many variables such that the variables can be permuted in any way without changing the function.

The introduction of chromatic symmetric functions propelled a cascade of research, much of which is centered around the following two properties that certain graphs exhibit.

1. A graph is called e-positive if its chromatic symmetric function can be written with only nonnegative coefficients in the classical elementary basis of the symmetric functions.

2. A graph is called Schur-positive if its chromatic symmetric function can be written with only nonnegative coefficients in the classical Schur basis of the symmetric functions.

It is an important fact that elementary symmetric functions expand nonnegatively in the Schur basis. Therefore, if a graph is e-positive, it is also Schur-positive.

Symmetric functions which expand nonnegatively in the e-basis or Schur basis often exhibit special representation theoretical or algebraic properties, such as intricate combinatorial interpretations of their coefficients. For example, all symmetric functions which expand nonnegatively in the Schur basis arise as the Frobenius image of some representation of the symmetric group, as discussed in [16]. A natural goal stated in [18] is to classify exactly which graphs are Schur-positive or e-positive. In particular, the following three major conjectures emerged from Stanley's studies of chromatic symmetric functions.

1. (The Stanley-Stembridge Conjecture, [20]): All claw-free incomparability graphs are e-positive.

2. (The Nonisomorphic Tree Conjecture, [18]): No two nonisomorphic trees have the same chromatic symmetric function.

3. (The Claw-free Conjecture, [19]): All claw-free graphs are Schur-positive.

Many partial results concerning these conjectures have emerged in the literature. Subfamilies of graphs have been shown to satisfy the Stanley-Stembridge Conjecture in [10], [4], and [2], among others. The Nonisomorphic Tree Conjecture has been computationally confirmed on trees of up to 29 vertices in [11]. Other results toward this conjecture can be found in [14], [15], and [8], for example.

Vesselin Gasharov made significant progress toward the Claw-free Conjecture by proving that all claw-free incomparability graphs are Schur-positive in [9]. Moreover, it was shown in [12] that all coefficients of chromatic symmetric functions corresponding to partitions of "hook" shapes are nonnegative. Most recently, David Wang and Monica Wang introduced a combinatorial formula which yields the Schur coefficients of chromatic symmetric functions, and used it to prove that certain graphs are not

Schur-positive in [21]. This formula is in terms of signed combinatorial objects known as special rim hook tabloids.

Due to Gasharov's result, the task of proving the Claw-free Conjecture has been reduced to showing the Schur-positivity of claw-free graphs which are not incomparability graphs. In this thesis, we focus on a particular family of graphs satisfying these properties known as generalized nets. Generalized nets consist of complete graphs with degree one vertices appended. We prove that all such graphs are Schur-positive, thus making progress toward the Claw-free Conjecture.

In order to achieve this result, we introduce a version of special rim hook tabloids which correspond to graphs. We then reinterpret the formula from [21] in terms of these objects, and construct sign-reversing maps on them to prove the Schur coefficients of the chromatic symmetric functions corresponding to generalized nets are nonnegative.

Accordingly, we develop a new method that can potentially be applied to other families of graphs to make more progress toward the Claw-free Conjecture.

The structure of this thesis is as follows. Chapter 2 provides some essential background on symmetric functions, chromatic symmetric functions, special rim hook tabloids, and the Claw-Free Conjecture. Chapter 3 introduces special rim hook G-tabloids and elucidates their role in computing Schur coefficients for chromatic symmetric functions. Chapter 4 presents a proof of the Schur-positivity of generalized nets. Finally, Chapter 5 is the conclusion, in which we summarize the significance of our results and outline avenues for future research in this area.

Chapter 2

Background

2.1 Preliminaries

We begin by fixing the notation for the sets of numbers over which we work. We set

1. $\mathbb{P}$ = the set of positive integers,

2. and $\mathbb{Q}$ = the set of rational numbers.

Next, we define our basic graph theoretical notation. All graphs we consider are undirected. Likewise, they are all simple, that is, they have a finite number of vertices and they do not have more than one edge between two vertices nor any edge from a vertex to itself.

We write $V(G)$ to denote the set of all vertices in a graph G. We say two vertices are *adjacent* if they are connected by an edge. The *degree* of a vertex v is the number of vertices adjacent to v. We also need the following definition, which describes a special type of subgraph.

2.1.1 Definition. An *induced subgraph* of a graph G is a subgraph H with vertices $U \subseteq V(G)$ such that

$$\epsilon \text{ is an edge between } u, v \in U \text{ in } H \iff \epsilon \text{ is an edge between } u, v \in U \text{ in } G.$$

We now establish notation for three famous families of graphs which we often reference or use as examples.

2.1.2 Definition. Let n be a positive integer.

1. The *path* P_n is the graph on n vertices $\{v_1, \ldots, v_n\}$ such that v_i is adjacent to v_{i+1} for $1 \leq i \leq n-1$ and no other edges are included.

2. The *cycle* C_n on $n \geq 3$ vertices is the graph consisting of a path P_n with an edge added between the two degree 1 vertices. We set $C_1 = P_1$ and $C_2 = P_2$.

3. The *complete graph* K_n is the graph on n vertices such that every pair of vertices is adjacent.

2.1.3 Example. In Figure 2.1, we depict the path P_4, the cycle C_4, and the complete graph K_4, from left to right respectively.

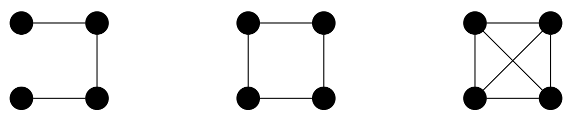


Figure 2.1: Examples of a path, a cycle, and a complete graph.

We note that K_3 appears as an induced subgraph of K_4 four distinct times. Likewise, P_3 appears as an induced subgraph of both P_4 and C_4 multiple times. On the other hand, P_4 does not appear as an induced subgraph of C_4, since an induced subgraph must include all edges between its vertices set which are included the original graph.

Next, we define a combinatorial object which is fundamental to the content of this thesis.

2.1.4 Definition. A *partition* of $n \geq 0$ is a sequence of weakly decreasing positive integers

$$\lambda = (\lambda_1, \ldots, \lambda_k) \qquad \text{where} \qquad \sum_{i=1}^{k} \lambda_i = n.$$

The integers λ_i are called the *parts* of λ. The *length* of λ is given by $\ell(\lambda) = k$ and the *size* of λ is given by $|\lambda| = n$. In the case where $n = 0$ and the sequence is empty, we write $\lambda = \emptyset$ and say λ is the *empty partition*.

We sometimes use exponents to denote repeated integers in a partition. For example,

$$\lambda = (3, 3, 2, 2, 2, 1) = (3^2, 2^3, 1).$$

2.1.5 Definition. The *Young diagram* or *diagram* of a partition λ of n is an array of n boxes (called *cells*) in left-justified rows such that row i contains λ_i boxes, where the rows are indexed from top to bottom and the columns are indexed from left to right.

We use English notation for Young diagrams, which means the rows of the diagrams weakly decrease in length from top to bottom. Some authors prefer French notation, in which rows weakly increase in length from top to bottom. Sometimes we refer to a partition λ and its Young diagram interchangeably.

2.1.6 Example. Figure 2.2 depicts the Young diagram of the partition $(5, 4, 3, 3, 2)$.

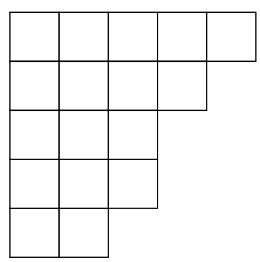


Figure 2.2: An example of a Young diagram.

The top-left cell has index $(1, 1)$, the top-right cell has index $(1, 5)$, and the bottom-left and bottom-right cells have indices $(5, 1)$ and $(5, 2)$ respectively.

Next, we introduce the ordered version of a partition because, in truth, a partition is unordered, and we write it in weakly decreasing order for convenience.

2.1.7 Definition. A *composition* of $n \geq 0$ is a sequence of positive integers

$$\kappa = [\kappa_1, \ldots, \kappa_j] \qquad \text{where} \qquad \sum_{i=1}^{j} \kappa_i = n.$$

The integers κ_i are called the *parts* of κ. The *length* of κ is given by $\ell(\kappa) = j$ and the *size* of κ is given by $|\kappa| = n$. In the case where $n = 0$ and the sequence is empty, we write $\kappa = \emptyset$ and say κ is the *empty composition*.

Given a composition κ, we use $\Lambda(\kappa)$ to denote the partition obtained by arranging the parts of κ in weakly decreasing order.

Again, we sometimes use exponents to denote repeated integers in a composition. For example,

$$\kappa = [1, 1, 3, 2, 4, 4, 4, 1] = [1^2, 3, 2, 4^3, 1].$$

In this case, we also have

$$\Lambda(\kappa) = (4^3, 3, 2, 1^3).$$

Next, we introduce the notion of a partial order on a set.

2.1.8 Definition. A *partially ordered set* or *poset* is a set P equipped with a binary relation $\leq$, called a *partial order*, satisfying

i. (*reflexivity*) $x \leq x$,

ii. (*antisymmetry*) if $x \leq y$ and $y \leq x$, then $x = y$, and

iii. (*transitivity*) if $x \leq y$ and $y \leq z$, then $x \leq z$

for all $x, y, z \in P$. We say that $x, y \in P$ are *comparable* if $x \leq y$ or $y \leq x$, and say they are *incomparable* otherwise. A *total order* is a partial order where any two elements are comparable. We write $x < y$ if $x \leq y$ and $x \neq y$.

We also use a special type of diagram to visualize posets.

2.1.9 Definition. Let P be a poset. For $x, y \in P$, we say x *covers* y and write $y \lessdot x$ if $y < x$ and there is no $z \in P$ such that $y < z < x$.

The *Hasse diagram* of P is the graph with vertices P and an edge from y up to x if $y \lessdot x$.

Accordingly, we use Hasse diagrams to visualize a poset by arranging the elements vertically based on the relation $\leq$.

2.1.10 Example. Consider the poset $P = \{a, b, c, d, e, f\}$ with relation $\leq$ satisfying

$$a \lessdot b \lessdot f, \qquad a \lessdot c \lessdot e, \qquad d \lessdot c, \qquad \text{and} \qquad b \lessdot e.$$

The Hasse diagram of P is depicted in Figure 2.3.

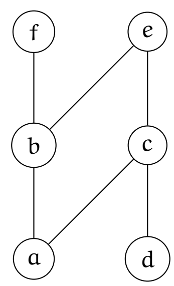


Figure 2.3: An example of a Hasse diagram of a poset P.

We also associate a special type of graph to all posets.

2.1.11 Definition. For some poset P, the *incomparability graph* of P is the graph $\mathrm{inc}(P)$ on vertices corresponding to the elements of P such that

$$x \text{ is adjacent to } y \iff x \text{ and } y \text{ are incomparable}$$

for all $x, y \in P$.

