Length-Minimizing Closed Curves on Manifolds with Boundary

by

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submitted by **Hannah Kohut** in partial fulfillment of the requirements for the degree of **Master of Science** in **Mathematics**.

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Abstract

The purpose of this thesis is to explore the properties of closed curves which are length-minimizing in a nontrivial homotopy class of a compact Riemannian manifold with boundary. This project was inspired by a question asked by Professor Liam Watson about the existence of length-minimizing curves on a torus with finitely many disks removed, and whether these curves leave the boundary components tangentially. In our research, we extend the question to compact smooth Riemannian manifolds with smooth boundary. We first discuss some preliminary results to determine an appropriate class of curves to minimize over. We then explore the properties of the length and energy functionals, showing that a minimizer of the energy is also a minimizer of the length. We directly minimize the energy functional to show the existence of a length-minimizing curve in any nontrivial homotopy class of a compact Riemannian manifold with boundary. Finally, we address the regularity of length-minimizing curves, showing that they are piecewise geodesics (possibly with infinitely many pieces).

Lay Summary

The purpose of this thesis is to explore the properties of loops which are shorter than all others on some space that prevents them from shrinking to a point. For example, imagine we have a peg board with wide round pegs and a very long rubber band. We place the rubber band around some number of pegs. If the rubber band is loose, we replace it with a shorter one (wrapping around the same pegs as before) until it pulls tightly against the pegs. We then examine the properties of this final loop. Namely, it is shorter than all others in this "class" and it does not have corners.

Preface

The topic of this thesis was jointly chosen by the author and her supervisors, Professor Ailana Fraser and Professor Liam Watson. It was inspired by a question posed by Professor Watson to Professor Fraser. Chapters 2 and 3 survey a collection of known results. To the best of the author's knowledge, the organization and presentation of this material is original. This thesis also contains new and unpublished research. This is the content of Chapter 4. This research was conducted through ongoing discussion and meetings with Professor Fraser. Each of us contributed to the proofs of the main results in this section, however, the author was responsible for writing this document in its entirety.

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Acknowledgments

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I am exceedingly grateful to my supervisors Professor Ailana Fraser and Professor Liam Watson. I have benefited greatly from their expertise and guidance throughout my degree. Professor Fraser's encouragement and patience was crucial to me while writing this thesis and advancing my mathematical research skills. Furthermore, I am thankful to Professor Watson for encouraging me to branch out and explore different types of mathematics over the course of my degree.

I would like to show appreciation for the mathematics community at the University of British Columbia. In particular, the geometry and topology groups enthusiastically welcomed me and helped to give me a sense of belonging in the UBC mathematics community during the COVID-19 pandemic.

I would like to thank my friends Nic Fellini, Arthur Lei Qiu, and Yuveshen Mooroogen for their unwavering support throughout my undergraduate and graduate studies. Our mathematical and non-mathematical conversations at all hours of the day were a constant source of inspiration and encouragement. They have become a second family to me. A special thanks to Arthur, who spent an unbelievable number of hours helping me edit this document.

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Chapter 1

Introduction

This thesis focuses on proving the existence and exploring properties of length-minimizing closed curves on Riemannian manifolds with boundary.

In Hanselman, Rasmussen, and Watson's *Bordered Floer Homology for Manifolds with Torus Boundary Via Immersed Curves* [7], the authors describe the following set-up, which resembles a peg board. Imagine we draw a closed curve γ on a 2-torus with finitely many points removed, as in Figure 1.1a, below. Call this space X. We would like to continuously deform the curve through closed curves so that it wraps around the punctures in the surface "as tightly as possible", as in Figure 1.1b. To better understand this problem, the authors lift the curves to a space that can be thought of as Euclidean $\mathbb{R}^2$ with a lattice of punctures. By widening each puncture point to an open disk of radius 2ε (Figure 1.1c) and then smoothly attaching an infinite tube of radius ε along the boundary of each disk (Figure 1.1d), we obtain a manifold M. The authors introduce a Riemannian metric g_ε on M that has non-positive curvature and smoothly interpolates between the flat metrics on the cylinders and the punctured plane.

In Lemma 7.1 of Hanselman, Rasmussen, and Watson [7], the authors prove the following:

(i) Any nontrivial free homotopy class of curves in X can be represented by a curve γ that minimizes length among curves in its homotopy class with respect to the metric g_ε.

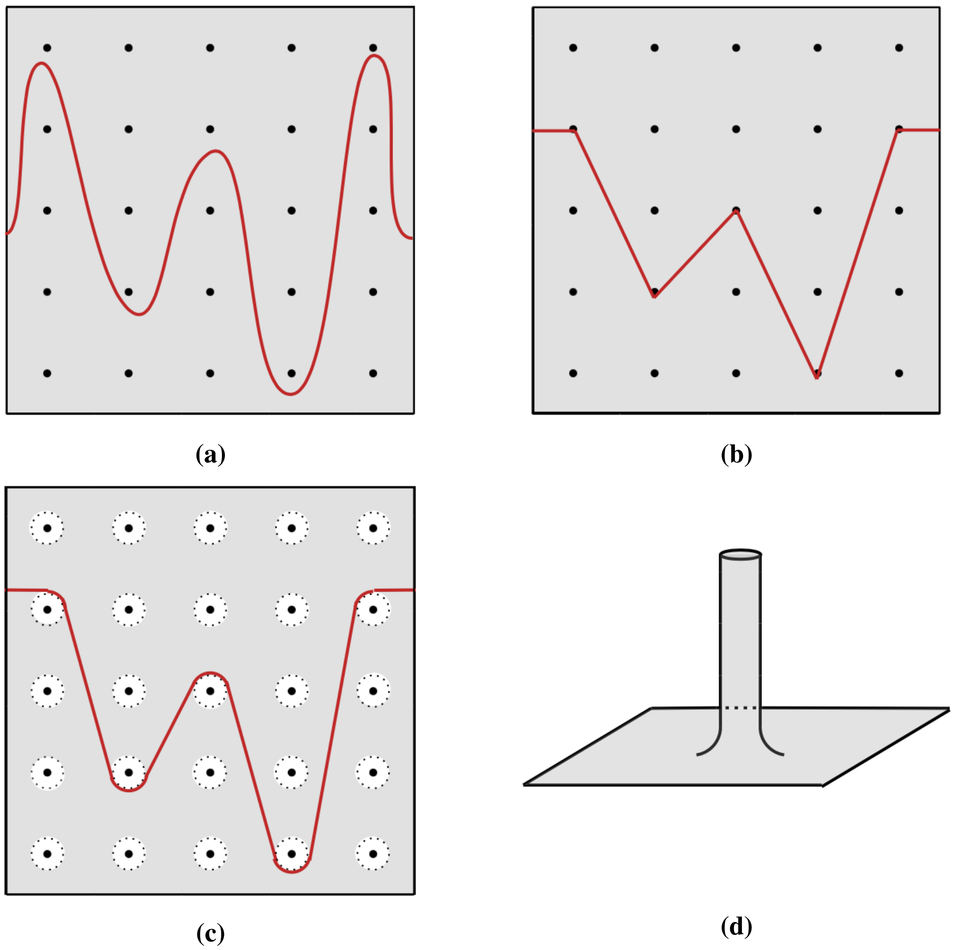


Figure 1.1: In panel (a), we draw a curve onto a torus with finitely many points removed. Panel (b) demonstrates a curve with has minimal length within its homotopy class. This curve has been "pulled tight". In panel (c), each lattice point is expanded to a disk of radius 2ε. Finally, in panel (d), we see a single disk of the lattice, where we smoothly attach a tube of radius ε to the boundary of the disk.

Moreover, by the work of Freedman–Hass–Scott [5], they conclude that

(ii) Any two distinct length minimizing curves intersect minimally and transversally.

Throughout this thesis we generalize Lemma 7.1 of [7] by asking if an analogous result holds on any manifold with boundary. Namely, we ask:

(I) On a manifold with boundary, does a length minimizing curve exist in any nontrivial homotopy class of closed curves?

Based on Lemma 7.1, we hope that this length minimizing curve is a piecewise geodesic. If the image of the minimizing curve resides in both the boundary and the interior of our manifold, we can also ask how the pieces of the curve interact at the boundary. Intuitively, we would expect that a curve that is "pulled tight" leaves the boundary tangentially. This leads us to ask:

(II) If a length minimizing curve exists, what properties does it exhibit?

As an answer to these questions, we will prove:

Theorem. *Let M be a compact Riemannian manifold with boundary. In any nontrivial free homotopy class of closed curves, there is a curve $\gamma\colon [0,1] \to M$ of least length. With the notation $A = \{t \in [0,1] : \gamma(t) \in M \setminus \partial M\}$ and $B = \{t \in [0,1] : \gamma(t) \in \partial M\}$,*

(i) γ is a geodesic in M on each connected component of A,

(ii) γ is a geodesic in ∂M on each connected component of B with non-empty interior, and

(iii) if t_0 is an isolated point of ∂B in $[0,1]$, then γ leaves ∂M tangentially at t_0. That is, γ is C^1 at t_0.

A more thorough review of the literature mentioned below can be found in Klingenberg's *Lectures on Closed Geodesics* [9]. Results regarding the existence of closed geodesics were first studied on manifolds without boundary. In 1898, Hadamard [6] proved that on surfaces of negative curvature, there is a length minimizing geodesic within each nontrivial homotopy class of closed curves. This geodesic is unique up to reparametrization. A similar result was shown by Poincaré [15] in 1905. He showed the existence of closed geodesics within a nontrivial homotopy class for analytic convex surfaces. It is also well known that in a compact manifold, there exists a closed geodesic within any nontrivial homotopy class [3, Theorem 12.2.2]. This result has been attributed to Cartan.

The case of a general Riemannian manifold was studied by Lyusternik and Fet [13]. In 1951, they proved the existence of a closed geodesic on any compact Riemannian manifold, regardless of the existence of a nontrivial homotopy class.

This problem was later generalized to a case of manifolds with boundary. In his 1985 paper, Scolozzi [16] proposes that in any compact, non-contractible manifold with boundary which can be written as a sublevel set, one can find a non-constant closed geodesic . Much of this paper relies on Scolozzi's results on fixed endpoint geodesics in manifolds with boundary, which can be found in [17]. In this thesis, we offer a strengthening of Scolozzi's result and show additional properties of the resulting length-minimizing curve.

In Chapter 2, we will discuss some background information that is needed to address questions (I) and (II). This includes: defining the length and energy functionals, together with reasonably broad classes of curves to minimize over; modes of convergence that we will make use of; and the homotopy class of a limiting curve. In Chapter 3, we will discuss the properties of the length and energy functionals, culminating in a result which shows that minimizers of the energy functional also minimize the length functional. Finally, in Chapter 4, we prove the existence of length minimizing curves in manifolds with possibly non-empty boundary, and discuss the properties of minimizers.

Chapter 2

Some Preliminary Results

2.1 Notation and conventions

Throughout, we will use M to denote a smooth (C^∞) Riemannian manifold with possibly non-empty boundary ∂M and metric g. Unless otherwise specified, we equip (M,g) with the Levi-Civita connection ∇ associated with g.

We use the word "curve" to mean a parametrized curve, i.e. a continuous map $\gamma\colon [a,b] \to M$. We use the phrase "image of the curve" to mean the set $\gamma([a,b]) \subset M$. We will specify further differentiability of our curves as needed, using the notation $C^q([a,b],M)$ to denote the class of q-times continuously differentiable curves $\gamma\colon [a,b] \to M$.

2.2 Length and energy: first definitions

In order to learn about length-minimizing curves, we will certainly need a notion of the length of a curve on a Riemannian manifold. We will introduce a *length functional* L associating to each suitably differentiable curve $\gamma\colon [a,b] \to M$ its length $L(\gamma)$ in a manner that is entirely determined by the image of the curve, i.e. is invariant under reparametrization. In fact, we will extend this length functional to curves which may only be differentiable in a weaker sense.

We will also introduce the notion of the *energy* of a curve. We will see that, unlike the length functional, the energy functional varies under reparametrization.

However, the energy functional is easier to work with in computation and has some notion of lower semicontinuity. Most importantly, energy-minimizing curves are also length-minimizing curves.

Definition 1. *Let M be a Riemannian manifold with or without boundary, and $\gamma\colon [a,b] \to M$ a piecewise C^1 curve. The* ***length of*** γ *is defined by*

$$L(\gamma) = \int_a^b |\gamma'(t)|\, dt,$$

and the ***energy of*** γ *is defined by*

$$E(\gamma) = \int_a^b |\gamma'(t)|^2\, dt.$$

In what follows, we will introduce some background results before rigorously addressing the properties of the length and energy functionals in Chapter 3.

2.3 The Nash embedding theorem

In the definition of the length and energy functionals, we considered curves in a Riemannian manifold M. We begin by noting that by embedding M into $\mathbb{R}^k$ for some large enough k, which is possible by the strong Whitney embedding theorem [11, Theorem 6.19], we can instead consider curves in an embedded submanifold of Euclidean space. However, arbitrary embeddings need not preserve length and energy, as they do not preserve the metrics.

