Variations on Sparsifier Constructions

by

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Abstract

We studied Joshua Batson, Daniel A. Spielman, and Nikhil Srivastava's Twice-Ramanujan Sparsifiers. On this basis, we try to generalize their potential function. Inspired by the idea of non-backtracking matrices, we designed the new potential formula that puts all eigenvalues on the complex plane. Following the construction of the twice Ramanujan sparsifier, we conclude some obstacles must be encountered to improve their bound.

Lay Summary

A graph is a data structure that is widely used in multiple areas such as communication networks and data science. In this era of increasing information, the notion of graph sparsification has been proposed. The point of graph sparsification is to approximate a graph by a sparser graph, which means that we want to remove some edges of the graph but still preserve some of its properties.

A major question in graph sparsification is how sparse a graph can be while its properties are still preserved. In 2008, Joshua Batson, Daniel A. Spielman, and Nikhil Srivastava constructed their so-called Twice-Ramanujan Sparsifiers, which gives an order n/ε^2 spectral sparsifier of a graph with n vertices, where epsilon is a parameter that specifies the degree of sparsification. In this thesis, we study their construction and some possible variations on it.

Inspired by the idea of non-backtracking matrices, we modify the potential functions they set and try to apply the methods in Twice-Ramanujan Sparsifiers to these potential functions. We identify some obstacles that need to be overcome to improve their construction.

Preface

The Section 4.1 is joint work with Joel Friedman. And the rest of dissertation is original, unpublished, independent work by the author, Ke Han Xiao.

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Chapter 1

Introduction

The purpose of this thesis is to present the sparsification algorithm of Batson et al. [1] and to explore possible ways to improve the construction.

1.1 Graph Sparsification

Graph sparsification is the task of approximating a graph by a sparse graph and is often useful in the design of efficient approximation algorithms.
The first notion of Sparsifier comes from Paul Chew's notion on graph spanners [3], is to is design a sparse graph in which the shortest-path distance between every pair of vertices is approximately preserved. However, Chew's defination [3] is very different from one we will be using, and we will not discuss it further.
Then, the idea of cut sparsifier has been invented by Benczur and Karger [2], which intends to preserve the cut between every pair of vertices. We now give an introduction, but precise definitions begin in Section 2.
Benczur and Karger use edge compression to sparsify the graph.

Definition 1.1.0.1 (Edge Compression (Benczur and Karger [2]))**.** *Given a graph $G = (V, E)$, a family $\{p_e\}_{e \in E}$ of reals $p_e \in (0, 1]$ indedexed on $e \in E$. Then, the compressed graph $G[p_e]_{e \in E}$ refers to the random weighted subgraph of G on vertex set V, and where each $e \in E$ appears in subgraph with probability $1/p_e$, independently of other edges.*

With this idea, Benczur and Karger proved the following theorem, which gives a sparsifier with expected O(nlog(n)) edges.

Theorem 1.1.0.2 (Basic Sampling theorem(Batson et al. [1])). *Let G be a graph in which the edges have mutually independent random weights, each distributed in the interval* $[0,1]$. *If the expected weight of every cut in G exceeds* $p = 3(d+2)(ln(n))/\varepsilon^2$ *for some* $\varepsilon \leq 1$ *and d, then with probability* $1 - O(n^{-d})$ *every cut in* $G[p_e]$ *has value within* $(1 \pm \varepsilon)$ *of its expectation.*

However, to verify the cut sparcifier requires checking every cut in the graph, which is clearly a NP-hard problem.
Then, based on the cut sparsifiers, Spielman and Teng have proposed a stronger notion of Spectral sparsifier.

Definition 1.1.0.3. *Given a graph G, we use* L_G *to denote its Laplacian, and* D_G *to denote its diagonal degree counting matrix.*

Theorem 1.1.0.4 (Sampling High-Conductance Graphs (Spielman and Teng [5])). *Let* $\varepsilon, p \in (0,1/2)$, *let G be an unweighted graph with n vertices whose smallest non-zero normalized Laplacian eigenvalue is at least* λ, *i.e. let* $\tilde{L} = D_G^{-1/2}(L_G)D_G^{-1/2}$,

$$\lambda_n(\tilde{L}) \geq ... \geq \lambda_2(\tilde{L}) \geq \lambda > \lambda_1((\tilde{L})) = 0 \tag{1.1}$$

Let S be a subset of the vertices of G, let F be the edges in $G(S)$, *and let* $H = E - F$ *be the rest of the edges. Sample the edges in subgraph* (S,F) *with probability* $p_{ij} = min(1, \frac{\gamma}{min(d_i,d_j)})$ *to get some sparsifier* $(S,\tilde{F})$, *with some* $\gamma > \Omega(logn)$. *Let* $\gamma = (\frac{12k}{\varepsilon\lambda})^2$, $k = max(log_2(3/p), log_2 n)$, *then with probability at least 1-k,* $\tilde{G} = (V, \tilde{F} + H)$ *will be a* $(1+\varepsilon)$*-approximation of G, and the number of edges in* $\tilde{F}$ *is at most:*

$$\frac{288max(log_2(3/p), log_2 n)^2}{(\varepsilon, \lambda)^2}|S| \tag{1.2}$$

This method also used the ramdomized approach, which also gives an expected $O(nlog(n))$ edges spectral sparsifier.
In this thesis, we will focus on another deterministic algorithm, which gives a $O(n/\varepsilon^2)$ edges spectral sparsifier, and some possible variance of it.

Chapter 2

Background

In this chapter, we give background materials; for more details and references, see [1].

2.1 Laplacian matrix

Definition 2.1.0.1. *A multiset a generalization of a set that allows the elements to have multiplicities.*

Definition 2.1.0.2. *An undirected graph $G = (V,E)$ is a pair, V is a set, E is a multi-set on $\{(u,v)|u,v \in V\}$.*

Definition 2.1.0.3. *A undirected weighted graph is a pair, $(G = (V,E),w)$ is a pair, where $G = (V,E)$ is a graph, and $w : E \to \mathbb{R}$. In this thesis, we are focusing on $w : E \to \mathbb{R}^+$.*

Definition 2.1.0.4. *For a undirected finite graph $G = (V,E)$, its adjacency matrix A_G is a $|V| \times |V|$ square matrix used to represent G. The elements of the matrix indicate whether pairs of vertices are adjacent or not in the graph. And in weighted case, the weighted adjacency matrix $A_{G,w}$ is a square matrix indexed on V, such that:*

$$(A_{G,w})_{u,v} = \sum_{e\ (u,v)} w(e) \tag{2.1}$$

If $u = v$, i.e. a loop, $w(e)$ is counted twice.

Definition 2.1.0.5. *The degree matrix D_G of an undirected graph $G = (V,E)$, is a diagonal matrix indexed on V, where $(D_G)_{u,v} = deg_G(v)$. $deg_G(v)$ is the degree of v in G, i.e. the number of edges upon which v is incident. For the weighted case, the degree matrix of a weighted graph (G,w) is given by:*

$$(D_{(G,w)})_{(v,v)} = \sum_{e \in E,\ with\ endpoint\ v} w(e) \tag{2.2}$$

If the edge is a loop, $w(e)$ is counted twice.

Definition 2.1.0.6. *Given a undirected graph $G = (V,E)$, its Laplacian matrix L_G is defined as:*

$$L_G = D_G - A_G \tag{2.3}$$

where D_G is the degree matrix of G and A_G is its adjacency matrix.

Definition 2.1.0.7. *For the weigthed graph $(G = (V,E), w)$, its weighted Laplacian matrix $L_{G,w}$ is defined as:*

$$L_{G,w} = D_{G,w} - A_{G,w} \tag{2.4}$$

where $D_{G,w}$ and $A_{G,w}$ are the weighted degree matrix and adjacency matrix of $(G = (V,E), w)$.

Theorem 2.1.0.8. *Given a undirected weighted graph $(G = (V,E), w)$ and its Laplacian matrix $L_{G,w}$, for any $\vec{x} : V \to \mathbb{R}$,*

$$\vec{x}^T L_G \vec{x} = \sum_{e=(u,v)\in E} w(e)(x(u) - x(v))^2 \tag{2.5}$$

Proof.

$$\begin{aligned}
\sum_{e=(u,v)\in E} w(e)(x(u) - x(v))^2 &= \sum_{e=(u,v)\in E} w(e)x(u)^2 + w(e)x(v)^2 - w(e)2x(u)x(v) \\
&= \sum_{e=(u,v)\in E} w(e)(x(u)^2 + x(v)^2) - \sum_{e=(u,v)\in E} w(e)2x(u)x(v) \\
&= \vec{x}^T D_{G,w}\vec{x} - \vec{x}^T A_{G,w}\vec{x} \\
&= \vec{x}^T (D_{G,w} - A_{G,w})\vec{x} \\
&= \vec{x}^T L_{G,w}\vec{x}
\end{aligned}$$

□

Definition 2.1.0.9. *Given a undirected graph $G = (V, E)$, we order it arbitrarily, and the incident matrix $B_{|E|\times|V|}$ of G will be:*

$$B(e,v) = \begin{cases} 1 & \text{if } v \text{ is } e\text{'s head} \\ -1 & \text{if } v \text{ is } e\text{'s tail} \\ 0 & \text{otherwise} \end{cases} \tag{2.6}$$

The Laplacian of G, $L_G = B_G^T B_G$. Moreover, if G is weighted, with the weight w, $L_{G,w} = B_G^T W B_G$, where W is the $|E| \times |E|$ diagonal matrix whose (e,e)-entry is $w(e)$.

2.2 For complete graph

Definition 2.2.0.1. *A complete graph is a graph in which each pair of graph vertices is connected by an edge. For the adjacency matrix of any complete graph K_n of n vertices, $A_{K_n} = E - I$, where E is the all-one matrix.*
As E has eigenvalue n with multiplicity 1 and 0 with multiplicity $n-1$, the adjacency matrix of any complete graph K_n, A_{K_n} has eigenvalue n-1 with multiplicity 1 and -1 with multiplicity n-1. Thus, L_{K_n} has eigenvalue 0 and n.