2.1.12 Example. The incomparability graph $\mathrm{inc}(P)$ of the poset P in Figure 2.3 is depicted in Figure 2.4.

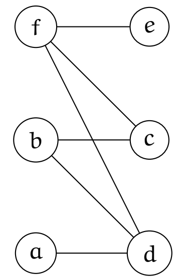


Figure 2.4: An example of an incomparability graph.

It is important to note that not every graph is an incomparability graph of some poset. For example, in Chapter 4, we discuss a family of graphs called generalized nets which are not incomparability graphs in almost all cases.

To finish this section, we define one more important graph that will feature prominently throughout the upcoming chapters.

2.1.13 Definition. The *claw* K_{13}, as depicted in Figure 2.5, is the graph consisting of P_3 with one vertex appended to the degree 2 vertex.

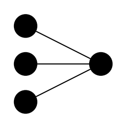


Figure 2.5: The claw.

2.2 Introduction to Symmetric Functions

In this section, we introduce the algebra of symmetric functions, along with its classical Schur basis.

2.2.1 Notation. We start by establishing some notation.

- We let $\mathbf{x} = \{x_1, x_2, x_3, \dots\}$ be a countably infinite set of commuting variables.
- We use $\mathbb{Q}[[\mathbf{x}]]$ to denote the *algebra of formal power series* in variables $\mathbf{x}$ over the rational numbers.

- We say a monomial $m = x_{i_1}^{d_1} x_{i_2}^{d_2} \cdots x_{i_q}^{d_q}$ has *degree*

$$\deg(m) = \sum_{j=1}^{q} d_j.$$

- We say that $f(\mathbf{x})$ is *homogeneous of degree* n if $\deg(m) = n$ for all monomials m appearing in $f(\mathbf{x})$.

2.2.2 Example. The power series

$$\begin{aligned} f(\mathbf{x}) &= x_1x_2x_3x_4 + x_1x_3x_4x_5 + \cdots + x_2x_3x_4x_5 + x_2x_4x_5x_6 + \cdots \\ &= \sum_{1 \le \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4} x_{\alpha_1} x_{\alpha_2} x_{\alpha_3} x_{\alpha_4} \end{aligned} \tag{2.2.3}$$

belongs to $\mathbb{Q}[[\mathbf{x}]]$ and is homogeneous of degree 4.

We have that any permutation π on the positive integers acts on $\mathbb{Q}[[\mathbf{x}]]$ by permuting the variables in $\mathbf{x}$. Explicitly, for some permutation π,

$$\pi \cdot f(x_1, x_2, x_3, \dots) = f(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}, \dots).$$

2.2.4 Example. We have

$$(1,2) \cdot \sum_{i=1}^{\infty} x_1 x_2^i x_3 = \sum_{i=1}^{\infty} x_1^i x_2 x_3. \tag{2.2.5}$$

2.2.6 Definition. We say $f(\mathbf{x})$ is *symmetric* if

$$\pi \cdot f(\mathbf{x}) = f(\mathbf{x})$$

for all permutations π on the positive integers.

We note that, because all permutations decompose into products of transpositions, a necessary and sufficient condition for $\mathbf{x}$ to be symmetric is that

$$(i, j) \cdot f(\mathbf{x}) = f(\mathbf{x})$$

for all transpositions (i, j). Accordingly, the power series in Equation 2.2.5 is not symmetric.

On the other hand, the power series $f(\mathbf{x})$ in Equation 2.2.3 is symmetric. Suppose we apply any transposition (i, j) to the variables of some monomial

$$m = x_{\alpha_1} x_{\alpha_2} x_{\alpha_3} x_{\alpha_4}$$

in the sum. The first case is that $i, j \notin \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, in which case m is unchanged by (i, j). If $i, j \in \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, permuting these two variables also does not change m. Without loss of generality, the last case is that $\alpha_1 = i$ and $j \notin \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. In this case,

$$(i, j) \cdot x_i x_{\alpha_2} x_{\alpha_3} x_{\alpha_4} = x_j x_{\alpha_2} x_{\alpha_3} x_{\alpha_4} \qquad \text{and} \qquad (i, j) \cdot x_j x_{\alpha_2} x_{\alpha_3} x_{\alpha_4} = x_i x_{\alpha_2} x_{\alpha_3} x_{\alpha_4}$$

so these two monomials in the sum are mapped to each other by (i, j). We conclude $f(\mathbf{x})$ is symmetric in this case.

We now let

$$\mathrm{Sym}_n(\mathbf{x}) = \{f \in \mathbb{Q}[[\mathbf{x}]] \mid f \text{ is symmetric and homogeneous of degree } n\}$$

and

$$\mathrm{Sym}(\mathbf{x}) = \bigoplus_{n \geq 0} \mathrm{Sym}_n(\mathbf{x}).$$

We refer to $\mathrm{Sym}(\mathbf{x})$ as *the algebra of symmetric functions*.

For the rest of this thesis, we primarily work over $\mathrm{Sym}(\mathbf{x})$. In [16, Section 4.9], it is verified that $\mathrm{Sym}(\mathbf{x})$ is indeed an algebra.

Since $\mathrm{Sym}(\mathbf{x})$ is an algebra, we can consider various bases of $\mathrm{Sym}(\mathbf{x})$, which usually span over all partitions λ. In this thesis, we focus on the classical Schur basis. In order to define this basis, we need a new combinatorial object.

2.2.7 Definition. A *semistandard Young tableau* (SSYT) of *shape* λ is a filling Q of the cells of the Young diagram λ with elements of $\mathbb{P}$ such that rows weakly increase from left to right and columns strictly increase from top to bottom.

We also associate a monomial to every semistandard Young tableau.

2.2.8 Definition. Given a semistandard Young tableau Q, we define the *weight* of Q to be

$$\mathrm{wt}(Q) = x_1^{\#1\mathrm{s}} x_2^{\#2\mathrm{s}} x_3^{\#3\mathrm{s}} \cdots .$$

Thus, the weight of a semistandard Young tableau Q records the number of times each positive integer appears in Q.

2.2.9 Example. In Figure 2.6, we portray several examples of SSYTs of shape (4,2,1).

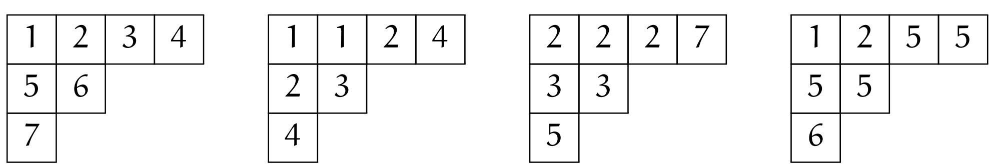


Figure 2.6: Examples of SSYTs of shape $(4, 2, 1)$.

From left to right, these SSYTs have weights

$$x_1x_2x_3x_4x_5x_6x_7, \qquad x_1^2x_2^2x_3x_4^2, \qquad x_2^3x_3^2x_5x_7, \qquad \text{and} \qquad x_1x_2x_5^4x_6. \tag{2.2.10}$$

We note that there are countably many SSYTs of shape λ for any nonempty partition λ. We now use the SSYT object to define a classical basis for the algebra of symmetric functions.

2.2.11 Definition. Given some partition λ, the *Schur function* associated to λ is

$$s_\lambda = \sum_Q \mathrm{wt}(Q)$$

where the sum spans over all semistandard Young tableaux Q of shape λ.

2.2.12 Example. For example, all the monomials in Equation 2.2.10 will appear as terms in the sum which is the Schur function $s_{(4,2,1)}$. Countably many other monomials will also appear, such as

$$x_{10}x_{20}x_{30}x_{40}x_{50}x_{60}x_{70}, \qquad x_2^2x_3^2x_4x_5^2, \qquad x_2^3x_5^2x_{10}x_{11}, \qquad \text{and} \qquad x_1x_2x_4^4x_6.$$

The following theorem shows the importance of Schur functions to the study of the algebra of symmetric functions.

2.2.13 Theorem. *All Schur functions are symmetric and*

$$\mathbf{s} = \{s_\lambda \mid \lambda \textit{ is a partition}\}$$

is a basis for Sym$(\mathbf{x})$.

Neither the fact that Schur functions are symmetric nor the fact that they form a basis is obvious from the definition. Proofs of both these statements can be found in [17, Section 7.2].

Throughout this thesis, we will investigate special types of symmetric functions which have expansions in the Schur basis with nonnegative coefficients.

2.2.14 Notation. We will use the notation

$$[s_\lambda]f(\mathbf{x}) = \text{the coefficient of } s_\lambda \text{ in the expansion of } f(\mathbf{x}) \text{ in } \mathbf{s}.$$

We will also sometimes refer to these coefficients as *Schur coefficients*.

2.2.15 Definition. We say a symmetric function $f(\mathbf{x}) \in \text{Sym}(\mathbf{x})$ is *Schur-positive* if

$$[s_\lambda]f(\mathbf{x}) \geq 0$$

for all partitions λ.

Some of the reasons for studying Schur-positivity, as well as some of the major results in this area, are discussed in [13]. Various methods for proving a symmetric function is Schur-positive are discussed in [1]. Chapters 4 is devoted to proving the Schur-positivity of a certain family of graphs called generalized nets.

2.3 Introduction to Chromatic Symmetric Functions

Building on the theory developed in Section 2.2, we now turn our focus to symmetric functions associated to graphs, known as chromatic symmetric functions.

We start by formally defining the notion of coloring the vertices of a graph such that an edge never connects two vertices of the same color.

2.3.1 Definition. Given a graph G with vertices $V(G)$, a *proper coloring* of G in q colors is a map

$$\mathcal{C} : V(G) \to \{1, 2, 3, \ldots, q\}$$

such that, if u and v are adjacent, then

$$\mathcal{C}(u) \neq \mathcal{C}(v).$$

We visualize a proper coloring by assigning each integer a distinct color then coloring all the vertices of our graph.

2.3.2 Example. In Figure 2.7, on the lefthand side, we depict a proper coloring of a certain graph. On the righthand side, we depict a non-proper coloring of the same graph, since some adjacent vertices are colored identically.

Figure 2.7: Examples of proper and non-proper colorings.

Building on this concept, we now associate to each graph a function that outputs the number of proper colorings in q colors for some nonnegative integer q.

2.3.3 Definition. The *chromatic polynomial* of a graph G is the function on nonnegative integers

$$\chi_G(q) = \#\text{ of proper colorings of } G \text{ in } q \text{ colors.}$$

2.3.4 Example. Figure 2.8 depicts the cycle graph C_3, which requires three colors for a proper coloring.

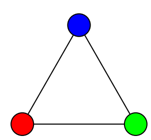


Figure 2.8: A proper coloring of the cycle graph C_3 requires 3 distinct colors.

If we have q colors to choose from, there are q choices for the first color, $q-1$ choices for the second color, and $q-2$ choices for the third color. Hence, the chromatic polynomial of C_3 is

$$\chi_{C_3}(q) = q(q-1)(q-2).$$

From the definition alone, it is not obvious that the chromatic polynomial is, in fact, a polynomial. A proof of this fact may be found in many texts, such as [17, Section 3.8].

