Combinatorial problems for intervals, fractal sets and tubes

by

Daniel Di Benedetto

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Combinatorial problems for intervals, fractal sets and tubes

Submitted by **Daniel Di Benedetto** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy** in **Mathematics**.

Examining Committee:

Joshua Zahl, Mathematics, UBC

Supervisor

József Solymosi, Mathematics, UBC

Co-supervisor

Young-Heon Kim, Mathematics, UBC

University Examiner

William Evans, Computer Science, UBC

University Examiner

Additional Supervisory Committee Members:

Izabella Łaba, Mathematics, UBC

Supervisory Committee Member

Andrew Rechnitzer, Mathematics, UBC

Supervisory Committee Member

Abstract

We discuss three problems related to combinatorial geometry, based on two separate works. The first work concerns arrangements of intervals in $\mathbb{R}^2$ for which there are many pairs forming trapezoids, meaning the convex hull of the pair is a trapezoid. We characterise arrangements forming more than a certain number of trapezoids, showing that all such sets have underlying algebraic structure. An important role is played in particular by conic curves. The proof uses a transformation from intervals in the plane to lines in $\mathbb{R}^3$ and then relies on a theorem of Guth and Katz on intersecting lines in $\mathbb{R}^3$.

The second work concerns combinatorial problems for discretised sets, where objects are only distinguishable up to some small scale. Discretised sets can be used to approximate fractal sets, and our results imply improved quantitative bounds for the 1/2-Furstenberg set problem in $\mathbb{R}^2$ and the upper Minkowski dimension of Besicovitch sets in $\mathbb{R}^3$, as well as slight generalisations of each of these problems. The techniques involved in this second work are mostly combinatorial and our main ingredient is the discretised sum-product theorem from additive combinatorics. In particular, we reduce the 1/2-Furstenberg set problem to the discretised-sum product problem and reduce the Besicovitch set problem to the Furstenberg set problem.

Lay Summary

Geometric objects are constrained in how they can behave. For example, a pair of lines can intersect in at most one point and this restricts the properties that are possible for arrangements of lines. The general theme considered in this thesis is to understand what arrangements are possible for certain types of geometric objects.

We first consider line segments in the plane. We show that there are essentially only three ways that an arrangement of line segments can have many pairs forming trapezoids. We then consider sets of points in the plane for which there are many lines that are close to many points from the set. We show that, under certain conditions, such a set of points must always be large. Lastly, we show that a set in 3-dimensional space containing a line segment of a fixed length in each direction must also be large.

Preface

This thesis is based on two papers, one of which is currently under review for publication by an academic journal and the other is yet to be submitted for publication. The work involved in these papers is the original intellectual product of the author, Daniel Di Benedetto, having worked on parts in collaboration with various coauthors.

Chapter 2 is based on the paper *Combinatorics of intervals in the plane I*. This is a joint work with Prof. József Solymosi and Ethan White and has been submitted for publication. The starting point for the paper was the map defined in the proof of Lemma 2.4, which was an idea of József Solymosi. The analysis of the various cases was performed by the author, in collaboration with Ethan White. Half of the figures were created by the author and half were created by Ethan White. The writing was done collaboratively by the author and Ethan White, under the guidance of József Solymosi.

Chapters 3 and 4 are based on the work *New results for Besicovitch sets and* $(\alpha, 2\alpha)$*-Furstenberg sets*. This is a joint work with Prof. Joshua Zahl and will be submitted for publication. The suggestion that the improvements for the two problems were possible using the techniques came from Joshua Zahl. The applications of the various techniques and the calculations that yielded the improvements were done by the author. All figures were created by the author. The writing was done by the author, under the guidance of Joshua Zahl.

Most of the content from these papers is unchanged in this thesis, except for stylistic changes.

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First and foremost, I would like to thank my supervisors Josh Zahl and József Solymosi for their guidance, support and encouragement over the course of my PhD. This thesis would not exist if it were not for them.

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I would like to thank my collaborator and friend, Ethan White. I am grateful for having had the chance to work together.

Finally, I would like to thank my friends, for helping me to have a life beyond mathematics over the past four years.

Dedication

To Anjali and my family.

Chapter 1

Introduction

Sets of geometric objects have natural associated combinatorial properties, such as the number of intersections of a certain type or the numbers of occurrences of a specific geometric configuration. In many cases, the geometry of the objects imposes constraints on the possible values of these combinatorial properties, and combinatorial geometry is broadly the study of these constraints.

We focus on extremal problems related to combinatorial geometry, which involve understanding how much combinatorial structure is possible for sets of geometric objects. A classical result of this type is the Szemerédi–Trotter theorem from [30], which gives an upper bound on the number of point-line incidences between sets of points and lines of a given size.

This thesis is based on two works. Chapter 2 is based on the first work, which studies a variant of a typical question in discrete geometry on arrangements with many occurrences of a particular configuration. In the second work, we apply combinatorial techniques to problems in geometric measure theory that have connections to harmonic analysis. This work contains two main results. Chapter 3 focuses on the first of these; chapter 4 focuses on the second.

Throughout, we use the notation $A \gtrsim B$ to mean there exists a universal constant $C > 0$ for which $A \geq CB$ and $A \lesssim B$ is defined similarly. If both directions hold, we write $A \sim B$.

1.1 Combinatorics of intervals

The first work concerns a combinatorial problem for sets of intervals, or line segments, in the plane. This is a natural analogue of extremal questions that have been studied for point sets, where one fixes a geometric configuration formed by some tuple of points and asks about the maximum number of realisations of this configuration for any point set of a given cardinality.

A well known example of a problem of this type is Erdős's unit distance problem, first posed in [13], which seeks an upper bound on the number of pair of points that can have unit distance within a set of n points. Similarly, Erdős and Purdy [14] attribute to Oppenheim an analogous question, where pairs of points with unit distance are replaced with triples of points forming a triangle with unit area. Also in [14], they raise questions on the maximum number of pairwise congruent triangles and the maximum number of equilateral triangles.

For any of these extremal problems, an interesting further question is: what can be said about the extremal arrangements? In many cases, the known constructions share some form of rigid structure and this suggests that such structure may be necessary. For example, a conjecture of Elekes [12] says that a point set with quadratically many collinear triples must have many points contained on a single cubic curve.

Elekes's conjecture is consistent with a more general theme in combinatorial geometry relating combinatorial structure to algebraic structure. This theme has driven many recent advances in the field through the use of algebraic methods. A particularly effective technique has been the polynomial method, which involves studying a polynomial related to the underlying set, such as a polynomial vanishing on a certain subset. In successful applications of this method, combinatorial properties of the set imply algebraic properties of the polynomial and these properties can in turn be used to analyse the set.

Combinatorial problems for arrangements of intervals in the plane have also been studied, for example in [29, 34, 35], with research focusing on their intersection or visibility properties. We instead consider a variant of the questions asked by Erdős and Purdy, focusing on intervals forming a particular geometric configuration.

For a pair of line segments in the plane, one can consider the shape formed by the convex hull of the two segments. The case where this shape is a trapezoid is somewhat special, in the sense that a generic pair of line segments will not have this property. We address a related extremal problem, showing that arrangements forming many trapezoids must have underlying structure. In particular, we show that sets of intervals forming more than a certain number of trapezoids must have a large subset that is algebraically structured.

Theorem 1.1. *Let $\mathfrak{I}$ be a set of N distinct intervals in $\mathbb{R}^2$. If more than $\gtrsim N^{3/2}\log N$ pairs of intervals form trapezoids then at least one of the following holds.*

0. There are $\gtrsim N^{1/2}\log N$ parallel intervals.

1. There is a centre S and a ratio λ such that for $\gtrsim N^{1/2}$ intervals from $\mathfrak{I}$, *one endpoint is the image of the other under a homothety with centre S and ratio λ.*

2. *There are two curves in $\mathbb{R}^2$ of degree at most 2 such that $\gtrsim N^{1/2}$ intervals from $\mathfrak{I}$ have an endpoint on each curve.*

Our main tool in proving Theorem 2.1 is an incidence theorem of Guth and Katz [15], obtained as part of their solution to Erdős's distinct distances problem in the plane using the polynomial method. Specifically, we use an incidence theorem on lines in $\mathbb{R}^3$, which asserts that if a set of lines has many pairwise incidences, then a large subset of the lines must be structured.

We show that intervals in the plane can be mapped to lines in $\mathbb{R}^3$ in such a way that pairs forming trapezoids correspond to intersecting lines. This allows us to apply the Guth–Katz theorem to determine that if a set of intervals forms many trapezoids, then the corresponding line set is structured. The main content of the proof involves understanding how this structure relates to the original set of intervals.

A feature of the proof is that it gives a way to generate examples of arrangements of intervals forming many trapezoids, which in many cases appear difficult to construct by other means. We present a number of these constructions.

1.2 Combinatorics of fractal sets

In the second work, we study two related problems in geometric measure theory. While these problems also have the flavour of combinatorial geometry, in this regime the sets of interest are infinite, with cardinality being replaced by measure or by some notion of fractal dimension.

A *Besicovitch set* is a compact set $E \subset \mathbb{R}^n$ containing a line segment in every possible direction. From the combinatorial perspective, one natural extremal question is how small such sets can be, given that a small Besicovitch set would have to contain a set of line segments with many intersections of high multiplicity.

In 1919, Besicovitch [2] showed that there exist such sets of Lebesgue measure zero. His motivation concerned Riemann integration: the existence of such sets proved that there are functions that are Riemann-integrable (by Lebesgue's criterion) in the plane but for which there exists no orthogonal coordinate system with respect to which the function could be evaluated using the repeated Riemann-integration in these two directions.

There are finer notions of size available in geometric measure theory, such as Hausdorff and Minkowski dimension. The idea behind these notions is to extend the usual notion of dimension to one that can assume non-integer values; in many cases, these notions can distinguish between 'small' and 'large' sets of Lebesgue measure zero.

Definition 1.2. *For a bounded set $E \subset \mathbb{R}^n$ and a small number $\delta > 0$, the* covering number *of E, which we denote by $\mathcal{E}_\delta(E)$, is the number of balls of radius δ needed to cover E. The* upper Minkowski dimension *of E is*

$$\overline{\dim}_M(E) = \limsup_{\delta \to 0} \frac{\log \mathcal{E}_\delta(E)}{\log(1/\delta)}$$

and the lower Minkowski dimension *of E is*

$$\underline{\dim}_M(E) = \liminf_{\delta \to 0} \frac{\log \mathcal{E}_\delta(E)}{\log(1/\delta)}.$$

If the two values are equal, we simply say Minkowski dimension*.*

Definition 1.3. *For a set $E \subset \mathbb{R}^n$, the* α-dimensional Hausdorff measure *of E is*

$$\mathcal{H}^\alpha(E) = \lim_{\delta \to 0} \inf_{\mathcal{U}} \left\{ \sum_i \operatorname{diam}(U_i)^\alpha : E \subset \bigcup_i U_i \right\},$$

where the infimum is taken over all countable covers $\mathcal{U} = \{U_i\}$ of E by balls of radius at most δ. The Hausdorff dimension *of E is*

$$\begin{aligned} \dim_H(E) &= \sup\{\alpha : \mathcal{H}^\alpha(E) > 0\} = \sup\{\alpha : \mathcal{H}^\alpha(E) = \infty\} \\ &= \inf\{\alpha : \mathcal{H}^\alpha(E) < \infty\} = \inf\{\alpha : \mathcal{H}^\alpha(E) = 0\}. \end{aligned}$$

See [25, Section 4] for an introduction to Hausdorff dimension and its properties. One property that follows from the fact that all balls have radius δ in the definition of Minkowski dimension, whereas in Hausdorff dimension the balls can be arbitrarily small, is that

$$\dim_H(E) \le \underline{\dim}_M(E) \le \overline{\dim}_M(E)$$

holds for any set E.

The Kakeya conjecture says that any Besicovitch set in $\mathbb{R}^n$, for $n \ge 2$, has Hausdorff (and therefore also Minkowski) dimension n. This has been proved in the affirmative for $n = 2$ by Davies [10] but is open for all other dimensions, though there has been partial progress in for all n.

This question is related to important conjectures in harmonic analysis such as the restriction conjecture and the Bochner–Riesz conjecture. See [26] for an introduction to these topics and their connections. In particular, the Kakeya conjecture is known to follow from the restriction conjecture, which in turn follows from the Bochner–Riesz conjecture [31]. Understanding Besicovitch sets is considered a major obstacle to progress on these problems.

In [38], Wolff introduced a new variant of the Kakeya conjecture related to work of Furstenberg. An *α-Furstenberg set* is a compact set $E \subset \mathbb{R}^2$ such that for every direction, there is a line with this direction whose intersection with E has Hausdorff dimension $\geq \alpha$. The Furstenberg set problem seeks a lower bound on the Hausdorff dimension of any such set.

Wolff [38] showed that any α-Furstenberg set has Hausdorff dimension at least

$$\max\left\{\alpha + \frac{1}{2}, 2\alpha\right\}, \tag{1.1}$$

and gave a construction showing that there exist α-Furstenberg sets of Hausdorff dimension $(3\alpha + 2)/2$. He conjectured that every α-Furstenberg set in fact has dimension at least $(3\alpha + 2)/2$, that is, that the construction given is essentially the smallest possible.

In [27], Molter and Rela generalised the concept of an α-Furstenberg set. In the original version, the associated set of lines each have different directions, so the set of lines is a one-dimensional set in the parameter space of lines; Molter and Rela considered *(α, β)-Furstenberg sets*, where the α-dimensional property need only be satisfied for a β-dimensional set of lines. They proved that any (α, β)-Furstenberg set has Hausdorff dimension at least

$$\max\left\{\alpha + \frac{\beta}{2}, 2\alpha + \beta - 1\right\}. \tag{1.2}$$

This was later improved by Lutz and Stull [24] in the range $\beta < 2\alpha$, who showed that the Hausdorff dimension must be at least

$$\alpha + \min\{\beta + \alpha\}. \tag{1.3}$$

When $\beta = 2\alpha$ and $\alpha \leq 1/2$, both bounds (1.2) and (1.3) give a lower bound of 2α. Similarly, Wolff's two lower bounds (1.1) both yield 1 for the dimension of a 1/2-Furstenberg set. Recently, Héra, Shmerkin and Yavicoli [23] improved the bound in this range to $1 + c$, for some constant $c = c(\alpha) > 0$. The value of c was not determined explicitly, and in particular the proof relies on a theorem of Bourgain for which an explicit bound has not been computed, though it is estimated to be extremely small.

Our first main theorem in this second work is an improvement to the lower bound on the Hausdorff dimension of $(\alpha, 2\alpha)$-Furstenberg sets, in the range $0 < \alpha \leq 1/2$.

Theorem 1.4. *Every $(\alpha, 2\alpha)$-Furstenberg set in $\mathbb{R}^2$ has Hausdorff dimension at least $2\alpha + c(\alpha)$, where*

$$c(\alpha) = \frac{\alpha(1-\alpha)}{504(2+\alpha)}.$$

In particular, every $1/2$-Furstenberg set in the plane has Hausdorff dimension at least $1 + \frac{1}{5040}$.

This result follows from an incidence theorem for discretised sets of points and lines. Our proof relies on combinatorial arguments and ultimately we apply the discretised sum-product theorem from additive combinatorics.

1.3 Besicovitch sets in $\mathbb{R}^3$

Our last main theorem concerns Besicovitch sets in $\mathbb{R}^3$. In [36], Wolff showed that such sets have Hausdorff dimension at least $5/2$. Obtaining bounds beyond $5/2$ is known to be a difficult problem due to the existence of $5/2$-dimensional sets that share properties with Besicovitch sets, such as the Heisenberg group example given in [20]. The existence of these sets means that the techniques required to go beyond this bound need to be able to distinguish these sets from genuine Besicovitch sets.

In [20], Katz, Łaba and Tao developed various techniques to show that Besicovitch sets in $\mathbb{R}^3$ have upper Minkowski dimension at least $5/2 + \varepsilon_0$, for some small but absolute constant $\varepsilon_0 > 0$. It is remarked in [20] that $\varepsilon_0 \geq 10^{-10}$ but the quantitative value is not optimised, and a careful analysis of the proof may lead to a substantially better, though still small, value.

At a qualitative level, we give a different proof of the theorem of Katz, Łaba and Tao, showing that Besicovitch sets have dimension greater than $5/2$, however our arguments also apply to slightly more general sets. We obtain a small quantitative improvement on the value of ε_0 and we present a fully explicit argument. This establishes a framework, so that subsequent improvements on any of the subproblems involved can easily be converted into improvements for the problem itself.

Theorem 1.5. *Every Besicovitch set in $\mathbb{R}^3$ has upper Minkowski dimension greater than $5/2 + 2.67 \times 10^{-8}$.*

The proof is essentially a reduction to the 1/2-Furstenberg problem from Chapter 3. Specifically, we show that the existence of a Besicovitch set in $\mathbb{R}^3$ with upper Minkowski dimension $5/2+\varepsilon_0$ implies the existence of a 1/2-Furstenberg set in $\mathbb{R}^2$ with dimension close to 1; if ε_0 is sufficiently small, this contradicts Theorem 1.4.

Chapter 2

Combinatorics of intervals

2.1 Introduction

An *interval* in $\mathbb{R}^2$ is a directed line segment. The convex hull of a pair of interval forms a particular shape and in this chapter we consider a combinatorial problem on arrangements of intervals for which many pairs form trapezoids.

We will denote an interval by an ordered four-tuple $(a, b; c, d) \in \mathbb{R}^4$, where $(a, b), (c, d) \in \mathbb{R}^2$ denotes the coordinates of the initial and terminal point, respectively. We require that $(a, b) \neq (c, d)$, so the interval has positive length and we call the interval $(c, d; a, b)$ the *reverse* of the interval $(a, b; c, d)$.

Formally, we say that a pair of distinct intervals $(a, b; c, d)$ and $(a', b'; c', d')$ *forms a trapezoid* if

$$(a - a')(d - d') = (b - b')(c - c'), \tag{2.1}$$

or

$$(a - c')(d - b') = (c - a')(b - d'), \tag{2.2}$$

or

$$(d - b)(c' - a') = (d' - b')(c - a). \tag{2.3}$$

Equations (2.1) and (2.2) represent the case when a pair of endpoints from each interval is parallel to the other pair. Whereas (2.3) represents the case when the two intervals are parallel. Note that these equations allow some acceptable degenerate cases, namely where the two intervals lie on the same line, or where they share an endpoint. Note also that both (2.1) and (2.2) can be satisfied simultaneously, as in the case when the two intervals form the diagonals of a parallelogram. All of these scenarios are illustrated in Figure 2.1.

Clearly there are arrangements of intervals for which every pair forms a trapezoid: for example if a set of intervals are all parallel to a single direction

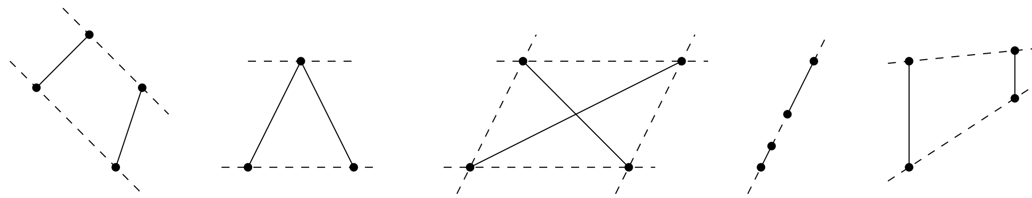


Figure 2.1: Intervals forming trapezoids: five cases

or if all the endpoints lie on two fixed parallel lines. These examples are very rigidly structured and one can therefore ask if all arrangements with many pairs forming trapezoids must have some similar structure. We answer this question, showing that if the number of pairs forming trapezoids is above a certain threshold, then many of the intervals must have rigid algebraic structure.

Theorem 2.1. *Let $\mathfrak{I}$ be a set of N distinct intervals in $\mathbb{R}^2$. If more than $\gtrsim N^{3/2} \log N$ pairs of intervals form trapezoids then at least one of the following holds.*

0. There are $\gtrsim N^{1/2} \log N$ parallel intervals.

1. There is a centre S and a ratio λ such that for $\gtrsim N^{1/2}$ intervals from $\mathfrak{I}$, one endpoint is the image of the other under a homothety with centre S and ratio λ.

2. There are two curves in $\mathbb{R}^2$ of degree at most 2 such that $\gtrsim N^{1/2}$ intervals from $\mathfrak{I}$ have an endpoint on each curve.

If a set of intervals are themselves parallel, then any pair will automatically form a trapezoid so the more interesting case is when this does not occur. It is also clear that for a set of intervals created by a single homothety, any pair will form a trapezoid, however the third possibility in Theorem 2.1 is more surprising. Based on our proof of this theorem, we construct examples of sets of N intervals forming $\gtrsim N^2$ trapezoids and with underlying conic curve structure. For examples of sets forming many trapezoids with homothety or conic curve structure, see Figure 2.2 and Figure 2.3, respectively.

2.2 Main lemma

In this section, we will prove the following technical lemma.

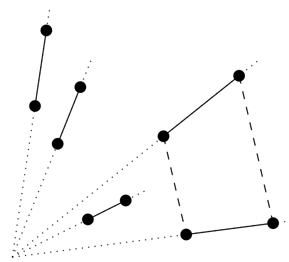


Figure 2.2: Case 2

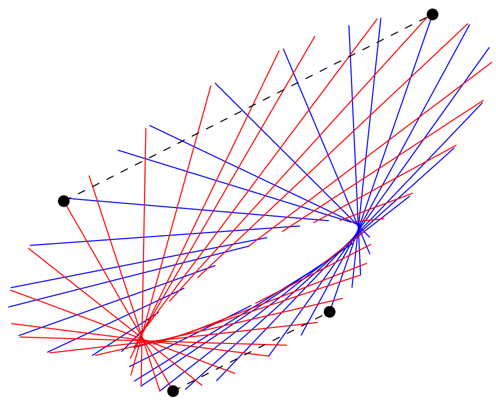


Figure 2.3: Case 3

Lemma 2.2. *Let $\mathfrak{I}$ be a set of N distinct intervals in $\mathbb{R}^2$. If more than $\gtrsim N^{3/2} \log N$ pairs of intervals form trapezoids then at least one of the following holds.*

0. *There are $\gtrsim N^{1/2}$ parallel intervals.*

1. *There are two parallel lines in $\mathbb{R}^2$ such that $\gtrsim N^{1/2}$ intervals have an endpoint on each line.*

2. *There are two parallel lines $\ell_1, \ell_2 \subset \mathbb{R}^2$ such that $\gtrsim N^{1/2}$ intervals $(a, b; c, d) \in \mathfrak{I}$ satisfy $(a, c) \in \ell_1$ and $(b, d) \in \ell_2$.*

3. *There are two subsets $\mathfrak{I}_1, \mathfrak{I}_2 \subset \mathfrak{I}$ such that for any $i_1 \in \mathfrak{I}_1$ and any $i_2 \in \mathfrak{I}_2$, the intervals i_1, i_2 form a trapezoid. In addition, $|\mathfrak{I}_1||\mathfrak{I}_2| \gtrsim N$.*

Note that in Item 2, the endpoints (a, b) and (c, d) of the intervals are not themselves contained on two lines; rather, the linear relationship is satisfied separately by the x-coordinates and the y-coordinates. Note also that Item 3 subsumes the others, but we separate them as they arise from distinct cases. For a typical set of intervals from Item 3, it will in fact be the case that no pair of intervals from within the same family forms a trapezoid, however there are some odd cases for which this is not true and yet the intervals do not satisfy any of Items 0, 1, 2.

Our first aim of this section is to set up a correspondence between intervals in $\mathbb{R}^2$ and lines in $\mathbb{R}^3$ such that a pair of intervals forms a trapezoid when the corresponding pair of lines intersect. In equations (2.1), (2.2), (2.3) we saw that there are three 'ways' for two intervals to form a trapezoid. If many pairs of intervals satisfy (2.3), then many intervals are parallel simply by pigeonholing. This accounts for conclusion 0 in Theorem 2.1. Therefore we are more interested in determining the structure of a set of intervals when (2.1) or (2.2) is satisfied for many pairs. With this in mind, we make the following definitions.

Definition 2.3. *Two intervals $I_1 = (a,b;c,d)$, $I_2 = (a',b';c',d')$ form a Type 1 trapezoid if they satisfy* (2.1)*. Similarly I_1, I_2 form a Type 2 trapezoid if they satisfy* (2.2)*.*

Note that Type 1 and Type 2 trapezoids are not mutually exclusive, see for example the middle pair of intervals in Figure 2.1. Moreover, if I_1' denotes the reverse of I_1, then I_1', I_2 form a Type 2 trapezoid if and only if I_1, I_2 form a Type 1 trapezoid. Also note the degenerate cases that I_1, I_1' form both a Type 1 and Type 2 trapezoid.

In the following lemma we show that there is a correspondence between intervals in $\mathbb{R}^2$ and lines in $\mathbb{R}^3$ such that a pair of intervals forms a Type 1 trapezoid if and only if the corresponding pair of lines in $\mathbb{R}^3$ intersect.

Lemma 2.4. *Let $\mathfrak{I}$ be a set of intervals in $\mathbb{R}^2$ such that any vertical line contains at most one distinct endpoint from the intervals in $\mathfrak{I}$. Then there is a bijection $\mathcal{L}$ from intervals in $\mathbb{R}^2$ to lines in $\mathbb{R}^3$ that are not parallel to the xy-plane, such that a pair of intervals forms a Type 1 trapezoid if and only if their images under $\mathcal{L}$ intersect.*

Proof. To every interval $(a,b;c,d)$ we associate the unique line

$$\mathcal{L}(a,b;c,d) = \left\{ \begin{pmatrix} b \\ d \\ 0 \end{pmatrix} + t \begin{pmatrix} a \\ c \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\} \subset \mathbb{R}^3. \tag{2.4}$$

Conversely, every line $\ell \subset \mathbb{R}^3$ that is not parallel to the xy-plane can be normalized to have the same form as the line in (2.4). Define $\mathcal{I}(\ell)$ to be this corresponding interval. Clearly $\mathcal{L}$ is a bijection from intervals in $\mathbb{R}^2$ to lines not parallel to the xy-axis in $\mathbb{R}^3$, and $\mathcal{I}$ is its inverse.

Let $(a, b; c, d)$ and $(a', b'; c', d')$ be intervals such that

$$(a, b; c, d) \neq (a', b'; c', d'),$$

though we allow the possibility that $(a', b'; c', d')$ is the reverse, $(c, d; a, b)$, of $(a, b; c, d)$. Observe that the lines $\mathcal{L}(a, b; c, d)$ and $\mathcal{L}(a', b'; c', d')$ intersect if and only if there is a solution $t \in \mathbb{R}$ to the system of equations

$$t(a - a') = b' - b; \quad t(c - c') = d' - d. \tag{2.5}$$

Note that (2.5) implies (2.1). Conversely, (2.1) implies there is a unique solution $t \in \mathbb{R}$ to (2.5), except in the case $a = a'$, $c = c'$, and either $b \neq b'$ or $d \neq d'$. But in this case either endpoints (a, b), (a', b') are distinct and on a vertical line, or (c, d), (c', d') are distinct and on a vertical line, contradicting our assumption about $\mathfrak{I}$.

$\square$

We remark that our assumption that vertical lines contain only one endpoint is easy to achieve for any finite set of intervals by applying a generic rotation of $\mathbb{R}^2$. Henceforth we will always assume a set of intervals has this property. Crucially, the intersection of $\mathcal{L}(a, b; c, d)$ and $\mathcal{L}(a', b'; c', d')$ does not imply (2.2). Instead by the argument of Lemma 2.4 we have that if $(a, b; c, d)$ and $(a', b'; c', d')$ are intervals without two endpoints on a single vertical line then they form a Type 2 trapezoid if and only if $\mathcal{L}(c, d; a, b)$ and $\mathcal{L}(a', b'; c', d')$ intersect. As a result, in order to 'capture' all of the pairs of intervals forming trapezoids as intersecting lines, each interval will correspond to two lines in $\mathbb{R}^3$, one for it and its reverse.

Definition 2.5. *Let $\mathfrak{I} = \{(a_i, b_i; c_i, d_i) \colon 1 \leq i \leq N\}$ be a set of N intervals in the plane. Define the following set of $2N$ lines in $\mathbb{R}^3$*

$$\mathfrak{L}(\mathfrak{I}) = \{\mathcal{L}(a_i, b_i; c_i, d_i) \colon 1 \leq i \leq N\} \cup \{\mathcal{L}(c_i, d_i; a_i, b_i) \colon 1 \leq i \leq N\},$$

where $\mathcal{L}$ is defined as in (2.4).

As remarked upon earlier, an interval and its reverse form a Type 1 and Type 2 trapezoid. The following lemma shows how the number of pairs of intervals in $\mathfrak{I}$ forming trapezoids compares to the number of pairs of intersecting lines in $\mathfrak{L}(\mathfrak{I})$.

Lemma 2.6. *Let $\mathfrak{I}$ be a set of N distinct intervals in $\mathbb{R}^2$ such that an interval and its reverse do not both appear in $\mathfrak{I}$. Let T be the number of pairs of intervals in $\mathfrak{I}$ that satisfy* (2.1) *or* (2.2)*, where a pair of intervals satisfying both* (2.1) *and* (2.2) *is counted with multiplicity two. Then*

$$2|T| = \#\{\textit{Pairs of intersecting lines in } \mathfrak{L}(\mathfrak{I})\} - N.$$

Proof. Let $\mathfrak{I} = \{(a_i, b_i; c_i, d_i) \colon 1 \leq i \leq N\}$. For every pair of intervals $(a, b; c, d), (a', b', c', d') \in \mathfrak{I}$ satisfying one (resp. both) of equation(s) (2.1), (2.2), there are two (resp. four) intersections among the lines

$$\mathcal{L}(a, b; c, d),\ \mathcal{L}(c, d; a, b),\ \mathcal{L}(a', b'; c', d'),\ \mathcal{L}(c', d'; a', b').$$

Note that $\mathcal{L}(a_i, b_i; c_i, d_i)$ and $\mathcal{L}(c_i, d_i; a_i, b_i)$ intersect exactly once for all $1 \leq i \leq N$. These intersections correspond to an interval forming a trapezoid with its reverse direction. All other pairs of intersecting lines in $\mathfrak{L}$ correspond to a trapezoid formed by distinct line segments. $\square$

So far we roughly have an equivalence between intervals forming trapezoids, and intersecting lines in $\mathbb{R}^3$. The main tool that we will use to prove that our sets of intervals have particular structure is a theorem of Guth and Katz on incidences of lines in $\mathbb{R}^3$. The following is a corollary of [15, Theorem 1.2] via a standard dyadic summation.

Theorem 2.7 (Guth and Katz [15])**.** *Let $\mathfrak{L}$ be a set of N lines in $\mathbb{R}^3$. If the number of pairs of intersecting lines in $\mathfrak{L}$ is $\gtrsim N^{3/2} \log N$, then at least one of the following holds.*

1. *There are $\gtrsim N^{1/2}$ concurrent lines in $\mathfrak{L}$.*
2. *There exists a plane containing $\gtrsim N^{1/2}$ lines of $\mathfrak{L}$.*
3. *There exists a regulus containing a subset of lines $\mathfrak{L}_R \subset \mathfrak{L}$ such that the number of pairs of intersecting lines in $\mathfrak{L}_R$ is $\gtrsim N$.*

The reason that reguli and planes appear here is that they are the only doubly ruled surfaces in $\mathbb{R}^3$; the plane is in fact infinitely ruled. Recall that a *doubly ruled surface* is one for which at every point on the surface, there are two distinct lines contained within the surface and containing the point. A regulus contains two families of lines such that there are no intersections within a family but any pair of lines from different families intersect. That is, if we select M lines from one ruling within a regulus and N lines from the other, then there will be precisely MN pairs of intersecting lines.

In [15], the authors proved a breakthrough result on the Erdős distinct distances problem, establishing the conjectured bound up to a multiplicative factor of $O(\sqrt{\log N})$. Theorem 2.7 was the main ingredient used, as well as a reduction from distinct distances to line incidences. Though Theorem 2.7 is tight up to the implicit multiplicative constant, the reduction appears not to be sharp due to a particular application of the Cauchy–Schwarz inequality. We note that our paradigm does not suffer from such a discrepancy, since each pair of intersecting lines in $\mathbb{R}^3$ corresponds to a pair of intervals in $\mathbb{R}^2$ forming a trapezoid, whereas in the distinct distances application such a pair corresponds to a quadruple of points, therefore requiring an additional step.

By combining our preliminary results with Theorem 2.7 we are able to prove Lemma 2.2.

Proof of Lemma 2.2. Let $\mathfrak{I}$ be a set of N distinct intervals such that $\mathfrak{I}$ does not contain both an interval and its reverse. Put $\mathfrak{L} = \mathfrak{L}(\mathfrak{I})$. For each pair of intervals in $\mathfrak{I}$ forming a trapezoid, one of the equations (2.1), (2.2), (2.3) must be satisfied by the corresponding coordinates.

Case 0: Parallel intervals. First suppose that at least half of the trapezoids are formed by pairs of intervals satisfying (2.3). For each such pair, the intervals are parallel so by pigeonholing we find a single direction to which $\gtrsim N^{1/2}\log N$ intervals are parallel.

We can now assume that at least half of the trapezoids are formed by pairs satisfying either (2.1) or (2.2). By Lemma 2.6, if there are at least $\gtrsim N^{3/2}\log N$ such trapezoids, then $\gtrsim N^{3/2}\log N$ pairs of lines intersect in $\mathfrak{L}$, excluding the interval-reverse interval intersections. By Theorem 2.7, many lines are concurrent, lie in a plane, or are contained in a regulus. We consider these three cases separately, which correspond to 1, 2, and 3 in Lemma 2.2.

Case 1: Concurrent lines. Suppose that $\gtrsim N^{1/2}$ lines of $\mathfrak{L}$ pass through the point $(u, v, w) \in \mathbb{R}^3$. Then $\gtrsim N^{1/2}$ lines of $\mathfrak{L}$ are of the form $[u-aw, v-cw, 0] + t[a, c, 1]$. These lines correspond to $\gtrsim N^{1/2}$ intervals with one endpoint on $y = u - wx$ and the other on $y = v - wx$. Note that if $u = v$, then many intervals are contained entirely on the line $y = u - wx$. In this case, an interval and its reverse might be represented in the set of concurrent lines passing through (u, v, w).

Case 2: Lines in a plane. Suppose that $\gtrsim N^{1/2}$ lines lie in the plane

$Ax + By + Cz + D = 0$. A line in $\mathfrak{L}$ of the form (2.4) belongs to this plane if

$$t(Aa + Bc + C) + Ab + Bd + D = 0,$$

for all $t \in \mathbb{R}$. Thus (a, c) is on the line $Ax + By + C = 0$ and (b, d) is on the line $Ax + By + D = 0$. Note that A, B are not both zero, since no line in $\mathfrak{L}$ is parallel to the xy-plane. We conclude that $\gtrsim N^{1/2}$ intervals $(a, b; c, d)$ of $\mathfrak{I}$ satisfy $Aa + Bc + C = Ab + Bd + D = 0$.

Case 3: Lines in a regulus. Suppose there exists a subset $\mathfrak{L}_R \subset \mathfrak{L}$ such that the number of pairs of intersecting lines in $\mathfrak{L}_R$ is $\gtrsim N$. As noted earlier, $\mathfrak{L}_R$ can be partitioned into $\mathfrak{L}_1$ and $\mathfrak{L}_2$, where the lines in $\mathfrak{L}_1$ belong to one ruling of the regulus and $\mathfrak{L}_2$ from the other. There are no intersecting pairs of lines within the same ruling, thus $|\mathfrak{L}_1||\mathfrak{L}_2| \gtrsim N$. Pulling back $\mathfrak{L}_1, \mathfrak{L}_2$ to intervals in $\mathbb{R}^2$ gives $\mathfrak{I}_1, \mathfrak{I}_2$ as described. □

2.3 Proof of Theorem 2.1

In this section, we prove Theorem 2.1 by analysing the geometry underlying each of the situations in the conclusion of Lemma 2.2.

Proof of Theorem 2.1. If the intervals in $\mathfrak{I}$ form more than $\gtrsim N^{3/2} \log N$ trapezoids, then we can apply Lemma 2.2. We will now consider each of the cases permitted by this lemma.

Case 0 (Parallel) or Case 1 (Concurrent): In both of these cases, there is nothing left to prove as they already give conclusions 0 and 2 of Theorem 2.1, respectively. An example of the Concurrent case, where the intervals have endpoints contained on two parallel lines, is given in Figure 2.4

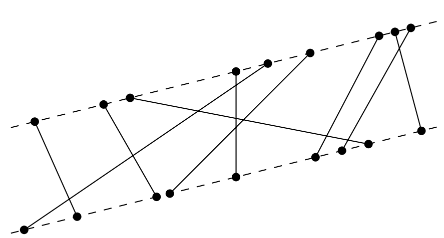


Figure 2.4: Case 1: Intervals corresponding to a set of concurrent lines in $\mathfrak{L}$

Case 2 (Coplanar): Suppose that many intervals $(a, b; c, d)$ of $\mathfrak{I}$ satisfy $Aa + Bc + C = Ab + Bd + D = 0$. Firstly, note that we cannot have

$(A, B) = (0, 0)$, since by Lemma 2.4 the plane $Ax+By+Cz+D = 0$ cannot be parallel to the xy-plane. We consider separately the case when $B = 0$. In this case, we have $Aa + C = 0$ and $Ab + D = 0$; since $A \neq 0$, this implies that $(a, b) = (-C/A, -D/A)$. Thus, for any such interval $(a, b; c, d)$, the endpoint (a, b) is formed by a homothety with centre $(-C/A, -D/A)$ and the ratio zero applied to (c, d).

Now suppose that $B \neq 0$. The equations $Aa+Bc+C = Ab+Bd+D = 0$ imply that $c = -(C + Aa)/B$ and $d = -(D + Ab)/B$. If $A + B = 0$, we have

$$c - a = -\frac{C + Aa}{B} - a = -\frac{C + (A + B)a}{B} = -\frac{C}{B},$$

and

$$d - b = -\frac{D + Ab}{B} - b = -\frac{D + (A + B)b}{B} = -\frac{D}{B}.$$

Thus, in this case, each endpoint (c, d) must be a translate of (a, b) by $(-C/B, -D/B)$. This is a special case of a homethety, where the centre is 'at infinity'. We will now assume that $A + B \neq 0$. The point $(x, y) \in \mathbb{R}^2$ is contained on the line containing the points (a, b) and (c, d) precisely when

$$(D + (A + B)b)\, x - (C + (A + B)a)\, y + bC - aD = 0.$$

Clearly the point $(-\frac{C}{A+B}, -\frac{D}{A+B})$ is contained on this line and since this is independent of a and b, this point must be contained on all such lines. This point will be the centre S of the homothety; it remains to be shown that there is a ratio λ such that

$$(c, d) - \left(-\frac{C}{A + B}, -\frac{D}{A + B}\right) = \lambda \left((a, b) - \left(-\frac{C}{A + B}, -\frac{D}{A + B}\right)\right). \quad (2.6)$$

Observe that

$$c + \frac{C}{A + B} = -\frac{A\left((A + B)a + C\right)}{(A + B)B}$$

and

$$a + \frac{C}{A + B} = \frac{(A + B)a + C}{A + B}.$$

Similarly,

$$d + \frac{D}{A + B} = -\frac{A\left((A + B)b + D\right)}{(A + B)B},$$

and

$$b + \frac{D}{A + B} = \frac{(A + B)b + D}{A + B}.$$

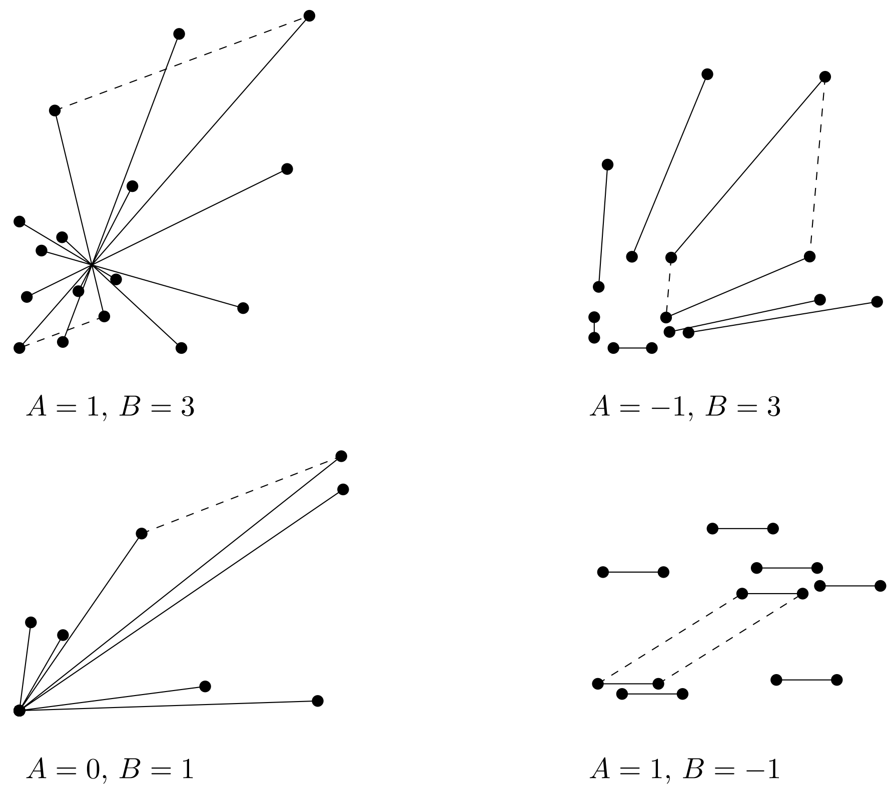


Figure 2.5: Intervals coming from lines contained in a plane

We conclude that (2.6) holds with $\lambda = -A/B$. See Figure 2.5 for some examples of arrangements of intervals formed in this way.

