The Local Trace Formula as a Motivic Identity

by

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Abstract

In 1991, James Arthur published a *local trace formula* ([Art91, Theorem 12.2]), which is an equality of distributions on the Lie algebra of a connected, reductive algebraic group G over a field F of characteristic zero. His approach was later used by Jean-Loup Waldspurger to give a slight reformulation, identifying the value of a particular distribution on a test function with that of its Fourier transform ([Wal95, Théorème V.2]). We show that this identity may be formulated as an identity of *motivic distributions* on definable manifolds. By so doing, we would make available the use of the *transfer principle* to establish the trace formula for groups defined over fields of positive characteristic.

Lay Summary

In this thesis we demonstrate that, due to the restrictions of the underlying symbolic calculus of the objects in question, a powerful theorem (Arthur's local trace formula for the Lie algebra of a connected reductive group) is true not only over extensions of p-adic number fields, as was previously known, but is in fact valid over all discretely valued local fields of sufficiently large residue characteristic.

Preface

This dissertation is original, unpublished, independent work by the author, Edward Alexander George Belk.

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Dedication

To my wife.

Chapter 1

Overview

> El mundo era tan reciente, que
> muchas cosas carecían de
> nombre, y para mencionarlas
> había que señalarlas con el dedo.
>
> —Gabriel García Márquez,
> *Cien años de soledad*

The purpose of this dissertation is to establish that Waldspurger's local trace formula on the Lie algebra of a connected, reductive algebraic group is a motivic identity, and therefore that it is true for all nonarchimedean local fields of sufficiently large residue characteristic. In this section, we give a brief outline of how this is to be accomplished.

It is known that, given an arbitrary field F and a connected, reductive group G defined over that field, the F-isomorphism class of G is determined by a combinatorial object known as the *indexed root datum*, which is to a great extent independent of choice of field F. More precisely, it consists of dual finitely-generated free abelian groups X and $X^\vee$, finite subsets $D_0 \subset D \subset X$ and $D^\vee$ dual to D, and a continuous action of the absolute Galois group $\Gamma = \mathrm{Gal}(F_s/F)$ on X, where F_s/F is a fixed separable closure. These objects are subject to certain restrictions, which we largely ignore for now.

The action of Γ factors through a finite quotient $\Gamma_E = \mathrm{Gal}(E/F)$ (say). Thus, the isomorphism class of G over F is determined by the finitely-presented information

$$(X, D, X^\vee, D^\vee, D_0, \tau),$$

where τ is the action of Γ_E on X. It is in fact true that this 6-tuple determines more than just the F-isomorphism class of G, but we will not need this for the moment.

Arthur's local trace formula on the Lie algebra g of G is, in some sense, a vast generalization of Parseval's identity, and consists in an identity of distributions on $C_c^\infty(g) \times C_c^\infty(g)$. As stated, it is known to be true only in the case that $\mathrm{char}\, F = 0$. One may desire to extend this result to nonarchimedean local fields of arbitrary characteristic, such as $\mathbf{F}_p(\!(t)\!)$; one method of doing so is by using the *transfer principle* from model theory.

Model theory, broadly, studies the relationships between formal theories (i.e. collections of sentences in a first-order language) and their models (i.e. sets with interpretations for all symbols in the language, for which the theory serves as a collection of axioms). For instance, in the language of rings, we can assert the *field axioms* (e.g. one of them, the assertion of commutativity of multiplication, is the sentence $\forall x \forall y (x \times y = y \times x)$), for which, for example, $\mathbf{Q}$ serves as a model.

In our context (of algebraic groups over discretely-valued local fields), one introduces a notion of *definable* sets, and a class of complex-valued functions on these sets that can be described in the first-order *Denef-Pas language*, called the class of *motivic functions*. By specializing these objects to nonarchimedean local fields, we obtain information about the behaviour of our functions on *all* fields of sufficiently large residue characteristic.

The *transfer principle* of Cluckers and Loeser states that, under certain conditions, the truth or falsity of a statement about motivic functions over a given field (i.e. in a given model) depends not on the field itself, but only on its residue field. Consequently, if $p \gg 0$, then a motivic function f will be zero when specialized to $\mathbf{F}_p((t))$ if and only if it is zero when specialized to $\mathbf{Q}_p$. In turn, this implies that equality of motivic functions in characteristic zero implies their equality in all fields of sufficiently large residue characteristic.

By first showing how the components of the local trace formula are determined by various definable and/or motivic constructions in the Denef-Pas language, we will show how the trace formula itself can be expressed as an equality of *motivic* distributions. Our desired result will then follow immediately.

We begin in Chapter 2 by recalling the classical theory of reductive groups and the various objects one associates with a connected, reductive group defined over a local field; we build the necessary machinery to state the local trace formula for the Lie algebra, with which we close the chapter. Next, in Chapter 3, we discuss the rôle Galois cohomology plays in classifying the objects we introduced in Chapter 2.

In Chapter 4, we introduce the essential notions of model theory, the Denef-Pas first-order language, and the basic objects of study (e.g. definable sets, motivic functions). In Chapter 5, we relate various established results showing that reductive groups can be specified in the Denef-Pas language using these objects. It is in this chapter that we begin the groundwork for the proof of our main theorem, namely, Theorem 6.7.1.

Finally, in Chapter 6, we demonstrate that the distribution used in the trace formula is a *motivic distribution*, and that consequently its truth or falsehood (over fields of sufficiently large residue characteristic) is determined by the residue field, and so in particular is independent of characteristic of the base field. We close with Chapter 7, where we collect explicit examples of the various constructions that appear in Chapter 2.

The main new results are Theorem 6.7.1 and Corollary 6.7.2; other original results are Theorem 3.4.2, Proposition 6.3.1, and the statements in Sections 6.4 and 6.5. In Section 2.5, we reformulate some classical results in a language more suitable for our purposes; as such, the results in that section may look novel, although there is little new content in them.

Chapter 2

Algebraic groups

Our chief goal is to prove that Waldspurger's local trace formula for the Lie algebra is an equality of motivic distributions. In order to state our goal precisely, we must review several aspects of the theory of algebraic groups. We close with Theorem 2.9.1, which is the formulation of Arthur's local trace formula that we will use.

In this chapter, we will review the classical theory of algebraic groups over a *fixed* (nonarchimedean) local field F. Depending on the situation, we may assume that, if the residue characteristic of F is positive, then it is *sufficiently large.*

2.1 Notation

Because the statement of the local trace formula has many ingredients and terms, we will want to begin by collecting the essential components of the theory, before we record any results; notation is mostly retained from [Wal95]. Familiarity is assumed with the definitions of algebraic tori and reductive linear algebraic groups.

Fix once and for all a nonarchimedean local field F, and a separable closure $F_s \supset F$. Denote by $|\cdot|_F$ the absolute value of F and v_F its normalized valuation, so that

$$|x|_F = q^{-v_F(x)}, \qquad x \in F,$$

where q is the cardinality of the residue field of F.

If V is any variety defined over F, we denote the F-points of that variety by $V(F)$. More generally, if $F \subset E \subset F_s$ any intermediate field, then we will denote by $V(E)$ the E-points of V.

2.2 Parabolic subgroups

Let G be a connected, reductive algebraic group defined over F, and let g denote its Lie algebra; as remarked in Section 2.1, we will denote the F-points of G (respectively, g) by $G(F)$ (respectively, $g(F)$).

A **parabolic subgroup** of G is an algebraic subgroup $P \subseteq G$ for which the variety G/P is complete; such subgroups obviously always exist (G, for instance), so we can meaningfully speak of a **minimal parabolic subgroup**, i.e. a parabolic subgroup not properly containing another. For an arbitrary parabolic subgroup P of G, we have the isomorphism of algebraic varieties

$$P \cong M_P \times N_P,$$

where N_P is the unipotent radical of P, and M_P is called its **Levi component**. We remark that we have also the isomorphism of groups of F-points $P(F) \cong M_P(F) \ltimes N_P(F)$.

The algebraic group M_P is itself a connected, reductive algebraic group, and so all constructions involving G can be done for M_P. This will be reflected in the notation simply by replacing G with the group in question, and so in the sequel we will not remark on each separate construction; as such, the reader can expect to encounter objects such as a_{M_P}, H_{M_P}, and so on. This will also apply to any torus T (see especially A_T, below.)

Fix a minimal parabolic subgroup $P_0 \subseteq G$; we will write M_0 for the Levi component of P_0, so that, if some parabolic subgroup $P \subseteq G$ contains P_0, then M_P contains M_0. By a **Levi subgroup** of G, we mean the Levi component of one of its parabolic subgroups; we will concern ourselves chiefly with Levi subgroups containing M_0. Denote by $\overline{P}$ the parabolic subgroup of G which is opposite to P, and which contains M_P; we recall that two parabolic subgroups P_1, P_2 are said to be **opposite** to each other if $P_1 \cap P_2$ is a Levi subgroup of both P_1 and P_2.

Suppose M is a Levi subgroup of G. We denote by $\mathscr{F}(M) = \mathscr{F}^G(M)$ the set of all parabolic subgroups of G which contain M, and by $\mathscr{L}(M) = \mathscr{L}^G(M)$ the set of Levi subgroups of G which contain M; we will usually abbreviate $\mathscr{F} = \mathscr{F}(M_0)$ and $\mathscr{L} = \mathscr{L}(M_0)$. In addition, we denote by $\mathscr{P}(M) = \mathscr{P}^G(M)$ the set of parabolic subgroups P of G such that $M_P = M$, and let $\mathscr{P} = \mathscr{P}(M_0)$.

Throughout this article, we will retain the convention from the literature of omitting the superscript G when this is one of the parabolic subgroups under consideration, writing $\mathscr{P}(M)$ for $\mathscr{P}^G(M)$, $\mathscr{L}$ for $\mathscr{L}^G(M)$, and (later) D_P for D_P^G, and so on. The sole exception to this rule will be the spaces a_P^G, defined below.

We will also (per [Wal95, p.43]) fix a special vertex in the apartment associated to A_{M_0}, and we will denote by K the stabilizer in $G(F)$ of this vertex; we refer the reader interested in this important construction to [BT72, §7.4]. For our purposes, we need only know that K is a maximal compact subgroup of $G(F)$ (*not* an algebraic group), with the property that [Art91, p.12]

$$G(F) = P(F)K \text{ for any parabolic subgroup } P \text{ of } G. \tag{2.1}$$

We will ultimately need to show that the K that we choose is *definable* (see Chapter 4), but we will not do so here.

Finally, if S is any subset of G, denote by $\mathrm{Norm}_G(S)$ the normalizer of S in G, and put

$$W^G := \mathrm{Norm}_G(M_0)(F)/M_0(F);$$

we will call W^G the **rational Weyl group** of M_0 in G, or more briefly, the **Weyl group** of M_0 in G.

We will similarly define, for every maximal subtorus T of G, the group

$$W(G,T) = \mathrm{Norm}_G(T(F))/T(F),$$

which we call the **Weyl group** of T in G. From [Kot05, §7.1] we have the inclusions

$$W(G,T) \subset \mathbf{W}(G)(F) \subset \mathbf{W}(G)(F_s),$$

where $\mathbf{W}(G) = \mathrm{Norm}_G(T)/T$ is a finite algebraic group defined over F. To borrow terminology from [Kot05], we will call the group $\mathbf{W}(G)(F_s)$ the **absolute Weyl group** of the pair (G,T), justifying our notation by the observation that the isomorphism class of $\mathbf{W}(G)$ is independent of choice of maximal torus T in G.

2.3 Roots and weights

Roots and characters can be used to classify isomorphism classes of reductive groups in a way which does not depend on the underlying field of definition. For this reason, we will take the opportunity to collect terminology and basic results for our later use. Familiarity is assumed with abstract (reduced and non-reduced) root systems (as outlined in e.g. [Hum12, Chapter III]).

The following result is a consequence of [Spr10, Theorem 15.2.6]:

Theorem 2.3.1. *Let G be a connected, reductive group defined over a field F as above, with fixed minimal parabolic subgroup P_0. Then:*

1. *There exists a maximal F-split F-torus $S \subset P_0$, whose centralizer in G is a maximal F-torus $T \subset P_0$;*
2. *There is a finite Galois subextension $F \subset E \subset F_s$ over which T splits; and*
3. *The rank of T and the F-rank of S are independent of choice of T and S.*

In the situation of point 2., we will say that G is **split** over E, and we may sometimes call it a **splitting field** of G (or T).

In Section 2.2 we introduced the notion of rational characters; now, we build on this idea.

Definition 2.3.2. [Spr10, §3.2.1 and §15.3.1] Let $X = X^*(T)$ be the group $\mathrm{Hom}(T, \mathbf{G}_m)$ of **absolute characters** of the torus T. Similarly, denote by ${}_FX = X^*(S)$ the group of (absolute) characters of S. We also have dually the groups of **absolute cocharacters**

$$X^\vee = X_*(T) = \mathrm{Hom}(\mathbf{G}_m, T) \text{ and } {}_FX^\vee = X_*(S) = \mathrm{Hom}(\mathbf{G}_m, S).$$

We emphasize that all morphisms in $X, X^\vee, {}_FX, {}_FX^\vee$ are defined *a priori* over F_s, although it happens [Spr10, Proposition 13.2.2] that elements of ${}_FX$ and ${}_FX^\vee$ are defined over F. After [Spr10, Theorem 3.2.11], we have

Theorem 2.3.3. *There is a perfect pairing $\langle , \rangle : X \times X^\vee \to \mathbf{Z}$ defined by the equation*

$$(\chi \circ \lambda)(t) = t^{\langle \chi, \lambda \rangle}, \quad t \in F^\times,$$

and similarly for ${}_FX$ and ${}_FX^\vee$, identifying X with $X^\vee$ and ${}_FX$ with ${}_FX^\vee$.

We will see a similar pairing in Equation (2.3). We use $\langle , \rangle$ to identify $\mathbf{R} \otimes_{\mathbf{Z}} X$ with $\mathbf{R} \otimes_{\mathbf{Z}} X^\vee$ and $\mathbf{R} \otimes_{\mathbf{Z}} ({}_FX)$ with $\mathbf{R} \otimes_{\mathbf{Z}} ({}_FX^\vee)$. We evidently have the inclusion

$$\mathbf{R} \otimes_{\mathbf{Z}} ({}_FX) \subset \mathbf{R} \otimes_{\mathbf{Z}} X,$$

and we equip $\mathbf{R} \otimes_{\mathbf{Z}} X$ (and hence $\mathbf{R} \otimes_{\mathbf{Z}} ({}_FX)$) with an inner product $(,)$ to turn them into Euclidean spaces. More precisely, we will pick these inner products to be invariant under the natural actions of $W(G, T)$ on X and ${}_FX$ (this can always be done, by "averaging" over the Weyl orbit; see [Spr10, §7.1.7]).

Recall the adjoint action of G, denoted $\mathrm{Ad}(x)$ for $x \in G(F)$; we want to consider the action of $\mathrm{Ad}(G)$ upon the Lie algebra $g(F)$. Obviously $\mathrm{Ad}(G)$ restricts to the adjoint action of T on g; the **weight spaces** of this action are the subspaces

$$g_\chi = \{Y \in g(F_s) : \mathrm{Ad}(t)Y = \chi(t)Y \text{ for all } t \in T(F_s)\},$$

where $\chi \in X$. The (finitely many) χ for which $g_\chi \neq 0$ are called the **weights** of T in G; the *weight space decomposition* $g = \bigoplus_\chi g_\chi$ is the eigenspace decomposition of $g(F_s)$ with respect to the commuting family of semisimple operators $\mathrm{Ad}(T)$.

The nontrivial weights of T in G are called the **roots** of T in G, and the collection of roots is denoted $R = R(G,T)$. It follows from the conjugacy of maximal tori over F_s that $R(G,T)$ does not depend on choice of maximal torus T. We put ${}_FR = {}_FR(G,S)$ for the roots of S in G (the so-called *relative roots*), which is similarly independent of choice of S.

Having these subsets R and ${}_FR$ of X and ${}_FX$, respectively, we might naturally want to ask if $X^\vee$ and ${}_FX^\vee$ also contain root systems.

Definition 2.3.4. The **coroot** $\alpha^\vee$ associated to the root $\alpha \in R$ (respectively, $\alpha \in {}_FR$) is the element of $\mathbf{R} \otimes_{\mathbf{Z}} X^\vee$ (respectively, $\mathbf{R} \otimes_{\mathbf{Z}} ({}_FX^\vee)$) associated to the vector

$$\frac{2}{(\alpha,\alpha)}\alpha,$$

the association being the one established in Theorem 2.3.3. The set of all coroots of T in G (respectively, S in G) is denoted by $R^\vee$ (respectively, ${}_F R^\vee$).

It is immediate from the definition that the set ${}_F R^\vee$ is a root system, and the set $R^\vee$ is a reduced root system. Moreover, the bijections $R \to R^\vee$ and ${}_F R \to {}_F R^\vee$ given by $\alpha \mapsto \alpha^\vee$ satisfy

$$\langle \alpha, \alpha^\vee \rangle = 2 \text{ for all } \alpha \in R.$$

In case G is not reductive, and the residue characteristic of F is 2, it is possible [Tit93, p. 128] that R and ${}_{F_s} R$ do not coincide; however, the following result tells us we need not worry about this situation.

Lemma 2.3.5. *[Spr10, Theorem 15.3.4] If G is a connected, reductive group, then $R = {}_{F_s} R$.*

We also quote the following result, which follows from [Spr10, §7.4.3 and Theorem 15.3.8].

Proposition 2.3.6. *With G, T, S as above, the set ${}_F R$ is (when nonempty) a root system, and the set R is a reduced root system.*

We may now define the **(absolute) root datum** associated to a pair (G, T) consisting of a connected, reductive algebraic group G and a maximal torus T: this is the 4-tuple $\Psi = (X, R, X^\vee, R^\vee)$ consisting of the character lattice $X = X^*(T)$ and its dual $X^\vee$, together with their respective root systems $R = R(G, T)$ and $R^\vee$.

More abstractly, we will use the term **root datum** to consist of any 4-tuple $(X, R, X^\vee, R^\vee)$, where X and $X^\vee$ are dual, finitely-generated free abelian groups, and $R \subset X, R^\vee \subset X^\vee$ are similarly dual (abstract) root systems. We distinguish these from the *indexed* root data ${}_i\Psi$, which we will introduce below in Section 2.6.

Finally, we define the notion of the *Weyl group of an abstract root system*, viz.

Definition 2.3.7. Let $\Psi = (X, R, X^\vee, R^\vee)$ be a root datum, let $\langle , \rangle$ be the pairing $X \times X^\vee \to \mathbf{Z}$, and for all $\alpha \in R$ put s_α for the endomorphism of X defined

$$s_\alpha(\chi) = \chi - \langle \chi, \alpha^\vee \rangle \alpha$$

for $\chi \in X$. The **Weyl group** $W(R)$ of the root system R is the group of automorphisms of X generated by $\{s_\alpha : \alpha \in R\}$.

By the observations in [Spr10, §7.4.1], we know that

$$W(R(G, T)) \cong \mathbf{W}(G)(F_s).$$

2.4 The Harish-Chandra homomorphism

The *Harish-Chandra homomorphism* attached to a parabolic subgroup P is an essential component of the trace formula, a function whose domain is M_P and whose codomain is a real vector space, depending on P. We begin this section by cataloguing these vector spaces for future reference.

Let G be a connected, reductive algebraic group defined over F, and let $X^*(G)_{\mathrm{rat}}$ be the group of **rational** characters of G; that is, the group of morphisms $G \to \mathbf{G}_m$ defined over F. We define a real vector space

$$a_G = \mathrm{Hom}(X^*(G)_{\mathrm{rat}}, \mathbf{R}),$$

and caution that a_G is *not* the Lie algebra of the split central torus $A_G = A_G(F)$; this unfortunate clash of notation is inherited from the literature. Observe, however, the isomorphism

$$a_G \cong \mathbf{R} \otimes_{\mathbf{Z}} X_*(A_G)_{\mathrm{rat}}, \tag{2.2}$$

where $X_*(A_G)_{\mathrm{rat}}$ denotes the group of rational cocharacters of the maximal split central torus A_G. Of course, as A_G is split, its characters are all defined over F and so $X_*(A_G)_{\mathrm{rat}} = X_*(A_G)$. We will also abuse notation by writing a_P for a_{M_P}.

Per [Art05, p. 24], there exist, for every inclusion $P \hookrightarrow Q$ of parabolic subgroups of G containing P_0, an induced inclusion $a_Q \hookrightarrow a_P$ and dual surjection $a_P \to a_Q$, whose kernel we will denote a_P^Q. As such, if $P \subseteq G$ is a parabolic subgroup of G containing M_0, we have

$$a_P = a_G \oplus a_P^G;$$

we emphasize again that the symbols a_P and a_P^G do *not* represent the same space, and that this is the only situation in this article where the omission of the superscript G alters the meaning (compare with $\mathscr{F}(M) = \mathscr{F}^G(M)$, etc.).

Put $a_0 = a_{P_0}$, and observe that our discussion above implies that a_0 includes as subspaces all a_P for which P is a parabolic subgroup of G containing P_0. As in Section 2.3, we fix an inner product $(,)$ on a_0 which is invariant under the natural action of W^G, to obtain a W^G-invariant metric on a_0. This in turn gives us a W^G-invariant notion of length, and thus volume, on each of the subspaces a_P, etc.

Although not needed for the definition of the Harish-Chandra homomorphism, we will take this opportunity to collect other facts about these real vector spaces that we will use later. Let us write $\mathrm{i}a_0^* \subset a_0 \otimes_{\mathbf{R}} \mathbf{C}$ for the real dual vector space of a_0; we emphasize that $\mathrm{i}a_0^*$ is a *real* vector space which is dual to a_0, in the same sense that the real dual to the additive group $\mathbf{R}$ is $\mathrm{i}\mathbf{R}$. As in Theorem 2.3.3, denote by $\langle \lambda, \chi \rangle$ the natural pairing given by

$$(\chi \circ \lambda)(t) = t^{\langle \lambda, \chi \rangle} \text{ for all } t \in F^\times, \tag{2.3}$$

for $\chi \in X^*(M_0)_{\mathrm{rat}}$ and $\lambda \in X_*(A_{M_0})_{\mathrm{rat}}$.

The isomorphism (2.2) naturally extends this pairing to the domain $a_0 \times X^*(M_0)_{\mathrm{rat}}$, and there is a canonical identification of $X^*(M_0)$ with a full-rank lattice in $\mathrm{i}a_0^*$ given by $\chi \mapsto (f \mapsto f(\chi))$. By tensoring with $\mathbf{R}$, we can use this isomorphism, and the pairing $\langle , \rangle$, to transfer the inner product $(,)$ from a_0 to $\mathrm{i}a_0^*$.

Thus, we have obtained compatible W^G-invariant measures on all vector spaces a_P and $\mathrm{i}a_P^*$ for $P \in \mathscr{F}$. We have one more concept before the main definition of this section, namely

Definition 2.4.1. The **facet** of a_M associated to the element $\lambda \in a_M$ is the set

$$F(\lambda) = \{\mu \in a_M : \mathrm{sgn}(\langle \eta, \mu \rangle) = \mathrm{sgn}(\langle \eta, \lambda \rangle) \text{ for all } \eta \in \mathrm{i}a_M^*\}.$$

The facets of maximal dimension are called the **chambers** of a_M, and are denoted $C(\lambda)$.

It is clear from the definition that a_M is the disjoint union of its facets.

Definition 2.4.2. Define the function $H_G : G(F) \to a_G$ by the condition

$$\langle H_G(x), \chi \rangle = \log(|\chi(x)|_F) = -v_F(\chi(x))$$

for all $x \in G(F), \chi \in X^*(G)_{\mathrm{rat}}$. The function H_G is known as the **Harish-Chandra homomorphism** attached to the algebraic group G.

More generally, let $P \in \mathscr{F}$, and recall the decomposition

$$G(F) = M_P(F)N_P(F)K$$

from Equation (2.1). This means that we have, for every $x \in G(F)$, elements $m \in M_P(F), n \in N_P(F), k \in K$ such that $x = mnk$. Define a homomorphism $H_P : G(F) \to a_P$ by $H_P(mnk) = H_{M_P}(m)$; although the decomposition $x = mnk$ need not be unique, it is not hard to see that this map is well defined. We call H_P the **Harish-Chandra homomorphism** attached to the parabolic subgroup P of

G.

Explicit computations of the Harish-Chandra homomorphism are accomplished in Chapter 7. From now on, we will consider $X = X^*(T)$ and $R = R(G,T)$ to be subsets of the codomain $\mathrm{i}a_0^*$, the canonical association arising from that of X with its double-dual.

We close by remarking that, for any character $\alpha \in X^*(G)_{\mathrm{rat}}$, we can divide the vector space a_G into open half-spaces

$$a_G^+(\alpha) = \{\lambda \in a_G : \langle \alpha, \lambda \rangle > 0\} \text{ and } a_G^-(\alpha) = \{\lambda \in a_G : \langle \alpha, \lambda \rangle < 0\},$$

with common boundary the hyperplane

$$a_G^0(\alpha) = \{\lambda \in a_G : \langle \alpha, \lambda \rangle = 0\}.$$

We will use this fact to diverse ends in the sequel.

2.5 Parameterizing parabolics

The trace formula contains summations indexed by several objects attached to the parabolic subgroups of the reductive group in question. In this section, we will use the tools of the previous sections to catalogue these objects. As usual, G is our connected reductive group with maximal torus T and maximal split torus $S \subset T$.

We begin by recalling the vector spaces a_G from Section 2.4 and the root system ${}_FR$ from Section 2.3. Let us call an element λ of a_0 **regular** if $\langle \alpha, \lambda \rangle \neq 0$ for all $\alpha \in {}_FR$; as in our observation at the end of Section 2.4, we can use a regular element to split ${}_F R$ into *positive* and *negative* halves.

More precisely: let $\lambda \in a_0$ be regular, and put

$$ {}_F R^+(\lambda) = \{\alpha \in {}_F R : \langle \lambda, \alpha \rangle > 0\} \text{ and } {}_F R^-(\lambda) = \{\alpha \in {}_F R : \langle \lambda, \alpha \rangle < 0\}. $$

If $\alpha \in {}_F R^+(\lambda)$, we will say that α is **λ-indecomposable** if it cannot be written as the sum of two or more elements of ${}_F R^+(\lambda)$; it follows from the proof of [Hum12, Theorem 10.1] that the set of λ-indecomposable roots of ${}_F R^+(\lambda)$ is a base of ${}_F R$.

Definition 2.5.1. For regular $\lambda \in a_0$, the **base of ${}_F R$ corresponding to λ** is the set ${}_F D(\lambda)$ of λ-indecomposable roots of ${}_F R^+(\lambda)$

For arbitrary $\lambda \in a_0$, we can define an action of $F^\times$ on G via

$$ t *_\lambda x := \lambda(t)^{-1} x \lambda(t) \tag{2.4} $$

for $t \in F^\times, x \in G$, and put

$$ P(\lambda) = \{x \in G : \lim_{t \to 0} t *_\lambda x \text{ exists}\}. $$

Then we have

Lemma 2.5.2. *[Spr10, Lemma 15.1.2] With notation as above:*

1. *Every $P(\lambda)$ is a parabolic subgroup of G;*

2. *$P(\lambda) = P(\lambda')$ if and only if λ and λ' lie in the same facet*

of a_0 (recall Definition 2.4.1); and

3. *If P is a parabolic subgroup of G which contains M_0, then $P = P(\lambda)$ for some $\lambda \in a_0$.*

The purpose of introducing the groups $P(\lambda)$ is to establish the following

Proposition 2.5.3. *The following sets are in bijection:*

1. *The set $\mathscr{P}$ of minimal parabolic subgroups of G.*
2. *The set of possible bases ${}_FD$ of ${}_FR$.*
3. *The set of chambers in a_0.*

Proof. Let us fix some regular $\lambda_0 \in a_0$; we have the associations

$$C(\lambda_0) \leftrightarrow P(\lambda_0) \leftrightarrow {}_FD(\lambda_0).$$

If some other regular λ lies in a chamber different from that of λ_0, then by Lemma 2.5.2 we know $P(\lambda) \neq P(\lambda_0)$, and from the definitions we know ${}_FR^+(\lambda) \neq {}_FR^+(\lambda_0)$ and so ${}_FD(\lambda) \neq {}_FD(\lambda_0)$. On the other hand, a different regular λ_0' in the same chamber as λ_0 will yield the same parabolic.

