Index estimates and compactness for constant mean curvature surfaces and Steklov eigenvalues

by

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Abstract

In this work, we focus on three problems. First, we give a relationship between the number of eigenvalues of the Jacobi operator below a certain threshold and the topology of closed constant mean curvature (CMC) surfaces in three-dimensional Riemannian manifolds. We then obtain that the (weak) Morse index of CMC surfaces in an arbitrary 3-manifold is bounded below by a linear function of the genus when the constant mean curvature is greater than a certain nonnegative value. In particular, this implies that stable CMC surfaces are topological spheres. Corresponding results for CMC surfaces with free boundary in 3-manifolds with boundary are obtained as well. Second, we consider the space of embedded free boundary CMC surfaces with bounded topology, bounded area, and bounded boundary length in a 3-manifold N with boundary. We show that this space is almost compact in the sense that any sequence of surfaces in this space has a convergent subsequence that converges to a free boundary CMC surface, graphically and smoothly except on a finite set of singularities. If in addition $Ric_N > 0$ and the boundary of N is convex, then the convergence is at most 2-sheeted. In particular, it is 1-sheeted if the limiting surface is not a minimal surface. Third, we consider the maximization of Steklov eigenvalues in higher dimensions. We show that for compact manifolds of dimension at least 3 with nonempty boundary, we can modify the manifold by performing surgeries of codimension 2 or higher, while keeping the Steklov spectrum nearly unchanged. This shows that certain changes in the topology of a domain do not have an effect when considering shape optimization questions for Steklov eigenvalues in dimension 3 and higher.

Lay Summary

CMC surfaces are critical of area functional for compactly supported variations that preserve the enclosed volume. Free boundary CMC surfaces are a special class of CMC surfaces that meet the boundary of the manifold orthogonally.

The Morse index is defined to be the number of "directions" in which one can perturb the surface to decrease its area up to second order. First, we relate the topology of a CMC surface with its Morse index. Second, we show that the space of free boundary CMC surfaces of 3-manifolds with bounded topology and mean curvature is weakly compact.

The Steklov eigenvalue is an important topic in spectral geometry and geometric analysis. Third, we show that certain aspects of the topology of a domain do not have an effect when considering maximization of Steklov eigenvalues in dimensions 3 or higher, in contrast to the case in dimension 2.

Preface

This thesis is based on three works, one of them has been published in an academic journal, and the other two are submitted for publication and under review.

The content discussed in Chapter 2 is based on the paper "Index estimates for surfaces with constant mean curvature in 3-dimensional manifolds" [4] published in the journal *Calculus of Variations and Partial Differential Equations*, Volume 60, 3 (2021), pages 1-20. This is a joint work with Nicolau S. Aiex under the guidance of my supervisors, Jingyi Chen and Ailana Fraser.

The content in Chapter 3 is based on the paper "Compactness of the space of free boundary CMC surfaces with bounded topology" [3], which has been submitted to an academic journal. This was a joint work with Nicolau S. Aiex under the guidance of my supervisors, Jingyi Chen and Ailana Fraser.

The content in Chapter 4 is based on the paper "Higher dimensional surgery and Steklov eigenvalues" [39], which has been submitted to an academic journal. I chose this problem under guidance of my supervisors, Jingyi Chen and Ailana Fraser, and was responsible for all aspects of this work.

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Acknowledgments

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Chapter 1

Introduction

This work is dedicated to problems related to constant mean curvature (CMC) surfaces and Steklov eigenvalues.

CMC surfaces are critical points of the area functional with respect to variations that preserve the enclosed volume, and satisfy certain nonlinear partial differential equations. Classical examples of CMC surfaces in $\mathbb{R}^3$ include the Delaunay surfaces, which are surfaces of revolution with constant mean curvature. Explicitly, these are the cylinder, the sphere, the catenoid, the unduloid and the nodoid. CMC surfaces are important objects in isoperimetric problems. In particular, the classical isoperimetric problem in $\mathbb{R}^3$ states that spheres have least area among all surfaces enclosing a fixed volume. In a compact Riemannian manifold M there exists a compact region whose boundary minimizes area among all regions with fixed volume. Moreover, when the dimension is less than or equal to seven, the boundary of any minimizing region is a smooth embedded hypersurface with constant mean curvature, and if M has nonempty boundary, then the boundary of the CMC hypersurface meets the boundary of M orthogonally.

In addition to problems in the theory of CMC surfaces, we study problems concerning the maximization of Steklov eigenvalues. The Steklov problem is an important and much studied eigenvalue problem. The study of Steklov eigenvalues, which can be seen as the spectrum of the Dirichlet-to-Neumann operator, has important applications in electrical impedance tomography that is used in medical and geophysical imaging. Just as for the classical Dirichlet and Neumann Laplacian eigenvalues, the shape optimization of Steklov eigenvalues has attracted a lot of attention, especially in recent years. The relationship between the existence of maximizing metrics of Steklov eigenvalues on surfaces and the existence of free boundary minimal surfaces of the unit ball in $\mathbb{R}^3$ was estabilished by Fraser and Schoen [30]. The existence of maximizing metrics for Steklov eigenvalues on surfaces, which can be considered as a maximization problem among all smooth metrics, has been used to construct free boundary minimal surfaces in the unit ball [24, 27, 28, 30]. In chapter 4 we study shape optimization for Steklov eigenvalues in higher dimension.

More specifically, we focus on three problems: finding index bounds for closed CMC surfaces and free boundary CMC surfaces in general three-dimensional Riemannian manifolds, proving compactness of the space of free boundary CMC surfaces with bounded topology in 3-manifolds, and showing that the certain changes in the topology of a Riemannian manifold of dimension 3 or higher do not have an effect when considering shape optimization for Steklov eigenvalues. Here, we outline the problems and state the main results, and outline the layout of the thesis.

1.1 Index estimates for CMC surfaces in 3-dimensional Riemannian manifolds

If M is a three-dimensional Riemannian manifold with nonempty boundary and Σ is a two-dimensional Riemannian surface with nonempty boundary $\partial\Sigma \subset \partial M$, then Σ is a free boundary surface with constant mean curvature (CMC) if it is a critical point for area functional with respect to all variations that leave the boundary of Σ in the boundary of M and preserve the enclosed volume of Σ (see definition for enclosed volume of CMC surfaces with boundary in Section 2.2). By the first variation formula, Σ is a free boundary CMC surface if and only if it has constant mean curvature and its boundary meets the boundary of M orthogonally. When the constant mean curvature is zero, Σ is a minimal surface. If Σ is a closed surface, we have similar concepts by ignoring the boundary. Note that CMC surfaces with nonzero mean curvature are always two-sided since the unit normal vector field induces a trivialization of the normal bundle.

Considering the second derivative of area functional, we obtain a quadratic form Q that acts on smooth functions on the surface. We are interested in the weak Morse index of this form, when it is restricted to the function space $S = \{u \in C^\infty(\Sigma) : \int_\Sigma u = 0\}$. Roughly speaking, the weak Morse index of a CMC surface measures the number of independent directions in which one can perturb the surface to decrease its area to the second order without changing the enclosed volume. More specifically, the weak index is the maximal dimension of a subspace of S such that $Q(u,u) < 0$ for all $u \in S \setminus \{0\}$. Here $Q(u,u) = \int_\Sigma |\nabla u|^2 - (|A|^2 + \mathrm{Ric}_M(\nu,\nu))u^2$ where $|A|$ is the length of the second fundamental form and $Ric_M(\nu,\nu)$ is the Ricci curvature of the ambient manifold M evaluated on the surface in the direction of the unit normal ν.

In [16], Cavalcante and de Oliveira obtained that the weak index of a closed CMC surface in a simply-connected space form with constant curvature $c = 0$ or $c = 1$ is bounded from below in terms of the genus g of the surface, i.e.,

$$\mathrm{Ind}_w(\Sigma) \geq \frac{g}{3+c}.$$

Their work was inspired by a paper of Ros [47]. To obtain their result, they first considered a new associated operator $\tilde{L} : S \to S$ (see [10]) given by

$$\tilde{L}u = Lu - \frac{\int_\Sigma Lu}{|\Sigma|}$$

where L is the usual Jacobi operator $L = \Delta + (|A|^2 + 2c)$. The weak index is the number of negative eigenvalues of the operator $\tilde{L}$. Their result follows by comparing the eigenvalues of the Hodge Laplacian $\Delta_{[1]}$ and the eigenvalues of the operator $\tilde{L}$. Corresponding results for free boundary CMC surfaces in mean convex subdomains of $\mathbb{R}^3$ and $\mathbb{S}^3$ were obtained in a similar manner in [15].

We generalize the results in [15, 16] to CMC surfaces in general 3-manifolds. Unlike the case of space forms, our surfaces sit in 3-manifolds in which we don't have global coordinates of harmonic one-forms to use as test functions in the second variation formula. However, with aid of the Nash embedding theorem, we can embed the ambient 3-manifold into Euclidean space $\mathbb{R}^d$, of some dimension d, and obtain the following theorem.

Theorem 1.1.1. *Let (M,g) be a 3-dimensional manifold without boundary isomet-*

rically embedded in $\mathbb{R}^d$ and Σ a closed orientable immersed CMC surface in M with $H_\Sigma \neq 0$. Assume there exist a real number τ and a q-dimensional vector space $\mathbb{V}^q$ of harmonic vector fields on Σ such that any non-zero $\xi \in \mathbb{V}^q$ satisfies

$$\int_\Sigma \sum_{i=1}^{2} \left(|II_M(e_i,\xi)|^2 + |II_M(e_i,\star\xi)|^2\right) - (R_M + H_\Sigma^2)|\xi|^2 \, dV_\Sigma < 2\tau \int_\Sigma |\xi|^2 \, dV_\Sigma.$$

Then

$$\#\{\text{eigenvalues of } \tilde{L}_\Sigma \text{ that are strictly smaller than } \tau\} \geq \frac{q}{2d}.$$

By setting $\tau = 0$, we are able to get weak index estimates.

Corollary 1.1.2. *With the same setting as in Theorem 2.3.5 . Suppose that every non-zero $\xi \in \mathscr{H}^1(\Sigma)$ satisfies*

$$\int_\Sigma \sum_{i=1}^{2} \left(|II_M(e_i,\xi)|^2 + |II_M(e_i,\star\xi)|^2\right) - R_M|\xi|^2 \, dV_\Sigma < \int_\Sigma H_\Sigma^2 |\xi|^2 \, dV_\Sigma. \tag{1.1.1}$$

Then

$$\text{Index}(\Sigma) \geq \frac{g}{d}.$$

Note that in dimension two, the dimension of space of harmonic vector fields is equal to the first Betti number, i.e., $q = 2g$.

The left-hand side of assumption (1.1.1) only involves the second fundamental form Π_M of the ambient manifold M embedded in $\mathbb{R}^d$ and the scalar curvature R_M of M. There are a number of three manifolds that satisfy condition (1.1.1), such as $\mathbb{R}^3$ and $\mathbb{S}^3$. One application of Corollary 1.1.2 is:

Corollary 1.1.3. *Let M be a closed Riemannian manifold of dimension 3. There exist constants $C > 0$ and $H_0 \geq 0$ depending on M such that every closed immersed CMC surface Σ of genus g in M and mean curvature $|H_\Sigma| > H_0$ satisfies*

$$\text{Index}(\Sigma) \geq Cg.$$

In particular, if Σ is stable then it must be a sphere.

Corollary 1.1.3 proves a CMC version of a conjecture of Schoen and Marques-Neves [44] for surfaces of sufficiently large constant mean curvature in arbitrary

3-manifolds. Comparing it to the conjecture of Schoen and Marques-Neves, this result has no condition on the curvature of the ambient 3-manifold. We also proved similar results for CMC surfaces with free boundary in general 3-manifolds.

1.2 Compactness of the space of free boundary CMC surfaces with bounded topology

We are also interested in the space of free boundary CMC surfaces possessing certain properties in general compact three-dimensional manifolds with boundary, such as the space of free boundary CMC surfaces with fixed genus and the number of boundary components in the unit ball in $\mathbb{R}^3$. It is an interesting open question whether this space has only finitely many elements up to congruence. We show that this space is compact in a weak sense.

The study of compactness of the space of minimal surfaces stems from the work by Choi and Schoen [19]. More specifically, they showed that the space of closed embedded minimal surfaces of fixed topological type in a three-dimensional manifold with positive Ricci curvature is compact in the C^k topology for any $k \geq 2$. As a consequence, a global curvature estimate for closed embedded minimal surfaces in terms of their genus was obtained. This compactness result was then generalized by Fraser and Li [26] to the space of free boundary embedded minimal surfaces of fixed topological type in compact three-dimensional manifold with positive Ricci curvature and strictly convex boundary. Recently, Sun [55] considered surfaces with constant mean curvature and proved a compactness result for the space of closed embedded CMC surfaces of fixed topological type in a compact three-dimensional manifold with positive Ricci curvature. Here, the definition of compactness is weaker than that defined in Choi-Schoen's work due to the appearance of a neck pinching phenomenon. In Chapter 3 we prove a compactness result in the case when the surfaces are free boundary CMC surfaces.

To prove compactness it is equivalent to show that any sequence of surfaces has a subsequence that converges to a limiting smooth surface locally, graphically, and smoothly. The idea for proving such convergence for surfaces with bounded topology originates from [19]. The key ingredients and steps are as follows: (1) Finite area: positive ambient Ricci curvature gives lower bound on the first Laplace eigenvalue of a minimal surface, this together with a topological upper bound on the first normalized Laplace eigenvalue shows that bounded topology implies boundedness of the area of the sequence of closed embedded minimal surfaces. (2) Curvature estimate: bounded topology, combined with the Gauss-Bonnent theorem and the area bound obtained from (1) implies finite total curvature; that is, the L^2 norm of the second fundamental form of the surface is finite. Using a scaling argument, the L^2 estimate implies a local L^∞ estimate. (3) Removable singularity: bounded topology of the minimal surfaces and classical harmonic function theory implies that isolated singularities are removable. (4) Multiplicity one: eigenvalue estimates can be used to show that the convergence is of multiplicity one. Finally the Arzela-Ascoli theorem and Allard's regularity can be used to show the desired smooth convergence. When the minimal surfaces have free boundary in an ambient manifold with convex boundary, Fraser and Li [26] used estimates for the first Steklov eigenvalue and an isoperimetric inequality for minimal surfaces to obtain an area bound as in step (1). The result follows by making appropriate adjustments to the other steps.

In the case of free boundary embedded CMC surfaces, the situation becomes more complicated. First, we don't have a lower bound on Steklov eigenvalues for CMC surfaces with boundary. We also lack corresponding isoperimetric inequalities. Thus we additionally assume that the sequence of surfaces has bounded area and perimeter. We then apply a standard blow-up argument to show that finite total curvature implies pointwise bounded second fundamental form. Second, since the surface is not minimal, we can't use harmonic function theory to prove that singularities are removable. However, we can use a blow-up argument to show that the tangent cone at a boundary singular point is a half plane, and as a consequence, this point can be filled in to the closure so that the surface is smooth up to the boundary. The interior removable singularity theorem can be proved in a similar manner. Last, we don't have Steklov eigenvalue estimates to prove that the multiplicity of convergence is one. Nevertheless, we are still able to prove this, since if the multiplicity of convergence is not one, we can construct a positive Jacobi field on the limiting surface, which implies strong stability. However, when the ambient space has positive Ricci curvature and convex boundary, no free boundary CMC surface can be stable, which thus leads to a contradiction.

More specifically, we prove the following compactness result.

Theorem 1.2.1. *Let N be a compact 3-dimensional manifold with boundary. Suppose $H_{\partial N} \geq H_0 > 0$ and let Σ_i be a sequence of free boundary embedded CMC surfaces with mean curvature H_i, genus g_i and number of ends r_i satisfying:*

(a) $|H_i| \leq H_0$;

(b) $g_i \leq g_0$;

(c) $r_i \leq r_0$;

(d) $\text{area}(\Sigma_i) \leq A_0$ *and*

(e) $\text{length}(\partial\Sigma_i) \leq L_0$.

Then there exists a smooth properly almost embedded CMC surface $\Sigma \subset N$ and a finite set $\Gamma \subset \Sigma$ such that, up to a subsequence, Σ_i converges to Σ locally graphically in the C^k topology on compact sets of $N \setminus \Gamma$ for all $k \geq 2$. Moreover, if Σ is minimal then it is properly embedded.

If in addition $(N, \partial N)$ satisfies either $Ric_N > 0$ and $A_{\partial N} \geq 0$ or $Ric_N \geq 0$ and $A_{\partial N} > 0$, then:

(i) when $H_\Sigma = 0$ the convergence is at most 2-sheeted;

(ii) when $H_\Sigma \neq 0$, then the convergence is 1-sheeted away from Γ.

The assumption that the mean curvature of the boundary of the ambient manifold is greater than that of the sequence of surfaces guarantees proper embededdness.

1.3 Higher dimensional surgery and higher Steklov eigenvalues

Another topic closely related to minimal surfaces is the Steklov eigenvalue problem on two-dimensional surfaces. Fraser and Schoen [29, 30] developed an existence theory for maximizing metrics on surfaces of genus zero and establised a connection between the existence of maximizing metrics for normalized Steklov eigenvalues and the existence of free boundary minimal surfaces of the unit ball in $\mathbb{R}^n$.

They showed that a metric that maximizes the k-th normalized Steklov eigenvalue arises geometrically as the induced metric on a free boundary minimal surface of a Euclidean ball, by showing that one can construct a conformal minimal immersion into a ball from the eigenfunctions corresponding to the Steklov eigenvalue.

