Points of small height on affine varieties defined over function fields of finite transcendence degree

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Abstract

The problem of this thesis concerns points of small height on affine varieties defined over arbitary function fields, and is based on published work with Prof. Dragos Ghioca (see [GN20]). The main result is as follows: the points lying outside the largest subvariety defined over the constant field cannot have arbitrarily small height.

Prior results of this type include [Ghi09], [Ghi14]. In particular, [Ghi14] answers this question for function fields of transcendence degree 1. It also captures the history of the subject and features an argument that was initially used by the author of this thesis to extend [Ghi14] to varieties defined over function fields of arbitrary (finite) transcendence degree.

The content of this thesis and the associated published paper not only extends [Ghi14] to arbitrary transcendence degree, but also provides a sharp lower bound for points which are not contained in the largest subvariety defined over the constant field. The argument here works directly with the defining polynomials of the variety (compare with [Ghi14]), and the lower bound only depends on their degrees.

Lay summary

Finding integer solutions to equations like $x^2 + y^2 = z^2$ is of great interest to number theorists. This is the same as studying rational points on certain curves and surfaces, and this perspective allows us to think geometrically. An important notion in studying rational points is the notion of height. Intuitively, one can think of height as measuring the complexity of a point, so that the lower the height is, the simpler the point. There is a collection of points with height 0, which we can think of as the simplest points.

We investigate the following question: other than the simplest points of height 0, how simple can the rational points on a curve/surface be? One can think of polynomials in the same way as integers and formulate the same question. It turns out that in this case the answer will depend on the equations that define the curve/surface.

Preface

This thesis is based on joint work with Professor Dragos Ghioca, [GN20]. Initially, the author of this thesis extended a Bogomolov type result of Prof. Ghioca ([Ghi14]) to function fields of arbitrary transcendence degree. Prof. Ghioca made this result effective by presenting an entirely novel argument, much different from [Ghi14], which resulted in this paper. The author of this thesis was responsible for validating the argument as well as ensuring the accuracy and completeness of the final draft prior to submission.

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Finally, I am thankful for my parents. They have brought me up to become the person I am today and I definitely would not be standing where I am without them.

Chapter 1

Introduction

1.1 Notation

Given an arbitrary field k and a function field K of transcendence degree $m \geq 1$ over k, we let $h : \mathbb{A}^N(\overline{K}) \longrightarrow \mathbb{Q}_{\geq 0}$ represent the Weil height for the points of the corresponding affine space (for any given $N \geq 1$); we refer the reader to the classical geometric construction of the Weil height for points on affine spaces defined over function fields as presented in Serre's book [Ser97], but we will also sketch briefly in Section 2 an algebraic construction of the Weil height.

1.2 Statement of our main results

Some of the most important theorems in arithmetic geometry in the past 30 years have been the proofs of the Bogomolov conjecture both for powers of the multiplicative group (see [Zha95, Bil97]) and also for abelian varieties (see [Zha98, Ull98]) defined over $\overline{\mathbb{Q}}$. In both cases, the fundamental principle has been that when G is either $\mathbb{G}_m^N$ or an abelian variety defined over $\overline{\mathbb{Q}}$, then the accumulating subvarieties of G for points of small canonical height are the torsion translates of algebraic subgroups of G; in other words, if $V \subseteq G$ is an irreducible subvariety with the property that for any $\epsilon > 0$, we have that the set

$$\left\{P \in V(\overline{\mathbb{Q}}) \colon \widehat{h}(P) < \epsilon\right\}$$

is Zariski dense in V, then $V = Q + W$, where $Q \in G_{\mathrm{tor}}$ and W is a connected algebraic subgroup of G.

Motivated by the above classical results, the first author studied in [Ghi09, Ghi14] a variant of the Bogomolov conjecture for points on affine varieties defined over function fields (alternatively, this could be interpreted as a variant of the Bogomolov conjecture for $\mathbb{G}_m^N$ defined over a function field). So, given a function field K/k, the accumulating subvarieties of $\mathbb{A}^N$ for points of small Weil height (with respect to the places of the function field K/k; see also Section 2 for the definition of the Weil height in this context) should be the subvarieties defined over $\overline{k}$ (the algebraic closure of the constant field) since they are the subvarieties containing a Zariski dense set of points from $\mathbb{A}^N(\overline{k})$ (which all have Weil height 0).

