Enumerative Geometry Problems for Calabi-Yau Manifolds with an Action

by

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Enumerative Geometry Problems for Calabi-Yau Manifolds with an Action

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Abstract

This dissertation consists of two main chapters, each pertaining to the enumerative geometry of Calabi-Yau manifolds with an action:

In Chapter 2 we study Euler characteristics of the G-invariant Hilbert schemes of points on an Abelian surface with a symplectic action by finite group G. One can package these Euler characteristics into generating series, whose reciprocal we prove is a holomorphic modular form for a particular congruence subgroup. For the standard involution of multiplication by -1, we prove an analogue of the Yau-Zaslow formula–that is, these Euler characteristics determine a weighted number of curves invariant under the involution and with rational quotient.

Motivated by the results of Chapter 2, in Chapter 3 we develop in more generality a theory counting invariant curves in Calabi-Yau threefolds with an involution. Our theory conjecturally results in analogues of the Gopakumar-Vafa invariants which count invariant curves of genus g and with genus h quotient. We prove the conjecture and compute all invariants in the case of a local Abelian surface with involution multiplication by -1, or a local Nikulin K3 surface together with the Nikuln involution.

Lay Summary

Certain geometrical spaces called *Calabi-Yau manifolds* came into the spotlight in physics as proposed models for extra-dimensions of spacetime. Given a Calabi-Yau manifold X, one interesting avenue is to count *curves* contained in X. Here, curve essentially means a closed surface (e.g. a sphere, the surface of a doughnut, or the surface of a pretzel). These counting invariants have rich mathematical structure, and dictate particle content in a certain sector of the corresponding physical theory.

Suppose now that our Calabi-Yau manifold X has certain geometrical symmetries (think, more exotic versions of symmetry under rotations or reflections). In this thesis, we begin to develop the more general theory counting curves in X that are themselves preserved by the symmetries of X. Ignoring any possible symmetries of X, our new theory reduces to the ordinary one.

Preface

This dissertation is an original intellectual product of the author, Stephen Pietromonaco, and, in the case of Chapter 3, collaboration with his coauthor, Jim Bryan.

- **Chapter 2** consists entirely of the article "G-invariant Hilbert Schemes on Abelian Surfaces and Enumerative Geometry of the Orbifold Kummer Surface" which is the original intellectual product of the author alone. It is published in *Research in the Mathematical Sciences*—volume 9, issue 1, pages 1-21, year 2022.

- **Chapter 3** consists entirely of the article "Counting invariant curves: a theory of Gopakumar-Vafa invariants for Calabi-Yau threefolds with an involution" which is the original intellectual product of the author along with his coauthor, Jim Bryan. The majority of the sections in this article are the result of mutual collaboration between the two coauthors. However, the author of this thesis is individually responsible for the lattice-theoretic results and theta function identities in our formulas, while Jim Bryan is responsible for our use of Nironi stability and wall-crossing in the $\imath$-MT theory.

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Chapter 1

Introduction

Geometry and physics have been intimately intertwined for centuries. In the past, it was arguably physicists who benefited more from the relationship, using robust mathematical frameworks to build theories. But in recent decades, the roles have been reversed and it is in fact physical insight which creates exiting new ideas and unexpected conjectures in mathematics.

Nowhere is this more true than with the ancient subject of *enumerative geometry*, where one attempts to count certain objects in a fixed ambient space. Enumerative geometry has undergone a renaissance over the last thirty years, thanks to an influx of ideas and conjectures from string theory and quantum field theory. Here are a few relevant examples:

- Gromov-Witten theory is the formalism defining counting invariants for stable maps from curves into a fixed target space X. This was inspired by the A-model topological string theory, where a curve represents a closed string worldsheet [**15**, **35**].

- The Donaldson-Thomas and Pandharipande-Thomas theories similarly produce counting invariants for certain stable sheaves or complexes of sheaves on X. In physics, these sheaves and complexes model certain gauge-theoretic objects called D-branes [**28**, **42**].

The generating series of these invariants that are so natural to form from a mathematical perspective, often end up coinciding with a partition function of a corresponding physical theory. As a consequence, physical insight may shed some light on hidden structure. For example, there may be reason to expect the generating series to have modular properties, not a priori obvious mathematically. Or perhaps there is a physically motivated re-packaging of the generating series in terms of far better invariants.

Broadly speaking, the original results to follow in this thesis represent a step in generalizing some of these well-studied ideas in enumerative geometry to the setting where our ambient geometry comes with a group action.

1.1 Background Topics on Enumerative Geometry

In this section we will briefly introduce some well-known topics in enumerative geometry, all of which will be relevant to the main thesis material of Chapters 2 and 3. Of course, we only mention a handful of topics, and numerous details are omitted, so we refer the interested reader to the survey [**38**].

1.1.1 Hilbert Schemes of Points on a $K3$ Surface and the Yau-Zaslow Formula

In one of the landmark early results of modern algebraic geometry, Göttsche computed the Betti numbers of the Hilbert scheme of points $\mathrm{Hilb}^d(S)$ for S a smooth quasi-projective surface [**23**]. The Hilbert scheme $\mathrm{Hilb}^d(S)$ should be thought of as a natural compactification of the configuration space of d points in S. More specifically, a point $Z \in \mathrm{Hilb}^d(S)$ represents a zero-dimensional subscheme $Z \subset S$ of length d.

When S is a $K3$ surface, Göttsche found the remarkable formula

$$\sum_{d=0}^{\infty} e(\mathrm{Hilb}^d(S))q^{d-1} = q^{-1}\prod_{n=1}^{\infty}(1-q^n)^{-24} \tag{1.1}$$

encoding the topological Euler characteristics of $\mathrm{Hilb}^d(S)$. The righthand side can be written as $\Delta(q)^{-1}$ where $\Delta(q) = \eta(q)^{24}$ is the unique modular cusp form of weight 12. It is expressed in terms of the Dedekind eta function

$$\eta(q) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n).$$

Formula (1.1) also appears in the enumerative geometry of rational curves in $K3$ surfaces. Let (S, L) be a $K3$ surface S together with an ample line bundle L of self-intersection $2d$. If the pair (S, L) is generic, then the number $n(d)$ of rational curves in the linear system $|L|$ is finite, and given explicitly by the Yau-Zaslow formula [**4, 12, 49**] as

$$\sum_{d=-1}^{\infty} n(d)q^d = q^{-1}\prod_{n=1}^{\infty}(1-q^n)^{-24}.$$

1.1.2 Pandharipande-Thomas Theory and the Gopakumar-Vafa Invariants

Given a Calabi-Yau threefold X and a curve class $\beta \in H_2(X, \mathbb{Z})$ one is interested in defining numbers that "count" curves in the class β with some fixed discrete data. In the previous subsection, rational curves in a fixed linear system on a $K3$ surface are isolated and finite in number, so they can be naïvely counted. The situation in general is far more complex, as curves can move in families. This leads to the notion of *virtual counting*. Gromov-Witten theory and Donaldson-Thomas theory first emerged defining virtual counting invariants, and are conjecturally equivalent [**38**].

But more relevant for us is an equivalent formulation of virtual counting invariants known as *Pandharipande-Thomas (PT) theory* [**40**]. For curve class $\beta \in H_2(X, \mathbb{Z})$ and $n \in \mathbb{Z}$, the central object is the moduli space of PT pairs:

$$\mathsf{PT}_{\beta,n}(X) = \left\{ (F, s) \,\middle|\, s \in H^0(X, F),\ [\mathrm{supp}(F)] = \beta,\ \chi(F) = n \right\}$$

where F is a coherent sheaf on X with proper support of pure dimension 1, and the cokernel of the section s has support of dimension 0. Intuitively, a PT pair models a curve in the class β along with finitely many points on the curve. From a physics perspective, a PT pair is a stable D2-D0 brane bound state, where β and n are essentially the D2-brane charge and D0-brane charge, respectively.

For any scheme S over $\mathbb{C}$, Behrend [**6**] defined a constructible function $\nu_S : S \to \mathbb{Z}$ and we define the *virtual Euler characteristic* to be the Behrend function weighted topological Euler characteristic:

$$e_{\mathsf{vir}}(S) = e(S, \nu_S) = \sum_{k \in \mathbb{Z}} k \cdot e\left(\nu_S^{-1}(k)\right).$$

We then define the *PT invariants* by

$$N^{\mathsf{PT}}_{\beta,n}(X) = e_{\mathsf{vir}}(\mathsf{PT}_{\beta,n}(X))$$

and the *PT partition function* by

$$Z^{\mathsf{PT}}(X) = \sum_{\beta,n} N^{\mathsf{PT}}_{\beta,n}(X)\, Q^\beta\, y^n.$$

For a given $\beta \neq 0$, there are infinitely many $n \in \mathbb{Z}$ such that $N^{\mathsf{PT}}_{\beta,n}(X) \neq 0$. Therefore, the enumerative meaning of the PT invariants for curves in the class β isn't immediately clear. However, using ideas from string theory, Gopakumar and Vafa [**22**] essentially conjectured that there are far better enumerative invariants encoded into $Z^{\mathsf{PT}}(X)$. This inspires the following definition:

Definition 1. The Gopakumar-Vafa invariants (via PT theory) $n_g^{\mathsf{PT}}(\beta)$ are defined via the following equation:

$$\log Z^{\mathsf{PT}}(X) = \sum_{k>0}\sum_{\beta,g} \frac{1}{k} \cdot Q^{k\beta} \cdot n_g^{\mathsf{PT}}(\beta) \cdot \psi_{-(-y)^k}^{g-1} \tag{1.2}$$

where

$$\psi_x = 2 + x + x^{-1}.$$

We interpret $n_g^{\mathsf{PT}}(\beta)$ as the virtual count of curves of geometric genus g in the class β. Though it is not clear from the definition, one expects that for fixed β, the invariants $n_g^{\mathsf{PT}}(\beta) = 0$ for all but finitely many g, which makes more enumerative sense.

Remark 2. Writing $\log Z^{\mathsf{PT}}(X)$ in the form given by the righthand side of Equation (1.2) uses the fact that the coefficient of Q^β in $Z^{\mathsf{PT}}(X)$ is the Laurant expansion of a rational function in y which is invariant under $y \leftrightarrow y^{-1}$ [**9**, **40**, **47**].

Notice that the definition of $n_g^{\mathsf{PT}}(\beta)$ is not geometrical—it is given purely via formula manipulation. Recently, Maulik-Toda [**32**] have proposed a geometrical definition of the Gopakumar-Vafa invariants, which we now briefly review. The moduli space $\mathsf{M}_\beta(X)$ of Maulik-Toda (MT) sheaves parameterizes Simpson stable one-dimensional coherent sheaves F such that $[\mathrm{supp}(F)] = \beta$ and $\chi(F) = 1$ [**26**]. The MT moduli space is a quasi-projective scheme and it has a proper morphism to the Chow variety given by the Hilbert-Chow morphism [**32**]:

$$\begin{aligned} \pi : \mathsf{M}_\beta(X) &\to \mathrm{Chow}_\beta(X) \\ [F] &\mapsto \mathrm{supp}(F) \end{aligned}$$

There is a perverse sheaf $\phi^\bullet$ on $\mathsf{M}_\beta(X)$ which is locally given by the perverse sheaf of vanishing cycles associated to the local super-potential (the moduli space is locally the critical locus of a holomorphic function on a smooth space, the so-called super-potential). The construction of $\phi^\bullet$ was done in [**7**], and requires the choice of "orientation data" : a squareroot of the virtual canonical line bundle on $\mathsf{M}_\beta(X)$. Maulik and Toda conjecture the existence of a canonical choice of orientation data (one that is compatible with the morphism π). Using that choice, the *Maulik-Toda* polynomial is defined as follows:

$$\mathsf{MT}_\beta(y) = \sum_{i\in\mathbb{Z}} \chi({}^pH^i(R^\bullet\pi_*\phi^\bullet))y^i$$

where ${}^pH^i(-)$ is the ith cohomology functor with respect to the perverse t-structure [**16**]. By self-duality of $\phi^\bullet$ and Verdier duality, $\mathsf{MT}_\beta(y)$ is an integer coefficient Laurent polynomial in y which is invariant under $y \leftrightarrow y^{-1}$. Noting that $\{\psi_y^g\}_{g\geq 0}$ forms an integral basis for such polynomials, we may write the MT polynomial as follows:

Definition 3. The GV invariants (via MT theory) $n_g^{\mathsf{MT}}(\beta)$ are defined by the equation

$$\mathsf{MT}_\beta(y) = \sum_{g\geq 0} n_g^{\mathsf{MT}}(\beta)\ \psi_y^g.$$

The main conjecture of Maulik and Toda is

Conjecture 4. $n_g^{\mathsf{MT}}(\beta) = n_g^{\mathsf{PT}}(\beta)$.

Remark 5. Compared to the definition via PT theory, the above definition of GV invariants is more directly tied to the geometry of curves in the class β and more closely matches the original physics definition. In particular, the invariants $n_g^{\mathsf{MT}}(\beta)$ only involve the single moduli space $\mathsf{M}_\beta(X)$. In contrast, the invariants $n_g^{\mathsf{PT}}(\beta)$ involve a subtle combination of the PT invariants associated to an infinite number of moduli spaces, namely the spaces $\mathsf{PT}_{\beta',n}(X)$ where $\beta = k\beta'$ and n is unbounded from above.

1.1.3 The Katz-Klemm-Vafa Formula

A *local $K3$ surface* is a non-compact Calabi-Yau threefold X built from a $K3$ surface S. The simplest model is to take $X = S \times \mathbb{C}$. Consider a curve class $\beta_d \in H_2(S,\mathbb{Z})$ with self-intersection $\beta_d^2 = 2d$. In this case, Conjecture 4 holds, and $n_g^{\mathsf{MT}}(\beta_d) = n_g^{\mathsf{PT}}(\beta_d)$ only depends on d and g (and not the divisibility of β_d). The invariants $n_g^{\mathsf{PT}}(\beta_d)$ and $n_g^{\mathsf{MT}}(\beta_d)$ were computed (in full generality) by Pandharipande and Thomas [**39**] and Shen and Yin [**46**, Thm 0.5] respectively. The Gopakumar-Vafa invariants of a local $K3$ surface are given by the famous KKV formula first conjectured by Katz, Klemm, and Vafa [**27**]:

$$\sum_{d=-1}^{\infty}\sum_{g=0}^{\infty} n_g(\beta_d)\psi_y^g q^d = -q^{-1}\prod_{n=1}^{\infty}(1-q^n)^{-20}(1+yq^n)^{-2}(1+yq^n)^{-2} \tag{1.3}$$

The right hand side can also be written as $\psi_y \cdot \phi_{10,1}(q,-y)^{-1}$ where

$$\phi_{10,1}(q,y) = q\,(y^{\frac{1}{2}} - y^{-\frac{1}{2}})^2 \prod_{n=1}^{\infty}(1-q^n)^{20}(1-yq^n)^2(1-yq^n)^2 \tag{1.4}$$

is the Fourier expansion of the unique Jacobi cusp form of weight 10 and index 1 [**17**].

The fact that $n_g(\beta_d)$ is independent of the divisibility of β_d is a deep and unusual feature of the local $K3$ geometry.

1.2 Introduction to Chapters 2 and 3

Having briefly reviewed some relevant background material above, in this section we give an overview of the original results appearing in the main body of this thesis.

1.2.1 Chapter 2: G-invariant Hilbert Schemes and Hyperelliptic Curves in Abelian Surfaces

In the case of an Abelian surface A, the analogue of Göttsche's formula (1.1) is trivial since $e(\mathrm{Hilb}^d(A)) = 0$ for $d > 0$. Alternatively, we can consider an Abelian surface A together with a (holomorphic) symplectic action by a finite group G, and define the analogous generating series

$$Z_{A,G}(q) := \sum_{d=0}^{\infty} e(\mathrm{Hilb}^d(A)^G) q^d. \tag{1.5}$$

Here $\mathrm{Hilb}^d(A)^G$ is the G-invariant Hilbert scheme, parameterizing finite G-invariant subschemes of A with length d. It is equivalently the fixed locus of the induced G action on $\mathrm{Hilb}^d(A)$. The G-invariant Hilbert scheme is disconnected, though each component is a smooth projective holomorphic symplectic variety of $K3$-type (unless G acts purely by translations). See also [**11**] where the G-invariant Hilbert scheme is studied on $K3$ surfaces.

I prove the following results in Chapter 2 (see Theorem 8, Proposition 10, and Appendix A for details on the modular forms):

Theorem.

1. *The function $Z^{-1}_{A,G}(q)$ is a holomorphic modular form of weight $\frac{1}{2}e(A/G)$ with multiplier system. If G acts without translations, then $Z^{-1}_{A,G}(q)$ is an explicit eta product presented in Table 1.1.*

2. *If $T \subset G$ is the subgroup of all elements acting by translations, then*

$$Z_{A,G}(q) = Z_{A/T,G/T}(q^{|T|}).$$

In particular, the modular form $Z^{-1}_{A,G}(q)$ in this case is an oldform—it is equal to a modular form from Table 1.1 with the variable change $q \mapsto q^{|T|}$.

Fujiki [**19**] classified translation-free symplectic actions on Abelian surfaces (Section 2.2). Each such action which arises corresponds to a row in Table 1.1 (see 2.1.1 for notation).

No.	G	Singularities of A/G	Modular form $Z_{A,G}^{-1}$	$\frac{1}{2}e(A/G)$
1	$\{e\}$	—	1	0
2	$\mathbb{Z}_2$	$16A_1$	$\frac{\eta^{16}(q)}{\eta^8(q^2)}$	4
3	$\mathbb{Z}_3$	$9A_2$	$\frac{\eta^9(q)}{\eta^3(q^3)}$	3
4	$\mathbb{Z}_4$	$4A_3 + 6A_1$	$\frac{\eta^6(q^2)\eta^4(q)}{\eta^4(q^4)}$	3
5	$\mathbb{Z}_6$	$A_5 + 4A_2 + 5A_1$	$\frac{\eta^5(q^3)\eta^4(q^2)\eta(q)}{\eta^4(q^6)}$	3
6	$\mathcal{Q}$	$2D_4 + 3A_3 + 2A_1$	$\frac{\eta^8(q^4)\eta^2(q)}{\eta^4(q^8)\eta(q^2)}$	$5/2$
7	$\mathcal{Q}$	$4D_4 + 3A_1$	$\frac{\eta^{15}(q^4)\eta^4(q)}{\eta^6(q^8)\eta^8(q^2)}$	$5/2$
8	$\mathcal{Q}$	$6A_3 + A_1$	$\frac{\eta(q^4)\eta^6(q^2)}{\eta^2(q^8)}$	$5/2$
9	$\mathcal{D}$	$D_5 + 3A_3 + 2A_2 + A_1$	$\frac{\eta^3(q^6)\eta^3(q^4)\eta^3(q^3)\eta(q)}{\eta^3(q^{12})\eta^2(q^2)}$	$5/2$
10	$\mathcal{T}$	$E_6 + D_4 + 4A_2 + A_1$	$\frac{\eta^5(q^{12})\eta^6(q^8)\eta(q^3)\eta(q)}{\eta^4(q^{24})\eta^2(q^6)\eta^2(q^2)}$	$5/2$
11	$\mathcal{T}$	$A_5 + 2A_3 + 4A_2$	$\frac{\eta^4(q^8)\eta^2(q^6)\eta(q^4)}{\eta^2(q^{24})}$	$5/2$

Table 1.1: The modular forms $Z_{A,G}^{-1}(q)$ for symplectic, translation-free actions.

By analogy with the Yau-Zaslow formula, one might expect the coefficients of $Z_{A,G}(q)$ in the q-expansion to "count" rational curves in the orbifold $[A/G]$—or equivalently, G-invariant curves $C \subset A$ whose quotient C/G is rational. We prove enumerative results of this nature in the case of $\langle\imath\rangle \cong \mathbb{Z}_2$ where $\imath : A \to A$ is the canonical involution $a \mapsto -a$. In other words, we give an enumerative interpretation of the coefficients of

$$Z_{A,\langle\imath\rangle}(q) = \prod_{n=1}^{\infty} \frac{(1-q^{2n})^8}{(1-q^n)^{16}}.$$

from the second row of Table 1.1.

Let A be a generic polarized Abelian surface of type $(1,d)$ with $\beta_d \in H_2(A,\mathbb{Z})$ the class of the primitive polarization. As in Section 1.1, consider the moduli space $\mathsf{M}_{\beta_d}(A)$ of Simpson stable pure one-dimensional sheaves F on A with $[\mathrm{supp}(F)] = \beta_d$ and $\chi(F) = 1$. The $\imath$ action lifts canonically to $\mathsf{M}_{\beta_d}(A)$, and we can restrict to the $\imath$-fixed locus $\mathsf{M}_{\beta_d}(A)^\imath$

which we prove is a disjoint union of holomorphic symplectic varieties of $K3$-type (Proposition 12 and Corollary 13).

We define quantities

$$\mathsf{n}_0(d) = e\big(\mathsf{M}_{\beta_d}(A)^{\imath}\big)$$

Because each component of $\mathsf{M}_{\beta_d}(A)^{\imath}$ is smooth and even dimensional, the Behrend function is trivial–therefore by [**26**, **32**], it is natural to expect that $\mathsf{n}_0(d)$ encodes information about $\imath$-invariant curves in the class β_d with rational quotient.

The following result which we prove in Chapter 2 is the analogue of the Yau-Zaslow formula in the case of $[A/\imath]$ (Theorem 16):

Theorem.

1. *The quantities* $\mathsf{n}_0(d)$ *are determined from the formula*

$$Z_{A,\langle\imath\rangle}(q) = \frac{1}{16}\sum_{d=0}^{\infty} \mathsf{n}_0(d)q^d. \tag{1.6}$$

2. $\mathsf{n}_0(d)$ *is a weighted count of rational curves in* $[A/\imath]$*. More specifically,*

$$\mathsf{n}_0(d) = \sum_{C\in\Pi} e(\overline{\mathrm{Jac}}(C)^{\imath}) \tag{1.7}$$

where Π *is the finite set of* $\imath$*-invariant curves in the class* β_d *with rational quotient. Here* $\overline{\mathrm{Jac}}(C)$ *is the compactified Jacobian of the integral curve* C*, see [**2**, Sec. 6.2.4].*

An $\imath$-invariant curve $C \subset A$ with rational quotient is an $\imath$-invariant hyperelliptic curve in A. Our results are consistent with the hyperelliptic counts of [**13**] (see Subsection 2.1.4).

Remark 6. Our proofs rely on an $\imath$-equivariant deformation equivalence between $\mathrm{Hilb}^d(A)$ and $\mathsf{M}_{\beta_d}(A)$, and it is unclear if this extends to other groups G. In other words, $e\big(\mathsf{M}_{\beta_d}(A)^G\big)$ should indeed be a weighted count of G-invariant curves in A with rational quotient, but it is unclear if these arise as coefficients of $Z_{A,G}(q)$ from Table 1.1.

1.2.2 Chapter 3: A Theory of Gopakumar-Vafa Invariants for Calabi-Yau Threefolds with an Involution

Our enumerative results summarized in the previous subsection were motivation to build a more general theory counting invariant curves in a Calabi-Yau threefold with an action. There are two main drawbacks to our invariants $\mathsf{n}_0(d)$ in the case of an Abelian surface:

1. Instead of enumerating the finite number of $\imath$-invariant curves with rational quotient, each curve contributes to $\mathsf{n}_0(d)$ with a weighting.

2. We don't keep track of the (geometric) genus g of the $\imath$-invariant curves with rational quotient.

With this in mind, let X be a Calabi-Yau threefold with an involution $\imath : X \to X$ preserving the holomorphic volume form, and let $\beta \in H_2(X, \mathbb{Z})^\imath$ be an $\imath$-invariant class.

Goal. *We want to define integers $n_{g,h}(\beta)$, called the $\imath$-Gopakumar-Vafa invariants, which represent a (virtual) count of $\imath$-invariant curves $C \subset X$ of genus g in the class β such that $C/\imath$ has genus h.*

In particular, $n_{g,0}(\beta)$ should count $\imath$-invariant hyperelliptic curves of genus g in the class β.

Just as in the ordinary theory reviewed above, we give two definitions of $n_{g,h}(\beta)$ which we conjecture are equal:

- Through a version of $\imath$-equivariant Pandharipande-Thomas theory, we define $n^{\mathsf{PT}}_{g,h}(\beta)$ in Subsection 3.2.1.

- Through an analog of the Maulik-Toda formalism applied to moduli spaces of Nironi stable sheaves on the stack $[X/\imath]$, we define $n^{\mathsf{MT}}_{g,h}(\beta)$ in Subsection 3.2.2.

Our main example is that of a local Abelian or Nikulin $K3$ surface $X = S \times \mathbb{C}$. Here S is either an Abelian surface with $\imath(a) = -a$ or a $K3$ surface together with a symplectic involution $\imath$ (this is known as a Nikulin $K3$ surface). In both cases, the action on $X = S \times \mathbb{C}$ acts trivially on the second factor. There are two deformation types of Nikulin $K3$ surfaces which we call Type (I) and Type (II), see Definition 74.

