Kakeya Maximal Function Conjecture For Semialgebraic Mappings

by

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BS-MS, Indian Institute of Science Education and Research, Pune, 2019

A THESIS SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF

Master of Science

in

THE FACULTY OF GRADUATE AND POSTDOCTORAL STUDIES
(Mathematics)

The University Of British Columbia
(Vancouver)

August 2021

The following individuals certify that they have read, and recommend to the Faculty of Graduate and Postdoctoral Studies for acceptance, the thesis entitled:

Kakeya Maximal Function Conjecture

For Semialgebraic Mappings

submitted by **Mukul Rai Choudhuri** in partial fulfillment of the requirements for the degree of **Master of Science** in **Mathematics**.

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Abstract

The Kakeya maximal function conjecture is a quantitative, single scale formulation of the Kakeya conjecture. Recently, algebraic methods have been leading to progress in the Kakeya family of problems. In 2018, Katz and Rogers proved a conjecture concerning the number of δ-tubes with δ-separated directions which intersect a semialgebraic set with proportion at least λ. We will discuss the proof of this result which involves real algebraic geometry. We will then use this result to prove the Kakeya maximal function conjecture for the special case when the mappings are semialgebraic.

Lay Summary

This thesis is in the subject of harmonic analysis. Harmonic analysis is the quantitative study of operators. An operator takes a complex valued function as its input and returns a transformed complex valued function as its output.

Interestingly harmonic analysis is connected to study of fractal like objects called Kakeya sets. A Kakeya set is a region of space which contains a unit line segment in every direction. We wish to understand how small we can make a Kakeya set and this question leads to the Kakeya conjecture.

Algebraic methods have been leading to progress on the Kakeya conjecture. This thesis discusses some of the tools and progress in this direction.

Preface

In chapter 1, we will discuss the Kakeya conjecture and the current status of progress towards proving it. Chapter 2 discusses the Kakeya maximal function conjecture and its connection to the Kakeya conjecture. We then will cover some real algebraic geometry in chapter 3 because it will be required in chapter 4 where we will discuss the proof of the result of Katz and Rogers [17]. Chapter 5 is dedicated to the proof of the main result of the thesis which is that the Kakeya maximal function conjecture holds for semialgebraic mappings. This is an original, unpublished result which is the joint work of the author and his supervisors, Drs. Izabella Łaba and Joshua Zahl.

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List of Notations

Notation	Meaning	Definition/first appearance
$\dim_H(E)$	Hausdorff dimension of E	Lemma 1.2.2
$\underline{\dim}_M(E)$	lower Minkowski dimension of E	Definition 1.2.1
$\overline{\dim}_M(E)$	upper Minkowski dimension of E	Definition 1.2.1
$T_e^\delta(a)$	δ-tube centered at $a \in \mathbb{R}^n$ with direction $e \in S^{n-1}$	Definition 1.3.1
$\mathbb{T}$	collection of δ-tubes	Definition 1.3.3
f_δ^*	Kakeya maximal function	Definition 2.1.1
$A \lesssim_{n,p,\epsilon} B$	$A \leq C_{n,p,\epsilon} B$	Section 2.1
$A \lesssim B$	$A \leq C_n B$	Section 2.1
$A \sim_{n,p} B$	$A \leq C_{n,p} B$ and $B \leq C_{n,p} A$	Section 2.1
S, S_k	semialgebraic sets	Definition 3.1.2

Chapter 1

Introduction

This chapter will introduce the basic objects of study in this thesis: Kakeya sets and the Kakeya conjecture. In 1917, Soichi Kakeya asked whether there is a minimum area of a region D in the plane, in which a needle of unit length can be rotated a full circle. This question came to be known as the Kakeya needle problem. In 1928, Besicovitch showed that we can find a set of arbitrarily small area in which this can be done and he also constructed a set of Lebesgue measure zero which has a unit segment pointing in every direction [1]. These sets are considered important because of their surprising connections to problems in oscillatory integrals (harmonic analysis), the analysis of dispersive and wave equations (PDE), combinatorics and number theory [3, 23, 26]. We will begin by constructing these sets by using a variant of Besicovitch's construction.

1.1 Kakeya sets of measure zero

Definition 1.1.1. A Kakeya set (sometimes called Besicovitch set) in $\mathbb{R}^n$ is a compact set $E \subset \mathbb{R}^n$ containing a unit line segment in every direction i.e for every $e \in S^{n-1}$, there exists $x \in \mathbb{R}^n$ such that $x + te \in E$ for all $t \in [-1/2, 1/2]$.

We now wish to prove the following theorem.

Theorem 1.1.1. There exist Kakeya sets of measure zero in $\mathbb{R}^n$ for every $n \geq 2$.

Suppose we have a Kakeya set K in $\mathbb{R}^2$, then $K \times [-1, 1]^{n-2}$ is clearly a Kakeya set in $\mathbb{R}^n$. Hence, it suffices to show existence of measure zero Kakeya sets in dimension two. Define a G-set to be a compact set $E \subset \mathbb{R}^2$ such that for any $m \in [0, 1]$, there is a line segment contained in E connecting $x = 0$ and $x = 1$ with slope m i.e. for all $m \in [0, 1]$, there exists $b \in \mathbb{R}$ such that $mx + b \in E$ $\forall x \in [0, 1]$. Clearly it suffices to prove existence of measure zero G-sets. We will explicitly construct such a set using a variant of Besicovitch's construction first done by Sawyer [22]. If $l = \{(x, y) : 0 \leq x \leq 1, y = mx+b\}$, then we define

$$S_l^\delta = \{(x, y) : 0 \leq x \leq 1, |y - (mx + b)| \leq \delta\}$$

which is essentially the δ neighborhood of l.

Before doing a rigorous proof, we now give an informal sketch to motivate the formal construction. The sketch involves the involves the sliding of triangles: we start from a right triangle with vertices (0,0), (0,1) and (1,0) which is clearly a G-set. Let N be a large integer. Subdivide the triangle into N first stage triangles by subdividing the vertical side in N equal intervals. Leave the top triangle alone and slide the others upward so that their intersections with the line $x = 0$ all coincide. Next, subdivide each of the first stage triangles into N second stage triangles as in the previous step. Now leave the top triangle in each group alone and slide the $N - 1$ others upward until the intersections of the N triangles in the group with the line $x = 1/N$ all coincide. Now subdivide again and repeat at $x = 2/N$. Keep repeating this process at $x = 3/N$, $x = 4/N$,..., $x = (N - 1)/N$. We end up with a shape which has a really small area. This completes our informal sketch.

With this sketch in mind we work towards a formal construction to prove the theorem. Define

$$\mathcal{A}_N = \left\{ \sum_{j=1}^{N} \frac{a_j}{N^j} : a_j \in \{0, 1, ..., N-1\} \right\}.$$

Further for $a \in \mathcal{A}_N$ define

$$\phi_a(t) = \sum_{j=1}^{N} \frac{(Nt - j + 1)a_j}{N^{j+1}}.$$

The line $l_a = \{(t, \phi_a(t)\}$ is the final position of the line with slope a in the subdividing and shifting of triangles described.

Lemma 1.1.2. For each $t \in [0, 1]$, there is an integer $k \in \{1, 2, ..., N\}$ and a set of N^{k-1} intervals each of length $2N^{-k}$ whose union contains the set $\{\phi_a(t) : a \in \mathcal{A}_N\}$.

Proof. Choose an integer k from the set $\{1, 2, \dots, N\}$ such that $\frac{k-1}{N} \leq t \leq \frac{k}{N}$. Define $a, b \in \mathcal{A}_N$ to be equivalent if $a_j = b_j$ when $j \leq k-1$. Then there will be N^{k-1} equivalence classes. Suppose a is equivalent to b, then

$$|\phi_a(t) - \phi_b(t)| \leq \sum_{j \geq k}^{N} \frac{|Nt - j + 1|(N-1)}{N^{j+1}}.$$

Since $k - 1 \leq Nt \leq k$, we have $|Nt - j + 1| \leq max(j - k, 1)$. Therefore

$$|\phi_a(t) - \phi_b(t)| \leq \frac{N-1}{N^{k+1}} \sum_{j \geq k}^{N} \frac{max(j-k, 1)}{N^{j-k}} \leq \frac{2}{N^k}.$$

□

Corollary 1.1.3. Let N be sufficiently large. Then there is a G-set $E_N \subset [0, 1] \times [-1, 1]$ which intersects every vertical line in measure $\leq 4/N$ and in particular $|E_N| \leq \frac{4}{N}$.

Proof. Let $E_N = \bigcup_{a \in \mathcal{A}_N} S_{l_a}^{N^{-N}}$. Using the previous lemma, it is easy to see that this is a G-set with the required properties. □

The above lemma would allow us to construct Kakeya sets of arbitrary small measure. We need a little more work to get one of measure zero.

Lemma 1.1.4. For every G-set E and every $\epsilon > 0$, $\eta > 0$, there is another G-set F, which is contained in the ϵ neighborhood of E and has measure $< \eta$.

Proof. Let $\{m_j\} = \{j\epsilon\}_{j=0}^{\lfloor 1/\epsilon \rfloor}$ and for each j, choose b_j such that we have $l_j = \{(x, y) : 0 \le x \le 1, y = m_j x + b_j\} \subset E$. Now define the function

$$A_j : [0,1] \times [-1,1] \to S_{l_j}^{\epsilon} \quad A_j(x,y) = (x, m_j x + b_j + \epsilon y).$$

Now define

$$F = \bigcup_{j=0}^{\lfloor 1/\epsilon \rfloor} A_j(E_N).$$

A_j maps slopes as $\mu \mapsto m_j + \epsilon\mu$. This implies that F is a G-set which is contained in the ϵ neighborhood of E. A_j contracts areas by a factor of ϵ, therefore

$$|F| \le \sum_{j=0}^{\lfloor 1/\epsilon \rfloor} |A_j(E_N)| \le \lfloor 1/\epsilon \rfloor \epsilon \frac{4}{N} \le \frac{4}{N}.$$

□

Proof of Theorem 1.1.1. Suppose we had a sequence $\{F_n\}_{n=0}^{\infty}$ of G-sets and a sequence of numbers $\{\epsilon_n\}_{n=0}^{\infty}$ converging to 0 such that the following properties hold when $n \ge 1$

(a) $F_n(\epsilon_n) \subset F_{n-1}(\epsilon_{n-1})$

(b) $|\overline{F_n(\epsilon_n)}| < 2^{-n}$.

where $F(\epsilon)$ is the open ϵ neighborhood of F. Let $K_n = \overline{F_n(\epsilon_n)}$, then the set $E = \cap_n K_n$ is compact with measure zero. Fix $m \in [0,1]$. Let $mx + b_j$ be a line of slope m in K_j. We can find a convergent subsequence of b_j which we also denote by b_j. The limit $b \in E$ as each K_j is closed. Fix $t \in [0,1]$, then $(0, mt + b_j) \to (0, mt + b)$ which lies in E by the same argument. Hence E is a G-set and so it is left to show such $\{F_n\}_{n=0}^{\infty}$ and $\{\epsilon_n\}_{n=0}^{\infty}$ with prescribed properties exist. We can construct such a sequence by induction quite easily. This completes the proof.

1.2 Hausdorff dimension and Minkowski dimension

While we have shown the existence of Kakeya sets of measure zero, Kakeya sets may be big in some other sense. To investigate this, we introduce the Hausdorff and Minkowski dimensions. These dimensions give us a way to measure the size of fractal like objects such as Kakeya sets of measure zero. First we define the Hausdorff dimension. Let $E \subset \mathbb{R}^n$ and fix $\alpha > 0$. For $\epsilon > 0$, we define $H_\alpha^\epsilon(E) = \inf\left(\sum_j r_j^\alpha\right)$ where the infimum is taken over all countable coverings of E by balls of radius less than ϵ. Now we define the α Hausdorff measure to be

$$H_\alpha(E) = \lim_{\epsilon \to 0^+} H_\alpha^\epsilon(E) = \lim_{\epsilon \to 0^+} \inf\left(\sum_j r_j^\alpha\right). \tag{1.1}$$

The above limit makes sense because H_α^ϵ increases as ϵ decreases, although the limit may be $+\infty$. Also $H_\alpha(E)$ is clearly a non increasing function of α. The following fact is useful: $H_\alpha(E) = 0$ iff $\sum_j r_j^\alpha$ can be made arbitrarily small.

Lemma 1.2.1. If $E \subset \mathbb{R}^n$, then we have $H_\alpha(E) = 0$ for all $\alpha > n$.

Proof. Let $\epsilon > 0$, then $r_i = \epsilon i^{-1/n} < \epsilon$ for all $i \geq 1$. Let C_i be a closed cube of diameter 2^{N_i} where N_i is chosen such that $2^{N_i} \leq r_i < 2^{N_i+1}$ (we will fix the center later). Since $\sum_i r_i^n = \infty$, we have $\sum_i |C_i| = \infty$ where $|C_i|$ indicates the Lebesgue measure in $\mathbb{R}^n$. If we have a collection of dyadic cubes such that the sum of the n-dimensional volumes of all the cubes in the collection is infinite, then we can pack $\mathbb{R}^n$ with this collection by choosing appropriate centers for the cubes. Further for any $\alpha > n$, the series $\sum_j r_j^\alpha$ converges and goes to 0 as $\epsilon \to 0$. As any cube (respectively ball) can be sandwiched between balls (respectively cubes) of comparable radii (respectively sidelength), we are done. $\square$

Lemma 1.2.2. There is a unique number α_0, called the Hausdorff dimension of E, such that $H_\alpha(E) = \infty$ if $\alpha < \alpha_0$ and $H_\alpha(E) = 0$ if $\alpha > \alpha_0$.