Our third problem will focus on the maximization for Steklov eigenvalues in higher dimensions. The theory of maximization for Steklov eigenvalues started from the theorem of Weinstock [57] which states that the unit disc uniquely maximizes the first nonzero Steklov eigenvalue among simply-connected domains with fixed boundary length 2π in $\mathbb{R}^2$. An interesting open question in higher dimensions is on which domain (or in the limit of which sequence of domains) with fixed boundary volume in $\mathbb{R}^n$ for $n \geq 3$ the first Steklov eigenvalue is maximized. Bucur, Ferone, Nitsch and Trombetti proved in [13] that the unit ball imaximizes the first normalized Steklov eigenvalue in the class of convex domains in $\mathbb{R}^n$ for $n \geq 3$. On the other hand, Fraser and Schoen [31] showed that the unit ball is no longer the maximizer in the wider class of contractible domains in $\mathbb{R}^n$ for $n \geq 3$. We generalized Fraser and Schoen's result to higher eigenvalues:

Theorem 1.3.1. *Let $n \geq 3$. For any $k \geq 1$, there exist a contractible domain Ω^* in $\mathbb{R}^n$ such that*

$$\bar{\sigma}_j(\Omega^*) > \bar{\sigma}_j(\mathbb{B}^n)$$

for $j = 1, \cdots, k$.

This result is a consequence of an interesting result on continuity of the spectrum under certain higher dimensional surgeries. It was shown in [31] that the number of boundary components of a compact manifold does not affect the supremum of the normalized Steklov eigenvalues. We prove more generally that the supremum of the normalized Steklov eigenvalues among all manifolds is the same as the supremum among manifolds with relatively simple topology. This is achieved by studying higher dimensional surgeries up to codimension 2, in the following theorem. If Σ is a compact properly embedded m-dimensional submanifold of Ω, given $\delta > 0$ small, we denote by $\Omega_{\Sigma,\delta}$ the Lipschitz domain obtained by removing the δ tubular neighbourhood of Σ from Ω.

Theorem 1.3.2. *The Steklov spectrum of a compact Riemannian manifold with* *boundary changes continuously under surgeries of codimension at least two, in the sense that, if $n-m \geq 2$ then*

$$\lim_{\delta \to 0^+} \bar{\sigma}_j(\Omega_{\Sigma,\delta}) = \bar{\sigma}_j(\Omega)$$

for $j = 0, 1, 2, \ldots$.

Here $\bar{\sigma}_j(\Omega) = \sigma_j(\Omega)|\partial\Omega|^{1/n-1}$ denotes the normalized j-th Steklov eigenvalue. Using a similar idea, we also show that in $\mathbb{R}^n$, for $n \geq 3$, the j-th normalized Steklov eigenvalue is not maximized in the limit by a sequence of contractible domains degenerating to the disjoint union of j unit balls, in contrast to the case in dimension two [34].

1.4 Layout

The focus of Chapter 2 is the proof of Theorem 1.1.3. We first introduce the problem by providing an overview of closed CMC surfaces and free boundary CMC surfaces as well as the definition of the Hodge Laplacian. We also provide all of the necessary calculations that are used to prove main theorem as well as its applications.

In Chapter 3, we concentrate on proving Theorem 1.2.1. We first explain that a pointwise curvature bound implies local smooth convergence and prove a curvature estimate and a removable singularity theorem. Finally, we conclude the chapter by providing a detailed proof of Theorem 1.2.1.

Chapter 4 is dedicated to proving Theorem 1.3.2 and Theorem 1.3.1. We introduce the Steklov eigenvalue problem and prove an L^2-noncentration lemma for Steklov eigenfunctions. We then prove continuity of Steklov spectrum under higher dimensional surgeries and apply this to prove Theorem 1.3.1.

Chapter 2

Index Estimates for CMC Surfaces in 3-Dimensional Riemannian Manifolds

2.1 Introduction

A closed hypersurface of constant mean curvature (CMC) may be variationally characterized as a critical point of the area functional under variations that preserve the enclosed volume. In a similar way, a free boundary constant mean curvature (free boundary CMC) hypersurface is a critical point where, in addition, the boundary is restricted to lie in a closed hypersurface. If such a hypersurface minimizes area for small perturbations then it is said to be *stable* for the corresponding problem. For example, solutions to the isoperimetric problem, that is, the problem of finding a hypersurface, with or without boundary, that has least area for a fixed enclosed volume, are stable CMC hypersurfaces.

In [11, 12] Barbosa-do Carmo and Barbosa-do Carmo-Eschenburg classified stable closed CMC hypersurfaces of Euclidean spaces, spheres and hyperbolic spaces. A similar result was obtained by Souam [53] for stable free boundary CMC hypersurfaces in a hemisphere and more recently for closed CMC surfaces in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$. Other classification results for stable free boundary CMC surfaces were obtained by Ros-Vergasta [48] and later improved by Nunes [45]. It is also natural to study CMC hypersurfaces of higher index. That is, those that have some small pertubations that decrease area with fixed enclosed volume.

In [56] Torralbo-Urbano made use of isometric embeddings of homogenous 3-manifolds into Euclidean space to study stable closed CMC surfaces and the isoperimetric problem in Berger spheres. The proof uses coordinates of hamornic vector fields to construct test functions for the second variation of area.

In the case of minimal surfaces, there have been multiple results establishing a connection between the topology of the surface and its index. For example, do Carmo-Peng [23], Fischer-Colbrie-Schoen [25] and Pogorelov [46] independently proved that stable two-sided minimal surfaces in $\mathbb{R}^3$ are planes.

In [47] Ros proved that the index of a minimal surface in $\mathbb{R}^3$ is bounded below by a linear function of its genus. This was later extended and improved by Chodosh-Maximo [17, 18]. A corresponding result was shown by Savo [50] for minimal hypersurfaces in n-spheres. In both situations the authors use harmonic vector fields to construct test functions but their constructions are fundamentally different. Ros uses the coordinates of harmonic vector fields on the surface with respect to the usual basis of $\mathbb{R}^3$, which works seamlessly since the ambient space is flat. On the other hand, Savo uses the coordinates of a bivector involving harmonic one-forms and later takes the average of the second variation for all such test functions. This allows him to overcome the extra term in the second variation given by the curvature of the ambient space.

Using a similar method to Savo [50], Sargent [49] proved corresponding index estimates for free boundary minimal hypersurfaces in convex bodies of Euclidean space. Ambrozio-Carlotto-Sharp [7] used both Ros' and Savo's approaches to relate the index of a closed minimal hypersurface in an arbitrary ambient manifold that can be suitably embedded into some Euclidean space to its first betti number. The authors later did the same for free boundary minimal hypersurfaces in 2-convex domains [8].

Our main results in this chapter follow the natural generalization of Torralbo-Urbano for higher index CMC surfaces, similar to Ambrozio-Carlotto-Sharp's approach. The main difference is that we are only allowed to use admissible functions. Fortunately, the coordinates of harmonic vector fields are admissible in the closed case. In the free boundary case we need the extra condition that these vector fields are tangential along the boundary. In either case we have to use both the harmonic vector field and its Hodge dual, so it only makes sense in the case of orientable CMC surfaces. It is not clear whether or not Savo's test functions preserve enclosed volume, so a generalization to higher dimensions doesn't seem to be straightforward.

A surprising difference between the CMC case and the minimal case, at least in the case of surfaces, is that the extra term involving the non-zero mean curvature allows us to have a wider variety of applications. More specifically, some of the ambient spaces that satisfy the conditions of our theorems for non-zero mean curvature do not work in the case of minimal surfaces, for example, flat ambient spaces. The same results in the case of zero mean curvature can be found in [7, 8], so we restrict our statements to the particular case when the mean curvature is constant and non-zero.

Our main theorem for closed orientable CMC surfaces is:

Theorem 2.3.5. *Let (M,g) be a 3-dimensional manifold without boundary isometrically embedded in $\mathbb{R}^d$ and Σ a closed orientable immersed CMC surface in M with $H_\Sigma \neq 0$. Assume there exist a real number τ and a q-dimensional vector space $\mathbb{V}^q$ of harmonic vector fields on Σ such that any non-zero $\xi \in \mathbb{V}^q$ satisfies*

$$\int_\Sigma \sum_{i=1}^{2} \left(|II_M(e_i,\xi)|^2 + |II_M(e_i,\star\xi)|^2\right) - (R_M + H_\Sigma^2)|\xi|^2 \, dV_\Sigma < 2\tau \int_\Sigma |\xi|^2 \, dV_\Sigma.$$

Then

$$\#\{\text{eigenvalues of } \tilde{L}_\Sigma \text{ that are strictly smaller than } \tau\} \geq \frac{q}{2d}.$$

And the corresponding index estimates:

Corollary 2.3.6. *Let (M,g) be a 3-dimensional Riemannian manifold without boundary isometrically embedded in $\mathbb{R}^d$ and Σ a closed orientable immersed CMC surface of genus g in M and $H_\Sigma \neq 0$. Suppose that every non-zero $\xi \in \mathscr{H}^1(\Sigma)$ satisfies*

$$\int_\Sigma \sum_{i=1}^{2} \left(|II_M(e_i,\xi)|^2 + |II_M(e_i,\star\xi)|^2\right) - R_M|\xi|^2 \, dV_\Sigma < \int_\Sigma H_\Sigma^2 |\xi|^2 \, dV_\Sigma.$$

Then

$$\text{Index}(\Sigma) \geq \frac{g}{d}.$$

As mentioned above, the free boundary case is slightly different since, a priori, only harmonic vector fields that are tangential along the boundary provide admissible test functions. However, it is still possible to find tangential vector fields whose dual vectors, despite not being tangential any more, have zero average along the boundary. This can be done as long as the dimension of the space of tangential harmonic vector fields is sufficiently large. As a consequence the index estimates are weaker.

The respective results for free boundary CMC surfaces are:

Theorem 2.3.8. *Let $(M, \partial M, g)$ be a 3-dimensional Riemannian manifold with boundary isometrically embedded in $\mathbb{R}^d$ and Σ a compact orientable immersed free boundary CMC surface in M with $H_\Sigma \neq 0$. Assume there exist a real number τ and a q-dimensional vector space $\mathbb{W}^q$ of harmonic vector fields on Σ that are tangential on the boundary $\partial\Sigma$, such that any non-zero $\xi \in \mathbb{W}^q$ satisfies*

$$\int_\Sigma \sum_{i=1}^{2} |II_M(e_i,\xi)|^2 + |II_M(e_i,\star\xi)|^2 \, dV_\Sigma \\ - \int_\Sigma (R_M + H_\Sigma^2)|\xi|^2 \, dV_\Sigma - 2\int_{\partial\Sigma} H_{\partial M}|\xi|^2 \, dV_{\partial\Sigma} < 2\tau \int_\Sigma |\xi|^2 \, dV_\Sigma.$$

Then

$$\#\{\textit{eigenvalues of } \tilde{L}_\Sigma \textit{ that are smaller than } \tau\} \geq \frac{q-d}{2d}.$$

Corollary 2.3.9. *Let $(M, \partial M, g)$ be a 3-dimensional Riemannian manifold with boundary isometrically embedded in $\mathbb{R}^d$ and Σ a compact orientable immersed free boundary CMC surface in M with genus g, r boundary components and $H_\Sigma \neq 0$. Suppose that every non-zero $\xi \in \mathscr{H}^1_T(\Sigma, \partial\Sigma)$ satisfies*

$$\int_\Sigma \sum_{i=1}^{2} |II_M(e_i,\xi)|^2 + |II_M(e_i,\star\xi)|^2 \, dV_\Sigma \\ - \int_\Sigma R_M|\xi|^2 \, dV_\Sigma - 2\int_{\partial\Sigma} H_{\partial M}|\xi|^2 \, dV_{\partial\Sigma} < \int_\Sigma H_\Sigma^2 |\xi| \, dV_\Sigma.$$

Then

$$\text{Index}(\Sigma) \geq \frac{2g+r-1-d}{2d}.$$

In the theorems above the operator $\tilde{L}_\Sigma$ corresponds to the twisted Dirichlet eigenvalue problem for the Jacobi operator on admissible functions [10]. The number of negative eigenvalues is the index of Σ in the CMC sense.

Let us mention that Cavalcande-de Oliveira in [15, 16] have obtained similar estimates independent of the inequality condition on the embedding of the ambient manifold for the particular case of closed CMC surfaces in $\mathbb{R}^3$, $\mathbb{S}^3$ and free boundary CMC surfaces in mean convex domains of these two cases.

A conjecture attributed to Schoen and Marques-Neves (see [20, 44]) which says that minimal surfaces in 3-manifolds of positive Ricci curvature have index bounded below by a linear function of their genus. In [7] Ambrozio-Carlotto-Sharp have confirmed this conjecture for a large class of ambient spaces. One may ask a similar question for closed CMC surfaces. In this case, it follows from our results that any closed CMC surface of sufficiently large mean curvature, depending only on the ambient space, in any closed 3-manifold has index bounded below by a linear function of the genus. In some examples we compute explicitly the lower bound necessary for the mean curvature.

As another application, in some examples we are able to prove that closed stable CMC surfaces of sufficiently large mean curvature have to be spheres. In particular, in $T^2 \times \mathbb{R}$ and in T^3 this partially supports a conjecture of Hauswirth-Perez-Romon-Ros regarding the isoperimetric profile of such ambient spaces. To be more specific, the authors conjecture that the isoperimetric profile is given by spheres, cylinders and pairs of planes. Since isoperimetric surfaces are stable CMC surfaces, our result supports the section of the profile that is given by spheres.

This chapter is divided as follows. In section 2 we cover some of the necessary background and notation used throughout the chapter. In section 3 we prove that Ros' test functions are admissible and prove our main theorems. In section 4 we discuss some applications and examples that satisfy the hypotheses of our main results.

2.2 Preliminaries on CMC Surfaces and Morse Index

In this section, we will give an overview some background material and establish notation that will be used throughout the chapter.

Let $(M, \partial M, g)$ is a complete Riemannian manifold of dimension $n+1$, with possibly empty boundary and let $(\Sigma^k, \partial\Sigma)$ be a compact immersed k-dimensional submanifold in M with boundary $\partial\Sigma$ in ∂M. In case ∂M is empty, we consider $\partial\Sigma$ to also be empty, in which case we say Σ^k is a closed (compact without boundary) immersed submanifold in M. When $k = n$, we say that Σ^n is two-sided if there is a global unit normal vector field N along Σ in M. Denote the connection on M and Σ by $\overline{\nabla}$ and ∇^Σ, respectively. The second fundamental form of Σ is defined by $A_\Sigma(X,Y) = \overline{\nabla}_X Y - \nabla^\Sigma_X Y$, its shape operator with respect to N is denoted $S_\Sigma(X)$, the mean curvature vector is given by $\vec{H}_\Sigma = \mathrm{tr}_\Sigma(A_\Sigma)$ and its mean curvature is $H_\Sigma = g(\vec{H}_\Sigma, N)$.

By Nash's embedding theorem we may assume M to be isometrically embedded into some Euclidean space $\mathbb{R}^d$ for some integer d. If we denote the Euclidean connection by D, then the second fundamental form of M in $\mathbb{R}^d$ is given by $II_M(X,Y) = D_X Y - \overline{\nabla}_X Y$, for any two tangent vectors X, Y on M. It follows that for any two tangent vectors X, Y on Σ, the following orthogonal decomposition holds

$$D_X Y = \nabla^\Sigma_X Y + A_\Sigma(X,Y) + II_M(X,Y). \tag{2.2.1}$$

Denote by $\Delta^{[p]}$ the Hodge Laplacian of Σ acting on p-forms and let $\nabla^*\nabla$ denote the rough Laplacian on vector fields. By choosing a local geodesic frame $\{e_1, \cdots, e_n\}$ of Σ, they are defined by

$$\Delta^{[p]} w = (d\delta + \delta d)w, \quad \nabla^*\nabla X = -\sum_{i=1}^{n} \nabla_{e_i}\nabla_{e_i} X$$

for p-forms w and vector fields X on Σ, where d is the exterior differential and δ is the codifferential. If Σ is closed, then w is harmonic if and only if $dw = 0$ and $\delta w = 0$.

Given a 1-form w, its dual vector field $w^\natural$ is defined by $g(w^\natural, X) = w(X)$ for any vector X. Reversely, given a vector field X we denote its dual 1-form $X^\flat(Y) = g(X,Y)$.

Let us now restrict ourselves to when Σ is an orientable surface. We may define $\star X = (\star X^\flat)^\natural$, where $\star$ is the Hodge operator with respect to the metric on Σ. The Laplacian acting on a vector field X may be defined as $\Delta_{[1]}X = \Delta^{[1]}X^\flat$. If X is a vector field on Σ, we have $\star dX^\flat = \mathrm{div}_\Sigma \star X$ and $\delta X^\flat = -\mathrm{div}_\Sigma X$. Hence, when Σ is closed, X is harmonic if and only if $\mathrm{div}_\Sigma X = \mathrm{div}_\Sigma \star X = 0$.

When Σ is a closed surface we denote the space of harmonic vector fields on Σ as

$$\mathscr{H}^1(\Sigma) = \{\xi \in T\Sigma : \ \mathrm{div}(\xi) = \mathrm{div}(\star\xi) = 0 \text{ on } \Sigma\ \}.$$

If Σ has genus g then $\dim \mathscr{H}^1(\Sigma) = 2g$.

