Decomposition of Topological Azumaya Algebras in the Stable Range

by

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Abstract

In this thesis, we establish decomposition theorems for topological Azumaya algebras, and topological Azumaya algebras with involutions of the first kind.

Decomposition of topological Azumaya algebras

Let $\mathscr{A}$ be a topological Azumaya algebra of degree mn over a CW complex X. We prove that if m and n are relatively prime, $m < n$, and the dimension of X is in the stable range for $\mathrm{GL}(m,\mathbb{C})$, then $\mathscr{A}$ can be decomposed as the tensor product of topological Azumaya algebras of degrees m and n. Moreover, if the dimension of X is outside of the stable range for $\mathrm{GL}(m,\mathbb{C})$, then $\mathscr{A}$ may not have such a decomposition.

Decomposition of topological Azumaya algebras with involution

Let $\mathscr{A}$ be a topological Azumaya algebra of degree mn with an orthogonal involution over a CW complex X. We prove that if m and n are relatively prime, $m < n$, and the dimension of X is in the stable range for $\mathrm{O}(m,\mathbb{C})$, then $\mathscr{A}$ can be decomposed as the tensor product of topological Azumaya algebras of degrees m and n with orthogonal involutions.

Let $\mathscr{A}$ be a topological Azumaya algebra of degree $2mn$ with a symplectic involution over a CW complex X. We prove that if n is odd, and $\dim(X) \leq 7$, then $\mathscr{A}$ can be decomposed as the tensor product of a topological Azumaya of degree $2m$ with a symplectic involution, and a topological Azumaya algebra of degree n with an orthogonal involution.

Lay Summary

One of the methods used to understand mathematical structures is to break them down into pieces, study those pieces, and recover the former structure by putting the pieces together with an operation. In primary school, we learn that any natural number bigger than 1 can be expressed as the product of powers of its prime factors. That is, prime numbers are the multiplicative building blocks for the natural numbers.

The research presented in this thesis lies in the field of *algebraic topology*. This branch of mathematics studies the concept of shape by using algebra, and its objects are spaces instead of numbers. We study objects called *topological Azumaya algebras*, and the operation among these objects is called *tensor product*. We provide conditions for topological Azumaya algebras so that we can express them as the tensor product of "smaller" algebras.

Preface

This dissertation is an original intellectual product of the author. The questions answered in Chapters 2, 3, and 4 were posed by Professor Ben Williams.

A version of Chapter 2 has been published [Arcila-Maya, N. (2021). Decomposition of topological Azumaya algebras. *Canadian Mathematical Bulletin*, 1-19. doi:10.4153/S000843952100045X].

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List of Symbols

G_{tors} Torsion subgroup of a group G

$Z(G)$ Center of a group G

$\text{Spin}(n)$ n-th spinor group

$\text{diag}(A_2,\cdots,A_n)$ Matrix having $A_2,\cdots,A_n$ in the diagonal and zeros otherwise

$\text{GL}(n,\mathbb{C})$ General linear group of order n

$\text{M}(n,\mathbb{C})$ Set of square matrices of order n

$\text{O}(n,\mathbb{C})$ Complex orthogonal group of order n

$\text{PGL}(n,\mathbb{C})$ Projective general linear group of order n

$\text{PO}(n,\mathbb{C})$ Projective complex orthogonal group of order n

$\text{PSp}(n,\mathbb{C})$ Projective complex symplectic group of order $3n$

$\text{PU}(n,\mathbb{C})$ Projective unitary group of order n

$\text{SO}(n,\mathbb{C})$ Special complex orthogonal group of order n

$\text{Sp}(n,\mathbb{C})$ Complex symplectic group of order $3n$

$\text{U}(n,\mathbb{C})$ Unitary group of order n

$[X,Y]$ Set of pointed homotopy classes of maps $X \to Y$

$\mathrm{B}G$	Classifying space of G
$\mathrm{Br}'(X)$	Cohomological Brauer group of a space X
$\mathrm{Br}(X)$	Brauer group of a space X
$\mathrm{K}(G,n)$	Eilenberg–MacLane space of type (G,n)
$\mathrm{cl}(\mathscr{A})$	Brauer class of a topological Azumaya algebra $\mathscr{A}$
$\pi_i(X)$	Homotopy group of a space X
$\tilde{\beta}_i$	Unreduced Bockstein
stab	Stabilization map
mult	Matrix multiplication homomorphism
$\oplus_*^r$	Homomorphism induced by r-fold direct sum on homotopy groups
$\oplus_*$	Homomorphism induced by direct sum on homotopy groups
$\boxplus_*$	Homomorphism induced by direct sum of symplectic groups on homotopy groups
$\otimes_*^r$	Homomorphism induced by r-fold tensor product on homotopy groups
$\otimes_*$	Homomorphism induced by tensor product on homotopy groups

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I would like to acknowledge that I wrote this thesis while residing on the traditional, ancestral, and unceded territory of the xʷməθkʷəy̓əm (Musqueam) people.

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And finally, my sincere appreciation goes to UBC for the financial support.

Dedication

A Dios.
A mis padres.
A mis hermanitas.
A mi esposo.

Chapter 1

Introduction

If I have seen farther it is by standing on the shoulders of Giants.
— Sir Isaac Newton (1855)

The aim of this thesis is to determine decomposition theorems in the theory of topological Azumaya algebras over a CW complex. We find sufficient conditions in terms of the dimension of a topological space X and the degrees of the algebras under which there is a tensor product decomposition.

1.1 Notation

We begin with some notation that we will use through this thesis.

Following standard set notation $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ represent the natural, integer, real, complex, and quaternion numbers, respectively.

Let G be a group. The center of G will be denoted by $Z(G)$. The torsion subgroup of G will be denoted by G_{tors}.

The notation we adopt for the classical Lie groups is as in [22]. We denote by $\mathrm{M}(n,\mathbb{C})$ the set of $n \times n$ complex matrices, this set is given a metric with respect to the coordinates of an $n \times n$ matrix and is homeomorphic with $\mathbb{C}^{n^2} \cong \mathbb{R}^{2n^2}$. The complex general linear group $\mathrm{GL}(n,\mathbb{C}) = \{M \in \mathrm{M}(n,\mathbb{C}) : M \text{ is invertible}\}$ is considered as a Lie group with the topology of an open subset of $\mathrm{M}(n,\mathbb{C})$. We use $\mathrm{U}(n,\mathbb{C})$, $\mathrm{O}(n,\mathbb{C})$, $\mathrm{SO}(n,\mathbb{C})$, and $\mathrm{Sp}(n,\mathbb{C})$ to denote the unitary group of order n, the complex orthogonal group of order n, the special complex orthogonal group of order n, and the complex symplectic group of order $2n$, respectively. These groups are closed subgroups of $\mathrm{GL}(n,\mathbb{C})$ in the unitary and orthogonal cases, and of $\mathrm{GL}(2n,\mathbb{C})$ in the symplectic case. In matrix terms these groups are defined as

$$\begin{aligned}
\mathrm{U}(n,\mathbb{C}) &= \left\{M \in \mathrm{M}(n,\mathbb{C}) : M^*M = MM^* = I_n\right\},\\
\mathrm{O}(n,\mathbb{C}) &= \left\{M \in \mathrm{M}(n,\mathbb{C}) : M^{\mathrm{tr}}M = MM^{\mathrm{tr}} = I_n\right\},\\
\mathrm{SO}(n,\mathbb{C}) &= \left\{M \in \mathrm{O}(n,\mathbb{C}) : \det(M) = 1\right\}, \text{ and}\\
\mathrm{Sp}(n,\mathbb{C}) &= \left\{M \in \mathrm{M}(2n,\mathbb{C}) : M^{\mathrm{tr}}J_{2n}M = J_{2n}\right\}
\end{aligned}$$

where tr denotes transposition, $*$ denotes the adjoint, and

$$J_{2n} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

All topological spaces will have the homotopy type of a CW complex. We fix basepoints for connected topological spaces, and for topological groups we take the identities as basepoints. We write $\pi_i(X)$ in place of $\pi_i(X,x_0)$. Homotopy groups of classical groups are taken from [22, Section 4.6].

1.2 Topological Azumaya Algebras

The concept of a central simple algebra over a field was generalized in 1951 by Azumaya [7] to the notion of an Azumaya algebra over a local ring, later in 1960, Auslander–Goldman [6] generalized this to an Azumaya algebra over a commutative ring, and then in 1966, Grothendieck [14] defined this concept in the context of a locally ringed topos, that is, in a category of sheaves equipped with a designated choice of sheaf of rings with enough points, $\mathscr{O}$, having local rings as stalks. One recovers the case of Auslander–Goldman when the category is $(\operatorname{Spec} R)_{\text{ét}}$ of étale sheaves over the spectrum of the ring R. That is, an *Azumaya algebra of degree n* is a sheaf of $\mathscr{O}$ algebras that is locally isomorphic to $\mathrm{M}(n,\mathscr{O})$.

Consequently, Grothendieck's definition not only generalizes a purely algebraic concept, but also leads to the definition of Azumaya algebras over objects of a different mathematical nature, like generalized rings and topological spaces.

Let k be a field. In the theory of central simple algebras over k, a theorem of Wedderburn states that any central simple algebra A/k has the form $\mathrm{M}(n,D)$, where D/k is a division algebra that is unique up to isomorphism, [24, Theorem 1.3]. This theorem reduces the classification problem for finite-dimensional central simple algebras over k to the classification problem for finite-dimensional central division algebras over k.

Let A and B be finite-dimensional central simple algebras over k. There exist division algebras D and E such that $A \cong \mathrm{M}(m,D)$ and $B \cong \mathrm{M}(n,E)$. We say A and B are *Brauer equivalent* if the division algebras D and E are isomorphic. The set of equivalence classes of finite-dimensional central simple algebras over k modulo Brauer equivalence has the group structure with the tensor product of algebras, [12, Proposition 4.3]. This group is called the Brauer group of k, and is denoted $\mathrm{Br}(k)$.

We see that each Brauer equivalence class contains a unique isomorphism class of finite-dimensional central division algebras, and each such division algebra is contained in a unique Brauer equivalence class. Therefore, the elements of the Brauer group of k are in one to one correspondence with the isomorphism classes of division algebras D/k.

If D/k is a finite-dimensional central division algebra, then $\dim_k(D)$ is a square, [12, Theorem 3.10]. The *degree* of D/k is defined by $\sqrt{\dim_k(D)}$. The theory of central division algebras is equipped with a structure theorem stating that every finite-dimensional central division algebra D can be broken up into pieces corresponding to the prime factorization of its degree $\deg(D) = p_1^{n_1} \cdots p_r^{n_r}$, i.e. D is isomorphic to $D_1 \otimes_k D_2 \otimes_k \cdots \otimes_k D_r$ where each D_i is a division algebra of degree $p_i^{n_i}$. This decomposition is unique up to isomorphism, [24, Theorem 5.7].

Definition 1.2.1. A *topological Azumaya algebra of degree n* over a topological space X is a bundle of associative and unital complex algebras over X that is locally isomorphic to the matrix algebra $\mathrm{M}(n,\mathbb{C})$, [14, 1.1].

Example 1.2.2. Let $\mathcal{V} : E \to X$ be a complex vector bundle of rank n. Let F

denote $\left\{(x,\varphi) \mid \varphi \in \mathrm{End}_{\mathbb{C}}\big(\mathscr{V}^{-1}(x)\big)\right\}$. Then the projection of F onto X is a topological Azumaya algebra of degree n over X. This is called the *endomorphism bundle* of $\mathscr{V}$ and we denote it by $\mathrm{End}(\mathscr{V}) : F \to X$.

Example 1.2.3. Let $\mathscr{A}$ be a topological Azumaya algebra of degree n over a space X. The fiber bundle $\mathscr{A}^{\mathrm{op}}$ defined by taking the opposite algebra $\mathscr{A}^{-1}(x)^{\mathrm{op}}$ for all $x \in X$ is a topological Azumaya algebra of degree n over X.

Example 1.2.4. Let $\rho : G \to \mathrm{PGL}(n,\mathbb{C})$ be a group representation. Let X be a space with a free G-action. Suppose the projection $p : X \to X/G$ is a principal G-bundle. Then the fiber bundle with fiber $\mathrm{M}(n,\mathbb{C})$ associated to p,

$$\mathrm{M}(n,\mathbb{C}) \longrightarrow X \times^{G} \mathrm{M}(n,\mathbb{C}) \longrightarrow X/G$$

is a topological Azumaya algebra of degree n over X/G.

Topological Azumaya algebras over a space X are classified by pointed homotopy classes of maps from X to $\mathrm{B}\,\mathrm{PGL}(n,\mathbb{C})$, as there is a bijective correspondence

$$\left\{\begin{array}{c}\text{Isomorphism classes of degree-}n\\ \text{topological Azumaya algebras over } X\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\text{Isomorphism classes of}\\ \text{principal } G\text{-bundles over } X\end{array}\right\},$$

where G is the topological group of automorphisms of $\mathrm{M}(n,\mathbb{C})$ as an algebra, [26, 8.2]. The Skolem–Noether theorem asserts that this is $\mathrm{PGL}(n,\mathbb{C})$; i.e., matrices acting by conjugation.

For brevity of notation we work with $\mathrm{U}(n,\mathbb{C})$ instead of $\mathrm{GL}(n,\mathbb{C})$. Our choice of notation does not affect our results because $\mathrm{U}(n,\mathbb{C})$ included in $\mathrm{GL}(n,\mathbb{C})$ as the maximal compact Lie subgroup is a deformation retract, in particular the inclusion is a homotopy equivalence, [22, Theorem I.4.11]. Hence, the homotopy type of $\mathrm{U}(n,\mathbb{C})$ is that of $\mathrm{GL}(n,\mathbb{C})$. The homotopy equivalence is more than an equivalence of spaces, it upgrades to one of topological groups, hence of classifiying spaces.

Example 1.2.5. Let $i = 0,1,\ldots,2n+1$, we can determine the number of isomorphism classes of topological Azumaya algebras of degree n over S^i. Indeed, let if $i = 0, 1, \dots, 2n+1$, then

$$[S^i, \mathrm{BPU}(n,\mathbb{C})] = \pi_i\big(\mathrm{BPU}(n,\mathbb{C})\big) = \begin{cases} 0 & \text{if } i = 0,1, \\ \mathbb{Z}/n & \text{if } i = 2, \\ 0 & \text{if } i > 2 \text{ and } i \text{ is odd}, \\ \mathbb{Z} & \text{if } i > 2 \text{ and } i \text{ is even}, \\ \mathbb{Z}/n! & \text{if } i = 2n+1. \end{cases}$$

A computation of the homotopy groups of the projective unitary groups is done in subsection 2.1.

There are two invariants associated to a topological Azumaya algebra of degree n over a space X.

Given two topological Azumaya algebras $\mathscr{A}$ and $\mathscr{A}'$ over X of degrees m and n, respectively, we can define the tensor product $\mathscr{A} \otimes \mathscr{A}'$ by applying the Kronecker product $\mathrm{M}(m,\mathbb{C}) \otimes_{\mathbb{C}} \mathrm{M}(n,\mathbb{C})$ fiberwise.

We say two topological Azumaya algebras $\mathscr{A}$ and $\mathscr{A}'$ are *Brauer equivalent* if there exist complex vector bundles $\mathscr{V}$ and $\mathscr{V}'$ such that $\mathscr{A} \otimes \mathrm{End}(\mathscr{V}) \cong \mathscr{A}' \otimes \mathrm{End}(\mathscr{V}')$ as bundles of $\mathbb{C}$-algebras.

The set of isomorphism classes of topological Azumaya algebras over X modulo Brauer equivalence with the tensor product operation has the group structure where the class $[\mathrm{End}(\mathscr{V})]$ for all complex vector bundles $\mathscr{V}$ is the identity. Moreover, since $\mathscr{A} \otimes \mathscr{A}^{\mathrm{op}} \cong \mathrm{End}(\mathscr{A})$ for all classes $[\mathscr{A}]$, then $[\mathscr{A}]^{-1} = [\mathscr{A}^{\mathrm{op}}]$. This group is called the *(topological) Brauer group of X*, and it is denoted by $\mathrm{Br}(X)$.

Let X and Y be topological spaces, and let $f : Y \to X$ be a continuous map. The map f induces a homomorphism on the Brauer groups $f^* : \mathrm{Br}(X) \to \mathrm{Br}(Y)$. In other words, $\mathrm{Br}(-)$ is a contravariant functor from the category of topological spaces to the category of abelian groups.

The *cohomological Brauer group of X* is defined by $\mathrm{Br}'(X) = \mathrm{H}^3(X;\mathbb{Z})_{\mathrm{tors}}$. See subsection 1.2.1 for a discussion about the relation between these invariants.

Let $\alpha \in \mathrm{Br}(X)$. The *period of α* is the order of α as an element of the Brauer group, and it is denoted by $\operatorname{per}(\alpha)$. Let $\mathcal{D}(\alpha)$ denote the set $\{\deg(\mathscr{A}) : \mathscr{A} \in \alpha\}$. The *index of* α is defined as $\operatorname{ind}(\alpha) := \gcd \mathcal{D}(\alpha)$.

The *Brauer class* of a topological Azumaya algebra $\mathscr{A} : X \to \mathrm{BPU}(n, \mathbb{C})$ is an element in $\mathrm{Br}(X)$ that is defined as follows. Let χ_n denote the composite of $\mathscr{A}$ with the projection of $\mathrm{BPU}(n, \mathbb{C})$ on the the first non-trivial stage of its Postnikov tower, $\mathrm{BPU}(n, \mathbb{C}) \to \mathrm{K}(\mathbb{Z}/n, 2)$, as illustrated in the diagram below. Then $\mathrm{cl}(\mathscr{A})$, the Brauer class of $\mathscr{A}$, is defined as the composite $\tilde{\beta}_n \circ \chi_n$, where $\tilde{\beta}_n : \mathrm{K}(\mathbb{Z}/n, 2) \to \mathrm{K}(\mathbb{Z}, 3)$ is the unreduced Bockstein map.

$$\begin{tikzcd}
 & \mathrm{BPU}(n,\mathbb{C}) \arrow[d] & \\
X \arrow[ur, bend left, "\mathscr{A}"] \arrow[r, "\chi_n"] & \mathrm{K}(\mathbb{Z}/n,2) \arrow[r, "\tilde{\beta}_n"] & \mathrm{K}(\mathbb{Z},3).
\end{tikzcd}$$

Let a and m be positive integers. Let $\mu_m \subset \mathrm{U}(am, \mathbb{C})$ be the cyclic subgroup of order m consisting of scalar matrices ζI_{am} for ζ a m-th root of unity. Since μ_m is a closed normal subgroup of $\mathrm{U}(am, \mathbb{C})$, then the quotient group $\mathrm{U}(am, \mathbb{C})/\mu_m$ is a Lie group. The space $\mathrm{U}(am, \mathbb{C})/\mu_m$ is equipped with a canonical degree-am topological Azumaya algebra $\mathscr{A}$ such that $\mathrm{cl}(\mathscr{A})$ is m-torsion in the Brauer group. In fact, the quotient homomorphism $q : \mathrm{U}(am, \mathbb{C})/\mu_m \to \mathrm{PU}(am, \mathbb{C})$ and the inclusion $\mathbb{Z}/m \to \mathbb{Z}/am$ induce the homotopy-pullback diagram in diagram (1.1).

$$\begin{tikzcd}
\mathrm{BU}(am,\mathbb{C})/\mu_m \arrow[r, "\mathrm{B}q"] \arrow[d] & \mathrm{BPU}(am,\mathbb{C}) \arrow[d] \\
\mathrm{K}(\mathbb{Z}/m,2) \arrow[r] & \mathrm{K}(\mathbb{Z}/am,2)
\end{tikzcd} \tag{1.1}$$

Therefore, a map $\mathscr{A}' : X \to \mathrm{BU}(am, \mathbb{C})/\mu_m$ induces a degree-am topological Azumaya algebra $\mathrm{B}q \circ \mathscr{A}' : X \to \mathrm{BPU}(am, \mathbb{C})$ such that $\mathrm{cl}\left(\mathrm{B}q \circ \mathscr{A}'\right)$ is m-

torsion.

$$
\begin{tikzcd}
 & \mathrm{BU}(am)/\mu_m \arrow[rr, "\mathrm{B}q"] \arrow[d] & & \mathrm{BPU}(am) \arrow[d] \\
 & \mathrm{K}(\mathbb{Z}/m,2) \arrow[rr] \arrow[dr, "\tilde{\beta}_m"'] & & \mathrm{K}(\mathbb{Z}/am,2) \arrow[dl, "\tilde{\beta}_{am}"] \\
X \arrow[rr, "\alpha"'] \arrow[uur, dotted, bend left, "\mathscr{A}'"] \arrow[uurrr, dotted, "\mathscr{A}"] & & \mathrm{K}(\mathbb{Z},3) &
\end{tikzcd}
\tag{1.2}
$$

In Proposition 2.2.4 we prove that if α is an m-torsion Brauer class of a space X, then it is represented by $\mathscr{A}$ a topological Azumaya algebra of degree am over X if and only if it is represented by a map $\mathscr{A}' : X \to \mathrm{BU}(am,\mathbb{C})/\mu_m$. In other words, there exists a lifting $\mathscr{A}$ of α if and only if there exists a lifting $\mathscr{A}'$ of α, see diagram (1.2). It is important to mention that these liftings are not unique. This means there is a surjective map of sets

$$
\left\{\begin{array}{c}\text{Isomorphism classes of principal}\\ \mathrm{U}(am,\mathbb{C})/\mu_m\text{-bundles over } X\end{array}\right\} \twoheadrightarrow \left\{\begin{array}{c}\text{Isomorphism classes}\\ \text{of degree-}am\text{ topological}\\ \text{Azumaya algebras over } X\\ \text{with } m\text{-torsion Brauer class}\end{array}\right\}.
$$

The *Brauer class of a map* $\mathscr{A}' : X \to \mathrm{BU}(am,\mathbb{C})/\mu_m$ is defined as the Brauer class of the topological Azumaya algebra $\mathrm{B}q \circ \mathscr{A}' : X \to \mathrm{BPU}(am,\mathbb{C})$, and it is denoted by $\mathrm{cl}(\mathscr{A}')$.

In what follows we describe the topological period-index problem and the topological decomposition problem, both of which are aspects of the study of the set $\mathcal{D}(\alpha)$ for α a Brauer class of a topological space.

1.2.1 The Topological Period-Index Problem

Let X be a space with finitely many connected components. The Brauer group is a subgroup of the cohomological Brauer group, $\mathrm{Br}(X) \subset \mathrm{Br}'(X)$. To see this consider the short exact sequence of compact Lie groups

$$1 \longrightarrow S^1 \longrightarrow \mathrm{U}(n,\mathbb{C}) \longrightarrow \mathrm{PU}(n,\mathbb{C}) \longrightarrow 1$$

which induces a exact sequence of pointed sets

$$\big[X,\mathrm{B}\,\mathrm{U}(n,\mathbb{C})\big] \xrightarrow{\mathcal{E}_n} \big[X,\mathrm{B}\,\mathrm{PU}(n,\mathbb{C})\big] \xrightarrow{\delta_n} \big[X,\mathrm{B}^2 S^1\big] \simeq \big[X,\mathrm{K}(\mathbb{Z},3)\big]. \tag{1.3}$$

As there is a bijective correspondence

$$\left\{\begin{matrix}\text{Isomorphism classes of rank-}n\\ \text{complex vector bundles over } X\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{Isomorphism classes of principal}\\ \mathrm{GL}(n,\mathbb{C})\text{-bundles over } X\end{matrix}\right\},$$

then the map $\mathcal{E}_n$ sends a complex vector bundle $\mathcal{V}$ over X to $\mathrm{End}(\mathcal{V})$.

The multiplication map $\mathrm{mult}: S^1 \times S^1 \to S^1$ induces a commutative operation on $\big[X,\mathrm{K}(\mathbb{Z},3)\big]$ which is the addition in $\mathrm{H}^3(X;\mathbb{Z})$, then the union of all $\mathrm{Im}\,\delta_n$ for $n \geq 1$ has the group structure as a subgroup of $\big[X,\mathrm{K}(\mathbb{Z},3)\big]$. However, it can also be endowed with another group structure. Let $\delta_m(\mathscr{A})$ and $\delta_n(\mathscr{A}')$ be elements of the union, we define

$$\delta_m(\mathscr{A}) \cdot \delta_n(\mathscr{A}') = \delta_{mn}\big(f_{\otimes} \circ (\mathscr{A} \times \mathscr{A}')\big),$$

where $f_{\otimes}$ is the map defined in (2.9). We show below that these group operations are equal.

$$\begin{tikzcd}
1 \arrow[r] & S^1 \times S^1 \arrow[r] \arrow[d, "\mathrm{mult}"] & \mathrm{U}(m,\mathbb{C}) \times \mathrm{U}(n,\mathbb{C}) \arrow[r] \arrow[d, "\otimes"] & \mathrm{PU}(m,\mathbb{C}) \times \mathrm{PU}(n,\mathbb{C}) \arrow[r] \arrow[d, "\otimes"] & 1 \\
1 \arrow[r] & S^1 \arrow[r] & \mathrm{U}(mn,\mathbb{C}) \arrow[r] & \mathrm{PU}(mn,\mathbb{C}) \arrow[r] & 1
\end{tikzcd} \tag{1.4}$$

$$\begin{tikzcd}
1 \arrow[r] & \mu_n \arrow[r] \arrow[d, hook] & \mathrm{SU}(n,\mathbb{C}) \arrow[r] \arrow[d, hook] & \mathrm{PU}(n,\mathbb{C}) \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & S^1 \arrow[r] & \mathrm{U}(n,\mathbb{C}) \arrow[r] & \mathrm{PU}(n,\mathbb{C}) \arrow[r] & 1
\end{tikzcd} \tag{1.5}$$

$$\begin{tikzcd}
{[X, \mathrm{BU}(m,\mathbb{C})] \times [X, \mathrm{BU}(m,\mathbb{C})]} \arrow[r] \arrow[d, "\mathcal{E}_m \times \mathcal{E}_n"] & {[X, \mathrm{BU}(mn,\mathbb{C})]} \arrow[d, "\mathcal{E}_{mn}"] \\
{[X, \mathrm{BPU}(m,\mathbb{C})] \times [X, \mathrm{BPU}(n,\mathbb{C})]} \arrow[r, "\otimes"] \arrow[d, "\delta_m \times \delta_n"] & {[X, \mathrm{BPU}(mn,\mathbb{C})]} \arrow[d, "\delta_{mn}"] \\
{[X, \mathrm{K}(\mathbb{Z},3)] \times [X, \mathrm{K}(\mathbb{Z},3)]} \arrow[r, "+"] & {[X, \mathrm{K}(\mathbb{Z},3)]}
\end{tikzcd} \tag{1.6}$$

$$\begin{tikzcd}
 & {[X, \mathrm{K}(\mathbb{Z}/n,2)]} \arrow[d] \\
{[X, \mathrm{BPU}(n,\mathbb{C})]} \arrow[ur] \arrow[r, "\delta_n"] & {[X, \mathrm{K}(\mathbb{Z},3)]}
\end{tikzcd} \tag{1.7}$$

The homomorphisms of short exact sequences in diagrams (1.4) and (1.5) induce the commutative diagrams (1.6) and (1.7), respectively.

The middle horizontal arrow in diagram (1.6) is the tensor product of topological Azumaya algebras defined by $\mathscr{A} \otimes \mathscr{A}' := f_{\otimes} \circ (\mathscr{A} \times \mathscr{A}')$. The commutativity of the lower square in diagram (1.6) implies that

$$\delta_{mn}(\mathscr{A} \otimes \mathscr{A}') = \delta_m(\mathscr{A}) + \delta_n(\mathscr{A}').$$

Hence $\delta_m(\mathscr{A}) \cdot \delta_n(\mathscr{A}') = \delta_m(\mathscr{A}) + \delta_n(\mathscr{A}')$ in $\bigcup_{n\geq 1} \mathrm{Im}\,\delta_n \subset [X, \mathrm{K}(\mathbb{Z},3)]$.

From diagram (1.7) the map δ_n factors through $[X, \mathrm{K}(\mathbb{Z}/n,2)] \cong \mathrm{H}^2(X;\mathbb{Z}/n)$, which is an n-torsion group by the universal coefficients theorem. Therefore, $\bigcup_{n\geq 1} \mathrm{Im}\,\delta_n$ is a subgroup of $\mathrm{H}^3(X;\mathbb{Z})_{\mathrm{tors}}$.

By the exactness of (1.3) we have that

$$\mathrm{Br}(X) \cong \bigcup_{n\geq 1} \mathrm{Im}\,\delta_n.$$

From this there is a map

$$\mathrm{Br}(X) \cong \bigcup_{n \geq 1} \mathrm{Im}\,\delta_n \hookrightarrow \mathrm{H}^3(X;\mathbb{Z})_{\mathrm{tors}} = \mathrm{Br}'(X). \tag{1.8}$$

Serre showed that if X is a finite CW complex, then the map above is surjective, [14, Theorem 1.6]. That is, every cohomological Brauer class $\alpha \in \mathrm{Br}'(X)$ is represented by topological Azumaya algebras over X of varying degrees. It can be shown that $\mathrm{per}(\alpha) | \deg(\mathscr{A})$, for all topological Azumaya algebras $\mathscr{A}$ representing α. Hence $\mathrm{per}(\alpha) | \mathrm{ind}(\alpha)$. Moreover, Antieau–Williams showed that the period and index of α have the same divisors, [3, Theorem 6].