In other words, given an affine subvariety $V \subseteq \mathbb{A}^N$, one considers $W \subseteq V$ be its largest subvariety defined over $\overline{k}$, which is simply the Zariski closure of the subset $V(\overline{k})$; then one expects to find some positive real number ϵ with the property that for each point $P \in V(\overline{K})$, if the Weil height of the point P satisfies the inequality $h(P) < \epsilon$, then we must have that $P \in W(\overline{K})$. The first author proved in [Ghi09] that this expectation is indeed met in the case when k is a finite field and then, later in [Ghi14], generalized his result to all function fields K/k of transcendence degree 1. In this paper, we prove the following result, which covers all function fields (of arbitrary finite transcendence degree).

Theorem 1.1. *Let k be a field and let K be a function field of transcendence degree $m \geq 1$ over k. We let $h : \mathbb{A}^N(\overline{K}) \longrightarrow \mathbb{Q}_{\geq 0}$ be the Weil height associated to the function field K/k. Let $V \subseteq \mathbb{A}^N$ be an affine subvariety defined over K and let $W \subseteq V$ be the Zariski closure of all points of V whose coordinates live in $\overline{k}$. Then there exists a positive real number c_0 with the property that for each $P \in (V \setminus W)(\overline{K})$, we have that $h(P) \geq c_0$.*

Remark 1.2. *The constant c_0 from the conclusion of Theorem 1.1 depends in an explicit way of the data defining the variety V. Indeed, as shown in the proof of Theorem 1.1, c_0 depends only on the total degrees of the polynomials from a finite set of generators for the vanishing ideal of V and on the degree of K over a rational function field K_0 (of transcendence degree m over k) used in the definition of the Weil height $h(\cdot)$ (for more details, see Section 2).*

The strategy of proof from [Ghi14] (which took inspiration from a clever trick the first author learned from the beautiful paper [BZ95] of Bombieri and Zannier) presented some natural obstructions to a generalization covering any function field, as explained in [Ghi14, Remark 2.7]. So, our proof of Theorem 1.1 (which stemmed from the second author's attempt of generalizing the results of [Ghi14] to arbitrary function fields) follows a different strategy than the one employed by the first author in [Ghi14, Ghi09]. Indeed, we are able to argue in a more direct way to prove the conclusion from Theorem 1.1 and, as a by-product of our method, we obtain also the following sharp lower bound for the Weil height of a point not contained on the largest subvariety of V defined over the constant field.

Theorem 1.3. *Let k be a field, let $m \in \mathbb{N}$, and let $K := k(t_1, \dots, t_m)$ be a function field of transcendence degree m over k. Let $V \subseteq \mathbb{A}^N$ be the zero* *locus of finitely many polynomials g_i in N variables with coefficients in K; we let $D := \max_i \deg(g_i)$ (where for any polynomial $g \in K[x_1, \dots, x_N]$, its degree $\deg(g)$ is defined to be its total degree in the variables $x_1, \dots, x_N$). We let $W \subseteq V$ be the Zariski closure of all points of V whose coordinates live in $\overline{k}$. Then for any point $P \in V(\overline{K})$, either $P \in W(\overline{K})$, or $h(P) \geq \frac{1}{D}$.*

The key result employed in the proof of Theorem 1.3 is our Proposition 3.1 (proven in Section 3), which is of independent interest and could potentially be useful for other applications. The lower bound of $\frac{1}{D}$ for the Weil height of a point $P \in (V \setminus W)(\overline{K})$ is the best possible, as shown by the following example.