This gives rise to the notion of an **isometric embedding** [12, p. 12]: an embedding $F\colon (N,\tilde{g}) \to (M,g)$ of Riemannian manifolds which satisfies $F^*g = \tilde{g}$. This type of embedding guarantees that the distance between points p and q as measured in the domain is the same as the distance between $F(p)$ and $F(q)$ as measured in the codomain.

With these definitions in mind, we may state the Nash embedding theorem.

Theorem 1 (Nash embedding theorem [14, Theorem 2])**.** *Let M be a Riemannian manifold of dimension n without boundary. Then there is a C^1 isometric embedding of (M,g) into $\mathbb{R}^{2n+1}$ with the standard Euclidean metric $\overline{g}$.*

We can also extend the Nash embedding theorem to a manifold with non-empty boundary M. For this, we introduce the double $D(M)$, which is a manifold without boundary containing M.

Definition 2. *Let M be an n-dimensional smooth manifold with or without boundary. The **double of** M is the n-dimensional manifold without boundary*

$$D(M) = M \sqcup_{\mathrm{id}} M,$$

where $\mathrm{id}\colon \partial M \to \partial M$ *is the identity map* [11, Example 9.32].

If $\partial M = \varnothing$, then $D(M)$ is simply the disjoint union of two copies of M. We will often distinguish between the two copies of M in $D(M)$ by labelling them as M_1 and M_2. Points in $D(M)$ will thus be labelled p_1 or p_2 depending on which of M_1 and M_2 they lie in. Some examples of the double can be found in Figure 2.1.

We may equip $D(M)$ with a smooth structure [10, p. 101]. Moreover, if M has a Riemannian metric g, then we can extend it smoothly to a metric G on $D(M)$ [12, Example 6.44]. That is, given the inclusion map $\iota : M \to D(M)$, we may define a smooth metric G such that $\iota^* G = g$. This allows us to extend the Nash embedding theorem to manifolds with boundary.

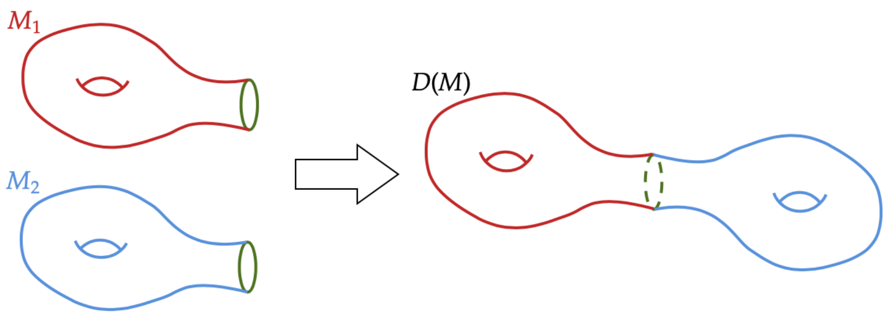
Corollary 2 (Nash embedding theorem for manifolds with boundary)**.** *Any n-dimensional Riemannian manifold with boundary, (M,g), admits a C^1 isometric embedding into $\mathbb{R}^{2n+1}$ with the standard Euclidean metric $\overline{g}$.*

Proof. Let G be a Riemannian metric on the double $D(M)$ which smoothly extends g. Applying the Nash embedding theorem to $D(M)$, we obtain an isometric embedding $F\colon (D(M),G) \to (\mathbb{R}^{2n+1},\overline{g})$. Thus, there is an embedding $(M,g) \to (\mathbb{R}^{2n+1},\overline{g})$ given by the composition

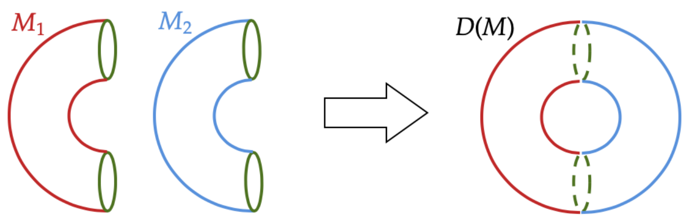
$$M \overset{\iota}{\hookrightarrow} D(M) \overset{F}{\hookrightarrow} \mathbb{R}^{2n+1},$$

where $\iota\colon M \to D(M)$ is the inclusion map. The embedding is isometric because

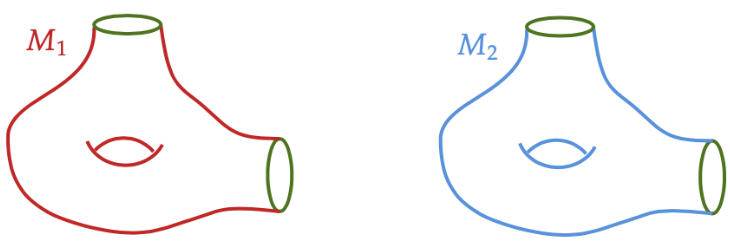
$$(F \circ \iota)^* \overline{g} = \iota^* F^* \overline{g} = \iota^* G = g. \qquad \square$$



(a) Let M be a torus with one disk removed. Then $D(M)$ is a genus two surface. If M_i is embedded into $(\mathbb{R}^3, \overline{g})$, as pictured, then $D(M)$ with metric $G = \overline{g}_i := \overline{g}|_{M_i}$ can be isometrically embedded into $\mathbb{R}^3$.



(b) Let $M = [0,1] \times S^1$ be a cylinder. Then $D(M)$ is a torus. If M_i is embedded as a "macaroni noodle" inside $\mathbb{R}^3$ with the Euclidean metric $\overline{g}$, then $D(M)$ with metric $G = \overline{g}_i$ can be isometrically embedded into $\mathbb{R}^3$.



(c) Let M be a torus with two disks removed. If M_i is embedded into $(\mathbb{R}^3, \overline{g})$, as pictured, then $D(M)$ with metric $G = \overline{g}_i$ cannot be isometrically embedded into $\mathbb{R}^3$, since identifying the boundary components would require us to change the shape of M_i for some i. However, since M is a 2-dimensional manifold, $D(M)$ can be isometrically embedded into $\mathbb{R}^5$.

Figure 2.1: Here are a few examples of the double of a manifold. Notice that in some special cases, the double of a 2-manifold can be isometrically embedded into $\mathbb{R}^3$, but in general, by the Nash embedding theorem, a 2-manifold is only guaranteed to have an isometric embedding into $\mathbb{R}^5$.

By Corollary 2, piecewise C^1 curves in Riemannian manifolds can be regarded as piecewise C^1 curves in Euclidean space in a way that preserves the notions of length and energy. We will often use this fact to simplify our arguments.

2.4 Function spaces

It turns out that to find curves minimizing length or energy, it is convenient to extend the definition of the length and energy functionals to a broader class of curves than those which are piecewise C^1. Throughout this section, we will define various classes of curves in manifolds in an attempt to find an appropriate space to minimize these functionals.

2.4.1 Lebesgue spaces

First, we recall the standard definition of $L^p([a,b],\mathbb{R}^k)$ for $p \in [1,\infty)$ as the space of functions $f\colon [a,b] \to \mathbb{R}^k$ such that

$$\|f\|_{L^p} \coloneqq \left(\int_a^b |f(x)|^p \, dx\right)^{1/p} < \infty.$$

On the other hand, $L^\infty([a,b],\mathbb{R}^k)$ is the space of functions $f\colon [a,b] \to \mathbb{R}^k$ such that

$$\|f\|_{L^\infty} \coloneqq \operatorname{ess\,sup}_{x\in[a,b]} |f(x)| < \infty.$$

We extend the Lebesgue space to manifolds as follows:

Definition 3. *For an embedded submanifold $M \subseteq \mathbb{R}^k$ with or without boundary, we define*

$$L^p([a,b],M) = \left\{\gamma \in L^p\left([a,b],\mathbb{R}^k\right) : \gamma(t) \in M \textit{ for a.e. } t \in [a,b]\right\}.$$

If $[a,b]$ and M are clear from context, we will use the notation L^p in place of $L^p([a,b],M)$.

We have seen that every Riemannian manifold M with or without boundary admits an isometric embedding $F\colon M \to \mathbb{R}^k$ for some k. The length and energy functionals on M can then be described in terms of the L^1 and L^2 norms of $\mathbb{R}^k$:

$$L(\gamma) = \|(F \circ \gamma)'\|_{L^1} \quad \text{and} \quad E(\gamma) = \|(F \circ \gamma)'\|_{L^2}^2.$$

We will often abuse notation by identifying γ with $F \circ \gamma$.

Since we aim to extend the definitions of L and E beyond piecewise C^1 curves in M, it is natural to ask: what is the broadest class of curves $\gamma \in L^p([a,b],\mathbb{R}^k)$ for which we can make sense of $\|\gamma'\|_{L^p}$? We will see that *Sobolev spaces* are the natural setting when working with the length and energy functionals.

2.4.2 Sobolev spaces

Sobolev spaces are particular subspaces of L^p, endowed with a norm which measures both the size of a function and its derivatives. However, we do not use the classical notion of the derivative in defining Sobolev spaces. In order to guarantee completeness, we must define a weaker notion of differentiability.

Definition 4. *Let $f \in L^1([a,b],\mathbb{R}^k)$. We say that $v \in L^1([a,b],\mathbb{R}^k)$ is a **weak derivative** of f if*

$$\int_a^b v(x)\varphi(x)\,dx = -\int_a^b f(x)\varphi'(x)\,dx,$$

for all $\varphi \in C^1([a,b],\mathbb{R}^k)$ such that $\varphi(a) = \varphi(b) = 0$. In this case, we write $v = Df$.

Notice that if f' exists in the classical sense, then it agrees with any weak derivative up to a measure zero set. This definition extends to higher order weak derivatives $D^q f$; in this case, we say that f is q-times weakly differentiable. We are now equipped to define Sobolev spaces on $\mathbb{R}^k$.

Definition 5. *For $p \in [1,\infty)$ and $q \in \mathbb{N}$, the **Sobolev space** $W^{q,p}([a,b],\mathbb{R}^k)$ is the space of functions $f \in L^p([a,b],\mathbb{R}^k)$ which are q-times weakly differentiable and such that*

$$\|f\|_{W^{q,p}} := \left(\sum_{i=0}^{q} \int_a^b |D^i f(x)|^p\,dx \right)^{\frac{1}{p}} < \infty.$$

Meanwhile, for $p = \infty$, we define $W^{q,\infty}([a,b],\mathbb{R}^k)$ as the space of functions $f \in$

$L^\infty([a,b],\mathbb{R}^k)$ *which are q-times weakly differentiable and such that*

$$\|f\|_{W^{q,\infty}} := \sum_{i=0}^{q} \operatorname{ess\,sup}_{x\in[a,b]} |D^i f(x)| < \infty.$$

Proposition 3. *Sobolev spaces satisfy the following properties:*

(a) If $1 \le p \le \infty$*, then* $W^{q,p}([a,b],\mathbb{R}^k)$ *is a Banach space* [8, Theorem A.1.6]*,*

(b) if $1 < p < \infty$*, then* $W^{q,p}([a,b],\mathbb{R}^k)$ *is reflexive (*[2, Proposition 8.1]*), and*

(c) $W^{q,2}([a,b],\mathbb{R}^k)$ *is a Hilbert space* [4, p. 302].

We extend this definition to curves on manifolds with or without boundary in a similar manner to the extension of Lebesgue spaces.

Definition 6. *For an embedded submanifold* $M \subseteq \mathbb{R}^k$ *with or without boundary, we define*

$$W^{q,p}([a,b],M) = \left\{\gamma \in W^{q,p}\left([a,b],\mathbb{R}^k\right) : \gamma(t) \in M \text{ for a.e. } t \in [a,b]\right\}.$$

If $[a,b]$ and M are clear from context, we will use the notation $W^{q,p}$ in place of $W^{q,p}([a,b],M)$.

Using Sobolev spaces, it is natural to attempt to minimize the length functional over $W^{1,1}([a,b],\mathbb{R}^k)$, or to minimize the energy functional over $W^{1,2}([a,b],\mathbb{R}^k)$. Because of this, we will interpret γ' to be a weak derivative when necessary.

Based on the properties of Sobolev spaces indicated in Proposition 3, it would be preferable to work in $W^{1,2}([a,b],\mathbb{R}^k)$ since it is a Hilbert space. For this reason, in Chapter 4 our strategy will be to minimize the energy over curves in $W^{1,2}$ instead of minimizing the length over curves in $W^{1,1}$.

2.4.3 Hölder spaces and the Sobolev embedding theorem

Another important aspect of Sobolev spaces is that, under certain conditions, they can be continuously embedded into *Hölder spaces*—spaces of functions that are classically differentiable and whose derivatives satisfy a certain "Hölder continuity" condition.