Theorem. *Let $X = S \times \mathbb{C}$ be as above, and let $\beta \in H_2(S)$ be an effective invariant curve class with $\beta^2 = 2d$. Then $n^{\mathsf{PT}}_{g,h}(\beta) = n^{\mathsf{MT}}_{g,h}(\beta)$. Moreover:*

1. *if S is a Type (II) Nikulin surface, $n^{\mathsf{PT}}_{g,h}(\beta)$ only depends on (g, h, d). In particular, it doesn't depend on the divisibility of β.*

2. *if S is an Abelian surface or a Type (I) Nikulin surface, $n^{\mathsf{PT}}_{g,h}(\beta)$ only depends on (g, h, d) as well as the parity of the divisibility of β.*

We denote these invariants by $n_{g,h}(d;\mathsf{type})$ *where* $\mathsf{type} \in \{\mathsf{A}^{\mathsf{ev}}, \mathsf{A}^{\mathsf{odd}}, \mathsf{N}_{\mathsf{I}}^{\mathsf{ev}}, \mathsf{N}_{\mathsf{I}}^{\mathsf{odd}}, \mathsf{N}_{\mathsf{II}}\}$ *distinguishes the cases in the obvious way. Then the invariants are determined from the formula:*

$$\sum_{g,h} n_{g,h}(d;\mathsf{type})\psi_y^{h-1}\psi_w^{g+1-2h} = \left[\frac{\Theta_T(q^2,w)}{\phi_{10,1}(q^2,-y)}\right]_{q^d}$$

where $[\cdots]_{q^d}$ *denotes the coefficient of* q^d *in the expression* $[\cdots]$. *Here,* T *is a lattice or shifted lattice depending on the type,* $\Theta_T(q^2,w)$ *is an explicitly determined Jacobi theta function (see Theorem 67), and* $\phi_{10,1}(q,y)$ *is the unique Jacobi cusp form of weight* 10 *and index* 1. *In particular, for types* $\mathsf{A}^{\mathsf{odd}}$ *and* $\mathsf{N}_{\mathsf{I}}^{\mathsf{odd}}$ *we get infinite product formulas:*

$$\sum_{g,h,d} n_{g,h}(d;\mathsf{A}^{\mathsf{odd}})\,\psi_y^{h}\,\psi_w^{g-1-2h}\,q^d = -4\prod_{n=1}^{\infty}\frac{(1+q^n)^8(1+wq^n)^4(1+w^{-1}q^n)^4}{(1-q^{2n})^4(1+yq^{2n})^2(1+y^{-1}q^{2n})^2}\ ,$$

$$\sum_{g,h,d} n_{g,h}(d;\mathsf{N}_{\mathsf{I}}^{\mathsf{odd}})\,\psi_y^{h}\,\psi_w^{g-2h}\,q^{d+1} = \ -\prod_{n=1}^{\infty}\frac{(1+q^n)^4(1+wq^n)^2(1+w^{-1}q^n)^2}{(1-q^{2n})^{12}(1+yq^{2n})^2(1+y^{-1}q^{2n})^2}\ .$$

Remark 7. In the case where β_d is a primitive polarization of type $(1,d)$ on an Abelian surface A, the invariants $n_{g,0}(d;\mathsf{A}^{\mathsf{odd}})$ are refinements of $\mathsf{n}_0(d)$ introduced in Section 1.2.1. The precise relationship is

$$\mathsf{n}_0(d) = -\sum_{g=1}^{d+1} 2^{2g}\,n_{g,0}(d;\mathsf{A}^{\mathsf{odd}})$$

where the minus sign is due to the fact that we use $A\times\mathbb{C}$ to define $n_{g,0}(d;\mathsf{A}^{\mathsf{odd}})$.

In addition to the case of a local Abelian/Nikulin $K3$ surface, we prove our conjecture and compute $n_{g,h}(\beta)$ in the following examples:

- An isolated smooth invariant curve (Section 3.3).
- The fiber class of elliptically fibered Calabi-Yau threefolds with certain involutions (Subsection 3.6.1).
- Arbitrary classes on the “local football”—a certain global quotient orbifold with two $B\mathbb{Z}_2$ points and coarse space $\mathbb{P}^1$ (Section 3.6.2).

Chapter 2

G-invariant Hilbert Schemes and Hyperelliptic Curves in Abelian Surfaces

2.1 Introduction

2.1.1 Göttsche's Formula on the Orbifold $[A/G]$

In one of the seminal results of modern algebraic geometry, Göttsche computed the Betti numbers of the Hilbert scheme of points $\mathrm{Hilb}^d(S)$ where S is a smooth quasi-projective surface [**23**]. When S is a $K3$ surface, he found the remarkable formula

$$\sum_{d=0}^{\infty} e(\mathrm{Hilb}^d(S))q^{d-1} = q^{-1}\prod_{n=1}^{\infty}(1-q^n)^{-24}. \tag{2.1}$$

The reciprocal of the righthand side equals the modular cusp form $\Delta(q) = \eta(q)^{24}$ of weight 12.

However, in the case of an Abelian surface A, the analogue of Göttsche's formula is trivial since $e(\mathrm{Hilb}^d(A)) = 0$ for $d > 0$. One can instead consider Abelian surfaces together with an action by a finite group. By studying the invariant loci—or equivalently, working on the orbifold—we will find a family of results analogous to (2.1).

Consider a complex Abelian surface A, and a finite group G acting on A preserving the holomorphic symplectic form. We will call such an action *symplectic*. The natural generating function produced from this data is

$$Z_{A,G}(q) := \sum_{d=0}^{\infty} e(\mathrm{Hilb}^d(A)^G)q^d \tag{2.2}$$

where $\mathrm{Hilb}^d(A)^G$ is the G-invariant Hilbert scheme, parameterizing finite G-invariant subschemes of length d. It is equivalently the fixed locus of the induced G action on $\mathrm{Hilb}^d(A)$.

The G-invariant Hilbert scheme is disconnected, though each component is a smooth projective holomorphic symplectic variety of $K3$-type (unless G acts purely by translations).

By definition of the orbifold $[A/G]$, we have

$$\mathrm{Hilb}(A)^G = \mathrm{Hilb}([A/G]).$$

So we regard $Z_{A,G}(q)$ as analogous to the lefthand side of (2.1) for $[A/G]$. Our following result can be understood as the analogue of Göttsche's formula for the orbifold $[A/G]$ (see Appendix A for details on the modular forms).

Theorem 8. *The function $Z^{-1}_{A,G}(q)$ is a modular form of weight $\frac{1}{2}e(A/G)$ for the congruence subgroup $\Gamma_0(|G|)$. Moreover, $Z^{-1}_{A,G}$ is an explicit eta product (see Table 2.1 and Proposition 10 below), and transforms with multiplier system induced from that of the Dedekind eta function. It is a holomorphic, non-cuspidal form, normalized with leading coefficient 1.*

Our proof of Theorem 8 relies on recent methods of Bryan-Gyenge [**11**] in the case of $K3$ surfaces.

Fujiki has completely classified symplectic actions by finite groups on Abelian surfaces [**19**]. In the case where the subgroup acting by translations is trivial, the only groups which arise, up to isomorphism, are

$$\mathbb{Z}_2,\ \mathbb{Z}_3,\ \mathbb{Z}_4,\ \mathbb{Z}_6,\ \mathcal{Q},\ \mathcal{D},\ \mathcal{T}$$

where we denote by $\mathbb{Z}_n$ the cyclic group of order n, and by $\mathcal{Q}, \mathcal{D}, \mathcal{T}$ the quaternion group of order 8, the binary diherdral group of order 12, and the binary tetrahedral group of order 24, respectively. Recall that these groups fall into the ADE classification: $\mathbb{Z}_n$ has ADE type A_{n-1} while $\mathcal{Q}, \mathcal{D}, \mathcal{T}$ have types D_4, D_5, E_6, respectively.

First consider actions by group homomorphisms; we call these *linear*. All Abelian surfaces carry a unique linear action by $\langle \imath \rangle \cong \mathbb{Z}_2$ where $\imath : A \to A$ is the canonical involution $a \mapsto -a$. Hence, A admits a symplectic linear $\mathbb{Z}_3$ action if and only if it does so for $\mathbb{Z}_6$. Then it suffices to study G isomorphic to one of $\mathbb{Z}_4, \mathbb{Z}_6, \mathcal{Q}, \mathcal{D}, \mathcal{T}$.

Remark 9. By the physical arguments of [**48**, Sec. 4], we understand why precisely these groups arise. Let G be isomorphic to one of the five groups listed above, and let $\overline{G} = G/\langle \imath \rangle$ be the quotient by the unique order 2 subgroup. Then the five $\overline{G}$ are precisely the subgroups of the even Weyl group $W^+(E_8)$ of the E_8 root lattice which pointwise fix a lattice of rank at least 4. This is in close analogy with the classification in the case of $K3$ surfaces.

Any group G with a symplectic action on A can be written uniquely as an extension

$$0 \to T \to G \to G_0 \to 0$$

where $T \subseteq G$ is the subgroup of all elements acting by translation, and the quotient G_0 acts linearly and symplectically on A. If T is trivial, we say the G action is *translation-free*. Note that $T \cong \mathbb{Z}_a \times \mathbb{Z}_b$ for some $a, b \geq 1$.

In Table 2.1 we present the modular form $Z_{A,G}^{-1}$ for all equivalence classes of translation-free actions. For such actions, G and G_0 are abstractly isomorphic. However, G might not act linearly. Notice there are translation-free actions by $\mathcal{Q}$ and $\mathcal{T}$ without fixed points (Nos. 8 and 11 in the table, respectively), so in particular, they do not preserve the origin.

No.	G	Singularities of A/G	Modular form $Z_{A,G}^{-1}$	$\frac{1}{2}e(A/G)$
1	$\{e\}$	—	1	0
2	$\mathbb{Z}_2$	$16A_1$	$\frac{\eta^{16}(q)}{\eta^8(q^2)}$	4
3	$\mathbb{Z}_3$	$9A_2$	$\frac{\eta^9(q)}{\eta^3(q^3)}$	3
4	$\mathbb{Z}_4$	$4A_3 + 6A_1$	$\frac{\eta^6(q^2)\eta^4(q)}{\eta^4(q^4)}$	3
5	$\mathbb{Z}_6$	$A_5 + 4A_2 + 5A_1$	$\frac{\eta^5(q^3)\eta^4(q^2)\eta(q)}{\eta^4(q^6)}$	3
6	$\mathcal{Q}$	$2D_4 + 3A_3 + 2A_1$	$\frac{\eta^8(q^4)\eta^2(q)}{\eta^4(q^8)\eta(q^2)}$	5/2
7	$\mathcal{Q}$	$4D_4 + 3A_1$	$\frac{\eta^{15}(q^4)\eta^4(q)}{\eta^6(q^8)\eta^8(q^2)}$	5/2
8	$\mathcal{Q}$	$6A_3 + A_1$	$\frac{\eta(q^4)\eta^6(q^2)}{\eta^2(q^8)}$	5/2
9	$\mathcal{D}$	$D_5 + 3A_3 + 2A_2 + A_1$	$\frac{\eta^3(q^6)\eta^3(q^4)\eta^3(q^3)\eta(q)}{\eta^3(q^{12})\eta^2(q^2)}$	5/2
10	$\mathcal{T}$	$E_6 + D_4 + 4A_2 + A_1$	$\frac{\eta^5(q^{12})\eta^6(q^8)\eta(q^3)\eta(q)}{\eta^4(q^{24})\eta^2(q^6)\eta^2(q^2)}$	5/2
11	$\mathcal{T}$	$A_5 + 2A_3 + 4A_2$	$\frac{\eta^4(q^8)\eta^2(q^6)\eta(q^4)}{\eta^2(q^{24})}$	5/2

Table 2.1: The modular forms $Z_{A,G}^{-1}(q)$ for symplectic, translation-free actions. The weight of the modular form is $\frac{1}{2}e(A/G)$, which is presented in the last column.

This reduces the problem to computing $Z_{A,G}$ when T is non-trivial. Interpreting T as a subgroup of A let $A' = A/T$, which is again an Abelian surface. We then get a symplectic translation-free action of $G' = G/T$ on the Abelian surface A'. In Section 2.3 we will prove the following result.[1]

Proposition 10. *With the notation as above, we have*

$$Z_{A,G}(q) = Z_{A',G'}(q^{|T|}).$$

In particular, the modular form $Z_{A,G}^{-1}$ where G has translations is an oldform: it is equal to a modular form $Z_{A',G'}^{-1}$ from Table 2.1 with the variable change $q \mapsto q^{|T|}$.

2.1.2 Refinement to χ_y-genus

We can refine our formulas by replacing the Euler characteristic with a more elaborate index. For our purposes, we will focus on the (normalized) χ_y-genus, which for a compact complex manifold M is defined in terms of the Hodge numbers as

$$\begin{aligned} \overline{\chi}_y(M) &= (-y)^{-\frac{1}{2}\dim(M)}\chi_y(M) \\ &= (-y)^{-\frac{1}{2}\dim(M)}\sum_{p,q}(-1)^p h^{p,q}(M)y^q. \end{aligned} \tag{2.3}$$

Notice that setting $y = -1$ recovers the Euler characteristic, $\overline{\chi}_{-1}(M) = e(M)$.

Our formulas will involve the function

$$\phi_{-2,1}(q,y) = \left(y^{\frac{1}{2}} - y^{-\frac{1}{2}}\right)^2 \prod_{n=1}^{\infty} \frac{(1-yq^n)^2(1-y^{-1}q^n)^2}{(1-q^n)^4} \tag{2.4}$$

which is the unique weak Jacobi form of weight -2 and index 1 [**17**, Thm. 9.3]. We define the generating function

$$Z_{A,G}^{\overline{\chi}}(q,y) = \sum_{d=0}^{\infty} \overline{\chi}_y(\mathrm{Hilb}^d(A)^G)q^d.$$

Proposition 11. *For all non-trivial translation-free symplectic actions we have*

$$Z_{A,G}^{\overline{\chi}}(q,y) = -\left(y^{\frac{1}{2}} + y^{-\frac{1}{2}}\right)^2 \frac{Z_{A,G}(q)}{\phi_{-2,1}(q^{|G|},-y)}.$$

Following [**11**], one can give similar formulas for the elliptic genus, the motivic class, and more generally, the birationality class, but we will not need those here.

[1] Note that G_0 and G' are abstractly isomorphic. But we distinguish them because they are different groups acting on different spaces.

2.1.3 Enumerative Geometry of the Orbifold Kummer Surface

The Katz-Klemm-Vafa (KKV) formula [**27**] was predicted by string theorists to compute the BPS states of D-branes moving in a $K3$ surface S.

In its modern mathematical formulation, the Maulik-Toda proposal [**32**] is applied to a (local) $K3$ surface to define BPS invariants $\mathsf{n}_\beta^{K3}(g)$ for each effective curve class β. These quantities, which we interpret as virtual counts of curves of geometric genus g in the class β, only depend on β through the self-intersection $\beta^2 = 2d - 2$, so we denote them by $\mathsf{n}_d^{K3}(g)$. The KKV formula is then

$$\begin{aligned}\sum_{d=0}^{\infty}\sum_{g=0}^{\infty} \mathsf{n}_d^{K3}(g)(y^{\frac{1}{2}} + y^{-\frac{1}{2}})^{2g} q^{d-1} &= -(y^{\frac{1}{2}} + y^{-\frac{1}{2}})^2 \frac{1}{\Delta(q)\phi_{-2,1}(q,-y)} \\ &= \frac{1}{q}\prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{20}(1+yq^n)^2(1+y^{-1}q^n)^2}.\end{aligned} \tag{2.5}$$

The coefficient of q^{d-1} in the formula is $\overline{\chi}_y(\mathrm{Hilb}^d(S))$, so the KKV formula relates the χ_y-genera of $\mathrm{Hilb}(S)$ and virtual counts of curves on a $K3$ surface. It has now been proven in full [**39**].

In this chapter we prove an analogue of the KKV formula for the *orbifold Kummer surface* $[A/\imath]$ by introducing a notion of $\imath$-BPS states. Here A is a polarized Abelian surface of type $(1, d)$ with $\beta_d \in H_2(A, \mathbb{Z})$ the class of the primitive polarization[2] and

$$\imath : A \to A$$

is the canonical involution $a \mapsto -a$.

As with ordinary BPS invariants, consider the moduli space $\mathsf{M}_{\beta_d}(A)$ of Simpson stable pure one-dimensional sheaves F on A with $[\mathrm{supp}(F)] = \beta_d$ and $\chi(F) = 1$. The $\imath$ action lifts canonically to $\mathsf{M}_{\beta_d}(A)$ by pullback.

The following is the Abelian surface-version of the fact that for a $K3$ surface S, a moduli space of stable sheaves with primitive Mukai vector and generic polarization is deformation equivalent to a Hilbert scheme of points on S.

Proposition 12. *There exists an $\imath$-equivariant deformation equivalence*

$$\mathsf{M}_{\beta_d}(A) \to \widehat{A} \times \mathrm{Hilb}^d(A) \tag{2.6}$$

where $\imath$ acts on both sides by pullback, and $\widehat{A} = \mathrm{Pic}^0(A)$ is the dual Abelian variety.

[2] Throughout, we must handle $d = 0$ separately. In this case, choose the product Abelian surface $A = E \times F$, with β_0 the class of $E \times \{\mathrm{pt}\}$.

This is essentially a result of Yoshioka [**50**]. Our observation is simply that his correspondence is $\imath$-equivariant, and we prove this in Section 2.4.3.

An immediate corollary is the following.

Corollary 13. *Restricting to the $\imath$-invariant locus, we get a component-wise deformation equivalence*

$$\mathsf{M}_{\beta_d}(A)^\imath \to \coprod_{i=1}^{16} \mathrm{Hilb}^d(A)^\imath. \tag{2.7}$$

In particular, each component of $\mathsf{M}_{\beta_d}(A)^\imath$ is a smooth holomorphic symplectic variety of $K3$-type.

The ordinary Hilbert-Chow morphism is $\imath$-equivariant, so we can restrict to the invariant locus

$$\pi_d : \mathsf{M}_{\beta_d}(A)^\imath \to \mathrm{Chow}_{\beta_d}(A)^\imath.$$

which is a disjoint union of Lagrangian fibrations. In Section 2.4 we apply the Maulik-Toda proposal[3] to this map in order to define $\imath$-BPS invariants $\mathsf{n}_h(d)$ of $[A/\imath]$ (see Definition 32). We interpret $\mathsf{n}_h(d)$ as the virtual number of $\imath$-invariant curves in the class β_d in A, whose quotient has geometric genus h. Equivalently, $\mathsf{n}_h(d)$ is a virtual count of genus h curves on the orbifold.

Remark 14. Our results to follow are coarse in the sense that the invariants $\mathsf{n}_h(d)$ do not individually track the geometric genus of the $\imath$-invariant curves in A. This is because we do not fully probe the K-theory of the orbifold $[A/\imath]$. In work in progress with J. Bryan [**14**] we give the refined formula, as well as propose a general framework defining equivariant BPS invariants on a Calabi-Yau threefold with an involution.

The following is an analogue of the KKV formula for the orbifold Kummer surface.

Theorem 15. *The $\imath$-BPS invariants $\mathsf{n}_h(d)$ (Definition 32) are determined by*

$$\begin{aligned}\frac{1}{16}\sum_{d=0}^{\infty}\sum_{h=0}^{\infty}\mathsf{n}_h(d)\big(y^{\frac12}+y^{-\frac12}\big)^{2h}q^d &= -\big(y^{\frac12}+y^{-\frac12}\big)^2\frac{Z_{A,\langle\imath\rangle}(q)}{\phi_{-2,1}(q^2,-y)}\\ &= \prod_{n=1}^{\infty}\frac{(1-q^{2n})^{12}}{(1-q^n)^{16}(1+q^{2n}y)^2(1+q^{2n}y^{-1})^2}.\end{aligned} \tag{2.8}$$

[3] The definition of the BPS invariants by Maulik-Toda applies to Calabi-Yau threefolds. But in the case of a local Calabi-Yau surface, the theory reduces to a theory of sheaves on the surface, see Section 2.4.1. Our results are therefore intrinsic to $[A/\imath]$.

We prove this in Section 2.4.4 using the work of Shen-Yin [**46**] on perverse Hodge numbers of Lagrangian fibrations to relate the Maulik-Toda polynomial to $\overline{\chi}_y(\mathrm{Hilb}(A)^\imath)$. We then apply our Proposition 11 which determines the χ_y-genera.

$\frac{1}{16}\mathsf{n}_h(d)$	$d=0$	1	2	3	4	5	6	7
$h=0$	1	16	144	960	5264	25056	106944	418176
1	0	0	-2	-32	-294	-2016	-11400	-56000
2	0	0	0	0	3	48	448	3136
3	0	0	0	0	0	0	-4	-64
4	0	0	0	0	0	0	0	0

Table 2.2: Values of $\frac{1}{16}\mathsf{n}_h(d)$ for small d and h.

We now want to specialize to counting rational curves. Making the specialization $y = -1$ in the KKV formula (2.5) results in

$$\sum_{d=0}^{\infty} \mathsf{n}_d^{K3}(0) q^{d-1} = \frac{1}{\Delta(q)}.$$

This is the Yau-Zaslow formula—one of the earliest and most foundational results in modern enumerative geometry [**49**]. It is a relationship between rational curves in $K3$ surfaces, modular forms, and Hilbert schemes of points. One remarkable feature of the Yau-Zaslow formula is that the invariants $\mathsf{n}_d^{K3}(0)$ give actual (not virtual) counts of rational curves. For the mathematical formulation of the theory, see [**4, 12, 18**].

In this spirit, we give an enumerative interpretation of our partition function $Z_{A,\langle\imath\rangle}$ from Table 2.1 where $G = \langle\imath\rangle \cong \mathbb{Z}_2$. Consistent with the perverse sheaf and vanishing cycle formalism of Maulik-Toda, Katz had previously defined the genus zero BPS invariants as Behrend-weighted Euler characteristics of the Simpson stable moduli space [**26**].

In the case of $[A/\imath]$, we apply this to the space $\mathsf{M}_{\beta_d}(A)^\imath$, which is a disjoint union of smooth holomorphic symplectic varieties of $K3$-type. Therefore, the Behrend weighting is trivial, and

$$\mathsf{n}_0(d) = e\big(\mathsf{M}_{\beta_d}(A)^\imath\big). \tag{2.9}$$

The following is an analogue of the Yau-Zaslow formula for $[A/\imath]$, which we will prove in Section 2.4.5.

Theorem 16. *The genus zero $\imath$-BPS invariants $\mathsf{n}_0(d)$ are determined by*

$$\frac{1}{16}\sum_{d=0}^{\infty}\mathsf{n}_0(d)q^d = Z_{A,\langle\imath\rangle}(q) = \prod_{n=1}^{\infty}\frac{(1-q^{2n})^8}{(1-q^n)^{16}}. \tag{2.10}$$

Moreover, if A is a $(1,d)$-polarized Abelian surface of Picard rank one and β_d is the unique primitive generator, then $\mathsf{n}_0(d)$ is a weighted count of rational curves on $[A/\imath]$. Specifically,

$$\mathsf{n}_0(d) = \sum_{C\in\Pi} e(\overline{\mathrm{Jac}}(C)^\imath) \tag{2.11}$$

where Π is the finite set of $\imath$-invariant curves in the class β_d with rational quotient, and $\overline{\mathrm{Jac}}(C)$ is the compactified Jacobian of the integral curve C.

Just as the Yau-Zaslow formula does for $K3$ surfaces, this theorem relates rational curves on the orbifold $[A/\imath]$ to the modular form $Z^{-1}_{A,\langle\imath\rangle}$ and hence, to Hilbert schemes of points on $[A/\imath]$ as well.

We note that in the case of a $K3$ surface with a symplectic action by a finite cyclic group, in [**51**] they are interested in enumerating orbits of curves with rational quotient. Though the perspective is somewhat different from us: they view Euler characteristics as a representation, whereas we take ordinary Euler characteristics of the invariant locus.

2.1.4 Hyperelliptic Curve Counting Invariants

Our formula can equivalently be interpreted as a weighted count of hyperelliptic curves. A study of the enumerative geometry of hyperelliptic curves in Abelian surfaces was initiated in [**45**], and fully solved in [**13**].

Definition 17. For a polarized Abelian surface A of type $(1,d)$, let $\mathsf{h}_g(d)$ be the finite number of geometric genus g hyperelliptic curves in the class of the polarization such that all Weierstrass points occur at a 2-torsion point of A.

Remark 18. Every hyperelliptic curve in A can be translated so that all Weierstrass points lie at 2-torsion points, in which case the curve becomes $\imath$-invariant. Therefore, $\mathsf{h}_g(d)$ are actual (not virtual) counts of $\imath$-invariant curves with rational quotient.

By [**13**, Prop. 4], the $\mathsf{h}_g(d)$ are computed from the formula[4]

$$\sum_{d=0}^{\infty}\sum_{g=1}^{\infty}\mathsf{h}_g(d)(w^{\frac{1}{2}}+w^{-\frac{1}{2}})^{2g+2}q^d = 4\,\phi^2_{-2,1}(q,-w) \tag{2.12}$$

[4]In [**13**, Sec. 5.4] what we are calling $\mathsf{h}_g(d)$ was denoted $\mathsf{h}^{A,\mathrm{Hilb}}_{g,\beta}$.

where $\phi_{-2,1}$ is the weak Jacobi form defined in (2.4). In [**13**], it is tacitly assumed that $d > 0$, but the formula correctly encodes the remaining invariant. If $d = 0$, the only non-vanishing invariant obtained from the formula is $\mathsf{h}_0(1) = 4$ which represents the four invariant genus one curves in the surface $A = E \times F$ in the class of $E \times \{\mathrm{pt}\}$, each of which have rational quotient.

The relationship between $Z_{A,\langle \imath \rangle}$ and the formula (2.12) arises by making the specialization $w = 1$. We have the straightforward identity of infinite products

$$\frac{\eta(q^2)^8}{\eta(q)^{16}} = \frac{1}{16}\phi^2_{-2,1}(q,-1)$$

and note that the lefthand side is precisely $Z_{A,\langle \imath \rangle}$. This along with (2.12) and the first claim of Theorem 16 immediately implies the following result.

Proposition 19. *For all $d \geq 0$, we have*

$$\mathsf{n}_0(d) = \sum_{g=1}^{d+1} \mathsf{h}_g(d) 2^{2g}.$$

The invariants $\mathsf{n}_0(d)$ are therefore less refined, as they do not individually track the geometric genus g (see Remark 14).

If A has Picard rank one (except for the case of $d = 0$) then by (2.11) we have

$$\sum_{C \in \Pi} e(\overline{\mathrm{Jac}}(C)^\imath) = \sum_{g=1}^{d+1} \mathsf{h}_g(d) 2^{2g}$$

But $|\Pi| = \sum_g \mathsf{h}_g(d)$, so we regard this as strong evidence for the following conjecture.

Conjecture 20. *If A is an Abelian surface of Picard rank one, and $C \subset A$ is an integral $\imath$-invariant curve of geometric genus g with rational quotient, then*

$$e(\overline{\mathrm{Jac}}(C)^\imath) = 2^{2g}.$$

In Section 2.4.6 we will prove that the conjecture holds in the case of smooth curves.

Remark 21. One should ask if there are similar enumerative interpretations of $e(\mathrm{Hilb}^d(A)^G)$ for the remaining groups in Table 2.1. To our knowledge, this breaks down outside of $G \cong \mathbb{Z}_2$ because the key deformation equivalence (2.6) is not G-equivariant. Of course, one can directly study $e(\mathsf{M}_{\beta_d}(A)^G)$ for the remaining G, but we do not pursue that here.

2.2 Symplectic Actions on Abelian Surfaces

2.2.1 Preliminaries

Let X be a complex torus of arbitrary dimension, with $0 \in X$ the origin. The group of biholomorphisms from X to itself is denoted $\mathrm{Aut}(X)$, while the subgroup of linear maps (automorphisms of X as a complex Lie group) is denoted $\mathrm{Aut}_0(X) \subset \mathrm{Aut}(X)$. Given any $x \in X$, let $t_x : X \to X$ be the biholomorphism translating by x.