Conversely, given a base ${}_FD$ of ${}_FR$, we put $C^+({}_FD)$ for the *positive chamber* in a_0 associated to ${}_FD$ defined

$$C^+({}_FD) = \{\lambda \in a_0 : \langle \alpha, \lambda \rangle > 0 \text{ for all } \alpha \in {}_FD\};$$

it is clear that, if nonempty, $C^+({}_FD)$ is a chamber in a_0; and indeed, we have by [Hum12, Exercise 10.7] that $C^+({}_F D) \neq \emptyset$. It is easy to check that

$$C^+({}_F D(\lambda_0)) = C(\lambda_0),$$

so the association of chambers to choices of base has an inverse and is bijective. The remaining correspondence is furnished by Lemma 2.5.2, and we are done. □

In Chapter 5, below, we will take advantage of the correspondence in Proposition 2.5.3. The same proof *mutatis mutandis* gives us

Corollary 2.5.4. *There is a one-to-one correspondence between elements of $\mathscr{P}(M)$ and the chambers in a_M, for any $M \in \mathscr{L}$.*

Recall that, given a Levi subgroup M of G, we denote by $\mathscr{P}(M)$ the set of parabolic subgroups P of G with $M_P = M$. Explicit examples of these correspondences are worked out in Chapter 7.

Our next goal is to parameterize the Levi subgroups of G by using sub-root systems of ${}_F R$; we begin by seeing how such a sub-root system relates to the absolute root system R "upstairs." Let us fix a base ${}_F D$ of ${}_F R$, and recall that we have ${}_F D \subset \mathrm{i}a_0^*$. Inside a_0, we have

$${}_F D^\vee := \{\alpha^\vee : \alpha \in {}_F D\},$$

and we see that a_0^G is the subspace spanned by ${}_F D^\vee$.

It can be shown [Spr10, 15.5.1–2] that the projection

$$\pi : \mathbf{R} \otimes_{\mathbf{Z}} X \ \to \mathbf{R} \otimes_{\mathbf{Z}} ({}_F X), \tag{2.5}$$

induced by the restriction of characters from T to S, maps R to ${}_F R \cup \{0\}$. Moreover, there exists a set R^+ of positive roots in R such that

$$\alpha \in R^+ \text{ if and only if } \pi(\alpha) \in {}_F R^+,$$

which in turn determines a base D of R. If D is a base of R, denote by D_0 the subset of D mapped to 0 under π. We quote

Lemma 2.5.5. *[Spr10, Proposition 15.5.3(iii)] Suppose $\alpha, \beta \in D - D_0$; then $\pi(\alpha) = \pi(\beta)$ if and only if α and β lie in the same Γ-orbit in D.*

We observe that for any $M \in \mathscr{L}$, it follows from the definitions that $R(M,T)$ is a subset of $R(G,T)$, that is itself a root system. Let us call such a set a *sub-root system* of $R(G,T)$.

Theorem 2.5.6. *Let G be a connected reductive group defined over F (with R, D, etc. as above), and let $P_0 \subset G$ be a minimal parabolic subgroup.*

1. *There is a one-to-one correspondence between the set of parabolic subgroups of G containing P_0, and the set of subsets of ${}_F D$, under which the subset ${}_F D_P \subset {}_F D$ corresponds to the parabolic subgroup $P(\lambda_P)$, for any λ_P such*

that $\langle \alpha, \lambda_P \rangle > 0$ if and only if $\alpha \in {}_F D_P$.

2. *There is a one-to-one, inclusion-preserving correspondence between the set of sub-root systems of ${}_F R$, and the set $\mathscr{L}$, under which the Levi subgroup $M \in \mathscr{L}$ corresponds to the sub-root system $R_M \subset {}_F R$, where*

$$R_M = \pi(R(M,T)) \subset {}_F R. \tag{2.6}$$

Proof. By Proposition 2.5.3, to each minimal parabolic subgroup P_0 we can associate a base ${}_F D$ of the root system ${}_F R$ such that

$$P_0 = \{x \in G : \lim_{t \to 0} \alpha^\vee(t) x \alpha^\vee(t)^{-1} \text{ exists for all } \alpha \in {}_F D\}.$$

To each subset ${}_F D_P$ of ${}_F D$ we associate the parabolic subgroup

$$P = \{x \in G : \lim_{t \to 0} \alpha^\vee(t) x \alpha^\vee(t)^{-1} \text{ exists for all } \alpha \in {}_F D_P\};$$

the notation is deliberate, for we will see shortly that subsets of ${}_F D$ will correspond to parabolic subgroups of G containing P_0. Furthermore, it is immediate from the definitions that

$${}_F D_P \subset {}_F D_{P'} \text{ if and only if } P' \subset P,$$

and so in particular that the parabolic subgroups we construct will contain P_0. The elements λ_P exist by Lemma 2.5.2; this proves point 1.

For the second point: all Levi subgroups of G are Levi factors of parabolic subgroups of G, all of which contain a minimal parabolic subgroup. Again by Proposition 2.5.3, it follows that we can obtain all elements of $\mathscr{L}$ by obtaining all Levi factors of all parabolic subgroups P containing the minimal parabolic P_0, as P_0 varies over all possible minimal parabolics.

Observe that the opposite of the parabolic P constructed above is

$$\bar{P} = \{x \in G : \lim_{t \to 0} \alpha^\vee(t)^{-1} x \alpha^\vee(t) \text{ exists for all } \alpha \in {}_F D_P\},$$

and that by definition, both P and $\bar{P}$ have Levi factor $M_P = P \cap \bar{P}$. By construction, the root system $R(M,T)$ of the pair (M,T) is a subsystem of $R(G,T)$ that projects under π to the sub-root system of ${}_F R$ spanned by ${}_F D_P$; denote this sub-root system of ${}_F R$ by R_M. As P varies over all parabolics containing P_0, we obtain all Levi factors of such parabolics in this way. Varying over all possible bases ${}_F D$ of ${}_F R$, we obtain all Levi subgroups of G containing M_0.

Furthermore, this construction exhausts all sub-root systems of ${}_F R$: if ${}_F\tilde{R} \subset {}_F R$ is such a sub-root system, then there is some $\tilde{\lambda} \in a_0$ such that, for all $\alpha \in {}_F R$, one has

$$\langle \alpha, \lambda \rangle > 0 \text{ if and only if } {}_F\tilde{R} \cap {}_F R^+.$$

By construction, we now have that $M(\tilde{\lambda}) := P(\tilde{\lambda}) \cap \bar{P}(\tilde{\lambda})$ is a Levi subgroup of G with root system ${}_F\tilde{R}$, and we are done. $\square$

We will call ${}_F D_P$ the **set of roots associated to** P. Finally, we close with the following

Definition 2.5.7. Let P be a parabolic subgroup of G. The **half-sum of roots** associated to P is the vector

$$\rho_P = \frac{1}{2} \sum_{\alpha \in_F D_P} \alpha.$$

Observe that $\rho_{\bar{P}} = -\rho_P$.

2.6 Galois actions

Let V be a variety defined over F and put $\Gamma := \mathrm{Gal}(F_s/F)$. The action of Γ on F_s induces an action on $V(F_s)$ through its natural (co-ordinate) action on the ring of functions $F_s[V]$; we will denote this action by $(\sigma, x) \mapsto \sigma_V.x$ for $\sigma \in \Gamma, x \in V(F_s)$. That is: we write σ_V for the image of σ under the map $\Gamma \to \mathrm{Aut}(V(F_s))$.

In particular, let us consider the case that $V = T$ is an algebraic torus. The action of Γ on the group $X = X^*(T)$ associated to the above action on $T(F_s)$ is then given by

$$(\sigma_X.\chi)(t) := \sigma_{\mathbf{G}_m}.(\chi(\sigma_T^{-1}.t))$$

for $\chi \in X, t \in T(F_s)$, making the diagram

$$\begin{tikzcd} T(F_s) \arrow[r, "\sigma_T."] \arrow[d, "\chi"'] & T(F_s) \arrow[d, "\sigma_X.\chi"] \\ \mathbf{G}_m \arrow[r, "\sigma_{\mathbf{G}_m}."] & \mathbf{G}_m \end{tikzcd}$$

commute. It is not hard to check that if we equip Γ with the profinite topology, and both X and $T(F_s)$ with the discrete topology, then these actions are continuous.

Furthermore, we see that Γ acts trivially on $\chi \in X$ (that is, $\sigma_X . \chi = \chi$ for all $\sigma \in \Gamma$) if and only if χ is defined over F, and so *a fortiori* Γ acts trivially on X if and only if T is F-split.

Thus, to each F-torus T we can associate a finitely-generated free abelian group X equipped with a continuous action of Γ; it turns out that this association is bijective. We have

Theorem 2.6.1. *Let $\mathcal{T}$ be the category of algebraic tori defined over F, and let $\mathcal{X}$ be the category of finitely-generated free abelian groups equipped with a continuous action of Γ. There is an anti-equivalence of categories $\mathcal{T} \to \mathcal{X}$ given by the map $T \mapsto X^*(T)$.*

The proof is fairly straightforward: the inverse functor to $T \mapsto X^*(T)$ is given by $X \mapsto X_* \otimes_{\mathbf{Z}} \mathbf{G}_m$, with the trivial Γ-action on the right-hand factor, where X_* is defined to be the Γ-module $\mathrm{Hom}(X, \mathbf{Z})$.

We remark that it is possible to translate facts about T to facts about $X^*(T)$. Building on our observations from above, we quote [Spr10, Proposition 13.2.2]:

Theorem 2.6.2. *Let T be an F-torus and let X be its character lattice with the associated Γ-action. Then:*

- *T is F-split if and only if Γ acts trivially on X, and*

- *T is F-anisotropic if and only if Γ has no nontrivial fixed point in X.*

Suppose $T \subset G$ is a maximal F-torus in a reductive group G defined over F; because T is defined over F, we know that $T(F_s)$ is Γ-stable and so restricts to an action of Γ on $T(F_s)$. Let us write $X = X^*(T)$, $R = R(G, T)$, and fix some set R^+ of positive roots in R; the action of $\sigma \in \Gamma$ on X maps the set R^+ to some other set of positive roots in R.

By [Spr10, §15.5.2], there is a unique element w_σ of the Weyl group $W(G, T)$ such that $w_\sigma(\sigma.R^+) = R^+$, and therefore $w_\sigma(D) = D$. Let us define a homomorphism $\tau : \Gamma \to \mathrm{Perm}(D)$ into the group of permutations of D via

$$\tau(\sigma)(\alpha) := w_\sigma(\sigma.\alpha) \tag{2.7}$$

for $\sigma \in \Gamma, \alpha \in D$; then by [Spr10, Proposition 15.5.3] $\tau = \tau(\Gamma)$ stabilizes D_0.

If we furthermore consider our choice of minimal parabolic P_0, one may actually associate to the triple (G, P_0, T) its **indexed root datum**, which is the 6-tuple

$$_i\Psi = (X, D, X^\vee, D^\vee, D_0, \tau),$$

where we have retained our notations from Sections 2.3 and 2.5. The triple (D, D_0, τ) is known as the **index** of the triple (G, P_0, T).

We remark here that, in the case that G is F-split, then its indexed root datum is determined up to isomorphism by its absolute root datum; in the general case, the F-points of our group are constructed as the group of fixed points of $G(F_s)$ under a particular action of the Galois group $\mathrm{Gal}(F_s/F)$, where F_s is some fixed separable closure of F. We discuss these matters in Chapter 3, below.

Again, each Levi subgroup of G is itself a connected, reductive algebraic group [Art91, p. 10], and so these same constructions apply not only to G, but to every Levi subgroup M of G.

2.7 Measures and integration

In this section, we explain how to choose measures which will allow us to integrate sensibly on non-compact linear algebraic groups. We follow the constructions of [Wal95, I.2–I.4].

The adjoint action of $G(F)$ on both itself and $g(F)$ induces actions of $G(F)$ on the spaces $C_c^\infty(G(F))$ and $C_c^\infty(g(F))$ of locally constant, compactly supported, complex-valued functions on $G(F)$, respectively $g(F)$ (the adjoint action of g on itself is denoted $\mathrm{ad}(X), X \in g$). To lighten notation, we will write $C_c^\infty(G)$ for $C_c^\infty(G(F))$, and similarly for $C_c^\infty(g)$.

For $P \in \mathscr{F}$, fix (left) Haar measure $\mathrm{d}n$ on $N_P(F)$ such that

$$\int_{N_P(F)} \mathrm{e}^{2\langle \rho_{\overline{P}}, H_{\overline{P}}(n)\rangle} \, \mathrm{d}n = 1,$$

where $\rho_{\overline{P}}$ is the half-sum of roots associated to $\overline{P}$, and $H_{\bar{P}}$ is the Harish-Chandra homomorphism associated to $\bar{P}$ (see Definitions 2.4.2 and 2.5.7).

We then fix an arbitrary (left) Haar measure $\mathrm{d}y$ on M_0, and it can be shown that there exists a Haar measure $\mathrm{d}x$ on G such that, for any $P_0 \in$

$\mathscr{P}(M_0)$, one has

$$\int_{G(F)} f(x)\,\mathrm{d}x = \int_{N_{P_0}(F)\times M_0(F)\times N_{\overline{P}_0}(F)} f(ny\overline{n})\delta_{P_0}(y)^{-1}\,\mathrm{d}\overline{n}\,\mathrm{d}y\,\mathrm{d}n$$

for all $f \in C_c^\infty(G)$, where δ_{P_0} is the **modulus character** of P_0; we equip G with this measure.

We recall that the modulus character δ_{P_0} is, roughly, used to turn a left Haar measure into a right Haar measure. More precisely [Sil79, §1.2]: for any topological group G, one writes $\mathrm{d}_L x, \mathrm{d}_R x$ for respective left and right Haar measure on G, and then defines δ_G to be the unique function $G \to \mathbf{R}_{>0}$ satisfying

$$\mathrm{d}_L(x^{-1}) = \mathrm{d}_R x = \delta_G(x)\mathrm{d}_L x$$

for all $x \in G$. As it happens [Kot05, §2.3], if G is a reductive group, then G is **unimodular**, meaning δ_G is identically 1 and so we need not specify left or right Haar measure on reductive groups in the future (as they are the same). We caution, however, that $\delta_P \neq 1$ for proper parabolic subgroups $P \subset G$ (although our remarks do imply that $\delta_M = 1$ for all $M \in \mathscr{L}$).

More generally: having now fixed $\mathrm{d}x$ on $G(F)$ and $\mathrm{d}n$ on $N_P(F)$ (for any $P \in \mathscr{F}$), we now define on $M_P(F)$ the Haar measure $\mathrm{d}y$ satisfying

$$\int_{G(F)} f(x)\,\mathrm{d}x = \int_{N_P(F)\times M_P(F)\times N_{\overline{P}}(F)} f(ny\overline{n})\delta_P(y)^{-1}\,\mathrm{d}\overline{n}\,\mathrm{d}y\,\mathrm{d}n$$

for all $f \in C_c^\infty(G)$. We record here also the fact that, if Haar measure on K

(i.e. our compact subgroup in $G(F)$ from Equation (2.1)) is chosen so that

$$\int_K \mathrm{d}k = 1,$$

then one also has (per [Wal95, I.4])

$$\int_{G(F)} f(x)\,\mathrm{d}x = \int_{M_P(F)\times N_P(F)\times K} f(ynk)\,\mathrm{d}k\,\mathrm{d}n\,\mathrm{d}y$$

for all $P \in \mathscr{F}$ and all $f \in C_c^\infty(G)$. We remark that $\mathrm{d}k$ and $\mathrm{d}x$ are *a priori* unrelated, but this is unimportant in the sequel.

If T is a split torus, then it is equipped with Haar measure for which the maximal compact subgroup of $T(F)$ has measure 1. If T is an arbitrary torus, it is equipped with Haar measure for which the induced measure on the (compact) quotient group T/A_T has total volume 1. As such, if $M_0 = T$ is itself a torus, we have actually defined on it *two* measures; one which allows us to define the compatible measure on $G = N_{P_0}M_0N_{\bar{P}_0}$, and another which allows us to integrate over T/A_M. It will be clear from context which one is intended.

For $X \in g(F)$, we denote by G_X the centralizer of X in G, so that

$$G_X(F) = \{x \in G(F) : \mathrm{Ad}(x)(X) = X\};$$

similarly, we write g_X for the centralizer of X in g (i.e., the kernel of $\mathrm{ad}(X)$). Denote by g_{reg} the set of regular semisimple elements of g; if T is a maximal subtorus of G, then T is said to be **elliptic** if $A_T = A_G$. Similarly $X \in g(F)$ is said to be **elliptic** if G_X is an elliptic maximal torus in G; denote by g_{ell}

the set of elliptic elements of g. It is a subset of g_{reg}.

Let $X = X_n + X_s$ be the Jordan decomposition of $X \in g(F)$, so X_n is nilpotent, X_s is semisimple, and $X_n X_s = X_s X_n$; then $\mathrm{ad}(X)$ descends to a linear operator on the quotient space g/g_{X_s}, whose determinant we denote $D^g(X)$.

Now: fix a neighbourhood V_G of 1 in G, as well as a neighbourhood V_g of 0 in g, both of which are open and invariant under $\mathrm{Ad}(G)$. We use the definition of g as the tangent space to G at the identity in order to fix a local algebraic isomorphism $V_g \to V_G$, then equip g with the unique Haar measure such that this isomorphism locally preserves measures. We do the same for each subgroup H of G, replacing V_G with $V_G \cap H$, etc. For the sake of completeness, let us specify that, given an embedding $G \hookrightarrow \mathrm{GL}(n)$, we will use the map $X \mapsto 1 + X$, where X is an element of the Lie algebra and 1 is the identity of the group.

For $P \in \mathscr{F}$, this process obtains for all $f \in C_c^\infty(g)$ the equality

$$\int_{g(F)} f(X)\,\mathrm{d}X = \int_{n_P(F)\times m_P(F)\times n_{\overline{P}}(F)} f(N + Y + \overline{N})\,\mathrm{d}\overline{N}\,\mathrm{d}Y\,\mathrm{d}N.$$

We fix a set $\mathscr{T}(G)$ of conjugacy class representatives of elliptic maximal tori of G, and we will assume without loss of generality that, for all $T \in \mathscr{T}(G)$, one has $A_T \subseteq A_{M_0}$ (see [Kot05, §§7.8–9] for a discussion of why this assumption can be made).

Next, we consider measures on the stabilizer subgroups: let $\mathrm{d}x$ and $\mathrm{d}\gamma$ be respective left Haar measures on $G(F)$ and $G_X(F)$, for $X \in g(F)$. The orbit $G_X(F)\backslash G(F)$ is then equipped with a unique $G(F)$-invariant measure

$\mathrm{d}\bar{x}$ satisfying, for all $f \in C_c^\infty(G)$, the equation

$$\int_{G(F)} f(x)\,\mathrm{d}x = \int_{G_X(F)\backslash G(F)} \int_{G_X(F)} f(\gamma x)\,\mathrm{d}\gamma\,\mathrm{d}\bar{x}. \tag{2.8}$$

This technique of integrating over $G(F)$ is referred to in the literature as *integration in stages* [Kot05, eqn. 2.4.1], and allows us to construct another measure we will use in the trace formula.

Namely: if T is a maximal elliptic torus in M and $A_M = A_T \subset M$ is the maximal split central torus, then [Kot05, §§7.2-3 and 7.11] by fixing a $G(F)$-invariant volume form on $T(F)\backslash G(F)$, we obtain a $G(F)$-invariant measure $\mathrm{d}\bar{x}$ on $T(F)\backslash G(F)$; indeed, this follows from Equation (2.8), because T is the centralizer of a regular semisimple element of g. We then define the unique $G(F)$-invariant measure $\mathrm{d}\dot{x}$ on $A_M(F)\backslash G(F)$ such that

$$\int_{A_M(F)\backslash G(F)} f(x)\,\mathrm{d}\dot{x} = \int_{T(F)\backslash G(F)} f(x)\,\mathrm{d}\bar{x} \tag{2.9}$$

for all $f \in C_c^\infty(T\backslash G)$; notice that this is possible because $A_M\backslash T = A_T\backslash T$ has volume 1.

Fix once and for all a non-degenerate, symmetric, bilinear form $B = B(X,Y)$, which is invariant with respect to the adjoint G-action, i.e.

$$B(\mathrm{ad}(x)X, \mathrm{ad}(x)Y) = B(X,Y) \text{ for all } X, Y \in g(F),\ x \in G(F),$$

as well as a nontrivial additive character ψ on F; the existence of such forms $B(X,Y)$ is established in [AR00, Proposition 4.1] for any field F of characteristic zero, or sufficiently large positive characteristic. Still following the construction of [Wal95], we use these to define two variations of the Fourier transform on functions $f \in C_c^\infty(g)$, namely,

$$\hat{f}(Y) = c_\psi(g) \int_{g(F)} f(X)\psi(B(Y,X))\,\mathrm{d}X$$

and

$$\check{f}(Y) = c_{\psi^{-1}}(g) \int_{g(F)} f(X)\psi^{-1}(B(Y,X))\,\mathrm{d}X,$$

where $c_\psi(g)$ is the unique constant such that, for all $X \in g(F)$, one has

$$\hat{\hat{f}}(X) = f(-X).$$

We use $B(X,Y)$ to identify g with its linear dual; as such, $f \mapsto \hat{f}$ gives an isomorphism $C_c^\infty(g) \to C_c^\infty(g)$, the inverse of which is given by $f \mapsto \check{f}$. We remark that exactly the same construction holds for any (algebraic) subgroup H of G such that the restriction of $B(X,Y)$ to h is nondegenerate.

We will return to a discussion of measures in Section 4.4, in which we pick such measures uniformly for families of fields.

2.8 The weight factor

We recall that our choice of minimal parabolic subgroup P_0 containing a maximal torus T determines bases D and ${}_FD$ of the root systems R and ${}_FR$, respectively, and that to each parabolic subgroup $P \in \mathscr{F}$ containing P_0, there corresponds a subset of ${}_FD$ denoted ${}_FD_P$. For parabolic subgroups $P \subseteq Q$ of G, we will denote by ${}_FD_P^Q$ the set of roots (with respect to T)

corresponding to the parabolic subgroup $P \cap M_Q$ of M_Q.

Inside a_P^G we have the lattice L_P^G, which is the $\mathbf{Z}$-span of $D_P^\vee$. We will write $\mathrm{vol}(a_P^G/L_P^G)$ for the volume of the (compact) quotient of the vector space by the lattice.

Definition 2.8.1. The **weight factor** $v_M(x,y)$ attached to a Levi subgroup $M \in \mathscr{L}$ is the function [Art81, §6]

$$v_M(x,y) = \mathrm{vol}\left(\mathrm{Hull}\{-H_P(y) + H_{\overline{P}}(x) : P \in \mathscr{P}(M)\}\right)$$

on $G(F) \times G(F)$, where $\mathrm{Hull}(U)$ denotes the convex hull of $U \subseteq a_P$.

This definition turns out to be inconvenient for our purposes. Instead, we will use an equivalent definition, inspired by the approach of [CHL11].

Let $M \in \mathscr{L}, P \in \mathscr{P}(M)$, and let $\mathrm{i}a_M^*$ be the vector space from Section 2.4. Define the function

$$\theta_P(\lambda) = \frac{1}{\mathrm{vol}(a_P^G/L_P^G)} \prod_{\alpha \in_F D_P} \lambda(\alpha^\vee), \tag{2.10}$$

where $\lambda \in ia_M^*$. According to [Art81, Lemma 6.2] and [Art91, p. 88], we have

$$v_M(x,y) = \lim_{\lambda \to 0} \sum_{P \in \mathscr{P}(M)} \frac{\exp\left(-\lambda(H_P(y) - H_{\overline{P}}(x))\right)}{\theta_P(\lambda)},$$

where $\lambda \in \mathrm{i}a_M^*$ is any regular element.

We can calculate the limit by replacing λ with $t\lambda, t \in \mathbf{R}_{>0}$, and then taking the limit as $t \to 0^+$, to obtain

Lemma 2.8.2. *For any regular* $\lambda \in \mathrm{i}a_M^*$, *one has*

$$v_M(x,y) = \sum_{P \in \mathscr{P}(M)} \frac{(-1)^p \left(\lambda\left(H_P(y) - H_{\overline{P}}(x)\right)\right)^p}{p!\theta_P(\lambda)}, \qquad (2.11)$$

where $p = \deg\theta_P = \dim a_P^G$.

Importantly: as long as λ is regular, the value of the right-hand side of Equation (2.11) is well-defined and independent of λ itself (a different regular λ' will give the same value), as in [CHL11, pp. 17-18]. We remark that the integer p need not be prime; examples of the weight factor are computed in Examples 7.2.3 and 7.3.3.

Therefore, let us fix once and for all a regular $\lambda \in \mathrm{i}a_M^*$ for every $M \in \mathscr{L}$; we will take Equation (2.11) as our definition of the weight factor v_M. The reason for this choice of definition is to make it easier to encode $v_M(x,y)$ as a *motivic function* (see Section 4.3, below).

2.9 The statement of the local trace formula

We may now define a distribution $J_G = J_G(f_1, f_2)$ on $C_c^\infty(g) \times C_c^\infty(g)$ as follows:

$$J_G(f_1,f_2) := \sum_{M \in \mathscr{L}} \frac{|W^M|}{|W^G|} \sum_{T \in \mathscr{T}_M} \frac{1}{|W(M,T)|} \int_{t_{\mathrm{reg}}(F)} |D^g(X)| \cdot \int_{A_M(F)\backslash G(F)} \int_{A_M(F)\backslash G(F)} f_1(\mathrm{Ad}(x)X) f_2(\mathrm{Ad}(y)X) v_M(x,y)\,\mathrm{d}\dot{x}\,\mathrm{d}\dot{y}\,\mathrm{d}X. \qquad (2.12)$$

We remark that the measures $\mathrm{d}\dot{x}$ and $\mathrm{d}\dot{y}$ are constructed as in Equation (2.9), above. The convergence of J_G on arbitrary $f_1, f_2 \in C_c^\infty(g)$ is not immediate, but follows from [Kot05, Proposition 20.1] (at least when the characteristic of F is zero). We are finally able to state the local trace formula.

Theorem 2.9.1. *[Wal95, théorème V.2] Let $\tilde{J}_G$ be the distribution on $C_c^\infty(g) \times C_c^\infty(g)$ defined on test functions $f = (f_1, f_2)$ by*

$$\tilde{J}_G(f_1, f_2) = J_G(\hat{f}_1, \check{f}_2);$$

if F has characteristic zero, then $\tilde{J}_G$ and J_G coincide.

We take a moment to explain the name *trace formula*: the group $G(F) \times G(F)$ acts on the space $L^2(G(F))$ by the action

$$G(F) \times G(F) \times L^2(G(F)) \to L^2(G(F)), \quad (x, y, f) \mapsto \big[g \mapsto f(x^{-1}gy)\big],$$

and extending this action to the tensor product of group algebras $C(G) \otimes_{\mathbf{C}} C(G)$ (where $C(G)$ is the $\mathbf{C}$-algebra of continuous $\mathbf{C}$-valued functions on $G(F)$, equipped with convolution) gives us an action

$$C(G) \otimes_{\mathbf{C}} C(G) \times L^2(G(F)) \to L^2(G(F))$$

defined on the generators by

$$((f_1 \otimes f_2)\phi)(g) = \int_{G(F)} \int_{G(F)} f_1(x) f_2(y) \phi(x^{-1}gy) \, \mathrm{d}x \, \mathrm{d}y.$$

This means that $f_1 \otimes f_2$ acts by an *integral operator* with kernel function $K = K(x,y)$ (say). When G is compact, the function K is trace class, meaning the integral

$$\int_{G(F)} K(x,x)\,\mathrm{d}x$$

converges; its value is the *trace* of the operator defined by $f_1 \otimes f_2$.

For general reductive groups, $K(x,y)$ is not of trace class, and so the trace cannot be computed in this way. Instead, Arthur [Art91] uses the Weyl integration formula (among other tools) to expand the integral in two ways, namely, the *geometric* expansion and the *spectral* expansion, and then to obtain from these expansions which are convergent. The trace formula, as stated there, consists in a formal identity

$$J_{\text{geom}}(f_1, f_2) = J_{\text{spec}}(f_1, f_2),$$

where J_{geom} is precisely our distribution J_G. Walspurger [Wal95] gave a form of the theorem for the Lie algebra, which instead took the form

$$J_{\text{geom}}(f_1, f_2) = J_{\text{geom}}(\hat{f}_1, \check{f}_2).$$

The Fourier transform of an orbital integral may be thought of as the Lie algebra analogue of characters of reductive groups; as such, the right-hand side plays the rôle of the spectral expansion J_{spec}.

Our task is to show that the distribution J_G is *motivic*. At its most basic level, this involves demonstrating that the "ingredients" of this distribution (i.e. the various measures, domains of integration, indices of summation, summands, integrands, etc.) can be specified in a particular first-order language. We will begin this work in Chapter 4; but in order to accomplish our work there, we will first need to understand some of these ingredients from a different perspective.

To that end, we devote the next chapter to providing a clean catalogue of our ingredients, and what parameterises them.

Chapter 3

Cohomology

Cohomology is often used to classify isomorphism classes in the theory of algebraic groups. In this chapter we will recall the necessary notions of cohomology, and collect the results relevant for our later work.

Let Γ be a group acting on another group A; depending on the context, we will impose certain further conditions on these objects, for instance that A is abelian, or that the action is a continuous action of topological groups.

Essentially, group cohomology is a systematic way to collect information on certain maps $\Gamma \to A, \Gamma \times \Gamma \to A$, etc. (called respectively 1-cocycles, 2-cocycles, and so on), which naturally encode important information about the objects under consideration.