Definition 7. *Let $q \in \mathbb{N}$ and $0 < \alpha \leq 1$. The* ***Hölder space*** *$C^{q,\alpha}([a,b],\mathbb{R}^k)$ is the set of maps $f \in C^q([a,b],\mathbb{R}^k)$ such that*

$$\|f\|_{C^{q,\alpha}} := \max_{0\leq i\leq q}\left\{\|f^{(i)}\|_{L^\infty}\right\} + \sup_{\substack{s,t\in[a,b],\\ s\neq t}} \frac{|f^{(q)}(s)-f^{(q)}(t)|}{|s-t|^\alpha} < \infty.$$

To prove one of our main results (Theorem 22), we will require the following case of the Sobolev embedding theorem.

Theorem 4 (Sobolev embedding theorem [1, Theorem 2.10])**.** *Let $p,\alpha \in \mathbb{R}$ with $0 < \alpha < 1$ and let q,r be non-negative integers such that $\frac{q-r-\alpha}{k} \geq \frac{1}{p}$. Then there is a continuous embedding*

$$W^{q,p}([0,1],\mathbb{R}^k) \hookrightarrow C^{r,\alpha}([0,1],\mathbb{R}^k).$$

2.5 Modes of convergence

In Chapter 4, we will take an energy-minimizing sequence of closed $W^{1,2}$ curves in some nontrivial homotopy class $\mathscr{H}$. Notice that by Theorem 4, $W^{1,2} \hookrightarrow C^0$, so considering these maps is reasonable. Using various modes of convergence, we are able to show that the minimizing sequence of curves has a limit in $\mathscr{H}$ attaining the minimal energy. In this section, we introduce the definitions of weak convergence and convergence in distribution, an important corollary of the Banach–Alaoglu theorem, a useful linear map, and the Rellich–Kondrachov compactness theorem.

2.5.1 Weak convergence and compactness theorems

Let X be a normed vector space and X^* the space of bounded linear functionals on X. We say that a sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ **converges weakly in** X to a point $x \in X$ if $\varphi(x_n) \to \varphi(x)$ for every $\varphi \in X^*$. In this case, we write $x_n \rightharpoonup x$. As the name suggests, norm convergence implies weak convergence. We also define a notion of convergence for sequences of operators in X^*, called *weak* convergence*. Namely, we say that a sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subset X^*$ **converges in the weak* topology to** φ if $\varphi_n(x) \to \varphi(x)$ for every $x \in X$. In this case, we write $\varphi_n \overset{*}{\rightharpoonup} \varphi$.

In general, we have $X \subset X^{**}$, so we see that it is easier for a sequence to converge in the weak* topology than the weak topology. However, if X is reflexive, then the notions of weak and weak* convergence are equivalent.

Eventually, we will be choosing a sequence of curves whose energies converge to the infimum of all possible energies attained in a fixed homotopy class; to show that this sequence of curves converges weakly in $W^{1,2}$, we will require the following corollary of the Banach–Alaoglu Theorem [2, Theorem 3.16].

Corollary 5 ([2, Theorem 3.18])**.** *Suppose that X is a reflexive Banach space. Then every bounded sequence has a weakly convergent subsequence.*

We will also state, without proof, the Rellich–Kondrachov compactness theorem. This result will help us to upgrade weak $W^{1,2}$ convergence to L^2 convergence.

Theorem 6 (Rellich–Kondrachov compactness theorem [8, Theorem A.1.8])**.** *Suppose $1 \leq p,q < \infty$. The Sobolev space $W^{1,p}([0,1],\mathbb{R}^k)$ is compactly embedded in $L^q([0,1],\mathbb{R}^k)$. That is, if $\{f_n\}_{n\in\mathbb{N}} \subset W^{1,p}$ satisfies*

$$||f_n||_{W^{1,p}} \leq B$$

for some $B \in \mathbb{R}$ independent of n, then $\{f_n\}_{n\in\mathbb{N}}$ has a subsequence that converges in L^q.

2.5.2 Convergence in distribution

Additionally, we will make use of a special case of convergence in the distributional sense in the proof of our main result. Given $U \subset \mathbb{R}^k$, we use the notation $\mathscr{D}(U,\mathbb{R}) \subset C_c^\infty(U,\mathbb{R})$ to denote the space of all smooth functions $\mathbb{R}^k \to \mathbb{R}$ with compact support contained in U which vanish on ∂U. Elements of $\mathscr{D}$ will be called *test functions*.

Definition 8. *The space $\mathscr{D}^*(U,\mathbb{R})$ is the set of* ***distributions*** *on U, which we equip with the weak* topology. Therefore,* ***convergence in distribution****, denoted $\{S_n\}_{n\in\mathbb{N}} \to S$, means that*

$$S_n(\varphi) \to S(\varphi)$$

for every $\varphi \in \mathscr{D}$. We can also extend this notion of convergence to vector valued linear maps. We say that linear maps $T_n\colon \mathscr{D} \to \mathbb{R}^k$ ***converge in distribution to*** *T if* *for every $1 \leq i \leq k$, the distributions*

$$\pi_i \circ T_n \to \pi_i \circ T,$$

where π_i is projection onto the i^{th} coordinate.

We are particularly interested in the following linear map.

Definition 9. *Given $g \in W^{1,2}([0,1],\mathbb{R}^k)$, we define $T_g \colon C_c^0([0,1],\mathbb{R}) \to \mathbb{R}^k$ by*

$$T_g(\varphi) = \int_0^1 \varphi(x) g(x)\, dx.$$

In the following series of lemmas, we will show that T_g defines a bounded linear map, and that if $g_n \to g$ either in L^2 norm or weakly in $W^{1,2}$, then $\{T_{g_n}\}_{n\in\mathbb{N}} \to T_g$ in the distributional sense.

Lemma 7. *If $g \in W^{1,2}([0,1],\mathbb{R}^k)$, then $T_g \colon C_c^0([0,1],\mathbb{R}) \to \mathbb{R}^k$ is a bounded linear map.*

Proof. Let $\varphi \in C_c^0([0,1],\mathbb{R}^k)$. By the definition of T_g,

$$|T_g(\varphi)| = \left| \int_0^1 \varphi(x) g(x)\, dx \right|.$$

Using the triangle inequality, we pull the norm inside the integral

$$\leq \int_0^1 |\varphi(x)||g(x)|\, dx.$$

As φ is a continuous function on a compact set, it attains a maximum value, which we extract from the integral

$$\leq \sup_{x\in[0,1]} |\varphi(x)| \int_0^1 |g(x)|\, dx.$$

Finally, we let $c = \int_0^1 |g(x)|\, dx$, and re-write the expression as

$$= c\|\varphi\|_{L^\infty([0,1],\mathbb{R}^k)},$$

where

$$0 \leq c \leq \int_0^1 |g(x)|^2 \, dx$$

by Hölder, and the latter is finite since $g \in W^{1,2}([0,1],\mathbb{R}^k)$. We conclude that $T_g \colon C_c^0([0,1],\mathbb{R}) \to \mathbb{R}^k$ is a bounded linear map. □

Lemma 8. *Suppose* $\{g_n\}_{n\in\mathbb{N}} \subset W^{1,2}([0,1],\mathbb{R}^k)$*. If* g_n *converges to* g *weakly in* $W^{1,2}([0,1],\mathbb{R}^k)$*, then* $T_{g_n}(\varphi) \to T_g(\varphi)$ *for any* $\varphi \in \mathscr{D}([0,1],\mathbb{R})$.

Proof. By Lemma 7, the map $T_g \colon C^0([0,1],\mathbb{R}) \to \mathbb{R}^k$ is bounded whenever $g \in W^{1,2}([0,1],\mathbb{R}^k)$. In particular, we may view $\mathscr{D}([0,1],\mathbb{R}) \subset C^0([0,1],\mathbb{R})$ to see that T_g is bounded for this class of functions. Composition with the projection onto the i^{th} component gives a bounded linear operator, and the result follows from the definition of weak convergence. □

Lemma 9. *Suppose* $\{g_n\}_{n\in\mathbb{N}} \subset W^{1,2}([0,1],\mathbb{R}^k)$*. If* $g_n \to g$ *in* $L^2([0,1],\mathbb{R}^k)$*, then* $T_{g_n}(\varphi) \to T_g(\varphi)$ *for any* $\varphi \in \mathscr{D}([0,1],\mathbb{R})$.

Proof. By the linearity of T,

$$|T_{g_n}(\varphi) - T_g(\varphi)| = \left| \int_0^1 \varphi(x)(g_n(x) - g(x)) \, dx \right|.$$

Using the triangle inequality, we pull the norm inside the integral

$$\leq \int_0^1 |\varphi(x)||g_n(x) - g(x)| \, dx.$$

Since φ is a continuous function on a compact set, it attains a maximum value, which we extract from the integral

$$\leq \max_{x\in[0,1]} (|\varphi(x)|) \int_0^1 |g_n(x) - g(x)| \, dx.$$

We now recognize the integral as the $L^1([0,1],\mathbb{R}^k)$ norm of $g_n - g$. By Hölder's inequality

$$\leq \max_{x\in[0,1]} (|\varphi(x)|) \|g_n(x) - g(x)\|_{L^2}.$$

Since $g_n \to g$ in L^2, we see that this converges to 0, as required. □

Lemma 10. *Let* $\{g_n\}_{n\in\mathbb{N}} \subset W^{1,2}([0,1],\mathbb{R}^k)$ *be a sequence of curves, and consider a subsequence* $\{g_{n_k}\}_{k\in\mathbb{N}} \subset \{g_n\}_{n\in\mathbb{N}}$*. Suppose*

$$T_{g_n} \to T_g \text{ and } T_{g_{n_k}} \to T_{\widetilde{g}}.$$

Then $g(x) = \widetilde{g}(x)$ *for almost every* $x \in [0,1]$.

Proof. Let φ be a test function. By the definition of convergence in distribution, the sequence $\{T_{g_n}(\varphi)\}_{n\in\mathbb{N}} \to T(\varphi)$ and $\left\{T_{g_{n_k}}(\varphi)\right\}_{n\in\mathbb{N}} \to T_{\widetilde{g}}(\varphi)$. In particular, since $\left\{T_{g_{n_k}}(\varphi)\right\}_{n\in\mathbb{N}} \subset \{T_{g_n}(\varphi)\}_{n\in\mathbb{N}} \subset \mathbb{R}^k$, we know that the limits are equal. That is,

$$T_g(\varphi) = T_{\widetilde{g}}(\varphi).$$

We consider the difference

$$0 = T_g(\varphi) - T_{\widetilde{g}}(\varphi) = \int_0^1 (g(x) - \widetilde{g}(x))\,\varphi(x)\,dx.$$

Since φ is an arbitrary test function, we have $g(x) = \widetilde{g}(x)$ for almost every $x \in [0,1]$, as desired. □

2.6 Homotopy classes are closed in the uniform topology

In Chapter 4, we will show that we have a sequence of closed curves in a nontrivial homotopy class $\mathscr{H}$ in M which converge to some limit γ in $C^{0,\beta}([a,b],\mathbb{R}^k)$ for a specific range of β's. Intuitively, since the sequence is getting "close enough" to γ, there cannot be a "hole" in the manifold preventing γ from being in the homotopy class. In Proposition 12, we will see that γ is an element of $\mathscr{H}$ by constructing an explicit homotopy. In order to do this, we require the notion of a *tubular neighbourhood.*

2.6.1 The existence of tubular neighbourhoods

Suppose that M is a Riemannian manifold without boundary, and $P \subset M$ an embedded submanifold. Let $\pi\colon NP \to P$ be the normal bundle of P in M, and $\mathscr{E}$ the domain of the exponential map of M. Then we can define $\mathscr{E}_P = \mathscr{E} \cap NP$, and let $E\colon \mathscr{E}_P \to M$ denote the restriction of the exponential map to $\mathscr{E}_P$.

A **normal neighbourhood** of P in M is an open subset $U \subset M$ that is the diffeomorphic image under E of an open subset $V \subset \mathscr{E}_P$ whose intersection with each fiber N_xP is star-shaped with respect to 0.

Definition 10 ([12, p. 133])**.** *A normal neighbourhood of P in M is called a* ***tubular neighbourhood*** *if it is the diffeomorphic image under E of a subset $V \subset \mathscr{E}_P$ of the form*

$$V = \{(x,v) \in NP\colon |v|_g < \delta(x)\}$$

for some positive continuous function $\delta\colon P \to \mathbb{R}$. If $\delta(x) \equiv \varepsilon$ for some positive constant ε, then U is called an ***ε-tubular neighbourhood****.*

An ε-tubular neighbourhood is illustrated in Figure 2.2. We now verify the existence of tubular neighbourhoods in manifolds without boundary.