Therefore, for a sufficiently large integer $l(\alpha)$ we have

$$\mathrm{ind}(\alpha) | \,\mathrm{per}(\alpha)^{l(\alpha)}. \tag{1.9}$$

The *topological period-index problem* is to find the sharpest bound on the integer $l(\alpha)$ in terms of the period of α and the dimension of X such that condition (1.9) holds.

For more about the progress that has been made towards solving the topological period-index problem, we refer the interested reader to [1], [2], [3], [4], [5], [10], [11], [15], and [16].

1.2.2 The Topological Decomposition Problem

Saltman asked in [24, page 35] whether the analogue to the prime decomposition theorem for central simple algebras over a field holds in general for Azumaya algebras over a commutative ring, that is whether an Azumaya algebra over a commutative ring can be decomposed as a tensor product of algebras of prime-power degree. In 2013, Antieau–Williams answered this question in [2, Corollary 1.3] by showing the following result:

Theorem 1.2.6. *For $n > 1$ an odd integer, there exist a 6-dimensional CW complex X and a topological Azumaya algebra $\mathscr{A}$ on X of degree $2n$ and period 2 such that $\mathscr{A}$ has no decomposition $\mathscr{A} \cong \mathscr{A}_2 \otimes \mathscr{A}_n$ into topological Azumaya algebras of degrees 2 and n, respectively.*

That is, prime decomposition of topological Azumaya algebras does not hold in general. Nonetheless, Antieau–Williams also showed that prime decomposition holds for the index of a Brauer class as stated in the theorem below, [5, Theorem 3].

Theorem 1.2.7. *Let X be a connected topological space, and let $\alpha = \alpha_1 + \cdots + \alpha_r$ be the prime decomposition of a Brauer class $\alpha \in \mathrm{Br}(X)$, so that each $\mathrm{per}(\alpha_i) = p_i^{\alpha_i}$ for distinct primes $p_1, \dots, p_r$. Then*

$$\mathrm{ind}(\alpha) = \mathrm{ind}(\alpha_1) \cdots \mathrm{ind}(\alpha_r).$$

The first goal of this thesis is to provide conditions on a positive integer n and a topological space X such that a topological Azumaya algebra of degree n on X has a tensor product decomposition. The main result of Chapter 2 is the following theorem:

Theorem 1.2.8. *Let m and n be positive integers such that m and n are relatively prime and $m < n$. Let X be a CW complex such that $\dim(X) \leq 2m+1$.*

If $\mathscr{A}$ is a topological Azumaya algebra of degree mn over X, then there exist topological Azumaya algebras $\mathscr{A}_m$ and $\mathscr{A}_n$ of degrees m and n, respectively, such that $\mathscr{A} \cong \mathscr{A}_m \otimes \mathscr{A}_n$.

Theorem 1.2.8 is a corollary of Theorem 2.2.5. We prove in Theorem 2.2.3 that a map $X \to \mathrm{BU}(abmn, \mathbb{C})/\mu_{mn}$ can be lifted to $\mathrm{BU}(am, \mathbb{C})/\mu_m \times \mathrm{BU}(bn, \mathbb{C})/\mu_n$ when the dimension of X is less than $2am+2$, the positive integers a, b, m and n are such that am is relatively prime to bn, and $am < bn$. The proof of Theorem 2.2.3 relies significantly in the description of the homomorphisms induced on homotopy groups by the r-fold direct sum of matrices $\oplus^r : \mathrm{U}(n, \mathbb{C}) \to \mathrm{U}(rn, \mathbb{C})$ in the range $\{0, 1, \dots, 2n\}$. We call this set "*the stable range for* $\mathrm{U}(n, \mathbb{C})$".

1.3 Topological Azumaya Algebras with Involution

Let A be a finite-dimensional central simple algebra over a field k. An involution on A is an anti-automorphism $\tau : A \to A$ satisfying $(\tau \circ \tau)(a) = a$ for all

$a \in A$. A central simple algebra with an involution is denoted by (A,τ). It can be checked that the center k is preserved under τ. The restriction of τ to k is therefore an automorphism which is either the identity or of order 2. Involutions that leave the center invariant are called involutions of the first kind. Involutions whose restriction to k is an automorphism of order 2 are called involutions of the second kind, [19].

Let $A \in \mathrm{GL}(n,\mathbb{C})$, we denote by $\mathrm{Int}_A : \mathrm{M}(n,\mathbb{C}) \to \mathrm{M}(n,\mathbb{C})$ the inner automorphism induced by A, $\mathrm{Int}_A(M) = AMA^{-1}$ for all $M \in \mathrm{M}(n,\mathbb{C})$.

Observe that transposition, $\mathrm{tr} : \mathrm{M}(n,\mathbb{C}) \to \mathrm{M}(n,\mathbb{C})$, is an involution of the first kind on $\mathrm{M}(n,\mathbb{C})$. All automorphisms of $\big(\mathrm{M}(n,\mathbb{C}),\mathrm{tr}\big)$ arise by conjugation by an invertible matrix: Let τ be an arbitrary involution on $\mathrm{M}(n,\mathbb{C})$. Since $\mathrm{tr}\circ\tau$ is an automorphism of $\big(\mathrm{M}(n,\mathbb{C}),\mathrm{tr}\big)$, then $\mathrm{tr}\circ\tau = \mathrm{Int}_A$ for some $A \in \mathrm{GL}(n,\mathbb{C})$. Hence $\tau(M) = A^{-\mathrm{tr}} M^{\mathrm{tr}} A^{\mathrm{tr}}$. Since τ is an involution, then

$$M = (A^{-\mathrm{tr}}A)M(A^{-1}A^{\mathrm{tr}}).$$

Therefore $\mathrm{Int}_{A^{-1}A^{\mathrm{tr}}} = \mathrm{id}_{\mathrm{M}(n,\mathbb{C})}$, this is $A^{-1}A^{\mathrm{tr}} = \lambda I_n$ for some $\lambda \in \mathbb{C}$. This last equation implies $A^{\mathrm{tr}} = \lambda A$, and thus $A = \lambda^2 A$. Hence $\lambda = 1$ or $\lambda = -1$, i.e. $A^{\mathrm{tr}} = A$ or $A^{\mathrm{tr}} = -A$.

Consider the subspaces

$$\begin{aligned}
\big(\mathrm{M}(n,\mathbb{C}),\tau\big)^{+} &= \big\{M \in \mathrm{M}(n,\mathbb{C}) :\ \tau(M) = M\big\} \quad \text{(symmetric elements), and}\\
\big(\mathrm{M}(n,\mathbb{C}),\tau\big)^{-} &= \big\{M \in \mathrm{M}(n,\mathbb{C}) :\ \tau(M) = -M\big\} \quad \text{(skew symmetric elements).}
\end{aligned}$$

Observe that

$$\big(\mathrm{M}(n,\mathbb{C}),\tau\big)^{+} = \begin{cases} A\big(\mathrm{M}(n,\mathbb{C}),\mathrm{tr}\big)^{+} & \text{if } A^{\mathrm{tr}} = A,\\ A\big(\mathrm{M}(n,\mathbb{C}),\mathrm{tr}\big)^{-} & \text{if } A^{\mathrm{tr}} = -A. \end{cases}$$

where $\dim_{\mathbb{C}}\big(\mathrm{M}(n,\mathbb{C}),\mathrm{tr}\big)^{+} = \frac{1}{2}n(n+1)$ and $\dim_{\mathbb{C}}\big(\mathrm{M}(n,\mathbb{C}),\mathrm{tr}\big)^{-} = \frac{1}{2}n(n-1)$.

In summary, we obtain the following proposition.

Proposition 1.3.1. *Let τ be an involution of the first kind on* $\mathrm{M}(n,\mathbb{C})$.

1. *The involution τ on* $\mathrm{M}(n,\mathbb{C})$ *has the form $\tau = \mathrm{Int}_A \circ \mathrm{tr}$ for some $A \in$*

$\mathrm{GL}(n,\mathbb{C})$ *such that* $A^{\mathrm{tr}} = \pm A$.

2. *The subspace* $\left(\mathrm{M}(n,\mathbb{C}),\mathrm{tr}\right)^{+}$ *has complex dimension* $\frac{1}{2}n(n+1)$ *if and only if* $A^{\mathrm{tr}} = A$.

3. *The subspace* $\left(\mathrm{M}(n,\mathbb{C}),\mathrm{tr}\right)^{+}$ *has complex dimension* $\frac{1}{2}n(n-1)$ *if and only if* $A^{\mathrm{tr}} = -A$.

Definition 1.3.2. Let τ be an involution on $\mathrm{M}(n,\mathbb{C})$. We define the *type* of τ as *orthogonal* if $A^{\mathrm{tr}} = A$, and as *symplectic* if $A^{\mathrm{tr}} = -A$.

Up to isomorphism, the matrix algebra $\mathrm{M}(n,\mathbb{C})$ can carry at most one involution (transposition) if n is odd; and up to two involutions (orthogonal and symplectic) if n is even, [19, Proposition I.2.20].

Knus–Parimala–Srinivas generalized the notion of a central simple algebras with an involution to Azumaya algebras over schemes, [18]. Saltman presented a classification of involutions of Azumaya algebras over commutative rings into kinds, [23, Section 3]. In this thesis we declare an involution on a degree topological Azumaya algebra to be an *involution of the first kind* or an *involution of the second kind* depending on whether or not there is a group action on the base space, respectively. All the involutions we discuss here are involutions of the first kind over, which are classified as orthogonal and symplectic involutions.

Definition 1.3.3. Let X be a connected topological space, and let $\mathscr{A}$ be a topological Azumaya algebra of degree n over X. An *involution on* $\mathscr{A}$ is a morphism of fiber bundles $\tau : \mathscr{A} \to \mathscr{A}$ such that $\tau \circ \tau = \mathrm{id}_{\mathscr{A}}$, and when restricted to fibers it is an anti-automorphism of complex algebras. In this case, $(\mathscr{A},\tau)$ is called a *topological Azumaya algebra with involution*.

Definition 1.3.4. Let X be a connected topological space, and let $(\mathscr{A},\tau)$ be a topological Azumaya algebra with involution over X. The involution τ is said to be *orthogonal* (*symplectic*) if the restriction $\tau|_{\mathscr{A}^{-1}(x)} : \mathscr{A}^{-1}(x) \to \mathscr{A}^{-1}(x)$ is an orthogonal (a symplectic) involution of complex algebras for all $x \in X$.

From [26, 8.2] there is a bijective correspondence

$$\left\{\begin{array}{c}\text{Isomorphism classes of degree-}n\\ \text{topological Azumaya algebras}\\ \text{over }X\text{ with an involution}\\ \text{locally isomorphic to }\tau\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\text{Isomorphism classes of principal}\\ \mathrm{Aut}\big(\mathrm{M}(n,\mathbb{C}),\tau\big)\text{-bundles}\\ \text{over }X\end{array}\right\},$$

where τ is an involution of the first kind on $\mathrm{M}(n,\mathbb{C})$. Proposition 1.3.1 implies that topological Azumaya algebras over X with involution are classified by $[X, B\mathrm{PO}(n,\mathbb{C})]$ in the orthogonal case, and $[X, B\mathrm{PSp}(n,\mathbb{C})]$ in the symplectic case.

The second goal of this thesis is to provide conditions on a positive integer n and a topological space X such that a topological Azumaya algebra of degree n over X with an orthogonal involution has a tensor product decomposition of topological Azumaya algebras with orthogonal involution. The main result of Chapter 3 is the following theorem:

Theorem 1.3.5. *Let m and n be positive integers such that m is even, and n is odd. Let X be a CW complex such that $\dim(X) \leq d$ where $d := \min\{m, n\}$.*

If $\mathscr{A}$ is a topological Azumaya algebra of degree mn over X with an orthogonal involution, then there exist topological Azumaya algebras $\mathscr{A}_m$ and $\mathscr{A}_n$ of degrees m and n, respectively, such that $\mathscr{A}_m$ and $\mathscr{A}_n$ have orthogonal involutions, $\mathscr{A}_n$ is Brauer-trivial and $\mathscr{A} \cong \mathscr{A}_m \otimes \mathscr{A}_n$.

We prove in Theorem 1.3.5 that a map $X \to B\mathrm{PO}(mn,\mathbb{C})$ can be lifted to $B\mathrm{PO}(m,\mathbb{C}) \times B\mathrm{SO}(n,\mathbb{C})$ when the dimension of X is less than $d+1$. The proof of Theorem 1.3.5 relies in the description of the homomorphisms induced on homotopy groups by the r-fold direct sum of matrices $\oplus^r : \mathrm{O}(n,\mathbb{C}) \to \mathrm{O}(rn,\mathbb{C})$ in the range $\{0,1,\ldots,n-1\}$. We call this set "*the stable range for* $\mathrm{O}(n,\mathbb{C})$".

The third goal of this thesis is to provide conditions on a positive integer n and a topological space X such that a topological Azumaya algebra of degree $2n$ over X with a symplectic involution has a tensor product decomposition of topological Azumaya algebras with involution. The main result of Chapter 4 is the following theorem:

Theorem 1.3.6. *Let X be a CW complex such that $\dim(X) \le 7$. Let m and n be positive integers such that $m > 1$, $n > 7$, and n is odd.*

If $\mathscr{A}$ is a topological Azumaya algebra of degree $2mn$ over X with a symplectic involution, then there exist topological Azumaya algebras $\mathscr{A}_{2m}$ and $\mathscr{A}_n$ of degrees $2m$ and n, respectively, such that $\mathscr{A}_{2m}$ has a symplectic involution, $\mathscr{A}_n$ has an orthogonal involution and is Brauer-trivial, and $\mathscr{A} \cong \mathscr{A}_{2m} \otimes \mathscr{A}_n$.

We prove in Theorem 1.3.6 that a map $X \to \mathrm{B}\,\mathrm{PSp}(mn,\mathbb{C})$ can be lifted to $\mathrm{B}\,\mathrm{PSp}(m,\mathbb{C}) \times \mathrm{B}\,\mathrm{SO}(n,\mathbb{C})$ when the dimension of X is less than 7. The proof of Theorem 1.3.6 relies in the description of the homomorphisms induced on homotopy groups by the r-fold direct sum of symplectic matrices $\boxplus^r : \mathrm{Sp}(m,\mathbb{C}) \to \mathrm{Sp}(rm,\mathbb{C})$, and by the homomorphism $\boxtimes : \mathrm{Sp}(m,\mathbb{C}) \times \mathrm{Sp}(n,\mathbb{C}) \to \mathrm{O}(4mn,\mathbb{C})$ in the range $\{0,1,\ldots,4m+2\}$. We call this set "*the stable range for* $\mathrm{Sp}(n,\mathbb{C})$".

1.4 Common Preliminaries

1.4.1 Matrix operations

For $m,n \in \mathbb{N}$ consider the following matrix operations:

1. The *direct sum of matrices*, $\oplus : \mathrm{M}(m,\mathbb{C}) \times \mathrm{M}(n,\mathbb{C}) \to \mathrm{M}(m+n,\mathbb{C})$ defined by
$$A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

2. The *r-fold direct sum*, $\oplus^r : \mathrm{M}(n,\mathbb{C}) \to \mathrm{M}(rn,\mathbb{C})$ given by
$$A^{\oplus r} = \underbrace{A \oplus \cdots \oplus A}_{r\text{-times}}.$$

3. The *tensor product of matrices*, $\otimes : \mathrm{M}(m,\mathbb{C}) \times \mathrm{M}(n,\mathbb{C}) \to \mathrm{M}(mn,\mathbb{C})$.

   Let $\mathbb{C}^m$ and $\mathbb{C}^n$ with the standard bases $\{e_1,\ldots,e_m\}$ and $\{e'_1,\ldots,e'_n\}$, respectively. The tensor product of spaces $\mathbb{C}^m \otimes \mathbb{C}^n$ has the standard basis
$$\{e_1 \otimes e'_1,\ldots,e_1 \otimes e'_n,\ldots,e_m \otimes e'_1,\ldots,e_m \otimes e'_n\}.$$

From this the tensor product of matrices is given by the *Kronecker product*

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mm}B \end{pmatrix},$$

for $A = (a_{ij}) \in \mathrm{M}(m, \mathbb{C})$.

4. The *r-fold tensor product*, $\otimes^r : \mathrm{M}(n, \mathbb{C}) \to \mathrm{M}(n^r, \mathbb{C})$ given by

$$A^{\otimes r} = \underbrace{A \otimes \cdots \otimes A}_{r\text{-times}}.$$

The operations above define maps of spaces. Let $\oplus_*$, $\oplus_*^r$, $\otimes_*$ and $\otimes_*^r$ denote the homomorphisms of homotopy groups induced by these maps of spaces.

1.4.2 Non-degenerate bilinear forms

Let V be a finite-dimensional complex vector space, and let B be a non-degenerate bilinear form on V. We denote such space as a pair (V, B).

If $\dim_{\mathbb{C}} V$ is even, V can be given both a skew-symmetric bilinear form and a symmetric bilinear form. If $\dim_{\mathbb{C}} V$ is odd, V can be given a symmetric bilinear form. These bilinear forms are unique up to isomorphism, given that $\mathbb{C}$ is algebraically closed, [9, Theorem 9.13].

The automorphism group of (V, B) is

$$\mathrm{Aut}(V, B) = \Big\{ T \in \mathrm{GL}(n, \mathbb{C}) : B(Tv, Tw) = B(v, w) \text{ for all } v, w \in V \Big\}.$$

Let $(\mathbb{C}^n, B)$ and $(\mathbb{C}^{2n}, B')$ denote the spaces $\mathbb{C}^n$ and $\mathbb{C}^{2n}$ with the standard symmetric bilinear form and the standard skew-symmetric bilinear form, respectively, then

$$\mathrm{Aut}(\mathbb{C}^n, B) \cong \mathrm{O}(n, \mathbb{C}) \quad \text{and} \quad \mathrm{Aut}(\mathbb{C}^{2n}, B') \cong \mathrm{Sp}(n, \mathbb{C}).$$

Tensor product of bilinear forms

Let (V,B) and (V,B') be finite-dimensional complex vector spaces with non-degenerate bilinear forms. We define the tensor product $(V,B)\otimes(V',B')$ as follows. There exist linear maps $\widetilde{B}: V\otimes V \to \mathbb{C}$ and $\widetilde{B'}: V'\otimes V' \to \mathbb{C}$ such that $\widetilde{B}(v\otimes w) = B(v,w)$ and $\widetilde{B'}(v'\otimes w') = B'(v',w')$. Then we have a linear map

$$(V\otimes V')\otimes(V\otimes V') \cong (V\otimes V)\otimes(V'\otimes V') \xrightarrow{\widetilde{B}\otimes\widetilde{B'}} \mathbb{C}\otimes\mathbb{C} \cong \mathbb{C}$$

given by $(v\otimes v')\otimes(w\otimes w') \mapsto \widetilde{B}(v\otimes w)\widetilde{B'}(v'\otimes w')$.

Let $B\otimes B'$ denote the composition of $\widetilde{B}\otimes\widetilde{B'}$ and the quotient map $(V\otimes V')\times(V\otimes V') \to (V\otimes V')\otimes(V\otimes V')$. Then $(V\otimes V', B\otimes B')$ is a non-degenerate bilinear form over $\mathbb{C}$, where $(B\otimes B')(v\otimes v', w\otimes w') = B(v,w)B'(v',w')$ (multiplication in $\mathbb{C}$), [13, Section 1.21, Section 1.27]. The tensor product is bifunctorial, so that one obtains induced maps

$$\otimes : \mathrm{Aut}(V,B)\times\mathrm{Aut}(V',B') \longrightarrow \mathrm{Aut}(V\otimes V', B\otimes B'). \tag{1.10}$$

The following tensor product operations are of interest in the subsequent chapters.

1. Let $(\mathbb{C}^m, B)$ and $(\mathbb{C}^n, B')$ be the spaces $\mathbb{C}^m$ and $\mathbb{C}^n$ with the standard symmetric bilinear forms. Let $\{e_1,\dots,e_m\}$ and $\{e'_1,\dots,e'_n\}$ be the standard bases of $\mathbb{C}^m$ and $\mathbb{C}^n$, respectively. Then B and B' are such that

   $$B\left(e_i, e_j\right) = \delta_{ij} \text{ and } B'\left(e'_k, e'_l\right) = \delta_{kl}$$

   for $1\le i,j\le m$ and $1\le k,l\le n$.

   The tensor product $(\mathbb{C}^m\otimes\mathbb{C}^n, B\otimes B')$ is a symmetric non-degenerate bilinear form such that $\{e_1\otimes e'_1,\dots,e_1\otimes e'_n,\dots,e_m\otimes e'_1,\dots,e_m\otimes e'_n\}$ is an ordered basis for $\mathbb{C}^m\otimes\mathbb{C}^n$ and

   $$(B\otimes B')\left(e_i\otimes e'_k, e_j\otimes e'_l\right) = B\left(e_i, e_j\right)B'\left(e'_k, e'_l\right) = \delta_{ij}\delta_{kl},$$

   this is, $B\otimes B'$ is the standard symmetric bilinear form on $\mathbb{C}^{mn}$. Therefore, the tensor product (1.10) induces a homomorphism

$$\otimes : \mathrm{O}(m,\mathbb{C}) \times \mathrm{O}(n,\mathbb{C}) \longrightarrow \mathrm{O}(mn,\mathbb{C}).$$

2. Let $(\mathbb{C}^{2m}, B)$ be the space $\mathbb{C}^{2m}$ with the standard skew-symmetric bilinear form, and $(\mathbb{C}^n, B')$ be the space $\mathbb{C}^n$ with the standard symmetric bilinear form. Let $\{e_1,\dots,e_{2m}\}$ and $\{e'_1,\dots,e'_n\}$ be the standard bases of $\mathbb{C}^m$ and $\mathbb{C}^n$, respectively. Then B and B' are such that

   $$B\left(e_i,e_j\right) = e_i^{\mathrm{tr}} J_{2m} e_j \text{ and } B'\left(e'_k,e'_l\right) = \delta_{kl}$$

   for $1 \le i,j \le 2m$ and $1 \le k,l \le n$.

   The tensor product of the bilinear forms above $(\mathbb{C}^{2m} \otimes \mathbb{C}^n, B \otimes B')$ is a skew-symmetric non-degenerate bilinear form such that

   $$\left\{e_1 \otimes e'_1,\dots,e_1 \otimes e'_n,\dots,e_{2m} \otimes e'_1,\dots,e_{2m} \otimes e'_n\right\}$$

   is an ordered basis for $\mathbb{C}^{2m} \otimes \mathbb{C}^n$ and

   $$(B \otimes B')\left(e_i \otimes e'_k, e_j \otimes e'_l\right) = B\left(e_i,e_j\right) B'\left(e'_k,e'_l\right) = \left(e_i^{\mathrm{tr}} J_{2m} e_j\right)\delta_{kl},$$

   this means $B \otimes B'$ is the standard skew-symmetric bilinear form on $\mathbb{C}^{2mn}$. Therefore, the tensor product (1.10) induces a homomorphism

   $$\otimes : \mathrm{Sp}(m,\mathbb{C}) \times \mathrm{O}(n,\mathbb{C}) \longrightarrow \mathrm{Sp}(mn,\mathbb{C}).$$

3. Let $(\mathbb{C}^{2m}, B)$ and $(\mathbb{C}^{2n}, B')$ be the spaces $\mathbb{C}^{2m}$ and $\mathbb{C}^{2n}$ with the standard skew-symmetric bilinear forms. Let $\{e_1,\dots,e_{2m}\}$ and $\{e'_1,\dots,e'_{2n}\}$ be the standard bases of $\mathbb{C}^{2m}$ and $\mathbb{C}^{2n}$, respectively. Then B and B' are such that

   $$B\left(e_i,e_j\right) = e_i^{\mathrm{tr}} J_{2m} e_j \text{ and } B'\left(e'_k,e'_l\right) = (e'_k)^{\mathrm{tr}} J_{2n} e'_l$$

   for $1 \le i,j \le 2m$ and $1 \le k,l \le 2n$.

   The tensor product $(\mathbb{C}^{2m} \otimes \mathbb{C}^{2n}, B \otimes B')$ is a symmetric non-degenerate bilinear form such that $\mathcal{B} = \{e_1 \otimes e'_1, \ldots, e_1 \otimes e'_{2n}, \ldots, e_{2m} \otimes e'_1, \ldots, e_{2m} \otimes e'_{2n}\}$ is an ordered basis for $\mathbb{C}^{2m} \otimes \mathbb{C}^{2n}$ and

$$(B \otimes B')\left(e_i \otimes e'_k, e_j \otimes e'_l\right) = B\left(e_i, e_j\right) B'\left(e'_k, e'_l\right) = \left(e_i^{\mathrm{tr}} J_{2m} e_j\right)\left((e'_k)^{\mathrm{tr}} J_{2n} e'_l\right).$$

Observe that $B \otimes B'$ is not the standard symmetric bilinear form on $\mathbb{C}^{4mn}$. From the tensor product (1.10) there is a homomorphism

$$\otimes : \mathrm{Sp}(m,\mathbb{C}) \times \mathrm{Sp}(n,\mathbb{C}) \longrightarrow G,$$

where $G \leq \mathrm{GL}(4mn,\mathbb{C})$ denotes $\mathrm{Aut}(\mathbb{C}^{4mn}, B \otimes B')$ with the basis $\mathcal{B}$. Let $P \in \mathrm{GL}(4mn,\mathbb{C})$ be the basis change matrix associated to changing the basis of $\mathbb{C}^{4mn}$ from $\mathcal{B}$ to an orthonormal basis. Then we have the commutative square below

$$\begin{tikzcd}
G \arrow[r, hook] \arrow[d, "\mathrm{Int}_P", "\cong"'] & \mathrm{GL}(4mn,\mathbb{C}) \arrow[d, "\mathrm{Int}_P", "\cong"'] \\
\mathrm{O}(4mn,\mathbb{C}) \arrow[r, hook] & \mathrm{GL}(4mn,\mathbb{C}).
\end{tikzcd}$$

Thus, the composite $\mathrm{Int}_P \circ \otimes : \mathrm{Sp}(m,\mathbb{C}) \times \mathrm{Sp}(n,\mathbb{C}) \to G \to \mathrm{O}(4mn,\mathbb{C})$ gives a homomorphism

$$\boxtimes : \mathrm{Sp}(m,\mathbb{C}) \times \mathrm{Sp}(n,\mathbb{C}) \longrightarrow \mathrm{O}(4mn,\mathbb{C}).$$

1.4.3 Auxiliary lemmas

Lemma 1.4.1. *Let G be a Lie group and let G_0 be the component of the identity. If $r : G \to G$ is conjugation by $P \in G_0$, then there is a basepoint preserving homotopy H from r to id_G such that for all $t \in [0,1]$, $H(-,t)$ is a homomorphism.*

Proof. Since G is path-connected, there exists a path α from P to I_m in G. Define $H : G \times [0,1] \to G$ by $H(A,t) = \alpha(t) A \alpha(t)^{-1}$. Observe that $H(-,t) : G \to$

G, $A \mapsto H(A,t)$ is a homomorphism. Moreover, H is such that

$$H(e_G,t) = e_G,\ \ H(A,0) = r(A) \ \text{ and } \ H(A,1) = A.$$

Therefore, the result follows. □

Lemma 1.4.2. *Let a and b be positive integers such that a and b are relatively prime. There exist positive integers u and v such that $|vb - ua| = 1$.*

Proof. Since a and b are relatively prime they generate the unit ideal in $\mathbb{Z}$. That is, the equation $ax + by = 1$ has integer solutions x_0, y_0. Given that $ax_0 + by_0 = 1$ and a, b are positive, then exactly one of x_0, y_0 is negative. By changing the sign of the negative one we obtain u, v both positive integers satisfying $ua - vb = \pm 1$. □

Chapter 2

Decomposition of Topological Azumaya Algebras

This chapter is organized as follows. The second section presents preliminaries on the effect of direct sum and tensor product operations on homotopy groups of compact Lie groups related to the unitary groups $\mathrm{U}(n,\mathbb{C})$. The third section is devoted to the proof of Theorem 2.2.3. We explain in Remark 2.2.7 why the decomposition in Theorem 1.2.8 is not unique up to isomorphism.

From now on, we are going to drop the $\mathbb{C}$ that comes in the notation of $\mathrm{U}(n,\mathbb{C})$.