1.3 Examples

Example 1.4. *Let $D \in \mathbb{N}$ and let $y = tx^D$ be a plane curve V defined over the function field $K = k(t)$ (for any given field k). Then each point $(a, b) \in V(\overline{K})$ where $b \in \overline{k}^*$ has its Weil height (see Section 2) equal to $\frac{1}{D}$, which is precisely the lower bound from Theorem 1.3. Furthermore, the only point on V with both coordinates in $\overline{k}$ is $(0, 0)$, i.e., with the notation as in Theorem 1.3, we have that $W = \{(0, 0)\}$.*

We also note (see our next example) that in the conclusion of either Theorem 1.1 or 1.3, one does indeed have to exclude the subvariety $W \subseteq V$, which is the largest subvariety defined over $\overline{k}$, in order to obtain a positive lower bound for the Weil height of the remaining points in $V(\overline{K})$.

Example 1.5. *Consider the plane $V \subset \mathbb{A}^3$ given by the equation $z = tx + y$, defined over the function field $k(t)$. The largest subvariety of V defined over $\overline{k}$ is the line given by the equations: $x = 0$ and $z = y$. Clearly, W contains infinitely many points of arbitrarily small Weil height (not only the ones defined over $\overline{k}$); so, in order to obtain a uniform positive lower bound for the Weil height of the points on the plane V, one would have to exclude the entire line W. Furthermore, there exist a Zariski dense set of points in $V \setminus W$ of Weil height 1 (which is the smallest height predicted by the conclusion of Theorem 1.3); indeed, any point on V of the form*

$$(a, b, ta + b) \text{ with } a, b \in \overline{k} \text{ and } a \neq 0$$

would have Weil height precisely equal to 1.

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Chapter 2

Heights in function fields

In this Section we construct the places for arbitrary function fields and define the corresponding Weil height; we refer the reader also to [Ser97] for additional details (including for a more geometric approach).

Let K be a function field of transcendence degree $m \geq 1$ over a field k. We let $t_1, \ldots, t_m$ be algebraically independent (over k) functions in K and consider the function field $K_0 := k(t_1, \ldots, t_m)$. Then the places Ω_{K_0} correspond to irreducible hypersurfaces of the projective space $\mathbb{P}_k^m$ (whose function field equals K_0). More explicitly, the places of the function field K_0/k correspond:

- either to the irreducible polynomials $Q \in k[t_1, \ldots, t_m]$, in which case we associate an (exponential) valuation, denoted v_Q, which is defined to be
$$v_Q\left(\frac{P_1}{P_2}\right) = \exp_Q(P_1) - \exp_Q(P_2),$$
for any nonzero $\frac{P_1}{P_2} \in k(t_1, \ldots, t_m)$ (where $\exp_Q(P_i)$ simply refers to the exponent of the irreducible polynomial Q appearing in the factorization of P_i in prime factors). Also, we let $n_{v_Q} := \deg(Q)$.

- or to the (negative) total degree function (which itself corresponds geometrically to the hyperplane at infinity from $\mathbb{P}_k^m$), in which case we associate a valuation, denoted v_∞, which is defined to be
$$v_\infty\left(\frac{P_1}{P_2}\right) = \deg(P_2) - \deg(P_1),$$
for any nonzero $\frac{P_1}{P_2} \in k(t_1, \ldots, t_m)$. Also, we let $n_{v_\infty} := 1$.

Then for each nonzero rational function $R \in k(t_1, \ldots, t_m)$, we have the product formula (or moreover, the sum formula since we work with exponential valuations)
$$\sum_{v \in \Omega_{K_0}} n_v \cdot v(R) = 0. \tag{2.0.1}$$

Given a finite extension L of K_0, we let Ω_L be the set of places of L lying above the places from Ω_{K_0}. For each place $w \in \Omega_L$ lying above a place $v \in \Omega_{K_0}$, we let $e(v \mid v_0)$ be the ramification index, i.e., normalizing the exponential valuation v on L so that its value group is $\mathbb{Z}$, then $e(v \mid v_0) := v(U)$, where $U \in K_0$ is a uniformizer for the valuation v_0 (i.e., $v_0(U) = 1$). Also, we let $n_v := n_{v_0} \cdot f(v \mid v_0)$, where $f(v \mid v_0)$ is the degree of the residue field extension corresponding to the two places. Then once again we have a product formula:

$$\sum_{v \in \Omega_L} n_v \cdot v(x) = 0, \tag{2.0.2}$$

for any nonzero $x \in L$.

