# Gopakumar-Vafa invariants of Banana type manifolds

by

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A THESIS SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF

**Doctor of Philosophy**

in

THE FACULTY OF GRADUATE AND POSTDOCTORAL STUDIES
(Mathematics)

The University of British Columbia
(Vancouver)

July 2021

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**Gopakumar-Vafa invariants of Banana type manifolds**

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# Abstract

This research develops a method to compute Katz's genus 0 Gopakumar-Vafa invariants of fiber curve classes of the Banana manifold, a smooth compact Calabi-Yau threefold, and its generalizations, the multi-Banana configurations. The invariants turn out to have geometric significance, being an actual count of certain local configurations of rational curves on a related threefold. In both these cases, the partition functions for the invariants exhibit a modularity and can be expressed in terms of Jacobi forms or theta functions.

# Lay Summary

Calabi-Yau threefolds first attracted attention in modern theoretical physics where they play a central role in attempts to explain different forces of nature. These spaces are also mathematically interesting geometric objects. One way of classifying different Calabi-Yau threefolds is by enumerating the different ways that various closed surfaces can lie inside the space. This thesis studies a particular Calabi-Yau threefold called a Banana manifold, which contains a peculiar surface consisting of three spheres joined at two points, and resembles of bunch of bananas. These banana structures ultimately determine how many different spheres are contained in the space. The counts of the different kinds of spheres also have a rather mysterious connection to number theory.

# Preface

This dissertation is an original intellectual product of the author, N. Morishige.

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# Acknowledgments

With special thanks to my supervisor, Professor Jim Bryan, for providing unwavering guidance and feedback throughout this project, and whose invaluable insight and knowledge steered me through this research.

# Chapter 1

# Introduction

## 1.1 Background

Enumerative geometry is an old branch of mathematics that studies geometric objects by counting geometric sub-objects. It flourished at the end of the nineteenth century, but progress was slow throughout the next century until thirty years ago, when physicists announced startling new conjectures [9], which were based on reinterpreting these problems using string theory. Mathematicians put these ideas into a rigorous framework, starting with Kontsevich's formula [25] for the number of rational curves of degree $d$ passing through $3d - 1$ points in the plane. This has led to a plethora of new and unexpected discoveries of independent mathematical interest which is ongoing.

The first and most studied curve counting theory in modern enumerative geometry is the Gromov-Witten invariant [17, 36]. These invariants try to count maps of curves that pass through prescribed subvarieties. The invariants are usually packaged into a formal power series, which often has a modular interpretation. In the earliest example, the Gromov-Witten invariants were computed from explicit recursive formulas seeded by a few known easy cases, but one of the main computational tools now is localization. However, in most cases it is very hard to compute these invariants. Another disadvantage of Gromov-Witten invariants is that they are rarely enumerative. They are often rational numbers instead of integers because of finite automorphisms of multiple covers, and can have degenerate contributions in higher genus from collapsing components.

In the case of a smooth three-dimension Calabi-Yau variety $X$, there are other curve counting theories, such as Donaldson-Thomas and Gopakumar-Vafa invariants, which roughly correspond to different choices of compactifications of the moduli space of curves. There are conjectural relationships between all these invariants which have been proved in some cases. The Gopakumar-Vafa invariants $n^g_\beta(X)$ were first proposed by physicists [15] as invariants whose generating function satisfied the following relation to the generating function of the Gromov-Witten invariants $N^g_\beta(X)$, for curves of genus $g$ and curve class $\beta \in H_2(X, \mathbf{Z})$:

$$\sum_{\substack{\beta\neq 0,\\ g}} N^g_\beta(X) q^\beta \lambda^{2g-2} = \sum_{\substack{\beta\neq 0,\\ g,m}} \frac{n^g_\beta(X)}{m} \left(2\sin\left(\frac{m\lambda}{2}\right)\right)^{2g-2} q^{m\beta}. \tag{1.1}$$

They also conjectured that these new invariants $n^g_\beta(X)$ would be integer valued, and for a fixed $\beta$, only finitely of them would be nonzero, and that these invariants would be more enumerative than the Gromov-Witten invariants.

Ten years later, Katz [23] gave the mathematical definition of the genus zero Gopakumar-Vafa invariants as follows. For a smooth Calabi-Yau threefold $X$, consider the moduli space $\mathcal{M}^X_\beta$ of one dimensional stable sheaves $E$ on $X$, with support $[E] = \beta$ for a fixed $\beta \in H_2(X, \mathbf{Z})$, and holomorphic Euler characteristic $\chi(E) = 1$. Then the genus 0 Gopakumar-Vafa invariants, $n^0_\beta(X)$ are defined as a weighted Euler characteristic:

$$n^0_\beta(X) = \chi(\mathcal{M}^X_\beta, \nu) \coloneqq \sum_{n\in\mathbf{Z}} n \cdot \chi_{\text{top}}(\nu^{-1}(n)),$$

where $\nu : \mathcal{M}^X_\beta \to \mathbf{Z}$ is Behrend's constructible function [3] and $\chi_{\text{top}}$ is the topological Euler characteristic.

Finally, after another decade, Maulik and Toda [28] gave the mathematical definition of Gopakumar-Vafa invariants on a smooth Calabi-Yau threefold $X$ for any genus. They consider the same moduli space $\mathcal{M}^X_\beta$ as before, of one dimensional stable sheaves $E$ on $X$, with $[E] = \beta \in H_2(X, \mathbf{Z})$, and $\chi(E) = 1$, together with the Hilbert-Chow map $\pi$, which sends a sheaf to its support one-cycle,

$$\pi : \mathcal{M}_\beta^X \to \mathrm{Chow}_\beta^X \,.$$

They construct a perverse sheaf of vanishing cycles $\phi$ on $\mathcal{M}_\beta^X$, and use the perverse cohomology of its pushforward to define the invariants as follows:

**Definition 1.** [28] The Gopakumar-Vafa invariants $n_\beta^g(X)$ are defined by the identity:

$$\sum_{i \in \mathbf{Z}} \chi(\,{}^p\mathcal{H}^i(\mathbf{R}\pi_*\phi))y^i = \sum_{g \geq 0} n_\beta^g(X)(y^{\frac{1}{2}} + y^{-\frac{1}{2}})^{2g}, \tag{1.2}$$

In view of this precise mathematical definition of the Gopakumar-Vafa invariants, Equation (1.1) is now interpreted as a conjectural relation between the Gromov-Witten and Gopakumar-Vafa curve counting theories, rather than a definition. The mathematical definition also makes it clear that the Gopakumar-Vafa invariants are integer valued.

The key feature of the Banana manifold is a Banana configuration, a local configuration of rational curves. There is a long and rich history of computing enumerative invariants for local configurations of rational curves in Calabi-Yau threefolds. One of the earliest works in this direction, which predates Gromov-Witten theory, is the paper of Aspinwall and Morrison [2], in which they computed the genus $g = 0$ Gopakumar-Vafa invariants of local $\mathbf{P}^1$, known as the conifold. Later, after the mathematical foundations of Gromov-Witten invariants and virtual classes became established, this result was generalized to compute all genus $g \geq 0$ invariants of local $\mathbf{P}^1$ by Faber and Pandharipande [14].

This inspired many other works computing invariants of local configurations of rational curves. Nodal curves and contractible curves were studied by Bryan, Katz and Leung [8]. In their paper, they introduced the idea of computing invariants via the universal cover, which is used in this thesis. Bryan and Karp [7] treated Gromov-Witten invaraints of the closed topological vertex, a configuration of three $\mathbf{P}^1$s meeting in a single triple point, and related configurations. These results were generalized by Karp, Liu, and Mariño [22]. Choi [10] considered local $\mathbf{P}^1$ with more general normal bundles.

Local configurations of rational curves were also studied using a refined curve counting theory, known as motivic Donaldson-Thomas theory. The conifold case was done by Morrison, Mozgovoy, Nagao, and Szendrői [30], while Davison and Meinhardt [12] computed refined invariants of contractible curves.

Via SYZ mirror symmetry, local configurations of rational curves have also attracted attention from the symplectic geometers. Mirror symmetry proposes a duality between the complex structures of one family of Calabi-Yau threefolds and the symplectic structures of a mirror family of Calabi-Yau threefolds. For example, the conifold is mirror to the cotangent bundle of the the 3-sphere $S^3$, the closed topological vertex is mirror to $\mathbf{P}^1\backslash\{3 \text{ points}\}$, and the Banana curve that appears in Chapter 2 of this thesis is mirror to a genus 2 curve. From the symplectic perspective, Abouzaid, Auroux, and Katzarkov [1] studied the Banana curves, whilst Ruddat [32] examined what he termed 'perverse curves', which are the multi-Banana configurations of Chapter 3.

## 1.2 Overview of results

We will give a brief informal discussion highlighting the main ideas of this thesis.

There are few examples where Gopakumar-Vafa invariants have been computed directly from the mathematical definition. In this thesis, we study a specific smooth compact Calabi-Yau threefold, the Banana manifold, and its generalizations. We find a method to directly compute their genus 0 Gopakumar-Vafa invariants for certain curve classes. We also find that the invariants do have an enumerative interpretation, counting curves on the universal cover, and that the generating function can be expressed in terms of Jacobi forms.

We first describe the geometry of the Banana manifold $X_{\mathrm{BAN}}$. We construct $X_{\mathrm{BAN}}$ starting from a generic rational elliptic surface $S$. We can think of $S \subset \mathbf{P}^1 \times \mathbf{P}^2$ as a hypersurface of degree (1,3). Alternatively, $S$ is also the blow up of the 9 base points of a pencil of cubics in $\mathbf{P}^2$. In either case, a general rational elliptic surface is an elliptic fibration, $S \to \mathbf{P}^1$ with twelve singular nodal fibers.

The fiber product of such a generic rational elliptic surface $S$ with itself, $S \times_{\mathbf{P}^1} S$, is a singular threefold with 12 conifold singularities.

The Banana manifold $X_{\text{BAN}}$ is defined to be the conifold resolution of a fiber product of a generic rational elliptic surface $S$ with itself, given by blowing up the diagonal, $\Delta \subset S \times_{\mathbf{P}^1} S$,

$$X_{\text{BAN}} := \text{Bl}_{\Delta}(S \times_{\mathbf{P}^1} S).$$

It has the structure of an Abelian surface fibration $\pi : X_{\text{BAN}} \to \mathbf{P}^1$, whose general fiber is a product of an elliptic curve with itself, $E \times E$.

The Banana manifold also has twelve singular fibers $F_{\text{BAN}}$. These singular fibers are non-normal surfaces. The normalization $\text{NRM}(F_{\text{BAN}})$ of each $F_{\text{BAN}}$ is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$ blown up at two anti-diagonal torus-fixed points, and the normalization map $\text{NRM}(F_{\text{BAN}}) \to F_{\text{BAN}}$ is given by gluing the six toric boundary divisors in opposite pairs. This toric boundary curve is called a Banana curve, $C_{\text{BAN}}$. So we have that each singular surface $F_{\text{BAN}}$ is the union of a 2-torus and the Banana curve,

$$F_{\text{BAN}} \cong (\mathbf{C}^* \times \mathbf{C}^*) \amalg C_{\text{BAN}}.$$

The Banana curve itself is isomorphic to three rational curves $C_{\text{BAN}} \cong C_1 \cup C_2 \cup C_3$, joined at two points, like a bunch of bananas. Each of the $C_i$ has normal bundle

$$N_{C_i/X_{\text{BAN}}} = \mathcal{O}(-1) \oplus \mathcal{O}(-1).$$

The $F_{\text{BAN}}$ surface and $C_{\text{BAN}}$ curves play a central role in our computation. One reason is that our techniques only computes Gopakumar-Vafa invariants for certain curve classes in the fibers. Such classes are spanned by the Banana curve classes, $[C_i]$.

The other reason is that one of the main techniques for the computation takes advantage of the ideas of localization and the motivic nature of Behrend's constructible function. This allows a number of reductions to be made, so that the problem can be simplified into what is eventually a combinatorial one.

The Banana manifold $X_{\mathrm{BAN}}$ possesses a natural group scheme action coming from the non-singular locus of the fibration which acts on the respective fiber. The smooth fibers act on themselves by translation, and the smooth locus of the Banana fibers, $F_{\mathrm{BAN}} \backslash C_{\mathrm{BAN}}$, act on $F_{\mathrm{BAN}}$, by multiplication.

The crucial lemma which allows the first important reduction states that the stable sheaves in the moduli space $\mathcal{M}_{\beta}^{X_{\mathrm{BAN}}}$ are supported on fibers of $\pi : X_{\mathrm{BAN}} \to \mathbf{P}^1$, which gives rise to the fibration of the moduli space $\mathcal{M}_{\beta}^{X_{\mathrm{BAN}}} \to \mathbf{P}^1$. Then the group scheme action described above can be lifted to the moduli space, and we show that it suffices to look at the fixed points of this action, which are the only nonzero contributions to the Behrend function weighted Euler characteristic. Thus we need to consider only invariant sheaves supported on the Banana fiber.

The next step is to recast the problem to one of sheaves on the universal cover of a formal neighborhood of the Banana fiber, $\widehat{F}_{\mathrm{BAN}}$. In order to relate the sheaves on the original threefold with those on the cover, we introduce a second group action, that of tensoring by degree 0 line bundles. This produces a further stratification of the moduli space, and we may restrict attention to the fixed points of this action. In turn, this allows us to find a one-to-one correspondence between the sheaves in the original moduli space with those of a corresponding moduli space of sheaves with support on the universal cover.

Finally, using stability and Euler characteristic arguments, we can show that the invariant sheaves in our moduli space are all structure sheaves of nonreduced genus 0 curves on the universal cover, which are subject to constraints that govern the allowed multiplicities of the irreducible components of the support curve. These constraints can be reinterpreted as a combinatorial problem of counting partitions whose odd parts are distinct.

Finally, when one combines the Gopakumar-Vafa invariants into a generating function, this yields $\phi_{-2,1}$, the Jacobi form of weight -2 and index 1 for reasons that are not entirely understood mathematically.

I then generalized this method to compute Gopakumar-Vafa invariants for formal neighborhoods of more general singular fibers that occur in resolutions of fiber products of special elliptic surfaces. Namely, I study a formal Calabi-Yau threefold $\widehat{F}_{\text{MB}}$, which is a conifold resolution of $\widehat{I}_v \times_{\mathbf{D}} \widehat{I}_w$, where $\widehat{I}_v \to \mathbf{D}$ is a surface over a formal disc with central $I_v$ fiber. Many of the earlier results and theorems from the original work on the Banana manifold can be carried over intact, but the counting problem is more complicated here, with larger numbers of curve classes to track. Although it is possible to obtain formulas for the invariants for general $v$ and $w$, I only explicitly calculate the invariants in the cases $v = w = 2$ and $v = 1$ because they illustrate the ideas well without becoming notationally cumbersome. In these cases, the resulting generating functions can be expressed in terms of theta functions and Jacobi forms.

The main content of this thesis consists of my preprints. Chapter 2 is my first work, where I study the Banana manifold $X_{\text{BAN}}$ and compute the genus 0 Gopakumar-Vafa invariants of its fiber curve classes. Chapter 3 is my follow-up work, where I generalize this technique to $\widehat{F}_{\text{MB}}$.

## 1.3 Further directions

There are a number of natural directions that the work in this thesis can be expanded to in the future. One of these would be to determine the refined motivic invariants of a Banana type configuration or its lift to the universal cover. A first step towards this goal would be to develop suitable a quiver model for this geometry, perhaps through understanding how to glue conifold quiver models to give an appropriate global quiver. Another generalization to take would be to directly compute the higher genus $g > 0$ Gopakumar-Vafa invariants for the Banana manifold. Since the details of the fixed points of the moduli space $\mathcal{M}_\beta^{X_{\text{BAN}}}$ are well understood, the Banana is a valuable prototype for understanding the shape that a localization formula for the higher genus invariants should take.

# Chapter 2

# Genus Zero Gopakumar-Vafa invariants of the Banana manifold

The Banana manifold $X_{\text{BAN}}$ is a compact Calabi-Yau threefold constructed as the conifold resolution of the fiber product of a generic rational elliptic surface with itself, first studied in [6]. We compute Katz's genus 0 Gopakumar-Vafa invariants [23] of fiber curve classes on the Banana manifold $X_{\text{BAN}} \to \mathbf{P}^1$. The weak Jacobi form of weight -2 and index 1 is the associated generating function for these genus 0 Gopakumar-Vafa invariants. The invariants are shown to be an actual count of structure sheaves of certain possibly nonreduced genus 0 curves on the universal cover of the singular fibers of $X_{\text{BAN}} \to \mathbf{P}^1$.

## 2.1 Introduction

### 2.1.1 Background

The genus zero Gopakumar-Vafa invariants are integer valued deformation invariants of Calabi-Yau threefolds that appeared in physics as a virtual count of rational curves on $X$ [15].

Mathematically Katz defined the genus 0 Gopakumar-Vafa invariants as follows [23].

**Definition 2.** Let $X$ be a projective Calabi-Yau threefold over $\mathbf{C}$, together with a fixed curve class $\beta \in H_2(X)$. By a Calabi-Yau threefold $X$, we mean a smooth threefold with trivial canonical bundle $K_X \cong \mathcal{O}_X$. We define $M_\beta^X$ to be the moduli space of Simpson semistable [33] pure 1-dimensional sheaves $\mathcal{F}$ on $X$ with $\mathrm{ch}_2(\mathcal{F}) = \beta^\vee$ and $\chi(\mathcal{F}) = 1$.

**Definition 3.** The genus 0 Gopakumar-Vafa (GV) invariants $n_\beta^0(X)$ of $X$ in curve class $\beta$ are defined as the Behrend function weighted Euler characteristics of this moduli space:

$$n_\beta^0(X) = e(M_\beta^X, \nu) := \sum_{k \in \mathbf{Z}} k \cdot e_{top}(\nu^{-1}(k)) \tag{2.1}$$

where $e_{top}$ is topological Euler characteristic and $\nu : M_\beta^X \to \mathbf{Z}$ is Behrend's constructible function [3].

*Remark* 4. The moduli space $M_\beta^X$ contains no strictly semi-stable sheaves, and so the moduli space is a projective scheme. (See Lemma 11).

*Remark* 5. The stability condition is equivalent to a condition on the Euler characteristic, namely, that a coherent sheaf $E \in M_\beta^X$ is stable if and only if any subsheaf $E' \subset E$ has nonpositive Euler characteristic $\chi(E') \leq 0$. This makes the moduli space manifestly independent of the choice of an ample class. (See Lemma 11).

More recently, an interpretation of all genus GV invariants $n_\beta^g(X), g \geq 0$, in terms of a sheaf of vanishing cycles on $M_\beta^X$ is given in [28]. In the case of genus 0 invariants, this reduces to the previous definition. Toda [34, Thm 6.9] has also shown that the genus 0 GV invariants can be extracted from the usual Donaldson-Thomas (DT) partition function. In particular when $X$ satisfies the conjecture of Maulik, Nekrasov, Okounkov, and Pandharipande [27], then the genus 0 GV invariants and genus 0 Gromov-Witten invariants $N_\beta^0(X)$ satisfy the relation:

$$N_\beta^0(X) = \sum_{k \mid \beta} \frac{n_{\beta/k}^0(X)}{k^3}. \tag{2.2}$$

In practice, these GV invariants can be hard to compute, particularly when $X$ is compact, and have been computed explicitly in very few cases.

In this paper, we directly compute genus 0 GV invariants of certain fiber cohomology classes of curves on a compact Calabi-Yau threefold $X = X_{\mathrm{BAN}}$, see Theorem 7. The full DT partition function for this threefold has been computed recently by Bryan [6]. This provides a prediction for all the GV invariants, and our computation agrees with this. The generating function for the invariants is given by a Jacobi form.

#### 2.1.2 Definition of the Banana manifold $X_{\mathrm{BAN}}$

Let $S$ be a generic rational elliptic surface. We view $S \subset \mathbf{P}^1 \times \mathbf{P}^2$ as a generic hypersurface of degree $(1,3)$. Then $S \to \mathbf{P}^1$ is an elliptic fibration with 12 singular nodal fibers. The fiber product $S \times_{\mathbf{P}^1} S$ is a singular threefold which has 12 conifold singularities. We describe the construction of the Banana manifold $X_{\mathrm{BAN}}$, and refer the reader to [6] for more details.

**Definition 6.** Given $S$ as above, we define the Banana manifold $X_{\mathrm{BAN}}$ to be

$$X_{\mathrm{BAN}} = \mathrm{Bl}_{\Delta}(S \times_{\mathbf{P}^1} S),$$

the conifold resolution of the fiber product $S \times_{\mathbf{P}^1} S$ given by blowing up along the diagonal $\Delta \subset S \times_{\mathbf{P}^1} S$.

The Banana manifold is a smooth compact Calabi-Yau threefold that has the structure of an Abelian surface fibration $\pi : X_{\mathrm{BAN}} \to \mathbf{P}^1$ with exactly 12 singular fibers which are each isomorphic to a surface we call $F_{\mathrm{SING}}$. The surface $F_{\mathrm{SING}}$ is $\mathbf{P}^1 \times \mathbf{P}^1$ blown up at two points and glued along opposite edges,

$$F_{\mathrm{SING}} \cong \mathrm{Bl}_{\Delta}(N \times N) \subset X_{\mathrm{BAN}}, \tag{2.3}$$

where $N \subset S$ is a singular fiber of the rational elliptic surface.

Each singular fiber $F_{\mathrm{SING}}$ contains a curve that we call a Banana configuration, or Banana curve $C_{\mathrm{BAN}}$, Figure 2.1. The Banana curve is a union of 3 rational curves intersecting in two points:

$$C_{\text{BAN}} = C_1 \cup C_2 \cup C_3, \qquad C_i \cong \mathbf{P}^1, \qquad C_i \cap C_j = \{p, q\} \tag{2.4}$$

$$N_{C_i/X_{\text{BAN}}} = \mathcal{O}(-1) \oplus \mathcal{O}(-1).$$

The rational components $C_1$ and $C_2$ are the proper transforms of the nodal curves

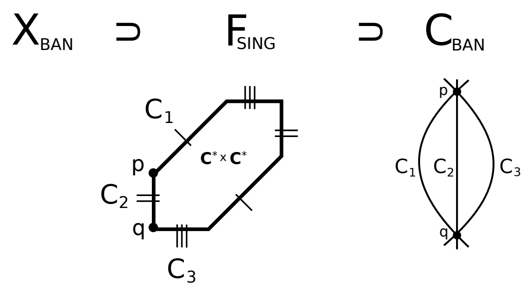


**Figure 2.1:** A singular fiber $F_{\text{SING}}$ containing the eponymous Banana curve $C_{\text{BAN}}$.

on respectively the first and second rational elliptic surfaces $S$ in the fiber product $S \times_{\mathbf{P}^1} S$, while $C_3$ is the exceptional curve from the conifold resolution.

The singular locus of the map $\pi : X_{\text{BAN}} \to \mathbf{P}^1$ is the disjoint union of the twelve copies of $C_{\text{BAN}}$, each of which lies on one of the twelve singular fibers isomorphic to $F_{\text{SING}}$ of $X_{\text{BAN}}$. We denote this collection of singular curves as

$$\amalg\, C_{\text{BAN}} := \amalg_{i=1}^{12} (C_{\text{BAN}})_i,$$

and the twelve singular fibers as

$$\amalg\, F_{\text{SING}} := \amalg_{i=1}^{12} (F_{\text{SING}})_i.$$

The geometry of the fibration $\pi : X_{\text{BAN}} \to \mathbf{P}^1$ gives a group scheme structure to its smooth locus, which we call $X^0_{\text{BAN}}$:

$$X^0_{\text{BAN}} = X_{\text{BAN}} \setminus \amalg\, C_{\text{BAN}} \to \mathbf{P}^1. \tag{2.5}$$

This action extends to an action of $X^0_{\text{BAN}} \to \mathbf{P}^1$ on all of $X_{\text{BAN}} \to \mathbf{P}^1$, see [6, §4.5]

Moreover, let $\Theta \subset H_2(X_{\text{BAN}}, \mathbf{Z})$ be the sublattice of fiber classes, namely classes represented by cocycles supported on a fiber. Then $\Theta$ is spanned by $[C_1]$, $[C_2]$, and $[C_3]$:

$$\Theta = \mathbf{Z}\,[C_1] \oplus \mathbf{Z}\,[C_2] \oplus \mathbf{Z}\,[C_3].$$

### 2.1.3 Main Result

Our main result is the following:

**Theorem 7.** *Let $X_{\text{BAN}}$ be as above. Fix a curve class $\beta_{\mathbf{d}}$,*

$$\beta_{\mathbf{d}} = d_1[C_1] + d_2[C_2] + [C_3] \in H_2(X_{\text{BAN}}), \qquad \mathbf{d} = (d_1, d_2) \in \mathbf{Z}^2_{\geq 0}.$$

*The genus 0 Gopakumar-Vafa invariants $n^0_{\beta_{\mathbf{d}}}(X_{\text{BAN}})$ are determined by the following equation:*

$$\sum_{d_1,d_2} n^0_{\beta_{\mathbf{d}}}(X_{\text{BAN}}) x^{d_1} y^{d_2} = 12 \prod_{m=1}^{\infty} \frac{(1 - x^m y^{m-1})^2 (1 - x^{m-1} y^m)^2}{(1 - x^m y^m)^4}. \qquad (2.6)$$

**Corollary 8.** *After the change of variables,*

$$q = xy, \qquad p = y,$$

*the genus 0 GV invariants satisfy the identity:*

$$\sum_{d_1,d_2} n^0_{\beta_{\mathbf{d}}}(X_{\text{BAN}}) q^{d_1} p^{d_2 - d_1 - 1} = 12\phi_{-2,1}(q, p),$$

*where $\phi_{-2,1}(q, p)$ is the unique weak Jacobi form of weight -2 and index 1:*

$$\phi_{-2,1}(q, p) = p^{-1}(1 - p)^2 \prod_{m=1}^{\infty} \frac{(1 - q^m p^{-1})^2 (1 - q^m p)^2}{(1 - q^m)^4}.$$

$$q = \exp(2\pi i \tau), \qquad p = \exp(2\pi i z), \qquad (\tau, z) \in \mathrm{H} \times \mathbf{C}.$$

In particular, this Jacobi form is one of the two generators of the ring of weak Jacobi forms. Furthermore, the index 1 weak Jacobi forms have a Fourier expansion $\sum c(4n - r^2)q^n p^r$ whose coefficients $c(4n - r^2)$ depend only on a quadratic expression in the degrees [13]. We get the immediate consequence:

**Corollary 9.** *The genus 0 GV invariants depend only on a quadratic function of the curve class. Namely, they satisfy* $n^0_{\beta_{\mathbf{d}}}(X_{\mathrm{BAN}}) = n^0_{||\beta_{\mathbf{d}}||}(X_{\mathrm{BAN}})$*, where* $||\beta_{\mathbf{d}}|| := 2d_1 + 2d_2 + 2d_1 d_2 - d_1^2 - d_2^2 - 1$.