2.1 Stabilization of operations on the unitary group

We begin by recalling the low degree homotopy groups of the unitary groups and the special unitary groups. Let $n \geq 1$ and $i < 2n$, the first homotopy groups of $\mathrm{U}(n)$ are given by Bott periodicity.

$$\pi_i(\mathrm{U}(n)) \cong \begin{cases} 0 & \text{if } i \text{ is even,} \\ \mathbb{Z} & \text{if } i \text{ is odd.} \end{cases}$$

The determinant map, $\det : \mathrm{U}(n) \to S^1$, is split so that $\pi_1(\mathrm{U}(n))$ maps isomorphically onto $\pi_1(S^1)$ and $\pi_1(\mathrm{SU}(n))$ is simply connected. It follows that

$$\pi_i(\mathrm{SU}(n)) \cong \begin{cases} 0 & \text{if } i = 1, \\ \pi_i(\mathrm{U}(n)) & \text{otherwise.} \end{cases}$$

We now compute the low degree homotopy groups of the spaces $\mathrm{U}(am)/\mu_m$ and $\mathrm{SU}(am)/\mu_m$. As $\mathrm{SU}(am)$ is a simply connected m-cover of $\mathrm{SU}(am)/\mu_m$ we have

$$\pi_i(\mathrm{SU}(am)/\mu_m) \cong \begin{cases} \mathbb{Z}/m & \text{if } i = 1, \\ \pi_i(\mathrm{SU}(am)) & \text{otherwise.} \end{cases}$$

All columns as well as the two top rows of diagram (2.1) are short exact. By the third isomorphism theorem the bottom row is also short exact.

$$\begin{tikzcd}
\mu_m \arrow[r, equal] \arrow[d, hook] & \mu_m \arrow[r] \arrow[d, hook] & \{1\} \arrow[d] \\
\mathrm{SU}(am) \arrow[r, hook] \arrow[d, two heads] & \mathrm{U}(am) \arrow[r, two heads, "\det"] \arrow[d, two heads] & S^1 \arrow[d, equal] \\
\mathrm{SU}(am)/\mu_m \arrow[r, "i"] & \mathrm{U}(am)/\mu_m \arrow[r, "\det"] & S^1.
\end{tikzcd} \tag{2.1}$$

Therefore, $\pi_i(\mathrm{U}(am)/\mu_m) \cong \pi_i(\mathrm{SU}(am)/\mu_m)$ for all $i > 1$. It remains to compute the fundamental group of $\mathrm{U}(am)/\mu_m$.

By exactness of the bottom row of diagram (2.1), the induced sequence on fundamental groups is exact,

$$0 \longrightarrow \pi_1\big(\mathrm{SU}(am)/\mu_m\big) \xrightarrow{i_*} \pi_1\big(\mathrm{U}(am)/\mu_m\big) \xrightarrow{\det_*} \pi_1\big(S^1\big) \longrightarrow 0. \tag{2.2}$$

The map $\det : \mathrm{U}(am) \to S^1$ has a section $t : S^1 \to \mathrm{U}(am)$ defined by

$$t(\omega) = \begin{pmatrix} \omega & 0 \\ 0 & I_{am-1} \end{pmatrix}.$$

The section t is one of groups; in fact $\mathrm{U}(am)$ is a semi-direct product of S^1 by $\mathrm{SU}(am)$. This section induces a section of $\det : \mathrm{U}(am)/\mu_m \to S^1$, which we also denote by t,

$$
\begin{tikzcd}
1 \arrow[r] & \mathrm{SU}(am)/\mu_m \arrow[r, "i"] & \mathrm{U}(am)/\mu_m \arrow[r, "\det"] & S^1 \arrow[r] \arrow[l, bend left, dotted, "t"] & 1.
\end{tikzcd} \tag{2.3}
$$

Since $\pi_1\left(S^1\right) \cong \mathbb{Z}$, sequence (2.2) splits. We describe $\pi_1\left(\mathrm{U}(am)/\mu_m\right)$ in terms of the homomorphisms $i_* : \pi_1\left(\mathrm{SU}(am)/\mu_m\right) \to \pi_1\left(\mathrm{U}(am)/\mu_m\right)$ and $t_* : \pi_1\left(S^1\right) \to \pi_1\left(\mathrm{U}(am)/\mu_m\right)$ as $\pi_1\left(\mathrm{U}(am)/\mu_m\right) = \mathrm{Im}\, i_* \oplus \mathrm{Im}\, t_* \cong \mathbb{Z}/m \oplus \mathbb{Z}$.

2.1.1 First unstable homotopy group of $\mathrm{U}(n,\mathbb{C})$

The standard inclusion of the orthogonal group $i : \mathrm{U}(n) \hookrightarrow \mathrm{U}(n+1)$ is $2n$-connected, hence it induces an isomorphism on homotopy groups in degrees less than $2n$ and an epimorphism in degree $2n$. This can be seen by observing the long exact sequence for the fibration $\mathrm{U}(n) \hookrightarrow \mathrm{U}(n+1) \to \mathrm{U}(n+1)/\mathrm{U}(n) \simeq S^{2n+1}$.

The first unstable homotopy group of $\mathrm{U}(n)$ happens in degree $2n$. Bott proves in [8] that $\pi_{2n}(\mathrm{U}(n)) \cong \mathbb{Z}/n!$.

Let $G \in \left\{\mathrm{U}(am), \mathrm{SU}(am), \mathrm{PU}(am), \mathrm{U}(am)/\mu_m\right\}$. Tables 2.1 and 2.2 summarize the previous results.

G	$\mathrm{U}(am)$	$\mathrm{SU}(am)$	$\mathrm{PU}(am)$	$\mathrm{U}(am)/\mu_m$	$\mathrm{SU}(am)/\mu_m$
$\pi_0(G)$	$*$	$*$	$*$	$*$	$*$
$\pi_1(G)$	$\mathbb{Z}$	0	$\mathbb{Z}/am$	$\mathbb{Z}/m \oplus \mathbb{Z}$	$\mathbb{Z}/m$

Table 2.1: Connected components and fundamental group of compact Lie groups related to the unitary group

i	even	odd	$2am$
$\pi_i(G)$	0	$\mathbb{Z}$	$\mathbb{Z}/(am)!$

Table 2.2: Homotopy groups and first unstable homotopy group of compact Lie groups related to the unitary group for $i = 2, \dots, 2am-1$.

2.1.2 Stabilization

Let $m,n \in \mathbb{N}$ and $m \leq n$. Define the map

$$\begin{aligned} \mathrm{s} : \mathrm{U}(m) &\longrightarrow \mathrm{U}(m+n) \\ A &\longmapsto A \oplus I_n. \end{aligned}$$

Since the map s is equal to the consecutive composite of standard inclusions, it follows that s is $2m$-connected. Hence, s induces a surjection in degree $2m$ and an isomorphism on homotopy groups in degrees less than $2m$.

Notation 2.1.1. Let stab denote $\pi_i(\mathrm{s})$, the homomorphism the map s induces on homotopy groups. The following isomorphism for $i < 2m$ will be used throughout the paper

$$\mathrm{stab} : \pi_i(\mathrm{U}(m)) \xrightarrow{\cong} \pi_i(\mathrm{U}(m+n)) \tag{2.4}$$

to identify $\pi_i(\mathrm{U}(m))$ with $\pi_i(\mathrm{U}(m+n))$ for all $i < 2m$.

Lemma 2.1.2. *Let $n,r \in \mathbb{N}$. For all $j = 1,\dots,r$ define $\mathrm{s}_j : \mathrm{U}(n) \to \mathrm{U}(rn)$ by*

$$\mathrm{s}_j(A) = \mathrm{diag}(I_n,\dots,I_n,A,I_n,\dots,I_n),$$

where A is in the j-th position. The maps s_j and s_{j+1} are pointed homotopic for all $j = 1,\dots,r-1$.

Proof. The block matrix

$$P_j = \begin{pmatrix} I_{(j-1)n} & & & \\ & 0 & I_n & \\ & I_n & 0 & \\ & & & I_{(r-j-1)n} \end{pmatrix}$$

is such that $P_j P_j = I_{rn}$ for $j = 1,\dots,r-1$. Moreover, if $A,B \in \mathrm{U}(n)$, then

$$P_j\,\mathrm{diag}(I_n,\dots,I_n,A,B,I_n,\dots,I_n)P_j = \mathrm{diag}(I_n,\dots,I_n,B,A,I_n,\dots,I_n),$$

where A and B are in positions (j,j), $(j+1,j+1)$, and $(j+1,j+1)$, (j,j), respectively.

From Lemma 1.4.1, s_j and s_{j+1} are pointed homotopic. $\square$

Notation 2.1.3. We call the s_j maps *stabilization maps*. As s_1 is equal to $s : U(n) \to U(n+(r-1)n)$, it follows that s_j is $2n$-connected for all $j = 1, \dots, r$. From Lemma 2.1.2 the homomorphisms induced on homotopy groups by the stabilization maps are equal, hence stab also denotes $\pi_i(s_1) = \cdots = \pi_i(s_r)$. Thus we identify $\pi_i(U(n))$ with $\pi_i(U(rn))$ for $i < 2n$ through stab. The identification allows one to introduce a slight abuse of notation, namely to identify x and $\mathrm{stab}(x)$ for $x \in \pi_i(U(n))$ and $i < 2n$.

2.1.3 Operations on homotopy groups

Proposition 2.1.4. *Let $i \in \mathbb{N}$, the homomorphism $\oplus_* : \pi_i(U(m)) \times \pi_i(U(n)) \to \pi_i(U(m+n))$ is given by*

$$\oplus_*(x,y) = \mathrm{stab}(x) + \mathrm{stab}(y)$$

for $x \in \pi_i(U(m))$ and $y \in \pi_i(U(n))$.

Proof. It is enough to observe that the direct sum factors as

$$\begin{array}{ccccc} U(m) \times U(n) & \xrightarrow{s_1 \times s_2} & U(m+n) \times U(m+n) & \xrightarrow{\mathrm{mult}} & U(m+n) \\ (A,B) & \longmapsto & \left(\begin{pmatrix} A & 0 \\ 0 & I_n \end{pmatrix}, \begin{pmatrix} I_m & 0 \\ 0 & B \end{pmatrix}\right) & \longmapsto & \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}. \end{array}$$

Thus $\oplus_*(x,y) = \mathrm{mult}_* \circ (\mathrm{stab} \times \mathrm{stab})(x,y) = \mathrm{stab}(x) + \mathrm{stab}(y)$, where the last equality is true by the Eckmann-Hilton argument, [25, Theorem 1.6.8]. $\square$

Corollary 2.1.5. *If $m < n$ and $i < 2m$, then $\oplus_*(x,y) = x + y$ for $x \in \pi_i(U(m))$ and $y \in \pi_i(U(n))$.*

Proof. Since s_1 and s_2 are $2m$-connected, the homomorphisms

$$\mathrm{stab} : \pi_i(U(m)) \longrightarrow \pi_i(U(m+n)) \qquad \text{and} \qquad \mathrm{stab} : \pi_i(U(n)) \longrightarrow \pi_i(U(m+n))$$

are isomorphisms $i < 2m$ and $i < 2n$, respectively. We use these isomorphisms to identify source and target.

From Proposition 2.1.4, $\oplus_*(x, y) = \mathrm{stab}(x) + \mathrm{stab}(y) = x + y$ for $i < 2m$. $\square$

Proposition 2.1.6. *Let $i \in \mathbb{N}$, the homomorphism $\oplus^r_* : \pi_i(\mathrm{U}(n)) \to \pi_i(\mathrm{U}(rn))$ is given by*

$$\oplus^r_*(x) = r\,\mathrm{stab}(x)$$

for $x \in \pi_i(\mathrm{U}(n))$.

Proof. Let $\Delta : \mathrm{U}(n) \to (\mathrm{U}(n))^{\times r}$ denote the diagonal map. The r-block summation factors as

$$\begin{array}{ccccccc} \mathrm{U}(n) & \xrightarrow{\Delta} & (\mathrm{U}(n))^{\times r} & \xrightarrow{\mathrm{s}_1\times\cdots\times\mathrm{s}_r} & (\mathrm{U}(rn))^{\times r} & \xrightarrow{\quad\mathrm{mult}\quad} & \mathrm{U}(rn) \\ A & \longmapsto & (A,\dots,A) & \longmapsto & \big(\mathrm{s}_1(A),\dots,\mathrm{s}_r(A)\big) & \longmapsto & \mathrm{s}_1(A)\cdots\mathrm{s}_r(A). \end{array}$$

By the Eckmann–Hilton argument $\mathrm{mult}_* : \pi_i(\mathrm{U}(rn))^r \to \pi_i(\mathrm{U}(rn))$ is given by

$$\mathrm{mult}_*(x_1,\dots,x_r) = x_1 + \cdots + x_r$$

for $x_j \in \pi_i(\mathrm{U}(rn))$ and $j = 1,\dots,r$. From this $\oplus^r_*$ takes the form

$$\begin{array}{cc} \pi_i(\mathrm{U}(n)) & x \\ \Big\downarrow{\scriptstyle\Delta} & \Big\downarrow \\ \pi_i(\mathrm{U}(n))^{\times r} & (x,\dots,x) \\ \Big\downarrow{\scriptstyle\mathrm{stab}^{\times r}} & \Big\downarrow \\ \pi_i(\mathrm{U}(rn))^{\times r} & \big(\mathrm{stab}(x),\dots,\mathrm{stab}(x)\big) \\ \Big\downarrow{\scriptstyle\mathrm{mult}_*} & \Big\downarrow \\ \pi_i(\mathrm{U}(rn)) & \underbrace{\mathrm{stab}(x)+\cdots+\mathrm{stab}(x)}_{r\text{ times}} \end{array}$$

This proves the statement. $\square$

Corollary 2.1.7. *If $i < 2n$, then $\oplus^r_*(x) = rx$ for $x \in \pi_i(\mathrm{U}(n))$.*

Proof. The homomorphism $\mathrm{stab}^{\times r} : \pi_i(\mathrm{U}(n))^{\times r} \to \pi_i(\mathrm{U}(rn))^{\times r}$ is an isomorphism for all $i < 2n$ because so is $\mathrm{stab} : \pi_i(\mathrm{U}(n)) \to \pi_i(\mathrm{U}(rn))$. By Proposition 2.1.6 we conclude $\oplus^r_*(x) = r\,\mathrm{s}_*(x) = rx$ for $i < 2n$. □

Lemma 2.1.8. *Let $L, R : \mathrm{U}(m) \to \mathrm{U}(mn)$ be the maps $L(A) = A \otimes I_n$ and $R(A) = I_n \otimes A$. There is a basepoint preserving homotopy H from L to R such that for all $t \in [0,1]$, $H(-,t)$ is a homomorphism.*

Proof. Let $A \in \mathrm{U}(m)$.

$$L(A) = \begin{pmatrix} a_{11}I_n & \cdots & a_{1m}I_n \\ \vdots & \ddots & \vdots \\ a_{m1}I_n & \cdots & a_{mm}I_n \end{pmatrix} \quad \text{and} \quad R(A) = \begin{pmatrix} A & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A \end{pmatrix} = A^{\oplus n}.$$

Let $P_{m,n}$ be the permutation matrix

$$\begin{aligned} P_{m,n} = [&e_1, e_{n+1}, e_{2n+1}, \dots, e_{(m-1)n+1}, e_2, e_{n+2}, e_{2n+2}, \dots, e_{(m-1)n+2}, \\ &\dots, \\ &e_{n-1}, e_{2n-1}, e_{3n-1}, \dots, e_{mn-1}, e_n, e_{2n}, e_{3n}, \dots, e_{(m-1)n}, e_{mn}] \end{aligned}$$

where, e_i is the i-th standard basis vector of $\mathbb{C}^{mn}$ written as a column vector. Observe that $L(A) = P_{m,n}R(A)P_{m,n}^{-1}$. The result follows from Lemma 1.4.1. □

Proposition 2.1.9. *Let $i \in \mathbb{N}$, the homomorphism $\otimes_* : \pi_i(\mathrm{U}(m)) \times \pi_i(\mathrm{U}(n)) \to \pi_i(\mathrm{U}(mn))$ is given by*

$$\otimes_*(x, y) = n\,\mathrm{stab}(x) + m\,\mathrm{stab}(y)$$

for $x \in \pi_i(\mathrm{U}(m))$ and $y \in \pi_i(\mathrm{U}(n))$.

Proof. By the mixed-product property of the tensor product of matrices

$$A \otimes B = (A \otimes I_n)(I_m \otimes B) = L(A)R(B).$$

Lemma 2.1.8 gives $L_* = \oplus^n_* : \pi_i(\mathrm{U}(m)) \to \pi_i(\mathrm{U}(mn))$. Proposition 2.1.6 yields $\otimes_*(x,y) = n\,\mathrm{stab}(x) + m\,\mathrm{stab}(y)$. $\square$

Corollary 2.1.10. *If $m < n$ and $i < 2m$, then $\otimes_*(x,y) = nx+my$ for $x \in \pi_i(\mathrm{U}(m))$ and $y \in \pi_i(\mathrm{U}(n))$.*

Proof. The statement follows from Corollary 2.1.7 and Proposition 2.1.9. $\square$

Proposition 2.1.11. *Let $i \in \mathbb{N}$, the homomorphism $\otimes^r_* : \pi_i(\mathrm{U}(n)) \to \pi_i(\mathrm{U}(n^r))$ is given by*

$$\otimes^r_*(x) = rn^{r-1}\,\mathrm{stab}(x)$$

for $x \in \pi_i(\mathrm{U}(n))$.

Corollary 2.1.12. *If $i < 2n$, then $\otimes^r_*(x) = rn^{r-1}x$ for $x \in \pi_i(\mathrm{U}(n))$.*

Proof. Corollary 2.1.7 and Proposition 2.1.11 yield the result. $\square$

Tensor product on the quotient

Let a, b, m and n be positive integers so that $m < n$. The tensor product operation $\otimes : \mathrm{U}(am) \times \mathrm{U}(bn) \to \mathrm{U}(abmn)$ sends the group $\mu_m \times \mu_n$ to μ_{mn}. Consequently, the operation descends to the quotient

$$\otimes : \mathrm{U}(am)/\mu_m \times \mathrm{U}(bn)/\mu_n \longrightarrow \mathrm{U}(abmn)/\mu_{mn}. \tag{2.5}$$

Proposition 2.1.13. *If $i > 1$, the homomorphism*

$$\otimes_* : \pi_i(\mathrm{U}(am)/\mu_m) \times \pi_i(\mathrm{U}(bn)/\mu_n) \longrightarrow \pi_i(\mathrm{U}(abmn)/\mu_{mn})$$

is given by

$$\otimes_*(x,y) = bn\,\mathrm{stab}(x) + am\,\mathrm{stab}(y)$$

for $x \in \pi_i(\mathrm{U}(am)/\mu_m)$ and $y \in \pi_i(\mathrm{U}(bn)/\mu_n)$.

Proof. There is a map of fibrations

$$\begin{tikzcd}
\mu_m \times \mu_n \arrow[r, hook] \arrow[d, "\mathrm{mult}"] & \mathrm{U}(am)\times \mathrm{U}(bn) \arrow[r] \arrow[d, "\otimes"] & \mathrm{U}(am)/\mu_m \times \mathrm{U}(bn)/\mu_n \arrow[d, "\otimes"] \\
\mu_{mn} \arrow[r, hook] & \mathrm{U}(abmn) \arrow[r] & \mathrm{U}(abmn)/\mu_{mn}.
\end{tikzcd} \tag{2.6}$$

From the homomorphism of long exact sequences associated to the fibrations in diagram (2.6) we obtain a commutative square

$$\begin{tikzcd}
\pi_i(\mathrm{U}(am))\times \pi_i(\mathrm{U}(bn)) \arrow[r, "\cong"] \arrow[d, "\otimes_*"] & \pi_i(\mathrm{U}(am)/\mu_m)\times \pi_i(\mathrm{U}(bn)/\mu_n) \arrow[d, "\otimes_*"] \\
\pi_i(\mathrm{U}(abmn)) \arrow[r, "\cong"] & \pi_i(\mathrm{U}(abmn)/\mu_{mn})
\end{tikzcd}$$

for $i > 1$. This diagram and Proposition 2.1.9 gives $\otimes_*(x, y) = \oplus_*^{bn}(x)+\oplus_*^{am}(y) = bn\,\mathrm{stab}(x) + am\,\mathrm{stab}(y)$ for all $i > 1$. $\square$

In the following proposition, we identify $\pi_1(\mathrm{U}(am)/\mu_m)$ with $\mathrm{Im}\, i_* \oplus \mathrm{Im}\, t_* \cong \mathbb{Z}/m \oplus \mathbb{Z}$, where i and t are the maps in diagram (2.3). We also identify $\mathbb{Z}/m$ and $\mathbb{Z}/n$ with the following subgroups of $\mathbb{Z}/mn$,

$$\mathbb{Z}/m \cong \{0, n, 2n, \ldots, (m-1)n\} \quad \text{and} \quad \mathbb{Z}/n \cong \{0, m, 2m, \ldots, (n-1)m\}.$$

Proposition 2.1.14. *The homomorphism*

$$\otimes_* : \pi_1(\mathrm{U}(am)/\mu_m)\times \pi_1(\mathrm{U}(bn)/\mu_n) \longrightarrow \pi_1(\mathrm{U}(abmn)/\mu_{mn})$$

is given by

$$\otimes_*(\alpha + x, \beta + y) = (\alpha+\beta) + (bnx + amy)$$

for $\alpha \in \mathbb{Z}/m \subset \mathbb{Z}/mn$, $\beta \in \mathbb{Z}/n \subset \mathbb{Z}/mn$, and $x, y \in \mathbb{Z}$.

Proof. Since the determinant of a tensor product is the product of powers of the determinants, we define $\phi : S^1 \times S^1 \to S^1$ by $\phi(v, \omega) = v^{bn}\omega^{am}$ so that the diagram below is a map of fibrations.

$$
\begin{tikzcd}
\mathrm{SU}(am)/\mu_m \times \mathrm{SU}(bn)/\mu_n \arrow[r, hook, "i\times i"] \arrow[d, "\otimes"] & \mathrm{U}(am)/\mu_m \times \mathrm{U}(bn)/\mu_n \arrow[r, "\det\times\det"] \arrow[d, "\otimes"] & S^1 \times S^1 \arrow[d, "\phi"] \arrow[l, dotted, bend right, "t\times t"'] \\
\mathrm{SU}(abmn)/\mu_{mn} \arrow[r, hook, "i"] & \mathrm{U}(abmn)/\mu_{mn} \arrow[r, "\det"] & S^1. \arrow[l, dotted, bend left, "t"]
\end{tikzcd}
$$

This map of fibrations induces a homomorphism of short exact sequences

$$
\begin{tikzcd}
\pi_1(\mathrm{SU}(am)/\mu_m) \times \pi_1(\mathrm{SU}(bn)/\mu_n) \arrow[r, "\otimes_*"] \arrow[d, hook, "i_*\times i_*"] & \pi_1(\mathrm{SU}(abmn)/\mu_{mn}) \arrow[d, hook, "i_*"] \\
\pi_1(\mathrm{U}(am)/\mu_m) \times \pi_1(\mathrm{U}(bn)/\mu_n) \arrow[r, "\otimes_*"] \arrow[d, two heads] & \pi_1(\mathrm{U}(abmn)/\mu_{mn}) \arrow[d, two heads] \\
\pi_1(S^1) \times \pi_1(S^1) \arrow[r, "\phi_*"] \arrow[u, dotted, bend left, "t_*\times t_*"] & \pi_1(S^1). \arrow[u, dotted, bend right, "t_*"']
\end{tikzcd}
\tag{2.7}
$$

We want to determine the homomorphism $\otimes_*$ in the middle of diagram (2.7). In order to do this, we will determine the homomorphism $\otimes_*$ at the top of diagram (2.7), and show that the short exact sequences in diagram (2.7) split compatibly so that $\otimes_* : \pi_1(\mathrm{U}(am)/\mu_m) \times \pi_1(\mathrm{U}(bn)/\mu_n) \to \pi_1(\mathrm{U}(abmn)/\mu_{mn})$ is equal to

$$
\begin{tikzcd}
\Big(\pi_1(\mathrm{SU}(am)/\mu_m) \times \pi_1(\mathrm{SU}(bn)/\mu_n)\Big) \oplus \Big(\pi_1(S^1) \times \pi_1(S^1)\Big) \arrow[d, "\otimes_* \oplus \phi_*"] \\
\pi_1(\mathrm{SU}(abmn)/\mu_{mn}) \oplus \pi_1(S^1).
\end{tikzcd}
$$

We begin by observing that there exists a similar map of fibrations to the one in diagram (2.6), but with the spaces $\mathrm{SU}(am)$ and $\mathrm{SU}(bn)$ instead of $\mathrm{U}(am)$ and $\mathrm{U}(bn)$, respectively. In this case we obtain the commutative square,

$$\begin{tikzcd}
\pi_1(\mathrm{SU}(am)/\mu_m)\times\pi_1(\mathrm{SU}(bn)/\mu_n) \arrow[r, "\cong"] \arrow[d, "\otimes_*"] & \mathbb{Z}/m\times\mathbb{Z}/n \arrow[d, "\psi"] \\
\pi_1(\mathrm{SU}(abmn)/\mu_{mn}) \arrow[r, "\cong"] & \mathbb{Z}/mn,
\end{tikzcd}$$

where $\mathbb{Z}/m$ and $\mathbb{Z}/n$ are considered as subgroups of $\mathbb{Z}/mn$, and $\psi:\mathbb{Z}/m\times\mathbb{Z}/n\to\mathbb{Z}/mn$ is addition. From this, $\otimes_*:\pi_1(\mathrm{SU}(am)/\mu_m)\times\pi_1(\mathrm{SU}(bn)/\mu_n)\to\pi_1(\mathrm{SU}(abmn)/\mu_{mn})$ is equal to the addition.

In order to prove the compatibility, we observe that even though diagram (2.8) does not commute, Claim 2.1.15 shows that it is commutative up to a pointed homotopy. Therefore, the induced diagram on homotopy groups does commute

$$\begin{tikzcd}
\pi_1(S^1)\times\pi_1(S^1) \arrow[r, "t_*\times t_*"] \arrow[d, "\phi_*"] & \pi_1(\mathrm{U}(am)/\mu_m)\times\pi_1(\mathrm{U}(bn)/\mu_n) \arrow[d, "\otimes_*"] \\
\pi_1(S^1) \arrow[r, "t_*"] & \pi_1(\mathrm{U}(abmn)/\mu_{mn}).
\end{tikzcd}$$

Consequently, the diagram below commutes

$$\begin{tikzcd}
\pi_1(\mathrm{U}(am)/\mu_m)\times\pi_1(\mathrm{U}(bn)/\mu_n) \arrow[r, "\cong"] \arrow[d, "\otimes_*"] & \mathrm{Im}(i_*\times i_*)\oplus\mathrm{Im}(u_*\times u_*) \arrow[d, "\psi\oplus\phi_*"] \\
\pi_1(\mathrm{SU}(abmn)/\mu_{mn}) \arrow[r, "\cong"] & \mathrm{Im}\, i_*\oplus\mathrm{Im}\, u_*,
\end{tikzcd}$$

this is, $\otimes_*(\alpha+x,\beta+y)=\psi(\alpha,\beta)+\phi_*(x,y)$ for $\alpha+x\in\mathbb{Z}/m\oplus\mathbb{Z}$ and $\beta+y\in\mathbb{Z}/n\oplus\mathbb{Z}$. By the Eckmann-Hilton argument and Corollary 2.1.10, $\otimes_*(\alpha+x,\beta+y)=\psi(\alpha,\beta)+(bnx+amy)$. □

Claim 2.1.15. Diagram (2.8) commutes up to a pointed homotopy,

$$\begin{tikzcd}
S^1\times S^1 \arrow[r, "t\times t"] \arrow[d, "\phi"] & \mathrm{U}(am)/\mu_m\times\mathrm{U}(bn)/\mu_n \arrow[d, "\otimes"] \\
S^1 \arrow[r, "t"] & \mathrm{U}(abmn)/\mu_{mn}.
\end{tikzcd} \tag{2.8}$$

Proof. Let $j=1,\dots,abmn$, and consider the stabilization maps $\mathrm{s}_j:\mathrm{U}(1)\to$

$\mathrm{U}(abmn)$. Let $v, \omega \in S^1$, then

$$t(\phi(v,\omega)) = \begin{pmatrix} v^{bn}\omega^{am} & 0 \\ 0 & I_{abmn-1} \end{pmatrix} \text{ and } t(v) \otimes t(\omega) = \begin{pmatrix} vt(\omega) & 0 & \cdots & 0 \\ 0 & t(\omega) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t(\omega) \end{pmatrix}.$$

Observe that $t \circ \phi$ is equal to the composite

$$\begin{tikzcd} S^1 \times S^1 \arrow[d, "\Delta \times \Delta"] \\ (S^1)^{\times bn} \times (S^1)^{\times am} \arrow[d, "g"] \\ \mathrm{U}(abmn)^{\times bn} \times \mathrm{U}(abmn)^{\times am} \arrow[d, "\mathrm{mult}"] \\ \mathrm{U}(abmn) \end{tikzcd}$$

where $g = (\mathrm{s}_1 \times \cdots \times \mathrm{s}_1, \mathrm{s}_1 \times \cdots \times \mathrm{s}_1)$, and $t \otimes t$ is equal to

$$\begin{tikzcd} S^1 \times S^1 \arrow[d, "\Delta \times \Delta"] \\ (S^1)^{\times bn} \times (S^1)^{\times am} \arrow[d, "h"] \\ \mathrm{U}(abmn)^{\times bn} \times \mathrm{U}(abmn)^{\times am} \arrow[d, "\mathrm{mult}"] \\ \mathrm{U}(abmn) \end{tikzcd}$$

where $h = (\mathrm{s}_1 \times \mathrm{s}_2 \times \cdots \times \mathrm{s}_{bn}, \mathrm{s}_1 \times \mathrm{s}_{bn+1} \times \cdots \times \mathrm{s}_{(am-1)bn+1})$. By Lemma 2.1.2, $t \circ \phi$ and $t \otimes t$ are pointed homotopic. $\square$

2.2 Proof of Theorem 1.2.8

Let a, b, m and n be positive integers. By applying the classifying-space functor to the homomorphism (2.5), we obtain a map

$$F_{\otimes} : \mathrm{BU}(am)/\mu_m \times \mathrm{BU}(bn)/\mu_n \longrightarrow \mathrm{BU}(abmn)/\mu_{mn}. \tag{2.9}$$

If we take the quotient by μ_{am} and μ_{bn} in (2.9), we write $f_{\otimes}$ instead of $F_{\otimes}$.