For any positive integer N we define the Weil height of a point $P := (a_1, \dots, a_N) \in \mathbb{A}^N(L)$ as follows:

$$h(P) := \frac{1}{[L : K_0]} \sum_{v \in \Omega_L} n_v \cdot \max\{0, -v(a_1), -v(a_2), \dots, -v(a_N)\}. \tag{2.0.3}$$

As proven in [Ser97] (see also [Ghi05, Chapter 4] for a comprehensive discussion regarding valuations on arbitrary function fields and heights for points on varieties defined over function fields), the Weil height is well-defined and it is independent of the choice of particular field L which contains the coordinates a_i of the point P. Using the geometric definition of the Weil height as in [Ser97] (see also [Lan83, Proposition 3.2, p. 63]), each point $P \in \mathbb{P}^N(K)$ corresponds a rational function $\psi : X \dashrightarrow \mathbb{P}^N$, where $X \subset \mathbb{P}^r$ is a projective variety defined over k, regular in codimension 1, whose function field is K; then

$$h(P) := \deg(\psi^{-1}(L)) \tag{2.0.4}$$

for a generic hyperplane L of $\mathbb{P}^N$, where the degree of $\psi^{-1}(L)$ is computed with respect to the embedding of X into $\mathbb{P}^r$. Also, we note that the normalization in our valuations depend on our initial choice of the functions $t_1, \dots, t_m$, but obviously, once these functions are fixed, the definition of all valuations and in turn of the corresponding Weil height is uniquely determined; furthermore, we note that a different choice for the rational functions t_i would lead to another Weil height h_2 which will be comparable with respect to the first Weil height, i.e., there would be positive constants c_1 and c_2 such that $c_1 h_1(P) \le h_2(P) \le c_2 h_1(P)$ for any point $P \in \mathbb{A}^N(\overline{K})$.

Chapter 3

Proof of our main results

Proposition 3.1. *Let k be an arbitrary field, let K be a function field over k, let $\overline{K}$ be the algebraic closure of K and $\overline{k}$ be the algebraic closure of k inside $\overline{K}$. Let $d, N, D \geq 1$ be integers, and let $f_i \in k[x_1, \ldots, x_N]$ for $i = 0, \ldots, d$ be polynomials of total degree at most D. Let $t \in \overline{K} \setminus \overline{k}$ and let $f \in k[t][x_1, \ldots, x_N]$ be defined as:*

$$f(x_1, \ldots, x_N) := \sum_{i=0}^{d} t^i f_i(x_1, \ldots, x_N). \tag{3.1.1}$$

We let $h : \mathbb{A}^N(\overline{K}) \longrightarrow \mathbb{Q}_{\geq 0}$ be the Weil height corresponding to the function field K/k. Then for any point $P \in \mathbb{A}^N(\overline{K})$ for which $f(P) = 0$, we have that

$$\text{either } h(P) \geq \frac{h(t)}{D}, \tag{3.1.2}$$

$$\text{or } f_i(P) = 0 \text{ for each } i = 0, \ldots, d. \tag{3.1.3}$$

Proof. First we note that $h(t) > 0$ since the only elements of $\overline{K}$ of height equal to 0 are the elements of $\overline{k}$.

Let $P := (a_1, \ldots, a_N) \in \mathbb{A}^N(\overline{K})$ such that $f(P) = 0$. Also, we let $L := K(t, a_1, \ldots, a_N)$. As in Section 2, we let Ω_L be the set of places of the function field L/k.

We assume that (3.1.3) does not hold; in particular, this means that the set

$$I := \{0 \leq i \leq d \colon f_i(P) \neq 0\}$$

contains at least two such indices. We will prove that $h(P) \geq \frac{h(t)}{D}$, i.e. that (3.1.2) must hold.

