THE MOTIVIC WEIGHT OF THE STACK OF 
AZUMAYA ALGEBRAS 
OVER AN ELLIPTIC CURVE 

by 

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The following individuals certify that they have read, and recommend to the Faculty of Graduate and Postdoctoral Studies for acceptance, the thesis entitled:

The Motivic Weight of the Stack of Azumaya Algebras over an Elliptic Curve

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Abstract

This thesis studies the motivic weight of the stack of Azumaya algebras on an elliptic curve, with the goal to express it in terms of the motivic zeta function. This proves an interesting case of the conjecture by Behrend and Dhillon. The general conjecture is notable not only for computing the motive of the stack of $G$-torsors, but also for its implication in defining the motivic Tamagawa number.
Lay Summary

Remember how we draw with pencils in kindergarten. In high school, we also draw curves, such as a circle using the equation $x^2 + y^2 = 1$, or a parabola using $y = x^2$. Algebraic curves are just like those curves above, potentially defined by more complex equations. So we have two ways to look at our objects: geometrically via the figures we draw, and algebraically via the equations. That’s where Algebraic Geometry lies: at the interface between geometry and algebra.

My research is in Algebraic Geometry and within it, moduli spaces. These are spaces that serve as maps classifying certain kinds of objects. You choose a type of object, such as the triangles that we learn in high school geometry. Then every point of this moduli space corresponds to exactly one triangle. The moduli spaces then are special, like a bird’s eye view. Thus, studying them reveals very intrinsic structures central to the theory.
Preface

This thesis is an original unpublished work done in collaboration with and under the guidance of Professor Kai Behrend.

He explained that the conjecture was resolved for $\mathbb{P}^1$ because all the vector bundles on this curve has been classified. After some Google search, I realized that Atiyah also classified all the vector bundles on elliptic curves as well. This spurred our direction.

My initial intention was to work with the vector bundles, i.e $GL_2$-torsors. He suggested that $PGL_2$-torsors, i.e Azumaya algebras, are more challenging and hence more interesting. He explained that some Azumaya algebras arise as Endomorphism groups of vector bundles. But is this all the Azumaya algebras there are? Three days of Google searches later, I brought him Tsen’s theorem, which says yes for a big class of cases. After verifying this for half an hour, he agreed. And so we were on our merry way.

The hardest case of the thesis is on the rank 2 decomposable vector bundle whose line bundles both have degree 0. He suggested a solution for this, using his past experience with the case. After thinking about this solution for a weekend, I realized that we needed a different group action for this argument to work. We spent a day trying out different actions but didn’t get what we want. Then we independently found that the adjoin action would resolve this (me by looking for all possible actions of the group on vector spaces). So the thesis was completed.
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I appreciate my friends at the math department who have made the time here fun and enjoyable. I am thankful for the home-stay family that has given me a home in Vancouver, and for Billie the dog who is always up for a walk.
1 Introduction and context

We aim to express the motivic weight of the moduli stack $\mathfrak{Bun}_{\text{PGL}_2}$ of PGL$_2$-torsors on elliptic curves in terms of the motivic zeta function. More precisely, we will show that for any elliptic curve $X$ over an algebraically closed field,

$$\mu(\mathfrak{Bun}_{\text{PGL}_2} X) = 2Z(X, q^{-2}). \quad (1.1)$$

This proves an interesting case of the much more general conjectural formula by Behrend and Dhillon in [B-D2006]. The proposed formula expresses the motive of the stack of $\mathfrak{Bun}_G$ in terms of the motivic zeta function:

$$\mu(\mathfrak{Bun}_G C) = |\pi_1(G)| q^{(g-1)\dim G} \prod_{i=1}^r Z(C, q^{-d_i}),$$

where $C$ is any smooth projective geometrically connected curve over any ground field $k$, and $G$ is any split connected semisimple algebraic group over $k$.

The generality of the proposed conjecture, together with its implications on defining the motivic Tamagawa number ([B-D2006]), suggests that there is plenty to be done.
2 Motivic zeta function

Given a curve $X$, the motivic zeta function is the generating function for the motivic weight of $X^{(n)}$, where

\[ X^{(n)} = \text{the scheme of effective divisors of degree } n \text{ on } X \]
\[ = \text{the Hilbert scheme of subschemes of length } n \text{ of } X \]
\[ = X^n/S_n, \text{ the n-th symmetric power of } X. \]

The general form of the motivic zeta function is

\[ Z(X, t) = 1 + \sum_{n>0} [X^{(n)}] t^n. \]

It has been shown that $Z(X, t)$ is a rational function in $t$ with denominator $\frac{1}{1-t} \cdot \frac{1}{1-Lt}$. For example, it is an easy exercise to show that for the projective line,

\[ Z(\mathbb{P}^1, t) = \frac{1}{1-t} \cdot \frac{1}{1-Lt}. \]

Similarly, the motivic zeta function for an elliptic curve $X$ can be computed to be

\[ Z(X, t) = 1 + \frac{\mu(X)t}{(1-t)(1-Lt)}. \]

Thus, the right hand side of Equation (1.1) is

\[ 2Z(X, q^{-2}) = 2 + 2 \frac{\mu(X)q}{(q-1)(q^2-1)} \]

\[(2.1)\]
3 PGL$_2$-torsors as Azumaya algebras

The projective group PGL$_n$ can also be interpreted as the automorphism group Aut($M_n$) of the square matrices $M_n$. The group $M_n$ can be further interpreted as the endomorphism algebra of the rank $n$ vector space. Over topological spaces, algebraic varieties, or schemes, the generalization of the endomorphism algebras of vector bundles are Azumaya algebras. This gives an equivalence of categories and stacks between the PGL$_n$-torsors and the Azumaya algebras.

