Parahoric Ext-algebras of a p-adic special linear group

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Abstract

Let F be a finite extension of $\mathbb{Q}_p$, k the algebraic closure of $\mathbb{F}_{p^r}$ for some r, and $G = \mathrm{SL}_2(F)$. In the context of the mod-p Local Langlands correspondence, it is natural to study the pro-p Iwahori Hecke algebra $H := \mathrm{End}_{k[G]}(k[G/I], k[G/I])$ attached to the pro-p Iwahori subgroup $I \leq G$. One reason is that when $F = \mathbb{Q}_p$, there is an equivalence between the category of H-modules and the category of smooth k-representations of G generated by their I-fixed vectors. Unfortunately, this equivalence fails when F is a proper extension of $\mathbb{Q}_p$.

We overcome this obstacle somewhat by passing to the derived setting. When F is a proper extension and I is a torsion free group, it was shown by Schneider that we can obtain an equivalence between the derived category of smooth representations and a certain derived category associated to H. Relatively little is known about this equivalence. In understanding more, we can consider the cohomology algebra $E^* := \mathrm{Ext}^*(k[G/I], k[G/I])$. The goal of this thesis is to study the related algebra $E^*_{\mathcal{P}} := \mathrm{Ext}^*_{\mathrm{Rep}_k(G)}(k[G/\mathcal{P}], k[G/\mathcal{P}])$ when $\mathcal{P}$ is taken to be either the Iwahori subgroup J of G or the maximal compact subgroup $K = \mathrm{SL}_2(\mathbb{Z}_p)$, both of which contain I. We are able to give explicit descriptions of these algebras, including the full product. Surprisingly, we deduce that E^*_K is commutative (not graded commutative) and that it is isomorphic to the centre of E^*_J.

Lay Summary

A square is a highly symmetric object; it has both rotational and reflectional symmetry. In mathematics, the set of symmetries of an object is called a *group*. Given a group, can we represent the symmetries as matrices? If so, how? These are the basic questions which underlie the branch of mathematics known as representation theory.

The mod-p Local Langlands conjecture is an important unsolved problem which connects representation theory to other areas of mathematics. It posits that there is a correspondence between certain mod-p representations of the group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$ and certain mod-p representations of the group $\mathrm{GL}_n(F)$. Recently it has become clear that there is a family of algebraic objects, called the parahoric Hecke Ext-algebras, which may be useful in studying this conjecture. The goal of this thesis is to write down explicit descriptions of two distinguished parahoric Hecke Ext-algebras.

Preface

This thesis is comprised of three chapters. Chapter 1 gives a very brief historical account of the mod-p Local Langlands program, and explains how the work presented here fits into the bigger picture. Chapter 2 mostly consists of background material. Its purpose is twofold: first, to ensure that the thesis is somewhat self-contained, and second, to define the notation which we shall often refer to in Chapter 3. Much of this chapter is standard material and was drawn from [BH06], [Her08], [Her13], [Hum90], [Mil20], and [Ser73], though the exposition is original. Finally, in Chapter 3 we present the original work of the author. These results build on previous work of Ollivier and Schneider (mainly [OS19] and [OS21]) as well as work of Ollivier currently in preparation. We plan to submit a version of Chapter 3 for publication as a part of [OS].

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Chapter 1

Introduction and background

Let $\mathbb{Q}_p$ be the field of p-adic numbers, where p is a prime number, and let $\mathbb{F}_p$ be the field with p elements. Denote by $\overline{\mathbb{Q}}_p$ and $\overline{\mathbb{F}}_p$ their respective algebraic closures. We fix F to be a finite extension of $\mathbb{Q}_p$. Then F admits a discrete valuation extending the one on $\mathbb{Q}_p$; in particular, the residue field of F is a finite extension of $\mathbb{F}_p$. Roughly speaking, the goal of the classical *local Langlands correspondence* for $G = \mathrm{GL}_n(F)$ is to establish some sort of "natural" correspondence

$$\left\{\begin{array}{c}\text{certain } n\text{-dimensional representations}\\ \text{of } \mathrm{Gal}(\overline{\mathbb{Q}}_p/F) \text{ over } \overline{\mathbb{Q}}_\ell\text{-vector spaces}\\ \text{(up to isomorphism)}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{certain representations}\\ \text{of } \mathrm{GL}_n(F) \text{ over } \overline{\mathbb{Q}}_\ell\text{-vector}\\ \text{spaces (up to isomorphism)}\end{array}\right\}$$

where ℓ is a fixed prime. Part of the challenge is that we do not know *a priori* exactly what form the correspondence should take. For example, which representations do we actually want to consider on both sides? And what does "natural" mean in this context?

The case $\ell \neq p$ was essentially solved in the early 2000's through the work of Henniart [Hen00] and Harris and Taylor [HT01], who describe a correspondence between isomorphism classes of certain n-dimensional continuous representations of $\mathrm{Gal}(\mathbb{Q}_p/F)$ over $\overline{\mathbb{Q}}_\ell$-vector spaces and isomorphism classes of certain smooth and irreducible representations of $\mathrm{GL}_n(F)$ over $\overline{\mathbb{Q}}_\ell$-vector spaces. Here, smooth (sometimes called locally constant) representations of $G = \mathrm{GL}_n(F)$ are representations for which the set $\mathrm{Stab}_G(v) := \{g \in G \mid g \cdot v = v\}$ is open in G for all v. Their correspondence is also compatible with "reduction modulo ℓ", namely, we retain a correspondence after reducing to representations over $\overline{\mathbb{F}}_\ell$-vector spaces (see [Vig01]).

In contrast, when $\ell = p$ the situation is much more complicated and no analogous correspondence is fully understood in general beyond the case of $G = \mathrm{GL}_2(\mathbb{Q}_p)$. The $\ell = p$ case is often called the *p-adic* local Langlands correspondence and, just like in the $\ell \neq p$ case, we would also like to find a correspondence which is compatible with reduction modulo p, this latter correspondence being called the *mod-p* local Langlands correspondence. The mod-p version is largely what we concern ourselves with here; specifically, we would like a

correspondence of the form

$$\left\{\begin{array}{c}\text{certain } n\text{-dimensional } \overline{\mathbb{F}}_p\text{-}\\ \text{representations of } \mathrm{Gal}(\overline{\mathbb{Q}}_p/F)\\ \text{(up to isomorphism)}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{certain } \overline{\mathbb{F}}_p\text{-representations of}\\ \mathrm{GL}_n(F) \text{ (up to isomorphism)}\end{array}\right\}.$$

Generally speaking, classifying the objects on the left-hand side is "easy", and all of the hard work must be done on the right-hand side. In 1995, Barthel and Livné [BL95] made significant progress in the case where $n = 2$ and $F = \mathbb{Q}_p$ by classifying the smooth irreducible representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ over $\overline{\mathbb{F}}_p$ (really, over any algebraically closed field of characteristic p) arising from parabolic induction. In passing, they also come across an exotic family of irreducible smooth representations called "supersingular" for which they were unable to say much about. In 2003, Breuil [Bre03] finished off the classification of smooth irreducible representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ by describing explicitly what these aforementioned supersingular representations look like. Moreover Breuil was able to establish a mod-p correspondence between the supersingular $\overline{\mathbb{F}}_p$-representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ and 2-dimensional irreducible $\overline{\mathbb{F}}_p$-representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$. Finally, in 2013, Colmez, Dospinescu, and Paskunas [CDP14] determine the p-adic version of the correspondence, essentially finishing off the case $F = \mathbb{Q}_p$ and $n = 2$. Remarkably, the p-adic and mod-p correspondences for $\mathrm{GL}_2(\mathbb{Q}_p)$ are induced by a functor.

The cases $F \neq \mathbb{Q}_p$ or $n \geq 3$ are more difficult and remain largely a mystery. For example, the irreducible smooth representations of $\mathrm{GL}_n(F)$ have not yet been classified. In the next section we discuss tools valid for any p-adic reductive group G over a p-adic field F.

Hecke algebras Let G be the F-rational points of a p-adic reductive group over a p-adic field F. We fix a pro-p-Iwahori subgroup I of G contained in a maximal compact subgroup K. Denote by $\mathrm{Rep}_k(G)$ the abelian category of smooth representations of G over a fixed algebraically closed field k of characteristic p. Part of the difficulty in understanding $\mathrm{Rep}_k(G)$ for $F \neq \mathbb{Q}_p$ comes from the fact that there are more supersingular representations than when $F = \mathbb{Q}_p$ (in the sense that they outnumber the 2-dimensional irreducible representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$, see [BP12]), and they have not been classified as of now. One approach is to instead study the category $\mathrm{Mod}(H)$ of modules over the pro-p Iwahori Hecke algebra $H = \mathrm{End}_{k[G]}(k[G/I], k[G/I])$ attached to I. The reason for doing so is that there exists a left exact functor $\mathfrak{h}\colon \mathrm{Rep}_k(G) \to \mathrm{Mod}(H)$, which, when $F = \mathbb{Q}_p$, becomes an equivalence of categories between $\mathrm{Mod}(H)$ and $\mathrm{Mod}^I(G)$ ([Oll09], [Koz16]). (Here, $\mathrm{Mod}^I(G)$ is the subcategory of all smooth representations of G generated by their I-fixed vectors.) Even when the functor is not an equivalence of categories, it appears that the category $\mathrm{Mod}(H)$ encodes a lot of information about $\mathrm{Rep}_k(G)$ and may play an important role in the local Langlands correspondence: for example, when $G = \mathrm{GL}_n(F)$ there is a numerical correspondence between the set of n-dimensional irreducible supersingular H-modules and the set of n-dimensional irreducible Galois representations ([Oll10]).

The Spherical Hecke algebra H_K is also of particular importance in the representation theory of p-adic reductive groups. It was used by Barthel and Livné in their classification of irreducible smooth representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. Their idea was that any such representation in characteristic p is a quotient of the compact induction $\text{c-ind}_{KZ}^{G}(\sigma)$ (where σ is an irreducible representation of K and Z is the centre of G) by a Hecke operator T. This was eventually generalized by Herzig in [Her11]. It should also be pointed out that both the Iwahori and Spherical Hecke algebras are natural objects considered in geometric representation theory, so their study may enlighten a geometric perspective of the mod-p local Langlands correspondence.

Hecke Ext-algebras The functor $\mathfrak{h}$ introduced previously fails to induce an equivalence of categories in general. Since $\mathfrak{h} = \mathrm{Hom}_G(\mathbf{X}, -)$, we see that $\mathfrak{h}$ is left exact and its left adjoint is

$$\begin{aligned} \mathfrak{t}\colon \mathrm{Mod}(H) &\longrightarrow \mathrm{Mod}^I(G) \\ M &\longmapsto \mathbf{X} \otimes_H M, \end{aligned}$$

where $\mathbf{X}$ denotes the compact induction of the trivial character of I. In general, $\mathfrak{h}$ will not be right exact, so it makes sense to study its right derived functors. Recently, Schneider [Sch15] justified this approach by showing that, when I is a torsionfree pro-p group (so, for example, for I the pro-p Iwahori subgroup of $\mathrm{SL}_2(\mathbb{Q}_p)$ and $p \geq 5$), there exist derived versions of $\mathfrak{h}$ and $\mathfrak{t}$ which give an equivalence between the derived category of smooth representations of G and the derived category of differential graded modules over a certain differential graded pro-p Iwahori-Hecke algebra $H^\bullet$.

Unfortunately, not much is known about the derived categories in question, and understanding them is an active area of research. In [OS19], the graded cohomology algebra

$$E^* := \mathrm{Ext}^*_{\mathrm{Rep}_k(G)}(\mathbf{X}, \mathbf{X})$$

of $H^\bullet$, dubbed the pro-p Iwahori Hecke Ext-algebra, is studied and the first results about its structure emerged. Its product is described explicitly and it is deduced that E^* is only supported in degrees $0 \leq i \leq d$ when I is an Iwahori pro-p subgroup with dimension d as a p-adic Lie group (really, for I any Poincaré group of dimension d). When $F = \mathbb{Q}_p$, for example, one has $d = 3$. Moreover, a certain duality is present between the E^i and E^{d-i} graded pieces whereby E^{d-i} can be realized as the finite vector space dual of E^i.

Let $\mathcal{P}$ be a parahoric subgroup of $G = \mathrm{SL}_2(\mathbb{Q}_p)$ containing the Iwahori subgroup J. The main goal of this thesis is to study the parahoric Hecke Ext-algebra

$$E^*_{\mathcal{P}} := \mathrm{Ext}^*_{\mathrm{Rep}_k(G)}(\mathbf{X}_{\mathcal{P}}, \mathbf{X}_{\mathcal{P}})$$

where $\mathbf{X}_{\mathcal{P}}$ is the compact induction of the trivial character of $\mathcal{P}$.

The structure of this thesis

Chapter 2 is expository in nature. In particular, we introduce the field of p-adic numbers $\mathbb{Q}_p$, discuss smooth representations of locally profinite groups, and define some of the language used to describe the modern theory. It should be noted that our treatment of locally profinite groups includes a heavy emphasis on the examples $\mathrm{GL}_n(\mathbb{Q}_p)$ and $\mathrm{SL}_n(\mathbb{Q}_p)$ even though things can often be stated in greater generality; we attempt to point out some of these generalizations when appropriate. Finally, Chapter 3 contains the main new results. We are able to describe fully the Iwahori and Spherical Hecke Ext-algebras E_J^* and E_K^* in the case $G = \mathrm{SL}_2(\mathbb{Q}_p)$, $p \geq 5$, when k is algebraically closed of characteristic p. We find many familiar properties inherited from the pro-p Iwahori Hecke Ext-algebra; for example, E_K^* and E_J^* are supported in d degrees, there is a duality, and so on. At the same time, many intriguing properties are discovered: one of the main results we present is that E_K^* is commutative and moreover we deduce that it is isomorphic to the centre of E_J^*.

Further questions

As work progresses, we are also met with many new and exciting questions. It is believed that several results obtained here can be extended much beyond the case $G = \mathrm{SL}_2(\mathbb{Q}_p)$. It should be pointed out that in characteristic different from p, Venkatesh [Ven19] introduced the Spherical Hecke Ext-algebra in the context of cohomology of arithmetic groups. Relatively little is yet known about the Spherical algebra in our context, although there is certainly some promise. For instance Ronchetti [Ron19] recently described a derived version of the Satake isomorphism. Furthermore, we suspect that understanding the Spherical Hecke Ext-algebra will allow us to extend the notion of supersingularity to $\mathrm{Mod}(E_I^*)$, and perhaps even to the derived category of differential graded modules over $H^\bullet$. An eventual goal is to have a concrete understanding of Schneider's equivalence in the derived setting and its role in the mod-p Langlands correspondence.

Chapter 2

Smooth representations

2.1 Representations of finite groups

We begin with the basics. Let G be a group, k a field, and V a finite-dimensional vector space over k. Denote by $\mathrm{GL}(V)$ the group of linear automorphisms of V. For now, we impose no restrictions (e.g. finiteness) on G.

Definition (Group representation)**.** A *k-representation* of G is a pair (ρ, V) where V is a k-vector space and ρ is a group homomorphism $\rho\colon G \to \mathrm{GL}(V)$.

We will often omit the k in k-representation when the coefficient field is clear from context. A morphism of k-representations $T\colon (\rho, V) \to (\tau, W)$ is a linear map $V \to W$ such that the following diagram commutes for all $g \in G$:

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ \downarrow{\scriptstyle \rho(g)} & & \downarrow{\scriptstyle \tau(g)} \\ V & \xrightarrow{T} & W \end{array} \tag{2.1}$$

Such a map T is also often called a *G-equivariant* map. In any case, we can now make sense of the category of all k-representations of G, which we denote $\mathrm{Rep}_k(G)$. It is an abelian category.

On the other hand, let $k[G]$ be the group algebra of G over k. As a vector space, it is the set of finite formal k-linear combinations of elements of G. Its product is induced by the product in the group; in other words, we set $g \cdot h = gh$ and extend linearly. Letting $\mathrm{Mod}(k[G])$ denote the category of (left) $k[G]$-modules, we have a functor

$$\begin{aligned} \mathrm{Rep}_k(G) &\longrightarrow \mathrm{Mod}(k[G]) \\ (\rho, V) &\longmapsto V, \end{aligned} \tag{2.2}$$

where we define the $k[G]$-module structure on V via $g \cdot v := \rho(g)(v)$, extending linearly. Conversely, given a $k[G]$-module M, we obtain a k-representation of (τ, M) by defining $\tau(g)(m) := g \cdot m$. One surmises that the functor (2.2) is an equivalence of categories (it is, though what we have here is even a little stronger). With this in mind, we will often identify the representation (ρ, V) with its underlying vector space V and write $g \cdot v$ for $\rho(g)(v)$. A *subrepresentation* of V is then simply a submodule of the $k[G]$-module V.

Remark 1. Given a group G, we always have the representation of G defined by viewing $k[G]$ as a module over itself. This is called the (left) *regular representation*. The field k can also be made into a representation by defining $g \cdot \alpha = \alpha$ for all $\alpha \in k$; this is the *trivial representation* (or *trivial character*) and we denote it 1_G.

For the remainder of this section, we suppose that G is finite. Given a subgroup H of a group G, there is a functor turning representations of H into representations of G.

Definition (Induced representation)**.** Let G be a finite group, H a subgroup of G, and V a representation of H. Then

$$k[G] \otimes_{k[H]} V$$

is a representation of G, with $g \cdot (h \otimes v) = gh \otimes v$. It is called the *induced representation* and is denoted $\mathrm{ind}_H^G(V)$.