The focus of this thesis will be a generalization of the chromatic polynomial to the algebra of symmetric functions. To work towards this definition, we first associate a monomial to every proper coloring of a graph G.

2.3.5 Definition. For some graph G on vertices $v_1, v_2, \dots, v_n$ and a proper coloring $\mathcal{C}$ of G, we define

$$\mathbf{x}^{\mathcal{C}} = x_{\mathcal{C}(v_1)} x_{\mathcal{C}(v_2)} \cdots x_{\mathcal{C}(v_n)}.$$

We can now define the special type of symmetric function which is most important to the content of this thesis.

2.3.6 Definition. The *chromatic symmetric function* (CSF) of a graph G is the formal power series

$$X_G(\mathbf{x}) = \sum_{\mathcal{C}} \mathbf{x}^{\mathcal{C}},$$

where the sum ranges over all proper colorings $\mathcal{C}$ of G.

The chromatic symmetric function is symmetric since permuting the variables of the function amounts to permuting the colors in each proper coloring. Such a permutation of colors always results in another proper coloring, which will already correspond to a monomial in $X_G(\mathbf{x})$.

Moreover, the chromatic symmetric function is a generalization of the chromatic polynomial in the sense that

$$X_G(1^q, 0, \dots) = \chi_G(q)$$

where the lefthand side denotes $X_G(\mathbf{x})$ with the first q variables set as 1 and all other variables set as 0. This fact, along with the fact that all CSFs are symmetric, is formally verified in [17, Section 7.8], for example.

It is also the case that any chromatic symmetric function $X_G(\mathbf{x})$ will be homogeneous of degree n, where n is the number of vertices of the graph G. This holds since the degree of each monomial $\mathbf{x}^{\mathcal{C}}$ is the sum of the number of times each color is used in $\mathcal{C}$. This sum is always equal to the number of vertices of G.

2.3.7 Example. As seen in Figure 2.8, any proper coloring of C_3 requires 3 distinct colors. Accordingly, we have

$$\begin{aligned} X_{C_3}(\mathbf{x}) &= x_1x_2x_3 + x_1x_3x_4 + \cdots + x_2x_3x_4 + x_2x_4x_5 + \cdots \\ &= \sum_{1 \leq \alpha_1 < \alpha_2 < \alpha_3} x_{\alpha_1}x_{\alpha_2}x_{\alpha_3} \qquad (2.3.8) \\ &= s_{(1^3)}, \end{aligned}$$

since the SSYTs of shape (1^3) are exactly those which are filled with three strictly increasing integers.

Another important fact is that chromatic symmetric functions behave well with respect to disjoint unions of graphs. This is proved in [18].

2.3.9 Proposition. *For graphs* G *and* H, *we have*

$$X_{G \cup H} = X_G \cdot X_H$$

where $G \cup H$ *denotes the disjoint union of* G *and* H.

Since CSFs belong to $\mathrm{Sym}(\mathbf{x})$, a natural question is which graphs have CSFs which expand in the Schur basis nonnegatively.

2.3.10 Definition. We say that a graph G is *Schur-positive* if $X_G(\mathbf{x})$ is Schur-positive. For brevity, we sometimes refer to the Schur coefficients of $X_G(\mathbf{x})$ as the *Schur coefficients of* G.

2.3.11 Example. By Equation 2.3.8, we have that C_3 is Schur-positive.

By Proposition 2.3.9, we have that the disjoint union of two Schur-positive graphs is Schur-positive. A famous conjecture regarding the Schur-positivity of graphs is discussed in Section 2.5.

2.4 Special Rim Hook Tabloids

In order to discuss an important result from [21], we now introduce a combinatorial object known as a special rim hook tabloid. We start with some basic definitions regarding subsets of vertices of graphs.

2.4.1 Definition. Let λ be a partition and G be a graph.

- If S is a subset of $V(G)$, we say S is a *stable set* if all pairs of vertices in S are nonadjacent.
- We say G has a *stable partition* of type λ if we can divide $V(G)$ into k stable sets (called *parts*) having cardinalities $\lambda_1, \ldots, \lambda_k$.
- A *semi-ordered stable partition* of G is a stable partition in which parts of the same cardinality are ordered.

2.4.2 Example. Consider the graph on the lefthand side of Figure 2.7. This graph has independence number 3.

If we divide the vertices into three subsets corresponding to each of the three colors, we have a stable partition of type $(3, 1, 1)$. If we order the two parts of size 1, we obtain a semi-ordered stable partition. For example, we could order these parts by assigning the red and green subsets the integers 1 and 2 respectively.

We now proceed with the main definitions of this section.

2.4.3 Definition. A *rim hook* of *length* k is a sequence of k connected cells in a Young diagram, each of which lie on the southeast boundary, and whose removal results in a smaller Young diagram.

For any rim hook, we call each path between consecutive cells a *step*. If the cells are in different rows, we use the term *north step* (N-step), whereas if the cells are in different columns, we use the term *east step* (E-step).

The next definition describes a combinatorial object obtained by filling a Young diagram with sequences of connected cells in a way which satisfies a specific property.

2.4.4 Definition. Let $\kappa = [\kappa_1, \ldots, \kappa_j]$ be a composition and $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition. A *rim hook tabloid* of shape λ and content κ is a filling of the cells of the Young diagram of λ with j sequences of connected cells r_i such that r_1 is a rim hook of length κ_1 and, for all $1 \leq i \leq j-1$, if $r_1, \ldots, r_i$ are removed from λ to form $\tilde{\lambda}$, r_{i+1} is a rim hook of $\tilde{\lambda}$ of length κ_{i+1}.

In this thesis, we focus on an important subclass of rim hook tabloids, as described by the next definition.

2.4.5 Definition. A *special rim hook tabloid* (SRH tabloid) is a rim hook tabloid such that every rim hook intersects the first column.

We define the *sign* of an SRH tabloid to be

$$\mathrm{sgn}(T) = (-1)^{\text{\# north steps in diagram}}.$$

Moreover, we use the notation $\mathcal{T}_\lambda$ to denote the set of all special rim hook tabloids of shape λ.

Given an SRH tabloid T, we denote by κ_T the *content* of T, which is the composition given by the rim hook lengths read from the bottom to the top of the diagram.

In this thesis, we exclusively focus on SRH tabloids, rather than rim hook tabloids in general.

2.4.6 Example. In Figure 2.9, we portray all the possible SRH tabloids of shape $\lambda = (4, 2, 2)$, that is, all elements of the set $\mathcal{T}_{(4,2,2)}$.

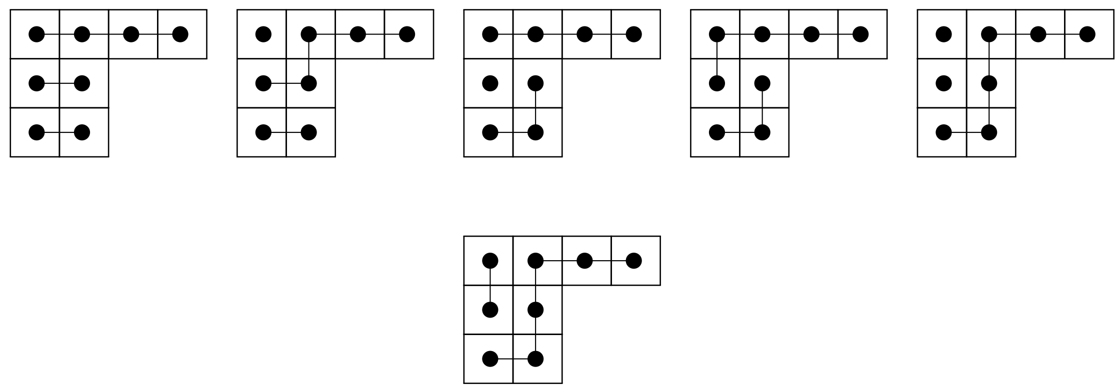


Figure 2.9: All SRH tabloids corresponding to the partition $(4, 2, 2)$.

The second, third, and sixth tabloids have negative sign because, computing the parity of the number of north steps in these diagrams, we find

$$(-1)^1 = (-1)^1 = (-1)^3 = -1.$$

On the other hand, the first, fourth, and fifth tabloids have positive sign. Computing the parity of the number of north steps in the diagrams, we find

$$(-1)^0 = (-1)^2 = (-1)^2 = 1.$$

Considering SRH tabloids allows us to state the following result from [21].

2.4.7 Theorem. *[21] For any graph* G *and any partition* λ *of* $|V(G)|$, *we have*

$$[s_\lambda]X_G = \sum_{T \in \mathcal{T}_\lambda} \mathrm{sgn}(T) \mathrm{so}_G(T),$$

where $\mathrm{so}_G(T)$ *denotes the number of semi-ordered stable partitions of* G *of type* $\Lambda(\kappa_T)$.

Theorem 2.4.7 provides a combinatorial formula for all the Schur coefficients of any graph. In [21], the formula is utilized to prove the non-Schur-positivity of subclasses of complete tripartite graphs, squid graphs, and pineapple graphs.

In Chapter 4, we show that a variation of this formula can be used to prove or disprove the Schur-positivity of certain graphs.

2.5 The Claw-free Conjecture

In this section, we address the most famous conjecture in the study of the Schur-positivity of graphs. We begin with an important definition.

2.5.1 Definition. A graph G is H-*free* if it does not contain a copy of H as an induced subgraph.

2.5.2 Example. The graphs depicted in Figure 2.7 are claw-free.

The following major conjecture, which is sometimes referred to as the Claw-free Conjecture is stated in [19].

2.5.3 Conjecture. *All claw-free graphs are Schur-positive.*

A large class of claw-free graphs are known to be Schur-positive, due to the following theorem.

2.5.4 Theorem ([9])**.** *All claw-free incomparability graphs are Schur-positive.*

All of Chapter 4 will be devoted to making progress toward this conjecture by proving a different subclass of claw-free graphs are Schur-positive. In order to achieve this goal, we employ a new version of Theorem 2.4.7, which will be introduced in Chapter 3.

Chapter 3

Special Rim Hook G-tabloids

In this chapter, we define an altered version of an SRH tabloid, which corresponds to a graph G. This new definition leads to a corollary of Theorem 2.4.7 that offers a different combinatorial interpretation of Schur coefficients.

3.0.1 Definition. Consider a graph G and a partial order $\leq$ on the vertices of G satisfying: if vertices u and v are nonadjacent, then u and v are comparable.

We can then define an *SRH* G-*tabloid* to be an SRH tabloid such that every cell is filled with a vertex of G and the following conditions are met.

- Cells spanned by the same rim hook contain vertices which form a stable set.
- For each rim hook, reading the corresponding vertices from southwest to northeast order results in an increasing sequence with respect to the partial order.

The *sign* and *shape* of an SRH G-tabloid are respectively the sign and shape of the underlying SRH tabloid. We denote the set of all SRH G-tabloids of shape λ by $\mathcal{T}_{\lambda,G}$.