Definition 2.2.0.2. *Given some integer d, a d-regular undirected graph is an undirected graph where each vertex has d neighbors.*

Definition 2.2.0.3. *An unweighted, d-regular graph G, is Ramanujan if the eigenvalues of L_G.*

$$0 = \lambda_1(G) < \lambda_2(G) \leq ... \leq \lambda_n(G) \tag{2.7}$$

satisfy:

$$d - 2\sqrt{d-1} \leq \lambda_2(G) \leq \lambda_n(G) \leq d + 2\sqrt{d-1} \tag{2.8}$$

Theorem 2.2.0.4. *([5]) To sparsify the complete graph K_n, we can use a d-regular Ramanujan graph with $\frac{dn}{2}$ edges, by giving weight n/d on each edge, it becomes a $(1 - 2\sqrt{d-1}/d)^{-1}$ approximation of K_n.*

Proof. All the non-zero eigenvalue of the d-regular Ramanujan graph lie between $d-2\sqrt{d-1}$and $d+2\sqrt{d+1}$. By giving weight n/d on each edge, we have a $\vec{x}L_{\tilde{G}}\vec{x}^T \in [n-2n\sqrt{d-1}/d, n+2n\sqrt{d-1}/d]$, for every unit x orthogonal to all 1-vector. Thus, it becomes a $(1-2\sqrt{d-1}/d)^{-1}$-approximation of K_n. $\square$

Definition 2.2.0.5. *Let $(G=(V,E),w)$ be a weighted, undirected graph, $\tilde{G}$ is a σ-spectral approximation of G, if for all $x \in \mathbb{R}^{|V|}$, $\frac{1}{\sigma}\vec{x}^T L_{\tilde{G}}\vec{x} \leq \vec{x}^T L_G \vec{x} \leq \sigma \vec{x}^T L_{\tilde{G}}\vec{x}$. This expression is equivalent as $\frac{1}{\sigma}\tilde{G} \preceq G \preceq \sigma\tilde{G}$*

If we restrict $x \in \{0,1\}^{|V|}$ (this is equivalent to separate the vertices into 2 groups), then $\vec{x}^T L_G \vec{x}$ measures exactly how many edges across those 2 groups, which is then equivalent to the notion of cut sparsifiers by Benczur and Karger.

2.2.1 Interlacing and rank-one update

Theorem 2.2.1.1. *Suppose $A \in \mathbb{R}^{n\times n}$ invertible square matrix and $u, v \in \mathbb{R}^n$. Then $A+uv^T$ is invertible iff $1+v^T A^{-1}u \neq 0$. Moreover, we have the **Sherman–Morrison formula** to describes this rank-one update:*

$$(A+uv^T)^{-1} = A^{-1} + \frac{A^{-1}uv^T A^{-1}}{1+v^T A^{-1}u} \tag{2.9}$$

2.2.2 Non-backtracking matrix

Definition 2.2.2.1. *Given an undirected graph $G=(V,E)$, the non-backtracking matrix H is an $|E|\times|E|$ matrix designed as, for all edges $(u,v),(x,y) \in E$,*

$$H((u,v),(x,y)) = \begin{cases} 1 & \textit{if } v = x \textit{ and } u \neq y \\ 0 & \textit{otherwise} \end{cases} \tag{2.10}$$

If G is a random d-regular graph on n vertices, with d fixed and n large (tending to infinity), there are trace methods to get a high probability lower bound on $\lambda_2(G)$= the second largest eigenvalue of the Adjancency matrix $A(G)$ of G. These methods compute:

$$E_{G\in\mathbb{G}_{n,d}}[trace(A_G^k)] \tag{2.11}$$

for some k, where $\mathbb{G}_{n,d}$ is a probability space of a d-regular graph on n vertices. Then, one can use a Markov bound to get a high probability lower bound on $\lambda_2(G)$. It is known that this lower bound can be improved by writing an equivalent estimate of:

$$E_{G\in\mathbb{G}_{n,d}}[trace(H_G^k)] \tag{2.12}$$

where H_G= "non-backtracking" matrix of G, applying a Markov bound to estimate the second largest eigenvalue of H_G, and then converting bound to a bound on $\lambda_2(G)$ of the Laplacian matrix $L(G)$ of G.[4]

Each eigenvalue λ of A_G corresponds to an eigenvalue $d-\lambda$ of L_G and two eigenvalues μ_1,μ_2 of H_G that are the solutions to the equation:

$$\mu^2-\mu\lambda+(d-1)=0 \tag{2.13}$$

So, the transformation λ to μ_1,μ_2 takes some of the real eigenvalues, λ, and place them on the complex circle

$$\{\mu\in\mathbb{C}||\mu|=\sqrt{d-1}\} \tag{2.14}$$

This motivates our research for variants of [1] that might improve the construction there.

Chapter 3

Twice Ramanujan Sparsifier

In this section, we describe Batson, Joshua and Spielman, Daniel A. and Srivastava, Nikhil's construction of Twice Ranmanujan Sparsifier, base on their paper Twice Ramanujan Sparsifiers([1]). We use this material to explore variants of this construction in Chapter 4 and 5.

3.1 Main result

Theorem 3.1.0.1. *([1]) Suppose $d > 1$, and $v_1, v_2, ..., v_m$ are vectors in $\mathbb{R}^N$ with*

$$\sum_{i \leq m} v_i v_i^T = id_{\mathbb{R}^n} \tag{3.1}$$

Then, there exist scalars $s_i \geq 0$ with $|\{i : s_i > 0\}| \leq dn$, with

$$id_{\mathbb{R}^n} \preceq \sum_{i \leq m} s_i v_i v_i^T \preceq \frac{d+1+2\sqrt{d}}{d+1-2\sqrt{d}} id_{\mathbb{R}^n} \tag{3.2}$$

Definition 3.1.0.2. *Let L be a symmetric Laplacian matrix, it can be diagonalized by:*

$$L = \sum_{i=1}^{n-1} \lambda_i u_i u_i^T \tag{3.3}$$

here $\lambda_1, \lambda_2, ...\lambda_{n-1}$ are the nonzero eigenvalues of L and $u_1, u_2, ..., u_{n-1}$ are the corresponding set of orthonormal eigenvectors. Then, we can define the $Moore-$

Penrose Pseudoinverse of L as:

$$L^+ = \sum_{i=1}^{n-1} \frac{1}{\lambda_i} u_i u_i^T \tag{3.4}$$

Note that, since the eigenvector preserves, $ker(L) = ker(L^+)$, and

$$L^+ L = L L^+ = \sum_{i=1}^{n-1} u_i u_i^T = id_{Im(L)} \tag{3.5}$$

Then, by using the theorem above, we can deduce the following main theorem:

Theorem 3.1.0.3. *([1]) For every $\varepsilon \in (0,1)$, every undirected weighted graph $G = (V,E,w)$ on n vertices contains a weighted subgraph $H = (V,F,\tilde{w})$ with $\lceil (n-1)/\varepsilon^2 \rceil$ edges, such that:*

$$(1-\varepsilon)^2 L_G \preceq L_H \preceq (1+\varepsilon)^2 L_G. \tag{3.6}$$

Proof. Given the incident matrix B_G and the weight matrix W corresponds to w. The Laplacian matrix $L_G = B_G^T W B_G$ of G. Let L_G^+ be the pseudo-inverse of L_G. As L_G and L_G^+ are positive semi-definate, taking the square root L_G^+, $(L_G^+)^{1/2}$, and let consider the $|V| \times |E|$ matrix $V_{n\times m}$ to be:

$$V_{n\times m} = (L_G^+)^{1/2} B_G^T W^{1/2} \tag{3.7}$$

Let the columns of $V_{n\times m}$ to be $v_1, v_2, ..., v_m$, we have:

$$\begin{aligned}\sum_i v_i v_i^T &= VV^T \\ &= (L_G^+)^{1/2} B_G^T W^{1/2} W^{1/2} B_G (L_G^+)^{1/2} \\ &= (L_G^+)^{1/2} L_G (L_G^+)^{1/2} = id_{im(L_G)}\end{aligned}$$

By Theorem 3.1.0.2., we shall have some $\{s_i\}_{i\leq m}$, such that:

$$id_{im(L_G)} \preceq \sum_{i\leq m} s_i v_i v_i^T \preceq \frac{d+1+2\sqrt{d}}{d+1-2\sqrt{d}} id_{im(L_G)} \tag{3.8}$$

Let S be a $|E| \times |E|$ diagonal matrix with $S(i,i) = s_i$, and let $L_H = B_G^T W^{1/2} S W^{1/2} B_G$, by the Min-Max theorem of eigenvalue, $\forall y \in im(L_G)$:

$$1 \leq \frac{y^T V S V^T y}{y^T y} \leq \frac{d+1+2\sqrt{d}}{d+1-2\sqrt{d}}$$
$$1 \leq \frac{y^T L_G^+ L_H L_G^+ y}{y^T y} \leq \frac{d+1+2\sqrt{d}}{d+1-2\sqrt{d}}$$

Consider some $x \in \mathbb{R}^n$, $x \perp 1$, $x L_G^{1/2} \in im(L_G^{1/2}) = im(L_G)$, and thus:

$$1 \leq \frac{(x L_G^{1/2})^T L_G^+ L_H L_G^+ x L_G^{1/2}}{(x L_G^{1/2})^T x L_G^{1/2}} \leq \frac{d+1+2\sqrt{d}}{d+1-2\sqrt{d}}$$
$$1 \leq \frac{x^T L_H x}{x^T L_G x} \leq \frac{d+1+2\sqrt{d}}{d+1-2\sqrt{d}}$$