Remark 2. Note that one can still make sense of the above definition even if G is not finite. In the context of smooth representations, for example, this functor is called *compact induction*. Refer to §2.3.

If H is a subgroup of G and V is a representation of G, then denote by $V|_H$ the representation of H obtained by restricting V to H.

Proposition 3 (Frobenius reciprocity for finite groups)**.** *Let G be a finite group, H a subgroup of G, and V a representation of H and W a representation of G. We have*

$$\mathrm{Hom}_{k[G]}(\mathrm{ind}_H^G V, W) \cong \mathrm{Hom}_{k[H]}(V, W|_H) \tag{2.3}$$

as vector spaces.

Proof. Consider the map

$$\begin{aligned} T\colon \mathrm{Hom}_{k[G]}(\mathrm{ind}_H^G V, W) &\longrightarrow \mathrm{Hom}_{k[H]}(V, W|_H) \\ \varphi &\longmapsto (v \mapsto \varphi(1 \otimes v)). \end{aligned} \tag{2.4}$$

First, note that the map $v \mapsto \varphi(1 \otimes v)$ is H-equivariant since $\varphi(1 \otimes \alpha h \cdot v) = \varphi(\alpha h \otimes v) = \alpha h \cdot \varphi(1 \otimes v)$ for any $h \in H$ and $\alpha \in k$. Therefore T is well-defined. Linearity of T is clear. If $\varphi \in \ker(T)$, then $\varphi(1 \otimes v) = 0$ for all $v \in V$ which implies that $\varphi(\alpha g \otimes v) = 0$ for all $\alpha \in k$ and $g \in G$ since φ is G-equivariant. So T is injective. For surjectivity, if $\psi \in \mathrm{Hom}_{k[H]}(V, W|_H)$ then $T(\varphi) = \psi$ where φ is defined by $\varphi(\alpha g \otimes v) = \alpha g \cdot \psi(v)$. It is straightforward to verify that this φ is well-defined. □

Remark 4. It is not so hard to check that in fact the isomorphism (2.4) is functorial in both V and W. In other words, $\mathrm{ind}_H^G(-)$ is left adjoint to $(-)_H$. Alternatively, notice that for V a representation of G there is a natural isomorphism of $k[H]$-modules $V_H \cong \mathrm{Hom}_{k[G]}(k[G], V)$ sending $v \in V_H$ to the map $1 \mapsto v$. (Here, $k[G]$ is viewed as a $(k[G], k[H])$-bimodule.) Then, by the hom-tensor adjunction, one recovers the content of Frobenius reciprocity.

Remark 5. More generally, let R and S be arbitrary rings and let $f\colon R \to S$ be a morphism of rings. There is a functor $\mathrm{Hom}_R(S, -)\colon \mathrm{Mod}(R) \to \mathrm{Mod}(S)$, sometimes called *coinduction*. It is the right adjoint of the restriction functor $(-)_R\colon \mathrm{Mod}(S) \to \mathrm{Mod}(R)$ which takes an S-module M and defines $r \cdot m := f(r) \cdot m$ for $r \in R$. Taking f to be the inclusion $k[H] \hookrightarrow k[G]$, where $H \leq G$, we have the adjunctions

$$\mathrm{ind}_H^G(-) \;\dashv\; (-)_H \;\dashv\; \mathrm{coind}_H^G(-) = \mathrm{Hom}_{k[H]}(k[G], -).$$

When G is finite, the coinduction and induction functors coincide, so that induction is both left and right adjoint to restriction. This is not true in general.

2.2 The field $\mathbb{Q}_p$

In this section we introduce the field of p-adic numbers $\mathbb{Q}_p$. We first briefly recall the notion of an inverse limit of a sequence rings. Given a sequence of rings $(A_n)_{n=1}^{\infty}$ together with ring homomorphisms

$$\cdots \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} A_1$$

for all n, we define the *inverse limit* of $(A_n, f_n)_{n\in\mathbb{N}}$ to be

$$\varprojlim (A_n, f_n) := \left\{ a = (a_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} A_n \mid f_n(a_n) = a_{n-1} \text{ for all } n \geq 2 \right\}.$$

The inverse limit $A = \lim_{\leftarrow} (A_n, f_n)$ is in particular a subring of $\prod_{n=1}^{\infty} A_n$, with addition and multiplication defined component-wise. It comes with the map $\varepsilon_n\colon A \to A_n$ which extracts the nth component of $a \in A$. We will often abbreviate $\epsilon_n(a)$ by a_n.

We now let $A_n = \mathbb{Z}/p^n\mathbb{Z}$ for a fixed prime p. For all $n \geq 1$, there is a natural (surjective) map $\pi_n\colon \mathbb{Z}/p^{n+1}\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$.

Definition (The p-adic integers)**.** Let p be a prime. The ring of p-adic integers $\mathbb{Z}_p$ is the inverse limit of $(\mathbb{Z}/p^n\mathbb{Z}, \pi_n)$.

Remark 6. There is a natural inclusion $\mathbb{Z} \hookrightarrow \mathbb{Z}_p$. Its image is the set of all eventually constant sequences.

Proposition 7. *We have $\mathbb{Z}_p^{\times} = \mathbb{Z}_p \setminus p\mathbb{Z}_p$, i.e. $x \in \mathbb{Z}_p$ is a unit if and only if it is not divisible by p.*

Proof. It is clear that $\mathbb{Z}_p^\times \subset \mathbb{Z}_p \setminus p\mathbb{Z}_p$. Conversely, suppose $x \notin p\mathbb{Z}_p$. Then x_n is invertible in $\mathbb{Z}/p^n\mathbb{Z}$ for any n. Hence let $y_n \in \mathbb{Z}/p^n\mathbb{Z}$ be such that $x_n y_n = 1$. It remains only to show that the sequence $(\ldots, y_2, y_1)$ lies in $\mathbb{Z}_p$, namely, that $\pi_n(y_n) = y_{n-1}$ for all n. Indeed, since $x_n y_n = 1$ we have $\pi_n(x_n y_n) = 1$ in $\mathbb{Z}/p^{n-1}\mathbb{Z}$ i.e. $x_{n-1}\pi_n(y_n) = 1$. Since inverses are unique, $\pi_n(y_n) = y_{n-1}$, as desired. □

Proposition 8. *We have $\mathbb{Z}_p/p^n\mathbb{Z}_p \cong \mathbb{Z}/p^n\mathbb{Z}$ for $n \geq 0$.*

Proof. Simply note that the map $\varepsilon_n \colon \mathbb{Z}_p \to \mathbb{Z}/p^n\mathbb{Z}$ has kernel $p^n\mathbb{Z}_p$. □

In $\mathbb{Z}_p$ we have the infinite chain of ideals

$$\mathbb{Z}_p \supset p\mathbb{Z}_p \supset p^2\mathbb{Z}_p \supset p^3\mathbb{Z} \supset \cdots \tag{2.5}$$

with $\bigcap_{n\geq 0} p^n\mathbb{Z}_p = \{0\}$. For any nonzero $x \in \mathbb{Z}_p$, there exists an integer N such that $x \in p^N\mathbb{Z}_p$ but $x \notin p^k\mathbb{Z}_p$ for all $k > N$. This integer N is called the *p-adic valuation* of x. More precisely, we define the p-adic valuation $\nu_p \colon \mathbb{Z}_p \to \mathbb{Z}$ to be

$$\nu_p(x) := \begin{cases} \max\{n \in \mathbb{N} : p^n \mid x\}, & \text{if } x \neq 0 \\ \infty, & \text{otherwise.} \end{cases}$$

The p-adic valuation determines an absolute value $|\cdot|_p$ on $\mathbb{Z}_p$ by setting

$$|x|_p = p^{-\nu_p(x)} \tag{2.6}$$

and $|0|_p = 0$. This, in turn, defines a topology on $\mathbb{Z}_p$, where two elements of $\mathbb{Z}_p$ are considered close if their difference is divisible by a large power of p. The sets $p^n\mathbb{Z}_p$ for $n \in \mathbb{N}$ are therefore open balls centred at 0; we see that the sequence (2.5) forms a neighbourhood basis of 0, meaning that any neighbourhood of 0 contains $p^n\mathbb{Z}_p$ for some n. In fact, any point $x \in \mathbb{Z}_p$ has the neighbourhood basis $\{x + p^n\mathbb{Z}\}_{n\in\mathbb{N}}$.

We also remark that the copy of $\mathbb{Z}$ lying in $\mathbb{Z}_p$ from Remark 6 is dense, and that $\mathbb{Z}_p$ is the completion of $\mathbb{Z}$ with respect to the p-adic absolute value.

Proposition 9. *The p-adic valuation satisfies the following properties: for all $x, y \in \mathbb{Z}_p$,*

(i) $\nu_p(xy) = \nu_p(x) + \nu_p(y)$*; and*

(ii) $\nu_p(x + y) \geq \min(\nu_p(x), \nu_p(y))$.

Proof. This follows immediately by the definition of ν_p. □

Proposition 10. *$\mathbb{Z}_p$ is an integral domain.*

Proof. Suppose $xy = 0$ in $\mathbb{Z}_p$. By property (i) above, $\infty = \nu_p(x) + \nu_p(y)$ so that either $x = 0$ or $y = 0$. □

Definition (The p-adic numbers)**.** The field of p-adic numbers $\mathbb{Q}_p$ is defined to be the field of fractions of $\mathbb{Z}_p$.

One extends ν_p to $\mathbb{Q}_p$ by defining $\nu_p(x/y) = \nu_p(x) - \nu_p(y)$ for all $x \in \mathbb{Z}_p$ and $y \in \mathbb{Z}_p \setminus \{0\}$. This defines a topology on $\mathbb{Q}_p$ via the p-adic absolute value $|\cdot|_p$ defined in (2.6). The set $p^n\mathbb{Z}_p$ corresponds to the closed ball $\{x \in \mathbb{Q}_p : |x|_p \leq p^n\}$ which is also open $\mathbb{Q}_p$.

Proposition 11. *$\mathbb{Q}_p$ with $|\cdot|_p$ is locally compact.*

Proof. Every point $x \in \mathbb{Q}_p$ has the open neighbourhood $x + \mathbb{Z}_p$. It remains only to show that $x + \mathbb{Z}_p$ is compact; though it suffices to only show that $\mathbb{Z}_p$ is compact as translation preserves compactness. Since $\mathbb{Z}_p$ is complete, this amounts to showing that $\mathbb{Z}_p$ is totally bounded.

To that end, let $\epsilon > 0$. Choose n so that $p^n\mathbb{Z}_p$ is a ball of radius less than ϵ. For any $x \in \mathbb{Z}_p$, the ball $x + p^n\mathbb{Z}_p$ is also of radius less than ϵ. The set of open balls $\{x + p^n\mathbb{Z}_p : x \in \mathbb{Z}_p\}$ evidently covers $\mathbb{Z}_p$; moreover, it is finite by Proposition 8. This shows that $\mathbb{Z}_p$ is totally bounded, as desired. □

A brief note on local fields The field $\mathbb{Q}_p$, as well as its finite extensions, are important examples of what are called *local fields*. A local field is a topological field[1] F with a non-trivial absolute value $|\cdot|$ such that F is locally compact in the induced topology. Here we have shown that $\mathbb{Q}_p$ is a local field with respect to the p-adic absolute value $|\cdot|_p$. Furthermore, $\mathbb{Q}_p$ is *non-Archimedean* as a local field. This means that $|\cdot|_p$ satisfies a stronger version of the triangle inequality: namely, $|x + y|_p \leq \max(|x|_p, |y|_p)$ for all $x, y \in \mathbb{Q}_p$. This fact follows from Proposition 9 (ii).

For any non-Archimedean local field F, its ring of integers $\mathcal{O}$ is a local ring with unique maximal ideal $\mathfrak{m}$ and finite residue field $\mathcal{O}/\mathfrak{m}$. In the case of $\mathbb{Q}_p$ we can see this directly. Its ring of integers, $\mathbb{Z}_p$, is a local ring with maximal ideal $p\mathbb{Z}_p$ (c.f. Proposition 7). Then $\mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{Z}/p\mathbb{Z}$ by Proposition 8.

If $\pi \in \mathcal{O}$ generates the maximal ideal $\mathfrak{m}$, then π is called a *uniformizing parameter*, or simply a *uniformizer*, of $\mathcal{O}$. With $\mathbb{Q}_p$ we will almost always take $\pi = p$.

As an aside, it is of note that there are only three classes of local fields: the non-Archimedean local fields of characteristic zero ($\mathbb{Q}_p$ and its finite extensions), the non-Archimedean local fields of positive characteristic (the Formal Laurent series $\mathbb{F}_q((T))$ with q a prime power), and the Archimedean local fields $\mathbb{R}$ and $\mathbb{C}$.

[1] A topological field is a field in which the field operations (addition, multiplication, inversion) are continuous.

2.3 Smooth representations of $\mathrm{GL}_n(F)$

Definition (Topological group)**.** A *topological group* is a group G together with a topology on G such that the maps

$$\begin{aligned} G \times G &\longrightarrow G & \qquad G &\longrightarrow G \\ (g,h) &\longmapsto gh, & \qquad g &\longmapsto g^{-1} \end{aligned}$$

are continuous. (Here, $G \times G$ is given the product topology.)

Remark 12. Continuity of the maps $(g,h) \mapsto gh$ and $g \mapsto g^{-1}$ is equivalent to the continuity of the map $(g,h) \mapsto gh^{-1}$.

The groups $(\mathbb{C}^\times, \cdot)$ and $(\mathbb{C}^n, +)$ are both familiar examples of topological groups (with respect to the standard topology on $\mathbb{C}$). Note that any ordinary group can be made into a topological group by giving it the discrete topology. Our main example of a topological group, however, is $\mathrm{GL}_n(\mathbb{Q}_p)$. We have already seen that $(\mathbb{Q}_p, +)$ and $(\mathbb{Z}_p, +)$ are topological groups with the topology induced by the p-adic valuation. If $G = \mathrm{GL}_n(\mathbb{Q}_p)$ or $\mathrm{SL}_n(\mathbb{Q}_p)$, then we give G the topology induced by its inclusion in $\mathbb{Q}_p^{n^2}$. Since inversion in G is a continuous operation, G is a topological group.

One of the goals of this section is to lay the groundwork needed in order to study the so-called *smooth* representations of $G = \mathrm{GL}_n(\mathbb{Q}_p)$ or $\mathrm{SL}_n(\mathbb{Q}_p)$.

Definition. A representation V of G is *smooth* if

$$V = \bigcup_{\substack{H \leq G \\ H \text{ open}}} V^H$$

where $V^H := \{v \in V : h \cdot v = v \text{ for all } h \in H\}$ is the subspace of H-fixed vectors.

One checks that if $T\colon V_1 \to V_2$ is a G-equivariant map between smooth representations (i.e. it is linear and satisfies (2.1)), then its kernel and image will again be smooth. The class of smooth representations together with G-equivariant maps forms an abelian category. It is a full subcategory of $\mathrm{Rep}_k(G)$. By abuse of notation, we refer to the category of smooth representations by $\mathrm{Rep}_k(G)$, replacing our earlier notation. All representations are assumed to be smooth from this point forward.

Lemma 13. *If H is an open subgroup of a topological group G, then so are the cosets gH and Hg for all $g \in G$.*

Proof. Let $g \in G$ and let φ_g be the continuous map $G \times G \to G$ defined by $\varphi_g(h) = gh$. Then $Hg = \varphi_{g^{-1}}^{-1}(H)$ is open. Similarly, gH is open. □

Proposition 14. *Let V be a representation of G. The following are equivalent:*

(i) V is smooth,

(ii) the map $\varphi\colon G \times V \to V$ given by $\varphi(g, v) = g \cdot v$ is continuous, where V is given the discrete topology,

(iii) the set $\mathrm{Stab}_G(v) := \{g \in G \mid g \cdot v = v\}$ is open in G for all v.

Proof.

(ii) $\implies$ (iii). Let $v \in V$. The set $\mathrm{Stab}_G(v)$ is the preimage of the open set $\{v\} \subset V$ under the restriction of $G \times V \to V$ to $G \times \{v\}$.

(iii) $\implies$ (i). If $v \in V$, then $v \in V^{\mathrm{Stab}_G(v)}$ and $\mathrm{Stab}_G(v)$ is open.

(i) $\implies$ (ii). Suppose $W \subseteq V$ and $(g, v) \in \varphi^{-1}(W)$. By assumption, there exists an open subgroup H such that $h \cdot v = v$ for all $h \in H$. Then $(g, v) \in gH \times \{v\} \subset \varphi^{-1}(W)$. But $gH \times \{v\}$ is open by Lemma 13. Therefore, $\varphi^{-1}(W)$ contains a neighbourhood of each of its points, and so it is open. □

Definition. A *locally profinite group* is a topological group G in which every neighbourhood of the identity contains a compact open subgroup.