Proposition 2.2.1. *Let a, b, m and n be positive integers such that am and bn are relatively prime and $am < bn$. There exist positive integers u and v satisfying $\left|vn(bn)^n - um(am)^m\right| = 1$, so that there exist a positive integer N and a homomorphism $\mathrm{T} : \mathrm{U}(am) \times \mathrm{U}(bn) \to \mathrm{U}(N)$ such that*

1. *the homomorphism T factors through $\widetilde{\mathrm{T}} : \mathrm{U}(am)/\mu_m \times \mathrm{U}(bn)/\mu_n \to \mathrm{U}(N)$, and*

2. *the homomorphisms induced on homotopy groups*

   $$\widetilde{\mathrm{T}}_i : \pi_i(\mathrm{U}(am)/\mu_m) \times \pi_i(\mathrm{U}(bn)/\mu_n) \longrightarrow \pi_i(\mathrm{U}(N))$$

   are given by

   $$\begin{cases} \widetilde{\mathrm{T}}_i(x, y) = um(am)^{m-1}x + vn(bn)^{n-1}y & \text{if } 1 < i < 2am, \\ \widetilde{\mathrm{T}}_i(\alpha + x, \beta + y) = um(am)^{m-1}x + vn(bn)^{n-1}y & \text{if } i = 1, \end{cases}$$

   where $\alpha \in \mathbb{Z}/m$, $\beta \in \mathbb{Z}/n$ and $x, y \in \mathbb{Z}$.

Proof. We first construct T.

Since am and bn are relatively prime, so are $m(am)^m$ and $n(bn)^n$. Hence there exist positive integers u and v such that $vn(bn)^n - um(am)^m = \pm 1$. Let

N denote $u(am)^m + v(bn)^n$ by Lemma 1.4.2. We define T as the composite

$$\begin{tikzcd}
\mathrm{U}(am) \times \mathrm{U}(bn) \arrow[d, "(\otimes^m, \otimes^n)"] \\
\mathrm{U}((am)^m) \times \mathrm{U}((bn)^n) \arrow[d, "(\oplus^u, \oplus^v)"] \\
\mathrm{U}(u(am)^m) \times \mathrm{U}(v(bn)^n) \arrow[d, "\oplus"] \\
\mathrm{U}(N).
\end{tikzcd}$$

1. We must show that $\mu_m \times \mu_n$ is contained in Ker(T). Let α and β be m-th and n-th roots of unity, respectively. Note that the element $\left(\alpha I_{am}, \beta I_{bn}\right)$ is sent to $\left(I_{u(am)^m}, I_{v(bn)^n}\right)$ by $(\otimes^m, \otimes^n)$, hence to the identity by the composite T defined above.

2. We first observe that Corollaries 2.1.5, 2.1.7 and 2.1.12 imply

$$\begin{aligned}
\mathrm{T}_i : \pi_i(\mathrm{U}(am)) \times \pi_i(\mathrm{U}(bn)) &\longrightarrow \pi_i(\mathrm{U}(N)) \\
(x, y) &\longmapsto um(am)^{m-1}x + vn(bn)^{n-1}y
\end{aligned}$$

for all $i < 2am$.

From part (1) there is a map of fibrations

$$\begin{tikzcd}
\mu_m \times \mu_n \arrow[r, hook] \arrow[d] & \mathrm{U}(am) \times \mathrm{U}(bn) \arrow[r] \arrow[d, "\mathrm{T}"] & \mathrm{U}(am)/\mu_m \times \mathrm{U}(bn)/\mu_n \arrow[d, "\widetilde{\mathrm{T}}"] \\
\{I_N\} \arrow[r] & \mathrm{U}(N) \arrow[r, equal] & \mathrm{U}(N).
\end{tikzcd}$$

Case 1. Let $i > 1$. From the long exact sequence, diagram (2.10) commutes.

$$\begin{tikzcd}
\pi_i(\mathrm{U}(am)) \times \pi_i(\mathrm{U}(bn)) \arrow[r, "\cong"] \arrow[d, "\mathrm{T}_i"] & \pi_i(\mathrm{U}(am)/\mu_m) \times \pi_i(\mathrm{U}(bn)/\mu_n) \arrow[d, "\widetilde{\mathrm{T}}_i"] \\
\pi_i(\mathrm{U}(N)) \arrow[r, equal] & \pi_i(\mathrm{U}(N))
\end{tikzcd} \tag{2.10}$$

Thus, $\widetilde{\mathrm{T}}_i(x, y) = \mathrm{T}_i(x, y) = um(am)^{m-1}x + vn(bn)^{n-1}y$ for $1 < i < 2m$.

Case 2. Let $i = 1$. From the long exact sequence there is a homomorphism of short exact sequences

$$\begin{tikzcd}
\pi_1(\mathrm{U}(am)) \times \pi_1(\mathrm{U}(bn)) \arrow[r, "\mathrm{T}_1"] \arrow[d, hook] & \pi_1(\mathrm{U}(N)) \arrow[d, equal] \\
\pi_1(\mathrm{U}(am)/\mu_m) \times \pi_1(\mathrm{U}(bn)/\mu_n) \arrow[r, "\widetilde{\mathrm{T}}_1"] \arrow[d, two heads] & \pi_1(\mathrm{U}(N)) \arrow[d] \\
\mathbb{Z}/m \times \mathbb{Z}/n \arrow[r] & 0.
\end{tikzcd}$$

The top short exact sequence splits. By direct inspection we obtain $\widetilde{\mathrm{T}}_1(\alpha + x, \beta + y) = \mathrm{T}_1(x, y) = um(am)^{m-1}x + vn(bn)^{n-1}y$.

$\square$

2.2.1 A $(2am+1)$-connected map

Let J be the map

$$J : \mathrm{BU}(am)/\mu_m \times \mathrm{BU}(bn)/\mu_n \longrightarrow \mathrm{BU}(abmn)/\mu_{mn} \times \mathrm{BU}(N)$$
$$(x, y) \longmapsto \left(F_{\otimes}(x, y), \mathrm{B}\widetilde{\mathrm{T}}(x, y)\right)$$

where the integer N is the one provided by Proposition 2.2.1.

Proposition 2.2.2. *Let a, b, m and n be positive integers such that am and bn are relatively prime and $am < bn$. The map J is $(2am+1)$-connected.*

Proof. We want to prove that the induced homomorphism on homotopy groups

$$\begin{array}{c} \pi_i(\mathrm{BU}(am)/\mu_m) \times \pi_i(\mathrm{BU}(bn)/\mu_n) \\ \Big\downarrow J_i \\ \pi_i(\mathrm{BU}(abmn)/\mu_{mn}) \times \pi_i(\mathrm{BU}(N)) \end{array} \tag{2.11}$$

is an isomorphism for all $i < 2am + 1$ and an epimorphism for $i = 2am + 1$.

Observe that the homotopy groups of the spaces involved are trivial in odd degrees below $2am+2$, hence it suffices to prove that J_i is an isomorphism for all i even and $i < 2am+1$.

We divide the proof into two cases.

Case 1. Let $i < 2am+1$ and $i \neq 2$. For this case computations can be done at the level of the universal covers of the groups $\mathrm{U}(am)/\mu_m, \mathrm{U}(bn)/\mu_n$ and $\mathrm{U}(abmn)/\mu_{mn}$.

The homomorphism (2.11) takes the form $J_i : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$. Propositions 2.1.14 and 2.2.1 yield

$$J_i(x,y) = \left(bnx + amy, um(am)^{m-1}x + vn(bn)^{n-1}y\right).$$

Thereby, the homomorphism (2.11) is represented by the matrix

$$\begin{pmatrix} bn & am \\ m(am)^{m-1}u & n(bn)^{n-1}v \end{pmatrix},$$

which is invertible. This proves J_i is an isomorphism.

Case 2. Let $i = 2$. The homomorphism (2.11) takes the form

$$J_2 : (\mathbb{Z}/m \oplus \mathbb{Z}) \times (\mathbb{Z}/n \oplus \mathbb{Z}) \longrightarrow (\mathbb{Z}/mn \oplus \mathbb{Z}) \times \mathbb{Z}.$$

Propositions 2.1.14 and 2.2.1 yield

$$J_2(x+\alpha, y+\beta) = \left((\alpha+\beta) + (bnx + amy), um(am)^{m-1}x + vn(bn)^{n-1}y\right).$$

Recall that $\psi : \mathbb{Z}/m \times \mathbb{Z}/n \to \mathbb{Z}/mn$ is addition where $\mathbb{Z}/m$ and $\mathbb{Z}/n$ are considered as subgroups of $\mathbb{Z}/mn$, see proof of Proposition 2.1.14. The homomorphism ψ is an isomorphism. From this and the invertibility of the matrix above, J_2 is an isomorphism. $\square$

2.2.2 Factorization through $F_\otimes : \mathrm{BU}(am,\mathbb{C})/\mu_m \times \mathrm{BU}(bn,\mathbb{C})/\mu_n \to \mathrm{BU}(abmn,\mathbb{C})/\mu_{mn}$

Theorem 2.2.3. *Let a, b, m and n be positive integers such that am and bn are relatively prime and $am < bn$. Let X be a topological space with the homotopy type of a finite dimensional CW complex such that* $\dim(X) \leq 2am+1$.

Every map $\mathscr{A} : X \to \mathrm{BU}(abmn)/\mu_{mn}$ can be lifted to the space $\mathrm{BU}(am)/\mu_m \times \mathrm{BU}(bn)/\mu_n$ along the map $F_\otimes$ up to a pointed homotopy.

Proof. Diagrammatically speaking, we want to find a map

$$\mathscr{A}_m \times \mathscr{A}_n : X \longrightarrow \mathrm{BU}(am)/\mu_m \times \mathrm{BU}(bn)/\mu_n$$

such that diagram (2.12) commutes up to homotopy.

$$\begin{tikzcd} & \mathrm{BU}(am)/\mu_m \times \mathrm{BU}(bn)/\mu_n \arrow[d, "F_\otimes"] \\ X \arrow[ur, dotted, bend left, "\mathscr{A}_m \times \mathscr{A}_n"] \arrow[r, "\mathscr{A}"] & \mathrm{BU}(abmn)/\mu_{mn} \end{tikzcd} \tag{2.12}$$

Proposition 2.2.2 yields a map

$$J : \mathrm{BU}(am)/\mu_m \times \mathrm{BU}(bn)/\mu_n \longrightarrow \mathrm{BU}(abmn)/\mu_{mn} \times \mathrm{BU}(N)$$

where N is some positive integer so that $N \gg bn > am$.

Observe that $F_\otimes$ factors through $\mathrm{BU}(abmn)/\mu_{mn} \times \mathrm{BU}(N)$, so we can write $F_\otimes$ as the composite of J and the projection proj_1 shown in diagram (2.13).

$$\begin{tikzcd} \mathrm{BU}(am)/\mu_m \times \mathrm{BU}(bn)/\mu_n \arrow[r, "J"] \arrow[d, "F_\otimes"] & \mathrm{BU}(abmn)/\mu_{mn} \times \mathrm{BU}(N) \arrow[dl, "\mathrm{proj}_1"] \\ \mathrm{BU}(abmn)/\mu_{mn} & \end{tikzcd} \tag{2.13}$$

Since J is $(2am+1)$-connected and $\dim(X) \leq 2am+1$, then by Whitehead's theorem

$$J_{\#} : \left[X, \mathrm{BU}(am)/\mu_m \times \mathrm{BU}(bn)/\mu_n\right] \longrightarrow \left[X, \mathrm{BU}(abmn)/\mu_{mn} \times \mathrm{BU}(N)\right]$$

is a surjection [25, Corollary 7.6.23].

Let s denote a section of proj_1. The surjectivity of $J_{\#}$ implies $s \circ \mathscr{A}$ has a preimage $\mathscr{A}_m \times \mathscr{A}_n : X \to \mathrm{BU}(am)/\mu_m \times \mathrm{BU}(bn)/\mu_n$ such that $J \circ (\mathscr{A}_m \times \mathscr{A}_n) \simeq s \circ \mathscr{A}$.

The commutativity of diagram (2.12) follows from commutativity of diagram (2.13). Thus, the result follows. □

2.2.3 Factorization through $f_{\otimes} : \mathrm{BPU}(am, \mathbb{C}) \times \mathrm{BPU}(bn, \mathbb{C}) \to \mathrm{BPU}(abmn, \mathbb{C})$

Proposition 2.2.4. *Let X be a finite CW complex. Let $\alpha \in \mathrm{Br}(X)$ be a class of period m. There exists a map $\mathscr{A}' : X \to \mathrm{BU}_{am}/\mu_m$ such that $\mathrm{cl}(\mathscr{A}') = \alpha$ if and only if α is represented by $\mathscr{A}$ a topological Azumaya algebra of degree am over X.*

Proof. Since $\alpha \in \mathrm{H}^3(X;\mathbb{Z})_{\mathrm{tors}}$ is m-torsion, there exists a class $\xi \in \mathrm{H}^2(X;\mathbb{Z}/m)$ such that $\tilde{\beta}_m(\xi) = \alpha$. Diagrammatically, there exists a lifting $\xi : X \to \mathrm{K}(\mathbb{Z}/m, 2)$ such that $\tilde{\beta}_m \circ \xi$ is pointed homotopic to α, see diagram (2.14).

$$\begin{tikzcd}
& X \arrow[dl, dotted, bend right, "\xi"'] \arrow[d, "\alpha"] & \\
\mathrm{K}(\mathbb{Z}/m, 2) \arrow[r, "\tilde{\beta}_m"] & \mathrm{K}(\mathbb{Z}, 3) \arrow[r, "\times m"] & \mathrm{K}(\mathbb{Z}, 3)
\end{tikzcd} \tag{2.14}$$

The map of fibrations below

$$\begin{tikzcd}
\mu_m \arrow[r, hook] \arrow[d, hook] & \mathrm{U}(am) \arrow[r, hook] \arrow[d, equal] & \mathrm{U}(am)/\mu_m \arrow[d, "q"] \\
S^1 \arrow[r, hook] & \mathrm{U}(am) \arrow[r] & \mathrm{PU}(am)
\end{tikzcd}$$

induces a commutative diagram

$$\begin{tikzcd}
\mathrm{BU}(am)/\mu_m \arrow[r, "\mathrm{B}q"] \arrow[d] & \mathrm{BPU}(am) \arrow[d] \\
\mathrm{K}(\mathbb{Z}/m,2) \arrow[r, "\tilde{\beta}_m"] & \mathrm{K}(\mathbb{Z},3).
\end{tikzcd}$$

In order to prove the proposition, we show that there exists a lifting $\mathscr{A}'$ of ξ if and only if there exists a lifting $\mathscr{A}$ of ξ, see diagram (2.15) below.

$$\begin{tikzcd}
 & \mathrm{BU}(am)/\mu_m \arrow[rr, "\mathrm{B}q"] \arrow[d] & & \mathrm{BPU}(am) \arrow[d] \\
 & \mathrm{K}(\mathbb{Z}/m,2) \arrow[rr] \arrow[dr, "\tilde{\beta}_m"'] & & \mathrm{K}(\mathbb{Z}/am,2) \arrow[dl, "\tilde{\beta}_{am}"] \\
X \arrow[ur, "\xi"] \arrow[uur, dotted, bend left, "\mathscr{A}'"] \arrow[uurrr, dotted, "\mathscr{A}"] \arrow[rr, "\alpha"'] & & \mathrm{K}(\mathbb{Z},3) &
\end{tikzcd} \tag{2.15}$$

If there exists a lifting $\mathscr{A}': X \to \mathrm{BU}(am)/\mu_m$, then the composite $\mathrm{B}q \circ \mathscr{A}'$ is a topological Azumaya algebra of degree am that represents the Brauer class α.

Conversely, suppose there exists an Azumaya algebra $\mathscr{A}$ of degree am making the outer square in the diagram below commute up to homotopy.

$$\begin{tikzcd}
X \arrow[dr, dotted, "\mathscr{A}'"] \arrow[drr, bend left, "\mathscr{A}"] \arrow[ddr, bend right, "\xi'"'] & & \\
 & \mathrm{BU}(am)/\mu_m \arrow[r, "\mathrm{B}q"] \arrow[d] & \mathrm{BPU}(am) \arrow[d] \\
 & \mathrm{K}(\mathbb{Z}/m,2) \arrow[r, "\tilde{\beta}_m"] & \mathrm{K}(\mathbb{Z},3)
\end{tikzcd}$$

In the inner square, the induced map on the homotopy fibers of $\mathrm{B}q$ and $\tilde{\beta}_m$ is a homotopy equivalence. An application of the 5-lemma implies that the inner square is a homotopy pullback square. Therefore, there exists a lifting $\mathscr{A}'$ representing α. $\square$

Theorem 2.2.5. *Let a, b, m and n be positive integers such that am and bn are relatively prime and $am < bn$. Let X be a CW complex such that* $\dim(X) \leq 2am+1$.

If $\mathscr{A}$ is a topological Azumaya algebra of degree $abmn$ such that $\mathrm{cl}(\mathscr{A})$ *has period mn, then there exist topological Azumaya algebras $\mathscr{A}_m$ and $\mathscr{A}_n$ of degrees am and bn, respectively, such that* $\mathrm{per}(\mathrm{cl}(\mathscr{A}_m)) = m$, $\mathrm{per}(\mathrm{cl}(\mathscr{A}_n)) = n$ *and $\mathscr{A} \cong \mathscr{A}_m \otimes \mathscr{A}_n$.*

Proof. In this case we want to solve the lifting problem shown in diagram (2.16) up to homotopy, with $\mathrm{per}(\mathrm{cl}(\mathscr{A}_m)) = m$, $\mathrm{per}(\mathrm{cl}(\mathscr{A}_n)) = n$.

$$\begin{tikzcd} & \mathrm{BPU}(am)\times \mathrm{BPU}(bn) \arrow[d, "f_\otimes"] \\ X \arrow[ur, dotted, bend left, "\mathscr{A}_m\times\mathscr{A}_n"] \arrow[r, "\mathscr{A}"] & \mathrm{BPU}(abmn) \end{tikzcd} \tag{2.16}$$

By Proposition 2.2.4 there exists a map $\mathscr{A}' : X \to \mathrm{BU}(abmn)/\mu_{mn}$ such that $\mathrm{per}(\mathrm{cl}(\mathscr{A}')) = \mathrm{per}(\mathrm{cl}(\mathscr{A})) = mn$. Then, by Theorem 2.2.3 there exists a map $\mathscr{A}'_m \times \mathscr{A}'_n : X \to \mathrm{BU}(am)/\mu_m \times \mathrm{BU}(bn)/\mu_n$ such that $F_\otimes \circ (\mathscr{A}'_m \times \mathscr{A}'_n) \simeq \mathscr{A}'$.

Since $\mathrm{cl}(\mathscr{A}'_m)\mathrm{cl}(\mathscr{A}'_n) = \mathrm{cl}(\mathscr{A}'_{mn})$, m and n are relatively prime, and the period of $\mathrm{cl}(\mathscr{A}')$ is mn then $\mathrm{per}(\mathrm{cl}(\mathscr{A}'_m)) = m$ and $\mathrm{per}(\mathrm{cl}(\mathscr{A}'_n)) = n$.

By Proposition 2.2.4 there exists a map

$$\mathscr{A}_m \times \mathscr{A}_n : X \longrightarrow \mathrm{BPU}(am)\times \mathrm{BPU}(bn)$$

such that $\mathrm{per}(\mathrm{cl}(\mathscr{A}_m)) = \mathrm{per}(\mathrm{cl}(\mathscr{A}'_m))$ and $\mathrm{per}(\mathrm{cl}(\mathscr{A}_n)) = \mathrm{per}(\mathrm{cl}(\mathscr{A}'_n))$.

It remains to show that diagram (2.16) commutes. Consider the diagram below

$$\begin{tikzcd} \mathrm{BU}(am)/\mu_m \times \mathrm{BU}(bn)/\mu_n \arrow[rr, "\mathrm{B}q\times\mathrm{B}q"] \arrow[dd, "F_\otimes"] & & \mathrm{BPU}(am)\times\mathrm{BPU}(bn) \arrow[dd, "f_\otimes"] \\ & X \arrow[ul, "\mathscr{A}'_m\times\mathscr{A}'_n"'] \arrow[ur, "\mathscr{A}_m\times\mathscr{A}_n"] \arrow[dl, "\mathscr{A}'"] \arrow[dr, "\mathscr{A}"'] & \\ \mathrm{BU}(abmn)/\mu_{mn} \arrow[rr] & & \mathrm{BPU}(abmn) \end{tikzcd} \tag{2.17}$$

Observe that the square, as well as top, bottom and left triangles, of diagram (2.17) commute. Hence, the right triangle commutes. $\square$

Theorem 1.2.8 is a corollary of Theorem 2.2.5.

Theorem 2.2.6. *Let a, b, m and n be positive integers such that am and bn are relatively prime and $am < bn$. The map $F_\otimes : \mathrm{BU}(am)/\mu_m \times \mathrm{BU}(bn)/\mu_n \to \mathrm{BU}(abmn)/\mu_{mn}$ does not have any section.*

Proof. Suppose there exists a section σ of $F_\otimes$.

By Proposition 2.1.14 the map $F_\otimes$ induces a homomorphism on homotopy groups which is given by $(x, y) \mapsto bn\,\mathrm{stab}(x) + am\,\mathrm{stab}(y)$ for $i > 2$. In degree $2am+2$ the homomorphism $(F_\otimes)_*$ takes the form $(F_\otimes)_* : \pi_{2am+2}(\mathrm{BU}(am)/\mu_m)\times \mathbb{Z} \to \mathbb{Z}$, where

$$\pi_{2am+2}(\mathrm{BU}(am)/\mu_m) \cong \pi_{2am+2}(\mathrm{BU}(am))$$

and $\pi_{2am+2}(\mathrm{BU}(am))$ is trivial when am is odd, and $\mathbb{Z}/2$ when am is even, see [20, Page 971]. Therefore, $(F_\otimes)_*(x, y) = am\,\mathrm{stab}(y)$. Thus $\mathrm{Im}(F_\otimes)_* = am\mathbb{Z}$.

On the other side, since σ is a section of $F_\otimes$, the composite

$$\begin{tikzcd}
\pi_{2am+2}(\mathrm{BU}(abmn)/\mu_{mn}) \arrow[d, "\sigma_*"] \\
\pi_{2am+2}(\mathrm{BU}(am)/\mu_m) \times \pi_{2am+2}(\mathrm{BU}(bn)/\mu_n) \arrow[d, "(F_\otimes)_*"] \\
\pi_{2am+2}(\mathrm{BU}(abmn)/\mu_{mn})
\end{tikzcd}$$

is the identity. This contradicts the fact that $\mathrm{Im}((F_\otimes)_* \circ \sigma_*) \subset am\mathbb{Z}$. $\square$

Remark 2.2.7. Under the hypotheses of Theorem 1.2.8, the topological Azumaya algebras $\mathscr{A}_m$ and $\mathscr{A}_n$ are not necessarily unique up to isomorphism. In order to see this, we consider the Moore-Postnikov tower for $f_\otimes$:

$$\begin{tikzcd}
 & \mathrm{BPU}(m)\times \mathrm{BPU}(n) \arrow[d] & \\
 & \vdots \arrow[d] & \\
\mathrm{K}(\pi_2 F, 2) \arrow[r] & Y[2] \arrow[d] & \\
\mathrm{K}(\pi_1 F, 1) \arrow[r] & Y[1] \arrow[d] \arrow[r, "k_1"] & \mathrm{K}(\pi_2 F, 3) \\
 & \mathrm{BPU}(mn) \arrow[r, "k_0"] & \mathrm{K}(\pi_1 F, 2),
\end{tikzcd}$$

where F is the homotopy fiber of $f_\otimes$, and $k_{i-1}: Y[i-1] \to \mathrm{K}(\pi_i F, i+1)$ is the k-invariant that classifies the fiber sequence $Y[i] \to Y[i-1]$, for $i > 0$, [17, Theorem 4.71].

Since the map $f_\otimes$ induces an isomorphism on π_2, and π_{2i+1} for $0 < i < m$, it follows that $\mathrm{BPU}(mn) \simeq Y[i]$ for $i = 1,2,3$, and $Y[2i] \simeq Y[2i+1]$ for $1 < i < m$.

The long exact sequence of $F \to \mathrm{BPU}(m) \times \mathrm{BPU}(n) \to \mathrm{BPU}(mn)$ yields

$$\pi_i F \cong \begin{cases} 0 & \text{if } i = 2 \text{ or } i \text{ is odd and } i < 2m+1, \\ \mathbb{Z} & \text{if } i \neq 2,\ i \text{ is even and } i < 2m+1. \end{cases}$$

Hence the Moore-Postnikov tower of $f_\otimes$ takes the form

$$\begin{tikzcd}
 & \mathrm{BPU}(m)\times \mathrm{BPU}(n) \arrow[d] & \\
 & \vdots \arrow[d] & \\
\mathrm{K}(\mathbb{Z}, 6) \arrow[r] & Y[6] \arrow[d] & \\
\mathrm{K}(\mathbb{Z}, 4) \arrow[r] & Y[4] \arrow[d] \arrow[r, "k_5"] & \mathrm{K}(\mathbb{Z}, 7) \\
 & \mathrm{BPU}(mn) \arrow[r, "k_3"] & \mathrm{K}(\mathbb{Z}, 5).
\end{tikzcd}$$

Let X be a CW complex of $\dim(X) \leq 6$. Let m and n be as in the hypothesis of Theorem 1.2.8, and $m > 3$. Let $\mathscr{A}$ be a topological Azumaya algebra of degree mn.

Observe that the k-invariant k_3 is trivial because $\mathrm{H}^5(\mathrm{BPU}(mn);\mathbb{Z})$ is trivial, [1, Proposition 4.1]. Hence there is no obstruction to lifting $\mathscr{A}$ to $Y[4]$. Similarly, we can lift the identity map $\mathrm{id}_{\mathrm{BPU}(mn)}$ to $Y[4]$, in this case we obtain the splitting $Y[4] \simeq \mathrm{BPU}(mn) \times \mathrm{K}(\mathbb{Z},4)$. Then the lifting of $\mathscr{A}$ takes the form $(\mathscr{A},\xi): X \to \mathrm{BPU}(mn) \times \mathrm{K}(\mathbb{Z},4)$.

The cohomology groups of X vanish for all degrees greater than 6, given that X is 6-dimensional. Thus $(\mathscr{A},\xi)$ can be lifted up the Moore-Postnikov tower to $\mathrm{BPU}(m) \times \mathrm{BPU}(n)$. See diagram (2.18).

$$\begin{tikzcd}
 & \mathrm{BPU}(m) \times \mathrm{BPU}(n) \arrow[d] & \\
 & \vdots \arrow[d] & \\
 & Y[4] \simeq \mathrm{BPU}(mn) \times \mathrm{K}(\mathbb{Z},4) \arrow[r, "k_5"] \arrow[d] & \mathrm{K}(\mathbb{Z},7) \\
X \arrow[ruuu, bend left, "\mathscr{A}_m \times \mathscr{A}_n"] \arrow[ru, bend left, "(\mathscr{A},\xi)"] \arrow[r, "\mathscr{A}"] & \mathrm{BPU}(mn) \arrow[r, "k_3"] & \mathrm{K}(\mathbb{Z},5)
\end{tikzcd} \tag{2.18}$$

This proves that $\mathscr{A}$ can be decomposed as $\mathscr{A}_m \otimes \mathscr{A}_n$. The lifting $(\mathscr{A},\xi)$ is not necessarily unique. In fact, every cohomology class $\xi \in \mathrm{H}^4(X;\mathbb{Z})$ gives rise to a lifting $(\mathscr{A},\xi)$.

Chapter 3

Decomposition of Topological Azumaya Algebras with Orthogonal Involution

This chapter is organized as follows. The first section presents preliminaries on the effect of direct sum and tensor product operations on homotopy groups of compact Lie groups related to the complex orthogonal groups $\mathrm{O}(n,\mathbb{C})$. The second section is devoted to the proof of Theorem 1.3.5.