The appearance of the weak Jacobi form $\phi_{-2,1}(q, p)$ in our expression of the GV invariants is somewhat surprising and not well understood. This Jacobi form has appeared, for instance, in the DT partition function for certain elliptically fibered Calabi-Yau threefolds, as well as in other examples.

### 2.1.4 Outline of method

Our method of proof ultimately reduces the computation of the Behrend function weighted Euler characteristic of the moduli space $M = M^{X_{\mathrm{BAN}}}_{\beta_{\mathbf{d}}}$ to an actual count of structure sheaves of genus 0 curves. These curves are possibly nonreduced curves in the universal cover $U(F_{\mathrm{SING}})$ of $F_{\mathrm{SING}}$. This is a sheaf theoretic analogue of the Gromov-Witten technique of passing to counts of genus 0 curves on the universal cover [8].

The main idea behind the reductions is to use the motivic nature of weighted Euler characteristics. This allows us to compute using stratification and fixed point sets, even though we do not have a global $\mathbf{C}^*$ action on our moduli space. An outline of the proof is as follows.

We begin in section 2.2 by proving that a stable sheaf is scheme-theoretically supported on a single fiber. This gives us a map $M \to \mathbf{P}^1$. It then suffices to compute the Behrend function weighted Euler characteristic $e(M, \nu)$ fiberwise. The group scheme action of $X^0_{\mathrm{BAN}} \to \mathbf{P}^1$ on $X_{\mathrm{BAN}} \to \mathbf{P}^1$ induces a fiberwise action on the moduli space, where the group of each fiber of $X^0_{\mathrm{BAN}} \to \mathbf{P}^1$ acts on the corresponding fiber of $M \to \mathbf{P}^1$. This fiberwise group action preserves $\nu$. Thus, $e(M, \nu)$ can be computed on orbits of this action.

The generic smooth fibers of $X_{\text{BAN}}$ are non-singular Abelian surfaces where the group action is transitive and support no invariant curves. Consequently, the sheaves supported on the smooth fibers contribute zero to $e(M, \nu)$. On the singular fibers $F_{\text{SING}}$, the group action gives a natural $\mathbf{C}^* \times \mathbf{C}^*$ torus action on the moduli space. The fixed points of this action are the only stable sheaves that contribute to $e(M, \nu)$. These sheaves are scheme-theoretically supported on the singular fibers $F_{\text{SING}}$ with set-theoretic support on the Banana curve configuration $C_{\text{BAN}}$. Thus we reduce the problem of computing $n^0_{\beta_{\mathbf{d}}}(X_{\text{BAN}})$ to that of counting torus-invariant stable sheaves on $F_{\text{SING}}$ of curve class $\beta_{\mathbf{d}}$ and Euler characteristic 1.

This count corresponds to the naive Euler characteristic of our moduli space, $\widetilde{n}^0_{\beta_{\mathbf{d}}}(X_{\text{BAN}})$. This is defined as the Euler characteristic without the Behrend function weighting:

$$\widetilde{n}^0_{\beta_{\mathbf{d}}}(X_{\text{BAN}}) := e(M^{X_{\text{BAN}}}_{\beta_{\mathbf{d}}}).$$

We begin by determining these.

We show in section 2.4 that, in fact, it suffices to count those invariant stable sheaves on $F_{\text{SING}}$ that push forward from the universal cover $U(F_{\text{SING}})$. This involves considering the action on the moduli space given by tensoring by line bundles on $F_{\text{SING}}$. Any sheaf fixed under this action must pull back to an equivariant sheaf on $U(F_{\text{SING}})$ which contains a distinguished subsheaf isomorphic under pushforward to the original. Now we need to determine how many of these distinguished stable torus-invariant sheaves there are on $U(F_{\text{SING}})$.

These distinguished sheaves on $U(F_{\text{SING}})$ can be counted using a combinatorial argument detailed in sections 2.5 and 2.6. The assumption of Euler characteristic 1 is very restrictive, and together with some elementary stability arguments, we show that the only torus invariant stable sheaves that push forward to invariant sheaves in our moduli space are structure sheaves of arithmetic genus 0 curves that satisfy certain constraints on adjoining components. Such curves can be classified by combinatorics in terms of the number of integer partitions whose odd parts are distinct, and we obtain a closed form generating function for $\widetilde{n}^0_{\beta_{\mathbf{d}}}(X_{\text{BAN}})$.

Finally, in section 2.7, we prove that $n^0_{\beta_{\mathbf{d}}}(X_{\mathrm{BAN}})$ is related to $\widetilde{n}^0_{\beta_{\mathbf{d}}}(X_{\mathrm{BAN}})$ by a sign change. We use the result of [4, Corollary 3.5] that given a $\mathbf{C}^*$ action with isolated fixed points $[\mathcal{F}] \in M^{\mathbf{C}^*}$, the weighted Euler characteristic depends only on the parities of the dimension of the tangent spaces at those points:

$$e(M,\nu) = \sum_{[\mathcal{F}]\in M^{\mathbf{C}^*}} (-1)^{\dim T_{[\mathcal{F}]} M}.$$

In [4], this comes from the computation of the weighted Euler characteristic of the Milnor fiber in the presence of a $\mathbf{C}^*$ action. However, the dimension of the tangent space at isolated fixed points may also be computed using virtual localization, as in [27]. We use the formula given in [27] to calculate the dimension of the groups $\mathrm{Ext}^1(\mathcal{F},\mathcal{F})$ for our fixed points $[\mathcal{F}]$, and thus the parity of $\dim(T_{[\mathcal{F}]}M)$. This finishes the proof of our main result.

Our method is limited to curve classes $d_1[C_1] + d_2[C_2] + [C_3]$ since a simple count of combinations of structure sheaves does not appear to suffice in the general case. For $d_1[C_1] + d_2[C_2] + d_3[C_3]$ with $d_1, d_2, d_3 > 1$, there are corresponding sheaves on the universal cover which are stable with Euler characteristic 1, but are not structure sheaves. At present, we do not know how to analyze the moduli space in these cases.

## 2.2 Setup to counting on $F_{\mathrm{SING}}$

Throughout the rest of this section, we let $X = X_{\mathrm{BAN}} \xrightarrow{\pi} \mathbf{P}^1$ and $M = M^X_\beta$, as given by Definitions 2 and 6 in the introduction.

We begin with two observations that hold for Simpson semistable pure 1-dimensional sheaves $\mathcal{F}$ with $\chi(\mathcal{F}) = 1$. First, all semistable sheaves are in fact stable. Second, the stability condition can be restated in terms of Euler characteristic of subsheaves or quotient sheaves.

Recall the definition of semistable and stable sheaves [19].

**Definition 10.** Let $Y$ be a complex projective scheme and $\mathcal{F}$ a pure coherent sheaf of dimension $d$ on $Y$. Fix an ample line bundle $H = \mathcal{O}(1)$ on $Y$. Define the Hilbert polynomial $P(\mathcal{F}, m)$ and reduced Hilbert polynomial $p(\mathcal{F}, m)$ of $\mathcal{F}$ as follows:

$$P(\mathcal{F}, m) := \chi(\mathcal{F} \otimes \mathcal{O}(m)) = \sum_{i=0}^{d} \frac{\alpha_i(\mathcal{F})}{i!} m^i,$$
$$p(\mathcal{F}, m) := \frac{P(\mathcal{F}, m)}{\alpha_d(\mathcal{F})}.$$

We say $\mathcal{F}$ is semistable if for any proper subsheaf $\mathcal{F}' \subset \mathcal{F}$, $p(\mathcal{F}') \leq p(\mathcal{F})$. A semistable sheaf is called stable if the inequality is strict.

**Lemma 11.** *There are no strictly semistable sheaves in $M$. Moreover, the stability condition for $\mathcal{E} \in M$ is equivalent to the following: $\mathcal{E}$ is stable if and only if $\chi(\mathcal{E}') \leq 0$ for any proper subsheaf $\mathcal{E}' \hookrightarrow \mathcal{E}$. Equivalently, $\mathcal{E}$ is stable if and only if $\chi(\mathcal{E}'') > 0$ for any quotient sheaf $\mathcal{E} \twoheadrightarrow \mathcal{E}'' \neq 0$.*

*Proof.* This follows from the definition of stability. □

**Corollary 12.** *$M$ is independent of the choice of polarization $H$.*

We begin by showing that the moduli space $M$ has the structure of a scheme over $\mathbf{P}^1$.

**Proposition 13.** *Suppose $\beta$ is a curve class such that $\pi_*\beta = 0$. Let $\mathcal{E} \in M$. Then $\mathcal{E}$ is scheme theoretically supported on a single fiber.*

*Proof.* Let $C = (\operatorname{Supp} \mathcal{E})_{\text{RED}}$ be the reduced support of $\mathcal{E}$. Since $\beta$ is a fiber class, $\operatorname{Supp} \pi_*\mathcal{E} = \{p_i\}$ is a finite set of points, so $C$ is a collection of fibers. But direct sums are necessarily unstable so the support of $\mathcal{E}$ must be connected. Hence, the set theoretic support of $\mathcal{E}$ is contained in a single fiber $F = F_x$, for some $x \in \mathbf{P}^1$.

Now $i : F \hookrightarrow X$ is a closed subscheme so we have the exact sequence:

$$0 \to \mathcal{I}_{F/X} \to \mathcal{O}_X \to i_*\mathcal{O}_F \to 0.$$

Since $F$ is an effective Cartier divisor on the nonsingular $X$, $\mathcal{O}_X(F)$ is locally free and we can tensor by $\mathcal{O}_X(F)$ to get the short exact sequence:

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(F) \to i_*\mathcal{O}_F(F) \to 0.$$

The normal bundle of the fiber class $F$ is trivial, so $\mathcal{O}_F(F) \cong \mathcal{O}_F$ and we get:

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(F) \to i_*\mathcal{O}_F \to 0,$$

which we can tensor with $\mathcal{E}$,

$$\mathcal{E} \to \mathcal{E}(F) \to \mathcal{E}_F \to 0.$$

Again, because $\mathcal{E}$ is supported on the fiber class $F$, and $\mathcal{O}_X(F)$ is a trivial line bundle when restricted to $F$, $\mathcal{E}(F) \cong \mathcal{E}$:

$$\mathcal{E} \to \mathcal{E} \to \mathcal{E}_F \to 0.$$

By stability, $\mathcal{E} \to \mathcal{E}$ is either the zero map or an isomorphism, which implies $\mathcal{E}_F = 0$ or $\mathcal{E} \cong \mathcal{E}_F$. By assumption, $\mathcal{E}$ and hence $\mathcal{E}_F$ is nonzero, so $\mathcal{E} \cong \mathcal{E}_F$ and $\mathcal{E}$ is scheme-theoretically supported on $F$. □

This gives us a natural map $\rho : M \to \mathbf{P}^1$ which allows us to compute the Behrend function weighted Euler characteristic fiberwise. Recall that for any constructible morphism, the weighted Euler characteristic can be computed as a push-forward [26]. So the map $\rho : M \to \mathbf{P}^1$ allows us to compute the Behrend function weighted Euler characteristics fiberwise. Thus, we get:

$$e(M, \nu) = e(\mathbf{P}^1, \rho_*\nu),$$

where $(\rho_*\nu)(t) = e(M_t, \nu_t)$ for $t \in \mathbf{P}^1$, $M_t := \rho^{-1}(t)$, and $\nu_t := \nu|_{M_t}$.

Recall (Eq. 2.5) that $X^0_{\text{BAN}} =: X^0$, the smooth locus of the fiber map of the Banana manifold is a group scheme $X^0 \to \mathbf{P}^1$ that acts on the Banana manifold $\pi : X \to \mathbf{P}^1$.

Let $X_t$ be a fiber of $X$, and $G_t$ the group of the fiber:

$$X_t \coloneqq \pi^{-1}(t) \subset X,$$
$$G_t \coloneqq X^0 \cap X_t, \qquad t \in \mathbf{P}^1.$$

The nonsingular fibers are Abelian surfaces which are products of an elliptic curve $E$ with itself. In this case, the group of the fiber is the fiber itself,

$$G_t = X_t, \quad \text{when } X_t = E \times E,$$

and $G_t$ acts by translation in the group law.

On the singular fibers $F_{\text{SING}}$ (Eq. 2.3), the group of the fiber is a torus acting by translation,

$$G_t = F_{\text{SING}} \backslash C_{\text{BAN}} \cong \mathbf{C}^* \times \mathbf{C}^*, \quad \text{when } X_t = F_{\text{SING}}.$$

**Proposition 14.** *To compute $n^0_\beta(X)$, it suffices to count those sheaves of $M$ with scheme theoretic support contained in $\amalg F_{\text{SING}}$ and set-theoretic support contained in $\amalg C_{\text{BAN}}$, which are also invariant under the action of the group scheme $X^0 \to \mathbf{P}^1$.*

*Proof.* On each fiber, $X_t = \pi^{-1}(t) \subset X, t \in \mathbf{P}^1$, the group of the fiber $G_t$ also acts on sheaves supported on $X_t$, which in turn, induces an action of $G_t$ on $M_t = \rho^{-1}(t)$. We will show in Section 2.7 that the group scheme action is trivial on $K_X$ and preserves the Behrend function $\nu$. This algebraic group action of $G_t$ on $M_t$, gives us a stratification of $M_t$ into locally closed equivariant subsets. By [3, 4], $e(M_t, \nu_t)$ can be computed on orbits of this action.

In particular, if the topological Euler characteristic of the $G_t$-orbit vanishes, then $e(M_t) = e(M_t^{G_t})$ [6, Lemma 4.1]. The action on the smooth fibers are fixed point free, hence the orbits, which are quotients of the group of the fiber, are positive dimensional Abelian varieties and have zero Euler characteristic. On the singular fibers, the group of the fiber is $G_t \cong \mathbf{C}^* \times \mathbf{C}^*$ and we apply [5, Corollary 2]. The fixed points of the group action on the moduli space corresponds to an isomorphism class of sheaves, $[E]$ such that $[E] \cong [g^*E]$, where $g : X_t \to X_t$ is the action on the underlying space given by the group element $g \in G_t$. In particular, the support of the sheaf has to be preserved by the group action. Over general points $t \in \mathbf{P}^1$, the fiber $X_t$ is smooth and the group action is that of the Abelian surface acting transitively on itself through translation in the group law. So these fibers contain no invariant curves. Consequently, the sheaves supported on smooth fibers do not contribute to $e(M, \nu)$.

On the singular fibers isomorphic to $F_{\text{SING}}$, subcurves of the Banana curve $C_{\text{BAN}}$ are the only curves preserved by the action of $\mathbf{C}^* \times \mathbf{C}^*$. Thus we have reduced our problem of computing $n^0_\beta(X)$ to counting only those sheaves $\mathcal{F} \in M$ which are $\mathbf{C}^* \times \mathbf{C}^*$-invariant and with $(\operatorname{Supp} \mathcal{F})_{\text{RED}} \subset \amalg\, C_{\text{BAN}}$. By Proposition 13, these sheaves are scheme-theoretically supported on $\amalg\, F_{\text{SING}}$. $\square$

Since each of the twelve singular fibers are isomorphic to, and disjoint from, each other, it suffices to count the torus-invariant sheaves in $M$ supported on only one of these fibers. Multiplying this count by twelve then gives the invariant $n^0_\beta(X)$. For the remainder of the paper, we will focus on such sheaves supported on one of the singular fibers.

**Definition 15.** Fix one of the singular fibers of $X$, which we will also call $F_{\text{SING}}$.

Define $M_{\text{SG}} \subset M$ to be sheaves in $M$ supported on $F_{\text{SING}}$,

$$M_{\text{SG}} = \{[\mathcal{F}] \in M \,|\, \operatorname{Supp} \mathcal{F} \subset F_{\text{SING}}\} \subset M.$$

Let $T$ be the 2-torus which acts on the fiber $F_{\text{SING}}$ and thus on $M_{\text{SG}}$:

$$T := X^0 \cap F_{\text{SING}} \cong \mathbf{C}^* \times \mathbf{C}^*,$$
$$T \text{ acts on } M_{\text{SG}}.$$

Define $M^T_{\text{SG}}$ to be sheaves in $M_{\text{SG}}$ invariant under the action of $T$:

$$M^T_{\text{SG}} := \{[\mathcal{F}] \in M_{\text{SG}} \,|\, [\mathcal{F}] \text{ invariant under } T\}.$$

With this notation in place, the following corollary to Proposition 14 is immediate.

**Corollary 16.** *The Gopakumar-Vafa invariants of $X$ can be computed from the Behrend function weighted count of $M^T_{\mathrm{SG}}$:*

$$e(M,\nu) = 12 e(M^T_{\mathrm{SG}}, \nu|_{M^T_{\mathrm{SG}}}).$$

## 2.3 Geometry

We want to convert our problem into one of counting sheaves on the universal cover $U(F_{\mathrm{SING}}) \xrightarrow{pr} F_{\mathrm{SING}}$. In this section, we explain some of its geometry that we will need in the rest of the paper.

### 2.3.1 Geometry of $U(F_{\mathrm{SING}})$

First we discuss some of the geometry of the universal cover, although we will not need the description of the formal neighborhood until Section 2.7.

**Notation 17.** *Denote by*

- $\widehat{F}_{\mathrm{SING}}$*: the formal completion of $X$ along $F_{\mathrm{SING}}$,*
- $U(F_{\mathrm{SING}})$*: the universal cover of the singular fiber $F_{\mathrm{SING}}$,*
- $U(\widehat{F}_{\mathrm{SING}})$ *: the universal cover of $\widehat{F}_{\mathrm{SING}}$,*
- $\mathrm{NRM}(F_{\mathrm{SING}})$*: the normalization of $F_{\mathrm{SING}}$,*
- $\widehat{\mathrm{NRM}(F_{\mathrm{SING}})}$*: the formal completion of the total space of the canonical bundle of the blow up of $\mathbf{P}^1 \times \mathbf{P}^1$ at the two torus fixed antidiagonal points, along the zero section,*

$$\widehat{\mathrm{NRM}(F_{\mathrm{SING}})} \cong \widehat{\mathrm{Bl}_{a,b}(\mathbf{P}^1 \times \mathbf{P}^1)} \longrightarrow \mathrm{TOT}\, K(\mathrm{Bl}_{a,b}\, \mathbf{P}^1 \times \mathbf{P}^1),$$
$$\{a,b\} = \{(0,\infty),(\infty,0)\} \in \mathbf{P}^1 \times \mathbf{P}^1. \tag{2.7}$$

We regard $\widehat{F}_{\text{SING}}$ as a formal Calabi-Yau threefold. In [6, Proposition 4.10] it is shown that $\widehat{\text{NRM}(F_{\text{SING}})}$ is an étale cover of $\widehat{F}_{\text{SING}}$,

$$\widehat{\text{NRM}(F_{\text{SING}})} \xrightarrow{\text{étale}} \widehat{F}_{\text{SING}}.$$

The momentum polytope of $\text{NRM}(F_{\text{SING}})$ and its toric fan are pictured in Figures 2.2 and 2.3.

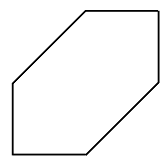

**Figure 2.2:** Momentum polytope of $\text{NRM}(F_{\text{SING}})$

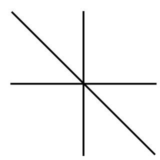

**Figure 2.3:** Toric fan of $\text{NRM}(F_{\text{SING}})$

Then $\widehat{\text{NRM}(F_{\text{SING}})}$, the formal neighborhood of the normalization of the singular fiber, is formally locally isomorphic to the total space of the canonical bundle of the blow up of $\mathbf{P}^1 \times \mathbf{P}^1$ at two points, which is the toric three-fold associated to the fan depicted in Figure 2.4. This fan comes from constructing cones over the two-dimensional polytope of Figure 2.3 placed at height 1 in $\mathbf{R}^3$.

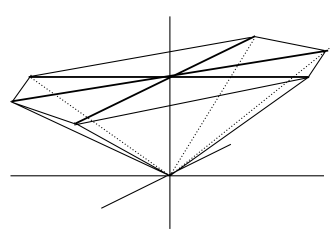

**Figure 2.4:** Toric fan of $\text{TOT}\ K(\text{Bl}_{a,b}(\mathbf{P}^1 \times \mathbf{P}^1)) \xleftarrow{\text{formal}} \widehat{\text{NRM}(F_{\text{SING}})}$

The map $\text{NRM}(F_{\text{SING}}) \to F_{\text{SING}}$ can be described by identifying opposite edges of the momentum polytope. In the case of $\widehat{\mathrm{NRM}(F_{\mathrm{SING}})} \xrightarrow{\text{étale}} \widehat{F}_{\mathrm{SING}}$, the gluing is done along a formal open neighborhood of the edges of the polytope.

From this geometry, we see that the moment polytope of $U(F_{\mathrm{SING}})$ has a planar projection given by an infinite tiling with hexagons, as shown in Figure 2.5. The cones over the dual tiling in Figure 2.6, placed at height 1, is then the fan associated to a non-finite type toric Calabi-Yau three-fold $\mathfrak{W}$ Figure 2.7 to which $U(\widehat{F}_{\mathrm{SING}})$ is formally locally isomorphic. We will return to this viewpoint in Section 2.7. From

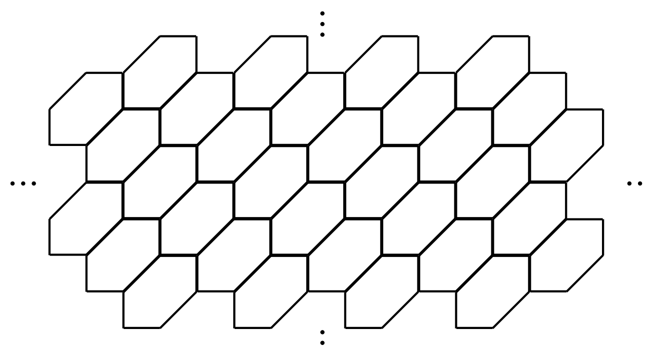

**Figure 2.5:** Momentum polytope of $U(F_{\mathrm{SING}})$

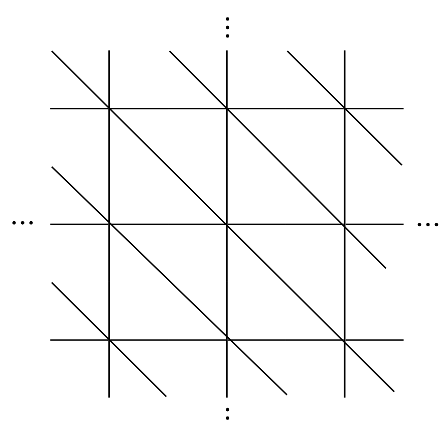

**Figure 2.6:** Dual tiling of $U(F_{\mathrm{SING}})$

this description in terms of the momentum polytope, we see that the group of deck transformations of $U(F_{\mathrm{SING}})$ is free abelian on two generators, $\pi_1(F_{\mathrm{SING}}) \cong \mathbf{Z} \times \mathbf{Z}$.

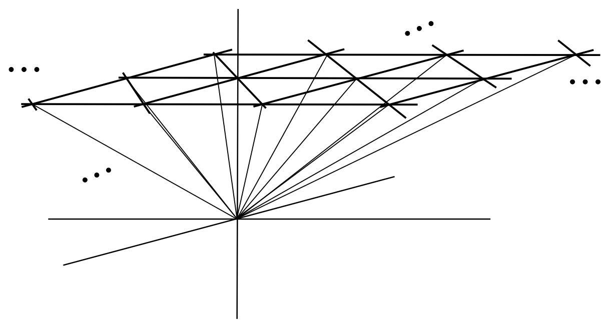

**Figure 2.7:** Non-finite type toric Calabi-Yau three-fold $\mathfrak{W}$ formally locally isomorphic to $U(\widehat{F}_{\text{SING}})$

### 2.3.2 Local geometry of $C_{\text{BAN}}$

As a consequence of Proposition 14, we only need to consider sheaves with scheme-theoretic support on $F_{\text{SING}}$. We will thus restrict our discussion in the following sections to studying sheaves on the surface $F_{\text{SING}}$. In particular, the support of such sheaves can only have thickenings in this surface, and not more generally in other $X_{\text{BAN}}$ directions.

The geometry of $C_{\text{BAN}} \subset F_{\text{SING}}$ near a node $p$ or $q$ is such that it locally looks like the union of the coordinate axes of $\mathbf{C}^3$ inside the union of the three coordinate planes. Away from the nodes, $C_{\text{BAN}} \subset F_{\text{SING}}$ looks formally locally like the zero section inside the total space of the union of the two $\mathbf{C}^* \times \mathbf{C}^*$-invariant subbundles of $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbf{P}^1$, see Figure 2.8.