While working with SRH G-tabloids, it is often helpful to distinguish between different parts of the underlying diagram.

3.0.2 Definition. Let T be an SRH G-tabloid. We define the *tail* of T, denoted $\text{tl}(T)$, to be the part of T containing all rows of length 1. Likewise, we define the *head* of T, denoted $\text{hd}(T)$ to be the part of T containing all rows of length strictly greater than 1. Either $\text{tl}(T)$ or $\text{hd}(T)$ could be empty, depending on the shape of T.

Let T and T′ be SRH G-tabloids. We say that

$$\mathrm{tl}(T) = \mathrm{tl}(T') \qquad \text{or} \qquad \mathrm{hd}(T) = \mathrm{hd}(T')$$

if and only if the tails, or respectively the heads, of T and T′ are equal as SRH G′-tabloids for some subgraph G′ of G.

There is some freedom afforded by the choice of a partial order in considering SRH G-tabloids. In this thesis, we address two approaches to choosing a partial order.

First, if G is the incomparability graph of some poset P, it is natural to use the partial order of the underlying poset. Since G is an incomparability graph, two vertices are non-adjacent if and only if they are comparable so the necessary condition is satisfied.

3.0.3 Example. Consider the poset P depicted in Figure 2.3 and the corresponding incomparability graph $G = \mathrm{inc}(P)$ depicted in Figure 2.4. In Figure 3.1, we list some SRH G-tabloids of shape $(2, 1^4)$, that is, elements of the set $\mathcal{T}_{(2,1^4),G}$.

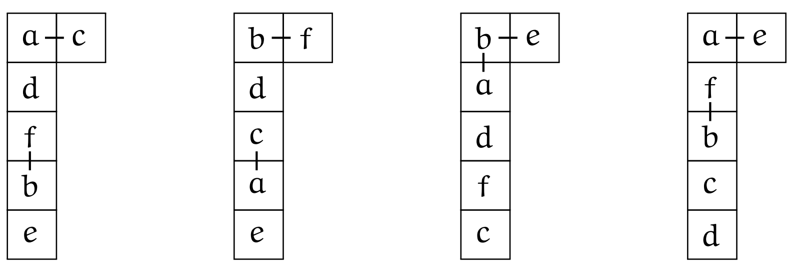


Figure 3.1: Examples of SRH G-tabloids corresponding to an incomparability graph.

If G is not an incomparability graph, we must take a different approach to choosing the partial order. The most direct technique is to choose a total order on G via a numerical labeling of the vertices. In Chapter 4, different choices of labelings are advantageous for counting SRH G-tabloids in different contexts.

3.0.4 Example. Consider the graph G in Figure 3.2, for which we have chosen a labeling of the vertices.

G =

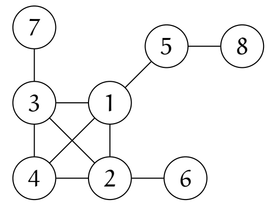


Figure 3.2: A labeled graph which is not an incomparability graph.

In this case, G is not an incomparability graph; for justification, see Section 4.1. In Figure 3.3, we list some SRH G-tabloids of shape $(3, 2, 1^3)$, that is, elements of $\mathcal{T}_{(3,2,1^3),G}$, using our numerical labeling of the vertices as our partial order.

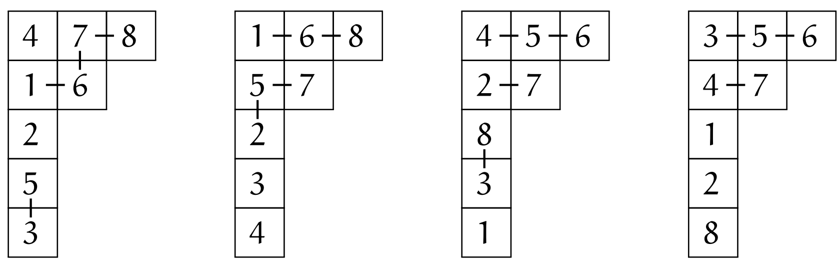


Figure 3.3: Examples of SRH G-tabloids corresponding to a non-incomparability graph.

By considering SRH G-tabloids, we may now obtain an alternative combinatorial interpretation of Schur coefficients.

3.0.5 Corollary. *Consider any graph* G*, a partition* λ *of* $|V(G)|$*, and a partial order on the vertices of* G *such that nonadjacent vertices are comparable. We have*

$$[s_\lambda]X_G = \sum_{T \in \mathcal{T}_{\lambda,G}} \mathrm{sgn}(T).$$

Proof. Each summand in the formula of Theorem 2.4.7 counts pairings of an SRH tabloid $T \in \mathcal{T}_\lambda$ with semi-ordered stable partitions of G of type $\Lambda(\kappa_T)$. The summand is then assigned a sign equal to sgn(T).

For any $T \in \mathcal{T}_\lambda$, we claim the set of pairings of T with semi-ordered stable partitions is in bijection with all SRH G-tabloids T' of shape λ and content κ_T.

To obtain the map

$$\Psi_T : \{(T, \Omega) \mid \Omega \text{ is a semi-ordered stable partition of } G \text{ of type } \Lambda(\kappa_T)\}$$
$$\longrightarrow$$
$$\{T' \in \mathcal{T}_{\lambda,G} \mid \kappa_{T'} = \kappa_T\},$$

we construct an SRH G-tabloid from a given (T, Ω) by identifying each stable set of size t in Ω with a rim hook of length t in T.

For any $t \geq 1$, we have the same number of stable sets of size t as rim hooks of length t in T because Ω is of type $\Lambda(\kappa_T)$. Moreover, the rim hooks of length t in T can be assigned an ordering based on the position (from bottom to top) of their southwest-most cell in the diagram.

Since Ω is also semi-ordered, we can thus uniquely associate the stable sets of size t to rim hooks of length t such that the ordering on the stable sets is consistent with the ordering on the rim hooks.

To obtain an inverse map

$$\Psi_T^{-1} : \{T' \in \mathcal{T}_{\lambda,G} \mid \kappa_{T'} = \kappa_T\}$$
$$\longrightarrow$$
$$\{(T, \Omega) \mid \Omega \text{ is a semi-ordered stable partition of } G \text{ of type } \Lambda(\kappa_T)\},$$

we let T be the underlying SRH tabloid structure of a given T' and construct a semi-ordered partition Ω of G by taking parts equal to the sets of vertices associated with each rim hook. We order the parts of equal size according to the order of the corresponding rim hooks by southwest-most cells (as we defined previously). Since $\kappa_{T'} = \kappa_T$, Ω will be of type

$$\Lambda(\kappa_{T'}) = \Lambda(\kappa_T).$$

Moreover, this map preserves the signs of the associated tabloids because, by definition, the sign of T' is that of its underlying SRH tabloid T so

$$\mathrm{sgn}(T) = \mathrm{sgn}(T').$$

Lastly, we have

$$\mathcal{T}_{\lambda,G} = \bigsqcup_{T \in \mathcal{T}_\lambda} \{T' \in \mathcal{T}_{\lambda,G} \mid \kappa_{T'} = \kappa_T\}$$

so constructing bijections Ψ_T for every $T \in \mathcal{T}_\lambda$ results in the equality

$$\sum_{T \in \mathcal{T}_\lambda} \mathrm{sgn}(T) so_G(T) = \sum_{T \in \mathcal{T}_{\lambda,G}} \mathrm{sgn}(T).$$

□

3.0.6 Remark. In the case where $G = \mathrm{inc}(P)$, SRH G-tabloids of shape λ are in sign-preserving bijection with P-arrays of shape $\pi(\lambda)$, as discussed in [9]. Accordingly, Corollary 3.0.5 can also be obtained as a corollary to the proof of Theorem 3 in [9].

Chapter 4

Generalized Nets

4.1 Definitions and Background for Generalized Nets

In this chapter, we classify a family of graphs known as generalized nets in terms of Schur-positivity. We begin with the definition of these graphs.

4.1.1 Definition. A *generalized net* $GN_{n,m}$, $n \geq 1$, $n \geq m \geq 0$, is a complete graph on n vertices with m degree one vertices appended to distinct vertices in the complete graph.

The vertices in the complete graph (of degree $n-1$ and n) are collectively referred to as the *body*.

The degree 1 vertices are referred to as *pendants*, the degree n vertices are referred to as *anchors*, and the degree $n-1$ vertices are referred to as *buoys*.

We note that our definition of generalized nets is more general than the definition of generalized nets given in [7] since we allow a variable number of pendants.

4.1.2 Example. The generalized net $GN_{5,3}$ is depicted in Figure 4.1.

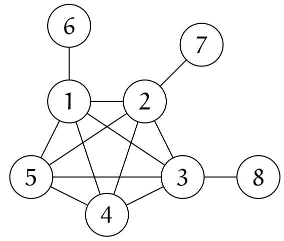
$GN_{5,3} =$ 

Figure 4.1: An example of a generalized net.

In this graph, vertices 6, 7, and 8 are pendants, vertices 1, 2, and 3 are anchors, and vertices 4 and 5 are buoys.

All generalized nets are claw-free. This can be observed directly from the graphs since there are no stable sets of three vertices which are all adjacent to another vertex.

In this chapter, we prove that all generalized nets are Schur-positive. This is consistent with Conjecture 2.5.3 that all claw-free graphs are Schur-positive.

Generalized nets are not incomparability graphs whenever there are three or more pendants. One way to see this is that these graphs contain the subgraph $GN_{3,3}$, depicted in Figure 4.2.

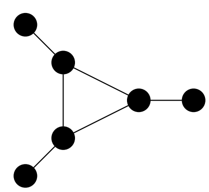


Figure 4.2: The graph $GN_{3,3}$.

In [18], this graph is used as one of the simplest examples of a claw-free non-incomparability graph. Accordingly, no generalized net $GN_{n,m}$, $m \geq 3$, is an incomparability graph since, if it were, the underlying poset would contain a substructure whose incomparability graph is $GN_{3,3}$.

Thus, we can conclude that generalized nets with at least three pendants are conjectured to be Schur-positive by Conjecture 2.5.3 but, unlike all incomparability graphs, are not proven Schur-positive by [9].

The fact that generalized nets are not incomparability graphs also means that, while counting SRH $GN_{n,m}$-tabloids, we assign a labeling on the vertices that acts as a total order. Throughout this chapter, we work with two different choices of labelings, each of which is advantageous in distinct situations.

4.1.3 Definition. In a *pendant-first labeling* for a generalized net, we label the pendant vertices $1, \ldots, m$, we label the anchors $m + 1, \ldots, 2m$ (so that each anchor $m + i$ is adjacent to the pendant labeled i for $1 \leq i \leq m$), and we label the buoys $2m+1, \dots, n+m$.