□

3.2 Barrier Functions

Definition 3.2.0.1. *([1]) Given $u, l \in \mathbb{R}$, and A a $n \times n$ symmetric matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Defines:*

$$\Phi^u(A) = Trace((uI - A)^{-1}) = \sum_i \frac{1}{u - \lambda_i} \tag{3.9}$$
$$\Phi^l(A) = Trace((A - lI)^{-1}) = \sum_i \frac{1}{\lambda_i - l} \tag{3.10}$$

In [1], the authors use those barrier functions to let the eigenvalues remain inside the boundary u, l, i.e. $\lambda_{max}(A) < u$, $\lambda_{min}(A) > l$.
Let $A^{(0)}$ to be the all zero matrix, with

$$\Phi^u(A^{(0)}) = \varepsilon_u \tag{3.11}$$
$$\Phi^l(A^{(0)}) = \varepsilon_l \tag{3.12}$$

for some $\varepsilon_u, \varepsilon_l \geq 0$. At some step q, with the matrix $A^{(q)}$, let $A^{(q+1)} = A^{(q)} + t v v^T$ with some given vector v and step-size t, we want the updated potential to be non-increasing with the upper and lower bound $u^{(q+1)} = u^{(q)} + \delta_u$ and $l^{(q+1)} = l^{(q)} + \delta_l$.

$$\Phi^{u^{(q+1)}}(A + tv^T v) \leq \Phi^{u^{(q)}}(A^{(q)}) \leq \varepsilon_u \tag{3.13}$$

$$\Phi^{L^{(q+1)}}(A + tv^T v) \leq \Phi^{L^{(q)}}(A^{(q)}) \leq \varepsilon_l \tag{3.14}$$

3.3 Upper Barrier Shift

At some step q, with the matrix $A^{(q)}$, let $A^{(q+1)} = A^{(q)} + tvv^T$ with some given vector v and step-size t. We bound $A^{(q+1)}$ by the new upper bound $u^{(q+1)} = u^{(q)} + \delta_u$.

By the Sherman–Morrison formula, we can write the updated potential as:

$$\begin{aligned}
\Phi^{u^{(q+1)}}(A^{(q+1)}) &= Trace((u^{(q+1)}I - A^{(q)} - tvv^T)^{-1}) \\
&= Trace((u^{(q+1)}I - A^{(q)})^{-1} + \frac{t(u^{(q+1)}I - A^{(q)})^{-1}vv^T(u^{(q+1)}I - A^{(q)})^{-1}}{1 - tv^T(u^{(q+1)}I - A^{(q)})^{-1}v}) \\
&= Trace((u^{(q+1)}I - A^{(q)})^{-1}) + \frac{tTrace(v^T(u^{(q+1)}I - A^{(q)})^{-1}(u^{(q+1)}I - A^{(q)})^{-1}v)}{1 - tv^T(u^{(q+1)}I - A^{(q)})^{-1}v} \\
&= \Phi^{u^{(q+1)}}(A^{(q)}) + \frac{tv^T(u^{(q+1)}I - A^{(q)})^{-2}v}{1 - tv^T(u^{(q+1)}I - A^{(q)})^{-1}v} \\
&= \Phi^{u^{(q)}}(A^{(q)}) - (\Phi^{u^{(q)}}(A^{(q)}) - \Phi^{u^{(q+1)}}(A^{(q)})) + \frac{v^T(u^{(q+1)}I - A^{(q)})^{-2}v}{1/t - v^T(u^{(q+1)}I - A^{(q)})^{-1}v}
\end{aligned}$$

Lemma 3.3.0.1 (Lemma 3.3 in [1])**.** *Suppose that $\lambda_{max(A)} < u$, and v is any vector. If*

$$\frac{1}{t} \geq v^T(u^{(q+1)}I - A^{(q)})^{-1}v + \frac{v^T(u^{(q+1)}I - A^{(q)})^{-2}v}{\Phi^{u^{(q)}}(A^{(q)}) - \Phi^{u^{(q+1)}}(A^{(q)})} = U_{A^{(q)}}(v) \tag{3.15}$$

then

$$\Phi^{u^{(q+1)}}(A^{(q+1)}) \leq \Phi^{u^{(q)}}(A^{(q)}) \text{ and } \lambda_{max}(A^{(q)} + tvv^T) < u + \delta_u \tag{3.16}$$

Proof.

$$
\begin{aligned}
\Phi^{u^{(q+1)}}(A^{(q+1)}) &= \Phi^{u^{(q)}}(A^{(q)}) - (\Phi^{u^{(q)}}(A^{(q)}) - \Phi^{u^{(q+1)}}(A^{(q)})) + \frac{v^T(u^{(q+1)}I - A^{(q)})^{-2}v}{1/t - v^T(u^{(q+1)}I - A^{(q)})^{-1}v} \\
&\leq \Phi^{u^{(q)}}(A^{(q)}) - (\Phi^{u^{(q)}}(A^{(q)}) - \Phi^{u^{(q+1)}}(A^{(q)})) + \frac{v^T(u^{(q+1)}I - A^{(q)})^{-2}v}{U_{A^{(q)}}(v) - v^T(u^{(q+1)}I - A^{(q)})^{-1}v} \\
&= \Phi^{u^{(q)}}(A^{(q)}) - (\Phi^{u^{(q)}}(A^{(q)}) - \Phi^{u^{(q+1)}}(A^{(q)})) + (\Phi^{u^{(q)}}(A^{(q)}) - \Phi^{u^{(q+1)}}(A^{(q)})) \\
&= \Phi^{u^{(q)}}(A^{(q)})
\end{aligned}
$$

□

Note that, as the potential function remains finite during the update, $\lambda_{max}(A^{(q+1)}) < u^{(q+1)}$.

Lemma 3.3.0.2 (Lemma 3.5 in [1])**.**

$$
\sum_i U_{A^{(q)}}(v_i) \leq \frac{1}{\delta_u} + \varepsilon_u \tag{3.17}
$$

Proof.

$$
\begin{aligned}
\sum_i U_{A^{(q)}}(v_i) &= \sum_i v_i^T(u^{(q+1)}I - A^{(q)})^{-1}v_i + \frac{\sum_i v_i^T(u^{(q+1)}I - A^{(q)})^{-2}v_i}{\Phi^{u^{(q)}}(A^{(q)}) - \Phi^{u^{(q+1)}}(A^{(q)})} \\
&= (u^{(q+1)}I - A^{(q)})^{-1} \cdot \sum_i v_i v_i^T + \frac{(u^{(q+1)}I - A^{(q)})^{-2} \cdot \sum_i v_i v_i^T}{\Phi^{u^{(q)}}(A^{(q)}) - \Phi^{u^{(q+1)}}(A^{(q)})} \\
&= \sum_i (u^{(q+1)}I - \lambda_i^{(q)})^{-1} + \frac{\sum_i (u^{(q+1)} - \lambda_i^{(q)})^{-2}}{\sum_i (u^{(q)} - \lambda_i^{(q)})^{-1} - \sum_i (u^{(q+1)} - \lambda_i^{(q)})^{-1}} \\
&= \sum_i (u^{(q+1)} - \lambda_i^{(q)})^{-1} + \frac{\sum_i (u^{(q+1)} - \lambda_i^{(q)})^{-2}}{\delta_u \sum_i (u^{(q)} - \lambda_i^{(q)})^{-1}(u^{(q+1)} - \lambda_i^{(q)})^{-1}} \\
&\leq \sum_i (u^{(q)} - \lambda_i^{(q)})^{-1} + \frac{\sum_i (u^{(q+1)} - \lambda_i^{(q)})^{-2}}{\delta_u \sum_i (u^{(q+1)} - \lambda_i^{(q)})^{-2}} \\
&\leq \frac{1}{\delta_u} + \varepsilon_u
\end{aligned}
$$

□

3.4 Lower Barrier Shift

At some step q, with the matrix $A^{(q)}$, let $A^{(q+1)} = A^{(q)} + tvv^T$ with some given vector v and step-size t. We bound $A^{(q+1)}$ by the new lower bound $l^{(q+1)} = l^{(q)} + \delta_l$. By the Sherman–Morrison formula, we can write the updated potential as:

$$\begin{aligned}
\Phi^{l^{(q+1)}}(A^{(q+1)}) &= Trace((A^{(q)} + tvv^T - l^{(q+1)I})^{-1}) \\
&= Trace((A^{(q)} - l^{(q+1)}I)^{-1} - \frac{t(A^{(q)} - l^{(q+1)}I)^{-1}vv^T(A^{(q)} - l^{(q+1)}I)^{-1}}{1 + tv^T(A^{(q)} - l^{(q+1)}I)^{-1}v}) \\
&= Trace((A^{(q)} - l^{(q+1)}I)^{-1}) - \frac{tTrace(A^{(q)} - l^{(q+1)}I)^{-1}(A^{(q)} - l^{(q+1)}I)^{-1})v)}{1 + tv^T(A^{(q)} - l^{(q+1)}I)^{-1}v} \\
&= \Phi^{l^{(q+1)}}(A^{(q)}) - \frac{tv^T(A^{(q)} - l^{(q+1)}I)^{-2}v}{1 + tv^T(A^{(q)} - l^{(q+1)}I)^{-1}v} \\
&= \Phi^{l^{(q)}}(A^{(q)}) + (\Phi^{l^{(q+1)}}(A^{(q)}) - \Phi^{l^{(q)}}(A^{(q)})) - \frac{v^T(A^{(q)} - l^{(q+1)}I)^{-2}v}{1/t + v^T(A^{(q)} - l^{(q+1)}I)^{-1}v}
\end{aligned}$$