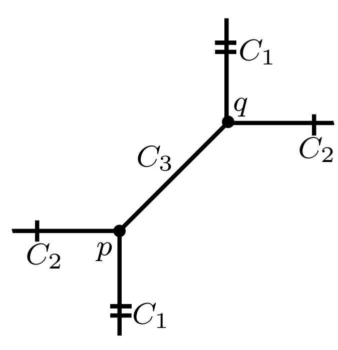


**Figure 2.8:** Local geometry of $C_{\text{BAN}}$

Let $e \subset U(F_{\text{SING}})$ be any possibly nonreduced, irreducible curve which is a lift of a multiple of a $C_{\text{BAN}}$ curve component,

$$[pr_* e] = d[C_i].$$

We will call such an $e$ an edge. Then $e$ is the intersection of two irreducible surface components in $U(F_{\text{SING}})$, that is, the intersection of two hexagons in the momentum polytope (Figure 2.5). Such an edge $e$ has possible thickenings that are determined by two numbers $m_e, n_e \geq 1$, which record the thickening in each of the two surface directions. More concretely, we can represent $U(F_{\text{SING}})$ near $e$ in local coordinates as the union of the $xy$ and the $xz$ coordinate planes in $\mathbf{C}^3 = \{(x, y, z)\}$. The $x$-axis where these planes meet then corresponds to the edge $e$. Since $e$ must lie in the surface $U(F_{\text{SING}})$, it has possible thickenings only in the $xy$ or $xz$ plane directions, and we can encode this information with two positive integers $m_e$ and $n_e$:

$$e \cong \operatorname{Spec} \mathbf{C}[x, y, z]/(y^{m_e}, z^{n_e}, yz).$$

More generally, suppose $\mathcal{C} \subset U(F_{\text{SING}})$ is a curve that lies over $C_{\text{BAN}}$, $(pr(\mathcal{C}))_{\text{RED}} \subset C_{\text{BAN}}$. We will call any point in $\mathcal{C}$ which is a preimage of a node of $C_{\text{BAN}}$ a vertex. Such a vertex is a point of intersection of up to three edges. These edges project to the three components $C_i$ of $C_{\text{BAN}}$ under the covering map.

In an affine neighborhood of a vertex with three edges, we can express $\mathcal{C}$ in local coordinates with the vertex as the origin of $\mathbf{C}^3$ and the edges as the coordinate axes. Then $\mathcal{C}$ has the structure:

$$\operatorname{Spec} \mathbf{C}[x, y, z]/(xyz, x^r y^m, z^n x^b, y^a z^s)$$

for some finite thickenings $a, b, m, n, r, s \geq 1$ as shown in Figure 2.9. If there are only two components meeting at a vertex, say with a missing $z$-axis, then locally $\mathcal{C}$ is given by:

$$\operatorname{Spec} \mathbf{C}[x, y, z]/(xyz, x^r y^m, z^n x, y z^s, z^{\max(n,s)}).$$

This is in fact the same as the degree 3 vertex case if we take the convention that the empty edge has thickenings of lengths 0 and 1, where the 0 length is taken in the direction of the axis with the larger thickening in their shared plane. For example, if $n \geq s$,

$$(xyz, x^r y^m, z^n x, yz^s, z^{\max(n,s)}) = (xyz, x^r y^m, z^n, yz^s).$$

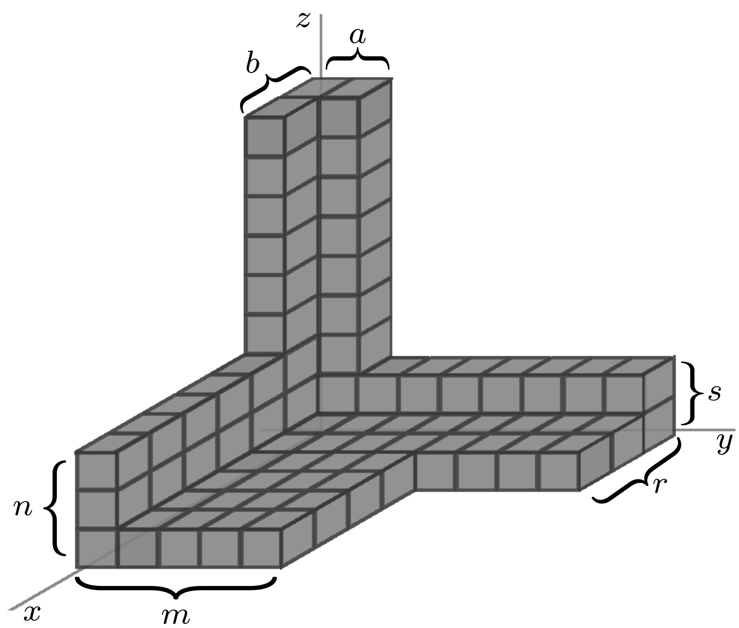


**Figure 2.9:** Multiple structure of a pure curve $\mathcal{C}$ near a vertex.

## 2.4 Reduction to counting on $U(F_{\text{SING}})$

In this section, we explain how to convert our problem into one of counting sheaves on the universal cover $U(F_{\text{SING}}) \xrightarrow{pr} F_{\text{SING}}$. We do this by considering a second $\mathbf{C}^* \times \mathbf{C}^*$ action on $M^T_{\text{SG}}$, that of tensoring with degree zero line bundles on $F_{\text{SING}}$.

Recall the categorical equivalence between equivariant sheaves on a covering space with a free group action and sheaves on the quotient. Given a discrete group $G$ and a $G$-space $X$, let $\text{Coh}^G(X)$ be the abelian category of coherent $G$-sheaves on $X$. These are pairs $(\mathcal{G}, \theta)$, where $\mathcal{G}$ is a coherent sheaf on $X$ and $\theta$ is a lift of the $G$-action.

Suppose the $G$ action on $X$ is free and let $Y$ be the quotient space, $\pi : X \to Y = X/G$. We have a categorical equivalence between coherent $G$-sheaves on $X$ and coherent sheaves on $Y$,

$$\mathrm{Coh}^G(X) \to \mathrm{Coh}(Y = X/G).$$

On one hand, given any coherent sheaf $\mathcal{F}$ on $Y$, its pullback $\pi^*\mathcal{F}$ is naturally a $G$-sheaf on $X$ and we have a functor $\mathrm{Coh}(Y) \to \mathrm{Coh}^G(X)$. We also have a functor in the other direction. Let $\mathcal{G} \in \mathrm{Coh}^G(X)$. Then $\pi_*\mathcal{G}$ has a natural action of $G$ induced from the action of $G$ on $\mathrm{Coh}^G(X)$ and the sheaf of $G$-invariants, $(\pi_*\mathcal{G})^G$ is a coherent sheaf on $Y$. This defines an inverse functor $\mathrm{Coh}^G(X) \to \mathrm{Coh}(Y)$.

We now determine $\mathrm{Pic}^0(F_{\text{SING}})$, which acts on $M^T_{\text{SG}}$ by tensoring.

**Definition 18.** Define $P$ to be the degree 0 line bundles on $F_{\text{SING}}$:

$$P := \mathrm{Pic}^0(F_{\text{SING}}).$$

**Proposition 19.** *Let* $P = \mathrm{Pic}^0(F_{\text{SING}})$ *as above. Then*

$$P \cong \mathbf{C}^* \times \mathbf{C}^*.$$

*Proof.* Let *pr* $: U(F_{\text{SING}}) \to F_{\text{SING}}$ be the universal cover of $F_{\text{SING}}$ and let $G = \pi_1(F_{\text{SING}})$ be the fundamental group of $F_{\text{SING}}$, which acts on $U(F_{\text{SING}})$ by deck transformations. Recall (Sec. 2.3.1) that the universal cover $U(F_{\text{SING}})$ is a non-finite type, non-normal toric variety with a countable number of irreducible components, each of which is isomorphic to the surface $\mathrm{Bl}_{a,b}(\mathbf{P}^1 \times \mathbf{P}^1)$, where $a$ and $b$ are two points in $\mathbf{P}^1 \times \mathbf{P}^1$. We will call this surface $S := \mathrm{Bl}_{a,b}(\mathbf{P}^1 \times \mathbf{P}^1)$ for this proof. Informally, the universal cover is an infinite union of toric surfaces which are moved around by the deck transformations.

In our case, $G = \mathbf{Z} \times \mathbf{Z}$ is generated by two elements $G = \langle e_1, e_2 \rangle$, so a lift of the $G$-action is determined by two commuting isomorphisms, $\mu_i : \mathcal{G} \to e_i^*\mathcal{G}$, $i = 1, 2$.

A degree zero line bundle $L$ on $F_{\text{SING}}$ corresponds to the $G$ line bundle $\widetilde{L} = (pr^*L, \mu_1, \mu_2)$. Here, $pr^*L$ is a degree zero line bundle. Because $\mathrm{H}^1(S, \mathcal{O}_S) = 0$, the line bundles on $S$ are determined by their degree. Since $pr^*L$ restricted to each irreducible surface component of $U(F_{\text{SING}})$ is degree zero, it is trivial on each component. Then on $U(F_{\text{SING}})$, which is connected and simply connected, $pr^*L$ is also trivial. If we choose a trivialization $pr^*L \cong U(F_{\text{SING}}) \times \mathbf{C}$, then $\mu_i(x, v) = (e_i(x), \mu_i(x)v)$. Since $\mu_i(x)$ is constant on each irreducible surface component of $U(F_{\text{SING}})$, $\mu_i(x) = \mu_i \in \mathbf{C}^*$.

Hence, $\widetilde{L}$ is the triple $(\mathcal{O}_{U(F_{\text{SING}})}, \mu_1, \mu_2)$, where $\mu_i$ is the map which acts on the fiber by multiplication by a constant, $\mu_i \in \mathbf{C}^*$.

These $(\mathcal{O}_{U(F_{\text{SING}})}, \mu_1, \mu_2)$, for $(\mu_1, \mu_2) \in \mathbf{C}^* \times \mathbf{C}^*$ are bijective with isomorphism classes of degree zero line bundles on $F_{\text{SING}}$, and we get $\mathbf{C}^* \times \mathbf{C}^* = \mathrm{Pic}^0(F_{\text{SING}})$.

$\square$

We will denote the line bundles on $F_{\text{SING}}$ constructed in the proof of Proposition 19 as $L_{\boldsymbol{\mu}}$:

$$L_{\boldsymbol{\mu}} = L_{\mu_1, \mu_2}, \qquad \text{for } \boldsymbol{\mu} = (\mu_1, \mu_2) \in P = \mathbf{C}^* \times \mathbf{C}^*.$$

We will prove in Proposition 23 that the fixed points of the action of tensoring by these degree zero line bundles will correspond in our moduli space to a sheaf given by the pushforward of a sheaf on the universal cover.

**Definition 20.** Consider the action of $P$ on $\mathrm{Coh}(F_{\text{SING}})$ given by $(L_{\mu_1, \mu_2}, \mathcal{F}) \mapsto L_{\mu_1, \mu_2} \otimes \mathcal{F}$. Define $M_{\text{SG}}^{\text{TP}}$ as those sheaves in the moduli space $M_{\text{SG}}^T$ which are also invariant under this action of the torus $P$,

$$M_{\text{SG}}^{\text{TP}} := \{[\mathcal{F}] \in M_{\text{SG}}^T | L_{\mu_1, \mu_2} \otimes \mathcal{F} \cong \mathcal{F} \text{ for all } (\mu_1, \mu_2) \in P.\}$$

We will also define a moduli space of sheaves on $U(F_{\text{SING}})$. There is an action on $U(F_{\text{SING}})$ induced naturally from the action of $T$ on $F_{\text{SING}}$ and we use the same notation for both.

**Definition 21.** Define $M^T_{\text{USG}}$ to be the moduli space

$$M^T_{\text{USG}} := \left\{ \widetilde{\mathcal{F}} \in \text{Coh}(U(F_{\text{SING}})) \middle| \begin{array}{l} \widetilde{\mathcal{F}} \text{ pure, one dimension, } T\text{-invariant,} \\ \text{stable, } [\text{Supp}(pr_* \mathcal{F})] = \beta,\, \chi(\widetilde{\mathcal{F}}) = 1 \end{array} \right\} / \text{ iso.}$$

To establish the correspondence between sheaves in $M^{\text{TP}}_{\text{SG}}$ supported on $F_{\text{SING}}$ and those in $M^T_{\text{USG}}$ on $U(F_{\text{SING}})$, we begin with the following observation.

*Remark* 22. Since $pr : U(F_{\text{SING}}) \to F_{\text{SING}}$ is a covering map, the purity and dimension of sheaf support is unchanged under pushforward and pullback. The torus $T$ action on $U(F_{\text{SING}})$ is by definition the pullback of the torus action on $F_{\text{SING}}$, so the notion of invariance under the torus action is also preserved. Also, note that the Euler characteristic is preserved under pushforward by this covering map, $\chi(\widetilde{\mathcal{F}}) = \chi(pr_*(\widetilde{\mathcal{F}}))$ for any sheaf $\widetilde{\mathcal{F}}$ on $U(F_{\text{SING}})$.

**Proposition 23.** *Let $\mathcal{E} \in M^{TP}_{\text{SG}}$. Then there is a $\widetilde{\mathcal{F}} \in M^T_{\text{USG}}$, unique up to translation by deck transformations, such that $pr_*\widetilde{\mathcal{F}} \cong \mathcal{E}$.*

*Proof.* Let $G = \mathbf{Z} \times \mathbf{Z} = \langle e_1, e_2 \rangle$ act on $U(F_{\text{SING}})$ by the deck transformations.

Now suppose $\mathcal{E} \in M^{\text{TP}}_{\text{SG}}$. Let $\widetilde{\mathcal{E}} = pr^*\mathcal{E}$. Then $\widetilde{\mathcal{E}}$ is a $G$-sheaf so it defines a triple, $\{\widetilde{\mathcal{E}}, \phi_1, \phi_2\}$, where $\phi_i : \widetilde{\mathcal{E}} \to e_i^*\widetilde{\mathcal{E}}$ covers the action of $e_i$ on $U(F_{\text{SING}})$, so that $[\phi_1, \phi_2] = 0$.

The line bundles $L_{\mu_1, \mu_2}$ pull back to

$$pr^* L_{\mu_1, \mu_2} \cong \{\mathcal{O}_{U(F_{\text{SING}})}, \mu_1, \mu_2\},$$

where each $\mu_i$ is the multiplication by scalar map.

The lift of $\mathcal{E} \otimes L_{\mu_1, \mu_2}$ is then the triple

$$pr^*(\mathcal{E} \otimes L_{\mu_1, \mu_2}) := \{\widetilde{\mathcal{E}} \otimes \mathcal{O}_{U(F_{\text{SING}})}, \mu_1\phi_1, \mu_2\phi_2\} = \{\widetilde{\mathcal{E}}, \mu_1\phi_1, \mu_2\phi_2\}.$$

By assumption, $\mathcal{E}$ satisfies $\mathcal{E} \otimes L_{\mu_1, \mu_2} \cong \mathcal{E}$, for all $\boldsymbol{\mu} = (\mu_1, \mu_2) \in \mathbf{C}^* \times \mathbf{C}^*$.

This means that we have an isomorphism of $G$-sheaves

$$\Psi_{\boldsymbol{\mu}} : \{\widetilde{\mathcal{E}}, \phi_1, \phi_2\} \cong \{\widetilde{\mathcal{E}}, \mu_1\phi_1, \mu_2\phi_2\},$$

which induces an automorphism

$$\psi_{\boldsymbol{\mu}} : \widetilde{\mathcal{E}} \to \widetilde{\mathcal{E}},$$

for all $\boldsymbol{\mu} = (\mu_1, \mu_2) \in \mathbf{C}^* \times \mathbf{C}^*$.

Combining these, we get a commutative diagram

$$\begin{array}{ccc} \widetilde{\mathcal{E}} & \xrightarrow{\phi_{\mathbf{i}}} & e_{\mathbf{i}}^*\widetilde{\mathcal{E}} \\ {\scriptstyle\psi_{\boldsymbol{\mu}}}\downarrow & & \downarrow{\scriptstyle e_{\mathbf{i}}^*\psi_{\boldsymbol{\mu}}} \\ \widetilde{\mathcal{E}} & \xrightarrow{\boldsymbol{\mu}\phi_{\mathbf{i}}} & e_{\mathbf{i}}^*\widetilde{\mathcal{E}} \end{array}$$

Since $(\mathcal{E} \otimes L_{\boldsymbol{\mu}}) \otimes L_{\boldsymbol{\lambda}} \cong \mathcal{E} \otimes (L_{\boldsymbol{\mu}} \otimes L_{\boldsymbol{\lambda}}) \cong \mathcal{E}$, both correspond to $\{\widetilde{\mathcal{E}}, \boldsymbol{\lambda\mu}\phi_{\mathbf{i}}\}$. In other words,

$$\Psi_{\boldsymbol{\mu}} \circ \Psi_{\boldsymbol{\lambda}} = \Psi_{\boldsymbol{\mu\lambda}}, \quad \text{and} \quad \psi_{\boldsymbol{\mu}} \circ \psi_{\boldsymbol{\lambda}} = \psi_{\boldsymbol{\mu\lambda}}$$

and this defines an action of $\mathbf{C}^* \times \mathbf{C}^*$ on $\widetilde{\mathcal{E}}$,

$$\begin{array}{ccc} \widetilde{\mathcal{E}} & \xrightarrow{\phi_{\mathbf{i}}} & e_{\mathbf{i}}^*\widetilde{\mathcal{E}} \\ {\scriptstyle\psi_{\boldsymbol{\mu}}}\downarrow & & \downarrow{\scriptstyle e_{\mathbf{i}}^*\psi_{\boldsymbol{\mu}}} \\ \widetilde{\mathcal{E}} & \xrightarrow{\boldsymbol{\mu}\phi_{\mathbf{i}}} & e_{\mathbf{i}}^*\widetilde{\mathcal{E}} \\ {\scriptstyle\psi_{\boldsymbol{\lambda}}}\downarrow & & \downarrow{\scriptstyle e_{\mathbf{i}}^*\psi_{\boldsymbol{\lambda}}} \\ \widetilde{\mathcal{E}} & \xrightarrow{\boldsymbol{\lambda\mu}\phi_{\mathbf{i}}} & e_{\mathbf{i}}^*\widetilde{\mathcal{E}} \end{array}$$

We define the subsheaves $\widetilde{\mathcal{E}}_{\mathbf{k}}$ of $\widetilde{\mathcal{E}}$, as follows:

$$\widetilde{\mathcal{E}}_{\mathbf{k}} \coloneqq \operatorname{Ker}(\psi_\mu - \mu_1^{k_1}\mu_2^{k_2}\operatorname{Id}),$$
$$\mathbf{k} \in \mathbf{Z} \times \mathbf{Z}\,,\ \boldsymbol{\mu} = (\mu_1, \mu_2) \neq (1,1) \in \mathbf{C}^* \times \mathbf{C}^*.$$

This is independent of choice of $\boldsymbol{\mu}$, from the definition given above of the action $\psi_{\mu}$.

We will call these subsheaves eigensheaves. We can decompose $\widetilde{\mathcal{E}}$ into eigensheaves of this torus action,

$$\widetilde{\mathcal{E}} = \bigoplus_{\mathbf{k}\in\mathbf{Z}\times\mathbf{Z}} \widetilde{\mathcal{E}}_{\mathbf{k}}.$$

Restricting to such an eigensheaf then gives the commuting diagram

$$\begin{array}{ccc} \widetilde{\mathcal{E}}_{\mathbf{k}} & \xrightarrow{\phi_{\mathbf{i}}} & e_{\mathbf{i}}^*\widetilde{\mathcal{E}}_{\mathbf{k}} \\ \psi_{\boldsymbol{\mu}}\downarrow & & \downarrow e_{\mathbf{i}}^*\psi_{\boldsymbol{\mu}} \\ \widetilde{\mathcal{E}}_{\mathbf{k}} & \xrightarrow{\boldsymbol{\mu}\phi_{\mathbf{i}}} & e_{\mathbf{i}}^*\widetilde{\mathcal{E}}_{\mathbf{k}} \end{array}$$

From this we see there are isomorphisms

$$(e_1^*)^i(e_2^*)^j\widetilde{\mathcal{E}}_{k_1,\,k_2} \cong \widetilde{\mathcal{E}}_{k_1-i,\,k_2-j},$$

and the eigensheaves are isomorphic to each other under the action of the deck transformations.

Consider one of these eigensheaves, say $\widetilde{\mathcal{E}}_{\mathbf{k}}$. From the construction, we see that its pushforward is isomorphic to the original $P$-invariant sheaf $pr_*\widetilde{\mathcal{E}}_{\mathbf{k}} \cong \mathcal{E}$ on $F_{\text{SING}}$ and $pr^*pr_*\widetilde{\mathcal{E}}_{\mathbf{k}} = \widetilde{\mathcal{E}}$. Conversely, if $\widetilde{\mathcal{F}}$ is such that $pr_*\widetilde{\mathcal{F}} = \mathcal{E}$, then by applying the eigenspace decomposition to $\widetilde{\mathcal{F}}$, $\widetilde{\mathcal{F}}$ must be a single summand by stability. This implies uniqueness up to translation.

We now establish that for our sheaves of interest, stability is preserved when moving between $U(F_{\text{SING}})$ and $F_{\text{SING}}$. We can use the Euler characteristic characterization of stability, Lemma 11.

Assume $\mathcal{F} \in M_{\text{SG}}^{\text{TP}}$. For any $\widetilde{\mathcal{F}} \in \text{Coh}(U(F_{\text{SING}}))$ such that $pr_*\widetilde{\mathcal{F}} = \mathcal{F}$, let $\widetilde{\mathcal{E}} \subset \widetilde{\mathcal{F}}$ be a subsheaf. Then its pushforward is a subsheaf of $\mathcal{F}$, $pr_*\widetilde{\mathcal{E}} \subset pr_*\widetilde{\mathcal{F}} = \mathcal{F}$. By stability of $\mathcal{F}$, we have $0 \geq \chi(pr_*(\widetilde{\mathcal{E}})) \Rightarrow 0 \geq \chi(\widetilde{\mathcal{E}})$. Thus $\widetilde{\mathcal{F}}$ is also stable.

For the converse, suppose $\widetilde{\mathcal{F}} \in M_{\text{USG}}^T$ with $\mathcal{F} = pr_*\widetilde{\mathcal{F}}$. From the construction in Proposition 23, $\mathcal{F}$ is fixed under the action of $P$. Let $\mathcal{E} \subset \mathcal{F}$ be a proper subsheaf. Define $\mathcal{E}_{\boldsymbol{\mu}} := \mathcal{E} \otimes L_{\boldsymbol{\mu}}$. Since $\deg(L_{\boldsymbol{\mu}}) = 0$, the Euler characteristic is independent of $\boldsymbol{\mu}$, $\chi(\mathcal{E}_{\boldsymbol{\mu}}) = \chi(\mathcal{E})$.

This gives a flat family of coherent sheaves over $(\mathbf{C}^* \times \mathbf{C}^*) \times F_{\text{SING}}$, whose restriction to $\boldsymbol{\mu} \times F_{\text{SING}}$ is $\mathcal{E}_{\boldsymbol{\mu}}$. Our $F_{\text{SING}}$ is proper, so coherent sheaves satisfy the existence part of the valuative criterion of properness. Thus, we have some limiting sheaf $\mathcal{E}_{\vec{0}}$, which is invariant under the action of $P$. Then by Proposition 23, there is some $\widetilde{\mathcal{E}}_0 \subset \widetilde{\mathcal{F}}$ such that $pr_* \widetilde{\mathcal{E}}_0 = \mathcal{E}_{\vec{0}}$. Since $\widetilde{\mathcal{F}}$ is assumed stable, $0 \geq \chi(\widetilde{\mathcal{E}}_0) = \chi(pr_* \widetilde{\mathcal{E}}_0) \Rightarrow 0 \geq \chi(\mathcal{E}_{\vec{0}}) = \chi(\mathcal{E})$, so $\mathcal{F}$ is also stable. □

In Section 2.7, we will show that this action of $P$ also preserves the symmetric obstruction theory of $M$, and calculate the parity of the tangent space dimensions at the fixed points of this action.

## 2.5 Counting Sheaves on $U(F_{\text{SING}})$

The main result we want to show in this section is the following:

**Proposition 24.** *Suppose $\mathcal{F} \in M^T_{\text{USG}}$. Then $\mathcal{F} \cong \mathcal{O}_{\mathcal{C}}$ for some $T$-invariant curve $\mathcal{C}$ with $\chi(\mathcal{O}_{\mathcal{C}}) = 1$.*

In order to prove Proposition 24, we need the following key lemma and its corollary, whose proofs we postpone until later.

**Lemma 25.** *Let $\mathcal{F} \in M^T_{\text{USG}}$ with support curve $\operatorname{Supp} \mathcal{F} = \mathcal{C}$. Then $\chi(\mathcal{O}_{\mathcal{C}}) \geq 1$.*

*Proof.* See Subsection 2.5.2. □

As a Corollary, we have

**Corollary 26.** *Let $\mathcal{F} \in M^T_{\text{USG}}$ with support curve $\operatorname{Supp} \mathcal{F} = \mathcal{C}$. Let $\mathcal{D}$ be any closed subscheme of $\mathcal{C}$. Then $\chi(\mathcal{O}_{\mathcal{D}}) \geq 1$.*

*Proof.* See Subsection 2.5.3. □

Using Corollary 26, we can prove Proposition 24.