For now, we will restrict our attention to 1-cocycles (or *crossed homomorphisms*) alone. We retain the notation of chapter 2.

3.1 Abelian cohomology

We will first treat the abelian case. Let Γ be a profinite group acting on an (additive) abelian group A; we will sometimes call A a **Γ-module**, and denote by A^Γ the subgroup of A fixed by the action of Γ. Recall that a group is said to be **profinite** if it is the inverse limit of an inverse system of finite topological groups.

Let $Z^1(\Gamma, A)$ be the set of all functions $c : \Gamma \to A$ satisfying

$$c(\sigma\tau) = c(\sigma) + \sigma.c(\tau)$$

where the action of $\sigma \in \Gamma$ on $a \in A$ is denoted $\sigma.a$. Note that if the action of Γ is trivial, then $Z^1(\Gamma, A)$ is just the set $\mathrm{Hom}(\Gamma, A)$ of group homomorphisms $\Gamma \to A$.

Equipping $Z^1(\Gamma, A)$ with the operation

$$(c + c')(\sigma) = c(\sigma) + c'(\sigma)$$

(for $c, c' \in Z^1(\Gamma, A), \sigma \in \Gamma$) yields an abelian group, the group of **1-cocycles** of Γ in A. We define the subgroup of **1-coboundaries**

$$B^1(\Gamma, A) = \{c \in Z^1(\Gamma, A) : c(\sigma) = \sigma.a - a \text{ for all } \sigma \in \Gamma\},$$

and put

$$H^1(\Gamma, A) := Z^1(\Gamma, A)/B^1(\Gamma, A).$$

The quotient group $H^1(\Gamma, A)$ is called the **1-cohomology group** of Γ in A. Clearly, if the action of Γ is trivial, we have $H^1(\Gamma, A) = \mathrm{Hom}(\Gamma, A)$.

We will typically write elements of $H^1(\Gamma, A)$ as $[c]$ for some $c \in Z^1(\Gamma, A)$. Observe from the definitions that $[c] = [d]$ if and only if there exists some $a \in A$ such that

$$d(\sigma) = -a + c(\sigma) + \sigma.a \tag{3.1}$$

for all $\sigma \in \Gamma$. We record the following standard result [Mil13, II.1.9]:

Theorem 3.1.1. *Let Γ be a profinite group and let*

$$1 \to A \to B \to C \to 1$$

be a short exact sequence of Γ-modules. Then there exists a long exact sequence of groups

$$\begin{aligned} 1 \to A^\Gamma \to B^\Gamma \to C^\Gamma \to H^1(\Gamma, A) \\ \to H^1(\Gamma, B) \to H^1(\Gamma, C) \to H^2(\Gamma, A) \to \cdots \end{aligned}$$

that continues infinitely far to the right, where $H^n(\Gamma, -)$ for $n > 1$ are the higher cohomology groups (whose definition we omit).

3.2 Nonabelian cohomology

In this section we will continue to assume that Γ is profinite, but we will allow A to be an arbitrary group equipped with a (left) action of Γ, which we will still denote by $\sigma.a$.

In this case, we define $Z^1(\Gamma, A)$, the **set of 1-cocycles**, to be the set of all maps $c : \Gamma \to A$ satisfying

$$c(\sigma\tau) = c(\sigma)(\sigma.c(\tau)) \tag{3.2}$$

for all $\sigma, \tau \in \Gamma$; the analogy with the abelian case is clear.

We stress that, because pointwise multiplication of cocycles does not in general yield a cocycle in the nonabelian case, $Z^1(\Gamma, A)$ has only the structure of a set, and not a group.

On the set $Z^1(\Gamma, A)$ we place the equivalence relation $\sim$, where $c \sim d$ if and only if there exists $a \in A$ such that

$$d(\sigma) = a^{-1}c(\sigma)(\sigma.a) \tag{3.3}$$

for all $\sigma \in \Gamma$. We denote by $H^1(\Gamma, A)$ the *pointed set* of equivalence classes in $Z^1(\Gamma, A)$, called the **1-cohomology set** of Γ in A. As a pointed set, its distinguished element [1] is the class consisting of all cocycles $c \in Z^1(\Gamma, A)$ satisfying

$$c(\sigma) = a^{-1}(\sigma.a)$$

for some $a \in A$.

Observe that, if A is abelian, then the 1-cohomology set we have just defined coincides with the 1-cohomology group defined in Section 3.1, with the distinguished class corresponding to the identity element of $H^1(\Gamma, A)$.

Definition 3.2.1. A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ of pointed sets is called **exact** at B if

$$g^{-1}(1_C) = f(A),$$

where 1_C is the distinguished element of C.

The corresponding result to Theorem 3.1.1 for the nonabelian case is familiar, but decidedly weaker. Per [Spr10, Proposition 12.3.4], we have:

Theorem 3.2.2. *Let Γ be a profinite group and let*

$$1 \to A \to B \to C \to 1$$

be a short exact sequence of groups equipped with a (left) Γ-action. Then the sequence of pointed sets

$$1 \to A^\Gamma \to B^\Gamma \to C^\Gamma \to H^1(\Gamma, A) \to H^1(\Gamma, B)$$

is exact. Moreover, if A is normal in B, then the sequence

$$1 \to A^\Gamma \to B^\Gamma \to C^\Gamma \to H^1(\Gamma, A) \to H^1(\Gamma, B) \to H^1(\Gamma, C)$$

is exact.

To close this section, we observe that, in both the abelian and nonabelian cases, an inclusion of groups $A \hookrightarrow B$ implies the existence of a map

$$H^1(\Gamma, A) \to H^1(\Gamma, B),$$

which is generally not injective.

3.3 Galois cohomology

In the theory of algebraic groups, isomorphism classes (of groups, tori, etc.) over a non-algebraically closed field are generally classified using *Galois descent.* Broadly speaking, this involves considering all F_s-isomorphism classes of the given object, and seeing how these isomorphism classes divide into F-isomorphism classes. We recall the Galois actions already described in Section 2.6.

This is easiest to see in the case in the situation of maximal tori. Obviously, there is a single F_s-isomorphism class of maximal F-tori in any algebraic group G, as they must all have the same rank; however, there will be more than one F-isomorphism class when we restrict to a non-algebraically closed field, as we will see shortly.

With G as before, let $S, S' \subset G$ be two maximal F-tori. Following the terminology of [Kot82], we will say that S and S' are **stably conjugate** if there exists $x \in G(F_s)$ such that

$$xS(F)x^{-1} = S'(F),$$

and we will say that they are **rationally conjugate** if there exists $y \in G(F)$ such that

$$yS(F)y^{-1} = S'(F).$$

These are clearly equivalence relations. Furthermore, because rational conjugacy implies stable conjugacy, we see that the set $\mathcal{T}(G)$ of all maximal F-tori of G is partitioned into stable conjugacy classes, which are in turn partitioned into rational conjugacy classes.

Theorem 3.3.1. *[Kot82, §3] Let G be an F-group, $T \subset G$ a maximal F-torus, and $N = \mathrm{Norm}_G(T)$ the normalizer of T in G; we recall the natural action of $\Gamma = \mathrm{Gal}(F_s/F)$ on*

$G(F_s)$. There is a one-to-one correspondence between the set of rational conjugacy classes of maximal tori of G, and the kernel of the map

$$H^1(\Gamma, N(F_s)) \to H^1(\Gamma, G(F_s)),$$

which we denote $\ker(G, T)$.

We include a proof of Theorem 3.3.1, as the construction contained therein is essential to our goal of establishing the trace formula as a motivic identity. Before we begin, we record the fact that, if H is any closed F-subgroup of G, then the restriction of the action of Γ to H induces a Γ-action on H, and so in particular $\sigma.(H(F_s)) = H(F_s)$ for all $\sigma \in \Gamma$.

Proof. We begin by remarking that the trivial class of the 1-cohomology set $H^1(\Gamma, G(F_s))$ is the set of all maps $\Gamma \to G(F_s)$ of the form $\sigma \mapsto x^{-1}x^{\sigma}$ for some $x \in G(F_s)$.

Let us lighten notation by temporarily writing x^{σ} for $\sigma.x$ (note that because this is a left-action, we have the awkward formula $x^{\sigma\tau} = (x^{\tau})^{\sigma}$). We emphasize that, with this notation, $G(F_s)^{\sigma}$ indicates the image of $G(F_s)$ under the action of $\sigma \in \Gamma$, and *not* the elements of $G(F_s)$ that are fixed by the element σ (a set which we do not encounter in this document).

Let S be a maximal F-torus of G and let $x \in G(F_s)$ satisfy

$$S(F_s) = xT(F_s)x^{-1};$$

such x must exist, as all maximal tori in G are conjugate over F_s; we may write $S = xTx^{-1}$ in this case.

Because S is defined over F we know that $S(F_s)^\sigma = S(F_s)$ for any $\sigma \in \Gamma$, and hence

$$\begin{aligned} T(F_s) = x^{-1}S(F_s)x = x^{-1}S(F_s)^\sigma x = x^{-1}(xT(F_s)x^{-1})^\sigma x \\ = x^{-1}x^\sigma T(F_s)^\sigma x^{-\sigma}x = (x^{-1}x^\sigma)T(F_s)(x^{-1}x^\sigma)^{-1}. \end{aligned}$$

It follows that $x^{-1}x^\sigma \in N(F_s)$. We can therefore define a map $c_x : \Gamma \to N(F_s)$ by the rule $\sigma \mapsto x^{-1}x^\sigma$, which satisfies the 1-cocycle condition:

$$c_x(\sigma\tau) = x^{-1}x^{\sigma\tau} = x^{-1}(x^\tau)^\sigma = x^{-1}x^\sigma x^{-\sigma}(x^\tau)^\sigma = x^{-1}x^\sigma(x^{-1}x^\tau)^\sigma,$$

and so $c_x \in Z^1(\Gamma, N(F_s))$. Composing with the natural map $Z^1(\Gamma, N(F_s)) \to H^1(\Gamma, N(F_s))$, we obtain the map

$$\mathcal{T}(G) \to H^1(\Gamma, N(F_s)), \qquad xTx^{-1} \mapsto [c_x].$$

We claim that this map is well-defined: indeed, suppose $xT(F_s)x^{-1} = \tilde{x}T(F_s)\tilde{x}^{-1}$. Then we have immediately that $x^{-1}\tilde{x} \in N(F_s)$, and so in $H^1(\Gamma, N(F_s))$ we have

$$[\sigma \mapsto x^{-1}x^\sigma] = [\sigma \mapsto (x^{-1}\tilde{x})^{-1}x^{-1}x^\sigma(x^{-1}\tilde{x})^\sigma] = [\sigma \mapsto \tilde{x}^{-1}\tilde{x}^\sigma]$$

and so $[c_x] = [c_{\tilde{x}}]$, as claimed.

Observe further that this map is invariant on rational conjugacy classes: if S, S' lie in the same rational conjugacy class, we can write $S(F) =$

$\gamma S'(F)\gamma^{-1}$ for some $\gamma \in G(F)$, and so if we write

$$S(F_s) = yT(F_s)y^{-1}, \quad S' = zT(F_s)z^{-1}$$

for some $y, z \in G(F_s)$, then we have

$$T(F_s) = y^{-1}S(F_s)y = y^{-1}\gamma S'(F_s)\gamma^{-1}y = y^{-1}\gamma zT(F_s)z^{-1}\gamma^{-1}y,$$

and so $n := y^{-1}\gamma z \in N(F_s)$. Using the fact that $\gamma^\sigma = \gamma$ for all $\sigma \in \Gamma$, we have

$$z^{-1}z^\sigma = (\gamma^{-1}yn)^{-1}(\gamma^{-1}yn)^\sigma = n^{-1}y^{-1}y^\sigma n^\sigma,$$

and so $[c_y] = [c_z]$ in $H^1(\Gamma, N(F_s))$.

By the definitions we know that $\ker(G,T)$ is the subset of $H^1(\Gamma_E, N(F_s))$ conisting of those classes whose representatives have the form $\sigma \mapsto y^{-1}y^\sigma$ for some $y \in G(F_s)$. It follows at once that the image of the map $xT(F_s)x^{-1} \mapsto [c_x]$ lies in $\ker(G,T)$, and we claim moreover that it is a surjection. But this is clear: every element of $\ker(G,T)$ has the form $[c_y]$ for some $y \in G(F_s)$, and the torus $T' = yT(F_s)y^{-1}$ maps to this element.

It remains only to show injectivity, and therefore suppose $[c_y] = [c_z]$ in $\ker(G,T)$; we aim to show that the rational class of yTy^{-1} is the same as that of zTz^{-1}. By assumption, there exists $n \in N(F_s)$ such that

$$y^{-1}y^\sigma = n^{-1}z^{-1}z^\sigma n^\sigma$$

for all $\sigma \in \Gamma$; equivalently,

$$1 = zny^{-1}y^{\sigma}n^{-\sigma}z^{-\sigma} = (yn^{-1}z^{-1})^{-1}(yn^{-1}z^{-1})^{\sigma},$$

which implies that $yn^{-1}z^{-1}$ is fixed by every $\sigma \in \Gamma$ and therefore lies in $G(F)$. It follows that the rational class of zTz^{-1} is the same as that of

$$(yn^{-1}z^{-1})zTz^{-1}(yn^{-1}z^{-1})^{-1} = yTy^{-1},$$

and we are done. □

There is also [Ser97, II.5.2, theorem 2] the following local duality theorem of Tate:

Theorem 3.3.2. *Let μ be the subgroup of $F_s^{\times}$ consisting of the roots of unity. If A is a Γ-module and*

$$A^{\vee} = \mathrm{Hom}(A, \mu)$$

is the dual module, then there exists a perfect pairing

$$H^i(\Gamma, A) \times H^{2-i}(\Gamma, A^{\vee}) \to H^2(\Gamma, \mu(F_s)) \cong \mathbf{Q}/\mathbf{Z}$$

for $i \in \{0, 1, 2\}$. In particular, if $A = T(F_s)$ is an F-torus of rank r, then there exists an isomorphism

$$H^1(\Gamma, T(F_s)) \cong H^1(\Gamma, X^*(T)).$$

Let us turn now to general reductive groups. We will say that one F-group G is an E**-form** of another G' if there exists an E-isomorphism of G onto G', and we will denote by $\mathrm{Aut}_E(G)$ the group of E-automorphisms of G. We may also speak of the **split form** of G, by which we will mean the F-isomorphism class of the split reductive group whose absolute root datum is that of G (or, occasionally, a representative of that class).

We recall that any action $(\sigma, x) \mapsto \sigma.x$ of Γ on $G(F_s)$ induces an action of Γ on $\mathrm{Aut}_{F_s}(G)$, namely

$$\Gamma \times \mathrm{Aut}_{F_s}(G) \to \mathrm{Aut}_{F_s}(G), (\sigma, f) \mapsto f^{\sigma},$$

such that the diagram

$$\begin{array}{ccc} G(F_s) & \xrightarrow{\sigma.} & G(F_s) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^{\sigma}} \\ G(F_s) & \xrightarrow{\sigma.} & G(F_s) \end{array}$$

commutes.

We quote [Spr10, Theorem 11.3.3]:

Theorem 3.3.3. *Let E/F be a Galois extension with $\Gamma_E =$ $\mathrm{Gal}(E/F)$. There is a one-to-one correspondence between the set of F-isomorphism classes of E-forms of G, and the 1-cohomology set $H^1(\Gamma_E, \mathrm{Aut}_E(G))$, under which the class of G corresponds to the distinguished class.*

We include a proof of Theorem 3.3.3, as it includes an important construction that we will use later.

Proof. As discussed in Section 2.6, any F-group G has a canonical action $\sigma \mapsto \sigma_G$ of Γ_E on $E[G]$, for any Galois subextension $F \subset E \subset F_s$. As in the diagram, this induces an action of Γ_E on $\mathrm{Aut}_E(G)$ is given by

$$\sigma.a = \sigma_G \circ a \circ \sigma_G^{-1},$$

where $\sigma \in \Gamma_E$ and $a \in \mathrm{Aut}_E(G)$. Moreover, this action is continuous when Γ_E has the Krull topology and $E[G]$ the discrete topology.

Let $\Pi(E/F, G)$ denote the set of F-isomorphism classes of E-forms of G. If G' is an E-form of G, then we can fix an E-isomorphism $\phi_{G'}$ (say) of $E[G']$ onto $E[G]$; we may then define a map $c_{G'} : \Gamma_E \to \mathrm{Aut}_E(G)$ via

$$c_{G'}(\sigma) = \phi_{G'} \circ \sigma_{G'} \circ \phi_{G'}^{-1} \circ \sigma_G^{-1}.$$

We remark that the right hand side of this equation is an automorphism of G (or, equivalently, an automorphism of $E[G]$).

A quick calculation shows that $c_{G'}$ satisfies the 1-cocycle condition, so $c_{G'} \in Z^1(\Gamma_E, \mathrm{Aut}_E(G))$; furthermore the equivalence class of this cocycle is independent of choice of isomorphism $\phi_{G'}$, and so we have a well-defined map

$$\mu : \Pi(E/F, G) \to H^1(\Gamma_E, \mathrm{Aut}_E(G)),$$

and it remains only to show that this map is bijective.

Injectivity is fairly straightforward: if G', G'' are E-forms such that $[c_{G'}] = [c_{G''}]$, then the isomorphism $E[G']$ onto $E[G'']$ must come from an isomorphism $F[G']$ onto $F[G'']$ (similar to the proof of Theorem 3.3.1, above).

In order to show surjectivity, we introduce a new symbol: for $c \in Z^1(\Gamma_E, \mathrm{Aut}_E(G))$, define the operation $\star_c$ via

$$\sigma \star_c f = c(\sigma)(\sigma.f),$$

for $\sigma \in \Gamma_E, f \in E[G]$. We can now define an action of Γ_E on $E[G]$ via

$$(\sigma, f) \mapsto \sigma \star_c f,$$

which is a group action as a consequence of the 1-cocycle condition. The set of elements fixed pointwise by this action is

$$F[G]_c = \{f \in E[G] : \sigma \star_c f = f \text{ for all } \sigma \in \Gamma_E\},$$

which is an F-algebra, and hence defines an E-form G_c of G. It is not hard to check that the cocycle arising from G_c is c itself, and we are done. □

We close with one last consequence of these results.

Corollary 3.3.4. *Let F be a field and let $\mathbf{Spl}(\Psi)$ be a split reductive group over F with root datum Ψ (such groups exist by [Spr10, Theorem 10.1.1]). There is a one-to-one correspondence between the set of F-isomorphism classes of E-forms of $\mathbf{Spl}(\Psi)$, and the 1-cohomology set*

$$H^1(\Gamma_E, \mathrm{Aut}_E(\mathbf{Spl}(\Psi))),$$

under which the class of $\mathbf{Spl}(\Psi)$ *corresponds to the distinguished class in* $H^1(\Gamma_E, \mathrm{Aut}_E(\mathbf{Spl}(\Psi)))$.

More precisely: there is a one-to-one correspondence between the set of F-isomorphism classes of connected reductive F-groups G which split over E, and the set $H^1(\mathrm{Gal}(E/F), \mathrm{Aut}_E(\mathbf{Spl}(\Psi)))$.

3.4 Tori in algebraic groups

In this section we establish a field-independent bound on the number of rational conjugacy classes of maximal tori in a given algebraic group, which we will need in Section 6.4. We retain our notation from the previous sections.

Suppose T is a maximal F-torus of G contained in the Levi subgroup M, and E/F is some Galois extension with $\Gamma_E = \mathrm{Gal}(E/F)$. The restriction of the action of Γ_E on $G(E)$ to $\mathrm{Norm}_M(T)$ (respectively, T) gives an action $\mathrm{Norm}_M(T)(E) \to \mathrm{Norm}_M(T)(E)$ (respectively, $T(E) \to T(E)$), allowing us to define an action on the quotient

$$\Gamma \times \mathrm{Norm}_M(T)(E)/T(E) \to \mathrm{Norm}_M(T)(E)/T(E),$$
$$(\sigma, nT(E)) \mapsto (\sigma.n)T(E).$$

This action is well-defined: if $nT(E) = n'T(E)$, then $n^{-1}n' \in T(E)$ and so

$$(\sigma.n)T(E) = \sigma.(n(n^{-1}n'))T(E) = (\sigma.n')T(E).$$

To lighten notation, let us temporarily write $N(M, T, E)$ for the group

$\mathrm{Norm}_M(T)(E)/T(E)$ equipped with this action of Γ_E; we then have the following short exact sequence of groups, all of which are equipped with a compatible Γ-action:

$$1 \longrightarrow T(E) \xrightarrow{i} \mathrm{Norm}_M(T)(E) \xrightarrow{\pi} N(M,T,E) \longrightarrow 1 \tag{3.4}$$

By Theorem 3.2.2, the short exact sequence (3.4) gives rise to the exact sequence of pointed sets

$$\begin{aligned} 1 \to T(F) \xrightarrow{i^0} \mathrm{Norm}_M(T)(F) \xrightarrow{\pi^0} N(M,T,E)^\Gamma \xrightarrow{\delta^0} H^1(\Gamma, T(E)) \\ \xrightarrow{i^1} H^1(\Gamma, \mathrm{Norm}_M(T)(E)) \xrightarrow{\pi^1} H^1(\Gamma, N(M,T,E)) \end{aligned} \tag{3.5}$$

where $N(M,T,E)^\Gamma$ denotes the set of elements of $N(M,T,E)$ that are fixed pointwise by Γ (note that, *a priori*, we do not know this set to be $N(M,T,F)$; nor, in fact, do we care).

The set $H^1(\Gamma, \mathrm{Norm}_M(T)(E))$ contains the set $\ker(M,T)$ (from Theorem 3.3.1) which parameterises our rational conjugacy classes of tori. This set depends *a priori* on the field in question; however, as we will show in Section 6.4, it is possible to define a set of representatives for $\ker(G,T)$ in a way which is independent of the field of definition of G, as long as its residue characteristic is sufficiently large.

The following result is the application of [Ser97, §5.5, Corollary 2] to our situation:

Proposition 3.4.1. *Let* $[c] \in H^1(\Gamma_E, \mathrm{Norm}_M(T)(E))$. *The elements of* $H^1(\Gamma_E, \mathrm{Norm}_M(T)(E))$ *that are mapped un-*

der π^1 *to* $\pi^1([c])$ *in* $H^1(\Gamma, N(M,T,E))$ *are in one-to-one correspondence with the elements of the orbit space* $N(M,T,E)^\Gamma \backslash H^1(\Gamma_E, T_c(E))$, *i.e. the quotient of* $H^1(\Gamma, T_c(E))$ *by the action of* $N(M,T,E)^\Gamma$.

We remark that we are able to *twist* the torus T by the cocycle $c \in Z^1(\Gamma_E, \mathrm{Norm}_M(T)(E))$ precisely because T is normal in $\mathrm{Norm}_M(T)$.

We can use this proposition do determine a bound on the size of $H^1(\Gamma_E, \mathrm{Norm}_M(T)(E))$; we will make this job easy by making several vast simplifications, to obtain an effective (if needlessly large) bound, which is not dependent on the cohomology sets $H^1(\Gamma_E, T_c(E))$ and $H^1(\Gamma, N(M,T,E))$.

Theorem 3.4.2. *Let* G *be a connected, reductive* F*-group, let* E/F *be a finite Galois extension with* $\Gamma_E = \mathrm{Gal}(E/F)$, *and put* $\Psi = (X, R, X^\vee, R^\vee)$ *for the absolute root datum of the pair* (G,T). *For any Levi subgroup* M *of* G, *one has*

$$|H^1(\Gamma_E, \mathrm{Norm}_M(T)(E))| \leq \Big(|W(R)| \cdot |\,\mathrm{Perm}(R)|\Big)^{|\Gamma_E|}.$$

We use the notation $\mathrm{Perm}(R)$ rather than $\mathrm{Aut}(R)$ to emphasize the fact that we are considering bijective *set maps* $R \to R$.

Proof. First, because each orbit space we consider is bounded by the size of the underlying set $H^1(\Gamma_E, T_c(E))$, we know by Proposition 3.4.1 that

$$|H^1(\Gamma_E, \mathrm{Norm}_M(T)(E))| \leq |H^1(\Gamma_E, N(M,T,E))| \cdot |H^1(\Gamma_E, T_c(E))|.$$

Because both sets Γ_E and $N(M,T,E)$ are finite, we know *a priori* that

$$|H^1(\Gamma_E, N(M,T,E))| \leq |\operatorname{Map}(\Gamma_E, N(M,T,E))| = |N(M,T,E)|^{|\Gamma_E|}$$

where $\operatorname{Map}(X,Y)$ denotes the set of all set maps between the (finite) sets X and Y.

From our remarks at the end of Section 2.2, we know that $|N(M,T,E)|$ is bounded by the size of the algebraic Weyl group $\mathbf{W}_G(F_s)$, which is exactly the Weyl group $W(R)$ of the (absolute) root system $R(G,T)$, because our group is reductive (recall Lemma 2.3.5).

Thus $|H^1(\Gamma_E, N(M,T,E))| \leq |W(R)|^{|\Gamma_E|}$ and it remains to place a bound on $|H^1(\Gamma_E, X^*(T_c))|$.

Because all the tori T_c have the same rank, all of our cohomology sets $H^1(\Gamma_E, X^*(T_c))$ can be written equivalently as $H^1(\Gamma_E^\star, X^*(T))$, as $\Gamma_E^\star$ varies over the actions of Γ_E on the finitely-generated, free abelian group $X^*(T)$.

Of course, not every action of Γ_E on $X^*(T)$ is compatible with the action of Γ_E on G; after all, the action of Γ_E on T (and hence on $X^*(T)$) is inherited from this action on G. In particular: if $Z = Z(G)$ is the centre of G, then we must have $Z \subset T$, as T is maximal.

The universal property of quotients identifies $X^*(T/Z)$ with the subgroup of $X^*(T)$ consisting of characters that are trivial on Z, and we have the short exact sequence of Γ_E-modules

$$1 \to X^*(T/Z) \to X^*(T) \to X^*(Z) \to 1$$

the second map being induced by restriction of characters from T to Z.

Recall the root system $R(G,T)$ from Section 2.3. By definition we have for any $\alpha \in R(G,T)$ that there exists some nonzero $Y \in g(F_s)$ such that

$$\alpha(t)Y = \mathrm{Ad}(t)Y \text{ for all } t \in T(F_s),$$

and so in particular $\alpha(Z(F_s)) = \mathrm{Ad}(Z(F_s)) = 0$ and thus $R(G,T) \subset X^*(T/Z)$.

The action of Γ_E on $X^*(T/Z)$ must map $R(G,T)$ to itself, and because $R(G,T)$ has full rank in $X^*(T/Z)$ we must have that Γ_E acts as a group of automorphisms in $\mathrm{Aut}(R(G,T))$. The action of Γ_E on $X^*(Z)$ is that which is inherited from the natural map $X^*(G) \to X^*(Z)$ induced by restriction of characters, and so is determined.

Thus, there cannot be more cocycles of Γ in $T(E)$ than there are permutations of the root system R, and we reduce the problem to conisdering *all* actions of Γ_E on R, i.e. group homomorphisms $\Gamma_E \to \mathrm{Perm}(R)$. It is clear that there are finitely many of these, and that their number depends only on Γ_E and R, and to be safe we take the crude bound

$$|H^1(\Gamma_E, X^*(T_c))| \leq |\,\mathrm{Map}(\Gamma_E, \mathrm{Perm}(R))| = |\,\mathrm{Perm}(R)|^{|\Gamma_E|},$$

and we are done. $\square$

This bound from Theorem 3.4.2 will arise again, so we will close with

Definition 3.4.3. The H^1**-bound** associated to the pair (Ψ, Γ_E)

consisting of a root datum Ψ and finite group Γ_E is the quantity

$$b(\Psi, \Gamma_E) := \Big(|W(R)| \cdot |\operatorname{Perm}(R)|\Big)^{|\Gamma_E|},$$

where $\operatorname{Perm}(R)$ denotes the group of permutations of the set R.

3.5 A pause for bookkeeping

We will close this chapter by collecting some of the consequences of our choices of objects attached to our reductive group G. More precisely, let us fix:

1. a connected, reductive algebraic group G defined over F.
2. a maximally F-split F-torus S in G.
3. a maximal F-torus T of G which contains S.
4. a minimal parabolic subgroup P_0 of G, defined over F and containing T.

Put $\Gamma = \operatorname{Gal}(F_s/F)$ and $\Gamma_E = \operatorname{Gal}(E/F)$ as usual (for any Galois subextension $F \subset E \subset F_s$). By [Spr10, Theorem 16.4.2], the F-isomorphism class of the triple (G, P_0, T) is determined by the indexed root datum ${}_i\Psi$.