Let j be the largest element of I.

We let S_∞ be the finite set of places of the function field L/k consisting of all places $v \in \Omega_L$ with the property that $v(t) < 0$.

Let $v \in S_\infty$. Since $f(P) = 0$, then there must exist an index $i \in I \setminus \{j\}$ (depending on v) such that

$$v\left(t^i \cdot f_i(a_1, \ldots, a_N)\right) \leq v\left(t^j \cdot f_j(a_1, \ldots, a_N)\right) \tag{3.1.4}$$

since otherwise, the ultrametric inequality yields that $|f(P)|_v = \left|t^j f_j(P)\right|_v \neq 0$, contradiction. Since $i \neq j$ and j was chosen to be the largest element of I, then $i < j$. We let

$$c_v := v\left(f_j(a_1, \ldots, a_N)\right). \tag{3.1.5}$$

Using (3.1.4) and (3.1.5) (along with the fact that $j - i \geq 1$ and that $v(t) < 0$), we get that

$$v\left(f_i(a_1, \ldots, a_N)\right) \leq v(t) + c_v \text{ and thus} \tag{3.1.6}$$

$$-v\left(f_i(a_1, \ldots, a_N)\right) \geq -v(t) - c_v. \tag{3.1.7}$$

Since f_i is a polynomial of degree at most D in the variables $x_1, \ldots, x_N$ with coefficients in the constant field k, inequality (3.1.7) yields that for each $v \in S_\infty$, we have

$$\max\{0, -v(a_1), \ldots, -v(a_N)\} \geq \frac{1}{D} \cdot \max\{0, -v(t) - c_v\}. \tag{3.1.8}$$

But since $f_j(a_1, \ldots, a_N) \neq 0$, then applying the product formula (2.0.2) for the nonzero element $f_j(a_1, \ldots, a_N)$ of L, we get that

$$\sum_{w \in \Omega_L} n_w \cdot \max\left\{0, -w\left(f_j(a_1, \ldots, a_N)\right)\right\} = \sum_{w \in \Omega_L} n_w \cdot \max\left\{0, w\left(f_j(a_1, \ldots a_N)\right)\right\}$$

and so, using (3.1.5), we get

$$\sum_{w \in \Omega_L} n_w \cdot \max\left\{0, -w\left(f_j(a_1, \ldots, a_N)\right)\right\} \geq \sum_{v \in S_\infty} n_v \cdot \max\{0, c_v\}. \tag{3.1.9}$$

Furthermore, (3.1.5) and (3.1.9) yield

$$\sum_{w \in \Omega_L \setminus S_\infty} n_w \cdot \max\left\{0, -w\left(f_j(a_1, \ldots, a_N)\right)\right\} \geq \sum_{v \in S_\infty} n_v \cdot \left(\max\{0, c_v\} - \max\{0, -c_v\}\right). \tag{3.1.10}$$

Using the fact that $f_j \in k[x_1, \ldots, x_N]$ is a polynomial of degree at most D, then we have that for each place $w \in \Omega_L \setminus S_\infty$, we have

$$\max\{0, -w(a_1), \ldots, -w(a_N)\} \geq \frac{\max\{0, -w\left(f_j(a_1, \ldots, a_N)\right)\}}{D}$$

and thus inequality (3.1.10) yields

$$\sum_{w \in \Omega_L \setminus S_\infty} n_w \cdot \max\left\{0, -w(a_1), \ldots, -w(a_N)\right\} \geq \frac{1}{D} \cdot \left(\sum_{v \in S_\infty} n_v \cdot \left(\max\{0, c_v\} - \max\{0, -c_v\}\right)\right). \tag{3.1.11}$$

Using the formula for the Weil height of the point $P \in \mathbb{A}^N$ and also, using the notation from Section 2 for the rational function field $K_0 \subseteq K$ with respect to which the Weil height is defined, we have the following:

$$h(P) = \frac{1}{[L : K_0]} \sum_{w \in \Omega_L} n_w \cdot \max\{0, -w(a_1), \dots, -w(a_N)\}.$$