*Proof (of Proposition 24).* Let $\mathcal{F} \in M^T_{\text{USG}}$ and $\mathcal{C} = \operatorname{Supp} \mathcal{F}$. By hypothesis, $\chi(\mathcal{F}) = 1$ so $\mathcal{F}$ has a nonzero global section $s$. Let $\mathcal{I}$ be the kernel of the map $s$. Then we have the exact sequence:

$$0 \to \mathcal{I} \to \mathcal{O}_C \xrightarrow{s} \mathcal{F}$$

Let $\mathcal{C}_s \subset \mathcal{C}$ be the support of $s$. Then $\mathcal{O}_{\mathcal{C}_s} = \mathcal{O}_{\mathcal{C}}/\mathcal{I}$ is a subsheaf of $\mathcal{F}$,

$$0 \to \mathcal{O}_{\mathcal{C}_s} \to \mathcal{F}$$

By Corollary 26, $\chi(\mathcal{O}_{\mathcal{C}_s}) \geq 1$, which contradicts Lemma 11 unless $\mathcal{O}_{\mathcal{C}_s} \cong \mathcal{O}_{\mathcal{C}} \cong \mathcal{F}$. □

### 2.5.1 Formula for Euler characteristic.

Before we present the proof of Lemma 25 and Corollary 26, we derive a formula to compute the Euler characteristic of structure sheaves of a certain type of curve in $U(F_{\text{SING}})$.

Since we are interested in sheaves on $X_{\text{BAN}}$ with support in class $\beta = d_1[C_1] + d_2[C_2] + [C_3]$, we can eliminate any isomorphisms induced by the deck transformations on $U(F_{\text{SING}})$ by fixing a curve that lies over $C_3 \subset C_{\text{BAN}}$. By Proposition 23, any point in $M^T_{\text{USG}}$ can be uniquely represented by a sheaf whose support contains this curve.

In the discusion of Section 2.3.2, we observed that the formal neighborhood of any irreducible component $e$ that covers $C_i$ is formally locally isomorphic to the total space of $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbf{P}^1$. As a consequence, we have a map from $e$ to the reduced curve $e_{\text{RED}}$ in our geometry.

We record these observations in the terminology that we will use in this section.

**Notation 27.** *Suppose* $\mathcal{F} \in M^T_{\text{USG}}$.

- $e_0 \cong \mathbf{P}^1 \subset U(F_{\text{SING}})$ *is a fixed curve such that* $pr_*(e_0) = [C_3]$.
- $\Gamma := \{\mathcal{C} \subset U(F_{\text{SING}}) \mid \dim \mathcal{C} = 1,$ *connected, $T$-fixed,* $e_0 \subset \mathcal{C}, [pr_*(\mathcal{C})] =$

*$d_1[C_1] + d_2[C_2] + [C_3], d_1, d_2 \geq 0\}$, see Figure 2.10.*

- *$\mathcal{C} := \operatorname{Supp} \mathcal{F}$ so $[(pr_* \mathcal{C})] = \beta$. Without loss of generality, we will let $\mathcal{F}$ be such that $\mathcal{C} \in \Gamma$.*
- *Edges $\{e_i\}$ are possibly nonreduced, irreducible components of $\mathcal{C}$.*
- *Vertices $\{v_j\}$ are points where two or more components of $\mathcal{C}$ intersect.*
- *$\phi : e \to e_{\text{RED}} \cong \mathbf{P}^1$ is the map that exists for edges $e$ in our geometry.*

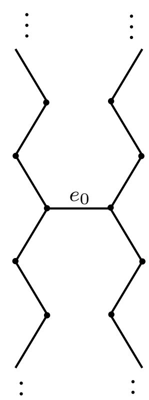


**Figure 2.10:** Example of a curve $\mathcal{C} \subset U(F_{\text{SING}}), \mathcal{C} \in \Gamma$

We can write the support of $\mathcal{F}$ as

$$\mathcal{C} := \operatorname{Supp} \mathcal{F} = \bigcup_{e_i \in \{\text{edges of } \mathcal{C}\}} e_i,$$

where each $e_i$ is a component with a unique torus invariant thickening on $C$ determined by two numbers on each edge, $m_e, n_e$ (see Section 2.3.2), and $(e_i)_{\text{red}} \cong \mathbf{P}^1$.

The following is a special case of [27, Lemma 5]. We give a self-contained proof for convenience, and tailor it to our situation and notation to obtain a formula to compute Euler characteristics of structure sheaves of support curves of $\mathcal{F} \in M^T_{\text{USG}}$.

**Lemma 28.** *[27] Let $\mathcal{C}$ be a connected pure one dimensional curve in $\Gamma$. Let $\{e_i\}$ be the edges and $\{v_j\}$ the vertices of $\mathcal{C}$.*

*Then $\chi(\mathcal{O}_\mathcal{C})$ satisfies*

$$\chi(\mathcal{O}_\mathcal{C}) = \sum_{e_i} E(\mathcal{O}_\mathcal{C}, e_i) - \sum_{v_j} V(\mathcal{O}_\mathcal{C}, v_j), \tag{2.8}$$

*where $E(\mathcal{O}_\mathcal{C}, e_i)$ and $V(\mathcal{O}_\mathcal{C}, v_j)$ are integer valued functions on the edges and vertices, respectively, and defined as follows:*

*Given an edge $e$ with thickening lengths $m$ and $n$, the integer $E(\mathcal{O}_\mathcal{C}, e)$ is given by*

$$E(\mathcal{O}_\mathcal{C}, e) = \left[\binom{m+1}{2} + \binom{n+1}{2} - 1\right].$$

*At a vertex $v$ with three incident edges and multiple structure as in Figure 2.9, then the integer $V(\mathcal{O}_\mathcal{C}, v)$ is given by*

$$V(\mathcal{O}_\mathcal{C}, v) = (mr + sa + bn - 1).$$

*If $v$ only has two incident edges, corresponding to, say, the $x$ and $y$ axes with thickenings as in Figure 2.9, then*

$$V(\mathcal{O}_\mathcal{C}, v) = (mr + \min(n, s) - 1).$$

*Proof.* This is a computation of Euler characteristic using the normalization sequence, and by pushing forward sheaves on irreducible components to their reduced counterparts.

Consider the normalization sequence,

$$0 \to \mathcal{O}_\mathcal{C} \to \bigoplus_i \mathcal{O}_\mathcal{C}|_{e_i} \to \bigoplus_{i \neq j} \mathcal{O}_\mathcal{C}|_{e_i \cap e_j} \to \bigoplus_{\substack{i,j,k \\ \text{distinct}}} \mathcal{O}_\mathcal{C}|_{e_i \cap e_j \cap e_k} \to 0.$$

Here, the edges $e_i$ might be non-reduced, and the restriction $\mathcal{O}_\mathcal{C}|_{e_i}$ is the structure sheaf of the unique minimal torus-fixed Cohen-Macaulay subscheme as constructed in [31, Section 2.3], whose ideal is given by localizing along the edge then extending to a 3-dimensional partition.

Then

$$\chi(\mathcal{O}_{\mathcal{C}}) = \bigoplus_{i} \chi(\mathcal{O}_{\mathcal{C}}|_{e_i}) \; - \bigoplus_{i \neq j} \chi(\mathcal{O}_{\mathcal{C}}|_{e_i \cap e_j}) \; + \bigoplus_{\substack{i,j,k \\ \text{distinct}}} \chi(\mathcal{O}_{\mathcal{C}}|_{e_i \cap e_j \cap e_k}).$$

First, we calculate the Euler characteristic of the restriction of our sheaf to a single edge. Let $e \subset \mathcal{C}$ be an edge with thickening lengths $m, n$ and map to the reduced curve, $\phi : e \to e_{\text{RED}} \cong \mathbf{P}^1$.

Then $\phi$ has zero dimensional fiber, so

$$\chi(\mathcal{O}_e) = \chi(\phi^* \mathcal{O}_{\mathbf{P}^1}) = \chi(\phi_* \mathcal{O}_e)$$

by the projection formula.

The normal bundle $\mathcal{N}$ of $e$ in $U(F_{\text{SING}})$ is formally locally isomorphic to a variety affine over $e_{\text{RED}} \cong \mathbf{P}^1$:

$$\mathcal{N} \coloneqq \operatorname{Tot}(\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbf{P}^1),$$

and its sheaf of algebras over $\mathbf{P}^1$ is given by

$$\phi_* \mathcal{O}_{\mathcal{N}} = \operatorname{Sym}^* \mathcal{N}^\vee = \bigoplus_{i,j} \mathcal{O}(i) \otimes \mathcal{O}(j).$$

So we can think of $\phi_* \mathcal{O}_e$ as a quotient of $\phi_* \mathcal{O}_{\mathcal{N}}$.

We may represent these summands graphically by boxes in the first quadrant labeled by monomial generators. The quotient sheaf $\phi_* \mathcal{O}_e$ with lengths $m$ and $n$ along the axes then corresponds to the diagram in Figure 2.11.

In other words,

$$\phi_* \mathcal{O}_e = \mathcal{O} \oplus \bigoplus_{i=1}^{m-1} \mathcal{O}(i) \oplus \bigoplus_{j=1}^{n-1} \mathcal{O}(j).$$

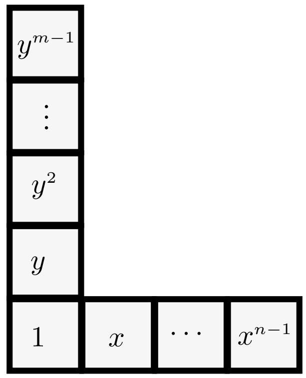


**Figure 2.11:** $\phi_* \mathcal{O}_e$ breaks up into a sum of line bundles on $\mathbf{P}^1$.

A straightforward application of Riemann-Roch gives,

$$\chi(\mathcal{O}_e) = \chi(\phi_* \mathcal{O}_e) = \left[ \binom{m+1}{2} + \binom{n+1}{2} - 1 \right].$$

Then $\bigoplus_{e_i} \chi(\mathcal{O}_{\mathcal{C}}|_{e_i}) = \sum_{e_i} E(\mathcal{O}_{\mathcal{C}}, e_i)$, where the edge contribution $E(\mathcal{O}_{\mathcal{C}}, e)$ to Equation 2.8 is of the required form.

In order to calculate the Euler characteristic contribution from the vertex terms, we need to express the lengths of the module at a vertex in terms of the thickenings of the incident edges. If we depict the multiple structure at a vertex in terms of boxes representing monomial ideals, we must count the number of common boxes shared by pairwise edges and subtract the contribution from boxes common to all three incident edges.

For example, suppose the vertex $v$ has three incident edges $e_x, e_y, e_z$, with multiple structure labeled as in Figure 2.9. Then for a given pair of edges, say $e_x, e_y$, the number of boxes they share is

$$\chi(\mathcal{O}_{e_x \cap e_y}) = mr - \min(n, s) - 1.$$

For a vertex with three incident edges, the pairwise intersections contribute the following total to the Euler characteristic:

$$\chi(\mathcal{O}_{e_x\cap e_y})+\chi(\mathcal{O}_{e_y\cap e_z})+\chi(\mathcal{O}_{e_z\cap e_x})$$
$$=(mr+\min(n,s)-1)+(sa+\min(r,b)-1)+(bn+\min(a,m)-1).$$

On the other hand, the boxes that are in the triple intersection have length:

$$\chi(\mathcal{O}_{e_x\cap e_y\cap e_z})=\min(n,s)+\min(r,b)+\min(a,m)-2.$$

Subtracting these expressions gives the contribution of a vertex with three edges.

$$\chi(\mathcal{O}_{e_x\cap e_y})+\chi(\mathcal{O}_{e_y\cap e_z})+\chi(\mathcal{O}_{e_z\cap e_x})-\chi(\mathcal{O}_{e_x\cap e_y\cap e_z})=rm+ns+ab-1.$$

Hence,

$$\bigoplus_{i\neq j}\chi(\mathcal{O}_{\mathcal{C}}|_{e_i\cap e_j})\;-\bigoplus_{\substack{i,j,k\\ \text{distinct}}}\chi(\mathcal{O}_{\mathcal{C}}|_{e_i\cap e_j\cap e_k})=\sum_{v_j}V(\mathcal{O}_{\mathcal{C}},v_j)$$

if we define the function $V(\mathcal{O}_{\mathcal{C}},v)=(rm+ns+ab-1)$ when $v$ has three incident edges, or $V(\mathcal{O}_{\mathcal{C}},v)=(mr-\min(n,s)-1)$ if it has two, as claimed. □

### 2.5.2 Proof of Lemma 25.

Let $\mathcal{C}$ be a connected curve in $\Gamma$ containing $e_0$. Then $\mathcal{C}$ naturally breaks up into a union of four branches characterized by their attachment type to $e_0$. Since the Euler characteristic computation on each branch is identical, we will use this decomposition of the curve to simplify the presentation of the proof of Lemma 25. To this end, we establish the following nomenclature conventions.

#### Terminology

*Branches:* The space $\overline{\mathcal{C}\setminus e_0}$ consists of edges that lie over $C_1$ or $C_2\subset C_{\mathrm{BAN}}$. We can write $\overline{\mathcal{C}\setminus e_0}$ as a disjoint union of (possibly empty) connected subcurves of four types:

$$\overline{\mathcal{C} \setminus e_0} = \mathcal{C}^\bullet \amalg \mathcal{C}_\bullet \amalg {}^\bullet\mathcal{C} \amalg {}_\bullet\mathcal{C}$$

These are distinguished by their attachment to $e_0$. The edge of the subcurve that intersects $e_0$ can cover either $C_1$ or $C_2$, and the intersection vertex can cover $p$ or $q$. For concreteness, we choose the identifications as indicated in Figure 2.12. We will call any of these four subcurves $\mathcal{C}^\bullet, \mathcal{C}_\bullet, {}^\bullet\mathcal{C}$, or ${}_\bullet\mathcal{C}$, the branches of $\mathcal{C}$.

Likewise, the notation $\mathcal{C}^\bullet_\bullet$ will mean the subcurve $\mathcal{C}^\bullet_\bullet = \mathcal{C}^\bullet \cup \mathcal{C}_\bullet$ and so forth. Edges and vertices will be decorated as needed to indicate the branch they are on.

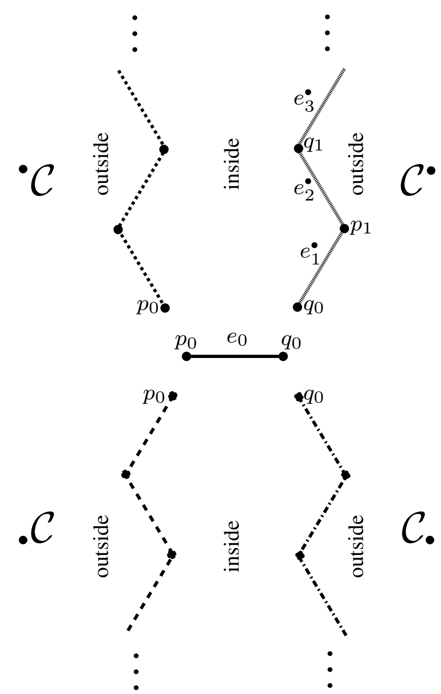


**Figure 2.12:** Schematic diagram of $\mathcal{C}$

*Numbering*: Let $|\mathcal{C}|$ denote the number of edges of any curve $\mathcal{C}$. We number the consecutive edges of each branch in increasing order away from $e_0$ and group them in *consecutive pairs*, labeled as $(e_{2k-1}, e_{2k})$, $k \geq 1$.

*Thickening*: Recall that as a consequence of Proposition 14, the sheaves we are interested in have scheme-theoretic support in the surface $F_{\text{SING}}$. Thus, any thickening of support curves for sheaves in $M^T_{\mathrm{USG}}$ will occur in the surface $U(F_{\mathrm{SING}})$. The fixed edge $e_0$ is the intersection of two irreducible surface components of $U(F_{\mathrm{SING}})$. Let $S$ be one of these irreducible surface components containing $e_0$, and let $g$ be the deck transformation which translates $S$ into the other component containing $e_0$. Then $g$ generates a $\mathbf{Z}$ subgroup of the deck transformations, $\langle g\rangle \cong \mathbf{Z} \subset \mathbf{Z} \times \mathbf{Z}$. We define the *inside hexagons* as those irreducible surface components of $U(F_{\mathrm{SING}})$ that are in the orbit of $S$ under the action of $\langle g\rangle$. Any other irreducible surface components will be called *outside hexagons*.

Every edge of $\mathcal{C}$, apart from $e_0$, is the intersection of an inside hexagon and an outside hexagon, and can be thickened in either of these two directions. We will call an edge thickening in the direction of the inside hexagon surface the *inside* direction. The thickening of an edge in the outside hexagon surface direction will be called the *outside* direction.

*One branch detail:* We choose one branch, say $\mathcal{C}^\bullet$, for detailed computations, Figure 2.13. Here, we will denote the lengths of the multiple structure on edges $e^\bullet_{2i-1}$ by $m_i$ on the inside and $n_i$ on the outside. Edges $e^\bullet_{2i}$ will have multiple structures of lengths $r_i$ on the inside and $s_i$ on the outside. The vertices will be numbered so that $p_i = e_{2i-1} \cap e_{2i}$ and $q_i = e_{2i} \cap e_{2i+1}$.

*Empty edge:* To make our formulas uniform, we will adopt the convention that an empty edge of $\mathcal{C}^\bullet$ will have inside multiplicity of 0, and outside multiplicity of 1. Also, if there are an odd number of edges in any branch so that the last of the consecutive pairs only contains a single element, $(e^\bullet_{2\alpha-1}, -)$, then we will append an empty edge to complete the pair.

### Euler characteristic of structure sheaf

Now that we have the notation in place, we first derive an expression for the Euler characteristic of the structure sheaf of a curve with only one nonempty branch, $\chi(\mathcal{O}_\mathcal{C})$ where $\mathcal{C} = e_0 \cup \mathcal{C}^\bullet$, and show that it is bounded below by 1. Furthermore, we will see the restrictions that equality imposes on the multiple structures $m_i, r_i, n_i, s_i$ that can appear in such a curve.

**Lemma 29.** *Let $\mathcal{C} = e_0 \cup \mathcal{C}^\bullet$, with $\emptyset \neq \mathcal{C}^\bullet$ and $|\mathcal{C}^\bullet| = 2\alpha$ or $2\alpha - 1$ for some*

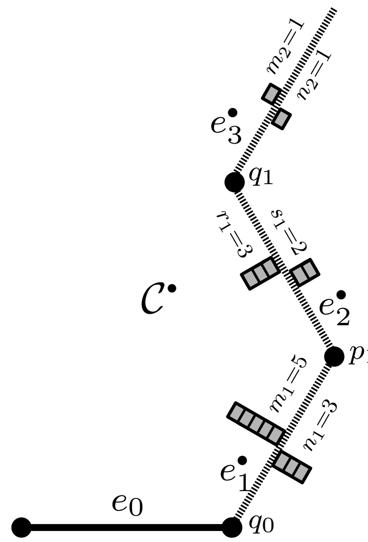


**Figure 2.13:** Example detail of $e_0 \cup \mathcal{C}^\bullet$

*positive integer $\alpha$. If $|\mathcal{C}^\bullet|$ is odd then we append an empty edge to $\mathcal{C}^\bullet$ in the formula below.*

*Then the Euler characteristic $\chi(\mathcal{O}_\mathcal{C})$ satisfies the following equality:*

$$\chi(\mathcal{O}_\mathcal{C}) = \frac{1}{2}n_1^2 + \frac{1}{2}\sum_{i=1}^{\alpha}((r_i - m_i)(r_i - m_i + 1)) + \frac{1}{2}\sum_{i=1}^{\alpha-1}(n_{i+1} - s_i)^2 + \frac{1}{2}\sum_{i=1}^{\alpha}(n_i + s_i - 2\min(n_i, s_i)) + \sum_{i=2}^{\alpha}(m_i - \min(r_{i-1}, m_i)) + \frac{1}{2}s_\alpha^2 \tag{2.9}$$

*In particular, $\chi(\mathcal{O}_\mathcal{C}) \geq 1$ with equality if and only if $n_1 = s_\alpha = 1$ and all the summation terms are zero.*

*Proof.* The last statement follows since $n_1, s_\alpha \geq 1$ and all the other summands are non-negative. Note that the second summation is always non-negative since the two factors of each summand never have opposite signs.

We will prove Eq. (2.9) using induction on $\alpha$, and a rearrangement of the formula in Lemma 28.

First, suppose $|\mathcal{C}^\bullet| = 2\alpha$. The formula in Lemma 28 for $\chi(\mathcal{O}_\mathcal{C})$ becomes:

$$\begin{aligned}\chi(\mathcal{O}_\mathcal{C}) &= \\ 1 &+ \sum_{i=1}^{\alpha}\left(\binom{m_i+1}{2} + \binom{n_i+1}{2} - 1\right) + \sum_{i=1}^{\alpha}\left(\binom{r_i+1}{2} + \binom{s_i+1}{2} - 1\right) \\ &-m_1 - \sum_{i=1}^{\alpha}(m_i r_i + \min(n_i, s_i) - 1) - \sum_{i=1}^{\alpha-1}(s_i n_{i+1} + \min(r_i, m_{i+1}) - 1)\end{aligned} \tag{2.10}$$

The summation terms are contributions from the $e_{2i-1}$ and $e_{2i}$ edges and vertex corrections from the $p_i$ and $q_i$, respectively.

In the odd case of $|\mathcal{C}^\bullet| = 2\alpha - 1$, the formula in Lemma 28 becomes:

$$\begin{aligned}\chi(\mathcal{O}_\mathcal{C}) &= \\ 1 &+ \sum_{i=1}^{\alpha}\left(\binom{m_i+1}{2} + \binom{n_i+1}{2} - 1\right) + \sum_{i=1}^{\alpha-1}\left(\binom{r_i+1}{2} + \binom{s_i+1}{2} - 1\right) \\ &-m_1 - \sum_{i=1}^{\alpha-1}(m_i r_i + \min(n_i, s_i) - 1) - \sum_{i=1}^{\alpha-1}(s_i n_{i+1} + \min(r_i, m_{i+1}) - 1)\end{aligned} \tag{2.11}$$

If we append an empty edge to $\mathcal{C}$ in this odd case, our convention dictates that we define $r_\alpha = 0$ and $s_\alpha = 1$. Then, we can rewrite Eq. (2.11) as:

$$\chi(\mathcal{O}_{\mathcal{C}_{2\alpha}}) = \\ 1 + \sum_{i=1}^{\alpha}\left(\binom{m_i+1}{2} + \binom{n_i+1}{2} - 1\right) + \sum_{i=1}^{\alpha}\left(\binom{r_i+1}{2} + \binom{s_i+1}{2} - 1\right) \\ -m_1 - \sum_{i=1}^{\alpha}(m_i r_i + \min(n_i, s_i) - 1) - \sum_{i=1}^{\alpha-1}(s_i n_{i+1} + \min(r_i, m_{i+1}) - 1) \tag{2.12}$$

This is now exactly the same as the even case, Eq. (2.10). So from here, we will assume $|\mathcal{C}^\bullet|$ is even, with empty edge appended, if needed, and in either case satisfies eq. (2.10).

To begin the induction, when $\alpha = 1$, Eq. (2.10) reduces to

$$\begin{aligned}\chi(\mathcal{O}_{\mathcal{C}_2}) = 1 &+ \binom{m_1+1}{2} + \binom{n_1+1}{2} - 1 \\ &+ \binom{r_1+1}{2} + \binom{s_1+1}{2} - 1 \\ &- m_1 - (m_1 r_1 + \min(n_1, s_1) - 1) \\ \\ &= \frac{1}{2}n_1^2 + \frac{1}{2}(r_1 - m_1)(r_1 - m_1 + 1) \\ &+ \frac{1}{2}(n_1 + s_1 - 2\min(n_1, s_1)) + \frac{1}{2}s_1^2,\end{aligned}$$

which satisfies Eq. (2.9).

Now suppose Eq. (2.9) is true for all $\mathcal{C} = e_0 \cup \mathcal{C}^\bullet$ with $|\mathcal{C}^\bullet| = 2k$ and $1 \le k \le \alpha$. Then for any $\mathcal{C} = e_0 \cup \mathcal{C}^\bullet$ with $|\mathcal{C}^\bullet| = 2\alpha + 2$, we can write this as a union $\mathcal{C} = \mathcal{C}_{2\alpha} \cup \mathcal{C}_2^\bullet$ where $\mathcal{C}_{2\alpha} = e_0 \cup \mathcal{C}_{2\alpha}^\bullet$ contains the first $2\alpha$ edges of $\mathcal{C}^\bullet$ and $\mathcal{C}_2^\bullet$ are the remaining edges of $\mathcal{C}^\bullet$. Then we have from eq. (2.10),

$$\begin{aligned}\chi(\mathcal{O}_{\mathcal{C}}) &= \chi(\mathcal{O}_{\mathcal{C}_{2\alpha}}) + \binom{m_{\alpha+1}+1}{2} + \binom{n_{\alpha+1}+1}{2} - 1 \\ &+ \binom{r_{\alpha+1}+1}{2} + \binom{s_{\alpha+1}+1}{2} - 1 \\ &- (s_\alpha n_{\alpha+1} + \min(r_\alpha, m_{\alpha+1}) - 1) \\ &- (m_{\alpha+1} r_{\alpha+1} + \min(n_{\alpha+1}, s_{\alpha+1}) - 1) \\ \\ &= \chi(\mathcal{O}_{\mathcal{C}_{2\alpha}}) - \frac{1}{2}s_\alpha^2 + \frac{1}{2}(r_{\alpha+1} - m_{\alpha+1})(r_{\alpha+1} - m_{\alpha+1} + 1) \\ &+ \frac{1}{2}(n_{\alpha+1} - s_\alpha)^2 + \frac{1}{2}(n_{\alpha+1} + s_{\alpha+1} - 2\min(n_{\alpha+1}, s_{\alpha+1})) \\ &+ \frac{1}{2}s_{\alpha+1}^2 + m_{\alpha+1} - \min(r_\alpha, m_{\alpha+1})\end{aligned}$$

Using the inductive step, we get Eq. (2.9)

Hence, the lemma follows. □

We can now formulate and prove a refined version of Lemma 25.