When Σ has nonempty boundary, a vector field being harmonic is not equivalent to $\mathrm{div}(\xi) = \mathrm{div}(\star\xi) = 0$. Instead we shall consider the following space:

$$\mathscr{H}^1_T(\Sigma, \partial\Sigma) = \left\{ \xi \in T\Sigma : \begin{array}{l} \mathrm{div}(\xi) = \mathrm{div}(\star\xi) = 0 \text{ on } \Sigma \\ \text{and } \xi \text{ is tangent along } \partial\Sigma \end{array} \right\}.$$

Any vector field in this space is harmonic and tangential along $\partial\Sigma$. By the Hodge theorem and Poincaré-Lefschetz duality, we know that $\mathscr{H}^1_T(\Sigma, \partial\Sigma)$ is isomorphic to $H_1(\Sigma, \partial\Sigma; \mathbb{R})$. Hence $\dim \mathscr{H}^1_T(\Sigma, \partial\Sigma) = 2g + r - 1$, where g is the genus and r is the number of boundary components of Σ. See for example [49, Lemma A.0.1] for a proof.

It is well know that Weitzenbock's formula relates the Hodge Laplacian and rough Laplacian of a vector field. That is, for a vector field ξ on Σ we have

$$\Delta_{[1]}\xi = \nabla^*\nabla\xi + \mathrm{Ric}_\Sigma(\xi), \tag{2.2.2}$$

where $\mathrm{Ric}_\Sigma(\xi)$ is defined by $\mathrm{Ric}_\Sigma(\xi, X) = g(\mathrm{Ric}_\Sigma(\xi), X)$ for every tangent vector X.

Now, let us define the Morse index of a CMC hypersurface. Let Σ be a two-sided hypersurface with boundary $\partial\Sigma$ in $(M, \partial M)$. The first variation formula of area with respect to a normal variation induced by a function u on Σ is given by

$$\left.\frac{d}{dt}\right|_{t=0} |\Phi_t(\Sigma)| = -n\int_\Sigma uH_\Sigma \, dV_\Sigma + \int_{\partial\Sigma} g(\eta, X)\, dV_{\partial\Sigma}, \tag{2.2.3}$$

where η is the outward pointing unit normal vector field along $\partial\Sigma$, X is the orthogonal projection of uN onto ∂M and Φ_t is the variation induced by u. A free boundary constant mean curvature hypersurface, henceforth denoted free boundary CMC hypersurface, is a critical point of the area functional with respect to variations that preserve enclosed volume. Here the enclosed volume represents the signed volume enclosed between the hypersurface Σ and $\Phi_t(\Sigma)$, see [48] for details. This means that u must satisfy $\int_\Sigma u\, dV_\Sigma = 0$, in which case we call it an admissible variation. Note that Σ must intersect ∂M orthogonally along $\partial\Sigma$, which is called the free boundary property, i.e., $\eta = -\nu$ where ν is the inward pointing normal vector field along ∂M. In this chapter we only consider CMC surfaces with non-zero mean curvature.

The quadratic form associated to the second variation of area of a free boundary CMC hypersurface with respect to u is

$$Q_\Sigma(u,u) = \int_\Sigma |\nabla u|^2 - (\mathrm{Ric}_M(N,N) + |A_\Sigma|^2)u^2\, dV_\Sigma - \int_{\partial\Sigma} h_{\partial M}(N,N)\, dV_{\partial\Sigma}, \tag{2.2.4}$$

where $\mathrm{Ric}_M(N,N)$ is the Ricci curvature of M in the direction N, $|A_\Sigma|^2 = \mathrm{tr}_\Sigma(S_\Sigma^T S_\Sigma)$ is the square norm of second fundamental form and $h_{\partial M}(X,Y) = g(\overline{\nabla}_X Y, \nu)$ denotes the scalar second fundamental form of ∂M in M with respect to ν.

By the divergence theorem we can also write (2.2.4) as

$$\begin{aligned} Q_\Sigma(u,u) = &- \int_\Sigma u\Delta u + (\mathrm{Ric}_M(N,N) + |A_\Sigma|^2)u^2\, dV_\Sigma \\ &+ \int_{\partial\Sigma} u\frac{\partial u}{\partial\eta} - h_{\partial M}(N,N)u^2\, dV_{\partial\Sigma}. \end{aligned}$$

We denote by $L_\Sigma = \Delta + \mathrm{Ric}_M(N,N) + |A_\Sigma|^2$ the Jacobi operator of Σ, where $\Delta u = \mathrm{div}_\Sigma(\nabla^\Sigma u)$.

The Morse index of a free boundary CMC hypersurface is defined as the index of the quadratic form Q_Σ restricted to the subspace of smooth functions u on Σ that are admissible, that is, such that $\int_\Sigma u dV_\Sigma = 0$. When the index is zero, we say that the hypersurface is stable in the CMC sense.

This notion of index coincides with the number of negative eigenvalues for the operator $\tilde{L}_\Sigma u = L_\Sigma u - \frac{1}{|\Sigma|}\int_\Sigma L_\Sigma u dV_\Sigma$ acting on admissible functions on Σ (see [10, Proposition 2.2]); that is, of the following eigenvalue problem:

$$\begin{cases} \tilde{L}_\Sigma u + \lambda u = 0, & \text{in } \Sigma \\ \dfrac{\partial u}{\partial \eta} - h_{\partial M}(N,N)u = 0, & \text{on } \partial\Sigma \\ \displaystyle\int_\Sigma u dV_\Sigma = 0. \end{cases} \tag{2.2.5}$$

The spectrum of $\tilde{L}_\Sigma$ consists of eigenvalues $\tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \dots$ corresponding to admissible eigenfunctions $\tilde{\phi}_1, \tilde{\phi}_2, \dots$ in $L^2(\Sigma)$, see [10] for details. Furthermore, the eigenvalues $\tilde{\lambda}_i$ satisfy the min-max characterization with respect to the quadratic form Q_Σ when restricted to admissible functions. That is, $\lambda_k = \inf \frac{Q_\Sigma(u,u)}{\int_\Sigma u^2 dV_\Sigma}$, where the infimum is taken over all smooth admissible functions that are orthogonal to the first $k-1$ eigenfunctions $\tilde{\phi}_1, \dots, \tilde{\phi}_{k-1}$.

In the special case where the boundaries ∂M and $\partial\Sigma$ are both empty, the corresponding variation formulas are obtained similarly by simply making the terms that involve the boundary as 0.

Similarly we can define the Morse index of a closed CMC hypersurface as the number of negative eigenvalues of the Jacobi operator with respect to admissible variations. Note that the Jacobi operators are the same in closed and boundary case, the only difference is the boundary condition in the eigenvalue problem.

Remark 2.2.1. *Since we are considering CMC surfaces Σ with nonzero mean curvature, Σ is automatically two-sided in the ambient manifold M because the mean curvature vector defines a trivialization of the normal bundle. As such, we only have the following three possible combinations of CMC surfaces and ambient space: (1) Σ is orientable and M is orientable; (2) Σ is orientable and M is non-orientable; (3) Σ is non-orientable and M is non-orientable. The idea in this chapter applies to the first two cases while it does not work in the third case, see* Remark 2.3.7 *for a more detailed discussion.*

2.3 Calculations and Main Results

Similarly to [47] and [7, 8] we intend to use the coordinates of harmonic vector fields as test functions to obtain our index estimates. However, we must first verify that such functions are admissible in the CMC sense.

The following is a general result in arbitrary codimension that follows from the divergence theorem.

Lemma 2.3.1. *Suppose $(\Sigma^k, \partial\Sigma, g)$ is a k-dimensional submanifold $(k \geq 2)$ isometrically embedded in some Euclidean space $\mathbb{R}^d$.*

- *If $\partial\Sigma = \emptyset$ and ξ is a harmonic vector field in $\mathscr{H}^1(\Sigma)$, then*

$$\int_\Sigma \langle V, \xi \rangle \, dV_\Sigma = 0,$$

where V is a constant vector in $\mathbb{R}^d$. In particular, if $k = 2$ and Σ is orientable, we also have

$$\int_\Sigma \langle V, \star\xi \rangle \, dV_\Sigma = 0.$$

- *If $\partial\Sigma \neq \emptyset$ and ξ is a tangential harmonic vector field in $\mathscr{H}^1_T(\Sigma, \partial\Sigma)$, then*

$$\int_\Sigma \langle V, \xi \rangle \, dV_\Sigma = 0.$$

Proof. Denote by $f(x) = \langle V, x \rangle$, then its gradient is given by $\nabla^\Sigma f = V^T$, where V^T denotes the projection of V onto Σ. If Σ is a closed submanifold and $\xi \in \mathscr{H}^1(\Sigma)$ we get

$$0 = \int_\Sigma \mathrm{div}_\Sigma(f\xi) = \int_\Sigma \langle \nabla^\Sigma f, \xi \rangle + f \,\mathrm{div}(\xi) = \int_\Sigma \langle \nabla^\Sigma f, \xi \rangle = \int_\Sigma \langle V, \xi \rangle.$$

When $k = 2$, the argument follows by substituting $\star\xi$ and observing that it is also a harmonic vector field.

In the case where Σ has nonempty boundary and $\xi \in \mathscr{H}^1_T(\Sigma)$ we get

$$\int_\Sigma \langle V, \xi \rangle = \int_\Sigma \mathrm{div}_\Sigma(f\xi) = \int_{\partial\Sigma} f \langle \xi, \eta \rangle = 0$$

The last equality is due to the tangential property of ξ. This completes the proof of the Lemma. □

Remark 2.3.2. *When an orientable surface Σ has nonempty boundary and dimension 2 this argument does not work for $\star\xi$ since it may not be tangential on the* *boundary when ξ is tangential. In fact, if ξ is tangential along the boundary then $\star\xi$ is necessarily orthogonal to the boundary. However, we may still be able to find tangential harmonic vector fields ξ such that $f\langle\xi,\eta\rangle$ has zero average along the boundary so its coordinates may still be admissible test functions.*

The following proposition involves calculations of coordinates of a harmonic vector field as test functions for the second variation formula both in closed case and boundary case. In fact, the calculations in the closed case directly follow from the result in the boundary case.

Proposition 2.3.3. *Let $(M,\partial M,g)$ be a 3-dimensional Riemannian manifold isometrically embedded in some Euclidean space $\mathbb{R}^d$. Let $(\Sigma,\partial\Sigma)$ be an immersed surface in M with free boundary $\partial\Sigma$ in ∂M.*

- *Given a vector field $\xi\in\mathscr{H}^1_T(\Sigma,\partial\Sigma)$, denote*

$$u_j=\langle\xi,E_j\rangle$$

where $\{E_j\}_{j=1}^d$ is the canonical basis of $\mathbb{R}^d$. Then

$$\begin{aligned}\sum_{j=1}^d Q_\Sigma(u_j,u_j)=&\int_\Sigma\sum_{i=1}^2|II_M(e_i,\xi)|^2+|A_\Sigma(e_i,\xi)|^2\,dV_\Sigma\\&-\int_\Sigma\left(\frac{|A_\Sigma|^2}{2}+\frac{R_M}{2}+\frac{H_\Sigma^2}{2}\right)|\xi|^2\,dV_\Sigma\\&-\int_{\partial\Sigma}H_{\partial M}|\xi|^2\,dV_{\partial\Sigma}\end{aligned}\tag{2.3.1}$$

where $\{e_i\}_{i=1}^2$ is an orthonormal frame on Σ, R_M is the scalar curvature of M and $H_{\partial M}$ denotes the mean curvature of ∂M with respect to the inner normal vector ν.

- *In particular, if ∂M and $\partial\Sigma$ are empty, for any vector field $\xi\in\mathscr{H}^1(\Sigma)$, we* *have*

$$\sum_{j=1}^{d} Q_\Sigma(u_j,u_j) = \int_\Sigma \sum_{i=1}^{2} |II_M(e_i,\xi)|^2 + |A_\Sigma(e_i,\xi)|^2 \, dV_\Sigma \\ - \int_\Sigma \left(\frac{|A_\Sigma|^2}{2} + \frac{R_M}{2} + \frac{H_\Sigma^2}{2} \right) |\xi|^2 \, dV_\Sigma \tag{2.3.2}$$

Proof. Using a local orthonormal basis $\{e_i\}_{i=1}^2$ on Σ we have

$$\nabla^\Sigma u_j = \sum_{i=1}^{2} e_i \langle \xi, E_j \rangle e_i = \sum_{i=1}^{2} \langle D_{e_i}\xi, E_j \rangle e_i.$$

Then (2.2.1) gives

$$D_{e_i}\xi = \nabla^\Sigma_{e_i}\xi + A_\Sigma(e_i,\xi) + II_M(e_i,\xi).$$

Thus,

$$\begin{aligned} \sum_{j=1}^{d} |\nabla^\Sigma u_j|^2 &= \sum_{j=1}^{d}\sum_{i=1}^{2} |\langle D_{e_i}\xi, E_j\rangle|^2 && (2.3.3) \\ &= \sum_{i=1}^{2}\sum_{j=1}^{d} \langle \nabla^\Sigma_{e_i}\xi, E_j\rangle^2 + \langle A_\Sigma(e_i,\xi), E_j\rangle^2 + \langle II_M(e_i,\xi), E_j\rangle^2 \\ &= |\nabla^\Sigma \xi|^2 + \sum_{i=1}^{2} |A_\Sigma(e_i,\xi)|^2 + |II_M(e_i,\xi)|^2. && (2.3.4) \end{aligned}$$

Gauss' equation for Σ in M gives us

$$2K_\Sigma = R_M - 2\,\mathrm{Ric}_M(N,N) - |A_\Sigma|^2 + H_\Sigma^2.$$

Hence,

$$\int_\Sigma \mathrm{Ric}_M(N,N) u_j^2 \, dV_\Sigma = \int_\Sigma \left(\frac{R_M}{2} - \frac{|A_\Sigma|^2}{2} + \frac{H_\Sigma^2}{2} - K_\Sigma \right) u_j^2 \, dV_\Sigma. \tag{2.3.5}$$

It follows from Weitzenbock's formula (2.2.2), ξ being a harmonic vector field and Σ being a surface that

$$\nabla^* \nabla \xi = -K_\Sigma \xi.$$

By computing the exterior derivative along $\partial\Sigma$ we have

$$d\xi^\flat(\eta, \xi) = \langle \nabla^\Sigma_\eta \xi, \xi \rangle - \langle \nabla^\Sigma_\xi \xi, \eta \rangle.$$

Since $d\xi^\flat = 0$,

$$\langle \nabla^\Sigma_\eta \xi, \xi \rangle = \langle \nabla^\Sigma_\xi \xi, \eta \rangle.$$

Then it follows from the divergence theorem that

$$\begin{aligned}
\int_\Sigma \Delta |\xi|^2 \, dV_\Sigma &= \int_{\partial\Sigma} \nabla^\Sigma_\eta |\xi|^2 \, dV_{\partial\Sigma} \\
&= 2 \int_{\partial\Sigma} \langle \nabla^\Sigma_\eta \xi, \xi \rangle \, dV_{\partial\Sigma} \\
&= 2 \int_{\partial\Sigma} \langle \nabla^\Sigma_\xi \xi, \eta \rangle \, dV_{\partial\Sigma} \\
&= -2 \int_{\partial\Sigma} \langle \overline{\nabla}_\xi \xi, \nu \rangle \, dV_{\partial\Sigma} \\
&= -2 \int_{\partial\Sigma} h_{\partial M}(\xi, \xi) \, dV_{\partial\Sigma}, \qquad (2.3.6)
\end{aligned}$$

which together with

$$\Delta |\xi|^2 = -2 \langle \nabla^* \nabla \xi, \xi \rangle + 2 |\nabla^\Sigma \xi|^2$$

implies that

$$\begin{aligned}
\int_\Sigma |\nabla^\Sigma \xi|^2 \, dV_\Sigma &= \int_\Sigma \frac{\Delta |\xi|^2}{2} \, dV_\Sigma + \int_\Sigma \langle \nabla^* \nabla \xi, \xi \rangle \, dV_\Sigma \\
&= -\int_{\partial\Sigma} h_{\partial M}(\xi, \xi) \, dV_{\partial\Sigma} - \int_\Sigma K_\Sigma |\xi|^2 \, dV_\Sigma. \qquad (2.3.7)
\end{aligned}$$

Note that

$$H_{\partial M} |\xi|^2 = h_{\partial M}(\xi, \xi) + h_{\partial M}(N, N) |\xi|^2. \qquad (2.3.8)$$

Then (2.3.9) in the proposition follows from (2.3.4), (2.3.5), (2.3.7), (2.3.8) and (2.2.4). The formula (2.3.10) directly follows by ignoring boundary terms. □

Similarly, for the dual harmonic vector field we have

Proposition 2.3.4. *Let $(M, \partial M, g)$ be a 3-dimensional Riemannian manifold isometrically embedded in some Euclidean space $\mathbb{R}^d$. Let $(\Sigma, \partial\Sigma)$ be an orientable immersed surface in M with free boundary $\partial\Sigma$ in ∂M.*

- *Given a vector field $\xi \in \mathscr{H}^1_T(\Sigma, \partial\Sigma)$, denote*

$$u_j^\star = \langle \star\xi, E_j \rangle$$

where $\{E_j\}_{j=1}^d$ is the canonical basis of $\mathbb{R}^d$. Then

$$\begin{aligned}\sum_{j=1}^d Q_\Sigma(u_j^\star, u_j^\star) = &\int_\Sigma \sum_{i=1}^2 |II_M(e_i, \star\xi)|^2 + |A_\Sigma(e_i, \star\xi)|^2 \, dV_\Sigma \\ &- \int_\Sigma \left(\frac{|A_\Sigma|^2}{2} + \frac{R_M}{2} + \frac{H_\Sigma^2}{2} \right) |\star\xi|^2 \, dV_\Sigma \\ &- \int_{\partial\Sigma} H_{\partial M} |\star\xi|^2 \, dV_{\partial\Sigma}\end{aligned} \tag{2.3.9}$$

where $\{e_i\}_{i=1}^2$ is an orthonormal frame on Σ, R_M is the scalar curvature of M and $H_{\partial M}$ denotes the mean curvature of ∂M with respect to the inner normal vector ν.