In a *pendant-last labeling* for a generalized net, we label the buoys $1, \dots, n-m$, we label the anchors $n-m+1, \dots, n$, and we label the pendants $n+1, \dots, n+m$ (so that each anchor i is adjacent to the pendant labeled $i+m$ for $n-m+1 \leq i \leq n$).

4.1.4 Example. In Figure 4.3, we depict a pendant-first labeling of $GN_{4,2}$ on the left and a pendant-last labeling of $GN_{4,2}$ on the right.

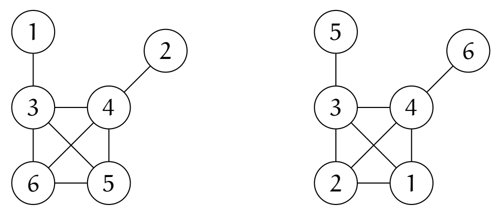


Figure 4.3: Examples of pendant-first and pendant-last labelings.

The goal of this chapter is to use Corollary 3.0.5 to prove that all generalized nets are Schur-positive.

4.2 A Recurrence Relation for Schur Coefficients of Generalized Nets

In order to demonstrate the Schur-positivity of generalized nets, we focus on the vertices in the tails of SRH $GN_{n,m}$-tabloids. In this section, we determine a recurrence relation on Schur coefficients for which the tabloids we are counting have a tail of length at least 1.

We start with a lemma concerning the number of pendants which may be in the tail.

4.2.1 Lemma. *Consider a partition $\lambda = (\lambda_1, \dots, \lambda_k)$ and assume $\lambda_k = 1$. For any $T \in \mathcal{T}_{\lambda, GN_{n,m}}$, $n \geq 1$, $n \geq m \geq 0$, we have that the tail cannot contain only pendants (regardless* *of the choice of labeling).*

Proof. Assume the tail of T has h pendants and no vertices from the body. The head must contain $n + m - h$ cells to include all the other vertices. Accordingly, the head may have at most $\lfloor \frac{n+m-h}{2} \rfloor$ rows and at most $\lfloor \frac{n+m-h}{2} \rfloor$ distinct rim hooks since every rim hook must intersect the first column. We then have

$$\#\text{vertices from the body in head} \leq \#\text{rim hooks in the head} \leq \left\lfloor \frac{n+m-h}{2} \right\rfloor < n.$$

The first inequality follows since every vertex from the head must be in its own rim hook. The last inequality follows since $m \leq n$ and $h \geq 1$. Hence, at least one of the n vertices from the body must be in the tail, contradicting the initial assumption. □

The next proposition shows that certain sets of SRH G-tabloids can be canceled out in terms of sign via an algorithm involving rearranging the pendants in the tails of these tabloids. The conditions for applying this algorithm are stated in as much generality as possible, in order to make this proposition applicable to other, similar, families of graphs.

We note that the terms "pendants" and "body vertices" are used this proposition in a more general context than in the definition of generalized nets. When we apply this algorithm to generalized nets, however, the two uses of the terms are consistent.

4.2.2 Proposition. *Let* G *be a graph, let* $\lambda = (\lambda_1, \ldots, \lambda_k)$ *be a partition such that* $\lambda_k = \lambda_{k-1} = 1$*, and let* $\mathcal{S} \subseteq \mathcal{T}_{\lambda,G}$ *be a subset of SRH* G*-tabloids such that*

$$hd(T) = hd(T') \text{ for all } T, T' \in \mathcal{S}.$$

Let $\mathcal{V}$ *denote the set of vertices of* G *appearing in* $tl(T)$ *for all* $T \in \mathcal{S}$*, and suppose*

$$\mathcal{V} = \mathcal{P} \sqcup \mathcal{U},$$

where $\mathcal{P}$ and $\mathcal{U}$ satisfy the following conditions.

(I) *We have that $\mathcal{P}$ is a stable set of vertices, called pendants, which are degree 1 in G and adjacent to at most 1 vertex in $\mathcal{U}$. If $p \in \mathcal{P}$ is adjacent to some $u \in \mathcal{U}$, we refer to u as the anchor corresponding to v.*

(II) *We have that $\mathcal{U}$ is a nonempty set of vertices, called body vertices, such that each $u \in \mathcal{U}$ is adjacent to at most one pendant.*

Lastly, suppose that $\mathcal{S}$ contains exactly the tabloids $T \in \mathcal{T}_{\lambda,G}$ for which the following conditions are satisfied.

(i) *All distinct body vertices $u, u' \in \mathcal{U}$ are in different rim hooks.*

(ii) *The bottom cell in T is filled by a pendant $p \in \mathcal{P}$ which is nonadjacent to the vertex in the cell above.*

We then have

$$\sum_{T \in \mathcal{S}} \operatorname{sgn}(T) = 0.$$

Proof. We label the vertices of G such that all pendants have labels smaller than the labels of the body vertices.

Consider all $T \in \mathcal{S}$ such that an N-step up from the bottom-most pendant is permissible. We can define a sign-reversing map on these tabloids by adding the N-step if it is not there and removing the N-step if it is there.

Thus, we are left counting tabloids for which no N-step up from the bottom-most vertex is allowed.

These tabloids fall under two cases.

Case (1) The pendant p in the bottom row is below a pendant q such that $p > q$.

Case (2) The pendant p in the bottom row is below a sequence of pendants $y_1, \dots, y_d$, ending with the anchor u corresponding to p such that

$$p < y_1 < \cdots < y_d < u,$$

$|\mathcal{P}| - 1 \geq d \geq 1$, and all consecutive vertices are connected by N-steps except p and y_1.

In any other case, an edge up from p is permissible. Recall that Condition (i) is that no two body vertices are in the same rim hook.

Moreover, Condition (ii) ensures p is not directly below its corresponding anchor u.

Consider any T which falls under Case (2). We map T to the otherwise identical tabloid with the new sequence

$$y_1 > p < y_2 < \cdots < y_d < u.$$

This new tabloid falls under Case (1). Moreover, since one N-step (from y_1 to y_2) is removed, this map is sign-reversing.

After applying this map, we consider the remaining Case (1) tabloids. If an N-step up from q is permissible, we apply a map which either adds or removes this N-step. Hence, we are left again with two subcases, for which no N-step up from q is permissible.

Case (a) Reading up from the bottom rows, we have the sequence $p > q > r$ for some pendant r.

Case (b) Reading up from the bottom rows, we have the sequence ending with the anchor u' corresponding to q such that

$$p > q < y_1 < \cdots < y_{d'} < u',$$

where $|\mathcal{P}|-2 \geq d' \geq 1$, all consecutive vertices from y_1 and on are connected by N-steps, and $p > y_1 > q$. We note that $d' \geq 1$ since the tabloids with sequence $p > q < u'$ are canceled out by tabloids with sequence $q < p < u'$ in Case (2). Likewise, the condition $p > y_1 > q$ holds since tabloids with $p < y_1$ are canceled by the Case (2) tabloids with sequence

$$q < p < y_1 < \cdots < y_{d'} < u.$$

Take any T which falls under Case (b). We map this tabloid to the otherwise identical tabloid with the new sequence

$$p > y_1 > q < \cdots < y_{d'} < u'.$$

Once again, this map removes one N-step (from y_1 to y_2) so it is sign-reversing. Moreover, the new tabloid falls under Case (a).

We then cancel out all Case (a) tabloids such that an N-step up from r is permissible by adding or removing that N-step. We proceed by applying the same method iteratively to the leftover tabloids which fall under Case (a).

On the jth step of this iteration for $2 \leq j \leq |\mathcal{P}| - 1$, we have tabloids of two cases.

Case (A) Reading up from the bottom rows, we have the sequence of pendants

$$p_1 > p_2 > \cdots > p_{j+1}.$$

Case (B) Reading up from the bottom rows, we have the sequence ending with the anchor u'' corresponding to p_j such that

$$p_1 > p_2 > \cdots > p_j < y_1 < \cdots < y_{d''} < u'',$$

where $|\mathcal{P}|-j \geq d'' \geq 1$, all consecutive vertices from y_1 and on are connected by N-steps, and $p_{j-1} > y_1 > p_j$. As before, these conditions arise from the other tabloids being canceled out by the map in the $j-1$th step.

We take any tabloid covered by Case (B) and map it to the otherwise identical tabloid with the new sequence

$$p_1 > p_2 > \cdots > p_{j-1} > y_1 > p_j < \cdots < y_{d''} < u''.$$

which removes one N-step (from y_1 to y_2) and thus is sign-reversing. We then apply a sign-reversing map to all Case (A) tabloids for which an N-step up from p_{j+1} is permissible (by adding or removing that N-step). We then proceed with the $j+1$th step to cancel out the remaining tabloids.

When $j = |\mathcal{P}|-1$, we consider tabloids with sequences (from the bottom up) ending with the anchor u''' corresponding to $p_{|\mathcal{P}|-1}$, such that

$$p_1 > p_2 > \cdots > p_{|\mathcal{P}|-2} > p_{|\mathcal{P}|-1} < y_1 < u''',$$

where y_1 is connected to u''' by an N-step and $p_{|\mathcal{P}|-2} > y_1 > p_{|\mathcal{P}|-1}$. We cancel these tabloids out with the otherwise identical tabloids with the new sequence

$$p_1 > p_2 > \cdots p_{|\mathcal{P}|-2} > y_1 > p_{|\mathcal{P}-1|} < u'''.$$

Accordingly, the only tabloids which remain have the sequence

$$p_1 > p_2 > \cdots > p_{|\mathcal{P}|},$$

read from the bottom up in their tails. For each of these tabloids, there are no more pendants in the tail besides in this sequence. Furthermore, the tabloids with the anchor u''' above $p_{|\mathcal{P}|}$ have been canceled and $\mathcal{U} \neq \emptyset$ so some other $u \in \mathcal{U}$ is above $p_{|\mathcal{P}|}$. Thus, an N-step up from $p_{|\mathcal{P}|}$ is permissible for all remaining tabloids. Hence we cancel these out via a sign-reversing map which adds or removes that N-step.

We note that at no point in this algorithm do we apply a map which results in two body vertices being in the same rim hook, so Condition (i) is satisfied for all the tabloids we considered. Moreover, all the tabloids we considered have pendants in the bottom row, nonadjacent to the vertex in the cell above, so Condition (ii) is always satisfied as well. Since $\mathcal{S}$ contains all tabloids satisfying the conditions of the proposition, all the maps send elements of $\mathcal{S}$ to other elements of $\mathcal{S}$.

We conclude

$$\sum_{T \in \mathcal{S}} \operatorname{sgn}(T) = 0.$$

□

In order to better illustrate the sign-reversing maps employed in the proof of Proposition 4.2.2, we include the following example with tabloid diagrams.

4.2.3 Example. Consider some $\mathcal{S} \subseteq \mathcal{T}_{\lambda,G}$ satisfying the conditions of Proposition 4.2.2, where

$$\lambda = (2^2, 1^8), \qquad G = GN_{6,6}, \qquad |\mathcal{P}| = 4, \qquad \text{and} \qquad |\mathcal{U}| = 4.$$

In this example, we depict sign-reversing maps on tabloids in $\mathcal{S}$, as used in the second step of the iteration in the proof of Proposition 4.2.2, in the case where $d' = 2$. First, we demonstrate how the tabloids which fall under Case (b) are canceled in

Figure 4.4.