Given any holomorphic map $f : X \to X'$ between complex tori, using that f is equivalent to a map between the corresponding universal covers, one can show that $h := t_{-f(0)} \circ f$ is linear. Therefore, holomorphic maps between complex tori can be uniquely factored as a linear map composed with a translation

$$f = t_{f(0)} \circ h.$$

This factorization induces a canonical surjective group homomorphism

$$\sigma : \mathrm{Aut}(X) \to \mathrm{Aut}_0(X)$$

mapping f to h, which restricts to the identity on $\mathrm{Aut}_0(X) \subset \mathrm{Aut}(X)$ and whose kernel is the subgroup of all translations of X. This proves the following.

Proposition 22. *The biholomorphism group of a complex torus X decomposes as $\mathrm{Aut}(X) = \mathrm{Aut}_0(X) \rtimes X$, where X is identified with the subgroup of translations.*

Given a subgroup $G \subseteq \mathrm{Aut}(X)$, we get an action of G on X by biholomorphisms. We will consider actions up to the following equivalence condition.

Definition 23. Consider pairs (X_i, G_i) for $i = 1, 2$ with $G_i \subseteq \mathrm{Aut}(X_i)$. We say the two pairs are *equivalent* if there exists a biholomorphism $w : X_1 \to X_2$ such that $G_2 = wG_1w^{-1}$ in $\mathrm{Aut}(X_2)$.

We specialize to the case where X is a 2-dimensional complex torus. A non-zero class $\alpha \in H^{2,0}(X)$ is called a *holomorphic symplectic form*. Since $h^{2,0}(X) = 1$, a holomorphic symplectic form is unique up to scale.

Definition 24. An automorphism $f \in \mathrm{Aut}(X)$ is holomorphic symplectic, or just *symplectic*, if f preserves a holomorphic symplectic form α. That is, if $f^*\alpha = \alpha$. An action by a group G on X is *symplectic* if each element of G defines a symplectic automorphism.

Lemma 25. *A group G with a symplectic action on X can be written uniquely as the (possibly non-split) extension*

$$0 \to T \to G \to \sigma(G) \to 0 \tag{2.13}$$

where $T \subseteq G$ is the subgroup of all elements acting by translation, and the induced action by $\sigma(G)$ is symplectic and linear.

Proof. The existence of the short exact sequence is clear from the definition of σ. Given $f \in G$ we can write $f = t_{f(0)} \circ h$, and then $\sigma(f) = h$, which acts linearly. The kernel is precisely elements of G acting by translation. Since the symplectic form can be taken to be constant—induced from $dz_1 dz_2$ on the universal cover—it is clearly invariant under translations. Therefore, $h = t_{-f(0)} \circ f$ is symplectic. $\square$

This proof illustrates the obstruction to the splitting of the extension. We have the unique factorization $f = t_{f(0)} \circ h$, but notice $t_{f(0)}$ might not be an element of G. The extension splits if and only if $t_{f(0)} \in G$ for all $f \in G$.

When T is trivial, we say the action is *translation-free*. Note that a translation-free action is not necessarily linear: if T is trivial, G and $\sigma(G)$ are abstractly isomorphic, but may act differently on X. We say that a translation-free action by G is *maximal* if there does not exist a translation-free group $H \subset \mathrm{Aut}(X)$ with $G \subsetneq H$ such that the G action is the restriction of the H action.

2.2.2 Fujiki's Classification

Fujiki has given a complete classification of symplectic actions by finite groups on two-dimensional complex tori [**19**][5]. The goal of this section is to condense the relevant results of Fujiki into a brief survey. Below, all actions are assumed to be symplectic, and we will narrow our focus to Abelian surfaces, even though the results apply also to non-algebraic tori.

By Lemma 3.3 of [**19**], the only groups with non-trivial linear actions on an Abelian surface A are (see Introduction for definitions) isomorphic to one of

$$\mathbb{Z}_2,\ \mathbb{Z}_3,\ \mathbb{Z}_4,\ \mathbb{Z}_6,\ \mathcal{Q},\ \mathcal{D},\ \mathcal{T}.$$

[5]This was equivalently carried out from a physics perspective in [**48**] by studying symmetry groups of certain non-linear sigma models on the underlying real torus T^4.

All Abelian surfaces carry a unique linear action by $\langle\imath\rangle \cong \mathbb{Z}_2$ where $\imath : A \to A$ is the canonical involution $a \mapsto -a$. Hence, A admits a symplectic linear $\mathbb{Z}_3$ action if and only if it does so for $\mathbb{Z}_6$. Then it suffices to study G isomorphic to one of $\mathbb{Z}_4, \mathbb{Z}_6, \mathcal{Q}, \mathcal{D}, \mathcal{T}$. If we let $\overline{G} = G/\langle\imath\rangle$, then we know from [**48**, Sec. 4] that the five isomorphism classes of $\overline{G}$ are precisely all subgroups of $W^+(E_8)$ which pointwise fix a lattice of rank at least 4.

One should understand which Abelian surfaces carry a linear action by a particular group. For the two cyclic groups $\mathbb{Z}_4$ and $\mathbb{Z}_6$, the full description of which tori admit such actions is given in [**19**, Prop. 3.7]. In particular, for all elliptic curves E, the product $E \times E$ admits an action by $\mathbb{Z}_4$ and $\mathbb{Z}_6$. These linear actions by cyclic groups, including $\mathbb{Z}_2$ and $\mathbb{Z}_3$, correspond to Nos. $2-5$ in Table 2.1.

We now discuss the non-cyclic case. Let $\mathbb{H} \cong \mathbb{R}^4$ denote the real quaternions. The space of complex structures on $\mathbb{H}$ is

$$M = \left\{ J \in \mathbb{H} \,\middle|\, J^2 = -1 \right\} \cong \mathbb{P}^1$$

and can be identified with the imaginary unit quaternions. Fujiki associates to $G \cong \mathcal{Q}, \mathcal{D}, \mathcal{T}$ a lattice $\Lambda_G \subset \mathbb{H}$ and forms the real torus $T_G = \mathbb{H}/\Lambda_G$. The space M parameterizes complex structures on T_G such that the group of units $\Lambda_G^\times \cong G$ induces a holomorphic linear G action on the complex torus. The lattices and their groups of units can be found in (and just above) Lemma 2.6 of Fujiki.

By Theorem 3.11 of Fujiki, all maximal linear actions by each G arise in this way, up to equivalence. These correspond to Nos. 6, 9, and 10 in Table 2.1. There is a non-maximal linear $\mathcal{Q}$ action corresponding to the restriction of the maximal $\mathcal{T}$ action to the unique normal subgroup $\mathcal{Q} \subset \mathcal{T}$. This is No. 7 in the Table.

In a very similar manner, Section 3.4 of Fujiki describes and classifies all non-linear translation-free actions. Only $\mathcal{T}$ can act maximally as such, which corresponds to No. 11 in the Table. But we can restrict to $\mathcal{Q} \subset \mathcal{T}$ giving a non-maximal non-linear action of $\mathcal{Q}$. This is No. 8 in the Table.

In all cases, the complex tori admitting a translation-free action by a non-cyclic group are parameterized by M. Those that are algebraic are of a special form.

Definition 26. A *singular Abelian surface* is an Abelian surface A whose Neron-Severi lattice $\mathrm{NS}(A)$ has rank 4, its largest possible value. Equivalently, A is a product $E \times F$ of isogenous elliptic curves with complex multiplication.

The following result combines Lemma 5.6 and Proposition 5.7 of Fujiki.

Proposition 27. *If A admits a translation-free action by $G \cong \mathcal{Q}, \mathcal{D}, \mathcal{T}$ then A is a singular Abelian surface. Moreover, A corresponds to a complex structure $J \in M$ such that $\mu J \in \Lambda_G$ for some real number $\mu \neq 0$, which depends on J.*

Because J must have unit norm, the set of $J \in M$ satisfying the second condition of Proposition 27 is countable and dense in M. Therefore, Abelian surfaces carrying an action by $\mathcal{Q}, \mathcal{D}$, and $\mathcal{T}$ are rigid—there are no infinitesimal deformations of the surface on which the group acts. In Theorems 7.2 and 7.4 of Fujiki, necessary and sufficient conditions are given for a singular Abelian surface to admit an action by one of the three non-cyclic groups.

2.2.3 Singularity Type

Let G be a finite group with a symplectic translation-free action on an Abelian surface A. The singularities of A/G are all of ADE type—that is, the stabilizer of an arbitrary point in A is a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$. Recall that $\mathbb{Z}_n$ has ADE type A_{n-1} while $\mathcal{Q}, \mathcal{D}, \mathcal{T}$ have types D_4, D_5, E_6, respectively. For a given action, let a_k denote the number of A_k singularities in A/G, let d_k denote the number of D_k singularities, and let e_k denote the number of E_k singularities. In the present case, we present the *singularity type* of an action as

$$a_1 A_1 + a_2 A_2 + a_3 A_3 + a_5 A_5 + d_4 D_4 + d_5 D_5 + e_6 E_6.$$

Proposition 28. *The singularity type of a symplectic translation-free action of G on A is precisely one of those listed in Table 2.1, with the corresponding value of $\frac{1}{2}e(A/G)$ listed in the final column.*

We will sketch the proof of this proposition, but see also Lemma 3.19 in [**19**][6]. Let $\mathrm{Sing}(A/G)$ and $(A/G)^\circ = A/G - \mathrm{Sing}(A/G)$ be the singular locus and smooth locus of A/G, respectively. If $\pi : Y \to A/G$ is the minimal resolution of singularities, then by standard properties of the Euler characteristic, we have

$$e(Y) = e\big((A/G)^\circ\big) + e\big(\pi^{-1}(\mathrm{Sing}(A/G))\big).$$

The first term is computed by noting that G acts freely on A away from points with stabilizers

$$e\big((A/G)^\circ\big) = -\left(\frac{a_1}{2} + \frac{a_2}{3} + \frac{a_3}{4} + \frac{a_5}{6} + \frac{d_4}{8} + \frac{d_5}{12} + \frac{e_6}{24}\right) \tag{2.14}$$

[6] We claim there are a few minor but relevant typos in Lemma 3.19 and equation (19) of [**19**]. Nonetheless, the ten actions described in Lemma 3.19 are precisely the ten non-trivial actions in our Table 2.1.

and since the exceptional locus of $\pi : Y \to A/G$ is a disjoint union of ADE configurations of smooth rational curves, the second term is

$$e\big(\pi^{-1}(\mathrm{Sing}(A/G))\big) = 2a_1 + 3a_2 + 4a_3 + 6a_5 + 5d_4 + 6d_5 + 7e_6.$$

Finally, since Y is a smooth $K3$ surface, $e(Y) = 24$. We therefore get a strong numerical constraint on the numbers of singular points of each type. For a given G, using that all subgroups define symplectic translation-free actions, one can systematically use this constraint to determine all allowed singularity types. The possible solutions correspond to the 11 columns in Table 2.1.

Given a singularity type, let r be the total number of singular points in A/G. From (2.14) and the obvious formula

$$e(A/G) = e((A/G)^{\circ}) + r,$$

we can easily verify the values in the final column of Table 2.1.

2.3 Computation of the Partition Functions $Z_{A,G}$

The goal of this section is to compute the partition functions

$$Z_{A,G}(q) := \sum_{d=0}^{\infty} e(\mathrm{Hilb}^d(A)^G)q^d$$

explicitly as an eta product, thereby proving Theorem 8. First, we handle the more elementary Proposition 10 on how translations affect the partition function.

Proof of Proposition 10. If G is a finite group acting symplectically on an Abelian surface A, and $T \subset G$ is the subgroup of translations, then $G' = G/T$ acts symplectically on $A' = A/T$ without translations. Since T acts freely, a G-invariant subscheme of A must have length dividing $|T|$. So $\mathrm{Hilb}^d(A)^G = \varnothing$ unless $d = m|T|$ for some integer $m \geq 0$, in which case $\mathrm{Hilb}^{m|T|}(A)^G \cong \mathrm{Hilb}^m(A')^{G'}$. Hence,

$$\begin{aligned} Z_{A,G}(q) = \sum_{d=0}^{\infty} e(\mathrm{Hilb}^d(A)^G)q^d &= \sum_{m=0}^{\infty} e(\mathrm{Hilb}^{m|T|}(A)^G)q^{m|T|} \\ &= \sum_{m=0}^{\infty} e(\mathrm{Hilb}^m(A')^{G'})q^{m|T|} = Z_{A',G'}(q^{|T|}). \end{aligned}$$

□

Therefore, the problem is reduced to translation-free actions. Our proof of Theorem 8 is based on a computation of Bryan-Gyenge [**11**], so we briefly review the relevant results. For a finite subgroup $G_\Delta \subset \mathrm{SU}(2)$ with associated ADE root system Δ, we can consider the natural action of G_Δ on $\mathbb{C}^2$, and define the local G_Δ-fixed partition function

$$Z_\Delta(q) = \sum_{n=0}^{\infty} e(\mathrm{Hilb}^n(\mathbb{C}^2)^{G_\Delta}) q^{n-\frac{1}{24}},$$

which Bryan-Gyenge compute for each allowed Δ.

Theorem 29 ([**11**, Thm. 1.2])**.** *For Δ of type A_n, the local partition function is given by*

$$Z_{A_n}(q) = \frac{1}{\eta(q)}$$

while for type D_n and E_n, it is

$$Z_\Delta(q) = \frac{\eta^2(q^2)\eta(q^{4E})}{\eta(q)\eta(q^{2E})\eta(q^{2F})\eta(q^{2V})}$$

*with (E, F, V) presented explicitly in each case in [**11**, Thm. 1.2].*

With this, we are ready to prove our main result.

Proof of Theorem 8. Suppose first that A is an Abelian surface with a translation-free symplectic action by a finite group G. To set notation, let $p_1, \ldots, p_r \in A/G$ be the singular points, each with stabilizer subgroup $G_i \subset G$ and corresponding ADE root system Δ_i. With $k = |G|$ and $k_i = |G_i|$, let $\{x_i^1, \ldots, x_i^{k/k_i}\}$ be the orbit in A corresponding to singular point p_i. The smooth locus of the quotient is

$$(A/G)^\circ = A/G - \{p_1, \ldots, p_r\}.$$

The same method from [**11**, Section 2] of stratifying the Hilbert scheme applies here, and gives the relation

$$\begin{aligned} \sum_{d=0}^{\infty} e(\mathrm{Hilb}^d(A)^G) q^d =& \left(\sum_{d=0}^{\infty} e(\mathrm{Hilb}^d((A/G)^\circ)) q^{kd} \right) \\ &\cdot \prod_{i=1}^{r} \left(\sum_{d=0}^{\infty} e(\mathrm{Hilb}^d(\mathbb{C}^2)^{G_i}) q^{\frac{dk}{k_i}} \right) \end{aligned} \tag{2.15}$$

By removing the singular points from A/G as well as their preimages in A, the restricted quotient map is unramified of degree k, which means

$$e(A) - \sum_{i=1}^{r} \#\{x_i^1, \ldots, x_i^{k/k_i}\} = k \cdot e((A/G)^\circ).$$

With $a := e((A/G)^\circ)$, and using that $e(A) = 0$, we get

$$a = -\sum_{i=1}^{r} \frac{1}{k_i}.$$

Since $(A/G)^\circ$ is a smooth quasi-projective surface, by Göttsche's formula [**23**]

$$\sum_{d=0}^{\infty} e(\mathrm{Hilb}^d((A/G)^\circ))q^{kd} = \prod_{n=1}^{\infty}(1-q^{kn})^{-a}.$$

Hence, using the definition of the local partition functions, as well as (2.15), we get

$$\begin{aligned} Z_{A,G}(q) &= \prod_{n=1}^{\infty}(1-q^{kn})^{-a} \cdot \prod_{i=1}^{r} q^{\frac{k}{24k_i}} Z_{\Delta_i}\big(q^{\frac{k}{k_i}}\big) \\ &= q^{\left(\frac{ka}{24}+\sum_i \frac{k}{24k_i}\right)}\eta^{-a}(q^k) \cdot \prod_{i=1}^{r} Z_{\Delta_i}\big(q^{\frac{k}{k_i}}\big). \end{aligned}$$

From the relation $a + \sum_i \frac{1}{k_i} = 0$, we see the exponent of the overall power of q vanishes. With the substitution $a = e(A/G) - r$ we see

$$Z_{A,G}(q) = \eta(q^k)^{r-e(A/G)} \cdot \prod_{i=1}^{r} Z_{\Delta_i}\big(q^{\frac{k}{k_i}}\big). \tag{2.16}$$

By Proposition 28, the ADE singularity type and Euler characteristic $e(A/G)$ can be read off of Table 2.1. Recall that $\mathbb{Z}_n$ has ADE type A_{n-1} while $\mathcal{Q}, \mathcal{D}, \mathcal{T}$ have types D_4, D_5, E_6, respectively. This determines the value of k_i for each Δ_i. From (2.16), we can use Theorem 29 to compute the function $Z_{A,G}^{-1}(q)$ as an eta product, which we record in the third column of Table 2.1. By Theorem 29, $Z_{\Delta_i}^{-1}$ transforms as a modular form of weight $\frac{1}{2}$ for all Δ_i. By (2.16), the weight of $Z_{A,G}^{-1}$ is therefore $\frac{1}{2}e(A/G)$. It is clear that the leading coefficient of $Z_{A,G}^{-1}$ is 1. Applying Proposition 93 case by case, we see that $Z_{A,G}^{-1}$ is a holomorphic modular form of level $k = |G|$.

In case G acts on A with translations, we apply Proposition 10. Since the G' action on A' is symplectic and translation-free, $Z_{A',G'}^{-1}$ is a holomorphic modular form of weight $\frac{1}{2}e(A'/G')$, with level $|G'|$, and normalized with leading coefficient 1. The weight, holomorphy, and normalization are invariant under the variable change $q \mapsto q^{|T|}$, but the new level is $|G'| \cdot |T| = |G|$. □

Proof of Proposition 11. The proof of the much more general Theorem 1.10 in [**11**, Sec. 5] goes through nearly verbatim in the case of Abelian surfaces, with the following minor adjustments. Let $Y \to A/G$ be the minimal resolution. We would use the definition $Z^{\mathsf{bir}}_{A,G}(q) = \sum_{d=0}^{\infty}[\mathrm{Hilb}^d(A)^G]_{\mathsf{bir}}\, q^d$. But since Y is a $K3$ surface, we define $Z^{\mathsf{bir}}_Y(q)$ just as in [**11**] with the extra power of q. In particular, this implies the result for the normalized χ_y-genera. $\square$

2.4 Enumerative Geometry of the Orbifold Kummer Surface

The purpose of this section is to prove the remaining results from Section 2.1.3 of the Introduction.

2.4.1 Review of Ordinary BPS Invariants

Let X be a Calabi-Yau threefold with curve class $\beta \in H_2(X, \mathbb{Z})$. The central quantity in the Maulik-Toda proposal defining the ordinary BPS states [**32**], is the Hilbert-Chow morphism[7]

$$\pi : \mathsf{M}_\beta(X) \to \mathrm{Chow}_\beta(X)$$

along with the perverse sheaf of vanishing cycles ϕ on $\mathsf{M}_\beta(X)$. We then define the Maulik-Toda polynomial

$$\mathsf{MT}(\pi) = \sum_{i,j\in\mathbb{Z}} (-1)^i y^j \dim \mathbb{H}^i\big(\,\mathrm{Chow}_\beta(X), {}^{\mathfrak{p}}\mathcal{H}^j(R\pi_*\phi)\big) \tag{2.17}$$

where $\mathbb{H}$ denotes hypercohomology, and ${}^{\mathfrak{p}}\mathcal{H}(\cdot)$ are the cohomology sheaves with respect to the perverse t-structure. For further details, see [**16, 32**]. By Verdier duality, $\mathsf{MT}(\pi)$ is a Laurent polynomial in y invariant under $y \leftrightarrow y^{-1}$. Therefore, we can write

$$\mathsf{MT}(\pi) = \sum_{g\geq 0} n_\beta(g)\big(y^{\frac{1}{2}} + y^{-\frac{1}{2}}\big)^{2g} \tag{2.18}$$

for uniquely determined integers $n_\beta(g)$, which we call the *(ordinary) BPS invariants*.

The example relevant to us is when the threefold X is a local $K3$ or Abelian surface S with effective class $\beta \in H_2(S, \mathbb{Z})$. In this case, one can define X as the total space of a fibration $f : X \to (\Delta, 0)$ by $K3$ or Abelian surfaces over a pointed disk $(\Delta, 0)$ such that:

1. All fibers of f are projective

[7] Here $\mathsf{M}_\beta(X)$ is the moduli space of Simpson stable one-dimensional sheaves F on X with $[\mathrm{supp}(F)] = \beta$ and $\chi(F) = 1$. Moreover, $\mathrm{Chow}_\beta(X)$ is the Chow variety of effective curves in the class β.

2. $f^{-1}(0) = S$

3. The class $\beta \in H_2(S, \mathbb{Z})$ does not deform algebraically off the central fiber to any order.

Because curves in the class β do not deform even scheme-theoretically off the central fiber, the theory localizes to studying sheaves on the surface S. Let $\mathsf{M}_\beta(S)$ be the moduli space of stable one-dimensional sheaves F with $[\mathrm{supp}(F)] = \beta$ and $\chi(F) = 1$. We know $\mathsf{M}_\beta(S)$ is a smooth projective holomorphic symplectic variety, and we additionally assume that it is of $K3$-type (this is not true when S is an Abelian surface, but by Corollary 13, the fixed locus under the standard involution is a disjoint union of varieties of this type).

Under the above assumptions, the Hilbert-Chow morphism

$$\pi : \mathsf{M}_\beta(S) \to \mathrm{Chow}_\beta(S)$$

has $\mathrm{Chow}_\beta(S) \cong \mathbb{P}^n$ where $\dim(\mathsf{M}_\beta(S)) = 2n$, by [**24**]. Hence, π is a Lagrangian fibration. Since $\mathsf{M}_\beta(S)$ is smooth, $\phi = \underline{\mathbb{Q}}[2n]$. For all $0 \leq p, q \leq 2n$, the perverse Hodge numbers of π are defined to be

$$ {}^{\mathfrak{p}}h^{p,q}(\pi) := \dim \mathbb{H}^{q-n}\big(\mathbb{P}^n, {}^{\mathfrak{p}}\mathcal{H}^{p-n}(R\pi_*\underline{\mathbb{Q}}[2n])\big). \tag{2.19}$$

Theorem 30 ([**46**, Thm. 0.2])**.** *Under the above hypotheses, the perverse Hodge numbers of π equal the ordinary Hodge numbers of $\mathsf{M}_\beta(S)$*

$$ {}^{\mathfrak{p}}h^{p,q}(\pi) = h^{p,q}(\mathsf{M}_\beta(S)).$$

Using (2.17) and (2.3), the following is then a small computation.

Corollary 31. *The Maulik-Toda polynomial associated to π can be expressed as the normalized χ_y-genus of $\mathsf{M}_\beta(S)$*

$$\mathsf{MT}(\pi) = \overline{\chi}_y(\mathsf{M}_\beta(S)).$$

2.4.2 $\imath$-BPS Invariants of the Orbifold Kummer Surface

There does not currently exist a definition of BPS invariants for orbifolds. We give here a coarse[8] definition in the case of the orbifold Kummer surface.

[8] See Remark 14

Let A be a polarized Abelian surface of type $(1, d)$ with $\beta_d \in H_2(A, \mathbb{Z})$ the class of the polarization. The orbifold Kummer surface is defined as the stack quotient $[A/\imath]$ where $\imath : A \to A$ is the canonical involution $a \mapsto -a$.

By the same argument as in Section 2.4.1, the Maulik-Toda proposal applied to a local orbifold Kummer surface will localize to a theory of sheaves on the surface itself. A natural guess to define orbifold BPS states is to use the proper map

$$\pi_d : \mathsf{M}_{\beta_d}(A)^\imath \to \mathrm{Chow}_{\beta_d}(A)^\imath \tag{2.20}$$

where the superscript $\imath$ denotes the fixed locus of the induced $\imath$ action.

By Corollary 13, the space $\mathsf{M}_{\beta_d}(A)^\imath$ is a disjoint union of smooth holomorphic symplectic varieties of $K3$-type. We can decompose it, along with the invariant Chow variety, into connected components

$$\mathsf{M}_{\beta_d}(A)^\imath = \coprod_{k,l} M_{k,l} \qquad \mathrm{Chow}_{\beta_d}(A)^\imath = \coprod_{l} B_l.$$

Restricting π_d to $M_{k,l}$, we get surjective maps $\pi_{k,l} : M_{k,l} \to B_l$. Since each B_l is smooth, by [**24**] each $\pi_{k,l}$ is a connected Lagrangian fibration over a projective space $B_l = \mathbb{P}^{d_l}$ where $\dim(M_{k,l}) = 2d_l$.

Associated to each $\pi_{k,l}$ with $\phi = \underline{\mathbb{Q}}_{M_{k,l}}[2d_l]$ we get the Maulik-Toda polynomial $\mathsf{MT}(\pi_{k,l})$ as in (2.17). The Maulik-Toda polynomial associated to the map in (2.20) is then

$$\mathsf{MT}(\pi_d) = \sum_{k,l} \mathsf{MT}(\pi_{k,l}).$$

Definition 32. The $\imath$-BPS invariants of the orbifold Kummer surface $[A/\imath]$ are integers $\mathsf{n}_h(d)$ defined for all $d \geq 0$ by

$$\mathsf{MT}(\pi_d) = \sum_{h=0}^{\infty} \mathsf{n}_h(d)(y^{\frac{1}{2}} + y^{-\frac{1}{2}})^{2h}. \tag{2.21}$$

2.4.3 Proof of Proposition 12

For all $d \geq 0$, we denote by $\mathsf{M}(A) = M_A(1, 0, -d)$ the moduli space of torsion-free sheaves on A with Chern character $(1, 0, -d) \in H^{2*}(A, \mathbb{Z})$.