**Lemma 30.** *Let* $\mathcal{F} \in M^T_{\mathrm{USG}}$ *with* $\mathcal{C} = \operatorname{Supp}\mathcal{F}$. *Then* $\chi(\mathcal{O}_{\mathcal{C}}) \geq 1$. *Equality holds if and only if* $\chi(\mathcal{O}_{e_0\cup\mathcal{C}^\bullet}) = \chi(\mathcal{O}_{e_0\cup\mathcal{C}_\bullet}) = \chi(\mathcal{O}_{e_0\cup{}^\bullet\mathcal{C}}) = \chi(\mathcal{O}_{e_0\cup{}_\bullet\mathcal{C}}) = 1$.

*Proof.* By symmetry, the Lemma 29 calculations done for the case of $e_0 \cup \mathcal{C}^\bullet$, also hold for structure sheaves of subcurves $e_0 \cup \mathcal{C}_\bullet$, $e_0 \cup {}^\bullet\mathcal{C}$, or $e_0 \cup {}_\bullet\mathcal{C}$, after an appropriate change of label.

Since $\mathcal{C} \cong e_0 \cup \mathcal{C}^\bullet \cup \mathcal{C}_\bullet \cup {}^\bullet\mathcal{C} \cup {}_\bullet\mathcal{C}$, in order to calculate $\chi(\mathcal{O}_{\mathcal{C}})$ using Lemma 29, we only need to see what correction terms are needed at points $p_0$ and $q_0$.

First, consider $\chi(\mathcal{O}|_{e_0\cup\mathcal{C}^\bullet_\bullet})$.

Then from Lemma 28, the only difference in the Euler characteristic calculation comes from the difference in the contribution at $q_0$. We have:

$$\chi(\mathcal{O}|_{e_0\cup\mathcal{C}^\bullet_\bullet}) = \chi(\mathcal{O}|_{e_0\cup\mathcal{C}^\bullet}) + \chi(\mathcal{O}|_{e_0\cup\mathcal{C}_\bullet}) - ns, \tag{2.13}$$

where $n$ is the outside multiplicity of $e_1^\bullet$ and $s$ is the outside multiplicity of $e_{1\bullet}$.

From Lemma 29, we know that

$$\chi(\mathcal{O}|_{e_0 \cup \mathcal{C}^\bullet}) - \frac{1}{2}n^2 > 0, \quad \chi(\mathcal{O}|_{e_0 \cup \mathcal{C}_\bullet}) - \frac{1}{2}s^2 > 0.$$

Combining this with

$$\frac{1}{2}(n-s)^2 \geq 0 \Rightarrow \frac{1}{2}n^2 + \frac{1}{2}s^2 \geq ns,$$

Eq. (2.13) becomes

$$\chi(\mathcal{O}|_{e_0 \cup \mathcal{C}^\bullet_\bullet}) > 0 \Rightarrow \chi(\mathcal{O}|_{e_0 \cup \mathcal{C}^\bullet_\bullet}) \geq 1. \tag{2.14}$$

Notice that Lemma 29 in fact gives us that $\chi(\mathcal{O}|_{e_0 \cup \mathcal{C}^\bullet}) - \frac{1}{2}n^2 \geq \frac{1}{2}$, with equality if and only if all the conditions which imply $\chi(\mathcal{O}|_{e_0 \cup \mathcal{C}^\bullet}) = 1$, including $n = 1$, are satisfied. Similarly, $\chi(\mathcal{O}|_{e_0 \cup \mathcal{C}_\bullet}) - \frac{1}{2}s^2 \geq \frac{1}{2}$, with equality if and only if all the conditions which imply $\chi(\mathcal{O}|_{e_0 \cup \mathcal{C}_\bullet}) = 1$ hold, including $s = 1$. So we see that equality in the right hand side of Eq. 2.14 holds if and only if $\chi(\mathcal{O}|_{e_0 \cup \mathcal{C}^\bullet}) = \chi(\mathcal{O}|_{e_0 \cup \mathcal{C}_\bullet}) = 1$.

Recall that from Lemma 29, we know that $\chi(\mathcal{O}|_{e_0 \cup \mathcal{C}^\bullet}) \geq 1$. By the remark at the beginning of this proof, by changing labels, a similar result holds for any of the branches, so $\chi(\mathcal{O}|_{e_0 \cup \mathcal{C}_\bullet}) \geq 1$ also.

Using this, we see that equality in Eq. 2.14 holds if and only if $\chi(\mathcal{O}|_{e_0 \cup \mathcal{C}^\bullet}) = \chi(\mathcal{O}|_{e_0 \cup \mathcal{C}_\bullet}) = 1$.

A similar argument holds for $\chi(\mathcal{O}|_{e_0 \cup {}^\bullet_\bullet\mathcal{C}})$, and we also have $\chi(\mathcal{O}|_{e_0 \cup {}^\bullet_\bullet\mathcal{C}}) \geq 1$, with equality if and only if $\chi(\mathcal{O}|_{e_0 \cup {}^\bullet\mathcal{C}}) = \chi(\mathcal{O}|_{e_0 \cup {}_\bullet\mathcal{C}}) = 1$.

Now, in general, we have $\mathcal{C} = e_0 \cup \mathcal{C}^\bullet_\bullet \cup {}^\bullet_\bullet\mathcal{C}$, so applying Euler characteristic on the normalization exact sequence,

$$\chi(\mathcal{O}_\mathcal{C}) = \chi(\mathcal{O}|_{e_0 \cup \mathcal{C}^\bullet_\bullet}) + \chi(\mathcal{O}|_{e_0 \cup {}^\bullet_\bullet\mathcal{C}}) - \chi(\mathcal{O}_{e_0}) = \chi(\mathcal{O}|_{e_0 \cup \mathcal{C}^\bullet_\bullet}) + \chi(\mathcal{O}|_{e_0 \cup {}^\bullet_\bullet\mathcal{C}}) - 1 \geq 1.$$

Equality holds if and only if $\chi(\mathcal{O}|_{e_0 \cup \mathcal{C}^\bullet_\bullet}) = \chi(\mathcal{O}|_{e_0 \cup {}^\bullet_\bullet\mathcal{C}}) = 1$, which proves the lemma. $\square$

*Remark* 31. Lemma 25 now immediately follows from Lemma 30.

### 2.5.3 Proof of Corollary 26.

*Proof.* Given $\mathcal{D}$ a closed subscheme of a curve $\mathcal{C}$, such that $\mathcal{C}$ is a curve in $\Gamma$, we first claim that we may assume that $\mathcal{O}_{\mathcal{D}}$ is pure 1-dimension. If not, by primary decomposition, there is a maximal pure 1-dimensional subscheme $\mathcal{D}_1 \subset \mathcal{D}$. Then we can write

$$0 \to K_0 \to \mathcal{O}_{\mathcal{D}} \to \mathcal{O}_{\mathcal{D}_1} \to 0,$$

where $K_0$ is some zero dimensional sheaf. Then $\chi(\mathcal{O}_{\mathcal{D}}) = \chi(\mathcal{O}_{\mathcal{D}_1}) + \chi(K_0) \geq \chi(\mathcal{O}_{\mathcal{D}_1})$ because any zero dimensional sheaf has nonnegative Euler characteristic.

We may also assume that $\mathcal{D}$ is connected. Indeed, if $\mathcal{D} = \amalg_i \mathcal{D}_i$, then $\chi(\mathcal{O}_{\mathcal{D}}) = \sum_i \chi(\mathcal{O}_{\mathcal{D}_i})$.

The only case left to consider that is not already covered by Lemma 30 is when $\mathcal{D}$ is a connected pure 1 dimension curve in $\Gamma$ that does not contain $e_0$. But then we can define a new curve $\mathcal{D}' := \mathcal{D} \cup e_0$ by attaching $e_0$ to a torus fixed point of valence 1 and apply Lemma 30 to $\mathcal{D}'$. This gives us $1 \leq \chi(\mathcal{O}_{\mathcal{D}'}) = \chi(\mathcal{O}_{\mathcal{D}}) + 1 - m_1$, where $m_1 \geq 1$ is the inside thickening of the chosen attaching edge. So $\chi(\mathcal{O}_{\mathcal{D}}) \geq 1$ in all cases as claimed. $\square$

## 2.6 Combinatorics

### 2.6.1 Discussion

We summarise the results of the previous section and show how this leads to a generating function for the naive count of curves in $M^T_{\text{USG}}$. In Proposition 24, we showed that the sheaves in $M^T_{\text{USG}}$ are torus fixed structure sheaves of curves $\mathcal{C}$ with $\chi(\mathcal{O}_{\mathcal{C}}) = 1$. In the proof of Lemma 29 and Lemma 30, we computed the constraints this imposes on the multiple structure of $\mathcal{C}$ in order for equality to hold. This leads to the following:

**Proposition 32.** *Let $\mathcal{F} \in M_{\text{USG}}^{T}$ and $\operatorname{Supp}(\mathcal{F}) = \mathcal{C} = e_0 \cup \mathcal{C}^{\bullet} \cup \mathcal{C}_{\bullet} \cup {}^{\bullet}\mathcal{C} \cup {}_{\bullet}\mathcal{C}$. Let $\{e_i\} \not\ni e_0$ be the edges of any one of the four branches of $\mathcal{C}$. Then the multiple structures of the $\{e_i\}$ satisfy the following properties.*

1. *The inside multiplicity of any edge that intersects $e_0$ is unrestricted.*
2. *All nonzero outside multiplicities must be 1.*
3. *For each consecutive pair $(e_{2k-1}, e_{2k})$, the inside multiplicities of the second edge is equal to or one less than that of the first.*
4. *The inside multiplicities are non-increasing on each branch.*

*Proof.* By Proposition 24, we must have $\chi(\mathcal{O}_{\mathcal{C}}) = 1$. By Lemma 30, this holds if and only if $\chi(\mathcal{O}|_C) = 1$ for all of the subcurves $C \in \{e_0 \cup \mathcal{C}^{\bullet}, e_0 \cup \mathcal{C}_{\bullet}, e_0 \cup {}^{\bullet}\mathcal{C}, e_0 \cup {}_{\bullet}\mathcal{C}\}$. By symmetry, it suffices to study the constraints this imposes on any one of these branches.

We will choose to let $\mathcal{C} = e_0 \cup \mathcal{C}^{\bullet}$ and continue to use the same notation as in Lemma 29. A consecutive pair $(e_{2k-1}, e_{2k})$ has inside multiplicity $m_k, r_k$, in that order, and outside multiplicity $n_k, s_k$. We will interpret the conclusion of the lemmas to see how they imply the conditions above.

In both lemmas, $n_1$ and $s_\alpha$ must be 1 in order that $\chi(\mathcal{O}|_C) = 1$.

Consider the four summation terms in Eq. 2.9. In order for the second and third summation to be 0, we must have all $s_i = n_i$ for $1 \leq i \leq \alpha$ and all $n_{i+1} = s_i$ for $1 \leq i \leq \alpha - 1$. Together with $n_1 = 1$, this implies that all $n_i = 1$ and $s_i = 1$ for $1 \leq i \leq \alpha$.

This shows condition (2).

In order for the first summation to be 0, we must have either $r_i = m_i$ or $r_i + 1 = m_i$ for all $i$. This is equivalent to condition (3).

The fourth summation term is equal to zero only when $r_{i-1} \leq m_i$ for all $i$. This, along with condition (3), gives condition (4). □

We would like to count the curves that satisfy these constraints. The constraints on each branch curve are independent of the other branches, so it suffices to count the possible subcurves for any one of the types $\{e_0 \cup \mathcal{C}^\bullet,\ e_0 \cup \mathcal{C}_\bullet,\ e_0 \cup {}^\bullet\mathcal{C},\ e_0 \cup {}_\bullet\mathcal{C}\}$, and then change labels as necessary to get the counts on the other types.

First, we count the allowed curves on some fixed branch. Since the outside multiplicities must always be 1, the only choice is in the inside multiplicities. We can represent these lengths as boxes, where the number of boxes in each row corresponds to the multiplicity of the corresponding edge, Figure 2.14.

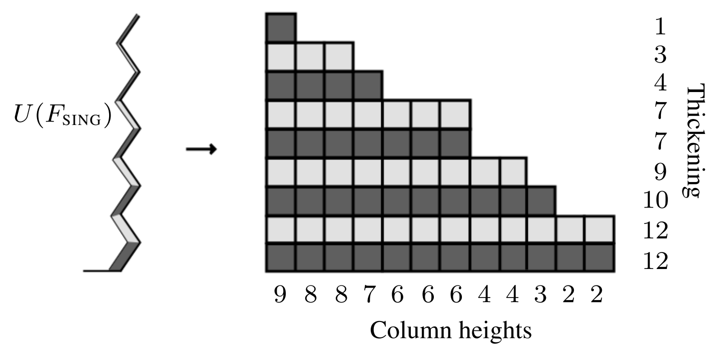


**Figure 2.14:** Multiple structure represented as a partition

Proposition 32 constrains the shape of this partition. Condition (1) says that the bottom row can be any length. Condition (4) means that the rows are non-increasing in length, so we have a Young diagram. Then if we view the Young diagram as a partition via its columns rather than its rows, condition (3) forces this partition to have odd parts distinct. We can visualize this by alternating row colors to highlight consecutive pairs as in Figure 2.14. Here, the dark capped columns give odd parts, and they occur singly since consecutive pairs have lengths that differ by at most one.

We need to keep track of the curve class that each partition represents. Edges along a given branch of $U(F_{\text{SING}})$ alternate between pushing forward to a multiple of $[C_1]$ and to $[C_2]$. In terms of our Young diagram, this means boxes of the same color correspond to the same curve class. The specific assignment of box color to curve class depends on the branch. The difference between the number of dark and light boxes is exactly the number of odd parts that appears in the partition.

We encode the previous discussion into a generating function. First, the number of integer partitions with *only* distinct odd parts $(\mathrm{ODOP})$ can be written using $q$ to track partitions and $t$ to track the number of odd parts [16, §2.5.21]:

$$\sum_{\lambda\in\mathrm{ODOP}} q^{|\lambda|}t^{OP(\lambda)} = \prod_{n=1}^{\infty}(1+tq^{2n-1}). \tag{2.15}$$

In this equation, $|\lambda|$ is the size of the ODOP partition $\lambda$, and $OP(\lambda)$ again denotes the number of odd parts in $\lambda$.

We are interested in partitions whose odd parts are distinct, but may have arbitrary even parts. The generating function for these odd parts distinct $(\mathrm{OPD})$ partitions is thus the following modification of Eq. (2.15):

$$\sum_{\lambda\in\mathrm{OPD}} q^{|\lambda|}t^{OP(\lambda)} = \prod_{n=1}^{\infty}\frac{(1+tq^{2n-1})}{(1-q^{2n})}. \tag{2.16}$$

Here, OPD are integer partitions with odd parts distinct, $|\lambda|$ is the size of the OPD partition $\lambda$, and $OP(\lambda)$ denotes the number of odd parts in $\lambda$.

On the other hand, we can express an $(\mathrm{OPD})$ partition using variables $x$ and $y$ that track the number of dark and light boxes, respectively, in our Young diagram:

$$\sum_{\lambda\in\mathrm{OPD}} q^{|\lambda|}t^{OP(\lambda)} = \sum_{\lambda\in\mathrm{OPD}} x^{\frac{1}{2}(|\lambda|+OP(\lambda))}y^{\frac{1}{2}(|\lambda|-OP(\lambda))}. \tag{2.17}$$

These expressions are related through the change of variables

$$q=\sqrt{xy}, \quad t=\sqrt{\frac{x}{y}}$$

so we can rewrite the right hand side of Eq. (2.16) as:

$$\prod_{n=1}^{\infty}\frac{1+x^ny^{n-1}}{1-x^ny^n}. \tag{2.18}$$

So far we have restricted the discussion to one branch. For the other branches, the counts have a similar expression, but the roles of $x$ and $y$ may be reversed, depending on whether the first edge covers $[C_1]$ or $[C_2]$. Therefore, the total count of curves satisfying Proposition 32 is

$$\prod_{n=1}^{\infty} \frac{(1+x^n y^{n-1})^2 (1+x^{n-1} y^n)^2}{(1-x^n y^n)^4}. \tag{2.19}$$

We have now proved the following:

**Proposition 33.** *The number of curves $\mathcal{C}$ satisfying the constraints in Proposition 32 can be expressed in terms of the number of partitions with distinct odd parts, namely,*

$$\sum_{d_1,d_2} \widetilde{n}^0_{\beta_{d_1,d_2}}(X_{\mathrm{BAN}}) x^{d_1} y^{d_2} = 12 \prod_{n=1}^{\infty} \frac{(1+x^n y^{n-1})^2 (1+x^{n-1} y^n)^2}{(1-x^n y^n)^4}. \tag{2.20}$$

*Remark* 34. The main result of the next section is to show that incorporating the Behrend function weighting into the Euler characteristic computation amounts to the following sign change:

$$n^0_{\beta_{d_1,d_2}}(X_{\mathrm{BAN}}) = (-1)^{d_1+d_2}\, \widetilde{n}^0_{\beta_{d_1,d_2}}(X_{\mathrm{BAN}}). \tag{2.21}$$

Together with the result of Proposition 33, this gives Eq. (2.6). This will then conclude the proof of our main result, Theorem 7.

## 2.7 Computing the Behrend function weighted Euler Characteristic

In this section we prove in Proposition 40 that the naive and Behrend function weighted Euler characteristics are related by a sign change as discussed in Remark 34.

Recall from Proposition 14 and the discussion preceding it, the sheaves in $M = M_\beta^{X_{\mathrm{BAN}}}$ are scheme-theoretically supported on fibers of $X_{\mathrm{BAN}} \to \mathbf{P}^1$, so, in particular, we have a fibration $\rho : M \to \mathbf{P}^1$. The group scheme $X^0 \to \mathbf{P}^1$ acts on $M \to \mathbf{P}^1$, preserving the Calabi-Yau form and the symmetric obstruction theory, and hence the Behrend function. So we only need to consider isolated fixed points in $[\mathcal{F}] \in M$, where $\mathcal{F}$ is supported on $F_{\mathrm{SING}}$.

In order to compute the Behrend function $\nu_M$, however, we need to study infinitesimal deformations into the whole space $X_{\text{BAN}}$. The tangent space to a fixed sheaf $[\mathcal{F}] \in M$, is given by

$$T_{[\mathcal{F}]}M = (\mathrm{Ext}^1_{X_{\text{BAN}}})_0(\mathcal{F}, \mathcal{F}).$$

Since we are only considering sheaves supported on $F_{\text{SING}}$, it suffices to compute this on $\widehat{F}_{\text{SING}}$, the formal completion of $X_{\text{BAN}}$ along $F_{\text{SING}}$, which is a formal toric Calabi-Yau threefold. The $\mathbf{C}^* \times \mathbf{C}^*$-action on $\widehat{F}_{\text{SING}}$ then allows us to use the Behrend-Fantechi result [4, Corollary 3.5].

More precisely, we have the following definition.

**Definition 35.** Let $\widehat{M}_{\text{SG}}$ be the formal scheme:

$$\widehat{M}_{\text{SG}} \text{ is the formal completion of } M \text{ along } M_{\text{SG}}.$$

Then the Behrend function satisfies [20],

$$(\nu_M)|_{M_{\text{SG}}} = (\nu_{\widehat{M}_{\text{SG}}})|_{M_{\text{SG}}}.$$

Recall that the action of the torus $T \cong \mathbf{C}^* \times \mathbf{C}^*$ on $F_{\text{SING}}$ came from the group scheme action on $X_{\text{BAN}}$. This torus action can be extended to an action on $\widehat{F}_{\text{SING}}$ [6, Lemma 4.5]. As a consequence, $\widehat{M}_{\text{SG}} \subset M$ inherits a $T$ action since $\widehat{M}_{\text{SG}}$ only depends on $\widehat{F}_{\text{SING}}$. Furthermore, this action is shown to preserve the symmetric obstruction theory on $\widehat{M}_{\text{SG}}$.

In the following Lemma 36, we show that the symmetric obstruction theory on $\widehat{M}_{\text{SG}}$ is also equivariant with respect to the group action induced from $P$. Then using [4, Corollary 3.5], the Behrend function weighted Euler characteristic of the moduli space depends only on the parity of the dimension of the tangent space at the fixed points $\widehat{M}^{\text{TP}}_{\text{SG}}$ of both actions,

$$e(\widehat{M}_{\text{SG}}, \nu_{\widehat{M}_{\text{SG}}}) = \sum_{[\mathcal{F}] \in \widehat{M}^{\text{TP}}_{\text{SG}}} (-1)^{\dim \mathrm{Ext}^1_{\widehat{F}_{\text{SING}}}(\mathcal{F}, \mathcal{F})}.$$

So all that will be left to do is to determine $\dim \operatorname{Ext}^1(\mathcal{F}, \mathcal{F})$ (mod 2) at the fixed points $[\mathcal{F}] \in \widehat{M}_{\text{SG}}^{\text{TP}}$.

**Lemma 36.** *The action of $P$ extends to an action on $\widehat{M}_{\text{SG}}$. Furthermore, the symmetric obstruction theory on $\widehat{M}_{\text{SG}}$ is equivariant with respect to this action.*

*Proof.* The action of $P \cong \mathbf{C}^* \times \mathbf{C}^*$ on $M_{\text{SG}}$ came from tensoring by degree 0 line bundles $L_{\boldsymbol{\mu}}$ supported on $F_{\text{SING}}$. By the same arguments as in Section 2.4, we also have $P \cong \mathbf{C}^* \times \mathbf{C}^* \subset \operatorname{Pic}^0(\widehat{F}_{\text{SING}})$. This induces an action of $P$ on the moduli space $\widehat{M}_{\text{SG}}$ as follows.

Given some $\boldsymbol{\mu} \in P$ corresponding to the flat line bundle $L_{\boldsymbol{\mu}}$ on $\widehat{F}_{\text{SING}}$, let $\mathcal{L}_{\boldsymbol{\mu}} := p_2^* L_{\boldsymbol{\mu}}$ where $p_i$ is projection to the $i$-th factor. Let $\mathcal{E}$ be the universal sheaf over $\widehat{M}_{\text{SG}} \times \widehat{F}_{\text{SING}}$.

$$\begin{array}{ccc} & \mathcal{E} & \\ & \downarrow & \\ & \widehat{M}_{\text{SG}} \times \widehat{F}_{\text{SING}} & \\ \swarrow^{p_1} & & \searrow^{p_2} \\ \widehat{M}_{\text{SG}} & & \widehat{F}_{\text{SING}} \end{array}$$

If we tensor $\mathcal{E}$ by $\mathcal{L}_{\boldsymbol{\mu}}$, this induces a map $\phi_\mu : \widehat{M}_{\text{SG}} \to \widehat{M}_{\text{SG}}$ by the universal property of $\mathcal{E}$ as in the diagram below. This gives an action of $P$ on $\widehat{M}_{\text{SG}}$ with $\mathcal{E}$ as an $P$-equivariant sheaf.

$$\begin{array}{ccc} \phi_\mu^* \mathcal{E} \cong \mathcal{E} \otimes \mathcal{L}_{\boldsymbol{\mu}} & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow \\ \widehat{M}_{\text{SG}} \times \widehat{F}_{\text{SING}} & \xrightarrow{\phi_\mu} & \widehat{M}_{\text{SG}} \times \widehat{F}_{\text{SING}}. \end{array}$$

From

$$\mathcal{H}om(\mathcal{E} \otimes \mathcal{L}_{\boldsymbol{\mu}}, \mathcal{E} \otimes \mathcal{L}_{\boldsymbol{\mu}}) \cong \mathcal{H}om(\mathcal{E}, \mathcal{E} \otimes \mathcal{L}_{\boldsymbol{\mu}} \otimes \mathcal{L}_{\boldsymbol{\mu}}^\vee) \cong \mathcal{H}om(\mathcal{E}, \mathcal{E}),$$

we get the canonical isomorphism

$$R\,\mathcal{H}om(\mathcal{E} \otimes \mathcal{L}_{\boldsymbol{\mu}}, \mathcal{E} \otimes \mathcal{L}_{\boldsymbol{\mu}}) \cong R\,\mathcal{H}om(\mathcal{E}, \mathcal{E}),$$

and thus

$$R\mathcal{H}om(\phi_\mu^*\mathcal{E}, \phi_\mu^*\mathcal{E}) \cong R\mathcal{H}om(\mathcal{E}, \mathcal{E}).$$

This implies that the shifted cone $\mathcal{F}$ of the trace map $R\mathcal{H}om(\mathcal{E}, \mathcal{E}) \to \mathcal{O}_{\widehat{M}_{\text{SG}} \times \widehat{F}_{\text{SING}}}$ in $D(\mathcal{O}_{\widehat{M}_{\text{SG}} \times \widehat{F}_{\text{SING}}})$ is preserved by $P$.

$$\begin{array}{ccc} & \mathcal{O}_{\widehat{M}_{\text{SG}} \times \widehat{F}_{\text{SING}}} & \\ {}^{+1}\swarrow & & \nwarrow^{tr} \\ \mathcal{F} & \longrightarrow & R\mathcal{H}om(\mathcal{E}, \mathcal{E}) \end{array}$$

All the constructions of the obstruction theory [4, Lemma 2.2]

$$E := R(p_1)_* R\mathcal{H}om(\mathcal{F}, \omega_{\widehat{F}_{\text{SING}}})[2] \to L_{\widehat{M}_{\text{SG}}},$$

as well as the nondegenerate symmetric bilinear form $\theta : E \to E^\vee[1]$ which is induced from $\omega_{\widehat{F}_{\text{SING}}} \cong \mathcal{O}_{\widehat{F}_{\text{SING}}} \to \mathcal{O}_{\widehat{F}_{\text{SING}}}$, are also equivariant. Hence the $P$-action is equivariant and symmetric, and preserves the symmetric obstruction theory on $\widehat{M}_{\text{SG}}$. □

### 2.7.1 Relating deformations of sheaves on $\widehat{F}_{\text{SING}}$ and $U(\widehat{F}_{\text{SING}})$

We will show that the dimension of $\text{Ext}^1(\mathcal{F}, \mathcal{F})$ for the fixed points $[\mathcal{F}] \in M_{\text{SG}}^{\text{TP}}$ has the same parity whether considered as sheaves on $\widehat{F}_{\text{SING}}$ or on $U(\widehat{F}_{\text{SING}})$. This implies that their Behrend function contributions to the Euler characteristic are the same, so we may calculate this on $U(\widehat{F}_{\text{SING}})$. We regard the fixed points as sheaves on the formal schemes, pushed forward under the respective inclusions $F_{\text{SING}} \hookrightarrow \widehat{F}_{\text{SING}}$ and $U(F_{\text{SING}}) \hookrightarrow U(\widehat{F}_{\text{SING}})$.