Lemma 3.4.0.1 (Lemma 3.4 in [1]). *Suppose that $\lambda_{min(A^q)} > l$, and $\Phi^{l^{(q)}}(A^{(q)}) \leq 1/\delta_l$. Let v be any vector. If*

$$0 \leq \frac{1}{t} \leq -v^T(A^{(q)} - l^{(q+1)}I)^{-1}v + \frac{v^T(A^{(q)} - l^{(q+1)}I)^{-2}v}{\Phi^{l^{(q+1)}}(A^{(q)}) - \Phi^{l^{(q)}}(A^{(q)})} = L_{A^{(q)}}(v) \quad (3.18)$$

then

$$\Phi^{l^{(q+1)}}(A^{(q+1)}) \leq \Phi^{l^{(q)}}(A^{(q)}) \text{ and } \lambda_{min}(A + tvv^T) < l + \delta_l \quad (3.19)$$

Proof. Firstly, $\lambda_{min(A^q)} > l$ and $\Phi^{l^{(q)}}(A^{(q)}) \leq 1/\delta_l$ implies all eigenvalues of A^q, $\lambda_i(A^q) \geq l^{(q)} + \delta_l = l^{(q+1)}$ for all i. Thus, $\lambda_i(A^{(q+1)}) \geq l^{(q+1)}$.

$$\begin{aligned}
\Phi^{l^{(q+1)}}(A^{(q+1)}) &= \Phi^{l^{(q)}}(A^{(q)}) + (\Phi^{l^{(q+1)}}(A^{(q)}) - \Phi^{l^{(q)}}(A^{(q)})) - \frac{v^T(A^{(q)} - l^{(q+1)}I)^{-2}v}{1/t + v^T(A^{(q)} - l^{(q+1)}I)^{-1}v} \\
&\leq \Phi^{l^{(q)}}(A^{(q)}) + (\Phi^{l^{(q+1)}}(A^{(q)}) - \Phi^{l^{(q)}}(A^{(q)})) - \frac{v^T(A^{(q)} - l^{(q+1)}I)^{-2}v}{L_{A^{(q)}}(v) + v^T(A^{(q)} - l^{(q+1)}I)^{-1}v} \\
&= \Phi^{l^{(q)}}(A^{(q)}) + (\Phi^{l^{(q+1)}}(A^{(q)}) - \Phi^{l^{(q)}}(A^{(q)})) - (\Phi^{l^{(q+1)}}(A^{(q)}) - \Phi^{l^{(q)}}(A^{(q)}))
\end{aligned}$$

$$= \Phi^{l^{(q)}}(A^{(q)})$$

□

Lemma 3.4.0.2 (Lemma 3.5, Lemma 3.6 in [1]).

$$\sum_i L_{A^{(q)}}(v_i) \geq \frac{1}{\delta_l} - \varepsilon_l \tag{3.20}$$

Proof.

$$\begin{aligned}
\sum_i L_{A^{(q)}}(v_i) &= -\sum_i v_i^T (A^{(q)} - l^{(q+1)} I)^{-1} v_i + \frac{\sum_i v_i^T (A^{(q)} - l^{(q+1)} I)^{-2} v_i}{\Phi^{l^{(q+1)}}(A^{(q)}) - \Phi^{l^{(q)}}(A^{(q)})} \\
&= -(A^{(q)} - l^{(q+1)} I)^{-1} \cdot \sum_i v_i v_i^T + \frac{(A^{(q)} - l^{(q+1)} I)^{-2} \cdot \sum_i v_i v_i^T}{\Phi^{l^{(q+1)}}(A^{(q)}) - \Phi^{l^{(q)}}(A^{(q)})} \\
&= -\sum_i (\lambda_i^{(q)} - l^{(q+1)})^{-1} + \frac{\sum_i (\lambda_i^{(q)} - l^{(q+1)})^{-2}}{\sum_i (\lambda_i^{(q)} - l^{(q+1)})^{-1} - \sum_i (\lambda_i^{(q)} - l^{(q)})^{-1}}
\end{aligned}$$

Now, to prove

$$-\sum_i (\lambda_i^{(q)} - l^{(q+1)})^{-1} + \frac{\sum_i (\lambda_i^{(q)} - l^{(q+1)})^{-2}}{\sum_i (\lambda_i^{(q)} - l^{(q+1)})^{-1} - \sum_i (\lambda_i^{(q)} - l^{(q)})^{-1}} \geq \frac{1}{\delta_l} - \sum_i (\lambda_i^{(q)} - l^{(q)})^{-1}$$

We have:

$$\begin{aligned}
-\sum_i (\lambda_i^{(q)} - l^{(q+1)})^{-1} + \frac{\sum_i (\lambda_i^{(q)} - l^{(q+1)})^{-2}}{\sum_i (\lambda_i^{(q)} - l^{(q+1)})^{-1} - \sum_i (\lambda_i^{(q)} - l^{(q)})^{-1}} &\geq \frac{1}{\delta_l} - \sum_i (\lambda_i^{(q)} - l^{(q)})^{-1} \\
\frac{\sum_i (\lambda_i^{(q)} - l^{(q+1)})^{-2}}{\sum_i (\lambda_i^{(q)} - l^{(q+1)})^{-1} - \sum_i (\lambda_i^{(q)} - l^{(q)})^{-1}} &\geq \frac{1}{\delta_l} - \sum_i (\lambda_i^{(q)} - l^{(q)})^{-1} + \sum_i (\lambda_i^{(q)} - l^{(q+1)})^{-1}
\end{aligned}$$

As $\sum_i(\lambda_i^{(q)} - l^{(q+1)})^{-1} - \sum_i(\lambda_i^{(q)} - l^{(q)})^{-1} > 0$:

$$\sum_i(\lambda_i^{(q)} - l^{(q+1)})^{-2} \geq (\frac{1}{\delta_l} - \sum_i \delta_l \frac{1}{(\lambda_i^{(q)} - l^{(q)})(\lambda_i^{(q)} - l^{(q+1)})})(\sum_i \delta_l \frac{1}{(\lambda_i^{(q)} - l^{(q)})(\lambda_i^{(q)} - l^{(q+1)})})$$
$$\sum_i(\lambda_i^{(q)} - l^{(q+1)})^{-2} \geq \sum_i \frac{1}{(\lambda_i^{(q)} - l^{(q)})(\lambda_i^{(q)} - l^{(q+1)})} + (\sum_i \delta_l \frac{1}{(\lambda_i^{(q)} - l^{(q)})(\lambda_i^{(q)} - l^{(q+1)})})^2$$
$$\sum_i(\lambda_i^{(q)} - l^{(q+1)})^{-2} - \sum_i \frac{1}{(\lambda_i^{(q)} - l^{(q)})(\lambda_i^{(q)} - l^{(q+1)})} \geq (\sum_i \delta_l \frac{1}{(\lambda_i^{(q)} - l^{(q)})(\lambda_i^{(q)} - l^{(q+1)})})^2$$
$$\sum_i \delta_l \frac{1}{(\lambda_i^{(q)} - l^{(q)})(\lambda_i^{(q)} - l^{(q+1)})^2} \geq (\sum_i \delta_l \frac{1}{(\lambda_i^{(q)} - l^{(q)})(\lambda_i^{(q)} - l^{(q+1)})})^2$$

By Cauchy–Schwarz inequality,

$$(\delta_l \sum_i \frac{1}{\lambda_i^{(q)} - l^{(q)}}) \sum_i \delta_l \frac{1}{(\lambda_i^{(q)} - l^{(q)})(\lambda_i^{(q)} - l^{(q+1)})^2} \geq (\sum_i \delta_l \frac{1}{(\lambda_i^{(q)} - l^{(q)})(\lambda_i^{(q)} - l^{(q+1)})})^2$$

As $\delta_l \Phi^{l^{(q)}}(A^{(q)}) \leq \delta_l \varepsilon_l \leq 1$, the inequality is valid. □

Theorem 3.4.0.3 (Lemma 3.5 in [1]). *Therefore, as long as*

$$\frac{1}{\delta_l} - \varepsilon_l \geq \frac{1}{\delta_u} + \varepsilon_u \geq 0 \tag{3.21}$$

we can always find such a v_i, with $U_{A^{(q)}}(v_i) \leq L_{A^{(q)}}(v_i)$. By using some t, $U_{A^{(q)}}(v_i) \leq t \leq L_{A^{(q)}}(v_i)$, we can update $A^{(q)}$ with the potential to be non-increasing.

In [1]'s construction, they set $A^{(0)}$ to be the all-zero matrix, and :

$$\delta_l = 1 \qquad \delta_u = \frac{\sqrt{d}+1}{\sqrt{d}-1}$$
$$\varepsilon_l = \frac{1}{\sqrt{d}} \qquad \varepsilon_u = \frac{\sqrt{d}-1}{d+\sqrt{d}}$$
$$l^0 = -n/\varepsilon_l \qquad u^0 = n/\varepsilon_u$$

Theorem 3.3.0.3 is satisfied as:

$$\frac{1}{\delta_u} + \varepsilon_u = \frac{\sqrt{d}-1}{\sqrt{d}+1} + \frac{\sqrt{d}-1}{d+\sqrt{d}} = \frac{d-1}{d+\sqrt{d}} = 1 - \frac{1}{\sqrt{d}} = \varepsilon_l - \frac{1}{\delta_l}$$

And after dn updates,

$$\begin{aligned}
\frac{\lambda_{max} A^{dn}}{\lambda_{min} A^{dn}} &\leq \frac{n/\varepsilon_u + dn\delta_u}{n/\varepsilon_l + dn\delta_l} \\
&= \frac{nd * \frac{\sqrt{d}+1}{\sqrt{d}-1} + n\frac{d+\sqrt{d}}{\sqrt{d}-1}}{nd - n\sqrt{d}} \\
&= \frac{d * \frac{\sqrt{d}+1}{\sqrt{d}-1} + \frac{d+\sqrt{d}}{\sqrt{d}-1}}{d - \sqrt{d}} \\
&= \frac{\sqrt{d} * (\sqrt{d}+1) + (\sqrt{d}+1)}{(\sqrt{d}-1)^2} \\
&= \frac{d + 2\sqrt{d} + 1}{d - 2\sqrt{d} + 1}
\end{aligned}$$

Chapter 4

Idea and Variation 1

4.1 Relaxation of the potential formula

Definition 4.1.0.1. *We generalize the potential* $\Phi^u(A) = \sum \frac{1}{u-\lambda_i}$ *by replacing* $u - \lambda_i$ *by:*

$$p(u, \lambda_i) = p_1(u) - \lambda_i p_2(u) \tag{4.1}$$

with some polynomial $p_1(u)$ *and* $p_2(u)$. *We denote this potential by* $\Phi_p^u(A)$.