Proposition 15. *$(\mathbb{Q}_p, +)$, $(\mathbb{Q}_p^\times, \cdot)$ and $\mathrm{GL}_n(\mathbb{Q}_p)$ are all locally profinite groups.*

Proof. In fact we have already seen that $(\mathbb{Q}_p, +)$ and $(\mathbb{Q}_p^\times, \cdot)$ are locally profinite; the sets $\{p^n\mathbb{Z}_p\}_{n\in\mathbb{N}}$ and $\{1 + p^n\mathbb{Z}_p\}_{n\in\mathbb{N}}$, respectively, form neighbourhood bases of the identity consisting of open compact subgroups. For $\mathrm{GL}_n(\mathbb{Q}_p)$, we have the following (open) neighbourhoods of the identity:

$$\mathrm{GL}_n(\mathbb{Z}_p) \supset 1 + p\,\mathrm{M}_n(\mathbb{Z}_p) \supset 1 + p^2\,\mathrm{M}_n(\mathbb{Z}_p) \supset \cdots . \tag{2.7}$$

These are open balls in $\mathrm{GL}_n(\mathbb{Q}_p)$, so any open neighbourhood of the identity must contain one of them. It remains to show that $\mathrm{GL}_n(\mathbb{Z}_p)$ is compact. Note that we can realize $\mathrm{GL}_n(\mathbb{Z}_p)$ as the inverse image of $\mathbb{Z}_p^\times$ under the continuous map $\det\colon \mathrm{GL}_n(\mathbb{Q}_p) \to \mathbb{Q}_p^\times$. By Proposition 11 and its proof, $\mathbb{Z}_p$ is compact, hence $\mathrm{GL}_n(\mathbb{Z}_p)$ is compact. □

Remark 16. Proposition 15 remains true if we replace $\mathbb{Q}_p$ with a finite extension of $\mathbb{Q}_p$ or GL_n with SL_n. The proof is similar.

Induction

Let G be a locally profinite group, H a closed subgroup of G, and V a smooth representation of H. One needs to be careful when attempting to define the induced representation of V. In the case of finite groups, for instance, recall that we have

$$\mathrm{ind}_H^G(V) := \underbrace{k[G] \otimes_{k[H]} V}_{(*)} \cong \underbrace{\{f\colon G \to V \mid f(hg) = h \cdot f(g) \text{ for all } h \in H\}}_{(**)}$$

where the action of G on $(**)$ is given by $g \cdot f(x) = f(xg)$. This isomorphism is non-canonical and indeed fails in general for infinite groups; we have a strict inclusion of $(*)$ in $(**)$ whose image is functions which are of finite support modulo H. This gives us two different versions of induction.

Definition (Induced representation)**.** Let G be a locally profinite group and H a closed subgroup of G. For a smooth representation V of H, define

$$\mathrm{Ind}_H^G(V) := \{f \colon G \to V \mid f(hg) = h \cdot f(g) \text{ for all } h \in H,\, g \in G\}^\infty$$

where $(-)^\infty$ denotes the largest smooth subrepresentation of $(-)$.

The action of G on $(**)$ is usually not smooth, which is why we instead take $(**)^\infty$ above.

Definition (Compactly induced representation)**.** Let G be a locally profinite group and H a closed subgroup of G. For a smooth representation V of H, define

$$\text{c-ind}_H^G(V) := \{f \in \mathrm{Ind}_H^G(V) \mid \text{the image of } \mathrm{supp}(f) \text{ in } H\backslash G \text{ is compact}\}.$$

Remark 17. The fact that $H\backslash \mathrm{supp}(f)$ is compact in the definition of compact induction implies that $\text{c-ind}_H^G(V) \cong k[G] \otimes_{k[H]} V$, just like with finite groups.

Proposition 18 (Frobenius reciprocity for smooth representations)**.** *Let H be a closed subgroup of G, V a smooth representation of H, and W a smooth representation of G. Then*

$$\mathrm{Hom}_{k[G]}(\text{c-ind}_H^G(V), W) \cong \mathrm{Hom}_{k[H]}(V, W|_H), \quad \textit{and}$$
$$\mathrm{Hom}_{k[G]}(W, \mathrm{Ind}_H^G V) \cong \mathrm{Hom}_{k[H]}(W|_H, V)$$

as vector spaces. Moreover, this isomorphism is functorial in both V and W.

Proof. The proof is similar to the case of finite groups; we refer the reader to Proposition 3 and its proof or to [BH06]. □

2.4 Decompositions of $\mathrm{GL}_n(F)$

The purpose of this section is to give a brief overview on Weyl groups as they relate to GL_n. Few proofs are given; we refer the reader to [IM65], [Vig05], [Hum90]. As usual, the main example we will have in mind is $G = \mathrm{GL}_2(\mathbb{Q}_p)$ or $G = \mathrm{SL}_2(\mathbb{Q}_p)$.

We begin with the following definition.

Definition (Coxeter system)**.** A *Coxeter system* is a pair (W, S) where W is a group and $S = \{s_1, s_2, \ldots, s_n\}$ generates W subject only to relations of the form $(s_i s_j)^{m_{ij}} = 1$ where $m_{ii} = 1$ and $m_{ij} \geq 2$ for $i \neq j$. If there is no relation involving $s_i s_j$, we write $m_{ij} = \infty$. The group W is called a *Coxeter group*.

An immediate consequence of the definition is that each generator has order 2. Reflection groups (groups generated by reflections in Euclidean space) are an important class of examples of Coxeter groups, and Coxeter groups are in some sense a generalization of this concept. Another prototypical example is S_n, the symmetric group on n letters, generated by the set of transpositions $(i, i+1)$ for $1 \leq i \leq n-1$.

Given a Coxeter system (W, S), there is a natural notion of "length" on W, described as follows. Let $1 \neq w \in W$. Since S generates W and every element of S has order 2, we may decompose w as the product $w = s_{i_1} s_{i_2} \cdots s_{i_n}$ for some $s_{i_j} \in S$. Define the length of w, denoted $\ell(w)$, to be the smallest possible value of n in such a decomposition of w. We then define $\ell(1) = 0$. Immediately, we see that $\ell(w) = \ell(w^{-1})$ and $\ell(ww') \leq \ell(w) + \ell(w')$ for all $w, w' \in W$.

2.4.1 The Bruhat decomposition

Let k be a field and $G = \mathrm{GL}_n(k)$ or $\mathrm{SL}_n(k)$. Let B be the subgroup of GL_n or SL_n consisting of upper triangular matrices, i.e.

$$B := \begin{pmatrix} * & * & * & \cdots & * \\ & * & * & \cdots & * \\ & & \ddots & \ddots & \vdots \\ & 0 & & \ddots & * \\ & & & & * \end{pmatrix} \tag{2.8}$$

(with non-zero entries on the diagonal, and which multiply to 1 in the case of SL_n). We call B the (standard) *Borel* subgroup of G.

Proposition 19 (Bruhat decomposition for GL_n)**.** *Let k be any field. We have the disjoint union*

$$\mathrm{GL}_n(k) = \bigcup_{w \in W} BwB$$

where W is the set of permutation matrices.

Proof. The process of decomposing a matrix g into the product $b_1 w b_2 \in BwB$ essentially reduces to row-reducing g using a sequence of elementary row operations. The finer details are left for the reader. □

Remark 20. The set of permutation matrices is not a subgroup of $\mathrm{SL}_n(k)$ since any odd permutation will have determinant -1. However, we can rectify this by replacing a single 1 with -1 in every odd permutation. This gives the Bruhat decomposition for SL_n.

Notice that in the Bruhat decomposition for GL_n, $W \cong S_n$ can be given the structure of a Coxeter group. It is in fact an example of a *Weyl group*, a class of Coxeter groups which

arise naturally in the study of Lie groups and Lie algebras. The goal for the remainder of this section will be to investigate extensions of the Bruhat decomposition, each indexed by a particular Weyl group, when $G = \mathrm{GL}_n(F)$ and F is a finite extension of $\mathbb{Q}_p$.

To that end, let F be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ and maximal ideal $\mathfrak{m}$. Let $G = \mathrm{GL}_n(F)$ (resp. $\mathrm{SL}_n(F)$), $K := \mathrm{GL}_n(\mathcal{O})$ (resp. $\mathrm{SL}_n(\mathcal{O})$), and T the subgroup of G of diagonal matrices. The subgroup K is a maximal compact subgroup of G.

Definition (Finite Weyl group)**.** Define the *finite Weyl group* W_0 of G to be $N(T)/T$, where $N(T)$ is the normalizer of T in G.

One computes that $N(T)$ is the set of generalized permutation matrices (with determinant one in the case of SL_n), i.e. $N(T) = W \ltimes T$. We conclude the following.

Proposition 21. *We have*

(i) $W_0 \cong S_n$*;*

(ii) The coset BwB *is well-defined for* $w \in W_0$*;*

(iii) $G = \bigcup_{w \in W_0} BwB$*.*

Proof. Since $N(T)$ is the set of generalized permutation matrices, the map $N(T) \to S_n$ sending a matrix to its permutation type has kernel T. For (ii), note that if $g_1 T = g_2 T$ in W_0 then $g_1 B = g_2 B$ since B contains T. In particular, $Bg_1B = Bg_2B$. Finally, (iii) follows from the Bruhat decomposition. □

Remark 22. When $G = \mathrm{GL}_n(F)$ we can view the finite Weyl group as a subgroup of G (via set of permutation matrices). This is not the case for $\mathrm{SL}_n(F)$. Indeed, if g is an odd permutation matrix, then the coset gT has order 2 in W_0 but any element of gT has order 4 in $\mathrm{SL}_n(F)$.

2.4.2 The Bruhat-Tits decompositions

For simplicity, we now focus on $G = \mathrm{GL}_n(F)$. We suppose that the field $\mathcal{O}/\mathfrak{m}$ has characteristic p. Fix the following unipotent subgroup of GL_n:

$$U := \begin{pmatrix} 1 & * & * & \cdots & * \\ & 1 & * & \cdots & * \\ & & \ddots & \ddots & \vdots \\ & 0 & & \ddots & * \\ & & & & 1 \end{pmatrix}.$$

It is a subgroup of the Borel subgroup B.

Definition. The Iwahori (resp. pro-p Iwahori) subgroup of G is the inverse image of $B(\mathcal{O}/\mathfrak{m})$ (resp. $U(\mathcal{O}/\mathfrak{m})$) under the map $\mathrm{GL}_n(\mathcal{O}) \to \mathrm{GL}_n(\mathcal{O}/\mathfrak{m})$.

Let J and I be the Iwahori and pro-p Iwahori subgroups of G, respectively. Our goal is to find an analog of the Bruhat decomposition for J and I. In other words, we want to decompose G into the disjoint union of the cosets JwJ and IwI.

Definition. The *extended affine Weyl group* W of G is defined to be $N(T)/(T \cap K)$.

We may view the finite Weyl group as a subgroup of the extended affine Weyl group. We have the following.

Proposition 23 (Bruhat-tits decomposition for J)**.** *For $w \in W$ the cosets JwJ is well-defined and we have the disjoint union*

$$\mathrm{GL}_n(F) = \bigcup_{w \in W} JwJ.$$

Proof. See [IM65, Theorem 2.16]. □

The extended affine Weyl group is generated by the transpositions $s_1, \ldots, s_{n-1}$ of W_0 along with the element

$$r := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & \ddots & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & & \ddots & 1 \\ p & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{2.9}$$

Despite no longer being a Coxeter group ($r^2 \neq 1$), we can still define a length function on W as follows. Let $s_1 \in W_0$ be a generator of W_0 and W_{aff} be the subgroup[2] of W generated by W_0 and the element rs_1r^{-1}. It is a Coxeter group. Furthermore, it is normal in W and we have the semi-direct product $W = W_{\mathrm{aff}} \rtimes R$ where R is the subgroup generated by r. For $wr^k \in W$ we then define $\ell(wr^k) = \ell(w)$.

We further extend W by defining $\widetilde{W} := N(T)/(T \cap I)$. Let $\Omega := (T \cap J)/(T \cap I)$. It can be viewed as a subgroup of $\widetilde{W}$ and is isomorphic to the set of diagonal matrices with entries in $\mathcal{O}/\mathfrak{m}$, so it is abelian and normal in $\widetilde{W}$. We have $\widetilde{W} = W \ltimes \Omega$. This is summarized by the existence of an exact sequence

$$0 \longrightarrow \Omega \longrightarrow \widetilde{W} \longrightarrow W \longrightarrow 0.$$

Although $\widetilde{W}$ is again not a Coxeter group (in the sense we have defined here), the length of W may be used to define a length on $\widetilde{W}$ via $\ell(w\omega) = \ell(w)$ for $w \in W$ and $\omega \in \Omega$ (see [Vig05]). All elements of Ω have length zero.

[2] W_{aff} is called the affine Weyl group.

Proposition 24 (Bruhat-tits decomposition for I)**.** *For $w \in \widetilde{W}$, the coset IwI is well-defined and we have the disjoint union*

$$\mathrm{GL}_n(F) = \bigcup_{w \in \widetilde{W}} IwI.$$

Proof. This follows by using the semi-direct product $\widetilde{W} = W \ltimes \Omega$ along with Proposition 23. □

Remark 25. Versions of Proposition 23 and 24 hold identically for $\mathrm{SL}_n(F)$. The generators of $\widetilde{W}$ are given by the transpositions $s_1, \ldots, s_{n-1}$ and r, appropriately modified to be elements of $\mathrm{SL}_n(F)$ instead of $\mathrm{GL}_n(F)$. (For example we replace a 1 in r by p^{-1}.) We investigate the example $\mathrm{SL}_2(\mathbb{Q}_p)$ below.

Example (Decompositions of $\mathrm{SL}_2(\mathbb{Q}_p)$)**.** Let us be explicit and determine the Bruhat-Tits decompositions of $G = \mathrm{SL}_2(\mathbb{Q}_p)$. We do so by computing the Weyl groups W and $\widetilde{W}$. In this case, we have

$$I = \begin{pmatrix} 1 + p\mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & 1 + p\mathbb{Z}_p \end{pmatrix} \subset \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix} = J$$

where all matrices are assumed to have determinant 1. The normalizer of T here is the subgroup of diagonal matrices together with the anti-diagonal matrices. From this, it is straightforward to compute that a set of coset representatives for $T \cap K$ in $N(T)$ is $(s_0 s_1)^{\mathbb{Z}} \cup (s_1 s_0)^{\mathbb{Z}}$ where

$$s_0 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{and} \quad s_1 := \begin{pmatrix} 0 & -p^{-1} \\ p & 0 \end{pmatrix}.$$

This gives us the extended affine Weyl group W. Meanwhile, recall that in general we have $\Omega \cong \mathbb{T}(\mathbb{F}_q)$ where $\mathbb{T}(\mathbb{F}_q)$ is the set of diagonal matrices in $\mathrm{SL}_2(\mathbb{F}_p)$. Specifically,

$$\Omega = \left\{ \begin{pmatrix} \omega_x & 0 \\ 0 & \omega_x^{-1} \end{pmatrix} : x \in \mathbb{F}_p^\times \right\}$$

where ω_x denotes a lift of $x \in \mathbb{F}_p$ in $\mathbb{Z}_p$. Then $\widetilde{W} = W \ltimes \Omega$.

2.5 The pro-p Iwahori Hecke algebra

Recall that in the representation theory of finite groups, there is an equivalence between the category of representations of G and the category of $k[G]$-modules. This equivalence of course fails if we replace the category of k-representations of G with the category of *smooth* k-representations. Indeed, there is no need for an arbitrary $k[G]$-module to correspond to a smooth representation. For smooth representations, it is useful to instead study modules over the Hecke algebra, as we shall see.

Definition. Let G be a locally profinite group and $\mathcal{U}$ an open subgroup of G. The Hecke algebra $H_{\mathcal{U}}$ associated to $\mathcal{U}$ is defined to be

$$H_{\mathcal{U}} := \mathrm{End}_{k[G]}(\mathbf{X}_{\mathcal{U}})^{\mathrm{op}},$$

where $\mathbf{X}_{\mathcal{U}} := \text{c-ind}_{\mathcal{U}}^{G}(1_{\mathcal{U}})$ is the compact induction of the trivial character of $\mathcal{U}$.

Consider the k-vector space $k[\mathcal{U}\backslash G/\mathcal{U}]$ with basis the set of double cosets $\mathcal{U}\backslash G/\mathcal{U}$. One may view it as the vector space of $\mathcal{U}$-biinvariant functions $G \to k$ (i.e. functions such that $f(u_1 g u_2) = f(g)$ for all $g \in G$ and $u_1, u_2 \in \mathcal{U}$) with compact support. We can turn it into a k-algebra by giving it the convolution product

$$(\varphi \cdot \psi)(x) := \sum_{g \in G/\mathcal{U}} \varphi(g)\psi(g^{-1}x) = \sum_{g \in \mathcal{U}\backslash G} \varphi(xg^{-1})\psi(g) \tag{2.10}$$

for $\varphi, \psi \in k[\mathcal{U}\backslash G/\mathcal{U}]$.