- *In particular, if ∂M and $\partial\Sigma$ are empty, for any vector field $\xi \in \mathscr{H}^1(\Sigma)$, we have*

$$\begin{aligned}\sum_{j=1}^d Q_\Sigma(u_j^\star, u_j^\star) = &\int_\Sigma \sum_{i=1}^2 |II_M(e_i, \star\xi)|^2 + |A_\Sigma(e_i, \star\xi)|^2 \, dV_\Sigma \\ &- \int_\Sigma \left(\frac{|A_\Sigma|^2}{2} + \frac{R_M}{2} + \frac{H_\Sigma^2}{2} \right) |\star\xi|^2 \, dV_\Sigma\end{aligned} \tag{2.3.10}$$

Proof. When Σ is orientable we have $|\star\xi| = |\xi|$ so the same result follows from

$$\int_\Sigma \Delta|\star\xi|^2 = \int_\Sigma \Delta|\xi|^2 \, dV_\Sigma = \int_{\partial\Sigma} h_{\partial M}(\xi, \xi) \, dV_{\partial\Sigma}. \tag{2.3.11}$$

This means that we have the same equation as (2.3.9) and (2.3.10) for $\star\xi$ whether or not its coordinate functions are admissible. □

The following is the main theorem for closed CMC surfaces in 3-dimensional manifolds embedded in Euclidean space. It estimates the number of eigenvalues of the $\tilde{L}_\Sigma$ below a certain threshold τ under certain conditions on the embedding and the geometry of M. This result is often referred to as a "concentration of spectrum inequality" (see [7]). Recall that $\tilde{L}_\Sigma u = L_\Sigma u - \int_\Sigma L_\Sigma u dV_\Sigma$.

Theorem 2.3.5. *Let (M,g) be a 3-dimensional manifold without boundary isometrically embedded in $\mathbb{R}^d$ and Σ a closed orientable immersed CMC surface in M with $H_\Sigma \neq 0$. Assume there exist a real number τ and a q-dimensional vector space $\mathbb{V}^q$ of harmonic vector fields on Σ such that any non-zero $\xi \in \mathbb{V}^q$ satisfies*

$$\int_\Sigma \sum_{i=1}^{2} \left(|II_M(e_i,\xi)|^2 + |II_M(e_i,\star\xi)|^2\right) - (R_M + H_\Sigma^2)|\xi|^2 \, dV_\Sigma < 2\tau \int_\Sigma |\xi|^2 \, dV_\Sigma.$$

Then

$$\#\{\textit{eigenvalues of } \tilde{L}_\Sigma \textit{ that are strictly smaller than } \tau\} \geq \frac{q}{2d}.$$

Proof. First, from Lemma 2.3.1 we have that u_j and $u_j^\star = \langle \star\xi, E_j\rangle$ for $j = 1, \cdots, d$ are admissible test functions. Applying the same calculation for $(2.3.10)$ to $u_j^\star$ we have that

$$\begin{aligned}\sum_{j=1}^{d} Q_\Sigma(u_j^\star, u_j^\star) = &\int_\Sigma \sum_{i=1}^{2} |II_M(e_i,\star\xi)|^2 + |A_\Sigma(e_i,\star\xi)|^2 \, dV_\Sigma \\ &- \int_\Sigma \left(\frac{|A_\Sigma|^2}{2} + \frac{R_M}{2} + \frac{H_\Sigma^2}{2}\right) |\star\xi|^2 \, dV_\Sigma.\end{aligned} \tag{2.3.12}$$

Now, whenever $\xi \neq 0$ we may pick $e_1 = \frac{\xi}{|\xi|}, e_2 = \frac{\star\xi}{|\star\xi|}$ as an orthonormal basis. Recalling that $|\xi| = |\star\xi|$, we have at every point

$$\sum_{i=1}^{2} |A_\Sigma(e_i,\xi)|^2 + |A_\Sigma(e_i,\star\xi)|^2 = \sum_{i,j=1,2} |A_\Sigma(e_i,e_j)|^2 |\xi|^2 = |A_\Sigma|^2|\xi|^2. \tag{2.3.13}$$

From this we obtain, by adding (2.3.10) and (2.3.12),

$$\sum_{j=1}^{d} Q_\Sigma(u_j,u_j)+Q_\Sigma(u_j^\star,u_j^\star)=\int_\Sigma \sum_{i=1}^{2}|II_M(e_i,\xi)|^2+|II_M(e_i,\star\xi)|^2\,dV_\Sigma \\ -\int_\Sigma (R_M+H_\Sigma^2)|\xi|^2\,dV_\Sigma.$$

For convenience, let $k=\#\{$ eigenvalues of $\tilde{L}_\Sigma$ that are strictly smaller than $\tau\}$ and let $\tilde{\phi}_1,\cdots,\tilde{\phi}_k$ be the eigenfunctions of the operator $\tilde{L}_\Sigma$ corresponding to the eigenvalues $\tilde{\lambda}_1\leq\cdots\leq\tilde{\lambda}_k<\tau$. Consider following the linear map defined by

$$\begin{aligned} F:\ \ \mathbb{V}^q &\longrightarrow\ \ \mathbb{R}^{2dk} \\ \xi &\longmapsto\ \ [\int_\Sigma u_j\tilde{\phi}_\alpha\,dV_\Sigma,\int_\Sigma u_j^\star\tilde{\phi}_\alpha\,dV_\Sigma], \end{aligned}$$

where $\alpha=1,\cdots,k$ and $j=1,\cdots,d$. By the Rank-Nullity theorem, if $2dk<q$, then there exists a nonzero harmonic vector field ξ such that $u_j,u_j^\star$ are orthogonal to the first k eigenfunctions of $\tilde{L}_\Sigma$. From the Min-max principle for the operator $\tilde{L}_\Sigma$ it follows that

$$\sum_{j=1}^{d} Q_\Sigma(u_j,u_j)+Q_\Sigma(u_j^\star,u_j^\star)\geq 2\lambda_{k+1}\int_\Sigma|\xi|^2\,dV_\Sigma\geq 2\tau\int_\Sigma|\xi|^2\,dV_\Sigma$$

which contradicts the assumption in the proposition. Thus $2dk\geq q$, or $k\geq\frac{q}{2d}$ as claimed. □

We remark that the same proof works to estimate the number of eigenvalues of the Jacobi operator. However, such an estimate would not directly give index estimates in the CMC sense.

As a corollary, letting $\tau=0$ we can get an estimate on numbers of negative eigenvalues of $\tilde{L}_\Sigma$, that is, an index estimate.

Corollary 2.3.6. *Let (M,g) be a 3-dimensional Riemannian manifold without boundary isometrically embedded in $\mathbb{R}^d$ and Σ a closed orientable immersed CMC sur-*

face of genus g in M with $H_\Sigma \neq 0$. Suppose that every non-zero $\xi \in \mathscr{H}^1(\Sigma)$ satisfies

$$\int_\Sigma \sum_{i=1}^{2} \left(|II_M(e_i,\xi)|^2 + |II_M(e_i,\star\xi)|^2\right) - R_M|\xi|^2 \, dV_\Sigma < \int_\Sigma H_\Sigma^2 |\xi|^2 \, dV_\Sigma.$$

Then,

$$\operatorname{Index}(\Sigma) \geq \frac{g}{d}$$

Remark 2.3.7. *In Theorem* 2.3.5 *and Corollary* 2.3.6*, the orientability assumption is essential for our method. Since in this case $\star\xi$ is a globally well-defined vector field. When Σ is non-orientable, in which case M has to be non-orientable, we could consider the oriented double covering $\hat{\Sigma}$ of Σ which would be a two-sided CMC immersed surface in M. The Hodge star operator $\star$ on $\hat{\Sigma}$ would be well defined but given a harmonic vector field ξ invariant under the change of sheets, the corresponding $\star\xi$ would be anti-invariant, or vice-versa. That is, only one of these can define a valid test function of Σ. However, our method requires both test functions to eliminate certain undesired terms involving the second fundamental form of Σ when computing the second variation in that direction.*

The next result is the corresponding theorem for free boundary CMC surfaces in 3-dimensional manifolds with boundary. From Remark 2.3.2 we observe $u_j^\star$ may not be admissible, however, it is possible to find suitable vector fields when the dimension of $\mathscr{H}_T^1(\Sigma, \partial\Sigma)$ is sufficiently large.

Theorem 2.3.8. *Let $(M, \partial M, g)$ be a 3-dimensional Riemannian manifold with boundary isometrically embedded in $\mathbb{R}^d$ and Σ a compact orientable immersed free boundary CMC surface in M with $H_\Sigma \neq 0$. Assume there exist a real number τ and a q-dimensional vector space $\mathbb{W}^q$ of harmonic vector fields on Σ that are tangential to $\partial\Sigma$ along the boundary $\partial\Sigma$, such that any non-zero $\xi \in \mathbb{W}^q$ satisfies*

$$\begin{aligned}\int_\Sigma \sum_{i=1}^{2} |II_M(e_i,\xi)|^2 + |II_M(e_i,\star\xi)|^2 \, dV_\Sigma \\ - \int_\Sigma (R_M + H_\Sigma^2)|\xi|^2 \, dV_\Sigma - 2\int_{\partial\Sigma} H_{\partial M}|\xi|^2 \, dV_{\partial\Sigma} < 2\tau \int_\Sigma |\xi|^2 \, dV_\Sigma.\end{aligned}$$

Then

$$\#\{\textit{eigenvalues of } \tilde{L}_\Sigma \textit{ that are smaller than } \tau\} \geq \frac{q-d}{2d}.$$

Proof. From Proposition 2.3.4 we know that

$$\begin{aligned}\sum_{j=1}^{d} Q_\Sigma(u_j^\star,u_j^\star) = &\int_\Sigma \sum_{i=1}^{2} |II_M(e_i,\star\xi)|^2 + |A_\Sigma(e_i,\star\xi)|^2\, dV_\Sigma \\ &- \int_\Sigma \left(\frac{|A_\Sigma|^2}{2} + \frac{R_M}{2} + \frac{H_\Sigma^2}{2}\right)|\xi|^2\, dV_\Sigma \\ &- \int_{\partial\Sigma} H_{\partial M}|\xi|^2\, dV_{\partial\Sigma}.\end{aligned} \tag{2.3.14}$$

Summing up (2.3.14) and (2.3.9), together with (2.3.13) gives

$$\begin{aligned}\sum_{j=1}^{d} Q_\Sigma(u_j,u_j) + Q_\Sigma(u_j^\star,u_j^\star) = &\int_\Sigma \sum_{i=1}^{2} |II_M(e_i,\xi)|^2 + |II_M(e_i,\star\xi)|^2\, dV_\Sigma \\ &- \int_\Sigma (R_M + H_\Sigma^2)|\xi|^2\, dV_\Sigma \\ &- 2\int_{\partial\Sigma} H_{\partial M}|\xi|^2\, dV_{\partial\Sigma}.\end{aligned}$$

Let $k = \#\{\text{eigenvalues of } \tilde{L}_\Sigma \text{ that are smaller than } \tau\}$ and let $\tilde{\phi}_1,\cdots,\tilde{\phi}_k$ be the eigenfunctions of $\tilde{L}_\Sigma$ corresponding to the eigenvalues $\tilde{\lambda}_1 \le \cdots \le \tilde{\lambda}_k < \tau$. Consider the linear map defined by

$$\begin{aligned} F:\quad \mathbb{W}^q &\longrightarrow \mathbb{R}^{2dk+d} \\ \xi &\longmapsto [\int_\Sigma u_j\phi_\alpha\, dV_\Sigma, \int_\Sigma u_j^\star\phi_\alpha\, dV_\Sigma, \int_\Sigma u_j^\star\, dV_\Sigma],\end{aligned}$$

where $\alpha = 1,\cdots,k$ and $j = 1,\cdots,d$. By the Rank-Nullity theorem, if $2dk+d<q$, then there exists a nonzero harmonic tangential vector field ξ such that $u_j, u_j^\star$ are orthogonal to the first k eigenfunctions of $\tilde{L}_\Sigma$. From the Min-max principle for $\tilde{L}_\Sigma$ it follows that

$$\sum_{j=1}^{d} Q_\Sigma(u_j,u_j) + Q_\Sigma(u_j^\star,u_j^\star) \ge 2\lambda_{k+1}\int_\Sigma |\xi|^2\, dV_\Sigma \ge 2\tau\int_\Sigma |\xi|^2\, dV_\Sigma$$

which contradicts the assumption in the proposition. Thus $2dk+d \ge q$, or $k \ge \frac{q-d}{2d}$ as claimed. □

Correspondingly, we have an index estimate for free boundary CMC surfaces.

Corollary 2.3.9. *Let $(M, \partial M, g)$ be a 3-dimensional Riemannian manifold with boundary isometrically embedded in $\mathbb{R}^d$ and Σ a compact orientable immersed free boundary CMC surface in M with genus g, r boundary components and $H_\Sigma \neq 0$. Suppose that every non-zero $\xi \in \mathscr{H}^1_T(\Sigma, \partial\Sigma)$ satisfies*

$$\int_\Sigma \sum_{i=1}^{2} |II_M(e_i,\xi)|^2 + |II_M(e_i,\star\xi)|^2 \, dV_\Sigma$$
$$- \int_\Sigma R_M|\xi|^2 \, dV_\Sigma - 2\int_{\partial\Sigma} H_{\partial M}|\xi|^2 \, dV_{\partial\Sigma} < \int_\Sigma H_\Sigma^2|\xi| \, dV_\Sigma.$$

Then

$$\operatorname{Index}(\Sigma) \geq \frac{2g+r-1-d}{2d}.$$

2.4 Applications in the Classification of Stable CMC Surfaces and Isoperimetric Problems

In this section we discuss some examples for which our index estimates for CMC surfaces apply. For some special manifolds that can be canonically embedded in $\mathbb{R}^n$ we characterize their stable CMC surfaces for a range of values of the mean curvature.

Let us first discuss some applications in the case of closed CMC surfaces in a Riemannian manifold without boundary.

Theorem 2.4.1. *Let M be a Riemannian manifold of dimension 3 immersed in $\mathbb{R}^d$ with second fundamental form II_M and scalar curvature R_M. If Σ is a closed orientable immersed CMC surface with mean curvature $H_\Sigma^2 > \sup|II_M|^2 - \inf R_M$ and $H_\Sigma \neq 0$ then*

$$\operatorname{Index}(\Sigma) \geq \frac{g}{d}.$$

In particular, if Σ is stable then it must be a sphere.

Proof. Whenever $\xi \neq 0$ we may pick $e_1 = \frac{\xi}{|\xi|}, e_2 = \frac{\star\xi}{|\star\xi|}$ as an orthonormal basis. Recalling that $|\xi| = |\star\xi|$, we have

$$\sum_{i=1}^{2} |II_M(e_i,\xi)|^2 + |II_M(e_i,\star\xi)|^2 = \sum_{i,j=1,2} |II_M(e_i,e_j)|^2|\xi|^2$$

at every point. Since $\sum_{i,j=1,2} |II_M(e_i,e_j)|^2 \leq |II_M|^2$ we obtain

$$\int_\Sigma \sum_{i=1}^{2} \left(|II_M(e_i,\xi)|^2 + |II_M(e_i,\star\xi)|^2\right) - R_M|\xi|^2 \, dV_\Sigma \leq$$
$$\left(\sup|II_M|^2 - \inf R_M\right) \int_\Sigma |\xi|^2 dV_\Sigma.$$

As long as $H_\Sigma^2 > \sup|II_M|^2 - \inf R_M$ the condition of Corollary 2.3.6 is satisfied. □

The following is a direct application of the Nash embedding theorem and the theorem above.

Corollary 2.4.2. *Let M be a closed Riemannian manifold of dimension 3. There exist constants $C > 0$ and $H_0 \geq 0$ depending on M such that for every closed orientable immersed CMC surface Σ of genus g in M and mean curvature $|H_\Sigma| > H_0$,*

$$\mathrm{Index}(\Sigma) \geq Cg$$

In particular, if Σ is stable then it must be a sphere.

Remark 2.4.3. *The result above proves a CMC version of a conjecture of Schoen and Marques-Neves for CMC surfaces of sufficiently large mean curvature in arbitrary 3-manifolds.*

First we would like to mention that Barbosa-do Carmo-Eschenburg [12] have studied CMC hypersurfaces in simply connected spaces of constant sectional curvature and proved that geodesic spheres are the only stable ones. In [16] Cavalcante-de Oliveira proved this using similar index estimates. Let us now present some examples that satisfy the conditions of the above results using canonical embeddings.

Closed CMC surfaces in $\mathbb{S}^2 \times \mathbb{R}$.

In [54] Souam proved that every stable CMC in $\mathbb{S}^2 \times \mathbb{R}$ is a rotational sphere. We are going to give an alternative proof using our index estimates.

Theorem 2.4.4 ([54, 4.1])**.** *Let Σ be a closed immersed CMC surface in $\mathbb{S}^2(r)\times\mathbb{R}$. Then,*

$$\mathrm{Index}(\Sigma) \geq \frac{g}{4}.$$

In particular, if in addition Σ is stable, then it is a sphere.

Proof. Consider the canonical embedding of $\mathbb{S}^2(r)\times\mathbb{R}$ in $\mathbb{R}^4$. Its second fundamental form is given by $II_{\mathbb{S}^2(r)\times\mathbb{R}}(X,Y) = \frac{1}{r}\langle \pi_* X, \pi_* Y\rangle$, where π denotes the projection onto $\mathbb{S}^2(r)$. In particular, $|II_{\mathbb{S}^2(r)\times\mathbb{R}}|^2 = \frac{2}{r^2}$. Since $R_{\mathbb{S}^2(r)\times\mathbb{R}} = \frac{2}{r^2}$, it follows from Theorem 2.4.1 that $\mathrm{Index}(\Sigma) \geq \frac{g}{4}$ whenever $H_\Sigma \neq 0$. It remains to consider the case $H_\Sigma = 0$. However, by moving spherical slices, it follows from the maximum principle that the only closed minimal surfaces in are slices $\mathbb{S}^2(r)\times\{t\}$ thus finishing the proof. □

Closed CMC surfaces in $T^2(\alpha,\beta)\times\mathbb{R}$.