Lemma 33. *There exists a canonical $\imath$-equivariant isomorphism*

$$\mathsf{M}(A) \to \widehat{A} \times \mathrm{Hilb}^d(A).$$

Proof. A sheaf $\mathcal{F} \in \mathsf{M}(A)$ can be uniquely expressed as $\mathcal{F} = L_0 \otimes \mathcal{I}_d$ where $L_0 \in \widehat{A}$ is a degree-zero line bundle and $\mathcal{I}_d$ is the ideal sheaf of a zero-dimensional subscheme of length d. The $\imath$-equivariance is immediate. $\square$

Let $A_0 = E \times F$ be a product Abelian surface with E and F two elliptic curves. Denote by σ the class of $E \times \{\mathrm{pt}\}$, and by f the class of $\{\mathrm{pt}\} \times F$. Lemma 33 reduces the proof of the proposition to constructing a $\imath$-equivariant deformation equivalence

$$\mathsf{M}(A) \to \mathsf{M}_{\beta_d}(A). \tag{2.22}$$

We do so by deforming to A_0, as summarized in the diagram

$$\begin{tikzcd} \mathsf{M}(A) \arrow[d] & \mathsf{M}_{\beta_d}(A) \\ \mathsf{M}(A_0) \arrow[r, "\sim"] & \mathsf{M}_{\sigma+df}(A_0) \arrow[u] \end{tikzcd} \tag{2.23}$$

Here, as we will explain, the vertical arrows are $\imath$-equivariant deformation equivalences and the horizontal arrow is a $\imath$-equivariant isomorphism induced from the relative Fourier-Mukai functor applied to the natural projection $A_0 \to E$.

Lemma 34. *The vertical arrows in diagram (2.23) are $\imath$-equivariant deformation equivalences.*

Proof. By Proposition 4.12 of [**50**], we have a deformation equivalence

$$\mathsf{M}(A) \to \mathsf{M}(A_0).$$

The proof involves a family of polarized Abelian surfaces. But clearly we have a fiberwise action on the family using the standard involution on each Abelian surface. It is in this sense that the deformation equivalence is $\imath$-equivariant. Precisely the same argument holds for $\mathsf{M}_{\sigma+df}(A_0) \to \mathsf{M}_{\beta_d}(A)$. $\square$

Consider the following diagram

$$\begin{tikzcd} & A_0 \times_E A_0 \arrow[dl, "p"'] \arrow[dr, "q"] & \\ A_0 & & A_0 \end{tikzcd} \tag{2.24}$$

where p, q are the two canonical projections. Let P denote the universal Poincaré line bundle on $A_0 \times_E A_0$, and define the relative Fourier-Mukai functor

$$\Phi_P : D^b(A_0) \to D^b(A_0)$$

by $\mathcal{E} \mapsto Rq_*\big(P \otimes p^*\mathcal{E}\big)$. For more details on relative Fourier-Mukai in more generality, see Section 3.2 of [**50**] or Chapter 6 of [**2**].

Lemma 35. *The relative Fourier-Mukai functor Φ_P is $\imath$-equivariant. More speecifically, for all $\mathcal{E} \in D^b(A_0)$*

$$\imath^*\big(\Phi_P(\mathcal{E})\big) \cong \Phi_P\big(\imath^*(\mathcal{E})\big).$$

Proof. The key is that the Poincaré bundle is $\imath_\Delta$-equivariant; that is, we can choose an isomorphism $P \to \imath_\Delta^* P$ where $\imath_\Delta$ is the induced diagonal action on $A_0 \times_E A_0$. This is because (see Definition 6.14 of [**2**])

$$P = \mathcal{O}_{A_0 \times_E A_0}\big(D + D' - \Delta\big)$$

where $D = A_0 \times \{0\}$, $D' = \{0\} \times A_0$ and Δ is the diagonal divisor in $A_0 \times_E A_0$. All three of these divisors are $\imath_\Delta$-invariant. The remainder of the argument follows from straightforward functorial properties of $\imath$, p, and q.

$\square$

Corollary 36. *The relative Fourier-Mukai functor Φ_P induces a $\imath$-equivariant isomorphism*

$$\mathsf{M}(A_0) \to \mathsf{M}_{\sigma+df}(A_0).$$

Proof. The isomorphism is a particular example of Theorem 3.15 in [**50**], noting that because our Chern character is primitive, there are no strictly semistable sheaves. The $\imath$-equivariance follows from the previous lemma. $\square$

This completes the proof of Proposition 12.

2.4.4 Proof of Theorem 15

By Corollary 31 we have

$$\mathsf{MT}(\pi_d) = \sum_{k,l} \overline{\chi}_y(M_{k,l}) = \overline{\chi}_y\big(\mathsf{M}_{\beta_d}(A)^\imath\big) \tag{2.25}$$

where the latter equality holds because the ordinary Hodge numbers are additive under disjoint unions. By Corollary 13, we have

$$\overline{\chi}_y\big(\mathsf{M}_{\beta_d}(A)^\imath\big) = 16\,\overline{\chi}_y\big(\operatorname{Hilb}^d(A)^\imath\big)$$

since the χ_y-genus is a deformation invariant. Therefore

$$\tfrac{1}{16}\mathsf{MT}(\pi_d) = \overline{\chi}_y\big(\operatorname{Hilb}^d(A)^\imath\big).$$

The result then follows using Proposition 11 along with the expression for $Z^{-1}_{A,\langle\imath\rangle}$ from Table 2.1.

2.4.5 Proof of Theorem 16

In this section we will prove Theorem 16 on the Yau-Zaslow formula for the orbifold Kummer surface. We first need to establish a few lemmas.

Lemma 37. *Let C be an integral curve with an involution $\imath : C \to C$. If the quotient $C/\imath$ is not rational, then $e(\overline{\operatorname{Jac}}(C)^\imath) = 0$.*

Proof. Let $\eta : \widetilde{C} \to C$ and $\delta : \widetilde{C/\imath} \to C/\imath$ be the corresponding normalization maps, and $\alpha : C \to C/\imath$ the quotient. We get a commuting diagram

$$\begin{tikzcd}
\widetilde{C} \arrow[r, "\eta"] \arrow[d, "\widetilde{\alpha}"'] & C \arrow[d, "\alpha"] \\
\widetilde{C/\imath} \arrow[r, "\delta"] & C/\imath
\end{tikzcd}$$

where the map $\widetilde{\alpha}$ is defined by the universal property of normalizations: since α is surjective and $C/\imath$ is integral, α factors uniquely via a map $C \to \widetilde{C/\imath}$. We get $\widetilde{\alpha}$ by composing with η. By pullback we get a short exact sequence of Abelian groups

$$0 \to \operatorname{Jac}(\widetilde{C/\tau}) \to \operatorname{Jac}(\widetilde{C}) \to \operatorname{Prym}(\widetilde{\alpha}) \to 0$$

where $\operatorname{Prym}(\widetilde{\alpha})$ is the Prym variety associated to $\widetilde{\alpha}$. Since $\widetilde{\alpha}$ is $\imath$-equivariant with the trivial action on $\widetilde{C/\imath}$, by restricting to the fixed locus we get an inclusion

$$\operatorname{Jac}(\widetilde{C/\imath}) \subset \operatorname{Jac}(\widetilde{C})^\imath. \tag{2.26}$$

Pulling back via η, we get a short exact sequence

$$0 \to G \to \operatorname{Jac}(C) \to \operatorname{Jac}(\widetilde{C}) \to 0$$

where G is a product of additive and multiplicative groups. By [**4**, Prop.2.2], this sequence splits canonically so that $\operatorname{Jac}(\widetilde{C}) \subset \operatorname{Jac}(C)$ corresponds to the subgroup of line bundles on C pushed forward from line bundles on $\widetilde{C}$. Because η is $\imath$-equivariant, this descends to an inclusion

$$\operatorname{Jac}(\widetilde{C})^{\imath} \subset \operatorname{Jac}(C)^{\imath}.$$

By [**4**, Lem.2.1] we thereby get a free action of $\operatorname{Jac}(\widetilde{C})^{\imath}$ on the invariant compactified Jacobian $\overline{\operatorname{Jac}}(C)^{\imath}$. But if $C/\imath$ is not rational, we know by (2.26) that a positive-dimensional Abelian variety therefore acts freely on $\overline{\operatorname{Jac}}(C)^{\imath}$. If a finite group of order n acts freely on $\overline{\operatorname{Jac}}(C)^{\imath}$, then n must divide $e(\overline{\operatorname{Jac}}(C)^{\imath})$. Because an abelian variety contains cyclic subgroups of all orders, $e(\overline{\operatorname{Jac}}(C)^{\imath}) = 0$. ☐

Lemma 38. *If $f : X \to Y$ is a surjective morphism of projective algebraic varieties and all fibers have vanishing Euler characteristic, then $e(X) = 0$.*

Proof. Consider first the case that f is a topological locally-trivial fibration with fiber F. A well-known result is that $e(X) = e(F) \cdot e(Y)$ from which the lemma follows. In the more general case, take a stratification of Y such that f is a topological locally-trivial fibration over each strata. The excision property of the Euler characteristic then reduces the problem to the previous case. ☐

Proof of Theorem 16. The genus $h = 0$ specialization of the Maulik-Toda polynomial is $y = -1$, as we see from (2.21). By the proof of Theorem 15,

$$\begin{aligned} \mathsf{n}_0(d) &= \mathsf{MT}_{-1}(\pi_d) \\ &= e\big(\mathsf{M}_{\beta_d}(A)^{\imath}\big) = 16\, e\big(\operatorname{Hilb}^d(A)^{\imath}\big). \end{aligned} \tag{2.27}$$

This proves the first assertion, noting the expression for $Z_{A,\langle\imath\rangle}$ (see Table 2.1).

For the final claim we assume A is a $(1, d)$-polarized Abelian surface with Picard rank one, and β_d the unique primitive generator. In this case,

$$\mathsf{M}_{\beta_d}(A) \cong \overline{\mathcal{J}\mathrm{ac}}^{\,d+1}_{\beta_d}(A)$$

where $\overline{\mathcal{J}\mathrm{ac}}^{\,d+1}_{\beta_d}(A) \to \operatorname{Chow}_{\beta_d}(A)$ is the relative compactified Jacobian of degree $d + 1$, which is the arithmetic genus of the class β_d. The fiber over a curve $C \in \operatorname{Chow}_{\beta_d}(A)$ is $\overline{\operatorname{Jac}}^{\,d+1}(C)$ parameterizing rank 1 torsion-free sheaves of degree $d + 1$ on C. We define

$$\Pi \subset \operatorname{Chow}_{\beta_d}(A)^{\imath}$$

to be the set of $\imath$-invariant curves in the class β_d with rational quotient. The set Π is finite because if it were not, the singular $K3$ surface $A/\imath$ would contain a positive-dimensional family of rational curves, which cannot occur. Let Y be the open subvariety of the invariant Chow variety parameterizing $\imath$-invariant curves with non-rational quotient.

$$Y = \mathrm{Chow}_{\beta_h}(A)^{\imath} - \Pi.$$

Given the map $\pi_d : \overline{\mathcal{J}\mathrm{ac}}_{\beta_d}^{d+1}(A)^{\imath} \to \mathrm{Chow}_{\beta_d}(A)^{\imath}$, we get the following decomposition of the total space

$$\overline{\mathcal{J}\mathrm{ac}}_{\beta_d}^{d+1}(A)^{\imath} = \pi_d^{-1}(Y) \cup \pi_d^{-1}(\Pi).$$

Since Π is finite, $\pi_d^{-1}(\Pi)$ is a closed subvariety of $\overline{\mathcal{J}\mathrm{ac}}_{\beta_d}^{d+1}(A)^{\imath}$ whose compliment is $\pi_d^{-1}(Y)$. Thus, by the excision property of the Euler characteristic

$$e(\overline{\mathcal{J}\mathrm{ac}}_{\beta_d}^{d+1}(A)^{\imath}) = e(\pi_d^{-1}(Y)) + e(\pi_d^{-1}(\Pi)).$$

The fiber of π_d over an invariant curve C is $\overline{\mathrm{Jac}}(C)^{\imath} \cong \overline{\mathrm{Jac}}^{d+1}(C)^{\imath}$, where the isomorphism is twisting by a fixed $\imath$-invariant line bundle of degree $d+1$. Therefore, all fibers of the restricted family $\pi_d^{-1}(Y) \to Y$ have vanishing Euler characteristic by Lemma 37, and $e(\pi_d^{-1}(Y)) = 0$ by Lemma 38. Finally, we have

$$\pi_d^{-1}(\Pi) = \coprod_{C \in \Pi} \overline{\mathrm{Jac}}(C)^{\imath},$$

from which it now follows that

$$e(\overline{\mathcal{J}\mathrm{ac}}_{\beta_d}^{d+1}(A)^{\imath}) = \sum_{C \in \Pi} e(\overline{\mathrm{Jac}}(C)^{\imath}).$$

Because $\mathsf{n}_0(d) = e(\overline{\mathcal{J}\mathrm{ac}}_{\beta_d}^{d+1}(A)^{\imath})$, this completes the proof. $\square$

2.4.6 Proof of Conjecture 20 for Smooth Curves

Let C be a smooth curve of genus g with an involution $\imath : C \to C$. Let $\alpha : C \to C/\imath$ be the quotient map, and let h be the genus of the smooth curve $C/\imath$. Just as in the proof of Lemma 37, we get a short exact sequence

$$0 \to \mathrm{Jac}(C/\imath) \to \mathrm{Jac}(C) \to \mathrm{Prym}(\alpha) \to 0$$

where $\mathrm{Prym}(\alpha)$ is an Abelian variety of dimension $g-h$ called the Prym variety. There is a canonical lift of the $\imath$ action to $\mathrm{Jac}(C)$ by pullback which acts trivially on $\mathrm{Jac}(C/\imath)$.

Moreover, the induced action on $\mathrm{Prym}(\alpha)$ is by the canonical involution $a \mapsto -a$, which has 2^{2g-2h} isolated fixed points. Therefore

$$\mathrm{Jac}(C)^{\imath} = \coprod_{i=1}^{2^{2g-2h}} \mathrm{Jac}(C/\imath)$$

If $C/\imath$ is a rational curve, then $h = 0$ and $\mathrm{Jac}(C/\imath)$ is a point, which proves the conjecture in this case.

Chapter 3

A Theory of Gopakumar-Vafa Invariants for Calabi-Yau Threefolds with an Involution

3.1 Ordinary GV invariants

Let X be a Calabi-Yau threefold (CY3) by which we mean a smooth quasi-projective variety over $\mathbb{C}$ of dimension three with $K_X \cong \mathcal{O}_X$. In 1998 [**22**], Gopakumar and Vafa (GV) defined via physics integer invariants $n_g(\beta)$ which give a virtual count of curves $C \subset X$ of genus g and class $[C] \in H_2(X)$.

Mathematically, there are two conjecturally equivalent sheaf theoretic approaches to defining $n_g(\beta)$, one by Pandharipande-Thomas (PT) via their stable pair invariants [**41**], and one more recently given by Maulik and Toda (MT) using perverse sheaves [**32**]. We begin by reviewing ordinary GV theory, and then we develop in a parallel fashion a theory of GV invariants for CY3s with an involution.

3.1.1 GV invariants via PT theory

Let $\mathsf{PT}_{\beta,n}(X)$ be the moduli space of PT pairs [**40**]:

$$\mathsf{PT}_{\beta,n}(X) = \left\{ (F,s) \,\middle|\, s \in H^0(X,F),\ [\mathrm{supp}(F)] = \beta,\ \chi(F) = n \right\}$$

where F is a coherent sheaf on X with proper support of pure dimension 1, and $\mathrm{coker}(s)$ has support of dimension 0.

For any scheme S over $\mathbb{C}$, Behrend [**6**] defined a constructible function $\nu_S : S \to \mathbb{Z}$ and we define the *virtual Euler characteristic* to be the Behrend function weighted topological Euler characteristic:

$$e_{\mathsf{vir}}(S) = e(S, \nu_S) = \sum_{k \in \mathbb{Z}} k \cdot e\left(\nu_S^{-1}(k)\right).$$

We then define the *PT invariants* by

$$N_{\beta,n}^{\mathsf{PT}}(X) = e_{\mathsf{vir}}(\mathsf{PT}_{\beta,n}(X))$$

and the *PT partition function* by

$$Z^{\mathsf{PT}}(X) = \sum_{\beta,n} N_{\beta,n}^{\mathsf{PT}}(X)\, Q^\beta\, y^n.$$

Definition 39. The Gopakumar-Vafa invariants (via PT theory) $n_g^{\mathsf{PT}}(\beta)$ are defined via the following equation:

$$\log Z^{\mathsf{PT}}(X) = \sum_{k>0} \sum_{\beta,g} \frac{1}{k} \cdot Q^{k\beta} \cdot n_g^{\mathsf{PT}}(\beta) \cdot \psi_{-(-y)^k}^{g-1} \tag{3.1}$$

where

$$\psi_x = 2 + x + x^{-1}$$

Remark 40. Writing $\log Z^{\mathsf{PT}}(X)$ in the form given by the righthand side of Equation (3.1) uses the fact that the coefficient of Q^β in $Z^{\mathsf{PT}}(X)$ is the Laurant expansion of a rational function in y which is invariant under $y \leftrightarrow y^{-1}$ [**9**, **40**, **47**]. Although it isn't clear from the definition, one expects that $n_g^{\mathsf{PT}}(\beta) = 0$ if $g < 0$.

Remark 41. Gopakumar and Vafa gave a formula relating their invariants to the Gromov-Witten (GW) invariants. Equation (3.1) is equivalent to the Gopakumar-Vafa formula after using the expected relationship between PT and GW invariants.

3.1.2 GV invariants via MT theory

We define the moduli space of Maulik-Toda (MT) sheaves to be

$$\mathsf{M}_\beta(X) = \left\{ F \,\middle|\, [\mathrm{supp}(F)] = \beta,\ \chi(F) = 1 \right\}$$

where F is a coherent sheaf on X with proper sheaf theoretic support of pure dimension 1 and where F is *Simpson stable*, which in this case is equivalent to the condition that if $F' \subsetneq F$, then $\chi(F') \leq 0$.

The MT moduli space is a quasi-projective scheme and it has a proper morphism to the Chow variety given by the Hilbert-Chow morphism:

$$\begin{aligned}\pi : \mathsf{M}_\beta(X) &\to \mathrm{Chow}_\beta(X)\\ [F] &\mapsto \mathrm{supp}(F)\end{aligned}$$

There is a perverse sheaf $\phi^\bullet$ on $\mathsf{M}_\beta(X)$ which is locally given by the perverse sheaf of vanishing cycles associated to the local super-potential (the moduli space is locally the critical locus of a holomorphic function on a smooth space, the so-called super-potential). The construction of $\phi^\bullet$ was done in [**7**], and requires the choice of "orientation data" : a squareroot of the virtual canonical line bundle on $\mathsf{M}_\beta(X)$. Maulik and Toda conjecture the existence of a canonical choice of orientation data (one that is compatible with the morphism π). Using that choice, the *Maulik-Toda* polynomial is defined as follows:

$$\mathsf{MT}_\beta(y) = \sum_{i\in\mathbb{Z}} \chi({}^pH^i(R^\bullet\pi_*\phi^\bullet))y^i$$

where ${}^pH^i(-)$ is the ith cohomology functor with respect to the perverse t-structure. By self-duality of $\phi^\bullet$ and Verdier duality, $\mathsf{MT}_\beta(y)$ is an integer coefficient Laurent polynomial in y which is invariant under $y \leftrightarrow y^{-1}$. Noting that $\{\psi_y^g\}_{g\geq 0}$ forms an integral basis for such polynomials, we may write the MT polynomial as follows:

Definition 42. The GV invariants (via MT theory) $n_g^{\mathsf{MT}}(\beta)$ are defined by the equation

$$\mathsf{MT}_\beta(y) = \sum_{g\geq 0} n_g^{\mathsf{MT}}(\beta)\ \psi_y^g.$$

The main conjecture of Maulik and Toda is

Conjecture 43. $n_g^{\mathsf{MT}}(\beta) = n_g^{\mathsf{PT}}(\beta)$.

Remark 44. Compared to the definition via PT theory, the above definition of GV invariants is more directly tied to the geometry of curves in the class β and more closely matches the original physics definition. In particular, the invariants $n_g^{\mathsf{MT}}(\beta)$ only involve the single moduli space $\mathsf{M}_\beta(X)$. In contrast, the invariants $n_g^{\mathsf{PT}}(\beta)$ involve a subtle combination of the PT invariants associated to an infinite number of moduli spaces, namely the spaces $\mathsf{PT}_{\beta',n}(X)$ where $\beta = k\beta'$ and n is unbounded from above.

3.1.3 The local $K3$ surface: the Katz-Klemm-Vafa (KKV) formula.

Suppose that S is a $K3$ surface and $\beta_d \in H_2(S)$ is a curve class with $\beta_d^2 = 2d$. The CY3 $X = S \times \mathbb{C}$ is sometimes called the local $K3$ surface. In this case, Conjecture 43 holds and $n_g^{\mathsf{MT}}(\beta_d) = n_g^{\mathsf{PT}}(\beta_d)$ only depends on d and g (and not the divisibility of β_d). The invariants $n_g^{\mathsf{PT}}(\beta_d)$ and $n_g^{\mathsf{MT}}(\beta_d)$ were computed (in full generality) by Pandharipande and Thomas [**39**] and Shen and Yin [**46**, Thm 0.5] respectively. The MT polynomials are given by the famous KKV formula first conjectured by Katz, Klemm, and Vafa [**27**]:

$$\sum_{d=-1}^{\infty} \mathsf{MT}_{\beta_d}(y) q^d = -q^{-1} \prod_{n=1}^{\infty} (1-q^n)^{-20}(1+yq^n)^{-2}(1+yq^n)^{-2} \qquad (3.2)$$

The right hand side can also be written as $\psi_y \cdot \phi_{10,1}(q,-y)^{-1}$ where $\phi_{10,1}(q,y)$ is the Fourier expansion of the unique Jacobi cusp form of weight 10 and index 1.

The fact that $n_g(\beta_d)$ is independent of the divisibility of β_d is a deep and unusual feature of the local $K3$ geometry.

3.2 GV invariants for CY3s with an involution

Let X be a CY3 equipped with an involution

$$\imath : X \to X$$

such that the induced action on K_X is trivial. The purpose of this article is to develop a theory of GV invariants which count $\imath$-invariant curves on X. Namely, we seek to define integers $n_{g,h}(\beta)$ which give a virtual count of genus g, $\imath$-invariant curves $C \subset X$ with $[C] = \beta \in H_2(X)$, and such that the genus of the quotient $C/\imath$ is h.

We develop this theory parallel to the presentation of the ordinary GV invariants given in Section 3.1. Namely we define invariants $n_{g,h}^{\mathsf{PT}}(\beta)$ and $n_{g,h}^{\mathsf{MT}}(\beta)$ in terms of a version of PT and MT theory respectively and we conjecture that they are equal.

We do not currently know how to include curves that are *fixed* by $\imath$ (as opposed to merely invariant) and so we make the following general assumption:

Assumption 45. Throughout, we assume that the curve classes $\beta \in H_2(X)$ that we consider do not admit an effective decomposition $\beta = \sum_i d_i C_i$ containing an $\imath$-fixed curve C_i.

3.2.1 $\imath$-GV invariants via PT theory.

We denote by R_+ and R_- the trivial and the nontrivial irreducible representation of the group of order 2 and we let $R_{\mathsf{reg}} = R_+ \oplus R_-$ denote the regular representation.

Recall that a sheaf F on X is *$\imath$-invariant* if $\imath^* F \cong F$ and that an *$\imath$-equivariant* sheaf is an $\imath$-invariant sheaf F along with a choice of a lift of $\imath$ to an isomorphism $\tilde{\imath} : \imath^* F \to F$.

If F is an $\imath$-equivariant sheaf then

$$\chi(F) = \sum_k (-1)^k H^k(X, F)$$

is naturally a virtual representation and so can be written in the form

$$\chi(F) = nR_{\mathsf{reg}} + \epsilon R_-$$

for some $n, \epsilon \in \mathbb{Z}$.

We define the space of $\imath$-equivariant PT pairs

$$\mathsf{PT}_{\beta,n,\epsilon}(X,\imath) = \left\{ (F,s) \,\middle|\, s \in H^0(X,F),\ [\mathrm{supp}(F)] = \beta,\ \chi(F) = nR_{\mathsf{reg}} + \epsilon R_- \right\}$$

where (F, s) is a PT pair such that F is an $\imath$-equivariant sheaf and s is an equivariant section. We note that

$$\mathsf{PT}_{\beta,d}(X)^{\imath} = \bigsqcup_{2n+\epsilon=d} \mathsf{PT}_{\beta,n,\epsilon}(X,\imath) \tag{3.3}$$

since the points of the $\imath$-fixed locus $\mathsf{PT}_{\beta,n}(X)^{\imath}$ corresponds to $\imath$-invariant PT pairs, but each $\imath$-invariant PT pair has a unique $\imath$-equivariant structure making the section equivariant.

Definition 46. We define the $\imath$-PT invariants and the $\imath$-PT partition function by

$$N^{\mathsf{PT}}_{\beta,n,\epsilon}(X,\imath) = e_{\mathsf{vir}}(\mathsf{PT}_{\beta,n,\epsilon}(X,\imath))$$
$$Z^{\mathsf{PT}}(X,\imath) = \sum_{\beta,n,\epsilon} N^{\mathsf{PT}}_{\beta,n,\epsilon}(X,\imath)\, Q^{\beta}\, y^n\, w^{\epsilon}$$

where in the sum, β is summed over the semi-group of classes satisfying Assumption 45.

These invariants are not new: they can be recovered from the orbifold PT invariants of the stack quotient $[X/\imath]$ studied for example in [**5**]. It follows from [**5**, Prop 7.19], that the coefficient of Q^{β} in $Z^{\mathsf{PT}}(X,\imath)$ is a Laurent expansion of a rational function in y and w which is invariant under $y \leftrightarrow y^{-1}$ and $w \leftrightarrow w^{-1}$. This allows us to make the following definition:

Definition 47. The $\imath$-GV invariants (via PT theory) $n^{\mathsf{PT}}_{g,h}(\beta)$ (for classes β satisfying Assumption 45) are defined by the formula

$$\log(Z^{\mathsf{PT}}(X,\imath)) = \sum_{k>0}\sum_{\beta,g,h} \frac{1}{k} Q^{k\beta} \cdot n^{\mathsf{PT}}_{g,h}(\beta) \cdot \psi^{h-1}_{-(-y)^k} \cdot \psi^{g+1-2h}_{w^k}$$

where as before $\psi_x = 2 + x + x^{-1}$.

Remark 48. The number $g+1-2h$ is half the number of fixed points on a smooth genus g curve with an involution having a quotient of genus h.