Proof. Define α_0 to be supremum of α such that $H_\alpha(E) = \infty$. Suppose $\alpha > \alpha_0$. Pick $\beta \in (\alpha_0, \alpha)$. Then $H_\beta(E) < \infty$. Let $M = H_\beta(E) + 1$. Now by definition of Hausdorff measure, for every $\epsilon > 0$ there exists covering of E such that $\sum_j r_j^\beta < M$ where $r_j < \epsilon$. For this covering $\sum_j r_j^\alpha \leq \epsilon^{\alpha-\beta} \sum_j r_j^\beta < \epsilon^{\alpha-\beta} M$. So $H_\alpha(E) = 0$. $\square$

Note that from Lemma 1.2.1, we have that the maximum Hausdorff dimension of a subset of $\mathbb{R}^n$ is n. We will denote Hausdorff dimension of E by $\dim_H(E)$. We will next define the Minkowski dimension. Before the rigorous definition, we informally discuss the idea behind the definition. Let E be a measure zero compact set. Denote $E_\delta = \{x \in \mathbb{R}^n : d(x, E) < \delta\}$. Now $|E_\delta| \to 0$ as $\delta \to 0$. The Minkowski dimension quantifies how fast $|E_\delta|$ approaches 0.

Definition 1.2.1. For any bounded subset $E \subset \mathbb{R}^n$, the upper Minkowski dimension of E is

$$\overline{\dim}_M(E) = \inf\{s > 0 : \limsup_{\delta \to 0^+} \delta^{s-n} |E_\delta| = 0\},$$

and the lower Minkowski dimension of E is

$$\underline{\dim}_M(E) = \inf\{s > 0 : \liminf_{\delta \to 0^+} \delta^{s-n} |E_\delta| = 0\}.$$

We will usually be concerned with lower Minkowski dimension and hence by default, Minkowski dimension will be lower Minkowski dimension. In accordance with this convention, $\dim_M(E)$ denotes lower Minkowski dimension of E.

Remark 1.2.1. Clearly we have $\overline{\dim}_M(E) \geq \underline{\dim}_M(E)$. Using arguments similar to the ones in the proof of lemma 1.2.2, we can show

$$\begin{aligned}\overline{\dim}_M(E) &= \inf\{s > 0 : \limsup_{\delta\to 0^+} \delta^{s-n}|E_\delta| < \infty\} \\ &= \sup\{s > 0 : \limsup_{\delta\to 0^+} \delta^{s-n}|E_\delta| = \infty\} \\ &= \sup\{s > 0 : \limsup_{\delta\to 0^+} \delta^{s-n}|E_\delta| > 0\}.\end{aligned}$$

Similar statements exist for $\underline{\dim}_M(E)$.

Remark 1.2.2. An alternative definition of Minkowski dimension is as follows. Let $E \subset \mathbb{R}^n$ be bounded. The lower Minkowski dimension is the supremum over α such that $|E_\delta| \geq C_\alpha \delta^{n-\alpha}$ for all $\delta \in (0, 1)$. The upper Minkowski dimension is the supremum over α such that $|E_\delta| \geq C_\alpha \delta^{n-\alpha}$ for a sequence of δ's that converges to 0. The proof of equivalence follows from the previous remark.

The Minkowski dimension is in some sense less sensitive than the Hausdorff dimension because in the Minkowski dimension we use balls of equal radius whereas in Hausdorff dimension the balls may have different radius. The following lemma is a formalization of this intuition.

Lemma 1.2.3. Let $E \subset \mathbb{R}^n$ be a bounded set. Then $\dim_H(E) \leq \dim_M(E)$.

Proof. Let N_δ be the maximum number of balls of radius δ centered at points in E that are disjoint. Now clearly $|E_\delta| \geq N_\delta C_n \delta^n$ where C_n is the Lebesgue measure of the unit ball. Pick $x_i \in E$ such that $B(x_1, \delta), \ldots, B(x_{N_\delta}, \delta)$ are disjoint. Then $B(x_1, 2\delta), \ldots, B(x_{N_\delta}, 2\delta)$ covers E since if there exists $x \in E \setminus \cup_{i=1}^{N_\delta} B(x_i, 2\delta)$, then the balls $B(x_1, \delta), \ldots, B(x_{N_\delta}, \delta), B(x, \delta)$ would be disjoint contradicting maximality of N_δ. If $\alpha < \dim_H(E)$, then by definition of Hausdorff dimension, $N_\delta(2\delta)^\alpha \to \infty$ as $\delta \to 0$. So $|E_\delta| \geq C_\alpha \delta^{n-\alpha}$ for δ small. □

1.3 Kakeya conjecture

Having defined the Hausdorff and Minkowski dimension, we are ready to state the Kakeya conjecture.

Conjecture 1.3.1. Kakeya Conjecture. If $E \subset \mathbb{R}^n$ is a Kakeya set ($n \geq 2$), E has Hausdorff dimension n and consequently Minkowski dimension n.

This is an open problem but there are partial results. We will outline some of the junctions towards proving the full conjecture. For $n = 2$, the conjecture was proved by Davies in 1971 [8] and the maximal function formulation (discussed in next chapter) was proved by Cordoba in 1977 [6] using L^2 arguments and bounds on volume of the intersection of tubes. For higher dimensions only partial results exist. Using the geometric "bush argument", Drury in 1983 [9] and Christ, Duoandikoetxea, Rubio de Francia in 1986 [5] were able to prove that Kakeya sets for $n \geq 3$ had dimension at least $(n+1)/2$. In 1995, Wolff [25] improved this bound using the geometric "hairbrush argument" to $(n+2)/2$.

Bourgain made the important contribution of using combinatorics to make progress on the conjecture which also inspired later progress [3]. For $n = 3$, Katz, Łaba, Tao in 2000 [16] proved that the Minkowski dimension of Kakeya sets is strictly greater than 5/2 and in 2017, Katz and Zahl [20] proved the same for the Hausdorff dimension.

There is also a finite field analogue of the Kakeya problem which was first proposed by Wolff [26]. In a major breakthrough Dvir [10] proved the finite field analogue of the Kakeya conjecture in 2008 using an algebraic argument involving polynomials. This breakthrough has inspired application of algebraic/polynomial methods to make progress on the Kakeya conjecture and related problems in Euclidean setting. We shall discuss more about this in chapter 3. Some developments have been skipped here for the sake of brevity and one can find a more comprehensive discussion in the survey articles by Wolff [26] and Katz, Tao [19].

The Kakeya problem may also be thought of as a packing problem of tubes which we now describe. First we define a tube.

Definition 1.3.1. We define the δ-tube centered at $a \in \mathbb{R}^n$ pointing in the direction $e \in S^{n-1}$ as the set $T_e^\delta(a) = \{x \in \mathbb{R}^n : |(x-a)\cdot e| \leq 1/2, |(x-a)^\perp| \leq \delta\}$.

Here $x^\perp = x - (x \cdot e)e$. So $T_e^\delta(a)$ is essentially the $\delta-$neighborhood of the unit line segment in the e direction centered at a. We will often denote tubes just by T. Any tube T has two elements $e, -e$ of S^{n-1} parallel to it. We call the set $\{e, -e\}$ the orientation of the tube T. The spherical distance $\mathrm{dist}_s(e, e')$ between the vectors $e, e' \in S^{n-1}$ is given by $\cos^{-1}(e \cdot e')$. We now define the separation between two tubes.

Definition 1.3.2. Let T, T' be two tubes with orientations $\{e_1, -e_1\}$ and $\{e_2, -e_2\}$ respectively. We define angular separation between two tubes $\angle(T, T')$ to be the minimum of $\mathrm{dist}_s(\pm e_1, \pm e_2)$.

Note that $\angle(T, T') \leq \pi/2$ for any two tubes T, T'. The spherical distance and Euclidean distance are comparable on S^{n-1} i.e for $x, y \in S^{n-1}$, we have $|x - y| \leq \mathrm{dist}_s(x, y) \leq \frac{\pi}{2}|x - y|$. So we could choose to work with Euclidean distance but the spherical distance is more convenient.

Definition 1.3.3. A collection $\mathbb{T}$ of δ-tubes is said to be pointing in δ-separated directions if $\angle(T, T') > \delta$ for every distinct pair $T, T' \in \mathbb{T}$. It is further said to be maximal if we cannot add another tube to it and preserve this property.

For the purpose of stating the Kakeya conjecture for tubes, we define the following quantity:

$$F(\delta) = \inf_{\mathbb{T}} \Big| \bigcup_{T \in \mathbb{T}} T \Big|$$

where we are taking infimum over all maximal collections of δ-tubes $\mathbb{T}$ pointing in δ-separated directions. If $\mathbb{T}$ consists of disjoint tubes, then clearly the size of the union is $\sim_n 1$. By arranging the tubes in a fractal like pattern as in the Besicovitch construction, we get the following lemma.

Lemma 1.3.2. $F(\delta) \lesssim_n \frac{1}{\log(\frac{1}{\delta})}$. In particular $F(\delta) \to 0$ as $\delta \to 0$.

The Kakeya conjecture in terms of tubes is a statement regarding the rate at which $F(\delta)$ approaches 0 as $\delta \to 0$.

Conjecture 1.3.3. Kakeya Conjecture for tubes. $F(\delta) \gtrsim_\epsilon \delta^\epsilon$ for every $\epsilon > 0$.

The above conjecture clearly would imply that Kakeya sets in $\mathbb{R}^n$ have Minkowski dimension n. One can interpret the above conjecture as saying that a maximal collection of δ-tubes pointing in δ-separated directions is essentially disjoint. We can drop the maximal condition to get a more general conjecture i.e if $\mathbb{T}$ is a collection of δ-tubes pointing in δ-separated directions, then

$$\Big|\bigcup_{T\in\mathbb{T}} T\Big| \gtrsim_\epsilon \delta^\epsilon(\delta^{n-1}|\mathbb{T}|).$$

Rearranging the above inequality, we get the following equivalent version which we will call the generalized Kakeya conjecture for tubes.

Conjecture 1.3.4. Generalized Kakeya conjecture for tubes. Let $E \subset \mathbb{R}^n$ be a Lebesgue measurable set. The number of δ-tubes pointing in δ-separated directions that can be contained in E is less than or equal to $C_\epsilon \delta^{1-n-\epsilon}|E|$.

Chapter 2

Kakeya Maximal Function Conjecture

2.1 Operator version of KMFC

The Kakeya conjecture is closely connected to a maximal operator which is consequently called the Kakeya maximal function. It looks similar to the Hardy-Littlewood maximal function but the geometry is fundamentally different and hence the methods do not translate. Now we define the Kakeya maximal function (operator).

Definition 2.1.1. Suppose $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $\delta > 0$, then we define the Kakeya maximal function $f^*_\delta : S^{n-1} \to [0, \infty]$ by

$$f^*_\delta(e) = \sup_{a \in \mathbb{R}^n} \frac{1}{|T^\delta_e(a)|} \int_{T^\delta_e(a)} |f|. \tag{2.1}$$

Remark 2.1.1. We have the following trivial bounds: $\|f^*_\delta\|_{L^\infty(S^{n-1})} \leq \|f\|_{L^\infty(\mathbb{R}^n)}$ and $\|f^*_\delta\|_{L^\infty(S^{n-1})} \leq C_n \delta^{-(n-1)} \|f\|_{L^1(\mathbb{R}^n)}$. Now if $f \in L^p(\mathbb{R}^n)$ for $1 < p < \infty$, then by Hölder's inequality, we get

$$\frac{1}{|T^\delta_e(a)|} \int_{T^\delta_e(a)} |f| \leq \frac{1}{|T^\delta_e(a)|} \left(\int_{\mathbb{R}^n} |f|^p \right)^{1/p} |T^\delta_e(a)|^{1/p'} = \frac{1}{|T^\delta_e(a)|^{\frac{1}{p}}} \left(\int_{\mathbb{R}^n} |f|^p \right)^{1/p}.$$

Hence $\|f_\delta^*\|_{L^\infty(S^{n-1})} \leq C_{n,p}\,\delta^{\frac{1-n}{p}}\,\|f\|_{L^p(\mathbb{R}^n)}$. As S^{n-1} has finite measure, we can replace the L^∞ norm on the left hand side with a L^q norm, for any $q \geq 1$. In particular for a fixed δ, the operator $f \mapsto f_\delta^*$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(S^{n-1})$ for any $1 \leq p, q \leq \infty$.

Remark 2.1.2. The following example shows that no bound of the form $\|f_\delta^*\|_{L^p(S^{n-1})} \leq C\,\|f\|_{L^q(\mathbb{R}^n)}$ where C was independent of δ and $1 \leq q < \infty$ can hold. Take E to be a Kakeya set of measure zero and set $f = \chi_{E_\delta}$, then clearly $f_\delta^* \equiv 1$. But the L^q norm of f goes to 0 as $\delta \to 0$. This shows that the operator norm blows up as we make δ small and hence we cannot have such a C independent of δ. The real questions are in understanding how the operator norms blow up as $\delta \to 0$.