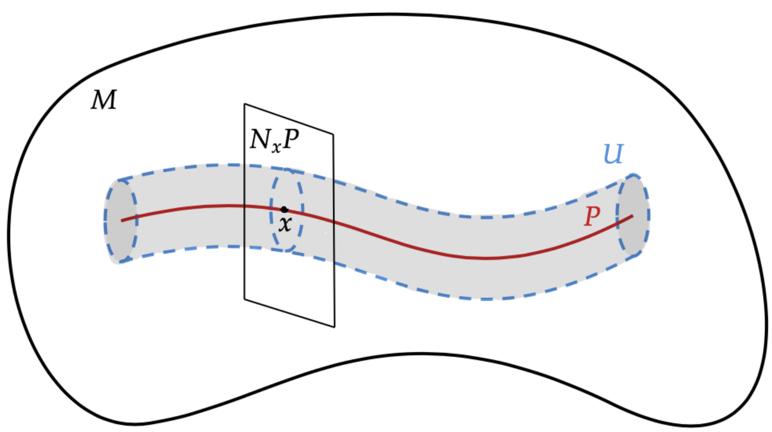


Figure 2.2: An ε-tubular neighbourhood in a manifold without boundary.

Theorem 11 (Tubular neighbourhood theorem [12, Theorem 5.25])**.** *Let (M,g) be a Riemannian manifold. Every embedded submanifold $P \subset M$ has a tubular neighbourhood in M, and every compact submanifold has an ε-tubular neighbourhood.*

We proceed as in Lee [12, Theorem 5.25].

Proof. We begin by showing that there is a neighbourhood $V_\delta \subset NP$ containing $(x,0)$ where the exponential map E is a diffeomorphism. By the inverse function theorem [11, Theorem 4.5], it is enough to check that $dE_{(x,0)}$ is bijective for each $x \in P$. Let $P_0 \subset NP$ be the image of the zero section of NP. We first consider the restriction

$$dE\big|_{P_0}\colon T_{(x,0)}P_0 \to TM$$

where $T_{(x,0)}P_0 \subseteq T_{(x,0)}NP$. The restriction of E to P_0 gives a diffeomorphsim

$$\begin{aligned} E\colon P_0 &\xrightarrow{\pi} P \hookrightarrow M \\ (x,0) &\mapsto x \mapsto x. \end{aligned}$$

By Lee [11, Theorem 3.6], we obtain an isomorphism $T_{(x,0)}P_0 \cong TP$. We now consider the restriction

$$dE\big|_{N_xP}\colon T_{(x,0)}N_xP \to TM$$

where $T_{(x,0)}N_xP \subseteq T_{(x,0)}NP$. In this region, we are sufficiently close to the zero section, so the exponential map E is a diffeomorphism. Again, using Theorem 3.6 of Lee [11], we see that $dE\colon T_{(x,0)}N_xP \to N_xP \subset T_xM$ is an isomorphism.

Finally, we see that $dE_{(x,0)}$ is surjective since $T_xM = T_xP \bigoplus N_xP$, and injective because TM and M have the same dimension. Therefore by the inverse function theorem, E is a diffeomorphism on a neighbourhood of the zero section. In particular, we let our neighbourhood take the form

$$V_\delta(x) = \{(x',v') \in NP\colon d_g(x,x') < \delta, |v'|_g < \delta\}.$$

Now, define a function $\Delta\colon P \to \mathbb{R}$ by

$$\Delta(x) = \sup\{\delta < 1\colon E \text{ is a diffeomorphism from } V_\delta(x) \text{ to its image}\}.$$

We show that Δ is continuous by showing that it is Lipshitz continuous. Let $x,x' \in M$. First, suppose that $d_g(x,x') < \Delta(x)$. Define $\lambda = \Delta(x) - d_g(x,x')$. Then we have

$V_\lambda(x') \subset V_{\Delta(x)}(x)$. Hence $\Delta(x') \geq \Delta(x) - d_g(x,x')$, so that

$$\Delta(x) - \Delta(x') \leq d_g(x,x'). \tag{2.1}$$

Now, suppose that $d_g(x,x') \geq \Delta(x)$. Since Δ is non-negative, equation (2.1) still holds. Reversing the roles of x and x', we also have

$$\Delta(x') - \Delta(x) \leq d_g(x,x'). \tag{2.2}$$

Combining equations (2.1) and (2.2), we see the desired condition

$$|\Delta(x) - \Delta(x')| \leq d_g(x,x')$$

and conclude that Δ is continuous.

Now, we show that E is a diffeomorphism on $V_{\Delta(x)}(x)$. To show injectivity, consider $(x_1,v_1),(x_2,v_2) \in V_{\Delta(x)}(x)$. Then there exists some $\delta' > 0$ with $\delta' < \Delta(x)$ such that $(x_1,v_1),(x_2,v_2) \in V_{\delta'}(x)$. From our work above, E is injective on $V_{\delta'}(x)$, hence injective on $V_{\Delta(x)}(x)$. Surjectivity of E on $V_{\Delta(x)}(x)$ follows from the definition of $\Delta(x)$. By Lee [11, Theorem 4.6], E is a diffeomorphism from $V_{\Delta(x)}(x)$ onto its image.

Define a neighbourhood $V = \{(x,v) \in NP \colon |v|_g < \frac{1}{2}\Delta(x)\} \subset NP$. We show that E is injective on V. Suppose $(x_3,v_3),(x_4,v_4) \in V$ such that $E(x_3,v_3) = E(x_4,v_4)$. Without loss of generality, we assume that $\Delta(x_3) \leq \Delta(x_4)$. We compute using the triangle inequality

$$d_g(x_3,x_4) \leq |v_3|_g + |v_4|_g < \frac{1}{2}\left(\Delta(x_3) + \Delta(x_4)\right) \leq \Delta(x_4).$$

In particular, this shows that both $(x_3,v_3),(x_4,v_4) \in V_{\Delta(x_4)}$. By the injectivity of E on $V_{\Delta(x_4)}$, we conclude that $(x_3,v_3) = (x_4,v_4)$.

Thus, we see that $E \colon V \to E(V)$ is a diffeomorphism, and conclude that $U = E(V)$ is a tubular neighbourhood. If $P \subset M$ is compact, then $\frac{1}{2}\Delta(x) > 0$ attains a minimum, so we may choose a uniform value ε, as desired. $\square$

2.6.2 Convergence of curves in a homotopy class

We now show that the C^0 limit of curves in a nontrivial homotopy class is a member of the same homotopy class.

Proposition 12. *Let (M,g) be a compact Riemannian manifold with or without boundary, and $\mathscr{H}$ a homotopy class of curves. If $\{\gamma_n\}_{n=0}^{\infty} \subset \mathscr{H}$ is a sequence such that $\gamma_n \to \gamma$ in $C^0([0,1],M)$, then $\gamma \in \mathscr{H}$.*

Proof. Consider $D(M)$, the double of M equipped with a metric G that smoothly extends g. By the Nash embedding theorem (Theorem 1), $D(M)$ can be regarded as an isometrically embedded submanifold of $\mathbb{R}^k$. By the tubular neighbourhood theorem (Theorem 11), there exists an ε-tubular neighbourhood $\mathscr{U}$ of $D(M)$ in $\mathbb{R}^k$. On this neighbourhood, the exponential map $E\colon \mathscr{U} \to E(\mathscr{U}) \subseteq \mathbb{R}^k$ is a diffeomorphism.

By the definition of $C^0\left([0,1],\mathbb{R}^k\right)$ convergence, since $\gamma_j \to \gamma$ in C^0, there exists some $K \in \mathbb{N}$ such that $\sup_{t\in[a,b]} |\gamma_j(t) - \gamma(t)| < \varepsilon$ for all $j \geq K$ (here we are identifying the curves with their images under the embedding). By discarding finitely many terms in the sequence, we may assume that $K = 1$. We will show that γ_1 is homotopic to γ by constructing an explicit homotopy through M.

First, define a straight line homotopy from γ_1 to γ through $\mathbb{R}^k$ by

$$h(s,t) = s\gamma(t) + (1-s)\gamma_1(t).$$

Now, we define a retraction $r\colon \mathscr{U} \to D(M)$ by

$$r(x) = \pi_{N(D(M))} \circ E^{-1}(x),$$

where $\pi_{N(D(M))}\colon N(D(M)) \to D(M)$ is the natural projection.

We also require the quotient map $q\colon D(M) \to M$, which is given by

$$q(p_i) = p,$$

for $i = 1,2$. We define a homotopy $H(s,t)\colon [0,1]\times[0,1] \to M$ by

$$H(s,t) = (q \circ r \circ h)(s,t),$$

which is a composition of continuous maps, so it is continuous. We also need to verify that $H(0,t) = \gamma_1(t)$ and $H(1,t) = \gamma(t)$.

Since $\gamma_1 \subset M$, both the retraction and quotient maps act as the identity. Therefore, using the definition of h,

$$H(0,t) = \gamma_1(t).$$

It remains to verify that $H(1,t) = \gamma(t)$. The manifold M is compact, hence closed. Therefore, as γ is the C^0 limit of a sequence of curves in M, $\gamma \subset M$. By the the same reasoning as above,

$$H(1,t) = \gamma(t).$$

We conclude that γ_1 is homotopic to γ. Thus, $\gamma \in \mathscr{H}$. □

Chapter 3

Properties of the Length and Energy Functionals

In this chapter, we explore some properties of the length and energy functionals. To begin, we consider the effect that reparametrizing a curve has on the length and energy functionals. We then turn our attention to the lower semicontinuity of the energy. Jost, in his book *Riemannian Geometry and Geometric Analysis*, shows that the energy is lower semicontinuous with respect to L^2 [8, Theorem 8.3.2]. Since we do not need quite so strong of a result, we introduce a weaker property that is akin to lower semicontinuity. Finally, we will show that the length and energy functionals have the same global minimizers up to some constraints on the parametrization of the curves.

3.1 Reparametrization of a curve

3.1.1 The length functional is invariant under reparametrization

Suppose that $\gamma : [a,b] \to M$ is a $W^{1,2}$ curve. A **reparametrization** of γ is a curve of the form $\widetilde{\gamma} = \gamma \circ \varphi : [c,d] \to M$, where $\varphi : [c,d] \to [a,b]$ is a monotone homeomorphism that is piecewise C^1 with finitely many pieces.

The curves γ and $\widetilde{\gamma}$ have the same image, so we would hope that our notion of length is unchanged by reparametrization. On the other hand, tracing along the image of the same curve at different speeds should take different amounts of energy. Indeed, we will see that the energy is dependent on parametrization.

Proposition 13. *The length functional is invariant under reparametrization.*

Proof. First, suppose that $\gamma\colon [a,b] \to \mathbb{R}^k$ is a $W^{1,2}$ curve, and $\varphi\colon [c,d] \to [a,b]$ is a monotone C^1 homeomorphism. We compute,

$$L(\gamma \circ \varphi) = \int_c^d |(\gamma \circ \varphi)'(s)|\, ds.$$

Using the chain rule,

$$= \int_c^d |\gamma'(\varphi(s))||\varphi'(s)|\, ds.$$

We change variables, letting $t = \varphi(s)$. If φ is increasing, then $dt = |\varphi'(s)|ds$ and $\varphi(d) > \varphi(c)$. Meanwhile, if φ is decreasing, then $dt = -|\varphi'(s)|ds$ and $\varphi(d) < \varphi(c)$. In either case we obtain

$$= \int_a^b |\gamma'(t)|\, dt.$$

By the definition of length,

$$= L(\gamma).$$

Now take φ to be piecewise C^1 with finitely many pieces. Then there exists a finite partition $[c_0, c_1], \ldots, [c_{m-1}, c_m]$ such that $\varphi|_{(c_i, c_{i+1})}$ is C^1. In this case, we write the length of $\gamma \circ \varphi$ as

$$L(\gamma \circ \varphi) = \sum_{i=0}^{m-1} \int_{c_i}^{c_{i+1}} |(\gamma \circ \varphi)'(s)|\, ds,$$

and repeat the above calculation for each term in the sum to conclude that the length is invariant under reparametrization. □

On the other hand, we show that we can change the energy of a curve whose image is the unit circle in $\mathbb{R}^2$ by changing the parametrization.