Remark 49. Although it is not apparent from this definition, we expect $n^{\mathsf{PT}}_{g,h}(\beta)$ to have good finiteness properties, namely that for fixed β we expect $n^{\mathsf{PT}}_{g,h}(\beta)$ to be non-zero for only a finite number of values of (g,h) and that those values should have $h \geq 0$ and $g \geq -1$. The possible occurence of non-zero counts for $g = -1$ is due to (for example) the possibility of invariant curves $C = C_1 \cup C_2$ consisting of a disjoint $\imath$-orbit of rational curves. Such a curve should be interpreted as having genus -1 via the formula $\chi(\mathcal{O}_C) = 1 - g$.

3.2.2 $\imath$-GV invariants via MT theory.

We define the moduli space of $\imath$-MT sheaves to be

$$\mathsf{M}^{\epsilon}_{\beta}(X,\imath) = \left\{ F \,\middle|\, [\mathrm{supp}(F)] = \beta,\ \chi(F) = R_{\mathsf{reg}} + \epsilon R_{-} \right\} \tag{3.4}$$

where F is an $\imath$-equivariant coherent sheaf on X with proper sheaf theoretic support of pure dimension one and where F is $\imath$-*stable*:

Definition 50. We say an $\imath$-equivariant sheaf F on X of pure dimension one with $\chi(F) = R_{\mathsf{reg}} + \epsilon R_{-}$ is $\imath$-*stable* if all $\imath$-equivariant subsheaves $F' \subsetneq F$, with $\chi(F') = kR_{\mathsf{reg}} + \gamma R_{-}$ satisfy $k \leq 1$ and if $k = 1$, then $\gamma < \epsilon$ and $[\mathrm{supp}(F')] = [\mathrm{supp}(F)] \in H_2(X)$.

Remark 51. We will show in Proposition 84 that $\imath$-stability can be reformulated in terms of Nironi stability for the corresponding sheaf on the stack quotient $[X/\imath]$. A consequence is that $\mathsf{M}^{\epsilon}_{\beta}(X,\imath)$ is a scheme and it is proper over $\mathrm{Chow}_{\beta}(X)$.

Let

$$\pi^{\epsilon} : \mathsf{M}^{\epsilon}_{\beta}(X,\imath) \to \mathrm{Chow}_{\beta}(X) \tag{3.5}$$

be the Hilbert-Chow morphism. Since $\mathsf{M}^{\epsilon}_{\beta}(X,\imath)$ parameterizes objects in the CY3 category of $\imath$-equivariant coherent sheaves on X, there exists a perverse sheaf of vanishing cycles[9] $\phi^{\bullet}$

[9] As in [**32**, Defn 2.7], we assume that our orientation is *strictly Calabi-Yau*.

on $\mathsf{M}^{\epsilon}_{\beta}(X,\imath)$ and we can define the $\imath$-MT polynomial in a fashion analogous to the ordinary MT polynomial:

$$\mathsf{MT}_{\beta}(y,w) = \sum_{i,\epsilon\in\mathbb{Z}} \chi({}^{p}H^{i}(R^{\bullet}\pi^{\epsilon}_{*}\phi^{\bullet}))\, y^{i}\, w^{\epsilon}. \tag{3.6}$$

As before, self-duality and Verdier duality imply that $\mathsf{MT}_{\beta}(y,w)$ is a Laurent polynomial in y invariant under $y \leftrightarrow y^{-1}$. We conjecture that in general $\mathsf{MT}_{\beta}(y,w)$ is also a Laurent invariant polynomial in w invariant under $w \leftrightarrow w^{-1}$. Assuming this conjecture, we can write $\mathsf{MT}_{\beta}(y,w)$ as a polynomial in ψ_y and ψ_w and make the following definition.

Definition 52. The $\imath$-GV invariants (via MT theory) $n^{\mathsf{MT}}_{g,h}(\beta)$ (for classes satisfying Assumption 45) are defined by the formula:

$$\mathsf{MT}_{\beta}(y,w) = \sum_{g,h} n^{\mathsf{MT}}_{g,h}(\beta)\ \psi_y^{h}\, \psi_w^{g+1-2h}.$$

Our main conjecture is that our two definitions of $\imath$-GV invariants are equivalent.

Conjecture 53. $n^{\mathsf{PT}}_{g,h}(\beta) = n^{\mathsf{MT}}_{g,h}(\beta)$.

3.2.3 Examples: local Abelian surfaces and local Nikulin $K3$ surfaces.

One of the main results of this paper are various $\imath$-equivariant versions of the KKV formula. Namely, we compute our invariants and prove our conjecture for the case $X = S\times\mathbb{C}$ where S is either an Abelian or $K3$ surface and where $\imath$ acts trivially on $\mathbb{C}$ and symplectically on S.

For the case of an Abelian surface, the involution is the natural one arising from the group structure: $\imath(a) = -a$. A $K3$ surface equipped with a symplectic involution is called a Nikulin surface and there are two distinct deformation types which we call Type (I) and Type (II) (see Definition 74).

As with the ordinary GV invariants of a local $K3$ surface, our invariants admit a surprising lack of dependency on the divisibility of the curve class and they are given by formulas involving Jacobi modular forms:

Theorem 54. *Let $X = S\times\mathbb{C}$ where S is an Abelian surface or a Nikulin $K3$ surface, and let $\beta\in H_2(S)$ be an effective invariant curve class with $\beta^2 = 2d$. Then:*

1. *if S is a Type (II) Nikulin surface, $n^{\mathsf{PT}}_{g,h}(\beta)$ only depends on (g,h,d). In particular, it doesn't depend on the divisibility of β.*

2. *if S is an Abelian surface or a Type (I) Nikulin surface, $n^{\mathsf{PT}}_{g,h}(\beta)$ only depends on (g,h,d) as well as the parity of the divisibility of β.*

We denote these invariants by $n_{g,h}(d;\mathsf{type})$ where $\mathsf{type} \in \{\mathsf{A}^{\mathsf{ev}}, \mathsf{A}^{\mathsf{odd}}, \mathsf{N}_{\mathsf{I}}^{\mathsf{ev}}, \mathsf{N}_{\mathsf{I}}^{\mathsf{odd}}, \mathsf{N}_{\mathsf{II}}\}$ distinguishes the cases in the obvious way. Then the invariants are determined from the formula:

$$\sum_{g,h} n_{g,h}(d;\mathsf{type})\psi_y^{h-1}\psi_w^{g+1-2h} = \left[\frac{\Theta_T(q^2,w)}{\phi_{10,1}(q^2,-y)}\right]_{q^d}$$

where $[\cdots]_{q^d}$ denotes the coefficient of q^d in the expression $[\cdots]$.

Moreover, T is a lattice or shifted lattice depending on the type, $\Theta_T(q^2,w)$ is an explicitly determined Jacobi theta function (see Theorem 67), and $\phi_{10,1}(q,y)$ is the unique Jacobi cusp form of weight 10 and index 1. In particular, for types $\mathsf{A}^{\mathsf{odd}}$ and $\mathsf{N}_{\mathsf{I}}^{\mathsf{odd}}$ we get infinite product formulas:

$$\sum_{g,h,d} n_{g,h}(d;\mathsf{A}^{\mathsf{odd}})\,\psi_y^h\,\psi_w^{g-1-2h}\,q^d = -4\prod_{n=1}^{\infty}\frac{(1+q^n)^8(1+wq^n)^4(1+w^{-1}q^n)^4}{(1-q^{2n})^4(1+yq^{2n})^2(1+y^{-1}q^{2n})^2}\,,$$
$$\sum_{g,h,d} n_{g,h}(d;\mathsf{N}_{\mathsf{I}}^{\mathsf{odd}})\,\psi_y^h\,\psi_w^{g-2h}\,q^{d+1} = -\prod_{n=1}^{\infty}\frac{(1+q^n)^4(1+wq^n)^2(1+w^{-1}q^n)^2}{(1-q^{2n})^{12}(1+yq^{2n})^2(1+y^{-1}q^{2n})^2}\,.$$

Remark 55. The specialization of the invariants $n_{g,h}(\beta)$ to $h=0$ count $\imath$-invariant hyperelliptic curves. The problem of counting the number of genus g hyperelliptic curves in a primitive class β_d on an Abelian surface A was first considered by Rose [**45**] and then solved by Bryan-Oberdieck-Pandharipande-Yin [**13**]. We may specialize our invariants $n^{\mathsf{PT}}_{g,h}\left(d;\mathsf{A}^{\mathsf{odd}}\right)$ to $h=0$ by setting $y=-1$. The above formula then becomes

$$\sum_{d=0}^{\infty}\sum_{g>0} n_{g,0}(d;\mathsf{A}^{\mathsf{odd}})\psi_w^{g-1}\,q^d = -4\prod_{n=1}^{\infty}\frac{(1+wq^n)^4(1+w^{-1}q^n)^4}{(1-q^n)^8}$$

We note that the invariant $n_{g,0}(d;\mathsf{A}^{\mathsf{odd}})$ is equal to $\mathsf{h}^{\mathsf{Hilb}}_{\mathsf{g},\mathsf{A},\beta_{\mathsf{d}}}$ in the notation of [**13**] and the above formula is equivalent to the equation in Proposition 4 of [**13**].

Remark 56. For the case where β_d is the primitive class on an Abelian surface, our invariants $n_{g,h}(d;\mathsf{A}^{\mathsf{odd}})$ are refinements of the invariants $\mathsf{n}_d(h)$ considered in [**43**]. The relationship is given by

$$\mathsf{n}_h(d) = -\sum_g 4^{g-2h}\cdot n_{g,h}(d,\mathsf{A}^{\mathsf{odd}}).$$

Our main technique to prove Theorem 54 / Theorem 67 is to use the Donaldson-Thomas Crepant Resolution Conjecture (DT-CRC) [**5**,**10**] to compute orbifold PT invariants in terms of the crepant resolution which in this case is a local $K3$ surface. We can then apply the KKV formula making crucial use of the description of the Picard lattice of Kummer $K3$ surfaces and Nikulin resolutions given by Garbagnati-Sarti [**20**, **21**]. This is carried out in Section 3.4.

In Section 3.5, we use the derived McKay correspondence to compute the MT versions of our invariants $n^{\mathsf{MT}}_{g,h}(\beta)$ for $X = S \times \mathbb{C}$. The final result is

Theorem 57. *The two definitions of $\imath$-GV invariants coincide for $X = S \times \mathbb{C}$ for S an Abelian surface or a Nikulin $K3$ surface:*

$$n^{\mathsf{PT}}_{g,h}(\beta) = n^{\mathsf{MT}}_{g,h}(\beta)\,.$$

3.2.4 Further examples

In Section 3.6, we give two other examples illustrating our theory and providing evidence of our conjecture. For the case where $X \to B$ is an elliptically fibered CY3 over a surface B with integral fibers and $\imath : X \to X$ is the composition of an involution on B and fiberwise multiplication by -1, we compute both $n^{\mathsf{PT}}_{g,h}(\beta)$ and $n^{\mathsf{MT}}_{g,h}(\beta)$ completely for $\beta = [F]$, the class of the fiber. Let $C \subset B$ be the fixed locus of the involution on the base and let $S = X\big|_C$ be the restriction of $X \to B$ to C. The result is the following:

Theorem 58. *Let $(X, \imath)$ be an elliptically fibered CY3 with notation as above. Then for all g and h, $n^{\mathsf{PT}}_{g,h}([F]) = n^{\mathsf{MT}}_{g,h}([F])$ and they are given by*

$$n_{g,h}([F]) = \begin{cases} -e(C) & \text{if } (g,h) = (1,0), \\ e(S) & \text{if } (g,h) = (0,0), \\ 0 & \text{otherwise.} \end{cases}$$

We also consider the case where $X = \mathrm{Tot}(\mathcal{O}(-1)\oplus\mathcal{O}(-1))$ is the conifold, $\imath : X \to X$ is the Calabi-Yau involution which acts non-trivially on the base, and $C \subset X$ is the zero section. We use the orbifold topological vertex to compute $n^{\mathsf{PT}}_{g,h}(d[C])$ and we use stability considerations to compute $n^{\mathsf{MT}}_{g,h}(d[C])$. The result is

Proposition 59. *For X the conifold with $\imath$ as above,*

$$n^{\mathsf{PT}}_{g,h}\left(d[C]\right) = n^{\mathsf{MT}}_{g,h}\left(d[C]\right) = \begin{cases} 1 & \textit{if } (g,h,d) = (0,0,1), \\ 0 & \textit{otherwise.} \end{cases}$$

We remark that despite the simplicity of the answer, the above proposition is the result of rather involved orbifold topological vertex computation which was done in [**10**] and it gives a non-trivial instance of our conjecture.

3.3 Motivating example of an isolated smooth invariant curve.

The simplest evidence of our conjecture is the case of a rigid local curve in the primitive class of the curve. Let C be a non-singular curve of genus g. Suppose there exists an involution $\imath : C \to C$ with fixed points

$$C^{\imath} = \{p_1, \ldots, p_{2m}\}.$$

If h is the genus of the quotient $C/\imath$, then by Riemann-Hurwitz we have

$$m = g + 1 - 2h.$$

Let N be an $\imath$-equivariant line bundle on C such that $H^0(N) = H^1(N) = 0$. Then

$$X = \mathrm{Tot}(N \oplus K_C N^{-1})$$

is a CY3 with an induced involution (also denoted $\imath$) acting trivially on K_X. Let $[C]$ be the class of the zero section $C \hookrightarrow X$ which we note is rigid.

The moduli spaces for the class $[C]$ can be determined explicitly:

Proposition 60. *The $\imath$-PT and $\imath$-MT moduli spaces in the class $[C]$ are given by*

$$\mathsf{PT}_{[C],n,\epsilon}(X,\imath) = \coprod_{\substack{T \subseteq \{1,\ldots,2m\} \\ |T| = \epsilon + m}} \mathrm{Sym}^{n+h-1}\left(C/\imath\right),$$

$$\mathsf{M}^{\epsilon}_{[C]}(X,\imath) = \coprod_{\substack{T \subseteq \{1,\ldots,2m\} \\ |T| = \epsilon + m}} \mathrm{Pic}^{h}\left(C/\imath\right).$$

As a consequence, we find there is a single non-zero $\imath$-GV invariant in the class $[C]$:

Corollary 61.

$$n^{\mathsf{PT}}_{g',h'}([C]) = n^{\mathsf{MT}}_{g',h'}([C]) = \begin{cases} 1 \textit{ if } (g',h') = (g,h), \\ 0 \textit{ if } (g',h') \neq (g,h). \end{cases}$$

Remark 62. We expect the invariants $n_{g',h'}(d[C])$ to be complicated for $d > 1$ and $g > 0$. For the related case of $X = \mathrm{Tot}(K_C \oplus \mathcal{O})$, Conjecture 43 is equivalent to the well known $P = W$ conjecture for the GL_d Hitchin system on C [**32**, § 9.3]. We expect that for $\imath : X \to X$, the natural lift of an involution on C, our Conjecture 53 should be equivalent to an orbifold version of $P = W$ for the orbifold curve $[C/\imath]$.

To prove Proposition 60 and Corollary 61, we need the following lemma.

Lemma 63. *Let $L = \mathcal{O}(D)$ be an $\imath$-equivariant line bundle on C admitting an $\imath$-invariant section $\mathcal{O}_C \to L$ vanishing on an $\imath$-invariant effective divisor D. Then D can be written*

$$D = \sum_j d_j(x_j + \imath(x_j)) + \sum_{i \in T} p_i$$

where $T \subseteq \{1 \cdots 2m\}$, $x_j \in C$, and

$$\chi(L) = nR_{\mathsf{reg}} + \epsilon R_-$$

with

$$n = 1 - h + \sum_j d_j, \quad \epsilon = |T| - m.$$

Moreover, the above formula still holds for $L = \mathcal{O}(D)$ with D $\imath$-invariant, but not necessarily effective.

Proof. Since the support of D is $\imath$-invariant, it must consist of free orbits and fixed points. Absorbing multiplicities which are not zero or one on the p_i's into the first term using $2p_i = p_i + \imath(p_i)$ we see D may be written in the given form. Assuming D is effective, we then get sequence

$$0 \to \mathcal{O}_C \to L \to \mathcal{O}_D(D) \to 0$$

and consequently we find

$$\chi(L) = \chi(\mathcal{O}_C) + (\sum_j d_j)R_{\mathsf{reg}} + \sum_{i \in T} \chi(\mathcal{O}_{p_i}(p_i)).$$

Using a local coordinate about p_i, it is easy to determine that $\chi(\mathcal{O}_{p_i}(p_i)) = R_-$. We also observe that

$$\chi(\mathcal{O}_C) = (1-h)R_+ + (h-g)R_- = (1-h)R_{\mathsf{reg}} - mR_-$$

and the formula for $\chi(L)$ follows. The general case is then obtained by writing $D = D' - D''$ with D' and D'' effective and using the sequence

$$0 \to \mathcal{O}(D' - D'') \to \mathcal{O}(D') \to \mathcal{O}_{D''}(D') \to 0.$$

$\square$

Proof of Proposition 60. An $\imath$-PT pair on X in the class $[C]$ must be supported on C and hence be an $\imath$-equivariant line bundle with a non-zero invariant section $\mathcal{O}_C \to L$. Such $\imath$-PT pairs are determined up to isomorphism by the associated invariant divisor D which by Lemma 63 is determined by the subset $T \subseteq \{1, \dots, 2m\}$ and $\sum_j d_j = n+h-1$ $\imath$-orbits on C or equivalently, $n+h-1$ points on $C/\imath$. The first equation in the proposition follows.

For the same reasons, an $\imath$-MT sheaf on X in the class $[C]$ must be an $\imath$-equivariant line bundle $L \to C$, or equivalently, a line bundle on the stack quotient $[C/\imath]$. Since $\chi(L) = R_{\mathsf{reg}} + \epsilon R_-$, L admits a non-zero $\imath$-invariant section and hence is of the form $L \cong \mathcal{O}_C(D)$ where D is an invariant divisor given as in Lemma 63 with $\sum_j d_j = h$.

The Picard group of an orbifold is given in general in [**44**, § B]. In particular for $[C/\imath]$ we have

$$\begin{aligned}\operatorname{Pic}([C/\imath]) &\cong \operatorname{Pic}^0(C/\imath) \oplus H^2_{\mathsf{orb}}([C/\imath]) \\ &\cong \operatorname{Pic}^0(C/\imath) \oplus \mathbb{Z} \oplus (\mathbb{Z}/2)^{2m} \\ &\cong \operatorname{Pic}(C/\imath) \oplus (\mathbb{Z}/2)^{2m}.\end{aligned}$$

Under the above isomorphism, the line bundle $\mathcal{O}_C(D)$ with D as in the lemma goes to $(\mathcal{O}_{C/\imath}(\sum_j d_j \overline{x}_j), 1_T)$ where $\overline{x}_j \in C/\imath$ is the point corresponding to the orbit $x_j + \imath(x_j)$ and $1_T = (t_1, \dots, t_{2m})$ where $t_i = 1$ if $i \in T$ and 0 otherwise. The second equation of Proposition 60 follows. $\square$

Proof of Corollary 61. Since $[C]$ is a primitive class, we have

$$\begin{aligned}\left[Z^{\mathsf{PT}}(X,\imath)\right]_{q^{[C]}} &= \left[\log Z^{\mathsf{PT}}(X,\imath)\right]_{q^{[C]}} \\ &= \sum_{g',h'} n^{\mathsf{PT}}_{g',h'}([C])\, \psi_y^{h'-1} \psi_w^{g'+1-2h'}.\end{aligned}$$

On the other hand, by Proposition 60 and using the fact that the Behrend function is $(-1)^d$ on a smooth scheme of dimension d we get

$$\begin{aligned}
\left[Z^{\mathsf{PT}}(X,\imath)\right]_{q^{[C]}} &= \sum_{T\subseteq\{1,\ldots,2m\}}\sum_{n}(-1)^{n+h-1}e\left(\mathrm{Sym}^{n+h-1}(C/\imath)\right)\, y^n w^{|T|-m}\\
&= \left(\sum_{k=0}^{2m}\binom{2m}{k}w^{k-m}\right) y^{1-h}\sum_{d=0}^{\infty} e\left(\mathrm{Sym}^{d}(C/\imath)\right)(-y)^d\\
&= \psi_w^m\, y^{1-h}(1+y)^{2h-2}\\
&= \psi_y^{h-1}\,\psi_w^{g+1-2h}
\end{aligned}$$

where we used MacDonald's formula [**31**] for the penultimate equality. The formula for $n^{\mathsf{PT}}_{g',h'}([C])$ then follows.

To compute $n^{\mathsf{MT}}_{g',h'}([C])$ we observe that $\mathrm{Chow}_{[C]}(X)$ is a point and that $\mathsf{M}^{\epsilon}_{[C]}(X,\imath)$ is smooth. Consequently, the Maulik-Toda polynomial is given by the (symmetrized) Poincare polynomial:

$$\mathsf{MT}_{[C]}(y,w) = \sum_{T\subseteq\{1,\ldots 2m\}} w^{|T|-m}\,\widetilde{P}_y\left(\mathrm{Pic}^h(C/\imath)\right).$$

Since $\mathrm{Pic}^h(C/\imath)$ is an Abelian variety of dimension h, its symmetrized Poincare polynomial is given by $y^{-h}(1+y)^{2h} = \psi_y^h$. Thus $\mathsf{MT}_{[C]}(y,w) = \psi_y^h\,\psi_w^{g+1-2h}$ and the formula for $n^{\mathsf{MT}}_{g',h'}([C])$ is proved. □

3.4 Local Abelian and Nikulin Surfaces (PT theory)

3.4.1 Overview

In this section we compute $Z^{\mathsf{PT}}(X,\imath)$, and thus determine all the invariants $n^{\mathsf{PT}}_{g,h}(\beta)$ for the case of $X = S\times\mathbb{C}$, where S is an Abelian surface with its symplectic involution $\imath(a) = -a$ or a Nikulin $K3$ surface which by definition comes with a symplectic involution (in both cases, $\imath$ acts trivially on the second factor). The main theorem is given by Theorem 67.

Our basic tool for computing the $\imath$-PT invariants of $S\times\mathbb{C}$ is the Donaldson-Thomas Crepant Resolution Conjecture (DT-CRC) which was conjectured in [**10**] and recently proven by Beentjes, Calabrese, and Rennemo in [**5**]. The idea is the following. Our $\imath$-PT partition function $Z^{\mathsf{PT}}(X,\imath)$ can be written in terms of the orbifold PT partition function $Z^{\mathsf{PT}}([X/\imath])$ and then the DT-CRC asserts that

$$Z^{\mathsf{PT}}([X/\imath]) = \frac{Z^{\mathsf{PT}}(Y)}{Z^{\mathsf{PT}}_{\mathsf{exc}}(Y)} \tag{3.7}$$

where $Y \to X/\imath$ is the crepant resolution, $Z^{\mathsf{PT}}(Y)$ is the ordinary PT partition function of Y, and $Z^{\mathsf{PT}}_{\mathsf{exc}}(Y)$ is the partition function for curve classes supported on the exceptional fibers. The variables in the above equality are identified via the Fourier-Mukai isomorphism in numerical K-theory. In the case of $X = S \times \mathbb{C}$,

$$Y = \widehat{S} \times \mathbb{C}$$

where $\widehat{S} \to S/\imath$ is the minimal resolution. In the case where S is an Abelian surface, $\widehat{S}$ is the associated Kummer $K3$ surface, and in the case where S is a Nikulin $K3$ surface, $\widehat{S}$ is a special kind of $K3$ surface which we call a Nikulin resolution.

We then can compute the right hand side of Equation (3.7) using the KKV formula (see Section 3.1.3). Doing this requires an explicit description of the Picard lattice of $\widehat{S}$, which was given by Garbagnati-Sarti [**20**, **21**]. Finally, to complete the computation, we will need some theta function identities which we prove in Section 3.4.4.

3.4.2 Using the DT-CRC

To use Equation (3.7), we will need to be explicit with our choice of variables. Let $X = S \times \mathbb{C}$ and let β_d be an effective, $\imath$-invariant, primitive curve class on S with $\beta_d^2 = 2d$ which we identify with the corresponding class on X. To determine our invariants $n^{\mathsf{PT}}_{g,h}(m\beta_d)$, we need to compute the partition function

$$Z^{\mathsf{PT}}(X, \imath) = \sum_{m \geq 0} \sum_{n,\epsilon} N^{\mathsf{PT}}_{m\beta_d, n, \epsilon}(X, \imath) Q^m y^n w^\epsilon.$$

Remark 64. Strictly speaking, when the rank of the invariant Picard group $\mathrm{Pic}(S)^\imath$ is greater than 1, the above is a *restricted* partition function: we don't sum over all invariant curve classes, but only over the semi-group generated by β_d. This suffices for determining the invariants $n^{\mathsf{PT}}_{g,h}(m\beta_d)$. A few of the statements made in this section require minor adjustments in the case where the invariant Picard rank is greater than 1. Note that for $d \leq 0$, the invariant Picard rank is necessarily greater than 1.

Our partition function $Z^{\mathsf{PT}}(X, \imath)$ can be determined from the PT partition function of the orbifold $[X/\imath]$. By definition (see [**5**]),

$$Z^{\mathsf{PT}}([X/\imath]) = \sum_{\alpha \in N_{\leq 1}([X/\imath])} N^{\mathsf{PT}}_\alpha([X/\imath]) \mathbf{Q}^\alpha$$

where the sum ranges over the numerical K-theory of $\mathrm{Coh}_{\leq 1}([X/\imath])$, the category of coherent sheaves on $[X/\imath]$ having proper support of dimension less than or equal to one.

We need to choose generators for the free $\mathbb{Z}$-modules

$$N_{\leq 1}([X/\imath]) \cong N_{\leq 1}([S/\imath])$$

and

$$N_{\leq 1}(Y) \cong N_{\leq 1}(\widehat{S}) \cong H_0(\widehat{S}) \oplus \mathrm{Pic}(\widehat{S})$$

in a way that is compatible with the Fourier-Mukai isomorphism

$$N_{\leq 1}([S/\imath]) \cong N_{\leq 1}(\widehat{S}).$$

It will be convenient to choose generators over $\mathbb{Q}$. Determining which linear combinations of our generators are integral classes is somewhat subtle and is addressed in Section 3.4.3. The generators, and their corresponding variables in the partition functions, are given in the following table:

Class in $N_{\leq 1}([S/\imath])$	Class in $H_0(\widehat{S}) \oplus \mathrm{Pic}(\widehat{S})$	Variable
$[\mathcal{O}_{\mathsf{pt}}]$	$[\mathsf{pt}]$	y
$[\mathcal{O}_{x_i} \otimes R_-]$	$[E_i]$	w_i
α_d	γ_d	Q

Table 3.1: Dictionary between K-theory classes on the stack $[S/\imath]$ and the minimal resolution $\widehat{S}$, along with the corresponding variable choice.