Then, consider the relaxed potential function:

$$\Phi_p^u(A) = \sum_i \frac{\frac{d}{du} p(u, \lambda_i)}{p(u, \lambda_i)} = \sum_i \frac{p_1'(u) - p_2' \lambda_i}{p_1(u) - p_2(u)\lambda_i} \tag{4.2}$$

In order to apply the Sherman–Morrison formula to the rank-one update, we use the Mobious transform:

$$\begin{aligned}\sum_i \frac{p_1'(u) - p_2' \lambda_i}{p_1(u) - p_2(u)\lambda_i} &= \sum_i \left(\frac{p_1'(u) - p_2' \lambda_i}{p_1(u) - p_2(u)\lambda_i} - \frac{(p_1(u) - p_2(u)\lambda_i) * \frac{p_2'}{p2(u)}}{p_1(u) - p_2(u)\lambda_i} + \frac{p_2'}{p2(u)} \right) \\ &= \sum_i \left(\frac{p_1'(u) - p_1(u) p_2' / p_2(u)}{p_1(u) - p_2(u)\lambda_i} + \frac{p_2'}{p_2(u)} \right)\end{aligned}$$

Let $C_1^u = p_1'(u) - p_1(u)p_2'/p_2(u)$

$$\Phi_p^u = Trace(C_1^u \frac{1}{p_1(u)I - p_2(u)A} + \frac{p_2'}{p_2(u)}I) \tag{4.3}$$

4.1.1 Sherman–Morrison

$$\Phi_p^{u+\delta u}(A + tv^T v) = Trace(C_1^{u+\delta u} \frac{1}{p_1(u+\delta u)I - p_2(u+\delta u)(A+tv^T v)} + \frac{p_2'}{p_2(u+\delta u)}I) \tag{4.4}$$

$$= Trace(C_1^{u+\delta u} \frac{1}{p_2(u+\delta u)} \frac{1}{\frac{p_1(u+\delta u)}{p_2(u+\delta u)}I - A - tv^T v} + \frac{p_2'}{p_2(u+\delta u)}I) \tag{4.5}$$

4.2 Variation 1

4.2.1 Upper potential

Definition 4.2.1.1. *Base on the notion of non-backtracking matrix, we consider the potential:*

$$\Phi^u(A) = \sum_i (\frac{1}{\mu_{i,1}} + \frac{1}{\mu_{i,2}}) = \sum \frac{2u - m - \lambda_i}{(u-m)u - (u-m)\lambda_i + \beta} \tag{4.6}$$

Where $\mu_{i,1}$ and $\mu_{i,2}$ are the roots of:

$$(u-m)^2 - (u-m)(\lambda_i - m) + \beta = 0 \tag{4.7}$$

related with u,m,λ_i and β. Solving the equation, we get:

$$\mu = \frac{(\lambda + m) \pm \sqrt{(\lambda + m)^2 - 4 * (m\lambda + \beta)}}{2} \tag{4.8}$$

$$= \frac{(\lambda + m) \pm \sqrt{(\lambda - m)^2 - 4 * \beta}}{2} \tag{4.9}$$

In this variation, we assume $(u-m)=\kappa$ and β are constant.

Claim 4.2.1.2.

$$\Phi^u(A)=\sum_i \frac{c_1}{u-\lambda_i+c_2}+c_3 \tag{4.10}$$

where c_1, c_2, c_3 are constants.

Proof. By applying the Mobious Transform to the potential function:

$$\Phi^u(A)=\sum_i \frac{2u-m-\lambda_i}{(u-m)u-(u-m)\lambda_i+\beta} \tag{4.11}$$

$$=\sum_i\left(\frac{2u-m-\lambda_i}{(u-m)u-(u-m)\lambda_i+\beta}-\sum_i \frac{\frac{(u-m)u-(u-m)\lambda_i+\beta}{u-m}}{(u-m)u-(u-m)\lambda_i+\beta}\right)+\frac{n}{u-m} \tag{4.12}$$

$$=\sum_i \frac{u-m-\frac{\beta}{u-m}}{(u-m)u-(u-m)\lambda_i+\beta}+\frac{n}{u-m} \tag{4.13}$$

$$=\sum_i \frac{1-\frac{\beta}{(u-m)^2}}{u-\lambda_i+\frac{\beta}{u-m}}+\frac{n}{u-m} \tag{4.14}$$

Since we assumed that $(u-m)=\kappa$ and β are constant, let $c_1=1-\frac{\beta}{u-m}^2$, $c_2=\frac{\beta}{u-m}$, $c_3=\frac{n}{u-m}$ be constants. The formula becomes:

$$\sum_i \frac{c_1}{u-\lambda_i+c_2}+c_3 \tag{4.15}$$

Therefore, the formula simply shifts u by a constant and multiplies the function by also a constant. Therefore, it doesn't bring us any change. □

Chapter 5

Variation 2

Inspired by the idea of non-backtracking matrices, we try to replace each eigenvalue λ_i of A by the corresponding non-backtracking eigenvalues $\mu_{i,1}, \mu_{i,2}$, and taking those values to the complex plane.

5.1 Upper potential function

Definition 5.1.0.1. *Given $u, l, c, \lambda_i \in \mathbb{R}$, with $\beta = c(u-l)^2$, we define $\mu_{i,1}, \mu_{i,2}$ to be the roots of the formula:*

$$(\mu - l)^2 - (\mu - l)(\lambda_i - l) + \beta = 0 \tag{5.1}$$

The equation could be simplified as:

$$\mu^2 - \mu(\lambda_i + l) + l\lambda_i + \beta = (\mu - l)(\mu - \lambda_i) + \beta \tag{5.2}$$

Moreover, we observe:

$$\mu_{i,1} + \mu_{i,2} = (\lambda_i + l) \tag{5.3}$$

$$\mu_{i,1}\mu_{i,2} = l\lambda_i + \beta \tag{5.4}$$

Definition 5.1.0.2. *Given $u, l \in \mathbb{R}$, and A a $n \times n$ symmetric matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.*

For each λ_i, we define the corresponding $\mu_{i,1}, \mu_{i,2}$ as above, and the upper potential Φ_l^u will be:

$$\Phi_l^u(A^{(q)}) = \sum_i (\frac{1}{u-\mu_{i,1}} + \frac{1}{u-\mu_{i,2}}) \tag{5.5}$$

Claim 5.1.0.3. *With $c < 1$, $u > l$, we can write the upper potential function as:*

$$\Phi_l^u = (1-c)Trace((u+c(u-l))I - A^{(q)})^{-1}) + \frac{n}{u-l}$$

Proof. Rewrite $\Phi_l^u(A^{(q)})$ as:

$$\Phi_l^u(A^{(q)}) = \sum_i \frac{(u-\mu_{i,1})+(u-\mu_{i,2})}{(u-\mu_{i,1})(u-\mu_{i,2})} = \sum_i \frac{2u-l-\lambda_i}{(u-l)u-(u-l)\lambda_i+\beta} \tag{5.6}$$

Apply the Mobius transform to this formula:

$$\Phi_l^u(A^{(q)}) = \sum_i \frac{2u-l-\lambda_i}{(u-l)u-(u-l)\lambda_i+\beta} \tag{5.7}$$

$$= \sum_i (\frac{2u-l-\lambda_i}{(u-l)u-(u-l)\lambda_i+\beta} - \frac{\frac{(u-l)u-(u-l)\lambda_i+\beta}{u-l}}{(u-l)u-(u-l)\lambda_i+\beta} + \frac{1}{u-l}) \tag{5.8}$$

$$= \sum_i \frac{u-l-\frac{\beta}{u-l}}{(u-l)u-(u-l)\lambda_i+\beta} + \frac{n}{u-l} \tag{5.9}$$

$$= \sum_i \frac{1-\frac{\beta}{(u-l)^2}}{u-\lambda_i+\frac{\beta}{u-l}} + \frac{n}{u-l} \tag{5.10}$$

$$= (1-c)Trace((u+c(u-l))I - A^{(q)})^{-1}) + \frac{n}{u-l} \tag{5.11}$$

□

As

$$\mu_i = \frac{(\lambda+l) \pm \sqrt{(\lambda-l)^2 - 4*\beta}}{2} \tag{5.12}$$

We tend to let $c > 0.25$ for putting most of those roots on the complex plane. Moreover, the constant c is strictly bounded above by 1, otherwise, the potential function becomes non-positive.