In Proposition 24, we showed that sheaves in $M_{\text{USG}}^T$ were possibly non-reduced structure sheaves $\mathcal{O}_{\mathcal{C}}$ of certain types of curves in $U(\widehat{F}_{\text{SING}})$. As explained in section 2.4, this corresponds to a point in $M_{\text{SG}}^{\text{TP}}$ by

$$M_{\text{SG}}^{\text{TP}} \ni \mathcal{F} \longleftrightarrow \widetilde{\mathcal{F}}_0 = \mathcal{O}_{\mathcal{C}} \in M_{\text{USG}}^T,$$

where the correspondence is given as

$$pr_*\widetilde{\mathcal{F}}_0 = \mathcal{F} \text{ and } pr^*\mathcal{F} = \widetilde{\mathcal{F}} = \bigoplus_{k,l\in \mathbf{Z}^2} (e_1^k e_2^l)^* \widetilde{\mathcal{F}}_0 \tag{2.22}$$

with the $G := \mathbf{Z} \times \mathbf{Z}$ action on $\mathrm{Coh}(U(\widehat{F}_{\mathrm{SING}}))$ covering the deck transformations.

**Proposition 37.** *For any $\mathcal{F} \in M_{\mathrm{SG}}^{TP}$, let $\mathcal{O}_{\mathcal{C}} \in M_{\mathrm{USG}}^{T}$ be the corresponding stable sheaf on $U(\widehat{F}_{\mathrm{SING}})$ so that $pr_*(\mathcal{O}_{\mathcal{C}}) = \mathcal{F}$. Then*

$$\mathrm{Ext}^1(\mathcal{F},\mathcal{F}) \cong \mathrm{Ext}^1(\mathcal{O}_{\mathcal{C}},\mathcal{O}_{\mathcal{C}}) \oplus \mathbf{C}^2.$$

*In particular, the dimensions of the deformation spaces have the same parity.*

*Proof.* Fix $\mathcal{F} \in M_{\mathrm{SG}}^{\mathrm{TP}}$. Recall the proof of Proposition 23. Under the general categorical equivalence of sheaves on $F_{\mathrm{SING}}$ with $G := \mathbf{Z} \times \mathbf{Z}$ equivariant sheaves on $U(\widehat{F}_{\mathrm{SING}})$, deformations of $\mathcal{F}$ correspond to deformations of the corresponding $G$-sheaf $pr^*\mathcal{F} = (\widetilde{\mathcal{F}}, \phi_1, \phi_2)$, $[\phi_1, \phi_2] = 0$. We can separate the deformations of the sheaf from the deformations of the lift of the action by considering the linear map between deformation spaces which forgets the equivariant part of the sheaf:

$$0 \to \mathrm{Ker} \to \mathrm{Def}(\widetilde{\mathcal{F}}, \phi_1, \phi_2) \to \mathrm{Def}(\widetilde{\mathcal{F}}_0) \to 0.$$

The kernel consists of deformations of the linear maps $\phi_i \in \mathrm{Hom}(\widetilde{\mathcal{F}}, e_i^*\widetilde{\mathcal{F}})$. These are given by pairs,

$$\{(\phi_1 + \epsilon\eta_1, \phi_2 + \epsilon\eta_2)\}, \qquad \eta_i \in \mathrm{Hom}(\widetilde{\mathcal{F}}, e_i^*\widetilde{\mathcal{F}}),\ \epsilon^2 = 0,$$

which cover the group action, so $[\phi_1 + \epsilon\eta_1, \phi_2 + \epsilon\eta_2] = 0$. In other words

$$\mathrm{Ker} = \mathrm{Def}(\phi_1, \phi_2) = \{(\eta_1, \eta_2) | [\eta_1, \phi_2] + [\eta_2, \phi_1] = 0\}$$

From Proposition 23 the sheaves in $M_{\mathrm{USG}}^{T}$ are of the special form satisfying Eq. (2.22), and so $\widetilde{\mathcal{F}} \cong e_i^*\widetilde{\mathcal{F}}$. Observe that in $\mathrm{Coh}^G U(\widehat{F}_{\mathrm{SING}})$ we can re-index, and then by equivariance and stability, we get

$$\mathrm{Hom}_G(\widetilde{\mathcal{F}}, e_i^*\widetilde{\mathcal{F}}) \cong \mathrm{Hom}_G(\widetilde{\mathcal{F}}, \widetilde{\mathcal{F}}) \cong \mathrm{Hom}(\widetilde{\mathcal{F}}_0, \widetilde{\mathcal{F}}_0) \cong \mathbf{C}.$$

So the commutator relation is trivial and $\{(\eta_1, \eta_2)\} = \mathbf{C} \times \mathbf{C}$. □

### 2.7.2 Computing deformations on $U(\widehat{F}_{\text{SING}})$

Let $\mathcal{C}$ be the support of a point $[\mathcal{O}_{\mathcal{C}}] \in M^T_{\text{USG}}$. To apply Proposition 37, we need to calculate the parity of the dimension of $\mathrm{Ext}^1(\mathcal{O}_{\mathcal{C}}, \mathcal{O}_{\mathcal{C}})$. We will do this by reducing our computation to the result in [27, Theorem 2]. We work in an ambient toric Calabi-Yau threefold, which we describe below.

For a fixed degree $\beta = (d_1, d_2, 1)$, the support $\mathcal{C}$ of any stable sheaf in $M^T_{\text{USG}}$ is contained in a finite type region of $U(\widehat{F}_{\text{SING}})$. Following the discussion in Subsection 2.3.1, such a region is formally locally isomorphic to some ambient smooth finite type toric Calabi-Yau threefold $W \subset \mathfrak{W}$, whose fan consists of the cones over the finitely many tiles of Figure 2.6 that contain $\mathrm{Supp}(\mathcal{C})$. We may thus compute the infinitesimal deformations of sheaves in $M^T_{\text{USG}}$ by considering them as sheaves on $W$.

**Definition 38.** Let $W$ be a smooth finite type toric Calabi-Yau threefold formally locally isomorphic to a formal neighborhood of $\mathrm{Supp}(\mathcal{C})$ for $\mathcal{C} \in M^T_{\text{USG}}$.

For the remainder of this section, we will work on the space $W$ for the computations of Ext and Hom groups. With this understanding, we will often suppress the subscript $W$ and write $\mathrm{Ext} := \mathrm{Ext}_W$ and $\mathrm{Hom} := \mathrm{Hom}_W$.

Furthermore, in [27] Maulik et al consider ideal sheaves, whereas we are interested in structure sheaves. So we will also need the following Lemma 39, but we defer its proof until after Proposition 40.

**Lemma 39.** $\mathrm{Ext}^1_W(\mathcal{O}_{\mathcal{C}}, \mathcal{O}_{\mathcal{C}}) \cong \mathrm{Ext}^1_W(\mathcal{I}_{\mathcal{C}}, \mathcal{I}_{\mathcal{C}})$*, where* $\mathcal{I}_{\mathcal{C}}$ *is the ideal sheaf of* $\mathcal{C}$ *in* $W$.

*Proof.* See following the proof of Proposition 40. □

**Proposition 40.** *The dimension of* $\mathrm{Ext}^1(\mathcal{O}_\mathcal{C}, \mathcal{O}_\mathcal{C})$ *is* $d_1 + d_2 \mod 2$.

*Proof.* We apply the formula of [27, Theorem 2] to compute the dimension of the tangent space. This result was proved with $T^3 \coloneqq (\mathbf{C}^*)^3$-equivariant cohomology. By equivariant Serre duality and restriction to the Calabi-Yau torus $T \cong (\mathbf{C}^*)^2 \subset T^3$ we get the equality:

$$\left.\frac{e(\mathrm{Ext}^1_W(\mathcal{I}_\mathcal{C}, \mathcal{I}_\mathcal{C}))}{e(\mathrm{Ext}^2_W(\mathcal{I}_\mathcal{C}, \mathcal{I}_\mathcal{C}))}\right|_T = (-1)^{\dim \mathrm{Ext}^1_W(\mathcal{I}_\mathcal{C}, \mathcal{I}_\mathcal{C})},$$

where

$$\dim \mathrm{Ext}^1_W(\mathcal{I}_\mathcal{C}, \mathcal{I}_\mathcal{C}) \equiv \chi(\mathcal{O}_\mathcal{C}) + \sum_{C_i} m_{C_i} d_{C_i} \pmod 2, \quad \mathcal{C} = \cup C_i.$$

The sum is taken over irreducible components $C_i$ in the support, each of which has normal bundle $\mathcal{O}(-m_i) \oplus \mathcal{O}(-2 + m_i)$ and length $d_{C_i}$. In our situation, we have that $\chi(\mathcal{C}) = 1$, all $m_{C_i} = 1$, and the total degree is $d_1 + d_2 + 1$. □

To complete the proof of Proposition 40, we need to prove Lemma 39. After some preliminary calculations, we will prove Lemma 39 by showing two separate isomorphisms, $\mathrm{Ext}^1(\mathcal{O}_\mathcal{C}, \mathcal{O}_\mathcal{C}) \cong \mathrm{Hom}(\mathcal{I}_\mathcal{C}, \mathcal{O}_\mathcal{C})$ and $\mathrm{Hom}(\mathcal{I}_\mathcal{C}, \mathcal{O}_\mathcal{C}) \cong \mathrm{Ext}^1(\mathcal{I}_\mathcal{C}, \mathcal{I}_\mathcal{C})$, which we deduce from different long exact sequences.

We begin with some preliminary observations that follow from our geometry.

**Lemma 41.** *With the notation as above, we have the following equations.*

$$\mathrm{Hom}(\mathcal{O}_W, \mathcal{O}_\mathcal{C}) = \mathrm{Hom}(\mathcal{O}_\mathcal{C}, \mathcal{O}_\mathcal{C}) = \mathbf{C}, \tag{2.23}$$

$$\mathrm{Ext}^1(\mathcal{O}_W, \mathcal{O}_\mathcal{C}) = 0, \tag{2.24}$$

$$\mathrm{Ext}^2(\mathcal{O}_\mathcal{C}, \mathcal{O}_W) = \mathrm{Ext}^1(\mathcal{O}_\mathcal{C}, \mathcal{O}_W) = \mathrm{Hom}(\mathcal{O}_\mathcal{C}, \mathcal{O}_W) = 0, \tag{2.25}$$

*Proof.* Eq. (2.23) follows from stability,

$$\mathrm{Hom}(\mathcal{O}_W, \mathcal{O}_\mathcal{C}) = \mathrm{Hom}(\mathcal{O}_\mathcal{C}, \mathcal{O}_\mathcal{C}) = H^0(\mathcal{O}_\mathcal{C}) = \mathbf{C}.$$

Then since our support curve $\mathcal{C}$ is assumed to have $\chi(\mathcal{O}_\mathcal{C}) = 1$, we get Eq. (2.24),

$$\mathrm{Ext}^1(\mathcal{O}_W, \mathcal{O}_\mathcal{C}) = H^1(\mathcal{O}_\mathcal{C}) = 0.$$

Next, for Eq. (2.25) we compute

$$\begin{aligned}
\mathrm{Ext}^2(\mathcal{O}_\mathcal{C}, \mathcal{O}_W) &= \mathrm{Ext}^1(\mathcal{O}_W, \mathcal{O}_\mathcal{C} \otimes K_W)^\vee && \text{by } T^3\text{-equivariant Serre duality} \\
&= \mathrm{Ext}^1(\mathcal{O}_W, \mathcal{O}_\mathcal{C})^\vee && \text{since } W \text{ is Calabi-Yau} \\
&= H^1(\mathcal{O}_\mathcal{C})^\vee && \\
&= 0 && \text{by Eq. (2.24).}
\end{aligned}$$

Similarly, we use $T^3$-equivariant Serre duality and $W$ being a Calabi-Yau threefold for the other equations:

$$\begin{aligned}
\mathrm{Ext}^1(\mathcal{O}_\mathcal{C}, \mathcal{O}_W) &= H^2(\mathcal{O}_\mathcal{C})^\vee = 0, && \text{since } C \text{ has dimension } 1. \\
\mathrm{Hom}(\mathcal{O}_\mathcal{C}, \mathcal{O}_W) &= H^3(\mathcal{O}_\mathcal{C})^\vee = 0 && \text{since } C \text{ has dimension } 1.
\end{aligned}$$

□

The first isomorphism we need to prove is the following.

**Lemma 42.** *Let the notation be as above. Then,*

$$\mathrm{Ext}^1(\mathcal{O}_\mathcal{C}, \mathcal{O}_\mathcal{C}) \cong \mathrm{Hom}(\mathcal{I}_\mathcal{C}, \mathcal{O}_\mathcal{C}) \tag{2.26}$$

*Proof.* We start with the exact sequence on $W$:

$$0 \to \mathcal{I}_\mathcal{C} \to \mathcal{O}_W \to \mathcal{O}_\mathcal{C} \to 0. \tag{2.27}$$

If we apply $\mathrm{Hom}(\,\bullet\,, \mathcal{O}_\mathcal{C})$ to Eq. (2.27), we get the long exact sequence:

$$\begin{aligned}
0 \to \mathrm{Hom}(\mathcal{O}_\mathcal{C}, \mathcal{O}_\mathcal{C}) &\to \mathrm{Hom}(\mathcal{O}_W, \mathcal{O}_\mathcal{C}) \to \mathrm{Hom}(\mathcal{I}_\mathcal{C}, \mathcal{O}_\mathcal{C}) \to \\
\to \mathrm{Ext}^1(\mathcal{O}_\mathcal{C}, \mathcal{O}_\mathcal{C}) &\to \mathrm{Ext}^1(\mathcal{O}_W, \mathcal{O}_\mathcal{C}) \to \cdots
\end{aligned} \tag{2.28}$$

Using Eq. (2.23) and Eq. (2.24) from Lemma 41 in the long exact sequence Eq. (2.28) gives

$$\mathrm{Hom}(\mathcal{I}_{\mathcal{C}}, \mathcal{O}_{\mathcal{C}}) \cong \mathrm{Ext}^1(\mathcal{O}_{\mathcal{C}}, \mathcal{O}_{\mathcal{C}})$$

as required. □

The second isomorphism is below.

**Lemma 43.** *Let the notation be as above. Then,*

$$\mathrm{Hom}(\mathcal{I}_{\mathcal{C}}, \mathcal{O}_{\mathcal{C}}) \cong \mathrm{Ext}^1(\mathcal{I}_{\mathcal{C}}, \mathcal{I}_{\mathcal{C}})$$

*Proof.* We start with the same exact sequence on $W$ as above:

$$0 \to \mathcal{I}_{\mathcal{C}} \to \mathcal{O}_W \to \mathcal{O}_{\mathcal{C}} \to 0. \tag{2.29}$$

This time we apply $\mathrm{Hom}(\,\bullet\,, \mathcal{O}_W)$ to Eq. (2.29) to get the long exact sequence:

$$\begin{aligned} 0 \to &\mathrm{Hom}(\mathcal{O}_{\mathcal{C}}, \mathcal{O}_W) \to \mathrm{Hom}(\mathcal{O}_W, \mathcal{O}_W) \to \mathrm{Hom}(\mathcal{I}_{\mathcal{C}}, \mathcal{O}_W) \to \\ \to &\mathrm{Ext}^1(\mathcal{O}_{\mathcal{C}}, \mathcal{O}_W) \to \mathrm{Ext}^1(\mathcal{O}_W, \mathcal{O}_W) \to \mathrm{Ext}^1(\mathcal{I}_{\mathcal{C}}, \mathcal{O}_W) \to \\ \to &\mathrm{Ext}^2(\mathcal{O}_{\mathcal{C}}, \mathcal{O}_W) \to \cdots \end{aligned} \tag{2.30}$$

Applying Lemma 41 to the long exact sequence Eq. (2.30) yields two isomorphisms,

$$\mathrm{Hom}(\mathcal{O}_W, \mathcal{O}_W) \cong \mathrm{Hom}(\mathcal{I}_{\mathcal{C}}, \mathcal{O}_W), \tag{2.31}$$

$$\mathrm{Ext}^1(\mathcal{O}_W, \mathcal{O}_W) \cong \mathrm{Ext}^1(\mathcal{I}_{\mathcal{C}}, \mathcal{O}_W). \tag{2.32}$$

Define the ring $R$ as

$$R \coloneqq \mathrm{Hom}(\mathcal{O}_W, \mathcal{O}_W) = H^0(\mathcal{O}_W).$$

Then Eq. (2.31) gives

$$\mathrm{Hom}(\mathcal{I}_{\mathcal{C}}, \mathcal{O}_W) \cong R. \tag{2.33}$$

The isomorphism $\mathrm{Hom}(\mathcal{I}_{\mathcal{C}}, \mathcal{O}_W) \cong R$ identifies the function $f \in R$ with the homomorphism given by multiplication by $f$,

$$\mathcal{I}_{\mathcal{C}} \xrightarrow{\bullet f} \mathcal{O}_W.$$

Also let $W_{\text{AFF}}$ be the affinization of $W$,

$$W_{\text{AFF}} \coloneqq \mathrm{Spec}\, R = \mathrm{Spec}\, H^0(\mathcal{O}_W).$$

In terms of the toric fans, the fan of $W$ is a refinement of that of $W_{\text{AFF}}$, and

$$W \xrightarrow{\pi} W_{\text{AFF}}$$

is a projective morphism. Hence,

$$\begin{aligned}
\mathrm{Ext}^1(\mathcal{O}_W, \mathcal{O}_W) &= H^1(W, \mathcal{O}_W) \\
&= H^1(W_{\text{AFF}}, \pi_*\mathcal{O}_W) \quad \text{by vanishing of higher direct image sheaves since} \\
&\qquad\qquad\qquad\qquad\quad \text{toric singularities are rational [11, Theorem 3.13]} \\
&= H^1(W_{\text{AFF}}, \mathcal{O}_{W_{\text{AFF}}}) \\
&= 0 \qquad\qquad\qquad\qquad\qquad\qquad \text{since } W_{\text{AFF}} \text{ is affine.}
\end{aligned} \tag{2.34}$$

Using Eq. (2.34) in Eq. (2.32), we get

$$\mathrm{Ext}^1(\mathcal{I}_{\mathcal{C}}, \mathcal{O}_W) = 0. \tag{2.35}$$

Finally, we apply $\mathrm{Hom}(\mathcal{I}_{\mathcal{C}}, \bullet)$ to Eq. (2.29) to get the long exact sequence:

$$\begin{aligned}
0 &\to \mathrm{Hom}(\mathcal{I}_{\mathcal{C}}, \mathcal{I}_{\mathcal{C}}) \to \mathrm{Hom}(\mathcal{I}_{\mathcal{C}}, \mathcal{O}_W) \to \mathrm{Hom}(\mathcal{I}_{\mathcal{C}}, \mathcal{O}_{\mathcal{C}}) \to \\
&\to \mathrm{Ext}^1(\mathcal{I}_{\mathcal{C}}, \mathcal{I}_{\mathcal{C}}) \to \mathrm{Ext}^1(\mathcal{I}_{\mathcal{C}}, \mathcal{O}_W) \to \cdots
\end{aligned} \tag{2.36}$$

Using Eq. (2.33), we have

$$\mathrm{Hom}(\mathcal{I}_{\mathcal{C}}, \mathcal{I}_{\mathcal{C}}) \hookrightarrow R \cong \mathrm{Hom}(\mathcal{I}_{\mathcal{C}}, \mathcal{O}_W).$$

But we also have $R \subset \mathrm{Hom}(\mathcal{I}_{\mathcal{C}}, \mathcal{I}_{\mathcal{C}})$ since any $f \in R$ gives a homomorphism $\mathcal{I}_{\mathcal{C}} \xrightarrow{\bullet f} \mathcal{I}_{\mathcal{C}}$. So we have,

$$\mathrm{Hom}(\mathcal{I}_{\mathcal{C}}, \mathcal{I}_{\mathcal{C}}) \cong R. \tag{2.37}$$

Now using Eq. (2.33), Eq.(2.37), and Eq.(2.35) in the long exact sequence Eq.(2.36), we conclude that

$$\mathrm{Hom}(\mathcal{I}_{\mathcal{C}}, \mathcal{O}_{\mathcal{C}}) \cong \mathrm{Ext}^1(\mathcal{I}_{\mathcal{C}}, \mathcal{I}_{\mathcal{C}})$$

□

*Proof (of Lemma 39).* Follows immediately from Lemma 42 and Lemma 43. □

# Chapter 3

# Genus Zero Gopakumar-Vafa invariants of multi-Banana configurations

The multi-Banana configuration $\widehat{F}_{\mathrm{MB}}$ is a local Calabi-Yau threefold of Schoen type. Namely, $\widehat{F}_{\mathrm{MB}}$ is a conifold resolution of $\widehat{I}_v \times_{\mathbf{D}} \widehat{I}_w$, where $\widehat{I}_v \to \mathbf{D}$ is an elliptic surface over a formal disc $\mathbf{D}$ with an $I_v$ singularity on the central fiber. An $I_1$ singular fiber in an elliptic fibration is a nodal rational curve, and an $I_v$ singular fiber, $v \geq 2$, is a cycle of $v$ rational curves with intersection matrix affine $A_{v-1}$.

We generalize the technique developed in our earlier paper to compute genus 0 Gopakumar-Vafa invariants of certain fiber curve classes. We illustrate the computation explicitly for $v = 1$ and $v = w = 2$. The resulting partition function can be expressed in terms of elliptic genera of $\mathbf{C}^2$, or classical theta functions, respectively.

## 3.1 Introduction

### 3.1.1 Background

Let $X$ be a quasi-projective Calabi-Yau threefold over $\mathbf{C}$, so that $X$ is smooth and $K_X \cong \mathcal{O}_X$. Fix a curve class $\beta \in H_2(X)$. Let $M = M_\beta^X$ be the moduli space of Simpson semistable [33], pure, 1-dimensional sheaves $\mathcal{F}$ with proper support on $X$ with $\mathrm{ch}_2(\mathcal{F}) = \beta^\vee$ and $\chi(\mathcal{F}) = 1$. The genus 0 Gopakumar-Vafa invariants $n_\beta^0(X)$ are defined mathematically by Katz [23]:

**Definition 44.** The genus 0 Gopakumar-Vafa (GV) invariants $n_\beta^0(X)$ of $X$ in curve class $\beta$ are defined as the Behrend function weighted Euler characteristics of the moduli space $M_\beta^X$.

$$n_\beta^0(X) = e(M_\beta^X, \nu) \coloneqq \sum_{k \in \mathbf{Z}} k \cdot e_{top}(\nu^{-1}(k)) \tag{3.1}$$

where $e_{top}$ is topological Euler characteristic and $\nu : M_\beta^X \to \mathbf{Z}$ is Behrend's constructible function [3].

In our previous paper [29], we computed the genus 0 Gopakumar-Vafa invariants of the Banana manifold, $X_{\mathrm{BAN}}$, a special kind of Schoen threefold, defined as the conifold resolution given by blowing up along the diagonal of the fiber product of a generic rational elliptic surface $S \to \mathbf{P}^1$ with itself :

$$X_{\mathrm{BAN}} \coloneqq \mathrm{Bl}_\Delta(S \times_{\mathbf{P}^1} S).$$

These results were consistent with the computation of the Donaldson-Thomas invariants of $X_{\mathrm{BAN}}$ obtained via topological vertex methods by Bryan [6].

In this paper, we use similar methods as before to obtain the genus 0 Gopakumar-Vafa invariants of certain fiber classes of related local Calabi-Yau threefolds, which we call multi-Banana configurations, and denote by $\widehat{F}_{\mathrm{MB}}$. Our motivation is to study the fiberwise contribution of these configurations, which exist as formal subschemes in special Schoen manifolds, (Section 3.2.1). Unlike in our previous paper, even the genus 0 Gopakumar-Vafa invariants associated to these configurations cannot be obtained by other methods at present. Additionally, the example configurations we study yield partition functions with modular properties that can be expressed succinctly.

We note the multi-Banana configuration $\widehat{F}_{\text{MB}}$ was studied by Ruddat in [32], where he calls them *perverse curves* and investigates their mirror symmetry. Our results appear to be compatible with results that appear in the physics literature [18, Section 3.3].

### 3.1.2 The multi-Banana configuration $\widehat{F}_{\text{MB}}$

The twelve singular fibers $F_{\text{BAN}}$ of the regular Banana manifold $X_{\text{BAN}}$ are normalizations of the product of $I_1$ singular fibers with themselves,

$$F_{\text{BAN}} \cong \text{Bl}_{\Delta}(I_1 \times I_1) \subset X_{\text{BAN}}.$$

Let $\widehat{F}_{\text{BAN}}$ be the formal completion of $X_{\text{BAN}}$ along $F_{\text{BAN}}$. Each $F_{\text{BAN}}$ is isomorphic to a non-normal toric variety whose normalization is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$ blown up at two points on the diagonal. We have $\pi_1(\widehat{F}_{\text{BAN}}) = \mathbf{Z} \times \mathbf{Z}$. See [29, Section 3.1] for details.