Proposition 26. *As k-algebras, $H_{\mathcal{U}}$ is isomorphic to $(k[\mathcal{U}\backslash G/\mathcal{U}], \cdot)^{\mathrm{op}}$, where $\cdot$ is the convolution product* (2.10)*.*

Proof. Consider the G-representation $k[G/\mathcal{U}]$; as a vector space it has basis the cosets $G/\mathcal{U}$, and the action of G is given by the usual action of G on $G/\mathcal{U}$. So $k[G/\mathcal{U}]$ identifies naturally with $\mathbf{X}_{\mathcal{U}}$ as representations of G. A function $f \in \mathbf{X}_U$ corresponds to

$$\sum_{g \in G/\mathcal{U}} f(g) g\mathcal{U} \in k[G/\mathcal{U}],$$

and the fact that we use compact induction ensures that this is a well-defined element of $k[G/\mathcal{U}]$. By Frobenius reciprocity, we have

$$\mathrm{End}_{k[G]}(\mathbf{X}_{\mathcal{U}}) \cong \mathrm{Hom}_{k[\mathcal{U}]}(1_{\mathcal{U}}, \mathbf{X}_{\mathcal{U}}|_{\mathcal{U}}) \cong \mathrm{Hom}_{k[U]}(1_{\mathcal{U}}, k[G/\mathcal{U}])$$

as vector spaces. However, $\mathrm{Hom}_{k[U]}(1_{\mathcal{U}}, k[G/\mathcal{U}])$ is just the space $k[G/\mathcal{U}]^{\mathcal{U}}$ of $\mathcal{U}$-fixed vectors; i.e. $k[\mathcal{U}\backslash G/\mathcal{U}]$. The fact that the products coincide follows by a straightforward computation. □

Remark 27. Recall that the group algebra $k[G]$ as the set of finite formal k-linear combinations of elements in G. There is, however, a natural identification (as vector spaces) between $k[G]$ and the the space of functions $G \to k$ with finite support. Tracing the product of $k[G]$ through this identification, we find that the product of two functions $\varphi, \psi \colon G \to k$ is given by the convolution

$$(\varphi \cdot \psi)(x) := \sum_{g \in G} \varphi(g)\psi(g^{-1}x), \tag{2.11}$$

which resembles the convolution product (2.10).

The pro-p Iwahori Hecke algebra We now let $G = \mathrm{SL}_n(\mathbb{Q}_p)$ and I to be its pro-p Iwahori subgroup. Suppose k has characteristic p. The pro-p Iwahori Hecke algebra is H_I; as is tradition we often write H for H_I and $\mathbf{X}$ for $\mathbf{X}_I$. Recall that the Weyl group $\widetilde{W} = W \ltimes \Omega$ indexes the set of double cosets $I\backslash G/I$. For $w \in \widetilde{W}$, denote by τ_w the characteristic function of IwI. We have

$$H = \bigoplus_{w \in \widetilde{W}} k\tau_w.$$

The extended affine Weyl group is generated by the transpositions $s_1, \ldots, s_{n-1}$ as well as the element r (see §2.4 and in particular Remark 25). If s is one of these generators and $v, w \in \widetilde{W}$, then

$$\begin{aligned} \tau_v \tau_w &= \tau_{vw}, \quad \text{if } \ell(vw) = \ell(v) + \ell(w) && \text{(braid relations)} \\ \tau_s^2 &= -e_1 \tau_s && \text{(quadratic relations)} \end{aligned}$$

where

$$e_1 := -\sum_{\omega \in \Omega} \tau_\omega.$$

The element e_1 is a central idempotent (see [OS19]).

In the context of the mod-p Local Langlands correspondence, the importance of the pro-p Iwahori Hecke algebra is due in part to the existence of the functor

$$\begin{aligned} \mathfrak{h} \colon \mathrm{Rep}_k(G) &\longrightarrow \mathrm{Mod}(H) \\ V &\longmapsto V^I, \end{aligned}$$

where V^I is the set of I-fixed vectors.

Proposition 28. *The functor $\mathfrak{h}$ sends non-zero representations to non-zero modules.*

Proof. We will prove this for $G = \mathrm{GL}_n(\mathbb{Q}_p)$ from which the result for $\mathrm{SL}_n(\mathbb{Q}_p)$ will follow. Write an element of I in the form $1 + A$ where $A \in \mathrm{M}_n(\mathbb{Z}_p)$ and define the map $I \to (\mathrm{M}_n(\mathbb{Z}/p\mathbb{Z}), +)$ by $1 + A \mapsto \overline{A}$ where $\overline{A}$ denotes the matrix A with entries reduced modulo $p\mathbb{Z}_p$. This map is well-defined with kernel $1 + p\,\mathrm{M}_n(\mathbb{Z}_p)$. Likewise, for $k \geq 1$, the map $1 + p^k\,\mathrm{M}_n(\mathbb{Z}_p) \to (\mathrm{M}_n(\mathbb{Z}/p\mathbb{Z}), +)$ defined by $1 + p^k A \mapsto \overline{A}$ has kernel $1 + p^{k+1}\,\mathrm{M}_n(\mathbb{Z}_p)$. We conclude that the subgroups

$$1 + p\,\mathrm{M}_n(\mathbb{Z}_p) \supset 1 + p^2\,\mathrm{M}_n(\mathbb{Z}_p) \supset \cdots \tag{2.12}$$

are open neighbourhoods of the identity, each normal in I and of p-power index.

Now, let V be a smooth representation of I. Fix $0 \neq v \in V$, and choose an open subgroup H of I which fixes v. Since H contains one of the neighbourhoods (2.12), we may simply assume H is normal and of p-power index. Then the representation V descends to a representation of the p-group I/H. Any representation of a p-group over a field of characteristic p admits fixed vectors (see [Ser77, Prop. 26]). Since a I/H-fixed vector will be an I-fixed vector of V, we are done. □

Remark 29. More generally, a *pro-p group* is a compact Hausdorff topological group which contains a neighbourhood basis of the identity consisting of normal subgroups of p-power index. Here we have shown that I is a pro-p group, hence its name.

Proposition 30 ([Koz16])**.** *Let $F = \mathbb{Q}_p$ and k be an algebraically closed field of characteristic p. Then the functor $\mathfrak{h}$ induces an equivalence of categories between $\mathrm{Rep}_k^I(G)$ and $\mathrm{Mod}(H)$, where $\mathrm{Rep}_k^I(G)$ is the category of smooth representations which are generated by their I-fixed vectors.*

Note that the left action of H on V^I is given by the following right action of $\mathrm{End}_{k[G]}(\mathbf{X})$ on V^I. We have $V^I \cong \mathrm{Hom}_{k[I]}(1_I, V)$, so $V^I \cong \mathrm{Hom}_{k[G]}(\mathbf{X}, V)$ by Frobenius reciprocity. Let ϕ_v be the element of $\mathrm{Hom}_{k[G]}(\mathbf{X}, V)$ corresponding to $v \in V^I$ (namely, so that $\phi_v(\mathbf{1}_I) = v$ where $\mathbf{1}_I$ is the characteristic function of I). Finally, for $T \in \mathrm{End}_{k[G]}(\mathbf{X})$, define $v \cdot T = \phi_v \circ T(\mathbf{1}_I)$.

In the next chapter, we discuss the failure of this functor to induce an equivalence for more general F, and introduce the parahoric Hecke Ext-algebras.

Chapter 3

Parahoric Hecke Ext-algebras

Let F be a finite extension of $\mathbb{Q}_p$, $G = \mathrm{GL}_n(F)$, and k a field of characteristic p. We saw earlier that it is helpful to study the functor

$$\begin{aligned} \mathfrak{h}\colon \mathrm{Rep}_k(G) &\longrightarrow \mathrm{Mod}(H) \\ V &\longmapsto V^I, \end{aligned}$$

where I is the pro-p Iwahori subgroup of G. Since $\mathfrak{h} = \mathrm{Hom}_{k[G]}(\mathbf{X}, -)$, this functor is left exact and has the right adjoint

$$\begin{aligned} \mathfrak{t}\colon \mathrm{Mod}(H) &\longrightarrow \mathrm{Rep}_k^I(G) \\ M &\longmapsto \mathbf{X} \otimes_H M \end{aligned}$$

where $\mathrm{Rep}_k^I(G)$ denotes the category of all smooth representations which are generated by their I-fixed vectors. In general, $\mathfrak{h}$ fails to be an equivalence and furthermore as it is not usually right exact. It makes sense, then, to consider the derived picture, where the situation improves somewhat. In [Sch15], Schneider shows when $p \geq 5$ that there are derived versions of $\mathfrak{h}$ and $\mathfrak{t}$ which induce an equivalence between the derived category of smooth representations and the derived category of differential graded modules over a certain differential graded pro-p Iwahori Hecke algebra $H^\bullet$. Unfortunately, the derived categories in question, and even the algebra $H^\bullet$, are still quite poorly understood.

One thing we can do is study its cohomology algebra $E^* := \mathrm{Ext}^*_{\mathrm{Rep}_k(G)}(\mathbf{X}, \mathbf{X})^{\mathrm{op}}$. In [OS19], Ollivier and Schneider describe its product explicitly and deduce, in particular, that it is only supported in degrees 0 to d when I is a pro-p Iwahori subgroup of dimension d as a p-adic Lie group.

For the remainder of this chapter, we let $G = \mathrm{SL}_2(\mathbb{Q}_p)$, $p \geq 5$, and take k to be the algebraic closure of a finite extension of $\mathbb{F}_p$. Let $\mathcal{P}$ be a parahoric subgroup containing J, which means that $\mathcal{P}$ is the union of double cosets JwJ. In [OS] the parahoric Hecke Ext-algebra

$$E^*_{\mathcal{P}} := \mathrm{Ext}^*_{\mathrm{Rep}_k(G)}(\mathbf{X}_{\mathcal{P}}, \mathbf{X}_{\mathcal{P}})^{\mathrm{op}}$$

is introduced. In this chapter we study the Iwahori and Spherical versions of $E^*_{\mathcal{P}}$, namely, the algebras

$$E^*_J := \mathrm{Ext}^*_{\mathrm{Rep}_k(G)}(\mathbf{X}_J, \mathbf{X}_J)^{\mathrm{op}}, \quad \text{and}$$
$$E^*_K := \mathrm{Ext}^*_{\mathrm{Rep}_k(G)}(\mathbf{X}_K, \mathbf{X}_K)^{\mathrm{op}}$$

where J is the Iwahori subgroup J and K is the maximal compact subgroup $\mathrm{SL}_2(\mathbb{Z}_p)$.

In [OS], the following result is shown.

Proposition 31 ([OS])**.** *Let $\mathcal{P}$ be a parahoric subgroup of G containing J. There exists an idempotent $e_{\mathcal{P}} \in H_{\mathcal{P}}$ such that $e_{\mathcal{P}} E^* e_{\mathcal{P}}$ and $E^*_{\mathcal{P}}$ are isomorphic as k-algebras.*

In light of this result, we are able to study E^*_K and E^*_J as algebras[1] inside E^*. The assumption $p \geq 5$ allows us to say, among other things, that E^* is supported in degrees 0 to 3. The same will be true for E^*_J and E^*_K.

3.1 Preliminaries

We first recall the key objects in this context. Refer to §2.4 and [OS21] §2.3.

Fix uniformizer $\pi = p$ of $\mathbb{Z}_p$. There is a nested sequence of open compact subgroups $I \subset J \subset K$ in G consisting of the Iwahori subgroup J, its pro-p Iwahori subgroup I, and the maximal compact subgroup K. When $G = \mathrm{SL}_2(\mathbb{Q}_p)$, we can be explicit and write $K = \mathrm{SL}_2(\mathbb{Z}_p)$,

$$I = \begin{pmatrix} 1+p\mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & 1+p\mathbb{Z}_p \end{pmatrix}, \quad \text{and} \quad J = \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix},$$

where (by abuse of notation) all matrices are understood to have determinant 1.

Let $T \subset G$ be the torus of diagonal matrices, T^0 its maximal compact subgroup, T^1 its maximal pro-p subgroup, and $N(T)$ its normalizer. We set $\widetilde{W} := N(T)/T^1$ and $W := N(T)/T^0$. These are the affine and extended affine Weyl groups, respectively, and they are related via the short exact sequence

$$0 \longrightarrow \Omega \longrightarrow \widetilde{W} \longrightarrow W \longrightarrow 0$$

where $\Omega := T^0/T^1$ can be identified with the torus of diagonal matrices in $\mathrm{SL}_2(\mathbb{F}_p)$. Here, W is generated by the two elements

$$s_0 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad s_1 := \begin{pmatrix} 0 & -p^{-1} \\ p & 0 \end{pmatrix}.$$

(By abuse of notation, here and throughout, we write elements of $\widetilde{W}$ as matrices in G.) Meanwhile, $\widetilde{W}$ is generated by s_0, s_1, and ω for ω ranging over Ω. Setting $\theta := s_0 s_1$, the

[1] We do not call $e_{\mathcal{P}} E^* e_{\mathcal{P}}$ a subalgebra of E^* as the units are not the same. The unit in $e_{\mathcal{P}} E^* e_{\mathcal{P}}$ is $e_{\mathcal{P}}$.

length function ℓ on W can be pulled back to $\widetilde{W}$ yielding $\ell(\theta^i) = |2i|$ and $\ell(s_0\theta^i) = |1-2i|$. Elements in $\Omega \subset \widetilde{W}$ have zero length.

We have the following decompositions into disjoint cosets:

$$G = \bigcup_{w\in W} JwJ = \bigcup_{w\in\widetilde{W}} IwI, \qquad K = J \cup Js_0J = \bigcup_{\substack{\omega\in\Omega \\ i\in\{0,1\}}} I\omega s_0^i I, \qquad J = \bigcup_{\omega\in\Omega} I\omega I. \tag{3.1}$$

Since the elements of $\widetilde{W}$ index a set of double coset representatives for $I\backslash G/I$, the characteristic functions of IwI for $w \in \widetilde{W}$ forms a basis for the pro-p Iwahori Hecke algebra H. We denote by τ_w the characteristic function of IwI.

One obtains the decomposition of vector spaces

$$E^* = H^*(I, \mathbf{X}) = \bigoplus_{w\in\widetilde{W}} H^*(I, \mathbf{X}(w))$$

where $\mathbf{X}(w) := \mathrm{ind}_I^{IwI}(1_I)$. For each i, define also the decreasing filtration

$$F^n E^i := \bigoplus_{\substack{w\in\widetilde{W} \\ \ell(w)\geq n}} H^i(I, \mathbf{X}(w)) \tag{3.2}$$

and the increasing filtration

$$F_n E^i := \bigoplus_{\substack{w\in\widetilde{W} \\ \ell(w)\leq n}} H^i(I, \mathbf{X}(w)) \tag{3.3}$$

(see [OS21] §2.2.4).

3.1.1 Idempotents

We also define the following class of idempotents. To a k-character $\lambda\colon \Omega \to k^\times$, define

$$e_\lambda := -\sum_{\omega\in\Omega} \lambda(\omega^{-1})\tau_\omega \in H.$$

Then e_λ is an idempotent. Recall that we have the isomorphism

$$\begin{aligned} \mathbb{F}_p^\times &\xrightarrow{\cong} \Omega \\ x &\longmapsto \begin{pmatrix} \omega_x & 0 \\ 0 & \omega_x^{-1} \end{pmatrix} T^1 \end{aligned} \tag{3.4}$$

where ω_x denotes a lift of x in $\mathbb{Z}_p$. When k has characteristic p, the inclusion $\mathbb{F}_p^\times \to k^\times$ composes with the inverse of (3.4) to produce the following family of k-characters of Ω (indexed by $m \in \mathbb{Z}$):

$$\begin{aligned} \mathrm{id}^m \colon \Omega \xrightarrow{\cong} \mathbb{F}_p^\times &\longrightarrow k^\times \\ x &\longmapsto x^m. \end{aligned}$$

One may verify that $\{e_{\mathrm{id}^m}\}_{m=0}^{p-2}$ is a family of orthogonal idempotents with sum equal to 1. Of particular importance to us here will be the central idempotent

$$e_{\gamma_0} := e_{\mathrm{id}^1} + e_{\mathrm{id}^{-1}}. \tag{3.5}$$

By orthogonality, we have $e_J \cdot e_{\gamma_0} = e_{\gamma_0} \cdot e_J = 0$ as long as $p \neq 2$.

The idempotents e_J and e_K as defined in Proposition 31 are given by

$$e_J = -\sum_{\omega \in \Omega} \tau_\omega, \qquad e_K = -\sum_{\substack{\omega \in \Omega \\ i \in \{0,1\}}} \tau_{\omega s_0^i} = e_J + \tau_{s_0} e_J. \tag{3.6}$$

The idempotent e_J is in fact central in H. It is the same as the idempotent e_1 of [OS19].

3.1.2 Finite duals

Recall the anti-automorphism $\mathcal{J}$ of E^* (see [OS19] §6) with the property that $\mathcal{J}(\alpha \cdot \beta) = (-1)^{ij}\mathcal{J}(\beta) \cdot \mathcal{J}(\alpha)$ for $\alpha \in E^i$ and $\beta \in E^j$. It acts on E^* by the following

$$\mathcal{J}(\tau_w) = \tau_{w^{-1}}, \qquad \mathcal{J}((0, \boldsymbol{c}^0, 0)_w) = (-1)^{\ell(w)}(0, \boldsymbol{c}^0, 0)_{w^{-1}} \tag{3.7}$$

$$\mathcal{J}(\phi_w) = \phi_{w^{-1}}, \qquad \mathcal{J}((0, \boldsymbol{\alpha}^0, 0)_w) = (-1)^{\ell(w)}(0, \boldsymbol{\alpha}^0, 0)_{w^{-1}}. \tag{3.8}$$

Also, $\mathcal{J}(e_J) = e_J$ and $\mathcal{J}(e_K) = e_K$.