5.2 Lower potential function

Definition 5.2.0.1. *Given $u, l, c, \lambda_i \in \mathbb{R}$, with $\beta = c(u-l)^2$, we define $\mu_{i,1}, \mu_{i,2}$ to be the roots of the formula:*

$$(\mu - u)^2 - (\mu - u)(\lambda_i - u) + \beta = 0 \tag{5.13}$$

The equation could be simplified as:

$$\mu^2 - \mu(\lambda_i + u) + u\lambda_i + \beta = (\mu - u)(\mu - \lambda_i) + \beta \tag{5.14}$$

Moreover, we observe:

$$\mu_{i,1} + \mu_{i,2} = (\lambda_i + u) \tag{5.15}$$

$$\mu_{i,1}\mu_{i,2} = u\lambda_i + \beta \tag{5.16}$$

Definition 5.2.0.2. *Given $u, l \in \mathbb{R}$, and A a $n \times n$ symmetric matrix with eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$.*
For each λ_i, we define the corresponding $\mu_{i,1}, \mu_{i,2}$ as above, and the upper potential Φ_l^u will be:

$$\Phi_u^l(A^{(q)}) = \sum_i \left(\frac{1}{\mu_{i,1} - l} + \frac{1}{\mu_{i,2} - l}\right) \tag{5.17}$$

Claim 5.2.0.3. *With $c < 1$, $u > l$, we can write the upper potential function as:*

$$\Phi_u^l = (1-c)Trace(A^{(q)} - (l - c(u-l))I)^{-1}) + \frac{n}{u-l}$$

Proof. Rewrite $\Phi_u^l(A^{(q)})$ as:

$$\Phi_u^l(A^{(q)}) = \sum_i \frac{(\mu_{i,1} - l) + (\mu_{i,2} - l)}{(\mu_{i,1} - l)(\mu_{i,2} - l)} = \sum_i \frac{\lambda_i + u - 2l}{(u-l)\lambda_i - (u-l)l + \beta} \tag{5.18}$$

Apply the Mobius transform to this formula:

$$\Phi_u^l(A^{(q)}) = \sum_i \frac{\lambda_i + u - 2l}{(u-l)\lambda_i - (u-l)l + \beta} \tag{5.19}$$

$$= \sum_i \left(\frac{\lambda_i + u - 2l}{(u-l)\lambda_i - (u-l)l + \beta} - \frac{\frac{(u-l)\lambda_i - (u-l)l + \beta}{u-l}}{(u-l)\lambda_i - (u-l)l + \beta} + \frac{1}{u-l} \right) \tag{5.20}$$

$$= \sum_i \frac{u - l - \frac{\beta}{u-l}}{(u-l)\lambda_i - (u-l)l + \beta} + \frac{n}{u-l} \tag{5.21}$$

$$= \sum_i \frac{1 - \frac{\beta}{(u-l)^2}}{\lambda_i - l + \beta} + \frac{n}{u-l} \tag{5.22}$$

$$= (1-c)Trace(A^{(q)} - (l - c(u-l))I)^{-1}) + \frac{n}{u-l} \tag{5.23}$$

Same as for the upper potential, we tend to let $1 > c > 0.25$ for putting most of those roots on the complex plane. □

5.3 Upper Barrier Shift

At some step q, with the matrix $A^{(q)}$, let $A^{(q+1)} = A^{(q)} + tvv^T$ with some given vector v and step-size t. We bound $A^{(q+1)}$ by the new upper bound $u^{(q+1)} = u^{(q)} + \delta_u$.

By the Sherman–Morrison formula, we can write the updated potential as:

$$\begin{aligned}
\Phi^{u^{(q+1)}}_{l^{(q+1)}}(A^{(q)}+tvv^T) &= (1-\frac{\beta^{(q+1)}}{(u^{(q+1)}-l^{(q+1)})^2})*Trace((u^{(q+1)}+\frac{\beta^{(q+1)}}{(u^{(q+1)}-l^{(q+1)})})*I-A^{(q)}-tvv^T)^{-1}) \\
&+\frac{n}{(u^{(q+1)}-l^{(q+1)})} \\
&= \frac{n}{(u^{(q+1)}-l^{(q+1)})}+Trace((u^{(q+1)}+\frac{\beta^{(q+1)}}{(u^{(q+1)}-l^{(q+1)})})*I-A^{(q)})^{-1} \\
&+\frac{t*(u^{(q+1)}+\frac{\beta^{(q+1)}}{(u^{(q+1)}-l^{(q+1)})})*I-A^{(q)})^{-1}vv^T(u^{(q+1)}+\frac{\beta^{(q+1)}}{(u^{(q+1)}-l^{(q+1)})})*I-A^{(q)})^{-1}}{1-t*v^T(u^{(q+1)}+\frac{\beta^{(q+1)}}{(u^{(q+1)}-l^{(q+1)})})*I-A^{(q)})^{-1}v})*(1-c)) \\
&= \Phi^{u^{(q+1)}}_{l^{(q+1)}}(A^{(q)})+(\frac{v^T(u^{(q+1)}+\frac{\beta^{(q+1)}}{(u^{(q+1)}-l^{(q+1)})})*I-A^{(q)})^{-2}v}{1/t-v^T(u^{(q+1)}+\frac{\beta^{(q+1)}}{(u^{(q+1)}-l^{(q+1)})})*I-A^{(q)})^{-1}v})*(1-c) \\
&= \Phi^u_l(A^{(q)})-(\Phi^u_l(A^{(q)})-\Phi^{u^{(q+1)}}_{l^{(q+1)}}(A^{(q)})) \\
&+(\frac{v^T(u^{(q+1)}+\frac{\beta^{(q+1)}}{(u^{(q+1)}-l^{(q+1)})})*I-A^{(q)})^{-2}v}{1/t-v^T(u^{(q+1)}+\frac{\beta^{(q+1)}}{(u^{(q+1)}-l^{(q+1)})})*I-A^{(q)})^{-1}v})*(1-c)
\end{aligned}$$

However, we see the first obstacle to modifying the potential function.

Theorem 5.3.0.1 (Obstacle 1)**.** *Due to the property of the Sherman Morrison formula, the rank-one update will always focus on the $p_1(u)I-p_2(u)A$ part of the function. Therefore, if we try to sum all the possible rank-one updates to make $\sum_i v_i^T v = I$, and apply the similar lemma 3.4.0.3 with a tight initial bound such as:*

$$\frac{1}{\delta_l}-\varepsilon_l \geq \sum_i L_{A^{(q)}}(v_i) \geq \sum_i U_{A^{(q)}}(v_i)\frac{1}{\delta_u}+\varepsilon_u \geq 0 \tag{5.24}$$

We must keep $Trace((u^{(q+1)}+\frac{\beta^{(q+1)}}{(u^{(q+1)}-l^{(q+1)})})*I-A^{(q+1)}))^{-1}$ non-increasing in our formula, which forces our potential to decrease during each update.

Claim 5.3.0.2. *To make $Trace((u^{(q+1)}+\frac{\beta^{(q+1)}}{(u^{(q+1)}-l^{(q+1)})})*I-A^{(q+1)})^{-1})$ non-increasing,* *we need*

$$U'_{A^{(q)}}(v) = v^T(u^{(q+1)} + \frac{\beta^{(q+1)}}{(u^{(q+1)} - l^{(q+1)})}) * I - A^{(q)})^{-1} v \tag{5.25}$$

$$+ \frac{v^T(u^{(q+1)} + \frac{\beta^{(q+1)}}{(u^{(q+1)} - l^{(q+1)})}) * I - A^{(q)})^{-2} v}{Trace((u^{(q)} + \frac{\beta^{(q)}}{(u^{(q)} - l^{(q)})}) * I - A^{(q)})^{-1}) - Trace((u^{(q+1)} + \frac{\beta^{(q+1)}}{(u^{(q+1)} - l^{(q+1)})}) * I - A^{(q)})^{-1})} \tag{5.26}$$

with $1/t \geq U'_{A^{(q)}}(v)$

Proof. To make $(u^{(q+1)} + \frac{\beta^{(q+1)}}{(u^{(q+1)} - l^{(q+1)})}) * I - A^{(q)})^{-1}$ non-increasing, since

$$\frac{n}{u^{(q+1)} - l^{(q+1)}} - \frac{n}{u^{(q)} - l^{(q)}} = \frac{-n(\delta_u - \delta_l)}{(u^{(q+1)} - l^{(q+1)}) * (u^{(q)} - l^{(q)})} \tag{5.27}$$

we need to cancel the decrement of $\frac{n}{u-l}$ in each update. So, instead of making the potential non-decreasing, we need:

$$\Phi_{l^{(q)}}^{u^{(q)}}(A^{(q)}) - \Phi_{l^{(q+1)}}^{u^{(q+1)}}(A^{(q)} + tvv^T) \geq \frac{n(\delta_u - \delta_l)}{(u^{(q+1)} - l^{(q+1)}) * (u^{(q)} - l^{(q)})} \tag{5.28}$$

Therefore, we want:

$$\begin{aligned}
\Phi_{l^{(q+1)}}^{u^{(q+1)}}(A^{(q)} + tvv^T) &\leq \Phi_l^u(A^{(q)}) - \frac{n(\delta_u - \delta_l)}{(u^{(q+1)} - l^{(q+1)}) * (u^{(q)} - l^{(q)})} \\
&= \Phi_l^u(A^{(q)}) - \frac{n(\delta_u - \delta_l)}{(u^{(q+1)} - l^{(q+1)}) * (u^{(q)} - l^{(q)})} \\
&\quad - (\Phi_l^u(A^{(q)}) - \Phi_{l^{(q+1)}}^{u^{(q+1)}}(A^{(q)}) - \frac{n(\delta_u - \delta_l)}{(u^{(q+1)} - l^{(q+1)}) * (u^{(q)} - l^{(q)})}) \\
&\quad + (\frac{v^T(u^{(q+1)} + \frac{\beta^{(q+1)}}{(u^{(q+1)} - l^{(q+1)})}) * I - A^{(q)})^{-2} v}{1/t - v^T(u^{(q+1)} + \frac{\beta^{(q+1)}}{(u^{(q+1)} - l^{(q+1)})}) * I - A^{(q)})^{-1} v}) * (1 - c)
\end{aligned}$$

As

$$\Phi_l^u(A^{(q)}) - \Phi_{l^{(q+1)}}^{u^{(q+1)}}(A^{(q)}) - \frac{n(\delta_u - \delta_l)}{(u^{(q+1)} - l^{(q+1)}) * (u^{(q)} - l^{(q)})}$$
$$= Trace((u^{(q)} + \frac{\beta^{(q)}}{(u^{(q)} - l^{(q)})}) * I - A^{(q)})^{-1} - (u^{(q+1)} + \frac{\beta^{(q+1)}}{(u^{(q+1)} - l^{(q+1)})}) * I - A^{(q)})^{-1}) * (1 - c)$$