Therefore, to compute the motivic weight of the moduli stack of PGL$_n$-torsors over an elliptic curve $X$, we can equivalently compute the motivic weight of the stack of Azumaya algebras of rank $n^2$ (the automorphisms are encoded in the structure of the stack).

A well known fact is that Azumaya algebras can arise from vector bundles:

**Lemma 3.1.** Let $E$ be a vector bundle of rank $n$ on a scheme $X$. Then End($E$) is an Azumaya algebra of rank $n^2$.

For curves over algebraically closed fields, it turns out that these are all the Azumaya algebras there are, as a consequence of Tsen’s theorem:

**Lemma 3.2.** Over algebraically closed fields, any rank $n^2$ Azumaya algebras on a curve $X$ is isomorphic to the endomorphism group of some rank $n$ vector bundles on $X$.

**Proof.** The short exact sequence of groups $G_m \to GL_n \to PGL_n$ gives rise to the long exact sequence of cohomology groups

$$
\begin{align*}
H^1(X, G_m) &\longrightarrow H^1(X, GL_n) \longrightarrow H^1(X, PGL_n) \longrightarrow H^2(X, G_m)
\end{align*}
$$

The group of $H^2(X, G_m)$ is also called the Brauer group $Br(X)$ of $X$. There is always an injection $Br(X) \hookrightarrow Br(k(X))$, where $Br(k(X))$ is the Brauer group of the function field $k(X)$ of $X$.

Tsen’s theorem states that the Brauer group of the function field of an algebraic curve over an algebraically closed field is trivial. Hence, when $k$ is algebraically closed, $Br(k(X)) = 0$, which implies that $Br(X) = 0$. 

3
The long exact sequence becomes

\[ H^1(X, G_m) \xrightarrow{f} H^1(X, GL_n) \xrightarrow{g} H^1(X, PGL_n) \xrightarrow{} 0 \quad (3.1) \]

Therefore, the group \( H^1(X, GL_n) \) of isomorphism classes of vector bundles surjects onto the group \( H^1(X, PGL_n) \) of isomorphism classes of Azumaya algebras. And all Azumaya algebras arise from vector bundles.

Exactness of (3.1) says that the kernel of the map \( g \) is the image of the group of isomorphism classes of line bundles \( H^1(X, G_m) \) under \( f \). This shows that:

**Lemma 3.3.** Over an algebraically closed field, two Azumaya algebras \( \text{End}(E) \) and \( \text{End}(E') \) over a curve are isomorphic if and only if there exists a line bundle \( L \) such that \( E \otimes L \cong E' \).

Henceforth, we denote \( E \sim E' \) if their corresponding Azumaya algebras are isomorphic.
4 Representative vector bundles for Azumaya algebra classes

Let $E(r, d)$ denote the set of isomorphism classes of vector bundles of rank $r$ and degree $d$ over a variety $X$. In accordance with Atiyah, let us denote $\mathcal{E}(r, d)$ the set of isomorphism classes of indecomposable vector bundles of rank $r$ and degree $d$ over $X$.

Given any line bundle $L$ of degree 1, we have the map $E(r, d) \rightarrow E(r, d + r)$ given by $E \rightarrow E \otimes L$. This map is bijective. Any vector bundle $E' \in E(r, d + r)$ is of the form $E \otimes L$ for some vector bundle $E \in E(r, d)$. Thus, Lemma 3.3 implies that $E(r, d)$ and $E(r, d + r)$ contribute the same isomorphism classes of Azumaya algebras. We denote this $E(r, d) \sim E(r, d + r)$.

Applying this to rank 2 vector bundles, we get $E(2, 0) \sim E(2, 2) \sim E(2, 4) \sim \ldots$, and $E(2, 1) \sim E(2, 3) \sim E(2, 5) \sim \ldots$. Therefore, $E(2, 0)$ and $E(2, 1)$ contribute the full set of isomorphism classes of rank 2 Azumaya algebras on $X$. We will calculate their motivic weights in the rest of the thesis.
5 Motivic weight of Automorphisms of Azumaya algebras on Elliptic curves

A vector bundle is either indecomposable or decomposable. The computations will follow this order.

5.1 Indecomposable vector bundles

Our calculation of motivic weights of automorphisms of Azumaya algebras on an elliptic curve uses the classification of vector bundles on elliptic curves by Atiyah in [Atiyah57].

For the indecomposable case, Theorem 7 in [Atiyah57] states that for elliptic curves, any $E(r,d)$ can be identified with the curve $X$ itself:

**Theorem 5.1.** Let $A$ be a fixed base point on an elliptic curve $X$. We may regard $X$ as an abelian variety with $A$ as the zero element. Then each set $E(r,d)$ of isomorphism classes of indecomposable vector bundles over $X$ of dimension $r$ and degree $d$ may be identified with $X$ in such a way that

\[
\det : E(r,d) \to E(1,d) \text{ corresponds to } H : X \to X
\]

where $H(x) = hx = x + x + \cdots + x$ ($h$ times), and $h = (r,d)$ is the highest common factor of $r$ and $d$.

Furthermore, Theorem 10 in [Atiyah57] completely shows the structure on the indecomposable vector bundles on elliptic curves:

**Theorem 5.2.** For an elliptic curve $X$, every indecomposable vector bundle $E \in E(r,d)$ of rank $r$ and degree $d$ is of the form $L \otimes E_A(r,d)$, where $L$ is some line bundle, and $E_A(r,d)$ is the indecomposable vector bundle in $E(r,d)$ identified with the zero element on $X$.

Furthermore, $L \otimes E_A(r,d) \cong E_A(r,d)$ if and only $L^{r/h} \cong \mathcal{O}$, where $h = (r,d)$.

Theorem 5.2 together with Lemma 3.3 shows that each set $E(r,d)$ gives rise to only one class of Azumaya algebra. In particular, we get one class of Azumaya algebra from the odd case $E(2,1)$, and one class from the even case $E(2,0)$.