Case 3 (Regulus): Let $\mathfrak{I}_1, \mathfrak{I}_2, \mathfrak{L}_1, \mathfrak{L}_2$ be as in the proof of Lemma 2.2. Let $\ell_j = \{[b_j, d_j, 0] + t[a_j, c_j, 1] \colon t \in \mathbb{R}\}$ for $j = 1, 2, 3$, be lines in $\mathfrak{L}_2$. A line $\ell = \{[b, d, 0] + t[a, c, 1] \colon t \in \mathbb{R}\} \in \mathfrak{L}_1$ intersects each ℓ_j, $j = 1, 2, 3$ and therefore satisfies

$$(a - a_j)(d - d_j) = (b - b_j)(c - c_j), \quad j = 1, 2, 3. \tag{2.7}$$

The above is a system of three degree two polynomial equations. Subtracting two equations of the system gives the two linear equations

$$\begin{aligned} &(d_1 - d_2)a + (c_2 - c_1)b + (b_2 - b_1)c + (a_1 - a_2)d = b_2c_2 - b_1c_1 \\ &(d_2 - d_3)a + (c_3 - c_2)b + (b_3 - b_2)c + (a_2 - a_3)d = b_3c_3 - b_2c_2. \end{aligned} \tag{2.8}$$

We consider three encompassing cases in the system (2.8).

Subcase (i): The 2×2 coefficient matrix induced by the coefficients of a and b in (2.8) is invertible, or the 2×2 coefficient matrix induced by the coefficients of c and d in (2.8) is invertible. If the 2×2 matrix induced by the coefficients of a and b in (2.8) is invertible, then Gaussian elimination on (2.8) gives a and b linearly in terms of c and d. Substituting these linear relations into one equation of (2.7) gives a degree two polynomial in c and d, i.e. the set of all (c, d) lie on a conic or line. Thus the set of all (a, b) also lies on a conic or line. An analogous result follows if the 2×2 matrix induced by the coefficients of c and d in (2.8) is invertible. See Figures 2.6 and 2.7 for examples.

Subcase (ii): Neither 2×2 coefficient matrix of Subcase (i) is invertible, but (2.8) has rank 2. Then all solutions (a, b) lie on one line, and solutions (c, d) on a line, i.e. b can be given linearly in a, and d linearly in c. Substituting these linear relations into (2.7) shows the set of all (a, d) lie on a conic. See Figure 2.9 for example.

Subcase (iii): (2.8) has rank 1. In this case, the points of all of the sets $\{(a_j, b_j)\}_{j=1}^3$, $\{(a_j, c_j)\}_{j=1}^3$, and $\{(a_j, d_j)\}_{j=1}^3$ are collinear. It follows that there exist

$$m_1, m_2, m_3, r_1, r_2, r_3 \in \mathbb{R}$$

such that $(a_j, b_j, c_j, d_j) = (a_j, m_1 a_j + r_1, m_2 a_j + r_2, m_3 a_j + r_3)$ for $j = 1, 2, 3$. Substituting this relation into (2.7) gives a quadratic equation in a_j^2 with a a_j^2 coefficient of $m_3 - m_1 m_2$. Since this quadratic equation has at least three solutions (namely a_1, a_2, a_3), the leading coefficient is zero, i.e. $m_3 - m_1 m_2 = 0$. This relation implies ℓ_1, ℓ_2, ℓ_3 all pass through the point $(r_1, -r_2 m_1 + r_3, -m_1)$. This cannot happen, since lines of the same ruling in a regulus do not intersect. We conclude that Subcase (iii) never occurs. The same argument also shows that in Subcase (i), the solution set of (a, b) lies on a conic, not a line.

□

2.4 Examples of sets forming many trapezoids

In this section, we give some specific constructions of sets forming many trapezoids based on our proof of Theorem 2.1. The conic curve case in particular leads to interesting examples that appear difficult to construct without using the correspondence established in Section 2.

2.4.1 Intervals from coplanar lines

As observed above, it is easy to see that for any set of intervals all satisfying the homothety conclusion of Theorem 2.1, every pair forms a trapezoid. We will nevertheless highlight some special cases of this case. Suppose that a set of intervals $(a, b; c, d)$ is contained in the plane $Ax + By + Cz + D = 0$. If $AB > 0$ then the point $(-\frac{C}{A+B}, -\frac{D}{A+B})$ is on the interval corresponding to $(a, b; c, d)$, and if $AB = 0$ it is an endpoint of the interval. A special case of this is when $A = B$ and pairs of intervals are the diagonals of a parallelogram. On the other hand, if $A + B = 0$, then $a + C/A = c$ and $b + D/A = d$. This corresponds to a set of intervals that are translates of each other. Hence, in this case the intervals are contained on a pencil of lines. See Figure 2.5 for examples.

2.4.2 Intervals from reguli

As was mentioned previously, the lines in a regulus have a complete bipartite structure with respect to intersections. The conic curve case therefore reveals an interesting class of examples of sets of intervals forming trapezoids in a complete bipartite way.

From Theorem 2.1, we know that one possibility is that the endpoints of $\mathfrak{I}_1$ and $\mathfrak{I}_2$ lie on a conic (Subcase (i)), or a line (Subcase (ii)). We will now showcase examples involving each type of conic, as well as the Subcase (ii) situation. To facilitate simpler computation, we examine axis parallel reguli. Up to a rigid motion (rotation and translation) of $\mathbb{R}^3$, any regulus takes the form

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} - \frac{z^2}{C^2} = 1, \tag{2.9}$$

or

$$z = \frac{x^2}{A^2} - \frac{y^2}{B^2}. \tag{2.10}$$

Equation (2.9) describes a hyperboloid of one sheet, and (2.10) describes a hyperbolic paraboloid. We determine what the corresponding interval set looks like in each case, and then discuss the arrangements that result from rigid transformations of these standard forms of reguli.

Axis parallel hyperboloids

The hyperboloid (2.9) is ruled by the following two families of lines, parameterized by $\theta \in [0, 2\pi]$.

$$\begin{pmatrix} A\sin\theta \\ -B\cos\theta \\ 0 \end{pmatrix} + t\begin{pmatrix} \frac{A}{C}\cos\theta \\ \frac{B}{C}\sin\theta \\ 1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} A\sin\theta \\ B\cos\theta \\ 0 \end{pmatrix} + t\begin{pmatrix} \frac{A}{C}\cos\theta \\ -\frac{B}{C}\sin\theta \\ 1 \end{pmatrix}, t \in \mathbb{R}.$$

This corresponds to the following two families of intervals parameterized by $\theta \in [0, 2\pi]$.

$$\left(\frac{A}{C}\cos\theta, A\sin\theta; \frac{B}{C}\sin\theta, -B\cos\theta\right),$$

and

$$\left(\frac{A}{C}\cos\theta, A\sin\theta; -\frac{B}{C}\sin\theta, B\cos\theta\right).$$

Observe that the above intervals have their first endpoints on the ellipse $Cx^2 + y^2 = A^2$ and their second on $Cx^2 + y^2 = B^2$. Furthermore, the second endpoint in the first family has a phase shift of $-\pi/2$ radians compared to the first endpoint. In the second family, the second endpoint has a phase shift of $+\pi/2$ compared to the first endpoint. If $A = B$ and $C = 1$, then the two families of intervals are the reverse of each other. In this case for any subset of the intervals, any pair of intervals forms a trapezoid. In all other cases, the two sets of intervals have bipartite structure. Examples of both of these cases are shown in Figure 2.6.

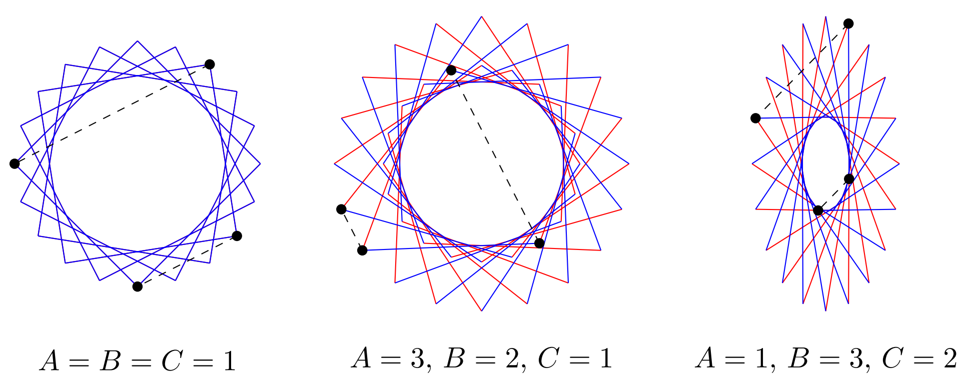


Figure 2.6: Intervals from the hyperboloid $x^2/A^2 + y^2/B^2 - z^2/C^2 = 1$.

Axis parallel paraboloids

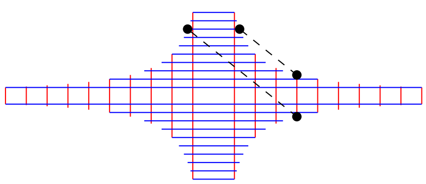
The hyperbolic paraboloid (2.10) is ruled by the following two families of lines, parameterized by $\lambda \in \mathbb{R} \setminus \{0\}$.

$$\begin{pmatrix} A/(2\lambda) \\ -B/(2\lambda) \\ 0 \end{pmatrix} + t \begin{pmatrix} A\lambda/2 \\ B\lambda/2 \\ 1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} A/(2\lambda) \\ B/(2\lambda) \\ 0 \end{pmatrix} + t \begin{pmatrix} A\lambda/2 \\ -B\lambda/2 \\ 1 \end{pmatrix}, t \in \mathbb{R}.$$

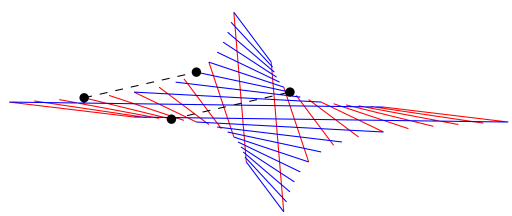
This corresponds to the following two families of intervals parameterized by $\lambda \in \mathbb{R} \setminus \{0\}$.

$$(A\lambda/2, A/(2\lambda); B\lambda/2, -B/(2\lambda)) \quad \text{and} \quad (A\lambda/2, A/(2\lambda); -B\lambda/2, B/(2\lambda)).$$

The left endpoint of all the above intervals lie on the hyperbola $y = \frac{A^2}{4x}$ and the right endpoint lies on $y = -\frac{B^2}{4x}$. The intervals in the first family intersect the x-axis, and the intervals in the second family intersect the y-axis. See Figure 2.7 for an example of intervals coming from hyperbolic paraboloids.



$A = 1,\ B = 2$



$A = 1,\ B = 2$

Figure 2.7: Intervals from the hyperbolic paraboloid $z = x^2/A^2 - y^2/B^2$

Reguli under affine transformation

It is easy to understand the effect of translations in $\mathbb{R}^3$ on sets of intervals in the plane, and by this we can completely describe the sets of intervals corresponding to axis parallel reguli in $\mathbb{R}^3$. A translation by the vector $(p, q, r) \in \mathbb{R}^3$ maps the line $(b, d, 0) + t(a, c, 1)$ to $(b + p, c + q, r) + t(a, c, 1)$. Hence the corresponding transformation on intervals maps $(a, b; c, d)$ to $(a, b + p - ra; c, d + q - rc)$. The effect of the translation by (p, q, r) in $\mathbb{R}^3$ on $\mathfrak{I}$ can therefore be described by the composition of three basic maps. First, a vertical shift of all left endpoints of intervals in $\mathfrak{I}$ by p. Second, a vertical shift of all right endpoints of intervals in $\mathfrak{I}$ by q. Third, an affine shear transformation acting on all of $\mathbb{R}^2$ by a factor r. See Figure 2.3 for an example of a set of intervals resulting from a translation of a hyperboloid, given by translating the hyperboloid of the form (2.9) with $A = 2, B = 1, C = 1/2$ by $(2, 0, 1/2)$.

Evidently, $\mathbb{R}^3$ translations do not change the type of conic that the endpoints lie on, so it is now clear that axis parallel reguli only produce intervals with endpoints lying on either ellipses or hyperbolas. In what follows, we will see that by rotating the axis parallel reguli it is also possible to produce parabolas and pairs of lines (which is a degenerate conic), thereby showing that each type of conic can be realised in this way.

If an interval is considered as a point in $\mathbb{R}^4$, then $\mathbb{R}^3$ translations induce an affine transformation on $\mathbb{R}^4$. The effect of $\mathbb{R}^3$ rotations on $\mathfrak{I}$ is more complicated: the map induced by rotations of $\mathbb{R}^3$ is in general not affine. Rotations in $\mathbb{R}^3$ are described by well-known matrices. For example, calculation with such a matrix shows that rotation around the x-axis by an angle α induces the map

$$(a, b; c, d)$$
$$\mapsto \left(\frac{a}{c\sin\alpha + \cos\alpha}, \frac{(bc - ad)\sin\alpha + b\cos\alpha}{c\sin\alpha + \cos\alpha}; \frac{c\cos\alpha - \sin\alpha}{c\sin\alpha + \cos\alpha}, \frac{d}{c\sin\alpha + \cos\alpha}\right),$$

on intervals in $\mathbb{R}^2$. Thus, the type of conic containing the endpoints of the intervals is in general not preserved under these transformations, and indeed there are examples where parabolas and pairs of lines arise as a result of rotating the axis parallel reguli – see Figure 2.8.

End points on a degenerate conic

All the Case 3 examples that we have seen so far belong to Subcase (i). We finish this section with an example from Subcase (ii). Recall that we want a family of intervals $\{(a_i, b_i; c_i, d_i)\}_i$ such that (a_i, b_i) and (c_i, d_i) lie on lines, and (a_i, c_i) lie on a conic, for all i. By rotating and scaling, we assume that (a_i, b_i) lie on the line $y = x$ and (a_i, c_i) lie on the hyperbola $y = 1/x$. These choices determine the following two families of intervals

$$\{(t, t; 1/t, u/t + v) \colon t \in \mathbb{R}\} \quad \text{and} \quad \{(t, ut; 1/t, v + 1/t) \colon t \in \mathbb{R}\},$$

where $u, v \in \mathbb{R}$, and $u \neq 1$. These intervals correspond to lines belonging to the hyperboloid

$$xy = z^2 + z(u + 1) + u + vx. \tag{2.11}$$

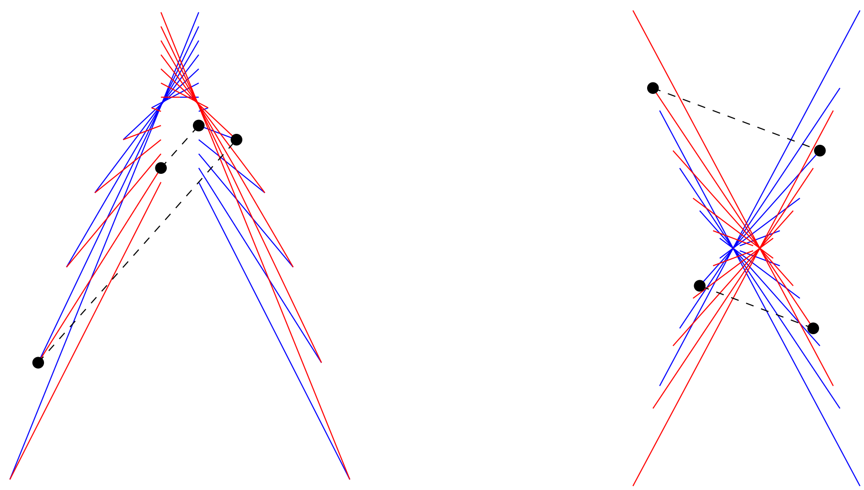


Figure 2.8: Rotating the hyperbolic paraboloid $z = x^2 - y^2$ by $\pi/2$ around the x-axis produces intervals with endpoints on the lines $x = \pm 1$ and the parabola $y = -\frac{x^2}{4}$; applying a further rotation by $\pi/4$ around the z-axis gives intervals with endpoints on the parabolas $y = \pm\frac{2x^2-1}{4\sqrt{2}}$

When $u = 1$, the two families are identical, and (2.11) is a cone. See Figure 2.9 for a drawing of this case.

2.5 Orthodiagonal quadrilaterals

An orthodiagonal quadrilateral is a convex quadrilateral with perpendicular diagonals. Several geometric and arithmetic characterizations of orthodiagonal quadrilaterals are known. For example, a convex quadrilateral is orthodiagonal if and only if the midpoints of the sides are the vertices of a rectangle. Another well known characterization is that the sum of the lengths of opposite sides is equal – see for example [19] and the references contained therein.

Our proof of Lemma 2.2 can easily be modified to deal with some variants of the the problem we have considered. An example is sets of intervals for which there are many pairs *forming orthodiagonal quadrilaterals*, meaning that the convex hull of the two intervals has perpendicular diagonals. This property is illustrated in the leftmost diagram of Figure 2.10. Arithmetically, two intervals $(a, b; c, d)$, $(a', b'; c',' d)$ forming an orthodiagonal quadrilateral

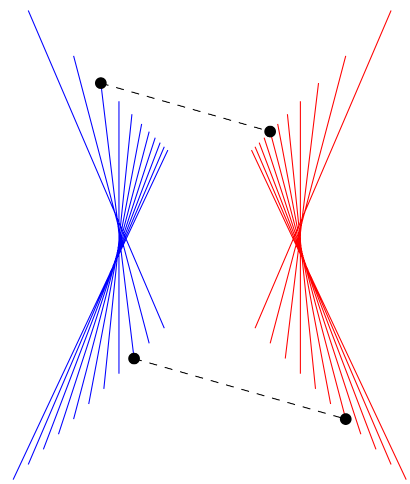


Figure 2.9: The hyperboloid (2.11) with $u = -1, v = 1$, produces intervals with endpoints on two pairs of lines

satisfy

$$(b - b')(d - d') = -(a - a')(c - c') \tag{2.12}$$

or

$$(b - d')(d - b') = -(a - c')(c - a'), \tag{2.13}$$

or

$$(d - b)(d' - b') = -(c - a)(c' - a'). \tag{2.14}$$

The arithmetic conditions (2.12),(2.13),(2.14) are not exclusive to orthodiagonal quadrilaterals, i.e. other pairs of intervals can satisfy them and we illustrate such possibilities in Figure 2.10.

The similarity of (2.12),(2.13),(2.14) to (2.1),(2.2),(2.3) allows a reuse of the previous techniques to create a result on orthodiagonal quadrilaterals, similar to Lemma 2.2. One notable difference is that two intervals coming from two different rulings of reguli may form any of the arrangements in Figure 2.10, instead of exclusively forming orthodiagonal quadrilaterals.

Theorem 2.8. *Let $\mathfrak{I}$ be a set of N distinct intervals in $\mathbb{R}^2$. If more than $\gtrsim N^{3/2} \log N$ pairs of intervals form orthodiagonal quadrilaterals, then one of the following holds.*

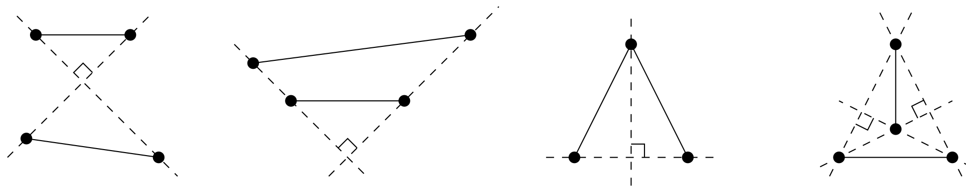


Figure 2.10: These four pair of intervals all satisfy one of (2.12),(2.13),(2.14) but only the first forms an orthodiagonal quadrilateral

0. *There exist subsets* $\mathfrak{I}_1, \mathfrak{I}_2 \subset \mathfrak{I}$ *such that all intervals within* $\mathfrak{I}_i$ *are parallel for* $i = 1, 2$ *and all intervals in* $\mathfrak{I}_1$ *are perpendicular to all intervals in* $\mathfrak{I}_2$*. Furthermore,* $|\mathfrak{I}_1||\mathfrak{I}_2| \gtrsim N \log^2 N$*.*

1. *There exist* $u, v, w \in \mathbb{R}$ *such that* $\gtrsim N^{1/2}$ *intervals* $(a, b; c, d)$ *satisfy*

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 & -w \\ w & 0 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} + \begin{pmatrix} -v \\ u \end{pmatrix}.$$

2. *There are two curves in* $\mathbb{R}^2$ *of degree at most* 2 *such that* $\gtrsim N^{1/2}$ *intervals from* $\mathfrak{I}$ *have an endpoint on each curve.*

In order to prove this theorem, one maps the interval (a, b, c, d) to the line

$$\mathcal{L}^{\perp}(a, b; c, d) = \left\{ \begin{pmatrix} b \\ -a \\ 0 \end{pmatrix} + t \begin{pmatrix} c \\ d \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\} \subset \mathbb{R}^3$$

instead of using the map $\mathcal{L}$ above. Under this alternative correspondence, a pair of lines $\mathcal{L}^{\perp}(a, b; c, d)$ and $\mathcal{L}^{\perp}(a', b'; c', d')$ intersect precisely when (2.12) is satisfied. An interesting specific instance of conclusion 2 in Theorem 2.8 is when a subset of the intervals comes from the pull back of a set of coplanar lines. In this case there are two perpendicular lines such that $\gtrsim N^{1/2}$ intervals have an endpoint on each line. In Figure 2.11 we showcase two instances of sets of intervals resulting from pulling back rulings of reguli by $\mathcal{L}^{\perp}$.

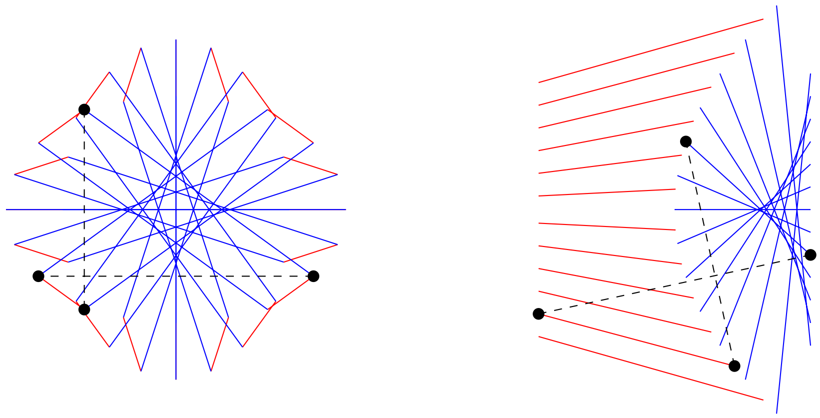


Figure 2.11: Two configurations of intervals with many pairs having endpoints on two perpendicular lines.

One can also adapt the method to treat a generalisation of the trapezoids problem considered above. Two intervals form a trapezoid if two of the edges of their convex hull are parallel, but our proof did not rely in an important way on this parallel property. Indeed by mapping the interval $(a, b; c, d)$ to the line

$$\mathcal{L}^\rho(a, b; c, d) = \left\{ \begin{pmatrix} b \\ d \\ 0 \end{pmatrix} + t \begin{pmatrix} a \\ \rho c \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\} \subset \mathbb{R}^3,$$

one obtains a correspondence under which a pair of lines $\mathcal{L}^\rho(a, b; c, d)$ and $\mathcal{L}^\rho(a', b'; c', d')$ intersect precisely when

$$(a - a')(d - d') = \rho(b - b')(c - c'),$$

i.e. the slopes formed by the endpoints of the intervals have ratio ρ. Thus, the arguments from Section 2 can now be applied, leading to an analogue of Lemma 2.2 in the case of a fixed ratio ρ.

Chapter 3

Furstenberg sets

3.1 Introduction

For $0 < \alpha \leq 1$, an *α-Furstenberg set* is a compact set $E \subset \mathbb{R}^2$ such that for every direction $\omega \in S^1$ there is a line with direction ω whose intersection with E has Hausdorff dimension at least α. In [38], Wolff considered the problem of estimating

$$\gamma(\alpha) = \inf\{\dim_H(E) : E \text{ is an } \alpha\text{-Furstenberg set}\},$$

and showed that

$$\max\left\{2\alpha, \alpha + \frac{1}{2}\right\} \leq \gamma(\alpha) \leq \frac{3\alpha}{2} + \frac{1}{2}. \tag{3.1}$$

He conjectured that the upper bound is sharp.

In [21], Katz and Tao showed that the discretised sum-product conjecture (later proved by Bourgain [4]) would imply improved lower bounds on the dimension of $1/2$-Furstenberg sets. Recently, Héra, Shmerkin, and Yavicoli [23] extended these arguments to obtain nontrivial estimates on the size of $(\alpha, 2\alpha)$-Furstenberg sets, where the corresponding set of line segments is 2α-dimensional rather than 1-dimensional. This variant was introduced earlier by Molter and Rela [27], who in fact considered the more general (α, β)-Furstenberg sets.

The main result in this chapter is a quantitative improvement to the bound of Héra, Shmerkin, and Yavicoli [23].

Theorem 3.1. *Every $(\alpha, 2\alpha)$-Furstenberg set in $\mathbb{R}^2$ has Hausdorff dimension at least $2\alpha + c(\alpha)$, where*

$$c(\alpha) = \frac{\alpha(1-\alpha)}{504(2+\alpha)}.$$

In particular, every $1/2$-Furstenberg set in the plane has Hausdorff dimension at least $1 + \frac{1}{5040}$.

We will prove a δ-discretised version of this statement, where $\delta > 0$ is any sufficiently small but fixed number; this is known to imply the Hausdorff dimension bound, for example by [23, Lemma 3.3].

We will consider lines that are of the form $y = mx + b$, with $0 \le m \le 1$ and $0 \le b \le 1$. We identify a line l of this form with the point $\iota(l) = (m, b) \in [0,1]^2$. If L is a set of lines, we define $\iota(L) = \{\iota(l) \colon l \in L\} \subset [0,1]^2$. For each $l \in L$, let $P_l \subset l \cap B(0,1)$ be a set of points. Let $0 < \alpha \le 1$ and $0 < \beta \le 2$. We say that the pair $(L, \{P_l\}_{l \in L})$ is a *discretised* (α, β)*-Furstenberg set* (at scale δ, with error ε) if the following properties hold.

- $\#L \ge \delta^{-\beta+\varepsilon}$, and for each ball $B \subset \mathbb{R}^2$ of radius $r \ge \delta$, we have
$$\#(B \cap \iota(L)) \le r^{\beta}\delta^{-\beta-\varepsilon}. \tag{3.2}$$
- For each $l \in \mathcal{L}$, $\#P_l \ge \delta^{-\alpha+\varepsilon}$, and for each ball $B \subset \mathbb{R}^2$ of radius $r \ge \delta$, we have
$$\#(B \cap P_l) \le r^{\alpha}\delta^{-\alpha-\varepsilon}. \tag{3.3}$$

The discretised version can now be stated as follows.

Proposition 3.2. *Let $(L, \{P_l\}_{l \in L})$ be a discretised $(\alpha, 2\alpha)$ Furstenberg set at scale δ, with error ε. Then*
$$\mathcal{E}_\delta\Big(\bigcup_{l \in L} P_l\Big) \gtrsim \delta^{-2\alpha - c(\alpha) + O(\varepsilon)}, \tag{3.4}$$
where
$$c(\alpha) = \frac{\alpha(1-\alpha)}{504(2+\alpha)}.$$

3.1.1 Proof strategy

At a qualitative level, the connection between sum-product estimates, point-line incidences, and Furstenberg sets is well known. In [8], Bourgain Katz and Tao proved a sum-product estimate over $\mathbb{F}_p$, and used this to obtain a point-line incidence bound in $\mathbb{F}_p^2$. The argument used by Héra, Shmerkin, and Yavicoli [23] is also based on these ideas, but uses the discretised projection theorem instead of the sum-product theorem. Our main contribution is that our arguments are carefully structured to obtain an effective quantitative relationship between the discretised sum-product theorem and the size of $(\alpha, 2\alpha)$-Furstenberg sets. One major difficulty is that while both the

$(\alpha, 2\alpha)$-Furstenberg problem and the discretised sum-product problem involve sets that satisfy non-concentration conditions, not all paths between these two non-concentration conditions are equally efficient.

At a qualitative level, the main argument is as follows. Suppose P is a set of N points, L is a set of N lines, and the pair (P, L) determine roughly $N^{3/2}$ incidences. Write $p \sim q$ if the points p and q are incident to a common line. A typical point will be incident to about $N^{1/2}$ lines, each of which is incident to about $N^{1/2}$ points, and these $N^{1/2}$ sets of points will be disjoint. Thus for a typical point $p \in P$, there are about $N = \#P$ points $q \in P$ with $p \sim q$. We will call this set of points the "bush" of p. The basic idea is that the intersection of a pair of these bushes, as illustrated in Figure 3.1 looks approximately like a Cartesian product set, up to a projective transformation. We find three collinear points p_1, p_2, p_3, and a fourth point p_4, so that there are about N points contained in the intersection of the bushes of $p_1, \dots, p_4$; call this set of points P'.

We apply a projective transformation that sends lines through p_1 to vertical lines; lines through p_2 to lines with slope -1; lines through p_3 to horizontal lines; and lines through p_4 to lines through the origin. In particular, there are four sets L_1, L_2, L_3, L_4 of lines, each of cardinality about $N^{1/2}$, so that the lines in L_1 are vertical; the lines in L_2 have slope -1; the lines in L_3 are horizontal; and the lines in L_4 contain the origin; these four sets of lines are precisely the images of the lines through $p_1, \dots, p_4$, respectively. Furthermore, if Q is the image of P' under the projective transformation described above, then every point in Q is contained in a line from each of these families. If we define X to be the set of x-intercepts of the lines in L_1 and Y to be the set of y-intercepts of lines in L_2, then X and Y have cardinality about $N^{1/2}$, the set Q has cardinality about N, and the sets $X \overset{Q}{-} Y$ and $X \overset{Q}{\div} Y$ each have cardinality about $N^{1/2}$. This violates the sum-product theorem. We conclude that the pair (P, L) must determine far fewer than $N^{3/2}$ incidences.

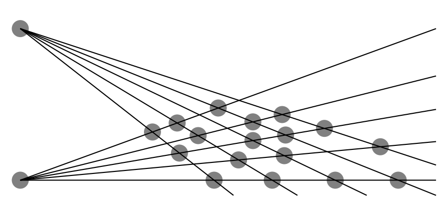


Figure 3.1: Points with approximate Cartesian product structure

3.2 Combinatorial tools

In this section, we record some combinatorial tools that we will use in later sections. We start with some basic averaging arguments.

Lemma 3.3 (Dyadic pigeonholing)**.** *Let A be a finite set on which there are functions $\mu, \nu_1, \ldots, \nu_k : A \to \mathbb{R}^+$. Suppose that we have*

$$\sum_{a \in A} \mu(A) \geq X, \tag{3.5}$$

and that for each $1 \leq i \leq k$, $\log \nu_i \lesssim \log X$. There is a subset $A' \subset A$ and numbers $m_1, \ldots, m_k$ such that for all $a \in A'$ and for each $1 \leq i \leq k$, we have

$$m_i \leq \nu_i(a) < 2m_i,$$

and

$$\sum_{a \in A'} \mu(a) \gtrsim X \log^{-k} X.$$

Proof. We have

$$\sum_{j=1}^{C \log_2 X} \sum_{\substack{a \in A: \\ 2^j \leq \nu_1(a) < 2^{j+1}}} \mu(A) \geq X,$$

so by the pigeonhole principle, there is a number $1 \leq j \lesssim \log X$ such that

$$\sum_{\substack{a \in A: \\ 2^j \leq \nu_1(a) < 2^{j+1}}} \mu(A) \gtrsim X \log^{-1} X.$$

Let

$$A_1 = \{a \in A : 2^j \leq \nu_1(a) < 2^{j+1}\}$$

and repeat the argument on the set A_1 relative to $\nu_2(\cdot)$. This yields a set $A_2 \subset A_1$ and we continue to repeat this process until we reach $A' := A_k$. At each step, we lose a $\lesssim \log X$ factor. $\square$

Generally, logarithmic losses will be acceptable for our purposes. This lemma therefore allows us to assume that any finite number of combinatorial properties are essentially uniform over the set. In particular, when we have a set satisfying an equation of the form (3.5), we will often apply this argument to obtain uniformity of $\mu(\cdot)$, by taking $\nu_1 = \mu$.

We will also use a popularity argument from [11], which serves a similar purpose in refining sets so that each element has high multiplicity relative to some relation.

Lemma 3.4. *Let $G = (A \sqcup B, E)$ be a bipartite graph. Then there are sets $A' \subset A$, $B' \subset B$, and $E' \subset E$ so that $\#E' \geq \#E/2$; each vertex in A' has degree at least $\frac{\#E}{4\#A}$; and each vertex in B' has degree at least $\frac{\#E}{4\#B}$.*

As we count subconfigurations in later sections, we start by considering simple objects and build on these to count richer objects. To do this, we will use the following version of Hölder's inequality.

Lemma 3.5. *Let A and B be finite sets and let $\sim$ be a relation on the pairs $A \times B$. If*

$$\#\{(a, b) \in A \times B : a \sim b\} \geq X,$$

and $k \geq 2$ is an integer,then

$$\#\{(a_1, \dots, a_k, b) \in A^k \times B : a_i \sim b \text{ for } i = 1, \dots, k\} \geq \frac{X^k}{(\#B)^{k/k'}},$$

where k' is the conjugate exponent of k, that is, $1/k + 1/k' = 1$.

Proof. By Hölder's inequality, we have

$$\begin{aligned} X &\leq \#\{(a, b) \in A \times B : a \sim b\} = \sum_{b \in B} 1 \cdot \#\{a \in A : a \sim b\} \\ &\leq (\#B)^{1/k'} \left(\sum_{b \in B} \#\{(a_1, \dots, a_k) \in A^k : a_i \sim b \text{ for } i = 1, \dots, k\} \right)^{1/k}, \end{aligned}$$

and rearranging gives the result. □

3.2.1 Additive combinatorics

We will also require some results from additive combinatorics. Before stating these, we introduce some relevant definitions. See [33] for a more thorough introduction to the area.

For finite sets A, B in an Abelian group, the *sum set* $A + B$ is the set of elements $a + b$, where $a \in A$ and $b \in B$. The *difference set* is defined similarly, as well as the *product set* and *ratio set* if the group operation is multiplication. The size of these sets represents the degree to which the sets A and B are closed under the group operation, and this can therefore be regarded as a measure of group-like structure.

For example, if $A = B$ is a generic finite set within the real numbers, each of the pairwise sums or differences will be distinct and the cardinality of the sum set and the difference set will be as large as possible. On the other hand, the closest thing to a finite subgroup of the real numbers under addition is an arithmetic progression, and if we let $A = B$ be an arithmetic progression, we have $\#(A + A) = 2(\#A) - 1$.

Sum sets and difference sets are central objects of study in the field of additive combinatorics. In particular, there are various extremal questions, seeking to understand under what conditions the sum set or difference set can be small [33].

Observe that in order for $A + B$ to be small, there must be many values of λ for which there are many pairs $(a, b) \in A \times B$ with

$$\lambda = a + b.$$

On the other hand, it is easy to see that this property does not guarantee a small sum set, since the union of an arithmetic progression and a generic set will still have this property despite having a large sum set due to the generic subset. In applications, it is often the case that only this weaker property holds, whereas one would like to know that the sum set or difference set is small. The Balog–Szemerédi–Gowers lemma roughly says that this stronger property can always be achieved from the weaker one by taking a suitable refinement of the sets.

The following generalisation of the sum set helps to make this idea precise. For a set of pairs $E \subset A \times B$, we can consider the *partial sum set* relative to E,

$$A \overset{E}{+} B = \{a + b : (a, b) \in E\}.$$

The *partial difference set* relative to E can be defined analogously.

Lemma 3.6 (Balog–Szemerédi–Gowers). *Let X, Y be finite subsets of an Abelian group G, and let $E \subset X \times Y$. Then there is a set $A \subset X$ such that*

$$\#A \gtrsim \frac{\#E}{\#Y},$$

and

$$\#(A - A) \lesssim \frac{(\#X)^4(\#Y)^3(X \overset{E}{-} Y)^4}{(\#E)^5}.$$

Proof. This result follows by making a very small modification to [6, Lemma 2.2]. The exact version can be found as Lemma 8 in [18]. $\square$

Similar to the partial sum and difference set is the idea of additive energy. For finite sets A and B, the *additive energy* is the number of quadruples

$(a, b, a', b') \in A \times B \times A \times B$ with $a + b = a' + b'$. The additive energy will be large precisely when the partial sum or difference set is small relative to a large set $E \subset A \times B$.

Since we will work with δ-discretised sets, this definition requires some modification. Indeed, for δ-discretised sets A and B, we define the *additive energy* of A and B to be

$$E_+(A, B) = \sum_{x \in A} \sum_{y \in B} \# \left(N_\delta(x + B) \cap N_\delta(y + A)\right),$$

where $N_\delta(\cdot)$ is the δ-neighbourhood of a set. If the group operation is multiplication, then *multiplicative energy* is defined analogously, that is,

$$E_\times(A, B) = \sum_{x \in A} \sum_{y \in B} \# \left(N_\delta(x \cdot B) \cap N_\delta(A \cdot y)\right).$$

When the ambient set is not just a group but a ring, we can compare the additive and multiplicative versions of each of these notions to study the interaction of addition and multiplication. The central theme along these lines is the sum-product phenomenon, which concerns the incompatibility of additive and multiplicative structure.

Our main ingredient in the proof of Theorem 3.2 is a quantitive statement on the sum-product phenomenon, known as the discretised sum-product theorem. We use the following variant of [16, Theorem 1.1].

Proposition 3.7. *Let $\delta > 0$ be a small number and let $A \subset [C^{-1}, C]$ be a δ-separated set. Suppose that*

1. *$\#A \geq K_1^{-1}\delta^{-\alpha}$,*

2. *for all finite intervals $J \subset \mathbb{R}$, we have $\#(A \cap J) \leq K_2|J|^\alpha \delta^{-\alpha}$,*

3. *$\mathcal{E}_\delta(A - A) \leq K_3 \mathcal{E}_\delta(A)$,*

4. *$E_\times(A, A) \geq K_4^{-1}|A|^3$.*

Then

$$K_1^3 K_2^3 K_3^{10} K_4^4 \gtrsim C^{-O(1)} |\log \delta|^{-O(1)} \delta^{-\frac{\alpha(1-\alpha)}{2+\alpha}}. \tag{3.6}$$

This result is essentially Theorem 1.1 from [16]. The statement of [16, Theorem 1.1] requires that $\mathcal{E}_\delta(A \cdot A)$ be small, which is slightly stronger than the assumption 4. However, in the proof of [16, Theorem 1.1], the assumption onthe size of $\mathcal{E}_\delta(A \cdot A)$ is only used to obtain a bound of the form 4, with $K_3 = K_4$. Furthermore, in [16, Theorem 1.1] the dependence on K_1 and K_2 is not specified. The proof of Proposition 3.7 is identical to the proof of [16, Theorem 1.1], except we must track the dependence on K_1, K_2, K_3, K_4 throughout the argument. We will briefly highlight the key steps in the proof of [16, Theorem 1.1], which illustrate the precise dependence on K_1, K_2, K_3, K_4.