Our foregoing discussion shows that these quantities determine:

i. the finitely-generated free abelian group X of characters of T defined over F_s.

ii. the finitely-generated free abelian group $X^\vee$ of cocharacters of T defined over F_s.

iii. an action of Γ (factoring through its quotient Γ_E) on $G(F_s)$, inducing actions on X and $X^\vee$.

iv. the finitely-generated free abelian group ${}_F X$ of characters of S defined over F_s.

v. the finitely-generated free abelian group ${}_F X^\vee$ of cocharacters of S defined over F_s.

vi. the root system $R \subset X$ of T in G, and the roots ${}_F R \subset {}_F X$ of S in G.

vii. dually, the coroot systems $R^\vee$ and ${}_F R^\vee$.

viii. the set R^+ of positive roots in R for which P_0 is the parabolic subgroup determined by R^+.

ix. the base D of R determined by R^+.

x. the projection

$$\pi : \mathbf{R} \otimes_{\mathbf{Z}} X \to \mathbf{R} \otimes_{\mathbf{Z}} ({}_F X)$$

induced by restriction of characters from T to S.

xi. the subset D_0 of D mapped to 0 under π.

xii. the Weyl group $W = W(G, T)$, together with its action on X.

xiii. the *index* of (G, T), i.e. the action of Γ on D which stabilizes D_0 given by

$$\tau(\gamma)(\alpha) = w_\gamma(\gamma.\alpha),$$

where $\gamma.\alpha$ denotes the usual action of Γ on X and w_γ is the unique element of $W(G,T)$ that maps $\gamma.R^+$ to R^+.

xiv. the sets $\mathscr{L}, \mathscr{F}$, and $\mathscr{P}(M)$ for every $M \in \mathscr{L}$ (recall their definitions from Section 2.2).

xv. the groups $X^*_{\mathrm{rat}}(M)$ of rational characters of M (i.e. those defined over F), for $M \in \mathscr{L}$.

xvi. the real vector spaces $a_M = \mathrm{Hom}(X^*_{\mathrm{rat}}(M), \mathbf{R})$ and their dual spaces $\mathrm{i}a^*_M \subset a^*_M \otimes_{\mathbf{R}} \mathbf{C}$, for $M \in \mathscr{L}$.

xvii. The lattices L_P^G for parabolic subgroups $P_0 \subset P \subset G$.

Chapter 4

Model theoretic constructions

It is our goal to establish that all of the objects in Theorem 2.9.1 may be stated using the language of model theory – specifically, the *Denef-Pas language* – which will show that it is an identity of *motivic distributions*. We now review the basic results about the Denef-Pas language.

Henceforth, when discussing an arbitrary nonarchimedean local field, we will use the letter k instead of F, which remains the field we fixed in Chapter 2; the residue field of k is denoted κ_k, and it is assumed that the valuation on k is $\mathbf{Z}$-valued. We mostly follow the approach of [CGH14b].

4.1 The Denef-Pas language

First of all, we introduce two first-order languages, namely the **Presburger language** and the **language of rings**. Both languages contain, as their symbols, a countably infinite set of variables, as well as logical symbols $\wedge, \neg$, and $\forall$, and parentheses. In practice, when dealing with models with standard interpretations, we will also include symbols $\vee, \exists, \implies, \iff$.

In addition, the Presburger language contains symbols for the constants 0 and 1, the binary function $+$, binary relations $=, \leq$, and, for every integer $d \geq 2$, the binary relation $\equiv_d$. In the model $\mathbf{Z}$ of the Presburger language, the formula $a \equiv_d b$ is valid if and only if $a \equiv b \bmod d$ (the remaining symbols have their usual meanings). The language of rings contains, in addition to the symbols listed in the previous paragraph, symbols for 0 and 1, binary relation $=$, and binary functions $+$ and $\times$. It does *not* include the symbols $\leq$ or $\equiv_d$.

In the Denef-Pas language, there are three sorts of variables, namely, those of the valued field (VF) sort, those of the residue field (RF) sort, and those of the value group sort; statements involving the first two sorts of variables are formulated using the language of rings, and statements involving variables of the value group sort are formulated using the Presburger language. In the intended interpretation (i.e. when using a discretely-valued local field k as a model), VF variables range over k, RF variables over κ_k, and value group variables over $\mathbf{Z}$.

The Denef-Pas language also contains the unary functions $\mathrm{ord} : k^\times \to \mathbf{Z}$ and $\overline{\mathrm{ac}} : k \to \kappa_k$; finally, for convenience, we include as constants of the valued field sort all elements of $\mathbf{Z}[\![t]\!]$ (that is, the field of formal power series with coefficients in $\mathbf{Z}$). The resulting language is the **Denef-Pas** language, and we denote it by $\mathcal{L}_\mathbf{Z}$.

4.2 Basic constructions

Given a formula $\phi \in \mathcal{L}_\mathbf{Z}$ – i.e., a syntactically correct concatenation of the symbols in $\mathcal{L}_\mathbf{Z}$ – we may interpret the formula by allowing the variables to run over the triple $k = (k, \kappa_k, \mathbf{Z})$; this naturally defines a subset of $k^m \times \kappa_k^n \times \mathbf{Z}^r$ (where ϕ contains m variables of the valued field sort, n of the residue field sort, and r of the value group sort) consisting of those elements for which the formula $\phi(k) = \phi(k, \kappa_k, \mathbf{Z})$ is true. For instance, the formula

$$\phi = (\exists y(xy = 1)),$$

where x, y are variables of the valued field sort, gives rise to the subset

$$\phi(k) = \{x \in k : \exists y \in k \text{ such that } xy = 1\} = k^\times,$$

that is, the set of units in k. The formula $h[m, n, r]$ consisting only of m free variables of the valued field sort, n free variables of the residue field sort, and r free variables of the value group sort, naturally gives rise to the set $k^m \times \kappa_k^n \times \mathbf{Z}^r$.

> **Remark.** We recall that variables in first-order formulas can be *free* or *bound*; a variable x occurring in a formula is **bound** if it is quantified over, and is **free** otherwise. That is: if P is any property and the formula takes the form $\forall x(P)$ or $\exists x(P)$, then x is bound in the formula, and is otherwise free.

Note that the variable x is free in the formula $x > 0$, but is bound in the formulas $\exists x(x > 0)$ and $\forall x(x > 0)$; furthermore, the same variable can be both free and bound in the same formula, as in the (syntactically correct, but false in every interpretation) formula

$$(x < 0) \wedge \forall x(x > 1).$$

Let $\mathcal{C}$ be the collection of all pairs (k, ϖ), where k is a nonarchimedean local field (with the canonical ring homomorphism $\mathbf{Z} \to k$), and ϖ is a uniformizer of k; for brevity, we will usually write simply $k \in \mathcal{C}$. Given an integer $M' > 0$, we denote by $\mathcal{C}_{M'}$ the collection of $k \in \mathcal{C}$ such that $\operatorname{char} \kappa_k > M'$.

The basic objects we will work with are the so-called *definable sets*. By a **definable set**, we mean a collection $X = (X_k)_k$ of subsets of some $h[m, n, r]$ (we will write $X \subset h[m, n, r]$), indexed by $k \in \mathcal{C}$, such that there exists a Denef-Pas formula ϕ such that $X_k = \phi(k)$ for every $k \in \mathcal{C}$.

We will use typical set-theoretic notations for definable sets, for instance $X \subset Y$ if and only if $X_k \subset Y_k$ for every k, and we will call a collection of functions $f_k : X_k \to Y_k$ between definable sets $X = (X_k)_k$ and $Y = (Y_k)_k$ a **definable function** if the collection of graphs

$$\{(x, f_k(x)) : x \in X_k\} \subset X_k \times Y_k, \quad k \in \mathcal{C},$$

is itself a definable set. While definable functions seem to be very general, they turn out to be analytic almost everywhere, a fact we will use below in Section 4.4. We cite

Lemma 4.2.1. *[CL08, Theorem 3.2.1] Let $N > 0$ be an integer and let Σ be a k-analytic definable submanifold of dimension d, $0 \leq d \leq N$. Then every definable function on Σ is k-analytic, outside a definable subset $\widetilde{\Sigma}$ of smaller dimension.*

By an **isomorphism** of definable sets, we mean ([CL15, §§4.1, 11.1]) a definable function $f : X \to Y$ which has a definable inverse.

Suppose S is a definable set; by a **family of definable sets** with parameter in S, we mean a definable set $X \subset S \times Y$ for some definable set Y; we will write $X = (X_s)_{s \in S}$ and define

$$X_{k,s} = \{y \in Y_k : (s, y) \in X_k\} \qquad (s \in S_k)$$

to be the **family members** of the family $(X_s)_{s \in S}$. Note that for every $k \in \mathcal{C}$ one has

$$X_k = \coprod_{s \in S_k} \{s\} \times X_{k,s}.$$

We remark that, by symmetry, a family of definable sets $X \subset S \times Y$ may be equally well considered as a family of definable sets with parameter in Y: indeed, if $X \subset S \times Y$ is any definable set, then

$$X_k = \coprod_{s \in S_k} \{s\} \times X_{k,s} = \coprod_{y \in Y_k} X_{k,y} \times \{y\}$$

for any $k \in \mathcal{C}$, where $X_{k,y} = \{s \in S_k : (s, y) \in X_k\}$.

4.3 Motivic functions

The *motivic functions* are a class of complex-valued functions, built from definable functions, which we define in this section. We have

Definition 4.3.1. [CGH14c, Definition 5] Let $X = (X_k)_k$ be a definable set. A collection $f = (f_k)_k$ of functions $f_k : X_k \to \mathbf{C}$ is called a **motivic function** on X if there exist integers N, N', and N'', such that, for all $k \in \mathcal{C}$, one has

$$f_k(x) = \sum_{i=1}^{N} q_k^{\alpha_{ik}(x)} \big(\#(Y_{ik})_x\big) \left(\prod_{j=1}^{N'} \beta_{ijk}(x) \right) \left(\prod_{\ell=1}^{N''} \frac{1}{1 - q_k^{a_{i\ell}}} \right)$$

for every $x \in X_k$, for some nonzero integers $a_{i\ell}$, definable functions $\alpha_i : X \to \mathbf{Z}$ and $\beta_{ij} : X \to \mathbf{Z}$, and definable sets $Y_i \subset X \times h[0, r_i, 0]$, where for $x \in X_k$ we denote by $(Y_{ik})_x$ the (finite) set $\{y \in \kappa_k^{r_i} : (x, y) \in Y_{ik}\}$.

Parallel to our above definition, we will define (for a definable set S) the notion of a **family of motivic functions** with parameter in S: this will be a motivic function on the family of definable sets $X \subset S \times Y$. The family members are functions

$$f_{k,s} : X_{k,s} \to \mathbf{C}.$$

In fact, because our tools will only allow us to prove results in sufficiently large residue characteristic, we will want to study functions which coincide with motivic functions in large residue characteristic.

For this reason, we will abuse terminology by calling a collection of functions $f_k : X_k \to \mathbf{C}$ *motivic* if there exists an integer $M' > 0$ and a motivic function $g = (g_k)_k$ such that $f_k = g_k$ for all $k \in \mathcal{C}_{M'}$ (and similarly for the terms *definable*, *isometry*, etc.)

4.4 Motivic integration

One thing we can do with motivic functions is integrate them; for instance, if f is a compactly-supported motivic function on a definable set X, then by picking a measure $\mathrm{d}\mu_k$ on each X_k we should obtain, for every field k of sufficiently large residue characteristic, a complex number $z_k = \int f_k \, \mathrm{d}\mu_k$. In order to do this in a uniform manner (rather than simply for each fixed k), we will need a uniformly given family of measures on each definable set $X = (X_k)_k$.

If X is an affine space, this is easily done: for $X = h[m, 0, 0]$, we use Haar measure on $X_k = k^m$ normalized so that the measure of $\mathcal{O}_F^m$ is 1 (where $\mathcal{O}_F$ denotes the ring of integers of F); for $X = h[0, 0, r]$ (and indeed on all definable subsets of $h[0, 0, r]$), we use the counting measure, as well as for subsets of $h[0, n, 0]$.

For arbitrary X, we follow the approach of [CL15, §9] and define a *motivic measure* $|\omega|$ on X to be a family of measures $(\omega_k)_k$ on each X_k that arises from a *definable volume form* on X. The construction is fairly technical, and we go over it in detail in Section 4.6; here, we will limit ourselves to a brief overview.

Cluckers and Loeser (see, for instance, [CL10]) have developed a theory of symbolic integration that specializes, in large positive characteristic, to the usual integral on each field. Integration with respect to a given definable volume form ω can also be constructed (as in [CL08]) in the same symbolic framework, and specializes to the corresponding integral with respect to $|\omega|_k$, for each valued field k.

Namely: if $\Sigma \hookrightarrow k^N$ is an embedding of a k-analytic, definable, orientable manifold of dimension d, and $f : k^N \to k$ any definable function, one obtains a definable differential d-form

$$\omega = f\,\mathrm{d}x_{i_1} \wedge \mathrm{d}x_{i_2} \wedge \cdots \wedge \mathrm{d}x_{i_d}, \quad 1 \le i_1 < i_2 < \cdots < i_d \le N.$$

Such a differential form ω induces a measure $|\omega|$ on Σ. By a **motivic measure** on Σ we will mean any measure which arises in this way. That is:

Definition 4.4.1. Let $X = (X_k)_k$ be a definable set. A **motivic measure** $\mu = (\mu_k)_k$ on X is a family of measures μ_k on X_k such that there exists some integer $M' > 0$ for which $k \in \mathcal{C}_{M'}$ implies that $\mu_k = |\omega_k|$, for some definable differential form ω.

4.5 Motivic distributions

Now that the notion of a motivic measure is established, we may state the following assertion about integration with respect to such measures:

Theorem 4.5.1. *[ST16, Theorem B.4] Let f be a motivic function on $X \times Y$, where X, Y are definable sets and Y is equipped with a motivic measure μ. Then there exists a motivic function g on X and an integer $M' > 0$ such that, for each $k \in \mathcal{C}_{M'}$ and each $x \in X_k$, one has*

$$g_k(x) = \int_{y \in Y_k} f_k(x, y)\,\mathrm{d}\mu_k(y),$$

whenever the function $y \mapsto f_k(x, y)$ on Y_k lies in $L^1(Y_k, \mu_k)$.

We remark that, given a motivic function f on a definable set $X \times Y$, the set of x for which $x \mapsto f_k(x, y)$ lies in $L^1(Y_k, \mu_k)$ need not be a definable set, but is necessarily the zero locus of a motivic function on X (by [CGH14a, §4]).

Very roughly, Theorem 4.5.1 states that *the class of motivic functions is closed under integration.* This will turn out to be very useful.

Definition 4.5.2. [CGH14b, Section 4.2] Let $X \subset h[m, 0, 0]$ be a definable set. A **Schwartz-Bruhat** function on X is a finite linear combination of characteristic functions of compact open balls in X, with $\mathbf{C}$ coefficients (such functions clearly specialize to locally constant, compactly-supported functions on each X_k). A **(complex) motivic distribution** on X is a collection of linear functionals

$$\Phi_k : C_c^\infty(X_k) \to \mathbf{C}$$

such that there exists an integer $M' > 0$ for which $k \in \mathcal{C}_{M'}$ implies that, for every definable set S and every family $(f_s)_{s \in S}$ of Schwartz-Bruhat motivic functions on X with parameter in S, there exists a motivic function $g : S \to \mathbf{C}$ such that

$$\Phi_k(f_{k,s}) = g_k(s) \text{ for all } s \in S_k.$$

When we speak of motivic functions in the context of motivic distributions, we will always assume they are Schwartz-Bruhat class. We close by extending this definition to families.

Definition 4.5.3. Let X and S be definable sets. A **family of mo-**

tivic distributions on X with parameter in S is a collection of linear functionals

$$\Phi_{k,s} : C_c^\infty(X_k) \to \mathbf{C}$$

such that there exists an integer $M' > 0$ for which $k \in \mathcal{C}_{M'}$ implies that, for every definable set S' and every family $(f_{s'})_{s' \in S'}$ of Schwartz-Bruhat motivic functions on X with parameter in S', there exists a motivic function $g : S \times S' \to \mathbf{C}$ such that

$$\Phi_{k,s}(f_{k,s'}) = g_k(s, s') \text{ for all } (s, s') \in S_k \times S'_k.$$

4.6 Integrating on definable sets

In this ancillary section, we explain in some detail the way in which one integrates *motivically* over definable sets, as defined above; we will mostly follow the approach of [CHL11] and [CGH14b, §3.5.1].

By an **isometry** of definable sets $f : X \to X'$, we mean a definable function such that

$$\overline{\mathrm{ord}}(x - x') = \overline{\mathrm{ord}}\big(f_k(x) - f_k(x')\big)$$

for every $x, x' \in X_k$ and $k \in \mathcal{C}$, where $\overline{\mathrm{ord}}$ is a suitable extension of the ord function defined as follows:

Extend the natural order on $\mathbf{Z}$ to the set $\mathbf{Z} \cup \{\pm\infty\}$, and define $\overline{\mathrm{ord}}$ on $h[1, 0, 0]$ to be the extension of ord by $\overline{\mathrm{ord}}(0) = +\infty$; for any $n, r > 0$, define

$\overline{\mathrm{ord}}$ on $h[0,n,r]$ via

$$\overline{\mathrm{ord}}(\xi,\alpha)=\begin{cases}+\infty & \text{if } \xi=\alpha=0,\\ -\infty & \text{otherwise,}\end{cases}$$

where $\xi \in h[0,n,0], \alpha \in h[0,0,r]$.

Finally, define $\overline{\mathrm{ord}}$ on $h[m,n,r]$ to be the infimum of the natural restriction of $\overline{\mathrm{ord}}$ to the component factors $h[m,0,0] = h[1,0,0]^m$ and $h[0,n,r]$. Clearly $\overline{\mathrm{ord}}(x) = +\infty$ if and only if $x = 0$, and $\overline{\mathrm{ord}}(x) = -\infty$ unless $x \in h[m,0,0]$.

Let us now introduce two special kinds of definable sets, which we collectively call *cells*.

Definition 4.6.1. By a 0-**cell** in $h[m+1,n,r]$ we mean a definable subset Z_S^0 in which the last variable of the valued field sort is the value of a definable function of the remaining variables, themselves elements of the definable set S. That is, if (x,z) are co-ordinates on $h[m+1,n,r]$, with $x \in h[m,n,r]$ and $z \in h[1,0,0]$, then

$$Z_S^0 = \{(x,z) \in h[m+1,n,r] : x \in S \text{ and } z = c(x)\}$$

for some definable function $c : S \to h[1,0,0]$. Retaining these co-ordinates, we will define a **1-cell** to be a definable set Z_S^1 of the form

$$\begin{aligned} Z_S^1 = \big\{(x,z) \in h[m+1,n,r] : x \in S \\ \text{and } \overline{\mathrm{ac}}(z - c(x)) = \xi(x) \\ \text{and } \operatorname{ord}(z - c(x)) = \alpha(x)\big\}, \end{aligned}$$

where

$$c : S \to h[1,0,0], \quad \xi : S \to h[0,1,0] \setminus \{0\}, \quad \alpha : S \to h[0,0,1]$$

are all definable functions.

Note that our restriction that ξ take nonzero values means that 1-cells are *not* 0-cells. Roughly speaking, 0-cells have one variable *equal to* the value of a definable function of the remaining variables, while 1-cells have (the residue class and valuation of) one variable *controlled by* values of definable functions of the remaining variables. We will see the importance of cells in Theorem 4.6.2.

Before that, we provide the promised description of definable differential forms; we review the construction from [CGH14b, §3.5.1].

Consider first the affine space k^N with co-ordinates $x_1, \dots, x_N$. By a **definable differential d-form** ω on k^N we mean a finite linear combination of summands of the form

$$f \, \mathrm{d}x_{i_1} \wedge \mathrm{d}x_{i_2} \wedge \cdots \wedge \mathrm{d}x_{i_d}, \quad 1 \le i_1 < i_2 < \cdots < i_d \le N,$$

where each f is a definable function $f : k^N \to k$. By Lemma 4.2.1, we have that if $\Sigma \subseteq k^N$ is a k-analytic, definable, orientable submanifold of dimension d, such a d-form ω induces a measure on Σ, which we denote by $|\omega|$. We have the following

Theorem 4.6.2. *[CHL11, Theorem 2.2.1] Let $S \subset h[m, n, r]$ be a definable set with $m > 0$. Then there exist nonnegative integers n', r' and an embedding (i.e. an isometric isomorphism onto the image)*

$$\lambda : h[m, n, r] \to h[m, n + n', r + r']$$

such that, if

$$\pi : h[m, n + n', r + r'] \to h[m, n, r]$$

is projection onto the first m, n, and r factors, respectively, then $\pi \circ \lambda$ is the identity on S, and $\lambda(S)$ is a finite disjoint union of cells.

Thus let us fix a definable differential d-form ω on k^N. Theorem 4.6.2 implies that for any definable set $\Sigma \subseteq k^N$ of dimension d, there exist integers s, t and a definable set $\Sigma' \subset \Sigma \times \kappa_k^s \times \mathbf{Z}^t$ such that there exists a definable bijection $j : \Sigma' \to \Sigma$; moreover, the restriction of j to the first factor coincides with the projection map onto Σ, and, for every $(\xi, \alpha) \in \kappa_k^s \times \mathbf{Z}^t$, the fibre

$$\Sigma'_{\xi,\alpha} = \{x \in \Sigma : (x, \xi, \alpha) \in \Sigma'\} \subseteq \Sigma$$

is (when nonempty) a k-analytic, definable, orientable manifold of dimension at most d. In turn, $\Sigma'_{\xi,\alpha}$ admits a definable, isometric isomorphism $\iota : \Sigma'_{\xi,\alpha} \to \Omega_{\xi,\alpha} \subseteq k^d$ which is induced by one of the co-ordinate projections $k^N \to k^d$. The restriction of ω to $\Sigma'_{\xi,\alpha}$ pulls back under ι^{-1} to a differential d-form $\omega_{\xi,\alpha}$ on $\Omega_{\xi,\alpha}$; because $\omega_{\xi,\alpha}$ is a top-degree form on $\Omega_{\xi,\alpha}$, we may write

$$\omega_{\xi,\alpha} = f \, \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_d$$

for some definable function $f : \Omega_{\xi,\alpha} \to k$, and we define the measure $|\omega|$ on Ω_ξ on every open subset U via

$$\int_U \mathrm{d}|\omega| = \int_{\Omega_{\xi,\alpha}} |f| \, \mathrm{d}x_1 \cdots \mathrm{d}x_d,$$

allowing us to integrate motivically.

Chapter 5

Early motivic constructions

In this chapter we explain how various objects (field extensions, reductive groups, etc.) are constructed in the Denef-Pas language. We mostly follow [ST16, §B.4.2-3] and [CHL11, §4.1-2]. Most of our work in this chapter will culminate in showing that connected reductive groups arise as a family of definable sets with parameter in a known definable set.

5.1 Notation

In this chapter (and subsequently) we will combine the notations of our previous chapters. It is therefore appropriate that we take a moment to elucidate precisely how we are going to do this.

Ultimately we are going to show that connected, reductive groups are definable by proving something like the following statement:

> **Claim.** Let Ψ be a root datum, and let G be a connected, reductive group with root datum Ψ and defined over a field of large residue characteristic. Then there exists a Denef-Pas formula specializing at F to G.

As F varies over all possible fields and Ψ over all root data, we will obtain our desired claim.

Therefore, we will continue to use the symbol F to denote the local field we fixed in Chapter 2.3, and E to denote certain separable extensions $F \subset E \subset F_s$ of that field. As in Chapter 4 we will use k for the arbitrary local field at which we may specialize a Denef-Pas formula, and we will fix once and for all a separable closure k^s of each field.

5.2 Fixed choices

After [GH16, §2], we mean by a **fixed choice** a fixed set which does not depend on the choice of local field in any way. These will be (field-independent) objects such as real vector spaces, finite groups, and root systems, and will be used to construct certain objects using the Denef-Pas language.

A word on terminology: we may use the term *determined by fixed choices* in two related, but subtly distinct ways, which we illustrate by an example: let X be a finite set and let n be a fixed positive integer, both of which we will consider to be *fixed choices*. Then the set of all set maps $X \to X^n$ is determined by these fixed choices, and can itself be taken as a fixed choice (as it has no dependence on any underlying field). By contrast, the set

$$Y_k = \{x \in k : x^n - 1 = 0\}$$

clearly is *not* a fixed choice – the nth roots of unity in a field obviously depend on the field in question; however, it *is* a definable set which is *determined* by the fixed choices. We will try to be as explicit as possible in distinguishing between these two usages.

The results in our previous chapter demonstrate that many of the objects with which we concern ourselves are determined by combinatorial data that can be expressed abstractly. For instance, a split reductive group over a given field is determined up to isomorphism by its absolute root datum, consisting of a finitely-generated free abelian group, a root system within it, and their duals. The actual field of definition has no bearing on this fact.

In the next few sections, the objects we will take as *fixed choices* will be the finite group Γ_{spl} (this will play the rôle of the Galois group of a splitting field of a maximal torus), and an absolute root datum $\Psi = (X, R, X^{\vee}, R^{\vee})$.

5.3 Field extensions

Let E/F be a Galois extension of our favourite field F with Galois group $\mathrm{Gal}(E/F) \cong \Gamma_{\mathrm{spl}}$; ideally, we would like to show that there exists a Denef-Pas statement which specializes at F to give the points of the field extension E/F. To that end, we will describe what will go into a Denef-Pas formula ϕ which, when true, defines at k a field extension $k_{[E:F]}/k$ of degree $[E : F]$ and Galois group isomorphic to Γ_{spl}.

Say $[E : F] = r$ and let k be an arbitrary field. Every Galois extension of k of degree r can be realized as the splitting field of a monic, degree-r polynomial with coefficients in k, although we will realize many other rings in this way. More precisely: for $\underline{a} = (a_{r-1}, a_{r-2}, \ldots, a_0) \in k^r$, define

$$P_{\underline{a}}(X) = X^r + a_{r-1}X^{r-1} + \cdots + a_1 X + a_0 \in k[X].$$

We may then define the quotient ring

$$k_{\underline{a}} = k[x]/(P_{\underline{a}}(X)), \tag{5.1}$$

which may or may not be a field. In order to restrict ourselves to the Galois extensions with Galois group isomorphic to Γ_{spl}, we add certain (definable) conditions on the r-tuples $\underline{a}$.

Specifically: the extension $k_{\underline{a}}/k$ will have the desired properties exactly when the following conditions are satisfied:

(1) The polynomial $P_{\underline{a}}(X)$ is irreducible over k.

(2) The polynomial $P_{\underline{a}}(X)$ is separable over k

(3) The extension $k_{\underline{a}}/k$ is normal.

(4) There is an isomorphism $\Gamma_{\text{spl}} \cong \text{Gal}(k_{\underline{a}}/k)$.

We see explicitly how these conditions can be defined in the Denef-Pas language:

(1) is defined by the concatenation of at most r statements of the form

$$\neg\big(\exists b_0, \ldots, b_{d-1}, c_0, \ldots, c_{m-d-1}$$
$$(X^d + b_{d-1}X^{d-1} + \cdots + b_0)(X^{m-d} + c_{m-d-1}X^{m-d-1} + \cdots + c_0) = P_{\underline{a}}(X)\big),$$

each of which is definable, as equality of polynomials is equivalent to equality of respective coefficients.

(2) is equivalent to the statement that $\gcd(P_{\underline{a}}(X), P'_{\underline{a}}(X)) = 1$; so, we can encode it in the Denef-Pas language similar to (1) simply by saying that $P_{\underline{a}}(X)$ and $P'_{\underline{a}}(X)$ have no common monic divisors in $k[X]$ of degree $1, \ldots, r-1$.

(3) is encoded in steps.

First, we observe that if the extension ℓ/k is known to be finite and separable, then the statement that ℓ/k is normal is equivalent to the statement that ℓ is the splitting field of a polynomial in $k[X]$. Thus, it is enough for us to include in ϕ the statement that $P_{\underline{a}}(X)$ splits over $k_{\underline{a}}$; having assumed $P_{\underline{a}}(X)$ to be irreducible, its splitting field must have degree at least r, and the statement that $P_{\underline{a}}$ splits over $k_{\underline{a}}$ reduces the problem to showing that $k_{\underline{a}}$ is definable.

Observe that the relation $X^r = -a_{r-1}X^{r-1} - \cdots - a_1 X - a_0$ on $k[X]$ allows us to construct a k-algebra structure on the vector space k^r; specifically, it is isomorphic to the k-subalgebra of $\mathrm{Mat}_{r\times r}(k)$ spanned by the r matrices $1, A, A^2, \ldots, A^{r-1}$, where

$$A := \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & -a_0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & -a_1 \\ 0 & 1 & 0 & \cdots & 0 & 0 & -a_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & -a_{r-3} \\ 0 & 0 & 0 & \cdots & 1 & 0 & -a_{r-2} \\ 0 & 0 & 0 & \cdots & 0 & 1 & -a_{r-1} \end{pmatrix}.$$

Thus, the ring $k_{\underline{a}}$ can be encoded as a definable subset of k^{r^2} in the Denef-Pas language, with k included as the subalgebra spanned by 1.