A useful proposition for computing in the indecomposable cases is the following:
Proposition 5.3. For any vector bundle $E$ on a locally Noetherian scheme, there is a short exact sequence

$$1 \to \text{Aut}(E)/k^* \to \text{Aut}(\text{End}(E)) \to \{\text{line bundles } | L \otimes E \cong E \} \to 1 \quad (5.1)$$

Proof. Consider the morphism from the sheaf of automorphisms $\text{Aut}(E)$ of the vector bundle $E$, to the sheaf $\text{Aut}(\text{End}(E))$ of automorphisms of the Azumaya algebra $\text{End}(E)$:

$$\text{Aut}(E) \to \text{Aut}(\text{End}(E))$$

This morphism sends an automorphism $u$ of the vector bundle to the associated inner automorphism $a \to uau^{-1}$ of $\text{End}(E)$. The morphism is sheaf-surjective because every automorphism of an Azumaya algebra on a locally Noetherian scheme is Zariski locally inner, by Skolem-Noether theorem (Proposition 3.2 in [Presotto15])

The kernel of the above morphism is the units of the trivial sheaf $\mathcal{O}^*$, essentially because the center of a matrix algebra consists of scalar matrices. So we have a short exact sequence of sheaves of groups on the variety:

$$1 \to \mathcal{O}^* \to \text{Aut}(E) \to \text{Aut}(\text{End}(E)) \to 1$$

Passing to the associated long exact sequence in cohomology by applying the global section functor, we get:

$$1 \to k^* \to \text{Aut}(E) \to \text{Aut}(\text{End}(E)) \to H^1(\mathcal{O}^*) \to H^1(\text{Aut}(E)) \quad (5.2)$$

Note that now the first two “Aut” stand for the automorphism groups, not sheaves, since they are the global section groups of the previous sheaves.

Now, notice that $H^1(\mathcal{O}^*)$ classifies the line bundles, and that a line bundle $L$ maps to $1$ in $H^1(\text{Aut}(E))$ if and only if $L \otimes E \cong E$. Thus after compacting (5.2), the desired short exact sequence follows.

5.1.1 Indecomposable odd vector bundles

Let $E \in \mathcal{S}(2,1)$ be an indecomposable vector bundle of rank 2 and degree 1 on an elliptic curve $X$. Then $E$ is stable, and hence has automorphism group $k^*$. 

Theorem 5.2 gives us that the set of line bundles $L$ such that $L \otimes E \cong E$ is $X[2] = \{\mathcal{O}, \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_1 \oplus \mathcal{E}_2\}$, i.e the four line bundles whose square is the trivial bundle $\mathcal{O}$.

Applying the exact sequence in Proposition 5.3, we have that the automorphism group of the Azumaya algebra $\text{End}(E)$ arising from the rank 2 indecomposable odd case is $\text{Aut}(\text{End}(E)) \cong X[2] \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. This then implies that the motivic contribution of this automorphism group is 1.

### 5.1.2 Indecomposable even vector bundles

Let $E \in \mathcal{E}(2,0)$ be an indecomposable vector bundle of rank 2 and degree 0 on an elliptic curve $X$. $E$ is now semi-stable. So the road to find its automorphism group is a bit longer.

By Theorem 5 in [Atiyah57], there exists a unique vector bundle $F_2 \in \mathcal{E}(2,0)$ with $\Gamma(F_2) \neq 0$. Choose this as the representative vector bundle of the Azumaya algebra class. Using Atiyah’s classification, we can compute its automorphism group:

**Lemma 5.4.** The unique indecomposable vector bundle $F_2 \in \mathcal{E}(2,0)$ has automorphism group isomorphic to $k \times k^*$.  

**Proof.** Theorem 5 (i) in [Atiyah57] states that there is an exact sequence

\[
0 \to \mathcal{O} \to F_2 \to \mathcal{O} \to 0 \tag{5.3}
\]

Applying the functor $\text{Hom}(-, F_2)$ to the short exact sequence (5.3), we get the long exact sequence of cohomology:

\[
0 \to \text{Hom}(\mathcal{O}, F_2) \to \text{End}(F_2) \to \text{Hom}(\mathcal{O}, F_2) \xrightarrow{\delta} \text{Ext}(\mathcal{O}, F_2) \tag{5.4}
\]

We can show that the boundary map $\delta : \text{Hom}(\mathcal{O}, F_2) \to \text{Ext}(\mathcal{O}, F_2)$ vanishes. Indeed, $\text{Hom}(\mathcal{O}, F_2)$ is generated by the canonical map $\mathcal{O} \to F_2$. The image of this canonical map is the extension of $\mathcal{O}$ by $F_2$

\[
\mathcal{O} \to M \to F_2 \tag{5.5}
\]

where $M$ is given by the push-out diagram

\[
\begin{array}{ccc}
\mathcal{O} & \longrightarrow & F_2 \\
\downarrow & & \downarrow \\
F_2 & \longrightarrow & M
\end{array}
\]
By the universal property of the push-out, the identity map $F_2 \to F_2$ induced a map $F_2 \to M$ that compose to the identity with the right hand side map of (5.5). Thus the short exact sequence (5.5) splits, and hence this extension is trivial in the extension group $	ext{Ext}(\mathcal{O}, F_2)$.

So the long exact sequence (5.4) becomes

$$0 \longrightarrow \text{Hom}(\mathcal{O}, F_2) \longrightarrow \text{End}(F_2) \longrightarrow \text{Hom}(\mathcal{O}, F_2) \overset{\delta}{\longrightarrow} 0 \quad (5.6)$$

Now, once we show that $\text{Hom}(\mathcal{O}, F_2) \cong k$, then it will follow that $\text{End}(F_2) \cong k^2$.