Here $\mathcal{O}_{\mathsf{pt}}$ is the structure sheaf of a generic point on $[S/\imath]$, $\mathcal{O}_{x_i} \otimes R_-$ is the structure sheaf of the i-th orbifold point x_i equipped with the non-trivial action of its stabilizer group, and

$$\alpha_d = \frac{1}{2} t_*(\mathrm{ch}^{-1}(\beta_d))$$

where $t : S \to [S/\imath]$ and $\mathrm{ch}^{-1}(\beta_d)$ is the class in $N_{\leq 1}(S)$ corresponding to $\beta_d \in \mathrm{Pic}(S)$ under the Chern character isomorphism. The generators of $H_0(\widehat{S})$ and $\mathrm{Pic}(\widehat{S})$ are given by the point class $[\mathsf{pt}]$, the classes of the exceptional divisors E_i, and

$$\gamma_d = c_1(FM(\alpha_d)),$$

the divisor class associated to the image of α_d under the Fourier-Mukai isomorphism.

The fact that the above choices are compatible with the Fourier-Mukai isomorphism uses the well-known fact that the isomorphism takes $\mathcal{O}_{x_i} \otimes R_-$ to $\mathcal{O}_{E_i}(-1)$.

With these variables, Equation (3.7) can be viewed as an equality of formal series[10] in the variables y, w_i, Q.

Lemma 65. $Z^{\mathsf{PT}}(X, \imath) = Z^{\mathsf{PT}}([X/\imath])\big|_{w_i=w}$

Proof. Our $\imath$-PT pairs on X are equivalent to PT pairs on the stack quotient $[X/\imath]$. However, keeping track of the K-theory class of the sheaf on $[X/\imath]$ is a refinement of the discrete data used for $\imath$-PT pairs. The lemma follows from observing that a sheaf on $[X/\imath]$ in the K-theory class

$$m\alpha_d + n[\mathcal{O}_{\mathsf{pt}}] + \sum_i v_i[\mathcal{O}_{x_i} \otimes R_-]$$

corresponds to an $\imath$-equivariant sheaf F with $[\mathrm{supp}(F)] = m\beta_d$ and

$$\chi(F) = nR_{\mathsf{reg}} + (\sum_i v_i)R_-.$$

$\square$

Next we define the *exceptional lattice*

$$\Lambda = \oplus_i \mathbb{Z}\langle E_i \rangle \subset \mathrm{Pic}(\widehat{S})$$

and we define

$$\Gamma_{m,d} = \left\{ v \in \Lambda \otimes \mathbb{Q} : \ m\gamma_d + v \text{ is a non-zero integral class in } \mathrm{Pic}(\widehat{S}) \right\}.$$

For $v = \sum_i v_i E_i$ we will use the following notation

$$l(v) = \sum_i v_i, \quad v^2 = -2\sum_i v_i^2. \tag{3.8}$$

We can write the log of the partition function on Y in terms of the Gopakumar-Vafa invariants of Y using Equation (3.1). We then specialize w_i to w to get:

$$\log Z^{\mathsf{PT}}(Y)\big|_{w_i=w} = \sum_{k>0}\sum_{m\geq 0}\sum_{v\in\Gamma_{m,d}}\sum_{h\geq 0} \frac{1}{k} n_h^{\mathsf{PT}}\left(m\gamma_d + v\right) Q^{km} w^{kl(v)} \psi_{-(-y)^k}^{h-1}.$$

[10] In general, the statement of the DT-CRC requires viewing the partition functions as rational functions in certain variables and the equality as an equality of rational functions. This issue does not arise in this case.

On the other hand, the invariants $n_{g,h}^{\mathsf{PT}}(m\beta_d)$ on X are by definition given by

$$\log Z^{\mathsf{PT}}(X,\imath) = \sum_{k,m>0}\sum_{g,h}\frac{1}{k}Q^{km}n_{g,h}^{\mathsf{PT}}(m\beta_d)\,\psi_{-(-y)^k}^{h-1}\psi_{w^k}^{g+1-2h}.$$

Taking the log of Equation (3.7), observing that $Z_{\mathsf{exc}}^{\mathsf{PT}}(Y) = Z^{\mathsf{PT}}(Y)\big|_{Q=0}$, and applying Lemma 65, we get

$$\log Z^{\mathsf{PT}}(X,\imath) = \log Z^{\mathsf{PT}}(Y)\big|_{w_i=w} - \log Z^{\mathsf{PT}}(Y)\big|_{w_i=w,Q=0}.$$

Combining this with the previous two equations, we arrive at

$$\sum_{k,m>0}\frac{Q^{km}}{k}\left(\sum_{g,h}n_{g,h}^{\mathsf{PT}}(m\beta_d)\,\psi_{-(-y)^k}^{h-1}\psi_{w^k}^{g+1-2h}\right)$$
$$= \sum_{k,m>0}\frac{Q^{km}}{k}\left(\sum_{h\geq 0}\sum_{v\in\Gamma_{m,d}}n_h^{\mathsf{PT}}(m\gamma_d+v)\,w^{kl(v)}\psi_{-(-y)^k}^{h-1}\right).$$

By Möbius inversion (or a simple induction argument), the quantities in the parenthesis in the above equation must be equal for all k and m. In particular, by setting $k=1$ we've proved

$$\sum_{g,h}n_{g,h}^{\mathsf{PT}}(m\beta_d)\,\psi_y^{h-1}\psi_w^{g+1-2h} = \sum_{h\geq 0}\sum_{v\in\Gamma_{m,d}}n_h^{\mathsf{PT}}(m\gamma_d+v)\,w^{l(v)}\psi_y^{h-1}. \qquad (3.9)$$

The invariants $n_h^{\mathsf{PT}}(m\gamma_d+v)$ are determined by the KKV formula since $Y=\widehat{S}\times\mathbb{C}$ is a local $K3$ surface. One formulation of the KKV formula from Section 3.1.3 is the following. For any effective curve class C, $n_h(C)$ is given by

$$\sum_{h\geq 0}n_h(C)\psi_y^{h-1} = \left[\frac{1}{\phi_{10,1}(q,-y)}\right]_{q^{\frac{C^2}{2}}}$$

where

$$\phi_{10,1}(q,-y) = -\psi_y\cdot q\prod_{n=1}^{\infty}(1+yq^n)^2(1+y^{-1}q^n)^2(1-q^n)^{20}$$

and where $[\cdots]_{q^a}$ denotes the coefficient of q^a in the expression $[\cdots]$. Applying this to Equation (3.9) and using the facts that $\gamma_d^2=d$ and $\gamma_d\in\Lambda^\perp$ we get

$$(m\gamma_d+v)^2 = m^2d+v^2$$

and so

$$\sum_{g,h} n^{\mathsf{PT}}_{g,h}\left(m\beta_d\right)\psi_y^{h-1}\psi_w^{g+1-2h} = \sum_{v\in\Gamma_{m,d}}\left[\frac{w^{l(v)}}{\phi_{10,1}(q,-y)}\right]_{q^{\frac{1}{2}(m^2d+v^2)}}$$
$$= \left[\frac{\Theta_{\Gamma_{m,d}}(q,w)}{\phi_{10,1}(q,-y)}\right]_{q^{\frac{1}{2}m^2d}}$$

where for any subset $T\subset\Lambda\otimes\mathbb{Q}$ we've defined

$$\Theta_T(q,w) = \sum_{v\in T} q^{-\frac{v^2}{2}}w^{l(v)}.$$

In Section 3.4.3 we compute $\Gamma_{m,d}$. The results are:

Proposition 66. *The subset $\Gamma_{m,d}\subset\Lambda\otimes\mathbb{Q}$ is given as follows:*

- *If S is an Abelian surface, or a Type (I) Nikulin surface, then*

$$\Gamma_{m,d} = \begin{cases} L & \text{if } m \text{ is even} \\ L+r_0 & \text{if } m \text{ is odd and } d \text{ is even} \\ L+r_1 & \text{if } m \text{ is odd and } d \text{ is odd}\end{cases}$$

  where in the Abelian surface case $L=K$, the so-called Kummer lattice, an even, negative definite rank 16 lattice, and in the Type (I) Nikulin case, $L=N$ is the so-called Nikulin lattice, an even, negative definite rank 8 lattice. The vectors r_0 and r_1 are particular vectors we will define in Section 3.4.3. See Section 3.4.3 for the definition of K and N.

- *If S is a Type (II) Nikulin surface, then d is even and*

$$\Gamma_{m,d} = N$$

  where N is the Nikulin lattice.

The shifted lattices $L+r_i$ have the property that all their vectors have squares which are congruent to i modulo 2. It follows that we may write

$$\Theta_{L+r_0}(q,w)+\Theta_{L+r_1}(q,w) = \Theta_{L_{\mathsf{sh}}}(q,w)$$

where

$$L_{\mathsf{sh}} = (L+r_0)\cup(L+r_1).$$

In summary, we've shown that

$$\sum_{g,h} n^{\mathsf{PT}}_{g,h}\,(m\beta_d)\,\psi_y^{h-1}\psi_w^{g+1-2h} = \left[\frac{\Theta_T(q,w)}{\phi_{10,1}(q,-y)}\right]_{q^{\frac{1}{2}m^2 d}} \tag{3.10}$$

where

$$T = \begin{cases} N & \text{if } S \text{ is Type (II) Nikulin,} \quad (\mathsf{N_{II}}) \\ N & \text{if } S \text{ is Type (I) Nikulin and } m \text{ is even,} \quad (\mathsf{N_I^{ev}}) \\ N_{\mathsf{sh}} & \text{if } S \text{ is Type (I) Nikulin and } m \text{ is odd,} \quad (\mathsf{N_I^{odd}}) \\ K & \text{if } S \text{ is Abelian and } m \text{ is even,} \quad (\mathsf{A^{ev}}) \\ K_{\mathsf{sh}} & \text{if } S \text{ is Abelian and } m \text{ is odd.} \quad (\mathsf{A^{odd}}) \end{cases} \tag{3.11}$$

We now make the crucial observation that the right hand side of the Equation (3.10) only depends on the curve class $m\beta_d$ through its square $(m\beta_d)^2 = 2m^2 d$ and (possibly) its divisibility modulo 2 (i.e. $m \bmod 2$). This leads to the main theorem of this section:

Theorem 67. *Let β be* any *effective $\imath$-invariant curve class (not necessarily primitive) on an Abelian or Nikulin surface S with $\beta^2 = 2d$. Then the invariants $n^{\mathsf{PT}}_{g,h}\,(\beta)$ of $S \times \mathbb{C}$ only depend on (g,h,d) in the case where S is a Type (II) Nikulin surface and on (g,h,d) and whether the divisibility of β is odd or even in the other cases. Denoting these invariants as $n^{\mathsf{PT}}_{g,h}\,(d;\mathsf{type})$ where $\mathsf{type} \in \{\mathsf{A^{odd}}, \mathsf{A^{ev}}, \mathsf{N_I^{odd}}, \mathsf{N_I^{ev}}, \mathsf{N_{II}}\}$. Then*

$$\sum_{g,h} n^{\mathsf{PT}}_{g,h}\,(d;\mathsf{type})\,\psi_y^{h-1}\psi_w^{g+1-2h} = \left[\frac{\Theta_T(q^2,w)}{\phi_{10,1}(q^2,-y)}\right]_{q^d}.$$

Moreover $\Theta_T(q^2,w)$ is given explicitly by

$$\Theta_T(q^2,w) = \begin{cases} \theta_0^{16} + 30\theta_0^8\theta_1^8 + \theta_1^{16} & \textit{if} \quad T = K, \textit{ (type } \mathsf{A^{ev}}\textit{),} \\[2ex] \theta_0^8 + \theta_1^8 & \textit{if} \quad T = N, \textit{ (types } \mathsf{N_I^{ev}} \textit{ and } \mathsf{N_{II}}\textit{),} \\[2ex] 4\cdot\frac{\Delta(q^2)}{\Delta(q)^2}\cdot\phi_{10,1}^2(q,-w) & \textit{if} \quad T = K_{\mathsf{sh}}, \textit{ (type } \mathsf{A^{odd}}\textit{),} \\[2ex] -\frac{\Delta(q^2)^{\frac{1}{2}}}{\Delta(q)}\cdot\phi_{10,1}(q,-w) & \textit{if} \quad T = N_{\mathsf{sh}}, \textit{ (type } \mathsf{N_I^{odd}}\textit{),} \end{cases} \tag{3.12}$$

where

$$\Delta(q) = q\prod_{n=1}^{\infty}(1-q^n)^{24}$$

is the unique modular cusp form of weight 12,

$$\phi_{10,1}(q,y) = -\psi_{-y} \cdot q \prod_{n=1}^{\infty}(1-yq^n)^2(1-y^{-1}q^n)^2(1-q^n)^{20}$$

is the unique Jacobi cusp form of weight 10 and index 1, and

$$\theta_i = \theta_i(q^2, w) = \sum_{k \in \mathbb{Z}+\frac{i}{2}} q^{2k^2} w^k$$

are the standard rank 1 theta functions.

We note that for $n^{\mathsf{PT}}_{g,h}(d; \mathsf{A}^{\mathsf{ev}})$ *and* $n^{\mathsf{PT}}_{g,h}(d; \mathsf{N}^{\mathsf{ev}}_{\mathsf{I}})$*,* d *is necessarily divisible by 4, and for* $n^{\mathsf{PT}}_{g,h}(d; \mathsf{N}_{\mathsf{II}})$*,* d *is necessarily even.*

Remark 68. It is straightforward to see that the above formulas for the case of $\mathsf{A}^{\mathsf{odd}}$ and $\mathsf{N}^{\mathsf{odd}}_{\mathsf{I}}$ lead to the product formulation given in Theorem 54 in the Introduction.

Remark 69. The theta functions $\Theta_K(q^2, w)$ and $\Theta_{K_{\mathsf{sh}}}(q^2, -w)$ are Jacobi forms of weight 8 and index 2 (for some congruence subgroup), while the theta functions $\Theta_N(q^2, w)$ and $\Theta_{N_{\mathsf{sh}}}(q^2, -w)$ are Jacobi forms of weight 4 and index 1 (for some congruence subgroup). It would be nice to have a direct, lattice theoretic explination of the identity $\Theta_{K_{\mathsf{sh}}} = 4\Theta^2_{N_{\mathsf{sh}}}$.

To complete the proof of Theorem 67, we must prove Proposition 66 and we must prove the formulas for $\Theta_T(q^2, w)$ given by Equation (3.12). This is carried out in the next two subsections.

3.4.3 The Picard lattice of $\widehat{S}$.

Recall that S is an Abelian or Nikulin surface and $\widehat{S} \to S/\imath$ is the associated Kummer $K3$ or Nikulin resolution respectively. In this section we describe $\mathrm{Pic}(\widehat{S})$ and in particular prove Proposition 66. Recall also that we defined the exceptional lattice:

$$\Lambda = \oplus_i \mathbb{Z} \langle E_i \rangle \subset \mathrm{Pic}(\widehat{S}).$$

Definition 70. Let $L \subset \mathrm{Pic}(\widehat{S})$ be the saturation of Λ in $\mathrm{Pic}(\widehat{S})$, i.e. the smallest primitive sublattice containing Λ such that Λ generates L over $\mathbb{Q}$. If $\widehat{S}$ is a Kummer surface, then L is by definition K, the Kummer lattice. If $\widehat{S}$ is a Nikulin resolution, then L is by definition N, the Nikulin lattice.

We note that by construction, we have the inclusions

$$\Lambda \subset L \subset L^\vee \subset \Lambda^\vee = \frac{1}{2}\Lambda.$$

Explicit descriptions of K and N are given in the following lemmas. The first is due to Nikulin, see for example [**34**, Lemma 5.2] .

Lemma 71. *The Nikulin lattice N is the overlattice of Λ generated by Λ and $\widehat{E} = \frac{1}{2}\sum_i E_i$.*

While the above shows that the Nikulin lattice is obtained from the exceptional lattice by adding a single vector, the situation for the Kummer lattice is more complicated. The pithiest way to state the result is as follows (see [**1**, § VIII.5])

Lemma 72. *Under the natural identification*

$$\Lambda^\vee/\Lambda \cong \mathrm{Maps}(\mathbb{F}_2^4, \mathbb{F}_2),$$

the Kummer lattice K is the overlattice of Λ such that the inclusion

$$K/\Lambda \subset \Lambda^\vee/\Lambda$$

corresponds to the inclusion

$$\mathrm{Aff}(\mathbb{F}_2^4, \mathbb{F}_2) \subset \mathrm{Maps}(\mathbb{F}_2^4, \mathbb{F}_2)$$

of affine linear maps (including the two constant maps) into all maps.

We next describe $\mathrm{Pic}(\widehat{S})$ which will allow us to prove Proposition 66. The class γ_d might not be an integral class, but it turns out that the embedding

$$\mathbb{Z}\langle 2\gamma_d\rangle \oplus L \subset \mathrm{Pic}(\widehat{S})$$

is always index two. Depending on the parity of d and the type of the surface S, the order two quotient group is generated either by γ_d, or by $\gamma_d + r_d$ where r_d is a certain vector only depending on $d \bmod 2$.

In the case where S is Abelian and $\widehat{S}$ is a Kummer $K3$, we quote Garbagnati-Sarti [**21**, Theorem 2.7] adapted to our notation.

Proposition 73. *The Picard lattice of the Kummer surface $\widehat{S}$ is the index 2 overlattice of $\mathbb{Z}\langle 2\gamma_d\rangle \oplus K$ generated by $\mathbb{Z}\langle 2\gamma_d\rangle \oplus K$ and $\gamma_d + r_d$ where*

- $r_d \in K^\vee - K$, $2r_d \in K$,
- $r_d^2 = d \bmod 2$.

The class $[r_d] \in \Lambda^\vee/\Lambda \cong \mathrm{Maps}(\mathbb{F}_2^4, \mathbb{F}_2)$ *only depends on* $d \bmod 2$ *and is unique up to isometries of* K*. For* d *even, the corresponding map* $\mathbb{F}_2^4 \to \mathbb{F}_2$ *is the characteristic function of a fixed linear 2 plane* $P_1 \subset \mathbb{F}_2^4$*. For* d *odd, the corresponding map is the characteristic function of* $P_1 \Delta P_2 \subset \mathbb{F}_2^4$ *where* P_1 *and* P_2 *are transversely intersecting 2 planes, and* Δ *denotes symmetric difference.*

There are two families of Nikulin $K3$ surfaces determined as follows. Let S be a Nikulin surface and recall that $\beta_d \in \mathrm{Pic}(S)$ is a primitive $\imath$-invariant effective class with $\beta_d^2 = 2d$. The existence of the Nikulin involution implies there is an inclusion

$$\mathbb{Z}\langle\beta_d\rangle \oplus E_8(-2) \subset \mathrm{Pic}(S).$$

The above is either (I) an isomorphism, or (II) an index 2 sublattice[11].

Definition 74. We say that S is Nikulin of Type (I) in the first case and of Type (II) in the second case. The latter can occur only when d is even.

Proposition 75. *The Picard lattice of a Type (II) Nikulin resolution* $\widehat{S}$ *is* $\mathbb{Z}\langle\gamma_d\rangle \oplus N$*. The Picard lattice of a Type (I) Nikulin resolution* $\widehat{S}$ *is the index 2 over lattice of* $\mathbb{Z}\langle 2\gamma_d\rangle \oplus N$ *generated by* $\mathbb{Z}\langle 2\gamma_d\rangle \oplus N$ *and* $\gamma_d + r_d$ *where*

- $r_d \in N^\vee - N$, $2r_d \in N$,
- $r_d^2 = d \bmod 2$.

The class $[r_d] \in \Lambda^\vee/\Lambda$ *only depends on* $d \bmod 2$ *and is unique up to isometries of* N *and is given by*

$$r_d = \begin{cases} \frac{1}{2}(E_1 + E_2) & \text{if } d \text{ is odd} \\ \frac{1}{2}(E_1 + E_2 + E_3 + E_4) & \text{if } d \text{ is even} \end{cases}$$

for a suitable numbering of the exceptional divisors $E_1, \ldots, E_8$.

Proof. See Proposition 2.1 and Corollary 2.2 of [**20**].

Propositions 73 and 75 then prove Proposition 66.

[11] This statement must be modified if the invariant Picard rank of S is greater than 1. c.f. Remark 64 .

3.4.4 Theta function identities

To finish the proof of Theorem 67, we must prove the formulas given in Equation (3.12). Recall that

$$\Lambda \subset L \subset L^{\vee} \subset \Lambda^{\vee} = \frac{1}{2}\Lambda$$

where $\Lambda = \oplus_i \mathbb{Z}\langle E_i\rangle$ is the exceptional lattice and L is either K or N. Since any element $\rho \in \Lambda^{\vee}/\Lambda$ may be uniquely written as

$$\rho = \frac{1}{2}\sum_i \rho_i E_i, \quad \rho_i \in \{0,1\}$$

we may define

$$c_1(\rho) = \sum_i \rho_i, \quad c_0(\rho) = \mathrm{rk}(\Lambda) - c_1(\rho),$$

i.e. the number of ρ_i's which are 1 or 0 respectively. The following lemma is our basic tool for computing theta functions[12]

Lemma 76. *Let π be the projection $\Lambda^{\vee} \to \Lambda^{\vee}/\Lambda$ and suppose that $T \subset \Lambda^{\vee}$ is a union of cosets: $T = \cup_{\rho\in\pi(T)}(\Lambda+\rho)$. Then*

$$\Theta_T(q^2, w) = \sum_{\rho\in\pi(T)} \theta_0^{c_0(\rho)}\theta_1^{c_1(\rho)}$$

where

$$\theta_i = \theta_i(q^2, w) = \sum_{k\in\mathbb{Z}+\frac{i}{2}} q^{2k^2} w^k.$$

Proof. Since the cosets $\Lambda+\rho$ are disjoint we have

$$\Theta_T(q^2, w) = \sum_{\rho\in\pi(T)} \Theta_{\Lambda+\rho}(q^2, w)$$

and then we observe that

$$\Lambda + \rho \cong \mathbb{Z}\langle E\rangle^{\oplus c_0(\rho)} \oplus \left(\mathbb{Z}\langle E\rangle + \frac{E}{2}\right)^{\oplus c_1(\rho)}$$

from which it follows that $\Theta_{\Lambda+\rho} = \theta_0^{c_0(\rho)}\theta_1^{c_1(\rho)}$. □

[12] The authors are very grateful to John Duncan who explained this to us.

Proposition 77. *The theta functions of the Nikulin lattice N and the shifted Nikulin lattice N_{sh} are given by*

$$\begin{aligned}\Theta_N(q^2, w) &= \theta_0^8 + \theta_1^8 \\ \Theta_{N_{\mathsf{sh}}}(q^2, w) &= \theta_0^2\theta_1^6 + 2\theta_0^4\theta_1^4 + \theta_0^6\theta_1^2\end{aligned}$$

Proof. It follows from Lemma 71 and Proposition 75 that

$$\pi(N) = \left\{0, \tfrac{1}{2}(E_1 + \cdots + E_8)\right\}$$

and that

$$\begin{aligned}\pi(N_{\mathsf{sh}}) &= \pi\left(N + \tfrac{1}{2}(E_1 + E_2)\right) \cup \pi\left(N + \tfrac{1}{2}(E_1 + \cdots + E_4)\right) \\ &= \left\{\tfrac{1}{2}(E_1 + E_2), \tfrac{1}{2}(E_3 + \cdots + E_8), \tfrac{1}{2}(E_1 + \cdots + E_4), \tfrac{1}{2}(E_5 + \cdots + E_8)\right\}.\end{aligned}$$

The value of c_1 on the above 4 elements is 2, 6, 4, and 4 respectively. The proposition then follows from Lemma 76. $\square$

Proposition 78. *The theta functions of the Kummer lattice K and the shifted Kummer lattice K_{sh} are given by*

$$\begin{aligned}\Theta_K(q^2, w) &= \theta_0^{16} + 30\theta_0^8\theta_1^8 + \theta_1^{16} \\ \Theta_{K_{\mathsf{sh}}}(q^2, w) &= 4\theta_0^4\theta_1^{12} + 16\theta_0^6\theta_1^{10} + 24\theta_0^8\theta_1^8 + 16\theta_0^{10}\theta_1^6 + 4\theta_0^{12}\theta_1^4\end{aligned}$$

Proof. As in the Nikulin case, we must determine the value of c_1 on all the elements of $\pi(K)$ and $\pi(K_{\mathsf{sh}})$. By Lemma 72, $\pi(K)$ is given by the 32 elements $\frac{1}{2}\sum_i \rho_i E_i$ where the corresponding map $\mathbb{F}_2^4 \to \mathbb{F}_2$ given by $i \mapsto \rho_i$ is an affine linear function. The value of c_1 on the two constant functions is 0 and 16 respectively, while the value of c_1 on the remaining 30 non-constant affine linear functions is 8. The formula for Θ_K then follows from Lemma 76. A simple but tedious way to determine the elements of $\pi(K_{\mathsf{sh}}) = \pi(K + r_0) \cup \pi(K + r_1)$ is to choose coordinates for $\mathbb{F}_2^4$ and write down all 64 elements explicitly. Doing so and reading the off the value of c_1 on each element we find that $\pi(K + r_0)$ has 4 elements with $c_1 = 4$, 24 elements with $c_1 = 8$, and 4 elements with $c_1 = 12$ and that $\pi(K + r_1)$ has 16 elements with $c_1 = 6$ and 16 elements with $c_1 = 10$. The formula for $\Theta_{K_{\mathsf{sh}}}$ then follows from Lemma 76. There is a more coordinate free approach to the same calculation using the affine geometry of $\mathbb{F}_2^4$. It requires analyzing the various symmetric differences of the affine hyperplanes and the two dimensional planes P_1 and P_2 appearing in Proposition 73.