By remark 2.1.1, we have $\|f_\delta^*\|_n \leq C_n\delta^{-(1-n^{-1})}\,\|f\|_n$. We can ask whether we can shrink the power of δ further.

Conjecture 2.1.1. Kakeya Maximal Function Conjecture (Operator version). For each $\epsilon > 0$, there exists C_ϵ such that

$$\|f_\delta^*\|_n \leq C_\epsilon\delta^{-\epsilon}\,\|f\|_n\,.$$

Remark 2.1.3. The following example shows that no bound of the above form with $p < n$ instead of n can hold. Assume to the contrary that such a bound does hold. Let $f = \chi_{B(0,\delta)}$, then $f_\delta^*(e) \sim_n \delta$. Therefore $\|f_\delta^*\|_p \sim_{n,p} \delta$ and $\|f\|_p \sim_{n,p} \delta^{n/p}$, which implies $\delta \leq C_{n,p,\epsilon}\delta^{-\epsilon}\delta^{n/p}$. This is a contradiction, hence $p \geq n$.

Theorem 2.1.2. The operator version of the Kakeya maximal function conjecture (conjecture 2.1.1) implies the Kakeya conjecture (conjecture 1.3.1).

Proof. Let us first do the easy case of the Minkowski dimension. Let $E \subset \mathbb{R}^n$ be a Kakeya set and $f = \chi_{E_\delta}$. Then $f_\delta^* \equiv 1$ and $\|f\|_n = |E_\delta|^{1/n}$. Applying the hypothesis to f, we get

$$1 \lesssim_\epsilon \delta^{-\epsilon}|E_\delta|^{1/n} \quad \text{i.e.} \quad |E_\delta| \gtrsim_\epsilon \delta^\epsilon.$$

This proves the Minkowski dimension of E is n. Now let us show the Hausdorff dimension of E is n. It is sufficient to prove $H_\alpha(E) \neq 0$ for all $\alpha < n$. Let $\{B(x_j, r_j)\}$ be a cover of E which we may assume to be finite and $r_j < 1/100$ without loss of generality. We need to show that there is a constant $c_\alpha > 0$ (independent of the cover $\{B(x_j, r_j)\}$ so that $\sum r_j^\alpha > c_\alpha$. We now dyadically pigeonhole the radii. So let $J_k = \{j : 2^{-k} < r_j \leq 2^{-k+1}\}$. For $e \in S^{n-1}$, let I_e denote a line segment in E parallel to e. Let

$$\Omega_k = \left\{ e \in S^{n-1} : |I_e \cap \bigcup_{j \in J_k} B(x_j, r_j)| \geq \frac{1}{100k^2} \right\}.$$

Now as $\sum_k \frac{1}{100k^2} < 1$, we have $\cup_{k=1}^\infty \Omega_k = S^{n-1}$. Let $E_k = \cup_{j \in J_k} B(x_j, 10r_j)$ and let $f = \chi_{E_k}$. Note that f does depend on k. Let a_e be the midpoint of I_e, then for $e \in \Omega_k$, we have

$$|T_e^{2^{-k}}(a_e) \cap E_k| \geq \frac{1}{100k^2} |T_e^{2^{-k}}(a_e)|.$$

Therefore

$$\left\| f^*_{2^{-k}} \right\|_n^n \geq \int_{\Omega_k} |f^*_{2^{-k}}|^n \gtrsim |\Omega_k| \frac{1}{k^{2n}}. \tag{2.2}$$

But by our hypothesis, we have

$$\left\| f^*_{2^{-k}} \right\|_n^n \lesssim_\epsilon 2^{k\epsilon n} |E_k| \lesssim 2^{k\epsilon n} |J_k| \, 2^{-kn} \tag{2.3}$$

for $0 < \epsilon < 1 - n^{-1}$. So combining 2.2 and 2.3, we get

$$|\Omega_k| \lesssim_\epsilon k^{2n} 2^{-kn(1-\epsilon)} |J_k|.$$

Now clearly $k^{2n} \lesssim_\epsilon, 2^{k\epsilon n}$. Thus we get

$$|\Omega_k| \lesssim_\epsilon 2^{-kn(1-2\epsilon)} |J_k|$$

for $0 < \epsilon < 1 - n^{-1}$. Using this, we obtain

$$\sum_j r_j^{n(1-2\epsilon)} \geq \sum_k 2^{-kn(1-2\epsilon)} |J_k| \gtrsim_\epsilon \sum_k |\Omega_k| \gtrsim 1,$$

since $\cup_{k=1}^{\infty}\Omega_k = S^{n-1}$. As the above inequality holds for arbitrarily small ϵ, we are done. $\square$

2.2 L^p versions of KMFC

We will now discuss some conjectures which do not involve any operator but involve getting upper bounds for L^p norms of functions which are given in terms of characteristic functions of δ-tubes pointing in δ-separated directions. We will call these conjectures L^p versions of Kakeya maximal function conjecture. We will explore the connection to the operator version of the Kakeya maximal function conjecture.

Conjecture 2.2.1. **L^p KMFC I.** Suppose $\mathbb{T}$ is a collection of δ-tubes pointing in δ-separated directions and suppose we have numbers $y_T \geq 0$ (where $T \in \mathbb{T}$) satisfying

$$\delta^{n-1} \sum_{T \in \mathbb{T}} y_T^{\frac{n}{n-1}} \leq 1.$$

Then we have $\left\| \sum_{T \in \mathbb{T}} y_T \chi_T \right\|_{\frac{n}{n-1}} \lesssim_\epsilon \delta^{-\epsilon}$.

We will show that the above conjecture implies the operator KMFC (conjecture 2.1.1) using the following technical lemma.

Lemma 2.2.2. Let $1 < p < \infty$ and let p' be the dual exponent of p. Suppose the following holds: if $\mathbb{T}$ is a maximal collection of δ-tubes pointing in δ-separated directions and we have numbers $y_T \geq 0$ (where $T \in \mathbb{T}$) satisfying $\delta^{n-1} \sum_{T \in \mathbb{T}} y_T^{p'} \leq 1$, then

$$\left\| \sum_{T \in \mathbb{T}} y_T \chi_T \right\|_{p'} \leq A_\delta.$$

Then we have a bound

$$\|f_\delta^*\|_p \lesssim A_\delta \|f\|_p .$$

Proof. Let $\{e_k\}$ be a maximal δ-separated subset of S^{n-1}. Suppose $\mathrm{dist}_s(e,e') \leq \delta$, then any tube of thickness δ pointing along e can be covered by C tubes again of thickness δ pointing in the direction e' where C depends only on n.

Now suppose $\mathrm{dist}_s(e,e') \leq \delta$ and fix a tube $T_e^\delta(a)$ parallel to e. Let $T_{e'}^\delta(a'_1), \ldots T_{e'}^\delta(a'_C)$ be a collection of C tubes parallel to e' that cover $T_e^\delta(a)$. Then we have

$$\frac{1}{|T_e^\delta(a)|}\int_{T_e^\delta(a)} |f| \leq \frac{1}{|T_e^\delta(a)|}\left[\int_{T_{e'}^\delta(a'_1)} |f| + \ldots + \int_{T_{e'}^\delta(a'_C)} |f|\right] \leq C f_\delta^*(e'). \quad (2.4)$$

So we have $f_\delta^*(e) \leq C f_\delta^*(e')$ and so

$$\begin{aligned}\|f_\delta^*\|_p^p &\lesssim \sum_k \int_{\mathrm{dist}_s(e_k,e)\leq\delta} |f_\delta^*(e)|^p \, de \\ &\lesssim \sum_k |f_\delta^*(e_k)|^p \delta^{n-1} \\ &= \delta^{n-1}\big(\sum_k |f_\delta^*(e_k)| z_k\big)^p,\end{aligned}$$

where $\sum_k z_k^{p'} = 1$. Let

$$y_k = \frac{z_k}{\delta^{\frac{n-1}{p'}}}, \quad \text{then, we have} \quad \delta^{n-1}\sum_k y_k^{p'} = 1.$$

Substituting $y_k \delta^{\frac{n-1}{p'}}$ for z_k, we get

$$\|f_\delta^*\|_p \lesssim \delta^{\frac{n-1}{p}} \sum_k |f_\delta^*(e_k)| y_k \delta^{\frac{n-1}{p'}} = \delta^{n-1}\sum_k y_k |f_\delta^*(e_k)|.$$

Using the definition of the Kakeya maximal function, we can find $a_k \in \mathbb{R}^n$

such that

$$\begin{aligned}
\|f_\delta^*\|_p &\lesssim \delta^{n-1} \sum_k y_k \frac{1}{|T_{e_k}^\delta(a_k)|} \int_{T_{e_k}^\delta(a_k)} |f| \\
&\lesssim \int_{\mathbb{R}^n} \Big(\sum_k y_k \chi_{T_{e_k}^\delta(a_k)} \Big) |f| \\
&\leq \Big\| \sum_k y_k \chi_{T_{e_k}^\delta(a_k)} \Big\|_{p'} \|f\|_p \leq A_\delta \|f\|_p \,.
\end{aligned}$$

□

Corollary 2.2.3. L^p KMFC I (conjecture 2.2.1) implies operator version of KMFC (conjecture 2.1.1).

Suppose $\mathbb{T}$ is a maximal collection of δ-tubes pointing in δ-separated directions and suppose we have numbers $y_T \geq 0$ (where $T \in \mathbb{T}$) satisfying $\delta^{n-1} \sum_{T \in \mathbb{T}} y_T^{\frac{n}{n-1}} = 1$. If we further impose that all y_T must be equal, then $y_T \sim 1$ for all $T \in \mathbb{T}$. This observation leads to the following apparently weaker conjecture.

Conjecture 2.2.4. L^p **KMFC II.** Suppose $\mathbb{T}$ is a maximal collection of δ-tubes pointing in δ-separated directions, then

$$\Big\| \sum_{T \in \mathbb{T}} \chi_T \Big\|_{\frac{n}{n-1}} \lesssim_\epsilon \delta^{-\epsilon}.$$

This conjecture is more geometric in nature. The function $\sum_{T \in \mathbb{T}} \chi_T$ counts the number of tubes passing through a point. We will establish that L^p KMFC II (conjecture 2.2.4) is equivalent to L^p KMFC I (conjecture 2.2.1), and hence it implies the operator KMFC (conjecture 2.1.1) and hence also the Kakeya conjecture (conjecture 1.3.1). But there is a easy and insightful way to see how L^p KMFC II (conjecture 2.2.4) implies the Kakeya conjecture for tubes (conjecture 1.3.3) and consequently the Minkowski dimension part of the Kakeya conjecture (conjecture 1.3.1). Suppose $\mathbb{T}$ is a maximal collection of δ-tubes pointing in δ-separated directions. Using

Hölder's inequality, we get

$$\left\| \sum_{T \in \mathbb{T}} \chi_T \right\|_{\frac{n}{n-1}} \left| \bigcup_{T \in \mathbb{T}} T \right|^{1/n} \geq \left\| \sum_{T \in \mathbb{T}} \chi_T \right\|_1 = \sum_{T \in \mathbb{T}} |T| \sim 1. \tag{2.5}$$

Assuming L^p KMFC II (conjecture 2.2.4), we get

$$\left| \bigcup_{T \in \mathbb{T}} T \right| \gtrsim_\epsilon \delta^\epsilon.$$

This proves the Kakeya conjecture for tubes (conjecture 1.3.3) and implies Minkowski dimension of Kakeya sets is n. We have another equivalent L^p version of KMFC which we now state and then our next goal will be to establish all the three versions of L^p KMFC are equivalent.

Conjecture 2.2.5. **L^p KMFC III.** Suppose $\mathbb{T}$ is a (not necessarily maximal) collection of δ-tubes pointing in δ-separated directions, then

$$\left\| \sum_{T \in \mathbb{T}} \chi_T \right\|_{\frac{n}{n-1}} \lesssim_\epsilon \delta^{-\epsilon} (\delta^{n-1} |\mathbb{T}|)^{\frac{n-1}{n}}.$$

L^p KMFC III (conjecture 2.2.5) is clealry stronger than L^p KMFC II (conjecture 2.2.4). But in fact both are equivalent, a theorem which we will only state but not prove. The proof can be found in the lecture notes of a course by Tao [24].

Theorem 2.2.6. L^p KMFC II (conjecture 2.2.4) implies L^p KMFC III (conjecture 2.2.5).

L^p KMFC I (conjecture 2.2.1) clearly implies L^p KMFC II (conjecture 2.2.4). But we also have that L^p KMFC III (conjecture 2.2.1) implies L^p KMFC I (conjecture 2.2.1), making all three equivalent. We will again state the theorem without proof.

Theorem 2.2.7. L^p KMFC III (conjecture 2.2.5) implies L^p KMFC I (conjecture 2.2.1).

A sketch of the proof is as follows: firstly note L^p KMFC III (conjecture 2.2.5) is special case of L^p KMFC I (conjecture 2.2.1) when some elements are equal to each other and the rest are zero. We can prove in general by dyadic pigeonholing.