In particular, it is now possible by this identification to evaluate the polynomial $P_{\underline{a}}(X)$ at $k_{\underline{a}}$ points; the condition that $k_{\underline{a}}/k$ is a normal extension is now encoded in the Denef-Pas language as the statement (additional to (1) and (2)) that there exist r roots of $P_{\underline{a}}(X)$ in $k_{\underline{a}}$.

(4) must be encoded using the language of *rings*, which does not allow us *a priori* to posit the existence of a group isomorphism. However, by explicitly appealing to its multiplication table we can write down in Denef-Pas the condition that there exist matrices $\sigma_1, \ldots, \sigma_r \in \mathrm{Mat}_{r\times r}(k)$ and a bijective correspondence

$$\psi : \Gamma_{\mathrm{spl}} \xrightarrow{\sim} \{\sigma_1, \ldots, \sigma_r\}$$

that respects the multiplication table.

More precisely, because $\Gamma_{\mathrm{spl}} = \{\gamma_1, \ldots, \gamma_r\}$ is finite, we will require only finitely many statements of the form $\psi(\gamma_i)\psi(\gamma_j) = \psi(\gamma_\ell)$, as $\gamma_i\gamma_j = \gamma_\ell$ varies over all possible products $\gamma_i\gamma_j = \gamma_\ell$ in Γ_{spl}. In particular, this condition is definable.

Summarizing: per [ST16, §B.4.2], there exists a Denef-Pas formula $S_{\Gamma_{\mathrm{spl}}} \subseteq k^{r+r^3}$ defining the set of all tuples $(\underline{a}, \sigma_1, \ldots, \sigma_r)$ such that $k_{\underline{a}}/k$ is a Galois extension with Galois group isomorphic to Γ_{spl} and there is a group isomorphism $\Gamma_{\mathrm{spl}} \xrightarrow{\sim} \{\sigma_1, \ldots, \sigma_r\}$.

We may now quantify over all Galois extensions having a given Galois group in a uniform way. Notice, however, that we will have much redundancy in this construction, as for instance the choice of σ_i is not unique.

Finally, suppose V is some variety defined over $\mathbf{Q}$; then in particular, there exists some integer $n_V > 0$ such that $V(\mathbf{Q})$ is the zero locus of finitely many polynomials $f_1, \dots, f_\ell$ in d variables with coefficients in $\mathbf{Z}[1/n_V]$.

For any local field k with characteristic $> n_V$ (or zero), the images of the integers $1, \dots, n_V$ under the canonical ring homomorphism $\mathbf{Z} \to k$ are invertible, and therefore have the well-defined map of polynomial rings

$$(\mathbf{Z}[1/n_V])[X_1, \dots, X_d] \to \mathcal{O}_k[X_1, \dots, X_d]$$

(where $\mathcal{O}_k \subset k$ is the ring of integers), which maps the set $\{f_1, \dots, f_\ell\}$ to the set $\{\bar{f}_1, \dots, \bar{f}_\ell\}$. In such a way, we have defined a variety over any k with residue characteristic $> n_V$.

Definition 5.3.1. Let V be a variety defined over $\mathbf{Q}$, and let $n_V > 0$ be such that the defining equations $f_1, \dots, f_\ell \in \mathbf{Q}[X_1, \dots, X_d]$ of V are defined over $\mathbf{Z}[1/n_V]$. For any $k \in \mathcal{C}_{n_V}$, define the k**-points** of V to be the set $V(k)$ of k-points of the variety defined by the equations $\bar{f}_1, \dots, \bar{f}_\ell$, where $\bar{f} \in k[X_1, \dots, X_d]$ is the image of $f \in \mathbf{Q}[X_1, \dots, X_d]$ under the canonical homomorphism

$$(\mathbf{Z}[1/n_V])[X_1, \dots, X_d] \to k[X_1, \dots, X_d].$$

The same construction allows us to define, for any Galois extension k'/k with $k \in \mathcal{C}_{n_V}$, the k'-points of V.

5.4 Split reductive groups

Let k be a field. Recall that split connected reductive groups over k are determined by their root datum; that is, there is a one-to-one correspondence between the set of k-isomorphism classes of split connected reductive groups G over k, and the set $\mathscr{D}$ of 4-tuples $\Psi = (X, R, X^\vee, R^\vee)$ consisting of finitely generated free abelian groups $X, X^\vee$ in perfect pairing $\langle , \rangle : X \times X^\vee \to \mathbf{Z}$, and root systems $R \subset X, R^\vee \subset X^\vee$ in bijection by $\langle , \rangle$ [Spr10, Theorem 9.6.2].

Observe that the objects in $\mathscr{D}$ can be described solely in terms of $\mathbf{Z}$. However, we caution that we will interpret $\Psi \in \mathscr{D}$ using the language of rings, and *not* the Presburger language (used for variables of the value group sort).

More precisely: for every $\Psi \in \mathscr{D}$, let us fix a split connected reductive group $\mathbf{Spl}(\Psi)$ over $\mathbf{Q}$ whose root datum is Ψ (again, we know such groups exist by [Spr10, Theorem 10.1.1]). There exists an algebraic embedding $\varphi_\Psi : \mathbf{Spl}(\Psi)(\mathbf{Q}) \hookrightarrow \mathrm{GL}(n_\Psi, \mathbf{Q})$ for some n_Ψ, and therefore there exists some $N_\Psi \in \mathbf{Z}$ such that φ_Ψ is defined by an equation with coefficients in $\mathbf{Z}[1/N_\Psi]$.

This implies that φ_Ψ defines an algebraic embedding $\mathbf{Spl}(\Psi) \hookrightarrow \mathrm{GL}(n_\Psi)$ for every field with sufficiently large residue characteristic (which depends only on Ψ). That is:

Lemma 5.4.1. *Let Ψ be a root datum. If G is any split reductive group with root datum Ψ, then there exist a definable set $\mathcal{G}$ and an integer $M' > 0$ such that $k \in \mathcal{C}_{M'}$ implies that $\mathcal{G}_k = G(k)$. Moreover, the constant M' depends only on the* *root datum* Ψ.

While we are at it, we will fix an algebraic embedding $\mathbf{G}_m(\mathbf{Q})^r$ into $\mathbf{Spl}(\Psi)(\mathbf{Q})$, where r is the rank of X. This will give us a distinguished maximal split torus $\mathbf{T}$ in $\mathbf{Spl}(\Psi)(\mathbf{Q})$, which is the Levi component of a minimal parabolic subgroup P_0 of $\mathbf{Spl}(\Psi)$ (in this case, a *Borel* subgroup). Thus, we can write $\mathbf{T} = M_0$, and consider the set $\mathscr{L} = \mathscr{L}(M_0)$ (from Section 2.2). The images of the Levi subgroups $M \in \mathscr{L}$ under our embedding φ_Ψ give us a "standard" set of embeddings into $\mathbf{Spl}(\Psi)$ (recall the definition of $\mathscr{L}$ from Section 2.2).

5.5 Connected reductive groups

We will obtain general connected reductive groups by exploiting the correspondence (established in Theorem 3.3.3) between the set of k-isomorphism classes of k^s-forms of connected reductive k-groups, and the elements of the 1-cohomology set $H^1(\Gamma_k, \mathrm{Aut}_{k^s}(G))$, where $\Gamma_k = \mathrm{Gal}(k^s/k)$ and $\mathrm{Aut}_{k^s}(G)$ is the group of k^s-automorphisms of the algebra $k^s[G]$ (identified with the k^s-automorphsims of the k^s-variety G). That is, we will "twist" the group $G(k)$ to obtain all of its k^s-forms.

We recall the construction from the proof of Theorem 3.3.3: having defined G, there is a natural action of Γ_k on $k^s[G]$ (hence on $G(k^s)$) through its action on the coordinates; if A^{Γ_k} denotes the points of the Γ_k-module A that are fixed by every element, then we have $G(k) = G(k^s)^{\Gamma_k}$. We *twist* this action by the cohomology class $[c]$ for $c \in Z^1(\Gamma_k, G(k^s))$ by defining a new action of Γ_k on $k^s[G]$, viz.

$$\gamma \star_c f := c(\gamma)(\gamma.f), \qquad \gamma \in \Gamma_k, f \in k^s[G],$$

where $\gamma.f$ is the canonical coordinate action; we put

$$k[G]_c = \{f \in k^s[G] : \gamma \star_c f = f \text{ for all } \gamma \in \Gamma_k\},$$

which is the function field of the k^s-form G_c of G. As remarked above (e.g. in the proof of Theorem 3.3.3), a different choice of representative c' of the class $[c]$ will give a k-isomorphic group.

In particular, our above remarks imply that if G is any connected reductive group with root datum Ψ, then there exists $c \in Z^1(\Gamma_k, \mathrm{Aut}_{k^s}(\mathbf{Spl}(\Psi)))$ such that $G = \mathbf{Spl}(\Psi)_c$. However, because we cannot encode the infinite extension k^s/k in the Denef-Pas language, we will instead make use of Lemma 3.3.4.

Definition 5.5.1. Let Ψ be a root datum and let Γ_{spl} be a finite group. The **definable set of 1-cocycles** of Γ_{spl} in $\mathrm{Aut}(\mathbf{Spl}(\Psi))$ over $k_{\underline{a}}$ is the set $Z_{\Psi,\Gamma_{\mathrm{spl}},\underline{a}}$ defined

$$Z_{\Psi,\Gamma_{\mathrm{spl}},\underline{a},k} = Z^1(\Gamma_{\mathrm{spl}}, \mathrm{Aut}_{k_{\underline{a}}}(\mathbf{Spl}(\Psi)(k_{\underline{a}}))) \tag{5.2}$$

for $k \in \mathcal{C}$, where $k_{\underline{a}}/k$ is an extension as in Equation (5.1) satisfying $\mathrm{Gal}(k_{\underline{a}}/k) \cong \Gamma_{\mathrm{spl}}$.

We remark that $Z_{\Psi,\Gamma_{\mathrm{spl}},\underline{a}}$ is indeed definable: we know $k_{\underline{a}}/k$ is definable from Section 5.3, and that $\mathbf{Spl}(\Psi)(k_{\underline{a}})$ is definable for sufficiently large residue characteristic (by Lemma 5.4.1, and recalling Definition 5.3.1). The automorphisms of this group defined over $k_{\underline{a}}$ are exactly those defined by $k_{\underline{a}}$-polynomial equations, and so are definable; and finally, the 1-cocycle condition is clearly definable (given our fixed choice Γ_{spl}).

Thus, rather than construct all connected, reductive algebraic groups as definable sets, we will instead fix the Galois group Γ_{spl} (say) of a finite Galois extension of nonarchimedean local fields, and show to be definable *all connected reductive algebraic k-groups which split over an extension $k_{\underline{a}}/k$ with* $\mathrm{Gal}(k_{\underline{a}}/k) \cong \Gamma_{\mathrm{spl}}$. As E and F vary over all finite Galois extensions of all such local fields of sufficiently large residue characteristic, this construction will give *all* connected reductive algebraic groups over such fields.

To avoid vacuous constructions, let us assume that Γ_{spl} is the Galois group of some finite, tamely ramified extension of local fields (in particular, Γ_{spl} must be solvable, though we will not use this fact).

Lemma 5.5.2. *Let G be a connected, reductive group over k, let Γ_{spl} be the Galois group of the splitting field $k_{\underline{a}}/k$ of a maximally split maximal k-torus of G, and for $c \in Z^1(\Gamma_{\mathrm{spl}}, \mathrm{Aut}_{k_{\underline{a}}}(G))$, let G_c be the group obtained by twisting G by c. The set*

$$\{G_c : c \in Z^1(\Gamma_{\mathrm{spl}}, \mathrm{Aut}_{k_{\underline{a}}}(G))\},$$

as $\underline{a}$ varies over all suitable coefficients in $k^{|\Gamma_{\mathrm{spl}}|}$ (as in Section 5.3), exhausts the set of k-groups with the same absolute *root datum as G.*

Proof. The lemma is an easy application of Corollary 3.3.4: the $k_{\underline{a}}$-forms of G are parameterised by the elements of $H^1(\Gamma_{\mathrm{spl}}, \mathrm{Aut}_{k_{\underline{a}}}(\mathbf{Spl}(\Psi)))$, with classes $[c]$ and $[c']$ giving rise to a k-isomorphic twisted group $G_c \cong G_{c'}$ if and only if $[c] = [c']$. $\square$

It is clear also that all Levi subgroups of a given connected, reductive group G are also twisted forms of Levi subgroups of the split form of G.

As such: to show that connected, reductive groups are definable in the Denef-Pas language, it suffices to show that $\mathrm{Aut}_{k_{\underline{a}}}(G)$ is a definable set; but this is clear, as an algebraic automorphism of G is defined by invertible polynomial equations. Our result now follows immediately, and we have that the set $Z_{\Psi,\Gamma_{\mathrm{spl}},\underline{a}}$ gives a parameterising set for the set of connected reductive groups over k which split over the extension $k_{\underline{a}}/k$ with $\mathrm{Gal}(k_{\underline{a}}/k) \cong \Gamma_{\mathrm{spl}}$ and with absolute root datum Ψ. Moreover, this set is definable whenever the residue characteristic of k is large enough (recall Lemma 5.4.1). As with field extensions, the parameterisation is not unique, and there will be many k-isomorphic groups arising from our construction.

We emphasize the independence of these results from the fields k to which we specialize; as in Lemma 5.4.1 the dependence is not on the field k, but the root datum Ψ. Finally, we close by quoting an earlier result, which also includes a statement about the Lie algebras of the groups we have constructed:

Lemma 5.5.3. *[CGH18, Proposition 4.1.1] For every root da-*

tum Ψ there exists a definable set Z_Ψ and families of definable sets $(\mathbf{G}_z)_{z\in Z_\Psi}$ and $(\mathbf{g}_z)_{z\in Z_\Psi}$ with parameter in Z_Ψ such that, for all k with sufficiently large residue characteristic, the set $\mathbf{G}_{z,k}$ is the set of k-points of a connected reductive group $\mathbf{G}_z$ with absolute root datum Ψ, and the set $\mathbf{g}_{z,k}$ is the set of k-points of the Lie algebra of the group $\mathbf{G}_z$. Moreover, all k-isomorphism classes of connected reductive groups with root datum Ψ arise in this way.

The set Z_Ψ in question is essentially constructed in the same way as our set $Z_{\Psi,\Gamma_{\mathrm{spl}},\underline{a}}$ from Equation (5.2). Thus: the same set Z_Ψ that parameterizes connected reductive groups also parameterizes their Lie algebras.

Chapter 6

Later motivic constructions

In this chapter we collect several new results on the definability or motivicity of various objects attached to the trace formula, culminating in a proof of our main Theorem 6.7.1.

6.1 Trivializing extensions

We begin with a short section describing a very large field extension k_Ψ/k, defined for any local field k, attached to a given root datum Ψ. It is constructed in such a way that all maximal k-tori in any connected reductive k-group G with absolute root datum Ψ split over k_Ψ, *and* such that $[k_\Psi : k] < \infty$ when the residue characteristic of k is sufficiently large.

We start by quoting the following classical result, as formulated by [KST20, Lemma 3.4]:

> **Theorem 6.1.1.** *Let Ψ be a root datum. There exist integers M' and $d_\Psi > 0$ such that, for every field $k \in \mathcal{C}_{M'}$ and every connected reductive group G defined over k with absolute root datum Ψ, the group G splits over an extension of k of degree at most d_Ψ. In fact, every maximal k-torus of G splits over an*

extension of degree at most d_Ψ.

Motivated by this theorem, we make the following

Definition 6.1.2. Let Ψ be a root datum, let d_Ψ be the constant from Theorem 6.1.1, and for any field k put

$$k_\Psi = \prod_{[k':k]\le d_\Psi} k' \tag{6.1}$$

for the compositum of all Galois subextensions $k \subset k' \subset k^s$ of degree $\le d_\Psi$.

We remark that, if k_1 and k_2 are two finite Galois subextensions of $k \subset k^s$, then their compositum k_1k_2 is definable: it is the degree-$\frac{[k_1:k][k_2:k]}{[(k_1\cap k_2):k]}$ extension of k that contains both k_1 and k_2 (of course, it is also possible to define it explicitly by means of polynomials, as in Section 5.3).

The motivation for Definition 6.1.2 is the following observation: because every k-torus of any reductive k-group G with root datum Ψ must split over k_Ψ, the action of $\mathrm{Gal}(k^s/k)$ on $G(k^s)$ must factor through $\mathrm{Gal}(k_\Psi/k)$, and therefore $\mathrm{Gal}(k^s/k_\Psi)$ acts trivially on the character lattice of any maximal torus of G. In this situation (i.e. when k^{triv}/k is a Galois extension for which $\mathrm{Gal}(k^s/k^{\mathrm{triv}})$ acts trivially on all character lattices of all maximal tori in G), we will say that the action of $\mathrm{Gal}(k^s/k)$ on G *trivializes* over k^{triv}, and we will call k^{triv}/k a **trivializing extension**.

Observe now that, if the residue characteristic of k is sufficiently large (in particular if it is greater than d_Ψ), then all of these subextensions $k \subset k' \subset k^s$ are tamely ramified, and so there are only finitely many of them; moreover, the number of such extensions is bounded independently of k. We give the following result, which combines [BHH$^+$15, Theorem 5.16] and [JR06, Proposition 2.2.1]:

Proposition 6.1.3. *Let $r > 0$ be an integer. There exists $M' > 0$ such that $k \in \mathcal{C}_{M'}$ implies that the number of tamely ramified extensions of k of degree r is finite. Moreover, this bound depends only on r.*

Proof. We will split into to cases, according as to whether the characteristic of k is 0, or positive.

Suppose first that $\operatorname{char} k = 0$, so that k is a finite extension of some $\mathbf{Q}_p$; by taking M' sufficiently large, we may assume further that k is tamely ramified over $\mathbf{Q}_p$, and that any degree-r extension of k is also tamely ramified. In particular: if k'/k is any extension of degree r, then there exists a subextension $k \subset k^u \subset k'$ such that k^u/k is unramified and k'/k^u is totally (tamely) ramified. For brevity, let us write $f = [k^u : k]$ and $e = [k' : k^u]$.

By [Has80, Chapter 16], there are exactly e totally tamely ramified extensions of k^u of degree e, and it is a classical result that there is a single unramified subextension of $k \subset k^s$ of degree f. As such, there are no more than

$$\sum_{e|r} e$$

tamely ramified extensions of k of degree r. We remark that this number is precisely the sum-of-divisors function $\sigma_1(r)$, but we will not use this notation to avoid confusion with our Galois actions.

Now, suppose $\operatorname{char} k = p$, so that k is a finite extension of some $\mathbf{F}_p(\!(t)\!)$, and suppose k'/k is a tamely ramified extension of degree r; we proceed as in the proof of [BHH+15, Theorem 5.6]. We recall the **relative discriminant** of the extension k'/k, defined analogously to the number field case.

That is: if $\mathcal{O}_k$ and $\mathcal{O}_{k'}$ are, respectively, the ring of integers of k and the ring of integers of k', fix a basis $(\delta_1, \ldots, \delta_r)$ of $\mathcal{O}_{k'}$ over $\mathcal{O}_k$, and let $\tau_1, \ldots, \tau_r$ denote the r distinct k-embeddings $k' \hookrightarrow k^s$. The **relative discriminant** of k'/k is the ideal of $\mathcal{O}_k$ defined

$$\operatorname{disc}(k'/k) := (\det(\tau_i(\delta_j)))^2,$$

where $(\tau_i(\delta_j))$ is the $r \times r$ matrix whose (i,j)-th entry is $\tau_i(\delta_j)$.

The extension k'/k again admits a subextension $k \subset k^u \subset k'$ such that k^u/k is unramified and k'/k^u is totally (tamely) ramified; as before put $e = [k' : k^u]$ and $f = [k^u : k]$; the extension k^u/k is unique and so there are as many such tamely ramified extensions k'/k as there are totally tamely ramified subextensions k'/k^u of degree e, as e varies over all divisors of r.

Therefore consider the totally tamely ramified extension k'/k^u, and let $f(X) \in k^u[X]$ be a minimal polynomial for this extension so that $k' = k^u[X]/(f(X))$. We know that any totally ramified extension of k^u can be obtained by adjoining the root of an Eisenstein polynomial; therefore, we may assume without loss of generality that we can write

$$f(X) = X^e + a_{e-1}X^{e-1} + \cdots + a_1 X + a_0 \in k^u[X],$$

with $v_{k^u}(a_i) \geq 1$ for all $i = 0, \ldots, e-1$ and $v_{k^u}(a_0) = 1$ (where v_{k^u} is, as usual, the normalized valuation $(k^u)^\times \to \mathbf{Z}$). By [BHH+15, Lemma 5.5], the discriminant of the extension k'/k^u is the same as the polynomial discriminant $\mathrm{disc}(f)$; moreover, we have $\mathrm{disc}(f) = e \cdot v_{k^u}(f'(\varpi))$ for any root ϖ of f, and (because such ϖ must be a uniformizer of k') we have also $v_{k^u}(\varpi) = \frac{1}{e}$.

The formal derivative of $f(X)$ is

$$f'(X) = eX^{e-1} + (e-1)a_{e-1}X^{e-2} + \cdots + a_1,$$

and we observe that $v_{k^u}(n) = 0$ whenever $n \in \mathbf{Z}$ (as k^u is an extension of $\mathbf{F}_p(\!(t)\!)$); furthermore, for $1 \leq i \leq r-1$ we observe that

$$v_{k^u}(ia_i\varpi^{i-1}) = v_{k^u}(i) + v_{k^u}(a_i) + v_{k^u}(\varpi^{i-1}) = v_{k^u}(a_i) + \frac{i-1}{e}$$

and

$$v_{k^u}(e\varpi^{e-1}) = \frac{e-1}{e},$$

so all the summands in the expression

$$f'(\varpi) = e\varpi^{e-1} + (e-1)a_{e-1}\varpi^{e-2} + \cdots + a_1$$

have distinct valuations. It follows that

$$v_{k^u}(f'(\varpi)) = \min_{1 \leq i \leq e-1} \left\{ v_{k^u}(a_i) + \frac{i-1}{e}, \frac{e-1}{e} \right\},$$

which is clearly $\frac{e-1}{e}$ because each $v_{k^u}(a_i) \geq 1$. That is,

$$\mathrm{disc}(k'/k^u) = \mathrm{disc}(f) = e \cdot v_{k^u}(f'(\varpi)) = \mathfrak{p}_{k^u}^{e-1},$$

where $\mathfrak{p}_{k^u}$ is the prime ideal of $\mathcal{O}_{k^u}$. Thus we see that all degree e totally tamely ramified extensions of k^u have discriminant $\mathfrak{p}_{k^u}^{e-1}$. By [BHH$^+$15, Theorem 5.16], there are exactly e such extensions, and so the total number of tamely ramified extensions k'/k is again

$$\sum_{e|r} e$$

and we are done. $\square$

We immediately have

Lemma 6.1.4. *Let Ψ be a root datum. There exists an integer $M' > 0$ such that $k \in \mathcal{C}_{M'}$ implies that k_Ψ/k is a finite extension, where k_Ψ is the extension defined in Equation (6.1). In particular, k_Ψ is definable when $k \in \mathcal{C}_{M'}$.*

Finally, we close with

Corollary 6.1.5. *Let Ψ be a root datum. There exists $M' > 0$ such that, for all $k \in \mathcal{C}_{M'}$, one has $|\operatorname{Gal}(k_\Psi/k)| \leq b(\Psi)$, where*

$$b_\Psi := \prod_{i=1}^{d_\Psi} \left(\sum_{e|r} e \right)^{\left(\sum_{e|r} e\right)},$$

and d_Ψ is the constant from Theorem 6.1.1.

Proof. For large enough residue characteristic, subextension of $k \subset k^s$ of degree $\leq d_\Psi$ is tamely ramified. Clearly $\mathrm{Gal}(k_\Psi/k)$ can be no larger than the product of all values $[k_1 : k]$, as k_1/k varies over all (tamely ramified) subextensions of k^s of degree at most d_Ψ.

As in the proof of Proposition 6.1.3, there are exactly $\sum_{e|r} e$ tamely ramified extensions of k of degree e, and so we have

$$\begin{aligned} |\,\mathrm{Gal}(k_\Psi/k)| &\leq \prod_{[k':k]\leq d_\Psi} |\,\mathrm{Gal}(k'/k)| \\ &= \prod_{i=1}^{d_\Psi} \left(\prod_{[k':k]=i} |\,\mathrm{Gal}(k'/k)| \right) = \prod_{i=1}^{d_\Psi} \left(\sum_{e|r} e \right)^{\left(\sum_{e|r} e\right)}, \end{aligned}$$

as claimed. □

6.2 Fixed choices

In this section we will collect the fixed choices which we will rely on to construct the trace formula in the Denef-Pas language. Our notation will be deliberately reminiscent of the constructions in Chapter 2.

We begin with a root datum: more precisely, we fix:

I. A finitely-generated free abelian group X and its dual $X^\vee = \mathrm{Hom}(X, \mathbf{Z})$.

II. A finite (abstract) root system $R \subset X$ and its dual $R^\vee \subset X^\vee$.

Fixed choices I and II determine a root datum $\Psi = (X, R, X^\vee, R^\vee)$; our remaining fixed choices will be made, so to speak, *compatibly* with this root datum.

Therefore let $M' > 0$ be such that the assumption of Lemma 6.1.4 is satisfied. We continue to fix:

III. A finite group Γ_{spl} together with an action on X (and hence on $X^\vee$).

IV. A subset $D \subset R$.

V. A subset $D_0 \subset D$.

VI. An action τ of Γ_{spl} on D preserving D_0 (this will give a void construction unless τ is compatible with the action defined in equation 2.7).

VII. The real vector space $V := \mathbf{R} \otimes_{\mathbf{Z}} X$ and its Γ_{spl}-fixed subspace ${}_FV$ spanned by ${}_FX := X^{\Gamma_{\text{spl}}}$.

VIII. For each sub-root system R' of R, the Weyl group $W = W(R')$ and its induced action on V, from Definition 2.3.7.

IX. A positive definite symmetric bilinear form $(,)$ on V which is W-invariant.

X. The real dual vector space V^*, identified with V by $(,)$.

XI. The orthogonal projection $\pi : V \to {}_FV$ and the set ${}_FD := \pi(D - D_0)$ (from Equation (2.5)).

XII. The set of all sub-root systems of ${}_FR$.

XIII. For every inclusion of subsets ${}_FD_P \hookrightarrow {}_FD_Q$ of ${}_FD$, an inclusion of real vector spaces $a_Q \hookrightarrow a_P$ of respective dimensions

$$\dim V - \operatorname{rk} R + |{}_FD_Q| \text{ and } \dim V - \operatorname{rk} R + |{}_FD_P|,$$

writing a_0 for the real vector space corresponding to the entire set ${}_FD$.

XIV. The real dual vector spaces $\mathrm{i}a_P^*$ of the spaces in item XIII.

XV. For each subset ${}_FD_P$ of ${}_FD$, an element $\lambda_P \in a_0$ satisfying $\langle \alpha, \lambda_P \rangle > 0$ if and only if $\alpha \in {}_FD_P$ (for $\alpha \in {}_FD$).

XVI. For every sub-root system R_M of ${}_FR$, a regular element λ_M of $\mathrm{i}a_M^*$.

XVII. For every sub-root system R_M of ${}_FR$, a system of roots ${}^MR$ in R mapped to ${}_FR$ under π (such a system exists by [Spr10, Lemma 15.5.1(ii)]).

XVIII. For every sub-root system R_M of ${}_FR$, its Weyl group $W(R_M)$, as defined in fixed choice VIII.

XIX. For every sub-root system R_M of ${}_FR$, the set of chambers in the vector space a_M, where $a_M = a_{M_P}$ for any parabolic P with Levi factor M.

XX. The integer $b(\Psi)$ from Corollary 6.1.5.

XXI. A set $FG(\Gamma_{\mathrm{spl}}, \Psi)$ of representatives $\tilde{\Gamma}$ of every isomorphism class $[H]$ of finite groups H with $|\Gamma_{\mathrm{spl}}| \le |H| \le b(\Psi)$ (this is evidently a finite set).

We remark that, in fixed choice XXI, we have many extraneous groups: the elements $\tilde{\Gamma}$ will play the rôle (in our constructions) of the Galois group of a finite extension of local fields, and as such $\tilde{\Gamma}$ will only generate non-vacuous sets if $\tilde{\Gamma}$ is isomorphic to such a Galois group. In particular, every non-solvable group $\tilde{\Gamma}$ will be irrelevant to our purposes (as will many solvable groups), but this will cause no harm.

In the remaining sections, we will explain how these fixed choices help us build the trace formula.

6.3 Parameterizing sets

In this section we construct definable sets which parameterize various terms in the trace formula; the notable exception is $\mathscr{T}_M$ itself, to which we devote Section 6.4. Lemma 5.5.3 is a good start, but does not give us everything we need: for instance, while this tells us how to construct the k-points of $\mathbf{g}_z$, it does *not* tell us how to obtain the points of $t_{\text{reg}} \subset \mathbf{g}_z$ for the Lie algebra t of every maximal elliptic torus T in $\mathbf{G}_z$.