If V is a vector space which decomposes into a direct sum indexed by $\widetilde{W}$, we denote by $V^{\vee,f}$ its finite dual as defined in [OS19, §7]. For $0 \leq i \leq 3$ we have an isomorphism of H-bimodules $\Delta^i \colon E^i \to ({}^{\mathcal{J}}(E^{d-i})^{\mathcal{J}})^{\vee,f}$ (see [OS19] §7.2.4). Here $({}^{\mathcal{J}}(E^{d-i})^{\mathcal{J}})^{\vee,f}$ is the finite dual of ${}^{\mathcal{J}}(E^{d-i})^{\mathcal{J}}$, and this latter space denotes the twisted H-bimodule obtained by taking the space E^{d-i} and defining the action of H via $(\tau, \beta, \tau') \mapsto \mathcal{J}(\tau') \cdot \beta \cdot \mathcal{J}(\tau)$ for $\tau, \tau' \in H$ and $\beta \in E^{d-i}$.

For any parahoric subgroup $\mathcal{P}$ containing J, the restriction map $(E^i)^{\vee,f} \to (e_{\mathcal{P}} E^i e_{\mathcal{P}})^\vee$ is a homomorphism of H-bimodules. We denote by $(e_{\mathcal{P}} E^0 e_{\mathcal{P}})^{\vee,f}$ the image of this map. From [OS] we have the following.

Lemma 32 ([OS])**.** *We have an isomorphism of $e_{\mathcal{P}} H e_{\mathcal{P}}$-bimodules*

$$\begin{aligned} e_{\mathcal{P}} E^i e_{\mathcal{P}} &\xrightarrow{\simeq} e_{\mathcal{P}} ({}^{\mathcal{J}}(E^{d-i})^{\mathcal{J}})^{\vee,f} e_{\mathcal{P}} \cong ({}^{\mathcal{J}}(e_{\mathcal{P}} E^{d-i} e_{\mathcal{P}})^{\mathcal{J}})^{\vee,f} \\ e_{\mathcal{P}} \cdot x \cdot e_{\mathcal{P}} &\longmapsto e_{\mathcal{P}} \cdot x^\vee \cdot e_{\mathcal{P}}. \end{aligned} \tag{3.9}$$

3.1.3 The H-bimodule structure of E^1

The H-bimodule structure of E^1 is studied in [OS21] §4 and §7. We recall a few results here.

The elements of E^1 can be represented as triples (refer to [OS21] §4). Briefly, for $w \in \widetilde{W}$, abbreviate the subspace $H^1(I, \mathbf{X}(w))$ of E^1 by $h^1(w)$. There is a decomposition

$$h^1(w) = h^1_-(w) \oplus h^1_0(w) \oplus h^1_+(w) \tag{3.10}$$

with

$$\begin{aligned} h^1_-(w) &\cong \mathrm{Hom}(\mathbb{Z}_p/p\mathbb{Z}_p, k) \\ h^1_0(w) &\cong \begin{cases} \mathrm{Hom}((1+p\mathbb{Z}_p)/(1+p^2\mathbb{Z}_p), k), & \text{if } \ell(w) \geq 1 \\ 0, & \text{if } \ell(w) = 0 \end{cases} \\ h^1_+(w) &\cong \mathrm{Hom}(\mathbb{Z}_p/p\mathbb{Z}_p, k) \end{aligned} \tag{3.11}$$

obtained via the Shapiro isomorphism $h^1(w) \cong \mathrm{Hom}((I_w)_\Phi, k)$ (see in particular [OS21] Remark 4.2). Here $(I_w)_\Phi$ refers to the Frattini quotient of I_w. For $c^\pm \in \mathrm{Hom}(\mathbb{Z}_p/p\mathbb{Z}_p, k)$ and $c_0 \in \mathrm{Hom}((1+p\mathbb{Z}_p)/(1+p^2\mathbb{Z}_p), k)$, we denote by $(c^-, c^0, c^+)_w$ the corresponding triple in $h^1(w)$ under the isomorphisms (3.11).

The H-bimodule action on these triples is given in [OS21] §4.4-4.6. In particular, we highlight the following. By equation (70) *loc. cit.* the right action of e_{id^m} on an element of E^1 satisfies the following identity:

$$(c^-, c^0, c^+)_w \cdot e_{\mathrm{id}^m} = e_{\mathrm{id}^{m(-1)^{\ell(w)}-2}} \cdot (c^-, 0, 0)_w + e_{\mathrm{id}^{m(-1)^{\ell(w)}}} \cdot (0, c^0, 0)_w + e_{\mathrm{id}^{m(-1)^{\ell(w)}+2}} \cdot (0, 0, c^+)_w. \tag{3.12}$$

We set some notation. There is an isomorphism

$$\begin{aligned} \iota \colon (1+p\mathbb{Z}_p)/(1+p^2\mathbb{Z}_p) &\xrightarrow{\cong} \mathbb{Z}_p/p\mathbb{Z}_p \\ 1+px &\longmapsto x \pmod{p\mathbb{Z}_p} \end{aligned}$$

which we use to fix once and for all elements $\boldsymbol{\alpha}, \boldsymbol{\alpha}^0, \boldsymbol{c}$, and $\boldsymbol{c^0}$ which satisfy

$$\boldsymbol{\alpha} \in \mathbb{Z}_p/p\mathbb{Z}_p \setminus \{0\}, \quad \boldsymbol{\alpha^0} = \iota^{-1}(\boldsymbol{\alpha}), \quad \boldsymbol{c} \in \mathrm{Hom}(\mathbb{Z}_p/p\mathbb{Z}_p, k) \text{ such that } \boldsymbol{c}(\boldsymbol{\alpha}) = 1, \quad \boldsymbol{c^0} := \boldsymbol{c}\iota \tag{3.13}$$

(see [OS21] §4.2.3). For $w \in \widetilde{W}$, we make note of the element $(0, \boldsymbol{c^0}, 0)_w \in h^1(w)$. We also set

$$\widetilde{W}^0 := \{w \in \widetilde{W} \mid \ell(s_0 w) = \ell(w) + 1\}, \qquad W^0 := \{w \in W \mid \ell(s_0 w) = \ell(w) + 1\}, \tag{3.14}$$

$$\widetilde{W}^1 := \{w \in \widetilde{W} \mid \ell(s_1 w) = \ell(w) + 1\}, \qquad W^1 := \{w \in W \mid \ell(s_1 w) = \ell(w) + 1\}. \tag{3.15}$$

For the remainder of this subsection, refer to [OS21] §7. The element

$$\zeta := (\tau_{s_0} + e_J)(\tau_{s_1} + e_J) + \tau_{s_1}\tau_{s_0} \tag{3.16}$$

is central in H. We use it to define the following H-bimodule endomorphisms f and g of E^*:

$$f(c) = \zeta \cdot c \cdot \zeta - c$$

and

$$g(c) = \zeta \cdot c - c \cdot \zeta$$

for all $c \in E^*$. Denote by f_i and g_i the maps f and g, respectively, when restricted to the graded pieces E^i. Evidently $\ker(f_0) = \{0\}$ and $\ker(g_0) = E^0$.

We now recall a few useful technical results regarding the two submodules $\ker(f_1)$ and $\ker(g_1)$ of E^1. First, when $p \neq 2, 3$, there is the exact sequence of H-bimodules

$$0 \longrightarrow \ker(f_1) \oplus \ker(g_1) \longrightarrow E^1 \longrightarrow E^1/(\ker(f_1) \oplus \ker(g_1)) \longrightarrow 0 \tag{3.17}$$

where $E^1/(\ker(f_1) \oplus \ker(g_1))$ is four-dimensional as an H-bimodule. It has k-basis the cosets of $x, y, x \cdot \tau_{s_1}$, $y \cdot \tau_{s_0}$ where

$$x = e_{\mathrm{id}} \cdot (0, 0, \boldsymbol{c})_1 \cdot e_{\mathrm{id}^{-1}} \tag{3.18}$$

$$y = e_{\mathrm{id}^{-1}} \cdot (\boldsymbol{c}, 0, 0)_1 \cdot e_{\mathrm{id}} \tag{3.19}$$

with $\boldsymbol{c}$ as in (3.13). Second, we have

$$(1 - e_{\gamma_0}) \cdot \ker(f_1) = (1 - e_{\gamma_0}) \cdot h^1_{\pm}(\widetilde{W}) \tag{3.20}$$

by [OS21] Remark 7.8. By the orthogonality of the idempotents e_{id^m}, multiplying both sides of (3.20) by $e_J = e_1$ gives $e_J \cdot \ker(f_1) = e_J \cdot h^1_{\pm}(\widetilde{W})$. However, the right-hand side here is zero by (3.12). Therefore $e_J \cdot \ker(f_1) = \ker(f_1) \cdot e_J = 0$.

A useful corollary is the following:

Lemma 33. *We have $e_{\mathcal{P}} \cdot \ker(f_1) = \ker(f_1) \cdot e_{\mathcal{P}} = 0$ for any parahoric subgroup $\mathcal{P}$ of G which contains J.*

Proof. This is immediate by the previous paragraph after using the fact that $e_{\mathcal{P}} = e_J \cdot e_{\mathcal{P}} = e_{\mathcal{P}} \cdot e_J$. □

We will also need:

Lemma 34 ([OS21] Proposition 7.3)**.** *The map*

$$\begin{aligned} F^1H &\longrightarrow \ker(g_1) \\ \tau_w \quad &\longmapsto \begin{cases} (0, \boldsymbol{c}^0, 0)_w, & \text{if } \ell(w) \geq 2 \text{ and } w \in W^0 \\ -(0, \boldsymbol{c}^0, 0)_w, & \text{if } \ell(w) \geq 2 \text{ and } w \in W^1 \\ (0, \boldsymbol{c}^0, 0)_{s_1\omega} - e_{\mathrm{id}} \cdot (0, 0, \boldsymbol{c}^0\iota^{-1})_\omega, & \text{if } w = s_1\omega \in s_1\Omega \\ -(0, \boldsymbol{c}^0, 0)_{s_0\omega} - e_{\mathrm{id}^{-1}} \cdot (\boldsymbol{c}^0\iota^{-1}, 0, 0)_\omega, & \text{if } w = s_0\omega \in s_0\Omega. \end{cases} \end{aligned} \tag{3.21}$$

is an isomorphism of H-bimodules.

Lemma 35. *For any parahoric subgroup $\mathcal{P}$ containing J, we have $e_{\mathcal{P}}E^1e_{\mathcal{P}} \cong e_{\mathcal{P}}(F^1H)e_{\mathcal{P}}$ as $e_{\mathcal{P}}He_{\mathcal{P}}$-bimodules.*

Proof. The exact sequence (3.17) yields the exact sequence

$$0 \longrightarrow e_{\mathcal{P}}(\ker(f_1) \oplus \ker(g_1))e_{\mathcal{P}} \longrightarrow e_{\mathcal{P}}E_1e_{\mathcal{P}} \longrightarrow e_{\mathcal{P}}(E^1/(\ker(f_1) \oplus \ker(g_1)))e_{\mathcal{P}} \longrightarrow 0.$$

Some simplification is possible. On the left, we have $e_{\mathcal{P}}(\ker(f_1) \oplus \ker(g_1))e_{\mathcal{P}} = e_{\mathcal{P}} \ker(g_1)e_{\mathcal{P}}$ as a consequence of Lemma 33. Meanwhile, recall that $E^1/(\ker(f_1) \oplus \ker(g_1))$ has basis x, y, $x \cdot \tau_{s_1}$, and $y \cdot \tau_{s_1}$ (refer to (3.18), (3.19)) which are all killed on the left (and right) by e_J (in particular by $e_{\mathcal{P}}$). Therefore $e_{\mathcal{P}}(E^1/(\ker(f_1) \oplus \ker(g_1)))e_{\mathcal{P}} = 0$. The resulting simplified exact sequence shows that $e_{\mathcal{P}}E^1e_{\mathcal{P}} = e_{\mathcal{P}} \ker(g_1)e_{\mathcal{P}}$. Conclude by applying Lemma 34. □

3.1.4 The H-bimodule structure of E^2

Recall the isomorphism $\Delta^2 \colon E^2 \to ({}^{\mathcal{J}}(E^1)^{\mathcal{J}})^{\vee,f}$. In light of (3.11), and given the natural identification between the dual of a $\mathrm{Hom}_{\mathbb{F}_p}$ space and tensor over $\mathbb{F}_p$, the element $\alpha \in (h^1(w))^\vee$ corresponding to $c = (c^-, c^0, c^+)_w \in h^1(w)$ can be viewed as the triple

$$(\alpha^-, \alpha^0, \alpha^+)_w \in \mathbb{Z}_p/p\mathbb{Z}_p \otimes_{\mathbb{F}_p} k \times (1 + p\mathbb{Z}_p)/(1 + p^2\mathbb{Z}_p) \otimes_{\mathbb{F}_p} k \times \mathbb{Z}_p/p\mathbb{Z}_p \otimes_{\mathbb{F}_p} k$$

for $\ell(w) \geq 1$ (with middle term zero if $\ell(w) = 0$). We have $\alpha(c) = c^-(\alpha^-) + c^0(\alpha^0) + c^+(\alpha^+)$. For $w \in \widetilde{W}$, the elements $(0, \boldsymbol{c}^0, 0)_w \in E^1$ and $(0, \boldsymbol{\alpha}^0, 0)_w \in ({}^{\mathcal{J}}(E^1)^{\mathcal{J}})^{\vee,f} \cong E^2$ are dual to each other. The left action of H on these triples is given in [OS21] Prop. 5.5.

Lemma 36. *The map*

$$\ker(g_2) \longrightarrow {}^{\mathcal{J}}((F^1H)^{\vee,f})^{\mathcal{J}} \tag{3.22}$$

$$(0, \boldsymbol{\alpha}^0, 0) \longmapsto \begin{cases} \tau_w^\vee|_{F^1H}, & \text{if } w \in W^0 \\ -\tau_w^\vee|_{F^1H}, & \text{if } w \in W^1 \end{cases} \tag{3.23}$$

is an isomorphism of H-bimodules.

Proof. See [OS21] Proposition 7.14 and its proof. □

We define $(F^1H)^{\vee,f}$ to be the image of the restriction map $H^{\vee,f} \to (F^1H)^\vee$. The following is the analog of Lemma 35 for E^2.

Lemma 37. *For any parahoric subgroup $\mathcal{P}$ containing J, we have $e_{\mathcal{P}}E^2e_{\mathcal{P}} \cong e_{\mathcal{P}}(F^1H)^{\vee,f}e_{\mathcal{P}}$ as $H_{\mathcal{P}}$-bimodules.*

Proof. Recall that $E^2 = \ker(f_2) \oplus \ker(g_2)$ by [OS21] Proposition 7.12. By Corollary 7.19 *loc. cit.* $\ker(f_1)$ and $\ker(f_2)$ are isomorphic as H_ζ-modules, where H_ζ denotes the localization of H in ζ. In particular, they are isomorphic as H-modules and so by Lemma 33 we have that $e_{\mathcal{P}} \cdot \ker(f_2) = \ker(f_2) \cdot e_{\mathcal{P}} = 0$. Therefore $e_{\mathcal{P}}E^2e_{\mathcal{P}} = e_{\mathcal{P}} \ker(g_2)e_{\mathcal{P}}$. However, by Lemma 36, we have

$$e_{\mathcal{P}} \ker(g_2)e_{\mathcal{P}} \cong e_{\mathcal{P}}{}^{\mathcal{J}}((F^1H)^{\vee,f})^{\mathcal{J}}e_{\mathcal{P}} \cong e_{\mathcal{P}}(F^1H)^{\vee,f}e_{\mathcal{P}}$$

where the second isomorphism is obtained by applying $\mathcal{J} \colon H \to H$. □

3.1.5 The H-bimodule structure of E^3

Given a basis $(\tau_w)_{w\in\widetilde{W}}$ for E^0 we naturally obtain the dual basis $(\tau_w^\vee)_{w\in\widetilde{W}}$ for $({}^\jmath(E^0)^\jmath)^{\vee,f}\cong E^3$. We denote by ϕ_w the element in E^3 corresponding to $\tau_w^\vee$. The formulas for the H-bimodule action on $({}^\jmath(E^0)^\jmath)^{\vee,f}$ are given in [OS19] Prop. 8.2.

Remark 38. For any $w\in\widetilde{W}$ with $\ell(w)\geq 1$ there exists a unique $\epsilon\in\{0,1\}$ with $w\in\widetilde{W}^\epsilon$. We define $\psi_w := \tau_{1-\epsilon}\cdot\phi_w = \phi_{s_{1-\epsilon}w} - e_J\cdot\phi_w$. The set of all ψ_w generates the subspace $\ker(\mathcal{S}^3)$ of E^3 (see [OS21] Remark 2.12). By the proof of Prop. 8.6 in [OS19] we have the decomposition $E^3=\ker(\mathcal{S}^3)\oplus ke_1\cdot\phi_1$.

3.2 The Iwahori Ext-algebra

Let $\mathcal{P}$ be a parahoric subgroup containing J. There exists a subgroup $\mathfrak{W}_\mathcal{P}$ of W such that $\mathcal{P}=\bigcup_{w\in\mathfrak{W}_\mathcal{P}}JwJ$.

Lemma 39 ([OS])**.** *Each double coset $\mathfrak{W}_\mathcal{P}w\mathfrak{W}_\mathcal{P}$ for $w\in W$ contains a unique element of minimal length d.*

We call the set of all such minimal length elements ${}_\mathcal{P}\mathcal{D}_\mathcal{P}$. They index the set of double cosets $\mathfrak{W}_\mathcal{P}\backslash W/\mathfrak{W}_\mathcal{P}$.

Let $\tau_w^\mathcal{P}$ denote the characteristic function of $\mathcal{P}w\mathcal{P}$ in $H_\mathcal{P}$. We then have the following.