We first outline the idea of the proof. The key object to consider is the set of ratios

$$\frac{A-A}{A-A} = \left\{ \frac{a_1 - a_2}{a_3 - a_4} : a_i \in A \text{ for } i = 1,2,3,4 \right\},$$

and the proof splits into two cases based on this set. The first case is when this set densely covers an interval of length ~ 1. This implies that $(A - A)/(A - A)$ must have a large covering number, and Plünnecke's inequality then implies that either $A - A$ or $A \cdot A$ have a large covering number. We call this the 'dense case'. Alternatively, it may be that the set $(A-A)/(A-A)$ has large gaps. In this case, the gaps can be exploited to find some other set related to A which must have a large covering number. By Plünnecke's inequality, this can again be converted into a lower bound for the covering number of either $A - A$ or $A \cdot A$. We call this second case the 'gap case'.

Observe that the denominators $a_3 - a_4$ in the set $(A - A)/(A - A)$ could be very small; the size of $a_3 - a_4$ affects the scale of the element $(a_1 - a_2)/(a_3 - a_4)$. For this reason, we use a threshold parameter γ and consider separately the case when the denominator $a_3 - a_4$ is or is not greater than δ^γ. In the end, γ can be optimised to give the best combined bound for the two cases.

Proof of Proposition 3.7. Proceeding as in the proof of [16, Theorem 1.1], using assumptions 3 and 4, we get

$$\mathcal{E}_\delta(aA' \pm bB') \lesssim K_3^2 K_4 (\#A).$$

Then following [16, Lemma 3.6], we get

$$\mathcal{E}_\delta(d_1 A' + d_2 A') \lesssim \big(\mathcal{E}_d(bA'')\big)^{-1} K_3^8 K_4^4 \rho^4 (\#A')$$

and

$$\mathcal{E}_d(bA'') \gtrsim \frac{\#A''}{K_1 K_2 \delta^{-\varepsilon} d^\alpha \#A'} \sim K_1^{-1} K_2^{-1} \delta^\varepsilon d^{-\alpha},$$

and therefore,

$$\mathcal{E}_\delta(d_1 A' + d_2 A') \lesssim K_1 K_2 K_3^8 K_4^4 \delta^{-\varepsilon} \rho^4 \max(|d_1|, |d_2|)^\alpha (\#A').$$

The argument from [16, Lemma 3.7] then gives

$$\begin{aligned}\mathcal{E}_\delta(d_1A_2 + d_2A_2 + \cdots + d_2A_2) \lesssim & K_1K_2K_3^{7+k}K_4^4|\log\delta|^{O(1)}\delta^{-O(\epsilon)} \\ & \cdot \rho^{5-k}\max(|d_1|,|d_2|)^\alpha(\#A').\end{aligned}$$

In the gap case, this leads to

$$K_1^2K_2K_3^8K_4^4 \gtrsim |\log\delta|^{-O(1)}\delta^{2\gamma+\alpha-1}$$

or

$$K_1^3K_3K_3^8K_4^4 \gtrsim |\log\delta|^{-O(1)}\delta^{-\gamma\alpha};$$

in the dense case, we get

$$K_1^3K_3K_3^{10}K_4^4 \gtrsim |\log\delta|^{-O(1)}\delta^{-\gamma\alpha}.$$

By choosing $\gamma = (1-\alpha)/(2+\alpha)$, we obtain the result. □

3.3 A discretised incidence theorem

Let $P \subset B(0,1) \subset \mathbb{R}^2$ be a set of points and let L be a set of lines in $\mathbb{R}^2$. We say that a point p is *incident* to a line l if $\mathrm{dist}(p,l) \leq \delta$. We write $I_\delta(P,L)$ to denote the set of pairs $(p,l) \in P \times L$ of incident points.

Suppose there exists a set $I \subset I_\delta(P,L)$ and a number $V \geq 1$ so that the following holds.

- For each line $l \in L$,
$$\#\{p \in P\colon (p,l) \in I\} \geq \delta^{-\alpha+\varepsilon}V. \tag{3.7}$$

- For each ball B of radius $r \geq \delta$,
$$\#\{p \in P \cap B\colon (p,l) \in I\} \leq r^\alpha\delta^{-\alpha-\varepsilon}V. \tag{3.8}$$

- Each point $p \in P$ is incident to approximately an average number of lines. More precisely,
$$\frac{(\#I)}{4(\#P)} \leq \#\{l \in L\colon (p,l) \in I\} \leq \delta^{-\varepsilon}\frac{(\#I)}{(\#P)}. \tag{3.9}$$

- For each point $p \in P$, the lines incident to p point in a set of directions that satisfies a“two-ends” condition. Specifically, for each nonzero vector $v \in \mathbb{R}^2$ and each $r \geq \delta$, we have
$$\#\{l \in L\colon (p,l) \in I, \angle(l,v) \leq r\} \leq Cr^\varepsilon\#\{l \in L\colon (p,l) \in I\}. \tag{3.10}$$

Proposition 3.8. *Let $P \subset B(0,1) \subset \mathbb{R}^2$ be a set of points, let L be a set of lines, and let $I \subset I_\delta(P,L)$. Suppose that P, L, and I satisfy properties* (3.7)–(3.10)*. If $\#L \leq \delta^{-2\alpha}$, then*

$$\#P \gtrapprox C^{-O(1)}\delta^{-2\alpha-c(\alpha)+O(\varepsilon)}(\delta^{2\alpha}\#L)^{\frac{169}{504}}V^{\frac{85}{168}}, \tag{3.11}$$

where

$$c(\alpha) = \frac{\alpha(1-\alpha)}{504(2+\alpha)}.$$

If A and B are quantities that depend on δ, we write $A \lessapprox B$ if there is an absolute constant C so that $A \leq \delta^{-C\varepsilon}B$ for all $\delta > 0$ sufficiently small, where ε is a fixed value for which we seek to prove Proposition 3.8. If $A \lessapprox B$ and $B \lessapprox A$, we write $A \approx B$.

Remark 3.9. *The most interesting case is when $V = 1$ and $\#L \approx \delta^{-2\alpha}$. However, in order to use Proposition 3.8 to obtain bounds on the size of $(\alpha, 2\alpha)$ Furstenberg sets, we will need to apply a two-ends reduction to ensure that* (3.10) *holds. Because of this, we must also consider other values of $V \geq 1$, and sets of lines with $\#L < \delta^{-2\alpha}$. For our applications, the specific exponent $V^{\frac{85}{168}}$ is not important; all that matters is that $\frac{85}{168}\alpha \geq c(\alpha)$ for all $\alpha \in (0,1)$.*

Remark 3.10. *When $V = 1$ and $\#L \approx \delta^{-2\alpha}$, a straightforward argument using the Cauchy-Schwarz inequality shows that $\#P \gtrapprox \delta^{-2\alpha}$, i.e.* (3.11) *holds with $c(\alpha) = 0$. The purpose of Proposition 3.8 is to establish* (3.11) *for an explicit $c(\alpha) > 0$.*

Before proving Proposition 3.8, we observe that the non-concentration condition (3.8) implies a similar type of non-concentration condition for the lines passing through a point.

Lemma 3.11. *Let $P \subset B(0,1) \subset \mathbb{R}^2$ be a set of points, let L be a set of lines, and let $I \subset I_\delta(P,L)$. Suppose that P, L, and I satisfy properties* (3.7)–(3.10)*. For each $p \in P$ and each nonzero vector v and each $r \geq \delta$, we have*

$$\#\{l \in L\colon (p,l) \in I,\ \angle(l,v) \leq r\} \lessapprox Cr^\alpha\delta^\alpha(\#P)V^{-1}. \tag{3.12}$$

Proof. Let $L' \subset L$ be the set of lines satisfying $(p,l) \in I,\ \angle(l,v) \leq r$. By (3.7) and (3.8), for each $l' \in L'$ there is a set $P_{l'} \subset P$ of size $\#P_{l'} \gtrapprox \delta^{-\alpha}V$ so that $|p - p'| \geq \delta^{O(\varepsilon)}$ for each $p' \in P_{l'}$. By (3.10), for each $p' \in P_{l'}$, there are $\gtrapprox C^{-1}\delta^{-\alpha}(\#L)(\#P)^{-1}V$ lines $l'' \in L$ with $(p', l'') \in I$ and $\angle(l', l'') \gtrapprox 1$. We conclude that

$$\begin{aligned}&\#\{(l', p', l'') \in L' \times P \times L \colon (p, l') \in I,\ (p', l') \in I, (p', l'') \in I; |p - p'| \gtrapprox 1,\ \angle(l', l'') \gtrapprox 1\}\\ &\quad \gtrapprox C^{-1}(\delta^{-\alpha}V)(\delta^{-\alpha}(\#L)(\#P)^{-1}V)(\#L')\end{aligned}$$

By pigeonholing, there exists $l''_0 \in L$ that is an element of at least

$$\gtrapprox C^{-1}\frac{(\delta^{-\alpha}V)(\delta^{-\alpha}(\#L)(\#P)^{-1}V)(\#L')}{\#L} = C^{-1}\delta^{-2\alpha}(\#L')(\#P)^{-1}V^2$$

triples. Denote this set of triples by $\mathcal{T}$. Since the lines in L' are δ-separated in parameter space and pass through a common point (up to uncertainty δ), the lines in L' point in δ-separated directions. Since $\angle(l', l''_0) \geq \delta^{\varepsilon}$ for each line $l' \in L'$, the intersection points $l' \cap l''_0$ are $\gtrapprox \delta$-separated. This implies that there are $\gtrapprox C^{-1}\delta^{-2\alpha}(\#L')(\#P)^{-1}V^2$ distinct δ-separated points p' so that $(l', p', l''_0) \in \mathcal{T}$. The points p' are contained in an interval of l''_0 of length $\leq r$, and thus by (3.8) we have

$$C^{-1}\delta^{-2\alpha}(\#L')(\#P)^{-1}V^2 \lessapprox r^{\alpha}\delta^{-\alpha}V.$$

Rearranging, we obtain (3.12). □

In our arguments below, we will make use of a technical lemma called two-ends reduction.

Lemma 3.12 (Two-ends reduction)**.** *Let $\delta > 0$ and let $E \subset [0, 1]$ be a set of δ-separated points. Then for each $\rho > 0$, there is an interval $J \subset [0, 1]$ so that*

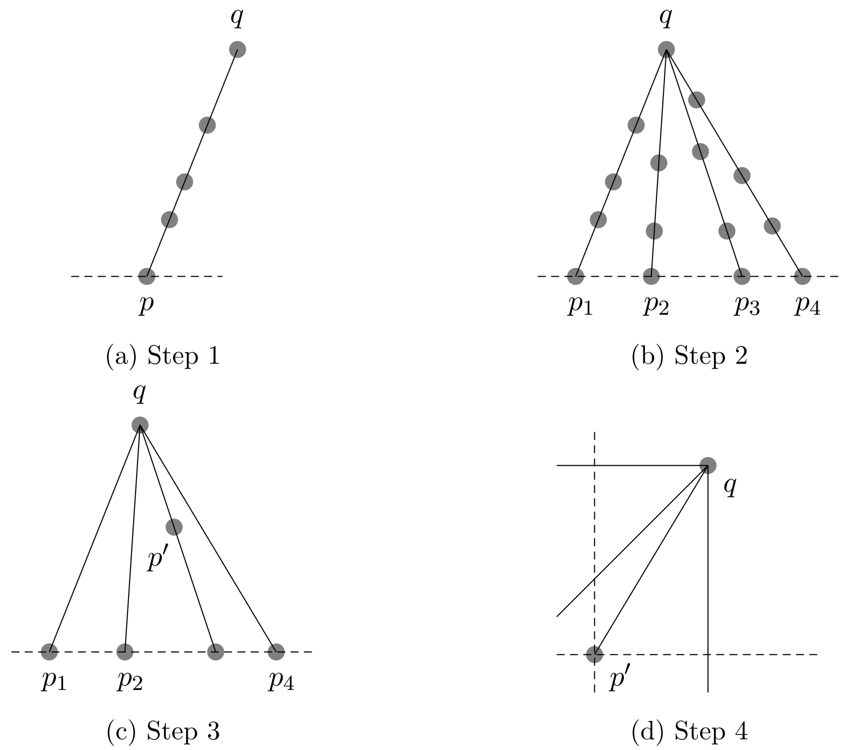
$$\#(E \cap J) \geq \delta^{\rho}(\#E),$$

and for each interval J' of length $r \geq \delta$, we have

$$\#(E \cap J') \leq (r/|J|)^{\rho}\#(E \cap J).$$

See e.g. [32] for details.

The forthcoming proof of Proposition 3.8 will serve as a template for the main argument in proving Theorem 1.5, where we apply the same arguments to more complicated objects. For this reason, we break the proof down into small parts to which we can easily refer.



(a) Step 1 (b) Step 2

(c) Step 3 (d) Step 4

Figure 3.2: The main steps in translating point-line incidences to a sum-product statement

Proof of Proposition 3.8. **Counting triples relative to a fixed line**

Let P, L, I be as in the statement of the Proposition. For each $l_0 \in L$, by (3.7), (3.10), (3.9), and (3.8), we have

$$\begin{aligned}
&\#\{(p,l,q) \in P \times L \times P : (p,l_0),(p,l),(q,l) \in I; |p-q| \gtrapprox 1; \angle(l_0,l) \gtrapprox 1; \\
&\quad \text{there are } \gtrapprox \delta^{-\alpha}V \text{ points incident to } l \text{ between } p \text{ and } q\} \\
&\quad \gtrapprox (\delta^{-\alpha}V)(C^{-1}\delta^{-\alpha}(\#L)(\#P)^{-1}V)(\delta^{-\alpha}V) = C^{-1}\delta^{-3\alpha}(\#L)(\#P)^{-1}V^2.
\end{aligned} \tag{3.13}$$

For each $q \in P$, let $L(q) \subset L$ be the set of lines so that $(\cdot, l, q)$ is an element of the set (3.13). For each $l \in L(q)$, there are $\lessapprox 1$ points $p \in P$ so that $(p, l, q) \in (3.13)$. After applying a two-ends reduction (see Lemma 3.12) to the set of directions $\{v(l) : l \in L(q)\}$, we can find an interval $J \subset S^1$ of directions so that a $\gtrapprox 1$ fraction of the lines $l \in L(q)$ satisfy $v(l) \in J$, and there are $\gtrapprox (\#L(q))^4$ quadruples of lines $l_1, \ldots, l_4$ with $v(l_i, l_j) \gtrapprox |J|$ when

$i \neq j$. Let d_{q,l_0} be the length of the interval J. After further pigeonholing, we can refine the set (3.13) by a $\lessapprox 1$ factor, and find a number d_{l_0} such that $d_{q,l_0} \sim d_{l_0}$ for every point $q \in P$ that contributes to at least one triple from (3.13).

Applying Hölder's inequality By applying Lemma 3.5 to the refinement of (3.13) described above, we have

$$\begin{aligned}\#\{(p_1,\ldots,p_4,l_1,\ldots,l_4,q) \in P^4 \times L^4 \times P : (p_i,l_0),(p_i,l_i),(q,l_i) \in I,\\ \text{for } i=1,\ldots,4 \in I; \operatorname{dist}(p_i,q) \gtrapprox 1;\ \angle(l_0,l_i) \gtrapprox 1,\ i=1,\ldots,4;\\ |p_1-p_2| \sim |p_2-p_3| \sim |p_3-p_4| \sim d_{l_0}\} \gtrapprox C^{-O(1)}\delta^{-12\alpha}(\#L)^4(\#P)^{-7}V^8,\end{aligned} \tag{3.14}$$

where in the above set we chose the labeling so that $p_1,\ldots,p_4$ appear in that order along l_0. Denote the above set of tuples by $\mathcal{N}_{l_0}$.

Note that if $(p_1,\ldots,p_4,l_1,\ldots,l_4,q) \in \mathcal{N}_{l_0}$, then $\operatorname{dist}(l_0,q) \gtrapprox 1$. By (3.13), we also have that there are $\gtrapprox \delta^{-\alpha}V$ points $p' \in P$ with $(p',l_3) \in I$ so that p' is between p_3 and q on the line l_3. Thus by (3.8) and the fact that $\angle(l_0,l_3) \gtrapprox 1$, we have that there are $\gtrapprox A\delta^{-\alpha}$ points $p' \in P$ so that $(p',l_3) \in I$, $\operatorname{dist}(l_0,p') \gtrapprox 1$, $|p'-q| \gtrapprox 1$, and p' is contained in the line segment between p_3 and q (i.e. p' is contained inside the triangle $\Delta_{p_1,p_4,q}$ spanned by p_1,p_4,q). This also implies that

$$\operatorname{dist}(p,\overline{p_1p'}) \gtrapprox d_{l_0}, \operatorname{dist}(p,\overline{p_4p'}) \gtrapprox d_{l_0}. \tag{3.15}$$

Thus,

$$\begin{aligned}\#\{(p_1,\ldots,p_4,l_1,\ldots,l_4,q,p') \in \mathcal{N}_{l_0} \times P : &(p',l_3) \in I;\ \operatorname{dist}(l_0,p') \gtrapprox 1;\\ &|p'-q| \gtrapprox 1\ p' \in \Delta_{p_1,p_4,q},\\ &\operatorname{dist}(p,\overline{p_1p'}) \gtrapprox d_{l_0},\ \operatorname{dist}(p,\overline{p_4p'}) \gtrapprox d_{l_0}\}\\ &\gtrapprox C^{-O(1)}\delta^{-13\alpha}(\#L)^4(\#P)^{-7}V^9.\end{aligned} \tag{3.16}$$

Summing over the lines Recall that the set described in (3.16) depends on the choice of line l_0. Let $\mathcal{D}$ be the union of such sets over all lines $l_0 \in L$, i.e.

$$\begin{aligned}\mathcal{D} = \{(l_0,p_1,\ldots,p_4,l_1,\ldots,l_4,q,p')\colon &(p_1,\ldots,p_4,l_1,\ldots,l_4,q) \in \mathcal{N}_{l_0};\ (p',l_3) \in I;\\ &\operatorname{dist}(l_0,p') \gtrapprox 1;\ |p'-q| \gtrapprox 1\ p' \in \Delta_{p_1,p_4,q};\\ &\operatorname{dist}(p,\overline{p_1p'}) \gtrapprox d_{l_0},\ \operatorname{dist}(p,\overline{p_4p'}) \gtrapprox d_{l_0}\}.\end{aligned}$$

We have

$$\#\mathcal{D} \gtrapprox C^{-O(1)}\delta^{-13\alpha}(\#L)^5(\#P)^{-7}V^9.$$

After dyadic pigeonholing, we can suppose there is a number d and a set $\mathcal{D}' \subset \mathcal{D}$ with $\#\mathcal{D}' \gtrapprox \mathcal{D}$ so that so that $d_{l_0} \sim d$ for each tuple in $\mathcal{D}'$.

Fixing four special points

Next, we will use pigeonholing to select a choice of (l_0, p_1, p_2, p_4, p') that occur in many tuples from $\mathcal{D}'$. There are $\#L$ choices for l_0. Once l_0 has been specified, there are at most $\delta^{-\alpha}V$ choices for p_1. Next, there are $\lessapprox d^\alpha\delta^{-\alpha}V$ choices for each of p_2 and p_4. Finally, there are $\leq \#P$ choices for p'. Thus there is a choice of (l_0, p_1, p_2, p_4, p') that appears in

$$\gtrapprox C^{-O(1)}\frac{\delta^{-13\alpha}(\#L)^5(\#P)^{-7}V^9}{(\#L)(\delta^{-\alpha}V)(d^\alpha\delta^{-\alpha}V)^2(\#P)} = C^{-O(1)}d^{-2\alpha}\delta^{-10\alpha}(\#L)^4(\#P)^{-8}V^6$$

tuples from $\#\mathcal{D}'$. Fix this choice of l_0, p_1, p_2, p_4, p'. Note that the point p_3 and line l_3 are determined up to multiplicity $\lessapprox 1$.

Write $p \sim q$ if there is a line $l \in L$ with $(p, l), (q, l) \in I$. Since $|p_i - q| \gtrapprox 1,\ i = 1, 2, 4$ and $|p' - q| \gtrapprox 1$, if $p_i \sim q$ then there is $\lessapprox 1$ line $l_i \in L$ incident to both p_i and q. Similarly for p' and q. Thus we have

$$\begin{aligned}\{q \in P\colon p_1 \sim q;\ p_2 \sim q;\ p_4 \sim q;\ p' \sim q;\ \operatorname{dist}(l_0, q) \gtrapprox 1;\ |p' - q| \gtrapprox 1;\\ p' \in \Delta_{p_1,p_4,q}; \operatorname{dist}(q, \overline{p_1p'}) \gtrapprox d;\ \operatorname{dist}(q, \overline{p_1p'}) \gtrapprox d\}\\ \gtrapprox C^{-O(1)}d^{-2\alpha}\delta^{-10\alpha}(\#L)^4(\#P)^{-8}V^6.\end{aligned} \tag{3.17}$$

Rescaling Cover $B(0, 1) \subset \mathbb{R}^2$ with finitely overlapping rectangles of length $1 \times 100d$, each of which contain the line segment p_1—p_4. Each point q in the above set is contained in $O(1)$ of these rectangles. By five applications of dyadic pigeonholing, we can assume that we have M rectangles, where each rectangle contains the same number of points, up to a factor of two and each rectangle contains the same number of lines respectively incident to each of p_1, p_2, p_4 and p, up to a factor of two. Moreover, (3.17) still holds up to a factor of $\lessapprox 1$. For each rectangle, we must have at least $\gtrapprox d^{-2\alpha}\delta^{-10\alpha}(\#L)^4(\#P)^{-8}V^6M^{-1}$ associated points q. But each point q is contained in $O(1)$ distinct rectangles and this implies that each rectangle contains at most a $\lessapprox 1/M$-fraction of the lines from each of p_1, p_2, p_4, p'. By

(3.12), each rectangle contains at most $\lessapprox \delta^\alpha(\#P)V^{-1}M^{-1}$ lines from each of these points. We will now fix one such rectangle R.

Note that once q has been specified, the lines l_1, l_2, l_3 are fixed up to multiplicity $\lessapprox 1$. The converse, however is not true; three lines l_1, l_2, l_3 intersect in a rectangle of dimensions roughly $\delta \times \delta/d$; the long axis of this rectangle is parallel to the long axis of the rectangle R. By (3.8), there could be as many as $\lessapprox d^{-\alpha}V$ points q in this rectangle that are incident to l_1, l_2 and l_3. We will refine the set (3.17) by a factor of $\lessapprox d^{-\alpha}V$, so that for each point q in this set, q is the only point in the corresponding rectangle of dimensions $\delta \times \delta/d$.

Let $\tilde{\delta} = \delta/d$. We will translate and rescale the rectangle R so that it becomes the unit square $[0,1]^2$. Let $\tilde{p}_1, \tilde{p}_2, \tilde{p}_4, \tilde{p}'$ be the images of p_1, p_2, p_4, p', respectively, under this transformation. We have that $\tilde{p}_1 = (0,0)$, $\tilde{p}_4 = (1,0)$, and $\tilde{p}_3$ is contained in the δ-neighborhood of the x-axis and has distance ≈ 1 from $\tilde{p}_1$ and $\tilde{p}_4$.

Each of the rectangular boxes described above becomes a square Q of dimensions $\tilde{\delta} \times \tilde{\delta}$, and each such square is connected to each of $\tilde{p}_1, \tilde{p}_2, \tilde{p}_4, \tilde{p}'$ by a line (up to uncertainty $\tilde{\delta}$). Denote this set of squares by $\mathcal{Q}$; we have

$$\#\mathcal{Q} \gtrapprox C^{-O(1)} d^{-\alpha} \delta^{-10\alpha} (\#L)^4 (\#P)^{-8} V^5 M^{-1}. \tag{3.18}$$

The lines incident to $\tilde{p}_1 = (0,0)$ satisfy an analogue of the non-concentration estimate (3.12), with $(dr)^\alpha$ in place of r^α (indeed; if v is a non-zero vector in $\mathbb{R}^2$, then the set of lines incident to $\tilde{p}_1$ and making angle $\le r$ with the vector v correspond to a set of lines in the original set L that are incident to p_1 and make angle $\le dr$ with the pre-image of v). Similarly for $\tilde{p}_2, \tilde{p}_3$, and $\tilde{p}'$.

Applying a projective transformation Next, we apply a projective transformation that sends the x-axis to the line at infinity. Lines containing $(0,0)$ are mapped to vertical lines; lines containing $(1,0)$ are mapped to horizontal lines. Let $\mathcal{Q}^\dagger$ denote the image of the boxes in $\mathcal{Q}$ under this transformation, and let $p^\dagger$ denote the image of $\tilde{p}'$ under this transformation. We have that each set in $\mathcal{Q}^\dagger$ is contained in $B(0,100)$ and $p^\dagger \in B(0,100)$. After a translation we can suppose that $p^\dagger$ is the origin. Let $\mathcal{Q}^{\dagger\dagger}$ be the image of $\mathcal{Q}^\dagger$ under this translation. The line (in the original configuration) passing through p_1 and p' is mapped to the y-axis, and the line passing through p_4 and p' is mapped to the x-axis. Since each point q from (3.17) had distance $\gtrapprox d$ from the lines $\overline{p_1p'}$ and $\overline{p_4p'}$, each square in $\mathcal{Q}$ had distance $\gtrapprox 1$ from the images of these lines, and thus each set in $\mathcal{Q}^{\dagger\dagger}$ has distance $\gtrapprox 1$ from the x and y axes.

The lines incident to p_1 become lines parallel to the y-axis; let X be the set of x-intercepts of these lines. The lines incident to $p_2 = (1, 0)$ become lines parallel to the x-axis; let Y be the set of y-intercepts of these lines. The sets X and Y are $\tilde{\delta}$ separated, and $X, Y \subset [C_0^{-1}, C_0]$ for some $C_0 \approx 1$. For each interval J we have

$$\begin{aligned} \#(X \cap J) &\lessapprox |J|^\alpha d^\alpha \delta^\alpha (\#P) V^{-1}, \\ \#(Y \cap J) &\lessapprox |J|^\alpha d^\alpha \delta^\alpha (\#P) V^{-1}. \end{aligned} \tag{3.19}$$

Applying a transformation of the form $(x, y) \mapsto (Tx, y)$ for $T \approx 1$, we can assume that lines incident to p_3 map to lines with slope -1. Call this set of lines Z. We also still have a set of lines, which we denote by W that are incident to the origin and the entire set E. Recall that by (3.12) and the fact that R contains at most a $\lessapprox 1/M$ fraction of the lines incident to each of p_1, p_2, p_4, p', we have

$$\begin{aligned} \#X &\lessapprox \delta^{-\alpha} (\#L)(\#P)^{-1} V M^{-1}, \\ \#Y &\lessapprox \delta^{-\alpha} (\#L)(\#P)^{-1} V M^{-1}, \\ \#Z &\lessapprox \delta^{-\alpha} (\#L)(\#P)^{-1} V M^{-1}, \\ \#W &\lessapprox \delta^{-\alpha} (\#L)(\#P)^{-1} V M^{-1}. \end{aligned} \tag{3.20}$$

Let $E \subset X \times Y$, with $(x, y) \in E$ if the corresponding box is an element of $\mathcal{Q}^{\dagger\dagger}$. By (3.18),

$$\#E \gtrapprox d^{-\alpha} \delta^{-10\alpha} (\#L)^4 (\#P)^{-8} V^5 M^{-1}.$$

Furthermore, if $(x, y) \in E$, then $x - y \in Z$, i.e.

$$\#(X \overset{E}{-} Y) \leq \#Z \lessapprox \delta^{-\alpha} (\#L)(\#P)^{-1} V M^{-1}.$$

Applying Balog–Szemerédi–Gowers Next, we will use the Balog-Szemerédi-Gowers lemma to find a large set $A \subset X$ whose sum-set is small. Indeed, by Lemma 3.6, we extract a set $A \subset X$ with

$$\begin{aligned} \#A &\gtrapprox C^{-O(1)} \frac{d^{-\alpha} \delta^{-10\alpha} (\#L)^4 (\#P)^{-8} V^5 M^{-1}}{\delta^{-\alpha} (\#L)(\#P)^{-1} V M^{-1}} \\ &= C^{-O(1)} \Big(d^{-2\alpha} \delta^{-8\alpha} (\#L)^3 (\#P)^{-7} V^4 \Big) \tilde{\delta}^{-\alpha}, \end{aligned}$$

and

$$\begin{aligned}\#(A-A) &\lessapprox C^{O(1)} \frac{\left(\delta^{-\alpha}(\#L)(\#P)^{-1}VM^{-1}\right)^{12}}{\left(d^{-\alpha}\delta^{-10\alpha}(\#L)^4(\#P)^{-8}V^5M^{-1}\right)^6}(\#A)\\ &= C^{O(1)}\left(d^{6\alpha}\delta^{48\alpha}(\#L)^{-12}(\#P)^{36}V^{-18}M^{-6}\right)(\#A).\end{aligned}$$

Since $E \cap (A \times Y) \subset E$ is still covered by $\#W$ lines through the origin, we have

$$E_\times(A,Y) \gtrapprox \frac{\left(\#(E\cap(A\times Y))\right)^2}{\#W}.$$

Thus, by Cauchy-Schwarz (see [33, Corollary 2.10]),

$$E_\times(A,A) \geq \frac{\left(E_\times(A,Y)\right)^2}{E_\times(Y,Y)} \gtrapprox \frac{\left(\#(E\cap(A\times Y))\right)^4}{(\#W)^2(\#Y)^3}.$$

But

$$\#\big(E\cap(A\times Y)\big) \gtrapprox (\#E)\frac{\#A}{\#X},$$

so we get

$$\begin{aligned}E_\times(A,A) &\gtrapprox \frac{(\#E)^4(\#A)^4}{(\#W)^2(\#X)^4(\#Y)^3} \gtrapprox \frac{(\#E)^5}{(\#W)^2(\#X)^4(\#Y)^4}(\#A)^3\\ &\gtrapprox C^{-O(1)}\frac{\left(d^{-\alpha}\delta^{-10\alpha}(\#L)^4(\#P)^{-8}V^5M^{-1}\right)^5}{\left(\delta^{-\alpha}(\#L)(\#P)^{-1}VM^{-1}\right)^{10}}(\#A)^3 \qquad (3.21)\\ &= C^{-O(1)}\left(d^{-5\alpha}\delta^{-40\alpha}(\#L)^{10}(\#P)^{-30}V^{15}M^5\right)(\#A)^3.\end{aligned}$$

Applying the discretised sum-product theorem To summarize, we have a set $A \subset [C_0^{-1}, C_0]$ for $C_0 \lessapprox 1$, with

$$\#A \gtrapprox C^{-O(1)}\left(d^{2\alpha}\delta^{8\alpha}(\#L)^{-3}(\#P)^7V^{-4}\right)^{-1}\tilde{\delta}^{-\alpha}, \qquad (3.22)$$

which satisfies the non-concentration condition

$$\#(A\cap J) \lessapprox \left(\delta^{2\alpha}(\#P)V^{-1}\right)|J|^\alpha\tilde{\delta}^{-\alpha}, \qquad (3.23)$$

and

$$\begin{aligned}\#(A-A) &\lessapprox C^{O(1)}\left(d^{6\alpha}\delta^{48\alpha}(\#L)^{-12}(\#P)^{36}V^{-18}M^{-6}\right)(\#A),\\ E_\times(A,A) &\gtrapprox C^{-O(1)}\left(d^{5\alpha}\delta^{40\alpha}(\#L)^{-10}(\#P)^{30}V^{-15}M^{-5}\right)^{-1}(\#A)^3.\end{aligned} \qquad (3.24)$$

Comparing this with Proposition 3.7 and using the bounds (3.22), (3.23), and (3.24), we conclude that

$$\begin{aligned}
&C^{O(1)}\left(d^{2\alpha}\delta^{8\alpha}(\#L)^{-3}(\#P)^{7}V^{-4}\right)^{3}\left(\delta^{2\alpha}(\#P)V^{-1}\right)^{3}\\
&\cdot\left(d^{6\alpha}\delta^{48\alpha}(\#L)^{-12}(\#P)^{36}V^{-18}M^{-6}\right)^{10}\left(d^{5\alpha}\delta^{40\alpha}(\#L)^{-10}(\#P)^{30}V^{-15}M^{-5}\right)^{4}\\
&= d^{86\alpha}\delta^{670\alpha}(\#L)^{-169}(\#P)^{504}V^{-255}M^{-80} = (\delta^{2\alpha}\#P)^{504}(\delta^{2\alpha}\#L)^{-169}V^{-255}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\gtrapprox \tilde{\delta}^{-\frac{\alpha(1-\alpha)}{2+\alpha}},
\end{aligned}$$

i.e.

$$(\delta^{2\alpha}\#P)^{504} \gtrapprox C^{-O(1)}d^{\frac{\alpha(1-\alpha)}{2+\alpha}-86\alpha}M^{80}\delta^{-\frac{\alpha(1-\alpha)}{2+\alpha}}(\delta^{2\alpha}\#L)^{169}V^{255}.$$

It is now clear that the worst case occurs when $d \sim 1$ and $M \sim 1$, which gives

$$\#P \gtrapprox C^{-O(1)}\delta^{-2\alpha-c(\alpha)}(\delta^{2\alpha}\#L)^{\frac{169}{504}}V^{\frac{85}{168}}, \quad c(\alpha) = \frac{\alpha(1-\alpha)}{504(2+\alpha)}.$$

In particular, $c(1/2) = \frac{1}{5040}$. This concludes the proof of Proposition 3.8. $\square$

Remark 3.13. *When $c = 1/2$, the discretised sum-product theorem (Proposition 3.7) gives a gain of roughly $\delta^{-1/100}$. The final gain in our discretised Furstenberg result is roughly 50 times, or two orders of magnitude, worse than this. Roughly speaking, one order of magnitude comes from repeated pigeonholing and use of Cauchy-Schwarz, while the other order of magnitude comes from the use of Balog-Szemerédi-Gowers. While some further optimizations are likely possible, it appears that such optimizations would yield rather modest gains. In particular, Proposition 3.7 requires that the difference set $A-A$ be small (rather than merely requiring that $E_+(A,A)$ be large), so it appears difficult to avoid using the Balog-Szemerédi-Gowers lemma). While a more efficient sequence of pigeonholing and Cauchy-Schwarz arguments might improve the final bound, since the current argument only looses (roughly) one order of magnitude at this step, any improvement here would at best yield a modest improvement in the final bound.*

3.4 From Proposition 3.8 to the Furstenberg set problem

In this section, we apply Proposition 3.8 to the Furstenberg set problem. We restate the results here for convenience.

Proposition 3.14. *Let $(L, \{P_l\}_{l\in L})$ be a discretised $(\alpha, 2\alpha)$ Furstenberg set at scale δ, with error ε. Then*

$$\mathcal{E}_\delta\Big(\bigcup_{l\in L} P_l\Big) \gtrsim \delta^{-2\alpha - c(\alpha) + O(\varepsilon)}, \tag{3.25}$$

where

$$c(\alpha) = \frac{\alpha(1-\alpha)}{504(2+\alpha)}.$$

Corollary 3.15. *Every $(\alpha, 2\alpha)$ Furstenberg set in $\mathbb{R}^2$ has Hausdorff dimension at least $2\alpha + c(\alpha)$, where*

$$c(\alpha) = \frac{\alpha(1-\alpha)}{504(2+\alpha)}.$$

In particular, every Furstenberg $1/2$-set in the plane has Hausdorff dimension at least $1 + \frac{1}{5040}$.

Again, if A and B are quantities that depend on δ, we write $A \lessapprox B$ if there is an absolute constant C so that $A \leq \delta^{-C\varepsilon} B$ for all $\delta > 0$ sufficiently small, where this time ε is a fixed value for which we seek to prove Proposition 3.2. If $A \lessapprox B$ and $B \lessapprox A$, we write $A \approx B$.

Proof of Proposition 3.2. **Some preliminary refinements**

Let P be a maximum δ-separated subset of $\bigcup_{l\in L} P_l$. Let $I_0 \subset P \times L$, where $(p, l) \in I_0$ if there is a point $p' \in P_l$ with $|p - p'| \leq \delta$. We have $\#I_0 \approx \delta^{-3\alpha}$.

Apply Lemma 3.4 to the bipartite graph $(P \sqcup L,\ I_0)$. We obtain sets $P_1 \subset P$, $L_1 \subset L$, $I_1 \subset I_0$, with $\#I_1 \geq \frac{1}{2}\#I_0$. Each $l \in L_1$ has degree $\gtrsim (\#I_1)(\#L)^{-1}$. Since each $l \in L$ contained $\approx \delta^{-\alpha}$ edges in the original graph $(P \sqcup L,\ I_0)$, we must have $\#L_1 \approx \delta^{-2\alpha}$, and each $l \in L_1$ must still contain $\approx \delta^{-\alpha}$ edges in the new graph $(P_1 \sqcup L_1,\ I_1)$. What we've gained, however, is that each $p \in P_1$ now contains $\gtrsim (\#I_0)(\#P)^{-1} \gtrapprox \delta^{-3\alpha}(\#P)^{-1}$ edges in $(P_1 \sqcup L_1,\ I_1)$.

For each $p \in P_1$, apply a two-ends reduction (Lemma 3.12) to the set of directions $\{v(l) : l \in L_1,\ (p, l) \in I_1\}$ with parameter $\varepsilon > 0$. This yields an interval $J_p \subset S^1$ of directions so that

$$\#\{l \in L_1 \colon (p, l) \in I_1,\ v(l) \in J_p\} \geq \delta^{\varepsilon}\#\{l \in L_1 \colon (p, l) \in I_1\},$$

and for every interval $J' \subset S^1$, we have

$$\#\{l \in L_1 \colon (p, l) \in I_1,\ v(l) \in J'\} \leq \Big(\frac{|J'|}{|J_p|}\Big)^{\varepsilon}\#\{l \in L_1 \colon (p, l) \in I_1,\ v(l) \in J_p\}. \tag{3.26}$$

After pigeonholing, there is a set $P_2 \subset P_1$, a number $0 < r \lesssim 1$, and a (large) number N so that $\#P_2 \gtrapprox \#P_1$, and for each $p \in P_2$ we have $r \leq |J_p| < 2r$ and

$$N \leq \#\{l \in L_1 \colon (p,l) \in I_1,\ v(l) \in J_p\} \leq 2N.$$

Define $L_2 = L_1$, and define

$$I_2 = \{(p,l) \in I_1 \cap (P_2 \times L_2) \colon v(l) \in J_p\}.$$

Apply Lemma (3.4) to $(P_2 \sqcup L_2, I_2)$, and let $(P_3 \sqcup L_3, I_3)$ be the resulting refinement. Since each point $p \in P_2$ had multiplicity between N and $2N$, we have $N\#P_2 \leq \#I_2 \leq 2N\#P_2$, and thus $\frac{1}{2}N(\#P_2) \leq \#I_3 \leq 2N(\#P_2)$. Since each element of P_3 has multiplicity at most $2N$, we have $\#P_3 \geq \frac{1}{4}\#P_2$, and each $p \in P_3$ has multiplicity at least $N/4$. In particular, for each point $p \in P_3$ and each interval $J' \subset S^1$ we have

$$\#\{l \in L_3 \colon (p,l) \in I_3,\ v(l) \in J'\} \leq 4\Big(\frac{|J'|}{2r}\Big)^{\varepsilon}\#\{l \in L_3 \colon (p,l) \in I_3\}.$$

We still have that each $l \in L_3$ has multiplicity $\approx \delta^{-\alpha}$ under the incidence relation I_3.