Here, we add some detail to the construction, in preparation for the next section. At the same time, we make a definable parameterization of Levi subgroups and parabolic subgroups.

In order to do this, we must appeal to the indexed root datum ${}_i\Psi$ coming from our fixed choices I–VI.

Proposition 6.3.1. *Let* ${}_i\Psi = (X, D, X^\vee, D^\vee, D_0, \tau)$ *be an indexed root datum and let* $\Lambda_{{}_i\Psi}$ *be the collection of elements* λ_M *from fixed choice XVI. There exist:*

- *an integer* $M' > 0$,

- *a definable set* $Z_{{}_i\Psi}$*, and*

- *families of definable sets* $(\mathbf{G}_z)_{z\in Z_{{}_i\Psi}}$, $(\mathbf{g}_z)_{z\in Z_{{}_i\Psi}}$, $(\mathbf{T}_z)_{z\in Z_{{}_i\Psi}}$*, and* $(\mathbf{P}_{0,z})_{z\in Z_{{}_i\Psi}}$ *with parameter in* $Z_{{}_i\Psi}$*, and*

- *a family of definable sets* $(\mathbf{M})_{(z,\lambda)\in Z_{{}_i\Psi}\times\Lambda_{{}_i\Psi}}$ *with parameter in* $Z_{{}_i\Psi}\times\Lambda_{{}_i\Psi}$

such that, for all $k \in \mathcal{C}_{M'}$ *and all* $z \in Z_{{}_i\Psi,k}$*:*

- $\mathbf{G}_{z,k}$ *is the set of* k*-points of a connected reductive group* $\mathbf{G}_z$ *with indexed root datum* ${}_i\Psi$*;*

- *the set* $\mathbf{g}_{z,k}$ *is the set of* k*-points of the Lie algebra of the group* $\mathbf{G}_z$*;*

- *the set* $\mathbf{T}_{z,k}$ *is the set of* k*-points of a maximally split maximal* k*-torus of* $\mathbf{G}_z$*;*

- *the set* $\mathbf{P}_{0,z,k}$ *is the set of* k*-points of a minimal parabolic* k*-subgroup of* $\mathbf{G}_z$ *containing* $\mathbf{T}_z$*;*

- *the set* $\mathbf{M}_{z,\lambda,k}$ *is the set of* k*-points of a Levi subgroup of* $\mathbf{G}_{z,k}$ *which contains* $\mathbf{T}_z$*, for every* $\lambda_M \in \Lambda_{{}_i\Psi}$*. Moreover, all Levi subgroups of* $\mathbf{G}_z$ *arise as such* $\mathbf{M}_{z,\lambda}$*.*

We will refer to the sets $Z_{{}_i\Psi}$ and $\Lambda_{{}_i\Psi}$ in the remaining sections, below, to help us prove our main theorem.

Proof. First of all: we already know that we can construct $\mathbf{G}_z$ and $\mathbf{g}_z$ to have a given root datum (by Lemma 5.5.3). We obtain a maximal torus $\mathbf{T}_z$ in $\mathbf{G}_z$ easily: we have seen that $\mathbf{G}_z$ is a twisted form of $\mathbf{Spl}(\Psi)$, and $\mathbf{T}_z$ is precisely the image of $\mathbf{T} \subset \mathbf{Spl}(\Psi)$ under this twisting.

Fixed choice XVI allows us to use the element $\lambda_0 \in \mathrm{i}a_0^*$ to construct the definable subset $\mathbf{P}_{0,z}$ of $\mathbf{G}_z$ via

$$\mathbf{P}_{0,z,k} = \{x \in \mathbf{G}_{z,k} : \lim_{t \to 0} \lambda_0(t)^{-1} x \lambda_0(t) \text{ exists.}\},$$

which by our remarks in Section 2.2 we know will be the k-points of a minimal parabolic subgroup of $\mathbf{G}_z$ defined over k. We remark that the predicate involving the limit is indeed definable: it may be stated in the form

$$\exists y \in \mathbf{G}_{z,k}(\forall n \in \mathbf{Z} \exists n' \in \mathbf{Z}(\mathrm{ord}(t) > n' \\ \implies \min_{i,j}\{\mathrm{ord}((\lambda_0(t)^{-1} x \lambda_0(t) - y)_{i,j})\} > n))),$$

where the minimum is taken over the (finitely) many entries of the matrix $\lambda_0(t)^{-1} x \lambda_0(t) - y$. It is now possible for us to compute the *index* of the triple $(\mathbf{G}_{z,k}, \mathbf{P}_{0,z,k}, \mathbf{T}_{z,k})$ (recall Section 2.6), which is evidently determined by the element z of $Z_{\Psi,\Gamma_{\mathrm{spl}}}$.

Recall the *index* of the triple (G, P_0, T) from Section 2.6; much like the root datum, it is determined by *definable* conditions on $X^*(T)$, and so having constructed the triple $(\mathbf{G}_{z,k}, \mathbf{P}_{0,z,k}, \mathbf{T}_{z,k})$ we can now compute its index in the Denef-Pas language. We can therefore make a definable condition on $z \in Z_{\Psi,\Gamma_{\mathrm{spl}},\underline{a}}$ to assert that the index of $(\mathbf{G}_{z,k}, \mathbf{P}_{0,z,k}, \mathbf{T}_{z,k})$ coincides with the index of ${}_i\Psi$ whenever the residue characteristic of k is sufficiently large.

It follows that, if we define $Z_{{}_i\Psi,\Gamma_{\mathrm{spl}},\underline{a}}$ to be the set of all $z \in Z_{\Psi,\Gamma_{\mathrm{spl}},\underline{a}}$ for which this condition is true, then

$$Z_{{}_i\Psi} = \bigcup_{\underline{a}} Z_{{}_i\Psi,\Gamma_{\mathrm{spl}},\underline{a}},$$

the union taken over all suitable tuples $\underline{a}$, is precisely the definable set asserted by the proposition.

Finally, we can perform this construction for all the elements λ_M from fixed choice XVI, and replacing λ_M by λ_M^{-1} in the above construction will result in constructing the opposite parabolic; we can then take their intersection to obtain a Levi subgroup of $\mathbf{G}_z$, as in Section 2.5. By Theorem 2.5.6, we know that in this way we will construct all Levi subgroups of $\mathbf{G}_z$, if the rational root system of $\mathbf{G}_z$ is the same as that from our fixed choices, and we are done. $\square$

6.4 Elliptic maximal tori

One of the indices of summation in the trace formula is $\mathscr{T}_M$, a set of representatives of rational conjugacy classes of elliptic maximal tori in M, for all $M \in \mathscr{L}$; this will prove to be one of the more challenging obstacles to overcome in establishing the trace formula to be motivic. As such, we devote the current section to establishing that the set $\mathscr{T}_M$ can be encoded in the Denef-Pas language.

To construct $\mathscr{T}_M$, we first construct a complete set of representatives for the rational conjugacy classes of maximal tori in M, and then show that ellipticity is a definable condition. Per Theorem 3.3.1, $\mathscr{T}_M$ is in bijection with $\ker(M, T)$ (for any maximal torus T in G), i.e. the set of equivalence classes in $H^1(\mathrm{Gal}(k_\Psi/k), \mathrm{Norm}_M(T)(k_\Psi))$ of cocycles of the form

$$\sigma \mapsto x^{-1}x^\sigma \text{ for some } x \in G(k_\Psi).$$

We cannot encode this set of equivalence classes directly in the Denef-Pas language. However, with our fixed choice of root datum Ψ, we *can* make a Denef-Pas statement which will help us construct a definable set of *representatives* of these equivalence classes.

Recall from Theorem 3.4.2 that the constant $b(\Psi, \mathrm{Gal}(k_\Psi/k))$ (from Definition 3.4.3) satisfies

$$\begin{aligned} |H^1(\mathrm{Gal}(k_\Psi), \mathrm{Norm}_G(T)(k_\Psi))| &\le b(\Psi, \mathrm{Gal}(k_\Psi)) \\ &= \Big(|W(R)| \cdot |\operatorname{Perm}(R)|\Big)^{[k_\Psi:k]}. \end{aligned}$$

From this, we obtain the valid (if crude) bound

$$|H^1(\mathrm{Gal}(k_\Psi), \mathrm{Norm}_G(T)(k_\Psi))| \le \Big(|W(R)| \cdot |\operatorname{Perm}(R)|\Big)^{b(\Psi)}$$

(where $b(\Psi)$ is the constant from Corollary 6.1.5), which is completely determined by our fixed choices I–III.

Now, recall from Section 5.4 that, if Ψ in any root datum, we can construct a split group $\mathbf{Spl}(\Psi)$ with root datum Ψ over $\mathbf{Z}[1/N_\Psi]$ with an embedding into $\mathrm{GL}(n_\Psi, \mathbf{Q})$. We may further assume, without loss of generality, the existence of a maximal torus $\mathbf{T}$ in $\mathbf{Spl}(\Psi)$ whose image in $\mathrm{GL}(n_\Psi, \mathbf{Q})$ is diagonal.

In this way, we may choose a "distinguished" torus $\mathbf{T}$ in $\mathbf{Spl}(\Psi)$; in the same way that every group G with root datum Ψ is obtained by twisting the group $\mathbf{Spl}(\Psi)$, their maximal tori can be obtained by twisting $\mathbf{T}$.

Having fixed $\mathbf{T} \subset \mathbf{Spl}(\Psi)$, we see that the twisted torus $\mathbf{T}_z$ (for $z \in Z_\Psi$) lies in $\mathbf{G}_z$. By Theorem 3.3.1, there is a one-to-one correspondence between the set of k-rational conjugacy classes of maximal tori in $\mathbf{G}_z$, and the kernel of the homomorphism

$$H^1(\mathrm{Gal}(k^s/k), \mathrm{Norm}_{\mathbf{G}_z}(\mathbf{T}_z)(k^s)) \to H^1(\mathrm{Gal}(k^s/k), \mathbf{G}_z(k^s)),$$

under which the rational class of $\mathbf{T}_z$ corresponds to the distinguished class. Because we cannot work over the infinite extension k^s/k, we will instead work over the trivializing extension k_Ψ/k. Having fixed Ψ, let us assume that the residue characteristic of k is large enough that the result of Proposition 6.1.3 holds, and therefore that $[k_\Psi : k] \leq b(\Psi)$.

Although we now know that k_Ψ/k is definable, we do *not* know that the Galois group of this extension is independent of k. As such, we must appeal to fixed choice XXI: suppose $\tilde{\Gamma}$ is one of our groups chosen there. The statement that $\mathrm{Gal}(k_\Psi/k) \cong \tilde{\Gamma}$ is definable, by the same argument given in Section 5.3.

Therefore, for each such $\tilde{\Gamma}$, we append this statement to all further constructions: then, given k, this statement is true for precisely one $\tilde{\Gamma}$, which we will then take in place of $\mathrm{Gal}(k_\Psi/k)$.

Thus, from now on, we will let $\tilde{\Gamma} \in FG(\Gamma_{\mathrm{spl}}, \Psi)$ and assume that k satisfies $\mathrm{Gal}(k_\Psi/k) \cong \Gamma_{\mathrm{spl}}$. Our remarks imply the existence of an integer $M' > 0$ such that, if $k \in \mathcal{C}_{M'}$, we have isomorphisms

$$H^1(\mathrm{Gal}(k^s/k), \mathrm{Norm}_{\mathbf{G}_z}(\mathbf{T}_z)(k^s)) \cong H^1(\tilde{\Gamma}, \mathrm{Norm}_{\mathbf{G}_z}(\mathbf{T}_z)(k_\Psi))$$

and

$$H^1(\mathrm{Gal}(k^s/k), \mathbf{G}_z(k^s)) \cong H^1(\tilde{\Gamma}, \mathbf{G}_z(k_\Psi)),$$

and we see that all maximal tori in $\mathbf{G}_z$ can be obtained by twisting $\mathbf{T}_z$ by elements of the cocycle space $Z^1(\tilde{\Gamma}, \mathrm{Norm}_{\mathbf{G}_z}(\mathbf{T}_z)(k_\Psi))$ (by Theorem 3.3.1). Thus:

Lemma 6.4.1. *There is a definable set $Y_{i\Psi} = (Y_z)_{z \in Z_{i\Psi}}$ with parameter in $Z_{i\Psi}$ and an integer $M' > 0$ such that, if $k \in \mathcal{C}_{M'}$, then there exists a unique $\tilde{\Gamma} \in FG(\Gamma_{\mathrm{spl}}, \Psi)$ (from fixed choice XXI) such that*

$$Y_{z,k} = Z^1(\tilde{\Gamma}, \mathrm{Norm}_{\mathbf{G}_z}(\mathbf{T}_z)(k_\Psi)).$$

Moreover, if we denote by $\mathbf{T}_{z,y,k}$ the k-points of the maximal torus of $\mathbf{G}_z$ obtained by twisting $\mathbf{T}_z$ by the cocycle $y \in (Y_{i\Psi})_{k,z}$, then all maximal k-tori in $\mathbf{G}_z$ arise as some such $\mathbf{T}_{z,y,k}$.

We remark that the group $\tilde{\Gamma}$ in question is exactly $\mathrm{Gal}(k_\Psi/k)$, which may depend on the field of specialization k. The second assertion of the lemma is an easy consequence of Theorem 3.3.1. We point out again that this is possible precisely because every torus in $\mathbf{G}_z$ splits over k_Ψ, which in our case is finite and definable.

Using the bound $b(\Psi)$, we can now make the following simplified statement in the Denef-Pas language:

> **Statement M-i.** Let G be a connected, reductive group over k with maximal torus T and root datum Ψ, and let $M \supset T$ be a Levi subgroup. Let i be an element of the set
>
> $$\left\{1, 2, \ldots, \Big(|W(R)| \cdot |\operatorname{Perm}(R)|\Big)^{b(\Psi)}\right\},$$
>
> let $\tilde{\Gamma}$ be an element of $FG(\Gamma_{\mathrm{spl}}, \Psi)$, and let $\mathbf{M}\text{-}\mathbf{i}(\tilde{\Gamma})$ be the statement "There exists a set of i inequivalent classes of cocycles of $\tilde{\Gamma}$ in $\operatorname{Norm}_M(T)(k_\Psi)$, and no set of $i+1$ inequivalent classes of cocycles of $\tilde{\Gamma}$ in $\operatorname{Norm}_M(T)(k_\Psi)$, which have the form $\sigma \mapsto x^{-1}x^\sigma$ for some $x \in G(k_\Psi)$".
>
> Given Γ_{spl}, statement $\mathbf{M}\text{-}\mathbf{i}(\tilde{\Gamma})$ is true for exactly one integer i in the set $\{1, 2, \ldots, (|W(R)| \cdot |\operatorname{Perm}(R)|)^{b(\Psi)}\}$.

Let us call such a set of cocycles a **maximal set** of cocycles of $\tilde{\Gamma}$ in $\operatorname{Norm}_M(T)(k_\Psi)$. Roughly speaking: we are able to posit, in the Denef-Pas language, the existence of a complete set of representatives of the set

$$\ker\left[H^1(\operatorname{Gal}(k_\Psi/k), \operatorname{Norm}_G(T)(k_\Psi)) \to H^1(\operatorname{Gal}(k_\Psi/k), G(k_\Psi))\right],$$

whenever the residue characteristic of k is sufficiently large.

Now, we can put this simplified statement to use in our construction of reductive groups and their attached objects. Recall the definable sets $Z_\Psi, \mathbf{T}_z, \mathbf{M}_{z,\lambda}$, etc. from Proposition 6.3.1. When true, Statement M-i lets us define a set of exactly i inequivalent cocycles of $\mathrm{Gal}(k_\Psi/k)$ in $\mathrm{Norm}_{\mathbf{M}_{z,\lambda}}(\mathbf{T}_z)(k_\Psi)$, to each element c of which we associate the twisted group $\mathbf{T}_{z,c}$ obtained by twisting $\mathbf{T}_z$ by c (recall Lemma 6.4.1). The resulting maximal set of cocycles $\Theta(\mathbf{M}_{z,\lambda})$ parameterises the elements of $\mathscr{T}_{\mathbf{M}_{z,\lambda}}$. We have

Proposition 6.4.2. *Let ${}_i\Psi$ be an indexed root datum and recall the notation of Proposition 6.3.1 and Lemma 6.4.1. There exists an integer $M' > 0$ such that, whenever $k \in \mathcal{C}_{M'}$, the following statement is true: "For all $z \in Z_{{}_i\Psi,k}$ and $\lambda \in \Lambda_{{}_i\Psi}$, there exists a maximal set $\Theta(\lambda) = \Theta(\lambda, {}_i\Psi)$ of pairwise inequivalent cocycles of* $\mathrm{Gal}(k_\Psi/k)$ *in* $\mathrm{Norm}_{\mathbf{M}_{z,\lambda}}(\mathbf{T}_z)(k_\Psi)$ *of the form $\sigma \mapsto x^{-1}x^\sigma$ for some $x \in \mathbf{M}_{z,\lambda}(k_\Psi)$, for all $z \in Z_{{}_i\Psi}$."*

This proposition in turn yields our desired result, viz.

Corollary 6.4.3. *Let ${}_i\Psi$ be an indexed root datum and let $(\mathbf{G}_z)_{z\in Z_{{}_i\Psi}}$ be as in Proposition 6.3.1. There exist an integer M' and a family of definable sets $(\vartheta_{z,\lambda})_{(z,\lambda)\in Z_{{}_i\Psi}\times\Lambda_{{}_i\Psi}}$ with parameter in $Z_{{}_i\Psi} \times \Lambda_{{}_i\Psi}$ such that, if $k \in \mathcal{C}_{M'}$, then $\vartheta_{z,\lambda,k}$ is a complete set of representatives of rational conjugacy classes of maximal elliptic tori in $\mathbf{M}_{z,\lambda,k}$.*

Proof. Retaining the notation of Proposition 6.4.2: let $y \in \Theta(\lambda)$ and let

$$\mathbf{T}_{z,\lambda,y} \subset \mathbf{M}_{z,\lambda} \subset \mathbf{G}_z$$

be a maximal torus in $\mathbf{M}_{z,\lambda}$ defined by the cocycle y. We aim to show that the statement "$\mathbf{T}_{z,\lambda,y}$ is elliptic in $\mathbf{M}_{z,\lambda}$" is Denef-Pas.

Observe by the definitions that $\mathbf{T}_{z,\lambda,y}$ is elliptic in $\mathbf{M}_{z,\lambda}$ if and only if the only characters of $\mathbf{T}_{z,\lambda,y}$ that are fixed by the action of $\tilde{\Gamma}$ are characters of the centre of $\mathbf{M}_{z,\lambda}$. The cocycle y defines an action $(\sigma, \chi) \mapsto \sigma.\chi$ of $\tilde{\Gamma}$ on $X^*(\mathbf{T}_z)$, which is definable; thus, the statement of ellipticity is

$$\forall \chi \in X^*(\mathbf{T}_z)(\neg(\sigma.\chi = \chi) \vee (\exists \chi' \in X^*(Z(\mathbf{M}_{z,\lambda}))(\chi = \chi'))),$$

which is clearly definable. $\square$

So we have shown that the set $\mathscr{T}_M$ for $M \subset \mathbf{G}_z$ is parameterized by the definable set ϑ_{z,λ_M}.

6.5 Measures

In this section we demonstrate that the measures $\mathrm{d}\dot{x}$, $\mathrm{d}\dot{y}$, and $\mathrm{d}X$, as well as the integration over $A_M(F)\backslash G(F)$, $t(F)$, and summations over $\mathscr{L}$ and $\mathscr{T}_M$ that appear in the trace formula may indeed be specified in the Denef-Pas language to give motivic integrals.

First, we deal with counting measures. By Theorem 2.5.6, the set $\mathscr{L}$ is in one-to-one correspondence with the set of sub-root systems of ${}_FR$, i.e. our fixed choice XII. On the finite set $\mathscr{L}$ we can place the measure

$$\mu(R_M) = \frac{|W(R_M)|}{|W({}_FR)|}$$

where R_M is the sub-root system of ${}_FR$ defined in Equation (2.6), which is a rational (i.e. $\mathbf{Q}$-valued) constant determined by the fixed choices XVIII. It is an integer scaling of the usual counting measure on $\mathscr{L}$, and so it follows from Theorem 4.5.1 that this measure is motivic.

The next term in the trace formula (reading from left-to-right) is the sum over $T \in \mathscr{T}_M$ of an integral scaled by $\frac{1}{|W(M,T)|}$. We know that $1 \leq |W(M,T)| \leq |W(R)|$, which is a fixed choice; therefore, as before, we will *stratify* the sum over those T with a Weyl group of a given cardinality.

More precisely: for each $n_W \in \{1, 2, \ldots, |W(R)|\}$, let $\vartheta_{z,\lambda_M,n_W} \subset \vartheta_{z,\lambda_M}$ be the subset consisting of those y satisfying

$$|W(M, T_y)| = n_W.$$

This condition can be written explicitly in the Denef-Pas language by asserting that there is a set of n_W pairwise inequivalent elements of $\mathrm{Norm}_M(T_y)$ modulo conjugation by $T_y(k)$, and no such set of n_W+1 pairwise inequivalent elements. It follows that $\vartheta_{z,\lambda_M,n_W}$ is definable, and we write

$$\sum_{T\in\mathscr{T}_M} a_T = \sum_{n_W=1}^{|W(R)|} \frac{1}{n_W} \sum_{|W(M,T)|=n_W} a_T,$$

where a_T is any summand indexed by $T \in \mathscr{T}_M$. The inner sum is a (motivic) integral over a finite set, with respect to the counting measure, while the outside sum is a sum of a bounded number of motivic functions. As $\mathscr{T}_M$ itself is definable, it follows that the measure

$$\mu(T) = \frac{1}{|W(M,T)|}$$

on $\mathscr{T}_M$ is motivic.

By Lemma 6.4.1 there is a definable set $Y_{{}_i\Psi}$ with parameter in $Z_{{}_i\Psi}$ parameterizing the set of maximal k tori in any group with indexed root datum ${}_i\Psi$ (for k with sufficiently large residue characteristic). The same construction as in Lemma 5.5.3 allows us to take the Lie algebras of these tori, and we have (using the notation established there)

Corollary 6.5.1. *Let ${}_i\Psi$ be an indexed root datum. There exist a definable set $Z_{{}_i\Psi}$, a family of definable sets $(Y_{{}_i\Psi,z})_{z\in Z_{{}_i\Psi}}$ with parameter in $Z_{{}_i\Psi}$, and families of definable sets $(\mathfrak{T}_y)_{y\in {}_i\Psi}$ and $(\mathfrak{t}_y)_{y\in Y_{{}_i\Psi}}$ with parameter in $Y_{{}_i\Psi}$ such that there exists $M' > 0$ for which $k \in \mathcal{C}_{M'}$ implies that, for every $z \in Z_{{}_i\Psi,k}$ and every $y \in Y_{{}_i\Psi,z,k}$, the set $\mathfrak{T}_{y,k}$ is the set of k-points of a maximal torus in $\mathbf{G}_z$ and $\mathfrak{t}_{y,k} \subset \mathbf{g}_{z,k}$ is the set of k-points of the Lie algebra of $\mathfrak{T}_{y,k}$. Moreover, all maximal tori of $\mathbf{G}_z$ arise in this way.*

Over $\mathbf{Q}$, we can fix an (algebraic) isomorphism on neighbourhoods of the identity $U_{\mathbf{g}} \to U_{\mathbf{G}}, X \mapsto 1 + X$, as in Section 2.7; this isomorphism will be defined for all fields of sufficiently large residue characteristic. The measure $\mathrm{d}X$ chosen on t_{reg} is the unique Haar measure for which this map locally preserves measures.

In Section 2.7 we chose Haar measure on M_0 arbitrarily; now, we will specify that the measure on $M_0(F)$ is motivic. It will then follow immediately that the measure $\mathrm{d}x$ on $G(F)$ is motivic, and hence also the measure $\mathrm{d}X$ on each $t_{\mathrm{reg}}(F)$.

Should there be any concern about the motivicity of the measure $\mathrm{d}n$ on N_P chosen in that section, we may freely replace it with any definable Haar measure $\mathrm{d}n'$ to obtain a new Haar measure $\mathrm{d}x'$ on G; the resulting functional $J_G(f_1, f_2)$ will then be scaled by the same constant as $J_G(\hat{f}_1, \check{f}_2)$, and the validity of the trace formula remains unchanged.

It remains only to discuss the integration spaces $A_M(F)\backslash G(F)$ and the measures $\mathrm{d}\dot{x}$. We recall from Equation (2.9) that the measure $\mathrm{d}\dot{x}$ on $A_M(F)\backslash G(F)$ is defined by the measure $\mathrm{d}\bar{x}$ on $T(F)\backslash G(F)$, and so we need only specify how to integrate over the orbit $T(F)\backslash G(F)$, which is identified with the orbit of a regular element of $T(F)$ under the adjoint action of G, and is hence definable. Thus, it suffices to show that the measure $\mathrm{d}\bar{x}$ on $T(F)\backslash G(F)$ is motivic.

This is not hard: maximal tori T in G arise as centralizers of regular elements of the Lie algebra t. That is: there exists a finite family of elements $\{X_T : T \in \mathscr{T}\}$ in t_{reg} such that

$$T = Z_G(X_T) = \{x \in G : \mathrm{Ad}(x)X_T = X_T\}$$

for all $T \in \mathscr{T}_M$.

It now follows immediately from [CGH18, Lemma 4.3.2] that there exists a motivic measure $\mathrm{d}\bar{x}$ on $T\backslash G$ – and hence a motivic measure $\mathrm{d}\dot{x}$ on $A_M\backslash G$ – for all $T \in \mathscr{T}_M$, when the residue characteristic of k is sufficiently large.

Following the construction provided in that lemma, we have the following

Lemma 6.5.2. *Let Ψ be a root datum and let $Z_{{}_i\Psi}, Y_{{}_i\Psi}, \mathbf{G}$ and $\mathbf{g}$ be as above. There exist an integer $M' > 0$ and a motivic function $c^{\mathbf{G}}$ on $Z_{{}_i\Psi}$ such that, for any definable family $\{f_s\}_{s\in S}$ of motivic functions on $\mathbf{g}$ with parameter in S, there exists a motivic function H on $\mathbf{g} \times S \times Y_{{}_i\Psi}$ such that*

$$\int_{\mathbf{T}_{z,y,k}\backslash\mathbf{G}_{z,k}} f(\mathrm{Ad}(x^{-1})X)\,\mathrm{d}\bar{x} = \frac{1}{c_k^{\mathbf{G}}(z)} H_k(X,s,y).$$

It follows immediately that the integrals over $A_M\backslash G$ appearing in the trace formula are motivic.

6.6 Two more factors

In this short section, we discuss the factors $D^g(X)$ and $v_M(x,y)$ that occur in the trace formula.

By [Kot05, §7.4], we have

$$D^g(X) = \prod_{\alpha\in R} \alpha(X),$$

so we will take this as the definition (evidently, determined by the fixed choice II). It remains to consider only the weight factor $v_M(x,y)$.

Recall that we defined the weight factor in Equation (2.11) as the sum over elements P of the set $\mathscr{P}(M)$ (or equivalently, by integrating against the discrete measure over the set of chambers in a_M, cf. fixed choice XIX) the quantity

$$v_M(x,y) = \frac{(-1)^{\dim a_P^G}(\lambda_M(H_P(y) - H_{\overline{P}}(x)))^{\dim a_P^G}\operatorname{vol}(a_P^G/L_P^G)}{(\dim a_P^G)!\prod_{\alpha\in_F D_P}\lambda_M(\alpha^\vee)},$$

where λ_M are as in the fixed choice XVI.

The denominator is evidently determined by fixed choices XI, XIII, and XVI; the numerator depends on the same fixed choices, as well as the function $H_P(y) - H_{\overline{P}}(x)$, which is easily seen from Definition 2.4.2 to be a definable function of x and y, as long as we can choose the subgroup K (from Equation 2.1) in a definable way.

This fact is guaranteed to us by [GR17, Propostion 5], which tells us that such maximal compact subgroups K can be definably chosen. Thus v_M is the sum of products of definable, rational-valued functions, and we deduce that $v_M(x,y)$ is itself a motivic function on $A_M\backslash G \times A_M\backslash G$.

6.7 Putting it together

We have now established the results which will allow us to prove our main theorem. We will want something like the following:

> **Approximate theorem 1.** *Let G be a connected, reductive group defined over a nonarchimedean local field F, let J_G be the distribution defined in Equation (2.12), and let $(f_s)_{s\in S}$ be a family of motivic functions with parameter in S. There exists*

a motivic function h on S such that $J_{G,k}(f_{s,k}) = h_k(s)$ for all $s \in S_k$, for all local fields k.

Proving this claim would prove that the local trace formula is true for arbitrary local fields. Unfortunately, we have no hope of proving this statement with our tools in such generality: having constructed our groups, Lie algebras, and other objects from our fixed choices and motivic functions, we have necessarily discarded finitely many residue characteristics from consideration.