We are now having our new $U'_{A^{(q)}}(v)$.
Since the potential function is not going to infinity, we know $\mu_{i,1}^{(q+1)}, \mu_{i,2}^{(q+1)} < u$ for all roots corresponding to λ_i of A. □

Thus, taking the $\sum vv^T = I$, and we have the upper bound similar to Lemma 3.3.0.2. to be:

Lemma 5.3.0.3.

$$\sum_i U'_{A^(q)}(v_i) \geq Trace(((u^{(q)} + \frac{\beta^{(q)}}{u^{(q)} - l^{(q)}}) * I - A^{(q)})^{-1}) + \frac{1}{\delta_u + c * (\delta_u - \delta_l)} \quad (5.29)$$

$$
\begin{aligned}
&\sum_i v_i^T (u^{(q+1)} + \frac{\beta^{(q+1)}}{(u^{(q+1)} - l^{(q+1)})}) * I - A^{(q)})^{-1} v_i \\
&+ \frac{\sum_i v_i^T (u^{(q+1)} + \frac{\beta^{(q+1)}}{(u^{(q+1)} - l^{(q+1)})}) * I - A^{(q)})^{-2} v_i}{Trace((u^{(q)} + \frac{\beta^{(q)}}{(u^{(q)} - l^{(q)})}) * I - A^{(q)})^{-1}) - Trace((u^{(q+1)} + \frac{\beta^{(q+1)}}{(u^{(q+1)} - l^{(q+1)})}) * I - A^{(q)})^{-1})} \\
&= (u^{(q+1)} + \frac{\beta^{(q+1)}}{(u^{(q+1)} - l^{(q+1)})}) * I - A^{(q)})^{-1} \cdot \sum_i v_i v_i^T \\
&+ \frac{(u^{(q+1)} + \frac{\beta^{(q+1)}}{(u^{(q+1)} - l^{(q+1)})}) * I - A^{(q)})^{-2} \cdot \sum v_i v_i^T}{Trace((u^{(q)} + \frac{\beta^{(q)}}{(u^{(q)} - l^{(q)})}) * I - A^{(q)})^{-1}) - Trace((u^{(q+1)} + \frac{\beta^{(q+1)}}{(u^{(q+1)} - l^{(q+1)})}) * I - A^{(q)})^{-1})} \\
&= Trace((u^{(q+1)} + \frac{\beta^{(q+1)}}{(u^{(q+1)} - l^{(q+1)})}) * I - A^{(q)})^{-1}) \\
&+ \frac{Trace((u^{(q+1)} + \frac{\beta^{(q+1)}}{(u^{(q+1)} - l^{(q+1)})}) * I - A^{(q)})^{-2})}{Trace((u^{(q)} + \frac{\beta^{(q)}}{(u^{(q)} - l^{(q)})}) * I - A^{(q)})^{-1}) - Trace((u^{(q+1)} + \frac{\beta^{(q+1)}}{(u^{(q+1)} - l^{(q+1)})}) * I - A^{(q)})^{-1})} \\
&= Trace((u^{(q+1)} + \frac{\beta^{(q+1)}}{(u^{(q+1)} - l^{(q+1)})}) * I - A^{(q)})^{-1}) \\
&+ \frac{Trace((u^{(q+1)} + \frac{\beta^{(q+1)}}{(u^{(q+1)} - l^{(q+1)})}) * I - A^{(q)})^{-2})}{(\delta_u + c * (\delta_u - \delta_l)) Trace((u^{(q)} + \frac{\beta^{(q)}}{(u^{(q)} - l^{(q)})}) * I - A^{(q)})^{-1}) * Trace((u^{(q+1)} + \frac{\beta^{(q+1)}}{(u^{(q+1)} - l^{(q+1)})}) * I - A^{(q)})^{-1})} \\
&\leq Trace((u^{(q+1)} + \frac{\beta^{(q+1)}}{(u^{(q+1)} - l^{(q+1)})}) * I - A^{(q)})^{-1}) \\
&+ \frac{Trace((u^{(q+1)} + \frac{\beta^{(q+1)}}{(u^{(q+1)} - l^{(q+1)})}) * I - A^{(q)})^{-2})}{(\delta_u + c * (\delta_u - \delta_l)) Trace((u^{(q+1)} + \frac{\beta^{(q+1)}}{(u^{(q+1)} - l^{(q+1)})}) * I - A^{(q)})^{-2})} \\
&= Trace(((u + \frac{\beta_u}{(u - l)}) * I - A^{(q)})^{-1}) + \frac{1}{(\delta_u + c * (\delta_u - \delta_l))}
\end{aligned}
$$

5.4 Lower Barrier Shift

At some step q, with the matrix $A^{(q)}$, let $A^{(q+1)} = A^{(q)} + tvv^T$ with some given vector v and step-size t. We bound $A^{(q+1)}$ by the new lower bound $l^{(q+1)} = l^{(q)} + \delta_l$. Moreover, we want $\mu_{min}^{(q)} > l^{(q)}$, and $\phi_{u^{(q)}}^{l^{(q)}}(A^{(q)}) \leq \frac{1}{\delta_l}$ to ensure the updated roots

$\mu_i^{(q+1)}$ always larger than $l^{(q)} + \delta_l$.
By the Sherman–Morrison formula, we can write the updated potential as:

$$\Phi_{u^{(q+1)}}^{l^{(q+1)}}(A^{(q+1)} + tvv^T) = (1-c)*Trace((A^{(q+1)} + tvv^T - (l^{(q+1)} - \frac{\beta^{(q+1)}}{(u^{(q+1)} - l^{(q+1)})})*I)^{-1}) + \frac{n}{(u^{(q+1)} - l^{(q+1)})}$$

$$= \frac{n}{(u^{(q+1)} - l^{(q+1)})} + Trace((A^{(q)} - (l^{(q+1)} - \frac{\beta^{(q+1)}}{(u^{(q+1)} - l^{(q+1)})})*I)^{-1}$$

$$- \frac{t*(A^{(q)} - (l^{(q+1)} - \frac{\beta^{(q+1)}}{(u^{(q+1)}-l^{(q+1)})})*I)^{-1}vv^T(A^{(q)} - (l^{(q+1)} - \frac{\beta^{(q+1)}}{(u^{(q+1)}-l^{(q+1)})})*I)^{-1}}{1+t*v^T(A^{(q)} - (l^{(q+1)} - \frac{\beta^{(q+1)}}{(u^{(q+1)}-l^{(q+1)})})*I)^{-1}v})*(1-c))$$

$$= \Phi_{u^{(q+1)}}^{l^{(q+1)}}(A^{(q)}) - (\frac{v^T(A^{(q)} - (l^{(q+1)} - \frac{\beta^{(q+1)}}{(u^{(q+1)}-l^{(q+1)})})*I)^{-2}v}{1/t + v^T(A^{(q)} - (l^{(q+1)} - \frac{\beta^{(q+1)}}{(u^{(q+1)}-l^{(q+1)})})*I)^{-1}v})*(1-c)$$

$$= \Phi_u^l(A^{(q)}) + (\Phi_{u^{(q+1)}}^{l^{(q+1)}}(A^{(q)}) - \Phi_{u^{(q)}}^{l^{(q)}}(A^{(q)})) + (\frac{v^T(A^{(q)} - (l^{(q+1)} - \frac{\beta^{(q+1)}}{(u^{(q+1)}-l^{(q+1)})})*I)^{-2}v}{1/t + v^T(A^{(q)} - (l^{(q+1)} - \frac{\beta^{(q+1)}}{(u^{(q+1)}-l^{(q+1)})})*I)^{-1}v})*(1-c)$$

Now, same as the upper potential, to keep $Trace((A^{(q+1)} - (l^{(q+1)} - \frac{\beta^{(q+1)}}{(u^{(q+1)}-l^{(q+1)})})I)^{-1})$ non-increasing, we need:

Claim 5.4.0.1. *Let $L'_{A^{(q)}}(v)$ to be:*

$$L'_{A^{(q)}}(v) = -v^T(A^{(q)} - (l^{(q+1)} - \frac{\beta^{(q+1)}}{(u^{(q+1)} - l^{(q+1)})})I)^{-1}v \tag{5.30}$$

$$+ \frac{v^T(A^{(q)} - (l^{(q+1)} - \frac{\beta^{(q+1)}}{(u^{(q+1)}-l^{(q+1)})})I)^{-2}v}{Trace((A^{(q)} - (l^{(q+1)} + \frac{\beta^{(q+1)}}{(u^{(q+1)}-l^{(q+1)})})I)^{-1}) - Trace((A^{(q)} - (l^{(q)} - \frac{\beta^{(q)}}{(u^{(q)}-l^{(q)})})I)^{-1})} \tag{5.31}$$

with $L'_{A^{(q)}}(v) \geq 1/t$, we can keep $Trace((A^{(q+1)} - (l^{(q+1)} - \frac{\beta^{(q+1)}}{(u^{(q+1)}-l^{(q+1)})})I)^{-1}$ non-increasing.