We define the local Calabi-Yau threefold $\widehat{F}_{\text{MB}}$ as follows:

**Definition 45.** The multi-Banana $F^{vw}_{\text{MB}}$ and the local multi-Banana configuration $\widehat{F}^{vw}_{\text{MB}}$ are the étale covers of $F_{\text{BAN}}$ and $\widehat{F}_{\text{BAN}}$, respectively,

$$F^{vw}_{\text{MB}} \to F_{\text{BAN}},$$

$$\widehat{F}^{vw}_{\text{MB}} \to \widehat{F}_{\text{BAN}},$$

associated to the subgroup $v\mathbf{Z} \times w\mathbf{Z} \subset \mathbf{Z} \times \mathbf{Z}$.

We sometimes suppress the decoration and write $F_{\text{MB}}$ and $\widehat{F}_{\text{MB}}$ instead. Observe that $F_{\text{BAN}} = F^{11}_{\text{MB}}$ and $\widehat{F}_{\text{BAN}} = \widehat{F}^{11}_{\text{MB}}$.

The geometry of multi-Banana configurations was studied by Kanazawa and Lau [21]. In particular, $\widehat{F}^{vw}_{\text{MB}}$ has $vw + 2$ curve classes, generated by three families of curves, $\{A_i\}$, $\{B_j\}$, and $\{C_k\}$, see section 3.2.2:

$$\beta \in \sum_{i=0}^{w-1} \mathbf{Z}\,[A_i] \oplus \sum_{j=0}^{v-1} \mathbf{Z}\,[B_j] \;\oplus \sum_{k=0}^{(v-1)(w-1)} \mathbf{Z}\,[C_k], \qquad \beta \in H_2(\widehat{F}_{\mathrm{MB}}^{vw}).$$

### 3.1.3 Main results

In some cases of small $v$ and $w$, the GV invariants have nice formulas. We can express the partition function in terms of $\phi_Q(p)$, the unique weak Jacobi form of weight -2 and index 1,

$$\phi_Q(p) = p^{-1}(1-p)^2 \prod_{m=1}^{\infty} \frac{(1-Q^m p^{-1})^2(1-Q^m p)^2}{(1-Q^m)^4},$$
$$Q = \exp(2\pi i \tau), \qquad p = \exp(2\pi i z), \qquad (\tau, z) \in \mathrm{H} \times \mathbf{C}.$$

and $\mathrm{Ell}_{Q,p}(\mathbf{C}^2, t)$, the equivariant elliptic genus of $\mathbf{C}^2$:

$$\mathrm{Ell}_{Q,p}(\mathbf{C}^2, t) = \frac{\sqrt{\phi_Q(pt)\phi_Q(p^{-1}t)}}{\phi_Q(t)}.$$

When $v = 1$, we have the following.

**Theorem 46.** *(See Theorem 53 for details and notation.) Fix a curve class $\beta_{(\boldsymbol{a},\, c)}$ in the local multi-Banana $\widehat{F}_{\mathrm{MB}} = \widehat{F}_{\mathrm{MB}}^{1w}$:*

$$\beta_{(\boldsymbol{a},\, c)} = \sum_{i=0}^{w-1} a_i[A_i] + c[C] + [B],$$
$$\boldsymbol{a} = (a_0, \ldots, a_{w-1}) \in \mathbf{Z}_{\geq 0}^{\,w},\; c \in \mathbf{Z}_{\geq 0}.$$

*Then the genus 0 Gopakumar-Vafa invariants $n^0_{\beta_{(\boldsymbol{a},c)}}(\widehat{F}_{\mathrm{MB}}^{1w})$ can be expressed* *as:*

$$\sum_{\boldsymbol{a},c} n^0_{\beta_{(\boldsymbol{a},\,c)}}(\widehat{F}_{\mathrm{MB}})\boldsymbol{r}^{\boldsymbol{a}} s^c = s\cdot\phi_Q(s)\sum_{i=0}^{w-1}\prod_{k=i}^{i+w-2}\mathrm{Ell}_{Q,s}(\mathbf{C}^2, R_{i;k}),$$

*where*

$$Q \coloneqq \prod_{i=0}^{w-1}(r_i s),$$
$$R_{a;b} \coloneqq r_a\cdot r_{a+1}\cdot r_{a+2}\cdots r_b\cdot s^{b-a+1},\quad a\le b,$$
$$r_{k+w}\coloneqq r_k.$$

In the case of $v = w = 2$, the curve classes are naturally labelled as $A_0, A_1, B_0, B_1, C_0, C_1$. We have the following result:

**Theorem 47.** *(See Theorem 52 for details and notation.) Let $v = w = 2$, and fix a curve class $\beta_{(\boldsymbol{a},\,\boldsymbol{c})}$ in the local multi-Banana $\widehat{F}_{\mathrm{MB}} = \widehat{F}^{22}_{\mathrm{MB}}$:*

$$\beta_{(\boldsymbol{a},\,\boldsymbol{c})} = a_0[A_0] + a_1[A_1] + c_0[C_0] + c_1[C_1] + [B_0],$$
$$\boldsymbol{a} = (a_0, a_1),\ \boldsymbol{c} = (c_0, c_1)\in \mathbf{Z}^{\,2}_{\ge 0}.$$

*Then the genus 0 Gopakumar-Vafa invariants $n^0_{\beta_{(\boldsymbol{a},\boldsymbol{c})}}(\widehat{F}_{\mathrm{MB}})$ are given by the following:*

$$\sum_{a_0,a_1,c_0,c_1} n^0_{\beta_{(\boldsymbol{a},\boldsymbol{c})}}(\widehat{F}_{\mathrm{MB}}) r_0^{a_0} r_1^{a_1} s_0^{c_0} s_1^{c_1} = 2\left[\frac{\phi_Q(r_0)\phi_Q(s_0)\phi_Q(r_1)\phi_Q(s_1)}{\phi_Q(r_0 s_0)\phi_Q(r_1 s_1)}\right]^{1/2},$$

*where*

$$Q \coloneqq r_0 r_1 s_0 s_1$$
$$\phi_Q(p) \coloneqq \phi_{-2,1}(Q,p).$$

*Remark* 48. We note that the appearance of the elliptic genera in the partition function of the multi-Banana suggests a correspondence via geometric engineering [24] to partition functions of Yang-Mills gauge theories on surfaces. This viewpoint is discussed further in the previously cited physics literature [18].

### 3.1.4 Outline of method

We recall the method we used in [29] to compute the genus 0 GV invariants of $X_{\mathrm{BAN}}$. The argument carries over largely unchanged for the local multi-Banana configurations $X = \widehat{F}_{\mathrm{MB}}$, apart from the final combinatorics computation, so we refer the reader to our previous paper for the details of the proofs of the statements in this summary of our method.

We have a $T \coloneqq \mathbf{C}^* \times \mathbf{C}^*$ torus action on $F_{\mathrm{MB}}$, given by translation on the smooth locus, and which extends to an action on all of $\widehat{F}_{\mathrm{MB}}$. This gives us an action on its coherent sheaves $\mathrm{Coh}(\widehat{F}_{\mathrm{MB}})$ and thus on the moduli space $M_\beta^{\widehat{F}_{\mathrm{MB}}}$. This action preserves the canonical class and is compatible with the symmetric obstruction theory. We can use the motivic nature of the Behrend function weighted Euler characteristic to stratify the moduli space under this group action [3, 4]. The nontrivial torus orbits make no contribution to $e(M_\beta^{\widehat{F}_{\mathrm{MB}}}, \nu)$, and we can reduce to considering only the $T$-fixed points of the moduli space $(M_\beta^{\widehat{F}_{\mathrm{MB}}})^T$.

We first count the fixed points of the moduli space. This gives us the naive Euler characteristic, $\widetilde{n}_\beta^0(\widehat{F}_{\mathrm{MB}})$, which we define as the Euler characteristic of the moduli space without the Behrend function weighting:

$$\widetilde{n}_\beta^0(\widehat{F}_{\mathrm{MB}}) \coloneqq e(M_\beta^{\widehat{F}_{\mathrm{MB}}}).$$

Using stability arguments we show that the sheaves in our moduli space have scheme-theoretic support on the multi-Banana surface $F_{\mathrm{MB}}$ [29, Proposition 12]. Thus, for computing $\widetilde{n}_\beta^0(\widehat{F}_{\mathrm{MB}})$, it suffices to count $T$-invariant sheaves of $F_{\mathrm{MB}}$.

We would like to work on the universal cover of a multi-Banana, $U(F_{\mathrm{MB}})$, to make the computations easier. This is an infinite type toric surface, whose irreducible components are isomorphic to the blow-up of $\mathbf{P}^1 \times \mathbf{P}^1$ at two torus fixed points. We give further details of the local geometry in Section 3.2.2. The universal cover $U(F_{\mathrm{MB}})$ is the same as that of the regular Banana fiber $U(F_{\mathrm{BAN}})$ considered in [29].

In order to relate the sheaves of $U(F_{\mathrm{MB}})$ with those of $F_{\mathrm{MB}}$, we introduce another torus action, which we denote by $P \coloneqq \mathbf{C}^* \times \mathbf{C}^*$. This $P$ action on $\mathrm{Coh}(F_{\mathrm{MB}})$

is defined by tensoring with degree 0 line bundles of $F_{\mathrm{MB}}$, as $\mathrm{Pic}^0(F_{\mathrm{MB}}) \cong \mathbf{C}^* \times \mathbf{C}^*$ [29, Section 4]. Again, the Euler characteristic contribution can be computed on orbits of the action, and it then suffices to consider only sheaves invariant under the two $\mathbf{C}^* \times \mathbf{C}^*$ actions, $T$ and $P$. The $T$ torus action also lifts to give an action on the universal cover and its sheaves.

Any sheaf fixed under the $P$ action can be realized as a direct image of an equivariant sheaf on $U(F_{\mathrm{MB}})$ [29, Proposition 22]. This equivariant sheaf contains a distinguished subsheaf which pushes forward to the original sheaf, and is unique up to deck transformations. Moreover stability, Euler characteristic 1, and invariance under the $T$ torus action is preserved in this correspondence.

The requirements of stability and Euler characteristic equal to 1 then puts restrictions on the allowed invariant stable sheaves. If we further specify that the curve class $\beta$ has degree exactly 1 in one of the curve families of $F_{\mathrm{MB}}^{vw}$, then all the $T$ and $P$ fixed sheaves in our moduli space correspond to structure sheaves of possibly non-reduced curves on $U(F_{\mathrm{MB}})$ [29, Proposition 23]. The multiplicity of each component is constrained [29, Proposition 31] by a condition, which is equivalent to requiring that the partition given by multiplicities of successive rational components from the fixed central degree 1 curve has a conjugate partition with odd parts that are distinct. We give the specific details of this condition in Section 3.4.

This count of the number of fixed points of the moduli space gives the naive Euler characteristic, $\widetilde{n}^0_\beta(\widehat{F}_{\mathrm{MB}})$. However, the Behrend function weighting amounts to a sign $(-1)^{\deg \beta}$, which depends on the total degree of the curve [29, Proposition 39], and this can be incorporated into the partition function.

We note that our technique is limited to computing invariants associated to fiber class curves such that the degree of one of these families is fixed to be 1. We do not yet know how to extend the technique to arbitrary degrees.

Our method allows us to calculate the partition function for $\widehat{F}_{\mathrm{MB}}^{vw}$ in the general case, for arbitrary $v$ and $w$. However, as $v$ and $w$ increase, there will be unavoidable linear relations among the curve classes, even after fixing the degree of one curve type to be 1. We will only present in detail the $2 \times 2$ case (Section 3.4) and the $1 \times w$ case (Section 3.5) as they illustrate the ideas sufficiently without the notation becoming burdensome.

## 3.2 Geometry

In this section, we give two examples of multi-Banana configurations $\widehat{F}_{\text{MB}}$ that exist as formal neighborhoods of surfaces inside compact Calabi-Yau threefolds. We then discuss some of the local geometry of multi-Banana configurations needed for the following sections.

### 3.2.1 Global geometry

**Definition 49.** A multi-Banana manifold $X_{\text{MB}}$ is a smooth Calabi-Yau threefold which is a conifold resolution of the fiber product of two rational elliptic surfaces, and such that the formal neighborhood of each singular fiber is a multi-Banana configuration.

*Example* 50. Let $S \xrightarrow{\pi} \mathbf{P}^1$ be a rational elliptic surface with singular fibers consisting of four $I_1$ and four $I_2$ singular fibers. Suppose $S$ has a 2-torsion section. This induces an order 2 automorphism $\phi_2$ that interchanges the nodes of each of the $I_2$ fibers.

We can then form the fiber product of $S$ with itself, $S \times_{\mathbf{P}^1} S$. In order to get a conifold resolution, we blow up the generalized diagonal $\widetilde{\Delta}$, consisting of the diagonal $\Delta$, as well as all its translates by iterations of $\phi_2$,

$$\widetilde{\Delta} \coloneqq (\phi_2^{(i)} \times \phi_2^{(j)})\Delta, \quad 0 \leq i, j < 2.$$

We will call this multi-Banana manifold $X_{22}$.

$$X_{22} \coloneqq \mathrm{Bl}_{\widetilde{\Delta}}(S \times_{\mathbf{P}^1} S)$$

In this case, the multi-Banana contains four $F_{\text{MB}}^{22}$ configurations, and four ordinary Banana fibers $F_{\text{BAN}}$.

Instead of taking the fiber product of $S$ with itself, we can also do the following construction.

*Example* 51. Let $S \xrightarrow{\pi} \mathbf{P}^1$ be a rational elliptic surface with two $I_1$ and two $I_5$ singular fibers. Then $S$ has a 5-torsion section, which induces an order 5 automorphism $\phi_5$ which acts on each $I_5$ fiber by cycling the nodes.

Now, let us take the quotient of $S$ by the action of $\phi_5$, and let $S'$ be the resolution of the quotient:

$$S' \coloneqq \text{Res}\,(S/\phi_5).$$

Notice that by construction, $S' \xrightarrow{\pi'} \mathbf{P}^1$ is another rational elliptic surface with singular fibers over the same base points:

$$\begin{aligned}\mathbf{P}^1_{\text{SING}} \coloneqq &\{p \in \mathbf{P}^1 | \pi^{-1}(p) \subset S \text{ singular}\} \\ =&\{p \in \mathbf{P}^1 | \pi'^{-1}(p) \subset S' \text{ singular}\}.\end{aligned}$$

We also have that the smooth fibers of $S$ and $S'$ are isogenous:

$$\begin{gathered}\phi_5|_p : \pi^{-1}(p) \to \pi'^{\,-1}(p), \\ \phi_5|_p \text{ is an isogeny}, \forall p \in \mathbf{P}^1 \backslash \mathbf{P}^1_{\text{SING}}.\end{gathered}$$

In this case, there is a conifold resolution of the fiber product, $S \times_{\mathbf{P}^1} S'$. From the construction, we have a rational map of schemes over $\mathbf{P}^1$,

$$S \dashrightarrow S' = \text{Res}\,(S/\phi_k),$$

so we get a graph

$$\overline{\Gamma} \subset S \times_{\mathbf{P}^1} S'.$$

Then the conifold resolution is given by blowing up this graph $\overline{\Gamma}$. We will call this multi-Banana threefold $X_{15}$:

$$X_{15} \coloneqq \text{Bl}_{\overline{\Gamma}}(S \times_{\mathbf{P}^1} S').$$

The multi-Banana manifold $X_{15}$ is a rigid Calabi-Yau threefold and contains two $F^{15}_{\text{MB}}$ multi-Banana configurations, and two $F^{51}_{\text{MB}}$ multi-Banana configurations.

### 3.2.2 Local geometry

We now examine the local geometry of the multi-Banana configurations in more detail and establish some notation we will need later.

We recall the construction from [21] and relate it to our discussion.

Let $L$ be a tiling of the plane $(x, y, 1) \subset \mathbf{R}^3$ given by:

$$\{x = m, z = 1\} \cup \{y = n, z = 1\} \cup \{y - x = r, z = 1\},$$
$$m, n, r \in \mathbf{Z}\,,$$
$$(x, y, z) \in \mathbf{R}^3.$$

Let $\mathcal{A}$ be the non-finite type toric threefold whose fan consists of all the cones over the proper faces of $L$ (Figure 3.1). Let $U(F_{\text{MB}})$ be the universal cover of $F_{\text{MB}}$, and $U(\widehat{F}_{\text{MB}})$ the universal cover of $\widehat{F}_{\text{MB}}$,

$$U(F_{\text{MB}}) \xrightarrow{pr} F_{\text{MB}}$$
$$U(\widehat{F}_{\text{MB}}) \xrightarrow{pr} \widehat{F}_{\text{MB}}.$$

Then $U(F_{\text{MB}}) \subset \mathcal{A}$ is the union of the toric divisors of $\mathcal{A}$, and $U(\widehat{F}_{\text{MB}})$ is the formal completion of $\mathcal{A}$ along $U(F_{\text{MB}})$.

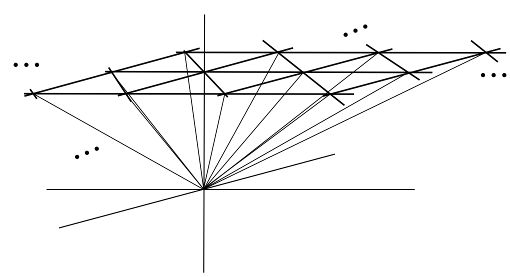

**Figure 3.1:** The fan of $\mathcal{A}$.

We have an action of $G = v\mathbf{Z} \times w\mathbf{Z}\,, v, w \in \mathbf{Z}_{\geq 0}$, on $L \subset \mathbf{R}^3$ by translation:

$$(v, w) \cdot (x, y, 1) = (x + v, y + w, 1).$$

which induces an automorphism $\psi_G : \mathcal{A} \to \mathcal{A}$ and also on $U(\widehat{F}_{\mathrm{MB}})$. These are then the deck transformations of the universal cover of the local multi-Banana configuration $\widehat{F}_{\mathrm{MB}}$:

$$U(\widehat{F}_{\mathrm{MB}}) \to U(\widehat{F}_{\mathrm{MB}})/\psi_G \cong \widehat{F}_{\mathrm{MB}}.$$

We denote by $\Xi$ the irreducible surface which is the momentum polytope of $\mathbf{P}^1 \times \mathbf{P}^1$ blown up at 2 points, and drawn as a hexagon in our diagrams:

$$\Xi := \text{momentum polytope of } \left(\mathrm{Bl}_{p_1,p_2}(\mathbf{P}^1 \times \mathbf{P}^1)\right), \quad p_1, p_2 \in \mathbf{P}^1 \times \mathbf{P}^1.$$

Then the momentum polytope of $U(F_{\mathrm{MB}})$ can be represented as a hexagonal tiling of the plane, and the momentum polytope of $F_{\mathrm{MB}}$ is given by $v \times w$ hexagons glued together along their toric boundary, as depicted in the example of Figure 3.2.

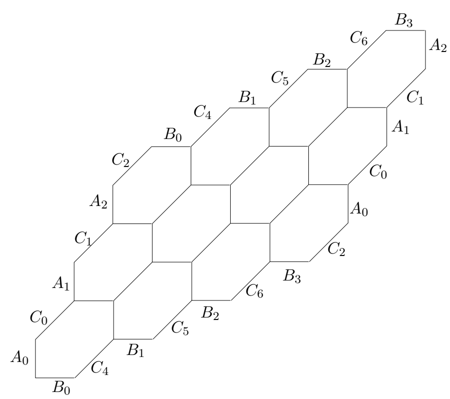


**Figure 3.2:** $F_{\mathrm{MB}}$ in the case $v = 3$ and $w = 4$. Here, the top boundary curves are identified with those along the bottom, and also the left edge with the right edge.

The irreducible components of the torus fixed curves in $F_{\mathrm{MB}}$ fall into three families of rational curves. Two of these families, $\{A_i\}$ and $\{B_j\}$, are proper transforms of the rational curves from the $I_v$ and $I_w$ singular fibers in $F_{\mathrm{MB}}$, respectively, and one family, $\{C_k\}$, are the exceptional curves of the conifold resolution.

We will draw these curves oriented as shown in Figure 3.3, so the vertical curves are in the $A$ family, the horizontal curves are $B$ family, and the diagonal curves are from the $C$ family.

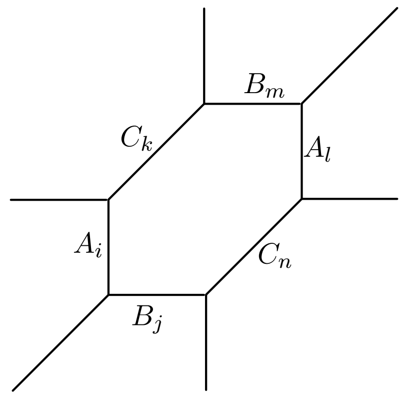


**Figure 3.3:** Curve labels for hexagon $\Xi$

When there is no confusion, we will also label irreducible components of the lifts to the universal cover of these torus invariant curves with the curve class of their projection to $F_{\text{MB}}$. That is, an irreducible component of $pr^{-1}(A_i) \subset U(F_{\text{MB}})$ will also be referred to as $A_i$ in the universal cover.

As each hexagon surface $\Xi$ is the momentum polytope of $\mathbf{P}^1 \times \mathbf{P}^1$ blown up at 2 points, their 6 boundary divisors have 2 relations. For example, in the labeled Figure 3.3, we have:

$$\begin{aligned} A_i + C_k &= A_l + C_n, \\ B_m + C_k &= B_j + C_n. \end{aligned} \tag{3.2}$$

from the equivalent ways to express the total transform of the rulings in each $\mathbf{P}^1$ factor.

A priori, there are $3vw$ torus invariant irreducible curves in $F_{\text{MB}}$, but these satisfy standard hexagon relations (Eq. 3.2), so it is possible to choose a basis of $vw+2$ curves, consisting of $v \times A$ curves, $w \times B$ curves, and $(v-1)(w-1)+1 \times C$ curves. For the small examples we consider, we will index the curves in each family in a simple way. It is possible to use a systematic choice of generators and labels for the general case [21, Section 5.1], but it would be notationally cumbersome for these examples, so we do not present that here.

As remarked in the introduction, our technique is limited to considering only curves where we restrict the degree of one family of curves to be exactly 1. For concreteness, we will assume our curves are of class

$$\beta = \sum a_i[A_i] + [B_0] + \sum c_k[C_k]. \tag{3.3}$$

The hexagonal tiling from $U(F_{\text{MB}})$ possesses a $v\mathbf{Z} \times w\mathbf{Z}$ periodicity from the deck transformations. In order to get rid of the ambiguity from the deck transformations, we will assume we have fixed a choice of fundamental domain $D$, and any curve we consider has its unique irreducible component covering $B_0$ inside $\overline{D}$. In other words, we will require that $T$-torus invariant curves $\mathcal{C} \subset U(F_{\text{MB}})$ with $[pr(\mathcal{C})] = \beta$ also satisfy $\mathcal{C} \cap pr^{-1}(B_0) \subset \overline{D}$.

From the arguments given in Subsection 3.1.4, in order to compute the naive Euler characteristic $\widetilde{n}^0_\beta(\widehat{F}_{\text{MB}})$, it suffices to count all configurations of possibly non-reduced $T$-torus invariant curves covering $\beta$ on the universal cover $U(F_{\text{MB}})$, subject to the constraint that the partition given by multiplicities of successive rational components of each tree emanating from $B_0$ has a conjugate partition which has odd parts that are distinct. In section 3.4 and 3.5, we will illustrate this count in two specific cases, namely when the fundamental domain consists of $2 \times 2$ hexagons, and also the case of $1 \times w$ hexagons. These configurations exist, for example, in $X_{22}$, and $X_{15}$, respectively, as described in the previous Section 3.2.1.

## 3.3 Notation and conventions

We gather in this section the conventions we use for product and sum expansions for Jacobi forms and elliptic genera.

Recall, the weak Jacobi form $\phi_{-2,1}(q,p)$ of weight -2 index 1 is defined as

$$\phi_{-2,1}(q,p) = p^{-1}(1-p)^2 \prod_{m=1}^{\infty} \frac{(1-q^m p^{-1})^2(1-q^m p)^2}{(1-q^m)^4},$$

The Jacobi theta function $\theta_1(q,p)$ function is given as

$$\begin{aligned}\theta_1(q,p) &= -\sum_{k\in\mathbf{Z}+\frac{1}{2}} q^{\frac{k^2}{2}}(-p)^k \\ &= -iq^{\frac{1}{8}}p^{-\frac{1}{2}}\prod_{m=1}^{\infty}(1-q^m)(1-q^{m-1}p)(1-q^m p^{-1})\end{aligned}$$

and the Dedkind $\eta$ function is

$$\eta(q) = q^{\frac{1}{24}}\prod_{m=1}^{\infty}(1-q^m).$$

Here,

$$q = \exp(2\pi i\tau), \qquad p = \exp(2\pi i z), \qquad (\tau,z)\in \mathrm{H}\times\mathbf{C}.$$

Since the first variable will be constant within our partition functions, we will use the shortened notation,

$$\begin{aligned}\phi_Q(p) &\coloneqq \phi_{-2,1}(Q,p) \\ \theta_Q(p) &\coloneqq \theta_1(Q,p) \\ \eta_Q &\coloneqq \eta(Q)\end{aligned}$$

Treating these expressions as formal power series, it is easy to verify the identities :

$$\begin{aligned}\sqrt{\phi_Q(p)} &= \frac{i\theta_Q(p)}{\eta_Q^3}, \\ \sqrt{p\,\phi_Q(p)} &= \sqrt{\frac{Q}{p}\,\phi_Q(\frac{Q}{p})}, \\ \sqrt{\phi_Q(p)} &= -\sqrt{\phi_Q(p^{-1})}.\end{aligned} \tag{3.4}$$

Suppose $M$ is a non-compact complex manifold of dimension $d$ with a $\mathbf{C}^*$ action with isolated fixed points $\{x\}$ of tangent weights $k_i$. We define the equivariant elliptic genus of $M$ to be:

$$\mathrm{Ell}_{q,y}(M,t) = \sum_{x \in M^{\mathbf{C}^*}} \prod_{j=1}^{d} y^{-\frac{1}{2}} \prod_{m=1}^{\infty} \frac{(1 - q^{m-1} y t^{-k_j(x)})(1 - q^m p^{-1} t^{k_j(x)})}{(1 - q^{m-1} t^{-k_j(x)})(1 - q^m t^{k_j(x)})}.$$

In particular ([35, Theorem 12]), we have:

$$\begin{aligned} \mathrm{Ell}_{q,y}(\mathbf{C}^2, t) &= \frac{\theta_1(q, yt)\theta_1(q, yt^{-1})}{\theta_1(q,t)\theta_1(q,t^{-1})} \\ &= \frac{\sqrt{\phi_{-2,1}(q, yt)\phi_{-2,1}(q, y^{-1}t)}}{\phi_{-2,1}(q,t)} \\ &= \frac{\sqrt{\phi_Q(yt)\phi_Q(y^{-1}t)}}{\phi_Q(t)}. \end{aligned} \tag{3.5}$$

## 3.4 Case $2 \times 2$

In this section, we study the case of $\widehat{F}_{22} := \widehat{F}^{22}_{\mathrm{MB}}$, and we use this example to illustrate in detail how the method described in the Section 3.1.4 leads to the computation of the Gopakumar-Vafa invariants.