Proposition 40 ([OS])**.** *The map $e_\mathcal{P}He_\mathcal{P}\to H_\mathcal{P}$ of k-algebras induced by the map*

$$\begin{aligned} H &\longrightarrow H_p \\ \tau_w &\longmapsto \begin{cases} 0, & \text{if } w\notin \widetilde{{}_\mathcal{P}\mathcal{D}_\mathcal{P}} \\ [\mathcal{P}:I]\tau_w^\mathcal{P}, & \text{if } w\in\widetilde{{}_\mathcal{P}\mathcal{D}_\mathcal{P}} \end{cases}\end{aligned}$$

is an isomorphism. Moreover, $[\mathcal{P}:I]\neq 0$ in k.

By Proposition 40, a basis for H_J is given by the set of all τ_d^J for $d\in{}_J\mathcal{D}_J$. Since $\mathfrak{W}_J=\{1\}$, we have ${}_J\mathcal{D}_J\cong W$. It follows that a basis for e_JHe_J is $\{e_J\tau_we_J\}_{w\in W}$.

A basis for $e_JE^1e_J$ By Lemma 35, we have that $e_JE^1_Je_J = e_J\ker(g_1)e_J\cong e_J(F^1H)e_J$ as H-bimodules. This isomorphism is induced by (3.21), which we temporarily call f. We may obtain a basis for $e_J\ker(g_1)e_J$ by studying the action of f on a basis of F^1H. The submodule F^1H of H has basis $\{\tau_w : w\in\widetilde{W},\ell(w)\geq 1\}$. The image of f simplifies somewhat when restricted to $e_JF^1He_J$. Since e_JHe_J has basis $\{e_J\tau_we_J : w\in W\}$, a basis element of $e_JF^1He_J$ is of the form $e_J\tau_we_J$ with $w\in W$ and $\ell(w)\geq 1$. For such an element,

we have

$$f(e_J\tau_w e_J) = e_J f(\tau_w) e_J$$
$$= \begin{cases} e_J(0,\boldsymbol{c}^0,0)_w e_J, & \text{if } \ell(w)\geq 2 \text{ and } \ell(s_0w)=\ell(w)+1 \\ -e_J(0,\boldsymbol{c}^0,0)_w e_J, & \text{if } \ell(w)\geq 2 \text{ and } \ell(s_1w)=\ell(w)+1 \\ e_J(0,\boldsymbol{c}^0,0)_w e_J, & \text{if } w\in s_1\Omega \\ -e_J(0,\boldsymbol{c}^0,0)_w e_J, & \text{if } w\in s_0\Omega. \end{cases} \tag{3.24}$$

It follows that a basis for $e_J E^1 e_J = e_J \ker(g_1) e_J$ is

$$\{e_J(0,\boldsymbol{c}^0,0)_w e_J \mid w\in W, \ell(w)\geq 1\}. \tag{3.25}$$

Remark 41. Let $w\in \widetilde{W}\setminus W$. Then by the above, $e_J(0,\boldsymbol{c}^0,0)_w e_J = 0$ and in fact $e_J h^1(w) e_J = 0$. In particular $H^1(J_w,k)=0$. Since $H^1(J_w,I)=\mathrm{Hom}((J_w)_\Phi,k)$, where $(J_w)_\Phi$ is the Frattini quotient of J_w, we conclude that $(J_w)_\Phi = 0$.

The above suggests that the Frattini quotient of J itself is trivial. We record this now and present an alternate (direct) proof.

Proposition 42. *The Frattini quotient of J is trivial.*

Proof. We proceed analogously to the proof of [OS18] Proposition 3.62. Recall the Iwahori factorization $J = \left(\begin{smallmatrix} 1 & 0 \\ p\mathbb{Z}_p & 1 \end{smallmatrix}\right) T^0 \left(\begin{smallmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{smallmatrix}\right)$. For $t\in\mathbb{Z}_p^\times$ and $b,c\in\mathbb{Z}_p$, one computes

$$\left[\left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{smallmatrix}\right)\right] = \begin{pmatrix} 1 & (t^2-1)\mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$$
$$\left[\left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ p\mathbb{Z}_p & 1 \end{smallmatrix}\right)\right] = \begin{pmatrix} 1 & 0 \\ (t^{-2}-1)p\mathbb{Z}_p & 1 \end{pmatrix}$$
$$\left[\left(\begin{smallmatrix} 1 & 0 \\ pc & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)\right] = \begin{pmatrix} 1-pbc & pb^2c \\ -p^2bc^2 & 1+pbc+p^2b^2c^2 \end{pmatrix} \in \begin{pmatrix} 1 & 0 \\ p^2\mathbb{Z}_p & 1 \end{pmatrix} \begin{pmatrix} 1-pbc & 0 \\ 0 & (1-pbc)^{-1} \end{pmatrix} \begin{pmatrix} 1 & p\mathbb{Z}_p \\ 0 & 1 \end{pmatrix}.$$

Since $p\geq 5$, $(t^2-1)\mathbb{Z}_p = \mathbb{Z}_p$ and $(t^{-2}-1)\mathbb{Z}_p = \mathbb{Z}_p$. (If $t=2$ then evidently t^2-1 is invertible in $\mathbb{Z}_p$ for $p\geq 5$. Moreover $t^{-2}-1\equiv 4^{-1}-1\not\equiv 0 \pmod{p\mathbb{Z}_p}$, so it too is invertible in $\mathbb{Z}_p$.) Therefore $\Phi(J)$ contains $I = \left(\begin{smallmatrix} 1 & 0 \\ p\mathbb{Z}_p & 1 \end{smallmatrix}\right) T^1 \left(\begin{smallmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{smallmatrix}\right)$. We then have the formula $[J:I] = [J:\Phi(J)][\Phi(J):I]$. The left-hand side is $p-1$, and since $[J:\Phi(J)]$ is a power of p, this forces $[J:\Phi(J)]=1$. □

A basis for $e_J E^2 e_J$ We proceed in an analogous way compared to $e_J E^1 e_J$. By Lemma 37, we have $e_J E^2 e_J \cong e_J (F^1H)^{\vee,f} e_J$ as H-bimodules. This isomorphism can be written explicitly using Lemma 36 and simplifying. Since a basis for $e_J(F^1H)^{\vee,f}e_J$ is the set of $e_J(\tau_w^\vee|_{F^1H})e_J$ with w ranging over W with $\ell(w)\geq 1$, a basis for $e_J E^2 e_J$ is

$$\{e_J(0,\boldsymbol{\alpha}^0,0)_w e_J \mid w\in W,\ \ell(w)\geq 1\}. \tag{3.26}$$

A basis for $e_J E^3 e_J$ By Lemma 32, the H-bimodule isomorphism $\Delta^3 \colon E^3 \to ({}^{\mathcal{J}}(E^0)^{\mathcal{J}})^{\vee,f}$ sending ϕ_w to $\tau_w^\vee$ restricts to an isomorphism $e_J E^3 e_J \to e_J({}^{\mathcal{J}}(E^0)^{\mathcal{J}})^{\vee,f} e_J$ and moreover we have the identification $e_J({}^{\mathcal{J}}(E^0)^{\mathcal{J}})^{\vee,f} e_J \cong ({}^{\mathcal{J}}(e_J E^0 e_J)^{\mathcal{J}})^{\vee,f}$. A basis for $e_J E^3 e_J$ is therefore given by

$$\{e_J \phi_w e_J \mid w \in W\}. \tag{3.27}$$

Remark 43. By Lemma 35 and Lemma 37, $e_J E^1 e_J \cong e_J F^1 H e_J$ and $e_J E^2 e_J \cong e_J (F^1 H)^{\vee,f} e_J$ as H-bimodules. We then have the following injective H-bimodule homomorphisms:

$$e_J E^1 e_J \xrightarrow{\simeq} e_J F^1 H e_J \hookrightarrow e_J H e_J \tag{3.28}$$

$$e_J E^2 e_J \xrightarrow{\simeq} e_J (F^1 H)^{\vee,f} e_J \tag{3.29}$$

$$e_J E^3 e_J \xrightarrow{\simeq} {}^{\mathcal{J}}(e_J H^{\vee,f} e_J)^{\mathcal{J}} \xrightarrow{\simeq} e_J H^{\vee,f} e_J \tag{3.30}$$

where the first arrow in (3.30) is the restriction of Δ^3 and the second arrow is induced by induced by the isomorphism $\mathcal{J} \colon H \to H$. The restriction map $e_J H^{\vee,f} e_J \to e_J (F^1 H)^{\vee,f} e_J$ shows that in fact $e_J (F^1 H)^{\vee,f} e_J$ is a quotient of $e_J H^{\vee,f} e_J$ modulo the subspace generated by $e_J \tau_1^\vee$. We conclude that for each $0 \le i \le 3$, there exists an injective H-bimodule map sending $e_J E^i e_J$ to one of $e_J H e_J$, $e_J H^{\vee,f} e_J$, or a quotient of $e_J H^{\vee,f} e_J$.

3.2.1 The product in $e_J E^* e_J$

The left action of $H = E^0$ on E^i for $i = 1, 2, 3$ is given in [OS21] Prop. 4.9, 5.5, and [OS19] Prop 8.2, respectively. The action simplifies when restricted to $e_J E^i e_J$. For $\epsilon \in \{0, 1\}$ and $w \in W$ with $\ell(w) \ge 1$, we have

$$\tau_{s_\epsilon} \cdot e_J(0, \boldsymbol{c}^0, 0)_w e_J = \begin{cases} -e_J(0, \boldsymbol{c}^0, 0)_{s_\epsilon w} e_J, & \text{if } w \in W^\epsilon \\ -e_J(0, \boldsymbol{c}^0, 0)_w e_J, & \text{if } w \in W^{1-\epsilon}, \end{cases} \tag{3.31}$$

$$\tau_{s_\epsilon} \cdot e_J(0, \boldsymbol{\alpha}^0, 0)_w e_J = \begin{cases} 0, & \text{if } w \in W^\epsilon \\ -e_J\left((0, \boldsymbol{\alpha}^0, 0)_w - (0, \boldsymbol{\alpha}^0, 0)_{s_\epsilon w}\right) e_J, & \text{if } w \in W^{1-\epsilon} \text{ with } \ell(w) \ge 2 \\ -e_J(0, \boldsymbol{\alpha}^0, 0)_w e_J & \text{if } w \in W^{1-\epsilon} \text{ with } \ell(w) = 1. \end{cases} \tag{3.32}$$

And for $w \in W$, $\omega \in \Omega$

$$\tau_{s_\epsilon} \cdot e_J \tau_w^\vee e_J = \begin{cases} e_J(\tau_{sw}^\vee - \tau_w^\vee) e_J, & \text{if } w \in W^{1-\epsilon} \\ 0, & \text{otherwise,} \end{cases} \tag{3.33}$$

where the element $\tau_w^\vee \in {}^{\mathcal{J}}(H^{\vee,f})^{\mathcal{J}} \cong E^3$ denotes the dual of τ_w. Using the anti-involution $\mathcal{J}$, one then readily obtains the right action of H on $e_J E^* e_J$.

We now investigate the full product in the algebra $e_J E^* e_J$. The product of E^* in $\mathcal{C}_{E^*}(Z)$, the commutator of the centre of H, is described explicitly in §9 of [OS21]. It is

Prop. 9.1 *loc. cit.* that $\mathcal{C}_{E^*}(Z)$ coincides with $\ker(g)$, in which case we see that $e_J E^* e_J$ is fully contained in $\mathcal{C}_{E^*}$. This implies in particular that e_J commutes with $\ker(g_1)$. Define the map

$$\begin{aligned} e_J F^1 H e_J \otimes_H e_J F^1 H e_J &\longrightarrow e_J{}^{\jmath}((F^1H)^{\vee,f})^{\jmath} e_J \\ e_J(\tau_v \otimes \tau_w) e_J &\longmapsto \begin{cases} -e_J(\tau_v \cdot \tau_w^{\vee}|_{F^1 H}) e_J, & \text{if } \ell(w) = 1 \\ 0, & \text{if } \ell(w) \geq 2 \end{cases} \end{aligned} \tag{3.34}$$

Remark 44 ([OS21] Remark 9.4)**.** In fact the map above has $e_J(\tau_v \otimes \tau_w)e_J \mapsto 0$ if $\ell(v) \geq 2$ or $\ell(w) \geq 2$.

We also introduce the map $e_J E^2 e_J \to e_J (F^1 H)^{\vee,f} e_J$ which is given via the composite

$$e_J E^2 e_J \xrightarrow{\Delta^2|_{e_J E^2 e_J}} e_J{}^{\jmath}((\ker(g_1))^{\vee,f})^{\jmath} e_J \xrightarrow{\cong} e_J{}^{\jmath}((F^1H)^{\vee,f})^{\jmath} e_J \tag{3.35}$$

Here the second map is induced by the inverse of (3.21). We see that the element $e_J \tau_w^{\vee}|_{F^1 H} e_J \in e_J{}^{\jmath}((F^1H)^{\vee,f})^{\jmath} e_J$ on the right-hand side corresponds to $e_J(0, \boldsymbol{\alpha}^0, 0)_w e_J \in e_J E^2 e_J$ on the left-hand side if $w \in W^0$ and $-e_J(0, \boldsymbol{\alpha}^0, 0)_w e_J$ otherwise.

We then have the following commutative diagrams of H-bimodules ([OS21] Prop. 9.5 and 9.6):

$$\begin{array}{ccc} e_J E^1 e_J \otimes e_J E^1 e_J & \xrightarrow{\text{Yoneda product}} & e_J E^2 e_J \\ \downarrow{\scriptstyle (3.21)^{-1} \otimes (3.21)^{-1}} & & \downarrow{\scriptstyle (3.35)} \\ e_J F^1 H e_J \otimes_H e_J F^1 H e_J & \xrightarrow{(3.34)} & e_J{}^{\jmath}((F^1H)^{\vee,f})^{\jmath} e_J \end{array} \tag{3.36}$$

$$\begin{array}{ccc} e_J E^1 e_J \otimes e_J E^2 e_J & \xrightarrow{\text{Yoneda product}} & e_J E^3 e_J \\ \downarrow{\scriptstyle (3.21)^{-1} \otimes (3.35)} & & \downarrow{\scriptstyle \Delta^3|_{e_J E^3}} \\ e_J F^1 H e_J \otimes_H e_J{}^{\jmath}((F^1H)^{\vee,f})^{\jmath} e_J & \xrightarrow{\tau \otimes \alpha \mapsto -\alpha(\jmath(\tau) -)} & e_J{}^{\jmath}((H)^{\vee,f})^{\jmath} e_J \end{array} \tag{3.37}$$

$$\begin{array}{ccc} e_J E^2 e_J \otimes e_J E^1 e_J & \xrightarrow{\text{Yoneda product}} & e_J E^3 e_J \\ \downarrow{\scriptstyle (3.35) \otimes (3.21)^{-1}} & & \downarrow{\scriptstyle \Delta^3|_{e_J E^3}} \\ e_J{}^{\jmath}((F^1H)^{\vee,f})^{\jmath} e_J \otimes_H e_J F^1 H e_J & \xrightarrow{\alpha \otimes \tau \mapsto -\alpha(-\jmath(\tau))} & e_J{}^{\jmath}((H)^{\vee,f})^{\jmath} e_J \end{array} \tag{3.38}$$

In the proofs of [OS21] Prop. 9.5 and 9.6, the product on the generators of $\mathcal{C}_{\widetilde{E^i}}(Z)$ is computed. Restricted to $e_J E^* e_J$, they are as follows. For $\epsilon \in \{0, 1\}$ and $w \in \widetilde{W}$ with

$\ell(w) \geq 1$, we have

$$e_J(0, \boldsymbol{c}^0, 0)_{s_\epsilon} \cdot (0, \boldsymbol{c}^0, 0)_{s_{1-\epsilon}} e_J = 0 \tag{3.39}$$

$$e_J(0, \boldsymbol{c}^0, 0)_{s_\epsilon} \cdot (0, \boldsymbol{c}^0, 0)_{s_\epsilon} e_J = (-1)^{1-\epsilon} e_J(0, \boldsymbol{\alpha}^0, 0)_{s_\epsilon} \tag{3.40}$$

$$e_J(0, \boldsymbol{c}^0, 0)_{s_\epsilon} \cdot (0, \boldsymbol{\alpha}^0, 0)_w e_J = \begin{cases} e_J(-\phi_{s_\epsilon w} + \phi_w), & \text{if } w \in \widetilde{W}^{1-\epsilon} \\ 0, & \text{otherwise} \end{cases} \tag{3.41}$$

$$e_J(0, \boldsymbol{\alpha}^0, 0)_w \cdot (0, \boldsymbol{c}^0, 0)_{s_\epsilon} e_J = \begin{cases} (-1)^{\ell(w)+1} e_J(-\phi_{w s_\epsilon} + \phi_w), & \text{if } w^{-1} \in \widetilde{W}^{1-\epsilon} \\ 0, & \text{otherwise} \end{cases} \tag{3.42}$$

In light of Remark 44, the product $e_J E^1 e_J \otimes_H e_J E^1 e_J \to e_J E^2 e_J$ is fully described by (3.39) and (3.40). We now write down the formula for the products $e_J E^2 e_J \otimes_H e_J E^1 e_J \to e_J E^3 e_J$ and $e_J E^1 e_J \otimes_H e_J E^2 e_J \to e_J E^3 e_J$.