We summarize the hierarchy and interrelations between the conjectures in the following diagram:

$$\begin{array}{ccccc}
L^p \text{ KMFC II(conjecture 2.2.4)} & \Longrightarrow & \text{KCT(conjecture 1.3.3)} & \Longleftrightarrow & \text{GKCT(conjecture 1.3.4)} \\
\Updownarrow & & & & \\
L^p \text{ KMFC I(conjecture 2.2.1)} & \Longrightarrow & \text{KMFC(conjecture 2.1.1)} & \Longrightarrow & \text{KC(conjecture 1.3.1)} \\
\Updownarrow & & & & \\
L^p \text{ KMFC III(conjecture 2.2.5)} & & & &
\end{array}$$

2.3 Interpolation and partial conjectures

We can interpolate between the Kakeya maximal function conjecture estimates and some trivial estimates to get some new conjectures. Proving these conjectures give partial progress on the Kakeya conjecture.

We have the trivial inequality

$$\|f_\delta^*\|_{L^\infty(S^{n-1})} \lesssim \delta^{-(n-1)} \|f\|_{L^1(\mathbb{R}^n)}.$$

Assume the operator version of KMFC (conjecture 2.1.1) is true, then we have

$$\|f_\delta^*\|_{L^n(S^{n-1})} \lesssim_\epsilon \delta^{-\epsilon} \|f\|_{L^n(\mathbb{R}^n)}.$$

As the Kakeya maximal function is a sublinear operator, we can apply Marcinkiewicz interpolation and get the following conjecture.

Conjecture 2.3.1. Partial KMFC (operator version). For $1 \leq p < n$, we have

$$\|f_\delta^*\|_{L^p(S^{n-1})} \lesssim_\epsilon \delta^{-\frac{n}{p}+1-\epsilon} \|f\|_{L^p(\mathbb{R}^n)}.$$

The interpolation actually yields a higher exponent on the LHS which we can reduce as S^{n-1} has finite measure. A way to see how this is the best possible bound for the Kakeya maximal function is by taking $f = \chi_{B(0,\delta)}$ and suppose $\|f^*_\delta\|_p \lesssim_a \delta^{-a}\|f\|_p$. Then we have $\delta \lesssim_a \delta^{-a}\delta^{n/p}$ i.e $\delta^{\frac{n}{p}-a-1} \gtrsim_a 1$ which implies $a \geq \frac{n}{p}-1$.

Lemma 2.3.2. The partial operator version of KMFC (conjecture 2.3.1) implies that Kakeya sets in $\mathbb{R}^n$ must have both Hausdorff and Minkowski dimension equal to at least p.

Proof. We first prove it for the Minkowski dimension. Let $E \subset \mathbb{R}^n$ be a Kakeya set and let $f = \chi_{E_\delta}$. Then applying this conjecture to f, we have

$$1 \lesssim_\epsilon \delta^{-\frac{n}{p}+1-\epsilon}|E_\delta|^{1/p}.$$

This implies $|E_\delta| \gtrsim \delta^{n-(p-\epsilon)}$. This completes the proof of the Minkowski dimension. Now we do the Hausdorff dimension. Let $\{B(x_j,r_j)\}$ be a finite cover of E and let $J_k = \{j : 2^{-k} < r_j \leq 2^{-k+1}\}$. For $e \in S^{n-1}$, let I_e denote a line segment in E parallel to e. Let

$$\Omega_k = \left\{ e \in S^{n-1} : |I_e \cap \bigcup_{j\in J_k} B(x_j,r_j)| \geq \frac{1}{100k^2} \right\}.$$

Now as $\sum_k \frac{1}{100k^2} < 1$, we have $\cup_{k=1}^\infty \Omega_k = S^{n-1}$. Let $E_k = \cup_{j\in J_k} B(x_j, 10r_j)$ and let $f = \chi_{E_k}$. Note that f does depend on k. Let a_e be the midpoint of I_e, then for $e \in \Omega_k$, we have

$$|T_e^{2^{-k}}(a_e) \cap E_k| \geq \frac{1}{100k^2}|T_e^{2^{-k}}(a_e)|.$$

Therefore

$$\left\|f^*_{2^{-k}}\right\|_p^p \geq \int_{\Omega_k} |f^*_{2^{-k}}|^p \gtrsim |\Omega_k|\frac{1}{k^{2p}}. \tag{2.6}$$

But by our hypothesis, we have

$$\left\|f^*_{2^{-k}}\right\|_p^p \lesssim_\epsilon 2^{kn-kp+k\epsilon p}|E_k| \lesssim 2^{kn-kp+k\epsilon p}|J_k|\, 2^{-kn}. \tag{2.7}$$

So combining inequalities 2.6 and 2.7, we get

$$|\Omega_k| \lesssim_\epsilon k^{2p}2^{-kp(1-\epsilon)}|J_k| \lesssim_\epsilon 2^{-kp(1-2\epsilon)}|J_k|.$$

Using this, we obtain

$$\sum_j r_j^{p(1-2\epsilon)} \geq \sum_k 2^{-kp(1-2\epsilon)}|J_k| \gtrsim_\epsilon \sum_k |\Omega_k| \gtrsim 1,$$

since $\cup_{k=1}^\infty \Omega_k = S^{n-1}$. As the above inequality holds for arbitrarily small ϵ, we are done. $\square$

Now let us look at some partial conjectures of the L^p version of KMFC. Suppose $\mathbb{T}$ is a maximal collection of δ-tubes pointing in δ-separated directions. Then, we clearly have

$$\left\|\sum_{T\in\mathbb{T}} \chi_T\right\|_\infty \lesssim \delta^{1-n}.$$

Interpolating the above with Conjecture 2.2.4 i.e L^p KMFC II, we get the following conjecture.

Conjecture 2.3.3. Partial L^p KMFC. Let $p > n/(n-1)$. Suppose $\mathbb{T}$ is a maximal collection of δ-tubes pointing in δ-separated directions. Then, we have

$$\left\|\sum_{T\in\mathbb{T}} \chi_T\right\|_p \lesssim_\epsilon \delta^{-\frac{n}{p'}+1-\epsilon}.$$

To see that this inequality is sharp (at least up to the $\delta^{-\epsilon}$ factor), look at the collection of tubes all passing through a sphere of radius δ. Then LHS $\geq (|\mathbb{T}|^p\delta^n)^{1/p} \sim \delta^{-\frac{n}{p'}+1}$.

Lemma 2.3.4. The partial L^p KMFC (conjecture 2.3.3) implies conjecture 2.3.1 with exponent p' and consequently that Kakeya sets in $\mathbb{R}^n$ must have both Hausdorff and Minkowski dimension equal to at least p'.

Proof. We first give a direct and elementary proof that Kakeya sets in $\mathbb{R}^n$ must have Minkowski dimension equal to at least p'. Applying Hölder's inequality, we get

$$\left\|\sum_{T\in\mathbb{T}} \chi_T\right\|_p \left|\bigcup_{T\in\mathbb{T}} T\right|^{1/p'} \geq \left\|\sum_{T\in\mathbb{T}} \chi_T\right\|_1 = \sum_{T\in\mathbb{T}} |T| \sim 1.$$

Now by applying the conjecture, we get

$$\left|\bigcup_{T\in\mathbb{T}} T\right| \gtrsim_{\epsilon} \delta^{n-p'+\epsilon p'}.$$

Hence Kakeya sets in $\mathbb{R}^n$ must have Minkowski dimension equal to at least p'. Now that is completed, we establish the full lemma. If $\mathbb{T}$ is a maximal collection of δ-tubes pointing in δ-separated directions and we have numbers $y_T \geq 0$ (where $T \in \mathbb{T}$) satisfying $\delta^{n-1}\sum_{T\in\mathbb{T}} y_T^p \leq 1$, then

$$\left\|\sum_{T\in\mathbb{T}} y_T \chi_T\right\|_p \lesssim_{\epsilon} \delta^{-\frac{n}{p'}+1-\epsilon}.$$

This follows from the proofs of theorems 2.2.6 and 2.2.7. The lemma now follows by applying lemma 2.2.2. $\square$

As p decreases to $n/(n-1)$, p' increases to n. So by proving the conjecture for values closer and closer to $n/(n-1)$, we get closer and closer to the full Kakeya conjecture.

Chapter 3

Real Algebraic Geometry

In 2008, Dvir [10] proved the finite field analogue of the Kakeya conjecture using an algebraic argument involving polynomials. This breakthrough has inspired application of algebraic/polynomial methods to make progress on the Kakeya conjecture and related problems in the Euclidean setting. In 2010, Guth proved the endpoint bound for the multilinear Kakeya conjecture using algebraic topology [13]. The restriction conjecture is a problem in harmonic analysis concerning oscillatory integrals which implies the Kakeya conjecture. In 2016, Guth ([11, 12]) made progress on the restriction conjecture using polynomial partitioning, a method originally used to solve the Erdős distinct distances problem by Guth and Katz in 2015 [14]. There have been some more algebraic developments which we will outline in the following chapter while discussing the polynomial Wolff axioms. Now we will discuss some basic theory of real algebraic geometry to develop the background required for the next chapter.

3.1 Semialgebraic sets and mappings

In this section we will discuss the definition of semi algbraic sets and mappings. First we define algebraic sets and ideal vanishing on a set.

Definition 3.1.1. Let B be a subset of $\mathbb{R}[x_1, \dots, x_n]$. Denote $V(B) = \{x \in \mathbb{R}^n : f(x) = 0 \text{ for all } f \in B\}$ i.e $V(B)$ is the set of simultaneous zeros of B

(sometimes called the set carved out by B). Such sets are called algebraic sets.

Definition 3.1.2. Given $A \subset \mathbb{R}^n$, we define $I(A) = \{f \in \mathbb{R}[x_1, \ldots, x_n] : f(x) = 0 \text{ for all } x \in A\}$ to be the ideal of $\mathbb{R}[x_1, \ldots, x_n]$ of polynomials vanishing on A.

We next define semialgebraic sets. A semialgebraic subset is a set of points satisfying a boolean combination of polynomial equations and inequalities. Using set algebra/boolean algebra we can write any semialgebraic set in a standard and simple form. This is why we make the following equivalent definition.

Definition 3.1.3. A semialgebraic subset of $\mathbb{R}^n$ is a subset of the form

$$\bigcup_{i=1}^{s} \bigcap_{j=1}^{r_i} \{x \in \mathbb{R}^n : f_{i,j} *_{i,j} 0\},$$

where $f_{i,j} \in \mathbb{R}[x_1, \ldots, x_n]$ and $*_{i,j}$ is either $=$ or $<$.

Any real algebraic variety can be written as the zero set of a single polynomial and the degree of this polynomial tells us something about the complexity of the variety. Motivated by this, we make the following definition.

Definition 3.1.4. The complexity of a semi algebraic set is the smallest sum of the degrees of the polynomials appearing in a complete description of the set.

We now define semialgebraic mappings.

Definition 3.1.5. Let $S_1 \subset \mathbb{R}^m$ and $S_2 \subset \mathbb{R}^n$ be two semialgebraic sets. A mapping $f : S_1 \to S_2$ is semialgebraic if its graph is semialgebraic in $\mathbb{R}^{n+m}$.

It is not hard to see that maps which are given by rational functions (quotient of polynomial functions) are semialgebraic. The complexity of a semialgebraic mapping is simply defined as the complexity of the graph. We will now discuss some properties of semialgebraic sets.

3.2 The Tarski-Seidenberg Principle and projection of semialgebraic sets

In this section, we will prove a fundamental property of semialgebraic sets which is that they are stable under projection i.e. the projection of semialgebraic sets is semialgebraic. But first we need to prove the Tarski-Seidenberg principle. To start with, for $a \in \mathbb{R}$ we affix the notation

$$\begin{aligned} \operatorname{sign}(a) &= 0 \quad \text{if } a = 0, \\ \operatorname{sign}(a) &= 1 \quad \text{if } a > 0, \\ \operatorname{sign}(a) &= -1 \quad \text{if } a < 0. \end{aligned}$$

Theorem 3.2.1. (The Tarski-Seidenberg principle) Let $f_i(x, Y) = h_{i,m_i}(Y)x^{m_i} + \ldots + h_{i,0}(Y)$, where $Y = (y_1, y_2, \ldots, y_n)$, for $i = 1, \ldots, s$ be a sequence of non constant polynomials in $n+1$ variables with coefficients in $\mathbb{Z}$. Let ϵ be a function from $\{1, 2, \ldots, s\}$ to $\{-1, 0, 1\}$. Then there exists a boolean combination $\mathcal{B}(Y)$ (i.e a finite composition of disjunctions, conjuctions and negations) of polynomial equations and inequalities in the variables Y with coefficients in $\mathbb{Z}$ such that for every $\widetilde{Y} \in \mathbb{R}^n$, the system

$$\begin{cases} \operatorname{sign}(f_1(x, \widetilde{Y})) & = \epsilon(1) \\ & \vdots \\ \operatorname{sign}(f_s(x, \widetilde{Y})) & = \epsilon(s) \end{cases} \tag{3.1}$$

has a solution x in $\mathbb{R}$ iff $\mathcal{B}(\widetilde{Y})$ holds true.