Indeed, applying the functor $\text{Hom}(\mathcal{O}, -)$ to the short exact sequence $\mathcal{O} \to F_2 \to \mathcal{O}$, we get the long exact sequence in cohomology

$$0 \longrightarrow \text{Hom}(\mathcal{O}, \mathcal{O}) \longrightarrow \text{Hom}(\mathcal{O}, F_2) \longrightarrow \text{Hom}(\mathcal{O}, \mathcal{O}) \overset{\delta}{\longrightarrow} \text{Ext}^1(\mathcal{O}, \mathcal{O}) \quad (5.7)$$

Of course $\text{Hom}(\mathcal{O}, \mathcal{O}) = k$. And $\text{Ext}^1(\mathcal{O}, \mathcal{O}) = k$ as well because there’s only one nontrivial extension given by $\mathcal{O} \to F_2 \to \mathcal{O}$. The boundary map $\delta$ maps the identity map $\mathcal{O} \to \mathcal{O}$ to this extension, and hence $\delta$ is an isomorphism.

Thus, compacting (5.7), we have $0 \longrightarrow \text{Hom}(\mathcal{O}, \mathcal{O}) \longrightarrow \text{Hom}(\mathcal{O}, F_2) \longrightarrow 0$. Hence $\text{Hom}(\mathcal{O}, F_2) \cong \text{Hom}(\mathcal{O}, \mathcal{O}) \cong k$.

With this, the exact sequence (5.6) implies that $\text{End}(F_2) \cong k^2$. We can explicitly write down the two linearly independent endomorphisms: the identity $F_2 \to F_2$ and the composition $F_2 \to \mathcal{O} \to F_2$. Every endomorphism of $F_2$ is a $k$-linear combination of these two. The units in this ring are $k \times k^*$.

Thus, $\text{Aut}(F_2) \cong k \times k^*$.

Theorem 5.2 gives us that the set of line bundles $L$ such that $L \otimes F_2 \cong F_2$ is trivial.

Applying the exact sequence in Proposition 5.3, we have that the automorphism group of the Azumaya algebra $\text{End}(F_2)$ arising from the rank 2 indecomposable even case is $\text{Aut}(\text{End}(F_2)) \cong k$. Therefore, the motivic contribution of this automorphism group is $1/q$.

5.2 Decomposable vector bundles

5.2.1 Odd degree decomposable vector bundles

Let $E \in E(2,1)$ be a decomposable vector bundle of rank 2 and degree 1 on an elliptic curve $X$. Then $E = M \oplus N$, for some line bundles $M$ and $N$, where $\deg M = m \geq 1$,
and \( \deg N = 1 - m \). Upon tensoring with \( N^{-1} \), we have \( E \sim \mathcal{O} \oplus (M \otimes N^{-1}) \). The line bundle \( M \otimes N^{-1} \) has degree \( 2m - 1 \). Thus, for each Azumaya algebra isomorphism class arising from such an \( E \), we can choose the representative vector bundle to be of the form \( E = \mathcal{O} \oplus P \), where \( P \) is a line bundle of degree \( 2m - 1 \), for some \( m \geq 1 \).

Let us denote by \( \mathcal{A}(2,1,m) \) the set of isomorphism classes of Azumaya algebra that is of the form \( \text{End}(\mathcal{O} \oplus P) \), where \( P \in \mathcal{E}(1, 2m - 1) \) is a line bundle of degree \( 2m - 1 \). For a fixed \( m \), \( \mathcal{A}(2,1,m) \) is parametrized by the line bundle \( P \). Using Theorem 5.1, \( \mathcal{A}(2,1,m) \) can be identified with the elliptic curve \( X \) itself. In other words, we have \( X \)-worth Azumaya algebra isomorphism classes for a fixed \( m \geq 1 \).

For \( E = \mathcal{O} \oplus P \), it can be shown directly that the set of line bundles \( L \) such that \( L \otimes E \cong E \) is trivial. Hence, all automorphisms of the Azumaya algebra \( \text{End}(E) \) arise from the automorphisms of the vector bundle \( E \). The latter can be directly computed. Thus we have that:

**Proposition 5.5.** Let \( E = \mathcal{O} \oplus P \), where \( P \) is a line bundle of degree \( 2m - 1 \) over an elliptic curve \( X \). Then \( \text{Aut}(\text{End}(E)) \cong k^* \times \Gamma(X,P) \).

The motivic weight of this automorphism group is \( \frac{1}{(q - 1)(q^{2m-1})} \).

**Proof.** An automorphism of \( E = \mathcal{O} \oplus P \) consists of four homomorphisms, from each line bundle in the source to each line bundles in the target.

Since \( \deg P = 2m - 1 \geq 1 > 0 = \deg \mathcal{O} \), there is only the zero map from \( P \) to \( \mathcal{O} \).

Homomorphisms from a line bundle to itself are precisely the scalar multiplications.

Homomorphisms from \( \mathcal{O} \) to \( P \) are \( \text{Hom}(\mathcal{O}, P) = \Gamma(X,P) \), which has dimension equal to \( \deg P = 2m - 1 \).

We can symbolically represent such automorphisms using matrix form:

\[
\begin{array}{c|cc}
\mathcal{O} & \mathcal{O} & P \\
\hline
\mathcal{O} & k^* & \Gamma(X,P) \\
P & 0 & k^*
\end{array}
\]

Automorphisms are units in the endomorphism ring, hence the determinant of the matrices need to be non-zero. So the scalar multiplications are in \( k^* \).

Therefore, \( \text{Aut}(E) \cong k^* \times k^* \times \Gamma(X,P) \).
Similar to that $\text{PGL}_2 = \text{GL}_2/k^*$, and since all automorphisms of the Azumaya algebras in this case arise from the automorphisms of the vector bundles, we have $\text{Aut}(\text{End}(E)) \cong \text{Aut}(E)/k^* \cong k^* \rtimes \Gamma(X, P)$.