Partitioning the arrangement For the remainder of the argument, we will fix r. Let $\mathcal{X}$ be a maximal r–separated subset of $[0,1]^2$, which we identify with the set of lines $y = mx+b$ with $(m,b) \in [0,1]^2$. For each $X \in \mathcal{X}$, define

$$L_X = \{l \in L_3 \colon |\iota(l) - X| \leq 3r\}.$$

For each $p \in P_3$, define l_p to be the line that contains p whose direction corresponds to the midpoint of the interval J_p. Define

$$P_X = \{p \in P_3 \colon |\iota(l_p) - X| \leq r\}.$$

Clearly each $l \in L_3$ is an element of at least one, and at most $O(1)$ sets L_X. Similarly, each $p \in P_3$ is an element of at least one, and at most $O(1)$ sets P_X. Furthermore, for each $X \in \mathcal{X}$, if $p \in P_X$ and $(p,l) \in I_3$, then $l \in L_X$. To see this, let $X \in \mathcal{X}$, let $p \in P_X$, and let $l \in L_3$ with $(p,l) \in I_3$. Then $(p,l) \in I_3$ implies $\operatorname{dist}(p,l) \leq \delta$, and $v(l) \in J_p$ implies $|v(l) - v(l_p)| \leq r$. Thus $|\iota(l), \iota(l_p)| \leq 2r$, so

$$|\iota(l), X| \leq |\iota(l), \iota(l_p)| + |\iota(l_p), X| \leq 3r.$$

In particular, for each $X \in \mathcal{X}$, each $p \in P_X$ is incident (according to the incidence relation I_3) to about N lines in L_X, and this set of lines still satisfies the non-concentration condition (3.26).

After three applications of dyadic pigeonholing, we can select a set $\mathcal{X}' \subset \mathcal{X}$ such that

$$\sum_{X \in \mathcal{X}'} \#\big(I_3 \cap (P_X \times L_X)\big) \gtrsim |\log \delta|^{-3} \# I_3,$$

and the respective quantities $\#P_X, \#L_X, \#\big(I_3 \cap (P_X \times L_X)\big)$ are essentially equal (within a factor of two) for each $X \in \mathcal{X}'$. Since each point in P_3 is contained in $O(1)$ such sets P_X, we have $\#P_X \lesssim \#P_3/\#\mathcal{X}'$ and similarly, $\#L_X \lesssim \#L_3/\#\mathcal{X}'$ for each $X \in \mathcal{X}'$. Moreover, each $X \in \mathcal{X}'$ satisfies

$$\#\big(I_3 \cap (P_X \times L_X)\big) \gtrsim |\log \delta|^{-3} \# I_3 / \#\mathcal{X}'.$$

Let $I_X = I_3 \cap (P_X \times L_X)$.

Averaging within the rectangle Cover R_X by interior-disjoint rectangles of dimensions $\delta r^{-1}/2 \times \delta/2$, whose long-axis points in the direction v_X. Note that if a line $l \in L_X$ is incident to a point in P_X contained in the interior of one such rectangle S, then its direction $v(l)$ is within distance r of the vector v_X, which implies that l is incident to any point contained in S. After a $O(1)$-refinement of the set of nonempty rectangles S, we can assume that any two rectangles have distance $\gtrsim \delta$ in the direction orthogonal to v_X, while still contributing the same total number of incidences, up to a $O(1)$-factor. After dyadic pigeonholing over the rectangles S, there is a number $1 \leq A_1 \lesssim \delta^{-\varepsilon} r^{-\alpha}$ and a $O(|\log \delta|)$ refinement I'_X of I_X so that for each $\delta r^{-1}/2 \times \delta/2$ rectangle S and any line $l \in L_X$, either $S \cap \{p \in P_X \colon (p,l) \in I'_X\}$ is empty, or the set contains $\sim A_1$ points; the condition $A \geq 1$ is trivial, while the condition $A_1 \lesssim \delta^{-\varepsilon} r^{-\alpha}$ comes from (3.3). Moreover, we still have $\#I'_X \gtrsim |\log \delta|^{-4} \# I_3/\#\mathcal{X}'$.

After dyadic pigeonholing again, this time over the set L'_X, we can assume that each line $l \in L''_X$ is incident to roughly the same number, A_2, of points in P_X under the relation I'_X, while still contributing $\gtrsim |\log \delta|^{-1} \# I'_X$ incidences in total. Let I''_X be the set of incidences from I'_X induced by taking the subset $L''_X \subset L'_X$. Note that since $L''_X \subset L'_X$, each line $l \in L''_X$ intersects $\sim A_2/A_1$ distinct rectangles S. Moreover, since $L''_X \subset L_X$ and $\#L_X \lesssim \#L_3/\#\mathcal{X}'$, we have $A_2 \gtrsim |\log \delta|^{-5} \# I_3/\#L_3$.

Rescaling We will now apply a translation and anisotropic rescaling by $2r^{-1}$ in the direction perpendicular to v_X and 2 in the direction of v_X;

this maps R_X to a square of dimensions $O(1) \times O(1)$. Each of the $\delta/r \times \delta$ rectangles described above is mapped to a $\delta/r \times \delta/r$ rectangle, and these rectangles are interior disjoint. Let $\tilde{L}_X$ denote the image of L''_X under this transformation. The lines in $\tilde{L}_X$ satisfy (3.2), with $\tilde{\delta} = \delta/r$ in place of δ. Specifically, for any ball B of radius $s \geq \delta/r$, we have

$$\#(B \cap \iota(\tilde{L}_X)) \lessapprox (rs)^{2\alpha}\delta^{-2\alpha-\varepsilon} = \delta^{-\varepsilon}s^{2\alpha}(\delta/r)^{-2\alpha}$$

and in particular, $\#\tilde{L}_X \lessapprox \delta^{-\varepsilon}(\delta/r)^{-2\alpha}$.

If Q is a $\delta/r \times \delta/r$ rectangle, we say that $l \in \tilde{L}_X$ is incident to Q if $l \cap Q \neq \emptyset$. Denote this set of incidences by $\tilde{I}_X \subset \mathcal{Q}_X \times \tilde{L}_X$. Clearly the number of incidences decreases by a factor of $\sim A_1$ compared to I''_X, so we have $\tilde{I}_X \gtrapprox |\log\delta|^{-5}\#I_3(\#\mathcal{X}')^{-1}A_1^{-1}$.

Apply Lemma 3.4 to each graph $(\mathcal{Q}_X \sqcup \tilde{L}_X, \tilde{I}_X)$, and let $(\mathcal{Q}'_X \sqcup \tilde{L}'_X, \tilde{I}'_X)$ be the resulting refinement. Then we have $\#\tilde{I}'_X \gtrapprox |\log\delta|^{-5}\#I_3(\#\mathcal{X}')^{-1}A_1^{-1}$. Since $\#\tilde{L}_X \lessapprox \#L_3/\#\mathcal{X}'$, this shows that each line in $\tilde{L}'_X$ must be incident to $\gtrapprox |\log\delta|^{-5}\#I_3(\#L_3)^{-1}A_1^{-1}$ rectangles under the relation $\tilde{I}'_X$. Similarly, since each nonempty rectangle in $\mathcal{Q}_X$ contains the images of at least $\sim A_1$ points from P_X, we must have $\#\mathcal{Q}_X \lessapprox \#P_X/A_1 \lessapprox \#P_3(\#\mathcal{X}')^{-1}A_1^{-1}$ and hence, each rectangle in $\mathcal{Q}'_X$ is incident to

$$\gtrapprox |\log\delta|^{-5}\#I_3/\#P_3 \gtrapprox |\log\delta|^{-5}N$$

lines under $\tilde{I}'_X$. Since every line in L_X incident to a point inside a rectangle is incident to all $\sim A_1$ relevant points within that rectangle, by fixing any such point p and recalling the condition (3.26), we get

$$\#\{l \in \tilde{L}_X : (p,l) \in I'_X, v(l) \in J'\} \lessapprox |\log\delta|^5 s^{\varepsilon}\#\{l \in \tilde{L}_X : (p,l) \in I'_X\}.$$

Let $l \in \tilde{L}_X$ and let $B(x,s)$ be a ball of radius $s \geq \delta/r$ whose intersection with l also intersects a rectangle from $\mathcal{Q}'_X$. Any such rectangle will be wholly contained within the slightly larger ball $B' = B(x,2s)$ and thus by the condition (3.3), we have

$$\#\big(\mathcal{Q}' \cap B\big) \lessapprox s^{\alpha}\delta^{-\alpha-\varepsilon}/A_1.$$

Applying Proposition 3.8 Let $V \sim \delta^{-\varepsilon}r^{-\alpha}/A_1$ be such that $V \geq 1$. Summarizing, we have a set $\tilde{I}'_X \subset \mathcal{Q}' \times \tilde{L}'_X$ where each rectangle $Q \in \mathcal{Q}'_X$ is incident to at least $\gtrapprox |\log\delta|^{-5}N$ and at most $\lessapprox N$ lines under $\tilde{I}'_X$. For every line $l \in \tilde{L}'_X$,

$$\begin{aligned}\#\{Q \in \mathcal{Q}'_X : (Q,l) \in \tilde{I}'_X\} &\gtrsim |\log\delta|^{-5} \# I_3 (\# L_3)^{-1} A_1^{-1} \\ &\gtrsim |\log\delta|^{-5} \delta^{\varepsilon} \# I_2 (\# L)^{-1} r^{\alpha} V \\ &\gtrsim |\log\delta|^{-5} \delta^{2\varepsilon} \# I_2 \delta^{2\alpha} r^{\alpha} V \\ &\gtrsim |\log\delta|^{-7} \delta^{4\varepsilon} (\delta/r)^{-\alpha} V.\end{aligned}$$

For any ball B of radius $s \geq \delta/r$,

$$\#\big(\mathcal{Q}' \cap B\big) \lesssim s^{\alpha} \delta^{-\alpha-\varepsilon}/A_1 \sim s^{\alpha} (\delta/r)^{-\alpha} V.$$

For each $Q \in \mathcal{Q}'_X$, and any nonzero vector $v \in \mathbb{R}$ and each $s \geq \delta/r$, we have

$$\#\{l \in \tilde{L}'_X : (Q,l) \in I'_X, \angle(l,v) \leq s\} \lesssim |\log\delta|^5 s^{\varepsilon} \#\{l \in \tilde{L}'_X : (Q,l) \in \tilde{I}'_X\}.$$

This is precisely the setting in which we can apply Proposition 3.8. We conclude that there are at least

$$\begin{aligned}&\gtrapprox (\delta/r)^{-2\alpha-c(\alpha)} \big((\delta/r)^{2\alpha} \# \tilde{L}_X) V^{\frac{85}{168}} \\ &\gtrapprox \delta^{-2\alpha-c(\alpha)} (\delta^{2\alpha} \# \tilde{L}_X)(r^{c(\alpha)} V^{\frac{85}{168}})\end{aligned}$$

interior-disjoint $\delta/r \times \delta/r$ squares in $\mathcal{Q}'$.

Summing over the rectangles Undoing the rescaling, we see that there are $\gtrapprox \delta^{-2\alpha-c(\alpha)}(\delta^{2\alpha}\#\tilde{L}_X)(r^{c(\alpha)}V^{\frac{85}{168}})$ interior-disjoint $\delta/r \times \delta$ rectangles, each of which contains at least $\sim A_1 \sim \delta^{-\varepsilon} r^{-\alpha}/V$ points from P_X. Thus,

$$\begin{aligned}\# P_X &\gtrapprox \Big(\delta^{-2\alpha-c(\alpha)} (\delta^{2\alpha} \# \tilde{L}_X)(r^{c(\alpha)} V^{\frac{85}{168}})\Big)\Big(r^{-\alpha}/V\Big) \\ &= \delta^{-2\alpha-c(\alpha)} (\delta^{2\alpha} \# \tilde{L}_X)\Big(r^{-\alpha+c(\alpha)} V^{\frac{85}{168}-1}\Big) \\ &\gtrapprox \delta^{-2\alpha-c(\alpha)} (\delta^{2\alpha} \# \tilde{L}_X)\Big(r^{-\alpha+c(\alpha)} (r^{-\alpha})^{\frac{85}{168}-1}\Big) \\ &= \delta^{-2\alpha-c(\alpha)} (\delta^{2\alpha} \# \tilde{L}_X) r^{c(\alpha)-\frac{85}{168}\alpha} \\ &\geq \delta^{-2\alpha-c(\alpha)} (\delta^{2\alpha} \# \tilde{L}_X).\end{aligned}$$

In the above equation we made crucial use of the fact that $\frac{85}{168}\alpha \geq c(\alpha)$, and thus the exponent of r is negative.

Recall that each point $p \in P$ is contained in at most $O(1)$ sets $\{P_X\}_{X \in \mathcal{X}'}$ and each line $l \in L$ is contained in at most $O(1)$ sets $\{L_X\}_{X \in \mathcal{X}'}$. Thus,

$$\# P \geq \sum_{X \in \mathcal{X}'} \# P_X \gtrapprox \delta^{-2\alpha-c(\alpha)} \sum_{X \in \mathcal{X}'} \Big(\delta^{2\alpha} \# \tilde{L}_X\Big) \geq \delta^{-2\alpha-c(\alpha)} \Big(\delta^{2\alpha} \# L\Big) \gtrapprox \delta^{-2\alpha-c(\alpha)}. \tag{3.27}$$

Since $P \subset \bigcup_{l \in L} P_l$ is δ-separated, we obtain (3.25). □

Remark 3.16. *The reduction from Proposition 3.2 to Proposition 3.8 is a variant of the two-ends reduction. For an argument of this type, there are scales $\delta \le r \le 1$. We partition our arrangement of points and lines into balls of radius r (in this case, the partitioning occurs in the parameter space of lines), and we rescale each ball to have radius 1. The original "fine" scale δ is rescaled to $\tilde{\delta} = \delta/r$.*

Compared to the usual two-ends reduction used for the Kakeya or restriction problem, however, we encounter an additional difficulty, which is that the points on each line satisfy an α-dimensional non-concentration condition at scale δ. If we had nearly matching upper and lower bounds on the number of points inside each ball of radius δ/r (such a bound would hold, for example, if the set of points on each line was Ahlfors-regular), then this would imply that the rescaled points on each line also satisfy an α-dimensional non-concentration condition at scale $\tilde{\delta}$. In our setting, however, we have an upper bound on the number of points inside each ball of radius δ/r, but we do not have a matching lower bound. In particular, the points could be very sparsely concentrated at scale δ/r, and then more densely concentrated at larger scales. The parameter A_1 in the proof of Proposition 3.2 was introduced to help measure the extent to which this phenomenon occurs.

Chapter 4

Besicovitch sets

4.1 Introduction

In this chapter, we apply the arguments from Theorem 3.1 to obtain a bound on the upper Minkowski dimension of Besicovitch sets in $\mathbb{R}^3$. Recall that a *Besicovitch set* in $\mathbb{R}^3$ is a compact set $E \subset \mathbb{R}^3$ that contains a unit line segment in every direction. The main result of this chapter is the following.

Theorem 4.1. *Every Besicovitch set in $\mathbb{R}^3$ has upper Minkowski dimension greater than $5/2 + 2.67 \times 10^{-8}$.*

In fact, our result applies more generally to certain discretised sets of tubes. For a small fixed parameter $\delta > 0$, a δ*-tube* is the δ-neighbourhood of a line segment of length 1. We say that a set of δ-tubes $\mathbb{T}$ satisfy the *Wolff axioms* up to error K_w if for any rectangular prism of dimensions $s \times t \times 2$, there are at most $\lesssim K_w st\delta^{-2}$ tubes from $\mathbb{T}$ fully contained in the prism. If $K_w \sim 1$, we simply say that $\mathbb{T}$ satisfy the Wolff axioms. For a set of δ-tubes $\mathbb{T}$, we will consider *shadings* $\{Y(\mathcal{T})\}_{\mathcal{T}\in\mathbb{T}}$, each of which is a subset $Y(\mathcal{T}) \subset \mathcal{T}$. We will prove the following discretised version of Theorem 4.1.

Theorem 4.2. *There is an absolute constant C such that for any $\varepsilon > 0$ there is a constant c_ε depending only on ε such that the following holds. Let $(\mathbb{T}, Y)$ be a set of δ-tubes satisfying the Wolff axioms and suppose that*

$$\sum_{\mathcal{T}\in\mathbb{T}} |Y(\mathcal{T})| \geq \delta^{\varepsilon}, \tag{4.1}$$

for every $\delta \leq \rho < 1$ we have

$$\left| \bigcup_{\mathcal{T}\in\mathbb{T}} N_\rho(Y(\mathcal{T})) \right| \leq c_\varepsilon \rho^{C\varepsilon} \rho^{1/2-\varepsilon_0}. \tag{4.2}$$

Then $\varepsilon_0 \geq 2.67 \times 10^{-8}$.

It is easy to see that the δ-neighbourhood of a Besicovitch set satisfies the Wolff axioms, however the tubes in a set merely satisfying the Wolff axioms need not point in δ-separated directions. This theorem therefore implies Theorem 4.1, by Definition 1.2.

In order to apply the arguments from Theorem 3.2, we require the tubes to satisfy various properties. After recording some geometric results concerning reguli in the next section, we will describe the required properties and then proceed with the main argument assuming these. Each of these properties has some associated parameters and we will therefore obtain a result in terms of these parameters. In the subsequent section, we will show that for any set of tubes satisfying the hypotheses of Theorem 4.2, there must be a refinement with these properties.

We finish this introductory section by listing some notation for various objects used in the proof of Theorem 4.2, along with the respective section where the notation is defined.

Notation	Meaning	Definition location
$\mathcal{T}$	δ-tube	Section 4.1
$\mathbb{T}$	Set of δ-tubes	Section 4.1
$Y(\mathcal{T})$	Shading of a tube	Section 4.1
$R_{\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3}$	Regulus generated by lines coaxial with $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$	Section 4.4
$\Pi(\mathcal{T}_1, \mathcal{T}_2)$	Plane generated by lines coaxial with $\mathcal{T}_1, \mathcal{T}_2$	Section 4.2
$v(\mathcal{T})$	Unit vector with the same direction as the line coaxial with $\mathcal{T}$	Section 4.3
$H_Y(\mathcal{T}_1, \dots, \mathcal{T}_k)$	The (joint) hairbrush of $\mathcal{T}_1, \dots, \mathcal{T}_k$	Subsection 4.3.1
$H'(\mathcal{T}_1, \mathcal{T}_2)$	The refined hairbrush of $\mathcal{T}_1$ and $\mathcal{T}_2$	Subsection 4.3.1
$R(\mathcal{T}_1, \mathcal{T}_2)$	The regulus containing the joint hairbrush of $\mathcal{T}_1$ and $\mathcal{T}_2$	Subection 4.5.2

Table 4.1: Notation

4.2 Reguli

Reguli play an important role in the incidence geometry of lines, as they are the only doubly ruled algebraic surface – except for the plane which is infinitely ruled. They will therefore also be relevant in the incidence geometry of tubes, however there are additional difficulties in this setting. For example, whereas the set of lines contained in a regulus is always 1-dimensional, this is not true in the case of tubes: in the extreme case, some portion of a 'regulus' may look essentially like a plane at scale δ and then the set of tubes contained in its δ-neighbourhood is two dimensional.

In this section, we collect some geometric lemmas that will be important in understanding how tubes intersect with neighbourhoods of reguli. Most of these are proved in [22] though the dependences of the various parameters are not given explicitly. We first recall some definitions from [22].

Definition 4.3 (Linear cone)**.** *Let L be a line in $\mathbb{R}^3$ and let Π be a plane containing L. A* linear cone *of angle α with vertex L is a set of the form*

$$\{p \in \mathbb{R}^3 \colon \operatorname{dist}(p, \Pi) \leq \alpha \operatorname{dist}(\pi(p), L)\},$$

where $\pi(p)$ is the orthogonal projection of p to Π.

Definition 4.4 (Quantitative skewness)**.** *Let L_1 and L_2 be two lines in $\mathbb{R}^3$ that intersect the unit ball. We define* $\operatorname{skew}(L_1, L_2)$ *to be the minimum value of α so that $L_2 \cap B(0,2)$ is contained in a linear cone of angle α with vertex L_1. We say that L_1 and L_2 are* $\geq c$-skew *if this number is at least $\geq c$.*

Definition 4.5 (Quantitative separation)**.** *Let L_1, L_2 be two distinct lines that intersect the unit ball. We say that L_1, L_2 are* t-separated (with error c) *if*

$$ct \leq \operatorname{dist}(p, L_2) \leq c^{-1} t \quad \text{for all } p \in L_1 \cap B(0,1). \tag{4.3}$$

We say that L_1 and L_2 are uniformly separated with error c *if they are t-separated with error c for some value of t.*

Lemma 4.6. *Let L_1 and L_2 be two lines intersecting the unit ball that are uniformly separated with error c_1 and $\geq c_2$-skew. For each point $p \in L_1$, let Π_p denote the plane spanned by L_2 and p. Then for any two points $p_1, p_2 \in L_1 \cap B(0,1)$, we have*

$$c_1 c_2 |p_2 - p_1| \lesssim \angle(\Pi_{p_1}, \Pi_{p_2}) \lesssim c_1 c_2 |p_2 - p_1|.$$

Proof. After applying a rigid motion we can assume that L_1 contains the point $(t, 0, 0)$ and is parallel to the yz-plane and that L_2 is the z-axis. Then L_1 can be written as

$$(t, 0, 0) + \mathbb{R}(0, a, 1)$$

for some value a. Observe that the uniform separation and skewness between L_1 and L_2 imply that $c_1 c_2 t \lesssim |a| \lesssim c_1^{-1} c_2^{-1} t$. Let $p_1 = (t, s_1 a, s_1)$ be a point on L_1. The closest point on L_2 to p_1 will be the point $(0, 0, s_1)$, so the plane Π_{p_1} will have normal vector

$$(0, 0, 1) \times (-t, -s_1 a, 0) = (s_1 a, -t, 0).$$

Thus, if $p_2 = (t, s_2 a, s_2)$ is another point on L_1, the angle θ between Π_{p_1} and Π_{p_2} will be

$$\theta \sim \sin(\theta) = \frac{\|((s_1 a, -t, 0) \times (s_2 a, -t, 0)\|}{\|(s_1 a, -t, 0)\| \|(s_2 a, -t, 0)\|}.$$

A simple calculation then gives

$$c_1 c_2 |p_2 - p_1| \lesssim c_1 c_2 |s_2 - s_1| \lesssim \theta \lesssim c_1^{-1} c_2^{-1} |s_2 - s_1| \leq c_1^{-1} c_2^{-1} |p_2 - p_1|.$$

□

Lemma 4.7. *Let L_1, L_2 be c_1-separated and c_2-skew lines that intersect the unit ball. Let L_1^*, L_2^* be lines intersecting L_1, L_2 inside the unit ball with*

$$c_3 \operatorname{dist}(L_2 \cap L_1^*, L_2 \cap L_2^*) \leq \operatorname{dist}(L_1 \cap L_1^*, L_1 \cap L_2^*) \leq c_3^{-1} \operatorname{dist}(L_2 \cap L_1^*, L_2 \cap L_2^*).$$

Then L_1^*, L_2^* are uniformly separated with error $c_1^2 c_2 c_3$ and $c_1^2 c_2$-skew. Furthermore, for each pair $i, j \in \{1, 2\}$, we have

$$c_1 \lesssim \angle(L_i, L_j^*) \lesssim 1.$$

Proof. Observe first of all that since L_1, L_2 are c_1-separated, we have

$$c_1 \lesssim \angle(L_i, L_j^*) \lesssim 1.$$

After applying a transformation that distorts angles by at most a factor of $\lesssim c_1^{-2} c_2^{-1}$, we can assume that L_1 is the x-axis, L_2^* is the z-axis, and L_2 is the line

$$(0, 0, 1) + \mathbb{R}(0, 1, 0).$$

Let $t = \operatorname{dist}(L_1 \cap L_1^*, L_1 \cap L_2^*)$. Then L_1^* contains the point $(t, 0, 0)$ as well as the point $(0, t', 1)$ for some value $c_3 t \lesssim t' \lesssim c_3^{-1} t$. That is, L_1^* is the line

$$(t, 0, 0) + \mathbb{R}(-t, t', 1),$$

which has distance $\max((1-s)t, st')$ from the z-axis. This shows that L_1^* and L_2^* are uniformly separated with error c_3. It is also clear that these two lines are ~ 1-skew. Combining these estimates with the loss of $\lesssim c_1^{-2} c_2^{-1}$ due to the transformation gives the result. □

Lemma 4.8. *Let L_1, L_2, L_3 be lines intersecting the unit ball such that L_1 and L_2 are uniformly separated with error c_1, L_1, L_3 and L_2, L_3 are $\geq c_2$-separated, and the three lines are pairwise $\geq c_3$-skew. Let L_1^*, L_2^* be two lines intersecting each L_i within the unit ball. Then L_1^*, L_2^* are uniformly separated with error $c_1 c_2^3 c_3^3$ and $c_2^2 c_3$-skew.*

Proof. Let $p_i = L_1 \cap L_i^*$ and let $t = \operatorname{dist}(p_1, p_2)$. Then if Π_i is the plane spanned by L_2 and p_i, Lemma 4.6 implies that

$$c_1 c_3 t \lesssim \angle(\Pi_1, \Pi_2) \lesssim c_1^{-1} c_3^{-1} t.$$

Since Π_i contains the line L_i^*, the intersections $\Pi_i \cap L_3$ take place within the unit ball. Since L_3 is $\geq c_2$-separated and $\geq c_3$-skew to L_2, we must have

$$c_2 c_3 \lesssim \angle(L_3, \Pi_i) \lesssim 1,$$

and this implies that

$$c_1 c_2 c_3 t \lesssim \operatorname{dist}(L_1^* \cap L_3, L_2^* \cap L_3) \lesssim c_1^{-1} c_2^{-1} c_3^{-2} t.$$

Finally, an application of Lemma 4.7 gives the result. □

Lemma 4.9. *Let L_1, L_2, L_3 be three lines that intersect the unit ball. Suppose that L_1, L_2 are uniformly separated with error c_1, L_1, L_3 and L_2, L_3 are 1-separated with error c_2, and the three lines are pairwise $\geq c_3$-skew. Let R be the regulus containing L_1, L_2, L_3 and let p be any point on R. Then there are lines L_1^*, L_2^*, L_3^*, L such that*

- *L_1^*, L_2^*, L_3^* and L are contained in R;*
- *L_1^*, L_2^*, L_3^* are pairwise 1-separated with error $c_2^2 c_3 \cdot \min(c_1^2 c_2 c_3, c_1^2 c_3^4)$ and $c_2^2 c_3$-skew;*
- *L intersects each of L_1^*, L_2^*, L_3^* and the points of intersection are contained in $B(0, (c_2^2 c_3 \cdot \min(c_1^2 c_2 c_3, c_1^2 c_3^4))^{-1})$;*

- $L \cap L_1^* = p$;
- $c_2^2 c_3 \cdot \min(c_1^2 c_2 c_3, c_1^2 c_3^4) \lesssim \angle(L, L_1^*) \lesssim 1$.

Proof. Let L_1^* be the unique line containing p and intersecting L_1, L_2, L_3. Then let L_2^* and L_3^* be any two lines intersecting L_1, L_2, L_3 and such that each pair of points $L_1 \cap L_i^*, L_1 \cap L_j^*$ is $\gtrsim 1$-separated. Now Lemma 4.8 implies that each pair L_i^*, L_j^* is 1-separated with error $c_2^2 c_3 \cdot \min(c_1^2 c_2 c_3, c_1^2 c_3^4)$ and $c_2^2 c_3$-skew. Now let L be the unique line containing p and intersecting L_1^*, L_2^*, L_3^*, and the last thing to note is that by the separation of each pair L_i^*, L_j^*, we must have

$$\angle(L, L_i^*) \gtrsim c_2^2 c_3 \cdot \min(c_1^2 c_2 c_3, c_1^2 c_3^4).$$

□

Lemma 4.10. *Let L_1, L_2, L_3, L_A be four lines such that*

- *L_i intersect L_A;*
- *L_1, L_2 are $\gtrsim 1$-separated and skew;*
- *L_1, L_3 are $\gtrsim c_1$-separated and $\gtrsim c_2$-skew;*
- *L_2, L_3 are $\gtrsim c_1$-separated and $\gtrsim c_2$-skew;*
- *L_3 makes angle $\gtrsim c_3$ with the plane parallel to L_1, L_2.*

Then R_{L_1,L_2,L_3} has gradient $\gtrapprox c_1 c_2 c_3$.

Proof. By applying an affine transformation that distorts angles by a $O(1)$ factor, we can assume that L_1 is the x-axis and the line coaxial with L_2 is

$$(1, 0, 1) + \mathbb{R}(0, 1, 0,).$$

Now by applying a further affine transformation that distorts angles by a factor of $\lessapprox c_1^{-1} c_2^{-1} c_3^{-1}$, we can assume that the line coaxial with $\mathcal{T}_3$ is

$$(0, 1, 0) + \mathbb{R}(0, 0, 1).$$

The regulus that vanishes on L_1^*, L_2^*, L_3^* is $Q = xy - xz - yz + z$, which has gradient $\geq 1/2$ on $Z(Q)$. The result follows. □

Lemma 4.11. *Let $L_1, L_2, L_3, L_A, L, L'$ be lines such that*

- *L_i intersect L_A;*
- *L_1, L_2 are $\gtrsim 1$-separated and skew;*
- *L_1, L_3 are $\gtrsim c_1$-separated and $\gtrsim c_2$-skew;*
- *L_2, L_3 are $\gtrsim c_1$-separated and $\gtrsim c_2$-skew;*
- *L_3 makes angle $\gtrsim c_3$ with the plane parallel to L_1, L_2;*
- *L and L' are $\gtrsim c_4$-separated and $\gtrsim 1$-skew with L_A;*
- *L_3 intersects the r-neighbourhood of L, L'.*

Then any other line intersecting L_A as well as the r-neighbourhood of L, L' is contained in the $\lesssim c_1^{-2}c_2^{-1}c_3^{-1}c_4^{-1}r$-neighbourhood of R_{L_1,L_2,L_3}.

Proof. By applying Lemma 4.10 to the lines L_1, L_2, L_3, L_A, we see that there is a monic polynomial Q vanishing on the lines L_i with $|\nabla Q| \gtrsim c_1c_2c_3$. We now have two $\gtrsim 1$-separated points and a third $\gtrsim c_1$-separated point along $\mathcal{T}_A$ that are δ-close to $Z(Q)$. This implies that the magnitude of the restriction of Q to $\mathcal{T}_A$ is at most $\lesssim c_1^{-1}\delta$. Similarly, the restriction of Q to the r-neighbourhood of L and L' has magnitude $\lesssim rc_1^{-1}$.

Fix a further line L^* intersecting L_A as well as the r-neighbourhood of L, L'. By restricting Q to L^*, we get a univariate polynomial which is bounded in magnitude by $\lesssim rc_1^{-1}$ at two points that are $\gtrsim 1$-separated and a third point that is $\gtrsim c_4$-separated. Now we can use Lagrange interpolation to recover the restriction of Q to this line. This shows that $|Q|$ is bounded by $\lesssim c_1^{-1}c_4^{-1}r$ on the L^*. Since we have $|\nabla Q| \gtrsim c_1c_2c_3$ on $Z(Q)$, we conclude that L^* is contained in the $\lesssim c_1^{-2}c_2^{-1}c_3^{-1}c_4^{-1}r$-neighbourhood of Q. □

4.3 Properties of $5/2+\varepsilon_0$-dimensional Besicovitch sets

In this section, we will describe some properties that a set of δ-tubes may satisfy. These properties will be important in enabling the reduction from Theorem 4.2 to Theorem 3.2.

Fix a set of tubes $(\mathbb{T}, Y)$ throughout this subsection.

(P1): Robust transversality I We say that $(\mathbb{T}, Y)$ is *s-robustly transverse* (with error $1/100$) if for each $x \in \bigcup Y(\mathcal{T})$, we have

$$\#\{\mathcal{T} \in \mathbb{T} : x \in Y(\mathcal{T}); \angle(v(\mathcal{T}), v_x) \le s\} \le \frac{1}{100}\#\{\mathcal{T} \in \mathbb{T} : x \in Y(\mathcal{T})\}.$$

If s is large, this property says that most pairs of tubes intersect at a large angle. This ensures, for example, that for most pairs of tubes, the volume of their intersection is small.

We will also a require robust transversality property to hold at a specific, slightly smaller scale.

(P1'): Robust transversality II If a set of tubes satisfies property (P1) with associated parameter s, we say that property (P1') holds if there is an absolute constant $C > 0$ such that $(\mathbb{T}, Y)$ is *s^C-robustly transverse* with error s^4.

(P2): Averaging reduction We say that $(\mathbb{T}, Y)$ satisfies the *averaging reduction* with multiplicity μ if there is a number μ such that for each $x \in \bigcup Y(\mathcal{T})$ we have

$$\mu \le \sum_{\mathcal{T} \in \mathbb{T}} \chi_{Y(\mathcal{T})}(x) < 2\mu. \tag{4.4}$$

With this property, we know almost exactly how many tubes pass through each point in our set. This gives us more control when analysing the combinatorial properties of objects within our set of tubes.

In [36], Wolff considered the *hairbrush* of a tube within a hypothetical Besicovitch set – this is the set of tubes from the set intersecting a single tube, which we call the *stem* of the hairbrush. This is a fruitful object to study because of the fact that distinct tubes in a hairbrush are mostly disjoint far away from the stem, which means that if a hairbrush contains many tubes then the union of its hairbrush must have large volume.

The next property controls the number of tubes from $\mathbb{T}$ that can intersect the neighbourhood of a line segment.

(P3): Few tubes in a fat hairbrush We say that property (P3) holds with associated parameter C_H if there is a number C_H such that for any tube $\mathcal{T} \in \mathbb{T}$ and any $\delta \le \rho < 1$,

$$\#\{\mathcal{T} \in \mathbb{T} : Y(\mathcal{T}) \cap N_\rho(L) \ne \varnothing\} \le C_H \rho^{1/2} \delta^{-2}.$$

(P4): Planiness We say that property (P4) holds if for every point $p \in \mathbb{R}^3$ there is a plane Π_p so that if $\mathcal{T} \in \mathbb{T}$ and $p \in Y(\mathcal{T})$, then

$$\angle(v(\mathcal{T}), \Pi_p) \lesssim \delta^{-\varepsilon_0}\delta^{1/2}.$$

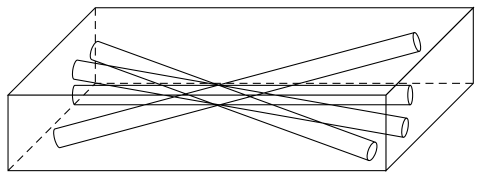


Figure 4.1: A plany intersection point

Given a tube $\mathcal{T}$, it could be the case that many of the tubes in its hairbrush are almost tangent to a single regulus. The last property, based on the *regulus map* introduced in [22], says that every hairbrush must contain many tubes for which this is not the case. We say that a regulus $R(\mathcal{T})$ containing the line coaxial with $\mathcal{T}$ is (c, c')-*non-degenerate* if there are three lines L_1, L_2, L_3 contained in $R(\mathcal{T})$ in the opposite ruling which are pairwise $\geq c$-separated and $\geq c'$-skew.

For any tube $\mathcal{T}$, we let $v(\mathcal{T})$ denote the unit vector with the same direction as the line coaxial with $\mathcal{T}$. Moreover, for a tube $\mathcal{T}'$ and a regulus R intersecting $\mathcal{T}'$, we let $\angle(v(\mathcal{T}'), R)$ denote the minimum angle between $v(\mathcal{T}')$ and the tangent plane T_pR of R at any point $p \in \mathcal{T} \cap R$.

(P5): Regulus map reduction We say that property (P5) holds with associated parameter c_R if for any $\delta \leq c, c' < 1$, there is a number $c_R = c_R(c, c')$ such that for any set of (c, c')-non-degenerate reguli $\{R(\mathcal{T})\}_{\mathcal{T}\in\mathbb{T}}$,

$$\#\{\mathcal{T}' \in H_Y(\mathcal{T}) : \angle(v(\mathcal{T}'), R(\mathcal{T})) > c_R\} \geq \frac{1}{2}\#H_Y(\mathcal{T})$$

for each $\mathcal{T} \in \mathbb{T}$.

4.3.1 Hairbrushes

In this subsection we record some lemmas about hairbrushes in $5/2+\varepsilon_0$-dimensional Besicovitch sets. For some of these, we will assume that certain properties (Pi) from the previous subsection are satisfied.

As in Chapter 3, if A and B are quantities that depend on δ, we write $A \lessapprox B$ if there is an absolute constant C so that $A \leq \delta^{-C\varepsilon}B$ for all $\delta > 0$

sufficiently small, where ε is the fixed value for which we seek to prove Theorem 4.2. If $A \lessapprox B$ and $B \lessapprox A$, we write $A \approx B$.

In the main argument, we will always be able to assume that the set of tubes $(\mathbb{T}', Y')$ that we are analysing satisfy

$$\sum_{\mathcal{T}\in\mathbb{T}'} |Y'(\mathcal{T})| \gtrapprox (\delta^2 \#\mathbb{T}'),$$

and this will lead to statements that hold up to a $\lessapprox 1$ factor. Losses of this type are acceptable in proving Theorem 4.2, so we generally will not be concerned with tracking the dependence explicitly. However, there is one point within the regulus map reduction argument at which we have a set of tubes for which we can only ensure

$$\sum_{\mathcal{T}\in\mathbb{T}'} |Y'(\mathcal{T})| \gtrapprox \delta^{O(\varepsilon_0)}(\delta^2 \#\mathbb{T}').$$

Here, the dependence on ε_0 will be important. For this reason, some of the lemmas that follow are explicit in the dependence on the size of the shading, whereas others are not in order to avoid unnecessary complexity.

When we are not able to ensure that

$$\sum_{\mathcal{T}\in\mathbb{T}'} |Y'(\mathcal{T})| \gtrapprox (\delta^2 \#\mathbb{T}'),$$

it will be necessary for the shadings to satisfy a “two-ends condition”, up to a $\lessapprox 1$ factor. This can always be obtained by applying Lemma 3.12 with exponent ε to each tube.

(P6) Two-ends condition We say that $(\mathbb{T}, Y)$ satisfies property (P6) holds if for each tube $\mathcal{T} \in \mathbb{T}$ there is a number r and ball B of radius r such that

$$|B \cap Y(\mathcal{T})| \geq \delta^{\varepsilon} |Y(\mathcal{T})|,$$

and for all balls B' of radius $\delta \leq r' \leq r$,

$$|B' \cap Y(\mathcal{T})| \geq (r'/r)^{\varepsilon} |B \cap Y(\mathcal{T})|.$$

Observe that if a set of tubes $(\mathbb{T}, Y)$ is s-robustly transverse with error $1/100$ and satisfies the averaging reduction with multiplicity μ, then if we let $H(\mathcal{T})$ consist only of intersecting tubes making angle $\geq s$ with $\mathcal{T}$, then any tube in $\mathcal{T}$ with $|Y(\mathcal{T})| \gtrapprox |\mathcal{T}|$ has $\#H(\mathcal{T}) \gtrapprox s\mu\delta^{-1}$.

We will use the following refinement lemma, which is essentially a version of Wolff's hairbrush lemma from [36], based on Cordóba's argument from [9].