Let $T^2(\alpha,\beta)$ denote the 2-torus defined by the quotient of $\mathbb{R}^2$ with respect to the lattice $\Gamma(\alpha,\beta)$ generated by $\{(0,1),(\alpha,\beta)\}$ satisfying $0\leq\alpha\leq\frac{1}{2}$, $\beta>0$ and $\alpha^2+\beta^2\geq 1$. We remark that any flat torus is isometric, up to dilations, to the above.

Let us consider the following map $f:\mathbb{R}^2\to\mathbb{R}^6$, $f=(f_1,f_2,f_3,f_4,f_5,f_6)$ defined by

$$\begin{aligned} f_1 &= C_1\cos(a_1u+b_1v)\\ f_2 &= C_1\sin(a_1u+b_1v)\\ f_3 &= C_2\cos(a_2u+b_2v)\\ f_4 &= C_2\sin(a_2u+b_2v)\\ f_5 &= C_3\cos(b_3v)\\ f_6 &= C_3\sin(b_3v). \end{aligned}$$

If we choose $a_i = 2\pi k_i$, $b_i = \frac{2\pi}{\sqrt{\alpha^2+\beta^2}}\left(l_i - k_i\frac{\alpha}{\beta}\sqrt{\alpha^2+\beta^2}\right)$ for positive integers k_i, l_i, i=1,2 and $b_3 = \frac{2\pi}{\sqrt{\alpha^2+\beta^2}}$ then f becomes an embedding of $T^2(\alpha,\beta)$ into $\mathbb{R}^6$.

Furthermore, as long as l_i, k_i satisfy

$$\frac{\alpha}{\beta}\sqrt{\alpha^2+\beta^2} < \frac{l_1}{k_1} < \left(\frac{\alpha}{\beta}+1\right)\sqrt{\alpha^2+\beta^2}$$
$$\left(\frac{\alpha}{\beta}-1\right)\sqrt{\alpha^2+\beta^2} < \frac{l_1}{k_1} < \frac{\alpha}{\beta}\sqrt{\alpha^2+\beta^2},$$

it is always possible to find positive constants C_1, C_2, C_3 so that f is an isometric embedding (see [41]).

Theorem 2.4.5. *Let k_i, l_i, $i = 1,2$ be positive integers and $C_1, C_2, C_3 > 0$ so that f defined above is an isometric embedding of $T^2(\alpha,\beta)$ into $\mathbb{R}^6$. If Σ is closed immersed CMC surface of genus g in $T^2(\alpha,\beta)\times\mathbb{R}$ with mean curvature $H_\Sigma^2 > C_1^2(a_1^2+b_1^2)^2 + C_2^2(a_2^2+b_2^2)^2 + C_3^2 b_3^4$, then*

$$\text{Index}(\Sigma) \geq \frac{g}{7}.$$

In particular, if in addition Σ is stable, then it is a sphere.

Proof. We embed $T^2(\alpha,\beta)\times\mathbb{R}$ in $\mathbb{R}^7$ using $f \times id$ and choose the following normal frame:

$$N_1 = (-\cos(a_1u+b_1v), -\sin(a_1u+b_1v), 0,0,0,0,0)$$
$$N_2 = (0,0,-\cos(a_2u+b_2v), -\sin(a_2u+b_2v), 0,0,0)$$
$$N_3 = (0,0,0,0,-\cos(b_3v), -\sin(b_3v), 0)$$
$$N_4 = M\Bigg(a_2C_2\sin(a_1u+b_1v), -a_2C_2\cos(a_1u+b_1v), -a_1C_1\sin(a_2u+b_2v),$$
$$a_1C_1\cos(a_2u+b_2v), -\frac{(a_2b_1-a_1b_2)C_1C_2}{b_3C_3}\sin(b_3v),$$
$$\frac{(a_2b_1-a_1b_2)C_1C_2}{b_3C_3}\cos(b_3v), 0\Bigg)$$

with $M = \frac{1}{a_1^2C_1^2 + a_2^2C_2^2 + \frac{C_1^2C_2^2}{b_3^2C_3^2}(a_2b_1-a_1b_2)^2}$.

The shape operators for each normal direction and basis $\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}$ are given by

$$S_1 = C_1 \begin{pmatrix} a_1^2 & a_1 b_1 \\ a_1 b_1 & b_1^2 \end{pmatrix}, S_2 = C_2 \begin{pmatrix} a_2^2 & a_2 b_2 \\ a_2 b_2 & b_2^2 \end{pmatrix}, S_3 = C_3 \begin{pmatrix} 0 & 0 \\ 0 & b_3^2 \end{pmatrix}, S_4 = 0.$$

So, we can compute $|S_1|^2 = C_1^2(a_1^2 + b_1^2)^2$, $|S_2|^2 = C_2^2(a_2^2 + b_2^2)^2$ and $|S_3|^2 = C_3^2 b_3^4$. From this it follows that

$$|II_{T^2\times\mathbb{R}}|^2 = C_1^2(a_1^2 + b_1^2)^2 + C_2^2(a_2^2 + b_2^2)^2 + C_3^2 b_3^4.$$

Since $R_{T^2\times\mathbb{R}} = 0$, the result follows from Theorem 2.4.1. □

In [37] Hauswirth-Perez-Romon-Ros study doubly periodic isoperimetric surfaces and conjecture that the isoperimetric profile of $T^2(\alpha,\beta)\times\mathbb{R}$ is given by spheres, cylinders around lines and pairs of planes. The authors then prove the conjecture for sufficiently large values of β. The result above provides support for the spherical part of the isoperimetric profile conjecture for any given values of α,β. It also says that any genus 1 isoperimetric surface, hence stable CMC, has a computable upper bound on the mean curvature, thus placing it further in the isoperimetric profile.

We would like to highlight two special cases. First, the hexagonal torus $T^2(\frac{1}{2},\frac{\sqrt{3}}{2})$ which is a critical case in Hauswirth-Perez-Romon-Ros' proof. In this case we may choose, for example, $a_1 = b_3 = 2\pi$, $a_2 = 4\pi$, $b_1 = 2\pi(1-\frac{1}{\sqrt{3}})$ and $b_2 = 2\pi(1-\frac{2}{\sqrt{3}})$. Then, to make f an isometric embedding we solve an underdetermined linear system so we may pick $C_1^2 = 1$, $C_2^2 = \frac{1}{2}(\sqrt{3}-1)$ and $C_3^2 = \frac{11+2\sqrt{3}}{6}$. However, these are not unique choices and are far from being optimal in the sense that the lower bound for H_Σ on the result above is minimal.

Secondly, the case of $\alpha = 0$, that is, rectangular tori. Rectangular tori can be embedded in $\mathbb{R}^4$ canonically and the estimate on H_Σ is better.

Corollary 2.4.6. *Let $T^2 = T^2(0,\beta)$, $\beta \geq 1$, be a rectangular torus. If Σ is a closed immersed CMC surface of genus g in $T^2\times\mathbb{R}$ with mean curvature $H_\Sigma^2 > (2\pi)^2\left(1+\frac{1}{\beta^2}\right)$, then*

$$\text{Index}(\Sigma) \geq \frac{g}{5}.$$

In particular, if in addition Σ is stable, then it is a sphere.

Proof. In the case of a rectangular tori the embedding reduces to

$$\begin{aligned} f_1 &= \frac{1}{2\pi}\cos(2\pi u) \\ f_2 &= \frac{1}{2\pi}\sin(2\pi u) \\ f_3 &= \frac{\beta}{2\pi}\cos(\frac{2\pi}{\beta}v) \\ f_4 &= \frac{\beta}{2\pi}\sin(\frac{2\pi}{\beta}v). \end{aligned}$$

We then embed $T^2 \times \mathbb{R}$ in $\mathbb{R}^5$ and using the canonical normal frame the shape operators satisfy $|S_1|^2 = (2\pi)^2$ and $|S_2|^2 = \left(\frac{2\pi}{\beta}\right)^2$. From this it follows that

$$|II_{T^2\times\mathbb{R}}|^2 = (2\pi)^2\left(1+\frac{1}{\beta^2}\right).$$

Since the scalar curvature is zero, the result follows from Theorem 2.4.1. □

Closed CMC surfaces in T^3.

Let us discuss the case of triply periodic CMC surfaces in $\mathbb{R}^3$. Similarly to the doubly periodic case, Hauswirth-Perez-Romon-Ros conjecture that the isoperimetric profile on T^3 is given only by spheres, cylinders around lines and pairs of planes. The following result supports the conjecture on the spherical branch in the same manner as above. Any rectangular torus is equivalent, up to dilations, to the quotient of $\mathbb{R}^3$ by the lattice generated by $\{\ (1,0,0),\ (0,A,0),\ (0,0,B)\ \}$. We identify such tori with $S^1(1)\times S^1(r_1)\times S^1(r_2)$ with $r_1 = \frac{2\pi}{A}$ and $r_2 = \frac{2\pi}{B}$.

Theorem 2.4.7. *Let $T^3 = S^1(1)\times S^1(r_1)\times S^1(r_2)$ be a rectangular 3-torus. If Σ is a closed immersed CMC surface of genus g in T^3 with mean curvature $H_\Sigma^2 > 1+\frac{1}{r_1^2}+\frac{1}{r_2^2}$, then*

$$\operatorname{Index}(\Sigma) \geq \frac{g}{6}.$$

In particular, if in addition Σ is stable, then it is a sphere.

Proof. Consider the canonical isometric embedding of T^3 into $\mathbb{R}^6$. Using the usual normal frame the corresponding shape operators satisfy $|S_1|^2 = 1$, $|S_2|^2 = \frac{1}{r_1^2}$ and $|S_3|^2 = \frac{1}{r_2^2}$. Thus,

$$|II_M|^2 = 1 + \frac{1}{r_1^2} + \frac{1}{r_2^2}.$$

Since the ambient space is flat, the statement follows from Theorem 2.4.1. $\square$

Closed CMC surfaces in Berger Spheres $\mathbb{S}_b^3(\kappa,\tau)$

Let $\mathbb{S}^3 \subset \mathbb{C}^2$ be the unit sphere. We denote by $\mathbb{S}_b^3(\kappa,\tau)$ the Berger Sphere with metric $g_{\kappa,\tau}(X,Y) = \frac{4}{\kappa}\left(g(X,Y + \left(\frac{4\tau^2}{\kappa} - 1\right) g(X,V)g(Y,V))\right)$, where g is the Euclidean metric and $V = (iz, iw)$ is a Killing vector field on $\mathbb{S}_b^3(\kappa,\tau)$.

Following Torralbo-Urbano [56, Section 2.1], as long as $\kappa - 4\tau^2 > 0$ the authors construct an isometric embedding of $\mathbb{S}_b^3(\kappa,\tau)$ into the complex projective space $\mathbb{C}P^2(\kappa - 4\tau^2)$ as a geodesic sphere. The space $\mathbb{C}P^2(\kappa - 4\tau^2)$ is the usual complex projective space endowed with the a dilation of the Fubini-Study metric $\frac{4}{\kappa - 4\tau^2} g_{FS}$ with constant holomorphic sectional curvature $\kappa - 4\tau^2$.

They then follow to embed $\mathbb{C}P^2(\kappa - 4\tau^2)$ into $\mathbb{R}^8$ and obtain a stability criterion similar to our index estimates in the case when the CMC surface is stable (see [56, Lemma 6.4]). By using the calculations already carried out in their paper we are able to improve their stability criterion in order to obtain the following index estimates.

Theorem 2.4.8. *Let $\mathbb{S}_b^3(\kappa,\tau)$ be a Berger Sphere with $\kappa - 4\tau^2 > 0$. If Σ is a closed immersed CMC surface of genus g in $\mathbb{S}_b^3(\kappa,\tau)$ with mean curvature $H_\Sigma^2 > \tau^2\left(\frac{\kappa}{4\tau^2} - 3\right)\left(\frac{\kappa}{4\tau^2} + 1\right)$ and $H_\Sigma \neq 0$, then*

$$\operatorname{Index}(\Sigma) \geq \frac{g}{8}.$$

In particular, if $\frac{\kappa}{4\tau^2} \in (1,3)$ then the index estimate is valid for all values of $H_\Sigma \neq 0$.

Proof. For short we write $\mathbb{S}_b^3 = \mathbb{S}_b^3(\kappa,\tau)$. Consider the isometric embeddings $\mathbb{S}_b^3 \subset \mathbb{C}P^2(\kappa - 4\tau^2) \subset \mathbb{R}^8$ constructed in [56, Section 2.1] and denote by $II_{\mathbb{S}_b^3}$ second fundamental form of $\mathbb{S}^3_b$ in $\mathbb{R}^8$. In the proof of [56, Theorem 6.5:Case(4)] Torralbo-Urbano compute the following:

$$\sum_{i,j=1}^{2} |II_{\mathbb{S}^3_b}(e_i,e_j)|^2 = -6\tau^2 + 2\kappa + \tau^2 \left(\frac{\kappa}{4\tau^2} - 1\right)^2 (1-C^2)^2,$$

where e_1, e_2 is an orthonormal frame on Σ and $C = \frac{\kappa}{4\tau} g_{\kappa,\tau}(N,V)$ satisfies $0 \leq C^2 \leq 1$.

Since $R_{S^3_b} = 2(\kappa - \tau^2)$, it follows from the proof of Theorem 2.4.1 that the index estimate is valid as long as $H^2_\Sigma > \tau^2 \left(\frac{\kappa}{4\tau^2} - 1\right)^2 - 4\tau^2$, which concludes the proof. □

Closed CMC surfaces in pinched manifolds.

Similar to [7] we would like to mention that the following examples remain valid in the CMC case.

Theorem 2.4.9. *Let M be an immersed manifold in $\mathbb{R}^d$ with $R_M > C|\vec{H}_M|^2$ for some $C > 0$. If Σ is a closed immersed orientable CMC surface of genus g in M with mean curvature $H^2_\Sigma \geq |\vec{H}_M|^2(1-2C)$ and $H_\Sigma \neq 0$, then*

$$\operatorname{Index}(\Sigma) \geq \frac{g}{d}.$$

In particular, if $R_M > \frac{1}{2}|\vec{H}_M|^2$ then any CMC surface satisfies the index estimate above.

Proof. Observe that Gauss' equation on M implies that $|II_M|^2 = |\vec{H}_M|^2 - R_M$. As a consequence, we have

$$\sup |II_M|^2 - \inf R_M < |\vec{H}_M|^2(1-2C).$$

The result follows from Theorem 2.4.1. □

In the case of closed hypersurfaces of $\mathbb{R}^4$ we have the following weaker pinching result. Compare to [7, Remark 5.3].

Theorem 2.4.10. *Let M be an embedded convex two-sided hypersurface of $\mathbb{R}^4$ with principal curvatures $0 \geq k_1 \geq k_2 \geq k_3$ with respect to the outward normal vector field. Suppose M satisfies a pinching condition $\frac{k_3}{k_1} < C$, for some $C > 0$. If Σ is a closed immersed CMC surface of genus g in M with mean curvature $H_\Sigma^2 \geq 3k_1^2(C^2 - 2)$ and $H_\Sigma \neq 0$, then*

$$\mathrm{Index}(\Sigma) \geq \frac{g}{4}.$$

In particular, if $\frac{k_3}{k_1} < \sqrt{2}$ then any CMC surface satisfies the index estimate above.

Proof. Firstly observe that $|II_M|^2 = k_1^2 + k_2^2 + k_3^2 \leq 3k_3^2$. Secondly, it follows from Gauss' equation that $R_M = 2(k_1k_2 + k_1k_3 + k_2k_3) \geq 6k_1^2$. That is,

$$\sup |II_M|^2 - \inf R_M \leq 3k_1^2 \left(\frac{k_3^2}{k_1^2} - 2 \right) < 3k_1^2(C^2 - 2).$$

The result follows from Theorem 2.4.1. □

Free boundary CMC surfaces

To avoid repetition, let us mention that a similar index estimate is valid for free boundary CMC surfaces in a mean convex domain of any of the above examples. More especifically, let Σ be a free boundary CMC surface of genus g and r boundary components in a domain with mean convex boundary, with respect to the inward normal direction, in any of the examples above. Suppose Σ satisfies the same mean curvature condition as in the closed case in its respective ambient manifold. Then $\mathrm{Index}(\Sigma) \geq \frac{2g+r-1-d}{2d}$ where d is the dimension of the Euclidean space where the respective ambient manifold is embedded.

Let us highlight the special case of mean convex domains in $\mathbb{R}^3$. The proof follows directly from Corollary 2.3.9. This particular case was first proved by Cavalcante-de Oliveira [15].

Theorem 2.4.11 ([15, Theorem 1.1])**.** *Let $(M, \partial M)$ be a mean convex, with respect to the inward normal vector, domain of $\mathbb{R}^3$. If Σ is a free boundary orientable CMC* *surface of M of genus g, r boundary components and* $H_\Sigma \neq 0$*, then*

$$\operatorname{Index}(\Sigma) \geq \frac{2g+r-4}{6}.$$

Chapter 3

Compactness of the Space of Free Boundary CMC Surfaces with Bounded Topology

3.1 Introduction

A constant mean curvature (CMC) surface Σ is a critical point of the area functional with respect to variations that preserve enclosed volume. As a consequence of the first variation of area, the scalar mean curvature has to be constant for a given choice of normal direction. If Σ is an immersed surface with boundary immersed in a manifold N with boundary and with $\partial\Sigma \subset \partial N$, we say the surface is free boundary CMC if it is a critical point with respect to variations that are in addition tangent along ∂N.