Claim 5.4.0.2. *As $\sum_i v_i v_i^T = I$,*

$$\sum_i L'_{A^{(q)}}(v_i) \leq \frac{1}{\delta_l - c*(\delta_u - \delta_l)} - \sum_i (\lambda_i^{(q)} - (l^{(q)} - \frac{\beta^{(q)}}{(u^{(q)} - l^{(q)})}))^{-1} \tag{5.32}$$

Proof.

$$
\begin{aligned}
&-\sum_i v_i^T (A^{(q)} - (l^{(q+1)} - \frac{\beta^{(q+1)}}{(u^{(q+1)} - l^{(q+1)})})I)^{-1} v_i \\
&+ \frac{\sum_i v_i^T (A^{(q)} - (l^{(q+1)} - \frac{\beta^{(q+1)}}{(u^{(q+1)} - l^{(q+1)})})I)^{-2} v_i}{Trace((A^{(q)} - (l^{(q+1)} - \frac{\beta^{(q+1)}}{(u^{(q+1)} - l^{(q+1)})})I)^{-1}) - Trace((A^{(q)} - (l^{(q)} - \frac{\beta^{(q)}}{(u^{(q)} - l^{(q)})})I)^{-1})} \\
&= -(A^{(q)} - (l^{(q+1)} - \frac{\beta^{(q+1)}}{(u^{(q+1)} - l^{(q+1)})})I)^{-1} \cdot \sum_i v_i v_i^T \\
&+ \frac{(A^{(q)} - (l^{(q+1)} - \frac{\beta^{(q+1)}}{(u^{(q+1)} - l^{(q+1)})})I)^{-2} \cdot \sum v_i v_i^T}{Trace((A^{(q)} - (l^{(q+1)} - \frac{\beta^{(q+1)}}{(u^{(q+1)} - l^{(q+1)})})I)^{-1}) - Trace((A^{(q)} - (l^{(q)} - \frac{\beta^{(q)}}{(u^{(q)} - l^{(q)})})I)^{-1})} \\
&= -Trace((A^{(q)} - (l^{(q+1)} - \frac{\beta^{(q+1)}}{(u^{(q+1)} - l^{(q+1)})})I)^{-1}) \\
&+ \frac{Trace((A^{(q)} - (l^{(q+1)} - \frac{\beta^{(q+1)}}{(u^{(q+1)} - l^{(q+1)})})I)^{-2})}{Trace((A^{(q)} - (l^{(q+1)} - \frac{\beta^{(q+1)}}{(u^{(q+1)} - l^{(q+1)})})I)^{-1}) - Trace((A^{(q)} - (l^{(q)} - \frac{\beta^{(q)}}{(u^{(q)} - l^{(q)})})I)^{-1})}
\end{aligned}
$$

Consider the inequality:

$$
\begin{aligned}
&-Trace((A^{(q)} - (l^{(q+1)} - \frac{\beta^{(q+1)}}{(u^{(q+1)} - l^{(q+1)})})I)^{-1}) \\
&+ \frac{Trace((A^{(q)} - (l^{(q+1)} - \frac{\beta^{(q+1)}}{(u^{(q+1)} - l^{(q+1)})})I)^{-2})}{\sum_i(\lambda_i - l^{(q+1)} + \frac{\beta^{(q+1)}}{(u^{(q+1)} - l^{(q+1)})}) - \sum_i(\lambda_i - l^{(q)} + \frac{\beta^{(q)}}{(u^{(q)} - l^{(q)})})} \\
&\geq \frac{1}{\delta_l - c * (\delta_u - \delta_l)} - \sum_i (\lambda_i^{(q)} - l^{(q)} + \frac{\beta^{(q)}}{(u^{(q)} - l^{(q)})})^{-1}
\end{aligned}
$$

This is equivalent as:

$$
\begin{aligned}
&\frac{\sum_i(\lambda_i^{(q)} - l^{(q+1)} + \frac{\beta^{(q+1)}}{(u^{(q+1)} - l^{(q+1)})})^{-2}}{\sum_i(\lambda_i - l^{(q+1)} + \frac{\beta^{(q+1)}}{(u^{(q+1)} - l^{(q+1)})}) - \sum_i(\lambda_i - l^{(q)} + \frac{\beta^{(q)}}{(u^{(q)} - l^{(q)})})} \\
&\geq \frac{1}{\delta_l - c * (\delta_u - \delta_l)} - \sum_i (\lambda_i^{(q)} - l^{(q)} + \frac{\beta^{(q)}}{(u^{(q)} - l^{(q)})})^{-1} + \sum_i (\lambda_i^{(q)} - l^{(q+1)} + \frac{\beta^{(q+1)}}{(u^{(q+1)} - l^{(q+1)})})^{-1}
\end{aligned}
$$

Then, we rewrite the inequality as:

$$\sum_i (\lambda_i^{(q)} - l^{(q+1)} + \frac{\beta^{(q+1)}}{(u^{(q+1)} - l^{(q+1)})})^{-2}$$
$$\geq (\frac{1}{\delta_l - c*(\delta_u - \delta_l)} - \sum_i \frac{\delta_l - c*(\delta_u - \delta_l)}{(\lambda_i^{(q)} - l^{(q)} + \frac{\beta^{(q)}}{(u^{(q)} - l^{(q)})})(\lambda_i^{(q)} - l^{(q+1)} + \frac{\beta^{(q+1)}}{(u^{(q+1)} - l^{(q+1)})})})$$
$$*(\sum_i \frac{\delta_l - c*(\delta_u - \delta_l)}{(\lambda_i^{(q)} - l^{(q)} + \frac{\beta^{(q)}}{(u^{(q)} - l^{(q)})})(\lambda_i^{(q)} - l^{(q+1)} + \frac{\beta^{(q+1)}}{(u^{(q+1)} - l^{(q+1)})})})$$
$$\sum_i (\lambda_i^{(q)} - l^{(q+1)} + \frac{\beta^{(q+1)}}{(u^{(q+1)} - l^{(q+1)})})^{-2}$$
$$\geq \sum_i \frac{1}{(\lambda_i^{(q)} - l^{(q)} + \frac{\beta^{(q)}}{(u^{(q)} - l^{(q)})})(\lambda_i^{(q)} - l^{(q+1)} + \frac{\beta^{(q+1)}}{(u^{(q+1)} - l^{(q+1)})})}$$
$$+(\sum_i \frac{\delta_l - c*(\delta_u - \delta_l)}{(\lambda_i^{(q)} - l^{(q)} + \frac{\beta^{(q)}}{(u^{(q)} - l^{(q)})})(\lambda_i^{(q)} - l^{(q+1)} + \frac{\beta^{(q+1)}}{(u^{(q+1)} - l^{(q+1)})})})^2$$
$$\sum_i (\lambda_i^{(q)} - l^{(q+1)} + \frac{\beta^{(q+1)}}{(u^{(q+1)} - l^{(q+1)})})^{-2} - \sum_i \frac{1}{(\lambda_i^{(q)} - l^{(q)} + \frac{\beta^{(q)}}{(u^{(q)} - l^{(q)})})(\lambda_i^{(q)} - l^{(q+1)} + \frac{\beta^{(q+1)}}{(u^{(q+1)} - l^{(q+1)})})}$$
$$\geq (\sum_i \frac{\delta_l - c*(\delta_u - \delta_l)}{(\lambda_i^{(q)} - l^{(q)} + \frac{\beta^{(q)}}{(u^{(q)} - l^{(q)})})(\lambda_i^{(q)} - l^{(q+1)} + \frac{\beta^{(q+1)}}{(u^{(q+1)} - l^{(q+1)})})})^2$$

Thus,

$$\sum_i \frac{\delta_l - c*(\delta_u - \delta_l)}{(\lambda_i^{(q)} - l^{(q)} + \frac{\beta^{(q)}}{(u^{(q)} - l^{(q)})})(\lambda_i^{(q)} - l^{(q+1)} + \frac{\beta^{(q+1)}}{(u^{(q+1)} - l^{(q+1)})})^2}$$
$$\geq (\sum_i \frac{\delta_l - c*(\delta_u - \delta_l)}{(\lambda_i^{(q)} - l^{(q)} + \frac{\beta^{(q)}}{(u^{(q)} - l^{(q)})})(\lambda_i^{(q)} - l^{(q+1)} + \frac{\beta^{(q+1)}}{(u^{(q+1)} - l^{(q+1)})})})^2$$

By Cauchy–Schwarz inequality,

$$(\delta_l - c*(\delta_u - \delta_l))(\sum_i \frac{1}{(\lambda_i^{(q)} - l^{(q)} + \frac{\beta^{(q)}}{(u^{(q)} - l^{(q)})})}) \sum_i \frac{\delta_l - c*(\delta_u - \delta_l)}{(\lambda_i^{(q)} - l^{(q)} + \frac{\beta^{(q)}}{(u^{(q)} - l^{(q)})})(\lambda_i^{(q)} - l^{(q+1)} + \frac{\beta^{(q+1)}}{(u^{(q+1)} - l^{(q+1)})})}$$
$$\geq (\sum_i \frac{\delta_l + c*(\delta_u - \delta_l)}{(\lambda_i^{(q)} - l^{(q)} + \frac{\beta^{(q)}}{(u^{(q)} - l^{(q)})})(\lambda_i^{(q)} - l^{(q+1)} + \frac{\beta^{(q+1)}}{(u^{(q+1)} - l^{(q+1)})})})^2$$

Thus, as long as $(\delta_l - c*(\delta_u - \delta_l))(\sum_i \frac{1}{(\lambda_i^{(q)} - l^{(q)} + \frac{\beta^{(q)}}{(u^{(q)} - l^{(q)})})}) \leq 1$, the inequality is valid. $\square$

Theorem 5.4.0.3 (Obstacle 2)**.** *In view of Obstacle 1, because of the restriction of the Sherman-Morrison formula, we must let our potential decrease during each update. In this situation, even though we have a better bound on step size for* $\mu_{i,1}, \mu_{i,2}$*, which are* $\frac{1}{\delta_u + c*(\delta_u - \delta_l)}, \frac{1}{\delta_l - c*(\delta_u - \delta_l)}$ *compared to* $\frac{1}{\delta_u}$ *and* $\frac{1}{\delta_l}$*, converting back to* λ_i*, its range is still between* $u + c*(\delta_u - \delta_l) > \lambda_i > l - c*(\delta_u - \delta_l)$*.*
Therefore, if we cannot overcome Obstacle 1, modifying the formula to $p_1(u) - p_2(u)\lambda_i$ *will not improve the Twice Ramanujan bound.*

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