Since $\Gamma(X, P)$ has dimension equal to $\deg P = 2m - 1$, the motivic weight of the automorphism group above is $\frac{1}{(q - 1)(q^{2m - 1})}$.

Summing up the motivic weights of the automorphism groups of $\mathcal{A}(2, 1, m)$ for $m$ from 1 to $\infty$, we get:

$$\mu(\mathcal{Bun}_{\text{PGL}_2}^{2, 1, \text{decomp}}, X) = \sum_{m=1}^{\infty} \frac{\mu(X)}{(q - 1)(q^{2m - 1})} = \frac{\mu(X)q}{(q - 1)(q^2 - 1)}.$$  

5.2.2 Even degree decomposable vector bundles

Let $E \in E(2, 0)$ be a decomposable vector bundle of rank 2 and degree 0 on an elliptic curve $X$. Then $E = M \oplus N$, for some line bundles $M$ and $N$, where $\deg M = m \geq 0$, and $\deg N = -m$. Upon tensoring with $N^{-1}$, we have $E \sim \mathcal{O} \oplus (M \otimes N^{-1})$. The line bundle $M \otimes N^{-1}$ has degree $2m$. Thus, for each Azumaya algebra isomorphism class arising from such an $E$, we can choose the representative vector bundle to be of the form $E = \mathcal{O} \oplus P$, where $P$ is a line bundle of degree $2m$, for some $m \geq 0$.

Case 1: $m > 0$

As before, let us denote by $\mathcal{A}(2, 0, m)$ the set of isomorphism classes of Azumaya algebra that is of the form $\text{End}(\mathcal{O} \oplus P)$, where $P \in \mathcal{E}(1, 2m)$ is a line bundle of degree $2m > 0$. For a fixed $m$, $\mathcal{A}(2, 1, m)$ is parametrized by the line bundle $P$. Using Theorem 5.1, $\mathcal{A}(2, 0, m)$ can be identified with the elliptic curve $X$ itself. In other words, we have $X$-worth Azumaya algebra isomorphism classes for a fixed $m \geq 1$.

Similar to the case of odd degree decomposable vector bundles, we have:

**Proposition 5.6.** Let $E = \mathcal{O} \oplus P$, where $P$ is a line bundle of degree $2m > 0$ over an elliptic curve $X$. Then $\text{Aut}(\text{End}(E)) \cong k^* \rtimes \Gamma(X, P)$.

The motivic weight of this automorphism group is $\frac{1}{(q - 1)q^{2m}}$.  

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The proof is similar to that of Proposition 5.5. And via the same logic,
\[ \mu(\text{Bun}_{\text{PGL}_2}^{2,0,\text{decomp},m>0}, X) = \sum_{m=1}^{\infty} \frac{\mu(X)}{(q-1)(q^2m)} = \frac{\mu(X)}{(q-1)(q^2-1)}. \]

Case 2: \( m = 0 \)

As in previous cases, to count the number of Azumaya algebras, we normalize the vector bundles so that the first line bundle is trivial. Analogously, there is \( X \)-worth Azumaya algebras of the form \( \text{End}(\mathcal{O} \oplus P) \), where \( P \) is a line bundle of degree 0.

The simplest case \( \text{End}(\mathcal{O} \oplus \mathcal{O}) \) has motivic weight contribution \( \mu(\text{B PGL}_2) \). This motivic weight is calculated by Bergh in [Bergh16]:

**Theorem 5.7.** Let \( k \) be any field in which 2 is invertible, and which contains all 2-nd roots of unity. Then we have \( \mu(\text{B PGL}_2) = 1/\mu(\text{PGL}_2) \) in \( K_0(\text{Stack}_k) \).

Thus in such fields where \( \text{char } k \neq 2 \), it is automatic that \( \mu(\text{B PGL}_2) = 1/\mu(\text{PGL}_2) = 1/q(q^2-1) \).

**Remark 5.8.** In such fields, Theorem 5.7 releases us from enforcing the torsor relations \( \mu(B G) = 1/\mu(G) \) on the ring of varieties for the case of \( \text{PGL}_2 \).

The case \( \text{End}(\mathcal{O} \oplus P) \), where \( P \neq \mathcal{O} \) is a non-trivial line bundle of degree 0, takes some more observations. The scheme \( X \setminus \{1 \text{ point}\} \) still gives rise to a versal family for the stack of Azumaya algebras of this kind. Indeed, any nontrivial line bundle of degree 0 on an elliptic curve has the form \( \mathcal{O}(p - p_0) \) for some point \( p \in X \) not the zero element \( p_0 \) on \( X \).

And vice versa, for a point \( p \neq p_0 \), take the corresponding vector bundle \( \mathcal{O} \oplus \mathcal{O}(p - p_0) \). Continuity of the versal family can be taken as the continuity of the points on \( X \). From here, it is only a matter of formality to check that this indeed satisfies the conditions of being a versal family.