The inclusion $\mathrm{O}(n,\mathbb{R}) \to \mathrm{O}(n,\mathbb{C})$ is an equivalence, [22, Corollary I.4.12 (3)]. From now on, we are going to drop the $\mathbb{C}$ that comes in the notation of $\mathrm{O}(n,\mathbb{C})$.

3.1 Stabilization of operations on the complex orthogonal group

We recall the homotopy groups of $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ in low degrees, and compute the homotopy groups of $\mathrm{PO}(n)$ and $\mathrm{PSO}(n)$ in low degrees.

When $n = 1$, then $\mathrm{O}(1) = S^0$, $\mathrm{SO}(1) = \{1\}$ and $\mathrm{PO}(1) = \mathrm{PSO}(1) = \{1\}$.

When $n = 2$, then $\mathrm{SO}(2) = \mathrm{U}_1 = S^1$, $\mathrm{PO}(2) \cong \mathrm{O}_2$ and $\mathrm{PSO}(2) \cong S^1$.

Let $n \geq 3$ and $i < n-1$, the first homotopy groups of the orthogonal group are given by Bott periodicity,

$$\pi_i(\mathrm{O}(n)) \cong \begin{cases} 0 & \text{if } i = 2,4,5,6 \pmod 8, \\ \mathbb{Z}/2 & \text{if } i = 0,1 \pmod 8, \\ \mathbb{Z} & \text{if } i = 3,7 \pmod 8. \end{cases}$$

The special orthogonal group is the connected component of the identity of $\mathrm{O}(n)$, then $\pi_0(\mathrm{SO}(n))$ is trivial. Moreover, for $n \geq 2$ there is a fiber sequence $\mathrm{SO}(n) \hookrightarrow \mathrm{O}(n) \xrightarrow{\det} \mathbb{Z}/2$ where $\mathbb{Z}/2$ has the discrete topology, then we can use the long exact sequence associated to it to see that $\pi_i(\mathrm{SO}(n)) \cong \pi_i(\mathrm{O}(n))$ for $i \geq 1$.

To calculate the homotopy groups of the projective orthogonal group and the projective special orthogonal group in low degrees when $n \geq 3$ consider the diagrams (3.1) and (3.2) below for $k \geq 2$.

$$\begin{tikzcd} \{\pm I_{2k}\} \arrow[r, equal] \arrow[d, hook] & \{\pm I_{2k}\} \arrow[r] \arrow[d, hook] & \{1\} \arrow[d] \\ \mathrm{SO}(2k) \arrow[r, hook] \arrow[d, two heads] & \mathrm{O}(2k) \arrow[r, two heads, "\det"] \arrow[d, two heads] & \{\pm 1\} \arrow[d, equal] \\ \mathrm{PSO}(2k) \arrow[r] & \mathrm{PO}(2k) \arrow[r, "\det"] & \{\pm 1\} \end{tikzcd} \tag{3.1}$$

$$\begin{tikzcd} \{I_{2k-1}\} \arrow[r, hook] \arrow[d, hook] & \{\pm I_{2k-1}\} \arrow[r, two heads] \arrow[d, hook] & \{\pm 1\} \arrow[d, equal] \\ \mathrm{SO}(2k-1) \arrow[r, hook] \arrow[d, two heads] & \mathrm{O}(2k-1) \arrow[r, two heads, "\det"] \arrow[d, two heads] & \{\pm 1\} \arrow[d] \\ \mathrm{PSO}(2k-1) \arrow[r] & \mathrm{PO}(2k-1) \arrow[r, "\det"] & \{1\} \end{tikzcd} \tag{3.2}$$

All columns as well as the two top rows of diagrams (3.1) and (3.2) are exact. The nine-lemma implies that the bottom rows in (3.1) and (3.2) are also exact. Therefore,

$$\begin{aligned} &\pi_i(\mathrm{PO}(2k)) \cong \pi_i(\mathrm{PSO}(2k)) \cong \pi_i(\mathrm{SO}(2k)) \quad \text{for all } i \geq 2, \text{ and} \\ &\mathrm{PO}(2k-1) \cong \mathrm{PSO}(2k-1) \cong \mathrm{SO}(2k-1) \end{aligned}$$

The only calculation left is the one of the fundamental group of $\mathrm{PSO}(2k)$ for $k \geq 2$. Let $n \geq 3$. By definition, the n-th spinor group $\mathrm{Spin}(n)$ is the universal double covering group of $\mathrm{SO}(n)$, [22, Page 74]. This group is also the universal covering of $\mathrm{PSO}(n)$, $\mathrm{Ker}(p) \to \mathrm{Spin}(n) \xrightarrow{p} \mathrm{PSO}(n)$, and

$$\pi_1(\mathrm{PSO}(n)) = \mathrm{Ker}(p) = Z(\mathrm{Spin}(n)).$$

From [22, Theorem II.4.4] the center of $\mathrm{Spin}(n)$ for $n \geq 3$ is given by

$$Z(\mathrm{Spin}(n)) = \begin{cases} \mathbb{Z}/2 & \text{if } n \text{ is odd,} \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \text{if } n \equiv 0 \pmod 4, \\ \mathbb{Z}/4 & \text{if } n \equiv 2 \pmod 4. \end{cases} \tag{3.3}$$

3.1.1 First unstable homotopy group of $\mathrm{O}(n, \mathbb{C})$

The standard inclusion of the orthogonal group $i : \mathrm{O}(n) \hookrightarrow \mathrm{O}(n+1)$ is $(n-1)$-connected, hence it induces an isomorphism on homotopy groups in degrees less than $n-1$ and an epimorphism in degree $n-1$. This can be seen by observing the long exact sequence for the fibration $\mathrm{O}(n) \hookrightarrow \mathrm{O}(n+1) \to \mathrm{O}(n+1)/\mathrm{O}(n) \simeq S^n$.

The first unstable homotopy group of $\mathrm{O}(n)$ happens in degree $n-1$. Consider the following segment of the long exact sequence

$$\pi_n(S^n) \xrightarrow{\partial} \pi_{n-1}(\mathrm{O}(n)) \xrightarrow{i_*} \pi_{n-1}(\mathrm{O}(n+1)) \longrightarrow 0.$$

By exactness of the sequence above, there is a short exact sequence

$$0 \longrightarrow \mathrm{Ker}\, i_* \hookrightarrow \pi_{n-1}(\mathrm{O}(n)) \xrightarrow{i_*} \pi_{n-1}(\mathrm{O}) \longrightarrow 0, \tag{3.4}$$

where $\mathrm{O} := \mathrm{colim}_{n \to \infty} \mathrm{O}(n)$ and $\pi_{n-1}(\mathrm{O}) \cong \pi_{n-1}(\mathrm{O}(n+1))$.

Let $n = 3, 7$. From [27, Corollary 10.6, Theorem 10.8], $\mathrm{Ker}\, i_*$ is trivial. Thus $\pi_{n-1}(\mathrm{O}(n)) \cong \pi_{n-1}(\mathrm{O})$, i.e. $\pi_{n-1}(\mathrm{O}(n))$ is trivial.

Let $n = 2$, then $\pi_1(\mathrm{O}(2)) = \pi_1(S^1) = \mathbb{Z}$.

Let $n \neq 1,2,3,7$. The short exact sequence (3.4) splits, [22, Corollary IV.6.14]. Therefore,

$$\pi_{n-1}(\mathrm{O}(n)) \cong \begin{cases} \mathbb{Z}\oplus\mathbb{Z} & \text{if } n \equiv 0,4 \pmod 8, \\ \mathbb{Z}/2\oplus\mathbb{Z}/2 & \text{if } n \equiv 1 \pmod 8, \\ \mathbb{Z}\oplus\mathbb{Z}/2 & \text{if } n \equiv 2 \pmod 8, \\ \mathbb{Z}/2 & \text{if } n \equiv 3,5,7 \pmod 8, \\ \mathbb{Z} & \text{if } n \equiv 6 \pmod 8. \end{cases} \tag{3.5}$$

Let $G \in \{\mathrm{O}(n),\mathrm{SO}(n),\mathrm{PO}(n),\mathrm{PSO}(n)\}$. Tables 3.1, 3.2, 3.3, and 3.4 summarize the previous results.

G	$\mathrm{O}(n)$	$\mathrm{SO}(n)$	$\mathrm{PO}(2k)$	$\mathrm{PSO}(2k)$
$\pi_0(G)$	$\mathbb{Z}/2$	$*$	$\mathbb{Z}/2$	$*$

Table 3.1: Connected components of compact Lie groups related to the complex orthogonal group

G	$\mathrm{O}(n)$	$\mathrm{SO}(n)$	$\mathrm{PO}(2k)$	$\mathrm{PSO}(2k)$
$\pi_1(G)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$Z(\mathrm{Spin}(2k))$	$Z(\mathrm{Spin}(2k))$

Table 3.2: Fundamental group of compact Lie groups related to the complex orthogonal group

$i > 1$ and $i \pmod 8$	0	1	2	3	4	5	6	7
$\pi_i(G)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

Table 3.3: Homotopy groups of compact Lie groups related to the complex orthogonal group for $i = 2,\ldots,n-2$.

n equals to								
$\pi_{n-1}(G)$			$\mathbb{Z}$	0	$\mathbb{Z}\oplus\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}$	0
$n>7$ and $n \pmod 8$	0	1	2	3	4	5	6	7
$\pi_{n-1}(G)$	$\mathbb{Z}\oplus\mathbb{Z}$	$\mathbb{Z}/2\oplus\mathbb{Z}/2$	$\mathbb{Z}/2\oplus\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}\oplus\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}$	$\mathbb{Z}/2$

Table 3.4: First unstable homotopy group compact Lie groups related to the complex orthogonal group

3.1.2 Stabilization

Let $m,n \in \mathbb{N}$ and $m \leq n$. Define the map $\mathrm{s}:\mathrm{O}(m) \to \mathrm{O}(m+n)$ by

$$\mathrm{s}(A) = \begin{pmatrix} A & 0 \\ 0 & I_n \end{pmatrix}.$$

Since the map s is equal to the composite of consecutive canonical inclusions, it follows that s is $(m-1)$-connected.

Notation 3.1.1. Let stab denote the homomorphisms the map s induces on homotopy groups. From now on, we will identify $\pi_i(\mathrm{O}(m))$ with $\pi_i(\mathrm{O}(m+n))$ for all $i < m-1$ through the isomorphism

$$\mathrm{stab}: \pi_i(\mathrm{O}(m)) \xrightarrow{\cong} \pi_i(\mathrm{O}(m+n)). \tag{3.6}$$

Lemma 3.1.2. *Let $n,r \in \mathbb{N}$. For all $j = 1,\dots,r$ define $\mathrm{s}_j:\mathrm{O}(n)\to\mathrm{O}(rn)$ by*

$$\mathrm{s}_j(A) = \mathrm{diag}(I_n,\dots,I_n,A,I_n,\dots,I_n),$$

where A is in the j-th position. Then the map s_j is pointed homotopic to s_{j+1} for $j = 1,\dots,r-1$.

Proof. Let P_j be the permutation matrix

$$P_j = \begin{pmatrix} I_{(j-1)n} & & & \\ & 0 & I_n & \\ & I_n & 0 & \\ & & & I_{(r-j-1)n} \end{pmatrix}.$$

We consider two cases according to the parity of n.

If n even, then $\det(P_j) = 1$. Moreover, P_j is such that $\mathrm{s}_{j+1}(A) = P_j \mathrm{s}_j(A) P_j$ for $j = 1, \dots, r-1$.

In the case n odd, $\det(P_j) = -1$. The matrices

$$W_d = \begin{pmatrix} I_{rn-2} & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix} \quad \text{and} \quad W_u = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & I_{rn-2} \end{pmatrix}$$

are such that $\det(W_d P_j) = \det(W_u P_j) = 1$. Moreover,

$$\mathrm{s}_2(A) = (W_d P_j)\mathrm{s}_1(A)(W_d P_j)^{-1}, \quad \text{and} \quad \mathrm{s}_{j+1}(A) = (W_u P_j)\mathrm{s}_j(A)(W_u P_j)^{-1}$$

for $j = 2, \dots, r-1$. By Lemma 1.4.1 the result follows. □

Notation 3.1.3. We call the s_j maps *stabilization maps*. As s_1 is equal to $\mathrm{s} : \mathrm{O}(n) \to \mathrm{O}(n + (r-1)n)$, it follows that s_j is $(n-1)$-connected for all $j = 1, \dots, r$. From Lemma 3.1.2 the homomorphisms induced on homotopy groups by the stabilization maps are equal, hence stab also denotes $\pi_i(\mathrm{s}_1) = \cdots = \pi_i(\mathrm{s}_r)$. Thus we identify $\pi_i(\mathrm{O}(n))$ with $\pi_i(\mathrm{O}(rn))$ for $i < n-1$ through stab. The identification allows one to introduce a slight abuse of notation, namely to identify x and $\mathrm{stab}(x)$ for $x \in \pi_i(\mathrm{O}(n))$ and $i < n-1$.

3.1.3 Operations on homotopy groups

We do not write proofs of some results in this section given that they are similar to the proofs of the results of Chapter 2.

Proposition 3.1.4. *Let $i \in \mathbb{N}$. The homomorphism $\oplus_* : \pi_i(\mathrm{O}(m)) \times \pi_i(\mathrm{O}(n)) \to \pi_i(\mathrm{O}(m+n))$ is equal to*

$$\oplus_*(x, y) = \mathrm{stab}(x) + \mathrm{stab}(y)$$

for $x \in \pi_i(\mathrm{O}(m))$ and $y \in \pi_i(\mathrm{O}(n))$.

Corollary 3.1.5. *If $m < n$ and $i < m-1$, then $\oplus_*(x, y) = x + y$ for $x \in \pi_i(\mathrm{O}(m))$ and $y \in \pi_i(\mathrm{O}(n))$.*

Proposition 3.1.6. *Let $i \in \mathbb{N}$. The homomorphism $\oplus^r_* : \pi_i(\mathrm{O}(n)) \to \pi_i(\mathrm{O}(rn))$ is equal to*

$$\oplus^r_*(x) = r\,\mathrm{stab}(x)$$

for $x \in \pi_i(\mathrm{O}(n))$.

Corollary 3.1.7. *If $i < n-1$, then $\oplus^r_*(x) = rx$ for $x \in \pi_i(\mathrm{O}(n))$.*

Lemma 3.1.8. *Let $L, R : \mathrm{O}(m) \to \mathrm{O}(mn)$ be the homomorphisms $L(A) = A \otimes I_n$ and $R(A) = I_n \otimes A$. There is a basepoint preserving homotopy H from L to R such that for all $t \in [0,1]$, $H(-,t)$ is a homomorphism.*

Proof. Let $A \in \mathrm{O}(m)$.

$$L(A) = \begin{pmatrix} a_{11}I_n & \cdots & a_{1m}I_n \\ \vdots & \ddots & \vdots \\ a_{m1}I_n & \cdots & a_{mm}I_n \end{pmatrix} \quad \text{and} \quad R(A) = \begin{pmatrix} A & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A \end{pmatrix} = A^{\oplus n}.$$

Let $P_{m,n}$ be the permutation matrix

$$\begin{aligned} P_{m,n} = [&e_1, e_{n+1}, e_{2n+1}, \dots, e_{(m-1)n+1}, e_2, e_{n+2}, e_{2n+2}, \dots, e_{(m-1)n+2}, \\ &\dots, \\ &e_{n-1}, e_{2n-1}, e_{3n-1}, \dots, e_{mn-1}, e_n, e_{2n}, e_{3n}, \dots, e_{(m-1)n}, e_{mn}] \end{aligned}$$

where e_i is the i-th standard basis vector of $\mathbb{C}^{mn}$ written as a column vector. Observe that $L(A) = P_{m,n}R(A)P^{-1}_{m,n}$.

If $\det(P_{m,n}) = 1$, the result follows from Lemma 1.4.1.

Suppose $\det(P_{m,n}) = -1$. Consider W_d and W_u from the proof of Lemma 3.1.2. Observe that $\det(P_{m,n}W_d) = \det(P_{m,n}W_u) = \det(W_d^{-1}W_u) = 1$, and $R(A) = \mathrm{s}_1(A)\mathrm{s}_2(A)\cdots\mathrm{s}_n(A)$. Then

$$\begin{aligned} L(A) &= P_{m,n}R(A)P^t_{m,n} \\ &= P_{m,n}\,\mathrm{s}_1(A)\mathrm{s}_2(A)\cdots\mathrm{s}_n(A)P^{-1}_{m,n} \\ &= P_{m,n}\Big(W_d\,\mathrm{s}_1(A)\cdots\mathrm{s}_{n-1}(A)W_d^{-1}\Big)\Big(W_u\,\mathrm{s}_n(A)W_u^{-1}\Big)P^{-1}_{m,n} \\ &= (P_{m,n}W_d)\,\mathrm{s}_1(A)\cdots\mathrm{s}_{n-1}(A)(P_{m,n}W_d)^{-1}(P_{m,n}W_u)\,\mathrm{s}_n(A)(P_{m,n}W_u)^{-1}. \end{aligned}$$

Applying Lemma 1.4.1 twice yields the result. □

Proposition 3.1.9. *Let $i \in \mathbb{N}$, the homomorphism $\otimes_* : \pi_i(\mathrm{O}(m)) \times \pi_i(\mathrm{O}(n)) \to \pi_i(\mathrm{O}(mn))$ is given by*

$$\otimes_*(x,y) = n\,\mathrm{stab}(x) + m\,\mathrm{stab}(y)$$

for $x \in \pi_i(\mathrm{O}(m))$ and $y \in \pi_i(\mathrm{O}(n))$.

Corollary 3.1.10. *If $m < n$ and $i < m-1$, then $\otimes_*(x,y) = nx + my$ for $x \in \pi_i(\mathrm{O}(m))$ and $y \in \pi_i(\mathrm{O}(n))$.*

Remark 3.1.11. Under the hypothesis of Corollary 3.1.10, the homomorphism

$$\otimes_* : \pi_{m-1}(\mathrm{O}(m)) \times \pi_{m-1}(\mathrm{O}(n)) \to \pi_{m-1}(\mathrm{O}(mn))$$

is given by $\otimes_*(x,y) = n\,\mathrm{stab}(x) + m\,\mathrm{stab}(y) = n\,\mathrm{stab}(x) + my$. This can be seen by observing that $\otimes_*$ is equal to the sum of the two paths around diagram (3.7).

$$\begin{tikzcd}
 & \pi_{m-1}(\mathrm{O}(m)) \times \pi_{m-1}(\mathrm{O}(n)) \arrow[dl, two heads, "\mathrm{proj}_1"'] \arrow[dr, two heads, "\mathrm{proj}_2"] & \\
\pi_{m-1}(\mathrm{O}(m)) \arrow[d, two heads, "\mathrm{stab}"] & & \pi_{m-1}(\mathrm{O}(n)) \arrow[d, "\cong"', "\mathrm{stab}"] \\
\pi_{m-1}(\mathrm{O}(mn)) \arrow[dr, "\times n"'] & & \pi_{m-1}(\mathrm{O}(mn)) \arrow[dl, "\times m"] \\
 & \pi_{m-1}(\mathrm{O}(mn)) \times \pi_{m-1}(\mathrm{O}(mn)) \arrow[d, "+"] & \\
 & \pi_{m-1}(\mathrm{O}(mn)) &
\end{tikzcd} \tag{3.7}$$

Proposition 3.1.12. *Let $i \in \mathbb{N}$. The homomorphism $\otimes_*^r : \pi_i(\mathrm{O}(n)) \to \pi_i(\mathrm{O}(n^r))$*

is given by

$$\otimes_*^r(x) = rn^{r-1}\operatorname{stab}(x)$$

for $x \in \pi_i(\mathrm{O}(n))$.

Corollary 3.1.13. *If* $i < n-1$, *the map* $\otimes_*^r(x) = rn^{r-1}x$ *for* $x \in \pi_i(\mathrm{O}(n))$.

3.1.4 Tensor product on the quotient

We want to describe the effect of the tensor product operation on the homotopy groups of the projective complex orthogonal group.

The methods we used in Chapter 2 to establish the decomposition of topological Azumaya algebras apply to those whose degrees are relatively prime. For this reason, we study the tensor product

$$\otimes : \mathrm{PO}(m) \times \mathrm{PO}(n) \longrightarrow \mathrm{PO}(mn) \tag{3.8}$$

in two cases: when m and n are odd, and when m is even and n is odd.

Case m and n odd

Let m and n be positive integers such that m and n are odd. Since $\mathrm{PO}(m) \times \mathrm{PO}(n) = \mathrm{SO}(m) \times \mathrm{SO}(n)$, the tensor product in (3.8) may be written as

$$\otimes : \mathrm{SO}(m) \times \mathrm{SO}(n) \longrightarrow \mathrm{SO}(mn). \tag{3.9}$$

Proposition 3.1.14. *Let* $i \in \mathbb{N}$. *The homomorphism*

$$\otimes_* : \pi_i(\mathrm{SO}(m)) \times \pi_i(\mathrm{SO}(n)) \longrightarrow \pi_i(\mathrm{SO}(mn))$$

is given by

$$\otimes_*(x, y) = n\operatorname{stab}(x) + m\operatorname{stab}(y)$$

for $x \in \pi_i(\mathrm{SO}(m))$ *and* $y \in \pi_i(\mathrm{SO}(n))$.

Case m even and n odd

Let m and n be positive integers such that m is even and n is odd. The tensor product operation $\otimes : \mathrm{O}(m)\times\mathrm{SO}(n)\to\mathrm{O}(mn)$ sends the center of $\mathrm{O}(m)\times\mathrm{SO}(n)$ to the center of $\mathrm{O}(mn)$. As a consequence, the operation descends to the quotient

$$\otimes : \mathrm{PO}(m)\times\mathrm{SO}(n) \longrightarrow \mathrm{PO}(mn). \tag{3.10}$$

Proposition 3.1.15. *Let $i>1$. The homomorphism*

$$\otimes_* : \pi_i(\mathrm{PO}(m))\times\pi_i(\mathrm{SO}(n)) \longrightarrow \pi_i(\mathrm{PO}(mn))$$

is given by

$$\otimes_*(x,y) = n\,\mathrm{stab}(x) + m\,\mathrm{stab}(y)$$

for $x\in\pi_i(\mathrm{PO}(m))$ and $y\in\pi_i(\mathrm{SO}(n))$.

Proof. There is a map of fibrations

$$\begin{tikzcd} Z(\mathrm{O}(m))\times\{I_n\} \arrow[r, hook] \arrow[d, "\otimes"] & \mathrm{O}(m)\times\mathrm{SO}(n) \arrow[r, two heads] \arrow[d, "\otimes"] & \mathrm{PO}(m)\times\mathrm{SO}(n) \arrow[d, "\otimes"] \\ Z(\mathrm{O}(mn)) \arrow[r, hook] & \mathrm{O}(mn) \arrow[r, two heads] & \mathrm{PO}(mn). \end{tikzcd} \tag{3.11}$$

From the homomorphism between the long exact sequences associated to the fibrations in diagram (3.11), we obtain a commutative square

$$\begin{tikzcd} \pi_i(\mathrm{O}(m))\times\pi_i(\mathrm{SO}(n)) \arrow[r, "\cong"] \arrow[d, "\otimes_*"] & \pi_i(\mathrm{PO}(m))\times\pi_i(\mathrm{SO}(n)) \arrow[d, "\otimes_*"] \\ \pi_i(\mathrm{O}(mn)) \arrow[r, "\cong"] & \pi_i(\mathrm{PO}(mn)), \end{tikzcd}$$

for $i>1$. From this diagram and Proposition 3.1.12 we have that for all $i>1$, $\otimes_*(x,y) = n\,\mathrm{stab}(x)+m\,\mathrm{stab}(y)$ for $x\in\pi_i(\mathrm{PO}(m))$ and $y\in\pi_i(\mathrm{SO}(n))$. ☐

Proposition 3.1.16. *The homomorphism*

$$\otimes_* : \pi_1(\mathrm{PO}(m))\times\pi_1(\mathrm{SO}(n)) \longrightarrow \pi_1(\mathrm{PO}(mn))$$

is given by the following expressions.

1. *If $m \equiv 0$ (mod 4) and n is odd, then $\otimes_*(x,\beta) = x$ for $x \in \mathbb{Z}/2 \oplus \mathbb{Z}/2$ and $\beta \in \mathbb{Z}/2$.*

2. *If $m \equiv 2$ (mod 4) and n is odd, then $\otimes_*(\alpha,\beta) = n\alpha$ for $\alpha \in \mathbb{Z}/4$ and $\beta \in \mathbb{Z}/2$.*

Proof. From the long exact sequence associated to diagram (3.11) there is a diagram of exact sequences

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\pi_1(\mathrm{O}(m)) \times \pi_1(\mathrm{SO}(n)) \arrow[r, "\otimes_*"] \arrow[d] & \pi_1(\mathrm{O}(mn)) \arrow[d] \\
\pi_1(\mathrm{PO}(m)) \times \pi_1(\mathrm{SO}(n)) \arrow[r, "\otimes_*"] \arrow[d] & \pi_1(\mathrm{PO}(mn)) \arrow[d] \\
\pi_0 Z(\mathrm{O}(m)) \arrow[r, equal] \arrow[d] & \pi_0 Z(\mathrm{O}(mn)) \arrow[d] \\
\pi_0(\mathrm{O}(m)) \arrow[r, "\otimes_*"] \arrow[d] & \pi_0(\mathrm{O}(mn)) \arrow[d] \\
\pi_0(\mathrm{PO}(m)) \arrow[r, "\otimes_*"] & \pi_0(\mathrm{PO}(mn)).
\end{tikzcd} \tag{3.12}$$

Since $Z(\mathrm{O}(m))$ is contained in the connected component of the identity of $\mathrm{O}(m)$, then the homomorphism $\pi_0 Z(\mathrm{O}(m)) \to \pi_0(\mathrm{O}(m))$ is trivial. Hence we have a diagram of short exact sequences

$$\begin{tikzcd}
\pi_1(\mathrm{O}(m)) \times \pi_1(\mathrm{SO}(n)) \arrow[r, hook] \arrow[d, "\otimes_*"] & \pi_1(\mathrm{PO}(m)) \times \pi_1(\mathrm{SO}(n)) \arrow[r, two heads] \arrow[d, "\otimes_*"] & \pi_0 Z(\mathrm{O}(m)) \arrow[d, equal] \\
\pi_1(\mathrm{O}(mn)) \arrow[r, hook] & \pi_1(\mathrm{PO}(mn)) \arrow[r, two heads] & \pi_0 Z(\mathrm{O}(mn)).
\end{tikzcd} \tag{3.13}$$

By Corollary 3.1.10, and the parities of m and n, the homomorphism $\otimes_* : \pi_1(\mathrm{O}(m)) \times \pi_1(\mathrm{SO}(n)) \to \pi_1(\mathrm{O}(mn))$ is the projection onto the first coordinate. Diagram (3.13) becomes

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}/2 \times \mathbb{Z}/2 \arrow[r] \arrow[d, "\mathrm{proj}_1"] & Z(\mathrm{Spin}(m)) \times \mathbb{Z}/2 \arrow[r] \arrow[d, "\otimes_*"] & \mathbb{Z}/2 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbb{Z}/2 \arrow[r] & Z(\mathrm{Spin}(mn)) \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0.
\end{tikzcd} \tag{3.14}$$

Part 1. If $m = 4k$, then diagram (3.14) takes the form

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}/2 \times \mathbb{Z}/2 \arrow[r, "i\times i"] \arrow[d, "\mathrm{proj}_1"] & (\mathbb{Z}/2 \oplus \mathbb{Z}/2) \times \mathbb{Z}/2 \arrow[r, "\varphi"] \arrow[d, "\otimes_*"] & \mathbb{Z}/2 \arrow[r] \arrow[d, equal] \arrow[l, bend right, dotted, "s"'] & 0 \\
0 \arrow[r] & \mathbb{Z}/2 \arrow[r, "i"] & \mathbb{Z}/2 \oplus \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0.
\end{tikzcd} \tag{3.15}$$

Let $s : \mathbb{Z}/2 \to (\mathbb{Z}/2 \oplus \mathbb{Z}/2) \times \mathbb{Z}/2$ be a section of φ, this section must satisfy $1 \mapsto (0 \oplus 1, 0)$. The composite $\otimes_* \circ s$, and s make the short exact sequences in diagram (3.15) split compatibly. Thus

$$\begin{tikzcd}
(\mathbb{Z}/2 \oplus \mathbb{Z}/2) \times \mathbb{Z}/2 \arrow[r, "\cong"] \arrow[d, "\otimes_*"] & \mathrm{Im}(i \times i) \oplus \mathrm{Im}\, s \arrow[d, "\mathrm{proj}_1 \oplus \mathrm{id}"] \\
\mathbb{Z}/2 \oplus \mathbb{Z}/2 \arrow[r, "\cong"] & \mathrm{Im}\, i \oplus \mathrm{Im}(\otimes_* \circ s).
\end{tikzcd}$$

Hence $\otimes_*(\alpha \oplus \beta, \gamma) = \alpha \oplus \beta$.