Then combining inequalities (3.1.8) and (3.1.11), we get that

$$h(P) \geq \frac{1}{D \cdot [L : K_0]} \cdot \sum_{v \in S_\infty} n_v \cdot (\max\{0, -v(t) - c_v\} + \max\{0, c_v\} - \max\{0, -c_v\}) . \tag{3.1.12}$$

Now, using that for any real numbers α and β with $\alpha > 0$, we have

$$\max\{0, \alpha - \beta\} + \max\{0, \beta\} - \max\{0, -\beta\} \geq \alpha,$$

then for each $v \in S_\infty$ we have

$$\max\{0, -v(t) - c_v\} + \max\{0, c_v\} - \max\{0, -c_v\} \geq -v(t). \tag{3.1.13}$$

Finally, using that

$$h(t) = \frac{1}{[L : K_0]} \cdot \sum_{v \in S_\infty} -v(t),$$

then inequalities (3.1.12) and (3.1.13) deliver the desired inequality from (3.1.2). This concludes our proof of Proposition 3.1. □

Proof of Theorem 1.3. We let $P := (a_1, \dots, a_N) \in V(\overline{K})$. We assume that

$$h(P) < \frac{1}{D} \tag{3.1.14}$$

and we will prove that $P \in W(\overline{K})$, as claimed in the conclusion of Theorem 1.3.

Let $f \in k(t_1, \dots, t_m)[x_1, \dots, x_N]$ be one of the finitely many generators of the vanishing ideal of V; in particular, according to our hypothesis, the total degree of f as a polynomial in $x_1, \dots, x_N$ is at most equal to $D \geq 1$. At the expense of multiplying f by a suitable nonzero polynomial in $k[t_1, \dots, t_m]$, we may assume from now on that $f \in k[t_1, \dots, t_m][x_1, \dots, x_n]$. We write

$$f(x_1, \dots, x_N) := \sum_{i=0}^{d_1} t_1^i \cdot f_i(x_1, \dots, x_N), \tag{3.1.15}$$

where $d_1 \geq 0$ is an integer and each $f_i \in k[t_2, \ldots, t_m][x_1, \ldots, x_N]$; in other words, d_1 is the largest power of t_1 appearing in any coefficient of f (each such coefficient being itself a polynomial in $k[t_1, \ldots, t_m]$).

We let $K_1 := k(t_2, \ldots, t_m)$ and let $\overline{K_1}$ be its algebraic closure inside $\overline{K}$. We let $h_1 : \mathbb{A}^N(\overline{K}) \longrightarrow \mathbb{Q}_{\geq 0}$ be the Weil height associated to the function field K/K_1. In particular, since h_1 counts only the degree in t_1 of any rational function in K (see also (2.0.4)), we have that

$$h(Q) \geq h_1(Q) \text{ for each } Q \in \mathbb{A}^N(\overline{K}). \tag{3.1.16}$$

Proposition 3.1 applied to the polynomial f from (3.1.15) yields that either $h_1(P) \geq \frac{1}{D}$ (note that $h_1(t_1) = 1$) and thus, using (3.1.16), we have

$$h(P) \geq \frac{1}{D}, \tag{3.1.17}$$

or $f_i(P) = 0$ for each $i = 0, \ldots, d_1$. Since we assumed the opposite inequality for the height of P as in (3.1.14), then it means that indeed, we must have that $f_i(P) = 0$ for each $i = 0, \ldots, d_1$.

Now, for each $i = 0, \ldots, d_1$, we write

$$f_i(x_1, \ldots, x_N) := \sum_{j=0}^{d_2} t_2^j \cdot f_{i,j}(x_1, \ldots, x_N), \tag{3.1.18}$$

where each polynomial $f_{i,j} \in k[t_3, \ldots, t_m][x_1, \ldots, x_N]$; furthermore, due to our original assumption on the total degree of the polynomial f, we also have that each $f_{i,j}$ has total degree in $x_1, \ldots, x_N$ at most equal to D.