The Tarski-Seidenberg principle proves semialgebraic sets are stable under projection for a special class of semialgebraic sets. To see this, observe that the set of points $(x, \widetilde{Y}) \in \mathbb{R}^{n+1}$ satisfying the system of equations (3.1) defines a semialgebraic set S and the set of all $\widetilde{Y} \in \mathbb{R}^n$ such that the system (3.1) has a solution x in $\mathbb{R}$ is the projection of S to its last n coordinates. The main difficulty lies in proving the Tarski-Seidenberg principle, the stability of general semialgebraic sets under projection follows easily from it as we will see later on.

We will need the following notation to prove the Tarski-Seidenberg principle. Let $f_1, \ldots, f_s$ be polynomials in $\mathbb{R}[x]$ and let $x_1 < \ldots < x_N$ be the roots in $\mathbb{R}$ of all f_i that are not identically zero. By convention we define $x_0 = -\infty, x_{N+1} = \infty$. If $I_k = (x_k, x_{k+1})$, then $\operatorname{sign}(f_i(x))$ is constant for $x \in I_k$, and is denoted as $\operatorname{sign}(f_i(I_k))$. The matrix with s rows and $2N+1$ colums whose ith row is

$$\operatorname{sign}(f_i(I_0), \operatorname{sign}(f_i(x_1)), \operatorname{sign}(f_i(I_1)), \ldots, \operatorname{sign}(f_i(x_N)), \operatorname{sign}(f_i(I_N))$$

is denoted $\operatorname{SIGN}(f_1, \ldots, f_s)$. If $m = \max(\{\deg(f_i) : i = 1, \ldots, s\})$ then $N \leq sm$. The set of matrices with entries in $\{-1, 0, 1\}$ having s rows and $2l+1$ columns, for $l = 0, \ldots, sm$ is denoted by $W_{s,m}$.

Next let ϵ be a function from $\{1, 2, \ldots, s\}$ to $\{-1, 0, 1\}$, then we define $W(\epsilon)$ to be the subset of $W_{s,m}$ whose elements are matrices having one of their columns coinciding with the sequence $\epsilon(1), \ldots, \epsilon(s)$. Then clearly for every sequence of polynomials $f_1, \ldots, f_s$ in $\mathbb{R}[x]$ of degree $\leq m$, the system

$$\begin{cases} \operatorname{sign}(f_1(x)) & = \epsilon(1) \\ & \vdots \\ \operatorname{sign}(f_s(x)) & = \epsilon(s) \end{cases}$$

has a solution x in $\mathbb{R}$ iff $\operatorname{SIGN}(f_1, \ldots, f_s) \in W(\epsilon)$.

Lemma 3.2.2. There exists a mapping ϕ from $W_{2s,m}$ to $W_{s,m}$ such that for every sequence of polynomials $f_1, \ldots, f_s$ in $\mathbb{R}[x]$ of degrees $\leq m$, with f_s nonconstant and none of the $f_1, \ldots, f_{s-1}$ identically zero, we have

$$\operatorname{SIGN}(f_1, \ldots, f_s) = \phi(\operatorname{SIGN}(f_1, \ldots, f_{s-1}, f_s', g_1, \ldots, g_s)),$$

where f_s' is the derivative of f_s, and $g_1, \ldots, g_s$ are remainders of the euclidean division of f_s by $f_1, \ldots, f_{s-1}, f_s'$

Proof. Let $x_1 < \ldots < x_N$ be the roots of those polynomials among $f_1, \ldots, f_{s-1}, f_s', g_1, \ldots, g_s$ that are not identically zero. Extract from these roots the subsequence

$x_{i_1} < \ldots < x_{i_M}$ of the roots of the polynomial $f_1, \ldots, f_{s-1}, f_s'$. The sequence $i_1, \ldots, i_M$ depends only on $w = \mathrm{SIGN}(f_1, \ldots, f_{s-1}, f_s', g_1, \ldots, g_s)$. By convention let $i_0 = 0$ with $x_0 = -\infty$ and let $i_{M+1} = N+1$ with $x_{N+1} = +\infty$. For $k = 1, \ldots, M$ at least one of the polynomials $f_1, \ldots, f_{s-1}, f_s'$ vanishes at x_{i_k} and this gives rise to a mapping $\theta : \{1, \ldots, M\} \to \{1, \ldots, s\}$. This mapping can be deduced from w alone and we also have $f_s(x_{i_k}) = g_{\theta(k)}(x_{i_k})$. In each of the intervals $(x_{i_k}, x_{i_{k+1}})$, the polynomial f_s may have a single root or no root at all since the f_s is either monotonically decreasing or increasing on each of the intervals. Suppose $M \neq 0$, we can say that f_s has a root

(a) in $(x_{i_k}, x_{i_{k+1}})$ for $k = 1, \ldots, M-1$ iff $\mathrm{sign}(g_{\theta(k)}(x_{i_k}))\,\mathrm{sign}(g_{\theta(k+1)}(x_{i_{k+1}})) = -1$,

(b) in $(-\infty, x_{i_1})$ iff $\mathrm{sign}(f_s'(-\infty, x_1))\,\mathrm{sign}(g_{\theta(1)}(x_{i_1})) = 1$,

(c) in $(x_{i_M}, +\infty)$ iff $\mathrm{sign}(f_s'(x_N, +\infty))\,\mathrm{sign}(g_{\theta(M)}(x_{i_M})) = -1$.

If $M = 0$, then we will have a single root in $(-\infty, \infty)$. Now let $y_1 < \ldots < y_L$ (with $L \leq sm$) be the roots of the polynomials $f_1, \ldots, f_s$. As before, let $y_0 = -\infty$ and $y_{L+1} = +\infty$. Define the function

$$\rho : \{0, \ldots, L+1\} \to \{0, \ldots, M+1\} \cup \{(k, k+1) : k = 0, \ldots, M\}$$
$$l \mapsto \begin{cases} k & \text{if } y_l = x_{i_k}, \\ (k, k+1) & \text{if } y_l \in (x_{i_k}, x_{i_{k+1}}). \end{cases}$$

We have already seen that L and the function ρ depends only on w. We can now verify that $\mathrm{SIGN}(f_1, \ldots, f_s)$ depends only on w. For $j = 1, \ldots, s-1$, we have

- if $\rho(l) = k$, then $\mathrm{sign}(f_j(y_l)) = \mathrm{sign}(f_j(x_{i_k}))$,
- if $\rho(l) = (k, k+1)$, then $\mathrm{sign}(f_j(y_l)) = \mathrm{sign}(f_j(x_{i_k}, x_{i_{k+1}}))$,
- also if $\rho(l) = k$ or $\rho(l) = (k, k+1)$, then $\mathrm{sign}(f_j(y_l, y_{l+1})) = \mathrm{sign}(f_j(x_{i_k}, x_{i_{k+1}})$.

Now we deal with the case $j = s$, we have

- if $\rho(l) = k$, then $\operatorname{sign}(f_s(y_l)) = \operatorname{sign}(g_{\theta(k)}(x_{i_k})$,
- if $\rho(l) = (k, k+1)$, then $\operatorname{sign}(f_s(y_l)) = 0$,
- if $l = 0$, then $\operatorname{sign}(f_s(-\infty, y_1)) = -\operatorname{sign}(f'_s(-\infty, x_1))$,
- if $l \neq 0$ and $\rho(l) = k$, then $\operatorname{sign}(f_s(y_l, y_{l+1}) = \operatorname{sign}(g_{\theta(k)}(x_{i_k})$ if this is nonzero, otherwise $\operatorname{sign}(f_s(y_l, y_{l+1}) = \operatorname{sign}(f'_s(x_{i_k}, x_{i_{k+1}}))$,
- if $l \neq 0$ and $\rho(l) = (k, k+1)$, then $\operatorname{sign}(f_s(y_l, y_{l+1}) = \operatorname{sign}(f'_s(x_{i_k}, x_{i_{k+1}}))$.

□

The Tarski-Siedenberg principle follows from the following proposition which we will prove using the above lemma.

Proposition 3.2.3. Let $f_i(x, Y) = h_{i,m_i}(Y)x^{m_i} + \ldots + h_{i,0}(Y)$, where $Y = (y_1, y_2, \ldots, y_n)$, for $i = 1, \ldots, s$ be a sequence of non constant polynomials in $n+1$ variables with coefficients in $\mathbb{Z}$. Further let $m = \max(\{m_i : i \leq s\})$. Let W' be a subset of $W_{s,m}$, then there exists a boolean combination $\mathcal{B}(Y)$ of polynomial equations and inequalities in the variables Y with coefficients in $\mathbb{Z}$ such that for every $\widetilde{Y} \in \mathbb{R}^n$, we have

$$\operatorname{SIGN}(f_1(x, \widetilde{Y}), \ldots, f_s(x, \widetilde{Y})) \in W' \iff \mathcal{B}(\widetilde{Y}) \text{ holds true.}$$

Proof. We will prove this by using the principle of transfinite induction. We associate to the sequence of polynomials $(f_1, \ldots, f_s)$ the multi-set $\{m_1, \ldots, m_s\}$ of their degrees in x. We define a strict total order on the collection of non empty finite sets of non negative integers now. Let $\alpha = \{m'_1, \ldots, m'_t\}$ and let $\beta = \{m_1, \ldots, m_s\}$, we say $\alpha \prec \beta$ if there exists a non negative integer p such that for every $q > p$ the number of times q appears in α is equal to the number of times q appears in β, and the number of times p appears in α is smaller than the number of times p appears in β.

Here are a few examples to illustrate the order. We have $\{3, 3, 2, 1, 1, 1\} \prec \{3, 3, 2, 2, 1\}$ (here $p = 2$) and $\{2, 2, 2, 1, 1, 0\} \prec \{3\}$ (here $p = 3$). This kind of ordering is used to rank the countries in Olympics based on their medal tallies (gold, silver, bronze).

The collection of nonempty finite sets of non negative integers, with that ordering, is a well ordered set as there is no infinite sequence $\sigma_1 \succ \sigma_2 \succ \sigma_3 \succ \ldots$, and therefore every subcollection has a least element. Now by the principle of transfinite induction it is sufficient to show that if the proposition holds all for all sequence of polynomials such that the corresponding multi-set lies in $\{\alpha : \alpha \prec \beta\}$, then it must hold for any sequence of polynomials with associated multi-set β.

Suppose we have polynomials $f_1, \ldots, f_s$ with associated multi set β such that $m = 0$, then for any $\widetilde{Y}$, the matrix $\mathrm{SIGN}(f_1(x, \widetilde{Y}), \ldots, f_s(x, \widetilde{Y}))$ has one column given by the signs of $h_{1,0}(\widetilde{Y}), \ldots, h_{s,0}(\widetilde{Y})$. From this, we can see that the proposition holds for any sequence of polynomials with associated multi-set β, since each of the finite elements of W' gives rise to a Boolean condition and these finitely many conditions can be concatenated by $\vee$. For example, if $s = 2$ and $W' = \{(0,1), (-1,0)\}$, then the Boolean conditions would be $[(h_{1,0}(\widetilde{Y}) = 0) \wedge (h_{2,0}(\widetilde{Y}) > 0)] \vee [(h_{1,0}(\widetilde{Y}) < 0) \wedge (h_{2,0}(\widetilde{Y}) = 0)]$. We did not even use the predecessors in this case.

Next suppose we have polynomials $f_1, ..., f_s$ with associated multi set β and suppose $m \geq 1$, without loss of generality we may assume that $m_s = m$. Let $f_s'(x, Y)$ be the partial derivative of $f_s(x, Y)$ with respect to x. Let F be the fraction field of $\mathbb{Z}[Y]$, then f_s may be divided by $f_1, f_2, \ldots, f_{s-1}, f_s'$ in $F[x]$ to obtain remainders $r_1, \ldots, r_s$. In fact for each $i = 1, 2, \ldots, s$, one can find even integers k_i such that $h_{i,m_i}(Y)^{k_i}(r_i(x, Y)) = g_i(x, Y)$ is a polynomial with integral coefficients. This is because $h_{i,m_i}(Y)$ is the leading coefficient of the divisor polynomial in $F[x]$.

Suppose that we are given a $\widetilde{Y} \in \mathbb{R}^n$ such that $h_{i,m_i}(\widetilde{Y}) \neq 0$ for all i and let $W'' = \phi^{-1}(W') \subset W_{2s,m}$, then

$$\mathrm{SIGN}(f_1(x, \widetilde{Y}), \ldots, f_s(x, \widetilde{Y})) \in W'$$

is equivalent to

$$\mathrm{SIGN}(f_1(x, \widetilde{Y}), \ldots, f_{s-1}(x, \widetilde{Y}), f_s'(x, \widetilde{Y}), r_1(x, \widetilde{Y}), \ldots, r_s(x, \widetilde{Y})) \in W''$$

by our previous lemma. Also since $h_{i,m_i}(\widetilde{Y}) \neq 0$, we have $\mathrm{sign}(r_i(x,\widetilde{Y})) = \mathrm{sign}(g_i(x,\widetilde{Y}))$. Thus the above is equivalent to

$$\mathrm{SIGN}(f_1(x,\widetilde{Y}),\ldots,f_{s-1}(x,\widetilde{Y}),f_s'(x,\widetilde{Y}),g_1(x,\widetilde{Y}),\ldots,g_s(x,\widetilde{Y})) \in W''.$$

The degree set of $f_1,\ldots,f_{s-1},f_s',g_1,\ldots,g_s$ precedes β, so we can find a boolean combination for above criterion by inductive hypothesis.