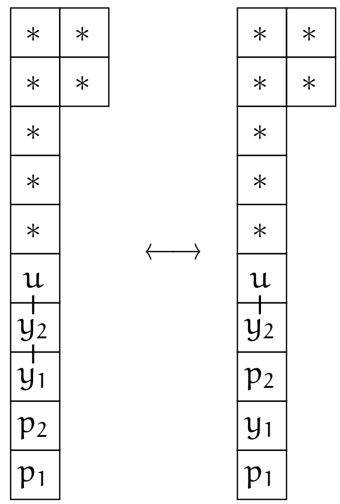


Figure 4.4: An illustration of the cancelation of Case (b) tabloids.

In Figure 4.4, u is the anchor corresponding to p_2, which is why an N-step up from p_2 is not permitted. We also have the relation $p_1 > y_1 > p_2$ (the cases where $y_1 > p_1 > p_2$ were canceled by the first step of the iteration). The tabloid on the left falls under Case (b) and the tabloid on the right falls under Case (a).

We illustrate in Figure 4.5 how we cancel out tabloids under Case (a) for which an N-step up from p_3 is permissible.

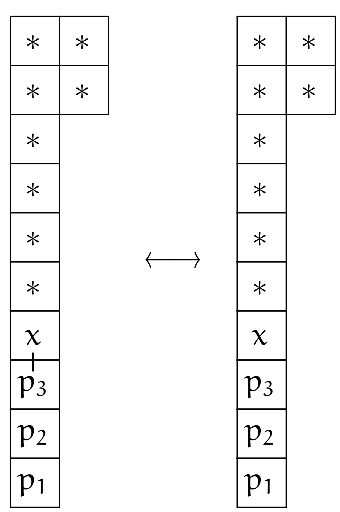


Figure 4.5: An illustration of the cancelation of Case (a) tabloids.

In this case, $p_1 > p_2 > p_3$. One possibility is that x is a pendant $p_4 > p_3$ which is in a rim hook that does not include the anchor u' corresponding to p_3. Otherwise, x is a body vertex other than u'.

After applying these maps, we are left with only Case (a) tabloids for which an N-step up from p_3 is not permissible. These tabloids fall under the two subcases depicted in Figure 4.6.

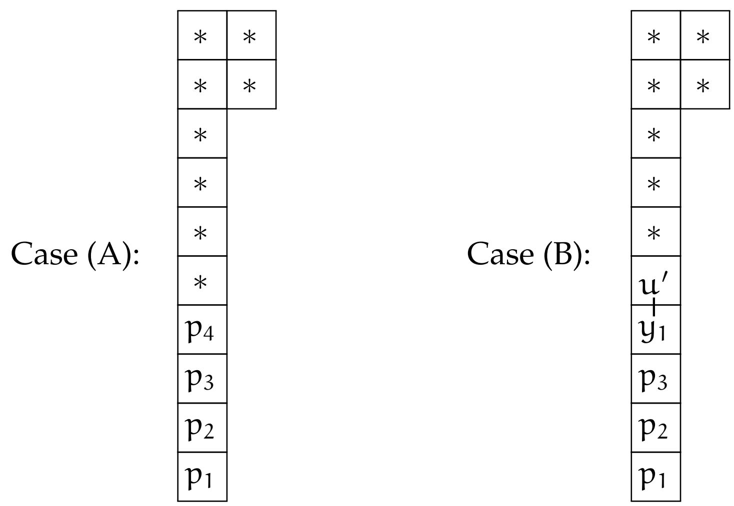


Figure 4.6: A depiction of the remaining cases after canceling Case (b) and some Case (a) tabloids.

In Case (A), we have that $p_1 > p_2 > p_3 > p_4$, and in Case (B), we have that $p_1 > p_2 > y_1 > p_3$. Since $|\mathcal{P}| = 4$, the next step of the algorithm will cancel all remaining tabloids out.

The final proposition of this section provides a key recurrence relation on most Schur coefficients of the chromatic symmetric functions of generalized nets. Before stating it, we introduce some helpful notation.

4.2.4 Notation.

1. We denote the subset of $\mathcal{T}_{\lambda,G}$ which has vertex v in the bottom-most cell of the first column by $\mathcal{T}_{\lambda,G}^{v}$. Likewise, for some subset S of vertices of G, we use the

notation

$$\mathcal{T}_{\lambda,G}^{S} = \bigcup_{\nu \in S} \mathcal{T}_{\lambda,G}^{\nu}.$$

2. Next, we define a function that outputs Schur coefficients of a graph's chromatic symmetric function but vanishes whenever either the graph or the coefficient is not properly defined. Explicitly, we let

$$\xi(\lambda, G) = \begin{cases} [s_\lambda]X_G & \text{if } G \text{ is a properly defined graph and } \lambda \text{ is a} \\ & \text{properly defined partition,} \\ 0 & \text{otherwise.} \end{cases}$$

3. We also define an operation on partitions corresponding to tabloids with tails of length at least 1.

   Given a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ such that $\lambda_h, \ldots, \lambda_k = 1$ for $1 \leq h \leq k$, we define

   $$\lambda \setminus 1^t = (\lambda_1, \ldots, \lambda_{k-t})$$

   for $t \leq k - h + 1$. We note if we consider $\lambda \setminus 1^t$ in the case where λ has strictly fewer than t parts equal to 1, we end up with an undefined partition.

We now provide examples to illustrate situations where $\xi(\lambda, G) = 0$.

4.2.5 Example. We have

$$\xi(\lambda, GN_{n-1,n}) = 0 \qquad \text{for all partitions } \lambda$$

since $GN_{n-1,n}$ is not a properly defined generalized net.

If $\lambda = (3, 2, 2, 1)$, we have

$$\xi(\lambda \setminus 1^2, G) = 0 \qquad \text{for all graphs } G$$

since $\lambda \setminus 1^2$ is not a properly defined partition.

On the other hand, we have

$$\xi(\lambda, GN_{n,n}) = [s_\lambda]X_{GN_{n,n}} \qquad \text{if } \lambda \text{ is a properly defined partition}$$

and

$$\xi(\lambda \setminus 1, G) = [s_{\lambda\setminus 1}]X_G \qquad \text{if } \lambda = (3,2,2,1) \text{ and } G \text{ is a properly defined graph.}$$

4.2.6 Proposition. *Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition and assume $\lambda_k = 1$. We have*

$$\begin{aligned} \xi(\lambda, GN_{n,m}) = {} & m\xi(\lambda \setminus 1, GN_{n-1,m-1} \cup P_1) + (n-m)\xi(\lambda \setminus 1, GN_{n-1,m}) \\ & + m\xi(\lambda \setminus 1^2, GN_{n-1,m-1}) \end{aligned} \tag{4.2.7}$$

for $n \geq m \geq 1$.

Proof. We denote the set of all pendants of $GN_{n,m}$ by P, the set of all anchors of $GN_{n,m}$ by A, and the set of all buoys of $GN_{n,m}$ by B. We count tabloids with a pendant-first labeling.

We observe that

$$\mathcal{T}_{\lambda, GN_{n,m}} = \mathcal{T}^A_{\lambda, GN_{n,m}} \sqcup \mathcal{T}^B_{\lambda, GN_{n,m}} \sqcup \mathcal{T}^P_{\lambda, GN_{n,m}}. \tag{4.2.8}$$

First, assume $n-1 \geq m$ so there is at least one buoy in $GN_{n,m}$.

In the case where an anchor $a \in A$ or buoy $b \in B$ is in the bottom row, there is no N-step up from the cell since we are using a pendant-first labeling. Hence, we can map these tabloids to tabloids for which the bottom cell is removed. These are sign-preserving bijections

$$\mathcal{T}^a_{\lambda, GN_{n,m}} \cong \mathcal{T}_{\lambda\setminus 1, GN_{n-1,m-1} \cup P_1} \qquad \text{and} \qquad \mathcal{T}^b_{\lambda, GN_{n,m}} \cong \mathcal{T}_{\lambda\setminus 1, GN_{n-1,m}}, \tag{4.2.9}$$

since removing an anchor detaches a pendant, decreasing the number of body vertices and the number of pendants by one, and removing a buoy just decreases the number of body vertices by one.

From Equation 4.2.8 and Equation 4.2.9, we then have

$$\begin{aligned}[s_\lambda]X_{GN_{n,m}} &= m \cdot \sum_{T \in \mathcal{T}_{\lambda\setminus 1, GN_{n-1,m-1} \cup P_1}} \operatorname{sgn}(T) + (n-m) \cdot \sum_{T \in \mathcal{T}_{\lambda\setminus 1, GN_{n-1,m}}} \operatorname{sgn}(T) \\ &\quad + \sum_{T \in \mathcal{T}^P_{\lambda, GN_{n,m}}} \operatorname{sgn}(T) \\ &= m[s_{\lambda\setminus 1}]X_{GN_{n-1,m-1}\cup P_1} + (n-m)[s_{\lambda\setminus 1}]X_{GN_{n-1,m}} + \sum_{T \in \mathcal{T}^P_{\lambda, GN_{n,m}}} \operatorname{sgn}(T) \qquad (4.2.10) \\ &= m\xi(\lambda \setminus 1, GN_{n-1,m-1} \cup P_1) + (n-m)\xi(\lambda \setminus 1, GN_{n-1,m}) \\ &\quad + \sum_{T \in \mathcal{T}^P_{\lambda, GN_{n,m}}} \operatorname{sgn}(T)\end{aligned}$$

since there are m anchors and $n - m$ buoys in the graph.

On the other hand, if $n = m$, we don't have the second bijection in Equation 4.2.9. Therefore, we also don't have the second terms in Equations 4.2.10. Thus, again we have

$$\begin{aligned}[s_\lambda]X_{GN_{n,m}} &= m\xi(\lambda \setminus 1, GN_{n-1,m-1} \cup P_1) + (n-m)\xi(\lambda \setminus 1, GN_{n-1,m}) \\ &\quad + \sum_{T \in \mathcal{T}^P_{\lambda, GN_{n,m}}} \operatorname{sgn}(T),\end{aligned}$$

since $\xi(\lambda \setminus 1, GN_{n-1,m}) = 0$ in this case.

If $\lambda_{k-1} \neq 1$, there are no pendants in the tail of any tabloid $T \in \mathcal{T}_{\lambda, GN_{n,m}}$ by Lemma 4.2.1. Therefore, we have

$$\mathcal{T}^P_{\lambda, GN_{n,m}} = \emptyset \qquad \text{so} \qquad \sum_{T \in \mathcal{T}^P_{\lambda, GN_{n,m}}} \operatorname{sgn}(T) = 0.$$

Moreover, if $\lambda_{k-1} \neq 1$,

$$\xi(\lambda \setminus 1^2, GN_{n-1,m-1}) = 0$$

also holds. Thus, in this case, Equation 4.2.7 holds.