Recalling Definition 4.5.3, we now rely only on our fixed choices and related constructions to prove

Theorem 6.7.1. *There exist an integer M', a definable set $Z_{{}_i\Psi}$, and a family of motivic distributions $(\Xi_z)_{z\in Z_{{}_i\Psi}}$ with parameter in $Z_{{}_i\Psi}$ on the family of definable sets $C_c^\infty(\mathfrak{g}_z)\times C_c^\infty(\mathfrak{g}_z)$ with parameter in $Z_{{}_i\Psi}$ such that, if $k \in \mathcal{C}_{M'}$, then*

$$\Xi_z(f_{1,z,k}, f_{2,z,k}) = J_{\mathbf{G}_z}(f_{1,z,k}, f_{2,z,k})$$

for all $(f_{1,z}, f_{2,z}) \in C_c^\infty(\mathbf{g}_z) \times C_c^\infty(\mathbf{g}_z)$, where $J_{\mathbf{G}_z}$ is the distribution on $C_c^\infty(\mathbf{g}_z) \times C_c^\infty(\mathbf{g}_z)$ defined in Equation (2.12), for all $z \in Z_{{}_i\Psi,k}$.

While the theorem seems extremely technical, it has an immediate corollary which is much more accessible, namely

Corollary 6.7.2. *There exists a prime number p such that*

Theorem 2.9.1 is true whenever the characteristic of F is at least p.

Proof (of Corollary 6.7.2). The corollary is essentially an application to Theorem 6.7.1 of the transfer principle of Cluckers and Loeser, with some extra work to overcome the fact that we are dealing with *distributions* rather than *functions*.

The transfer principle was first stated for exponential integrals in [CL10, Theorem 9.2.4] for a class of functions known as *IC-functions* (whose definition we will not give here). There, it took the following form:

Statement. Given IC-functions f, f' respectively on definable sets S, S', there exists an integer $M' > 0$ such that, if $k_1, k_2 \in \mathcal{C}_{M'}$ have isomorphic residue fields, then $f_{k_1} = f'_{k_1}$ if and only if $f_{k_2} = f'_{k_2}$.

As such, this result allows us to *transfer* results from characteristic zero to positive characteristic (and vice versa). Rather than address the question of what exactly are IC-functions, we instead appeal to [CGH18, Theorem 1.3.3], which tells us that the analogous statement is true *mutatis mutandis* for *all* motivic exponential functions, as long as their specializations to local fields of characteristic 0 are integrable (which is indeed the case here). In particular, this implies that the distribution J_G constructed over fields of sufficiently large residue characteristic converges if it is motivic (because it converges in characteristic zero; cf. Theorem 4.5.1).

More precisely: the Cluckers-Loeser transfer principle asserts that, if f is a motivic function on a definable set X, then there exists an integer $M' > 0$ such that, if $f_k(x) = 0$ for all $x \in X_k$, then $f_{k'}(x) = 0$ for all $x \in X_{k'}$ for any

$k' \in \mathcal{C}_{M'}$ whose residue field is isomorphic to that of k.

In order to extend this to motivic distributions, we follow the approach of [CCGS11]: suppose T, T' are motivic distributions on a definable set X, so that there exists some $M' > 0$ such that $k \in \mathcal{C}_{M'}$ implies that, for every definable set S and every family of Schwartz-Bruhat motivic functions $(f_s)_{s\in S}$ in $C_c^\infty(X)$ with parameter in S, there exist motivic functions g, g' on S such that

$$T_k(f_{s,k}) = g_k(s) \text{ and } T'_k(f_{s,k}) = g'_k(s) \text{ for all } s \in S_k. \tag{6.2}$$

We know nothing else *a priori* about the values of T and T' on other functions in $C_c^\infty(X)$; however, after the approach of [CCGS11, §3.2.1], we will see that their values on motivic functions is all that needs to be known. We briefly review the argument given there.

One constructs a family of motivic test functions of Schwartz-Bruhat class that is shown to be dense in the space of locally constant, compactly supported, complex-valued functions on each X_k. One begins by fixing a positive integer d, and constructing a family of motivic functions on $\mathbf{Q}[\![t]\!]^d \times \mathbf{Z}^d$ consisting of characteristic functions of compact open balls of all radii, centered on all points of $\mathbf{Q}[\![t]\!]^d$. These are shown to specialize (for sufficiently large residue characteristic) to functions on $\mathcal{O}_k[\![t]\!]^d$ (where $\mathcal{O}_k \subset k$ is the ring of integers).

One then expresses X as a union of a family of compact definable subsets $\Omega_k \subset X_k$, such that every element of X_k is conjugate to some element of Ω_k. Finally, a filtration indexed by $\mathbf{Z}$ is placed on $\Omega = \{\Omega_n\}_{n\in\mathbf{Z}}$, and from this definable family of definable compact subsets one creates a family $\{\xi_{a,n}\}, a \in \mathbf{Q}[\![t]\!]^d \times \mathbf{Z}^d, n \in \mathbf{Z}$ of definable functions which specialize (for $k \in \mathcal{C}_{M'}$) to a subset of $C_c^\infty(X)$ which is dense in the L^∞ topology ([CCGS11, Corollary 3.2]).

It follows that it is enough to check that the motivic functions g and g' arising in Equation (6.2) are the same for T and T'; by the argument just given, if these functions coincide, then the distributions T and T' will coincide as well.

By the transfer principle we quote above, the nullity of the motivic function $(g - g')_k$ on S_k (for $k \in \mathcal{C}_{M'}$) depends only on the isomorphism class of κ_k, and so in particular is independent of the characteristic of k. $\square$

We dedicate the remainder of this section to the proof of Theorem 6.7.1. We will refer to our fixed choices by the numbering in Section 6.2.

Proof (of Theorem 6.7.1). We will construct a motivic distribution Ξ_z and show that it coincides with the distribution J_G from the trace formula, in large residue characteristic. We refer to the fixed choices from Section 6.2 by the numbering established there.

The definable set $Z_{{}_i\Psi}$ whose existence we assert is the definable set we constructed in Proposition 6.3.1; we also have the set $\Lambda_{{}_i\Psi}$ of elements λ_M we named there. We used these sets to construct the groups $\mathbf{G}_z$, $\mathbf{M}_{z,\lambda}$ and $\mathbf{T}_z$; in Corollary 6.5.1, we constructed a family of definable sets $(Y_{{}_i\Psi,z})_{z \in Z_{{}_i\Psi}}$ which parameterized the sets of k-points of the regular elements of the Lie algebras of each $\mathbf{T}_z$, and in Corollary 6.4.3 constructed a definable set of representatives for each $\mathscr{T}_{\mathbf{M}_z}$.

In Section 6.5 we saw that the sums over $\mathscr{L}$ and $\mathscr{T}_M$ can be considered as motivic integrals over finite sets; as mentioned there, the set $\mathscr{L}$ is determined by our fixed choice XII, and we have just indicated the dependence of the set $\mathscr{T}_M = \mathscr{T}_{\mathbf{M}_z}$ on the element z of $Z_{{}_i\Psi,k}$.

In Section 6.6 we demonstrated that the factors $D^g(X)$ and $v_M(x,y)$ are definable and motivic, respectively. The measures $\mathrm{d}\dot{x}$ and $\mathrm{d}\dot{y}$ were shown to be motivic in Lemma 6.5.2, and the measure $\mathrm{d}X$ is motivic by the discussion following Corollary 6.5.1.

Thus, given a family $(f_{1,s}, f_{2,s})_{s\in S}$ of Schwartz-Bruhat motivic test functions on $C_c^\infty(\mathbf{g}_z) \times C_c^\infty(\mathbf{g}_z)$ with parameter in S, it now follows immediately from Theorem 4.5.1 that $J_{\mathbf{G}_z}(f_{1,s}, f_{2,s})$ is a motivic function of $s \in S$, which completes the proof. $\square$

Chapter 7

Examples

In this chapter, we collect examples of many of the objects we constructed in Chapter 2; we do *not* construct any examples of definable or motivic functions, apart from the weight factor v_M. We begin with a motivating example.

7.1 Three classical groups

We begin by calculating the root data of the three classical groups $\mathrm{GL}(2), \mathrm{SL}(2)$, and $\mathrm{PGL}(2)$; this is chiefly due to their comparitive simplicity, and the relationships between the root data that arise from the short exact sequences

$$1 \to \mathrm{SL}(2) \to \mathrm{GL}(2) \to \mathbf{G}_m \to 1$$

and

$$1 \to \mathbf{G}_m \to \mathrm{GL}(2) \to \mathrm{PGL}(2) \to 1.$$

Let us assume that the field of definition is F and that the residue characteristic of F is not 2.

Example 7.1.1. For $G = \mathrm{GL}(2)$ let us take P_0 to be the subgroup of upper-triangular matrices, containing the maximal (split) torus T of diagonal elements. The character lattice of T has the form $\mathbf{Z}\chi_1 \oplus \mathbf{Z}\chi_2$, where

$$\chi_i : T(F) \to \mathbf{G}_m, \quad \chi_i \begin{pmatrix} x_1 & \\ & x_2 \end{pmatrix} = x_i, .$$

Dually, the cocharacter lattice is $X^\vee = \mathbf{Z}\lambda_1 \oplus \mathbf{Z}\lambda_2$, where

$$\lambda_1(t) = \begin{pmatrix} t & \\ & 1 \end{pmatrix}, \lambda_2(t) = \begin{pmatrix} 1 & \\ & t \end{pmatrix}.$$

The usual dot product identifies X with $X^\vee$, making λ_i the dual element to χ_i.

The Lie algebra $g(F)$ is simply $\mathrm{Mat}_{2\times 2}(F)$, and the calculation

$$\begin{pmatrix} x_1 & \\ & x_2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1^{-1} & \\ & x_2^{-1} \end{pmatrix} = \begin{pmatrix} a & x_1 x_2^{-1} b \\ x_1^{-1} x_2 c & d \end{pmatrix} \tag{7.1}$$

tells us that the weight space g_χ is nonzero exactly when $\chi \in \{1, \pm(\chi_1 - \chi_2)\}$, and so $R(G,T) = \{\pm\alpha\}$, where $\alpha = \chi_1 - \chi_2$.

The coroot in $X^\vee$ which is dual to α is by definition the element of $X^\vee$ associated to the element

$$\alpha^\vee = \frac{2}{(\alpha, \alpha)}\alpha = \frac{2}{\|(1,-1)\|^2}\alpha = \alpha,$$

and so $\alpha^\vee = \lambda_1 - \lambda_2$. We confirm that

$$(\alpha \circ \alpha^\vee)(t) = \alpha((\lambda_1 - \lambda_2)(t)) = \alpha \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} = 2.$$

Thus, with $\alpha = \chi_1 - \chi_2, \alpha^\vee = \lambda_1 - \lambda_2$, we have that the root datum of GL(2) is

$$\Psi = (\mathbf{Z}\chi_1 \oplus \mathbf{Z}\chi_2, \{\pm\alpha\}, \mathbf{Z}\lambda_1 \oplus \mathbf{Z}\lambda_2, \{\pm\alpha^\vee\}).$$

Example 7.1.2. Now we will take $G = \mathrm{SL}(2)$ with P_0 again the subgroup of upper-triangular matrices; this is a subgroup of GL(2), and as such we must have that its cocharacter lattice lies in that of GL(2). The maximal torus T is still the set of diagonal matrices, although in this case it has rank 1.

It is not hard to check that $a\lambda_1 + b\lambda_2$ is a cocharacter of SL(2) if and only if $a = -b$, and so clearly a generator of the set of cocharacters is $\lambda' := \lambda_1 - \lambda_2 = \alpha^\vee$ (retaining the notation from Example 7.1.1); thus $X^\vee = \mathbf{Z}\alpha^\vee$, and we know that X must also have rank 1. Indeed, one can check that the group of characters of the maximal torus is generated by the character

$$\chi' \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} = t;$$

a quick calculation then shows us that $(\lambda')^\vee = 2\chi'$, which Equation (7.1) shows us to be one of two roots of T in G, the other being $-2\chi'$. Thus, using the notation of Example 7.1.1, we see that the root datum of SL(2) is

$$\Psi = (\mathbf{Z}(\tfrac{1}{2}(\chi_1 - \chi_2)), \{\pm\alpha\}, \mathbf{Z}(\lambda_1 - \lambda_2), \{\pm\alpha^\vee\}).$$

Note that $X/(\mathbf{Z}R) \cong \mathbf{Z}/2\mathbf{Z}$ and $X^\vee/(\mathbf{Z}R^\vee) \cong 0$ (i.e. the trivial group).

Example 7.1.3. The third of our related examples is not a subgroup, but a quotient, of one of the groups we have already treated. Let Z denote the centre of GL(2) (i.e. the algebraic subgroup of scalar matrices), and let us consider $G = \mathrm{PGL}(2) = \mathrm{GL}(2)/Z$.

By the universal property of quotients, a character on G is a character on GL(2) that is trivial on Z. Retaining our notation from Example 7.1.1, we see that the subgroup of characters that are trivial on Z is generated by the element $\chi_1 - \chi_2$; it follows at once that the character lattice of PGL(2) is naturally isomorphic to $\mathbf{Z}(\chi_1 - \chi_2) = \mathbf{Z}\alpha$ (retaining our notation from Example 7.1.1).

Finally, we consider the natural surjection from cocharacters of the maximal torus of GL(2) to cocharacters of the maximal torus of G, induced by the surjection $\mathrm{GL}(2) \to G$. It is not hard to check that cocharacters λ, λ' in GL(2) will map to the same cocharacter in G if and only if there exists some $n \in \mathbf{Z}$ with

$$\lambda(t) = \lambda'(t) \cdot \begin{pmatrix} t^n & \\ & t^n \end{pmatrix},$$

and so we have

$$X^\vee = (\mathbf{Z}\lambda_1 \oplus \mathbf{Z}\lambda_2)/\mathbf{Z}(\lambda_1 + \lambda_2) \cong \mathbf{Z}\big(\tfrac{1}{2}(\lambda_1 - \lambda_2)\big).$$

It is now clear that every element of $X^\vee$ has the form

$$t \mapsto \begin{pmatrix} t^n & \\ & 1 \end{pmatrix} \cdot Z \text{ for some } n \in \mathbf{Z},$$

and we have that the root datum of G is

$$\Psi = (\mathbf{Z}(\chi_1 - \chi_2), \{\pm\alpha\}, \mathbf{Z}(\tfrac{1}{2}(\lambda_1 - \lambda_2)), \{\pm\alpha^\vee\}).$$

We note in this case that $X/(\mathbf{Z}R) \cong 0$ and $X^\vee/(\mathbf{Z}R^\vee) \cong \mathbf{Z}/2\mathbf{Z}$.

We say that the root datum of SL(2) is *dual* to the root datum of PGL(2), in the sense that if $(X, R, X^\vee, R^\vee)$ is the root datum of SL(2), then $(X^\vee, R^\vee, X, R)$ is the root datum of PGL(2).

We summarize the relationship between the character lattices of these three groups in the figure below:

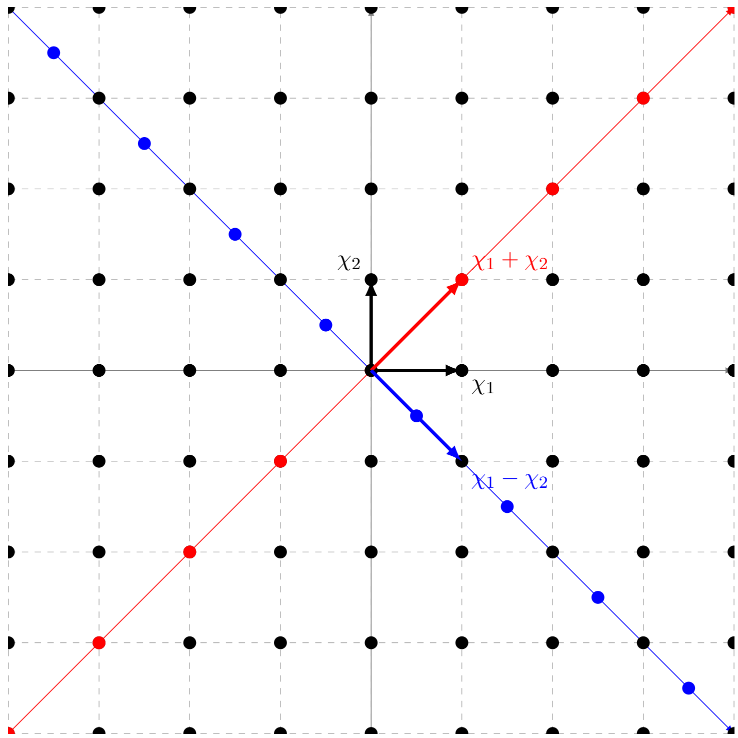

Figure 7.1: Characters of three classical groups

The (red and black) lattice points on the plane are all the characters of GL(2); those lying on the red line through the origin are trivial on both SL(2) and PGL(2). The black points on the blue line through the origin are the characters of PGL(2), while *all* drawn points (black and blue) on the blue line are characters of SL(2).

7.2 SL(2)

Throughout this section, the symbol G denotes the algebraic group SL(2), and the field F of definition of G is assumed to have characteristic not equal to 2.

Example 7.2.1. In this example we will compute the sets $\mathscr{F}$, $\mathscr{L}$, and $\mathscr{P}$, compute the parabolic subgroups as limits, and demonstrate the correspondence between choice of base D of the root system $R(G,T)$ and choice of minimal parabolic subgroup of G.

We already computed the root datum of G in Example 7.1.2. As before, choose P_0 to be the subgroup of upper-triangular matrices, so that $T = M_0$ is the split maximal torus

$$\left\{ \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} : t \in F^\times \right\}.$$

The opposite parabolic to P_0 is $\bar{P}_0 = P_0^t$, the set of lower-triangular matrices, as we verify by the observation that $M_0 = P \cap P_0^t$ (we emphasise that, in general, the opposite parabolic is *not* the transpose).

Thus, we can identify X and $X^\vee$ with $\mathbf{Z}$, which identifies α with 2 (or -2) and $\alpha^\vee$ with 1 or -1, respectively; under this identification, the pairing $\langle , \rangle$ is ordinary integer multiplication. We obviously have $A_G = 1, A_{M_0} \cong \mathbf{G}_m$, and we record the sets of subgroups of G we defined:

- $\mathscr{F}$ consists of $P_0, \bar{P}_0$, and G.
- $\mathscr{L}$ consists of M_0 and G itself.

- $\mathscr{P}$ consists of P_0 and $\bar{P}_0$.

From these observations, it follows easily that $\mathfrak{a}_0$ is a one-dimensional real vector space, in which any nonzero element is regular. Let $\lambda \in \mathfrak{a}_0$ be positive; by our observation that $P(\lambda)$ depends only on the Weyl facet of λ, we can and will assume that $\lambda = \alpha^\vee$. Then (recalling the action $*_\lambda$ from Equation (2.4))

$$t *_\lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & t^2 b \\ t^{-2} c & d \end{pmatrix},$$

and so clearly

$$\lim_{t \to 0} t *_\lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ exists } \iff c = 0;$$

that is, $P(\alpha^\vee) = P_0$ is the subgroup of upper-triangular matrices. A similar calculation shows that $P(-\alpha^\vee) = \bar{P}_0$, the subgroup of lower-triangular matrices. Finally, as $0(t) = 1$ (for $0 \in \mathfrak{a}_0$), it is easily seen that $P(0) = G$.

Because X is one-dimensional, the only possible choices of base for R are $\{\alpha\}$ and $\{-\alpha\}$; the first of these bases corresponds to our choice of minimal parabolic subgroup (containing T), as our above calculations show; in particular, for us, $D_0 = \{\alpha\}$.

Because every parabolic subgroup in question is $P_0, \bar{P}_0$, or G, this description includes all bases of the form D_P^Q where $P \subseteq Q \subseteq G$ are parabolic subgroups. The only proper subset of the base D_0 is the empty set, which corresponds to the parabolic subgroup G. Evidently, the positive chamber of $\mathfrak{a}_{M_0}$ is the positive ray $\mathbf{R}_{>0}$.

Example 7.2.2. In this example we compute the Weyl group of the root system of G, the Harish-Chandra homomorphism associated to each parabolic subgroup, and the modulus character associated to the minimal parabolic, and use these to compute the measures on the groups G, M_0, and N_{P_0}. Notation is retained from Example 7.2.1, with the exception that the field of definition is now $\mathbf{Q}_p$ (with $p \neq 2$).

By definition we know $\mathfrak{g} = \mathfrak{sl}_2$ is the Lie algebra of traceless 2×2 matrices, and if χ is any rational character on G, then

$$G/\ker\chi \cong \operatorname{Im}\chi \leq \mathbf{G}_m,$$

from which it follows that $[G,G] \leq \ker\chi$; but $[G,G] = G$ and so χ must be trivial. Dually, it follows that $a_G = 0$.

The Levi component of our minimal parabolic subgroup P_0 is the maximal torus

$$M_0 = \left\{ \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} : t \in F^\times \right\},$$

whose character group is isomorphic to $\mathbf{Z}$ and is generated by the homomorphism

$$\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \mapsto t.$$

Then $\mathfrak{a}_{M_0} = \mathfrak{a}_{M_0}^G \cong \mathbf{R}$, and the homomorphism H_{M_0} is defined

$$H_{M_0} \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} = (\log |t|_F) \in a_{M_0}.$$

The domain of H_{M_0} is extended to G by our decomposition $G = M_0N_0K$, where N_0 is the subgroup of upper-triangular unipotent matrices, and K can be taken to be $G(\mathcal{O})$, which contains the matrix $\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$.

This element together with the identity give a full set of coset representatives of the quotient group $\mathrm{Norm}_G(M_0)/M_0 \cong \mathbf{Z}/2\mathbf{Z}$; that is, W^G (clearly W^{M_0} is trivial). Indeed: we know that T equals its own centralizer in G, and so if $t = \begin{pmatrix} x & \\ & y \end{pmatrix} \in T(F_s)$ is conjugated by some element n of $\mathrm{Norm}_G(T)(F_s)$, we must have

$$ntn^{-1} = t \text{ or } ntn^{-1} = \begin{pmatrix} y & \\ & x \end{pmatrix},$$

which is exactly the image of $\begin{pmatrix} x & \\ & y \end{pmatrix}$ under conjugation by $\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$.

We remark here that, while $M_{P_0} = M_{\bar{P}_0}$, it is *not* the case that $H_P = H_{\bar{P}}$: this is due to the fact that

$$H_P(mnk) := H_{M_P}(m),$$

given a decomposition $x = mnk$ (recall Equation 2.1). Clearly, the decomposition of $x \in G(F)$ with respect to P is different from the decomposition with respect to $\bar{P}$.

Per Example 7.2.1, we know

$$D_0 = \{2\} \text{ and } D_0^\vee = \{1\}.$$

We see how the Harish-Chandra homomorphism allows us to calculate the measures on our groups (see Chapter 2.7, above): observe that, if $t = p^{-k}x \in \mathbf{Q}_p$ with $x \in \mathbf{Z}_p^\times$ and $k \geq 0$, then

$$\begin{pmatrix} 1 & t \\ & 1 \end{pmatrix} = \begin{pmatrix} p^{-k} & \\ & p^k \end{pmatrix} \begin{pmatrix} 1 & \\ p^{-k}x^{-1} & 1 \end{pmatrix} \begin{pmatrix} p^k & x \\ -x^{-1} & 1 \end{pmatrix}.$$

On the other hand, if $k < 0$ then $\begin{pmatrix} 1 & t \\ & 1 \end{pmatrix}$ lies in K and so

$$H_{\bar{P}_0} \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix} = H_{\bar{P}_0}(1) = 0.$$

From these observations it follows that

$$H_{\bar{P}_0} \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix} = \begin{cases} 0 & \text{if } t \in \mathbf{Z}_p, \\ \log |t|_F & \text{if } t \notin \mathbf{Z}_p. \end{cases}$$

(We remark that these values are in fact vectors in a_{M_0}, and not, strictly speaking, real numbers). The half-sum of roots (recall Definition 2.5.7) associated to P_0 is 1, whereas that associated to $\bar{P}_0$ is -1; thus, our measure on N_{P_0} is defined by the condition

$$1 = \int_{N_{P_0}} \exp\left(2\langle -1, H_{\bar{P}_0}(n)\rangle\right)\, \mathrm{d}n = \int_{N_{P_0}} \exp(-2H_{\bar{P}_0}(n))\, \mathrm{d}n;$$

the isomorphism $N_{P_0} \cong \mathbf{G}_a(\mathbf{Q}_p)$ allows us to write $n \in N_{P_0}$ as $n = \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix}$. These co-ordinates allow us to use the usual additive Haar measure $\mathrm{d}t$ on $\mathbf{Q}_p$; recall that this measure satisfies

$$\int_{\mathbf{Z}_p} \mathrm{d}t = 1,$$

from which it follows immediately that

$$\int_{\mathbf{Z}_p^\times} \mathrm{d}t = 1 - \frac{1}{p}.$$

Therefore, to compute the integral over N_{P_0} – that is, over $\mathbf{Q}_p$ – we need only add the integrals over $\mathbf{Z}_p$ and $\mathbf{Q}_p - \mathbf{Z}_p$. Having already computed the former, we now compute the latter: we have

$$\begin{aligned}
\int_{\mathbf{Q}_p - \mathbf{Z}_p} \exp\left(2\left\langle -1, H_{\bar{P}_0}\begin{pmatrix} 1 & t \\ & 1 \end{pmatrix}\right\rangle\right) \mathrm{d}t \\
= \sum_{k=1}^{\infty} \int_{p^{-k}\mathbf{Z}_p^\times} \exp\left(2\left\langle -1, H_{\bar{P}_0}\begin{pmatrix} 1 & t \\ & 1 \end{pmatrix}\right\rangle\right) \mathrm{d}t \\
= \sum_{k=1}^{\infty} \int_{p^{-k}\mathbf{Z}_p^\times} |p^{-2k}|\,\mathrm{d}t = \sum_{k=1}^{\infty} p^{-2k} \int_{p^{-k}\mathbf{Z}_p^\times} \mathrm{d}t = \sum_{k=1}^{\infty} p^{-k}\left(1 - \frac{1}{p}\right) \\
= \left(1 - \frac{1}{p}\right)\sum_{k=1}^{\infty} p^{-k} = \left(1 - \frac{1}{p}\right)\left(\frac{p^{-1}}{1 - p^{-1}}\right) = \frac{1}{p}.
\end{aligned}$$

It follows that using the usual Haar measure yields the equation

$$\int_{N_{P_0}} \exp\left(2\langle -1, H_{\bar{P}_0}(n)\rangle\right) \mathrm{d}n = 1 + \frac{1}{p},$$

and we deduce that the measure we want is that which has been scaled by this constant; namely, if $n \in N_{P_0}$ is written $n = \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix}$, then $\mathrm{d}n = \frac{\mathrm{d}t}{1+p^{-1}}$. The calculation

$$\begin{pmatrix} a & b \\ & a^{-1} \end{pmatrix}\begin{pmatrix} x & y \\ & x^{-1} \end{pmatrix} = \begin{pmatrix} ax & ay + bx^{-1} \\ & (ax)^{-1} \end{pmatrix}$$

shows that $\frac{\mathrm{d}x\,\mathrm{d}y}{x^2}$ is a left Haar measure on P_0; similarly, the equation

$$\begin{pmatrix} x & y \\ & x^{-1} \end{pmatrix}\begin{pmatrix} a & b \\ & a^{-1} \end{pmatrix} = \begin{pmatrix} ax & bx + a^{-1}y \\ & (ax)^{-1} \end{pmatrix}$$

shows $\mathrm{d}x\,\mathrm{d}y$ to be a right Haar measure on P_0. It follows at once that the modulus character is

$$\delta_{P_0}\begin{pmatrix} x & y \\ & x^{-1}\end{pmatrix} = x^2.$$

We can take K to be $\mathrm{SL}_2(\mathbf{Z}_p)$ and $\mathrm{d}k$ to be Haar measure on K so that $\int_K \mathrm{d}k = 1$. On

$$M_0 = \left\{\begin{pmatrix} y & \\ & y^{-1}\end{pmatrix} : y \in \mathbf{Q}_p^\times\right\}$$

we place the Haar measure $\mathrm{d}m = \frac{\mathrm{d}y}{y}$; Haar measure $\mathrm{d}x$ on G then satisfies

$$\int_G f(x)\,\mathrm{d}x = \int_{M_0\times N_{P_0}\times K} f\left(\begin{pmatrix} y & \\ & y^{-1}\end{pmatrix}\begin{pmatrix} 1 & t \\ & 1\end{pmatrix} k\right)\,\mathrm{d}k\frac{\mathrm{d}t}{1+p^{-1}}\frac{\mathrm{d}y}{y},$$

and we have our measures in co-ordinates.

Example 7.2.3. In this example we compute the weight factor associated to the minimal Levi subgroup of G. We retain the notation from Examples 7.2.1 and 7.2.2, allowing again the field of definition to be arbitrary (with residue characteristic not equal to 2).