If $h_{i,m_i}(\widetilde{Y}) = 0$ for some i, then we can truncate the corresponding polynomial f_i and obtain a sequence of polynomials, whose degree set in x precedes β. So we get another boolean combination for this case. Concatenating all of these boolean combinations we can find the required boolean condition. By the principle of transfinite induction, we are done. $\square$

The following corollary is the form in which the principle is most used. We are just replacing $\mathbb{Z}$ with $\mathbb{R}$.

Corollary 3.2.4. Let $f_i(x,Y) = h_{i,m_i}(Y)x^{m_i} + \ldots + h_{i,0}(Y)$ where $Y = (y_1,y_2,\ldots,y_n)$, for $i = 1,\ldots,s$ be a sequence of non constant polynomials in $n+1$ variables with coefficients in $\mathbb{R}$. Let ϵ be a function from $\{1,2,\ldots,s\}$ to $\{-1,0,1\}$. Then there exists a boolean combination $\mathcal{B}(Y)$ (i.e a finite composition of disjunctions, conjuctions and negations) of polynomial equations and inequalities in the variables Y with coefficients in $\mathbb{R}$ such that for every $\widetilde{Y} \in \mathbb{R}^n$, the system

$$\begin{cases} \mathrm{sign}(f_1(x,\widetilde{Y})) & = \epsilon(1) \\ & \vdots \\ \mathrm{sign}(f_s(x,\widetilde{Y})) & = \epsilon(s) \end{cases}$$

has a solution x in $\mathbb{R}$ iff $\mathcal{B}(\widetilde{Y})$ holds true.

Proof. There exists a positive integer M and polynomials $g_i \in \mathbb{Z}[x,Y,T]$ of $n+1+M$ variables such that $f_i(x,Y) = g_i(x,Y,a)$ where a contains all the coefficients of all of the f_i. Then we can apply the Tarski-Seidenberg principle to the polynomials $g_i(x,Y,T)$ to get a boolean combination $\mathcal{B}(Y,T)$

of polynomials in $n+M$ variables with coefficients in $\mathbb{Z}$. Fix $T = a$ on both sides of the equivalence to get the required relation. $\square$

Clearly every semialgebraic set is a finite union of sets of the form $\{(Y, x) \in \mathbb{R}^{n+1} : f_i(Y, x) = 0, i = 1, \ldots, l, g_j(Y, x) > 0, j = 1, \ldots, m\}$. The projection of such a set to the first n coordinates is semialgebraic by the above corollary of the Tarski-Seidenberg principle. This proves that semialgebraic sets are stable under projection. We can even bound the complexity of the projection, a fact we state without proof.

Theorem 3.2.5. Let $\Pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be the projection map to the first n coordinates. Then for any $E \geq 1$, there is a constant $C(n, E)$ so that, for every semialgebraic subset S of $\mathbb{R}^{n+1}$ of complexity at most E, the projection $\Pi(S)$ is a semialgebraic subset of $\mathbb{R}^n$ and has complexity at most $C(n, E)$.

3.3 Gromov's algebraic lemma

In this section, we discuss concepts like dimension of semialgebraic sets and also decomposition of semialgebraic sets. We will also state Gromov's algebraic lemma which will be used in the the following chapter. To begin with, it is easy to see that the semialgebraic subsets of $\mathbb{R}$ are exactly the finite unions of points and open intervals. We next state a theorem without proof regarding the decomposition of semialgebraic sets.

Theorem 3.3.1. Every semialgebraic subset of $\mathbb{R}^n$ is the disjoint union of a finite number of semialgebraic sets, each of them semialgebraically homeomorphic to an open hypercube $(-1, 1)^d$, where d is a non negative integer (with $(-1, 1)^0$ being a point).

The proof can be found in [2]. As one might guess, the dimension of an algebraic set is the largest d coming from the decomposition. We now give an algebraic definition which coincides with the just stated definition.

Definition 3.3.1. Let $S \subset \mathbb{R}^n$ be a semialgebraic set. Denote by $P(S) = \mathbb{R}[x_1, \ldots, x_n]/I(S)$ the ring of polynomial functions on S. The dimension of S is the dimension of the ring $P(S)$, i.e the maximal length of chains of prime ideals of $P(S)$.

In a similar vein to theorem 3.3.1, we have Gromov's algebraic lemma which we now state. A proof can be found in [4] and one of the key components of the proof is Tarski-Seidenberg principle.

Theorem 3.3.2. Gromov's algebraic lemma For all integers $n, E, r \geq 1$, there exists $M(n, E, r) < \infty$ with the following properties. For any compact semialgebraic set $S \subset [-1, 1]^n$ of dimension d and complexity at most E, there exists an integer $N \leq M(n, E, r)$ and C^r maps $\phi_1, \ldots, \phi_N : [-1, 1]^d \to [-1, 1]^n$ so that

$$\bigcup_{j=1}^{N} \phi_j([-1, 1]^d) = S$$

and

$$\|\phi_j\|_{C^r} = \max_{|\alpha| \leq r} \|\partial^\alpha \phi_j\|_\infty \leq 1.$$

Chapter 4

On the Polynomial Wolff axioms

The polynomial method was first used in the Kakeya family of problems by Dvir [10] in 2008 to fully prove the finite field version of Kakeya conjecture. This breakthrough inspired many more applications of the polynomial method to the Kakeya family of problems as we touched upon in the introduction of the previous chapter.

The restriction conjecture is a problem in Euclidean harmonic analysis concerning oscillatory integrals which implies the Kakeya conjecture. One of the most important applications of the polynomial method to the Kakeya family of problems was Guth's result on the restriction conjecture in 2016 [11, 12]. Guth was able to make progress on the restriction conjecture using the technique of polynomial partitioning, an algebraic technique that was introduced by Guth and Katz [14] in 2015 to solve the Erdős distinct distances problem. Since the restriction conjecture is stronger than the Kakeya conjecture, Guth's work led to progress on the Kakeya conjecture as well.

In chapter 1, we gave a formulation of the Kakeya conjecture (conjecture 1.3.3) that can be interpreted as saying δ-tubes pointing in δ-separated directions are essentially disjoint. We rearranged the inequality to obtain an equivalent formulation that seeks to estimate the number of δ-tubes pointing in δ-separated directions that can be contained in a set in terms of the

Lebesgue measure of that set (conjecture 1.3.4). In 2018, Guth and Zahl [15] showed how progress can be made on this formulation of the Kakeya conjecture using the technique of polynomial partitioning.

When applying polynomial partitioning to the restriction/Kakeya problem, the following important sub-problem naturally arises: does conjecture 1.3.4 for the special case of δ-neighborhoods of algebraic varieties hold? Guth conjectured that the answer was affirmative in [12]. Before we state Guth's conjecture precisely, we state Wongkew's lemma [21] as it will be needed to understand Guth's conjecture.

Lemma 4.0.1. (Wongkew) Let $Z \subset \mathbb{R}^n$ be a m-dimensional algebraic variety of degree at most D and let Z_δ denote the δ neighborhood of Z. Then we have

$$|Z_\delta \cap [-1,1]^n| \leq c(n,D)\delta^{n-m}.$$

Before we continue, we make a simplifying assumption. We are primarily interested in the Kakeya conjecture which can be formulated as saying that δ-tubes pointing in δ-separated directions are essentially disjoint. Since essentially disjoint is a local property, without loss of generality we may assume all our sets and tubes to be compact subsets of $[-1,1]^n$. We are now ready to state Guth's conjecture from [12].

Conjecture 4.0.2. (Guth) For all integers $n, D \geq 2$ and all $\epsilon > 0$, there is a constant $C(n,D,\epsilon) > 0$ so that the number of δ-tubes, pointing in δ-separated directions, contained in the δ-neighborhood of an m-dimensional algebraic variety $Z \subset \mathbb{R}^n$, of degree at most D is bounded by $C(n,D,\epsilon)\delta^{1-m-\epsilon}$.

From conjecture 1.3.4, we would expect the upper bound to be $C_\epsilon \delta^{1-n-\epsilon}|Z_\delta \cap [-1,1]^n|$. But Wongkew's lemma tells us that $|Z_\delta \cap [-1,1]^n| \leq c(n,D)\delta^{n-m}$, this explains the upper bound in conjecture 4.0.2.

Roughly speaking, conjecture 4.0.2 says that δ neighborhood of varieties cannot contradict the Kakeya conjecture. This conjecture was proven for $n = 3$ by Guth, which led to progress on the three-dimensional restriction conjecture [11, 18], and by Zahl for $n = 4$, which led to progress on the four-dimensional Kakeya conjecture [27], as well as the four-dimensional restriction conjecture [7].

Conjecture 4.0.2 was open in higher dimensions until 2018, when Katz and Rogers [17] proved conjecture 1.3.4 for semialgebraic sets i.e semialgebraic sets do not contradict the Kakeya conjecture. We will give the rigorous statement of their result shortly. In particular, their result implied conjecture 4.0.2 was true for all dimensions. This follows from Wongkew's lemma and because the δ-neighborhood of an algebraic variety of degree at most D is a semialgebraic set whose complexity is at most $c(n, D)$. Proving conjecture 4.0.2 fully leads to further improvements for the restriction conjecture in higher dimensions, as was noted in [12].

Actually Katz and Rogers proved something even stronger than conjecture 1.3.4 for semialgebraic sets. To explain and motivate their full result, we need to discuss the polynomial Wolff axioms. The Wolff axioms were introduced by Wolff [25] as the minimal set of conditions that one needed to impose on a collection of δ-tubes to make the hairbrush argument work. Critically δ-tubes pointing in δ-seperated directions satisfied the Wolff axioms. We now define the polynomial axioms first introduced by Guth and Zahl [15] which are a generalization of Wolff axioms.

Definition 4.0.1. Polynomial Wolff axioms We say that collections $\mathbb{T}$ of δ-tubes in $\mathbb{R}^n$ satisfy the polynomial Wolff axioms if, for every integer $E \geq 2$, there is a constant $C(n, E) > 0$ so that

$$\#(\{T \in \mathbb{T} : |T \cap S| \geq \lambda |T|\}) \leq C(n, E)|S|\delta^{1-n}\lambda^{-n}$$

whenever S is a semialgebraic set, of complexity at most E, and $\lambda \geq \delta > 0$.

Just as in the case of Wolff axioms we wish to show that δ-tubes pointing in δ-separated directions satisfy the polynomial Wolff axioms. Katz and Rogers [17] showed this was true up to a factor of $C_\epsilon \delta^{-\epsilon}$. We finally state the theorem proved by Katz and Rogers.

Theorem 4.0.3. (Katz-Rogers) Let $n, E \geq 2$ be integers and let $\epsilon > 0$. Then there is a constant $C(n, E, \epsilon) > 0$ so that, for every set $\mathbb{T}$ of δ-tubes in $\mathbb{R}^n$ pointing in δ-separated directions,

$$\#(\{T \in \mathbb{T} : |T \cap S| \geq \lambda |T|\}) \leq C(n, E, \epsilon)|S|\delta^{1-n-\epsilon}\lambda^{-n}$$

whenever S is a semialgebraic set, of complexity at most E, and $\lambda \geq \delta > 0$.

Firstly observe that when we put $\lambda = 1$, we essentially get conjecture 1.3.4 for semialgebraic sets as we discussed previously. The λ^{-n} factor may be rationalized as follows. Suppose we have a collection $\mathbb{T}$ of $\lambda \times \delta$ tubes pointing in δ-separated directions. Then we pick a maximal subcollection $\mathbb{T}'$ which is (δ/λ)-direction separated. We have $\#\mathbb{T}' \gtrsim \lambda^{n-1}\#\mathbb{T}$. The Kakeya conjecture leads us to expect that the collection $\mathbb{T}'$ is essentially disjoint (scale $\mathbb{R}^n$ up by a factor λ^{-1}). So we expect

$$\left|\bigcup_{T \in \mathbb{T}} T\right| \geq \left|\bigcup_{T \in \mathbb{T}'} T\right| \gtrsim \#\mathbb{T}'(\lambda\delta^{n-1}) \gtrsim \#\mathbb{T}(\lambda^n\delta^{n-1}).$$

Suppose S contains $\cup_{T \in \mathbb{T}} T$, then clearly we have $|S| \gtrsim \#\mathbb{T}(\lambda^n\delta^{n-1})$. Rearranging, we get

$$\#\mathbb{T} \lesssim |S|\delta^{1-n}\lambda^{-n}.$$

This resembles the statement of theorem 4.0.3. So we have a heuristic explanation for the λ^{-n} factor.

In this chapter we will only prove the $\lambda = 1$ case of theorem 4.0.3 which is essentially conjecture 1.3.4 for semialgebraic sets. So we will have proved conjecture 4.0.2. But the $\lambda < 1$ case is non-trivial and we will need the full strength of theorem 4.0.3 in the next chapter when we will prove KMFC holds for semialgebraic mapping. As mentioned previously, we will be using tools from real algebraic geometry including Gromov's algebraic lemma to prove theorem 4.0.3 for $\lambda = 1$.