Although most results about minimal surfaces have an equivalent version for CMC surfaces, there are many distinctions between their behaviour. For example, when the mean curvature is non-zero, the mean curvature vector defines a trivialization of the normal bundle. That is, every CMC surface in a 3-manifold is 2-sided. On the other hand, CMC surfaces may have tangential self-touching points as long as the mean curvature vector points in opposite directions at those points.

The goal of this chapter is to prove a CMC version of Fraser-Li's result [26] for free boundary minimal surfaces. We prove:

Theorem 3.1.1. *Let N be a compact 3-dimensional manifold with boundary. Suppose $H_{\partial N} \geq H_0$ and let Σ_i be a sequence of free boundary embedded CMC surfaces with mean curvature H_i, genus g_i and number of ends r_i satisfying:*

(a) $|H_i| \leq H_0$;

(b) $g_i \leq g_0$;

(c) $r_i \leq r_0$;

(d) $\text{area}(\Sigma_i) \leq A_0$ *and*

(e) $\text{length}(\partial\Sigma_i) \leq L_0$.

Then there exists a smooth properly almost embedded CMC surface $\Sigma \subset N$ and a finite set $\Gamma \subset \Sigma$ such that, up to a subsequence, Σ_i converges to Σ locally graphically in the C^k topology on compact sets of $N \setminus \Gamma$ for all $k \geq 2$. Moreover, if Σ is minimal then it is properly embedded.

If in addition $(N, \partial N)$ satisfies either $Ric_N > 0$ and $A_{\partial N} \geq 0$ or $Ric_N \geq 0$ and $A_{\partial N} > 0$, then:

(i) when $H_\Sigma = 0$ the convergence is at most 2-sheeted;

(ii) when $H_\Sigma \neq 0$, then the convergence is 1-sheeted away from Γ.

Let us highlight the main differences that justify the extra hypotheses and the weaker compactness for embedded CMC surfaces.

First, we mention that there are no Steklov eigenvalue estimates for free boundary CMC surfaces. Consequently, the hypothesis of length bound remains crucial. Secondly, there is no suitable isoperimetric inequality that allows us to remove the bound on the area. One could exchange the length bound condition by stability of the surface (see [43]) but as seen in [4], stable free boundary CMC surfaces have bounded topology, so this condition would be topologically restrictive. Thirdly, even under positive Ricci curvature of the ambient space, a sequence of free boundary CMC surfaces may have a neck-pinching phenomenum where the norm of the second fundamental form blows-up at a point in the limit. Naturally the convergence is not smooth along these points where curvature is accumulating. Finally, the maximum principle for CMC surfaces only applies when their mean curvature vectors point in the same direction. That is, a sequence of embedded free boundary CMC surfaces may touch tangentially in the limit as long as the limiting surface is not minimal.

Despite the lack of certain useful technical results, the proof of the theorem relies on the same main ideas, each of which is proved using a less optimal technique to compensate for the above limitations. These are: L^2-curvature bounds from the topological bound, improvement to local pointwise curvature estimates, a removable singularity theorem for interior and boundary points and construction of positive eigenfunctions for the Jacobi operator to study convergence under a curvature condition on the ambient space.

Let us briefly address our approach to each of the above tools. The integral curvature bounds follow directly from the Gauss-Bonnet Theorem, from which the fact the mean curvature is non-zero introduces an extra term of area and the geodesic curvature along the boundary brings in a term of boundary length. The local pointwise curvature estimate comes from a blow-up argument as in [58] and Schauder estimates. It becomes relevant that the blow-up of a CMC surface around a point is a minimal surface in $\mathbb{R}^3$. Unlike [26, Theorem 4.1] we do not know whether a CMC surface is conformally equivalent to a punctured Riemann surface so we do not have the branch point structure to prove the removable singularity theorem. This will follow from a blow-up argument to prove that tangent cones are in fact planes (or half-planes) and a local free boundary CMC foliation argument to prove that it is also unique. These are the same ideas as in [58] together with the methods in [9] to deal with boundary singularity points. In fact, most of the calculations are done in the latter reference and only minor adaptations are necessary for the non-minimal case. Finally, the construction of positive eigenfunctions for the Jacobi operator on the limiting surface is a well known method (see [22, 51, 52]) and again, most of the necessary calculations are done in [9].

The extra condition on the mean curvature of ∂N is only necessary to apply the maximum principle for CMC surfaces and ensure that the interior of the limiting surface is properly immersed in N. Removing this condition would allow for interior tangential touching points between ∂N and the limit surface.

Finally, we mention that the corresponding result for closed CMC surfaces was proved by Sun in [55]. Although the core ideas are the same, the approach to certain steps is different and we focus on the behaviour at the boundary.

This chapter is divided as follows. Section 2 establishes notations, necessary definitions and we prove the geometric version of Schauder estimates. In section 3 we prove the local pointwise curvature estimates from uniform integral curvature bounds. The Removable Singularity Theorem is proved in section 4 and we write the proof for the specific case of a singularity along the boundary. Section 5 is dedicated to prove the Compactness Theorem.

3.2 Preliminaries on Convergence of Surfaces

In this section we establish notation and preliminary results that will be used throughout this chapter.

Let $(N^3, \partial N, g)$ be a 3-dimensional manifold with non-empty boundary and a smooth Riemannian metric g.

Definition 3.2.1. *We say that an immersed surface $\Sigma \subset N$ with non-empty boundary is properly almost embedded in an open set $U \subset N$ if $(\Sigma \setminus \partial\Sigma) \cap U$ is properly immersed in $(N \setminus \partial N) \cap U$, $\partial\Sigma \subset \partial N \cap U$ and there exists a set $\mathscr{S} \subset \Sigma$ such that*

(a) $\Sigma \setminus \mathscr{S}$ is embedded;

(b) for each $p \in \mathscr{S}$ there exists a neighbourhood V of p in U such that $\Sigma \cap V$ is a union of connected components W_j, $j = 1, \dots, l_p$, each W_j is embedded, for each $j \neq j'$ we have W_j lying to one side of $W_{j'}$ and $W_j \cap W_{j'} \subset \mathscr{S} \cap V$.

The set $\mathscr{S}$ is called the self-touching set of Σ. We say Σ is free boundary if in addition Σ is orthogonal to ∂N.

The following lemma is a standard application of Schauder estimates in geometry. It will allow us to improve from $C^{1,\alpha}$ to $C^{2,\alpha}$ graphical convergence of surfaces, given pointwise curvature estimates.

Lemma 3.2.2. *Let N be a three dimensional compact manifold, $A > 0$ and $f \in C^{0,\alpha}(N)$ with $|f|_{0,\alpha} < H_0$. There exists $r_0(N,A) > 0$ and $C_0(N,A,H_0) > 0$ such that* *for any C^2 immersed surface with free boundary $\Sigma \subset N$ with $H_\Sigma = f$ and $x_0 \in \Sigma$ satisfying*

$$|A_\Sigma(x)| \leq A \text{ on } \{x \in \Sigma : d_\Sigma(x,x_0) < r_0\}$$

there exists a function u defined on a subset of $T_{x_0}\Sigma$ with the following properties:

(i) $\{x \in \Sigma : d_\Sigma(x,x_0) < r_0\}$ *is the graph of u over the exponential function;*

(ii) the domain of u contains a ball centered at 0 *and radius* $\frac{r_0}{2}$ *and*

(iii) $\|u|_{B_{\frac{r_0}{4}}(0)}\|_{2,\alpha} < C_0$.

Proof. We may assume, that $r_0 > 0$ is smaller than the convexity radius of N so that $\{x \in \Sigma : d_\Sigma(x,x_0) < r_0\}$ is contained in a geodesic ball in N. Hence, without loss of generality we may assume that $\{x \in \Sigma : d_\Sigma(x,x_0) < r_0\}$ is contained in a small ball in $\mathbb{R}^3$ with a metric g, $x_0 = 0$ and $T_{x_0}\Sigma = \mathbb{R}^2 \times \{0\}$. We may further assume that g is equivalent to the Euclidean metric g_0, that is, $c^{-1}g_0 \leq g \leq cg_0$ where the constant $c > 0$ depends only on the geometry of N.

Claim 1. *If r_0 is sufficiently small, depending only on the geometry of N and A, then for all $x \in \{x \in \Sigma : d_\Sigma(x,0) < r_0\}$ we have $|\nu(x) - \nu(0)|_{g_0} \leq CAd_\Sigma(x,0)$ for some $C > 0$ depending only on the geometry of N.*

Indeed, let $\gamma \subset \Sigma$ be a curve parametrised by arc-length joining 0 to x. If we denote by ∇^0 the Euclidean connection, then it follows that

$$|\nu(x) - \nu(0)|_{g_0} \leq \sup_t |\nabla^0_{\dot\gamma}\nu|_{g_0} l(\gamma)$$

Since we are working in geodesic coordinates then $\nabla^0_{\dot\gamma} = \nabla_{\dot\gamma} + O(|\gamma|)$, hence

$$|\nabla^0_{\dot\gamma}\nu|_{g_0} \leq c|\nabla^0_{\dot\gamma}\nu|_g \leq c|\nabla_{\dot\gamma}\nu|_g + |\gamma|C \leq cA + r_0 C,$$

for some $C > 0$ depending only on the geometry of N. If we pick $r_0 C < cA$ we obtain

$$|\nu(x) - \nu(0)|_{g_0} \leq 2cAl(\gamma).$$

Taking the limit as the length of γ tends to $d_\Sigma(x,0)$ we conclude the claim as desired.

Now, suppose that $CAr_0 \leq 1$, then $\{x \in \Sigma : d_\Sigma(x,0) \leq r_0\}$ is the graph of a function u, otherwise we would have $\nu(x)$ perpendicular to $\nu(0)$ for some x hence $|\nu(x) - \nu(0)| > 1$. Furthermore, it follows that $\sup|\nabla u| \leq 2CAr_0$ as long as $C^2A^2r_0^2 < \frac{3}{4}$.

Let us denote by $\Omega \subset T_0\Sigma$ the domain of u and take $\delta > 0$ the largest radius such that $B_\delta(0) \subset \Omega$. In particular there exists $\bar{y} \in \partial B_\delta(0)$ such that $d_\Sigma((\bar{y},u(\bar{y})),(0,0)) = r_0$.

Claim 2. *For any pair $y_1, y_2 \in B_\delta(0)$ we have $d_\Sigma((y_1,u(y_1)),(y_2,u(y_2))) \leq (1+(2CAr_0)^2)^{\frac{1}{2}}|y_1 - y_2|$ and, if $4CAr_0 < 1$ then $\frac{r_0}{2} \leq \delta$.*

We have $y_1 + t(y_2 - y_1) \in B_\delta(0) \subset \Omega$ for $t \in [0,1]$, from which follows that $d_\Sigma((y_1,u(y_1)),(y_2,u(y_2))) \leq \int_0^1 (1+|\nabla u(y_1+t(y_2-y_1))|^2)^{\frac{1}{2}}|y_1-y_2|dt$ and it proves the inequality.

For the second part observe that if $4CAr_0 < 1$ then $d_\Sigma((\bar{y},u(\bar{y})),(0,0)) \leq \frac{3}{2}|\bar{y}|$. Suppose that $\delta < \frac{r_0}{2}$, then $|\bar{y}| < \frac{r_0}{2}$ and $d_\Sigma((\bar{y},u(\bar{y})),0) \leq \frac{3r_0}{4}$ which is a contradiction and it proves the claim.

Claim 3. *If $4CAr_0 < 1$ then ∇u restricted to $B_{\frac{r_0}{2}}(0)$ is a Lipschitz function of Lipschitz constant less than $\frac{3}{2}A$.*

The length of the second fundamental form is a multiple of norm of the Hessian of u, thus $|A_\Sigma| \leq A$ implies $|Hess(u)| \leq (1+|\nabla u|^2)^{\frac{1}{2}}A < 2A$ whenever $4CAr_0 < 1$. It follows that for any $y_1, y_2 \in \Omega$, $|\nabla u(y_1) - \nabla u(y_2)| \leq 2A d_\Sigma((y_1,u(y_1)),(y_2,u(y_2)))$. From the previous claim we get that

$$d_\Sigma((y_1,u(y_1)),(y_2,u(y_2))) \leq (1+(2CAr_0)^2)^{\frac{1}{2}}|y_1 - y_2| \leq \frac{3}{2}|y_1 - y_2|,$$

thus proving the claim.

Finally, on $B_{\frac{r_0}{2}}(0)$ the function u satisfies the equation $\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = f$. The coefficients of the equation have $C^{0,\alpha}$ norm depending on the metric g and the Lipschitz constant of ∇u. In particular all coefficients have $C^{0,\alpha}$ norm depending only on N and A.

Let us consider the case without boundary components on $\{x \in \Sigma : d_\Sigma(x,x_0) < r_0\}$. Then Schauder estimates [33, Corollary 6.3] implies that there exists a constant $C = C(N,A)$ such that $\|u|_{B_{\frac{r_0}{4}}(0)}\|_{2,\alpha} < C(\|u|_{B_{\frac{r_0}{2}}(0)}\|_0 + \|f\|_{0,\alpha})$. Note that $|u(y) - u(0)| < 2Ar_0|y| < \frac{2Ar_0^2}{2}$ and $u(0) = 0$. That is, $\|u|_{B_{\frac{r_0}{4}}(0)}\|_{2,\alpha} < C_0$ for $C_0 = C_0(N,A,H_0)$.

In case there are boundary components of Σ on $\{x \in \Sigma : d_\Sigma(x,x_0) < r_0\}$ then Schauder estimates ([2, Theorem 7.1]) gives us the same inequality with an extra term depending on the $C^{1,\alpha}$ norm of the inward normal derivative along the boundary. Since Σ is free boundary, u satisfies homogenous Neumann boundary conditions in which case the extra term vanishes and we obtain the same result. □

In the following we make precise the notion of graphical convergence of surfaces (see also [51]).

Definition 3.2.3. *Let N be a 3-manifold, with or without boundary, Σ_i a sequence of smooth surfaces in N and Σ a smooth surface in N. Pick $p \in \Sigma$ and $r > 0$ sufficiently small so that we may identify $B_r^N(p)$ with an Euclidean ball (or half-ball if $p \in \partial N$) with the same metric as N. We say that Σ_i converges locally graphically in the C^k ($C^{k,\alpha}$) topology to Σ at p if for $r > 0$ sufficiently small we have:*

(a) $\Sigma \cap B_r^N(p)$ is the graph of a C^k ($C^{k,\alpha}$) function u defined on $B_r(0) \subset T_p\Sigma$;

(b) $\Sigma_i \cap B_r^N(p)$ is the graph of C^k ($C^{k,\alpha}$) functions $u_i^1, \ldots u_i^L$ defined on $B_r(0) \subset T_p\Sigma$ for i sufficiently large and

(c) u_i^j converges to u in the C^k ($C^{k,\alpha}$) topology as $i \to \infty$.

If L is constant for sufficiently large i, we say that the convergence is L-sheeted.

Remark 3.2.4. *Let $U \subset N$ be an open set. If Σ_i is a sequence of embedded surfaces converging locally graphically with L-sheets to a two-sided surface Σ on a compacts sets of U then for each compact set $\Omega \subset \Sigma \cap U$, $r > 0$ sufficiently small and i sufficiently large we may write $\Sigma_i \cap B_r^N(\Omega)$ as the graph of functions $u_{i,j} : \Omega \to \mathbb{R}$, $j = 1, \ldots, L$, under the exponential map in the normal direction of Σ. In other words, each connected component $\Sigma_{i,j}$ of $\Sigma_i \cap B_r^N(\Omega)$ can be written as $\{\exp_x(u_{i,j}(x)N_\Sigma(x)) : x \in \Omega\}$ for each $j = 1, \ldots, L$. In addition, $u_{i,j}$ must tend to 0 for each $j = 1, \ldots, L$.*

3.3 Curvature Estimate for Free Boundary CMC Surfaces

In this section we prove an improvement from uniformly small total curvature estimate to uniform local pointwise curvature estimate. The proof is inspired by [58, Theorem 1] and we focus on the local estimates around a boundary point. We point out as well that the same proof holds even if the surface is not CMC but has uniformly bounded $C^{0,\alpha}$ mean curvature.

Theorem 3.3.1. *Let N be a compact 3-manifold with boundary. There exists a small enough $r_0 > 0$ such that the following holds: whenever Σ is a properly immersed CMC surface in N, $Q \in \Sigma$, $\partial\Sigma \cap B^N_{r_0}(Q)$ is either empty or free boundary in $\partial N \cap B^N_{r_0}(Q)$ and the mean curvature of Σ satisfies $H_\Sigma \leq H_0$. Then there exists $\varepsilon_0 > 0$ depending on $B^N_{r_0}(Q)$ and H_0 such that if $\int_{\Sigma\cap B^N_{r_0}(Q)} |A|^2 \leq \varepsilon_0$, then*

$$\max_{0\leq\sigma\leq r_0}\left(\sigma^2 \sup_{\Sigma\cap B^N_{r_0-\sigma}(Q)} |A|^2\right) \leq C_0$$

where the constant C_0 only depends on geometry of $B^N_{r_0}(Q)$ and H_0.

Proof. Suppose false, that is, for $r_n \to 0$ and $\varepsilon_n \to 0$ there exist free boundary CMC surfaces $\Sigma_n \subset N$ and $Q_n \in \Sigma_n$ satisfying:

(i) $H_n \leq H_0$;

(ii) $\int_{\Sigma_n\cap B^N_{r_n}(Q_n)} |A_n|^2 \leq \varepsilon_n$ and

(iii) $\max_{0\leq\sigma\leq r_n}\left(\sigma^2 \sup_{\Sigma\cap B^N_{r_n-\sigma}(Q_n)} |A_n|^2\right) > n$.