For the remainder of the proof, we assume $\lambda_{k-1} = 1$.

Consider the subset of tabloids in $\mathcal{T}^{p}_{\lambda,GN_{n,m}}$ such that a certain pendant p is in the bottom cell and is directly below its anchor. Since there are no N-steps up from the anchor, we map these tabloids to tabloids where the bottom two cells are removed. We can accordingly obtain sign-preserving bijections for each pendant p. These bijections each map from the previously described subset corresponding to p to $\mathcal{T}_{\lambda\setminus 1^2,GN_{n-1,m-1}}$, since removing a pendant and corresponding anchor lowers the number of pendants and the number of body vertices by one.

We can thus count the aforementioned tabloids by adding the term

$$m \sum_{T \in \mathcal{T}_{\lambda\setminus 1^2, GN_{n-1,m-1}}} \operatorname{sgn}(T) = m[s_{\lambda\setminus 1^2}]X_{GN_{n-1,m-1}} = m\xi(\lambda \setminus 1^2, GN_{n-1,m-1})$$

to the sum we are computing (we have a factor of m since there are m pendants).

Denote the set of the remaining tabloids in $\mathcal{T}^{p}_{\lambda,GN_{n,m}}$ by $\mathcal{S}$. We can complete the proof by showing

$$\sum_{T \in \mathcal{S}} \operatorname{sgn}(T) = 0.$$

We partition $\mathcal{S}$ such that

$$\mathcal{S} = \bigsqcup_{H} \mathcal{S}_H,$$

where the disjoint union spans over all possible heads of tabloids $T \in \mathcal{S}$ and each $\mathcal{S}_H \subseteq \mathcal{S}$ is exactly the subset such that

$$\operatorname{hd}(T) = \operatorname{hd}(T') = H$$

for all $T, T' \in \mathcal{S}_H$.

For any $\mathcal{S}_H$, let $\mathcal{P}_H$ and $\mathcal{U}_H$ respectively denote the set of pendants and the set of body vertices appearing in the tail of every $T \in \mathcal{S}_H$. By Lemma 4.2.1 and the definition of a generalized net, Conditions (I) and (II) of Proposition 4.2.2 are satisfied for $\mathcal{P}_H$ and $\mathcal{U}_H$.

Moreover, all $u, u' \in U$ are all adjacent so Condition (i) of Proposition 4.2.2 is also satisfied. Lastly, recall $\mathcal{S}$ does not include any tabloids with a pendant in the bottom cell directly below its anchor. Therefore, Condition (ii) of Proposition 4.2.2 holds as well. We conclude

$$\sum_{T \in \mathcal{S}_H} \operatorname{sgn}(T) = 0$$

for all H so

$$\sum_{T \in \mathcal{S}} \operatorname{sgn}(T) = 0.$$

□

4.3 A Formula for Special Schur Coefficients of Generalized Nets

From Section 4.2, we have a recurrence relation for the Schur coefficients of generalized nets in the case where the coefficients correspond to partitions which end in 1. In this section, we address the case where coefficients correspond to partitions which do not end in 1.

4.3.1 Lemma. *Assume* $[s_\lambda]X_{GN_{n,m}} \neq 0$ *for some* $\lambda = (\lambda_1, \ldots, \lambda_k)$ *such that* $\lambda_k \neq 1$. *Then* $n = m$ *and* $\lambda = (2^n)$.

Proof. Let $GN_{n,m}$ have a pendant-last labeling. By definition, there may be at most one vertex from the body in each rim hook in T. Moreover, the vertices from the body must all be positioned at the beginning of their rim hooks since their labels are smaller than that of the pendants. Since every rim hook must intersect the first column, this implies all n anchors and buoys must be in the first column.

In order for the tail to have no cells, every row in T must have at least 2 cells. Hence, the second column of T must also have n cells. All the vertices from the body are in the first column so this implies we have n pendants to fill the second column. There cannot be any more vertices in the graph so $n = m$ and $\lambda = (2^n)$ must both hold. □

To address the case where the tail of $T \in \mathcal{T}_{G_{n,m},\lambda}$ has no cells, we compute a formula for the corresponding coefficients, which first requires a few lemmas.

4.3.2 Lemma. *We have that*

$$[s_{(2^C,1^D)}]X_{GN_{C+D,C-1} \cup P_1} = [s_{(2^{C-1},1^{D+1})}]X_{GN_{C+D,C-1}}$$

for $C \geq 1, D \geq 0$.

Proof. We label $GN_{C+D,C-1} \cup P_1$ with a pendant-last labeling and assign the degree 0 vertex (which we call x) the largest label. For any $T \in \mathcal{T}_{\lambda,GN_{C+D,C-1} \cup P_1}$, we have that the first column must be completely filled with the $C + D$ vertices from the body (since these vertices are each in distinct rim hooks and labeled minimally). Moreover, this implies there are no N-steps in the first column.

Consider any tabloid for which x is not in the bottom-most cell in the second column. Since x is nonadjacent to every other vertex and labeled maximally, an N-step from the vertex below x and an E-step from the vertex to the left of x are both permissible.

Accordingly, we map all tabloids with an N-step to x to the otherwise identical tabloids with an E-step to x and vice versa. This map is a sign-reversing involution (on the tabloids for which x is not in the bottom-most cell in the second column) since an N-step is always added or removed.

Thus, it remains to count the tabloids which have x in the bottom-most cell in the second column. Since x is labeled maximally, there is no N-step up from x so there must be an E-step to x from the vertex to the left of x.

Consider the map which sends these tabloids to tabloids in $\mathcal{T}_{(2^{C-1},1^{D-1}),GN_{C+D,C-1}}$ (under a pendant-last labeling) by removing the cell containing x. Since no N-steps are added or removed and since every $T \in \mathcal{T}_{(2^{C-1},1^{D-1}),GN_{C+D,C-1}}$ is in the image of the map, this is a sign-preserving bijection.

We conclude

$$[s_{(2^C,1^D)}]X_{GN_{C+D,C-1}\cup P_1} = [s_{(2^{C-1},1^{D+1})}]X_{GN_{C+D,C-1}}.$$

□

4.3.3 Example. In Figure 4.7, we illustrate examples of the two bijections in the proof of Lemma 4.3.2, in the case where $C = 3, D = 3$.

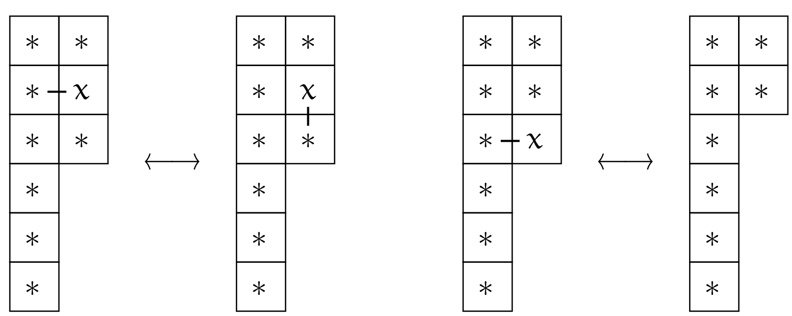


Figure 4.7: An illustration of the map in the proof of Lemma 4.3.2.

On the left, we observe two types of tabloids which are canceled out by the first sign-reversing involution in the proof. On the right, we see how tabloids in $\mathcal{T}_{(2^3,1^3),GN_{6,2}\cup P_1}$ with x in the bottom cell of the second column are mapped to tabloids in $\mathcal{T}_{(2^2,1^4),GN_{6,2}}$.

For the sake of simplicity, we now fix some notation for a certain type of Schur coefficient which satisfies a convenient recurrence relation (Lemma 4.3.5).

4.3.4 Notation. We let

$$f(C, D) = \begin{cases} [s_{(2^C,1^D)}]X_{GN_{C+D,C}} & \text{if } C, D \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

We note that

$$\ell((2^C, 1^D)) = C + D = \#\text{vertices in the body of } GN_{C+D,C}.$$

This equality makes the tabloids easier to count with a pendant-last labeling.

4.3.5 Lemma. *We have that*

$$f(C, D) = Cf(C-1, D) + Df(C, D-1)$$

for $C, D \geq 1$.

Proof. We count tabloids using a pendant-last labeling. All $C + D$ vertices from the body of $GN_{C+D,C}$ must be in separate rim hooks and are labeled minimally. Therefore they all lie in the first column (which accordingly has no N-steps).

Consider the vertex in the bottom-most cell in the first column, which we call x. Since $D \geq 1$, this vertex is in the tail of the diagram and is thus in a rim hook of length 1.

If x is an anchor, we have a sign-preserving bijection

$$\mathcal{T}^x_{(2^C,1^D),GN_{C+D,C}} \to \mathcal{T}_{(2^C,1^{D-1}),GN_{C+D-1,C-1} \cup P_1},$$

obtained by removing the bottom-most cell in the first column. Indeed, removing an anchor detaches one of the pendant vertices, yielding a generalized net with one less pendant and one less vertex in the body, along with a disjoint path of length 1.

Similarly, if x is a buoy, we have a sign-preserving bijection

$$\mathcal{T}^{x}_{(2^C,1^D),GN_{C+D,C}} \to \mathcal{T}_{(2^C,1^{D-1}),GN_{C+D-1,C}},$$

obtained by removing the bottom-most cell in the first column. Removing a buoy results in a generalized net with one less vertex in the body.

Since there are C anchors and D buoys, these bijections result in the relation

$$f(C,D) = C[s_{(2^C,1^{D-1})}]X_{GN_{C+D-1,C-1}\cup P_1} + D[s_{(2^C,1^{D-1})}]X_{GN_{C+D-1,C}}.$$

Applying Lemma 4.3.2, we obtain

$$\begin{aligned} f(C,D) &= C[s_{(2^{C-1},1^D)}]X_{GN_{C+D-1,C-1}} + D[s_{(2^C,1^{D-1})}]X_{GN_{C+D-1,C}} \\ &= Cf(C-1,D) + Df(C,D-1). \end{aligned}$$

□

To finish the section, we use the prior lemmas to prove that the coefficients $[s_{(2^n)}]X_{GN_{n,n}}$ satisfy a nonnegative formula.

4.3.6 Proposition. *We have that*

$$f(n,0) = \begin{cases} n! & \text{if } n \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

for $n \geq 1$.

Proof. We have that

$$f(n,0) = [s_{(2^n)}]X_{GN_{n,n}}.$$

The desired formula holds for $n = 1$ since there are no valid SRH $GN_{1,1}$-tabloids of shape (2). Likewise, the formula holds for $n = 2$ since there are two horizontal rim hooks of length 2 which may be ordered 2 ways. We proceed by induction and assume the formula holds for $f(n-2, 0)$ for some $n \geq 3$.

We count tabloids using a pendant-last labeling so that, as in the proof of Lemma 4.3.1, the first column must be filled with all the vertices from the body of the graph (which are all anchors).