However, we can no longer naively compute the automorphism groups of each Azumaya algebra of this kind and add them up, due to the extra symmetries of the versal family, namely the isomorphisms between distinct members of the versal family:

**Proposition 5.9.** Given an elliptic curve \( X \). Let \( X \setminus \{p_0\} \) parametrizes the stack of Azumaya algebras of the form \( \text{Aut}(\mathcal{O} \oplus \mathcal{O}(p - p_0)) \), where each point \( p \in X \) corresponds to
its obvious algebra. Then the symmetry groupoid of this versal family is

\[ N_{\text{PGL}_2} = N_{\text{GL}_2}/k^* = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \uplus \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \mid a, b, c, d \in k^* \right\} / k^* \]

**Proof.** A symmetry of a family of objects is an isomorphism of one member of the family with another member of the family. Suppose \( \text{Aut}(O \oplus O(p_1 - p_0)) \) is isomorphic to \( \text{Aut}(O \oplus O(p_2 - p_0)) \). For ease of notation, we denote \( M = O(p_1 - p_0) \), and \( N = O(p_2 - p_0) \). Then by Lemma 3.3, the isomorphism between the Azumaya algebra means that their respective vector bundles differ by tensoring with a line bundle \( L \):

\[
(O \oplus M) \otimes L \cong (O \oplus N) \iff \begin{cases}
O \otimes L \cong O \\
M \otimes L \cong N
\end{cases}
\]

\[
O \otimes L \cong O \\
M \otimes L \cong N
\]

\[
L \cong O \\
M \cong N
\]

In both cases, to count the isomorphisms between Azumaya algebras, we will first count the isomorphisms between the vector bundles:

**Case 1:** \( L \cong O \) and \( M \cong N \).

An isomorphism \( O \oplus M \to O \oplus N \) include four homomorphisms, from the two line bundles in the source to the two line bundles in the target. On an elliptic curve, homomorphisms from a line bundle to itself are precisely the scalar multiplications. Furthermore, the homomorphism group between two distinct line bundles of degree 0 has dimension 0, and hence consists only of the zero map. Thus, such an isomorphism can be symbolically represented via a matrix form:

\[
\begin{array}{ccc}
& O & M \\
O & k^* & 0 \\
M \cong N & 0 & k^*
\end{array}
\]

In other words, an isomorphism \( O \oplus M \to O \oplus N \) is a matrix of the form \( \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \), where \( a, d \in k^* \).

**Case 2:** \( L \cong N \) and \( M \cong N^{-1} \).

Then \( O \oplus M \cong O \oplus N^{-1} \). By tensoring the right hand side with \( N \) and using Lemma 3.3, we can replace the representative vector bundle \( O \oplus M \) with \( N \oplus O \) for the same Azumaya...
algebra. This enables us to consider isomorphisms $N \oplus \mathcal{O} \to \mathcal{O} \oplus N$. As before, such isomorphisms can be symbolically represented via a matrix form:

$$
\begin{array}{c|c|c}
N & \mathcal{O} & \mathcal{O} \\
\mathcal{O} & 0 & k^* \\
N & k^* & 0 \\
\end{array}
$$

In other words, an isomorphism $N \oplus \mathcal{O} \to \mathcal{O} \oplus N$ is a matrix of the form $\left(\begin{array}{cc}0 & b \\ c & 0 \end{array}\right)$, where $b, c \in k^*$.

The two cases above show that the symmetry groupoid of the stack of vector bundles of the form $\mathcal{O} \oplus \mathcal{O}(p - p_0)$, where each point $p \in X$ corresponds to its obvious vector bundle is

$$
N_{GL_2} = \left\{ \left(\begin{array}{cc}a & 0 \\ 0 & d \end{array}\right) \sqcup \left(\begin{array}{cc}0 & b \\ c & 0 \end{array}\right) \mid a, b, c, d \in k^* \right\}
$$

Similar to that $PGL_2 = GL_2/k^*$, and since all automorphisms of the Azumaya algebras arise from the automorphisms of the vector bundles, we found the symmetry groupoid of the stack of Azumaya algebras of the form $Aut(\mathcal{O} \oplus \mathcal{O}(p - p_0))$ to be $N_{PGL_2} = N_{GL_2}/k^*$.

**Remark 5.10.** The key ingredient in the proof is seeing that the stack of decomposable vector bundles $\mathcal{O} \oplus \mathcal{O}(p - p_0)$ has the symmetry groupoid $N_{GL_2}$. This is notated so because rank 2 vector bundles are $GL_2$-torsors. And Azumaya algebras are $PGL_2$-torsors. From here, noticing that $PGL_2$ is the quotient group of $GL_2$ by $k^*$ finishes the proof. This proof stays essentially the same when we work with other groups related to $GL_2$, such as $SL_2$.

**Remark 5.11.** Let $T_{PGL_2} = \left\{ \left(\begin{array}{cc}a & 0 \\ 0 & b \end{array}\right) \mid a, b \in k^* \right\} / k^*$ be the standard maximal torus of the algebraic group $PGL_2$. Then its normalizer is $N_{PGL_2}$, which is also the symmetry groupoid of the versal family above. This further explains our notation. Their quotient is the Weyl group $W = \{ \pm 1 \}$. This fact will help our proofs work out nicely in what follows.

With a versal family parametrized by $X - \{p_0\}$ and the symmetry groupoid $N_{PGL_2}$ acting on this family, the motivic weight contribution of Azumaya algebras of this case is $\mu(X - \{p_0\}/N_{PGL_2})$. We don’t yet have a direct formula to calculate this motivic weight. To work with this, we shall build a vector bundle on $X - \{p_0\}/N_{PGL_2}$ and take advantage of
the product formula. This method of calculating the motivic weight $\mu(X-\{p_0\}/_{\text{N}_{\text{PGL}_2}})$ is detailed in the rest of this section.

Up to now, we have normalized the decomposable vector bundles so that the first line bundle is the trivial one, and thus we were led to consider vector bundles of the form $\mathcal{O} \oplus \mathcal{O}(p-p_0)$. By tensoring with $\mathcal{O}(-p+p_0)$, we see that $\mathcal{O} \oplus \mathcal{O}(p-p_0) \sim \mathcal{O} \oplus \mathcal{O}(-p+p_0)$, i.e., they give rise to the same Azumaya algebra. In terms of the parametrizing space, this gives a double covering from $X-\{p_0\}$ to $\mathbb{P}^1-\{p_0\}$.