Unfortunately, the case by case analysis in this approach is not particularly less tedious than the direct enumeration. $\square$

To complete the proof of Theorem 67, it only remains to prove the following identities

$$\begin{aligned}\Theta_{N_{\mathsf{sh}}}(q^2,w) &= -\frac{\Delta(q^2)^{\frac{1}{2}}}{\Delta(q)}\cdot\phi_{10,1}(q,-w)\\ &= \psi_w\cdot q\cdot\prod_{n=1}^{\infty}(1+q^n)^{12}(1-q^n)^8(1+wq^n)^2(1+w^{-1}q^n)^2\end{aligned}$$

and

$$\begin{aligned}\Theta_{K_{\mathsf{sh}}}(q^2,w) &= 4\frac{\Delta(q^2)}{\Delta(q)^2}\cdot\phi_{10,1}^2(q,-w)\\ &= 4\psi_w^2\cdot q^2\cdot\prod_{n=1}^{\infty}(1+q^n)^{24}(1-q^n)^{16}(1+wq^n)^4(1+w^{-1}q^n)^4.\end{aligned}$$

Since the equations for $\Theta_{N_{\mathsf{sh}}}(q^2,w)$ and $\Theta_{K_{\mathsf{sh}}}(q^2,w)$ given in Propositions 77 and 78 can be factored as

$$\begin{aligned}\Theta_{N_{\mathsf{sh}}}(q^2,w) &= \theta_0^2\theta_1^2(\theta_0^2+\theta_1^2)^2\\ \Theta_{K_{\mathsf{sh}}}(q^2,w) &= 4\theta_0^4\theta_1^4(\theta_0^2+\theta_1^2)^4\end{aligned}\tag{3.13}$$

we see that it suffices to prove the identity:

$$\theta_0^2\theta_1^2(\theta_0^2+\theta_1^2)^2 = \psi_w\cdot q\cdot\prod_{n=1}^{\infty}(1+q^n)^{12}(1-q^n)^8(1+wq^n)^2(1+w^{-1}q^n)^2.\tag{3.14}$$

By the Jacobi triple product identity, we may write

$$\theta_0(q^2,w) = \prod_{n=1}^{\infty}(1-q^{4n})(1+wq^{4n-2})(1+w^{-1}q^{4n-2})\tag{3.15}$$

$$\theta_1(q^2,w) = q^{\frac{1}{2}}\cdot(w^{\frac{1}{2}}+w^{-\frac{1}{2}})\cdot\prod_{n=1}^{\infty}(1-q^{4n})(1+wq^{4n})(1+w^{-1}q^{4n}).\tag{3.16}$$

We also have the following

Lemma 79.

$$\theta_0(q^2,w)^2+\theta_1(q^2,w)^2 = \prod_{n=1}^{\infty}(1-q^{2n})^2(1+q^{2n-1})^2(1+wq^{2n-1})(1+w^{-1}q^{2n-1}).$$

Proof. The left hand side of the above equation is given by

$$\sum_{n,m\in\mathbb{Z}} q^{2m^2+2n^2}w^{n+m} + q^{2\left(m+\frac{1}{2}\right)^2+2\left(n+\frac{1}{2}\right)^2}w^{n+m+1}.$$

Letting $n = \frac{1}{2}(a-b)$ and $m = \frac{1}{2}(a+b)$ the sum rearranges to

$$\sum_{\substack{a,b\in\mathbb{Z}\\ a\equiv b \bmod 2}} q^{b^2}\left(q^{a^2}w^a + q^{(a+1)^2}w^{a+1}\right) = \sum_{a,b\in\mathbb{Z}} q^{b^2}q^{a^2}w^a = \theta_0(q,1)\theta_0(q,w).$$

Then applying the Jacobi triple product identity to the right hand side of the above proves the lemma. $\square$

Now applying Lemma 79 and Equations (3.15) to the left hand side of Equation (3.14), we get

$$\begin{aligned}
\theta_0^2\theta_1^2(\theta_0^2+\theta_1^2)^2 =& q\cdot\psi_w\cdot\prod_{n=1}^{\infty}(1-q^{4n})^2(1+wq^{4n})^2(1+w^{-1}q^{4n})^2\\
&\cdot(1-q^{4n})^2(1+wq^{4n-2})^2(1+w^{-1}q^{4n-2})^2\\
&\cdot(1-q^{2n})^4(1+q^{2n-1})^4(1+wq^{2n-1})^2(1+w^{-1}q^{2n-1})^2\\
=& q\cdot\psi_w\cdot\prod_{n=1}^{\infty}(1+q^n)^{12}(1-q^n)^8(1+wq^n)^2(1+w^{-1}q^n)^2
\end{aligned}$$

where in the last equality we have used the fact that

$$\prod_{n=1}^{\infty}(1+wq^{4n})(1+wq^{4n-2})(1+wq^{2n-1}) = \prod_{n=1}^{\infty}(1+wq^n)$$

and similar considerations.

This completes the proof of Theorem 67.

3.5 Local Abelian and Nikulin Surfaces (MT theory)

3.5.1 Overview.

In this section we prove some basic results about $\imath$-stability and we prove Conjecture 53 for local Abelian and Nikulin surfaces.

In Subsection 3.5.2 we show $\imath$-stability is equivalent to a certain kind of Nironi stability on the stack quotient $[X/\imath]$. In Subsection 3.5.3 we prove Conjecture 53 for $X = S\times\mathbb{C}$

where S is an Abelian or Nikulin surface. The basic idea is the following. Using the results of Subsection 3.5.2 and the projection $X \to \mathbb{C}$ we show that all $\imath$-stable MT sheaves on X are given by Nironi δ-stable sheaves on $[S/\imath] \times \{t\}$ for some $t \in \mathbb{C}$. We then show that Nironi δ-stability on $[S/\imath]$ is the large volume limit of a certain Bridgeland stability condition on $[S/\imath]$ constructed by Lim and Rota [**30**]. We then apply the derived Fourier-Mukai correspondence to show that our moduli spaces are given by, up to a factor of $\mathbb{C}$, moduli spaces of objects in the derived category of $\widehat{S}$ which are stable with respect to the large volume limit of one of the stability conditions on $K3$ surfaces constructed by Bridgeland [**8**]. Finally, we use the results of Bayer and Macri [**3**] to show these moduli spaces are deformation equvalent to moduli spaces of MT sheaves on $\widehat{S}$. This then allows us to apply Conjecture 43, which is known to hold for $\widehat{S} \times \mathbb{C}$ by [**39**, **46**]. The upshot is that we prove that Equation (3.9) holds with MT GV invariants replacing PT GV invariants on both sides and then the subsequent arguments of Section 3.4 apply word for word.

3.5.2 Nironi Stability

The category of $\imath$-equivariant sheaves on X and the category of sheaves on the stack $[X/\imath]$ are canonically equivalent and in this section we will not notationally differentiate between a sheaf on the stack and the corresponding $\imath$-equivariant sheaf.

In [**36**], Nironi developed a theory of slope stability for Deligne-Mumford stacks analougous to Simpson stability for schemes. Nironi stability for the stack $[X/\imath]$ involves a choice of an ample divisor H on the coarse space $X/\imath$ and the choice of a "generating bundle" V which we may take to be (see [**36**, Def. 2.2, Prop. 2.7])

$$V = (\mathcal{O}_X \otimes R_+)^{\oplus a} \oplus (\mathcal{O}_X \otimes R_-)^{\oplus b}$$

for any $a, b \in \mathbb{N}$.

Nironi's slope function is obtained from the generalized Hilbert polynomial of a sheaf F (i.e. the $\imath$-invariant part of $\chi(F \otimes V(mH))$) by dividing the second coefficient by the leading coefficient. For 1-dimensional sheaves F with $[\mathrm{supp}(F)] = \beta$, $\chi(F) = nR_{\mathsf{reg}} + \epsilon R_-$, and our choice of V, Nironi's slope function is given by

$$\mu(F) = \frac{(a+b)n + b\epsilon}{(a+b)H \cdot \beta}.$$

The slope function only depends on a and b through the number

$$\delta = \frac{b}{a+b} \in \mathbb{Q} \cap (0,1)$$

so we write

$$\mu_\delta(F) = \frac{n+\delta\epsilon}{H\cdot\beta}.$$

Definition 80. Let F be an $\imath$-equivariant sheaf on X with pure 1-dimensional support, $[\operatorname{supp}(F)] = \beta$, and $\chi(F) = nR_{\mathsf{reg}} + \epsilon R_{-}$. Then F is *Nironi δ (semi-)stable* if for all $\imath$-equivariant subsheaves $F' \subsetneq F$, $\mu_\delta(F') < \mu_\delta(F)$ (resp. $\mu_\delta(F') \leq \mu_\delta(F)$).

Let $\mathsf{M}^{\delta\text{-s}}_{\beta,n,\epsilon}([X/\imath])$ (resp. $\mathsf{M}^{\delta\text{-ss}}_{\beta,n,\epsilon}([X/\imath])$) be the moduli stack of Nironi δ (semi-)stable sheaves with β, n, ϵ as above. Nironi proves that the usual properties enjoyed by moduli stacks of Simpson (semi-)stable sheaves hold for moduli stacks of Nironi (semi-)stable sheaves [**36**, Theorems 6.21, 6.22]. In particular we have:

Theorem 81. *The stack $\mathsf{M}^{\delta\text{-s}}_{\beta,n,\epsilon}([X/\imath])$ is a $\mathbb{C}^*$-gerbe over its coarse moduli space. In particular, any Nironi δ stable sheaf is simple.*

Theorem 82. *Assume X is projective, then $\mathsf{M}^{\delta\text{-ss}}_{\beta,n,\epsilon}([X/\imath])$ has a projective coarse moduli space.*

The following corollary is then standard.

Corollary 83. *For X quasi-projective, the Hilbert-Chow morphism*

$$\mathsf{M}^{\delta\text{-ss}}_{\beta,n,\epsilon}([X/\imath]) \to \operatorname{Chow}_\beta(X)^\imath$$

given by $F \mapsto [\operatorname{supp}(F)]$ is proper.

Proposition 84. *Let F be an $\imath$-equivariant sheaf on X having proper pure 1-dimensional support and with $[\operatorname{supp}(F)] = \beta$ and $\chi(F) = R_{\mathsf{reg}} + \epsilon R_{-}$. Let $\delta > 0$ be a sufficiently small rational number. Then the following conditions are equivalent*

1. *F is Nironi δ semistable.*
2. *F is Nironi δ stable.*
3. *F is $\imath$-stable (see Definition 50).*

Proof. Let $F' \subsetneq F$ be an $\imath$-equivariant subsheaf with $\chi(F') = kR_{\mathsf{reg}} + \gamma R_{-}$ and $[\operatorname{supp}(F')] = \beta'$.

Suppose that F is Nironi δ semistable. Then the inequality

$$\frac{k+\delta\gamma}{H\cdot\beta'} \leq \frac{1+\delta\epsilon}{H\cdot\beta}$$

holds and is equivalent to

$$k \leq \left(\frac{H \cdot \beta'}{H \cdot \beta}\right)(1 + \delta\epsilon) - \delta\gamma. \tag{3.17}$$

Since $\operatorname{supp}(F') \subseteq \operatorname{supp}(F)$ and $\dim \operatorname{supp}(F') \neq 0$, we have

$$0 < \frac{H \cdot \beta'}{H \cdot \beta} \leq 1.$$

If $\frac{H \cdot \beta'}{H \cdot \beta} < 1$, then for sufficiently small δ we have $k < 1$. If $\frac{H \cdot \beta'}{H \cdot \beta} = 1$, then $\beta' = \beta$ and $k \leq 1 + \delta(\epsilon - \gamma)$. Thus we see that either $k < 1$ or $k = 1$ and $\beta' = \beta$ and $\gamma \leq \epsilon$. Finally, if $k = 1$, $\beta' = \beta$, and $\gamma = \epsilon$, then $\chi(Q) = 0$ where

$$0 \to F' \to F \to Q \to 0.$$

But since $\beta' = \beta$, $\dim Q = 0$ and so $\chi(Q) = 0$ implies that $Q = 0$ which implies that $F' = F$. Thus we've shown that (1) implies that either $k < 1$ or $k = 1$ and $\beta' = \beta$ and $\gamma < \epsilon$ which by Definition 50 means that F is $\imath$-stable. Thus (1) implies (3). Moreover, we've shown that the inequality (3.17) must be strict so that (1) implies (2). And of course (2) implies (1) so it remains to show that (3) implies (2).

To that end, suppose that F is $\imath$-stable, i.e. $k \leq 1$ and if $k = 1$ then $\gamma < \epsilon$ and $\beta' = \beta$. We need to prove that the inequality (3.17) holds. Since $\dim \operatorname{supp}(F') > 0$, $\frac{H \cdot \beta'}{H \cdot \beta} > 0$ which means the right hand side of (3.17) is positive for sufficiently small δ, and so if $k < 1$, (3.17) is true. If $k = 1$, then by hypothesis, $\gamma < \epsilon$ and $\beta' = \beta$ so (3.17) becomes $1 \leq 1 + \delta\epsilon - \delta\gamma$ which is true. $\square$

3.5.3 Proof of Conjecture 53 for Local Abelian and Nikulin surfaces

We now consider $X = S \times \mathbb{C}$ with S an Abelian or Nikulin surface and we adopt the notation of Section 3.4.

By Proposition 84 we may identify the moduli space of $\imath$-stable MT sheaves with the moduli space of Nironi δ stable sheaves:

$$\mathsf{M}^{\epsilon}_{m\beta_d}(X, \imath) = \mathsf{M}^{\delta\text{-ss}}_{m\alpha_d,1,\epsilon}([X/\imath]) = \mathsf{M}^{\delta\text{-s}}_{m\alpha_d,1,\epsilon}([X/\imath]).$$

Lemma 85. *Let F be an $\imath$-stable MT sheaf on X. Then F is scheme theoretically supported on $S \times \{t\}$ for some $t \in \mathbb{C}$.*

Proof. First suppose that the image of the support of F under the map $S \times \mathbb{C} \to \mathbb{C}$ is not a single point. Then F can be written as $F_1 \oplus F_2$ which violates stability since both factors are equivariant subsheaves. Thus we may suppose that F is set theoretically supported on some $S_t = S \times \{t\}$. Consider the short exact sequence of $\imath$-equivariant sheaves:

$$0 \to \mathcal{O}_X(-S_t) \to \mathcal{O}_X \to \mathcal{O}_{S_t} \to 0.$$

Noting that $\mathcal{O}_X(-S_t) \cong \mathcal{O}_X$ and tensoring with F, we get the right exact sequence:

$$F \to F \to F \otimes \mathcal{O}_{S_t} \to 0.$$

By Theorem 81, F is simple and hence the first map is either 0 or an isomorphism. Since $F \otimes \mathcal{O}_{S_t}$ is non-zero by construction, the first map must be zero and thus the second map induces an isomorphism $F \cong F \otimes \mathcal{O}_{S_t}$ by exactness. $\square$

By the lemma, every $\imath$-stable MT sheaf on X can be identified with a Nironi δ stable sheaf on $[S/\imath] \times \{t\}$ for some $t \in \mathbb{C}$. We let $\mathsf{M}^{\delta\text{-s}}_{\eta}([S/\imath])$ denote the moduli space of Nironi δ stable sheaves on $[S/\imath]$ in the K-theory class η. Then applying the above discussion and the analysis of K-theory classes done in Section 3.4 (and using the same notation) we get

$$\mathsf{M}^{\epsilon}_{m\beta_d}(X, \imath) = \coprod_{\substack{v \in \Gamma_{m,d} \\ l(v)=\epsilon}} \mathsf{M}^{\delta\text{-s}}_{\eta(m\alpha_d, v)}([S/\imath]) \times \mathbb{C} \tag{3.18}$$

where

$$\eta(m\alpha_d, v) = m\alpha_d + [\mathcal{O}_{\mathsf{pt}}] + \sum_i v_i[\mathcal{O}_{x_i} \otimes R_-].$$

The following proposition is key:

Proposition 86. *The moduli space $\mathsf{M}^{\delta\text{-s}}_{\eta(m\alpha_d,v)}([S/\imath])$ is deformation equivalent to $\mathsf{M}^{s}_{(0,m\gamma_d+v,1)}(\widehat{S})$, the moduli space of Simpson sheaves on $\widehat{S}$ with Mukai vector $(0, m\gamma_d + v, 1)$. Moreover, the deformation equivalence is compatible with the Hilbert-Chow morphism.*

Assuming the above Proposition and noting that

$$\mathsf{M}_{m\gamma_d+v}(\widehat{S} \times \mathbb{C}) = \mathsf{M}^{s}_{(0,m\gamma_d+v,1)}(\widehat{S}) \times \mathbb{C},$$

we may compute the Maulik-Toda polynomial of $(X, \imath)$ as follows. We use the definition, Equation (3.18), and the above Proposition to get

$$\mathsf{MT}_{m\beta_d}(y, w) = \sum_{v \in \Gamma_{m,d}} \mathsf{MT}_{m\gamma_d+v}(y) w^{l(v)}$$

where $\mathsf{MT}_{m\beta_d}(y,w)$ is the Maulik-Toda polynomial of $(X,\imath)$ in the class $m\beta_d$ and $\mathsf{MT}_{m\gamma_d+v}(y)$ is the Maulik-Toda polynomial of $\widehat{S}\times\mathbb{C}$ in the class $m\gamma_d+v$.

It then follows immediately from the above equation that the MT analog of Equation (3.9) holds:

$$\sum_{g,h} n^{\mathsf{MT}}_{g,h}\,(m\beta_d)\,\psi_y^{h-1}\psi_w^{g+1-2h} = \sum_{h\geq 0}\sum_{v\in\Gamma_{m,d}} n^{\mathsf{MT}}_{h}\,(m\gamma_d+v)\,w^{l(v)}\psi_y^{h-1}.$$

All the analysis in Section 3.4 subsequent to Equation (3.9) then goes through word for word with the MT versions of the invariants and we see that they are given by the same formulas (in Theorem 67) as the PT versions of the invariants. We thus conclude that

$$n^{\mathsf{PT}}_{g,h}\,(m\beta_d) = n^{\mathsf{MT}}_{g,h}\,(m\beta_d)$$

holds and thus Conjecture 53 holds for $(X,\imath)$.

It remains only to prove Proposition 86.

Proof of Proposition 86. We remark that although we don't directly use it, these moduli spaces are all hyperkahler manifolds of $K3[n]$ type and the map to Chow is a Lagrangian fibration.

In [**30**], Lim and Rota construct Bridgeland stability conditions on orbifold surfaces with Kleinian orbifold points. For notational simplicity, they assume that their orbifold surface has a single orbifold point, but their method easily applies to orbifold surfaces with multiple Kleinian orbifold points such as $[S/\imath]$.

Their stability condition depends (in the case of $[S/\imath]$) on parameters $\gamma\in(0,\frac{1}{2})$ and $w\in\mathbb{C}$ and has a central charge $Z_{w,\gamma}$ which in our situation takes values

$$Z_{w,\gamma}\left(m\alpha_d+n[\mathcal{O}_{\mathsf{pt}}]+\sum_i v_i[\mathcal{O}_{x_i}\otimes R_-]\right) = -n-\frac{1}{2}\gamma\sum_i v_i + iH\cdot(m\alpha_d).$$

The parameter w must be choosen satisfying two inequalities which in our situation read

$$\begin{aligned}\Re(w) &> -\frac{(\Im(w))^2}{H^2}+3-\gamma^2\\ 2\Re(w) &> \frac{\Im(w)}{H^2}-3\gamma > 0\end{aligned}$$

and the corresponding heart of the derived category is given by a certain tilt $\mathrm{Coh}^{-\Im(w)}([S/\imath])$ (see [**30**, § 4.2]).

Recalling that $\epsilon = \sum_i v_i$, we see that the slope function associated to the central charge is exactly the Nironi slope function μ_δ with $\gamma = 2\delta$. Consequently, in any limit with $\Im(w) \to \infty$ (and satisfying the necessary inequalities), the Lim-Rota stable objects become Nironi stable sheaves. Thus we may identify $\mathsf{M}^{\delta\text{-s}}_{\eta(m\alpha_d,v)}([S/\imath])$ with the moduli space of Lim-Rota stable objects for $\gamma = 2\delta$ and a choice of w with appropriately large $\Im(w)$.

We may now consider the derived Fourier-Mukai equivalence

$$FM : D^b([S/\imath]) \to D^b(\widehat{S}).$$

This equivalence will take the Lim and Rota's stability conditions on $[S/\imath]$ to some Bridgeland stability condition on the $K3$ surface $\widehat{S}$. We claim these stability conditions are in fact the stability conditions on $K3$ constructed by Bridgeland in [**8**]. These stability conditions are characterized by lying in the connected component of the stability conditions on $\widehat{S}$ where $\mathcal{O}_{\mathsf{pt}}$ is stable. In [**30**, § 5], Lim and Rota analyze the stability of $\mathcal{O}_{\mathsf{pt}}$ and show that for generic deformations of their stability conditions, it is stable. Our claim follows.

We thus can make the identification

$$\mathsf{M}^{\delta\text{-s}}_{\eta(m\alpha_d,v)}([S/\imath]) = \mathsf{M}^{\mathsf{Bridgeland}}_{(0,m\gamma_d+v,1)}(\widehat{S})$$

where the moduli space on the right is the moduli space of objects on $\widehat{S}$ with Mukai vector $(0, m\gamma_d + v, 1)$ which are stable with respect to the Bridgeland stability condition which is Fourier-Mukai equivalent to our choosen Lim-Rota condition on $[S/\imath]$.

We may now apply the results of Bayer and Macri [**3**], who analyze in great detail all of the moduli spaces of Bridgeland semistable objects on a $K3$ surface. They show that all moduli spaces of objects semistable with respect to one of Bridgeland's constructed stability conditions are deformation equivalent hyperkahler manifolds, provided that the moduli space has no strictly semistable objects. In particular, the moduli space $\mathsf{M}^{\mathsf{Bridgeland}}_{(0,m\gamma_d+v,1)}(\widehat{S})$ is deformation equivalent to $\mathsf{M}^{s}_{(0,m\gamma_d+v,1)}(\widehat{S})$ which proves the main assertion of the Proposition. Moreover, Bayer-Macri analyze the Hilbert-Chow morphism which they show to be a Lagrangian fibration (whenever it is well-defined) and compatible with the deformation equivalence. $\square$

3.6 Additional Examples

3.6.1 Elliptically Fibered Calabi-Yau Threefold

Let $\pi : X \to B$ be an elliptically fibered Calabi-Yau threefold with section $B \hookrightarrow X$ and with integral fibers. We additionally require the following conditions:

1. There is an involution $\bar{\imath} : B \to B$ whose fixed point locus is a smooth curve $C \subset B$.

2. $\pi|_S : S \to C$ is a smooth elliptic surface, where $S = \pi^{-1}(C)$.

3. $\bar{\imath}$ lifts to a Calabi-Yau involution $\imath : X \to X$ which restricts to a fiberwise action by -1 on S over C.

Let $[F] \in H_2(X)$ be the class of a fiber F of $\pi : X \to B$. The following is our main result for this example:

Theorem 87. *Conjecture 53 holds for the class $[F]$ and the invariants are given by*

$$n^{\mathsf{PT}}_{g,h}([F]) = n^{\mathsf{MT}}_{g,h}([F]) = \begin{cases} -e(C) & (g,h) = (1,0), \\ e(S) & (g,h) = (0,0), \\ 0 & \textit{otherwise} \end{cases}$$

We prove this theorem below by directly computing both the $\imath$-PT and $\imath$-MT invariants in the class $[F]$.

The $\imath$-fixed locus $X^\imath$ is given by $S[2]$, the 2-torsion points of the elliptic fibration $S \to C$. It admits a decomposition

$$X^\imath = C_0 \sqcup C_1$$

where $C_0 \cong C$ is the zero section, and C_1 is the locus of non-trivial 2-torsion points. It is a degree 3 cover $C_1 \to C$ which is simply ramified at the nodes of the nodal fibers and doubly ramified at the cusps of the cuspidal fibers.

The compactified Jacobian $\mathrm{J}\overline{\mathrm{ac}}^d(X/B)$ of the family of elliptic curves $X \to B$ can be identified with the moduli space of pure dimension 1 stable sheaves $\mathcal{E}$ with $ch(\mathcal{E}) = (0, 0, [F], d)$.

The following facts are standard for this situation[13]

- $\mathrm{J}\overline{\mathrm{ac}}^d(X/B) \cong X$.

- There is a rank d bundle $V_d \to \mathrm{J}\overline{\mathrm{ac}}^d(X/B)$ whose fiber over $\mathcal{E}$ is $H^0(\mathcal{E})$.

- The map $\mathsf{PT}_{[F],d}(X) \to \mathrm{J}\overline{\mathrm{ac}}^d(X/B)$ given by $[\mathcal{O}_X \xrightarrow{s} \mathcal{E}] \mapsto \mathcal{E}$ induces an isomorphism $\mathsf{PT}_{[F],d}(X) \cong \mathbb{P}(V_d)$.

[13]See for example [**33**, bottom of page 2] combined with the fact that the moduli space of stable pairs supported on a family of integral plane curves is isomorphic to the relative Hilbert scheme of points [**33**, top of page 3] and [**40**, Prop. 1.8].

The above constructions are $\imath$-equivariant and exhibit $\mathsf{PT}_{[F],d}(X)$ as a projective bundle over X so that we may identify the fixed locus as follows:

$$\mathsf{PT}_{[F],d}(X)^{\imath} = \mathbb{P}(V_d^+)|_{X^\imath} \sqcup \mathbb{P}(V_d^-)|_{X^\imath}$$

where

$$V_d|_{X^\imath} \cong V_d^+|_{X^\imath} \oplus V_d^-|_{X^\imath}$$

is the eigenbundle decomposition for the $\imath$ action. Since $X^\imath = C_0 \sqcup C_1$, we see

$$\mathsf{PT}_{[F],d}(X)^{\imath} = \mathbb{P}^+_{d,0} \sqcup \mathbb{P}^+_{d,1} \sqcup \mathbb{P}^-_{d,0} \sqcup \mathbb{P}^-_{d,1}$$

where we have abbreviated $\mathbb{P}(V_d^\pm)|_{C_i}$ as $\mathbb{P}^\pm_{d,i}$.