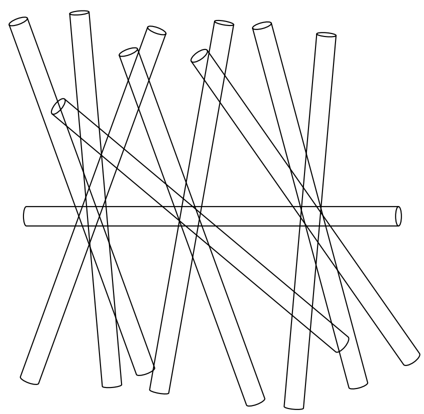


Figure 4.2: A hairbrush

Lemma 4.12. *Let $(\mathbb{T}_0, Y)$ be a set of δ-tubes satisfying the Wolff axioms up to error K_w and intersecting a single tube $\mathcal{T}_0 \in \mathbb{T}$ with*

$$\sum_{\mathcal{T} \in \mathbb{T}_0} |Y(\mathcal{T})| \geq \lambda(\delta^2 \# \mathbb{T}_0).$$

Suppose moreover that each tube in $\mathbb{T}_0$ makes angle $\geq s$ with the line coaxial with $\mathcal{T}_0$, and that the tubes $\mathbb{T}_0$ satisfy property (P6). Then there is a refinement Y' of Y such that

$$\sum_{\mathcal{T} \in \mathbb{T}_0} |Y'(\mathcal{T})| \gtrapprox \lambda(\delta^2 \# \mathbb{T}_0),$$

and for any $x \in \mathbb{R}^3$,

$$\#\{\mathcal{T} \in \mathbb{T}_0 : x \in Y'(\mathcal{T})\} \lessapprox \lambda^{-2} K_w s^{-1}.$$

Proof. Cover $B(0,1)$ by a family of $\lesssim 1$-overlapping δ-balls $\mathcal{B}$. We have

$$\sum_{B \in \mathcal{B}} \sum_{\mathcal{T} \in \mathbb{T}_0} |(Y(\mathcal{T}) \cap B) \setminus N_{\lambda s}(\mathcal{T}_0)| \approx \lambda \delta^2 \# \mathbb{T}_0,$$

so after dyadic pigeonholing over $\mathcal{B}$, we get a subset $\mathcal{B}' \subset \mathcal{B}$ and a number $\nu_{\mathcal{B}'}$ such that

$$\sum_{\mathcal{T} \in \mathbb{T}_0} |Y(\mathcal{T}) \cap B| \sim \nu_{\mathcal{B}'},$$

for each $B \in \mathcal{B}'$ and where each ball $B \in \mathcal{B}'$ is at distance $\gtrsim \lambda s$ from $\mathcal{T}_0$. For each $\mathcal{T} \in \mathbb{T}_0$, let

$$Y_{\mathcal{B}'}(\mathcal{T}) = \bigcup_{B \in \mathcal{B}'} Y(\mathcal{T}) \cap B,$$

and then let $\mathbb{T}_0'$ be the set of tubes in $\mathbb{T}_0$ with $|Y_{\mathcal{B}'}(\mathcal{T})| \gtrapprox \lambda|\mathcal{T}|$. Since

$$\sum_{B\in\mathcal{B}'}\sum_{\mathcal{T}\in\mathbb{T}_0}|Y_{\mathcal{B}'}(\mathcal{T})\cap B| \approx \lambda\delta^2\#\mathbb{T}_0,$$

we must have $\#\mathbb{T}_0' \gtrapprox \lambda\#\mathbb{T}_0$.

Let $\mathcal{T} \in \#\mathbb{T}_0'$ and let $p \in \mathcal{T}$ be a point that is $\gtrapprox \lambda s$-separated from $\mathcal{T}_0$. Then for any angle α, the number of tubes in $\mathbb{T}_0$ that can intersect both p and $\mathcal{T}_1$ and make angle $\leq \alpha$ with $v(\mathcal{T})$ is at most $\lessapprox \lambda^{-1}s^{-1}K_w\alpha\delta^{-1}$. To see this, observe that any such tube must be contained in the $\lessapprox \lambda^{-1}s^{-1}K_w\delta$-neighbourhood of the plane spanned by the lines coaxial with $\mathcal{T}_0$ and $\mathcal{T}$, so this bound simply follows from the Wolff axioms. Thus, by dyadic summation, we have

$$\begin{aligned}\lambda\delta^{-1}\nu_{\mathcal{B}'} &\approx \sum_{\substack{B\in\mathcal{B}':\\ B\cap Y_{\mathcal{B}'}(\mathcal{T})\neq\varnothing}}\sum_{\mathcal{T}'\in\#\mathbb{T}_0'}|Y_{\mathcal{B}'}(\mathcal{T}')|\\ &= \sum_{\substack{B\in\mathcal{B}':\\ B\cap Y_{\mathcal{B}'}(\mathcal{T})\neq\varnothing}}\sum_{k}\sum_{\substack{\mathcal{T}'\in\#\mathbb{T}_0':\\ 2^{-k-1}\leq\angle(v(\mathcal{T}'),L)<2^{-k}}}|Y_{\mathcal{B}'}(\mathcal{T}')|\\ &\lessapprox \lambda^{-1}s^{-1}K_w\delta^2,\end{aligned}$$

and this implies that $\nu_{\mathcal{B}'} \lessapprox \lambda^{-2}s^{-1}K_w\delta^3$. That is, for each $B \in \mathcal{B}'$, there are $\lessapprox \lambda^{-2}s^{-1}K_w$ tubes $\mathcal{T}$ from $\#\mathbb{T}_0'$ with $Y_{\mathcal{B}'}(\mathcal{T}) \cap B \neq \varnothing$. □

This lemma clearly implies the following bound on the volume of a hairbrush, which is a variant of [36, Lemma 3.4].

Lemma 4.13 (Wolff's hairbrush lemma)**.** *Let $(\mathbb{T}_0, Y)$ be a set of δ-tubes satisfying the Wolff axioms and intersecting a single line L_0 with*

$$\sum_{\mathcal{T}\in\mathbb{T}_0}|Y(\mathcal{T})| \geq \lambda(\delta^2\#\mathbb{T}_0).$$

Suppose moreover that each tube in $\mathbb{T}_0$ makes angle $\geq s$ with the L_0, and that the tubes $\mathbb{T}_0$ satisfy property (P6). Then

$$\left|\bigcup_{\mathcal{T}\in\mathbb{T}_0}Y(\mathcal{T})\setminus N_{\lambda s}(\mathcal{T}_0)\right| \gtrapprox \lambda^3K_w^{-1}s(\delta^2\#\mathbb{T}_0).$$

On the other hand, if the union of a set of tubes has small volume, then there must be many pairwise intersections between the tubes, so this leads to hairbrushes with large cardinality. The strategy in [36] involves comparing these two properties and yields the following bound on the volume of a set of tubes.

Theorem 4.14. *Let $(\mathbb{T}, Y)$ be a set of δ-tubes satisfying the Wolff axioms with*

$$\sum_{\mathcal{T}\in\mathbb{T}} |Y(\mathcal{T})| \geq \lambda.$$

Then

$$\left|\bigcup_{\mathcal{T}\in\mathbb{T}} Y(\mathcal{T})\right| \gtrapprox \lambda^{5/2}\delta^{1/2}(\delta^2\#\mathbb{T})^{3/4}.$$

Hairbrushes will be important for our argument, but we have to count more complicated objects in order to ensure certain properties. We denote the intersection of the hairbrushes $H(\mathcal{T}_1), \dots, H(\mathcal{T}_k)$ by $H(\mathcal{T}_1, \dots, \mathcal{T}_k)$. Moreover, we define the *refined hairbrush* $H'(\mathcal{T}_1, \mathcal{T}_2)$ to be a set of tubes in $H(\mathcal{T}_1, \mathcal{T}_2)$ that intersect $\mathcal{T}_2$ with multiplicity $\lesssim 1$. Most of these lemmas are stated in [22] but we require more explicit dependence of the various parameters.

Lemma 4.15. *Let $(\mathbb{T}, Y)$ be a set of δ-tubes satisfying the Wolff axioms with error K_w and with*

$$\sum_{\mathcal{T}\in\mathbb{T}} |Y(\mathcal{T})| \gtrapprox \lambda,$$

and

$$\left|\bigcup_{\mathcal{T}\in\mathbb{T}} Y(\mathcal{T})\right| \lessapprox \delta^{1/2-\varepsilon_0}.$$

Suppose also that the tubes satisfy properties (P1) and (P2) with associated parameters s and μ, and property (P6). Then we have

$$\#\{(\mathcal{T}, \mathcal{T}_1, \mathcal{T}_2) \in \mathbb{T}^3 : \mathcal{T} \in H'(\mathcal{T}_1, \mathcal{T}_2); \angle(\mathcal{T}, \mathcal{T}_i) \gtrsim s\} \gtrapprox \lambda^3 K_w^{-1} s^2 \mu^2 \delta^{-2}(\#\mathbb{T}).$$

Proof. By assumption, all tubes have at least $\gtrapprox s\mu\delta^{-1}$ and at most $\lessapprox \mu\delta^{-1}$ tubes in their hairbrush, each making angle $\gtrsim s$ with the hairbrush stem. Let $\mathcal{T}_1 \in \mathbb{T}$. Then by Lemma 4.13, we have

$$\left|\bigcup_{\mathcal{T}\in H(\mathcal{T}_1)} Y(\mathcal{T})\right| \gtrapprox \lambda^3 K_w^{-1} s^2 \mu\delta.$$

For each $\mathcal{T}' \in \mathbb{T}$, we can now define

$$Y_{\mathcal{T}_1}(\mathcal{T}') = Y(\mathcal{T}') \cap \bigcup_{\mathcal{T} \in H(\mathcal{T}_1)} Y(\mathcal{T}),$$

and by properties (P1) and (P2), we will have

$$\sum_{\mathcal{T}' \in \mathbb{T}} |Y_{\mathcal{T}_1}(\mathcal{T}')| \gtrapprox \lambda^3 K_w^{-1} s^3 \mu^2 \delta,$$

where each $\mathcal{T}'$ makes angle $\approx s$ with the corresponding tube $\mathcal{T}$. Dyadic pigeonhole to find a set of tubes $\mathbb{T}_2$ such that every $\mathcal{T}_2 \in \mathbb{T}_2$ satisfies

$$|Y_{\mathcal{T}_1}(\mathcal{T}_2)| \gtrapprox \lambda^3 K_w^{-1} s^3 \mu^2 \delta (\#\mathbb{T}_2)^{-1}.$$

Then we have

$$\#\{(\mathcal{T}, \mathcal{T}_1, \mathcal{T}_2) \in \mathbb{T}^3 : \mathcal{T} \in H'(\mathcal{T}_1, \mathcal{T}_2); \angle(\mathcal{T}, \mathcal{T}_i) \gtrsim s\} \gtrapprox \lambda^3 K_w^{-1} s^2 \mu^2 \delta^{-2} (\#\mathbb{T}).$$

$\square$

In fact, we can show that for most of these triples, the tubes $\mathcal{T}_1$ and $\mathcal{T}_2$ must be quantitatively separated and skew.

For two tubes $\mathcal{T}_1, \mathcal{T}_2$, there is a unique plane containing the line coaxial with $\mathcal{T}_1$ which is parallel to the line coaxial with $\mathcal{T}_2$ – denote this plane by $\Pi(\mathcal{T}_1, \mathcal{T}_2)$.

Lemma 4.16. *If $(\mathbb{T}, Y)$ is a set of δ-tubes satisfying the Wolff axioms with error K_w and with*

$$\sum_{\mathcal{T} \in \mathbb{T}} |Y(\mathcal{T})| \geq \lambda$$

and

$$\left| \bigcup_{\mathcal{T} \in \mathbb{T}} Y(\mathcal{T}) \right| \lessapprox \delta^{1/2-\varepsilon_0}.$$

Suppose also that the tubes satisfy properties (P1) and (P2) with associated parameters s and μ, and property (P6). Then we have

$$\begin{aligned}\#\{(\mathcal{T}, \mathcal{T}_1, \mathcal{T}_2) \in \mathbb{T}^3 : &\mathcal{T} \in H'(\mathcal{T}_1, \mathcal{T}_2); \angle(\mathcal{T}, \mathcal{T}_i) \gtrsim s; \mathcal{T}_i, \mathcal{T}_j \gtrapprox \lambda^{16} K_w^{-6} s^6 \delta^{4\varepsilon_0}\text{-sep}, \\ &\mathcal{T}_i, \mathcal{T}_j \gtrapprox \lambda^{13} K_w^{-5} s^4 \delta^{4\varepsilon_0}\text{-skew}\} \gtrapprox \lambda^3 K_w^{-1} s^2 \mu^2 \delta^{-2} (\#\mathbb{T}).\end{aligned} \tag{4.5}$$

Proof. By Lemma 4.15 we have

$$\#\{(\mathcal{T},\mathcal{T}_1,\mathcal{T}_2)\in\mathbb{T}^3:\mathcal{T}\in H'(\mathcal{T}_1,\mathcal{T}_2);\angle(\mathcal{T},\mathcal{T}_i)\gtrsim s\}\gtrapprox\lambda^3K_w^{-1}s^2\mu^2\delta^{-2}(\#\mathbb{T}).$$

Denote this set of triples by $\mathcal{S}$, and

Let C be a parameter to be determined and suppose that

$$\#\{(\mathcal{T},\mathcal{T}_1,\mathcal{T}_2)\in\mathcal{S}:\mathcal{T}_1,\mathcal{T}_2\text{ are not }C^{-1}\text{-skew}\}\gtrapprox\lambda^3K_w^{-1}s^2\mu^2\delta^{-2}(\#\mathbb{T})$$

Then we must have

$$\#\{(\mathcal{T},\mathcal{T}_1,\mathcal{T}_2)\in\mathcal{S}:\mathcal{T}_2\subset N_{C^{-1}}(\Pi(\mathcal{T},\mathcal{T}_1))\}\gtrapprox\lambda^3K_w^{-1}s^2\mu^2\delta^{-2}(\#\mathbb{T}).$$

By dyadic pigeonholing, we obtain a refinement $\mathbb{T}'$ consisting of popular tubes such that for each $\mathcal{T}\in\mathbb{T}'$, we have

$$\#\{(\mathcal{T}_1,\mathcal{T}_2)\in\mathbb{T}^2:(\mathcal{T},\mathcal{T}_1,\mathcal{T}_2)\in\mathcal{S};\mathcal{T}_2\subset N_{C^{-1}}(\Pi(\mathcal{T},\mathcal{T}_1))\}\gtrapprox\lambda^3K_w^{-1}s^2\mu^2\delta^{-2}.$$

By pigeonholing, we will select a tube $\mathcal{T}_1$ associated to each $\mathcal{T}$ such that

$$\#\{\mathcal{T}'\in H(\mathcal{T}):\mathcal{T}'\subset N_{C^{-1}}(\Pi(\mathcal{T},\mathcal{T}_1))\}\gtrapprox\lambda^3K_w^{-1}s\mu\delta^{-1}.$$

Fixing a tube $\mathcal{T}$, we will now delete the associated set $N_{C^{-1}}(\Pi(\mathcal{T},\mathcal{T}_1))$ from all shadings of tubes in $\mathbb{T}$, that is for each $\mathcal{T}'\in\mathbb{T}$ we define the new shading

$$Y'(\mathcal{T}')=Y(\mathcal{T}')\setminus N_{C^{-1}}(\Pi(\mathcal{T},\mathcal{T}_1)).$$

We will iterate this procedure.

Let $N\geq 1$. Observe that by the Wolff axioms, the number of tubes from $\mathbb{T}$ for which more than a N-fraction of their shading is contained in the C^{-1}-neighbourhood of a single plane is $\lessapprox N\lambda^{-1}C^{-1}K_w\delta^{-2}$. Thus, after iterating this procedure $\sim\lambda N^{-1}C$ times, at least half of the tubes in $\mathbb{T}'$ will have lost at most a $\lessapprox\lambda K_w^{-1}N^{-2}C$-fraction of their shading. By letting $N\sim\lambda^{1/2}K_w^{-1/2}C^{1/2}$, we can therefore iterate this procedure $\approx\lambda^{1/2}K_w^{-1/2}C^{1/2}$ times and at least half of the tubes will still have at least half of their shadings.

At each step, by Lemma 4.13 the volume of the set that we delete is at least

$$\gtrapprox\lambda^6K_w^{-2}s^2\mu\delta.$$

Hence, we must have

$$C^{1/2}\lambda^{13/2}K_w^{-6/2}s^2\mu\delta\lessapprox\delta^{1/2-\varepsilon_0}.$$

In other words, we get a contradiction by taking

$$C^{-1} \approx \lambda^{13} K_w^{-5} s^4 \delta^{4\varepsilon_0}.$$

It remains to prove the separation condition between $\mathcal{T}_1$ and $\mathcal{T}_2$. Observe that since the tubes satisfy property (P2) with multiplicity μ, for any $\delta \le d < 1$, we have

$$\#\{(\mathcal{T},\mathcal{T}_1,\mathcal{T}_2) \in \mathbb{T}^3 : \mathcal{T} \in H'(\mathcal{T}_1,\mathcal{T}_2); \mathrm{dist}(\mathcal{T}\cap\mathcal{T}_1, \mathcal{T}\cap\mathcal{T}_2) \le d\} \lessapprox d\mu^2\delta^{-2}(\#\mathbb{T}).$$

This implies that we must have

$$\begin{aligned}\#\{(\mathcal{T},\mathcal{T}_1,\mathcal{T}_2) \in \mathcal{S} : \mathcal{T}_1,\mathcal{T}_2 \ge C^{-1}\text{-skew}; \mathrm{dist}(\mathcal{T}\cap\mathcal{T}_1, \mathcal{T}\cap\mathcal{T}_2) \gtrapprox \lambda^3 K_w^{-1} s^2\}\\ \gtrapprox \lambda^3 K_w^{-1} s^2 \mu^2 \delta^{-2}(\#\mathbb{T})\end{aligned}$$

But now the skewness between each such pair $\mathcal{T}_1,\mathcal{T}_2$ implies that each $\mathcal{T}_1,\mathcal{T}_2$ are $\gtrapprox \lambda^{16} K_w^{-6} s^6 \delta^{4\varepsilon_0}$-separated. □

Lemma 4.17. *Let $(\mathbb{T}, Y)$ be a set of δ-tubes satisfying the Wolff axioms, such that*

$$\sum_{\mathcal{T}\in\mathbb{T}} |Y(\mathcal{T})| \gtrapprox 1,$$

and

$$\left|\bigcup_{\mathcal{T}\in\mathbb{T}} Y(\mathcal{T})\right| \le \delta^{1/2-\varepsilon_0}.$$

Suppose also that the tubes satisfy properties (P1) and (P2) with associated parameters s and μ. Then we have

$$\begin{aligned}\#\{(\mathcal{T},\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3) \in \mathbb{T}^4 : &\mathcal{T} \in H'(\mathcal{T}_1,\mathcal{T}_2); \mathcal{T}_3 \in H(\mathcal{T}); \angle(\mathcal{T},\mathcal{T}_i) \gtrsim s;\\ &\mathcal{T}_i,\mathcal{T}_j \gtrapprox s^6\delta^{4\varepsilon_0}\text{-}sep, \gtrapprox s^4\delta^{4\varepsilon_0}\text{-}skew;\\ &\mathrm{dist}(\mathcal{T}_1\cap\mathcal{T}, \mathcal{T}_3\cap\mathcal{T}), \mathrm{dist}(\mathcal{T}_2\cap\mathcal{T}, \mathcal{T}_3\cap\mathcal{T}) \gtrapprox 1\} \gtrapprox s^3\mu^3\delta^{-5}.\end{aligned} \tag{4.6}$$

Proof. By Lemma 4.16, we have

$$\begin{aligned}\#\{(\mathcal{T},\mathcal{T}_1,\mathcal{T}_2) \in \mathbb{T}^3 : &\mathcal{T} \in H'(\mathcal{T}_1,\mathcal{T}_2); \angle(\mathcal{T},\mathcal{T}_i) \gtrsim s;\\ &\mathcal{T}_1,\mathcal{T}_2 \gtrapprox s^6\delta^{4\varepsilon_0}\text{-sep}, \gtrapprox s^4\delta^{4\varepsilon_0}\text{-skew}\} \gtrapprox s^2\mu^2\delta^{-4}.\end{aligned} \tag{4.7}$$

Thus, by properties (P1) and (P2),

$$\begin{aligned}\#\{(\mathcal{T},\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3) \in \mathbb{T}^4 : &\mathcal{T} \in H'(\mathcal{T}_1,\mathcal{T}_2); \mathcal{T}_3 \in H(\mathcal{T}); \angle(\mathcal{T},\mathcal{T}_i) \gtrsim s;\\ &\mathcal{T}_1,\mathcal{T}_2 \gtrapprox s^6\delta^{4\varepsilon_0}\text{-sep}, \gtrapprox s^4\delta^{4\varepsilon_0}\text{-skew}\\ &\mathrm{dist}(\mathcal{T}_1\cap\mathcal{T}, \mathcal{T}_3\cap\mathcal{T}), \mathrm{dist}(\mathcal{T}_2\cap\mathcal{T}, \mathcal{T}_3\cap\mathcal{T}) \gtrapprox 1\} \gtrapprox s^3\mu^3\delta^{-5}.\end{aligned} \tag{4.8}$$

Denote this set of quadruples by $\mathcal{Q}$. Let $C \geq 1$ be a parameter to be determined, and suppose that either

$$\#\{(\mathcal{T}, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3) \in \mathcal{Q} : \mathcal{T}_3 \subset N_{C^{-1}}(\Pi(\mathcal{T}, \mathcal{T}_1))\} \gtrapprox s^3\mu^3\delta^{-5},$$

or

$$\#\{(\mathcal{T}, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3) \in \mathcal{Q} : \mathcal{T}_3 \subset N_{C^{-1}}(\Pi(\mathcal{T}, \mathcal{T}_2))\} \gtrapprox s^3\mu^3\delta^{-5};$$

there is no loss of generality in assuming the former holds. Then in particular, we have

$$\begin{aligned}\#\{(\mathcal{T}, \mathcal{T}_1, \mathcal{T}_3) \in \mathbb{T}^3 : \mathcal{T} \in H(\mathcal{T}_1, \mathcal{T}_2); \angle(\mathcal{T}_i, \mathcal{T}) \gtrsim s; \mathcal{T}_3 \subset N_{C^{-1}}(\Pi(\mathcal{T}, \mathcal{T}_1))\}& \\ \gtrapprox s^3\mu^2\delta^{-4}.&\end{aligned}$$

This estimate has the same form as the output of Lemma 4.15, up to a factor of $\lessapprox s^{-1}$ so by Lemma 4.16, we get a contradiction by taking $C^{-1} \approx s^5\delta^{4\varepsilon_0}$ and that $\mathcal{T}_3$ is $\gtrapprox s^5\delta^{4\varepsilon_0}$-separated from each of $\mathcal{T}_1, \mathcal{T}_2$. □

Lemma 4.18. *Let $(\mathbb{T}, Y)$ be a set of tubes with*

$$\sum_{\mathcal{T} \in \mathbb{T}} |Y(\mathcal{T})| \gtrapprox 1,$$

$$\left|\bigcup_{\mathcal{T} \in \mathbb{T}} Y(\mathcal{T})\right| \leq \delta^{1/2-\varepsilon_0},$$

Suppose also that the tubes satisfy properties (P1), (P1'), (P2), (P4). Then

$$\begin{aligned}\#\{(\mathcal{T}, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3) \in \mathbb{T}^4 :& \mathcal{T} \in H'(\mathcal{T}_1, \mathcal{T}_2); \mathcal{T}_3 \in H(\mathcal{T}); \angle(\mathcal{T}, \mathcal{T}_i) \gtrsim s;\\ & \mathcal{T}_i, \mathcal{T}_j \gtrapprox s^6\delta^{4\varepsilon_0}\text{-}sep, \gtrapprox s^4\delta^{4\varepsilon_0}\text{-}skew;\\ & \operatorname{dist}(\mathcal{T}_1 \cap \mathcal{T}, \mathcal{T}_3 \cap \mathcal{T}), \operatorname{dist}(\mathcal{T}_2 \cap \mathcal{T}, \mathcal{T}_3 \cap \mathcal{T}) \gtrapprox 1;\\ & \angle(v(\mathcal{T}_3), \Pi(\mathcal{T}_1, \mathcal{T}_2)) \gtrapprox s^{O(1)}\delta^{4\varepsilon_0}\} \gtrapprox s^3\mu^3\delta^{-5}.\end{aligned}$$

Proof. By Lemma 4.17 we have

$$\begin{aligned}\#\{(\mathcal{T}, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3) \in \mathbb{T}^4 :& \mathcal{T} \in H'(\mathcal{T}_1, \mathcal{T}_2); \mathcal{T}_3 \in H(\mathcal{T}); \angle(\mathcal{T}, \mathcal{T}_i) \gtrsim s;\\ & \mathcal{T}_i, \mathcal{T}_j \gtrapprox s^6\delta^{4\varepsilon_0}\text{-sep}, \gtrapprox s^4\delta^{4\varepsilon_0}\text{-skew};\\ & \operatorname{dist}(\mathcal{T}_1 \cap \mathcal{T}, \mathcal{T}_3 \cap \mathcal{T}), \operatorname{dist}(\mathcal{T}_2 \cap \mathcal{T}, \mathcal{T}_3 \cap \mathcal{T}) \gtrapprox 1\} \gtrapprox s^3\mu^3\delta^{-5}.\end{aligned}$$

Suppose there is a number $\delta \leq \rho < 1$ such that for at least half of these quadruples $(\mathcal{T}, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)$, we have $\angle(v(\mathcal{T}_3), \Pi(\mathcal{T}_1, \mathcal{T}_2)) \leq \rho$. Let $\rho < \alpha < 1$ be another parameter to be optimised later. We must have either

$$\begin{aligned}\#\{(\mathcal{T}, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3) \in \mathcal{Q} :& \mathcal{T}_i \in H(\mathcal{T}); \angle(v(\mathcal{T}_3), \Pi(\mathcal{T}_1, \mathcal{T}_2)) < \rho;\\ & \angle(v(\mathcal{T}), \Pi(\mathcal{T}_1, \mathcal{T}_2)) \leq \alpha\} \gtrapprox s^3\mu^3\delta^{-5},\end{aligned} \tag{4.9}$$

or

$$\begin{aligned}\#\{(\mathcal{T},\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3)\in\mathcal{Q}:&\mathcal{T}_i\in H(\mathcal{T});\angle(v(\mathcal{T}_3),\Pi(\mathcal{T}_1,\mathcal{T}_2))<\rho;\\ &\angle(v(\mathcal{T}),\Pi(\mathcal{T}_1,\mathcal{T}_2))>\alpha\}\gtrapprox s^3\mu^3\delta^{-5},\end{aligned}\tag{4.10}$$

If (4.9) holds, we get a contradiction by applying Lemma 4.18 and taking $\alpha\approx s^5\delta^{4\varepsilon_0}$.

We therefore assume that (4.10) holds, which implies that there exists a triple $(\mathcal{T},\mathcal{T}_1,\mathcal{T}_2)\in\mathbb{T}^3$ with

$$\#\{\mathcal{T}_3\in H(\mathcal{T}):\angle(v(\mathcal{T}_3),\Pi(\mathcal{T}_1,\mathcal{T}_2))<\rho;\angle(v(\mathcal{T}),\Pi(\mathcal{T}_1,\mathcal{T}_2))>\alpha\}\gtrapprox s^3\mu\delta^{-1},$$

so there is a point $p\in Y(\mathcal{T})$ with

$$\begin{aligned}\#\{\mathcal{T}_3\in H(\mathcal{T}):p\in Y(\mathcal{T}_3);\angle(v(\mathcal{T}_3),\Pi(\mathcal{T}_1,\mathcal{T}_2))<\rho;\angle(v(\mathcal{T}),\Pi(\mathcal{T}_1,\mathcal{T}_2))>\alpha\}\\ \gtrapprox s^3\mu.\end{aligned}$$

But since $v(\mathcal{T})$ makes an angle of less than $\lessapprox\delta^{1/2-\varepsilon_0}$ with the plane map Π_p at p, this implies that there is a vector v_p such that

$$\#\{\mathcal{T}_3\in H(\mathcal{T}):p\in Y(\mathcal{T}_3);\angle(v(\mathcal{T}_3),l_p)\lesssim\rho\alpha^{-1}\}\gtrapprox\mu.$$

Thus, taking $\rho\alpha^{-1}\approx s^C$ contradicts property (P1'), so we get the result with $\rho\approx s^{C'}\delta^{4\varepsilon_0}$. □

4.4 Main argument

In this section we prove the main argument, which implies Theorem 4.2 in the case when properties (P1)–(P5) are satisfied. We start by briefly outlining the argument; an outline of the reduction to this case is given in the subsequent section.

Ultimately, we will map one subset of tubes in $\mathbb{T}$ to points in the plane and another subset to lines, in such a way that a point-line pair is δ-incident if the corresponding tubes intersect. If the original set of tubes has small volume, then there must be many intersecting tubes and this will lead to a set of points and lines with many δ-incidences, to which we can apply the argument from Proposition 3.8.

In order to successfully argue that the resulting planar arrangement has too many incidences, this transformation needs to be approximately injective. To see this, observe that if the fibre of some point contains multiple tubes from $\mathbb{T}$, then we cannot control the number of tubes intersecting the

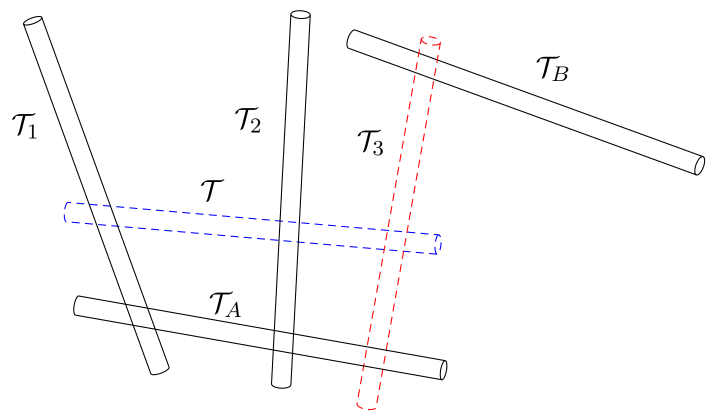


Figure 4.3: Four fixed tubes in $\mathbb{T}$ with many intersecting pairs $(\mathcal{T}, \mathcal{T}_3)$

tubes from this fibre, as any line incident to this point is incident to the image of each corresponding tube. Injectivity of the map will be related to tubes from $\mathbb{T}$ not being contained in neighbourhoods of lower dimensional algebraic varieties. Thus, properties (P1) and (P5) are crucial to the argument.

We will consider tubes in $\mathbb{T}$ intersecting some fixed, carefully chosen tubes, which effectively lowers the dimension of the set of possible tubes. For example, the set of lines intersecting two fixed lines is two-dimensional as it can be described by the intersection point on each of the fixed lines. This allows us to assign coordinates to the tubes in such a way that the set essentially looks like a planar configuration, up to a transformation, and restricts the ways in which the tubes can cluster close to lower dimensional sets.

Rather than mapping the subsets to the plane and repeating the arguments from Proposition 3.8, we will apply most of the arguments directly to the tubes in $\mathbb{R}^3$ as this leads to a better quantitative bound. The main idea of Proposition 3.8 was to find a subset of the points that looked approximately like a Cartesian product set, so that geometric structure could be interpreted arithmetically; we will find a subset of tubes that directly yields such a set, once mapped to the plane. This allows us to almost directly apply the discretised sum-product theorem.

Carrying out the combinatorial arguments on the tubes means that we have to count rather complicated subconfigurations, so it is helpful to keep the planar argument in mind. We start by finding four tubes $\mathcal{T}_A, \mathcal{T}_B, \mathcal{T}_1, \mathcal{T}_2$

with many pairs

$$\{(\mathcal{T}, \mathcal{T}_3) \in H'(\mathcal{T}_1, \mathcal{T}_2) \times H'(\mathcal{T}_A, \mathcal{T}_B) : \mathcal{T} \in H(\mathcal{T}_3)\}, \tag{4.11}$$

as illustrated in Figure 4.3. The tubes $\mathcal{T}_A, \mathcal{T}_B, \mathcal{T}_1, \mathcal{T}_2$ will then act as a frame according to which we can assign coordinates to the tubes $\mathcal{T}$ and $\mathcal{T}_3'$, based on their intersection points with the frame. Given a fixed frame $\mathcal{T}_A, \mathcal{T}_B, \mathcal{T}_1, \mathcal{T}_2$, the tubes $\mathcal{T}$ will be our 'points' and the tubes $\mathcal{T}_3$ will be our 'lines', thereby identifying the set of pairs (4.11) with point-line incidences.

As in the proof of Proposition 3.8, we will count subconfigurations of the forms in Figure 3.2. This results in us fixing four special 'points', as in Step 3 of Figure 3.2, with many further 'points' q incident to a common 'line' with each of the special 'points'. A crucial component in implementing this argument in Proposition 3.8 was ensuring the separation between various pairs of points among these quintuples and we require the analogous properties.

Proposition 4.19. *Let $(\mathbb{T}, Y)$ be a set of δ-tubes satisfying the Wolff axioms with*

$$\sum_{\mathcal{T} \in \mathbb{T}} |Y(\mathcal{T})| \gtrapprox 1,$$

and

$$\left| \bigcup_{\mathcal{T} \in \mathbb{T}} Y(\mathcal{T}) \right| \lessapprox \delta^{1/2 - \varepsilon_0}.$$

Suppose also that $(\mathbb{T}, Y)$ satisfy properties (P1)–(P5) with associated numbers s, μ, C_H, c_R. Then

$$\delta^{-1/10} \lessapprox \left((c_R(s^4 \delta^{4\varepsilon_0}, s^2 \delta^{4\varepsilon_0}) \right)^{-12007/10} \left(c_R(s^{22} \delta^{28\varepsilon_0}, s^{10} \delta^{12\varepsilon_0}) \right)^{-4551/5} \cdot \mu^{-3999} \delta^{-3999/2} C_H^{9102/5} s^{-O(1)} \delta^{-428797\varepsilon_0/5}. \tag{4.12}$$

Remark 4.20. *Assuming best case scenarios for the various parameters can give an idea of the limitations of our methods on the value of ε_0. In particular, assuming that the regulus case could be ruled out with constants $c_R^{-1} \lesssim 1$ and that $C_H \lesssim 1$, (4.12) would lead to a bound of approximately 10^{-6} for ε_0; significant improvements beyond this would require improvements to the sum-product technology that we use, or new ideas in how to apply the sum-product bound to Besicovitch sets. The limitations of our regulus map methods are less transparent and it is possible that significant improvements could be made, leading to improved constants c_R and thereby improving the* *value of ε_0 beyond what is presented herein. We discuss this briefly in the Conclusion.*

The proof splits into three parts. In the first part, we build on the lemmas from the previous section to estimate the number of configurations of the form in Figure 4.3. Then we apply two-ends reductions in analogy with the proof of Proposition 3.2, and the last part involves applying the arguments from Proposition 3.8 and mapping to the plane.

Proof. **Part I: Reduction to a two-dimensional incidence problem for tubes**

By Lemma 4.17, we have

$$\begin{aligned}\#\{(\mathcal{T},\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3)\in\mathbb{T}^4 :\,&\mathcal{T}\in H'(\mathcal{T}_1,\mathcal{T}_2);\mathcal{T}_3\in H(\mathcal{T});\angle(\mathcal{T},\mathcal{T}_i)\gtrsim s;\\ &\mathcal{T}_i,\mathcal{T}_j\gtrapprox s^6\delta^{4\varepsilon_0}\text{-sep},\gtrapprox s^4\delta^{4\varepsilon_0}\text{-skew};\\ &\operatorname{dist}(\mathcal{T}_1\cap\mathcal{T},\mathcal{T}_3\cap\mathcal{T}),\operatorname{dist}(\mathcal{T}_2\cap\mathcal{T},\mathcal{T}_3\cap\mathcal{T})\gtrapprox 1;\\ &\angle(v(\mathcal{T}_3),\Pi(\mathcal{T}_1,\mathcal{T}_2))\gtrapprox s^{O(1)}\delta^{4\varepsilon_0}\}\gtrapprox s^3\mu^3\delta^{-5}.\end{aligned}$$

Apply a dyadic pigeonholing argument to find a set of tubes $\mathbb{T}'\subset\mathbb{T}$ such that for each $\mathcal{T}\in\mathbb{T}'$, we have

$$\begin{aligned}\#\{(\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3)\in\mathbb{T}^3 :\,&\mathcal{T}\in H'(\mathcal{T}_1,\mathcal{T}_2);\mathcal{T}_3\in H(\mathcal{T});\angle(\mathcal{T},\mathcal{T}_i)\gtrsim s;\\ &\mathcal{T}_i,\mathcal{T}_j\gtrapprox s^6\delta^{4\varepsilon_0}\text{-sep},\gtrapprox s^4\delta^{4\varepsilon_0}\text{-skew};\\ &\operatorname{dist}(\mathcal{T}_1\cap\mathcal{T},\mathcal{T}_3\cap\mathcal{T}),\operatorname{dist}(\mathcal{T}_2\cap\mathcal{T},\mathcal{T}_3\cap\mathcal{T})\gtrapprox 1;\\ &\angle(v(\mathcal{T}_3),\Pi(\mathcal{T}_1,\mathcal{T}_2))\gtrapprox s^{O(1)}\delta^{4\varepsilon_0}\}\gtrapprox s^3\mu^3\delta^{-5}(\#\mathbb{T}')^{-1}.\end{aligned}$$

Denote this set of quadruples by $\mathcal{Q}_1$.

We claim that there is a number C such that for any number $\delta\le\rho<1$, any triple $(\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3)\in\mathbb{T}^3$ and any line L,

$$\#\{\mathcal{T}\in\mathbb{T}:(\mathcal{T},\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3)\in\mathcal{Q}_1;\mathcal{T}\subset N_\rho(L)\}<C\rho^{1/2}\delta^{-1/2}.$$

Indeed, suppose that for some number ρ, we have

$$\#\{\mathcal{T}\in\mathbb{T}:(\mathcal{T},\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3)\in\mathcal{Q}_1;\mathcal{T}\subset N_\rho(L)\}\ge C\rho^{1/2}\delta^{-1/2},$$

for some parameter C.

Let $\mathbb{T}_L$ be this set of tubes. Let $\mathcal{B}$ be a minimal covering of $B(0,1)$ by balls of radius δ. By Lemma 4.12, there is a refinement $Y'(\mathcal{T})$ of $Y(\mathcal{T})$ for each tube $\mathcal{T}\in\mathbb{T}_L$ such that

$$\sum_{\mathcal{T}\in\mathbb{T}_L}|Y'(\mathcal{T})|\gtrapprox\lambda(\delta^2\#\mathbb{T}_L),$$

and for any $x \in \mathbb{R}^3$,

$$\#\{\mathcal{T} \in \mathbb{T}_L : x \in Y'(\mathcal{T})\} \lessapprox s^{-1}.$$

Now, by property (P4), we have

$$\begin{aligned}\#\{(\mathcal{T}, B, \mathcal{T}_4) \in \mathbb{T}_L \times \mathcal{B}' \times \mathbb{T} :& (\mathcal{T}, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3) \in \mathcal{Q}_1; \mathcal{T}_4 \in H(\mathcal{T}); \mathcal{T} \subset N_\rho(L); \\ &\angle(\mathcal{T}, \mathcal{T}_4) \gtrsim s; \angle(v(\mathcal{T}_4), R_{\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3}) > c_{R,1}\} \\ &\gtrapprox C\rho^{1/2} s\mu\delta^{-3/2},\end{aligned} \tag{4.13}$$

where we let

$$c_{R,1} = c_R(s^6\delta^{4\varepsilon_0}, s^4\delta^{4\varepsilon_0}). \tag{4.14}$$

But since the tubes $\mathcal{T} \in \mathbb{T}_L$ are $\lessapprox s^{-1}$-overlapping for each such ball B, we get

$$\begin{aligned}\#\{(\mathcal{T}, \mathcal{T}_4) \in \mathbb{T}'_L \times \mathbb{T} :& (\mathcal{T}, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3) \in \mathcal{Q}_1; \mathcal{T}_4 \in H(\mathcal{T}); \mathcal{T} \subset N_\rho(L); \angle(\mathcal{T}, \mathcal{T}_4) \gtrsim s; \\ &\angle(v(\mathcal{T}_4), R_{\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3}) > c_{R,1}\} \gtrapprox C\rho^{1/2} s^2\mu\delta^{-3/2},\end{aligned}$$

Finally, note that any tube making angle greater than $c_{R,1}$ with $R_{\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3}$ intersects the δ-neighbourhood of $R_{\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3}$ in a set contained in at most $\lessapprox c_{R,1}^{-1}$ δ-balls. Hence,

$$\#\{\mathcal{T}_4 \in \mathbb{T} : \angle(v(\mathcal{T}_4), L) \gtrsim s; \mathcal{T}_4 \cap N_\rho(L) \neq \varnothing\} \gtrapprox c_{R,1} C\rho^{1/2} s^2\mu\delta^{-3/2},$$

which contradicts Corollary 4.29 by taking

$$C \approx c_{R,1}^{-1} C_H s^{-2}\mu^{-1}\delta^{-1/2}.$$

Note also that since $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ are pairwise $\gtrapprox s^6\delta^{4\varepsilon_0}$-separated and $\gtrapprox s^4\delta^{4\varepsilon_0}$-skew, Lemma 4.8 implies that

$$\begin{aligned}\#\{\mathcal{T} \in \mathbb{T} :& (\mathcal{T}, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3) \in \mathcal{Q}_1; \min\operatorname{dist}(L, \mathcal{T}) < \rho\} \\ &\lessapprox c_{R,1}^{-1} C_H s^{-20}(\mu^{-1}\delta^{-1/2})\delta^{-14\varepsilon_0}\rho^{1/2}\delta^{-1/2}.\end{aligned} \tag{4.15}$$

Apply a dyadic pigeonholing argument to the triples $(\mathcal{T}, \mathcal{T}_1, \mathcal{T}_2)$ within the set of quadruples $\mathcal{Q}_1$. For any remaing triple, we have

$$\#\{\mathcal{T}_3 \in \mathbb{T} : (\mathcal{T}, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3) \in \mathcal{Q}_1\} \gtrapprox s^3\mu\delta^{-1}.$$

Moreover, each of these tubes $\mathcal{T}_3$ satisfies

$$\angle(v(\mathcal{T}), v(\mathcal{T}_3)) \gtrsim s,$$

so by Lemma 4.13 we have

$$\left| \bigcup_{\mathcal{T}_3 \in \mathbb{T} : (\mathcal{T}, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3) \in \mathcal{Q}_1} Y_5(\mathcal{T}_3) \right| \gtrapprox s^4 \mu \delta.$$

For each tube $\mathcal{T}' \in \mathbb{T}$, define the shading

$$Y'(\mathcal{T}') = Y_5(\mathcal{T}') \cap \bigcup_{\mathcal{T}_3 : (\mathcal{T}, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3) \in \mathcal{Q}_1} Y_5(\mathcal{T}_3).$$

Since each point has multiplicity $\gtrapprox \mu$, we then have

$$\sum_{\mathcal{T}' \in \mathbb{T}} |Y'(\mathcal{T}')| \gtrapprox s^5 \mu^2 \delta,$$

where each $\mathcal{T}'$ makes angle $\gtrsim s$ with the corresponding tube $\mathcal{T}_3$ and satisfies $\angle(v(\mathcal{T}'), R(\mathcal{T}_3)) \gtrapprox c_{R,1}$ for the set of reguli $\{R_{\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3}\}_{\mathcal{T}_3 \in \mathbb{T}}$, each containing the line coaxial with the corresponding tube $\mathcal{T}_3$.