Part 2. If $m = 4k + 2$, then diagram (3.14) takes the form

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}/2 \times \mathbb{Z}/2 \arrow[r] \arrow[d, "\mathrm{proj}_1"] & \mathbb{Z}/4 \times \mathbb{Z}/2 \arrow[r] \arrow[d, "\otimes_*"] & \mathbb{Z}/2 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/4 \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0.
\end{tikzcd} \tag{3.16}$$

By direct inspection we have $\otimes_*(1,0) = n$ and $\otimes_*(0,1) = 0$. Therefore, $\otimes_*(\alpha, \beta) = n\alpha$. $\square$

Proposition 3.1.17. *The homomorphism*

$$\otimes_* : \pi_0(\mathrm{PO}(m)) \times \pi_0(\mathrm{SO}(n)) \longrightarrow \pi_0(\mathrm{PO}(mn))$$

is a bijection.

3.2 Proof of Theorem 1.3.5

Let m and n be positive integers. By applying the classifying-space functor to the homomorphism (3.8) we obtain a map

$$f_{\otimes} : \mathrm{BPO}(m) \times \mathrm{BPO}(n) \longrightarrow \mathrm{BPO}(mn).$$

Proposition 3.2.1. *Let m and n be positive integers such that m and n are odd, and relatively prime. Let d denote $\min\{m,n\}$. Then there exist positive integers u and v satisfying $|vn - um| = 1$, so that there exist a positive integer N and a homomorphism $\mathrm{T} : \mathrm{SO}(m) \times \mathrm{SO}(n) \to \mathrm{SO}(N)$ such that for all $i < d-1$ the homomorphisms induced on homotopy groups*

$$\mathrm{T}_i : \pi_i(\mathrm{SO}(m)) \times \pi_i(\mathrm{SO}(n)) \longrightarrow \pi_i(\mathrm{SO}(N))$$

are given by

$$\mathrm{T}_i(x,y) = ux + vy$$

for $x \in \pi_i(\mathrm{SO}(m))$ and $y \in \pi_i(\mathrm{SO}(n))$.

Proof. Assume without loss of generality $m < n$. Since m and n are relatively prime, there exist positive integers u and v such that $vn - um = \pm 1$ by Lemma 1.4.2. Let N denote $um + vn$, and let T denote the composite

$$\mathrm{SO}(m) \times \mathrm{SO}(n) \xrightarrow{(\oplus^u, \oplus^v)} \mathrm{SO}(um) \times \mathrm{SO}(vn) \xrightarrow{\quad \oplus \quad} \mathrm{SO}(N).$$

Let $i < m-1$. From Corollaries 3.1.5 and 3.1.7 we have that $\mathrm{T}_i(x,y) = ux + vy$. □

Proposition 3.2.2. *Let m and n be positive integers such that m is even, and n is odd. Then there exist positive integers u and v satisfying $\left|vn - 2um^2\right| = 1$, so that there exist a positive integer N and a homomorphism $\widetilde{\mathrm{T}} : \mathrm{PO}(m) \times \mathrm{SO}(n) \to \mathrm{SO}(N)$ such that the homomorphisms induced on homotopy groups*

$$\widetilde{\mathrm{T}}_i : \pi_i(\mathrm{PO}(m)) \times \pi_i(\mathrm{SO}(n)) \longrightarrow \pi_i(\mathrm{SO}(N))$$

are given by the following expressions. Let d denote $\min\{m,n\}$.

1. *If* $1 < i < d-1$,
$$\widetilde{\mathrm{T}}_i(x,y) = 2umx + vy$$
for $x \in \pi_i(\mathrm{PO}(m))$ *and* $y \in \pi_i(\mathrm{SO}(n))$.

2. *If* $i = 1$, *then*
$$\begin{cases} \widetilde{\mathrm{T}}_1(\alpha \oplus \beta, \gamma) = z\beta + \gamma & \text{if } m \equiv 0 \text{ (mod 4)}, \\ \widetilde{\mathrm{T}}_1(\delta, \gamma) = z'\delta + \gamma & \text{if } m \equiv 2 \text{ (mod 4)}. \end{cases}$$
for $z, z', \alpha, \beta, \gamma \in \mathbb{Z}/2$ *and* $\delta \in \mathbb{Z}/4$.

Proof. Without loss of generality, suppose $m < n$. Since m and n are relatively prime, there exist positive integers u and v such that $vn - 2um^2 = \pm 1$ by Lemma 1.4.2. Let N denote $um^2 + vn$, and let T denote the composite

$$\begin{tikzcd} \mathrm{O}(m) \times \mathrm{SO}(n) \arrow[d, "(\otimes^2, \mathrm{id})"] \\ \mathrm{SO}(m^2) \times \mathrm{SO}(n) \arrow[d, "(\oplus^u, \oplus^v)"] \\ \mathrm{SO}(um^2) \times \mathrm{SO}(vn) \arrow[d, "\oplus"] \\ \mathrm{SO}(N). \end{tikzcd}$$

Note that the elements $(\pm I_m, I_n)$ are sent to (I_{m^2}, I_{n^2}) by $(\otimes^2, \mathrm{id})$, hence to the identity by the composite T defined above. Hence T factors through $\mathrm{PO}(m) \times \mathrm{SO}(n)$

$$\begin{tikzcd} \mathrm{O}(m) \times \mathrm{SO}(n) \arrow[d, two heads] \arrow[dr, bend left, "\mathrm{T}"] & \\ \mathrm{PO}(m) \times \mathrm{SO}(n) \arrow[r, "\widetilde{\mathrm{T}}"'] & \mathrm{SO}(N). \end{tikzcd}$$

From Proposition 3.1.15, and Corollaries 3.1.5, 3.1.7 and 3.1.13 we have that $\mathrm{T}_i(x,y) = 2umx + vy$ for $i < m-1$.

The map of fibrations

$$\begin{tikzcd}
Z(\mathrm{O}(m)) \times \{I_n\} \arrow[r, hook] \arrow[d] & \mathrm{O}(m) \times \mathrm{SO}(n) \arrow[r, two heads] \arrow[d, "\mathrm{T}"] & \mathrm{PO}(m) \times \mathrm{SO}(n) \arrow[d, "\widetilde{\mathrm{T}}"] \\
\{I_N\} \arrow[r] & \mathrm{SO}(N) \arrow[r, equal] & \mathrm{SO}(N)
\end{tikzcd}$$

induces a commutative diagram

$$\begin{tikzcd}
\pi_i(\mathrm{O}(m)) \times \pi_i(\mathrm{SO}(n)) \arrow[r, "\cong"] \arrow[d, "\mathrm{T}_i"] & \pi_i(\mathrm{PO}(m)) \times \pi_i(\mathrm{SO}(n)) \arrow[d, "\widetilde{\mathrm{T}}_i"] \\
\pi_i(\mathrm{SO}(N)) \arrow[r, equal] & \pi_i(\mathrm{SO}(N))
\end{tikzcd}$$

for $i > 1$. Then $\widetilde{\mathrm{T}}_i(x, y) = \mathrm{T}_i(x, y) = 2umx + vy$ for $1 < i < m-1$.

For $i = 1$, the map of fibrations induces the commutative diagram below

$$\begin{tikzcd}
\pi_1(\mathrm{O}(m)) \times \pi_1(\mathrm{SO}(n)) \arrow[r, hook] \arrow[d, "\mathrm{T}_1"] & \pi_1(\mathrm{PO}(m)) \times \pi_1(\mathrm{SO}(n)) \arrow[d, "\widetilde{\mathrm{T}}_1"] \\
\pi_1(\mathrm{SO}(N)) \arrow[r, equal] & \pi_1(\mathrm{SO}(N)),
\end{tikzcd}$$

which takes the form

$$\begin{tikzcd}
\mathbb{Z}/2 \times \mathbb{Z}/2 \arrow[r, hook] \arrow[dr, "\mathrm{T}_1"'] & Z(\mathrm{Spin}(m)) \times \mathbb{Z}/2 \arrow[d, "\widetilde{\mathrm{T}}_1"] \\
 & \mathbb{Z}/2.
\end{tikzcd} \tag{3.17}$$

Observe that the equality $vn - 2um^2 = \pm 1$ implies v is odd, then $\mathrm{T}_1(\alpha, \beta) = 2um\alpha + v\beta = \beta$, i.e. T_1 is projection onto the second coordinate.

Subcase i: Suppose $m = 4k$ for some $k \in \mathbb{Z}$. Diagram (3.17) takes the form

$$\begin{tikzcd}
\mathbb{Z}/2 \times \mathbb{Z}/2 \arrow[r, hook] \arrow[dr, "\mathrm{T}_1"'] & (\mathbb{Z}/2 \oplus \mathbb{Z}/2) \times \mathbb{Z}/2 \arrow[d, "\widetilde{\mathrm{T}}_1"] \\
 & \mathbb{Z}/2
\end{tikzcd}$$

where the horizontal homomorphism is the inclusion $(\alpha, \beta) \mapsto (\alpha \oplus 0, \beta)$. Thus $\widetilde{\mathrm{T}}_1(1 \oplus 0, 0) = 0$ and $\widetilde{\mathrm{T}}_1(\mathbf{0}, 1) = 1$. Let z denote $\widetilde{\mathrm{T}}_1(0 \oplus 1, 0)$. Hence $\widetilde{\mathrm{T}}_1(\alpha \oplus \beta, \gamma) =$

$z\beta+\gamma$.

Subcase ii: Suppose $m = 4k+2$ for some $k \in \mathbb{Z}$. Diagram (3.17) takes the form

$$\begin{tikzcd} \mathbb{Z}/2\times\mathbb{Z}/2 \arrow[r, hook] \arrow[dr, "\mathrm{T}_1"'] & \mathbb{Z}/4\times\mathbb{Z}/2 \arrow[d, "\widetilde{\mathrm{T}}_1"] \\ & \mathbb{Z}/2 \end{tikzcd}$$

where the horizontal homomorphism is the inclusion $(\alpha,\beta)\mapsto(\alpha,\beta)$. Thus $\widetilde{\mathrm{T}}_1(0,1)=1$. Let z' denote $\widetilde{\mathrm{T}}_1(1,0)$. Hence $\widetilde{\mathrm{T}}_1(\delta,\gamma)=z'\delta+\gamma$. □

Let J denote the map

$$J : \mathrm{BPO}(m)\times\mathrm{BPO}(n) \longmapsto \mathrm{BPO}(mn)\times\mathrm{BSO}(N)$$
$$(x,y) \longmapsto \Big(f_{\otimes}(x,y), \mathrm{B}\widetilde{\mathrm{T}}(x,y)\Big).$$

Let J_i denote the homomorphism induced on homotopy groups by J.

$$J_i : \pi_i(\mathrm{BPO}(m))\times\pi_i(\mathrm{BPO}(n)) \to \pi_i(\mathrm{BPO}(mn))\times\pi_i(\mathrm{BSO}(N)) \tag{3.18}$$

3.2.1 A d-connected map where d is the minimum of two odd positive integers

Proposition 3.2.3. *Let m and n be positive integers such that m and n are odd, and relatively prime.. The map J is d-connected where d denotes* $\min\{m,n\}$.

Proof. Without loss of generality, let $m<n$. Let $i<m$. By Propositions 3.1.14 and 3.2.1, and Corollary 3.1.10 the homomorphism

$$J_i : \pi_i(\mathrm{BSO}(m))\times\pi_i(\mathrm{BSO}(n)) \to \pi_i(\mathrm{BSO}(mn))\times\pi_i(\mathrm{BSO}(N))$$

is represented by the matrix

$$\begin{pmatrix} n & m \\ u & v \end{pmatrix}.$$

Observe that this matrix is invertible for all $i<m$ because by Proposition

3.2.1 its determinant satisfies $nv - um = \pm 1$.

Let $i = m$. Propositions 3.1.6 and 3.2.1, and Corollary 3.1.7 show that the homomorphism $\mathrm{T}_{m-1} : \pi_{m-1}(\mathrm{SO}(m)) \times \pi_{m-1}(\mathrm{SO}(n)) \to \pi_{m-1}(\mathrm{SO}(N))$ is given by $\mathrm{T}_{m-1}(x, y) = \mathrm{stab}(u\,\mathrm{stab}(x)) + \mathrm{stab}(v\,\mathrm{stab}(y)) = u\,\mathrm{stab}(x) + vy$, where $vn - um = \pm 1$. This can be seen by observing that T_{m-1} is equal to the sum of the two paths around diagram (3.19).

$$\begin{tikzcd}
& \pi_{m-1}(\mathrm{SO}(m)) \times \pi_{m-1}(\mathrm{SO}(n)) \arrow[dl, two heads, "\mathrm{proj}_1"'] \arrow[dr, two heads, "\mathrm{proj}_2"] & \\
\pi_{m-1}(\mathrm{SO}(m)) \arrow[d, two heads, "\mathrm{stab}"] & & \pi_{m-1}(\mathrm{SO}(n)) \arrow[d, "\mathrm{stab}", "\cong"'] \\
\pi_{m-1}(\mathrm{SO}(um)) \arrow[dr, "\times u"'] & & \pi_{m-1}(\mathrm{SO}(vn)) \arrow[dl, "\times v"] \\
& \pi_{m-1}(\mathrm{SO}(um)) \times \pi_{m-1}(\mathrm{SO}(vn)) \arrow[d, "\mathrm{stab} \times \mathrm{stab}", "\cong"'] & \\
& \pi_{m-1}(\mathrm{SO}(N)) \times \pi_{m-1}(\mathrm{SO}(N)) \arrow[d, "+"] & \\
& \pi_{m-1}(\mathrm{SO}(N)) &
\end{tikzcd} \tag{3.19}$$

Hence J_m is given by

$$\begin{aligned}
J_m : \pi_m(\mathrm{B\,SO}(m)) \times \pi_m(\mathrm{B\,SO}(n)) &\longrightarrow \pi_m(\mathrm{B\,SO}(mn)) \times \pi_m(\mathrm{B\,O}_N) \\
(x, y) &\longmapsto \big(n\,\mathrm{stab}_1(x) + my, u\,\mathrm{stab}_2(x) + vy\big)
\end{aligned}$$

where

$$\mathrm{stab}_1 : \pi_{m-1}(\mathrm{SO}(m)) \twoheadrightarrow \pi_{m-1}(\mathrm{SO}(mn))$$

and

$$\mathrm{stab}_2 : \pi_{m-1}(\mathrm{SO}(m)) \twoheadrightarrow \pi_{m-1}(\mathrm{SO}(um))$$

are epimorphisms.

If $m = 1$ or $m \equiv 3, 5, 7 \pmod 8$, then J_m has trivial target.

Let $m \neq 1$ and $m \equiv 1 \pmod 8$, then

$$J_m : (\mathbb{Z}/2 \oplus \mathbb{Z}/2) \times \mathbb{Z}/2 \to \mathbb{Z}/2 \times \mathbb{Z}/2$$

is given by

$$J_m(x \oplus y, z) = \big(\mathrm{stab}_1(x \oplus y) + z, u\,\mathrm{stab}_2(x \oplus y) + vz\big)$$

Observe that J_m factors as

$$(\mathbb{Z}/2 \oplus \mathbb{Z}/2) \times \mathbb{Z}/2 \xrightarrow{(\mathrm{stab}, \mathrm{id})} \mathbb{Z}/2 \times \mathbb{Z}/2 \xrightarrow{\begin{pmatrix} n & m \\ u & v \end{pmatrix}} \mathbb{Z}/2 \times \mathbb{Z}/2.$$

Hence J_m is an epimorphism. □

3.2.2 A d-connected map where d is the minimum of two positive integers of opposite parity

Proposition 3.2.4. *Let m and n be positive integers such that m is even, and n is odd. Let d denote* $\min\{m, n\}$. *The homomorphism J_i is an isomorphism for all $i < d$.*

Proof. Without loss of generality, suppose $m < n$.

$$J_i : \pi_i(\mathrm{BPO}(m)) \times \pi_i(\mathrm{BSO}(n)) \to \pi_i(\mathrm{BPO}(mn)) \times \pi_i(\mathrm{BSO}(N)) \tag{3.20}$$

Let $i < m$. Given that the homotopy groups of the spaces involved are zero in degrees $i \equiv 3, 5, 6, 7 \pmod 8$, it suffices to prove that J_i is an isomorphism for $i = 1, 2$, and for $i \equiv 0, 1, 2, 4 \pmod 8$ with $i > 2$.

Let $i = 1$. By Proposition 3.1.17 the homomorphism $J_1 : \mathbb{Z}/2 \to \mathbb{Z}/2$ is the identity.

Let $i = 2$. The homomorphism (3.18) takes the form

$$J_2 : Z(\mathrm{Spin}(m)) \times \mathbb{Z}/2 \longrightarrow Z(\mathrm{Spin}(mn)) \times \mathbb{Z}/2.$$

Case i: Let $m = 4k$ for some $k \in \mathbb{Z}$. From Propositions 3.1.16 and 3.2.2, J_2 is given by

$$\begin{aligned} J_2 : (\mathbb{Z}/2 \oplus \mathbb{Z}/2) \times \mathbb{Z}/2 &\longrightarrow (\mathbb{Z}/2 \oplus \mathbb{Z}/2) \times \mathbb{Z}/2 \\ (\alpha \oplus \beta, \gamma) &\longmapsto \Big(\alpha \oplus \beta, z\beta + \gamma\Big), \end{aligned}$$

i.e. J_2 is represented by the invertible matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & z & 1 \end{pmatrix} \quad \text{where} \quad z \in \mathbb{Z}/2.$$

Hence J_2 is an isomorphism.

Case ii: Let $m = 4k+2$ for some $k \in \mathbb{Z}$. From Proposition 3.1.16, $f_{\otimes}(\alpha, \beta) = n\alpha$. Since $vn - 2um^2 = \pm 1$, then $f_{\otimes}(\alpha, \beta) = \alpha$. By Proposition 3.2.2, J_2 is given by

$$\begin{aligned} J_2 : \mathbb{Z}/4 \times \mathbb{Z}/2 &\longrightarrow \mathbb{Z}/4 \times \mathbb{Z}/2 \\ (\alpha, \beta) &\longmapsto \Big(\alpha, z'\alpha + \beta\Big), \end{aligned}$$

i.e. J_2 is represented by the invertible matrix

$$\begin{pmatrix} 1 & 0 \\ z' & 1 \end{pmatrix} \quad \text{where} \quad z' \in \mathbb{Z}/2.$$

Hence J_2 is an isomorphism.

Let $i > 2$. The homomorphism (3.18) takes the form $J_i : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ if $i \equiv 0, 4 \pmod 8$, and $J_i : \mathbb{Z}/2 \times \mathbb{Z}/2 \to \mathbb{Z}/2 \times \mathbb{Z}/2$ if $i \equiv 1, 2 \pmod 8$.

Note that both homomorphisms are represented by the matrix

$$\begin{pmatrix} n & m \\ 2um & v \end{pmatrix}$$

for $i < m$. By Proposition 3.2.2, the determinant of this matrix is $vn - 2um^2 = \pm 1$. This proves J_i is an isomorphism. $\square$

$$
\begin{tikzcd}
 & \pi_{m-1}(\mathrm{O}(m)) \times \pi_{m-1}(\mathrm{SO}(n)) \arrow[dl, "\mathrm{proj}_1"'] \arrow[dr, "\mathrm{proj}_2"] & \\
\pi_{m-1}(\mathrm{O}(m)) \arrow[d, twoheadrightarrow, "\mathrm{stab}"] & & \pi_{m-1}(\mathrm{SO}(n)) \arrow[dd, equal] \\
\pi_{m-1}(\mathrm{O}(m^2)) \arrow[d, "\times(2m)"] & & \\
\pi_{m-1}(\mathrm{O}(m^2)) \arrow[d, "\mathrm{stab}", "\cong"'] & & \pi_{m-1}(\mathrm{SO}(n)) \arrow[d, "\mathrm{stab}", "\cong"'] \\
\pi_{m-1}(\mathrm{O}(um^2)) \arrow[dr, "\times u"'] & & \pi_{m-1}(\mathrm{SO}(vn)) \arrow[dl, "\times v"] \\
 & \pi_{m-1}(\mathrm{O}(um^2)) \times \pi_{m-1}(\mathrm{SO}(vn)) \arrow[d, "\mathrm{stab}\times\mathrm{stab}", "\cong"'] & \\
 & \pi_{m-1}(\mathrm{SO}(N)) \times \pi_{m-1}(\mathrm{SO}(N)) \arrow[d, "+"] & \\
 & \pi_{m-1}(\mathrm{SO}(N)) &
\end{tikzcd}
\tag{3.21}
$$

Proposition 3.2.5. *Let m and n be positive integers such that m is even, and n is odd. Let d denote* $\min\{m,n\}$. *The induced homomorphism*

$$J_d : \pi_d(B\mathrm{PO}(m)) \times \pi_d(B\mathrm{SO}(n)) \longrightarrow \pi_d(B\mathrm{PO}(mn)) \times \pi_d(B\mathrm{SO}(N)) \tag{3.22}$$

is an epimorphism.

Proof. Suppose $m < n$. Propositions 3.1.6 and 3.2.2, and Corollaries 3.1.7 and

3.1.13 show that $\mathrm{T}_m : \pi_m(\mathrm{O}(m)) \times \pi_m(\mathrm{SO}(n)) \to \pi_m(\mathrm{SO}(N))$ is given by

$$\mathrm{T}_{m-1}(x,y) = \mathrm{stab}(u\,\mathrm{stab}(2m\,\mathrm{stab}(x))) + \mathrm{stab}(v\,\mathrm{stab}(y)) = 2um\,\mathrm{stab}(x) + vy,$$

where $vn - 2um^2 = \pm 1$. This can be seen by observing that T_{m-1} is equal to the sum of the two paths around diagram (3.21).

Hence (3.22) is given by

$$\begin{aligned} J_m : \pi_m(\mathrm{BPO}(m)) \times \pi_m(\mathrm{BSO}(n)) &\longrightarrow \pi_m(\mathrm{BSO}(mn)) \times \pi_m(\mathrm{BO}_N) \\ (x,y) &\longmapsto \big(n\,\mathrm{stab}_1(x) + my, 2um\,\mathrm{stab}_2(x) + vy\big) \end{aligned}$$

where

$$\mathrm{stab}_1 : \pi_{m-1}(\mathrm{O}(m)) \twoheadrightarrow \pi_{m-1}(\mathrm{O}(mn))$$

and

$$\mathrm{stab}_2 : \pi_{m-1}(\mathrm{O}(m)) \twoheadrightarrow \pi_{m-1}(\mathrm{O}(m^2))$$

are epimorphisms.

Observe that for $m = 1$ or $m \equiv 3,5,6,7 \pmod 8$, the homomorphism J_m has trivial target.

Let $m = 2$. Then $J_2 : \mathbb{Z} \times \mathbb{Z}/2 \to \mathbb{Z}/4 \times \mathbb{Z}/2$ is given by

$$J_2(x,y) = \big(n\,\mathrm{stab}_1(x) + 2y, y\big).$$

Thus J_2 is an epimorphism.

Let $i > 2$ and $m \equiv 0,4 \pmod 8$. Then $J_m : (\mathbb{Z} \oplus \mathbb{Z}) \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ is given by

$$J_m(x \oplus y, z) = \big(n\,\mathrm{stab}_1(x \oplus y) + mz, 2um\,\mathrm{stab}_2(x \oplus y) + vz\big).$$

Note that J_m factors as

$$(\mathbb{Z} \oplus \mathbb{Z}) \times \mathbb{Z} \xrightarrow{(\mathrm{stab},\mathrm{id})} \mathbb{Z} \times \mathbb{Z} \xrightarrow{\begin{pmatrix} n & 2um \\ m & v \end{pmatrix}} \mathbb{Z} \times \mathbb{Z}.$$

Hence J_m is an epimorphism.

Let $i > 2$ and $m \equiv 2 \pmod 8$. Then $J_m : (\mathbb{Z}/2 \oplus \mathbb{Z}) \times \mathbb{Z}/2 \to \mathbb{Z}/2 \times \mathbb{Z}/2$ is given by

$$J_m(x \oplus y, z) = \big(\mathrm{stab}_1(x \oplus y), z\big).$$

Then J_m is an epimorphism.

If we suppose $n < m$. In the same manner it can be proved that

$$\begin{aligned} J_n : \pi_n(\mathrm{BPO}(m)) \times \pi_n(\mathrm{BSO}(n)) &\longrightarrow \pi_n(\mathrm{BSO}(mn)) \times \pi_n(\mathrm{BO}_N) \\ (x, y) &\longmapsto \big(nx + m\,\mathrm{stab}_1(y), 2umx + v\,\mathrm{stab}_2(y)\big) \end{aligned}$$

is an epimorphism, where

$$\mathrm{stab}_1 : \pi_{n-1}(\mathrm{SO}(n)) \twoheadrightarrow \pi_{n-1}(\mathrm{SO}(mn))$$

and

$$\mathrm{stab}_2 : \pi_{n-1}(\mathrm{SO}(n)) \twoheadrightarrow \pi_{n-1}(\mathrm{SO}(vn)).$$

□

From Propositions 3.2.4, and 3.2.5 we obtain Corollary 3.2.6.

Corollary 3.2.6. *Let m and n be positive integers such that m is even, and n is odd. Let d denote* $\min\{m, n\}$. *The map J is d-connected.*

3.2.3 Factorization through $f_\otimes : \mathrm{BPO}(m, \mathbb{C}) \times \mathrm{BSO}(n, \mathbb{C}) \to \mathrm{BPO}(mn, \mathbb{C})$

Proof of Theorem 1.3.5. Diagrammatically speaking, we want to find a map

$$\mathscr{A}_m \times \mathscr{A}_n : X \longrightarrow \mathrm{BPO}(m) \times \mathrm{BSO}(n)$$

such that diagram (3.23) commutes up to homotopy

$$\begin{tikzcd} & \mathrm{BPO}(m) \times \mathrm{BSO}(n) \arrow[d, "f_\otimes"] \\ X \arrow[ur, dotted, bend left, "\mathscr{A}_m \times \mathscr{A}_n"] \arrow[r, "\mathscr{A}"] & \mathrm{BPO}(mn). \end{tikzcd} \tag{3.23}$$

Without loss of generality, let $m < n$. Corollary 3.2.6 yields a map $J : \mathrm{BPO}(m) \times \mathrm{BSO}(n) \to \mathrm{BPO}(mn) \times \mathrm{BSO}(N)$ where N is some positive integer so that $N \gg n > m$. Observe that $f_{\otimes}$ factors through $\mathrm{BPO}(mn) \times \mathrm{BSO}(N)$, so we can write $f_{\otimes}$ as the composite of J and the projection proj_1 shown in diagram (3.24).

$$\begin{tikzcd}
\mathrm{BPO}(m) \times \mathrm{BSO}(n) \arrow[r, "J"] \arrow[d, "f_{\otimes}"] & \mathrm{BPO}(mn) \times \mathrm{BSO}(N) \arrow[dl, "\mathrm{proj}_1"'] \\
\mathrm{BPO}(mn) &
\end{tikzcd} \tag{3.24}$$

Since J is m-connected and $\dim(X) \le m$, then by Whitehead's theorem

$$J_{\#} : \left[X, \mathrm{BPO}(m) \times \mathrm{BSO}(n)\right] \longrightarrow \left[X, \mathrm{BPO}(mn) \times \mathrm{BSO}(N)\right]$$

is a surjection, [25, Corollary 7.6.23].

Let s denote a section of proj_1. The surjectivity of $J_{\#}$ implies $s \circ \mathscr{A}$ has a preimage $\mathscr{A}_m \times \mathscr{A}_n : X \to \mathrm{BPO}(m) \times \mathrm{BSO}(n)$ such that $J \circ (\mathscr{A}_m \times \mathscr{A}_n) \simeq s \circ \mathscr{A}$.

Commutativity of diagram (3.23) follows from commutativity of diagram (3.24). Thus, the result follows. $\square$

Chapter 4

Decomposition of Topological Azumaya Algebras with Symplectic Involution

This chapter is organized as follows. The first section presents preliminaries on the effect of direct sum and tensor product operations on homotopy groups of compact Lie groups related to the complex orthogonal groups $\mathrm{Sp}(n,\mathbb{C})$. The second section is devoted to the proof of Theorem 1.3.6.

From now on, we are going to drop the $\mathbb{C}$ that comes in the notation of $\mathrm{Sp}(n,\mathbb{C})$.

4.1 Stabilization of operations on the complex symplectic group

We recall the homotopy groups of $\mathrm{Sp}(n)$ in low degrees, and compute the homotopy groups of $\mathrm{PSp}(n)$ in low degrees.

When $n = 1$, then $\mathrm{Sp}(1) = \mathrm{SL}(2,\mathbb{C})$ and $\mathrm{PSp}(1) = \mathrm{PSL}(2,\mathbb{C})$.