Then we let $K_2 := k(t_3, \ldots, t_m)$ and let $\overline{K_2}$ be its algebraic closure inside $\overline{K}$. We let $h_2 : \mathbb{A}^N(\overline{K}) \longrightarrow \mathbb{Q}_{\geq 0}$ be the Weil height corresponding to the function field K/K_2; once again, similar to inequality (3.1.16), since the height h_2 picks up the total degree in t_1 and t_2 of any rational function in $K = k(t_1, \ldots, t_m)$, as opposed to the total degree in all m variables as it is the case for the Weil height $h : \mathbb{A}^N(\overline{K}) \longrightarrow \mathbb{Q}_{\geq 0}$ which corresponds to the function field K/k, then we also have that

$$h_2(Q) \leq h(Q) \text{ for each } Q \in \mathbb{A}^N(\overline{K}). \tag{3.1.19}$$

Now, applying Proposition 3.1 to each polynomial f_i from (3.1.18), we conclude that either $h_2(P) \geq \frac{1}{D}$ (note that $h_2(t_2) = 1$), which in turn (due to inequality (3.1.19)) yields

$$h(P) \geq \frac{1}{D}, \tag{3.1.20}$$

or $f_{i,j}(P) = 0$ for each $j = 0, \dots, d_2$. Since we assumed that the opposite inequality (3.1.14) holds (which contradicts (3.1.20)), then we must have that indeed, $f_{i,j}(P) = 0$ for each $i = 0, \dots, d_1$ and each $j = 0, \dots, d_2$.

We continue the above process, this time applying Proposition 3.1 to each polynomial $f_{i,j}$ which we write in terms of the powers of t_3 appearing in its coefficients. For example, at step ℓ in our process (for some $\ell = 1, \dots, m$), we deal with polynomials of the form $f_{i_1,\dots,i_{\ell-1}} \in k[t_\ell, \dots, t_m][x_1, \dots, x_N]$, for some $i_j \in \{0, \dots, d_j\}$ for each $j = 1, \dots, \ell - 1$ (where the d_j's are the maximum degrees in t_j of the coefficients of the original polynomial $f \in k[t_1, \dots, t_m][x_1, \dots, x_N]$). Then we write each such polynomial as

$$f_{i_1,\dots,i_{\ell-1}}(x_1, \dots, x_N) := \sum_{j=0}^{d_\ell} t_\ell^j \cdot f_{i_1,\dots,i_{\ell-1},j}(x_1, \dots, x_N), \qquad (3.1.21)$$

where each $f_{i_1,\dots,i_{\ell-1},j} \in k[t_{\ell+1}, \dots, t_m][x_1, \dots, x_N]$ has total degree at most D in the variables $x_1, \dots, x_N$. Then letting $K_\ell := k(t_{\ell+1}, \dots, t_m)$ and $\overline{K_\ell}$ be its algebraic closure inside $\overline{K}$, we let $h_\ell : \mathbb{A}^N(\overline{K}) \longrightarrow \mathbb{Q}_{\geq 0}$ be the Weil height associated to the function field K/K_ℓ (with the choice $t_1, \dots, t_\ell$ for the algebraically independent functions generating the function field, as in Section 2). As before (see inequalities (3.1.16) and (3.1.19)), since h_ℓ counts only the total degree in the variables $t_1, \dots, t_\ell$, then we have that

$$h_\ell(Q) \leq h(Q) \text{ for each } Q \in \mathbb{A}^N(\overline{K}). \qquad (3.1.22)$$