The group $N_{\text{PGL}_2}$ acting on $X-\{p_0\}$ is equivariant to the Weyl group $W = \{\pm 1\}$ acting on $\mathbb{P}^1-\{p_0\}$, via the exact sequence $T_{\text{PGL}_2} \to N_{\text{PGL}_2} \to W$. This leads us to the covering:

$$
\frac{X-\{p_0\}}{N_{\text{PGL}_2}} \quad \downarrow \quad \frac{\mathbb{P}^1-\{p_0\}}{W}
$$

(5.8)

This covering gives a nontrivial motivic weight relation, via the following proposition:

**Proposition 5.12.** Let $\tilde{Y} \to Y$ be a degree 2 unrammified covering of smooth algebraic varieties. Suppose the normalizer group $N_{\text{PGL}_2}$ acts on $\tilde{Y}$ in an equivariant way to $W$ acting on $Y$ under the map induced by the exact sequence $T_{\text{PGL}_2} \to N_{\text{PGL}_2} \to W$.

$$
\frac{\tilde{Y}}{N_{\text{PGL}_2}} \quad \downarrow \quad \frac{Y}{W}
$$

(5.9)

Then there is a motivic weight relation:

$$
\mu(\frac{\tilde{Y}}{N_{\text{PGL}_2}}) = \frac{\mu(Y)}{q+1} + \frac{\mu(\tilde{Y})}{q^2-1}.
$$

(5.10)

**Proof.** Consider the adjoin action of $\text{PGL}_2$ on the matrix group $M(2 \times 2)$:

$$
\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}
$$

(5.11)

where $[a \ b] \ c \ d]$ denotes the coset in $\text{PGL}_2$ represented by the matrix.

Restricting this to the action of $N_{\text{PGL}_2}$ on $\mathbb{A}^2$:

$$
\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & y \\ z & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & y \\ z & 0 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & ay \\ a^{-1}z & 0 \end{bmatrix}
$$
This action, together with the given action of $N_{PGL_2}$ on $\tilde{Y}$, builds a rank 2 vector bundle 
\[
\tilde{Y} \times_{N_{PGL_2}} \mathbb{A}^2 \rightarrow \tilde{Y}/_{N_{PGL_2}}.
\]
As a $N_{PGL_2}$-variety, $\mathbb{A}^2$ decomposes as
\[
\mathbb{A}^2 = G_m^2 \sqcup (G_m \times 0 \sqcup 0 \times G_m) \sqcup 0
\]
Consequently, $\tilde{Y} \times_{N_{PGL_2}} \mathbb{A}^2$ decomposes as
\[
\tilde{Y} \times_{N_{PGL_2}} \mathbb{A}^2 = (\tilde{Y} \times_{N_{PGL_2}} G_m^2) \sqcup (G_m \times 0 \sqcup 0 \times G_m) \sqcup \tilde{Y}/_{N_{PGL_2}}
\]
where the last equality follows because $N_{PGL_2} \cong (G_m \times 0 \sqcup 0 \times G_m)$.

The left hand side item in the above decomposition is a vector bundle of rank 2 over $\tilde{Y}/_{N_{PGL_2}}$ and thus has motivic weight $\mu(\tilde{Y} \times_{N_{PGL_2}} \mathbb{A}^2) = q^2 \cdot \mu(\tilde{Y}/_{N_{PGL_2}})$. Therefore, in terms of motivic weight, the above decomposition gives us:
\[
q^2 \cdot \mu(\tilde{Y}/_{N_{PGL_2}}) = \mu(\tilde{Y} \times_{N_{PGL_2}} G_m^2) + \mu(\tilde{Y}) + \mu(\tilde{Y}/_{N_{PGL_2}})
\]
(5.12)

The only unfamiliar term in the expression above is $\tilde{Y} \times_{N_{PGL_2}} G_m^2$. The following lemma, which uses the application of Morita equivalence, gives its motivic weight:

**Lemma 5.13.**
\[
\mu(\tilde{Y} \times_{N_{PGL_2}} G_m^2) = \mu(\tilde{Y} \times_W G_m) = (q - 1)\mu(Y)
\]

We temporarily defer the proof of this lemma towards the end of this section.

Now, applying Lemma 5.13, we can rewrite Equation (5.12) as
\[
q^2 \cdot \mu(\tilde{Y}/_{N_{PGL_2}}) = \mu(\tilde{Y} \times_{N_{PGL_2}} G_m^2) + \mu(\tilde{Y}) + \mu(\tilde{Y}/_{N_{PGL_2}})
\]
\[
= (q - 1)\mu(Y) + \mu(\tilde{Y}) + \mu(\tilde{Y}/_{N_{PGL_2}})
\]

After some basic arithmetic manipulation, we get
\[
\mu(\tilde{Y}/_{N_{PGL_2}}) = \frac{\mu(Y)}{q+1} + \frac{\mu(\tilde{Y})}{q^2-1}.
\]
This concludes the proof of Proposition 5.12.
To complete the proof of Proposition 5.12, the last step is proving Lemma 5.13. Before proceeding, let us reintroduce a popular application of Morita equivalence, which will be used in the proof of the lemma.

**Theorem 5.14.** Let \( \pi : X \to Y \) be a homomorphism of algebraic varieties, and let \( \pi' : G \to H \) be a homomorphism of algebraic groups. Suppose \( G \) acts on \( X \) and \( H \) acts on \( Y \) in such a way that is compatible with the two maps \( \pi \) and \( \pi' \). That is, the actions satisfy \( \pi(g \cdot x) = \pi'(g) \cdot \pi(x) \). Thus we have a commutative diagram

\[
\begin{array}{ccc}
X \times G & \longrightarrow & X \\
\downarrow \pi \times \pi' & & \downarrow \pi \\
Y \times H & \longrightarrow & Y
\end{array}
\]

Then there is a morphism of the quotient stacks \( X_G \to Y_H \). This is an isomorphism of stacks if the following two conditions are satisfied:

(i) topological full faithfulness: the following commutative diagram is a pullback diagram

\[
\begin{array}{ccc}
X \times G & \xrightarrow{(\cdot, \text{id})} & X \times X \\
\downarrow \pi \times \pi' & & \downarrow \pi \\
Y \times H & \xrightarrow{(\cdot, \text{id})} & Y \times Y
\end{array}
\]

(ii) essential surjectivity: the map \( X \times H \to Y \) defined by \( (x, h) \to h \cdot \pi(x) \) admits local section.