Let $n \geq 1$ and $i < 4n+2$, the first homotopy groups of the symplectic group are given by Bott periodicity.

$$\pi_i(\mathrm{Sp}(n)) \cong \begin{cases} 0 & \text{if } i = 0,1,2,6 \pmod 8, \\ \mathbb{Z}/2 & \text{if } i = 4,5 \pmod 8, \\ \mathbb{Z} & \text{if } i = 3,7 \pmod 8. \end{cases}$$

$$\pi_{4n}(\mathrm{Sp}(n)) \cong \pi_{4n+1}(\mathrm{Sp}(n)) \cong \begin{cases} \mathbb{Z}/2 & \text{if } n \text{ is odd}, \\ 0 & \text{if } n \text{ is even}. \end{cases}$$

The quotient map $\mathrm{Sp}(n) \to \mathrm{PSp}(n)$ is a universal cover with fiber $\mathbb{Z}/2$, then $\pi_1(\mathrm{PSp}(n)) \cong \mathbb{Z}/2$. The symplectic group $\mathrm{Sp}(n)$ is a subgroup of $\mathrm{SL}(2n,\mathbb{C})$. For $n > 0$ there is a fibration $\{\pm I_{2n}\} \hookrightarrow \mathrm{Sp}(n) \to \mathrm{PSp}(n)$, then we can use the long exact sequence associated to it to see that $\pi_i(\mathrm{PSp}(n)) \cong \pi_i(\mathrm{Sp}(n))$ for $i \geq 2$.

4.1.1 First unstable homotopy group of $\mathrm{Sp}(n,\mathbb{C})$

Let $A \in \mathrm{Sp}(n)$, we write A as a block matrix

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

where the A_{ij}'s are $n \times n$ matrices for $i,j = 1,2$, and they satisfy the following conditions: $A_{11}^{\mathrm{tr}} A_{21}$ are $A_{12}^{\mathrm{tr}} A_{22}$ are symmetric, and $A_{11}^{\mathrm{tr}} A_{22} - A_{21}^{\mathrm{tr}} A_{12} = I_n$.

The standard inclusion of the symplectic group $i : \mathrm{Sp}(n) \hookrightarrow \mathrm{Sp}(n+1)$ is defined as

$$i(A) = \begin{pmatrix} & & & 0 & & & & 0 \\ & A_{11} & & \vdots & & A_{12} & & \vdots \\ & & & 0 & & & & 0 \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ & & & 0 & & & & 0 \\ & A_{21} & & \vdots & & A_{22} & & \vdots \\ & & & 0 & & & & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

The standard inclusion of the symplectic group is $(4n+2)$-connected, hence it induces an isomorphism on homotopy groups in degrees less than $4n+2$ and an epimorphism in degree $4n+2$. This can be seen by observing the long exact sequence for the fibration $\mathrm{Sp}(n) \hookrightarrow \mathrm{Sp}(n+1) \to \mathrm{Sp}(n+1)/\mathrm{Sp}(n) \simeq S^{4n+3}$.

The first unstable homotopy group of $\mathrm{Sp}(n)$ happens in degree $4n+2$. From [21, Section 2]

$$\pi_{4n+2}(\mathrm{Sp}(n)) \cong \begin{cases} \mathbb{Z}/(2n+1)! & \text{if } n \text{ is even,} \\ \mathbb{Z}/(2n+1)! \cdot 2 & \text{if } n \text{ is odd.} \end{cases}$$

Let $G \in \{\mathrm{Sp}(n), \mathrm{PSp}(n)\}$. We summarize these results in tables 4.1, 4.2, 4.3, and 4.4.

G	$\mathrm{Sp}(n)$	$\mathrm{PSp}(n)$
$\pi_0(G)$	$*$	$*$
$\pi_1(G)$	0	$\mathbb{Z}/2$

Table 4.1: Connected components and fundamental group of compact Lie groups related to the complex symplectic group

$i > 1$ and $i \pmod 8$	0	1	2	3	4	5	6	7
$\pi_i(G)$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$

Table 4.2: Homotopy groups of compact Lie groups related to the complex symplectic group for $i = 2, \ldots, 4n-1$

n	even	odd
$\pi_{4n}(G) \cong \pi_{4n+1}(G)$	0	$\mathbb{Z}/2$

Table 4.3: Homotopy groups of compact Lie groups related to the complex symplectic group for in degrees $4n$ and $4n+1$

4.1.2 Stabilization

In this chapter we use a different direct sum of matrices from the one used in Chapter 1.

Direct sum of symplectic matrices

n	even	odd
$\pi_{4n+2}(G)$	$\mathbb{Z}/(2n+1)!$	$\mathbb{Z}/(2n+1)!\cdot 2$

Table 4.4: First unstable homotopy group of compact Lie groups related to the complex symplectic group

Let $(\mathbb{C}^{2m},B)$ and $(\mathbb{C}^{2n},B')$ denote the standard skew-symmetric forms on the spaces $\mathbb{C}^{2m}$ and $\mathbb{C}^{2n}$. Let $\{e_1,\dots,e_{2m}\}$ and $\left\{e'_1,\dots,e'_{2n}\right\}$ be the standard bases of $(\mathbb{C}^{2m},B)$ and $(\mathbb{C}^{2n},B')$, respectively.

The set $\mathcal{B}=\left\{e_1,\dots,e_m,e'_1,\dots,e'_n,e_{m+1},\dots,e_{2m},e'_{n+1},\dots,e'_{2n}\right\}$ is an ordered basis for $\mathbb{C}^{2m}\oplus\mathbb{C}^{2n}$. Let $B\oplus B':(\mathbb{C}^{2m}\oplus\mathbb{C}^{2n})\times(\mathbb{C}^{2m}\oplus\mathbb{C}^{2n})\to\mathbb{C}$ be the bilinear form defined by

$$(B\oplus B')\left(v\oplus v',w\oplus w'\right)=B(v,w)+B'\left(v',w'\right)$$

for $v,w\in V$ and $v',w'\in V'$. Then $(\mathbb{C}^{2m}\oplus\mathbb{C}^{2n},B\oplus B')$ with respect to $\mathcal{B}$ is the standard skew-symmetric bilinear form. Thus the direct sum of symplectic matrices $\boxplus:\mathrm{Sp}(m)\times\mathrm{Sp}(n)\to\mathrm{Sp}(m+n)$ is given by

$$A\boxplus B=\left(\begin{array}{cc:cc} A_{11} & & A_{12} & \\ & B_{11} & & B_{12} \\ \hdashline A_{21} & & A_{22} & \\ & B_{21} & & B_{22} \end{array}\right).$$

r-fold direct sum of symplectic matrices

The r-fold direct sum of matrices $\boxplus^r:\mathrm{Sp}(n)\to\mathrm{Sp}(rn)$ is given by

$$A^{\boxplus r}=\begin{pmatrix} A_{11}^{\oplus r} & A_{12}^{\oplus r} \\ A_{21}^{\oplus r} & A_{22}^{\oplus r} \end{pmatrix}.$$

Let $m, n \in \mathbb{N}$ and $m \leq n$. Define the map $\mathrm{s} : \mathrm{Sp}(m) \to \mathrm{Sp}(m+n)$ by

$$\mathrm{s}(A) = \left(\begin{array}{cc:cc} A_{11} & & A_{12} & \\ & I_n & & I_n \\ \hdashline A_{21} & & A_{22} & \\ & I_n & & I_n \end{array}\right).$$

Since the map s is equal to the composite of consecutive canonical inclusions, it follows that s is $(4m+2)$-connected.

Notation 4.1.1. Let stab denote the homomorphisms the map s induces on homotopy groups. From now on, we will identify $\pi_i(\mathrm{Sp}(m))$ with $\pi_i(\mathrm{Sp}(m+n))$ for all $i < 4m+2$ through the isomorphism

$$\mathrm{stab} : \pi_i(\mathrm{Sp}(m)) \xrightarrow{\cong} \pi_i(\mathrm{Sp}(m+n)). \tag{4.1}$$

Lemma 4.1.2. *Let $n, r \in \mathbb{N}$. For all $j = 1, \ldots, r$ define $\mathrm{s}_j : \mathrm{Sp}(n) \to \mathrm{Sp}(rn)$ by*

$$\mathrm{s}_j(A) = \begin{pmatrix} \mathrm{s}_j(A_{11}) & \mathrm{s}_j(A_{12}) \\ \mathrm{s}_j(A_{21}) & \mathrm{s}_j(A_{22}) \end{pmatrix}.$$

where $\mathrm{s}_j(A_{kl}) = \mathrm{diag}(I_n, \ldots, I_n, A_{kl}, I_n, \ldots, I_n)$, where A_{kl} is in the j-th position for $k, l = 1, 2$. Then the map s_j is pointed homotopic to s_{j+1} for $j = 1, \ldots, r-1$.

Proof. Let P be the permutation matrix

$$P = \begin{pmatrix} I_{(j-1)n} & & & \\ & 0 & I_n & \\ & I_n & 0 & \\ & & & I_{(r-j-1)n} \end{pmatrix}.$$

Observe that $\det(P_j) = 1$, and $\mathrm{s}_{j+1}(A_{kl}) = P\,\mathrm{s}_j(A_{kl})P$ for $j = 1, \ldots, r-1$ and $k, l = 1, 2$. Then

$$\begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix} \begin{pmatrix} \mathrm{s}_j(A_{11}) & \mathrm{s}_j(A_{12}) \\ \mathrm{s}_j(A_{21}) & \mathrm{s}_j(A_{22}) \end{pmatrix} \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix} = \begin{pmatrix} \mathrm{s}_{j+1}(A_{11}) & \mathrm{s}_{j+1}(A_{12}) \\ \mathrm{s}_{j+1}(A_{21}) & \mathrm{s}_{j+1}(A_{22}) \end{pmatrix}.$$

Since $\operatorname{diag}(P,P) \in \mathrm{Sp}(n)$ and $\mathrm{Sp}(n)$ is path-connected, the result follows from Lemma 1.4.1. □

Notation 4.1.3. We call the s_j maps *stabilization maps*. As s_1 is equal to $\mathrm{s} : \mathrm{Sp}(n) \to \mathrm{Sp}(n+(r-1)n)$, it follows that s_j is $(4n+2)$-connected for all $j = 1, \dots, r$. From Lemma 4.1.2 the homomorphisms induced on homotopy groups by the stabilization maps are equal, hence stab also denotes $\pi_i(\mathrm{s}_1) = \cdots = \pi_i(\mathrm{s}_r)$.Thus we identify $\pi_i(\mathrm{Sp}(n))$ with $\pi_i(\mathrm{Sp}(rn))$ for $i < 4n+2$ through stab. The identification allows one to introduce a slight abuse of notation, namely to identify x and $\mathrm{stab}(x)$ for $x \in \pi_i(\mathrm{Sp}(n))$ and $i < 4n+2$.

4.1.3 Operations on homotopy groups

Proposition 4.1.4. *Let $i \in \mathbb{N}$. The homomorphism $\boxplus_* : \pi_i(\mathrm{Sp}(m)) \times \pi_i(\mathrm{Sp}(n)) \to \pi_i(\mathrm{Sp}(m+n))$ is equal to*

$$\boxplus_*(x,y) = \mathrm{stab}(x) + \mathrm{stab}(y)$$

for $x \in \pi_i(\mathrm{Sp}(m))$ and $y \in \pi_i(\mathrm{Sp}(n))$.

Corollary 4.1.5. *If $m < n$ and $i < 4m+2$, then $\boxplus_*(x,y) = x+y$ for $x \in \pi_i(\mathrm{Sp}(m))$ and $y \in \pi_i(\mathrm{Sp}(n))$.*

Proposition 4.1.6. *Let $i \in \mathbb{N}$. The homomorphism $\boxplus^r_* : \pi_i(\mathrm{Sp}(n)) \to \pi_i(\mathrm{Sp}(rn))$ is equal to*

$$\boxplus^r_*(x) = r\,\mathrm{stab}(x)$$

for $x \in \pi_i(\mathrm{Sp}(n))$.

Corollary 4.1.7. *If $i < 4n+2$, then $\boxplus^r_*(x) = rx$ for $x \in \pi_i(\mathrm{Sp}(n))$.*

We want to understand the effect on homotopy groups of the tensor product operations that were described in items 2 and 3 in Section 1.4.2, $\otimes : \mathrm{Sp}(m,\mathbb{C}) \times \mathrm{O}(n,\mathbb{C}) \to \mathrm{Sp}(mn,\mathbb{C})$ and $\boxtimes : \mathrm{Sp}(m,\mathbb{C}) \times \mathrm{Sp}(n,\mathbb{C}) \to \mathrm{O}(4mn,\mathbb{C})$.

Effect of $\otimes : \mathrm{Sp}(m,\mathbb{C}) \times \mathrm{O}(n,\mathbb{C}) \to \mathrm{Sp}(mn,\mathbb{C})$ on homotopy groups in the stable range of $\mathrm{Sp}(m,\mathbb{C})$

Let L and R denote the restrictions of $\otimes$ to $\mathrm{Sp}(m) \times \{I_n\}$ and $\{I_{2m}\} \times \mathrm{O}(n)$, respectively.

Lemma 4.1.8. *Let $L : \mathrm{Sp}(m) \to \mathrm{Sp}(mn)$ be the homomorphism $L(A) = A \otimes I_n$. There is a basepoint preserving homotopy H from L to the n-fold direct sum map of symplectic matrices $\boxplus^n : \mathrm{Sp}(m) \to \mathrm{Sp}(mn)$ such that for all $t \in [0,1]$, $H(-,t)$ is a homomorphism.*

Proof. Let $A \in \mathrm{Sp}(m)$,

$$L(A) = \begin{pmatrix} A_{11} \otimes I_n & A_{12} \otimes I_n \\ A_{21} \otimes I_n & A_{22} \otimes I_n \end{pmatrix} \quad \text{and} \quad A^{\boxplus n} = \begin{pmatrix} A_{11}^{\oplus n} & A_{12}^{\oplus n} \\ A_{21}^{\oplus n} & A_{22}^{\oplus n} \end{pmatrix}.$$

Let $P_{m,n}$ be the permutation matrix

$$\begin{aligned} P_{m,n} = [&e_1, e_{n+1}, e_{2n+1}, \ldots, e_{(m-1)n+1}, e_2, e_{n+2}, e_{2n+2}, \ldots, e_{(m-1)n+2}, \\ &\ldots, \\ &e_{n-1}, e_{2n-1}, e_{3n-1}, \ldots, e_{mn-1}, e_n, e_{2n}, e_{3n}, \ldots, e_{(m-1)n}, e_{mn}] \end{aligned}$$

where e_i is the i-th standard basis vector of $\mathbb{C}^{mn}$ written as a column vector. Observe that $A_{ij} \otimes I_n = P_{m,n} A_{ij}^{\oplus n} P_{m,n}^{-1}$ for $i,j = 1,2$. This implies

$$L(A) = \begin{pmatrix} P_{m,n} & 0 \\ 0 & P_{m,n} \end{pmatrix} A^{\boxplus n} \begin{pmatrix} P_{m,n}^{-1} & 0 \\ 0 & P_{m,n}^{-1} \end{pmatrix}.$$

The result follows from Lemma 1.4.1. □

The 2-fold direct sum $\oplus^2 : \mathrm{O}(n) \to \mathrm{O}(2n)$ is such that $\mathrm{Im} \oplus^2 \subset \mathrm{Sp}(n)$. Let d denote this homomorphism $d : \mathrm{O}(n) \to \mathrm{Sp}(n)$

$$d(A) = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}.$$

Lemma 4.1.9. *Let $R : \mathrm{O}(n) \to \mathrm{Sp}(mn)$ be the homomorphism $R(A) = I_{2m} \otimes A$. Then $R(A) = d(A)^{\oplus m}$ for all $A \in \mathrm{O}(n)$.*

In view of Lemma 4.1.9, we focus our attention on determining the effect of the homomorphism d on homotopy groups in the stable range. We want to prove Proposition 4.1.14.

Theorem 4.1.10. *([22, I.4.11, I.4.12]) The inclusions*

1. $\mathrm{U}(n,\mathbb{C}) \to \mathrm{GL}(n,\mathbb{C})$
2. $\mathrm{O}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R})$
3. $\mathrm{O}(n,\mathbb{R}) \to \mathrm{O}(n,\mathbb{C})$
4. $\mathrm{U}(2n,\mathbb{C}) \cap \mathrm{Sp}(n,\mathbb{C}) \to \mathrm{Sp}(n,\mathbb{C})$

are homotopy equivalences.

There are field inclusions $\mathbb{R} \hookrightarrow \mathbb{C}$ and $\mathbb{C} \hookrightarrow \mathbb{H}$. The field $\mathbb{C}$ can be seen as a subfield of $\mathbb{H}$ by including $1 \mapsto 1$ and $i \mapsto i$. The conjugation on $\mathbb{H}$ restricts to that of $\mathbb{C}$. Let c, c', and q be the inclusions

$$c : \mathrm{O}(n,\mathbb{R}) \to \mathrm{U}(n,\mathbb{C}) \qquad c' : \mathrm{Sp}(n,\mathbb{C}) \cap \mathrm{U}(2n,\mathbb{C}) \to \mathrm{Sp}(n,\mathbb{C})$$
$$q : \mathrm{GL}(n,\mathbb{C}) \to \mathrm{GL}(n,\mathbb{H}).$$

Theorem 4.1.11. *([22, Theorem IV.5.12]) Let i be in the stable range for $\mathrm{O}(n,\mathbb{R})$. Then the induced homomorphism*

$$c_* : \pi_i(\mathrm{O}(n,\mathbb{R})) \longrightarrow \pi_i(\mathrm{U}(n,\mathbb{C}))$$

is an isomorphism onto $2\pi_i(\mathrm{U}(n,\mathbb{C}))$ if $i \equiv 3$ (mod 8) and is an isomorphism if $i \equiv 7$ (mod 8). In all other cases, it is trivial because the source is torsion and the target is torsion-free.

Observe that we can replace c by a map $\mathrm{O}(n,\mathbb{C}) \to \mathrm{GL}(n,\mathbb{C})$ without changing the effect in homotopy groups.

Lemma 4.1.12. *([22, pp. 22–23]) There exists a monomorphism of algebras $c' : \mathrm{M}(n,\mathbb{H}) \to \mathrm{M}(2n,\mathbb{C})$ given by*

$$c'(A + jB) = \begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix}.$$

The restriction of c' to $\mathrm{U}(n,\mathbb{H})$ is an isomorphism

$$c' : \mathrm{U}(n,\mathbb{H}) \xrightarrow{\cong} \mathrm{U}(2n,\mathbb{C}) \cap \mathrm{Sp}(n,\mathbb{C}).$$

Since the conjugation on $\mathbb{H}$ restricts that on $\mathbb{C}$, the inclusion q restricts to $q : \mathrm{U}(n,\mathbb{C}) \to \mathrm{U}(n,\mathbb{H})$, i.e.

$$q : \mathrm{U}(n,\mathbb{C}) \longrightarrow \mathrm{U}(2n,\mathbb{C}) \cap \mathrm{Sp}(n,\mathbb{C})$$

by Lemma 4.1.12.

The composite $c' \circ q : \mathrm{U}(n,\mathbb{C}) \to \mathrm{U}(2n,\mathbb{C}) \cap \mathrm{Sp}(n,\mathbb{C}) \xrightarrow{\cong} \mathrm{Sp}(n,\mathbb{C})$ is given by

$$(c' \circ q)(A) = \begin{pmatrix} A & 0 \\ 0 & \overline{A} \end{pmatrix}.$$

Theorem 4.1.13. *([22, IV.6.1.(2)]) Let i be in the stable range for $\mathrm{GL}(n,\mathbb{C})$. Then the induced homomorphism*

$$q_* : \pi_i(\mathrm{U}(n,\mathbb{C})) \longrightarrow \pi_i(\mathrm{U}(2n,\mathbb{C}) \cap \mathrm{Sp}(n,\mathbb{C}))$$

is an isomorphism onto $2\pi_i(\mathrm{U}(2n,\mathbb{C}) \cap \mathrm{Sp}(n,\mathbb{C}))$ if $i \equiv 3$ (mod 8), an epimorphism if $i \equiv 5$ (mod 8) (the source is infinite cyclic and the target has order 2) and is an isomorphism if $i \equiv 7$ (mod 8).

Proposition 4.1.14. *Let $i < n-1$. The homomorphism*

$$d_* : \pi_i(\mathrm{O}(n)) \longrightarrow \pi_i(\mathrm{Sp}(n))$$

is an isomorphism onto $2\pi_i(\mathrm{Sp}(n))$ if $i \equiv 3, 7$ (mod 8). In all other cases, either the source or the target is trivial.

Proof. Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{O}(n,\mathbb{C}) \arrow[rr, "d"] & & \mathrm{Sp}(n,\mathbb{C}) \\
\mathrm{O}(n,\mathbb{R}) \arrow[u, "\simeq"] \arrow[r, "c"] & \mathrm{U}(n,\mathbb{C}) \arrow[r, "q"] & \mathrm{Sp}(n,\mathbb{C}) \cap \mathrm{U}(2n,\mathbb{C}). \arrow[u, "\simeq", "c'"']
\end{tikzcd}$$

The effect of d_* on homotopy groups in the stable range is identified with the effect of $q_* \circ c_*$. From Theorems 4.1.10, 4.1.11, and 4.1.13 the result follows. $\square$

Proposition 4.1.15. *Let $4m+2 < n-1$ and $i < 4m+2$, the homomorphism $\otimes_* : \pi_i(\mathrm{Sp}(m)) \times \pi_i(\mathrm{O}(n)) \to \pi_i(\mathrm{Sp}(mn))$ is given by*

$$\otimes_*(x, y) = nx + 2my$$

for $x \in \pi_i(\mathrm{Sp}(m))$ and $y \in \pi_i(\mathrm{O}(n))$.

Proof. Lemmas 4.1.8 and 4.1.9, and Proposition 4.1.14 yield the result. $\square$

Effect of $\boxtimes : \mathrm{Sp}(m,\mathbb{C}) \times \mathrm{Sp}(n,\mathbb{C}) \to \mathrm{O}(4mn,\mathbb{C})$ on homotopy groups in the stable range of $\mathrm{Sp}(m,\mathbb{C})$

Recall that $\boxtimes : \mathrm{Sp}(m) \times \mathrm{Sp}(n) \to \mathrm{O}(4mn)$ is equal to the composite

$$\mathrm{Sp}(m) \times \mathrm{Sp}(n) \xrightarrow{\otimes} G \xrightarrow{\mathrm{Int}_P} \mathrm{O}(4mn),$$

where $G \leq \mathrm{GL}(4mn)$ and $P \in \mathrm{GL}(4mn)$ is a basis change matrix, item 3 in Section 1.4.2. Let L and R denote the restrictions of $\boxtimes$ to $\mathrm{Sp}(m) \times \{I_{2n}\}$ and $\{I_{2m}\} \times \mathrm{Sp}(n)$, respectively.

Let c' and c'' denote the inclusions

$$c' : \mathrm{Sp}(n,\mathbb{C}) \longrightarrow \mathrm{GL}(2n,\mathbb{C}) \qquad \text{and} \qquad c'' : \mathrm{O}(n,\mathbb{C}) \longrightarrow \mathrm{GL}(2n,\mathbb{C}).$$

Proposition 4.1.16. *Let $m \leq n$ and $i < 4m+2$, the homomorphism $\left(c''_* \circ \boxtimes_*\right) : \pi_i(\mathrm{Sp}(m)) \times \pi_i(\mathrm{Sp}(n)) \to \pi_i(\mathrm{GL}(4mn))$ is given by*

$$\left(c''_* \circ \boxtimes_*\right)(x, y) = 2nc'_*(x) + 2mc'_*(y)$$

for $x \in \pi_i(\mathrm{Sp}(m))$ and $y \in \pi_i(\mathrm{Sp}(n))$.

Proof. By the mixed-product property of the tensor product of matrices, if $(A, B) \in \mathrm{Sp}(m) \times \mathrm{Sp}(n)$ then $A \otimes B = (A \otimes I_{2n})(I_{2m} \otimes B)$. Hence $A \boxtimes B = L(A)R(B)$.

Consider the commutative diagrams

$$\begin{tikzcd}
\mathrm{Sp}(m) \arrow[r, "c'"] \arrow[d, "L"] & \mathrm{GL}(2m) \arrow[d, "L"] \\
G \arrow[r, hook] \arrow[d, "\mathrm{Int}_P", "\cong"'] & \mathrm{GL}(4mn) \arrow[d, "\mathrm{Int}_P", "\cong"'] \\
\mathrm{O}(4mn) \arrow[r, "c''"] & \mathrm{GL}(4mn)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Sp}(n) \arrow[r, "c'"] \arrow[d, "R"] & \mathrm{GL}(2n) \arrow[d, "R"] \\
G \arrow[r, hook] \arrow[d, "\mathrm{Int}_P", "\cong"'] & \mathrm{GL}(4mn) \arrow[d, "\mathrm{Int}_P", "\cong"'] \\
\mathrm{O}(4mn) \arrow[r, "c''"] & \mathrm{GL}(4mn).
\end{tikzcd}$$

From these we obtain the commutative squares below.

$$\begin{tikzcd}
\pi_i(\mathrm{Sp}(m)) \arrow[r, "c'_*"] \arrow[d, "L_*"] & \pi_i(\mathrm{GL}(2m)) \arrow[d, "L_*"] \\
\pi_i(\mathrm{O}(4mn)) \arrow[r, "c''_*"] & \pi_i(\mathrm{GL}(4mn))
\end{tikzcd} \tag{4.2}$$

$$\begin{tikzcd}
\pi_i(\mathrm{Sp}(n)) \arrow[r, "c'_*"] \arrow[d, "R_*"] & \pi_i(\mathrm{GL}(2n)) \arrow[d, "R_*"] \\
\pi_i(\mathrm{O}(4mn)) \arrow[r, "c''_*"] & \pi_i(\mathrm{GL}(4mn))
\end{tikzcd} \tag{4.3}$$

Recall that $R : \mathrm{GL}(2n) \to \mathrm{GL}(4mn)$ is equal to $(2n)$-fold direct sum, then for $y \in \pi_i(\mathrm{GL}(2n))$ the homomorphism $R_* : \pi_i(\mathrm{GL}(2n)) \to \pi_i(\mathrm{GL}(4mn))$ is such that $R_*(y) = 2m\,\mathrm{stab}(y)$. Moreover, $R, L : \mathrm{GL}(2n) \to \mathrm{GL}(4mn)$ are pointed homotopic, then L_* is equal to $R_* : \pi_i(\mathrm{GL}(2n)) \to \pi_i(\mathrm{GL}(4mn))$, and is given by $L_*(x) = 2n\,\mathrm{stab}(x)$ for $x \in \pi_i(\mathrm{GL}(2m))$.

Therefore, by the commutativity of the squares (4.2) and (4.3), we have that $L_* : \pi_i(\mathrm{Sp}(m)) \to \pi_i(\mathrm{O}(4mn))$ and $R_* : \pi_i(\mathrm{Sp}(n)) \to \pi_i(\mathrm{O}(4mn))$ satisfy the following equalities

$$(c''_* \circ L_*)(x) = 2nc'_*(x) \quad \text{and} \quad (c''_* \circ R_*)(y) = 2mc'_*(y)$$

for $x \in \pi_i(\mathrm{Sp}(m))$, $y \in \pi_i(\mathrm{Sp}(n))$ and $i < 4m+2$. □

Theorem 4.1.17. *([22, Theorem IV.5.16]) Let i be in the stable range for*

$\mathrm{Sp}(n,\mathbb{C})$. *Then the induced homomorphism*

$$c'_* : \pi_i(\mathrm{Sp}(n,\mathbb{C})) \longrightarrow \pi_i(\mathrm{GL}(n,\mathbb{C}))$$

is an isomorphism if $i \equiv 3$ *(mod 8) and is an isomorphism onto* $2\pi_i(\mathrm{GL}(2n,\mathbb{C}))$ *if* $i \equiv 7$ *(mod 8). In all other cases, it is trivial because the source is torsion and the target is torsion-free.*

Corollary 4.1.18. *Let* $m \le n$ *and* $i < 4m+2$*, the homomorphism*

$$\boxtimes_* : \pi_i(\mathrm{Sp}(m)) \times \pi_i(\mathrm{Sp}(n)) \longrightarrow \pi_i(\mathrm{O}(4mn))$$

is given by

$$\boxtimes_*(x,y) = \begin{cases} nx+my & \text{if } i \equiv 3 \text{ (mod 8)}, \\ 4(nx+my) & \text{if } i \equiv 7 \text{ (mod 8)}, \\ 0 & \text{otherwise}, \end{cases}$$

for $x \in \pi_i(\mathrm{Sp}(m))$ *and* $y \in \pi_i(\mathrm{SO}(n))$.