### 3.4.1 $T$-Torus fixed curves on $\widehat{F}_{22}$

We first fix a choice of a fundamental domain $D$ in $U(F_{22})$ so that there is no ambiguity in our counts due to deck transformations.

We label the curves of the $2 \times 2$ hexagons of the momentum polytope of the fundamental domain in $U(F_{22})$ with our convention explained in Section 3.2.2. The vertical curves cover curves in the $A$ family, the horizontal curves those in the $B$ family, and the diagonal curves cover the $C$ family as shown in Figure 3.4. There is a $2\mathbf{Z} \times 2\mathbf{Z}$ periodicity of this fundamental domain in the universal cover.

We can choose a basis for the homology classes of curves of $F_{22}$ given by the 6 curves $A_0, A_1, B_0, B_1, C_0$, and $C_1$, as labelled in Figure 3.4. This can be shown using simple applications of the standard hexagon relations (Eq. 3.2) as follows.

First observe that the sum of the the $C$ curves in each row is constant, as is the

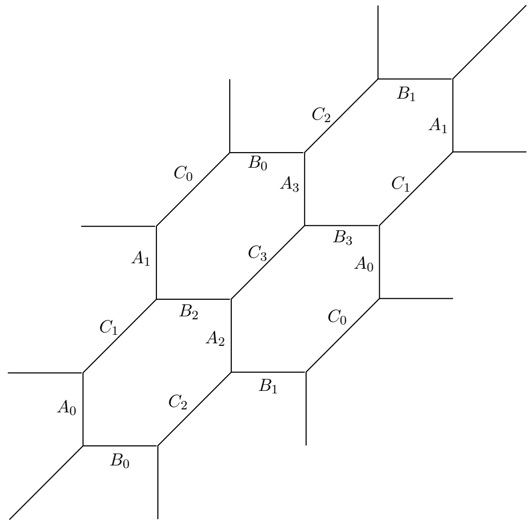


**Figure 3.4:** $2 \times 2$ hexagon momentum polytope of the fundamental domain in $U(F_{22})$.

sum in each column,

$$\begin{aligned} C_0 + C_2 &= C_1 + C_3, \\ C_0 + C_1 &= C_2 + C_3. \end{aligned} \tag{3.6}$$

This follows by combining two hexagon relations to the bottom row of hexagons in Fig. 3.4. For example,

$$\begin{aligned} A_0 + C_1 &= A_2 + C_2, \\ A_0 + C_0 &= A_2 + C_3. \end{aligned}$$

yields $C_0 + C_2 = C_1 + C_3$. The other relation is derived similarly.

We can also write the sum of all the diagonal curves in two ways, grouped as rows or as columns. In this case when $I = J = 2$, we have:

$$(C_0 + C_1) + (C_2 + C_3) = (C_0 + C_2) + (C_1 + C_3),$$

Together with the previous Eq. 3.6, this implies that

$$C_1 = C_2,$$
$$C_0 = C_3.$$

In a similar fashion, it is easy to deduce that

$$A_0 = A_2,$$
$$A_1 = A_3,$$
$$B_0 = B_2,$$
$$B_1 = B_3.$$

We can thus choose a basis for the homology classes of curves given by the 6 curves $A_0, A_1, B_0, B_1, C_0, C_1$.

We are interested in sheaves invariant under the action of the torus $T$, so their support must be contained in the $T$-torus fixed curves. We will assume from now on that the support curve $C$ of our sheaf has $\deg B_0 = 1$ and $\deg B_1 = 0$ so that it is in the homology class:

$$[C] = \beta \in \sum_{i=0,1} a_i[A_i] + [B_0] + \sum_{j=0,1} c_j[C_j]. \tag{3.7}$$

Recall there is a 1-1 correspondence between the $\pi_1(F_{22})$-equivariant sheaves on $U(F_{\text{MB}})$ and the $P$-fixed sheaves on $F_{\text{MB}}$ up to deck transformations [29, Proposition 22]. To remove the ambiguity, we will further require that the corresponding equivariant sheaf in $U(F_{\text{MB}})$ has support curve $\mathcal{C} \subset U(F_{\text{MB}})$, whose reduced irreducible component covering the curve in class $[B_0]$ is in our chosen fundamental domain $D$. In this example, notice that there are two possible choices for $B_0$, since $[B_0] = [B_2]$.

The irreducible components of the curve $\mathcal{C}$ may be nonreduced, so we need to keep track of the multiplicity of each component to determine the curve class of $pr(\mathcal{C})$. We let the variable $r_i$ track the number of curves, counted with multiplicity, which cover $A_i$, $i \in \{0, 1\}$. Let $s_j$ track the number of curves, counted with multiplicity, which cover $C_j$, $j \in \{0, 1\}$. A typical curve $\mathcal{C}$ which covers a curve in class Eq. (3.7) is pictured in Figure 3.5.

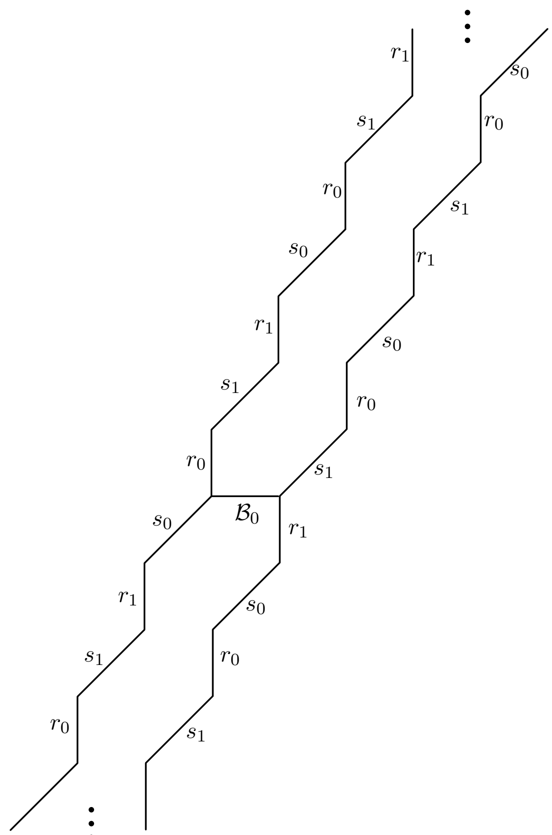


**Figure 3.5:** A typical torus fixed curve in $U(F_{22})$ which covers a curve with $\deg[B] = \deg[B_0] = 1$. Here $r_i$ and $s_j$ are used to track the multiplicity of curves which cover $A_i$ and $C_j$ curves, respectively.

### 3.4.2 Translating to combinatorics

We explain the details of our method of converting the count of invariant stable sheaves into a combinatorics problem.

Recall from the discussion in Section 3.1.4 the naive Euler characteristic $\widetilde{n}^0_\beta(\widehat{F}_{22}) = e(M_\beta^{\widehat{F}_{22}})$ for curves in class $\beta$ of the form Eq. (3.7) equals a count of $T$-torus invariant structure sheaves of genus 0 curves on the universal cover $\mathcal{C} \subset U(F_{\text{MB}})$ that cover class $\beta$ [29, Proposition 23], subject to the certain conditions [29, Proposition 31] on their multiplicity that we explain below.

First we introduce some terminology. We will refer to irreducible components of $\mathcal{C}$ as edges, and the intersection of two or more edges as vertices. Let $\mathcal{B}_0 \subset \mathcal{C}$ be the edge that covers class $[B_0]$ and lies in the fundamental domain $D$ by assumption.

We will call the union of $\mathcal{B}_0$ with any one of the four disjoint subcurves of $\overline{\mathcal{C}\backslash\mathcal{B}_0}$ a branch of $\mathcal{C}$. Then the curve counts can be done on each branch separately, and will be the same on each branch, up to relabeling.

The edge $\mathcal{B}_0$ is the intersection of two irreducible surface component hexagons isomorphic to $\Xi$ in $U(F_{\text{MB}})$. Let $S$ be one of these and let $g$ be the deck transformation that translates $S$ into the other. The hexagons $g^m S$, $m \in \mathbf{Z}$, in the orbit of $S$ under the group of deck transformations $\langle g \rangle \cong \mathbf{Z}$ will be called inside hexagons. Any other hexagons will be called outside hexagons.

Any edge of $\overline{\mathcal{C}\backslash\mathcal{B}_0}$ covers $A_i$ or $C_j$, $i, j \in \{0, 1\}$, and will be the intersection of an inside hexagon and an outside one. These $T$-invariant edges can can have monomial thickening in these two directions. Any thickenings in the direction of the inside hexagon will be called inside thickenings, and those in the direction of the outside hexagon will be called outside thickenings. However, because of stability and the requirement of the Euler characteristic of $\mathcal{O}_{[\mathcal{C}]}$ to be 1, the possible thickenings can only be of a particular form.

Thickenings on the edges that cover $A_i$ or $C_j$ are subject to the following properties [29, Proposition 31]:

1. Inside thickenings of any edge that intersects $\mathcal{B}_0$ is unrestricted.

2. All nonzero outside thickenings must be 1.

3. Inside thickenings are non-increasing on components along a branch in the direction moving away from $\mathcal{B}_0$.

4. Inside thickenings for two adjacent edges contained in a common inside

hexagon can either be the same or differ by one.

We can interpret the inside multiplicity of each edge as length of a part in a partition. These constraints are independent on each branch, so we examine one branch at a time. Along each branch, the non-increasing length condition says that the allowed multiplicities of edges form a Young diagram. If we examine the conjugate partition of the branch, the fourth condition can be interpreted as saying that any odd parts that appear in the conjugate partition are distinct, with no restriction on the even parts.

The generating function that counts the number of partitions $p(n)$ with odd parts distinct can be written as the product of generating functions for partitions with arbitrary even parts with that of partitions with unique odd parts:

$$\sum p(n)q^n = \prod \frac{1}{1-q^{2n}} \prod (1+q^{2n-1})$$

We must further refine the parts, because we have four curve classes $A_i$ and $C_j$, for $i, j \in \{0, 1\}$ to keep track of. In other words, we need to keep track of the residue classes mod 4 in our partition.

Consider for example the northeast branch of the curve shown above, which we reproduce in Figure 3.6. Suppose we number the edges consecutively, starting with the first edge $e_1$ that intersects $\mathcal{B}_0$. Then the every odd-numbered edge will contribute to an odd part, and the even-numbered ones to an even part. The first edge $e_1$ in this example curve covers $C_1$, and we assign the variable $s_1$ to track this curve. The second edge $e_2$ covers $A_0$ and we use the variable $r_0$ to track this.

So we refine the generating function above, and replace powers of the variable $q$, by

$$\begin{aligned} q^1 &\mapsto s_1 \\ q^2 &\mapsto s_1 r_0 \\ q^3 &\mapsto s_1 r_0 s_0 \\ q^4 &\mapsto s_1 r_0 s_0 r_1, \end{aligned}$$

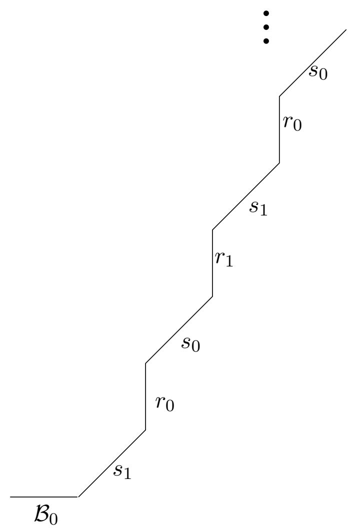


**Figure 3.6:** Detail of the northeast branch of the curve shown in Figure 3.5.

and for higher powers of $q$, we continue the pattern, so that

$$q^{4m+i} = q^{4m}q^i \mapsto (s_1 r_0 s_0 r_1)^m q^i$$

We also use the notation

$$Q \coloneqq r_0 s_0 r_1 s_1.$$

Now, the generating function that counts the number of partitions with odd parts distinct can be expressed for the northeast branch as:

$$\frac{(1+s_1)(1+s_0r_0s_1)(1+Qs_1)(1+Qs_0r_0s_1)\cdots}{(1-r_0s_1)(1-Q)(1-Qr_0s_1)(1-Q^2)\cdots}$$
$$= \frac{(1+s_1)\prod_{m=1}^{\infty}(1+Q^m s_1)(1+Q^m r_1^{-1})}{(1-r_0s_1)\prod_{m=1}^{\infty}(1-Q^m)(1-r_0s_1Q^m)}$$

We do this for each branch and multiply the contribution from all four branches. Notice also that there are two distinct possible locations for the curve $B_0$. However, they give the same contribution to the generating function, since the four branches in either location consist of the same sequence of curves, up to renaming of the branches. This accounts for the '2' that appears in the formula below.

Hence, using the identities (3.4), the partition function can be expressed in terms of the theta function $\theta_Q(p)$ as:

$$2i\eta_Q^{-6}\,\frac{\theta_Q(-r_0)\theta_Q(-s_0)\theta_Q(-r_1)\theta_Q(-s_1)}{\theta_Q(r_0s_0)\theta_Q(r_1s_1)}. \tag{3.8}$$

or in terms of the weak Jacobi form $\phi_Q(p)$ as:

$$2\left[\frac{\phi_Q(-r_0)\phi_Q(-s_0)\phi_Q(-r_1)\phi_Q(-s_1)}{\phi_Q(r_0s_0)\phi_Q(r_1s_1)}\right]^{1/2} \tag{3.9}$$

As we explained in the introduction, this count of the fixed points corresponds to the naive Euler characteristic contribution, $\widetilde{n}^0_\beta(\widehat{F}_{\mathrm{MB}})$,

$$\sum_{\substack{a_0,a_1\\c_0,c_1}} \widetilde{n}^0_{\beta_{(\mathbf{a},\mathbf{c})}}(\widehat{F}_{\mathrm{MB}})r_0^{a_0}r_1^{a_1}s_0^{c_0}s_1^{c_1} = 2\left[\frac{\phi_Q(-r_0)\phi_Q(-s_0)\phi_Q(-r_1)\phi_Q(-s_1)}{\phi_Q(r_0s_0)\phi_Q(r_1s_1)}\right]^{1/2}. \tag{3.10}$$

However, the Behrend function weighting amounts to a sign that depends on the degree of the curve class [29, Remark 33]:

$$\widetilde{n}^0_{\beta_{(\mathbf{a},\mathbf{c})}}(\widehat{F}_{\mathrm{MB}}) = (-1)^{a_0+a_1+c_0+c_1}n^0_{\beta_{(\mathbf{a},\mathbf{c})}}(\widehat{F}_{\mathrm{MB}}).$$

We can incorporate this sign by replacing our tracking variables by their negatives.

Hence, we have shown the following.

**Theorem 52.** *Fix a curve class $\beta_{(\boldsymbol{a},\boldsymbol{c})}$ in the local multi-Banana $\widehat{F}_{\mathrm{MB}} = \widehat{F}^{22}_{\mathrm{MB}}$,*

$$\beta_{(\boldsymbol{a},\boldsymbol{c})} = a_0[A_0] + a_1[A_1] + c_0[C_0] + c_1[C_1] + [B_0],$$
$$\boldsymbol{a} = (a_0, a_1),\ \boldsymbol{c} = (c_0, c_1) \in \mathbf{Z}^2_{\geq 0}.$$

*Then the genus 0 Gopakumar-Vafa invariants $n^0_{\beta_{(\boldsymbol{a},\boldsymbol{c})}}(\widehat{F}_{\mathrm{MB}})$ are given by the following:*

$$\sum_{\substack{a_0,a_1\\c_0,c_1}} n^0_{\beta_{(\boldsymbol{a},\boldsymbol{c})}}(\widehat{F}_{\mathrm{MB}}) r_0^{a_0} r_1^{a_1} s_0^{c_0} s_1^{c_1} = 2\left[\frac{\phi_Q(r_0)\phi_Q(s_0)\phi_Q(r_1)\phi_Q(s_1)}{\phi_Q(r_0 s_0)\phi_Q(r_1 s_1)}\right]^{1/2},$$

*where we use the notation*

$$Q := r_0 r_1 s_0 s_1$$
$$\phi_Q(p) := \phi_{-2,1}(Q, p).$$

*and $\phi_Q(p) = \phi_{-2,1}(Q,p)$ is the unique weak Jacobi form of weight -2 and index 1:*

$$\phi_Q(p) = p^{-1}(1-p)^2 \prod_{m=1}^{\infty} \frac{(1-Q^m p^{-1})^2 (1-Q^m p)^2}{(1-Q^m)^4},$$
$$Q = \exp(2\pi i \tau), \qquad p = \exp(2\pi i z), \qquad (\tau, z) \in \mathrm{H} \times \mathbf{C}.$$

## 3.5 Case $1 \times w$

In this section, we look in detail at the case of $F^{1w}_{\mathrm{MB}}$, when $v = 1$ and $w \geq 1$.

In this case, the fundamental domain in $U(F^{1w}_{\mathrm{MB}})$ has a $\mathbf{Z} \times w\mathbf{Z}$ periodicity in the universal cover. Its momentum polytope is given by $1 \times w$ hexagons. Using the hexagon relations (Eq. 3.2) as in the previous section, it is easy to see there is only one horizontal curve class, which we call $B$, and one diagonal curve class, which we call $C$. There are $w$ distinct vertical curve classes, $A_i, 0 \leq i \leq w-1$, (Figure 3.7).

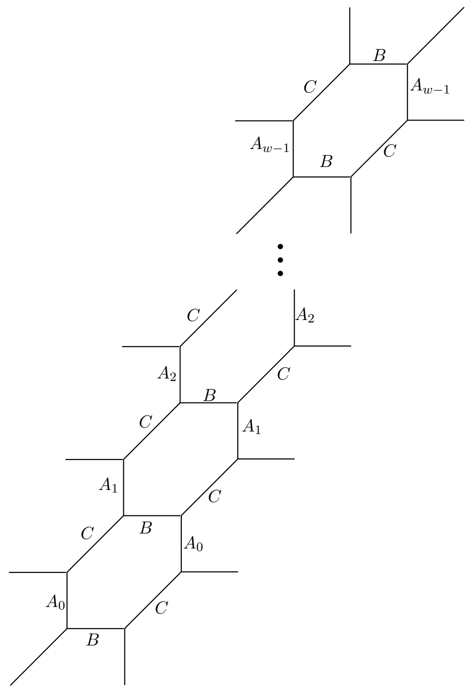


**Figure 3.7:** $1 \times w$ hexagon momentum polytope of the fundamental domain in $U(F_{\text{MB}}^{1w})$.

Let us assume that the support curve $\mathcal{C}$ of our sheaf has $\deg B = 1$ so that

$$[\mathcal{C}] \in \sum_{i=0}^{w} a_i[A_i] + [B] + c[C].$$

Let $r_i$ track the number of $A_i$ curves and $s$ track the number of $C$ curves. For this case, we define the variable $Q$ to be:

$$Q \coloneqq \prod_{i=0}^{w-1} (r_i s). \tag{3.11}$$

We will also use the following multi product notation:

$$R_{a;b} \coloneqq r_a \cdot r_{a+1} \cdot r_{a+2} \cdots r_b \cdot s^{b-a+1}, \quad a \leq b, \tag{3.12}$$

where the subscript of $r$ is interpreted mod $w$:

$$r_{k+w} \coloneqq r_{[k]}, \quad [k] \in \mathbf{Z}/w\mathbf{Z}.$$

In particular,

$$R_{0;b} \coloneqq \prod_{i=0}^{b} (r_i s).$$

First, suppose the single $B$ curve is located connected to an $A_0$ curve. Then, we can count the number of partitions with odd parts distinct in the same way as before, and the generating function for these configurations is expressed as follows:

$$(1+s)^2 \prod_{m=1}^{\infty} \frac{(1+sQ^m)^2(1+s^{-1}Q^m)^2}{(1-Q^m)^4} \times \prod_{k=0}^{w-2} \left[ \frac{(1+sR_{0;k})(1+s^{-1}R_{0;k})}{(1-R_{0;k})^2} \right.$$
$$\left. \times \prod_{m=1}^{\infty} \frac{(1+sR_{0;k}Q^m)(1+s^{-1}R_{0;k}Q^m)(1+sR_{0;k}^{-1}Q^m)(1+s^{-1}R_{0;k}^{-1}Q^m)}{(1-R_{0;k}Q^m)^2(1-R_{0;k}^{-1}Q^m)^2} \right]$$

We can write this more succinctly using the weak Jacobi form $\phi_Q(p)$ as:

$$(-s)\phi_Q(-s) \prod_{k=0}^{w-2} \left[ \frac{\sqrt{\phi_Q(-sR_{0;k})}\sqrt{\phi_Q(-sR_{0;k})}}{\phi_Q(R_{0;k})} \right]; \tag{3.13}$$

or alternatively, in terms of the theta function $\theta_Q(p)$ as:

$$(-s)\phi_Q(-s) \prod_{k=0}^{w-2} \left[ \frac{\theta_Q(-sR_{0;k})\theta_Q(-sR_{0;k}^{-1})}{\theta_Q(R_{0;k})\theta_Q(R_{0;k}^{-1})} \right]. \tag{3.14}$$

Notice that, from Eq. (3.5), the product can be expressed in terms of the equivariant elliptic genus of $\mathbf{C}^2$,

$$(-s)\phi_Q(-s) \prod_{k=0}^{w-2} \mathrm{Ell}_{Q,-s}(\mathbf{C}^2, R_{0;k}). \tag{3.15}$$

There are $w$ different locations possible for the $B$ curve in the $1 \times w$ hexagon, characterized by the choice of which $A_i$ curve, $0 \le i \le w-1$, that the $B$ curve is connected to. Although the generating function for the partitions with distinct odd parts associated to these other configurations depends on the particular location of $B$, it is easy to see that it differs from the previous formula only by a cyclic shift of indices in the $R_{0;k}$ variable.

The total partition function counts the contribution from all possible locations of the $B$ curve, and is thus expressed as a sum over the generating functions from each location,

$$(-s)\phi_Q(-s)\sum_{i=0}^{w-1}\prod_{k=i}^{i+w-2}\mathrm{Ell}_{Q,-s}(\mathbf{C}^2, R_{i;k}). \tag{3.16}$$

As in the previous section, this count of fixed points corresponds to the naive Euler characteristic. To take account of the Behrend function weighting, we can incorporate a sign based on the degree of the curve class by simply replacing our tracking variables by their negatives.

Thus, for the case of $\widehat{F}^{1w}_{\mathrm{MB}}$, we have the following partition function.

**Theorem 53.** *Fix a curve class $\beta_{(\boldsymbol{a},\,c)}$ in the local multi-Banana $\widehat{F}_{\mathrm{MB}} = \widehat{F}^{1w}_{\mathrm{MB}}$:*

$$\beta_{(\boldsymbol{a},\,c)} = \sum_{i=0}^{w-1} a_i[A_i] + c[C] + [B],$$
$$\boldsymbol{a} = (a_0, \ldots, a_{w-1}) \in \mathbf{Z}^{\,w}_{\geq 0},\ c \in \mathbf{Z}_{\geq 0}.$$

*Then the genus 0 Gopakumar-Vafa invariants $n^0_{\beta_{(\boldsymbol{a},c)}}(\widehat{F}^{1w}_{\mathrm{MB}})$ can be expressed as:*

$$\sum_{\boldsymbol{a},c} n^0_{\beta_{(\boldsymbol{a},c)}}(\widehat{F}_{\mathrm{MB}})\boldsymbol{r}^{\boldsymbol{a}} s^c = s\cdot\phi_Q(s)\sum_{i=0}^{w-1}\prod_{k=i}^{i+w-2}\mathrm{Ell}_{Q,s}(\mathbf{C}^2, R_{i;k}).$$

*where* $\mathrm{Ell}_{q,y}(\mathbf{C}^2, t)$ *is the equivariant elliptic genus of* $\mathbf{C}^2$*, and we use the notation:*

$$\boldsymbol{r}^{\boldsymbol{a}} \coloneqq r_0^{a_0} r_1^{a_1} \dots r_{w-1}^{a_{w-1}},$$
$$Q \coloneqq \prod_{i=0}^{w-1} (r_i s),$$
$$R_{a;b} \coloneqq r_a \cdot r_{a+1} \cdot r_{a+2} \cdots r_b \cdot s^{b-a+1}, \quad a \le b,$$
$$r_{k+w} \coloneqq r_{[k]}, \quad [k] \in \mathbf{Z}/w\mathbf{Z}.$$

We also mention that it is possible to choose to fix the degree of the $A$ family of curve classes or the $C$ class to be 1 instead of the $B$ curve. However, in this $1 \times w$ case, doing so reduces to the ordinary Banana configuration $F_{\text{BAN}}$ case and yields the same formula as in the earlier paper [29].

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