Pick $0 < \sigma_n < r_n$ such that

$$\sigma_n^2 \sup_{\Sigma\cap B^N_{r_n-\sigma_n}(Q_n)} |A_n|^2 = \max_{0\leq\sigma\leq r_n}\left(\sigma^2 \sup_{\Sigma\cap B^N_{r_n-\sigma}(Q_n)} |A_n|^2\right)$$

and write $\lambda_n^2 = \sup_{\Sigma\cap B^N_{r_n-\sigma_n}(Q_n)} |A_n|^2$. For each n there exists $z_n \in \Sigma \cap B^N_{r_n-\sigma_n}(Q_n)$ such that $|A_n(z_n)| > \frac{\lambda_n}{2}$.

By taking a subsequence we have $Q_n \to Q$ and $B^N_{r_n}(Q_n)$ contained in a geodesic ball of N. Without loss of generality we may assume that $\Sigma_n \cap B^N_{r_n}(Q_n) \subset \mathbb{R}^3$ with a metric g. Henceforth we denote by $B_r(p)$ the ball in $\mathbb{R}^3$ with respect to the metric g.

Now, define $\tilde{\Sigma}_n = \lambda_n(\Sigma_n - z_n)$, it satisfies:

(a) $|\tilde{A}_n(\tilde{x}_n)| \leq 2$ for all $\tilde{x}_n \in \tilde{\Sigma}_n \cap B_1(0)$ and n sufficiently large;

(b) $|\tilde{A}_n(0)| > \frac{1}{2}$ and

(c) $\int_{\tilde{\Sigma}_n} |\tilde{A}_n|^2 \leq \varepsilon_n$.

Indeed, if $\tilde{x}_n \in B_1(0)$ then $\tilde{x}_n = \lambda_n(x_n - z_n)$ with $x_n \in \Sigma_n \cap B_{\frac{1}{\lambda_n}}(z_n)$. It follows that $x_n \in \Sigma_n \cap B_{r_n - (\sigma_n - \frac{1}{\lambda_n})}(Q_n)$. Since

$$\left(\sigma_n - \frac{1}{\lambda_n}\right)^2 \sup_{\Sigma \cap B^N_{r_n - (\sigma_n - \frac{1}{\lambda_n})}(Q_n)} |A_n|^2 \leq \max_{0 \leq \sigma \leq r_n} \left(\sigma^2 \sup_{\Sigma \cap B^N_{r_n - \sigma}(Q_n)} |A_n|^2\right) = \sigma_n^2 \lambda_n,$$

it implies that

$$|A_n(x_n)|^2 < \left(\frac{1}{1 - \frac{1}{\sigma_n \lambda_n}}\right)^2 \lambda_n^2.$$

We know that $\sigma_n^2 \lambda_n^2 > n$ thus $|A_n(x_n)| < 2\lambda_n$ for n sufficiently large, which proves (a). Property (b) follows from rescaling and (c) holds because the total curvature is scale invariant.

Let $\hat{\Sigma}_n$ denote the connected component of $\tilde{\Sigma}_n \cap B_1(0)$ containing 0. We may further assume, after an ambient rotation and translation, that $T_0\hat{\Sigma}_n = \{x_3 = 0\}$.

Using property (a), Lemma 3.2.2 implies that there exists $\hat{r}_0$ independent of n such that $\hat{\Sigma}_n \cap B_{\hat{r}_0}(0)$ is the graph of a function u_n satisfying $\|u_n|_{B_{\frac{\hat{r}_0}{4}}(0)}\|_{2,\alpha} < \hat{C}_0$, where $\hat{C}_0$ is independent of n. If 0 is a boundary point then the domain of u_n is a half ball but Lemma 3.2.2 remains true.

Finally, u_n converges, up to a subsequence, in the C^2 topology to a function u_∞. If we denote its graph by $\hat{\Sigma}_\infty$ then it satisfies $|\hat{A}_\infty(0)| \geq \frac{1}{2}$ from (b) and $\int_{\hat{\Sigma}_\infty} |\hat{A}_\infty|^2 = 0$ from (c), which is a contradiction and completes the proof of the theorem. $\square$

3.4 Removable Singularities

As we will see later, the compactness result does not give us smooth convergence everywhere. The points in which we do not have sufficient curvature estimates are potential singularities either because of a neckpinching phenomenum where the curvature may blow up, or self-touching points where the convergence is not single sheeted. However, we are still able to prove that these are removable singularities so the limiting surface is still a smooth object.

We are going to prove that if the total curvature is bounded on a CMC surface, then isolated singularities are removable. This is an adaptation of [58, Theorem 2] and the arguments are the same except for the foliation argument to prove uniqueness of the tangent cone. We are going to focus on the case in which the singularity is along the boundary, but the same result holds for interior singularities and the proof follows the exact same arguments.

The idea of the proof is to improve the integral curvature bound to a pointwise curvature decay near the singularity to show that the tangent cones are totally geodesic. By adapting the foliation argument of White [58] we prove that the tangent cone is unique from which we can show that near the singularity the surface is indeed the graph of a C^1 function. We can then improve it further using elliptic regularity.

Firstly, let us prove the existence of a local CMC foliation with free boundary which will be needed later. This is a straightforward adaptation of [9, Section 3] together with White's approach to deal with a family of functionals [58, Appendix].

Let $\theta \in (0, \frac{\pi}{4})$ and define $D_\theta = \{x \in \mathbb{R}^2 : (x_1 + a)^2 + x_2^2 \leq 1 \text{ and } x_1 \geq 0\}$, where $a = \cos^{-1}(\theta) \in (\frac{1}{\sqrt{2}}, 1)$. This is the part of the disk of radius 1 centered on the x_1-axis that intersects the line $x_1 = 0$ at angle θ. Its boundary components are denoted by $\partial_0 D_\theta = \partial D_\theta \cap \{x \in \mathbb{R}^2 : x_1 = 0\}$ and $\partial_+ D_\theta = \overline{\partial D_\theta \setminus \partial_0 D_\theta}$. The regular cylinder over D_θ in $\mathbb{R}^3$ is denoted by $C_\theta = D_\theta \times \mathbb{R}$, with corresponding boundary components $\partial_0 C_\theta = \partial_0 D_\theta \times \mathbb{R}$ and $\partial_+ C_\theta = \partial_+ D_\theta \times \mathbb{R}$.

Given a function $f \in C^{2,\alpha}(D_\theta)$, we define $N_g^+(f)$ to be the normal vector over graph(f) with respect to g pointing in the positive direction of the x_3-axis, that is, $g(N_g^+(f), \frac{\partial}{\partial x_3}) > 0$. We write $H_g^+(f) = g(\vec{H}_g(f), N_g^+(f))$ as the scalar mean curvature with respect to $N_g^+(f)$. In particular, $\vec{H}_g(f)$ points in the positive direction of

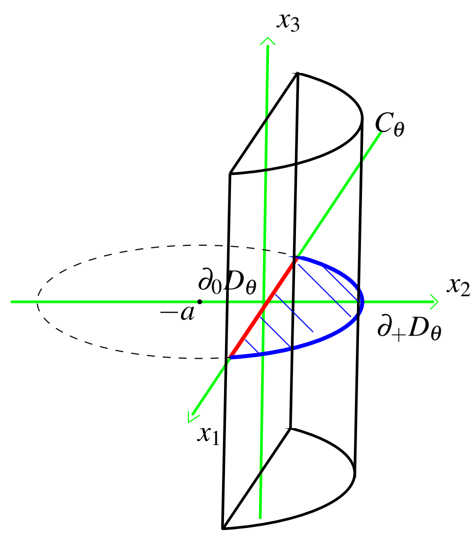

Figure 3.1: Blow up picture of a free boundary CMC surface near singularity on the boundary

the x_3-axis when $H_g^+(f) > 0$ and in the negative direction otherwise.

Let us denote by X the space of $C^{2,\alpha}$ metrics on C_θ and define the map

$$\Phi : \mathbb{R} \times \mathbb{R} \times X \times C^{2,\alpha}(\partial_+ D_\theta) \times C^{2,\alpha}(D_\theta) \to C^{0,\alpha}(D_\theta) \times C^{1,\alpha}(\partial_0 D_\theta) \times C^{2,\alpha}(\partial_+ D_\theta)$$

by

$$\Phi(h,t,g,w,u) = \left(H_g^+(t+u) - h, \frac{\partial}{\partial \eta_g}(t+u), u_{|_{\partial_+ D_\theta}} - w \right),$$

where η_g is the inward conormal vector along $\partial_0 D_\theta$.

Proposition 3.4.1 ([9, Proposition 21])**.** *For every $t_0 \in \mathbb{R}$, there exist a neighbourhood U_{t_0} of the Euclidean metric δ in X, $\varepsilon_{t_0} > 0$ and*

$$u : (-\varepsilon_{t_0}, \varepsilon_{t_0}) \times (t_0 - \varepsilon_{t_0}, t_0 + \varepsilon_{t_0}) \times U_{t_0} \times B_{\varepsilon_{t_0}}^{C^{2,\alpha}(\partial_+ D_\theta)}(0) \to C^{2,\alpha}(D_\theta)$$

so that $t \mapsto$ graph$(t + u(h,t,g,w))$ defines a $C^{2,\alpha}$ foliation of $D_\theta \times [t_0 - \frac{\varepsilon_{t_0}}{2}, t_0 + \frac{\varepsilon_{t_0}}{2}]$ by surfaces with constant mean curvature h with respect to the metric g, free boundary along $\partial_0 C_\theta$ and $(t+u)_{|_{\partial_+ D_\theta}} = t + w$.

Furthermore, if $h > 0$ then $\vec{H}_g(t+u)$ points in the positive direction of the x_3-axis and in the negative direction when $h < 0$.

Proof. Observe that Φ defined above is a C^1 function and $D_5\Phi(0,t_0,\delta,0,0)$ defines the same isomorphism as in [9, Appendix B]. The result then follows from the Implicit Function Theorem. $\square$

Let us denote by $B_1^+ = \{x \in \mathbb{R}^3 : \|x\| \leq 1, x_1 \geq 0\}$ the upper half ball in $\mathbb{R}^3$ and $\partial_0 B_1^+ = \partial B_1^+ \cap \{x \in \mathbb{R}^3 : x_1 = 0\}$.

Theorem 3.4.2. *Let g be a Riemannian metric on B_1^+ and Σ be a smooth, properly embedded, CMC surface in $B_1^+ \setminus \{0\}$, $\partial\Sigma \subset \partial B_1^+$, free boundary on $\partial_0 B_1^+ \setminus \{0\}$ and $0 \in \overline{\partial\Sigma}$. Suppose $\int_\Sigma |A_\Sigma|^2 \leq C$ then $\Sigma \cup \{0\}$ is a smooth properly embedded CMC surface in B_1^+.*

Proof. Let $r_0 > 0$ and $\varepsilon_0 > 0$ be as in Theorem 3.3.1. Pick $\delta > 0$ sufficiently small so that $\int_{\Sigma \cap B_\delta^+} |A_\Sigma|^2 \leq \varepsilon_0$. It follows from Theorem 3.3.1 that

$$|A_\Sigma(x)| d_g(x,0) \leq C_0,$$

whenever $d_g(x,0) < \delta$.

Claim 1. *Every tangent cone of Σ at 0 is a union of half-planes in $\mathbb{R}^3 \setminus \{0\}$.*

Let $r_i \to \infty$ be any sequence and $\Sigma_i = r_i\Sigma$ its corresponding blow-up around 0.

Observe that the curvature estimate above is scale invariant, so Σ_i satisfies the same curvature bounds whenever $d_g(x,0) < r_i\delta$. It follows that, up to a subsequence, Σ_i converges locally graphically in the $C^{1,\alpha}$ topology on compact sets to a complete surface Σ_∞ in T_0N, which we identify with $\mathbb{R}^3$ with the Euclidean metric. Lemma 3.2.2 implies that Σ_i in fact converges locally graphically in the $C^{2,\alpha}$ topology on compact sets of $\mathbb{R}^3 \setminus \{0\}$. In particular, Σ_∞ has free boundary on $\{x \in \mathbb{R}^3 : x_1 = 0\} \setminus \{0\}$ and for any compact set $K \subset \mathbb{R}^3 \setminus \{0\}$ we have

$$\int_{K \cap \Sigma_\infty} |A_\infty|^2 = \lim_{i \to \infty} \int_{K \cap \Sigma_i} |A_i|^2 = \lim_{i \to \infty} \int_{(r_i^{-1}K) \cap \Sigma} |A_\Sigma|^2 = 0$$

That is, $A_\infty = 0$. Hence Σ_∞ is an union of half-planes perpendicular to $\{x \in \mathbb{R}^3 : x_1 = 0\}$.

Claim 2. *If $\delta > 0$ is sufficently small then $\Sigma \cap B^+_\delta \setminus \{0\}$ is topologically a finite and disjoint union of disks, half disks or half-disks punctured at 0 with free boundary on $\partial_0 B^+_\delta \setminus \{0\}$.*

Firstly, we improve the curvature estimates. Fix $y \in \Sigma_i \cap B^+_{r_i\delta}$ and put $x = r_i^{-1} y \in \Sigma \cap B^+_\delta$. Then $|A_\Sigma(x)| d_g(x,0) = |A_i(y)| d_g(y,0) \to 0$ as $i \to \infty$ from the previous claim. Thus $\lim_{x \to 0} |A_\Sigma(x)| d_g(x,0) = 0$.

Secondly, we use a standard Morse Theory argument. Let $f : \Sigma \cap B^+_\delta \to \mathbb{R}$ be defined by $f(x) = \frac{1}{2} d_g(x,0)^2$. We can write its Hessian at a point x and direction v as $\mathrm{Hess} f(v,v)_x = A_\Sigma(v,v)_x g(N_\Sigma(x), \dot\gamma(1))_x + Q_x(v,v)$, where N_Σ is the normal vector field on Σ, γ is the minimizing geodesic in B^+_1 from 0 to x and Q is a quadratic form satisfying $Q \geq \frac{1}{2} g$ as long as $d_g(x,0) < \delta$ is sufficiently small. Hence, any critical point of f in a small neighbourhood of 0 is a strict local minimum, even if the critical point is along the boundary. It follows from Morse Theory for manifolds with boundary [40] that every connected component of $\Sigma \cap B^+_\delta$ is a disk with a single critical point, a free boundary half disk with a single critical point or a free boundary half disk punctured at 0 without critical points. Since critical points cannot accumulate, it must be finite for δ sufficiently small. This proves the claim.

Next, we shall prove that for each punctured half-disk its tangent cone is unique. Pick $\hat\Sigma \subset \Sigma \cap B^+_\delta \setminus \{0\}$ a connected component that corresponds to a punctured half-disk. Let $\hat\Sigma_i$ be the corresponding dilation by $r_i \to \infty$ and $\hat\Sigma_\infty \subset \Sigma_\infty$ its tangent cone at 0. As we have seen above, $\hat\Sigma_\infty$ is a half-plane perpendicular to $\{x \in \mathbb{R}^3 : x_1 = 0\}$.

Claim 3. *The tangent cone $\hat\Sigma_\infty$ is unique, that is, it is independent of the blow-up sequence $r_i \to \infty$.*

Without loss of generality let us identify $\hat\Sigma_\infty$ with the half-plane $P_+ = \{x \in \mathbb{R}^3 : x_3 = 0\} \cap \{x \in \mathbb{R}^3 : x_1 \geq 0\}$. We may also assume, possibly taking another subsequence, that the mean curvature vector of $\hat\Sigma_i$ points in the positive direction of the x_3-axis for i sufficiently large.

Let D_θ and C_θ be as in Proposition 3.4.1 Since $\hat\Sigma_i$ is embedded, then for i sufficiently large $\hat\Sigma_i \cap \partial_+ C_\theta$ is the graph of a function $w_i = w_{r_i}$ over $\partial_+ D_\theta$. We know that $\hat\Sigma_i$ converges in $C^{2,\alpha}$ graphical sense to P_+ away from zero, from which follows that $\|w_i\|_{2,\alpha} \to 0$ where the norm is defined on $\partial_+ D_\theta$.

The mean curvature of $\hat{\Sigma}_i$ is given by $\hat{H}_i = r_i{}^{-1} H_\Sigma$ which is constant at each i and tends to 0 as i tends to infinity. If we denote by g_i the corresponding blow up of the metric g in a neighbourhood of 0 in B_1^+, it follows from Proposition 3.4.1 that for all i sufficiently large there exists $\varepsilon_0 > 0$ and a unique function $u_{i,t} = u(\hat{H}_i, t, g_i, w_i) : D_\theta \to \mathbb{R}$ for each $|t| < \varepsilon_0$ such that $u_{i,t} = t + w_i$ on $\partial_+ D_\theta$, the graph of $u_{i,t}$ over D_θ meets $\partial_0 C_\theta$ orthogonally, it has constant mean curvature equal to $\hat{H}_i$ and its mean curvature vector points in the same direction as the mean curvature vector of $\hat{\Sigma}_i$. Furthermore, $u_{i,t}$ varies smoothly on t and it defines a foliation of a region $D_\theta \times [-c, c]$ for some $c > 0$ independent of i.

Let $t_i \in (-\varepsilon_0, \varepsilon_0)$ be such that $u_{i,t_i}(0) = 0$. Since their mean curvature vector points in the same direction, we may apply the maximum principle to the graph of u_{i,t_i} and $\hat{\Sigma}_i$ to conclude that $\hat{\Sigma}_i$ must be contained entirely to one side of the graph of u_{i,t_i}. The side itself will depend on whether t_i is positive or negative. In case the mean curvature vector of $\hat{\Sigma}_i$ were to point in the negative direction of the x_3-axis, we would have used the same argument with $u(-\hat{H}_i, t, g_i, w_i)$ instead.