In Example 7.2.1 we computed

$$D_0 = D_0^G = \{\alpha\} \text{ where } \alpha \leftrightarrow 2; \text{ all other } D_P^Q = \emptyset.$$

There are precisely two subsets of D_0, namely $\emptyset$ and D_0 itself; these correspond respectively to the parabolic subgroups G and P_0. The only interesting spaces of the form a_P^Q are $a_{P_0} = a_{P_0}^G$ and its linear dual, both of which are one-dimensional.

Because $\alpha^\vee = 1$ we have that $\mathrm{vol}(a_{P_0}^G/L_{P_0}^G) = 1$ and thus $\theta_{P_0}(\lambda) = \lambda$ (as defined in equation (2.10)). Finally, for the weight factor, we fix any nonzero λ and consider separately the cases $M = G$ and $M = M_0$. The first case is simple: $v_G(x, y) = 0$, because H_G is identically zero (this is clear by considering its codomain).

In the second case, we have

$$\begin{aligned} v_M(x,y) &= \frac{(-1)\left(\lambda\left(H_{P_0}(y) - H_{\bar{P}_0}(x)\right)\right)}{\lambda} + \frac{(-1)\left(\lambda\left(H_{\bar{P}_0}(y) - H_{P_0}(x)\right)\right)}{\lambda} \\ &= -\left(H_{P_0}(y) + H_{\bar{P}_0}(y) - (H_{P_0}(x) + H_{\bar{P}_0}(x))\right), \end{aligned}$$

which is evidently independent of λ. We remark that, while λ is technically a linear functional, we can identify it with a real number (having identified a_0 with $\mathbf{R}$).

7.3 GL(3)

In this section we repeat many of the examples from Section 7.2 for the group $G = \mathrm{GL}(3)$. Due to the computational complexity of this case over the case $G = \mathrm{SL}(2)$, as well as its close similarity with the computation in Example 7.1.1, we will assert the root datum of G without performing the full calculations we saw there.

Throughout this section, the field of definition F of G is assumed to have residue characteristic not equal to 2 or 3. We caution that the large diagrams that arise in our examples necessarily give rise to unwieldy typesetting.

Example 7.3.1. In this example we will compute the sets $\mathscr{F}$, $\mathscr{L}$, and $\mathscr{P}$, compute the parabolic subgroups as limits, and demonstrate the correspondence between choice of base D of the root system $R(G,T)$ and choice of minimal parabolic subgroup of G.

Let P_0 be the subgroup of upper-triangular matrices. We will take $T = M_0$ to be the split maximal torus

$$\left\{ \begin{pmatrix} x & & \\ & y & \\ & & z \end{pmatrix} : x, y, z \in \mathbf{Q}_p^\times \right\};$$

then the pair (G,T) has root datum $(X^*, R, X_*, R^\vee)$, where:

- X^* is the free abelian group generated by the morphisms $\chi_i : T \to$

$\mathbf{G}_m, i \in \{1,2,3\}$, given by

$$\chi_i \begin{pmatrix} x & & \\ & y & \\ & & z \end{pmatrix} = \begin{cases} x & \text{if } i = 1; \\ y & \text{if } i = 2; \\ z & \text{if } i = 3. \end{cases}$$

- R is the subset $\{\pm\alpha, \pm\beta, \pm(\alpha+\beta)\}$ of X^*, where $\alpha = \chi_1 - \chi_2$ and $\beta = \chi_2 - \chi_3$; that is,

$$\alpha \begin{pmatrix} x & & \\ & y & \\ & & z \end{pmatrix} = xy^{-1} \text{ and } \beta \begin{pmatrix} x & & \\ & y & \\ & & z \end{pmatrix} = yz^{-1};$$

- X_* is the free abelian group generated by the morphisms $\lambda_i : \mathbf{G}_m \to T, i \in \{1,2,3\}$, given by

$$\lambda_1(t) = \begin{pmatrix} t & & \\ & 1 & \\ & & 1 \end{pmatrix}, \lambda_2(t) = \begin{pmatrix} 1 & & \\ & t & \\ & & 1 \end{pmatrix}, \lambda_3(t) = \begin{pmatrix} 1 & & \\ & 1 & \\ & & t \end{pmatrix};$$

- $R^\vee$ is the subset of X_* consisting of $\{\pm\alpha^\vee, \pm\beta^\vee, \pm(\alpha^\vee + \beta^\vee)\}$, where $\alpha^\vee = \lambda_1 - \lambda_2$ and $\beta^\vee = \lambda_2 - \lambda_3$; that is,

$$\alpha^\vee(t) = \begin{pmatrix} t & & \\ & t^{-1} & \\ & & 1 \end{pmatrix} \text{ and } \beta^\vee(t) = \begin{pmatrix} 1 & & \\ & t & \\ & & t^{-1} \end{pmatrix}.$$

This time, the pairing between X^* and X_* is given by the dot product (in our co-ordinates), and so we identify X^* with $\mathbf{Z}^3$ in the obvious way, so that

$$\alpha \leftrightarrow (1,-1,0) \text{ and } \beta \leftrightarrow (0,1,-1);$$

thus the pairing $\langle , \rangle$ identifies α with $\alpha^\vee$ and β with $\beta^\vee$ in this case. The calculation

$$\begin{pmatrix} t^\ell & & \\ & t^m & \\ & & t^n \end{pmatrix} \begin{pmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{pmatrix} \begin{pmatrix} t^{-\ell} & & \\ & t^{-m} & \\ & & t^{-n} \end{pmatrix}$$
$$= \begin{pmatrix} \gamma_{11} & t^{\ell-m}\gamma_{12} & t^{\ell-n}\gamma_{13} \\ t^{m-\ell}\gamma_{21} & \gamma_{22} & t^{m-n}\gamma_{23} \\ t^{n-\ell}\gamma_{31} & t^{n-m}\gamma_{32} & \gamma_{33} \end{pmatrix}$$

shows us that the subgroup $P_{\alpha,\beta}$ of upper-triangular matrices equals $P(\lambda)$ if and only if

$$\lambda(t) = \begin{pmatrix} t^\ell & & \\ & t^m & \\ & & t^n \end{pmatrix}, \text{ where } \ell > m > n;$$

taking $(1,0,-1)$ as our representative in this Weyl facet, we have $P_{\alpha,\beta} = P(1,0,-1)$, and similarly $P_{-\alpha,-\beta} = P(-1,0,1)$. The positive chamber of $a_{M_{\alpha,\beta}}$ (corresponding to our choice $P_{\alpha,\beta}$ of minimal parabolic subgroup) therefore corresponds to the subset

$$\{(x,y,z) \in a_{M_{\alpha,\beta}} : x > y > z\} \subset a_{M_{\alpha,\beta}}.$$

The base corresponding to this chamber is evidently $D_{P_{\alpha,\beta}} = \{\alpha, \beta\}$; in fact, we have labelled our subgroups so that the base corresponding to minimal parabolic P_{α_1,α_2} is precisely $\{\alpha_1, \alpha_2\}$.

A quick calculation shows that there are thirteen parabolic subgroups of G containing M_0, each of which is characterized by the condition that certain entries be zero, or arbitrary. As such, we can represent them symbolically: for instance, the subgroup of upper-triangular matrices is the subgroup of matrices of the form

$$\begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix},$$

where $*$ denotes an arbitrary entry (as usual, blank entries are zero). In the below image, we name six parabolic subgroups besides G, whose opposites coincide (in this case) with the subgroups consisting of their transposes (of course, this is also trivially true of G):

$$P_{\alpha,\beta} = \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix}, \quad \bar{P}_{\alpha,\beta} = P_{-\alpha,-\beta} = \begin{pmatrix} * & & \\ * & * & \\ * & * & * \end{pmatrix},$$

$$P_{\alpha+\beta,-\alpha} = \begin{pmatrix} * & & * \\ * & * & * \\ & & * \end{pmatrix}, \quad \bar{P}_{\alpha+\beta,-\alpha} = P_{-\alpha-\beta,\alpha} = \begin{pmatrix} * & * & \\ & * & \\ * & * & * \end{pmatrix},$$

$$P_{\beta,-\alpha-\beta} = \begin{pmatrix} * & & \\ * & * & * \\ * & & * \end{pmatrix}, \quad \bar{P}_{\beta,-\alpha-\beta} = P_{-\beta,\alpha+\beta} = \begin{pmatrix} * & * & * \\ & * & \\ & * & * \end{pmatrix},$$

$$P_{\alpha,\alpha+\beta} = \begin{pmatrix} * & * & * \\ & * & * \\ & * & * \end{pmatrix}, \quad \bar{P}_{\alpha,\alpha+\beta} = P_{-\alpha,-\alpha-\beta} = \begin{pmatrix} * & & \\ * & * & * \\ * & * & * \end{pmatrix},$$

$$P_{\alpha+\beta,\beta} = \begin{pmatrix} * & * & * \\ * & * & * \\ & & * \end{pmatrix}, \quad \bar{P}_{\alpha+\beta,\beta} = P_{-\alpha-\beta,-\beta} = \begin{pmatrix} * & * & \\ * & * & \\ * & * & * \end{pmatrix},$$

$$P_{\beta,-\alpha} = \begin{pmatrix} * & & * \\ * & * & * \\ * & & * \end{pmatrix}, \quad \bar{P}_{\beta,-\alpha} = P_{-\beta,\alpha} = \begin{pmatrix} * & * & * \\ & * & \\ * & * & * \end{pmatrix}.$$

We illustrate the inclusions of parabolic subgroups by arrows in the diagram below:

$$\begin{tikzcd}
 & & G & & & \\
P_{\alpha+\beta,\beta} \arrow[urr] & P_{\alpha,\alpha+\beta} \arrow[ur] & P_{\beta,-\alpha} \arrow[u] & P_{-\beta,\alpha} \arrow[ul] & P_{-\alpha,-\alpha-\beta} \arrow[ull] & P_{-\alpha-\beta,-\beta} \arrow[ulll] \\
P_{\alpha,\beta} \arrow[u] \arrow[ur] & P_{\alpha+\beta,-\alpha} \arrow[ul] \arrow[ur] & P_{\beta,-\alpha-\beta} \arrow[u] \arrow[urr] & P_{-\beta,\alpha+\beta} \arrow[ull] \arrow[u] & P_{-\alpha-\beta,\alpha} \arrow[ul] \arrow[ur] & P_{-\alpha,-\beta} \arrow[ul] \arrow[u]
\end{tikzcd}$$

This diagram makes clear the following statements:

- $\mathscr{F}$ consists of $P_{\alpha,\beta}, P_{\alpha+\beta,-\alpha}, P_{\beta,-\alpha-\beta}, P_{\alpha,\alpha+\beta}, P_{\alpha+\beta,\beta}, P_{\beta,-\alpha}$, their opposites, and G itself.
- $\mathscr{L}$ consists of $M_{\alpha,\beta}, M_{\alpha,\alpha+\beta}, M_{\alpha+\beta,\beta}, M_{\alpha,-\beta}$ and G itself.
- $\mathscr{P}$ consists of $P_{\alpha,\beta}, P_{\alpha+\beta,-\alpha}, P_{\beta,-\alpha-\beta}$, and their opposites.

Retaining the notation $P(\lambda)$ from above, we have:

- $P_{\alpha,\beta} = P(1,0,-1)$ and $P_{-\alpha,-\beta} = P(-1,0,1)$;
- $P_{\alpha+\beta,-\alpha} = P(0,1,-1)$ and $P_{-\alpha-\beta,\alpha} = P(0,-1,1)$;
- $P_{\beta,-\alpha-\beta} = P(-1,1,0)$ and $P_{-\beta,\alpha+\beta} = P(1,-1,0)$;
- $P_{\alpha,\alpha+\beta} = P(1,0,0)$ and $P_{-\alpha,-\alpha-\beta} = P(-1,0,0)$;
- $P_{\beta,-\alpha} = P(0,1,0)$ and $P_{-\beta,\alpha} = P(0,-1,0)$;
- $P_{\alpha+\beta,\beta} = P(0,0,-1)$ and $P_{-\alpha-\beta,-\beta} = P(0,0,1)$; and

- $G = P(0,0,0) = \bar{G}$.

The Levi factors are easily categorized:

- The Levi factor of each of the minimal parabolic subgroups, i.e. of

  $$P_{\alpha,\beta},\ P_{\alpha+\beta,-\alpha},\ P_{\beta,-\alpha-\beta},\ P_{-\alpha,-\beta},\ P_{-\alpha-\beta,+\alpha},\ \text{and}\ P_{-\beta,\alpha+\beta},$$

  is $M_{\alpha,\beta} = M_0$, i.e. the subgroup of diagonal matrices.

- The Levi factor of $P_{\alpha,\alpha+\beta}$ and $P_{-\alpha,-\alpha-\beta}$ is $M_{\alpha,\alpha+\beta} = \begin{pmatrix} * & & \\ & * & * \\ & * & * \end{pmatrix}$.

- The Levi factor of $P_{\alpha+\beta,\beta}$ and $P_{-\alpha-\beta,-\beta}$ is $M_{\alpha+\beta,\beta} = \begin{pmatrix} * & * & \\ * & * & \\ & & * \end{pmatrix}$.

- The Levi factor of $P_{\beta,-\alpha}$ and $P_{-\beta,\alpha}$ is $M_{\alpha,-\beta} = \begin{pmatrix} * & & * \\ & * & \\ * & & * \end{pmatrix}$.

Again we can take $K = G(\mathcal{O})$; in this case, M_0 is normalized by the subgroup generated by the elementary matrices

$$\begin{pmatrix} & -1 & \\ 1 & & \\ & & 1 \end{pmatrix},\ \begin{pmatrix} 1 & & \\ & & -1 \\ & 1 & \end{pmatrix},\ \text{and}\ \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix}.$$

It is not hard to show in this case that W^G is isomorphic to the permutation group on three letters.

Finally, the space a_{M_0} is the real vector space generated by all maps of the form

$$\left(t \mapsto \begin{pmatrix} t^{e_1} & & \\ & t^{e_2} & \\ & & t^{e_3} \end{pmatrix}\right) \mapsto \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \cdot \begin{pmatrix} \ell \\ m \\ n \end{pmatrix},$$

as (ℓ, m, n) varies over all elements of $\mathbf{Z}^3$; identifying this map with the vector $(\ell, m, n) \in a_{M_0} \cong \mathbf{R}^3$, we see that a_G is the subspace of $\mathbf{R}^3$ generated by the element $(1, 1, 1)$. Thus we have an identification

$$a_{M_0}^G = \{(x, y, z) \in \mathbf{R}^3\}/\sim,$$

where

$$(x_1, y_1, z_1) \sim (x_2, y_2, z_2) \iff x_1 - x_2 = y_1 - y_2 = z_1 - z_2;$$

equivalently, $a_{M_0}^G$ is the quotient of $\mathbf{R}^3$ by $\mathbf{R}(1, 1, 1)$.

Example 7.3.2. In this example we compute the Harish-Chandra homomorphism associated to our minimal Levi subgroup M_0 of G. Notation is retained from Example 7.3.1.

Denote by P_0 again the subgroup of upper-triangular matrices (so again $\bar{P}_0$ is the subgroup of lower-triangular matrices and M_0 is the subgroup of diagonal matrices). The only rational characters on G are integer exponents of the determinant: this is a consequence of the fact that any rational character on G corresponds to a character on the abelianization, which in this case is isomorphic to $\mathbf{G}_m$ (recall the similar observation in Example 7.2.2).

Because M_0 is F-split of rank 3, it admits an F-isomorphism $M_0 \cong \mathbf{G}_m^3$, and so its rational characters have the form

$$\begin{pmatrix} x & & \\ & y & \\ & & z \end{pmatrix} \mapsto x^\ell y^m z^n, \qquad \ell, m, n \in \mathbf{Z}. \tag{7.2}$$

It follows that $a_G \cong \mathbf{R}$, and thus that the function H_G is defined

$$H_G(\gamma) = (\log |\det \gamma|_F) \in a_G, \qquad \gamma \in G.$$

Similarly, an easy calculation gives $a_{M_0} \cong \mathbf{R}^3$ and

$$H_{M_0} \begin{pmatrix} x & & \\ & y & \\ & & z \end{pmatrix} = \begin{pmatrix} \log |x|_F \\ \log |y|_F \\ \log |z|_F \end{pmatrix}.$$

We have $A_G = Z(G) \cong \mathbf{G}_m$ (i.e. the scalar matrices), and $A_{M_0} = M_0$ because M_0 is split, as noted above.

Example 7.3.3. In this example we calculate the weight factor $v_0 = v_{M_0}$ associated to the Levi subgroup $M_0 = T$. We will use notation established in Examples 7.3.1 and 7.3.2.

As before, identify a_0 with $\mathbf{R}^3$ under the identification

$$\lambda_1 \leftrightarrow (1,0,0), \quad \lambda_2 \leftrightarrow (0,1,0), \quad \lambda_3 \leftrightarrow (0,0,1),$$

as before, so that $a_G = \mathbf{R}(1,1,1)$ and $a_0^G = a_{P_0}^G$ is the plane Π defined by the equation $x + y + z = 0$. The roots α and β which span the root system of the pair (G, T) are identified, respectively, with the vectors $(1, -1, 0)$ and $(0, 1, -1)$, which evidently lie in Π; the dual statement *mutatis mutandis* is true for $\alpha^\vee$ and $\beta^\vee$, and the pairing $\langle , \rangle$ coincides with the dot product in these coordinates.

Usual Lebesgue measure on $\mathbf{R}^3$ satisfies our hypothesis of Weyl group invariance, and the condition of regularity for $\lambda \in \mathrm{i}a_0^*$ is equivalent to the condition that λ_1, λ_2, and λ_3 are all distinct, and distinct modulo $\mathbf{R}(1,1,1)$.

In the definition of v_M, the index of summation is the set $\mathscr{P}(M_0)$ of minimal parabolic subgroups of G which contain M_0; these are

$$P_{\alpha,\beta}, \quad P_{-\beta,\alpha+\beta}, \quad P_{-\alpha-\beta,\alpha}, \quad P_{-\alpha,-\beta}, \quad P_{\beta,-\alpha-\beta}, \quad P_{\alpha+\beta,-\alpha}.$$

Let us consider first $P_{\alpha,\beta}$, i.e. the subgroup of upper-triangular matrices. The space a_P^G is exactly Π, for every $P \in \mathscr{P}(M_0)$; the lattice $L_{P_{\alpha,\beta}}^G$ is the $\mathbf{Z}$-span of the set ${}_F D_{P_{\alpha,\beta}}^\vee$. A fundamental domain for $L_{P_{\alpha,\beta}}$ in Π is the parallelogram with vertices at

$$(0,0,0), \quad (0,-1,1), \quad (1,0,-1), \quad \text{and } (1,-1,0),$$

whose volume (area) is $\sqrt{3}$. Evidently, this volume is independent of choice of $P \in \mathscr{P}(M_0)$, as all of the lattices L_P^G coincide.

From the definitions we have

$$\theta_P(\lambda) = \frac{1}{\mathrm{vol}(a_P^G / L_P^G)} \prod_{\alpha \in {}_F D_P} \lambda(\alpha^\vee),$$

from which – writing $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ – we deduce immediately

$$\begin{aligned}
\theta_{P_{\alpha,\beta}}(\lambda) &= \frac{1}{\sqrt{3}}(\lambda \cdot \alpha^\vee)(\lambda \cdot \beta^\vee) = \frac{(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)}{\sqrt{3}}, \\
\theta_{P_{-\beta,\alpha+\beta}}(\lambda) &= \frac{1}{\sqrt{3}}(\lambda \cdot (-\beta^\vee))(\lambda \cdot (\alpha^\vee + \beta^\vee)) = \frac{(\lambda_3 - \lambda_2)(\lambda_1 - \lambda_3)}{\sqrt{3}}, \\
\theta_{P_{-\alpha-\beta,\alpha}}(\lambda) &= \frac{1}{\sqrt{3}}(\lambda \cdot (-\alpha^\vee - \beta^\vee))(\lambda \cdot \alpha^\vee) = \frac{(\lambda_3 - \lambda_1)(\lambda_1 - \lambda_2)}{\sqrt{3}}, \\
\theta_{P_{-\alpha,-\beta}}(\lambda) &= \frac{1}{\sqrt{3}}(\lambda \cdot (-\alpha^\vee))(\lambda \cdot (-\beta^\vee)) = \frac{(\lambda_3 - \lambda_2)(\lambda_2 - \lambda_1)}{\sqrt{3}}, \\
\theta_{P_{\beta,-\alpha-\beta}}(\lambda) &= \frac{1}{\sqrt{3}}(\lambda \cdot \beta^\vee)(\lambda \cdot (-\alpha^\vee - \beta^\vee)) = \frac{(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_3)}{\sqrt{3}}, \\
\theta_{P_{\alpha+\beta,-\alpha}}(\lambda) &= \frac{1}{\sqrt{3}}(\lambda \cdot (\alpha^\vee + \beta^\vee))(\lambda \cdot (-\alpha^\vee)) = \frac{(\lambda_2 - \lambda_1)(\lambda_1 - \lambda_3)}{\sqrt{3}},
\end{aligned}$$

and $\deg \theta_P = 2$ for all P. Observe that

$$\begin{aligned}
\theta_{P_{-\alpha,-\beta}}(\lambda) = \theta_{P_{\alpha,\beta}}(\lambda) &= \frac{(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)}{\sqrt{3}}, \\
\theta_{P_{-\beta,\alpha+\beta}}(\lambda) = \theta_{P_{\beta,-\alpha-\beta}}(\lambda) &= \frac{(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_1)}{\sqrt{3}}, \\
\theta_{P_{-\alpha-\beta,\alpha}}(\lambda) = \theta_{P_{\alpha+\beta,-\alpha}}(\lambda) &= \frac{(\lambda_3 - \lambda_1)(\lambda_1 - \lambda_2)}{\sqrt{3}},
\end{aligned}$$

and so we can simplify the resulting expression for v_0. So doing, we calculate (for regular $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ in $\mathrm{i}\mathfrak{a}_0^*$) that $v_0(x, y)$ equals

$$\begin{aligned}
&\frac{\sqrt{3}}{2}\Bigg(\frac{\big(\lambda(H_{P_{\alpha,\beta}}(y) - H_{P_{-\alpha,-\beta}}(x))\big)^2 + \big(\lambda(H_{P_{-\alpha,-\beta}}(y) - H_{P_{\alpha,\beta}}(x))\big)^2}{(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)} \\
&+ \frac{\big(\lambda(H_{P_{-\beta,\alpha+\beta}}(y) - H_{P_{\beta,-\alpha-\beta}}(x))\big)^2 + \big(\lambda(H_{P_{\beta,-\alpha-\beta}}(y) - H_{P_{-\beta,\alpha+\beta}}(x))\big)^2}{(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_1)} \\
&+ \frac{\big(\lambda(H_{P_{-\alpha-\beta,\alpha}}(y) - H_{P_{\alpha+\beta,-\alpha}}(x))\big)^2 + \big(\lambda(H_{P_{\alpha+\beta,-\alpha}}(y) - H_{P_{-\alpha-\beta,\alpha}}(x))\big)^2}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_1)}\Bigg).
\end{aligned}$$

In case x and y both lie in $M_0(F)$, all maps H_{P_i} coincide with the map H_{M_0} and the weight factor vanishes; in general, this is not the case.

To illustrate, let us choose some $t = p^{-\ell}u \in F^\times$ with $v_F(u) = 0$ and put

$$x = \begin{pmatrix} 1 & & t \\ & 1 & \\ & & 1 \end{pmatrix} \text{ and } y = \begin{pmatrix} 1 & & \\ & 1 & t \\ & & 1 \end{pmatrix}.$$

We have the equations

$$x = \begin{pmatrix} p^{-\ell} & & \\ & 1 & \\ & & p^{\ell} \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ p^{-\ell}u^{-1} & & 1 \end{pmatrix} \begin{pmatrix} p^{\ell} & & u \\ & 1 & \\ -u^{-1} & & \end{pmatrix}$$

and

$$y = \begin{pmatrix} 1 & & \\ & p^{-\ell} & \\ & & p^{\ell} \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & p^{-\ell}x^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & p^{\ell} & x \\ & -x^{-1} & \end{pmatrix},$$

and we compute:

$$H_{P_{\alpha,\beta}}(x) = H_{P_{-\beta,\alpha+\beta}}(x) = H_{P_{\alpha+\beta,-\alpha}}(x) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

$$H_{P_{-\alpha,-\beta}}(x) = H_{P_{\beta,-\alpha-\beta}}(x) = H_{P_{-\alpha-\beta,\alpha}}(x) = \begin{pmatrix} \ell \\ 0 \\ -\ell \end{pmatrix},$$

and similarly

$$H_{P_{\alpha,\beta}}(y) = H_{P_{\alpha+\beta,-\alpha}}(y) = H_{P_{\beta,-\alpha-\beta}}(y) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

$$H_{P_{-\alpha,-\beta}}(y) = H_{P_{-\alpha-\beta,\alpha}}(y) = H_{P_{-\beta,\alpha+\beta}}(y) = \begin{pmatrix} 0 \\ \ell \\ -\ell \end{pmatrix}.$$

The terms we exponentiate in the weight factor are

$$\lambda\left(H_{P_{\alpha,\beta}}(y) - H_{P_{-\alpha,-\beta}}(x)\right) = \begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{pmatrix} \cdot \begin{pmatrix}-\ell\\ 0\\ \ell\end{pmatrix} = \ell(\lambda_3 - \lambda_1),$$

$$\lambda\left(H_{P_{-\alpha,-\beta}}(y) - H_{P_{\alpha,\beta}}(x)\right) = \begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{pmatrix} \cdot \begin{pmatrix}0\\ \ell\\ -\ell\end{pmatrix} = \ell(\lambda_2 - \lambda_3),$$

$$\lambda\left(H_{P_{-\beta,\alpha+\beta}}(y) - H_{P_{\beta,-\alpha-\beta}}(x)\right) = \begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{pmatrix} \cdot \begin{pmatrix}-\ell\\ \ell\\ 0\end{pmatrix} = \ell(\lambda_2 - \lambda_1),$$

$$\lambda\left(H_{P_{\beta,-\alpha-\beta}}(y) - H_{P_{-\beta,\alpha+\beta}}(x)\right) = \begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{pmatrix} \cdot \begin{pmatrix}0\\ 0\\ 0\end{pmatrix} = 0,$$

$$\lambda\left(H_{P_{-\alpha-\beta,\alpha}}(y) - H_{P_{\alpha+\beta,-\alpha}}(x)\right) = \begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{pmatrix} \cdot \begin{pmatrix}0\\ \ell\\ -\ell\end{pmatrix} = \ell(\lambda_2 - \lambda_3),$$

$$\lambda\left(H_{P_{\alpha+\beta,-\alpha}}(y) - H_{P_{-\alpha-\beta,\alpha}}(x)\right) = \begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{pmatrix} \cdot \begin{pmatrix}-\ell\\ 0\\ \ell\end{pmatrix} = \ell(\lambda_3 - \lambda_1).$$

Thus

$$\begin{aligned}
v_0(x,y) &= \frac{\sqrt{3}\ell^2}{2}\bigg(\frac{(\lambda_3-\lambda_1)^2+(\lambda_2-\lambda_3)^2}{(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)} \\
&\qquad + \frac{(\lambda_2-\lambda_1)^2}{(\lambda_2-\lambda_3)(\lambda_3-\lambda_1)} + \frac{(\lambda_2-\lambda_3)^2+(\lambda_3-\lambda_1)^2}{(\lambda_3-\lambda_1)(\lambda_1-\lambda_2)}\bigg) \\
&= \frac{\sqrt{3}\ell^2}{2(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_3-\lambda_1)}\Big((\lambda_3-\lambda_1)^3+(\lambda_3-\lambda_1)(\lambda_2-\lambda_3)^2 \\
&\qquad + (\lambda_1-\lambda_2)^3+(\lambda_3-\lambda_1)^2(\lambda_2-\lambda_3)+(\lambda_2-\lambda_3)^3\Big) \\
&= \frac{2\sqrt{3}\ell^2(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_3-\lambda_1)}{2(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)(\lambda_3-\lambda_1)} \\
&= \sqrt{3}\ell^2.
\end{aligned}$$

Thus $v_0(x,y)$ is indeed well-defined for any regular $\lambda \in \mathrm{i}a_0^*$, and is independent of choice of such λ, as claimed. Finally, we confirm that this is, in fact, the volume of the convex hull of the set

$$\{-H_P(y)+H_{\overline{P}}(x) : P \in \mathscr{P}(M)\};$$

our above work shows these points to be

$$\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}-\ell\\0\\\ell\end{pmatrix}, \begin{pmatrix}0\\\ell\\-\ell\end{pmatrix}, \text{ and } \begin{pmatrix}-\ell\\\ell\\0\end{pmatrix},$$

which describe a parallelogram in the plane Π, whose area is therefore

$$\left|\begin{pmatrix} -\ell \\ 0 \\ \ell \end{pmatrix} \times \begin{pmatrix} 0 \\ \ell \\ -\ell \end{pmatrix}\right| = |(-\ell^2, -\ell^2, -\ell^2)^t| = \sqrt{3}\ell^2,$$

as claimed.

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