We will also dyadic pigeonhole over the set $\mathbb{T}$ within $\mathcal{Q}_1$ to get a set of tubes $\mathbb{T}'$ for which

$$\sum_{\mathcal{T}' \in \mathbb{T}'} |Y'(\mathcal{T}')| \gtrapprox s^3 \mu^2 \delta,$$

and each tube $\mathcal{T}' \in \mathbb{T}'$ satisfies

$$|Y'(\mathcal{T}')| \gtrapprox s^3 \mu^2 \delta (\#\mathbb{T}')^{-1}.$$

We thus have

$$\#\{(\mathcal{T}_3, \mathcal{T}') \in \mathbb{T} \times \mathbb{T}' : (\mathcal{T}, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3) \in \mathcal{Q}_1; Y(\mathcal{T}_3) \cap Y'(\mathcal{T}') \neq \varnothing\} \gtrapprox s^2 \mu^2 \delta^{-2},$$

and therefore,

$$\#\{(\mathcal{T}_A, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_B) \in \mathcal{Q}_1 \times \mathbb{T}' : \mathcal{T}_3 \in H'(\mathcal{T}_A, \mathcal{T}_B)\} \gtrapprox s^5 \mu^4 \delta^{-6}. \tag{4.16}$$

Let $\mathcal{Q}_2$ denote this set of quintuples.

Part II: Two-ends reductions

For each quadruple $(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_B)$ occurring in $\mathcal{Q}_2$, we have the sets

$$\{\mathcal{T}_A \cap \mathcal{T}_3 : (\mathcal{T}_A, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_B) \in \mathcal{Q}_2\}.$$

By applying the two-ends reduction (see Lemma 3.12) to the tube $\mathcal{T}_3$, we get a number $r_{1,\mathcal{T}_3}$ and a ball $B_{\mathcal{T}_3}$ of radius $r_{1,\mathcal{T}_3}$ containing a $r_{1,\mathcal{T}_3}^{\varepsilon}$-fraction of these intersections and such that the portion of these intersections occurring inside $B_{\mathcal{T}_3}$ satisfy the two-ends condition. By dyadic pigeonholing over the quadruples $(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_B)$, we can assume that every remaining quadruples has corresponding diameter $\sim r_1$. Let $\mathcal{Q}_3$ be the corresponding refinement of $\mathcal{Q}_2$.

After applying dyadic pigeonholing to the set of triples $(\mathcal{T}_A, \mathcal{T}_3, \mathcal{T}_B)$ appearing in $\mathcal{Q}_3$, we obtain a popular subset $\mathcal{V} \subset \mathbb{T}^3$ with approximately equal multiplicity. We now want to further refine this set so that each of the pairs $(\mathcal{T}_A, \mathcal{T}_B)$ is quantitatively separated and skew. As a first step, we will estimate the cardinality of $\mathcal{V}$ from below. Observe that for any fixed triple $(\mathcal{T}_A, \mathcal{T}_3, \mathcal{T}_B)$, the number of pairs $(\mathcal{T}_1, \mathcal{T}_2)$ that can appear in a quintuple in $\mathcal{Q}_3$ with $(\mathcal{T}_A, \mathcal{T}_3, \mathcal{T}_B)$ is at most $\lessapprox \mu^2\delta^{-2}$. Thus, by (4.16) we have $\#\mathcal{V} \gtrapprox s^5\mu^2\delta^{-4}$. Hence, by Lemma 4.16,

$$\#\{(\mathcal{T}_A, \mathcal{T}_3, \mathcal{T}_B) \in \mathcal{V} : \mathcal{T}_A, \mathcal{T}_B \gtrapprox s^{O(1)}\delta^{4\varepsilon_0}\text{-sep}, \gtrapprox s^{O(1)}\delta^{4\varepsilon_0}\text{-skew}\} \gtrapprox s^5\mu^2\delta^{-4}$$

and

$$\#\{(\mathcal{T}_A, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_B) \in \mathcal{Q}_3 : \mathcal{T}_A, \mathcal{T}_B \gtrapprox s^6\delta^{4\varepsilon_0}\text{-sep}, \gtrapprox s^4\delta^{4\varepsilon_0}\text{-skew}\} \gtrapprox s^5\mu^4\delta^{-6}. \tag{4.17}$$

Denote this refinement of $\mathcal{Q}_3$ by $\mathcal{Q}_4$. By the Lemma 3.5, we get

$$\begin{aligned} \#\{(\mathcal{T}_A, \mathcal{T}, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_B) \in \mathbb{T}^6 :&(\mathcal{T}_A, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_B), (\mathcal{T}, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_B) \in \mathcal{Q}_4\} \\ &\gtrapprox s^8\mu^8\delta^{-12}(s\mu\delta^{-7})^{-1} = s^7\mu^7\delta^{-5}, \end{aligned} \tag{4.18}$$

where $\text{dist}(\mathcal{T}_A \cap \mathcal{T}_3, \mathcal{T} \cap \mathcal{T}_3) \sim r_1$. Note that since $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ are pairwise $\gtrapprox s^{O(1)}\delta^{4\varepsilon_0}$-separated and $\gtrapprox s^{O(1)}\delta^{4\varepsilon_0}$-skew, by Lemma 4.8 each pair $\mathcal{T}_A$ and $\mathcal{T}$ must be uniformly separated with error $\approx s^{O(1)}\delta^{28\varepsilon_0}$ and $\approx s^{O(1)}\delta^{12\varepsilon_0}$-skew. Denote this set of tuples by $\mathcal{S}$.

We claim that there is a number C such that for any number $\delta < \rho < 1$, any triple $(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3) \in \mathbb{T}^3$ and any line L',

$$\#\{\mathcal{T}_3 \in \mathbb{T} : \mathcal{T}_3 \subset N_\rho(L')\} \le C\rho^{1/2}\delta^{-1/2}.$$

Indeed, suppose that for some number ρ, we have

$$\#\{\mathcal{T}_3 \in \mathbb{T} : \mathcal{T}_3 \subset N_\rho(L')\} > C\rho^{1/2}\delta^{-1/2}.$$

Let $\mathcal{B}$ be a minimal covering of $B(0,1)$ by balls of radius δ and let $\mathbb{T}_{L'}$ denote this set of tubes $\mathcal{T}_3$. By Lemma 4.12, we can find a subset $\mathcal{B}''$ such that each ball is intersected with multiplicity $\lessapprox s^{-1}$ by tubes in $\mathbb{T}_{L'}$. Thus, by properties (P1), (P2) and (P4), we get

$$\begin{aligned}\#\{(B,\mathcal{T}')\in\mathcal{B}''\times\mathbb{T}:&(\mathcal{T}_A,\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3,\mathcal{T}_B)\in\mathcal{Q}_3;(\mathcal{T},\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3,\mathcal{T}_B)\in\mathcal{Q}_3;\\ &\mathcal{T}_3\subset N_\rho(R);\mathcal{T}'\in H(\mathcal{T}_3);\angle(v(\mathcal{T}'),R_{\mathcal{T}_A,\mathcal{T},\mathcal{T}_B})>c_{R,2}\}\\ &\gtrapprox C^{-1}\rho^{1/2}s^2\mu\delta^{-3/2},\end{aligned}$$

where we let

$$c_{R,2}=c_R(s^{O(1)}\delta^{28\varepsilon_0},s^{O(1)}\delta^{12\varepsilon_0}). \tag{4.19}$$

But any tube making angle greater than $c_{R,2}$ with $R_{\mathcal{T}_A,\mathcal{T},\mathcal{T}_B}$ intersects the δ-neighbourhood of $R_{\mathcal{T},\mathcal{T}_A,\mathcal{T}_B}$ in a set contained in at most $\lessapprox c_{R,2}^{-1}$ δ-balls. Hence, we get

$$\#\{\mathcal{T}'\in\mathbb{T}:\angle(v(\mathcal{T}_4),L')\gtrapprox s;\mathcal{T}_4\cap N_\rho(L')\neq\varnothing\}\gtrapprox c_{R,2}C^{-1}\rho^{1/2}s^2\mu\delta^{-3/2},$$

which contradicts Corollary 4.29 by taking

$$C\approx C_H c_{R,2}^{-1}s^{-2}\mu^{-1}\delta^{-1/2}.$$

Recall that $\mathcal{T}_A$ and $\mathcal{T}$ are uniformly separated with error $\approx s^{O(1)}\delta^{28\varepsilon_0}$ and $\approx s^{O(1)}\delta^{12\varepsilon_0}$-skew, and that each of $\mathcal{T}$ and $\mathcal{T}_A$ is $\approx s^{O(1)}\delta^{4\varepsilon_0}$-separated and $\approx s^{O(1)}\delta^{4\varepsilon_1}$-skew. Lemma 4.8 therefore implies that

$$\#\{\mathcal{T}_3\in\mathbb{T}:\min\operatorname{dist}(\mathcal{T}_3,L')<\rho\}\lessapprox C_H c_{R,2}^{-1}s^{-O(1)}\delta^{-38\varepsilon_0}(\mu^{-1}\delta^{-1/2})\rho^{1/2}\delta^{-1/2}.$$

After dyadic pigeonholing over the quadruples $(\mathcal{T}_A,\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_B)$ in the set $\mathcal{S}$, we can assume that there is a subset $\mathcal{Q}^*$, with each quadruple satisfying

$$\#\{(\mathcal{T},\mathcal{T}_3)\in\mathbb{T}^2:(\mathcal{T}_A,\mathcal{T},\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3,\mathcal{T}_B)\in\mathcal{S}\}\gtrapprox s^7\mu^7\delta^{-5}(\#\mathcal{Q}^*)^{-1}. \tag{4.20}$$

Denote this set of pairs by $\mathcal{I}^*$. By property (P2), we have $\#\mathcal{Q}^*\lessapprox\mu^2\delta^{-6}$, so

$$\mathcal{I}^*\gtrapprox s^7\mu^5\delta. \tag{4.21}$$

After dyadic pigeonholing over the tubes $\mathcal{T}$ appearing in this set of pairs, we can assume that there is a set $\mathbb{T}^*\subset\mathbb{T}$ such that for each $\mathcal{T}\in\mathbb{T}^*$, we have

$$\#\{\mathcal{T}_3\in\mathbb{T}:(\mathcal{T},\mathcal{T}_3)\in\mathcal{I}^*\}\gtrapprox s^7\mu^7\delta^{-5}(\#\mathcal{Q}^*)^{-1}(\#\mathbb{T}^*)^{-1}. \tag{4.22}$$

Now let $\mathcal{I}^{**}$ be the set of pairs from $\mathcal{I}^*$ for which the corresponding tube $\mathcal{T}$ is in the set $\mathbb{T}^*$, and apply Lemma 3.4 to the graph $(\mathbb{T}^*\sqcup\mathbb{T},\mathcal{I}^{**})$. This yields two subsets, which we denote by $\mathbb{T}_P^*$ and $\mathbb{T}_L^*$, respectively.

Recall that for any L, we have

$$\#\{\mathcal{T} \in \mathbb{T} : (\mathcal{T}, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3) \in \mathcal{Q}^*; \min \operatorname{dist}(L, \mathcal{T}) < \rho\} \lessapprox C_H c_{R,1}^{-1} s^{-13} \delta^{-15\varepsilon_0} \rho^{1/2} \delta^{-1/2}$$

and this in particular shows that we have

$$r_1 \gtrapprox C_H^{-2} c_{R,1}^2 s^{36} \delta^{40\varepsilon_0}. \tag{4.23}$$

Partitioning the arrangement At this point, the argument closely follows the two-ends reduction in the proof of Proposition 3.2

We will decompose the set of incidences into neighbourhoods of lines intersecting $\mathcal{T}_1$ and $\mathcal{T}_2$. If we take a minimal r_1-covering $\mathcal{X}_1$ of the set of lines intersecting $\mathcal{T}_1$ and $\mathcal{T}_2$, then each tube in $\mathbb{T}_P^*$ will be fully contained in at least one and at most $O(1)$ tubes in $\mathcal{X}_1$. Thus, we have

$$\#\mathcal{I}^{**} \approx \sum_{X_1 \in \mathcal{X}_1} \mathcal{I}^{**} \cap (\mathbb{T}_P^*[X_1] \times \mathbb{T}_L^*).$$

After dyadic pigeonholing over $\mathcal{X}_1$, we can assume that for each $X_1 \in \mathcal{X}_1' \subset \mathcal{X}_1$, the quantities $\mathcal{I}^{**} \cap (\mathbb{T}_P^*[X_1] \times \mathbb{T}_L^*)$ and $\#\mathbb{T}_P^*[X_1]$ are respectively equal up to a factor of two, and that for each $X_1 \in \mathcal{X}_1'$, the number of tubes from $\mathbb{T}_L^*$ intersecting X_1 is approximately equal.

We claim that any intersection from $\mathcal{I}^{**}$ for a tube in $\mathcal{T}_L^*$ can only occur inside a small number of tubes in $\mathcal{X}_1'$. To see this, recall that by the two-ends condition, all of the relevant incidences occur within a single $\sim r_1$-ball B. By Lemma 4.12, we can assume that the tubes in $\mathcal{X}_1'$ intersect each of these balls with multiplicity $\lessapprox s^{-1}$. Hence, for each $X_1 \in \mathcal{X}_1'$, we have

$$I_{X_1} \gtrapprox \mathcal{I}^{**} (\#\mathcal{X}_1')^{-1},$$

$$\mathbb{T}_{P,X_1} \lessapprox \#\mathbb{T}_P^* (\#\mathcal{X}_1')^{-1},$$

$$\mathbb{T}_{L,X_1} \lessapprox s^{-1} \#\mathbb{T}_L^* (\#\mathcal{X}_1')^{-1}.$$

Averaging within the r_1-tube Fix one such X_1. Continuing in analogy with Proposition 3.2, we will further partition X_1 into rectangular prisms of dimensions $\delta/2 \times \delta/2 \times \min(\delta r_1^{-1}, \delta)$, with the long axis in the direction of the line coaxial with X_1. After dyadic pigeonholing over this set of prisms, we obtain a number A_1 such that each prism intersects $\sim A_1$ tubes from $\mathbb{T}_{L,X_1}$. Observe that any tube from $\mathbb{T}_{P,X_1}$ that intersects this box must intersect all tubes from $\mathbb{T}_{L,X_1}$ that intersect this prism and that we have

$$1 \leq A_1 \lessapprox C_H c_{R,2}^{-1} s^{-34} \delta^{-39\varepsilon_0} r_1^{-1/2}. \tag{4.24}$$

Rescaling By rescaling X_1 by $\sim r_1^{-1}$ in the two directions orthogonal to the line coaxial with X_1, each tube from $\mathbb{T}_P$ contained in X_1 becomes essentially a δr_1^{-1}-tube. We denote the image of $\mathbb{T}_P$ by $\mathbb{T}_P^\dagger$ and the image of $\mathbb{T}_L$ by $\mathbb{T}_L^\dagger$. Note that for any $\mathcal{T}_3 \in \mathbb{T}_L^\dagger$ we still have that for any line L and any number $\delta r_1^{-1} < r < 1$,

$$\#\{\mathcal{T} \in \mathbb{T}_P^\dagger : \min \operatorname{dist}(\mathcal{T}, L) < r\} \lessapprox C_H c_{R,1}^{-1} s^{-O(1)} \delta^{-15\varepsilon} r^{1/2} (\delta r_1^{-1})^{-1/2}, \tag{4.25}$$

but now for each $\mathcal{T} \in \mathbb{T}_P^\dagger$, we have

$$\#\{\mathcal{T}_3 \in \mathbb{T}_L^\dagger : \min \operatorname{dist}(\mathcal{T}_3, L) < r\} \lessapprox C_H c_{R,2}^{-1} s^{-O(1)} \delta^{-39\varepsilon} A_1^{-1} r_1^{-1/2} r^{1/2} (\delta r_1^{-1})^{-1/2}. \tag{4.26}$$

The next step involves repeating this two-ends argument to the other family of tubes. For each $\mathcal{T} \in \mathbb{T}_P^\dagger$, apply Lemma 3.12 to the set

$$\{\mathcal{T} \cap \mathcal{T}_3 : (\mathcal{T}, \mathcal{T}_3) \in I^\dagger\}.$$

This gives a number $r_{2,\mathcal{T}}$ and a ball $B_\mathcal{T}$ of radius $r_{2,\mathcal{T}}$ containing at least a $\gtrapprox r_{2,\mathcal{T}}^\varepsilon$-fraction of these intersections and such that the portion of these intersections occurring inside $B_\mathcal{T}$ satisfy the two-ends condition. By dyadic pigeonholing over the tubes $\mathcal{T} \in \mathbb{T}_P^\dagger$, we can assume that each of these diameters is $\sim r_2$. Note that we must have

$$r_2 \gtrapprox C_H^{-2} c_{R,2}^2 s^{74} \delta^{88\varepsilon_0}. \tag{4.27}$$

Partitioning the arrangement By covering the set of lines intersecting $\mathcal{T}_A$ and $\mathcal{T}_B$ by $\sim r_2$-neighbourhoods of lines $\mathcal{X}_2$, we can carry out an analogous rescaling argument to the one applied to $\mathbb{T}_P$. We again have that each tube in $\mathbb{T}_L$ is contained in at least one and at most $O(1)$ tubes in $\mathcal{X}_2$ and after dyadic pigeonholing we can assume that for each $X_2 \in \mathcal{X}_2'$, the quantities $\mathcal{I}^\dagger \cap (\mathbb{T}_L^\dagger[X_2] \times \mathbb{T}_P^\dagger)$ and $\#\mathbb{T}_L^\dagger[X_2]$ are respectively equal up to a factor of two and similarly for the number of tubes from $\mathbb{T}_P^\dagger$ intersecting X_2 is approximately equal.

Averaging within the r_2-tube and rescaling On the other hand, each tube in $\mathcal{T}_P^\dagger$ intersects a single ball of radius $\sim r_2$ by the two-ends condition and we show that not many tubes from $\mathcal{X}_2'$ can intersect this ball. Recalling that

$$\operatorname{dist}(\mathcal{T} \cap \mathcal{T}_3, \mathcal{T}_A \cap \mathcal{T}_3) \approx 1,$$

and that $\mathcal{T}_A$ and $\mathcal{T}_B$ are $\gtrapprox s^6\delta^{4\varepsilon_0}$ and $\gtrapprox s^4\delta^{4\varepsilon_0}$-skew and repeating the application of Lemma 4.12, we get that each tube in $\mathbb{T}_P^{\dagger}$ can intersect at most $\lessapprox s^{-1}$ tubes from $\mathcal{X}_2'$.

After further partitioning the tubes in $\mathcal{X}_2'$ by prisms of dimensions $\delta r_1^{-1}/2\times \delta r_1^{-1}/2 \times \delta r_1^{-1} r_2^{-1}/2$ and applying the analogous averaging and rescaling arguments, we end up with a number

$$1 \le A_2 \lessapprox C_H c_{R,1}^{-1} s^{-O(1)} \delta^{-15\varepsilon_0} r_2^{-1/2} \tag{4.28}$$

and two sets $\mathbb{T}_P^{\dagger\dagger}$ and $\mathbb{T}_L^{\dagger\dagger}$ of $\delta r_1^{-1} r_2^{-1}$-tubes with

$$\#\mathbb{T}_P^{\dagger\dagger} \lessapprox s^{-1} \#\mathbb{T}_P (\#X_1)^{-1} (\#X_2)^{-1} A_2^{-1},$$

$$\#\mathbb{T}_L^{\dagger\dagger} \lessapprox s^{-1} \#\mathbb{T}_L (\#X_1)^{-1} (\#X_2)^{-1} A_1^{-1},$$

and

$$I^{\dagger\dagger} \gtrapprox (\#\mathcal{I})(\#\mathcal{X}_1)^{-1}(\#\mathcal{X}_2)^{-1} A_1^{-1} A_2^{-1},$$

where $\mathcal{I}^{\dagger\dagger} = \mathcal{I}^{\dagger} \cap (\mathbb{T}_P^{\dagger\dagger} \times \mathbb{T}_L^{\dagger\dagger})$. Moreover, for any tube in either of these sets, the intersections in $\mathcal{I}^{\dagger\dagger}$ along that tube satisfy the two-ends condition and for any line L and any number $\delta r_1^{-1} r_2^{-1} < r < 1$, we have

$$\#\{\mathcal{T}_3 \in \mathbb{T}_L^{\dagger\dagger} : \min \operatorname{dist}(\mathcal{T}_3, L) < r\} \lessapprox C_H c_{R,2}^{-1} s^{-34} \delta^{-39\varepsilon_0} A_1^{-1} r_1^{-1/2} r^{1/2} (\delta r_1^{-1} r_2^{-1})^{-1/2} \tag{4.29}$$

and

$$\#\{\mathcal{T} \in \mathbb{T}_P^{\dagger\dagger} : \min \operatorname{dist}(\mathcal{T}, L) < r\} \lessapprox C_H c_{R,1}^{-1} s^{-13} \delta^{-15\varepsilon_0} A_2^{-1} r_2^{-1/2} r^{1/2} (\delta r_1^{-1} r_2^{-1})^{-1/2}. \tag{4.30}$$

For convenience, let $C_1 = C_H c_{R,2}^{-1} s^{-34} \delta^{-39\varepsilon_0}$ and $C_2 = C_H c_{R,1}^{-1} s^{-13} \delta^{-15\varepsilon_0}$.

By dyadic pigeonholing, we can assume that each tube in a subset of $\mathbb{T}_P^{\dagger\dagger}$ has approximately the same multiplicity in $\mathcal{I}^{\dagger\dagger}$, while still contributing the same number of intersections, up to a $\lessapprox 1$-factor. We apply Lemma 3.4 to the graph $(\mathbb{T}_P^{\dagger\dagger} \sqcup \mathbb{T}_L^{\dagger\dagger}, \mathcal{I}^{\dagger\dagger})$ to obtain refinements of $\mathbb{T}_P^{\dagger\dagger}$ and $\mathbb{T}_L^{\dagger\dagger}$, where each tube has at least average multiplicity. Let $\mu_P = (\#\mathcal{I}^{\dagger\dagger})(\#\mathbb{T}_P^{\dagger\dagger})^{-1}$ and $\mu_L = (\#\mathcal{I}^{\dagger\dagger})(\#\mathbb{T}_L^{\dagger\dagger})^{-1}$ denote the respective averages. Since the graph refinement lemma prunes at most half of the edges and the multiplicity for each point can only decrease, the previous application of pigeonholing ensures that each point also has multiplicity $\lessapprox \mu_P$. Note that we have

$$\mu_P \gtrapprox s(\#\mathcal{I})(\#\mathbb{T}_P)^{-1} A_1^{-1} \tag{4.31}$$

and

$$\mu_L \gtrapprox s(\#\mathcal{I})(\#\mathbb{T}_L)^{-1} A_2^{-1}. \tag{4.32}$$

Part III: A two-dimensional incidence theorem for tubes

Fix a tuple $(\mathcal{T}_A, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_{l_0}, \mathcal{T}_B) \in \mathcal{Q}^*$ appearing in this refinement of $\mathcal{S}$; we will now apply the arguments from Proposition 3.8.

Counting triples relative to a fixed line We have

$$\begin{aligned}\#\{(\mathcal{T}_p, \mathcal{T}_l, \mathcal{T}_q) \in & \mathbb{T}_P^{\dagger\dagger} \times \mathbb{T}_L^{\dagger\dagger} \times \mathbb{T}_P^{\dagger\dagger} : (\mathcal{T}_p, \mathcal{T}_{l_0}), (\mathcal{T}_p, \mathcal{T}_l), (\mathcal{T}_q, \mathcal{T}_l) \in \mathcal{I}^{\dagger\dagger}; \\ & \operatorname{dist}(\mathcal{T}_p \cap \mathcal{T}_l, \mathcal{T}_q \cap \mathcal{T}_l) \gtrapprox 1; \operatorname{dist}(\mathcal{T}_p \cap \mathcal{T}_{l_0}, \mathcal{T}_p \cap \mathcal{T}_l) \gtrapprox 1\} \gtrapprox \mu_L^2 \mu_P.\end{aligned} \tag{4.33}$$

For each tube $\mathcal{T}_q$ occurring in this set of triples, apply Lemma 3.12 to the corresponding sets $\mathcal{T}_p \cap \mathcal{T}_l$, where $(\mathcal{T}_p, \mathcal{T}_l, \mathcal{T}_q)$ is a triple in (4.33). This results in a number $r_{3,\mathcal{T}_q}$ for each $\mathcal{T}_q$, which is the diameter of a two-ends subset of the tube. After dyadic pigeonholing over the tubes $\mathcal{T}_q$, we obtain a uniform diameter r_{3,l_0} for each remaining tube $\mathcal{T}_q$.

Applying Hölder's inequality By Lemma 3.5 applied to (4.33), we get

$$\begin{aligned}\#\{(\mathcal{T}_{p_1}, \dots, \mathcal{T}_{p_4}, \mathcal{T}_{l_1}, \dots, \mathcal{T}_{l_4}, \mathcal{T}_q) \in (\mathbb{T}_P^{\dagger\dagger})^4 \times (\mathbb{T}_L^{\dagger\dagger})^4 \times \mathbb{T}_P^{\dagger\dagger} : & \\ (\mathcal{T}_{p_i}, \mathcal{T}_{l_0}), (\mathcal{T}_{p_i}, \mathcal{T}_{l_i}), (\mathcal{T}_q, \mathcal{T}_{l_i}) \in \mathcal{I} & \\ \text{for all } i = 1, \dots, 4\} \gtrapprox \mu_L^8 \mu_P^4 (\#\mathbb{T}_P^{\dagger\dagger})^{-3}. &\end{aligned} \tag{4.34}$$

Since each tube $\mathcal{T}_{l_3}$ has degree $\gtrapprox \mu_L$ in the graph $I^{\dagger\dagger}$, we then get

$$\begin{aligned}\#\{(\mathcal{T}_{p_1}, \dots, \mathcal{T}_{p_4}, \mathcal{T}_{l_1}, \dots, \mathcal{T}_{l_4}, \mathcal{T}_q, \mathcal{T}_{p'}) \in (\mathbb{T}_P^{\dagger\dagger})^4 \times (\mathbb{T}_L^{\dagger\dagger})^4 \times (\mathbb{T}_P^{\dagger\dagger})^2 : & \\ (\mathcal{T}_{p_i}, \mathcal{T}_{l_0}), (\mathcal{T}_{p_i}, \mathcal{T}_{l_i}), (\mathcal{T}_q, \mathcal{T}_{l_i}) \in \mathcal{I} \text{ for all } i = 1, \dots, 4; (\mathcal{T}_{p'}, \mathcal{T}_{l_3}) \in \mathcal{I}\} & \\ \gtrapprox \mu_L^9 \mu_P^4 (\#\mathbb{T}_P^{\dagger\dagger})^{-3}, &\end{aligned} \tag{4.35}$$

where each tube $\mathcal{T}_{p'}$ has $\operatorname{dist}(\mathcal{T}_{p'} \cap \mathcal{T}_{l_3}, \mathcal{T}_{p_3} \cap \mathcal{T}_{l_3}) \gtrapprox r_{3,l_0}$ and $\operatorname{dist}(\mathcal{T}_{p'} \cap \mathcal{T}_{l_3}, \mathcal{T}_q \cap \mathcal{T}_{l_3}) \gtrapprox r_{3,l_0}$.

Summing over the 'lines' Recall that this set of tuples depends on $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_A, \mathcal{T}_B, \mathcal{T}_{l_0}$. Summing over all quadruples $(\mathcal{T}_A, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_B) \in \mathcal{Q}^*$ and $\mathcal{T}_{l_0}$ in the associated set $\mathbb{T}_L^{\dagger\dagger}$, we get

$$\begin{aligned}\#\mathcal{U} &\gtrapprox \mu_L^9 \mu_P^4 (\#\mathbb{T}_P^{\dagger\dagger})^{-3} (\#\mathcal{Q}^*)(\#\mathbb{T}_L^{\dagger\dagger}) \gtrapprox \mu_L^8 \mu_P^4 \mathbb{T}_P^{-3} (\#\mathcal{Q}^*) I^{\dagger\dagger} \\ &\gtrapprox \mu_L^8 \mu_P^4 (\#\mathbb{T}_P^{\dagger\dagger})^{-3} s^7 s^7 \delta^{-5} (\#X_1)^{-1} (\#X_2)^{-1} A_1^{-1} A_2^{-1},\end{aligned} \tag{4.36}$$

where $\mathcal{U}$ is the set of 15-tuples

$$(\mathcal{T}_A, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_{l_0}, \mathcal{T}_B, \mathcal{T}_{p_1}, \mathcal{T}_{p_2}, \mathcal{T}_{p_3}, \mathcal{T}_{p_4}, \mathcal{T}_{l_1}, \mathcal{T}_{l_2}, \mathcal{T}_{l_3}, \mathcal{T}_{l_4}, \mathcal{T}_q, \mathcal{T}_{p'}),$$

with $(\mathcal{T}_A, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_B) \in \mathcal{Q}^*$ and the remaining tubes form a tuple of the form (4.35). After dyadic pigeonholing, we can assume that for each remaining tuple, we have $r_{3,l_0} \sim r_3$ for a single number r_3.

Fixing four special 'points' We now want to pigeonhole to fix a 9-tuple $(\mathcal{T}_A, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_{l_0}, \mathcal{T}_B, \mathcal{T}_{p_1}, \mathcal{T}_{p_2}, \mathcal{T}_{p_4}, \mathcal{T}_{p'})$ occurring in many 15-tuples in $\mathcal{U}$. To this end, we first bound the number of possible 9-tuples. There are $\lesssim \delta^{-2}$ choices for $\mathcal{T}_{l_0}$; once this tube is fixed, there are $\lessapprox (\mu\delta^{-1})^4$ choices for $\mathcal{T}_B, \mathcal{T}_A, \mathcal{T}_1, \mathcal{T}_2$, by properties P1 and P2. Having fixed these five tubes, there are $\lessapprox \mu_L$ choices for $\mathcal{T}_{p_1}$ and then $\lessapprox ((C_2 r_2^{-1/2}/A_2) r_3^{1/2} (\delta r_1^{-1} r_2^{-1})^{-1/2})^2$ choices for $\mathcal{T}_{p_2}, \mathcal{T}_{p_3}$. Finally, there are $\lessapprox \#\mathbb{T}_P^{\dagger\dagger}$ choices for $\mathcal{T}_{p'}$. In total, this gives at most

$$\begin{aligned} &\lessapprox \delta^{-2}(\mu\delta^{-1})^4 \mu_L ((C_2 r_2^{-1/2}/A_2) r_3^{1/2} (\delta r_1^{-1} r_2^{-1})^{-1/2})^2 \#\mathbb{T}_P^{\dagger\dagger} \\ &= C_2^2 r_1 r_3 A_2^{-2} \mu^4 \mu_L \#\mathbb{T}_P^{\dagger\dagger} \delta^{-7} \end{aligned} \tag{4.37}$$

possible 9-tuples $(\mathcal{T}_A, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_{l_0}, \mathcal{T}_B, \mathcal{T}_{p_1}, \mathcal{T}_{p_2}, \mathcal{T}_{p_4}, \mathcal{T}_{p'})$. By pigeonholing, we thus fix a tuple

$$(\mathcal{T}_A, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_{l_0}, \mathcal{T}_B, \mathcal{T}_{p_1}, \mathcal{T}_{p_2}, \mathcal{T}_{p_4}, \mathcal{T}_{p'})$$

with

$$\begin{aligned} &\#\{(\mathcal{T}_{p_3}, \mathcal{T}_{l_1}, \dots, \mathcal{T}_{l_4}, \mathcal{T}_q) \in \mathbb{T}^6 : (\mathcal{T}_A, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_{l_0}, \mathcal{T}_B, \mathcal{T}_{p_1}, \mathcal{T}_{p_2}, \mathcal{T}_{p_4}, \mathcal{T}_{p'}) \in \mathcal{U}\} \\ &\quad \gtrapprox \mu_L^8 \mu_P^4 (\#\mathbb{T}_P^{\dagger\dagger})^{-3} s^7 \mu^7 \delta^{-5} (\#X_1)^{-1} (\#X_2)^{-1} A_1^{-1} A_2^{-1} (C_2^2 r_1 r_3 A_2^{-2} \mu^4 \mu_L \#\mathbb{T}_P^{\dagger\dagger} \delta^{-7})^{-1} \\ &\qquad = C_2^{-2} \mu_L^7 \mu_P^4 (\#\mathbb{T}_P^{\dagger\dagger})^{-4} s^7 \mu^3 \delta^2 (\#X_1)^{-1} (\#X_2)^{-1} A_1^{-1} A_2^{-1} r_1^{-1} r_3^{-1}. \end{aligned} \tag{4.38}$$

It remains to transform this subconfiguration to the plane and then we can conclude the proof.

Mapping to the plane At this point, we have an additional step compared to the argument in Chapter 3. This step involves describing the transformation that will map the subconfiguration (4.38) to the plane.

In order to facilitate the transformation to the plane, we first apply an affine transformation $T(x) = Ax + b$ such that (up to error of approximately $\lessapprox \delta r_1^{-1} r_2^{-1}$)

$$L(\mathcal{T}_1) \mapsto (0,0,0) + \mathbb{R}(1,0,0);$$

$$L(\mathcal{T}_2) \mapsto (0,0,1) + \mathbb{R}(0,1,0);$$
$$L(\mathcal{T}_A) \mapsto (0,0,0) + \mathbb{R}(0,0,1).$$

We can obtain this via a sequence of simple transformations. Firstly, by applying a translation and rotation, we can assume that (up to error $\lessapprox \delta r_1^{-1} r_2^{-1}$) the line coaxial with $\mathcal{T}_1$ is the x-axis, that the centre of $\mathcal{T}_1 \cap \mathcal{T}_A$ is the origin, and that the plane spanned by the lines coaxial with $\mathcal{T}_1$ and $\mathcal{T}_A$ is the xz-plane. Now, by applying a linear transformation that distorts angles by at most a factor of $\lessapprox s^{-1}$, we can also assume that $\mathcal{T}_A$ is the z-axis. A further linear transformation that distorts angles by at most a factor of $\lessapprox s^{-6}\delta^{-4\varepsilon_0}$ preserves these properties and takes $\mathcal{T}_2 \cap \mathcal{T}_A$ to (up to error $\lessapprox s^{-7}\delta^{-4\varepsilon_0}\delta r_1^{-1} r_2^{-1}$) the point $(0,0,1)$. Finally, a transformation that distorts angles by at most a factor of $\lessapprox s^{-4}\delta^{-4\varepsilon_0}$ takes the line coaxial with $\mathcal{T}_2$ to $(0,0,1) + \mathbb{R}(0,1,0)$. Let d be the distortion error, so in particular we have $d \gtrapprox s^{11}\delta^{8\varepsilon_0}$.

This transformation distorts the picture, stretching some angles and lengths so that the image of a tube is no longer exactly the $\delta r_1^{-1} r_2^{-1}$-neighbourhood of a line segment. Nevertheless, the image of a tube will be contained in a genuine $\approx d^{-1}\delta r_1^{-1} r_2^{-1}$-tube, so the estimates from before still hold as estimates about a set of $\approx d^{-1}\delta r_1^{-1} r_2^{-1}$-tubes. Note however that some pairs of tubes may still only be $\approx \delta$-separated.

Finally, the tubes may no longer be contained in the unit ball; by pigeonholing over a boundedly overlapping set of $\lessapprox d^{-3}$ balls we can find a ball of radius ≈ 1 containing at least a $\gtrapprox d^3$-fraction of the intersections between tubes $(\mathcal{T}, \mathcal{T}_3)$, where $\mathcal{T} \in U'$. Without loss of generality, we will assume that this ball is centred at the origin. We now have

$$\begin{aligned} \#\{(\mathcal{T}_{p_3}, \mathcal{T}_{l_1}, \ldots, \mathcal{T}_{l_4}, \mathcal{T}_q) \in \mathbb{T}^6 : (\mathcal{T}_A, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_{l_0}, \mathcal{T}_B, \mathcal{T}_{p_1}, \mathcal{T}_{p_2}, \mathcal{T}_{p_4}, \mathcal{T}_{p'}) \in \mathcal{U}\} \\ \gtrapprox d^3 C_2^{-2} \mu_L^7 \mu_P^4 (\#\mathbb{T}_P^{\dagger\dagger})^{-4} s^7 \mu^3 \delta^2 (\#X_1)^{-1} (\#X_2)^{-1} A_1^{-1} A_2^{-1} r_1^{-1} r_3^{-1}. \end{aligned} \tag{4.39}$$

For each $\mathcal{T} \in \mathbb{T}_P^{\dagger\dagger}$, the sets $\mathcal{T} \cap \mathcal{T}_1$ and $\mathcal{T} \cap \mathcal{T}_2$ are each contained in a ball of radius $\lessapprox s^{-1} d^{-1}\delta$ centred at points $(x_0, 0, 0)$ and $(0, y_0, 1)$ respectively, where

$$|x_0|, |y_0| \approx 1,$$

and the sets $\mathcal{T}' \cap \mathcal{T}_2$ are $\lessapprox d^{-1}$-overlapping. Observe that if we have $|y_0|, |y_1| \lessapprox 1$ and $|y_1 - y_0| \geq r$, then

$$\left|\frac{1}{y_0} - \frac{1}{y_1}\right| = \left|\frac{(y_1 - y_0)}{y_0 y_1}\right| \geq r \left|\frac{1}{y_0 y_1}\right| \gtrapprox r.$$

On the other hand, each $\mathcal{T}_3 \in \mathbb{T}_L^{\dagger\dagger}$ intersects $\mathcal{T}_A$ in a $s^{-1}d^{-1}\delta r_1^{-1}r_2^{-1}$-ball centred at some point $(0,0,z_0)$, where $|z_0| \approx 1$ and $|1-z_0| \approx 1$. Thus the line coaxial with $\mathcal{T}_3$ has the form

$$(-az_0, -bz_0, 0) + \mathbb{R}(a,b,1),$$

for some values of a,b, determined up to error $O(s^{-1}d^{-1}\delta r_1^{-1}r_2^{-1})$. Since $\mathcal{T}_3$ is $\gtrapprox s^4\delta^{4\varepsilon_0}$-separated from $\mathcal{T}_1$ and $\mathcal{T}_2$, we must have $|bz_0| \gtrapprox s^4\delta^{4\varepsilon_0}$ Since $|z_0| \lessapprox 1$, this implies that $|b| \gtrapprox s^4\delta^{4\varepsilon_0}$, and then $|z_0 - 1| \gtrapprox 1$ gives $|b - bz_0| \gtrapprox s^4\delta^{4\varepsilon_0}$. We also have

$$|a - az_0| \gtrapprox s^4\delta^{4\varepsilon_0}.$$

Since $|z_0 - 1| \lessapprox 1$, this gives $|a| \gtrapprox s^4\delta^{4\varepsilon_0}$, and therefore $|az_0| \gtrapprox s^4\delta^{4\varepsilon_0}$.