Proof. Let $i < 4m+2$ and $i \equiv 3,7$ (mod 8), then by Proposition 4.1.16 diagrams (4.2) and (4.3) take the form

$$\begin{tikzcd} \pi_i(\mathrm{Sp}(m)) \arrow[r, "c'_*"] \arrow[d, "L_*"] & \pi_i(\mathrm{GL}(2m)) \arrow[d, "\times(2n)"] \\ \pi_i(\mathrm{O}(4mn)) \arrow[r, "c''_*"] & \pi_i(\mathrm{GL}(4mn)) \end{tikzcd}$$

$$\begin{tikzcd} \pi_i(\mathrm{Sp}(n)) \arrow[r, "c'_*"] \arrow[d, "R_*"] & \pi_i(\mathrm{GL}(2n)) \arrow[d, "\times(2m)"] \\ \pi_i(\mathrm{O}(4mn)) \arrow[r, "c''_*"] & \pi_i(\mathrm{GL}(4mn)). \end{tikzcd}$$

By Theorems 4.1.11 and 4.1.17, if $i \equiv 3$ (mod 8), then c'_* is an isomorphism and c''_* is an isomorphism onto $2\pi_i(\mathrm{GL}(4mn,\mathbb{C}))$. Therefore, $2L_*(x) = 2nx$ and $2R_*(y) = 2my$, i.e. $L_*(x) = nx$ and $R_*(x) = my$.

By Theorems 4.1.11 and Theorem 4.1.17, if $i \equiv 7$ (mod 8), then c'_* is an isomorphism onto $2\pi_i(\mathrm{GL}(2l,\mathbb{C}))$ for $l = m,n$ and c''_* is an isomorphism. Therefore, $L_*(x) = 4nx$ and $R_*(y) = 4my$.

In all other cases, either the source or the target is trivial. $\square$

Corollary 4.1.19. *Let $i < 4m+2$, the homomorphism*

$$\boxtimes^2_* : \pi_i(\mathrm{Sp}(m)) \longrightarrow \pi_i(\mathrm{O}(4m^2))$$

is given by

$$\boxtimes^2_*(x) = \begin{cases} 2mx & \text{if } i \equiv 3 \text{ (mod 8)}, \\ 8mx & \text{if } i \equiv 7 \text{ (mod 8)}, \\ 0 & \text{otherwise}, \end{cases}$$

for $x \in \pi_i(\mathrm{Sp}(m))$.

4.1.4 Tensor product on the quotient

We want to describe the effect of the tensor product operation on the homotopy groups of the projective complex symplectic group.

The methods we used in Chapter 2 to establish the decomposition of topological Azumaya algebras apply to those whose degrees are relatively prime. For this reason, we only study the tensor product (4.4) where n is odd. The tensor product operation $\otimes : \mathrm{Sp}(m) \times \mathrm{SO}(n) \to \mathrm{Sp}(mn)$ sends $Z(\mathrm{Sp}(m)) \times \{I_n\}$ to $Z(\mathrm{Sp}(mn))$. As a consequence, the operation descends to the quotient

$$\otimes : \mathrm{PSp}(m) \times \mathrm{SO}(n) \longrightarrow \mathrm{PSp}(mn). \tag{4.4}$$

Proposition 4.1.20. *Let $i < 4m+1$. The homomorphism*

$$\otimes_* : \pi_i(\mathrm{PSp}(m)) \times \pi_i(\mathrm{SO}(n)) \longrightarrow \pi_i(\mathrm{PSp}(mn))$$

is given by $\otimes_*(x,y) = nx + 2my$ for $x \in \pi_i(\mathrm{PSp}(m))$, $y \in \pi_i(\mathrm{SO}(n))$.

Proof. There exits a map of fibrations

$$\begin{tikzcd}
Z(\mathrm{Sp}(m)) \times \{I_n\} \arrow[r, hook] \arrow[d, "\otimes"] & \mathrm{Sp}(m) \times \mathrm{SO}(n) \arrow[r, two heads] \arrow[d, "\otimes"] & \mathrm{PSp}(m) \times \mathrm{SO}(n) \arrow[d, "\otimes"] \\
Z(\mathrm{Sp}(mn)) \arrow[r, hook] & \mathrm{Sp}(mn) \arrow[r, two heads] & \mathrm{PSp}(mn).
\end{tikzcd} \tag{4.5}$$

Then there exits a homomorphism between the long exact sequences associated to the fibrations in diagram (4.5). For $i > 1$ we obtain a commutative square

$$\begin{tikzcd}
\pi_i(\mathrm{Sp}(m)) \times \pi_i(\mathrm{SO}(n)) \arrow[r, "\cong"] \arrow[d, "\otimes_*"] & \pi_i(\mathrm{PSp}(m)) \times \pi_i(\mathrm{SO}(n)) \arrow[d, "\otimes_*"] \\
\pi_i(\mathrm{Sp}(mn)) \arrow[r, "\cong"] & \pi_i(\mathrm{PSp}(mn)).
\end{tikzcd}$$

From this diagram and Proposition 4.1.15 we have that for all $1 < i < 4m+1$, $\otimes_*(x, y) = nx + 2my$ where $x \in \pi_i(\mathrm{PSp}(n))$ and $y \in \pi_i(\mathrm{SO}(n))$.

If $i = 0, 1$, then $\otimes_*$ is trivial. $\square$

4.2 Proof of Theorem 1.3.6

Let m and n be positive integers such that n is odd.

By applying the classifying-space functor to the homomorphism (4.4) we obtain a map

$$f_\otimes : \mathrm{B\,PSp}(m) \times \mathrm{B\,SO}(n) \longrightarrow \mathrm{B\,PSp}(mn).$$

Proposition 4.2.1. *Let m and n be positive integers such that n is odd. There exist positive integers u and v satisfying $\left|vn - 4um^2\right| = 1$, so that there exist a positive integer N and a homomorphism $\widetilde{\mathrm{T}} : \mathrm{PSp}(m) \times \mathrm{SO}(n) \to \mathrm{SO}(N)$ such that the homomorphisms induced on homotopy groups*

$$\widetilde{\mathrm{T}}_i : \pi_i(\mathrm{PSp}(m)) \times \pi_i(\mathrm{SO}(n)) \longrightarrow \pi_i(\mathrm{SO}(N))$$

are given by the following expressions. Let d denote $\min\{4m+2, n-1\}$.

1. *If* $1 < i < d$, *then*

$$\widetilde{\mathrm{T}}_i(x,y) = \begin{cases} 2umx + vy & \text{if } i \equiv 3 \text{ (mod 8)}, \\ 8umx + vy & \text{if } i \equiv 7 \text{ (mod 8)}, \\ 0 & \text{otherwise}, \end{cases}$$

   for $x \in \pi_i(\mathrm{PSp}(m))$ *and* $y \in \pi_i(\mathrm{SO}(n))$.

2. *If* $i = 1$, *then* $\widetilde{\mathrm{T}}_1(x,y) = zx + y$ *for* $x, y \in \mathbb{Z}/2$ *and some* $z \in \mathbb{Z}/2$.

Proof. Without loss of generality suppose $4m+2 < n-1$. There exist positive integers u and v such that $vn - 4um^2 = \pm 1$ by Lemma 1.4.2. Let N denote $4um^2 + vn$, and let T denote the composite

$$\begin{tikzcd} \mathrm{Sp}(m) \times \mathrm{SO}(n) \arrow[d, "(\boxtimes^2, \mathrm{id})"] \\ \mathrm{SO}(4m^2) \times \mathrm{SO}(n) \arrow[d, "(\oplus^u, \oplus^v)"] \\ \mathrm{SO}(4um^2) \times \mathrm{SO}(vn) \arrow[d, "\oplus"] \\ \mathrm{SO}(N). \end{tikzcd}$$

Note that the elements $(\pm I_m, I_n)$ are sent to (I_{4m^2}, I_n) by $(\otimes^2, \mathrm{id})$, hence to the identity by the composite T defined above. Hence T factors through $\mathrm{PSp}(m) \times \mathrm{SO}(n)$

$$\begin{tikzcd} \mathrm{Sp}(m) \times \mathrm{SO}(n) \arrow[d, two heads] \arrow[drr, bend left, "\mathrm{T}"] & & \\ \mathrm{PSp}(m) \times \mathrm{SO}(n) \arrow[rr, "\widetilde{\mathrm{T}}"'] & & \mathrm{SO}(N). \end{tikzcd}$$

From Corollaries 4.1.5 and 4.1.19 we have that for $i < 4m+2$

$$\mathrm{T}_i(x,y) = \begin{cases} 2umx + vy & \text{if } i \equiv 3 \pmod 8, \\ 8umx + vy & \text{if } i \equiv 7 \pmod 8, \\ 0 & \text{otherwise.} \end{cases}$$

The map of fibrations

$$\begin{tikzcd} Z(\mathrm{Sp}(m)) \times \{I_n\} \arrow[r, hook] \arrow[d] & \mathrm{Sp}(m) \times \mathrm{SO}(n) \arrow[r, two heads] \arrow[d, "\mathrm{T}"] & \mathrm{PSp}(m) \times \mathrm{SO}(n) \arrow[d, "\widetilde{\mathrm{T}}"] \\ \{I_N\} \arrow[r] & \mathrm{SO}(N) \arrow[r, equal] & \mathrm{SO}(N) \end{tikzcd}$$

induces a commutative diagram

$$\begin{tikzcd} \pi_i(\mathrm{Sp}(m)) \times \pi_i(\mathrm{SO}(n)) \arrow[r, "\cong"] \arrow[d, "\mathrm{T}_i"] & \pi_i(\mathrm{PSp}(m)) \times \pi_i(\mathrm{SO}(n)) \arrow[d, "\widetilde{\mathrm{T}}_i"] \\ \pi_i(\mathrm{SO}(N)) \arrow[r, equal] & \pi_i(\mathrm{SO}(N)) \end{tikzcd}$$

for $i > 1$. Then $\widetilde{\mathrm{T}}_i(x,y) = \mathrm{T}_i(x,y)$ for $1 < i < 4m+2$.

For $i = 1$, the map of fibrations induces the commutative diagram below

$$\begin{tikzcd} \pi_1(\mathrm{Sp}(m)) \times \pi_1(\mathrm{SO}(n)) \arrow[r, hook] \arrow[d, "\mathrm{T}_1"] & \pi_1(\mathrm{PSp}(m)) \times \pi_1(\mathrm{SO}(n)) \arrow[d, "\widetilde{\mathrm{T}}_1"] \\ \pi_1(\mathrm{SO}(N)) \arrow[r, equal] & \pi_1(\mathrm{SO}(N)), \end{tikzcd}$$

which takes the form

$$\begin{tikzcd} 0 \times \mathbb{Z}/2 \arrow[r, hook] \arrow[dr, "\mathrm{T}_1"'] & \mathbb{Z}/2 \times \mathbb{Z}/2 \arrow[d, "\widetilde{\mathrm{T}}_1"] \\ & \mathbb{Z}/2. \end{tikzcd} \tag{4.6}$$

Observe that the equality $vn - 4um^2 = \pm 1$ implies v is odd, then $\mathrm{T}_1(0,\beta) = \beta$, i.e. T_1 is the identity. Thus, $\widetilde{\mathrm{T}}_1(0,1) = 1$. Let z denote $\widetilde{\mathrm{T}}_1(1,0)$. Then $\widetilde{\mathrm{T}}_1(x,y) = zx + y$. $\square$

4.2.1 A d-connected map where $d = \min\{4m+3, n\}$

Let J denote the map

$$\begin{aligned} J : \mathrm{BPSp}(m) \times \mathrm{BSO}(n) &\longmapsto \mathrm{BPSp}(mn) \times \mathrm{BSO}(N) \\ (x,y) &\longmapsto \left(f_{\otimes}(x,y), \mathrm{B}\widetilde{\mathrm{T}}(x,y)\right). \end{aligned} \tag{4.7}$$

Let J_i denote the homomorphism induced on homotopy groups by J.

$$J_i : \pi_i(\mathrm{BPSp}(m)) \times \pi_i(\mathrm{BSO}(n)) \longrightarrow \pi_i(\mathrm{BPSp}(mn)) \times \pi_i(\mathrm{BSO}(N)) \tag{4.8}$$

Proposition 4.2.2. *Let m and n be positive integers such that n is odd. Let d denote $\min\{4m+3, n\}$ and let $i < d$, then the homomorphism J_i is an isomorphism if $i > 0$ and $i \not\equiv 0$ (mod 8).*

Proof. Without loss of generality we can assume $4m+3 < n$.

Let $0 < i < 4m+1$ and $i \not\equiv 0 \pmod 8$. Observe that J_i is trivial for $i = 0, 1$ and, $i \equiv 3, 7 \pmod 8$ with $i > 2$. Then we study the cases $i = 2$ and, $i \equiv 1, 2, 4, 5, 6 \pmod 8$ with $i > 2$.

Let $i = 2$. By Propositions 4.1.20 and 4.2.1 the homomorphism $J_2 : \mathbb{Z}/2 \times \mathbb{Z}/2 \to \mathbb{Z}/2 \times \mathbb{Z}/2$ is represented by the invertible matrix

$$\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}.$$

Let $2 < i < 4m+1$ and $i \equiv 4 \pmod 8$. By Propositions 4.1.20 and 4.2.1 the homomorphism $J_i : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ is represented by the invertible matrix

$$\begin{pmatrix} n & 2m \\ 2um & v \end{pmatrix}.$$

Let $2 < i < 4m+1$ and $i \equiv 1, 2 \pmod 8$. By Propositions 4.1.20 and 4.2.1 the homomorphism $J_i : 0 \times \mathbb{Z}/2 \to \mathbb{Z}/2 \times 0$ is given by $J_i(0, y) = (2my, vy) = (0, y)$.

Let $2 < i < 4m+1$ and $i \equiv 5, 6 \pmod 8$. By Propositions 4.1.20 and 4.2.1 the homomorphism $J_i : \mathbb{Z}/2 \times 0 \to 0 \times \mathbb{Z}/2$ is given by $J_i(x, 0) = (nx, 2umx) = (x, 0)$.

Let m be even, then $4m+1 \equiv 1$ and $4m+2 \equiv 2 \pmod 8$. Thus, J_{4m+1} and J_{4m+2} take the form $0 \times \mathbb{Z}/2 \to \mathbb{Z}/2 \times 0$. As in the case $i \equiv 1,2$, we have $J_{4m+1}(0,y) = J_{4m+2}(0,y) = (0,y)$.

Let m be odd, then $4m+1 \equiv 5$ and $4m+2 \equiv 6 \pmod 8$. Thus, J_{4m+1} and J_{4m+2} take the form $\mathbb{Z}/2 \times 0 \to 0 \times \mathbb{Z}/2$. As in the case $i \equiv 5,6$, we have $J_{4m+1}(x,0) = J_{4m+2}(x,0) = (x,0)$. □

As corollary of Propositions 4.2.2 we obtain the following.

Corollary 4.2.3. *Let m and n be positive integers such that $m > 1$, $n > 7$, and n is odd. Then the map J is 7-connected.*

4.2.2 Factorization through $f_\otimes : \mathrm{BPSp}(m,\mathbb{C}) \times \mathrm{BSO}(n,\mathbb{C}) \to \mathrm{BPSp}(mn,\mathbb{C})$

Theorem 1.3.6 is a Corollary of Theorem 4.2.4.

Theorem 4.2.4. *Let X be a CW complex such that $\dim(X) \le 7$. Let m and n be positive integers such that $m > 1$, $n > 7$, and n is odd. Every map $\mathscr{A} : X \to \mathrm{BPSp}(mn)$ can be lifted to $\mathrm{BPSp}(m) \times \mathrm{BSO}(n)$ along the map $f_\otimes$.*

Proof. Diagrammatically speaking, we want to find a map

$$\mathscr{A}_{2m} \times \mathscr{A}_n : X \longrightarrow \mathrm{BPSp}(m) \times \mathrm{BSO}(n)$$

such that diagram (4.9) commutes up to homotopy

$$\begin{tikzcd} & \mathrm{BPSp}(m) \times \mathrm{BSO}(n) \arrow[d, "f_\otimes"] \\ X \arrow[ur, dotted, bend left, "\mathscr{A}_{2m} \times \mathscr{A}_n"] \arrow[r, "\mathscr{A}"] & \mathrm{BPSp}(mn). \end{tikzcd} \tag{4.9}$$

Without loss of generality supposse $4m+2 < n-1$. Corollary 4.2.3 yields a map $J : \mathrm{BPSp}(m) \times \mathrm{BSO}(n) \to \mathrm{BSO}(N)$ where N is some positive integer so that $N \gg n-1 > 4m+2$. Observe that $f_\otimes$ factors through $\mathrm{BPSp}(mn) \times \mathrm{BSO}(N)$, so we can write $f_\otimes$ as the composite of J and the projection proj_1 shown in diagram (4.10).

$$\begin{tikzcd} \mathrm{BPSp}(m) \times \mathrm{BSO}(n) \arrow[r, "J"] \arrow[d, "f_{\otimes}"] & \mathrm{BPSp}(mn) \times \mathrm{BSO}(N) \arrow[dl, "\mathrm{proj}_1"'] \\ \mathrm{BPSp}(mn) & \end{tikzcd} \tag{4.10}$$

Since J is 7-connected and $\dim(X) \leq 7$, then by Whitehead's theorem

$$J_{\#} : [X, \mathrm{BPSp}(m) \times \mathrm{BSO}(n)] \longrightarrow [X, \mathrm{BPSp}(mn) \times \mathrm{BSO}(N)]$$

is a surjection, [25, Corollary 7.6.23].

Let s denote a section of proj_1. The surjectivity of $J_{\#}$ implies $s \circ \mathscr{A}$ has a preimage $\mathscr{A}_{2m} \times \mathscr{A}_n : X \to \mathrm{BPSp}(2) \times \mathrm{BSO}(n)$ such that $J \circ (\mathscr{A}_{2m} \times \mathscr{A}_n) \simeq s \circ \mathscr{A}$.

Commutativity of diagram (4.9) follows from commutativity of diagram (4.10). Thus, the result follows. $\square$

Chapter 5

Conclusions

In this chapter we summarize the contributions to the discipline, and discuss lingering questions and further research directions arising from this work.

5.1 Contributions

We know that there is no prime decomposition of topological Azumaya algebras in general. Furthermore, there is no prime decomposition for Azumaya algebras over a commutative ring as there is for central simple algebras, [2, Corollary 1.3]. In Chapter 2, we show that there exists a not-neccesarily-unique tensor product decomposition for topological Azumaya algebras over low dimensional CW complexes, and that such decomposition does not exist for topological Azumaya algebras over an arbitrary CW complex. The proof of Theorem 2.2.6 implies that for positive integers m and n where $m < n$, if $\mathscr{A}$ is a topological Azumaya algebra of degree mn over a finite CW complex of dimension higher than $2m+1$, then $\mathscr{A}$ may not be decomposable as $\mathscr{A}_m \otimes \mathscr{A}_n$. In fact, consider the unit $(2m+2)$-sphere, and let $\mathscr{S} : S^{2m+2} \to \mathrm{BPU}(mn,\mathbb{C})$ be a degree-mn topological Azumaya algebra on S^{2m+2} such that $\mathscr{S}$ generates $\pi_{2m+2}(\mathrm{BPU}(mn,\mathbb{C}))$, then $\mathscr{S}$ cannot be decomposed as the tensor product of topological Azumaya algebras of degrees m and n.

In Chapter 3, we prove that a topological Azumaya algebra over a low dimensional CW complex carrying an orthogonal involution is decomposable as the tensor product of a topological Azumaya algebra with an orthogonal involution and a Brauer-trivial topological Azumaya algebra with an orthogonal involution.

In Chapter 4, we prove that a topological Azumaya algebra over a CW complex of dimension less than 8 with a symplectic involution can be decomposed as the tensor product of a topological Azumaya algebra with a symplectic involution and a Brauer-trivial topological Azumaya algebra with an orthogonal involution.

5.2 Future Directions

5.2.1 Azumaya algebras over smooth complex varieties

As mentioned in Chapter 1, Grothendieck generalized the notion of an Azumaya algebra over a commutative ring to the concept of an Azumaya algebra over a smooth complex variety $(X,\mathscr{O})$. Grothendieck's definition specializes to the topological case by taking the sheaf $\mathscr{O}$ to be the sheaf of continuous functions with value $\mathbb{C}$.

We prove in Chapter 2 that topological Azumaya algebras over a low dimensional CW complex are decomposable as the tensor product of topological Azumaya algebras of lower degrees. Then one can consider the decomposition question in the more general context of smooth complex varieties. Let X be a smooth complex variety. A *degree-n Azumaya algebra over X* is a locally-free sheaf of algebras $\mathscr{A}$ such that there is an étale cover $\pi : U \to X$ such that $\pi^*\mathscr{A} \cong \mathrm{M}(n,\mathscr{O}_U)$.

Question 5.2.1. Let $\mathscr{A}$ be a degree-mn Azumaya algebra over a smooth complex variety X. Can $\mathscr{A}$ be decomposed as the tensor product $\mathscr{A}_m \otimes \mathscr{A}_n$, where $\mathscr{A}_m$ and $\mathscr{A}_n$ are Azumaya algebras of degrees m and n, respectively?

5.2.2 Classification of topological Azumaya algebras

Conjecture 5.2.2. Let $i \in \{1,\dots,r\}$. Let n_i and p_i be positive integers such that p_i is prime for all i and $p_1 < \cdots < p_r$. There exist a CW complex of dimension at least $2p_1+2$ and a topological Azumaya algebra $\mathscr{A}$ on X of degree $p_1^{n_1}\cdots p_r^{n_r}$

such that $\mathscr{A}$ has no decomposition $\mathscr{A} \cong \mathscr{A}_{p_1} \otimes \cdots \otimes \mathscr{A}_{p_r}$ where $\deg(\mathscr{A}_{p_i}) = p_i^{n_i}$ for all i.

Partial results have been found. In fact, Theorem 1.2.6 provides an example of a topological Azumaya algebra of degree $2n$ over a space X where $n > 1$ is an odd integer, $\dim(X) > 5$ and $\mathscr{A}$ does not decompose as $\mathscr{A} \cong \mathscr{A}_2 \otimes \mathscr{A}_n$.

5.2.3 Bijectivity of $\mathcal{D}$ sets given a map of spaces

Let X and Y be simply connected spaces, and let $f : Y \to X$ be a continuous map of finite-dimensional CW complexes. Let $\alpha \in \mathrm{Br}(X)$ such that $\mathrm{per}(\alpha) = n$.

$$\begin{tikzcd}
Y \arrow[r, "\mathscr{A}'"] \arrow[ddddddd, "f"'] & \mathrm{B}\,\mathrm{PU}(n,\mathbb{C}) \arrow[d] & \\
& W[2n+1] \arrow[d] & \\
& W[2i+2] \arrow[d] & \\
& W[2i] \arrow[r, "k_{2i+3}"] \arrow[d] & \mathrm{K}(\mathbb{Z}, 2i+3) \\
& \vdots \arrow[d] & \\
& W[4] \arrow[r, "k_7"] \arrow[d] & \mathrm{K}(\mathbb{Z}, 7) \\
& \mathrm{K}(\mathbb{Z}/n, 2) \arrow[r, "\mathbf{0}"] \arrow[d, "\tilde{\beta}_n"] & \mathrm{K}(\mathbb{Z}, 5) \\
X \arrow[r, "\alpha"'] \arrow[ur, bend left] \arrow[uur, dotted, bend left] \arrow[uuuur, dotted, bend left] & \mathrm{K}(\mathbb{Z}, 3) &
\end{tikzcd} \tag{5.1}$$

The map f induces a homomorphism $f^* : \mathrm{Br}(X) \to \mathrm{Br}(Y)$ which produces a Brauer class of Y, $f^*(\alpha)$. Observe that f also induces an injective map of sets

$$\begin{aligned}
\tilde{f} : \mathcal{D}(\alpha) &\longrightarrow \mathcal{D}\big(f^*(\alpha)\big) \\
\deg(\mathscr{A}) &\longmapsto \deg(\mathscr{A} \circ f).
\end{aligned}$$

Conjecture 5.2.3. If $\dim(X)$ is in the stable range of $\mathrm{BPU}(n,\mathbb{C})$ and f induces an isomorphism on $\mathrm{H}_*(-;\mathbb{Z}_{(n)})$, then $\tilde{f}$ is surjective.

In what follows we refer the reader to diagram (5.1). In order to prove that $\tilde{f}$ is surjective, we must take a topological Azumaya algebra of degree n over Y representing $f^*(\alpha)$, namely $\mathscr{A}'$, and find a lifting of α along the Moore-Postnikov tower of $\mathrm{BPU}(n,\mathbb{C})$ so that the rectangle in diagram (5.1) commutes up to homotopy. This leads us to "smaller" lifting problems given by the stages of the Moore-Postnikov tower.

We believe that under the hypothesis of the conjecture above we can apply the methods used in Chapters 2 to find a lifting of α.

5.2.4 Lifting a topological Azumaya algebra at the boundary

Let X be a CW complex. Let m and n be positive integers such that $m < n$. Consider the relative Postnikov tower of the map $f_\otimes$ in diagram (5.2) for large values of m

$$\begin{tikzcd}
 & \mathrm{BPU}(m,\mathbb{C})\times\mathrm{BPU}(n,\mathbb{C}) \arrow[d] & \\
F_{2m+1} \arrow[r] & Y[2m+1] \arrow[d] & \\
 & Y[2m] \arrow[r, "k_{2m+1}"] \arrow[d] & \mathrm{K}\big(\pi_{2m+1}F_{2m+1}, 2m+2\big) \\
\mathrm{sk}_{2m+1}X \arrow[r, "\mathscr{A}"] \arrow[ur, bend left, "\xi"] \arrow[uur, dotted, bend left, "\xi'"] & \mathrm{BPU}(mn,\mathbb{C}) &
\end{tikzcd}$$

where $k_{i+1} : Y[i] \to \mathrm{K}\big(\pi_{i+1}F_{i+1}, i+2\big)$ is the k-invariant that classifies the fiber sequence $F_{i+1} \to Y[i+1] \to Y[i]$, and $\mathrm{sk}_{2m+1}X$ is the $(2m+1)$st skeleton of X. The map T from Theorem 1.2.8 guarantees the existence of the lifting ξ. The k-invariant k_{2m+1} is the obstruction to lifting ξ to $Y[2m+1]$.

There exists a lifting ξ' of ξ if and only if the composite

$$k_{2m+1}\circ\xi \in \mathrm{H}^{2m+2}\big(\mathrm{sk}_{2m+1}X; \pi_{2m+1}F_{2m+1}\big)$$

vanishes. Hence it becomes relevant to understand the k-invariant k_{2m+1} in order to solve the lifting problem in diagram (5.2).

$$\begin{tikzcd} & \mathrm{BPU}(m,\mathbb{C}) \times \mathrm{BPU}(n,\mathbb{C}) \arrow[d, "f_\otimes"] \\ X \arrow[ur, dotted, "\mathscr{A}_m \times \mathscr{A}_n"] \arrow[r, "\mathscr{A}"] & \mathrm{BPU}(mn,\mathbb{C}) \end{tikzcd} \tag{5.2}$$

Question 5.2.4. When is $k_{2m+1} \circ \xi$ nullhomotopic? Can the cohomology class $k_{2m+1} \in \mathrm{H}^{2m+2}\big(Y[2m]; \pi_{2m+1} F_{2m+1}\big)$ be computed?

Since k_{2m+1} has cohomological degree $(2m+2)$ (the first number out of the stable range), we refer to this problem as the lifting problem at the boundary.

5.2.5 Decomposition of topological Azumaya algebras with symplectic involution

We studied the decomposition of topological Azumaya algebras with symplectic involution in Chapter 4. We proved in this chapter a decomposition theorem for topological Azumaya algebras over CW complexes of dimension less than 8, Theorem 1.3.6. This result differs from those in Theorems 1.2.8 and 1.3.5 because the dimension of the space is more restricted. The reason for this restriction is that the homomorphism $\boxtimes^2_* : \pi_i(\mathrm{Sp}(m,\mathbb{C})) \to \pi_i(\mathrm{SO}(4m^2,\mathbb{C}))$ is not consistent in the stable range of $\mathrm{Sp}(m,\mathbb{C})$, Corollary 4.1.19. Therefore, using our method, it is not possible to construct the map $\widetilde{\mathrm{T}}$ in Proposition 4.2.1 so that the map J in (5.3) is $(4m+3)$-connected.

$$\begin{aligned} J : \mathrm{BPSp}(m,\mathbb{C}) \times \mathrm{BSO}(n,\mathbb{C}) &\longrightarrow \mathrm{BPSp}(mn,\mathbb{C}) \times \mathrm{BSO}(N,\mathbb{C}) \\ (x,y) &\longmapsto \Big(f_\otimes(x,y), \mathrm{B}\widetilde{\mathrm{T}}(x,y)\Big) \end{aligned} \tag{5.3}$$

We may overcome this impasse by pre-composing or post-composing the map $\mathrm{B}\boxtimes^2 : \mathrm{BSp}(m,\mathbb{C}) \to \mathrm{BSO}(4m^2,\mathbb{C})$ with self-maps

$$\mathrm{BSp}(m,\mathbb{C}) \longrightarrow \mathrm{BSp}(m,\mathbb{C}) \qquad \text{or} \qquad \mathrm{BSO}(4m^2,\mathbb{C}) \longrightarrow \mathrm{BSO}(4m^2,\mathbb{C}),$$

respectively, so that the induced map on homotopy groups has a consistent effect in the stable range of $\mathrm{Sp}(m,\mathbb{C})$. We might be able to construct these maps using a combination of unstable Adams operations.

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