For any $T \in \mathcal{T}_{(2^n),GN_{n,n}}$, we consider the bottom-most rim hook which necessarily starts with an E-step then includes N-steps up the second column spanning i cells where i ranges from 1 to $n-1$. Note this rim hook cannot span all n cells in the second column since there are no stable $n+1$-sets in $G_{n,n}$.

For each choice of a bottom-most rim hook spanning i cells in the second column, we consider a mapping in which this rim hook is removed, leaving the rest of the tabloid unchanged. In each case, this results in a bijection from the subset of tabloids with the given rim hook to the set

$$\mathcal{T}_{(2^{n-i}1^{i-1}),GN_{n-1,n-i-1}\cup P_1}.$$

Indeed, each such map can be inverted by simply adding the given rim hook back to the tabloid. Under such a mapping, the sign of the tabloid changes by a factor of $(-1)^{i-1}$ since there are $i-1$ N-steps being removed. There are n choices for the anchor at the beginning of the rim hook, then $\binom{n-1}{i}$ choices for the i pendants in the rim hook (their order is determined by the labeling and the pendant corresponding to the anchor is excluded).

Accordingly, via these bijections, we obtain the formula

$$f(n,0) = n \sum_{i=1}^{n-1} \binom{n-1}{i} (-1)^{i-1} [s_{(2^{n-i},1^{i-1})}] X_{GN_{n-1,n-i-1}\cup P_1}.$$

By Lemma 4.3.2, we can convert this to

$$\begin{aligned} f(n,0) &= n\sum_{i=1}^{n-1}\binom{n-1}{i}(-1)^{i-1}[s_{(2^{n-i-1},1^i)}]X_{GN_{n-1,n-i-1}} \\ &= n\sum_{i=1}^{n-1}\binom{n-1}{i}(-1)^{i-1}f(n-i-1,i). \end{aligned} \tag{4.3.7}$$

We then have

$$\begin{aligned} f(n,0) &= n(-1)^{n-2}f(0,n-1) + n\sum_{i=1}^{n-2}\binom{n-1}{i}(-1)^{i-1}f(n-i-1,i) \\ &= (-1)^n n! + n\sum_{i=1}^{n-2}\binom{n-1}{i}(-1)^{i-1}f(n-i-1,i) \end{aligned}$$

since $f(0,n-1) = (n-1)!$ as it simply counts tabloids of shape (1^{n-1}) with no N-steps. We then apply Lemma 4.3.5 and Equation 4.3.7 to obtain

$$\begin{aligned} f(n,0) &= (-1)^n n! + n\sum_{i=1}^{n-2}\binom{n-1}{i}(-1)^{i-1}(n-i-1)f(n-i-2,i) \\ &+ n\sum_{i=1}^{n-2}\binom{n-1}{i}(-1)^{i-1}if(n-i-1,i-1) \\ &= (-1)^n n! + n(n-1)\sum_{i=1}^{n-2}\frac{(n-2)!}{(n-2-i)!i!}(-1)^{i-1}f(n-i-2,i) \\ &+ n(n-1)\sum_{i=1}^{n-2}\frac{(n-2)!}{(n-1-i)!(i-1)!}(-1)^{i-1}f(n-i-1,i-1) \\ &= (-1)^n n! + nf(n-1,0) \\ &+ n(n-1)\sum_{j=0}^{n-3}\frac{(n-2)!}{(n-2-j)!j!}(-1)^{j}f(n-j-2,j) \\ &= (-1)^n n! + nf(n-1,0) + n(n-1)f(n-2,0) \\ &- n(n-1)\sum_{j=1}^{n-2}\binom{n-2}{j}(-1)^{j-1}f(n-j-2,j) + n(n-1)(-1)^{n-3}f(0,n-2) \\ &= (-1)^n n! + nf(n-1,0) + n(n-1)f(n-2,0) - nf(n-1,0) + (-1)^{n-1}n! \end{aligned}$$

$$= n(n-1)f(n-2,0)$$
$$= \begin{cases} n(n-1)\cdot(n-2)! & \text{if } n \text{ is even} \\ n(n-1)\cdot 0 & \text{if } n \text{ is odd} \end{cases}$$

so the desired formula holds by induction. □

4.4 Generalized Nets are Schur-Positive

In this section, we apply the results of Sections 4.2 and 4.3 to prove the main theorem of this chapter.

4.4.1 Theorem. *All generalized nets* $GN_{n,m}$ *are Schur-positive for* $n \geq 1$ *and* $n \geq m \geq 0$.

Proof. We proceed by induction on the number of vertices.

If $m = 0$, 1, or 2, $GN_{n,m}$ is a claw-free incomparability graph, as discussed in [10]. Thus, these graphs are Schur-positive by Theorem 2.5.4 so these base cases are satisfied.

Assume $GN_{n,m-1}$ are Schur-positive for all $n \geq m-1$ for some $m \geq 3$.

We first show $G_{m,m}$ is Schur-positive. Consider any $\lambda = (\lambda_1, \ldots, \lambda_k)$ such that $[s_\lambda]X_{GN_{m,m}} \neq 0$. If $\lambda_k \neq 1$, by Lemma 4.3.1, $\lambda = 2^m$. Then, by Proposition 4.3.6,

$$[s_{(2^m)}]X_{GN_{m,m}} > 0.$$

Otherwise, we may assume $\lambda_k = 1$. By Proposition 4.2.6,

$$[s_\lambda]X_{GN_{m,m}} = \xi(\lambda, GN_{m,m}) = m\xi(\lambda \setminus 1, GN_{m-1,m-1} \cup P_1) + m\xi(\lambda \setminus 1^2, GN_{m-1,m-1}) \geq 0$$

where the righthand side is nonnegative by the inductive hypothesis (recall also disjoint unions of Schur-positive graphs are Schur-positive by Proposition 2.3.9).

Thus, $GN_{m,m}$ is Schur-positive.

Assume $GN_{n-1,m}$ is Schur-positive for some $n \geq m+1$. Consider any $\lambda = (\lambda_1, \ldots, \lambda_k)$ such that $[s_\lambda]X_{GN_{n,m}} \neq 0$. By Lemma 4.3.1, $\lambda_k = 1$ since $n \neq m$. We have

$$\begin{aligned}[s_\lambda]X_{GN_{n,m}} = \xi(\lambda, GN_{n,m}) &= m\xi(\lambda \setminus 1, GN_{n-1.m-1} \cup P_1) + (n-m)\xi(\lambda \setminus 1, GN_{n-1,m}) \\ &+ m\xi(\lambda \setminus 1^2, GN_{n-1,m-1}) \\ &\geq 0\end{aligned}$$

by Proposition 4.2.6 and the two inductive hypotheses.

We conclude all generalized nets are Schur-positive. □

Since generalized nets are claw-free, the prior theorem makes progress toward Conjecture 2.5.3 by confirming these graphs are Schur-positive.

Chapter 5

Conclusion

Since Gasharov's result that claw-free incomparability graphs are Schur-positive, there have been relatively few proofs demonstrating the Schur-positivity of graphs. This is largely due to the absence of established tools for proving such results. Instead, more progress has been made toward disproving Schur-positivity for graphs, such as in [6].

The introduction of a combinatorial formula for Schur coefficients of chromatic symmetric functions in [21] represents a promising step toward a more extensive classification of the Schur-positivity of graphs. In [21], this formula is applied to specific coefficients of chromatic symmetric functions to prove that some families of graphs are not Schur-positive.

The proof of Theorem 4.4.1 illustrates that this formula can also be employed to prove that all the Schur coefficients of certain chromatic symmetric functions are non-negative. Proposition 4.2.2 is stated in generality in order to extend these techniques to other families of graphs which resemble generalized spiders. For example, these methods should be applicable to the family of graphs obtained from generalized nets by removing edges between body vertices.

Thus, besides ensuring the Schur-positivity of the generalized nets family, the proof of Theorem 4.4.1 also introduces a new approach for proving Schur-positivity in general.

A natural future direction for this research would be to consider the family of generalized spiders, which each consist of a complete graph with paths of various lengths appended to distinct vertices of the complete graph. An example of a generalized spider is depicted in Figure 5.1.

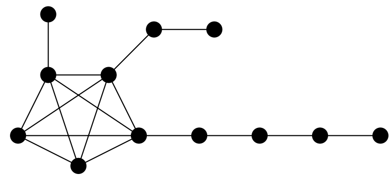


Figure 5.1: An example of a generalized spider.

Generalized spiders are a large class of claw-free graphs which usually are not incomparability graphs, and generalized nets are the simplest members of this family.

As shown in [7], generalized spiders are precisely the line graphs of spiders (trees with exactly one vertex of degree 3 or greater).

An even more ambitious goal would be to prove the Schur-positivity of all line graphs, which is conjectured to hold since line graphs are claw-free. Since generalized spiders are one of the simplest families of line graphs, guaranteeing their Schur-positivity would be a major step toward this objective.

In [7], it is shown that generalized nets $GN_{n,3}$ ($n \geq 3$) are never e-positive. Accordingly, Theorem 4.4.1 also implies that $GN_{n,3}$ ($n \geq 3$) is an infinite family of graphs that are Schur-positive but not e-positive. Since this type of family has rarely been studied, this is another significant contribution to the literature.

Another family of claw-free non-incomparability graphs is the family of triangular towers. Triangular towers consist of two copies of C_3 such that the vertices in the first copy are each connected to distinct vertices in the second copy via paths of equal length. For example, Figure 5.2 depicts two examples of triangular tower graphs on different number of vertices.

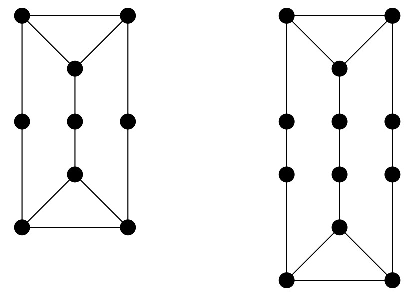


Figure 5.2: Examples of triangular tower graphs.

Triangular towers are shown to be non-e-positive in [5]. On the other hand, these graphs are conjectured to be Schur-positive by the Claw-free Conjecture. Applying similar methods to this family to demonstrate Schur-positivity would be a meaningful continuation of this line of research.

More broadly, the introduction of special rim hook tabloids in Chapter 3 represents an alternate lens on the Schur-positivity of chromatic symmetric functions. We show that the main result from [21] can be reframed so that every Schur coefficient of a chromatic symmetric function can be computed as a sum of the signs of special rim hook G-tabloids. Using this interpretation, we can also rewrite Gasharov's proof in [9] in terms of these new combinatorial objects. If the Claw-free Conjecture holds, it could be possible to construct sign-reversing maps on special rim hook G-tabloids corresponding to any claw-free graph to show its Schur-positivity. In this way, one could hope to generalize Gasharov's result to non-incomparability graphs. This thesis represents a small step toward developing methods to attain such a goal.

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