We will now map each tube $\mathcal{T}$ in $\mathbb{T}_P^{\dagger\dagger}$ to the point $(1/x_0, 1/y_0)$ in the plane, which is determined up to error $\approx s^{-1}d^{-1}\delta_0$ and each tube $\mathcal{T}_3$ in $\mathbb{T}_L^{\dagger\dagger}$ to the line

$$\{(x,y) \in \mathbb{R}^2 : 1 + (b_0z_0 - b_0)y + a_0z_0x = 0\},$$

which is determined up to error $\approx s^{-1}d^{-1}\delta r_1^{-1}r_2^{-1}$.

Let us first check that incidences are preserved under this transformation. If $\mathcal{T}$ and $\mathcal{T}_3$ intersect, then for some value $t \in \mathbb{R}$, we have

$$|(1-t)x_0 - a(t - z_0)| = O(s^{-1}d^{-1}\delta r_1^{-1}r_2^{-1})$$

and

$$|ty_0 - b(t - z_0)| = O(s^{-1}d^{-1}\delta r_1^{-1}r_2^{-1}),$$

where a, b, x_0, y_0, z_0 are the numbers associated to $\mathcal{T}$ and $\mathcal{T}_3$ as above. Rearranging this latter equation gives

$$t = \frac{bz_0 + O(s^{-1}d^{-1}\delta r_1^{-1}r_2^{-1})}{b - y},$$

and by substituting this into the former we get

$$|a(bz_0 - z_0(b-y_0)) - x_0(b - y_0 - bz_0)| = |ay_0z_0 + (bz_0 - b)x_0 + x_0y_0| = O(s^{-1}d^{-1}\delta r_1^{-1}r_2^{-1}).$$

Finally, we divide through by x_0y_0 to get

$$\left|1 + \frac{az_0}{x_0} + \frac{bz_0 - b}{y_0}\right| = O(s^{-1}d^{-1}\delta r_1^{-1}r_2^{-1}|x_0|^{-1}|y_0|^{-1}).$$

Since $|x_0|, |y_0| \gtrapprox 1$, we conclude that the distance between the image of $\mathcal{T}$ and the image of $\mathcal{T}_3$ is at most $\lessapprox s^{-1}d^{-1}\delta r_1^{-1}r_2^{-1}$. Thus, two tubes that intersected at scale $\delta r_1^{-1} r_2^{-1}$ become a point-line pair that is $s^{-1}d^{-1}\delta r_1^{-1} r_2^{-1}$-incident. It remains to check the non-concentration and separation conditions.

Suppose that two points $(1/x_1, s/y_1)$ and $(1/x_2, 1/y_2)$ are at distance $\leq r$, and let $\mathcal{T}, \mathcal{T}'$ be the associated tubes. Then we must have

$$\operatorname{dist}(\mathcal{T} \cap \mathcal{T}_1, \mathcal{T}' \cap \mathcal{T}_1) \lessapprox r$$

and

$$\operatorname{dist}(\mathcal{T} \cap \mathcal{T}_2, \mathcal{T}' \cap \mathcal{T}_2) \lessapprox r.$$

By Lemma 4.8, this implies that

$$\operatorname{dist}(\mathcal{T} \cap \mathcal{T}_3, \mathcal{T}' \cap \mathcal{T}_3) \lessapprox s^{-22}\delta^{-28\varepsilon_0} r,$$

where $\mathcal{T}_3$ is any tube in $\mathbb{T}_L$ intersecting both $\mathcal{T}$ and $\mathcal{T}'$. In particular, this shows that the points p_1, p_2, p_4 are pairwise $\gtrapprox s^{22}\delta^{28\varepsilon_0} r_3$-separated and the points q are $\gtrapprox s^{22}\delta^{28\varepsilon_0}$-separated from each of the points p_1, p_2, p_4, p'. Let $\alpha = s^{22}\delta^{28\varepsilon_0}$.

Fix a point p that is the image of a tube $\mathcal{T}$ and suppose that two lines

$$\{(x, y) \in \mathbb{R}^2 : 1 + (b_1 z_1 - b_1)y + a_1 z_1 x = 0\}$$

and

$$\{(x, y) \in \mathbb{R}^2 : 1 + (b_2 z_2 - b_2)y + a_2 z_2 x = 0\}$$

are both $s^{-1}d^{-1}\delta r_1^{-1} r_2^{-1}$-incident to p and have angular separation $\leq r$. Let $\mathcal{T}_3, \mathcal{T}_3'$ be the tubes associated to these two lines. Observe that

$$|a_2 z_2 - a_1 z_1| \lessapprox r$$

implies that $\mathcal{T}_3'$ intersects the plane $z = 0$ within $N_r(L)$, where

$$L:\ (-a_1 z_1, 0, 0) + \mathbb{R}(0, 1, 0).$$

Moreover,

$$|(b_2 z_2 - b_2) - (b_1 z_1 - b_1)| \lessapprox r,$$

so $\mathcal{T}_3'$ intersects the plane $z = 1$ within $N_r(L')$, where

$$L':\ (0, -b_1(z_1 - 1), 1) + \mathbb{R}(1, 0, 0),$$

Observe that $L, L', \mathcal{T}_A$ define a regulus R which contains $\mathcal{T}_3$. Thus, by Lemma 4.11, $\mathcal{T}_3'$ is contained in the $\lessapprox s^{-17}\delta^{-20\varepsilon_0} r$-neighbourhood of $R_{\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3}$.

But we know that $\angle(v(\mathcal{T}_B), R_{\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3}) \gtrapprox c_{R,1}$ and this implies that any such tube $\mathcal{T}_3'$ intersects B in a segment of length $\lessapprox c_{R,1}^{-1} s^{-17}\delta^{-20\varepsilon_0} r$. Recall that $\mathcal{T}$ and $\mathcal{T}_A$ are uniformly separated with error $\gtrapprox s^{22}\delta^{28\varepsilon_0}$ and $\gtrapprox s^{10}\delta^{12\varepsilon_0}$ skew. Moreover, $\mathcal{T}_B$ is $\gtrapprox s^6\delta^{4\varepsilon_0}$-separated and $\gtrapprox s^4\delta^{4\varepsilon_0}$-skew with each of $\mathcal{T}, \mathcal{T}_A$. Thus, all such tubes $\mathcal{T}_3'$ are contained in the $\lessapprox s^{-O(1)}\delta^{-56\varepsilon_0} c_{R,1}^{-1} r$-neighbourhood of a line. In particular, this implies that each of the lines l_1, l_2, l_3, l_4 makes angle $\gtrapprox s^{O(1)}\delta^{56\varepsilon_0} c_{R,1}$ with the line l_0. Furthermore, this shows that for any point p and any vector v_p, we have

$$\begin{aligned}\#\{l \in L : &\operatorname{dist}(l,p) \le \tilde{\delta}; \angle(l, v_p) \le r\} \\ &\lessapprox C_H c_{R,2}^{-1} s^{-O(1)}\delta^{-39\varepsilon_0} A_1^{-1} r_1^{-1/2} (s^{-O(1)}\delta^{-56\varepsilon_0} r)^{1/2} (\delta r_1^{-1} r_2^{-1})^{-1/2}.\end{aligned} \tag{4.40}$$

We now know that for each fixed line l_3, the set of possible points p_3 appearing in (4.39) with l_3 is contained in a single ball of radius $\lessapprox s^{-O(1)}\delta^{-56\varepsilon_0} c_{R,1}^{-1} d^{-1}\delta r_1^{-1} r_2^{-1}$. Thus, there is a fixed tube $\mathcal{T}_{p_3}$ such that any other valid tube must intersect $\mathcal{T}_1$ and $\mathcal{T}_2$ within distance $\lessapprox s^{-45}\delta^{-56\varepsilon_0} c_{R,1}^{-1} d^{-1}\delta r_1^{-1} r_2^{-1}$ of their intersections with $\mathcal{T}_{p_3}$. By Lemma 4.8, this implies that all such tubes intersect $\mathcal{T}_{l_0}$ within a single ball of radius $\lessapprox s^{-O(1)}\delta^{-84\varepsilon_0} c_{R,1}^{-1} d^{-1}\delta r_1^{-1} r_2^{-1}$. This shows that for each fixed line l_3, there can be at most $\lessapprox C_2 s^{-O(1)}\delta^{-84\varepsilon_0} c_{R,1}^{-1} d^{-1}$ points p_3 occurring together in (4.39) and therefore,

$$\begin{aligned}\#\{(l_1,\dots,l_4,q) \in L^4 \times P :&(\mathcal{T}_A, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_{l_0}, \mathcal{T}_B, \mathcal{T}_{p_1}, \mathcal{T}_{p_2}, \mathcal{T}_{p_4}, \mathcal{T}_{p'}) \in \mathcal{U}\} \\ &\gtrapprox d^4 C_2^{-3} c_{R,1} \mu_L^7 \mu_P^4 (\#\mathbb{T}_P^{\dagger\dagger})^{-4} \delta^{84\varepsilon_0} s^{O(1)} \\ &\cdot \mu^3 \delta^2 (\#X_1)^{-1} (\#X_2)^{-1} A_1^{-1} A_2^{-1} r_1^{-1} r_3^{-1}.\end{aligned}$$

Furthermore, we know that the distance between each of p_1, p_2, p_4, p' and q is $\gtrapprox s^{22}\delta^{28\varepsilon_0}$, so for each q the set of possible lines l_i makes angle $\lessapprox s^{-23}\delta^{-28\varepsilon_0} d^{-1}\delta r_1^{-1} r_2^{-1}$ with a fixed vector. This implies that the set of related possible tubes $\mathcal{T}_{l_i}$ are contained within the $\lessapprox s^{-67}\delta^{-84\varepsilon_0} c_{R,1}^{-1} d^{-1}\delta r_1^{-1} r_2^{-1}$-neighbourhood of a single line. Hence,

$$\begin{aligned}\#\{q \in P :&(\mathcal{T}_A, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_{l_0}, \mathcal{T}_B, \mathcal{T}_{p_1}, \mathcal{T}_{p_2}, \mathcal{T}_{p_4}, \mathcal{T}_{p'}) \in \mathcal{U}\} \\ &\gtrapprox C_1^{-4} C_2^{-3} C_{R,1}^{-3} d^6 \mu_L^7 \mu_P^4 (\#\mathbb{T}_P^{\dagger\dagger})^{-4} \delta^{252\varepsilon_0} s^{O(1)} \\ &\cdot \mu^3 \delta^2 (\#X_1)^{-1} (\#X_2)^{-1} A_1^3 A_2^{-1} r_1 r_3^{-1}.\end{aligned}$$

Note that since each pair l_1, l_2 make angle $\gtrapprox s^{O(1)}\delta^{56\varepsilon_0} c_{R,1} r_3$, the set of points q within distance $s^{-1} d^{-1}\delta r_1^{-1} r_2^{-1}$ of both lines must have corresponding tubes $\mathcal{T}_q$ intersecting $\mathcal{T}_{l_1}$ within a single ball of radius

$$\lessapprox s^{-45}\delta^{-56\varepsilon_0} C_{R,1} d^{-1} r_3 (\delta r_1^{-1} r_2^{-1}),$$

by a similar argument as for the points p_3. Thus, there are at most

$$\begin{aligned}&\lessapprox C_2(s^{-45}\delta^{-56\varepsilon_0}C_{R,1}d^{-1}r_3)^{1/2}A_2^{-1}r_2^{-1}(\delta r_1^{-1}r_2^{-1})^{-1/2}\\ &= C_2 s^{-45/2}\delta^{-28\varepsilon_0}C_{R,1}^{1/2}d^{-1/2}A_2^{-1}r_2^{-1}r_3^{1/2}\end{aligned}$$

such points q so by refining (4.39) by at most this number we can ensure that each pair l_1, l_2 is mutually incident to at most one point q. In this refined set, we have

$$\begin{aligned}\#\{q \in P :&(\mathcal{T}_A, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_{l_0}, \mathcal{T}_B, \mathcal{T}_{p_1}, \mathcal{T}_{p_2}, \mathcal{T}_{p_4}, \mathcal{T}_{p'}) \in \mathcal{U}\}\\ &\gtrapprox C_1^{-4}C_2^{-4}C_{R,1}^{-7/2}d^{13/2}\mu_L^7\mu_P^4(\#\mathbb{T}_P^{\dagger\dagger})^{-4}\delta^{308\varepsilon_0}s^{O(1)}\\ &\cdot \mu^3\delta^2(\#X_1)^{-1}(\#X_2)^{-1}A_1^3 r_1 r_2 r_3^{-1/2}.\end{aligned}$$

Rescaling We are now almost exactly in the same situation as in Proposition 3.8. Apply a α^{-1}-scaling in the direction orthogonal to l_0 so that each point q is ≈ 1-separated from l_0. Now cover $B(0,1)$ by $100r_3 \times 1$-rectangles containing the line segment p_1–p_2. After dyadic pigeonholing, we can assume that each of M rectangles contains $\gtrapprox \#EM^{-1}$ points q and that each rectangle contains $\lessapprox \mu_P A_1^{-1}M^{-1}$ lines incident to each of p_1, p_2, p_4, p'. Fix one such rectangle R. After rescaling by $\lessapprox r_3^{-1}$ in the direction of l_0, we can assume that R has side lengths ≈ 1. Incidences now occur at scale

$$\tilde{\delta} \lessapprox \alpha^{-1}s^{-1}d^{-1}\delta r_1^{-1}r_2^{-1}r_3^{-1} \lessapprox s^{-23}\delta^{-28\varepsilon_0}d^{-1}\delta r_1^{-1}r_2^{-1}r_3^{-1}. \tag{4.41}$$

Finally, by refining $\lessapprox s^{-23}\delta^{-28\varepsilon_0}d^{-1}$ points q, we can ensure that every point is $\tilde{\delta}$-separated.

Applying a projective transformation After applying a projective transformation, we end up with two sets X, Y of $\tilde{\delta}$-separated points, each contained in the interval $[C_0^{-1}, C_0]$, with

$$\#X, \#Y \lessapprox \mu_P A_1^{-1}M^{-1}.$$

For any interval J, the set X satisfies

$$\#(X \cap J) \lessapprox C_H c_{R,2}^{-1}s^{-O(1)}\delta^{-67\varepsilon_0}A_1^{-1}r_1^{-1/2}|J|^{1/2}(\tilde{\delta})^{-1/2} \tag{4.42}$$

and we have

$$\begin{aligned}\mathcal{E}_{\tilde{\delta}}(X \overset{E}{-} Y) &\lessapprox \mu_P A_1^{-1}M^{-1},\\ \mathcal{E}_{\tilde{\delta}}(X \overset{E}{\div} Y) &\lessapprox \mu_P A_1^{-1}M^{-1},\end{aligned}$$

where

$$\#E \gtrapprox C_1^{-4}C_2^{-4}C_{R,1}^{-7/2}d^{15/2}\mu_L^7\mu_P^4(\#\mathbb{T}_P^{\dagger\dagger})^{-4}\delta^{336\varepsilon_0}s^{O(1)}\mu^3\delta^2(\#X_1)^{-1}(\#X_2)^{-1}A_1^3 r_1 r_2 r_3^{-1/2}.$$

Concluding the proof By applying Lemma 3.6 and Proposition 3.7 in the same way as in Proposition 3.8, this gives

$$\begin{aligned}\tilde{\delta}^{-1/10} \lessapprox &(\mu_P A_1^{-1} M^{-1})^{163} (C_1^{-4} C_2^{-4} C_{R,1}^{-7/2} d^{15/2} \mu_L^7 \mu_P^4 (\#\mathbb{T}_P^{\dagger\dagger})^{-4} \\ &\cdot \delta^{336\varepsilon_0} s^{O(1)} \mu^3 \delta^2 (\#X_1)^{-1} (\#X_2)^{-1} A_1^3 r_1 r_2 r_3^{-1/2} M^{-1})^{-83} \\ &\cdot (C_H c_{R,2}^{-1} s^{-O(1)} \delta^{-67\varepsilon_0} A_1^{-1} r_1^{-1/2})^3 \tilde{\delta}^{-3/2}.\end{aligned} \tag{4.43}$$

After plugging in the estimates (4.41), (4.42), (4.31), (4.32), (4.23), (4.24), (4.27)(4.28) and rearranging, we obtain

$$\begin{aligned}\delta^{-1/10} \lessapprox &\left((c_R(s^4\delta^{4\varepsilon}, s^2\delta^{4\varepsilon})\right)^{-12007/10} \left(c_R(s^{22}\delta^{28\varepsilon}, s^{10}\delta^{12\varepsilon})\right)^{-4551/5} \\ &\cdot \mu^{-3999} \delta^{-3999/2} C_H^{9102/5} s^{-O(1)} \delta^{-428797\varepsilon/5}.\end{aligned} \tag{4.44}$$

□

4.5 From Proposition 4.19 to Theorem 4.2

In this section, we complete the proof of Theorem 4.2, and therefore Theorem 4.1. We restate the theorem to be proved for convenience.

Theorem 4.2 . *There is an absolute constant C such that for any $\varepsilon > 0$ there is a constant c_ε depending only on ε such that the following holds. Let $(\mathbb{T}, Y)$ be a set of δ-tubes satisfying the Wolff axioms and suppose that*

$$\sum_{\mathcal{T}\in\mathbb{T}} |Y(\mathcal{T})| \geq \delta^\varepsilon, \tag{4.1}$$

for every $\delta \leq \rho < 1$ we have

$$\left|\bigcup_{\mathcal{T}\in\mathbb{T}} N_\rho(Y(\mathcal{T}))\right| \leq c_\varepsilon \rho^{C\varepsilon} \rho^{1/2-\varepsilon_0}. \tag{4.2}$$

Then $\varepsilon_0 \geq 2.67 \times 10^{-8}$.

To achieve this, we will perform a sequence of refinements to the hypothetical set of tubes until each of the properties (P1)–(P5) is satisfied, at which point we can apply Proposition 4.19. The properties (P1)–(P4) are standard reductions for Besicovitch sets; the main difficulty is establishing property (P5).

We establish property (P5) by considering the 'regulus map' introduced in [22] as well as the sticky reduction from [20]. The idea is that if property

(P5) does not hold, then after changing scales we can assume that the tubes have very strong regulus structure. By comparing this with other properties satisfied by the tubes. we reach a contradiction if the original regulus structure was too strong. This yields a parameter for which property (P5) must hold.

To establish properties (P1) and (P1'), we use induction on scales. For a given value of ε for which we seek to prove Theorem 4.2, this allows us to assume that the conclusion holds at larger scales δ.

Base case For the base case of the induction, it suffices to prove Theorem 4.2 for a fixed value of $0 < \delta < 1$. Suppose that c_ε is a small number for which we will prove the theorem. If (4.1) holds, then there is a tube $\mathcal{T}$ in $\mathbb{T}$ with $|Y(\mathcal{T})| \geq \delta^\varepsilon |\mathcal{T}| \geq c\delta^{2+\varepsilon}$ for some absolute constant c. By choosing δ large enough so that $c\delta^{2+\varepsilon} \geq c_\varepsilon$, we get

$$\left| \bigcup_{\mathcal{T} \in \mathbb{T}} N_\delta(Y(\mathcal{T})) \right| \geq c_\varepsilon \geq c_\varepsilon \delta^{1/2-\varepsilon_0},$$

as required.

Inductive hypothesis For any $\delta' > \delta$, Theorem 4.2 holds.

For the remainder of this subsection, we show how to obtain some of the properties described in Section 4.3 for a given set of tubes; we will apply the forthcoming lemmas in the next subsection.

Lemma 4.21 (Robust transversality I)**.** *Let $(\mathbb{T}, Y)$ be a set of δ-tubes satisfying the Wolff axioms and with*

$$\sum_{\mathcal{T} \in \mathbb{T}} |Y(\mathcal{T})| \gtrapprox 1.$$

Suppose also that

$$\left| \bigcup_{\mathcal{T} \in \mathbb{T}} Y(\mathcal{T}) \right| \lessapprox \delta^{1/2-\varepsilon_0}.$$

Then there is a refinement $(\mathbb{T}', Y')$ of $(\mathbb{T}, Y)$ that is s-robustly transverse with error $1/100$, where $s \gtrapprox 1$.

Proof. This follows from a standard argument, so we only describe the idea; a full proof in the four-dimensional case can be found as Proposition 2.3 in

[17]. The idea is that either the conclusion of the lemma holds, or the set can be essentially partitioned into disjoint s-tubes. By rescaling each s-tube by $\sim s^{-1}$ in the directions orthogonal to its coaxial line, we can then apply the inductive hypothesis and translate this into a lower bound on the volume of the original δ-tubes. The resulting bound is $\gtrapprox \delta^{1/2-\varepsilon_0} s^{-1/2+\varepsilon_0}$, so we must have $s = \delta^{O(\varepsilon)}$ or we get an improvement over the desired conclusion. $\square$

A similar argument gives property (P1').

Lemma 4.22 (Robust transversality II)**.** *Let $(\mathbb{T}, Y)$ be a set of δ-tubes satisfying the Wolff axioms and with*

$$\sum_{\mathcal{T}\in\mathbb{T}} |Y(\mathcal{T})| \gtrapprox 1.$$

Suppose also that

$$\left|\bigcup_{\mathcal{T}\in\mathbb{T}} Y(\mathcal{T})\right| \lessapprox \delta^{1/2-\varepsilon_0},$$

and that $(\mathbb{T}, Y)$ satisfies (P1) with associated number s. Then there is a refinement $(\mathbb{T}', Y')$ of $(\mathbb{T}, Y)$ that is $s^{O(1)}$-robustly transverse with error s^4.

Property (P2) can be obtained by a simple dyadic pigeonholing argument.

Lemma 4.23 (Averaging reduction)**.** *Let $(\mathbb{T}, Y)$ be a set of δ-tubes satisfying the Wolff axioms and with*

$$\sum_{\mathcal{T}\in\mathbb{T}} |Y(\mathcal{T})| \gtrapprox 1.$$

Suppose also that

$$\left|\bigcup_{\mathcal{T}\in\mathbb{T}} Y(\mathcal{T})\right| \lessapprox \delta^{1/2-\varepsilon_0}.$$

There is a refinement Y' of Y and a number

$$\delta^{\varepsilon_0}\delta^{-1/2} \lessapprox \mu \lessapprox \delta^{-1/2},$$

such that

$$\sum_{\mathcal{T}\in\mathbb{T}} |Y'(\mathcal{T})| \gtrapprox 1 \tag{4.45}$$

and for each $x \in \bigcup Y'(\mathcal{T})$ we have

$$\mu \le \sum_{\mathcal{T}\in\mathbb{T}} \chi_{\mathcal{T}}(x) < 2\mu. \tag{4.46}$$

Proof. By dyadic pigeonholing applied to the points in $\bigcup Y(\mathcal{T})$, we can assume the existence of a number μ and a refinement Y' of Y satisfying (4.45) and (4.46). The lower bound on μ follows from (4.2), whereas the upper bound follows from Theorem 4.14 applied to any set of tubes satisfying (4.45). ☐

Notice that if a set of tubes $(\mathbb{T}, Y)$ has property (P1) and Lemma 4.23 is applied, then the resulting set still has property (P1). This is because for each point $x \in \mathbb{R}^3$, the quantities

$$\#\{\mathcal{T} \in \mathbb{T} : x \in Y'(\mathcal{T})\}$$

and

$$\#\{\mathcal{T} \in \mathbb{T} : x \in Y(\mathcal{T}); \angle(v(\mathcal{T}), v_x) \le \rho\}$$

either remain unchanged or are both equal to zero.

Property (P4) requires Bennett, Carbery, and Tao's multilinear Kakeya theorem from [1]; we will use a variant of this theorem due to Bourgain and Guth.

Theorem 4.24 (Bourgain–Guth, [7], Theorem 6)**.** *Let $\mathbb{T}$ be a set of δ-tubes in $\mathbb{R}^3$. Then for each $\varepsilon > 0$, there exists a constant C_ε so that*

$$\int \Big(\sum_{\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \in \mathbb{T}} \chi_{\mathcal{T}_1} \chi_{\mathcal{T}_2} \chi_{\mathcal{T}_3} \ \big| v(\mathcal{T}_1) \wedge v(\mathcal{T}_2) \wedge v(\mathcal{T}_3) \big| \Big)^{1/2} \le C_\varepsilon \delta^{-\varepsilon} (\delta^2 |\mathbb{T}|)^{3/2}. \tag{4.47}$$

Lemma 4.25 (Plany reduction)**.** *Let $(\mathbb{T}, Y)$ be a set of δ-tubes satisfying the Wolff axioms and with*

$$\sum_{\mathcal{T} \in \mathbb{T}} |Y(\mathcal{T})| \gtrapprox 1.$$

Suppose also that

$$\Big| \bigcup_{\mathcal{T} \in \mathbb{T}} Y(\mathcal{T}) \Big| \lessapprox \delta^{1/2 - \varepsilon_0}.$$

There is a refinement Y' of Y such that

$$\sum_{\mathcal{T} \in \mathbb{T}} |Y'(\mathcal{T})| \gtrapprox 1$$

and for every point $p \in \mathbb{R}^3$ there is a plane Π_p so that if $p \in Y'(\mathcal{T})$, then

$$\angle(v(\mathcal{T}), \Pi_p) \lessapprox \lambda^{-3} \delta^{-\varepsilon_0} \delta^{1/2}.$$

Proof. This is a simple consequence of Theorem 4.24. A proof can be found in [22, Lemma 3.5]. ☐

4.5.1 Sticky reduction

In order to obtain a better bound for property (P5), we will use a variant of the sticky reduction from [20]. This reduction follows from Wolff's X-ray estimate from [37] and allows us to assume that the tubes must pack together within thicker tubes. This is the the key point at which we use the upper Minkowski dimension property that the covering number must be small at all scales.

In [37], Wolff proved the following theorem (the statement therein is slightly different but the proof still applies).

Theorem 4.26 (Wolff [37])**.** *Let $\mathbb{T}$ be a set of δ-tubes satisfying the Wolff axioms up to error M. Then*

$$\left\| \sum_{T\in\mathbb{T}} \chi_T \right\|_{d'} \lessapprox \delta^{-1/5} M^{1/5} (\delta^2 \#\mathbb{T})^{7/10}.$$

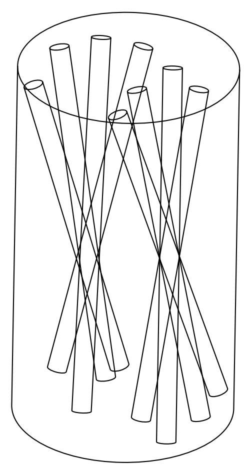


Figure 4.4: Sticky tubes

We apply this estimate along with the reductions from the previous subsection to obtain the following result for a fixed scale $\delta \leq< \rho < 1$.

Lemma 4.27. *For a fixed number $\delta \leq \rho < 1$, there is a number $1 \leq K_{w,\rho} \lessapprox \rho^{-4\varepsilon_0}$, a number*

$$\delta^{\varepsilon_0}\delta^{-1/2} \lessapprox \mu \lessapprox \delta^{-1/2},$$

and a number $s \gtrsim 1$ such that the following holds. There is a refinement $(\mathbb{T}', Y')$ of $(\mathbb{T}, Y)$ such that

$$\sum_{\mathcal{T} \in \mathbb{T}'} |Y'(\mathcal{T})| \gtrapprox 1,$$

and there is a set of $\gtrapprox \rho^{-2}$ ρ-tubes $(\mathbb{T}_\rho, Y_\rho)$ such that:

- *$(\mathbb{T}_\rho, Y_\rho)$ satisfy the Wolff axioms with multiplicity $K_{w,\rho}$;*
- *$(\mathbb{T}_\rho, Y_\rho)$ satisfy property (P1) with associated parameter $s \approx 1$;*
- *$(\mathbb{T}_\rho, Y_\rho)$ satisfy property (P1');*
- *$(\mathbb{T}_\rho, Y_\rho)$ satisfy property (P3);*
- *each ρ-tube $\mathcal{T}_\rho \in \mathbb{T}_\rho$ contains $\approx K_{w,\rho}^{-1} \delta^{-2} \rho^2$ tubes from $\mathbb{T}$, which we denote by $\mathbb{T}[\mathcal{T}_\rho]$, and for any $\mathcal{T} \in \mathbb{T}'$ there is a ρ-tube $\mathcal{T}_\rho \in \mathbb{T}_\rho$ with $\mathcal{T} \subset \mathcal{T}_\rho$;*
- *there is a number $\mu_\rho^* \lessapprox K_{w,\rho}^{-1/4} (\delta/\rho)^{-1/2}$ such that for every $\mathcal{T}_\rho \in \mathbb{T}_\rho$ and $x \in \bigcup_{\mathcal{T} \in \mathbb{T}[\mathcal{T}_\rho]} Y'(\mathcal{T})$, we have*

$$\#\{\mathcal{T}_\rho \in \mathbb{T}_\rho : \exists \mathcal{T} \in \mathbb{T}[\mathcal{T}_\rho] \text{ with } N_\rho(x) \cap Y'(\mathcal{T}) \neq \varnothing\} \approx \mu/\mu_\rho^*.$$

Proof. Take a minimal ρ-covering of $\bigcup Y(\mathcal{T})$ and let $(\mathbb{T}_\rho, Y_\rho)$ be a set of essentially distinct ρ-tubes where each $\mathcal{T}_\rho \in \mathbb{T}_\rho$ fully contains at least one tube from $\mathbb{T}$. By dyadic pigeonholing over the set of tubes in $\mathbb{T}_\rho$, we can assume that the quantities $\#\mathbb{T}[\mathcal{T}_\rho]$ are approximately equal for each $\mathcal{T}_\rho \in \mathbb{T}'_\rho$. Since $\mathbb{T}$ satisfy the Wolff axioms, we must have $\#\mathbb{T}[\mathcal{T}_\rho] \lesssim \rho^2 \delta^{-2}$; let $K_{w,\rho} \geq 1$ be such that $\#\mathbb{T}[\mathcal{T}_\rho] \sim K_{w,\rho}^{-1} \rho^2 \delta^{-2}$. Note that this implies that $\mathbb{T}'_\rho$ satisfy the Wolff axioms up to error $\lessapprox K_{w,\rho}$.

We decompose the set into boundedly overlapping balls of radius ρ, which cover $B(0,2)$. Within each such ball B, we have the further decomposition of the set into the sets $\mathbb{T}[\mathcal{T}_\rho]$ and we apply the averaging reduction within each these sets. By dyadic pigeonholing over all such sets, we find a subset with corresponding multiplicities approximately equal to a single number μ_ρ^*. For each $\mathcal{T} \in \mathbb{T}$, let $Y'(\mathcal{T})$ be the corresponding refined shading. Then we still have

$$\sum_{\mathcal{T} \in \mathbb{T}} |Y'(\mathcal{T})| \gtrapprox 1,$$

and
$$\left|\bigcup_{\mathcal{T}\in\mathbb{T}} Y'(\mathcal{T})\right| \lessapprox \delta^{1/2-\varepsilon_0},$$
so we can now apply Lemmas 4.21, 4.22, 4.23 and 4.25 to $(\mathbb{T}, Y')$. This yields a refinement Y'' of Y' and a number
$$\delta^{\varepsilon_0}\delta^{-1/2} \lessapprox \mu \lessapprox \delta^{-1/2},$$
such that for each $\mathcal{T} \in \mathbb{T}$ and every $x \in Y''(\mathcal{T})$,
$$\#\{\mathcal{T}_\rho \in \mathbb{T}_\rho : \exists \mathcal{T} \in \mathbb{T}[\mathcal{T}_\rho] \text{ with } x \in Y''(\mathcal{T})\} \approx \mu/\mu_\rho^*.$$
This set of tubes still satisfy (P1), (P1') and the plany reduction, since for the latter two lemmas each point $x \in \mathbb{R}^3$ is either selected or deleted from all shadings containing the point.

We repeat this argument for each remaining k in the specified range. For any scale that is not of this form, the property will still hold, up to $\lessapprox 1$ factors.

Let $\delta \le \rho \le \delta^{C\varepsilon}$ be an arbitrary scale. By Theorem 4.26, we have
$$\left\|\sum_{\mathcal{T}\in\mathbb{T}_\rho} \chi_{Y''(\mathcal{T})}\right\|_{d'} \lesssim \rho^{-1/5} K_{w,\rho}^{1/5}(\rho^2 \#\mathbb{T}'_\rho)^{7/10}.$$
On the other hand, by Hölder's inequality we have
$$\left\|\sum_{\mathcal{T}\in\mathbb{T}_\rho} \chi_{Y''(\mathcal{T})}\right\|_{d'} \ge \left\|\sum_{\mathcal{T}\in\mathbb{T}_\rho} \chi_{Y''(\mathcal{T})}\right\|_{1} \left|\bigcup_{\mathcal{T}\in\mathbb{T}_\rho} Y''(\mathcal{T})\right|^{-2/5} \gtrapprox \rho^{-1/5+2\varepsilon_0/5}(\rho^2\#\mathbb{T}'_\rho).$$
It follows that
$$(\rho^2\#\mathbb{T}'_\rho)^{3/10} \lessapprox K_{w,\rho}^{1/5}\rho^{-2\varepsilon_0/5}$$
and thus, $K_{w,\rho} \lessapprox \rho^{-4\varepsilon_0}$.

Fix a tube $\mathcal{T}_\rho \in \mathbb{T}_\rho$ and let $x \in Y''(\mathcal{T})$ be any point contained in the shading of some tube $\mathcal{T} \in \mathbb{T}[\mathcal{T}_\rho]$. Apply a uniform scaling by ρ^{-1} to the set $B(x,\rho)$, so that the ρ-segments of δ-tubes in $\mathbb{T}[\mathcal{T}_\rho]$ now look essentially like δ/ρ-tubes. Then by Theorem 4.14,
$$\left|\bigcup_{\mathcal{T}\in\mathbb{T}} Y''(\mathcal{T})\right| \gtrapprox \rho^2(\delta/\rho)^{1/2}\left(\#\mathbb{T}[\mathcal{T}_\rho]\delta^{-2}\rho^2\right)^{3/4} \gtrapprox K_{w,\rho}^{-3/4}\rho^2(\delta/\rho)^{1/2}.$$

On the other hand,

$$\mu_\rho^* \cdot \left| \bigcup_{\mathcal{T} \in \mathbb{T}} Y''(\mathcal{T}) \right| \lessapprox K_{w,\rho}^{-1},$$

and comparing these estimates gives

$$\mu^* \lessapprox K_{w,\rho}^{-1/4} (\delta/\rho)^{-1/2}.$$

$\square$

By applying this lemma at a range of scales, we can ensure that the conclusion holds concurrently for any $\delta \leq \rho < 1$.

Lemma 4.28. *For any $\delta \leq \rho < 1$, there is a number $1 \leq K_{w,\rho} \lessapprox \rho^{-4\varepsilon_0}$, a number*

$$\delta^{\varepsilon_0} \delta^{-1/2} \lessapprox \mu \lessapprox \delta^{-1/2},$$

and a number $s \gtrsim 1$ such that the following holds. There is a refinement $(\mathbb{T}', Y')$ of $(\mathbb{T}, Y)$ such that

$$\sum_{\mathcal{T} \in \mathbb{T}'} |Y'(\mathcal{T})| \gtrapprox 1,$$

and there is a set of $\gtrapprox \rho^{-2}$ ρ-tubes $(\mathbb{T}_\rho, Y_\rho)$ such that:

- *$(\mathbb{T}_\rho, Y_\rho)$ satisfy the Wolff axioms with multiplicity $K_{w,\rho}$;*
- *$(\mathbb{T}_\rho, Y_\rho)$ satisfy property (P1) with associated parameter $s \approx 1$;*
- *$(\mathbb{T}_\rho, Y_\rho)$ satisfy property (P1');*
- *$(\mathbb{T}_\rho, Y_\rho)$ satisfy property (P3);*
- *each ρ-tube $\mathcal{T}_\rho \in \mathbb{T}_\rho$ contains $\approx K_{w,\rho}^{-1} \delta^{-2} \rho^2$ tubes from $\mathbb{T}$, which we denote by $\mathbb{T}[\mathcal{T}_\rho]$, and for any $\mathcal{T} \in \mathbb{T}'$ there is a ρ-tube $\mathcal{T}_\rho \in \mathbb{T}_\rho$ with $\mathcal{T} \subset \mathcal{T}_\rho$;*
- *there is a number $\mu_\rho^* \lessapprox K_{w,\rho}^{-1/4} (\delta/\rho)^{-1/2}$ such that for every $\mathcal{T}_\rho \in \mathbb{T}_\rho$ and $x \in \bigcup_{\mathcal{T} \in \mathbb{T}[\mathcal{T}_\rho]} Y'(\mathcal{T})$, we have*

$$\#\{\mathcal{T}_\rho \in \mathbb{T}_\rho : \exists \mathcal{T} \in \mathbb{T}[\mathcal{T}_\rho] \text{ with } N_\rho(x) \cap Y'(\mathcal{T}) \neq \varnothing\} \approx \mu / \mu_\rho^*.$$

Proof. We first apply Lemma 4.27 at each scale of the form $\rho' = \delta^{k\varepsilon}$, where $k = 1, \ldots, \lceil 1/\varepsilon \rceil$. Any remaining scale ρ is within a $\lessapprox 1$-factor of one such scale, so the desired conclusion still holds up to a factor of $\lessapprox 1$.Note also that there are $\lessapprox 1$ scales at which we apply a logarithmic refinement, so in total we refine the set by a factor of $\lessapprox 1$. ☐

Corollary 4.29. *For any $\delta \leq \rho < 1$, let $(\mathbb{T}, Y')$ and $(\mathbb{T}_\rho, Y_\rho)$ be the tubes output by Lemma 4.28. Then for any tube $\mathcal{T} \in \mathbb{T}$ and any $\delta \leq \rho < 1$, we have*

$$\#\{\mathcal{T}' \in \mathbb{T} : Y'(\mathcal{T}') \cap N_\rho(Y'(\mathcal{T})) \neq \varnothing\} \lessapprox s^{-1}\rho^{-\varepsilon_0}\rho^{1/2}\delta^{-2}.$$

Proof. Fix ρ. By Lemma 4.28, the tubes in

$$\{\mathcal{T}' \in \mathbb{T} : Y'(\mathcal{T}') \cap N_\rho(\mathcal{T}) \neq \varnothing\}$$

are contained within ρ-tubes from $\mathbb{T}_\rho$, each containing $\approx K_{w,\rho}^{-1}\delta^{-2}\rho^2$ tubes from $\mathbb{T}$. Thus, we have

$$\#\{\mathcal{T}_\rho \in \mathbb{T}_\rho : Y_\rho(\mathcal{T}_\rho) \cap N_\rho(Y'(\mathcal{T}))\} \gtrapprox K_{w,\rho}\delta^2\rho^{-2}\#\{\mathcal{T}' \in \mathbb{T} : Y'(\mathcal{T}') \cap N_\rho(\mathcal{T}) \neq \varnothing\}.$$

On the other hand, Lemma 4.13 and (4.2) imply that

$$\#\{\mathcal{T}_\rho \in \mathbb{T}_\rho : Y_\rho(\mathcal{T}_\rho) \cap N_\rho(Y'(\mathcal{T}))\} \lessapprox K_{w,\rho}s^{-1}\rho^{-2}\rho^{1/2-\varepsilon_0},$$

and combining these two bounds gives the result. ☐

4.5.2 Regulus map reduction

In this subsection we show that we can assume that the tubes $\mathbb{T}$ lack regulus structure in the following sense. Given a tube $\mathcal{T}$, it could be the case that many of the tubes in its hairbrush are almost tangent to a single regulus. By analysing a set of tubes obeying the 'regulus map' introduced in [22], we will reduce to the situation in which this never occurs. Specifically, we will prove the following lemma.