4.1 Katz-Rogers' theorem for λ=1

Before we begin the proof, we need to switch to a more convenient setup. For notational convenience we work in $\mathbb{R}^{n+1}$ instead of $\mathbb{R}^n$. We also work with modified δ-tubes which are defined as follows

$$T_{a,d}(\delta) = \{(x,t) \in \mathbb{R}^n \times [0,1] : |x - a - td| \leq \delta\},$$

where $d \in [-1,1]^n$ and a is only allowed to take values such that $T_{a,d}(\delta) \subset [-1,1]^n \times [0,1]$. The axis of such a tube is clearly $\{(a,0)+t(d,1) : t \in [0,1]\}$. Hence $(a,0)$ is the base point of the tube and $(d,1)$ is the direction of the tube. We also note that $\{(a,d) \in \mathbb{R}^{2n} : T_{a,d}(\delta) \subset [-1,1]^n \times [0,1]\} \subset [-1,1]^{2n}$. We define the angle/direction separation between the tubes $T_{a,d}(\delta)$ and $T_{a',d'}(\delta)$ to be $|d-d'|$. Now we are ready to state the appropriate version of theorem 4.0.3 with $\lambda = 1$. This can alternately be viewed as Kakeya conjecture (conjecture 1.3.4) for semialgebraic sets.

Theorem 4.1.1. (Katz-Rogers' theorem for $\lambda = 1$) Let $S \subset [-1,1]^n \times [0,1]$ be a semialgebraic set of complexity at most E, then the number of δ-tubes pointing in δ-separated directions that can be contained in S is at less than or equal to $C(n,E,\epsilon)|S|\delta^{-n-\epsilon}$.

Gromov's algebraic lemma is a key component of the proof of the theorem. Before we prove the theorem, we need some lemmas which we now state and prove. The Tarski–Seidenberg principle (theorem 3.2.5) will play a crucial play in the proof of two of the lemmas. The first lemma shows the set of parameters for which a δ-tube can be contained in a semialgebraic set S is semialgebraic.

Lemma 4.1.2. Let $S \subset [-1,1]^n \times [0,1]$ be a semialgebraic set of complexity at most E and let $\delta > 0$ be fixed. Then $L = \{(a,d) \in [-1,1]^{2n} : T_{a,d}(\delta) \subset S\}$ is a semialgebraic set of complexity at most $C(n,E)$, a constant depending only on n and E.

Proof. We first define the semialgebraic set $Y = \{(a,d,x,t) \in [-1,1]^{3n} \times [0,1] : (x,t) \notin S, (x,t) \in T_{a,d}(\delta)\}$. This set is semialgebraic because $(x,t) \in T_{a,d}(\delta)$ iff $|x-a-td|^2 \leq \delta^2$ (the left hand size of this inequality is a polynomial with variables a,d,x,t). Writing $Z = \Pi(Y)$ where Π is the projection $(a,d,x,t) \mapsto (a,d)$, by theorem 3.2.5 we conclude that Z is semialgebraic of complexity depending only on n and E. Finally we observe $L = [-1,1]^{2n} \setminus Z$ which completes the proof. $\square$

It will be useful to extract a semialgebraic section of L in which for every direction $d \in [-1,1]^n$, there is at most one associated base point a.

Lemma 4.1.3. Let $L \subset [-1,1]^{2n}$ be a compact semialgebraic set of complexity at most E. Let Π be the orthogonal projection into the final n coordinates $(a,d) \mapsto d$. Then there is a constant $C(n,E) > 0$ and a semialgebraic set L' of complexity at most $C(n,E)$, so that $L' \subset L$ and $\Pi(L') = \Pi(L)$ and so that for each $d \in [-1,1]^n$, there is at most one a with $(a,d) \in L'$.

Proof. It suffices to show that for the projection Π_1 defined by $(a,d) \mapsto (a_2,\ldots,a_n,d)$, there is a constant $C(E) > 0$ and a semialgebraic set L_1 of complexity at most $C(E)$ so that $L_1 \subset L$ and $\Pi_1(L_1) = \Pi_1(L)$, so that for any $(a_2,\ldots,a_n,d)$ there is at most a_1 with $(a_1,\ldots,a_n,d) \in L_1$. Having done that, we obtain L_2 by applying the same result to L_1 with the first coordinate replaced by the second. Similarly we obtain L_j from L_{j-1} with the first coordinate replaced by the jth, and finally setting $L' = L_n$.

So we now show how to construct L_1. Whenever $(a_2,\ldots,a_n,d) \in \Pi(L)$ we let $(a_1,\ldots,a_n,d) \in L_1$ with a_1 the maximal value so that $(a_1,\ldots,a_n,d) \in L$. We now need to show this set is semialgebraic of bounded complexity. For this, let $Y = \{(x,a,d) \in \mathbb{R} \times L : x > a_1, (x,a_2,\ldots,a_n,d) \in L\}$. Then we get L_1 if we project Y to its last $2n$ coordinates and take its complement in L. By theorem 3.2.5, the complexity will be bounded. This completes the proof. $\square$

The final lemma is a calculation which concerns the integral of a monic polynomial.

Lemma 4.1.4. For $n \geq 1$, there exists constants $C(n) > 0$ such that

$$\int_0^1 |t^n + c_{n-1}t^{n-1} + \ldots + c_0|\, dt \geq C(n),$$

where $c_0,\ldots,c_{n-1}$ are arbitary complex numbers.

Proof. Using the fundamental theorem of algebra, we can factorize $|t^n + c_{n-1}t^{n-1} + \ldots + c_0| = |t - \alpha_1| \ldots |t - \alpha_n|$. By plane geometry given any arbitrary $\alpha \in \mathbb{C}$, we have $|t - \alpha| \geq \frac{1}{4n}$ for $t \in [0,1]$ except possibly on an subinterval of length $\frac{1}{2n}$. So $|t - \alpha_1| \ldots |t - \alpha_n| \geq \frac{1}{(4n)^n}$ for $t \in [0,1]$ except possibly on n subintervals each of length at most $\frac{1}{2n}$. So we have $|t - \alpha_1| \ldots |t - \alpha_n| \geq \frac{1}{(4n)^n}$ on a subset of $[0,1]$ of measure at least $1/2$. This proves the lemma. $\square$

The crucial fact in the above lemma is that the lower bound is independent of the coefficients $c_0, \ldots, c_{n-1}$. We are now ready to prove theorem 4.1.1 i.e Katz-Rogers' theorem for $\lambda = 1$.

Proof of Theorem 4.1.1. We prove by contradiction. Suppose the theorem is false. Then there exist n, E, ϵ with the following property: for every $C > 0$, there exists a semialgebraic set of complexity E, a collection of δ-tubes $\mathbb{T}$ pointing in δ-separated directions contained in S such that $\#\mathbb{T} > C|S|\delta^{-n-\epsilon}$. Also $\delta \to 0$ as $C \to \infty$ since the theorem holds for $\delta > c > 0$. Further we must have $|S| \gtrsim_n \delta^n$ as S by hypothesis must contain at least one tube. In this proof as n, E, ϵ are fixed, we will shorten $\lesssim_{n,E,\epsilon}$ to just $\lesssim$.

Now using lemma 4.1.2, we have $L = \{(a, d) \in [-1,1]^{2n} : T_{a,d}(\delta/2) \subset S\}$ is semialgebraic of bounded complexity. As before, let Π be the orthogonal projection into the final n coordinates $(a, d) \mapsto d$. Next using lemma 4.1.3, we extract a semialgebraic section $L' \subset L$ i.e. $\Pi(L') = \Pi(L)$ and for every $d \in [-1,1]^n$, there is at most one $(a, d) \in L'$. Further lemma 4.1.3 assures us that L' has bounded complexity. Every tube $T_{a,d}(\delta) \in \mathbb{T}$ has radius δ, hence we can wiggle around tubes of radius $\delta/2$ inside $T_{a,d}(\delta)$. The set of directions of tubes of radius $\delta/2$ that can be contained in $T_{a,d}(\delta)$ is $B(d, c\delta)$. Since the tubes in $\mathbb{T}$ point in δ-separated directions, the sets $B(d, c\delta)$ are finitely intersecting as we vary d over the directions of tubes in $\mathbb{T}$. Using this and our hypothesis, we get

$$|\Pi(L')| \gtrsim \#\mathbb{T}\delta^n > C|S|\delta^{-\epsilon}. \tag{4.1}$$

We now wish to apply Gromov's algebraic lemma (theorem 3.3.2) to the semialgebraic set $L' \subset [-1,1]^{2n}$. Since L' is in bijection with $\Pi(L') \subset [-1,1]^n$, the dimension of L' is n. We pick r to be the greatest integer greater than $\frac{4n^2}{\epsilon}$. By Gromov's algebraic lemma, there exists an integer $N \lesssim 1$ and C^r maps $(F_1, G_1), \ldots, (F_N, G_N) : [-1,1]^n \to [-1,1]^{2n}$ such that

$$\bigcup_{j=1}^{N} (F_i, G_i)\big([-1,1]^n\big) = L' \quad \text{and} \quad \|(F_j, G_j)\|_{C^r} \leq 1. \tag{4.2}$$

Observe that $\bigcup_{j=1}^{N} G_j([-1,1]^n) = \Pi(L')$. Since $N \lesssim 1$, the pigeonhole principle and inequality (4.1) imply the existence of j such that $|G_j([-1,1]^n)| \gtrsim C|S|\delta^{-\epsilon}$. A ball of radius $\delta^{\frac{\epsilon}{2n}}$ in $[-1,1]^n$ has volume $\delta^{\epsilon/2}$. So we can decompose $[-1,1]^n$ into $\delta^{-\epsilon/2}$ many balls of radius $\delta^{\frac{\epsilon}{2n}}$ which are finitely intersecting. By applying pigeonhole principle again, we can find a ball $B \subset [-1,1]^n$ of radius $\delta^{\frac{\epsilon}{2n}}$ centered at x_0 such that

$$|G_j(B)| \gtrsim \frac{C|S|\delta^{-\epsilon}}{\delta^{-\epsilon/2}} = C|S|\delta^{-\epsilon/2}. \tag{4.3}$$

The next step involves replacing the C^r-function (F_j, G_j) by its $(r-1)$-th degree Taylor polynomial (F, G) at x_0. By the derivative estimates from Gromov's algebraic lemma (inequality 4.2) and our choice of r, for $x \in B$ we have

$$|(F_j, G_j)(x) - (F, G)(x)| \leq |x - x_0|^r \lesssim (\delta^{\frac{\epsilon}{2n}})^{\frac{4n^2}{\epsilon}} = \delta^{2n}. \tag{4.4}$$

Consequently $|G_j(x) - G(x)| \lesssim \delta^{2n}$ for all $x \in B$. So G maps B into the δ^{2n}-neighborhood of $G_j(B)$. By inequality (4.3) and recalling that $|S| \geq \delta^n$, we can conclude that

$$|G(B)| \gtrsim C|S| \tag{4.5}$$

whenever C is sufficiently large so that δ is sufficiently small.

Having establish this upper bound on S, our goal now is to come up with a lower bound on S which will contradict the upper bound. For $x \in [-1,1]^n$, the line segment $\{(F_j(x) + tG_j(x), t) : t \in [0,1]\}$ is the axis of a tube of the form $T_{a,d}(\delta/2)$ contained in S. So the line segment is contained in S. Using inequality (4.4), we can conclude that $\{(F(x) + tG(x), t) : t \in [0, 1]\} \subset S$ whenever $x \in B$. Hence

$$|S| \geq \int_0^1 |(F + tG)(B)|\, dt. \tag{4.6}$$

We ultimately want only $|G(B)|$ on the left hand side as this would contradict inequality (4.5). This will take a few more steps. We would like to apply change of variable formula to the expression $|(F + tG)(B)|$. But as $F + tG$ may not be one to one, we need an extra factor. By Bézout's theorem, the function $F+tG$ maps at most $(r-1)^n$ points of B to the same place. So by change of variables, we have

$$|F + tG(B)| \geq \frac{1}{(r-1)^n} \int_B |\det[(DF + tDG)(x)]|\, dx. \tag{4.7}$$

Combining the inequalities (4.6) and (4.7), we obtain

$$|S| \gtrsim \int_0^1 \int_B |\det[(DF + tDG)(x)]|\, dx\, dt \tag{4.8}$$

since r depends only on n and ϵ. We now want to remove F from the left hand side. Lemma 4.1.4 will help us do this. Observe that $\det[(DF + tDG)(x)]$ is a degree n polynomial in t with leading coefficient $\det[(DG)(x)]$. So we can write $\det[(DF + tDG)(x)] = \det[(DG)(x)]P_x(t)$ where $P_x(t)$ is a monomial in t of degree n whose lower order coefficients depend on x. Now using Fubini's theorem and lemma 4.1.4, we get

$$\begin{aligned}
|S| &\gtrsim \int_0^1 \int_B |\det[(DF + tDG)(x)]|\, dx\, dt \\
&= \int_B \int_0^1 |\det[(DF + tDG)(x)]|\, dt\, dx \\
&= \int_B |\det[(DG)(x)]| \left(\int_0^1 |P_x(t)|\, dt \right) dx \\
&\gtrsim \int_B |\det[(DG)(x)]|\, dx \\
&= |G(B)|
\end{aligned}$$

The above inequality, together with inequality (4.5) implies $1 \gtrsim C$. Recall $\gtrsim$ means $\gtrsim_{n,E,\epsilon}$ and n, E, ϵ are fixed throughout the proof. Hence, since C is allowed to be any positive number, we have arrived at a contradiction. This completes the proof. $\square$

Chapter 5

KMFC for semialgebraic mappings

In this chapter, we will use theorem 4.0.3 of Katz and Rogers [17] to prove the operator version of Kakeya maximal function conjecture (conjecture 2.1.1) for semialgebraic mappings. This is an original, unpublished result which is the joint work of the author and his supervisors, Drs. Izabella Łaba and Joshua Zahl.