Example 1. *The energy is not necessarily preserved under reparametrization. Consider the curve* $\gamma\colon [0,2\pi] \to \mathbb{R}^2$ *given by*

$$\gamma(t) = (\sin(t), \cos(t)).$$

Let $\varphi\colon [0,1] \to [0,2\pi]$ *be given by* $\varphi(s) = 2\pi s$. *We first compute the energy of* γ *using the Euclidean norm of* $\mathbb{R}^2$.

$$\begin{aligned} E(\gamma) &= \int_0^{2\pi} |\gamma'(t)|^2 \, dt \\ &= \int_0^{2\pi} \left(\cos^2(t) + \sin^2(t)\right) dt \\ &= \int_0^{2\pi} 1 \, dt \\ &= 2\pi. \end{aligned}$$

To contrast,

$$\begin{aligned} E(\gamma \circ \varphi) &= \int_0^1 |(\gamma \circ \varphi)'(s)|^2 \, ds \\ &= \int_0^1 \left((2\pi)^2 \cos^2(2\pi s) + (2\pi)^2 \sin^2(2\pi s)\right) ds \\ &= \int_0^1 4\pi^2 \, ds \\ &= 4\pi^2. \end{aligned}$$

Thus, the energy of γ *is less than that of* $\gamma \circ \varphi$, *even though they have the same image.*

3.1.2 Parametrization by arc length

The proof of Proposition 21 below relies on the ability to reparametrize a curve with unit speed. Recall that a piecewise regular curve $\gamma\colon [a,b] \to M$ is said to be **parametrized by arc length** if $|\gamma'(t)| = 1$ for every $t \in [a,b]$. It is possible to reparametrize any curve by arc length in the *forward direction*. That is, such that the reparametrization is an increasing function.

Lemma 14. *Any piecewise regular curve has a unique forward reparametrization by arc length* [12, Proposition 2.49].

For this proof, we follow the methods of Lee [12, Proposition 2.49].

Proof. First, we suppose that $\gamma\colon [a,b] \to M$ is a regular curve. That is, $\gamma'(t) \neq 0$ for any $t \in [a,b]$. Fix $t_0 \in [a,b]$, and define a function

$$s(t) = \int_{t_0}^{t} |\gamma'(u)|\, du.$$

This is a local diffeomorphism mapping the interval $[a,b]$ into some $[c,d] \subset \mathbb{R}$. By the fundamental theorem of calculus and since γ is regular, $s'(t) = |\gamma'(t)| > 0$, so $s(t)$ is a strictly increasing diffeomorphism. Therefore, $\widetilde{\gamma} = \gamma \circ s^{-1}$ is a reparametrization of γ in the forward direction. Moreover, using the chain rule we compute

$$\begin{aligned} |\widetilde{\gamma}'(t)| &= \left| \left(s^{-1}(t)\right)' \gamma'\left(s^{-1}(t)\right) \right| \\ &= \left| \frac{1}{s'(s^{-1}(t))} \right| \left| \gamma'\left(s^{-1}(t)\right) \right|. \end{aligned}$$

By the definition of s, we have

$$\begin{aligned} &= \frac{1}{|\gamma'(s^{-1}(t))|} \left| \gamma'\left(s^{-1}(t)\right) \right| \\ &= 1. \end{aligned}$$

Thus, $\widetilde{\gamma}$ is a forward reparametrization of γ by arc length.

The result follows for piecewise regular curves γ by inducting on the number of regular pieces.

To show uniqueness, suppose that $\widetilde{\gamma} = \gamma \circ \varphi$ and $\overline{\gamma} = \gamma \circ \psi$ are both unit speed forward reparametrizations of a piecewise regular curve γ. Notice that since $\widetilde{\gamma}$ and $\overline{\gamma}$ are parametrized by arc length and trace out the same image, they must have the same domain up to translation. Let us call this domain $[0,c]$.

We define a piecewise regular homeomorphism $\eta = \varphi \circ \psi^{-1}\colon [0,c] \to [0,c]$. Thus, $\overline{\gamma} = \widetilde{\gamma} \circ \eta$. We compute within each regular piece of γ using the fact that

$1 = |\overline{\gamma}'(t)| = |\widetilde{\gamma}'(t)|$ within each of the finitely many regular sections.

$$\begin{aligned} 1 &= \left|\overline{\gamma}(t)'\right| \\ &= |\widetilde{\gamma}'(\eta(t))\eta'(t)| \\ &= |\widetilde{\gamma}'(\eta(t))|\eta'(t) \\ &= \eta'(t). \end{aligned}$$

Since $\eta(0) = 0$ and η is continuous, we see that $\eta(t) = t$ for every $t \in [0,c]$. Hence, $\widetilde{\gamma} = \overline{\gamma}$, and the forward reparametrization by arc length is unique. $\square$

Remark 15. *The above argument can be adapted to show that for any positive constant c, we can construct a forward reparametrization of a curve with constant speed c.*

3.2 A condition akin to lower semicontinuity of the energy functional

Another property of interest is lower semicontinuity. Recall that for a topological space X, a function $F\colon X \to (-\infty,\infty]$ is **lower semicontinuous** if for every $\lambda \in \mathbb{R}$, the set $\{x \in X\colon F(x) \leq \lambda\}$ is closed in X. In particular, if F is lower semicontinuous, then

$$\liminf_{n\to\infty} F(x_n) \geq F(x) \tag{3.1}$$

for any sequence $\{x_n\}_{n=1}^{\infty} \subset X$ converging to x. If X is a metric space, then the converse holds as well [2, p. 10].

We would like to obtain an inequality of the same type as Equation (3.1) for the energy functional. Although one can show that the energy functional is lower semicontinuous with respect to L^2, the proof is quite involved (for example, see Jost [8, Theorem 8.3.1]). In this section, we will prove a weaker result that is sufficient for our purposes. Namely,

Proposition 16. *Suppose* $\{\gamma_n\}_{n\in\mathbb{N}} \subset W^{1,2}([0,1],\mathbb{R}^k)$ *converges to some limit* γ *in*

L^2 and weakly in $W^{1,2}$. Then

$$\liminf_{n\to\infty} E(\gamma_n) \geq E(\gamma).$$

In order to prove this proposition, we require the following result about lower semicontinuity of the norm with respect to weak convergence.

Lemma 17. *Let X be a Hilbert space. The norm $\|\cdot\|$ on X is lower semicontinuous with respect to the weak topology on X.*

Proof. It is enough to check that equation (3.1) is satisfied to show lower semicontinuity. Let $\{x_n\}_{n\in\mathbb{N}} \subset X$ be a sequence that converges weakly to x. That is, $\varphi(x_n) \to \varphi(x)$ for any $\varphi \in X^*$. By the definition of the operator norm,

$$|\varphi(x_n)| \leq \|\varphi\|_{\mathrm{op}}\|x_n\|.$$

Taking the limit inferior on each side gives

$$\liminf_{n\to\infty} |\varphi(x_n)| \leq \liminf_{n\to\infty} \|\varphi\|_{\mathrm{op}}\|x_n\|.$$

Since $\varphi(x_n) \to \varphi(x)$, the limit on the left hand side is simply $|\varphi(x)|$. Thus,

$$|\varphi(x)| \leq \liminf_{n\to\infty} \|\varphi\|_{\mathrm{op}}\|x_n\|. \tag{3.2}$$

Now we make use of the fact that X is a Hilbert space. Namely, we use the fact that we may view x as a linear operator acting on $\varphi \in X^*$. Therefore,

$$\|x\| = \sup_{\|\varphi\|_{\mathrm{op}}\leq 1} |\varphi(x)|.$$

Using the identity from equation (3.2),

$$\leq \sup_{\|\varphi\|_{\mathrm{op}}\leq 1} \left(\liminf_{n\to\infty} \|\varphi\|_{\mathrm{op}}\|x_n\|\right).$$

Since $\|\varphi\|_{\text{op}}$ is independent of n, we take it outside of the inferior limit to achieve

$$= \left(\sup_{\|\varphi\|_{\text{op}} \leq 1} \|\varphi\|_{\text{op}} \right) \left(\liminf_{n\to\infty} \|x_n\| \right).$$

Finally, we use the upper bound $\sup_{\|\varphi\|_{\text{op}} \leq 1} \|\varphi\|_{\text{op}} \leq 1$ to conclude

$$\leq \liminf_{n\to\infty} \|x_n\|.$$

We have shown that $\|x\| \leq \liminf_{n\to\infty} \|x_n\|$ for any weakly convergent sequence $x_n \rightharpoonup x$. Thus, the norm is lower semicontinuous with respect to weak convergence. □

We now have all of the tools we need to prove Proposition 16.

Proof of Proposition 16. Suppose $\{\gamma_n\}_{n\in\mathbb{N}} \subset W^{1,2}([0,1],\mathbb{R}^k)$ converges to γ in L^2 and in weak $W^{1,2}$. By Lemma 17, the Sobolev norm $\|\cdot\|_{W^{1,2}}$ is lower semicontinuous with respect to weak $W^{1,2}$ convergence. Therefore,

$$\|\gamma\|_{W^{1,2}}^2 \leq \liminf_{n\to\infty} \|\gamma_n\|_{W^{1,2}}^2.$$

We expand, using the definition of the norm to achieve:

$$\|\gamma\|_{L^2}^2 + \|D\gamma\|_{L^2}^2 \leq \liminf_{n\to\infty} \|\gamma_n\|_{L^2}^2 + \|D\gamma_n\|_{L^2}^2.$$

By assumption, $\gamma_n \to \gamma$ in L^2 so we have

$$\|\gamma\|_{L^2}^2 + \|D\gamma\|_{L^2}^2 \leq \|\gamma\|_{L^2}^2 + \liminf_{n\to\infty} \|D\gamma_n\|_{L^2}^2.$$

Finally, subtracting $\|\gamma\|_{L^2}^2$ from each side, we obtain the desired result:

$$\|D\gamma\|_{L^2}^2 \leq \liminf_{n\to\infty} \|D\gamma_n\|_{L^2}^2.$$ □

3.3 The length and energy functionals have a positive lower bound

In this section, we show that if $\mathscr{H}$ is a nontrivial homotopy class of curves in M, then both the length and energy functionals have positive lower bounds over curves in $\mathscr{H}$. First, we will compare the size of the length and energy of a given curve.

Recall that, upon isometrically embedding M into $\mathbb{R}^k$, we may write

$$L(\gamma) = \|\gamma'\|_{L^1} \quad \text{and} \quad E(\gamma) = \|\gamma'\|_{L^2}^2.$$

Then by Hölder's inequality,

$$(L(\gamma))^2 \leq (b-a)E(\gamma),$$

with equality if and only if $|\gamma'|$ is constant. Although this is a simple application of Hölder, this inequality, and in particular its equality case, will be important in determining when the length and energy have the same minimizers. We now state the main result of this section.

Lemma 18. *Let M be a compact Riemannian manifold with or without boundary, and $\mathscr{H}$ a nontrivial class of closed $C^1([0,1],M)$ curves. Then the length and energy functionals have a lower bound over $\mathscr{H}$.*

Proof. First, suppose that M is a manifold without boundary. Since M is compact, its injectivity radius $\mathrm{inj}(M) > 0$ [12, Lemma 6.16]. Fix $\varepsilon = \mathrm{inj}(M)$. For any point $p \in M$, the exponential map $\exp_p \colon T_pM \to M$ is a diffeomorphism from $B_\varepsilon(0)$ onto its image.

Suppose that the lengths of the curves in $\mathscr{H}$ cannot be bounded below by a positive constant. Then there exists a sequence $\{\gamma_n\}_{n\in\mathbb{N}}$ such that $L(\gamma_n) \to 0$. In particular, there exists an $N \in \mathbb{N}$ such that $L(\gamma_N) < \varepsilon$. We will construct an explicit homotopy through M between γ_N and $\gamma_N(0)$. Since the exponential map is a radial isometry on the ball of radius ε about 0 in $T_{\gamma_N(0)}M$, we see that γ_N is contained in the geodesic ball of radius ε about $\gamma_N(0)$. In particular, γ_N is in the domain of

$\exp^{-1}_{\gamma_N(0)}$. Consider the map $h\colon [0,1]\times[0,1]\to M$ given by

$$h(s,t) = \exp_{\gamma_N(0)}\left(s\exp^{-1}_{\gamma_N(0)}(\gamma_N(t))\right).$$

This is a continuous map, as it is a composition of continuous maps. Moreover, we see that

$$h(0,t) = \exp_{\gamma_N(0)}(0) = \gamma_N(0), \text{ and}$$

$$h(1,t) = \exp_{\gamma_N(0)}\left(\exp^{-1}_{\gamma_N(0)}(\gamma_N(t))\right) = \gamma_N(t).$$

That is, we have a homotopy between $\gamma_N(0)$ and γ_N. This contradicts the fact that $\mathscr{H}$ is a nontrivial homotopy class. Thus, the length functional has a positive lower bound over $\mathscr{H}$.

Now, suppose that M is a manifold with boundary. We may run the same argument as above on $D(M)$. In this case, we define a homotopy $H\colon [0,1]\times[0,1]\to M$ by

$$H(s,t) = q\circ h(s,t),$$

where $q\colon D(M)\to M$ is the quotient map given by $q(p_k) = p_1$. The map H takes $\gamma_N(0)$ to γ_N through curves in M. Once again, this yields a contradiction and we see that the length functional has a positive lower bound over $\mathscr{H}$.