Proposition 45. *Let $v \in W^\epsilon$, $v^{-1} \in W^\delta$, $w \in W$ with $\ell(w) \geq 1$. We have*

$$e_J(0, \boldsymbol{c}^0, 0)_{s_\epsilon v} \cdot (0, \boldsymbol{\alpha}^0, 0)_w e_J \tag{3.43}$$
$$= \begin{cases} (-1)^{\ell(s_\epsilon v)+1} e_J(-\phi_{s_\epsilon v w} + \phi_{vw}) e_J, & \text{if } \ell(s_\epsilon v w) < \ell(s_\epsilon v) + \ell(w) \text{ and } \ell(s_\epsilon v) \leq \ell(w) \\ 0, & \text{otherwise} \end{cases}$$

$$e_J(0, \boldsymbol{\alpha}^0, 0)_w \cdot (0, \boldsymbol{c}^0, 0)_{v s_\delta} e_J \tag{3.44}$$
$$= \begin{cases} (-1)^{\ell(w)+1} e_J(-\phi_{w v s_\delta} + \phi_{wv}) e_J, & \text{if } \ell(w v s_\delta) < \ell(w) + \ell(v s_\delta) \text{ and } \ell(v s_\delta) \leq \ell(w) \\ 0, & \text{otherwise} \end{cases}$$

Proof. We check (3.43) first. Proceed by induction on $\ell(v)$. The case $\ell(v) = 0$ is already covered by (3.41). Next, assume that (3.43) holds for $s_\epsilon v$; we will show it holds for $s_{1-\epsilon} s_\epsilon v$. We have

$$\begin{aligned} &e_J \cdot (0, \boldsymbol{c}^0, 0)_{s_{1-\epsilon} s_\epsilon v} \cdot (0, \boldsymbol{\alpha}^0, 0)_w \\ &\quad = -e_J \tau_{s_{1-\epsilon}} \cdot (0, \boldsymbol{c}^0, 0)_{s_\epsilon v} \cdot (0, \boldsymbol{\alpha}^0, 0)_w \\ &\quad = \begin{cases} -(-1)^{\ell(s_\epsilon v)+1} e_J \tau_{s_{1-\epsilon}} \cdot (-\phi_{s_\epsilon v w} + \phi_{vw}), & \text{if } \ell(s_\epsilon v w) < \ell(s_\epsilon v) + \ell(w) \text{ and } \ell(s_\epsilon v) \leq \ell(w) \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

This can be simplified using [OS21] §2.3.9 and equation (40) therein. We have three cases.

Case 1 ($\ell(s_\epsilon v) < \ell(w)$). In this case we have $s_\epsilon v w \in W^\epsilon$ and $vw \in W^{1-\epsilon}$. We get

$$-(-1)^{\ell(v)} e_J \tau_{s_{1-\epsilon}} \cdot (-\phi_{s_\epsilon v w} + \phi_{vw}) = (-1)^{\ell(s_\epsilon v)} e_J(-\phi_{s_{1-\epsilon} s_\epsilon v w} + \phi_{s_\epsilon v w}),$$

as desired.

Case 2 ($\ell(s_\epsilon v) = \ell(w)$). In this case we wish to show $e_J \tau_{s_{1-\epsilon}} \cdot (-\phi_{s_\epsilon vw} + \phi_{vw}) = 0$. Indeed, $\ell(s_\epsilon vw) = 0$ so that both $s_\epsilon vw \in W^{1-\epsilon}$ and $vw \in W^{1-\epsilon}$. This proves the result.

Case 3 ($\ell(s_\epsilon v) > \ell(w)$). Evidently the product is zero here by the induction hypothesis.

This shows (3.43). The equation (3.44) follows by using the fact that $\mathcal{J}(\alpha\cdot\beta) = \mathcal{J}(\beta)\cdot\mathcal{J}(\alpha)$ for $\alpha \in E^2$ and $\beta \in E^1$ along with the equations $\mathcal{J}((0, \boldsymbol{\alpha}^0, 0)_w) = (-1)^{\ell(w)}(0, \boldsymbol{\alpha}^0, 0)_{w^{-1}}$ and $\mathcal{J}((0, \boldsymbol{c}^0, 0)_w) = (-1)^{\ell(w)}(0, \boldsymbol{c}^0, 0)_{w^{-1}}$

□

3.2.2 The centre of $e_J E^* e_J$

In this section we will compute the centre (not the graded centre) of $e_J E^* e_J$. Recall the central element ζ in (3.16). The element $e_J\zeta$ is again central and in fact the centre of $e_J H e_J$ is $e_J k[\zeta]$.

Recall the elements $\psi_w \in E^3$ defined in Remark 38. We denote by E^3_{odd} the linear subspace of E^3 generated by elements of the form $\psi_{s_1(s_0s_1)^n} + \psi_{s_0(s_1s_0)^n}$ for $n \geq 0$.

Lemma 46. *The component of the centre of $e_J E^* e_J$ lying in $e_J E^3 e_J$ is $e_J(E^3_{odd} \oplus k\phi_1)e_J$.*

Proof. By Remark 38, it suffices to show that the component of the centre in $e_J \ker(\mathcal{S}^3)e_J$ is generated by the elements $e_J(\psi_{s_1(s_0s_1)^n} + \psi_{s_0(s_1s_0)^n})e_J$ for $n \geq 0$. Suppose that $x \in e_J \ker(\mathcal{S}^3)e_J$ is a central element of $e_J E^*$. We write

$$x = \sum_{i=1}^{n} e_J \alpha_i \psi_{w_i}, \tag{3.45}$$

with $\alpha_i \in k$ and $w_i \in W$. For $w \in W$ and $\epsilon \in \{0, 1\}$, the element τ_{s_ϵ} acts on ψ_w as follows

$$e_J(\tau_{s_\epsilon} \cdot \psi_w) = \begin{cases} -e_J\psi_w, & \text{if } w \in W^{1-\epsilon} \\ 0, & \text{if } w \in W^\epsilon \text{ and } \ell(w) = 1 \\ e_J\psi_{s_{1-\epsilon}w}, & \text{if } w \in W^\epsilon \text{ and } \ell(w) \geq 2 \end{cases} \tag{3.46}$$

$$e_J(\psi_w \cdot \tau_{s_\epsilon}) = \begin{cases} 0, & \text{if } w^{-1} \in W^\epsilon \\ -e_J\psi_w, & \text{if } w^{-1} \in W^{1-\epsilon} \text{ and } \ell(w) = 1 \\ e_J(\psi_{ws_\epsilon} - \psi_w), & \text{if } w^{-1} \in W^{1-\epsilon} \text{ and } \ell(w) \geq 2. \end{cases} \tag{3.47}$$

To simplify the notation of what follows, for $\epsilon, \eta \in \{0, 1\}$ let W^ϵ_η denote the subset of W^ϵ consisting of elements w with $w^{-1} \in W^\eta$.

As x is central and $\tau_{s_1}^2 = -e_J\tau_{s_1}$ we have $-\tau_{s_1}\cdot x = \tau_{s_1}\cdot x\cdot\tau_{s_1}$. Expanding both sides,

$$e_J\left(-\sum_{\substack{w_i \text{ s.t.}\\ w_i\in W^1\\ \ell(w_i)\geq 2}}\alpha_i\psi_{s_0w_i}+\sum_{\substack{w_i \text{ s.t.}\\ w_i\in W^0}}\alpha_i\psi_{w_i}\right)$$

$$=e_J\left(\sum_{\substack{w_i \text{ s.t.}\\ w_i\in W_0^1\\ \ell(w_i)\geq 3}}\alpha_i\psi_{s_0w_is_1}-\sum_{\substack{w_i \text{ s.t.}\\ w_i\in W_0^1\\ \ell(w_i)\geq 2}}\alpha_i\psi_{s_0w_i}-\sum_{\substack{w_i \text{ s.t.}\\ w_i\in W_0^0}}\alpha_i\psi_{w_is_1}+\sum_{\substack{w_i \text{ s.t.}\\ w_i\in W_0^0}}\alpha_i\psi_{w_i}\right)$$

and canceling,

$$e_J\left(-\sum_{\substack{w_i \text{ s.t.},\\ w_i\in W_1^1\\ \ell(w_i)\geq 2}}\alpha_i\psi_{s_0w_i}+\sum_{\substack{w_i \text{ s.t.}\\ w_i\in W_1^0}}\alpha_i\psi_{w_i}\right)=e_J\left(\sum_{\substack{w_i \text{ s.t.}\\ w_i\in W_0^1\\ \ell(w_i)\geq 3}}\alpha_i\psi_{s_0w_is_1}-\sum_{\substack{w_i \text{ s.t.}\\ w_i\in W_0^0\\ \ell(w_i)\geq 2}}\alpha_i\psi_{w_is_1}\right). \quad (3.48)$$

Similarly, using the equation $\tau_{s_0}\cdot x = \tau_{s_0}\cdot x\cdot\tau_{s_1}$, we obtain

$$e_J\left(-\sum_{\substack{w_i \text{ s.t.}\\ w_i\in W_0^0\\ \ell(w_i)\geq 2}}\alpha_i\psi_{s_1w_i}+\sum_{\substack{w_i \text{ s.t.}\\ w_i\in W_0^1}}\alpha_i\psi_{w_i}\right)=e_J\left(\sum_{\substack{w_i \text{ s.t.}\\ w_i\in W_1^0\\ \ell(w_i)\geq 3}}\alpha_i\psi_{s_1w_is_0}-\sum_{\substack{w_i \text{ s.t.}\\ w_i\in W_1^1\\ \ell(w_i)\geq 2}}\alpha_i\psi_{w_is_0}\right). \quad (3.49)$$

Let $n\geq 0$. From (3.48), we conclude that the coefficient of $\psi_{(s_1s_0)^n}$ in (3.45) must be the same as the coefficient of $\psi_{s_0(s_1s_0)^ns_1}$; likewise, (3.49) shows that the coefficient of $\psi_{(s_0s_1)^n}$ must be the same as the coefficient of $\tau_{s_1(s_0s_1)^ns_0}$. Therefore if $\ell(w)$ is even then the coefficient of ψ_w is zero.

Now, using (3.48) again, we see that the coefficient of $\psi_{s_0(s_1s_0)^n}$ must be the same as the coefficient of $\psi_{s_1(s_0s_1)^n}$. This shows that the component of the centre in $e_JE^3e_J$ is contained in $e_JE^3_{\text{odd}}e_J$. The opposite inclusion holds as one readily checks using the formulas (3.46) and (3.47). For $n\geq 0$ and $\epsilon\in\{0,1\}$, we see that

$$\tau_{s_\epsilon}\cdot(\psi_{s_1(s_0s_1)^n}+\psi_{s_0(s_1s_0)^n})=\psi_{(s_\epsilon s_{1-\epsilon})^n}-\psi_{s_\epsilon(s_{1-\epsilon}s_\epsilon)^n}=(\psi_{s_1(s_0s_1)^n}+\psi_{s_0(s_1s_0)^n})\cdot\tau_{s_\epsilon}.$$

□

Lemma 47. *We have $e_J E^2 e_J \cong e_J \ker(\mathcal{S}^3) e_J$ as H-bimodules.*

Proof. We chain together the following H-bimodule isomorphisms

$$e_J E^2 e_J \xrightarrow[(3.29)]{\simeq} (e_J F^1 H e_J)^{\vee,f} \cong (e_J E^0 e_J)^{\vee,f}/k e_J \tau_1^\vee \xrightarrow[(\Delta^3)^{-1}]{\simeq} e_J E^3 e_J / k e_J \phi_1 \cong e_J \ker(\mathcal{S}^3) e_J, \tag{3.50}$$

where the last isomorphism is due to the fact that there is the decomposition of H-bimodules $e_J E^3 e_J = e_J \ker(\mathcal{S}^3) e_J \oplus k e_J \phi_1$. □

Remark 48. We denote by $e_J E^2_{\text{odd}} e_J$ the preimage of $e_J E^3_{\text{odd}} e_J$ in $e_J E^2 e_J$ via (3.50). For $w \in W^\epsilon$ and $w^{-1} \in W^\eta$, the isomorphism acts on basis elements as follows

$$e_J(0, \boldsymbol{\alpha}^0, 0)_w e_J \mapsto (-1)^{1-\epsilon} e_J(\psi_w + \psi_{s_{1-\epsilon} w} + \psi_{s_\epsilon s_{1-\epsilon} w} + \cdots + \psi_{s_{1-\eta}}) e_J,$$

with inverse $e_J \psi_w e_J \mapsto (-1)^{1-\epsilon} e_J[(0, \boldsymbol{\alpha}^0, 0)_w + (0, \boldsymbol{\alpha}^0, 0)_{s_{1-\epsilon} w}] e_J$. The subspace $e_J E^2_{\text{odd}} e_J$ therefore has as a basis the set of elements

$$e_J \left(-(0, \boldsymbol{\alpha}^0, 0)_{s_1 (s_0 s_1)^n} - (0, \boldsymbol{\alpha}^0, 0)_{(s_0 s_1)^n} + (0, \boldsymbol{\alpha}^0, 0)_{s_0 (s_1 s_0)^n} + (0, \boldsymbol{\alpha}^0, 0)_{(s_1 s_0)^n} \right) e_J \tag{3.51}$$

for $n \geq 0$.

Let f be the isomorphism $e_J F^1 H \to e_J E^1$ induced by (3.21).

Proposition 49. *For $0 \leq i \leq 3$, let $Z_i(e_J E^* e_J)$ denote the component of the centre of $e_J E^* e_J$ lying in $e_J E^i e_J$. The centre of $e_J E^* e_J$ is given component-wise as follows*

(i) $Z_0(e_J E^* e_J) = e_J k[\zeta]$,

(ii) $Z_1(e_J E^* e_J) = e_J f(\zeta - 1) k[\zeta]$,

(iii) $Z_2(e_J E^* e_J) = e_J E^2_{odd} e_J$,

(iv) $Z_3(e_J E^* e_J) = e_J(E^3_{odd} \oplus k\phi_1) e_J$.

Proof. (i) Evidently $Z_0(e_J E^* e_J) \subset e_J k[\zeta]$ as $e_J k[\zeta]$ is the centre of $e_J H e_J \subset e_J E^* e_J$. On the other hand, let $x \in e_J k[\zeta]$. By Remark 43, for any $z \in e_J E^* e_J$ there is an injective H-bimodule homomorphism sending z to an element in either $e_J H e_J$, $(e_J H e_J)^{\vee,f}$, or a quotient of $(e_J H e_J)^{\vee,f}$. It remains only to check that $e_J \zeta$ commutes with $(e_J H e_J)^{\vee,f}$. Using the formula (3.33) we compute that for any $e_J \tau_w^\vee \in e_J H^{\vee,f}$ with $w \in W^\epsilon$ and $w^{-1} \in W^\eta$,

$$e_J \tau_w^\vee \cdot \zeta_J = \begin{cases} \tau_w^\vee, & \text{if } \ell(w) = 0, \\ \tau_1^\vee, & \text{if } \ell(w) = 1, \\ \tau_{w s_{1-\eta} s_\eta}^\vee, & \text{if } \ell(w) \geq 2, \end{cases} \qquad \zeta_J \cdot e_J \tau_w^\vee = \begin{cases} \tau_w^\vee, & \text{if } \ell(w) = 0, \\ \tau_1^\vee, & \text{if } \ell(w) = 1, \\ \tau_{s_\epsilon s_{1-\epsilon} w}^\vee, & \text{if } \ell(w) \geq 2. \end{cases}$$

Indeed, the elements $w s_{1-\eta} s_\eta$ and $s_\epsilon s_{1-\epsilon} w$ are equal for $\ell(w) \geq 2$.