Then since $\mathbb{P}^\pm_{d,i}$ is a smooth projective bundle over a curve we have

$$e_{\mathsf{vir}}\left(\mathbb{P}^\pm_{d,i}\right) = (-1)^{\dim V_d^\pm} \cdot \dim V_d^\pm \cdot e\left(C_i\right).$$

Using the decomposition given in Equation (3.3) :

$$\mathsf{PT}_{[F],d}(X)^{\imath} = \bigsqcup_{d=2n+\epsilon} \mathsf{PT}_{[F],n,\epsilon}(X,\imath)$$

we see that we need only compute $\dim V_d^\pm$ and ϵ for each component. To do so, we pick $[s : \mathcal{O}_F \to \mathcal{E}] \in \mathbb{P}^\pm_{d,i}$ where $\mathcal{E}$ is an $\imath$-invariant sheaf supported on a smooth fiber F and $s \in H^0(\mathcal{E})^\pm$. Let $p_0 = F \cap C_0$ and $\{p_1, p_2, p_3\} = F \cap C_1$ be the origin and non-trivial two torsion points of F respectively. Up to renumbering, we may assume $\mathcal{E} \mapsto p_i$ under the map $\mathbb{P}^\pm_{d,i} \to C_i$. The corresponding $\imath$-PT pair is $[s : \mathcal{O}_F \to \mathcal{E} \otimes R_\pm]$ and so by definition

$$\begin{aligned} H^0(\mathcal{E} \otimes R_\pm) &= nR_{\mathsf{reg}} + \epsilon R_- \\ &= nR_+ + (n+\epsilon)R_-. \end{aligned}$$

On the other hand,

$$H^0(\mathcal{E} \otimes R_\pm) = H^0(\mathcal{E}) \otimes R_\pm = (V_d^+ \oplus V_d^-) \otimes R_\pm$$

and so

$$\dim V_d^\pm = n = \frac{1}{2}(d-\epsilon).$$

Moreover, since s is an invariant section of $\mathcal{E} \otimes R_\pm$, it determines an isomorphism $\mathcal{E} \otimes R_\pm \cong \mathcal{O}_F(D)$ where D is an $\imath$-invariant effective divisor. Such divisors are in the form given by Lemma 63 which by the same lemma shows

$$\epsilon = |T| - 2$$

where $T \subset \{0,1,2,3\}$. Finally, the map $\mathbb{P}^{\pm}_{d,i} \to C_i$ which sends $\mathcal{E} \mapsto p_i$ is given by summing the points in the divisor D in the group law of F. In particular we must have

$$\sum_{j \in T} p_j = p_i$$

in the group law. For $p_i = p_0$ the only possible subsets are

$$T = \emptyset, \{0\}, \{1,2,3\}, \{0,1,2,3\} \text{ with } \epsilon = -2, -1, 1, 2 \text{ respectively,}$$

and for $p_i = p_1$ the only possible subsets are

$$T = \{1\}, \{0,1\}, \{2,3\}, \{0,2,3\} \text{ with } \epsilon = -1, 0, 0, 1 \text{ respectively.}$$

We then can compute the $Q^{[F]}$ coefficient of $Z^{\mathsf{PT}}(X,\imath)$ by taking e_{vir} of the components of $\mathsf{PT}_{[F],d}(X)^{\imath}$, multiplying by the appropriate $y^n w^{\epsilon}$, and then summing over n and ϵ:

$$\begin{aligned}
\Big[Z^{\mathsf{PT}}(X,\imath)\Big]_{Q^{[F]}} &= \sum_{n=1}^{\infty} (-1)^n n\, y^n \left((w^{-2} + w^{-1} + w + w^2) e(C_0) + (w^{-1} + 2 + w^1) e(C_1) \right) \\
&= \frac{-y}{(1+y)^2} \left((\psi_w^2 - 3\psi_w) e(C_0) + \psi_w e(C_1) \right) \\
&= -e(C_0)\psi_y^{-1}\psi_w^2 + (3e(C_0) - e(C_1))\psi_y^{-1}\psi_w.
\end{aligned}$$

Noting that $[\log Z^{\mathsf{PT}}(X,\imath)]_{Q^{[F]}} = [Z^{\mathsf{PT}}(X,\imath)]_{Q^{[F]}}$ since $[F]$ is primitive, the formula for $n^{\mathsf{PT}}_{g,h}([F])$ in Theorem 87 then follows from the fact that

$$e(S) = 3e(C_0) - e(C_1).$$

The above formula is easily proved by observing that the equality holds when restricted to any fiber of $S \to C$ (smooth or singular).

To compute the $\imath$-MT invariants, we use the following

Lemma 88. $\mathsf{M}^{\epsilon}_{[F]}(X,\imath) \cong \mathsf{PT}_{[F],1,\epsilon}(X,\imath)$.

Proof. It suffices to show that for all $\mathcal{E} \in \mathsf{M}^{\epsilon}_{[F]}(X,\imath)$ there is a unique (up to scale) $\imath$-invariant section s. Then the isomorphism in the lemma is given by

$$\mathcal{E} \mapsto [s : \mathcal{O}_X \to \mathcal{E}].$$

Any $\imath$-MT sheaf $\mathcal{E}$ admits some $\imath$-invariant section s since $\chi(\mathcal{E}) = R_{\mathsf{reg}} + \epsilon R_-$. We then get an $\imath$-equivariant short exact sequence

$$0 \to \mathcal{O}_F \xrightarrow{s} \mathcal{E} \to Q \to 0$$

where $F = \operatorname{supp}(\mathcal{E})$ is some fiber of $S \to C$ and Q is necessarily 0 dimensional since F is integral. Moreover, $\chi(\mathcal{O}_F) = R_+ - R_-$ for any fiber F and so $\chi(Q) = H^0(Q) = (2+\epsilon)R_-$. Taking the long exact sequence associated to the above short exact sequence, and then restricting to the $\imath$-invariant part yields an isomorphism

$$H^0(\mathcal{O}_F) \cong H^0(\mathcal{E})^\imath$$

so that $H^0(\mathcal{E})^\imath$ is 1 dimensional and hence generated by s. □

The lemma then allows us to apply our previous analysis of the components of $\mathsf{PT}_{[F],d,\epsilon}(X,\imath)$ specialized to the case $d = 1$ where $\mathbb{P}^{\pm}_{1,i} \cong C_i$. As before, the components and the corresponding ϵ is determined by the subset $T \subset \{0,1,2,3\}$. The result is

$$\mathsf{M}^{\epsilon}_{[F]}(X,\imath) \cong \begin{cases} C_0 & \text{if } \epsilon = \pm 2, \\ C_0 \sqcup C_1 & \text{if } \epsilon = \pm 1, \\ C_1 \sqcup C_1 & \text{if } \epsilon = 0. \end{cases}$$

Since $\mathsf{M}^{\epsilon}_{[F]}(X,\imath)$ is a smooth curve, the perverse sheaf of vanishing cycles is the shifted constant sheaf, $\phi^\bullet = \mathbb{Q}[1]$. Furthermore, $\mathrm{Chow}_{[F]}(X)^\imath$ here is C with $\pi^\epsilon : \mathsf{M}^{\epsilon}_{[F]}(X,\imath) \to C$ given by projection $C_i \to C$ on each component.

In general, if

$$\pi : C'' \to C'$$

is a proper surjective morphism of smooth curves, then π is semi-small and consequently $R^\bullet\pi_*\mathbb{Q}[1]$ is a perverse sheaf on C' [**16**, Theorem 4.2.7] and so

$${}^pH^i(R^\bullet\pi_*\mathbb{Q}[1]) = \begin{cases} R^\bullet\pi_*\mathbb{Q}[1] & i = 0, \\ 0 & i \neq 0. \end{cases}$$

Consequently we have

$$\begin{aligned} \sum_i \chi({}^pH^i(R^\bullet\pi_*\mathbb{Q}[1]))y^i &= \chi(R^\bullet\pi_*\mathbb{Q}[1]) \\ &= -e(C'') \end{aligned}$$

where the last equality follows from the perverse Leray spectral sequence.

Applying this to $\pi^\epsilon : \mathsf{M}^\epsilon_{[F]}(X, \imath) \to C$ we can then compute the Maulik-Toda polynomial:

$$\begin{aligned}\mathsf{MT}_{[F]}(y,w) &= -e(C_0)(w^{-2}+w^{-1}+w^1+w^2) - e(C_1)(w^{-1}+2+w^1) \\ &= -e(C)\psi_w^2 + e(S)\psi_w\end{aligned}$$

and the formula for $n^{\mathsf{MT}}_{g,h}([F])$ follows. The proof of Theorem 87 is then complete.

3.6.2 The Local Football

Let $\imath$ be an involution of $C \cong \mathbb{P}^1$ fixing two points z_0 and z_∞. The line bundles $\mathcal{O}_C(-z_0)$ and $\mathcal{O}_C(-z_\infty)$ are naturally $\imath$-equivariant and consequently the CY3

$$X = \mathrm{Tot}(\mathcal{O}_C(-z_0) \oplus \mathcal{O}_C(-z_\infty))$$

has a natural involution which we also call $\imath$.

The global stack quotient $[X/\imath]$ is called a *local football*, though here we use this term to mean the pair $(X, \imath)$. The purpose of this section is to give a proof of Proposition 59 which we restate here:

Proposition 89. *For all $d > 0$, we have*

$$n^{\mathsf{PT}}_{g,h}(d[C]) = n^{\mathsf{MT}}_{g,h}(d[C]) = \begin{cases} 1 & (d,g,h) = (1,0,0) \\ 0 & \textit{otherwise} \end{cases}$$

We start with $\imath$-PT theory. As we did in Lemma 65, we may compute the $\imath$-PT invariants of $(X, \imath)$ by computing orbifold PT invariants of the stack quotient $[X/\imath]$. Since $[X/\imath]$ is toric, we may use the orbifold topological vertex [**10**]. The partition function $Z^{\mathsf{PT}}([X/\imath])$ is computed in Section 4.2 of [**10**] where it is given by $DT'(\mathcal{X}_{2,2})$ in the notation of [**10**]. Here we are using [**5**, Theorem 6.12] which states that the reduced Donaldson-Thomas partition function is equal to the Pandharipande-Thomas partition function. By the formula just below [**10**, Prop. 3] we have

$$Z^{\mathsf{PT}}([X/\imath]) = \prod_{u \in \{v, vp_0, vr_0, vp_0r_0\}} M(u, -q)^{-1} \tag{3.19}$$

where

$$M(x,q) = \prod_{n=1}^{\infty} (1 - xq^n)^{-n}.$$

The variables v, p_0, r_0, and q track classes in the K-theory of sheaves on $[X/\imath]$ which we can equivalently regard as $\imath$-equivariant sheaves on X. Recall that the variables in $Z^{\mathsf{PT}}(X,\imath)$ track the curve class and χ of the sheaf. Thus to specialize $Z^{\mathsf{PT}}([X/\imath])$ to $Z^{\mathsf{PT}}(X,\imath)$, we must compute the curve class and χ of each of these classes. The results are given in the following table:

$Z^{\mathsf{PT}}([X/\imath])$ variable	Class in $[X/\imath]$ (see [**10**, § 3.3])	Equivariant class on X	(χ,β) of class	$Z^{\mathsf{PT}}(X,\imath)$ variable
p_0	$[\mathcal{O}_{z_0}\otimes R_-]$	$[\mathcal{O}_{z_0}\otimes R_-]$	$(R_-,0)$	w
r_0	$[\mathcal{O}_{z_\infty}\otimes R_-]$	$[\mathcal{O}_{z_\infty}\otimes R_-]$	$(R_-,0)$	w
q	$[\mathcal{O}_{\overline{p}}]$	$[\mathcal{O}_p\oplus\mathcal{O}_{\imath(p)}]$	$(R_{\mathsf{reg}},0)$	y
v	$[\mathcal{O}_{[C/\imath]}(-\overline{p})]$	$[\mathcal{O}_C(-p-\imath(p))]$	$(-R_-,[C])$	Qw^{-1}

Table 3.2: Change of variables between those of $Z^{\mathsf{PT}}(X,\imath)$ to those of $Z^{\mathsf{PT}}([X/\imath])$ in [**10**].

In the above table, $\overline{p}\in[C/\imath]$ is a generic point corresponding to the $\imath$-orbit $\{p,\imath(p)\}\subset C$ and the formula $\chi(\mathcal{O}_C(-p-\imath(p)))=-R_-$ is obtained by applying Lemma 63.

Equation (3.19) then specializes to

$$Z^{\mathsf{PT}}(X,\imath)=M(Qw^{-1},-y)^{-1}M(Q,-y)^{-2}M(Qw,-y)^{-1}.$$

It is straightforward to show that

$$\log M(x,-y)^{-1}=\sum_{k=1}^{\infty}\frac{x^k}{k}\,\psi^{-1}_{-(-y)^k}.$$

and so

$$\begin{aligned}\log Z^{\mathsf{PT}}(X,\imath)&=\sum_{k=1}^{\infty}\frac{1}{k}Q^k\cdot\left(w^{-k}+2+w^k\right)\cdot\psi^{-1}_{-(-y)^k}\\&=\sum_{k=1}^{\infty}\frac{1}{k}Q^k\cdot\psi^{-1}_{-(-y)^k}\cdot\psi_{w^k}\end{aligned}$$

and the formula for $n^{\mathsf{PT}}_{g,h}(d[C])$ then follows.

Turning now to the $\imath$-MT theory, we begin with the following key result.

Lemma 90. *$\mathsf{M}^{\epsilon}_{d[C]}(X,\imath)$ is empty for $d>1$.*

Proof. Since $\mathsf{M}^{\epsilon}_{d[C]}(X,\imath)$ is proper over $\mathrm{Chow}_{d[C]}(X)$ which is a point, if it is non-empty, it admits a fixed point for the torus action induced from the action on X. In order to obtain a contradiction, we suppose there exists a torus invariant $\imath$-MT sheaf $\mathcal{E}$ in the class $d[C]$ with $d>1$.

Then the scheme-theoretic support of $\mathcal{E}$ is the thickened curve C_λ determined from a 2-dimensional partition λ. Since $\chi(\mathcal{E})=R_{\mathsf{reg}}+\epsilon R_-$, there is a $\imath$-invariant section

$$0\to\mathcal{O}_{C_\lambda}\xrightarrow{s}\mathcal{E}\to Q\to 0,$$

and then since $\mathcal{E}$ is $\imath$-stable, $\chi(\mathcal{O}_{C_\lambda})=kR_{\mathsf{reg}}+mR_-$ with $k\le 1$.

Let $\pi:X\to C$ be the projection, then $\chi(\mathcal{O}_{C_\lambda})=\chi(\pi_*\mathcal{O}_{C_\lambda})$. We have

$$\pi_*\mathcal{O}_{C_\lambda}=\bigoplus_{(i,j)\in\lambda}\mathcal{O}_C(z_0)^i\otimes\mathcal{O}_C(z_\infty)^j=\bigoplus_{(i,j)\in\lambda}\mathcal{O}_C(iz_0+jz_\infty).$$

Writing $i=2a+i'$ and $j=2b+j'$ with $a,b\ge 0$ and $(i',j')\in\{(0,0),(0,1),(1,0),(1,1)\}$ and applying Lemma 63, we get

$$\chi(\mathcal{O}_C(iz_0+jz_\infty))=(1+a+b)R_{\mathsf{reg}}+(i'+j'-1)R_-.$$

Since we are taking sums of these terms, the only way to guarantee $k\le 1$ as above, is if $\lambda=\square$ is the unique partition of length 1.

Thus $\mathcal{E}$ is scheme-theoretically supported on C, and hence it must a torus invariant, $\imath$-equivariant, rank d vector bundle on C. Such vector bundles split as a direct sum of $\imath$-equivariant line bundles which then contradicts the $\imath$-stability of $\mathcal{E}$. $\square$

It follows from the lemma that $n^{\mathsf{MT}}_{g,h}(d[C])=0$ for $d>1$ and for $d=1$ we apply Corollary 61 which then finishes the proof of Proposition 59.

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Appendix A

Modular Forms and Eta Products

This appendix will be devoted to giving a brief overview of the modular objects relevant to our results. An excellent reference is Chapters 1 and 2 of [**29**]. We are interested in modular forms of integral or half-integral weight with multiplier system for the congruence subgroup

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, c \equiv 0 \pmod{N} \right\} \subset \mathrm{SL}_2(\mathbb{Z})$$

for an integer $N \geq 1$. A multiplier system on $\Gamma_0(N)$ is a function $v : \Gamma_0(N) \to \mathbb{C}^*$ satisfying some consistency conditions. We will not need the details, so we refer the reader to [**25**, Sec. 2.6].

Let $\mathfrak{H} = \{z \in \mathbb{C} \,|\, \Im(z) > 0\}$ be the upper-half plane in $\mathbb{C}$.

Definition 91. A holomorphic function $f : \mathfrak{H} \to \mathbb{C}$ is called a *modular form* of weight $k \in \mathbb{R}$ and multiplier system v on $\Gamma_0(N)$ if f is holomorphic at all cusps $\mathbb{Q} \cup \{\infty\}$, and if for all $L \in \Gamma_0(N)$, f transforms as

$$f(L\tau) = f\left(\frac{a\tau + b}{c\tau + d}\right) = v(L)(c\tau + d)^k f(\tau), \quad L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{A.1}$$

We call f a *cusp form* if additionally, f vanishes at all cusps.

We employ the change of variables $q = \exp(2\pi i\tau)$ and with it, the abuse of notation writing $f(\tau)$ and $f(q)$ interchangeably. The fundamental building block of a large class of modular forms is the Dedekind eta function (or just eta function)

$$\eta(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n),$$

which is a modular form of weight $\frac{1}{2}$ and multiplier system v_η on $\mathrm{SL}_2(\mathbb{Z})$; see [**25**, Sec. 2.8] where v_η is given explicitly.[14]

Definition 92. An *eta product* of level $N \geq 1$ is a function $f : \mathfrak{H} \to \mathbb{C}$ of the form

$$f(q) = \prod_{m|N} \eta(q^m)^{a_m}$$

such that $a_m \in \mathbb{Z}$ (possibly negative, or zero) for all $m|N$, and where the product is over positive divisors of N.

From the modular properties of the Dedekind eta function, one can show that an eta product f of level N transforms as a modular form on $\Gamma_0(N)$ of weight

$$k = \frac{1}{2}\sum_{m|N} a_m \in \tfrac{1}{2}\mathbb{Z},$$

and with multiplier system

$$v_f(L) = \prod_{m|N}\left(v_\eta\begin{pmatrix} a & mb \\ c/m & d \end{pmatrix}\right)^{a_m}, \qquad L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

When we say "transforms as" we mean that f satisfies (A.1) for all $L \in \Gamma_0(N)$. An eta product f is automatically holomorphic on $\mathfrak{H}$. This is because the form of $\eta(q)$ indicates that any poles of f must occur at $q = 0$ or $|q| = 1$. All that is left to consider is when an eta product is holomorphic at the cusps. The following proposition gives necessary and sufficient conditions.

Proposition 93 ([**29**, Cor. 2.3])**.** *An eta product f of level N is holomorphic at the cusps if and only if the following holds for all positive divisors c of N*

$$\sum_{m|N} \frac{(\gcd(c,m))^2)}{m} a_m \geq 0.$$

Moreover, f vanishes at all cusps if and only if each inequality is strict. An eta product is therefore a modular form of weight k for $\Gamma_0(N)$ if and only if each inequality is satisfied, and it is a cusp form if and only if each is strictly satisfied.

[14]Equivalently, $\eta(q)$ is a modular form of weight $\frac{1}{2}$ on the metaplectic double cover $\mathrm{Mp}_2(\mathbb{Z})$ of $\mathrm{SL}_2(\mathbb{Z})$.

Appendix B

Tables of Values for $n_{g,h}(d, \mathsf{A}^{\mathsf{odd}})$

In this appendix, we list explicitly the values of $n_{g,h}(d, \mathsf{A}^{\mathsf{odd}})$ for $d \leq 7$. This case includes the primitive class β_d on a Picard rank one Abelian surface S where these numbers have some enumerative significance. The $h = 0$ numbers are (up to the minus sign due to the second factor in $X = S \times \mathbb{C}$) actual counts of $\imath$-invariant hyperelliptic curves on S; they coincide with the counts computed in [**13**]. For each d, the highest genus occuring is $g = d + 1$, the arithmetic genus of the class β_d. Let

$$\mathrm{Chow}_{\beta_d}(X)_h \subset \mathrm{Chow}_{\beta_d}(X)^{\imath}$$

be the dimension h components of the $\imath$-fixed point locus. Then one can show that

$$n_{d+1,h}(d, \mathsf{A}^{\mathsf{odd}}) = e_{\mathsf{vir}}\left(\mathrm{Chow}_{\beta_d}(X)_h\right).$$

The right hand side can be computed directly since $\mathrm{Chow}_{\beta_d}(X)^{\imath} = \mathrm{Chow}_{\beta_d}(S)^{\imath} \times \mathbb{C}$ and that the first factor here is the disjoint union of the $\imath$-invariant linear systems of the $\imath$-invariant line bundles in the class β_d.

For $d \leq 1$, the only non-zero values of $n_{g,h}(d, \mathsf{A}^{\mathsf{odd}})$ are given by

$$n_{1,0}(0, \mathsf{A}^{\mathsf{odd}}) = -4, \quad n_{2,0}(1, \mathsf{A}^{\mathsf{odd}}) = -16.$$

For $2 \leq d \leq 7$, see the table below:

Table B.1: Non-zero values of $n_{g,h}(d, \mathsf{A}^{\mathsf{odd}})$ for $2 \leq d \leq 7$.

	$\mathbf{d = 2}$		$\mathbf{d = 3}$	
	$h = 0$	$h = 1$	$h = 0$	$h = 1$
$g = 2$	-48		-64	
$g = 3$	-24	8	-160	
$g = 4$			-16	32

	$\mathbf{d = 4}$			$\mathbf{d = 5}$		
	$h = 0$	$h = 1$	$h = 2$	$h = 0$	$h = 1$	$h = 2$
$g = 2$	-112			-96		
$g = 3$	-456	24		-1056		
$g = 4$	-192	96		-912	224	
$g = 5$	-4	48	-12	-96	320	
$g = 6$					32	-48

	$\mathbf{d = 6}$				$\mathbf{d = 7}$			
	$h = 0$	$h = 1$	$h = 2$	$h = 3$	$h = 0$	$h = 1$	$h = 2$	$h = 3$
$g = 2$	-192				-128			
$g = 3$	-1920	32			-3264			
$g = 4$	-2992	512			-7776	704		
$g = 5$	-736	1056	-64		-3424	3072		
$g = 6$	-16	384	-144		-240	1920	-448	
$g = 7$		8	-72	16		192	-480	
$g = 8$							-48	64

Appendix C

Shifted Nikulin and Kummer lattices in terms of a shifted D_4 lattice

Let $\Lambda = \oplus_{i=1}^{n} \mathbb{Z}\langle e_i \rangle$ be the trivial lattice with inner-product $e_i \cdot e_j = \delta_{ij}$. We define the D_n lattice by

$$D_n = \big\{ v \in \Lambda \;\big|\; l(v) \equiv 0 \bmod 2 \big\},$$

recalling the definition of $l(v)$ in (3.8). It is straightforward to show that the dual lattice is given by

$$D_n^\vee = \Lambda \cup \left(\Lambda + \tfrac{1}{2} \sum_{i=1}^{n} e_i \right).$$

In the case of $n = 8$, the following observation about the Nikulin lattice N follows immediately from Lemma 71:

Proposition 94. $N \cong D_8^\vee(-2)$.

Let λ be any root in the D_4 root system. Define the shifted D_4 lattice $D_{4,\mathsf{sh}} = D_4 + \frac{1}{2}\lambda$ along with the corresponding Jacobi theta function

$$\Theta_{D_{4,\mathsf{sh}}}(q, w) = \sum_{v \in D_{4,\mathsf{sh}}} q^{v^2} w^{l(v)}$$

which we can explicitly evaluate:

Proposition 95.

$$\Theta_{D_{4,\mathsf{sh}}}(q^2, w) = (q\,\psi_w)^{\frac{1}{2}} \cdot \prod_{n=1}^{\infty} (1+q^n)^6 (1-q^n)^4 (1+wq^n)(1+w^{-1}q^n)$$

We note that this formula also essentially appeared in [**37**, Ch. 2, Prop. 3].

The above proposition explains an interesting connection we observed between the shifted Nikulin lattice N_{sh} and shifted Kummer lattice K_{sh}:

Corollary 96.

$$\Theta_{N_{\mathsf{sh}}}(q^2, w) = \Theta_{D_{4,\mathsf{sh}}}(q^2, w)^2$$
$$\Theta_{K_{\mathsf{sh}}}(q^2, w) = 4\,\Theta_{D_{4,\mathsf{sh}}}(q^2, w)^4$$

Proof. Equations (3.13) and (3.14) reduce the above claims to Proposition 95. $\square$

Proof of Proposition 95. Without loss of generality, suppose the root of D_4 we specify is $\lambda = e_1 + e_2$. We define the lattice isomorphism $f : D_4^\vee(2) \to D_4$ by

$$f(e_1) = \lambda \qquad f(e_3) = e_3 + e_4$$
$$f(e_2) = e_1 - e_2 \qquad f(e_4) = e_3 - e_4$$

and observe that f identifies the two shifted lattices $D_{4,\mathsf{sh}}$ and $D_{4,\mathsf{sh}}^\vee = D_4^\vee(2) + \frac{1}{2}e_1$. Therefore defining the theta function

$$\Theta_{D_{4,\mathsf{sh}}^\vee}(q, w) = \sum_{v \in D_{4,\mathsf{sh}}^\vee} q^{\frac{1}{2}v^2} w^{l(v)}$$

we have the formula

$$\Theta_{D_{4,\mathsf{sh}}}(q, w) = \Theta_{D_{4,\mathsf{sh}}^\vee}(q, w)$$

By the same methods as in Subsection 3.4.4, the description

$$D_{4,\mathsf{sh}}^\vee = \left(\Lambda + \tfrac{1}{2}e_1\right) \cup \left(\Lambda + \tfrac{1}{2}(e_2 + e_3 + e_4)\right)$$

allows us to compute the theta function explicitly to be

$$\Theta_{D_{4,\mathsf{sh}}}(q^2, w) = \theta_0^3\theta_1 + \theta_0\theta_1^3 = \theta_0\theta_1(\theta_0^2 + \theta_1^2).$$

By (3.15) and Lemma 79, this completes the proof. $\square$

Remark 97. Alternatively, one can give formulas for N_{sh} and K_{sh} in terms of shifts of the D_4 lattice, from which Corollary 96 follows. Let $\lambda_k = f(e_k)$ denote the four roots from the proof of Proposition 95 and define the shifted lattices $D_{4,k} = D_4 + \frac{1}{2}\lambda_k$. One can show that via the isomorphism f

$$N_{\mathsf{sh}} \cong D_{4,1}(-1) \oplus D'_{4,1}(-1)$$
$$K_{\mathsf{sh}} \cong \coprod_{k=1}^{4} D_{4,k}(-1) \oplus D'_{4,k}(-1) \oplus D''_{4,k}(-1) \oplus D'''_{4,k}(-1)$$

where the primes signify that the different D_4 summands have different generators, e.g. $D_4' \subset \oplus_i \mathbb{Z}\langle e_i' \rangle$ and $D_4'' \subset \oplus_i \mathbb{Z}\langle e_i'' \rangle$, etc. In terms of the two-variable theta functions

$$\Theta_{D_{4,k}}(q,w) = \Theta_{D_{4,\mathsf{sh}}}(q,w)$$

for all $1 \leq k \leq 4$. One can then show that the above lattice formulas imply Corollary 96.