Before we begin, we recall the definition of Kakeya maximal function. Let $f \in L^1_{loc}(\mathbb{R}^n)$ and $\delta > 0$, then the Kakeya maximal function $f^*_\delta : S^{n-1} \to [0, \infty]$ is defined by

$$f^*_\delta(e) = \sup_{a \in \mathbb{R}^n} \frac{1}{|T^\delta_e(a)|} \int_{T^\delta_e(a)} |f|. \tag{5.1}$$

5.1 KMFC for characteristic functions of semialgebraic sets

We first prove the operator version of KMFC (conjecture 2.1.1) for characteristic functions of semialgebraic sets.

Theorem 5.1.1. Let $S \subset [-1, 1]^n$ be a semialgebraic set of complexity at most E. There exists constants $C(n, E, \epsilon) > 0$ such that

$$\|(\chi_S)^*_\delta\|_{L^n(S^{n-1})} \leq C(n, E, \epsilon)\delta^{-\epsilon}|S|^{1/n}.$$

Proof. Let us denote $\chi_S = f$. We split the proof into two cases. In the first case, suppose $|S| \leq \delta^n$. We have the trivial inequality $f^*_\delta(e) \leq C_n\delta^{1-n}|S|$ for every $e \in S^{n-1}$, and hence for this case we have the inequality

$$\|f^*_\delta\|_{L^n(S^{n-1})} \lesssim \delta^{1-n}|S| \leq \delta^{1-n}|S|^{\frac{n-1}{n}}|S|^{1/n} \leq |S|^{1/n}.$$

So this case is done. Now for the second case, suppose $|S| > \delta^n$. Let $\{e_k\}$ be a maximal δ-separated subset of S^{n-1}. Let $\theta_k = \{e \in S^{n-1} : \mathrm{dist}_s(e, e_k) \leq \delta\}$, so we can decompose $S^{n-1} = \cup_k \theta_k$ into finitely intersecting caps. The value of f^*_δ is approximately equal on each of these caps. To be more precise, using equation (2.4), we can find a constant C_n such that when $\mathrm{dist}_s(e, e') \leq \delta$, we have $f^*_\delta(e) \leq C_n f^*_\delta(e')$. Choose a collection of tubes $\mathbb{T}$ with directions $\{e_k\}$ such that the tubes of $\mathbb{T}$ maximize intersection with S. So if $T^\delta_{e_k}(a_k) \in \mathbb{T}$, then

$$\frac{|T^\delta_{e_k}(a_k) \cap S|}{|T^\delta_{e_k}(a_k)|} = f^*_\delta(e_k).$$

The existence of such a_k is assured by the dominated convergence theorem.

To get an upper bound on $\|f^*_\delta\|_{L^n(S^{n-1})}$, we aim to get an upper bound on the distribution function $\mu(\{e \in S^{n-1} : f^*_\delta(e) \geq \lambda\})$ where μ denotes the surface measures on S^{n-1}. Since we have seen that the value of f^*_δ is approximately equal on each cap, we essentially only need to count the number of k such that $f^*_\delta(e_k) \geq \lambda$. The last step is exactly what the result of Katz and Rogers [17] i.e theorem 4.0.3 is about.

We now carry out this argument more rigorously. Let $\lambda > C_n\delta$. Now suppose $f^*_\delta(e_k) < \frac{\lambda}{C_n}$. When $\mathrm{dist}_s(e, e_k) \leq \delta$, we have $f^*_\delta(e) < \lambda$ i.e f^*_δ is less than λ on θ_k. So we have

$$\{e \in S^{n-1} : f_\delta^*(e) \geq \lambda\} \subset \bigcup_{\{k: f_\delta^*(e_k) \geq \frac{\lambda}{C_n}\}} \theta_k.$$

Since θ_k are finitely intersecting we have

$$\begin{aligned}
\mu(\{e \in S^{n-1} : f_\delta^*(e) \geq \lambda\}) &\leq \sum_{\{k: f_\delta^*(e_k) \geq \frac{\lambda}{C_n}\}} \mu(\theta_k) \\
&\lesssim \#\left(\left\{k : f_\delta^*(e_k) \geq \frac{\lambda}{C_n}\right\}\right)\delta^{n-1} \\
&= \#\left(\left\{T \in \mathbb{T} : |T \cap S| \geq \frac{\lambda}{C_n}|T|\right\}\right)\delta^{n-1} \\
&\leq C(n, E, \epsilon)|S|\delta^{-\epsilon}\lambda^{-n}.
\end{aligned}$$

We used the result of Katz and Rogers [17] i.e theorem 4.0.3 in the last line of the above calculation, observe that $\frac{\lambda}{C_n} > \delta$ so it is applicable.

Using the above inequality we are ready to compute $\|f_\delta^*\|_{L^n(S^{n-1})}$. Consider

$$\begin{aligned}
\|f_\delta^*\|_{L^n(S^{n-1})}^n &= n \int_0^\infty \lambda^{n-1} \mu(\{e \in S^{n-1} : f_\delta^*(e) \geq \lambda\})\, d\lambda \\
&\leq n\left[\mu(S^{n-1}) \int_0^{C_n\delta} \lambda^{n-1}\, d\lambda + C(n, E, \epsilon)|S|\delta^{-\epsilon} \int_{C_n\delta}^1 \lambda^{-1}\, d\lambda\right] \\
&\leq C(n)\delta^n + C(n, E, \epsilon)|S|\delta^{-\epsilon} \\
&\leq C(n, E, \epsilon)|S|\delta^{-\epsilon}
\end{aligned}$$

The last line above follows because we have assumed $|S| > \delta^n$, we have also used $C(n, E, \epsilon)$ to denote multiple constants which is a slight abuse of notation. Taking n-th root on both sides, the proposition is established. $\square$

5.2 KMFC for general semialgebraic mappings

Now that we have established the operator version of KMFC (conjecture 2.1.1) for the characteristic function of a semialgebraic set, we use dyadic pigeonholing to extend our result to general semialgebraic functions.

Theorem 5.2.1. For every $n, E \geq 2$ and $\epsilon > 0$, there exists constants $C(n, E, \epsilon) > 0$ such that

$$\|f_\delta^*\|_{L^n(S^{n-1})} \leq C(n, E, \epsilon)\delta^{-\epsilon} \|f\|_{L^n(\mathbb{R}^n)}$$

whenever f is a semialgebraic function of complexity at most E supported in $[-1, 1]^n$.

Proof. Without loss of generality, we may assume $\|f\|_{L^n(\mathbb{R}^n)} = 1$. Let $S_h = \{x \in [-1, 1]^n : |f(x)| > \delta^{-1})\}$ and $S_l = \{x \in [-1, 1]^n : |f(x)| \leq \delta^{-1})\}$. Further let $f_h = f\chi_{S_h}$ and $f_l = f\chi_{S_l}$. Then clearly $f = f_h + f_l$. Since $\|f\|_{L^n(\mathbb{R}^n)} = 1$, we would expect the measure of S_h to be small. To be more precise, we have

$$1 = \int |f|^n \geq \int_{S_h} |f_h|^n \geq (\delta^{-1}))^n |S_h|,$$

rearranging, we get $|S_h| \leq \delta^n$. This allows us to get an L^∞ bound on $(f_h)_\delta^*$ by using Hölder's inequality as follows

$$\begin{aligned}
(f_h)_\delta^*(e) &= \sup_{a \in \mathbb{R}^n} \frac{1}{|T_e^\delta(a)|} \int_{T_e^\delta(a)} |f_h| \\
&\leq C_n \delta^{1-n} \int_{S_h} |f_h| \cdot 1 \\
&\leq C_n \delta^{1-n} \|f_h\|_{L^n(\mathbb{R}^n)} |S_h|^{\frac{n-1}{n}} \\
&\leq C_n.
\end{aligned}$$

Since the surface measure of S^{n-1} depends only on n, we get

$$\|(f_h)_\delta^*\|_{L^n(S^{n-1})} \leq C_n. \tag{5.2}$$

So we have sufficient control over the f_h part. We now need to focus on the f_l piece. Let $S_0 = \{x \in [-1,1]^n : |f(x)| \leq 1\}$. Further let $S_j = \{x \in [-1,1]^n : 2^{j-1} < |f(x)| \leq 2^j\}$ for $j = 1, 2, \ldots, \log(\delta^{-1})$. Then we can write $f_l = \sum_{j=0}^{\log(\delta^{-1})} f_j$ where $f_j = f\chi_{S_j}$. Further we have $\|f_l\|^n_{L^n(\mathbb{R}^n)} = \sum_{j=0}^{\log(\delta^{-1})} \|f_j\|^n_{L^n(\mathbb{R}^n)} \leq 1$. Now using sublinearity and theorem 5.1.1, we get

$$\begin{aligned} \|(f_l)^*_\delta\|_{L^n(S^{n-1})} &\leq \sum_{j=0}^{\log(\delta^{-1})} \|(f_j)^*_\delta\|_{L^n(S^{n-1})} \\ &\leq \sum_{j=0}^{\log(\delta^{-1})} 2^j \left\|(\chi_{S_j})^*_\delta\right\|_{L^n(S^{n-1})} \\ &\leq C(n,E,\epsilon)\delta^{-\epsilon}\left[|S_0|^{1/n} + \sum_{j=1}^{\log(\delta^{-1})} 2^j |S_j|^{1/n}\right]. \end{aligned} \tag{5.3}$$

By Hölder's inequality, we have $\sum_{i=1}^M a_i \leq M^{\frac{n-1}{n}} \left(\sum_{i=1}^M a_i^n\right)^{\frac{1}{n}}$. Therefore we get

$$\sum_{j=1}^{\log(\delta^{-1})} 2^j |S_j|^{1/n} \leq (\log(\delta^{-1}))^{\frac{n-1}{n}} \left(\sum_{j=1}^{\log(\delta^{-1})} 2^{jn}|S_j|\right)^{1/n}. \tag{5.4}$$

Now let us find a lower bound for $\|f_l\|^n_{L^n(\mathbb{R}^n)}$.

$$1 \geq \|f_l\|^n_{L^n(\mathbb{R}^n)} = \sum_{j=0}^{\log(\delta^{-1})} \|f_j\|^n_{L^n(\mathbb{R}^n)} \geq \sum_{j=1}^{\log(\delta^{-1})} 2^{(j-1)n}|S_j| = 2^{-n} \sum_{j=1}^{\log(\delta^{-1})} 2^{jn}|S_j|. \tag{5.5}$$

Putting together equations (5.4) and (5.5), we get that

$$\sum_{j=1}^{\log(\delta^{-1})} 2^j |S_j|^{1/n} \leq C(n,\epsilon)\delta^{-\epsilon}. \tag{5.6}$$

Since $S_0 \subset [-1,1]^n$, we have the trivial inequality $|S_0|^{1/n} \leq 2$. Using this along with equations (5.3) and (5.6), we get that

$$\|(f_l)^*_\delta\|_{L^n(S^{n-1})} \leq C(n,E,\epsilon)\delta^{-\epsilon}. \tag{5.7}$$

Obviously we have $\|f^*_\delta\|_{L^n(S^{n-1})} \leq \|(f_h)^*_\delta\|_{L^n(S^{n-1})} + \|(f_l)^*_\delta\|_{L^n(S^{n-1})}$. Chaining this inequality with inequalities (5.2), (5.7), we get

$$\|f^*_\delta\|_{L^n(S^{n-1})} \leq C(n,E,\epsilon)\delta^{-\epsilon}.$$

As we had assumed $\|f\|_{L^n(\mathbb{R}^n)} = 1$, we are done.

Chapter 6

Conclusion

This thesis discusses applications of algebraic methods in the Kakeya family of problems. We went into the details of an important result in this area, namely theorem 4.0.3 proved by Katz and Rogers [17]. Using this result, the author along with Drs. Izabella Łaba and Joshua Zahl proved the Kakeya maximal function conjecture for semialgebraic mappings (theorem 5.2.1). This is an original and unpublished result. The Kakeya maximal function conjecture was discussed in Chapter 2 and it is a quantitative, single scale formulation of the Kakeya conjecture.

There is potential for further research and to improve upon the original work done in this thesis. One goal is thinking about whether one actually needs the $\delta^{-\epsilon}$ in theorem 5.1.1. The usual example showing that a $\delta^{-\epsilon}$ term is needed in the general case comes from a Kakeya set that has Lebesgue measure zero. But the usual construction of a measure zero Kakeya set isn't semi-algebraic. Katz and Rogers have $\delta^{-\epsilon}$ in their bound, but it might be possible to remove it. If this could be done, then it would have a nice interpretation: while measure zero Kakeya sets exist, they can't be semi-algebraic. The author will pursue this goal along with his advisors Drs. Izabella Łaba and Joshua Zahl.

$\square$

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