(ii) Let $y \in Z_1(e_J E^* e_J)$. Then the pre-image of y in $e_J F^1 H$ under f is central in $e_J H e_J$ and thus lies in $e_J k[\zeta]$. It is a straightforward computation to verify that $e_J F^1 H \cap e_J k[\zeta] = e_J(\zeta_J - 1)k[\zeta]$, which gives $Z_1(e_J E^* e_J) \subseteq e_J f(\zeta_J - 1)k[\zeta]$. On the other hand, note that

$$\begin{aligned} f(e_J(\zeta - 1)) &= e_J\big[-(0, \boldsymbol{c}^0, 0)_{s_0} + (0, \boldsymbol{c}^0, 0)_{s_1} - \tau_{s_1}(0, \boldsymbol{c}^0, 0)_{s_0} + \tau_{s_0}(0, \boldsymbol{c}^0, 0)_{s_1}\big] \\ &= e_J\big[-(0, \boldsymbol{c}^0, 0)_{s_0} + (0, \boldsymbol{c}^0, 0)_{s_1} - (0, \boldsymbol{c}^0, 0)_{s_0}\tau_{s_1} + (0, \boldsymbol{c}^0, 0)_{s_1}\tau_{s_0}\big] \end{aligned}$$

Using (3.39) and (3.40), one computes

$$\begin{aligned} e_J(0, \boldsymbol{c}^0, 0)_{s_\epsilon} \cdot \big[-(0, \boldsymbol{c}^0, 0)_{s_0} + (0, \boldsymbol{c}^0, 0)_{s_1} &- (0, \boldsymbol{c}^0, 0)_{s_0}\tau_{s_1} + (0, \boldsymbol{c}^0, 0)_{s_1}\tau_{s_0}\big] \\ &= e_J(0, \boldsymbol{\alpha}^0, 0)_{s_\epsilon} + e_J(0, \boldsymbol{\alpha}^0, 0)_{s_\epsilon}\tau_{s_{1-\epsilon}} \\ &= e_J(0, \boldsymbol{\alpha}^0, 0)_{s_\epsilon}, \end{aligned}$$

and

$$\begin{aligned} \big[-(0, \boldsymbol{c}^0, 0)_{s_0} + (0, \boldsymbol{c}^0, 0)_{s_1} - \tau_{s_1}(0, \boldsymbol{c}^0, 0)_{s_0} &+ \tau_{s_0}(0, \boldsymbol{c}^0, 0)_{s_1}\big] \cdot e_J(0, \boldsymbol{c}^0, 0)_{s_\epsilon} \\ &= e_J(0, \boldsymbol{\alpha}^0, 0)_{s_\epsilon} + e_J\tau_{s_{1-\epsilon}}(0, \boldsymbol{\alpha}^0, 0)_{s_\epsilon} \\ &= e_J(0, \boldsymbol{\alpha}^0, 0)_{s_\epsilon}. \end{aligned}$$

Meanwhile, for $w \in W^\epsilon$, $w^{-1} \in W^\delta$ we have

$$\begin{aligned} e_J(0, \boldsymbol{\alpha}^0, 0)_w \cdot \big[-(0, \boldsymbol{c}^0, 0)_{s_0} &+ (0, \boldsymbol{c}^0, 0)_{s_1} - (0, \boldsymbol{c}^0, 0)_{s_0}\tau_{s_1} + (0, \boldsymbol{c}^0, 0)_{s_1}\tau_{s_0}\big] \\ &= \begin{cases} e_J \cdot (\phi_{ws_{1-\delta}} - \phi_w + \phi_{ws_{1-\delta}s_\delta} - \phi_{ws_{1-\delta}}), & \text{if } \ell(w) \geq 2 \\ e_J \cdot (-\phi_w + \phi_1), & \text{if } \ell(w) = 1 \end{cases} \\ &= \begin{cases} e_J \cdot (-\phi_w + \phi_{ws_{1-\delta}s_\delta}), & \text{if } \ell(w) \geq 2 \\ e_J \cdot (-\phi_w + \phi_1), & \text{if } \ell(w) = 1 \end{cases} \end{aligned}$$

and

$$\begin{aligned} \big[-(0, \boldsymbol{c}^0, 0)_{s_0} + (0, \boldsymbol{c}^0, 0)_{s_1} &- \tau_{s_1}(0, \boldsymbol{c}^0, 0)_{s_0} + \tau_{s_0}(0, \boldsymbol{c}^0, 0)_{s_1}\big] \cdot e_J(0, \boldsymbol{\alpha}^0, 0)_w \\ &= \begin{cases} e_J \cdot (\phi_{s_{1-\epsilon}w} - \phi_w + \phi_{s_{1-\epsilon}s_\epsilon w} - \phi_{s_{1-\epsilon}w}), & \text{if } \ell(w) \geq 2 \\ e_J \cdot (-\phi_w + \phi_1), & \text{if } \ell(w) = 1 \end{cases} \\ &= \begin{cases} e_J \cdot (-\phi_w + \phi_{s_{1-\epsilon}s_\epsilon w}), & \text{if } \ell(w) \geq 2 \\ e_J \cdot (-\phi_w + \phi_1), & \text{if } \ell(w) = 1 \end{cases} \end{aligned}$$

The elements $ws_{1-\delta s_\delta}$ and $s_{1-\epsilon}s_\epsilon w$ are of course the same. Therefore the element $e_J f(\zeta - 1)$ is central.

(iii) If $z \in e_J E^2 e_J$ is central, then its image in $e_J E^3 e_J$ via (3.50) is central in $e_J E^* e_J$; hence $Z_2(e_J E^* e_J) \subseteq e_J E^2_{\text{odd}} e_J$. To show the opposite inclusion, we simply check that the basis of $e_J E^2_{\text{odd}} e_J$ commutes with $e_J E^1 e_J$. Because $e_J E^1 e_J$ is generated by $e_J(0, \boldsymbol{c}^0, 0)_{s_0} e_J$ and $e_J(0, \boldsymbol{c}^0, 0)_{s_1} e_J$ as an H-bimodule, it suffices to only check commutativity on these elements. For $n \geq 0$, we have

$$\begin{aligned}
&(0, \boldsymbol{c}^0, 0)_{s_0} \cdot [-(0, \boldsymbol{\alpha}^0, 0)_{s_1(s_0 s_1)^n} - (0, \boldsymbol{\alpha}^0, 0)_{(s_0 s_1)^n} + (0, \boldsymbol{\alpha}^0, 0)_{s_0(s_1 s_0)^n} + (0, \boldsymbol{\alpha}^0, 0)_{(s_1 s_0)^n}] \\
&= e_J(\phi_{s_1(s_0 s_1)^{n-1}} - \phi_{(s_0 s_1)^n} - \phi_{(s_1 s_0)^n} + \phi_{s_0(s_1 s_0)^n}) \\
&= [-(0, \boldsymbol{\alpha}^0, 0)_{s_1(s_0 s_1)^n} - (0, \boldsymbol{\alpha}^0, 0)_{(s_0 s_1)^n} + (0, \boldsymbol{\alpha}^0, 0)_{s_0(s_1 s_0)^n} + (0, \boldsymbol{\alpha}^0, 0)_{(s_1 s_0)^n}] \cdot (0, \boldsymbol{c}^0, 0)_{s_0}
\end{aligned}$$

and

$$\begin{aligned}
&(0, \boldsymbol{c}^0, 0)_{s_1} \cdot [-(0, \boldsymbol{\alpha}^0, 0)_{s_1(s_0 s_1)^n} - (0, \boldsymbol{\alpha}^0, 0)_{(s_0 s_1)^n} + (0, \boldsymbol{\alpha}^0, 0)_{s_0(s_1 s_0)^n} + (0, \boldsymbol{\alpha}^0, 0)_{(s_1 s_0)^n}] \\
&= e_J(-\phi_{s_0(s_1 s_0)^{n-1}} + \phi_{(s_0 s_1)^n} + \phi_{(s_1 s_0)^n} - \phi_{s_1(s_0 s_1)^n}) \\
&= [-(0, \boldsymbol{\alpha}^0, 0)_{s_1(s_0 s_1)^n} - (0, \boldsymbol{\alpha}^0, 0)_{(s_0 s_1)^n} + (0, \boldsymbol{\alpha}^0, 0)_{s_0(s_1 s_0)^n} + (0, \boldsymbol{\alpha}^0, 0)_{(s_1 s_0)^n}] \cdot (0, \boldsymbol{c}^0, 0)_{s_1}.
\end{aligned}$$

(iv) By Lemma 46.

□

Remark 50. We have really computed the centre here rather than the graded centre, and the fact that the centre contains a nontrivial square (see (3.40)) shows that they are not the same.

3.3 The Spherical Ext-algebra

In this section we describe the Spherical Ext-algebra E^*_K.

A basis for H_K is the set of all τ^K_d for $d \in {}_K\mathcal{D}_K$, where ${}_K\mathcal{D}_K$ consists of elements $d \in W$ which have minimal length in their coset $\mathfrak{W}_K \, d \, \mathfrak{W}_K$. We compute ${}_K\mathcal{D}_K$ explicitly.

Lemma 51. *We have* ${}_K\mathcal{D}_K = \{w \in W : \ell(s_0 w) = \ell(w s_0) = \ell(w) + 1\} = \{s_1(s_0 s_1)^n \mid n \geq 0\} \cup \{1\}$.

Proof. Recall that $\mathfrak{W}_K = \{1, s_0\}$. Therefore $w \in W$ has minimal length in the coset $\mathfrak{W}_K \, w \, \mathfrak{W}_K$ if and only if $\ell(s_0 w) = \ell(w s_0) = \ell(w) + 1$. Moreover, each coset has exactly one element such that $\ell(s_0 w) = \ell(w s_0) = \ell(w) + 1$. This proves the first equality. For the second equality, simply note that an arbitrary element $w \in W$ is either of the form $(s_0 s_1)^n$ or $(s_0 s_1)^n s_0$ for $n \in \mathbb{Z}$. □

Recall that E^*_K identifies with $e_K E^* e_K$ where e_K is the idempotent defined in (3.6). The degree 0 component of the isomorphism is given explicitly in Proposition 40. It follows that a basis for $e_K H e_K$ is $\{e_K \tau_w e_K\}_{w \in \widetilde{{}_K\mathcal{D}_K}}$.

A basis for $e_K E^1 e_K$ We proceed in an analogous way to §3.2. By Lemma 35, we have that $e_K E^1_K e_K = e_K \ker(g_1) e_K \cong e_K(F^1 H)e_K$ as H-bimodules. This isomorphism is induced by the isomorphism $f\colon F^1 H \to \ker(g_1)$ described in (3.21). A basis for $e_K F^1 H e_K$ is $\{e_K \tau_w e_K \mid w \in \widetilde{{}_K\mathcal{D}_K}, \ell(w) \geq 1\}$. Since all elements $w \in \widetilde{{}_K\mathcal{D}_K}$ have $\ell(s_0 w) = \ell(w s_0) = \ell(w) + 1$, for $w \in \widetilde{{}_K\mathcal{D}_K}$ with $\ell(w) \geq 1$ we get

$$\begin{aligned} f(e_K \tau_w e_K) &= e_K f(\tau_w) e_K \\ &= \begin{cases} e_K(0, \boldsymbol{c}^0, 0)_w e_K, & \text{if } \ell(w) \geq 2 \text{ and } \ell(s_0 w) = \ell(w) + 1 \\ -e_K(0, \boldsymbol{c}^0, 0)_w e_K, & \text{if } \ell(w) \geq 2 \text{ and } \ell(s_1 w) = \ell(w) + 1 \\ e_K(0, \boldsymbol{c}^0, 0)_w e_K, & \text{if } w \in s_1 \Omega \\ -e_K(0, \boldsymbol{c}^0, 0)_w e_K, & \text{if } w \in s_0 \Omega \end{cases} \\ &= e_K(0, \boldsymbol{c}^0, 0)_w e_K. \end{aligned}$$

We conclude that a basis for $e_K E^1 e_K = e_K \ker(g_1) e_K$ is

$$\{e_K(0, \boldsymbol{c}^0, 0)_w e_K \mid w \in \widetilde{{}_K\mathcal{D}_K}\}. \tag{3.52}$$

Remark 52. Similar to Remark 41, we conclude that $H^1(K_w, k) = 0$ for any $w \in W \backslash \widetilde{{}_K\mathcal{D}_K}$ and that the Frattini quotient of K is trivial.

We record this latter fact here and provide an alternate proof.

Proposition 53. *The Frattini quotient of K is trivial.*

Proof. By Proposition 42, we have $\Phi(J) = J$ which yields the sequence of nested subgroups $J \subseteq \Phi(K) \subseteq K$. Therefore $[K : J] = [K : \Phi(K)][\Phi(K) : J]$. The left-hand side is $p + 1$ which forces $[K : \Phi(K)] = 1$. □

Remark 54. The proof above generalizes readily to any parahoric $\mathcal{P}$ subgroup containing J due to the fact that $[\mathcal{P} : J] \equiv 1 \pmod{p}$ for any $\mathcal{P}$.

The following is again analogous to §3.2.

A basis for $e_K E^2 e_K$ A basis for $e_K (F^1 H)^{\vee, f} e_K$ is $\{e_K(\tau_w|_{F^1 H}) e_K \mid w \in \widetilde{{}_K\mathcal{D}_K}, \ell(w) \geq 1\}$. The isomorphism $e_K E^2 e_K \cong e_K (F^1 H)^{\vee, f} e_K$ from Lemma 37 then naturally defines the following basis for $e_K E^2 e_K$:

$$\{e_K(0, \boldsymbol{\alpha}^0, 0)_w e_K \mid w \in \widetilde{{}_K\mathcal{D}_K}, \ell(w) \geq 1\}.$$

A basis for $e_K E^3 e_K$ Using the isomorphism $\Delta^3 \colon E^3 \to ({}^\jmath(E^0)^\jmath)^{\vee,f}$ and restricting, we obtain the following basis for $e_K E^3 e_K$:

$$\{e_K \phi_w e_K \mid w \in \widetilde{{}_K\mathcal{D}_K}\}.$$

Proposition 55. *The map*

$$\begin{aligned} Z(e_J E^* e_J) &\longrightarrow e_K E^* e_K \\ z &\longmapsto e_K z e_K = e_K z \end{aligned} \tag{3.53}$$

is a k-algebra isomorphism, where $Z(e_J E^* e_J)$ is the centre of $e_J E^* e_J$.

Proof. This map is a homomorphism of k-algebras. We show that it is bijective on each graded piece, i.e. on $Z_i(e_J E^* e_J)$ for $0 \leq i \leq 3$.

- For $i = 0$, one computes that

  $$e_K \zeta^n e_K = e_K(1 + \tau_{s_1} + \cdots + \tau_{s_1(s_0 s_1)^{n-1}}) e_K.$$

  Recall that the elements $e_K \tau_{s_1(s_0 s_1)^n} e_K$, $n \geq 0$, together with e_K define a basis for $e_K E^* e_K$. Therefore the map (3.53) restricted to $Z_0(e_J E^*)$ is injective with image $e_K E^0 e_K$. Its inverse is $e_K \tau_{s_1(s_0 s_1)^n} e_K \mapsto e_J \zeta^n (\zeta - 1)$.

- For $i = 1$, recall that $e_K E^1 e_K \cong e_K F^1 H e_K$ as H-bimodules by Lemma 35. The $i = 1$ component of (3.53) then factors through the following H-bimodule homomorphisms

  $$Z_1(e_J E^* e_J) = e_J f(\zeta - 1) k[\zeta] e_J \xrightarrow{\simeq} e_J(\zeta - 1) k[\zeta] \xrightarrow{e_K \cdot -} e_K F^1 H e_K \xrightarrow{\simeq} e_K E^1 e_K \tag{3.54}$$

  By the proof of the $i = 0$ case, we know that the second arrow is indeed bijective.

- We now check $i = 3$ (as we will need this for $i = 2$). Recall that we have $e_K \phi_w e_K = 0$ when $w \in W^1$ or $w^{-1} \in W^1$. Therefore, for $n \geq 0$, (3.53) has

  $$e_J \left(\psi_{s_1(s_0 s_1)^n} + \psi_{s_0(s_1 s_0)^n}\right) \longmapsto e_K \psi_{s_1(s_0 s_1)^n} e_K = -e_K \phi_{s_1(s_0 s_1)^n} e_K.$$

  The elements $e_K \tau^\vee_{s_1(s_0 s_1)^n} e_K$ for $n \geq 0$ define a basis for $(e_K F^1 H e_K)^{\vee,f}$. Therefore (3.53) restricted to E^3_{odd} is injective with image $(e_K F^1 H e_K)^{\vee,f}$. Hence the restriction to $Z_3(e_J E^*)$, i.e. the map $Z_3(e_J E^* e_J) = E^3_{\mathrm{odd}} \oplus k e_J \phi_1 \to (e_K E^0 e_K)^{\vee,f}$, is bijective.

- For $i = 2$, we proceed analogously to the $i = 1$ case. We have that $e_K E^2 e_K \cong (e_K F^1 H e_K)^{\vee,f}$ by Lemma 37, so the $i = 2$ component of (3.53) factors through

  $$Z_2(e_J E^* e_J) = E^2_{\mathrm{odd}} \xrightarrow[(3.50)]{\simeq} E^3_{\mathrm{odd}} \xrightarrow{e_K \cdot -} (e_K F^1 H e_K)^{\vee,f} \xrightarrow{\simeq} e_K E^2 e_K.$$

  The second arrow is bijective by the $i = 3$ case.

□

3.3.1 The anti-involution on $e_K E^* e_K$

Recall the anti-involution $\mathcal{J}\colon E^* \to E^*$ we introduced in §3.1.2. Its action on the basis elements of $e_K E^* e_K$ is given therein. We denote by $\mathcal{J}_K$ the map $\mathcal{J}$ restricted to $e_K E^* e_K$. It acts almost trivially:

Proposition 56. *For $\alpha \in e_K E^i e_K$, we have*

$$\mathcal{J}_K(\alpha) = \begin{cases} \alpha, & \text{if } i = 0, 3 \\ -\alpha, & \text{if } i = 1, 2. \end{cases}$$

Proof. Let $\alpha \in H^i(I, \mathbf{X}(d))$ where $d \in \widetilde{{}_K\mathcal{D}_K}$. Since $d^{-1} = -d$, the formulas (3.7) and (3.8) then imply that $e_K \mathcal{J}(\alpha) e_K = e_K \alpha e_K$ if $i = 0, 3$ or $-e_K \alpha e_K$ if $i = 1, 2$. Therefore the claim holds on a basis for $e_K E^* e_K$. □

Remark 57. The somewhat surprising fact that E_K^* is commutative (rather than graded commutative) is implied directly by Proposition 56 as follows. If $\alpha \in e_K E^i e_K$ and $\beta \in e_K E^j e_K$ then we can proceed by cases based on the values of i and j. For example, if $i = j = 1$, then $\alpha \cdot \beta \in H^2(K, \mathbf{X}_K)$ and

$$\begin{aligned} \alpha \cdot \beta &= \mathcal{J}_K(\alpha) \cdot \mathcal{J}_K(\beta) \\ &= -\mathcal{J}_K(\beta \cdot \alpha) \\ &= \beta \cdot \alpha \end{aligned}$$

where we use Proposition 56 for the first and third equalities. The remaining cases are similar.

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