In the etale topology, any smooth surjective map of varieties admits local section. This condition can be verified in place of (ii).

This theorem is a direct application of Morita equivalence, which can be found in most texts discussing stacks, for example exercise 1.94 in [Behrend11].

We now give the proof of Lemma 5.13.

**Proof of lemma 5.13.** Under the adjoin action of \( N_{\text{PGL}_2} \) on \( G_m^2 \) as described in (5.11), the determinant of the matrix acted on does not change.
To say the same thing in a more elaborate way, let \( \det : \mathbb{G}_m^2 \to \mathbb{G}_m \) maps the matrix \( \begin{pmatrix} 0 & y \\ z & 0 \end{pmatrix} \) to (the negative of) its determinant \( yz \). And consider the quotient map \( q : N_{\text{PGL}_2} \to W = N_{\text{PGL}_2}/T_{\text{PGL}_2} \). Then the adjoin group action of \( N_{\text{PGL}_2} \) on \( \mathbb{G}_m^2 \) is compatible with the trivial group action of \( W \) on \( \mathbb{G}_m \), under the two maps \( \det \) and \( q \). Thus we have the following commutative diagram:

\[
\begin{array}{ccc}
(\tilde{\mathcal{Y}} \times \mathbb{G}_m^2) \times N_{\text{PGL}_2} & \longrightarrow & \tilde{\mathcal{Y}} \times \mathbb{G}_m^2 \\
(id, \det), q \downarrow & & \downarrow (id, \det) \\
(\tilde{\mathcal{Y}} \times \mathbb{G}_m) \times W & \longrightarrow & \tilde{\mathcal{Y}} \times \mathbb{G}_m
\end{array}
\] (5.13)

where the upper arrow is the group action of \( N_{\text{PGL}_2} \) on \( \tilde{\mathcal{Y}} \) and \( \mathbb{G}_m^2 \), and the lower arrow is the action of \( W \) that is an involution on \( \tilde{\mathcal{Y}} \) and the trivial action on \( \mathbb{G}_m \).

Apply Theorem 5.14 on this diagram, where the two conditions can be readily verified, we have an isomorphism of the quotient stacks

\[
\tilde{\mathcal{Y}} \times_{N_{\text{PGL}_2}} \mathbb{G}_m^2 = \tilde{\mathcal{Y}} \times \mathbb{G}_m^2 / N_{\text{PGL}_2} = \tilde{\mathcal{Y}} \times \mathbb{G}_m / W = \tilde{\mathcal{Y}} \times_W \mathbb{G}_m
\]

Isomorphic stacks have the same motivic weight. The right hand side is a line bundle over \( \tilde{\mathcal{Y}} / W = \mathcal{Y} \), and thus by the product formula has motivic weight \( \mu(\tilde{\mathcal{Y}} \times W \mathbb{G}_m) = (q - 1) \cdot \mu(\tilde{\mathcal{Y}} / W) = (q - 1) \cdot \mu(\mathcal{Y}) \). So we arrive at the desired conclusion that

\[
\mu(\tilde{\mathcal{Y}} \times_{N_{\text{PGL}_2}} \mathbb{G}_m^2) = \mu(\tilde{\mathcal{Y}} \times W \mathbb{G}_m) = (q - 1)\mu(\mathcal{Y})
\]

With Lemma 5.13 proven, the proof of Proposition 5.12 is now complete.

Let us return to our quest of computing the motivic weight contribution \( \mu(X - \{p_0\} / N_{\text{PGL}_2}) \) from Azumaya algebras of the form \( \mathcal{O} \oplus \mathcal{O}(p - p_0) \). Apply Proposition 5.12 to the double covering (5.8), we see that the motivic weight is

\[
\mu(X - \{p_0\} / N_{\text{PGL}_2}) = \frac{\mu(\mathbb{P}^1 - \{p_0\})}{q + 1} + \frac{\mu(X - \{p_0\})}{q^2 - 1}
\]

\[
= \frac{q}{q + 1} + \frac{\mu(X) - 1}{q^2 - 1}
\]

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6 Summary and Result

A summary of the motivic weight contribution from the various cases:

- Indecomposable odd degree vector bundles: 1

- Indecomposable even degree vector bundles: $1/q$

- Decomposable odd degree vector bundles: $\frac{\mu(X)q}{(q - 1)(q^2 - 1)}$

- Decomposable even degree vector bundle where $m > 0$: $\frac{\mu(X)}{(q - 1)(q^2 - 1)}$

- Decomposable trivial bundle $\mathcal{O} \oplus \mathcal{O}$: $\frac{1}{q(q^2 - 1)}$

- Decomposable nontrivial even degree vector bundle where $m = 0$: $\frac{q}{q + 1} + \frac{\mu(X) - 1}{q^2 - 1}$

Summing up the contributions, we see that the motivic weight $\mathcal{Bun}_{\text{PGL}_2}$ of PGL$_2$-torsors on an elliptic curve $X$ is

$$\mu(\mathcal{Bun}_{\text{PGL}_2}X) = 2 + 2 \frac{\mu(X)q}{(q - 1)(q^2 - 1)} = 2Z(X, q^{-2})$$

This is all that needs to be shown for the main result of the thesis.
References