Finally, we extend this result to the energy functional. Let $\gamma\in\mathscr{H}$. The preceding argument shows that

$$L(\gamma)\geq A>0$$

for some constant A. By Hölder's inequality,

$$(b-a)E(\gamma)\geq (L(\gamma))^2\geq A^2>0,$$

and we see that the energy functional also has a positive lower bound over $\mathscr{H}$. □

3.4 Minimizers of the length and energy functionals

One of the aims of this thesis is to show the existence of length-minimizing curves in manifolds with boundary. In Chapter 2, we saw that the natural space of curves to minimize the length functional over is $W^{1,1}$. However, we also defined the energy functional, whose natural domain is $W^{1,2}$ curves. In Section 2.4, we discussed the advantages of working over $W^{1,2}$ rather than $W^{1,1}$. Thus, we seek a way to answer our question using the energy functional rather than the length functional.

First we will use a calculus of variations argument to determine where the length and energy functionals have the same critical points. Then we will show that, up to a certain constraint on the parametrization, the length and energy have the same minimizers. Throughout this section, we assume that minimizers of the length and energy exist. This will be addressed in Chapter 4.

Some of the methods in this section can be applied to show that if a curve maximizes the length, then it maximizes the energy. Similarly, if a curve maximizes the energy, then it maximizes the length. However, it is easy to see that maximizers of the length and energy functionals do not exist. Figure 3.1, demonstrates that we can always increase the length of a curve by adding more "bumps".

3.4.1 Critical points of the length and energy functionals

In this section, we compute the first variation formulas for the length and energy functionals. We will then compare the two formulas to determine where the critical points of the length and energy functionals coincide.

Proposition 19. *Let M be a Riemannian manifold, and $\gamma \in W^{1,2}([0,1],M)$. We define a family of $W^{1,2}$ curves by $F\colon (-\varepsilon,\varepsilon)\times[0,1] \to M$ such that $F(s,t) = \gamma_s(t)$ with $\gamma_0(t) = \gamma(t)$. Moreover, for ease of notation, we define $T := dF\left(\frac{\partial}{\partial t}\right) = \gamma_s{}'$ and $V := dF\left(\frac{\partial}{\partial s}\right)$. The first variation formula for the length functional is given by*

$$\frac{\partial}{\partial s}L(\gamma_s)\bigg|_{s=0} = \int_0^1 \frac{1}{|\gamma'|}\left(\frac{\partial}{\partial t}\langle V,\gamma'\rangle - \langle \nabla_{\frac{\partial}{\partial t}}\gamma', V\rangle\right) dt. \tag{3.3}$$

Proof. We begin by differentiating under the integral

$$\frac{\partial}{\partial s}L(\gamma_s)\bigg|_{s=0} = \int_0^1 \frac{1}{2|\gamma_s{}'|}2\left\langle \nabla_{\frac{\partial}{\partial s}}\gamma_s{}', \gamma_s{}'\right\rangle dt\bigg|_{s=0}.$$

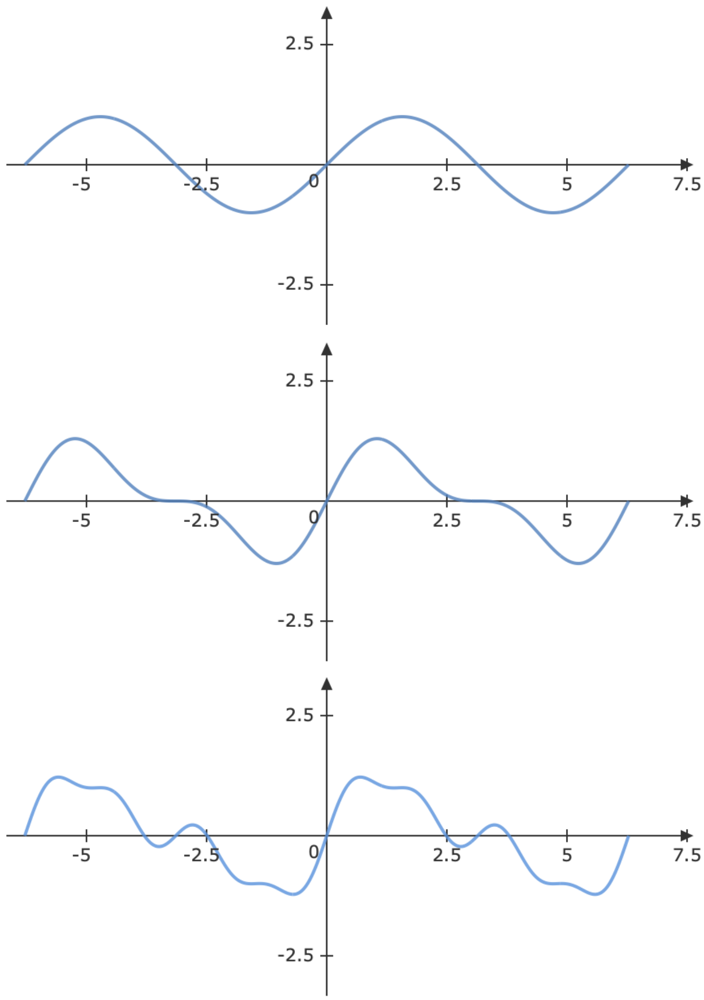


Figure 3.1: The length and energy functionals do not have an upper bound. You can always increase the length and energy of the curve by adding more "bumps". From top to bottom, we increase the length of a curve with fixed endpoints.

Using the definition of T, we replace $\gamma_s{}'$ in the first entry of the metric by $dF\left(\frac{\partial}{\partial t}\right)$

$$= \int_0^1 \frac{1}{|\gamma_s{}'|} \left\langle \nabla_{\frac{\partial}{\partial s}} dF\left(\frac{\partial}{\partial t}\right), \gamma_s{}' \right\rangle dt \bigg|_{s=0}.$$

By the torsion free property of the connection, $\nabla_{\frac{\partial}{\partial s}} dF\left(\frac{\partial}{\partial t}\right) = \nabla_{\frac{\partial}{\partial t}} dF\left(\frac{\partial}{\partial s}\right)$. Applying this identity to the first entry of the metric, we obtain

$$= \int_0^1 \frac{1}{|\gamma_s{}'|} \left\langle \nabla_{\frac{\partial}{\partial t}} dF\left(\frac{\partial}{\partial s}\right), \gamma_s{}' \right\rangle dt \Big|_{s=0}.$$

We use the definitions of T and V to simply our expression to

$$= \int_0^1 \frac{1}{|\gamma_s{}'|} \langle \nabla_{\frac{\partial}{\partial t}} V, T \rangle \, dt \Big|_{s=0}.$$

Now, using the metric compatibility of the connection

$$= \int_0^1 \frac{1}{|\gamma_s{}'|} \left(\frac{\partial}{\partial t} \langle V, T \rangle - \langle \nabla_{\frac{\partial}{\partial t}} T, V \rangle \right) dt \Big|_{s=0}.$$

Evaluating at $s = 0$ gives

$$= \int_0^1 \frac{1}{|\gamma'|} \left(\frac{\partial}{\partial t} \langle V, \gamma' \rangle - \langle \nabla_{\frac{\partial}{\partial t}} \gamma', V \rangle \right) dt. \qquad \square$$

We now remark on a few cases in which the first variation formula can be simplified. Suppose that γ is C^1 and is parametrized with a constant speed $|\gamma'| = c$ and length ℓ. Then equation (3.5) simplifies to

$$\frac{\partial}{\partial s} L(\gamma_s) \Big|_{s=0} = \frac{1}{c} \left(\langle V, \gamma' \rangle \Big|_0^{\ell/c} - \int_0^{\ell/c} \left(\langle \nabla_{\frac{\partial}{\partial t}} \gamma', V \rangle \right) dt \right). \tag{3.4}$$

Furthermore, if γ is C^1 except on $\{t_1, \ldots, t_m\}$ and parametrized with a constant speed c, the first variation formula becomes

$$\frac{\partial}{\partial s} L(\gamma_s) \Big|_{s=0} = \frac{1}{c} \left(-\sum_{i=1}^{m} \langle \gamma'(t_i^+) - \gamma'(t_i^-), V(t_i) \rangle - \int_0^{\ell/c} \langle \nabla_{\frac{\partial}{\partial t}} \gamma', V \rangle \, dt \right), \tag{3.5}$$

where $\gamma'(t_i^-) := \lim_{t \to t_i^-} \gamma'(t)$ and $\gamma'(t_i^+) := \lim_{t \to t_i^+} \gamma'(t)$.

Proposition 20. *Let M be a smooth Riemannian manifold, and $\gamma \in W^{1,2}([0,1], M)$. Define a family of $W^{1,2}$ curves by $F\colon (-\varepsilon, \varepsilon) \times [0,1] \to M$ such that $F(s,t) = \gamma_s(t)$ with $\gamma_0(t) = \gamma(t)$. As before, we define $T := dF\left(\frac{\partial}{\partial t}\right) = \gamma_s{}'$ and $V := dF\left(\frac{\partial}{\partial s}\right)$. Then* *the first variation formula for the energy is given by*

$$\frac{\partial}{\partial s}E(\gamma_s)\bigg|_{s=0} = 2\int_0^1 \left(\frac{\partial}{\partial t}\langle V, \gamma'\rangle - \langle \nabla_{\frac{\partial}{\partial t}}\gamma', V\rangle \, dt\right). \tag{3.6}$$

Proof. By the definition of the energy functional,

$$\frac{\partial}{\partial s}E(\gamma_s)\bigg|_{s=0} = \frac{\partial}{\partial s}\int_0^1 \langle {\gamma_s}', {\gamma_s}'\rangle \, dt\bigg|_{s=0}.$$

Differentiating under the integral gives

$$= 2\int_0^1 \langle \nabla_{\frac{\partial}{\partial s}} {\gamma_s}', {\gamma_s}'\rangle \, dt\bigg|_{s=0}.$$

We replace the term ${\gamma_s}'$ in the first slot of the metric by $dF\left(\frac{\partial}{\partial t}\right)$ to obtain

$$= 2\int_0^1 \left\langle \nabla_{\frac{\partial}{\partial s}} dF\left(\frac{\partial}{\partial t}\right), {\gamma_s}'\right\rangle dt\bigg|_{s=0}.$$

By the torsion free property of the connection,

$$= 2\int_0^1 \left\langle \nabla_{\frac{\partial}{\partial t}} dF\left(\frac{\partial}{\partial s}\right), {\gamma_s}'\right\rangle dt\bigg|_{s=0}.$$

We now begin using T and V for ease of notation

$$= 2\int_0^1 \langle \nabla_{\frac{\partial}{\partial t}} V, T\rangle \, dt\bigg|_{s=0}.$$

Using the metric compatibility of the connection, we have

$$= 2\int_0^1 \left(\frac{\partial}{\partial t}\langle V, T\rangle - \langle V, \nabla_{\frac{\partial}{\partial t}} T\rangle\right) dt\bigg|_{s=0}.$$

Evaluating at $s = 0$ and using the symmetry of the metric gives

$$= 2\int_0^1 \left(\frac{\partial}{\partial t}\langle V, \gamma'\rangle - \langle \nabla_{\frac{\partial}{\partial t}} \gamma', V\rangle\right) dt. \qquad \square$$

Furthermore, if γ is C^1, then integrating the first term of Equation (3.6) gives

$$\frac{\partial}{\partial s}E(\gamma_s)\bigg|_{s=0} = 2\left(\langle V,\gamma'\rangle\bigg|_{t=0}^{1} - \int_0^1 \langle \nabla_{\frac{\partial}{\partial t}}\gamma', V\rangle\, dt\right).$$

Through an inductive argument, we see that if γ is C^1 for $t \in [0,1] \setminus \{t_1,\ldots,t_m\}$, then the first variation formula for the energy is given by

$$\frac{\partial}{\partial s}E(\gamma_s)\bigg|_{s=0} = 2\left(-\sum_{i=1}^{m}\langle \gamma'(t_i^+) - \gamma'(t_i^-), V(t_i)\rangle - \int_0^1 \langle \nabla_{\frac{\partial}{\partial t}}\gamma', V\rangle\, dt\right), \tag{3.7}$$

where $\gamma'(t_i^-) := \lim_{t\to t_i^-}\gamma'(t)$ and $\gamma'(t_i^+) := \lim_{t\to t_i^+}\gamma'(t)$.

Equations (3.5) and (3.7) only differ by a factor of $2c$, a positive constant. Thus under these conditions, the length and energy functionals have the same critical points. In summary,

(i) If γ is piecewise C^1, parametrized with constant speed, and is a critical point of L, then it is a critical point of E, and

(ii) if γ is piecewise C^1 and a critical point of E then it is a critical point of L.

Finally, we introduce a definition:

Definition 11 ([12, p. 103])**.** *A smooth curve γ is a **geodesic** if $\nabla_{\frac{\partial}{\partial t}}\gamma' = 0$. That is, if it has zero acceleration.*

Notice that if γ is a geodesic, then it is a critical point of both the length and energy functionals.