Proposition 3.1 applied to each polynomial $f_{i_1,\dots,i_{\ell-1}}$ as in (3.1.21) yields that either $h_\ell(P) \geq \frac{1}{D}$ (note that $h_\ell(t_\ell) = 1$), which would actually contradict inequality (3.1.14) according to (3.1.22), or we must have that

$$f_{i_1,\dots,i_{\ell-1},j}(P) = 0 \text{ for each } j = 0, \dots, d_\ell,$$

which allows us to continue our process. After m steps, we conclude that we can write the original polynomial $f \in k[t_1, \dots, t_m][x_1, \dots, x_N]$ as

$$f(x_1, \dots, x_N) = \sum_{\substack{0 \leq i_j \leq d_j \\ \text{for each} \\ 1 \leq j \leq m}} \left(\prod_{j=1}^m t_j^{i_j} \right) \cdot f_{i_1,\dots,i_m}(x_1, \dots, x_N),$$

where each $f_{i_1,\dots,i_m} \in k[x_1, \dots, x_N]$ (while d_j is the maximum degree of t_j appearing in the coefficients of f, which are themselves polynomials in

$k[t_1, \dots, t_m]$). Furthermore, repeated applications of Proposition 3.1 (as explained above), coupled with our assumption from (3.1.14) yields that

$$f_{i_1,\dots,i_m}(P) = 0 \text{ for each } i_j = 0, \dots, d_j, \text{ for each } j = 1, \dots, m. \tag{3.1.23}$$

Equations (3.1.23) yield that indeed $P \in W(\overline{K})$, where W is the largest subvariety of V defined over k. This concludes our proof of Theorem 1.3. □

Proof of Theorem 1.1. First of all, at the expense of replacing k by $\overline{k}$ and also replacing K by $\overline{k} \cdot K$, we may assume from now on, that k is algebraically closed (and K is a function field over k).

Since K is a function field over k of transcendence degree m, then we may pick algebraically independent functions $t_1, \dots, t_m$ in the function field K/k such that K is algebraic over $k(t_1, \dots, t_m)$. Furthermore, we assume the Weil height for the points in $\mathbb{A}^N(\overline{K})$ were constructed with respect to the places of the function field $k(t_1, \dots, t_m)/k$ (see Section 2).

In the case when k has characteristic $p > 0$, then at the expense of replacing each t_i by t_i^{1/p^ℓ} for a suitable integer $\ell \geq 0$ (and also adjoining each t_i^{1/p^ℓ} to K), we may assume that K is actually separable over $k(t_1, \dots, t_m)$ (where we prefer to keep the notation for t_i rather than formally replacing t_i by t_i^{1/p^ℓ}). Also, note that constructing the Weil height for the points in $\mathbb{A}^N(\overline{K})$ using the normalization for the places of the function field $k\left(t_1^{1/p^\ell}, \dots, t_m^{1/p^\ell}\right)/k$ simply introduces a factor of p^ℓ (and thus would change the absolute constant c_0 from the conclusion in Theorem 1.1 only by a factor of $\frac{1}{p^\ell}$). Therefore, from now on, we assume $K/k(t_1, \dots, t_m)$ is a finite, separable extension and that the Weil height of the points in $\mathbb{A}^N(\overline{K})$ was constructed with respect to the places of the function field $k(t_1, \dots, t_m)/k$.

We let $K_0 := k(t_1, \dots, t_m)$ and replacing K by a finite extension, we may as well assume that K/K_0 is a Galois extension. Then we let $X \subseteq \mathbb{A}^N$ be the union of all Galois conjugates of V over K_0, i.e.,

$$X := \bigcup_{\sigma \in \mathrm{Gal}(K/K_0)} V^\sigma;$$

then X is an affine variety defined over $K_0 = k(t_1, \dots, t_m)$. Furthermore, the subvariety $W \subseteq V$ being defined over k is invariant under $\mathrm{Gal}(K/K_0)$ and therefore, W is also the largest subvariety of X defined over k (since each point in $X(k)$ is actually a point in $V(k)$). Then Theorem 1.3 yields that there exists a positive constant c_0 (simply depending on the maximum total degree of the polynomials from a minimal generating set for the vanishing ideal of X) such that for each point $P \in X(\overline{K})$ we have that either $h(P) \geq c_0$ or $P \in W(\overline{K})$. Since $V \subseteq X$, we obtain the desired conclusion in Theorem 1.1. □

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