Lemma 4.30. *Let $(\mathbb{T}, Y')$ be the tubes output by Lemma 4.28. For any $\delta \leq c, c' < 1$, there is a number*

$$\begin{aligned} c_R \gtrapprox \min\Bigg(&\left(\delta^{890\varepsilon}c^{-24}c'^{-21}\right)^{1/(1-1672\varepsilon)}, \left(\delta^{578\varepsilon}c^{48}c'^{57}\right)^{1/(1-2072\varepsilon)}, \\ &\left(\delta^{742\varepsilon}c^{12}c'^{9}\right)^{1/(1-4072\varepsilon)}, \left(\delta^{694\varepsilon}c^{36}c'^{45}\right)^{1/(1-2532\varepsilon)}\Bigg), \end{aligned} \tag{4.48}$$

such that for any set of (c, c')-non-degenerate reguli $\{R(\mathcal{T})\}_{\mathcal{T}\in\mathbb{T}}$, we have

$$\#\{\mathcal{T}' \in H(\mathcal{T}) : \angle(v(\mathcal{T}'), R(\mathcal{T})) > c_R\} \geq \frac{1}{2}\#H(\mathcal{T})$$

for each $\mathcal{T} \in \mathbb{T}$.

We will prove this lemma by contradiction. Suppose that $(\mathbb{T}, Y')$ has a set of related (c, c')-non-degenerate reguli $\{R(\mathcal{T})\}_{\mathcal{T}\in\mathbb{T}}$ such that

$$\#\{\mathcal{T}' \in H(\mathcal{T}) : \angle(v(\mathcal{T}'), R(\mathcal{T})) \leq c_R\} \geq \frac{1}{2}\#H(\mathcal{T}). \tag{4.49}$$

For each $\mathcal{T} \in \mathbb{T}$, redefine $H(\mathcal{T})$ to consist only of these tubes. We will show that one reaches a contradiction if c_R is sufficiently small. This subsection will therefore consist of a sequence of statements that must hold if Lemma 4.30 fails.

Lemma 4.31. *There is a number*

$$\delta^{1/2-\varepsilon_0} \leq \theta \leq c_R,$$

a refinement Y'' of Y', and a set of reguli $\{R(\mathcal{T})\}_{\mathcal{T}\in\mathbb{T}}$ such that the

$$\sum_{\mathcal{T}\in\mathbb{T}} |Y''(\mathcal{T})| \gtrapprox 1,$$

and for each $\mathcal{T} \in \mathbb{T}$, if $p \in Y''(\mathcal{T})$ then

$$\angle(\Pi_p, T_pR(\mathcal{T})) \sim \theta.$$

Proof. Since the tubes satisfy property (P3), for each tube $\mathcal{T}$ and each point $p \in Y'(\mathcal{T})$, we can define

$$\theta_{p,\mathcal{T}} = \angle(\Pi_p, T_pR(\mathcal{T})),$$

which is well-defined up to uncertainty $\delta^{1/2-\varepsilon_0}$. By (4.49),

$$\sum_{\mathcal{T}\in\mathbb{T}} |\{p \in Y'(\mathcal{T}) : \delta^{1/2-\varepsilon_0} \leq \angle(\Pi_p, T_pR(\mathcal{T})) \leq c_R\} \gtrapprox \sum_{\mathcal{T}\in\mathbb{T}} |Y'(\mathcal{T})| \gtrapprox 1,$$

and after dyadic pigeonholing we can assume that there is an angle $\delta^{1/2-\varepsilon_0} \leq \theta \leq c_R$ such that

$$\sum_{\mathcal{T}\in\mathbb{T}} |\{p \in Y'(\mathcal{T}) : \theta/2 \leq \angle(\Pi_p, T_pR(\mathcal{T})) \leq \theta\} \gtrapprox 1.$$

For each $\mathcal{T} \in \mathbb{T}$, let $Y''(\mathcal{T})$ be the shading consisting of points $p \in \mathcal{T}$ for which $\angle(\Pi_p, T_p R(\mathcal{T})) \sim \theta$. We then have

$$\sum_{\mathcal{T}\in\mathbb{T}} |Y''(\mathcal{T})| \gtrapprox 1.$$

$\square$

In general, the set of lines intersecting three skew lines form one ruling of a unique regulus, whereas the set of lines intersecting only two skew lines is a two-dimensional family. If a set of tubes obey the regulus map, then the degrees of freedom of the tubes are reduced, so that at scales close to θ, the set of tubes intersecting only two skew tubes behaves like a set of tubes intersecting three skew tubes. We can use the sticky reduction to change scales and obtain relatively stronger regulus structure. We quantify this statement in the following lemma

Lemma 4.32. *There is a number $1 \leq K_{w,\theta} \lessapprox \theta^{-4\varepsilon_0}$, a number*

$$\delta^{\varepsilon_0}\delta^{-1/2} \lessapprox \mu \lessapprox \delta^{-1/2},$$

a number

$$\mu^*_\rho \lessapprox K_{w,\theta}^{-1/4}(\delta/\theta)^{-1/2},$$

and a number $s \gtrapprox 1$ such that the following holds. There is a set of $\gtrapprox \theta^{-2}$ θ-tubes $(\mathbb{T}_\theta, Y_\theta)$ such that:

- *$(\mathbb{T}_\theta, Y_\theta)$ satisfy the Wolff axioms with multiplicity $K_{w,\theta}$;*
- *$(\mathbb{T}_\theta, Y_\theta)$ are s-robustly transverse with error $1/100$;*
- *each $\mathcal{T}_\theta \in \mathbb{T}_\theta$ has a tube $\mathcal{T} \in \mathbb{T}[\mathcal{T}_\theta]$ such that for every $x \in Y_\theta(\mathcal{T}_\theta)$, we have*
$$\#\{\mathcal{T}'_\theta \in \mathbb{T}_\theta : \exists \mathcal{T}' \in \mathbb{T}[\mathcal{T}_\theta] \text{ with } x \in Y''(\mathcal{T}')\} \approx \mu/\mu^*_\theta.$$
- *for any $\mathcal{T}^1_\theta \in \mathbb{T}_\theta$, there is a further θ-tube $V(\mathcal{T}^1_\theta)$ (which need not be in $\mathbb{T}$) such that $\mathcal{T}^1_\theta, V(\mathcal{T}^1_\theta)$ are uniformly separated with error $c^4c'^3$ and $c^2c'^3$-skew. This tube has the property that for any $\theta \leq c_1, c_2 < 1$ and any tube $\mathcal{T}^2_\theta \in \mathbb{T}_\theta$ that is 1-separated with error c_1 and c_2-skew with $\mathcal{T}^1_\theta$, every tube in $H(\mathcal{T}^1_\theta, \mathcal{T}^2_\theta)$ also intersects $V(\mathcal{T}^1_\theta)$. Here, c, c' are the non-degeneracy parameters associated to the regulus map.*

Proof. The first two properties follow directly from Lemma 4.28; the only property that remains to be proven is the last one. Given a pair $\mathcal{T}_\theta^1, \mathcal{T}_\theta^2 \in \mathbb{T}_\theta$ of tubes that are 1-separated with error $\gtrapprox c_1$ and c_2-skew, and a tube $\mathcal{T}_\theta \in H(\mathcal{T}_\theta^1, \mathcal{T}_\theta^2)$, there is a δ-tube $\mathcal{T} \in \mathbb{T}[\mathcal{T}_\theta]$ such that $Y_6'(\mathcal{T})$ intersects the Y_6' shading of the distinguished δ-tube $\mathcal{T}_1 \in \mathbb{T}[\mathcal{T}_\theta^1]$.

The regulus $R(\mathcal{T}_1)$ is defined by three c-separated and c'-skew lines L', L'', L''' that intersect the line coaxial with $\mathcal{T}_1$. Let $p \in L'$ be a point at distance d from $L' \cap \mathcal{T}_1$, for some parameter d to be determined, and let $L = L(\mathcal{T}_1)$ be the line contained in $R(\mathcal{T}_1)$ containing p and in the opposite ruling to L'. By Lemma 4.8, the line coaxial with $\mathcal{T}_1$ and L are then uniformly separated with error $c^4c'^3$ and $c^2c'^3$-skew. Let $V(\mathcal{T}_\theta^1) = N_\theta(L) \cap B(0,2)$. Then $\mathcal{T}$ is guaranteed to intersect $V(\mathcal{T}_\theta^1)$ as long as $s^{-1}c^{-6}c'^{-6}d^2 \leq \theta$. Thus, we can take $d \gtrapprox s^{1/2}c^3c'^3\theta^{1/2}$ and this property will hold for all tubes $\mathcal{T}$. We now know that $\mathcal{T}_\theta$ intersects the three θ-tubes $\mathcal{T}_\theta^1, V(\mathcal{T}_\theta^1), \mathcal{T}_\theta^2$. □

The θ-tubes now have very strong regulus structure, as well as still satisfying properties (P1)–(P4). This is almost enough to obtain a contradiction.

We will obtain a contradiction by estimating the number of occurrences of a specific object within our set of tubes. The object that we are interested in is illustrated in Figure 4.5.

Lemma 4.33. *Let c, c' be the non-degeneracy parameters of the regulus map and let $s, K_{w,\rho}, \mu$ and μ^* be the parameters output by Lemma 4.32. Let $\mathcal{B}$ be a minimal covering of $B(0,1)$ by balls of radius θ. Then there are numbers*

$$c_1 \gtrapprox s^8 K_{w,\rho}^{-8} \theta^{4\varepsilon_0} (\mu/\mu^*)^8 \theta^{-4} (\#\mathbb{T}_\theta)^{-4}$$

and

$$c_2 \gtrapprox s^5 K_{w,\rho}^{-6} \theta^{4\varepsilon_0} (\mu/\mu^*)^8 \theta^{-4} (\#\mathbb{T}_\theta)^{-4}$$

such that

$$\#\{(\mathcal{T}_\theta^0, \mathcal{T}_\theta^1, \mathcal{T}_\theta^2, B) \in \mathbb{T}_\theta \times \mathbb{T}_\theta \times \mathbb{T}_\theta \times \mathcal{B} : \mathcal{T}_\theta^1, \mathcal{T}_\theta^0 c_1\text{-sep}, c_2\text{-skew}; \exists \mathcal{T} \in H(\mathcal{T}_\theta^0) \text{ with } |\mathcal{T} \cap B| \gtrapprox \theta^3$$

$$; \exists \mathcal{T}' \in H(\mathcal{T}_\theta^1, \mathcal{T}_\theta^2) \text{ with } \mathcal{T}' \cap B \gtrapprox \theta^3; \operatorname{dist}(B, \mathcal{T}_\theta^i) \gtrsim s \text{ for } i = 0, 1\} \gtrapprox K_{w,\rho}^{-8} s^{15} (\mu/\mu^*)^{14} \theta^{-1}.$$

Proof. Fix a tube $\mathcal{T}_\theta^0 \in \mathbb{T}_\theta$ with $|Y_\theta(\mathcal{T}_\theta^0)|/|\mathcal{T}_\theta^0| \gtrapprox 1$. By Lemma 4.13, we have

$$\left| \bigcup_{\mathcal{T}_\theta \in H(\mathcal{T}_\theta^0)} Y_\theta(\mathcal{T}_\theta) \setminus N_s(\mathcal{T}_\theta^0) \right| \gtrapprox s K_{w,\rho}^{-1} \mu/\mu^* \theta.$$

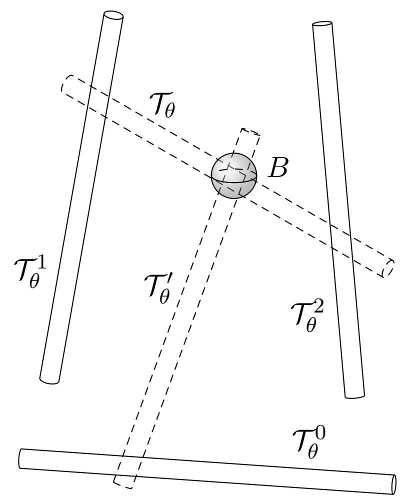


Figure 4.5: A quadruple $(\mathcal{T}_\theta^0, \mathcal{T}_\theta^1, \mathcal{T}_\theta^2, B)$ with the associated tubes $\mathcal{T}_\theta, \mathcal{T}_\theta'$.

Let

$$Y_\theta'(\mathcal{T}_\theta) = \left(Y_\theta(\mathcal{T}_\theta) \cap \bigcup_{\mathcal{T}_\theta \in H(\mathcal{T}_\theta^0)} Y_\theta(\mathcal{T}_\theta) \right) \setminus N_s(\mathcal{T}_\theta^0).$$

Since each point $x \in Y_\theta(\mathcal{T}_\theta)$ has multiplicity $\approx \mu/\mu^*$, we thus have

$$\sum_{\mathcal{T}_\theta \in \mathbb{T}_\theta} |Y_\theta'(\mathcal{T}_\theta)| \gtrapprox s^2 K_{w,\rho}^{-1} (\mu/\mu^*)^2 \theta.$$

After dyadic pigeonholing, we can assume that each tube $\mathcal{T}_\theta$ in a refinement $\mathbb{T}_\theta^0 \subset \mathbb{T}_\theta$ satisfies

$$|Y_\theta'(\mathcal{T}_\theta)|/|\mathcal{T}_\theta| \gtrapprox K_{w,\rho}^{-1} s^2 (\mu/\mu^*)^2 \theta^{-1} (\#\mathbb{T}_\theta^0)^{-1}.$$

We will now apply Lemma 4.16 with λ replaced by

$$\lambda_0 = K_{w,\rho}^{-1} s^2 (\mu/\mu^*)^2 \theta^{-1} (\#\mathbb{T}_\theta^0)^{-1},$$

which gives

$$\begin{aligned} \#\{(\mathcal{T}_\theta, \mathcal{T}_\theta^1, \mathcal{T}_\theta^2) \in (\mathbb{T}_\theta^0)^3 : &\mathcal{T}_\theta \in H'(\mathcal{T}_\theta^1, \mathcal{T}_\theta^2); \angle(\mathcal{T}_\theta, \mathcal{T}_\theta^i) \gtrsim s; \\ &\mathcal{T}_\theta^1, \mathcal{T}_\theta^2 \gtrapprox c_1\text{-sep and } c_2\text{-skew}\} \\ &\gtrapprox \lambda_0^3 K_{w,\rho}^{-1} s^2 (\mu/\mu^*)^2 \theta^{-2} (\#\mathbb{T}_\theta^0), \end{aligned} \tag{4.50}$$

where

$$c_1 \gtrapprox \lambda_0^{16} K_{w,\rho}^{-6} s^4 \theta^{4\varepsilon_0}$$

and

$$c_2 \gtrapprox \lambda_0^{13} K_{w,\rho}^{-5} s^4 \theta^{4\varepsilon_0}.$$

Thus, for any covering $\mathcal{B}$ of $B(0,1)$ by balls of radius δ, we get

$$\begin{aligned}\#\{(\mathcal{T}_\theta, \mathcal{T}_\theta^1, \mathcal{T}_\theta^2, B) \in (\mathbb{T}_\theta^0)^3 \times \mathcal{B} : & \mathcal{T}_\theta \in H'(\mathcal{T}_\theta^1, \mathcal{T}_\theta^2); \angle(\mathcal{T}_\theta, \mathcal{T}_\theta^i) \gtrsim s; \\ & \mathcal{T}_\theta^1, \mathcal{T}_\theta^2 \gtrapprox c_1\text{-sep and } c_2\text{-skew}; \\ & \operatorname{dist}(B, \mathcal{T}_\theta^1) \gtrapprox \lambda_0 s\} \gtrapprox \lambda_0^4 K_{w,\rho}^{-1} s^2 (\mu/\mu^*)^2 \theta^{-3} (\#\mathbb{T}_\theta^0).\end{aligned}$$

By Lemma 4.12, we can refine the balls $\mathcal{B}$ to a subset $\mathcal{B}'$ such that the tubes $\mathcal{T}_\theta$ are $\lessapprox \lambda_0^{-2} s^{-1} K_{w,\rho}$-overlapping. Thus, if we let $\mathcal{Q}$, be the set of resulting quadruples, we have

$$\#\mathcal{Q} \gtrapprox K_{w,\rho}^{-8} s^{15} (\mu/\mu^*)^{14} \theta^{-1}.$$

□

Note that for a typical triple $(\mathcal{T}_\theta^0, \mathcal{T}_\theta^1, \mathcal{T}_\theta^2)$ within this set of quadruples, at least a $\theta^{O(\varepsilon_0)}$-fraction of the regulus $R(\mathcal{T}_\theta^1, \mathcal{T}_\theta^2)$ must be covered by the hairbrush of $\mathcal{T}_\theta^0$, so a typical tube in the hairbrush of $\mathcal{T}_\theta^0$ will intersect $\gtrapprox \theta^{O(\varepsilon_0)} \theta^{-1/2}$ distinct balls from $\mathcal{B}$ that intersect $R(\mathcal{T}_\theta^1, \mathcal{T}_\theta^2)$.

On the other hand, we will show that the due to the curvature of the regulus and the fact that it is algebraic of bounded degree, a typical tube in the hairbrush of $\mathcal{T}_\theta^0$ will intersect the regulus transversely, thereby only intersecting $\lessapprox \theta^{-O(\varepsilon_0)}$ distinct balls from $\mathcal{B}$; this will be a contradiction for small values of θ. In the following lemma we refine the set of quadruples to obtain a subset for which each tuple is typical in the required sense.

Lemma 4.34. *Let $\mathcal{Q}$ denote the set of quadruples output by Lemma 4.33. There are numbers*

$$b_1 \gtrapprox K_{w,\rho}^{-11} s^{15} (\mu/\mu^*)^{14} \theta^7,$$

$$b_2 \gtrapprox (c^4 c'^3) c_1^5 \min(c_2, c^2 c'^3)^4 K_{w,\rho}^{-11} s^{15} (\mu/\mu^*)^{14} \theta^7,$$

such that we still have $\#\mathcal{Q}' \gtrapprox \#\mathcal{Q}$, where $\mathcal{Q}'$ is the set of quadruples $(\mathcal{T}_\theta^0, \mathcal{T}_\theta^1, \mathcal{T}_\theta^2, B)$ satisfying the following additional conditions.

For every triple $(\mathcal{T}_\theta^1, \mathcal{T}_\theta^2, B)$, no tube $\mathcal{T}_\theta^0$ occurring together in a quadruple with $(\mathcal{T}_\theta^1, \mathcal{T}_\theta^2, B)$ is contained in the b_1-neighbourhood of $R = R(\mathcal{T}_\theta^1, \mathcal{T}_\theta^2)$. For every triple $(\mathcal{T}_\theta^0, \mathcal{T}_\theta^1, \mathcal{T}_\theta^2)$, no ball B contains a point $p \in R$ with

$$\angle(T_p R, v(\mathcal{T}_\theta^0)) \lessapprox b_2.$$

Proof. For any fixed triple $(\mathcal{T}_\theta^1, \mathcal{T}_\theta^2, B)$ and for any $\theta \le d < 1$, the number of possible tubes $\mathcal{T}_\theta^0$ contained in the d-neighbourhood of the regulus $R(\mathcal{T}_\theta^1, \mathcal{T}_\theta^2)$ is $\lessapprox dK_{w,\rho}\theta^{-2}$. By letting

$$d \lessapprox K_{w,\rho}^{-11} s^{15} (\mu/\mu^*)^{14} \theta^7,$$

we get that at most half of the quadruples in $\mathcal{Q}$ have this property, so we prune these quadruples and still retain at least half of the set.

Suppose now that there is a point $p_0 \in R$ for which $\angle(v(\mathcal{T}_\theta^0), T_{p_0}R) \le t$, for some parameter $\theta \le t < 1$ to be determined. Since R is a regulus, there are two lines L_1 and L_2 contained in R that contain the point p_0, one from each of the two rulings in R. One of these two lines must intersect $\mathcal{T}_\theta^1, V(\mathcal{T}_\theta^1), \mathcal{T}_\theta^2$ inside the unit ball; we will show that this implies angular separation between L_1 and L_2. Indeed, suppose that $\angle(L_1, L_2) \le \alpha$ for some $\theta \le \alpha < 1$. Then by Lemma 4.6, we have

$$\alpha \gtrapprox (c^4 c'^3)^2 (c_1)^2 \min(c_2^2, c^4 c'^6) \min(c_1, \min(c_2^3, c^6 c'^9)).$$

Since the angles between these two lines is at least α, one of the two lines, say L_1, must make an angle of at least $\gtrsim \alpha$ with $v(\mathcal{T}_\theta^0)$. If we move along this line, the tangent plane to R rotates around the line; we can quantify the rotation based on the pairwise skewness of lines contained in R. We consider first the case when L_1 is in the same ruling of R as $\mathcal{T}_\theta^1$. Then by Lemma 4.6, for any parameter $\theta \le d < 1$, as we move distance d along L_1, the angle between the tangent planes is $\gtrapprox \max(c_1 c_2, c^6 c'^6) d$. Similarly, if L_1 is in the opposite ruling, then as we move distance d, the angle changes by $\gtrapprox (c^4 c'^3) c_1^5 \min(c_2, c^2 c'^3)^4 d$.

In either case, if $d \approx (c^{-4} c'^{-3}) c_1^{-5} \min(c_2, c^2 c'^3)^{-4} t$, then the associated tangent plane must make angle at least $> t$ with $v(\mathcal{T}_\theta^0)$. This implies that the set of points p in R for which $\angle(v(\mathcal{T}_\theta^0), T_p R) \le t$ has covering number at most $\lessapprox (c^{-4} c'^{-3}) c_1^{-5} \min(c_2, c^2 c'^3)^{-4} t \theta^{-2}$. But the covering number must in fact be at least

$$\gtrapprox K_{w,\rho}^{-11} s^{15} (\mu/\mu^*)^{14} \theta^5,$$

so by taking

$$t \approx (c^4 c'^3) c_1^5 \min(c_2, c^2 c'^3)^4 K_{w,\rho}^{-11} s^{15} (\mu/\mu^*)^{14} \theta^7,$$

we get the desired bound on b_2. $\square$

We can now show that a contradiction occurs for sufficiently small values of θ, thereby proving Lemma 4.30.

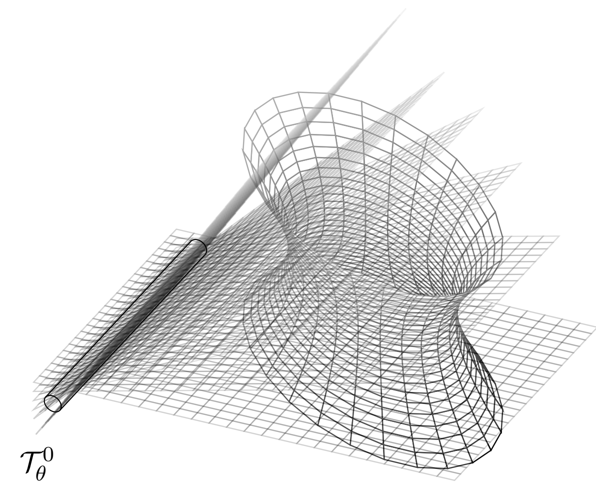


Figure 4.6: Decomposing R according to its intersections with a family of planes containing the line coaxial with $\mathcal{T}_\theta^0$

Proof of Lemma 4.30. By Lemma 4.34 and bt pigeonholing, we fix a triple $(\mathcal{T}_\theta^0, \mathcal{T}_\theta^1, \mathcal{T}_\theta^2)$ for which

$$\#\{B \in \mathcal{B}' : (\mathcal{T}_\theta^0, \mathcal{T}_\theta^1, \mathcal{T}_\theta^2, B) \in \mathcal{Q}'\} \gtrapprox K_{w,\rho}^{-11} s^{15} (\mu/\mu^*)^{14} \theta^5.$$

Let K denote the cardinality of this set. Let Π_i be a set of $\sim \theta$-separated planes containing $L(\mathcal{T}_\theta^0)$. We will now partition the regulus R according to its intersection with the θ-neighbourhoods of these planes, as in Figure 4.6.

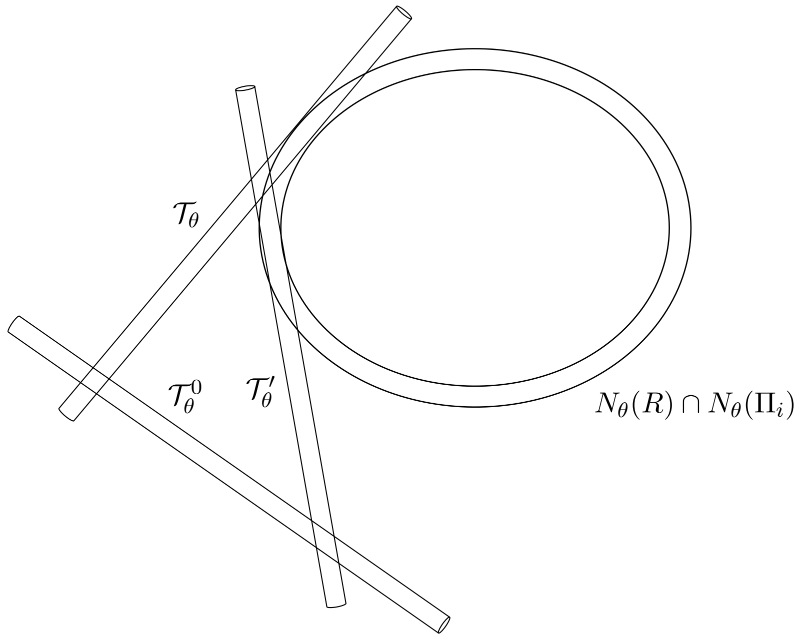


Figure 4.7: Inside each plane

Indeed, we have

$$\#\{B \in \mathcal{B}' : (\mathcal{T}_\theta^0, \mathcal{T}_\theta^1, \mathcal{T}_\theta^2, B) \in Q\}$$
$$\approx \sum_{\Pi_i} \#\{B \in \mathcal{B}' : (\mathcal{T}_\theta^0, \mathcal{T}_\theta^1, \mathcal{T}_\theta^2, B) \in Q; \exists \mathcal{T}_\theta \in H(\mathcal{T}_\theta^0) \text{ with } B \cap \mathcal{T}_\theta \neq \varnothing\}.$$

After a dyadic pigeonholing argument, we can assume that there is a number M such that for each of M planes the θ-covering number of $R \cap \Pi_i$ is $\approx M^{-1}K$.

Each tube in $H(\mathcal{T}_\theta^0)$ must be contained in the $\approx \theta$-neighbourhood of one of the planes Π_i, so partition $H(\mathcal{T}_\theta^0)$ according to the planes. Since we have refined the set to only include points p with $\angle(v(\mathcal{T}_\theta^0), T_pR) \gtrapprox b_2$, each set $N_\theta(R) \cap N_\theta(\Pi_i)$ is contained in the $b_2^{-1}\theta$-neighbourhood of a degree 2 algebraic curve. In particular, for each plane, the intersection of the corresponding set with $B(0,2)$ has θ-covering number $\lessapprox b_2^{-1}\theta^{-1}$ and this implies that there are $M \gtrapprox b_2\theta K$ distinct planes.

We will first show that if L intersects $N_\theta(R)$ in a line segment of length $> d$, then L must make small angle with a line that is fully contained in $N_\theta(R)$. Indeed, suppose that L intersects $N_\theta(R)$ in a line segment of length d and makes angle $> \alpha$ with any line contained in R. Fix a point $p \in R$

within distance θ of $L \cap N_\theta(R)$ and let L_1 and L_2 be the lines contained in R that contain the point p. We can assume without loss of generality that L makes smaller angle with L_1 than L_2. After traveling a distance of $\sim d$ along both L and L_1, we get the points p_L and p_1, respectively. Along with p, these points form a triangle such that the interior is contained in $N_\theta(R)$. Note that p_L and p_1 are $\gtrapprox d\alpha$-separated.

By Lemma 4.6, in the worst case we get

$$(c^4 c'^3) c_a^5 \min(c_2, c^2 c'^3)^4 d^2 \alpha < \theta$$

$$\alpha < (c^{-4} c'^{-3}) c_1^{-5} \min(c_2, c^2 c'^3)^{-4} d^{-2} \theta.$$

Since L also intersects L_1, this implies that $L \cap B(0,2)$ is contained in the $\lessapprox (c^{-4} c'^{-3}) c_1^{-5} \min(c_2, c^2 c'^3)^{-4} d^{-2}\theta$-neighbourhood of L_1, as claimed.

For each plane, we will consider a quantity which is effectively the curvature of the conic curve in the intersection of the plane with R. For each such curve γ_i, let $m(\gamma_i)$ be the length of the largest intersection of $\gamma \cap B(0,1)$ with the θ-neighbourhood of a line. After dyadic pigeonholing over the set of planes, we can assume that for each plane, the corresponding curve γ has $m(\gamma_i) \approx C^{-1}$. We will now show that we can assume C to be small. Fix one such plane Π_i. The θ-neighbourhood of the corresponding curve within $B(0,2)$ can be partitioned into $\approx C^{-1}\theta^{-1/2}$ parts, each of which is contained in the $\approx \theta$-neighbourhood of a line segment of length $\approx C\theta^{1/2}$. Thus, there is a line L intersecting $\mathcal{T}_\theta^0$ that intersects $N_\theta(R) \cap N_\theta(\Pi_i)$ in a line segment of length $\approx C\theta^{1/2}$. The previous two paragraphs show that L is contained in the $\lessapprox (c^{-4} c'^{-3}) c_1^{-5} \min(c_2, c^2 c'^3)^{-4} C^{-2}$-neighbourhood of R.

We can repeat this argument for each of the remaining planes Π_i, of which there are $\gtrapprox b_2 \theta K$, to find such a line L for each plane. For at least $1/2$ of these planes, the corresponding lines come from a single ruling of R. Among these, we can find a triple of planes with pairwise angular separation $\gtrapprox b_2 \theta^2 K$. Let L_1, L_2, L_3 be the corresponding lines.

We have $\angle(L_i, \mathcal{T}_\theta^0) \gtrapprox b_2$, so each pair L_i, L_j satisfies $\angle(L_i, L_j) \gtrapprox b_1 b_2 \theta^2 K$. Note that these lines intersect $\mathcal{T}_\theta^1, V(\mathcal{T}_\theta^1), \mathcal{T}_\theta^2$ or are contained in the same ruling as them. By Lemma 4.8, we therefore conclude that the lines L_i are pairwise $\gtrapprox \max(c_1 c_2, c^6 c'^6) b_2 \theta^2 K$-separated

Each of the three associated lines is contained in the

$$\lessapprox (c^{-4} c'^{-3}) c_1^{-5} \min(c_2, c^2 c'^3)^{-4} C^{-2}$$

-neighbourhood of R; since R is of degree two, this implies that there is a segment of $\mathcal{T}_\theta^0$ of length $\approx d$ that is contained in the d_1-neighbourhood of

R, for some
$$d_1 \lessapprox (c^{-4}c'^{-3})c_1^{-5}\min(c_2, c^2c'^3)^{-4}C^{-2}.$$
Thus, $\mathcal{T}_\theta^0$ is contained in the d_2-neighbourhood of R, for some
$$d_2 \lessapprox d^{-2}(c^{-4}c'^{-3})c_1^{-5}\min(c_2, c^2c'^3)^{-4}C^{-2},$$
which is a contradiction if
$$d^{-2}(c^{-4}c'^{-3})c_1^{-5}\min(c_2, c^2c'^3)^{-4}C^{-2} \lessapprox b_1.$$
We can therefore assume that
$$C^{-2} \approx (c^4c'^3)c_1^5\min(c_2, c^2c'^3)^4\max(c_1c_2, c^6c'^6)^2 b_1 b_2^2 K^2\theta^4.$$

Partition $\mathcal{T}_\theta^0$ into subtubes of length $\theta^{1/2}$. By pigeonholing over these subtubes, we fix one contributing at least $\gtrapprox K\theta^{1/2}$ balls in $\mathcal{B}$. On one hand, property (P4) implies that the set of tubes in $H(\mathcal{T}_\theta^0)$ intersecting this subtube is contained in the $\lessapprox \theta^{1/2-\varepsilon_0}$-neighbourhood of a single plane; this will lead to a contradiction. First, by dyadic pigeonholing, we can assume that each tube in $H(\mathcal{T}_\theta^0)$ intersecting the fixed subtube contributes essentially the same number of balls in $\mathcal{B}$. Furthermore, this number must be at least $\gtrapprox K\theta\mu_\theta^{-1}$ and at most $\lessapprox C\theta^{-1/2}$.

By definition of the quantity $m(\cdot)$ and the fact that any relevant point is at distance $\gtrapprox s$ from $\mathcal{T}_\theta^0$, as we move along the curve, the points at which the corresponding tubes intersect $\mathcal{T}_\theta^0$ jumps by $\gtrapprox s\theta^{1/2}$, as illustrated in Figure 4.7. Thus, there are at most $\lessapprox s^{-1}$ such tubes intersecting the fixed subtube. Hence, we have
$$K\theta^{1/2} \lessapprox \theta^{-1/2-\varepsilon_0}s^{-1}C\theta^{-1/2}.$$
Plugging in the estimates on K and C and rearranging leads to
$$\begin{aligned}
&K_{w,\rho}^{-647/2}s^{647/2}(\mu/\mu^*)^{289}\theta^{144+31\varepsilon}\\
&\qquad\lessapprox c^{-6}c'^{-9/2}\min((K_{w,\rho}^{-32}s^{30}\theta^{4\varepsilon}(\mu/\mu^*)^{26}\theta^{13}), c^2c'^3)^{-6}\\
&\qquad\cdot\max((K_{w,\rho}^{-70}s^{66}\theta^{8\varepsilon_0}(\mu/\mu^*)^{58}\theta^{29}), c^6c'^6)^{-1}.
\end{aligned}$$
Considering the various possible cases in the maximum and minimum and plugging in the estimates on μ/μ^* and $K_{w,\rho}$ results in the desired bound,
$$\begin{aligned}
c_R \geq \theta \gtrapprox \min\Bigg(&\left(\delta^{890\varepsilon}c^{-24}c'^{-21}\right)^{1/(1-1672\varepsilon)}, \left(\delta^{578\varepsilon}c^{48}c'^{57}\right)^{1/(1-2072\varepsilon)},\\
&\left(\delta^{742\varepsilon}c^{12}c'^{9}\right)^{1/(1-4072\varepsilon)}, \left(\delta^{694\varepsilon}c^{36}c'^{45}\right)^{1/(1-2532\varepsilon)}\Bigg).
\end{aligned}$$

□

We can now complete the proof of Theorem 4.2 by applying these reductions and then Proposition 4.19.

Proof of 4.2. By Lemma 4.28, we obtain a set of tubes $(\mathbb{T}, Y')$ satisfying properties (P1) with associated parameter ≈ 1; satisfying property (P1'); satisfying property (P2) with multiplicity $\mu \gtrapprox \delta^{-1/2+\varepsilon_0}$; satisfying property (P3) with associated number $C_H \lessapprox \rho^{-\varepsilon_0}$; and satisfying property (P4). Then by Lemma 4.30, we can assume that $(\mathbb{T}, Y')$ satisfy property (P5) with associated number

$$c_R \geq \theta \gtrapprox \min\left(\left(\delta^{890\varepsilon_0} c^{-24} c'^{-21}\right)^{1/(1-1672\varepsilon_0)}, \left(\delta^{578\varepsilon_0} c^{48} c'^{57}\right)^{1/(1-2072\varepsilon_0)}, \right.$$
$$\left. \left(\delta^{742\varepsilon_0} c^{12} c'^{9}\right)^{1/(1-4072\varepsilon_0)}, \left(\delta^{694\varepsilon_0} c^{36} c'^{45}\right)^{1/(1-2532\varepsilon_0)} \right).$$

We can now apply Proposition 4.19 to this set of tubes. After rearranging, this leads to

$$\delta^{-1/10} \lessapprox \delta^{-\frac{\varepsilon_0(1530904360832\varepsilon_0^2 - 34794228584\varepsilon_0 + 18725549)}{5(1672\varepsilon_0 - 1)(2072\varepsilon_0 - 1)}},$$

and hence,

$$\varepsilon_0 \geq 2.67 \times 10^{-8}.$$

□

Chapter 5

Conclusion

Many open problems remain related to the topics considered in this thesis. In this section, we mention some questions that arise from our results.

5.1 Combinatorics of intervals

In Chapter 2, we considered the geometric property whereby a pair of intervals forms a trapezoid. We showed that if a set of intervals has many such pairs, then the set must have rigid structure, which is of one of three types.

Theorem 1.1 gives a threshold on the number of trapezoids, beyond which there must necessarily be structure. One natural question is whether this threshold can be lowered, while still obtaining a nontrivial (though weaker) conclusion.

Problem 5.1. *Let $0 < \varepsilon_1 \leq 1/2$. Is there a constant $c = c(\varepsilon_1) > 0$ such that the following holds? Any set of N distinct intervals in $\mathbb{R}^2$ with $\lesssim N^{1/2-\varepsilon_1}$ intervals in any of the arrangements 0, 1, 2, forms $\lesssim N^{3/2-c}$ trapezoids.*

The transformation used in the proof of Theorem 1.1 from intervals in the plane to lines in $\mathbb{R}^3$ still applies, however progress on this problem would require progress on the analogous variant of Theorem 2.7. This is an interesting problem itself, which would likely have other implications in combinatorial geometry.

Problem 5.2. *Let $0 < \varepsilon_1 \leq 1/2$. Is there a constant $c = c(\varepsilon_1) > 0$ such that the following holds? Any set of N lines in $\mathbb{R}^3$ with $\lesssim N^{1/2-\varepsilon_1}$ lines containing a common point, contained in a single plane or contained in a single regulus has $\lesssim N^{3/2-c}$ pairwise incidences.*

The polynomial method used in [15] is sensitive to that particular threshold, so it appears new techniques are required. Nevertheless, algebraic methods may still have some part to play.

5.2 Furstenberg sets and Besicovitch sets

Improving the bounds further on the $(\alpha, 2\alpha)$-Furstenberg set problem in $\mathbb{R}^2$ and the Besicovitch set problem in $\mathbb{R}^3$ remains an interesting problem. Some additional optimisations of our techniques may well be possible and some bounds for intermediate results are likely far from sharp.

The reduction from geometric incidences to a sum-product problem that was used in both Theorem 1.4 and Theorem 1.5 was expensive at the quantitative level, in particular due to the application of the Balog–Szemerédi–Gowers lemma. It would be interesting to obtain a purely geometric proof of the discretised point-line incidences theorem, Proposition 3.8, which avoids this route.

Problem 5.3. *Is there a purely geometric proof of Proposition 3.8?*

If possible, this could yield improved bounds for the $(\alpha, 2\alpha)$-Furstenberg set problem and the upper Minkowski dimension of Besicovitch sets in $\mathbb{R}^3$. Such a result may also lead to improved bounds for the discretised sum-product theorem.

In our reduction from Besicovitch sets to Furstenberg sets, one important step was the regulus map reduction, Lemma 4.30, which allows us to assume that the tubes lack regulus structure in a quantified sense. The degree to which the tubes lack regulus structure depends on the value ε_0, and the dependence given by our argument is rather weak. This loss ends up being the main contribution to the size of the final bound. This means that, for example, an improvement by an order of magnitude in the dependence of the regulus map parameter should translate into an order of magnitude improvement to the value of ε_0, up to potentially around 10^{-6}, as explained in Remark 4.20. This therefore constitutes an interesting problem.

Problem 5.4. *Is it possible to obtain improved bounds on Lemma 4.30?*

We suspect that some improvement may be possible by using our techniques in a more efficient manner, but there could also be a different argument that leads to a more substantial improvement.

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