3.4.2 When do the minimizers of the length and energy coincide?

In the previous section, we saw a few places where the length and energy functionals have the same critical points, so it would be reasonable to ask whether there are minima that coincide. To begin, we ask the following:

> Is it true that a curve minimizes the length functional if and only if it minimizes the energy functional?

Unfortunately, the answer is no. We have seen that the length functional is invariant under reparametrization. Therefore, if a curve γ minimizes the length functional, every reparametrization of γ also minimizes the length functional. On the other hand, the energy varies with reparametrization. If γ is a curve that minimizes the energy functional, a reparametrization of γ will not, in general, minimize the energy functional. In this sense it is more difficult to minimize the energy than the length. Hence we may revise the above question, splitting it into two parts:

(a) Under what conditions does a minimizer of the length functional minimize the energy?

(b) Is a minimizer of the energy functional a minimizer of the length?

In the following proposition, we answer question (a) with a sufficient condition, and show that the answer to question (b) is "yes".

Proposition 21. *Let M be a Riemannian manifold, and $\mathscr{H}$ a nontrivial class of smooth closed curves. Assume that global minimizers of the length and energy functionals in $\mathscr{H}$ exist. Then*

(i) a constant speed minimizer of the length functional is a minimizer of the energy, and

(ii) a minimizer of the energy functional is a minimizer of the length.

Proof. We prove each part in turn.

(i) Suppose that γ_1 is a constant speed curve minimizing the length functional. Then for any other curve $\lambda \in \mathscr{H}$, we have

$$L(\gamma_1) \leq L(\lambda).$$

By Hölder's inequality,

$$(L(\gamma_1))^2 \leq (L(\lambda))^2 \leq (b-a)E(\lambda).$$

Moreover, since γ_1 is parametrized with constant speed, the equality case of Hölder's inequality gives

$$(b-a)E(\gamma_1) = (L(\gamma_1))^2 \leq (L(\lambda))^2 \leq (b-a)E(\lambda).$$

Therefore, γ_1 is a minimizer of the energy functional.

(ii) Now, we will show that a minimizer of the energy functional is also a minimizer of the length functional. Suppose that γ_2 minimizes the energy functional in $\mathscr{H}$. Let $\lambda \in \mathscr{H}$, and consider $\widetilde{\lambda}$, a reparametrization of λ by arc length. Then we have,

$$E(\gamma_2) \leq E(\widetilde{\lambda}).$$

By Hölder's inequality,

$$(L(\gamma_2))^2 \leq (b-a)E(\gamma_2) \leq (b-a)E(\widetilde{\lambda}).$$

Since $\widetilde{\lambda}$ is parametrized by arc length, the equality case of Hölder's inequality gives

$$(L(\gamma_2))^2 \leq (b-a)E(\gamma_2) \leq (b-a)E(\widetilde{\lambda}) = (L(\widetilde{\lambda}))^2.$$

Finally, since the length is invariant under reparametrization,

$$(L(\gamma_2))^2 \leq (b-a)E(\gamma_2) \leq (b-a)E(\widetilde{\lambda}) = (L(\widetilde{\lambda}))^2 = (L(\lambda))^2.$$

Therefore, γ_2 is a minimizer of the length functional, as desired. □

Chapter 4

The Existence of Length-Minimizing Curves

Throughout this chapter, we regard all Riemannian manifolds M as being isometrically embedded into $\mathbb{R}^k$. Given a nontrivial homotopy class $\mathscr{H}$ of closed curves in M, we define $\Omega = \mathscr{H} \cap W^{1,2}([0,1],M)$.

4.1 The existence of a length-minimizing curve

In section 3.4, we saw that any minimizer of the energy functional is a minimizer of the length functional. Thus, if we wish to find a length-minimizing curve in $\mathscr{H}$, it is enough to find an energy-minimizing curve. In this section, we show the existence of a curve γ such that $E(\gamma)$ takes on the infimum of the energy over Ω.

It is always possible to take an energy-minimizing sequence $\{\gamma_n\}_{n\in\mathbb{N}}$ such that

$$E(\gamma_n) \to \inf_{\omega\in\Omega} E(\omega).$$

A natural candidate for γ would be a limit of the sequence $\{\gamma_n\}_{n\in\mathbb{N}}$. However, it is not clear whether such a limit exists or in which sense we can ask for convergence to γ. This will be addressed in the following theorem.

Theorem 22. *Let M be a compact Riemannian manifold with or without boundary, and $\mathscr{H}$ a nontrivial homotopy class of closed curves in M. Then there exists a* *sequence* $\{\gamma_n\}_{n\in\mathbb{N}} \subset \Omega$ *and a closed curve* γ *such that*

(i) $E(\gamma_n) \to \inf_{\omega\in\Omega} E(\omega)$*;*

(ii) $\gamma_n \rightharpoonup \gamma$ *in* $W^{1,2}([0,1],\mathbb{R}^k)$*;*

(iii) $\gamma_n \to \gamma$ *in* $L^2([0,1],\mathbb{R}^k)$*; and*

(iv) $\gamma_n \to \gamma$ *in* $C^{0,\beta}([0,1],\mathbb{R}^k)$ *for each* $0 < \beta < \frac{1}{2}$.

Moreover, γ *is in* $\mathscr{H}$ *and minimizes length among all curves in* Ω.

Proof. By Lemma 18, $\inf_{\omega\in\Omega} E(\omega) > 0$. Choose a sequence $\{\lambda_i\}_{i\in\mathbb{N}} \subset \Omega$ such that

$$E(\lambda_i) \to \inf_{\omega\in\Omega} E(\omega).$$

Step 1: Finding a uniform upper bound of the elements λ_i. We begin by finding an uniform upper bound of the sequence elements in the $W^{1,2}$ norm. Possibly by taking a subsequence, we may assume that for every $i \in \mathbb{N}$,

$$E(\lambda_i) = \|\lambda_i'\|_{L^2} \leq B_1$$

for some uniform upper bound $B_1 > \inf_{\omega\in\Omega} E(\omega)$. By an abuse of notation, we will continue to call this sequence $\{\lambda_i\}_{i\in\mathbb{N}}$. Moreover, since M is compact, there exists a constant B_2 such that M is contained in the $(k-1)$-dimensional ball of radius B_2 about 0. Hence for every $i \in \mathbb{N}$, we have

$$\|\lambda_i\|_{L^2}^2 = \int_0^1 |\lambda_i(t)|^2\, dt \leq \int_0^1 \sup_{x\in M} |x|^2\, dt = \sup_{x\in M} |x|^2 \leq B_2^2.$$

Finally, letting $B = B_1 + B_2^2$, we see that $\|\lambda_i\|_{W^{1,2}} \leq B$ for every $i \in \mathbb{N}$.

Step 2: Extracting convergent subsequences. By Theorem 4, a special case of the Sobolev embedding theorem, there exists a continuous embedding $W^{1,2}([0,1],\mathbb{R}^k) \hookrightarrow C^{0,\alpha}([0,1],\mathbb{R}^k)$ whenever $\alpha \in \left(0,\frac{1}{2}\right)$. The continuity of the embedding preserves the upper bound B in the Hölder norm. That is $\|\lambda_i\|_{C^{0,\alpha}} \leq B$ for every $i \in \mathbb{N}$. Since our sequence is bounded above, Corollary 5 of the Banach–Alaoglu theorem tells us that there exists a subsequence $\{\mu_j\}_{j\in\mathbb{N}} \subset \{\lambda_i\}_{i\in\mathbb{N}}$ that converges weakly in $W^{1,2}$ to a closed curve $\mu \in W^{1,2}([0,1],\mathbb{R}^k)$.

Notice that the sequence $\{\mu_j\}_{j\in\mathbb{N}}$ is still uniformly bounded above in the $W^{1,2}$ norm. Thus by Theorem 6, there exists a further subsequence $\{\nu_k\}_{k\in\mathbb{N}} \subset \{\mu_j\}_{j\in\mathbb{N}}$ such that $\nu_k \to \nu$ in $L^2([0,1],\mathbb{R}^k)$.

Finally, by the Arzelà–Ascoli theorem, there exists a subsequence $\{\gamma_\ell\}_{\ell\in\mathbb{N}} \subset \{\nu_k\}_{k\in\mathbb{N}}$ such that $\gamma_\ell \to \gamma$ in the $C^{0,\beta}([0,1],\mathbb{R}^k)$ norm for some closed curve γ and for each $\beta < \alpha$.

Step 3: Uniqueness of the limit. We would like to show that the three limits, μ, ν, and γ are equal. To show $\mu = \nu$, we appeal to distributions. By Lemmas 8 and 9, we know that

$$T_{\mu_j}(\varphi) \to T_\mu(\varphi) \quad \text{and} \quad T_{\nu_k}(\varphi) \to T_\nu(\varphi).$$

where $\varphi \in \mathscr{D}([0,1],\mathbb{R})$ is an arbitrary test function. Then by Lemma 10, $\mu(t) = \nu(t)$ for almost every $t \in [0,1]$. By the continuity of μ and ν, we see that $\mu = \nu$.

We prove that $\nu = \gamma$ by showing $\gamma_\ell \to \gamma$ in the L^2 norm. We compute

$$\begin{aligned} \lim_{\ell\to\infty} \|\gamma_\ell - \gamma\|_{L^2}^2 &= \lim_{\ell\to\infty} \int_0^1 |\gamma_\ell(t) - \gamma(t)|^2 \, dt \\ &\leq \lim_{\ell\to\infty} \sup_{t\in[0,1]} \left(|\gamma_\ell(t) - \gamma(t)|\right)^2 \\ &= 0, \end{aligned}$$

where the last line follows from the fact that $\gamma_\ell \to \gamma$ in the C^0 norm. By almost everywhere uniqueness of limits in L^2, we see that $\gamma(t) = \nu(t)$ for almost every $t \in [0,1]$. By the continuity of γ and ν, we conclude that $\gamma = \nu$.

Hence, we have constructed a sequence of closed curves $\{\gamma_\ell\}_{\ell\in\mathbb{N}}$ such that $E(\gamma_\ell) \to \inf_{\omega\in\Omega} E(\omega)$ and $\gamma_\ell \to \gamma$ in the spaces: weak $W^{1,2}$; L^2; and $C^{0,\beta}$ for each $0 < \beta < \frac{1}{2}$.

Step 4: The limiting curve γ is in $\mathscr{H}$ and minimizes length over Ω. By Proposition 12 and condition (iv), γ is in $\mathscr{H}$. By Proposition 16, conditions (ii) and (iii) tell us that

$$E(\gamma) \leq \liminf_{n\to\infty} E(\gamma_n) = \inf_{\omega\in\Omega} E(\omega).$$

In particular, since $\gamma \in \Omega$ and $E(\gamma) = \inf_{\omega\in\Omega} E(\omega)$, γ minimizes the energy over

Ω. Finally, by Proposition 21, a minimizer of the energy functional is a minimizer of the length functional, so γ is length-minimizing among curves in Ω. □

4.2 Length-minimizing curves are geodesics in manifolds without boundary

We now restrict our attention to the case of manifolds without boundary. We would like to show that the length-minimizing curve constructed in Section 4.1 is a geodesic. This result is well known in Riemannian geometry (see, for example [8, Theorem 1.5.1], [12, Theorem 6.4], or [18, Theorem 4.2]). We will prove an analogous result for manifolds with boundary in the following section.

Theorem 23. *If M is a compact Riemannian manifold without boundary and $\mathscr{H}$ and γ are as in Theorem 22, then γ is a geodesic.*

Proof. By Theorem 22, there exists a closed curve γ which minimizes length among all curves in Ω. We would like to show that γ is a geodesic.

Consider a point $t_0 \in [0,1]$. Since $\gamma \in C^0$, there exists an interval $(t_1,t_2) \subset [0,1]$ with $t_0 \in (t_1,t_2)$ such that $\gamma|_{(t_1,t_2)}$ is contained in a geodesic ball about $\gamma(t_0)$. Since γ is length-minimizing, γ is the shortest curve between $\gamma(t_1)$ and $\gamma(t_2)$. Indeed, suppose instead that there exists a curve μ such that $L(\mu|_{(t_1,t_2)}) < L(\gamma|_{(t_1,t_2)})$. We define a new closed curve $\Gamma\colon [0,1] \to M$ by

$$\Gamma(t) = \begin{cases} \gamma(t), & t \in [0,t_1] \cup [t_2,1] \\ \mu(t), & t \in [t_1,t_2]. \end{cases}$$

We then compute:

$$L(\Gamma) = \int_0^1 |\Gamma'(t)|\, dt.$$

By the definition of Γ,

$$= \int_0^{t_0} |\gamma'(t)|\, dt + \int_{t_0}^{t_1} |\mu'(t)|\, dt + \int_{t_1}^{1} |\gamma'(t)|\, dt.$$

Since μ is strictly shorter than γ between $\gamma(t_1)$ and $\gamma(t_2)$, we have

$$< \int_0^{t_0} |\gamma'(t)|\,dt + \int_{t_0}^{t_1} |\gamma'(t)|\,dt + \int_{t_1}^{1} |\gamma'(t)|\,dt$$
$$= \int_0^1 |\gamma'(t)|\,dt.$$

Finally, by the construction of γ

$$= \inf_{\omega\in\Omega} L(\omega).$$

Therefore

$$L(\Gamma) < L(\gamma) = \inf_{\omega\in\Omega} L(\omega),$$

a contradiction. Hence, $\gamma|_{(t_1,t_2)}$ is length-minimizing.

By Proposition 6.11 of Lee's *Introduction to Riemannian Manifolds* [12], there is a unique geodesic λ from $\gamma(t_1)$ to $\gamma(t_2)$, and λ is the unique minimizing curve between these points. Thus, $\lambda(t) = \gamma(t)$ for $t \in (t_1,t_2)$, so we see that γ is locally a geodesic. Since we chose t_0 arbitrarily, we conclude that γ is a geodesic. □

4.3 Properties of length-minimizing curves in manifolds with boundary

In the previous section, we were able to conclude that a length-minimizing curve in a Riemannian manifold without boundary is a geodesic. We would like to replicate this result for Riemannian manifolds with boundary. Unfortunately, the notion of geodesics cannot be directly extended to these. In general, we cannot even ask that a least length curve is more than C^1 in a manifold with smooth boundary. The smoothness of the curve can fail at the points where γ switches between the interior of the manifold and the boundary.

Example 2. *Let* $a = \frac{-1-\sqrt{7}}{4}$*, and* $m = \frac{-a}{\sqrt{1-a^2}}$*. We define a set*

$$U = B_1((0,0)) \cup B_{\frac{3}{4}}\left(\left(6,\tfrac{1}{4}\right)\right) \cup B_{\frac{1}{2}}\left(\left(\frac{m}{2\sqrt{1+m^2}} - 2, -\frac{1}{2\sqrt{1+m^2}} - 2\right)\right),$$

where $B_r(a)$ *is the ball of radius* r *centred at* a*. Let* $M = \mathbb{R}^2 \setminus U$ *and choose* $\mathscr{H}$

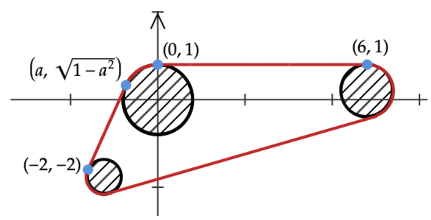
to be the set of closed curves in M encircling all three removed disks. In Figure 4.1a, we illustrate the least length curve in this class. The path between $(-2,-2)$ and $(6,1)$ travelling along the upper portion of our closed curve is given by the function

$$f(x) = \begin{cases} mx + 2(m-1), & x \leq a \\ \sqrt{1-x^2}, & a < x \leq 0 \\ 1, & 0 < x. \end{cases}$$

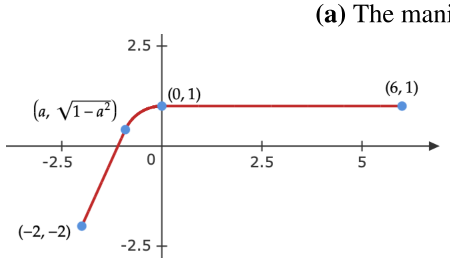
The graph of this function can be seen in Figure 4.1b. Differentiating, we obtain

$$\frac{df}{dx}(x) = \begin{cases} m, & x \leq a \\ \frac{-x}{\sqrt{1-x^2}}, & a < x \leq 0 \\ 0, & 0 < x. \end{cases}$$

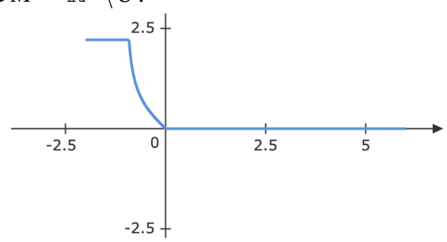
In Figure 4.1c, we see that the derivative has corners. Thus, the length-minimizing curve is C^1, but not C^2.



(a) The manifold $M = \mathbb{R}^2 \setminus U$.



(b) The graph of $f(x)$.



(c) The graph of $f'(x)$.

Figure 4.1: An illustration of Example 2. Panel (a) shows the shortest closed curve in M encircling the three removed disks. Panels (b) and (c) represent the graphs of f and f', respectively. We see that f' has corners, hence f' is continuous, but not differentiable.

We have seen that a length-minimizing curve can fail to be smooth at the "touching points" where it switches between the interior and boundary of the manifold. We now ask: on a manifold with boundary, is a length-minimizing curve a geodesic away from the touching points? In the following theorem, we will see that the answer is "yes". Moreover, we will see that in some cases, the length-minimizing curve leaves the touching points tangentially. This fits with our intuition from $\mathbb{R}^2$, as seen below in Figure 4.2.

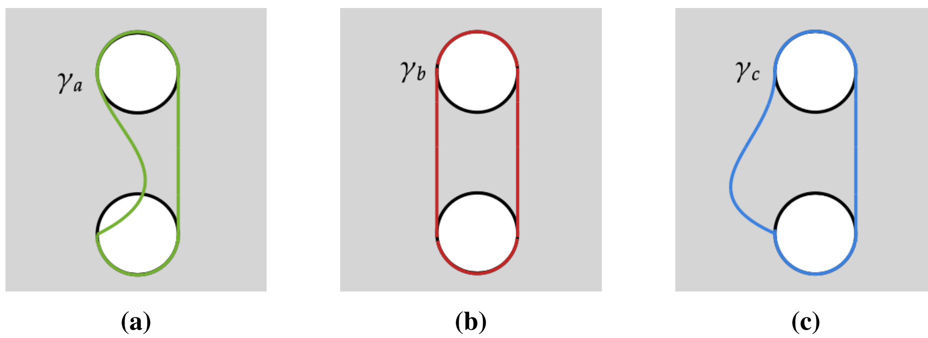


Figure 4.2: A length-minimizing curve must leave the boundary of the manifold tangentially. In panel (a), we see a curve with a corner pushing it out of the manifold. This curve cannot be in $\mathscr{H}$, so it is inadmissible. Panel (b) depicts a curve that leaves the boundary tangentially. This looks like a rubber band wrapped tightly around the two disks, and is length-minimizing. Panel (c) shows a curve whose corner pushes it into the manifold. This curve is too "loose", and could be shortened by pulling it tighter against the two disks.

Theorem 24. *Let M be a compact Riemannian manifold with boundary and let $\mathscr{H}$ and γ be as in Theorem 22. With the notation $A = \{t \in [0,1] \colon \gamma(t) \in M \setminus \partial M\}$ and $B = \{t \in [0,1] \colon \gamma(t) \in \partial M\}$,*

(i) γ is a geodesic in M on each connected component of A,

(ii) γ is a geodesic in ∂M on each connected component of B with non-empty interior, and

(iii) if t_0 is an isolated point of ∂B in $[0,1]$, then γ leaves ∂M tangentially at t_0. That is, γ is C^1 at t_0.

Proof. On connected components of A, we argue as in the proof of Theorem 23 to see that γ is a geodesic in M and establish (i). Similarly, we apply this argument to the connected components of B with non-empty interior to achieve property (ii). In particular, γ is smooth on open, connected components of A and B.

We now prove property (iii). Taking t_0 as in the statement of the theorem, there exists an interval $(t_1,t_2) \subset [0,1]$ containing t_0 such that $(t_1,t_2) \cap \partial B = \{t_0\}$. By construction, the curve $\gamma|_{(t_1,t_2)}$ is energy-minimizing (this can be seen in detail in the proof of Theorem 23).

First, we will show that for each $t \in (t_1, t_2) \setminus \{t_0\}$

$$\nabla_{\frac{\partial}{\partial t}} \gamma'(t) = 0, \tag{4.1}$$

where ∇ is the Levi-Civita connection of M. Let $\varphi \in C^\infty([0,1],\mathbb{R})$ be a bump function such that $\varphi(t) > 0$ for $t \in (t_1,t_2)$, and 0 otherwise. Set $V = \varphi \nabla_{\frac{\partial}{\partial t}} \gamma'$, and let γ_s be a variation of γ with variation field V. Since γ is energy-minimizing, deformations of γ in M are non-decreasing in energy. Hence

$$0 \leq \frac{\partial}{\partial s} E(\gamma_s)\bigg|_{s=0}.$$

By the first variation formula for the energy (Equation (3.7)), we expand this as

$$= \langle \gamma'(t_2), V(t_2) \rangle - \langle \gamma'(t_1), V(t_1) \rangle - \int_{t_1}^{t_2} \langle \nabla_{\frac{\partial}{\partial t}} \gamma'(t), V(t) \rangle \, dt.$$

By the definition of V, the leading terms vanish, leaving

$$= -\int_{t_1}^{t_2} \langle \nabla_{\frac{\partial}{\partial t}} \gamma'(t), \varphi \nabla_{\frac{\partial}{\partial t}} \gamma'(t) \rangle \, dt.$$

Since $\varphi > 0$ on (t_1,t_2), the integrand is non-negative, and we conclude that $\nabla_{\frac{\partial}{\partial t}} \gamma'(t)$ must be 0 for almost every $t \in (t_1,t_2)$. Moreover, since γ is smooth on $(t_1,t_2) \setminus \{t_0\}$, we conclude that $\nabla_{\frac{\partial}{\partial t}} \gamma'(t) = 0$ for each t in $(t_1,t_2) \setminus \{t_0\}$.

We are now ready to show that $\gamma(t_0)$ is not a corner. Let $\gamma'(t_0^-) := \lim_{t \to t_0^-} \gamma'(t)$ and $\gamma'(t_0^+) := \lim_{t \to t_0^+} \gamma'(t)$. Choose a variation field $\overline{V}$ that fixes the endpoints $\gamma(t_1)$

and $\gamma(t_2)$, and such that
$$\overline{V}(t_0) = \gamma'(t_0^+) - \gamma'(t_0^-).$$

Let $\overline{\gamma}_s$ be a variation of γ with variation field $\overline{V}$. Since γ is energy-minimizing, deformations in M are non-decreasing in the energy, so

$$0 \leq \frac{\partial}{\partial s} E(\overline{\gamma}_s)\bigg|_{s=0}.$$

By the first variation formula of the energy (3.7),

$$\begin{aligned}
&= \langle \gamma'(t_2), \overline{V}(t_2)\rangle - \langle \gamma'(t_1), \overline{V}(t_1)\rangle \\
&- \langle \gamma'(t_0^+) - \gamma'(t_0^-), \overline{V}(t_0)\rangle - \int_{t_1}^{t_2} \langle \nabla_{\frac{\partial}{\partial t}} \gamma'(t), \overline{V}(t)\rangle \, dt.
\end{aligned}$$

By the definition of $\overline{V}$, $\overline{V}(t_1) = \overline{V}(t_2) = 0$, and the first two terms vanish. This leaves

$$= -\langle \gamma'(t_0^+) - \gamma'(t_0^-), \overline{V}(t_0)\rangle - \int_{t_1}^{t_2} \langle \nabla_{\frac{\partial}{\partial t}} \gamma'(t), \overline{V}(t)\rangle \, dt.$$

By Equation (4.1), $\gamma|_{(t_1,t_2)}$ is a geodesic in M except at t_0, and we have $\nabla_{\frac{\partial}{\partial t}} \gamma'(t) = 0$ at these points. Thus, the integral term vanishes

$$= -\langle \gamma'(t_0^+) - \gamma'(t_0^-), \overline{V}(t_0)\rangle.$$

Finally, using the definition of $\overline{V}$ at t_0,

$$= -|\gamma'(t_0^+) - \gamma'(t_0^-)|^2.$$

The last line is non-positive. Thus, in order for it to be greater than or equal to 0, we must have
$$\gamma'(t_0^+) = \gamma'(t_0^-).$$

We conclude that γ is C^1 at t_0, hence it leaves the boundary tangentially. □

Remark 25. *We stated condition (iii) of this theorem in the case that $t_0 \in \partial B$ can be separated from other points of ∂B. However, if (t_0) is an accumulation point of*

∂B in $[0,1]$, we still expect that $\gamma(t_0)$ will not be a corner.

4.4 Extensions to other classes of curves

The methods used in this thesis can be generalized to any nontrivial homotopy class of curves (for example, fixed endpoint homotopies). In particular, in Section 4.2 we have given an alternative proof that in a complete Riemannian manifold, there exists a minimizing geodesic between any two points.

These methods can also be adapted to show the existence of length-minimizing curves in manifolds that are not complete. For example, the proof of Theorem 23 can be adapted to show that in any open convex subset of $\mathbb{R}^k$, a length-minimizing curve exists between any pair of points. The proof of Theorem 24 can also be adapted to show the existence and properties of length-minimizing curves in a more general setting; for instance, in an open convex subset of $\mathbb{R}^k$ with finitely many open disks removed.

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