Finally, pick another sequence $r_i' \to \infty$ and let $\hat{\Sigma}_i'$, P_+' be its corresponding blow-up and tangent cone respectively. For each k we pick i_k so that $r_{i_k}' > k r_k$. Observe that $\hat{\Sigma}_{i_k} = \frac{r_{i_k}'}{r_k} \hat{\Sigma}_k$ which is contained to one side of the graph of $\frac{r_{i_k}'}{r_k} u_{k,t_k}$. Since each u_{k,t_k} is regular at 0, $\frac{r_{i_k}'}{r_k} u_{k,t_k}$ must converge to a unique tangent cone, that is, P_+. If P_+' were different from P_+, then it would imply that $\frac{r_{i_k}'}{r_k} \hat{\Sigma}_k$ contains points on both sides of the graph of $\frac{r_{i_k}'}{r_k} u_{k,t_k}$. Notice that P_+' must also be a half plane with free boundary on $\{x \in \mathbb{R}^3 : x_1 = 0\}$. This concludes the claim.

It follows from uniqueness of the tangent cone that $\hat{\Sigma} \cup \{0\}$ is the graph of a C^1 function u around a neighbourhood of 0 over the tangent cone. Since u is also a solution to the CMC equation with smooth coefficients, then by elliptic regularity it must also be smooth.

To conclude the proof, now that we know that $\hat{\Sigma} \cup \{0\}$ is regular, it follows from the maximum principle that it is the unique connected component containing 0. Thus $\Sigma \cup \{0\}$ is properly embedded. $\square$

We now state the equivalent result for removable interior singularities without repeating the proof.

Theorem 3.4.3. *Let g be a Riemannian metric on B_1 and Σ be a smooth, properly embedded, CMC surface in $B_1 \setminus \{0\}$ with $H_\Sigma \leq H_0$, and $0 \in \overline{\Sigma}$. Suppose $\int_\Sigma |A_\Sigma|^2 \leq C$ then $\Sigma \cup \{0\}$ is a smooth properly embedded CMC surface in B_1.*

Remark 3.4.4. *The proof is exactly the same with a minor modification in the construction of the foliation. That is, by dropping the Neumann component in the definition of Φ in Proposition 3.4.1.*

3.5 Compactness Theorem

In this section we will prove our main theorem for free boundary embedded CMC surfaces. As we shall see, the limiting surface may not be embedded because CMC surfaces may have tangential self-intersection as long as the normal vector points at opposite directions.

Theorem 3.5.1. *Let N be a compact 3-dimensional manifold with boundary. Suppose $H_{\partial N} \geq H_0$ and let Σ_i be a sequence of free boundary embedded CMC surfaces with mean curvature H_i, genus g_i and number of ends r_i satisfying:*

(a) $|H_i| \leq H_0$;

(b) $g_i \leq g_0$;

(c) $r_i \leq r_0$;

(d) $\text{area}(\Sigma_i) \leq A_0$ *and*

(e) $\text{length}(\partial \Sigma_i) \leq L_0$.

Then there exists a smooth properly almost embedded CMC surface $\Sigma \subset N$ and a finite set $\Gamma \subset \Sigma$ such that, up to a subsequence, Σ_i converges to Σ locally graphically in the C^k topology on compact sets of $N \setminus \Gamma$ for all $k \geq 2$. Moreover, if Σ is minimal then it is properly embedded.

If in addition $(N, \partial N)$ satisfies either $Ric_N > 0$ and $A_{\partial N} \geq 0$ or $Ric_N \geq 0$ and $A_{\partial N} > 0$, then:

(i) *when $H_\Sigma = 0$ the convergence is at most 2-sheeted;*

(ii) when $H_\Sigma \neq 0$, then the convergence is 1*-sheeted away from* Γ.

Proof. Let us dente by A_i the second fundamental form of Σ_i. Given $x \in \Sigma$, it follows from the Gauss equation that $|A_i|^2(x) = H_i^2 + 2K_N(T_x\Sigma) - 2K_i(x)$, where K_N, K_i are the sectional curvatures of N and Σ_i respectively. From the Gauss-Bonnet theorem we have

$$\int_{\Sigma_i} |A_i|^2 = H_i^2 \text{area}(\Sigma_i) + 2\int_{\Sigma_i} K_N(T_xN) + 2\int_{\partial\Sigma_i} \kappa_g + 4\pi(2g_i + r_i - 2),$$

where κ_g denotes the geodesic curvature of $\partial\Sigma_i$. Because Σ_i is free boundary, we have that that $\kappa_g = A_{\partial N}(\tau_{\partial\Sigma}, \tau_{\partial\Sigma})$, where $A_{\partial N}$ is the second fundamental form of ∂N with respect to the inner normal vector and $\tau_{\partial\Sigma}$ is the unit tangent vector of $\partial\Sigma$. Since N is compact, there exists a constant $C = C(N) > 0$ such that

$$\int_{\Sigma_i} |A_i|^2 \leq C(H_i^2 \text{area}(\Sigma_i) + g_i + r_i + \text{area}(\Sigma_i) + \text{length}(\partial\Sigma_i)).$$

Hence, from hypotheses (a)-(e) we have that the total curvature is uniformly bounded by a constant $C_0 = C_0(N, H_0, g_0, r_0, A_0, L_0) > 0$.

Denote by μ_i the Radon measure on N defined by $\mu_i(U) = \int_{\Sigma_i \cap U} |A_i|^2$, for a subset $U \subset N$. It follows from the above that there exists a Radon measure μ in N such that, up to a subsequence, μ_i converges weakly to μ. Furthermore, the set $\Gamma = \{p \in N : \mu(\{p\}) \geq 1\}$ has at most C_0 elements.

For each $x \in N \setminus \Gamma$ there exists $r > 0$ such that $\mu(B_r^N(x)) < 1$. Hence, for each i sufficiently large $\mu_i(B_r^N(x)) < 1$, that is, $\int_{\Sigma_i \cap B_r^N(x)} |A_i|^2 < 1$. By possibly choosing a smaller value of r, it follows from Theorem 3.3.1 that

$$\sup_{\Sigma_i \cap B^N_{\frac{r}{2}}} |A_i|^2 \leq C,$$

for some constant $C > 0$ independent of i.

Let $r < r_0$ as in Lemma 3.2.2 and suppose that, up to a subsequence, $\Sigma_i \cap B^N_{\frac{r}{4}}(x)$ is non-empty for all i sufficiently large. Since Σ_i is embedded then $\Sigma_i \cap B^N_{\frac{r}{4}}(x)$ is the union of disjoint embedded connected components $\Sigma_{i,1}, \ldots, \Sigma_{i,L}$ each of which is the graph of a function defined on an open ball of fixed radius on $T_{y_{i,j}}\Sigma_{i,j}$ for some $y_{i,j} \in \Sigma_{i,j}$, $j = 1, \ldots, L$. Because Σ_i is compact, the number of sheets L must be finite and thus constant for i sufficiently large. Hence, under the appropriate identifications we may further assume that $\Sigma_{i,j} \cap B^N_{r'}(x)$ is the graph of a function $u_{i,j}$ defined on $B_{r'}(0) \subset T_{y_i}\Sigma_i$, for some $0 < r' < \frac{r}{8}$ depending only on L and a fixed $y_i \in \Sigma_i \cap B^N_{r'}(x)$. Furthermore $u_{i,j}$ has uniform $C^{2,\alpha}$ bounds as in Lemma 3.2.2.

We may assume that, up to a subsequence, y_i converges to y' and $T_{y_i}\Sigma_i$ converges to a plane $P \subset T_{y'}N$. In which case, under further identifications, we have that $\Sigma_{i,j} \cap B^N_{\frac{r'}{2}}(y')$ is the graph of a function $u'_{i,j}$ defined on an open ball on P (or half-ball in case y' is on the boundary of N) and uniform $C^{2,\alpha}$ estimates for all i sufficiently large, and each $j = 1,\ldots,L$. Hence, up to a subsequence $u'_{i,j}$ converges to a function u'_j in the $C^{2,\beta}$ topology for all $\beta < \alpha$ and $\Sigma_i \cap B^N_{\frac{r'}{2}}(y')$ converges to $\Sigma'_j = \mathrm{graph}(u'_j)$ for each $j = 1,\ldots,L$. From this it follows that $y' \in \Sigma' = \cup_j \Sigma'_j$ and P is in fact $T_{y'}\Sigma'$.

Now, given any compact set $K \subset N \setminus \Gamma$ we cover it by finitely many open balls $\{B^N_{\frac{r}{4}}(x_k)\}_{k=0,\ldots,m}$ as above so that, up to a subsequence, $\Sigma_i \cap (\cup_k B^N_{\frac{r}{4}}(x_k))$ converges to a surface $\Sigma' \subset N \setminus \Gamma$ locally graphically in the $C^{2,\beta}$ topology on K. Taking a countable exhaustion by compact sets and using a diagonal argument we have that, up to a subsequence, Σ_i converges to Σ' locally graphically in the $C^{2,\beta}$ topology on compact sets of $N \setminus \Gamma$. Smooth convergence away from Γ follows from Allard's regularity Theorem [5, 6, 36].

Since Σ_i is a properly embedded CMC surface with free boundary along $\partial N \setminus \Gamma$, then Σ' is a properly almost embedded CMC surface in $N \setminus \Gamma$ with free boundary along $\partial N \setminus \Gamma$. Observe that on a neighbourhood of any self-touching point of Σ' the surface can be written as connected components that lie to one side of one another. A transversal self-intersection is an open condition so it would contradict the fact that Σ_i is embedded.

Define Σ to be the closure of Σ', so $\Gamma = \Sigma \setminus \Sigma'$. Since Γ is finite, there exists $r > 0$ so that $B^N_r(p) \setminus \{p\}$ contains no points of Γ. From the graphical convergence on compact sets of $B^N_r(p) \setminus \{p\}$, each sheet must converge to an embedded component of $\Sigma \cap B^N_r(p) \setminus \{p\}$. Since there are only finitely many sheets, we may pick $r > 0$ sufficiently small so that every component of $\Sigma \cap B^N_r(p) \setminus \{p\}$ contains p in its closure.

Claim 1. *The limit surface Σ is regular.*

We know that $\int_{\Sigma\cap(B_r^N(p)\setminus\{p\})}|A_\Sigma|^2 \leq C_0$. Thus we may apply the Removable Singularity Theorems 3.4.2 or 3.4.3 to each embedded component of $\Sigma\cap(B_r^N(p)\setminus\{p\})$ depending on whether p belongs to the boundary of N or to the interior. Hence $\Sigma = \Sigma'\cup\Gamma$ is a regular surface.

Now suppose that $H_\Sigma = 0$. Then the maximum principle for minimal surfaces implies that there are no self-touching points, so Σ is embedded.

We now prove properties (i) and (ii). Henceforth, let us assume that either $Ric_N > 0$ and $A_{\partial N} \geq 0$ or $Ric_N \geq 0$ and $A_{\partial N} > 0$.

The following argument is the same as in [51] and [9] and we include the main idea without repeating the calculations.

Claim 2. *If $H_\Sigma \neq 0$ then the convergence is 1-sheeted away from Γ.*

Suppose by contradiction that Σ_i converges to Σ with at least 2 sheets. Since the convergence is graphical over compact sets of $N\setminus\Gamma$, for any compact set $\Omega\subset\Sigma\setminus\Gamma$ and a sufficiently small tubular neighbourhood $V_r(\Omega)$ of Ω, $\Sigma_i\cap V_r(\Omega)$ contains at least 2 connected components Σ_i^0 and Σ_i^1 each of which can be written as follows, for i sufficiently large:

$$\Sigma_i^\nu = \{exp_p(u_i^\nu(p)N_\Sigma(p)) : p\in\Omega\},$$

for $u_i^\nu : \Omega\to\mathbb{R}$, $\nu = 0,1$.

We may assume without loss of generality that $u_i^1 > u_i^0$ for all i sufficiently large since the surface Σ_i is embedded.

Let us denote $v_i(t) = u_i^0 + t(u_i^1 - u_i^0)$ and $\Phi_i(p,t) = exp_p(v_i(t)(p)N_\Sigma(p))$ for $p\in\Omega$. Consider the variation $\Sigma_i(t) = \{\Phi_i(p,t) : p\in\Omega\}$ so that $\Sigma_i(\nu) = \Sigma_i^\nu$, $\nu = 0,1$. Take a function $w\in C_c^\infty(\Omega)$ with compact support and define $w^\nu\in C_c^\infty(\Sigma_i^\nu)$ by $w^\nu(\Phi(p,\nu)) = u(p)$ for each $\nu = 0,1$. Now, consider any compactly supported vector field X in $V_r(\Omega)$ such that $g(X,N_{\Sigma_i^\nu}) = w^\nu$ along Σ_i^ν, for each $\nu = 0,1$, and its associated flow $\Psi(q,s)$ on $V_r(\Omega)$. Finally we consider the following 2-parameter variation:

$$\Sigma_i(t,s) = (\Psi_s)_\sharp\Sigma_i(t).$$

From the first variation formula it follows that

$$\frac{\partial}{\partial s}\bigg|_{s=0} area(\Sigma_i(\nu,s)) = -H_i \int_{\Sigma_i^\nu} w^\nu,$$

for each $\nu = 0,1$. Since the mean curvature vectors of each sheet converge to $\vec{H}_\Sigma$, we have that their normal vectors point in the same direction for i sufficiently large. That is, their scalar mean curvature has the same sign, so we have

$$\frac{\partial}{\partial s}\bigg|_{s=0} area(\Sigma_i(1,s)) - \frac{\partial}{\partial s}\bigg|_{s=0} area(\Sigma_i(0,s)) = -H_i \int_{\Sigma} (J\Phi_i^1 - J\Phi_i^0) w,$$

where for each $\nu = 0,1$, $\Phi_i^\nu(p) = \Phi_i(p,\nu)$ and $J\Phi_i^\nu$ its corresponding Jacobian.

Following the same idea and calculations as in [9], using the Mean Value Theorem and taking the second variation of area, for some $t_i \in (0,1)$, we obtain that $\tilde{h}_i = u_i^1 - u_i^0$ satisfies

$$-\int_\Omega w\left(L_i\tilde{h}_i + H_iF_i(\tilde{h}_i)\right) + \int_{\partial N\cap\Omega} B_i(w,\tilde{h}_i) = 0.$$

Here L_i is an elliptic operator in divergence form, L_i converges uniformly to the Jacobian operator L_Σ of Σ as $i \to \infty$, $F_i(\tilde{h}_i) = (J\Phi_i^1 - J\Phi_i^0)$ and $B_i(w_1,w_2)$ converges uniformly to $w_1(\frac{\partial w_2}{\partial \eta} + A_{\partial N}(N_\Sigma,N_\Sigma)w_2)$, where η denotes the outward conormal vector of $\partial\Sigma$ and $A_{\partial N}$ the second fundamental form with respect to the outward conormal of ∂N. Since u_i^ν tends to 0 in C^2, Φ_i^ν tends to the identity uniformly; that is, $F_i(\tilde{h}_i)$ tends to 0.

Fix a point $q_0 \in \Omega$ and define $h_i(p) = \frac{\tilde{h}_i(p)}{h_i(q_0)}$. Using Harnack estimates and a blow-up contradiction argument we have that h_i converges smoothly to a function h on Ω satisfying

$$\begin{cases} L_\Sigma h = 0, \text{ on } \Omega; \\ \dfrac{\partial h}{\partial \eta} + A_{\partial N}(N_\Sigma,N_\Sigma)h = 0, \text{ on } \partial\Sigma\cap\Omega. \end{cases}$$

Since the blow up of the elliptic equation for h_i is the one above, the proof is the same as in [9, Claim 1, p.20]. It follows from the maximum principle for elliptic equations that $h > 0$ on Ω. By taking an exhaustion of $\Sigma \setminus \Gamma$ by compact sets and using a diagonal sequence argument, we have that h_i converges to $h > 0$ on compact sets of $\Sigma \setminus \Gamma$. Again, it follows from the exact same argument as in [9] that h is uniformly bounded on $\Sigma \setminus \Gamma$, thus it extends to a smooth function on Σ, which is positive by the maximum principle.

Finally, we observe that the conditions $Ric_N > 0$ and $A_{\partial N} \geq 0$ or $Ric_N \geq 0$ and $A_{\partial N} > 0$ imply that the quadratic form

$$Q_\Sigma(w_1, w_2) = -\int_\Sigma w_1 L_\Sigma w_2 + \int_{\partial\Sigma} w_1 \left(\frac{\partial w_2}{\partial \eta} + A_{\partial N}(N_\Sigma, N_\Sigma) w_2 \right)$$

is positive definite. Hence its first eigenvalue is positive. However, h is a positive eigenfunction so it must correspond to the first eigenvalue, which is a contradiction since the corresponding eigenvalue is 0. We conclude that the convergence must be 1-sheeted away from Γ.

The case $H_\Sigma = 0$ follows the exact same arguments as above. Unlike the previous case, the mean curvature vector of each graph converges to 0 so they may have opposite orientation. As a consequence, the scalar mean curvature may have opposite sign, hence the function F_i would not tend to 0 as desired.

Claim 3. *If $H_\Sigma = 0$ then the convergence is at most 2-sheeted.*

Suppose the convergence is at least 3-sheeted. Then at least two sheets have the same orientation; that is, the normal vectors of the sheets point in the same direction. Following the same argument as in the previous case, the functions F_i will converge to 0, so we can construct a positive eigenfunction $h > 0$ for the Jacobi operator L_Σ on Σ corresponding to the eigenvalue 0. This is a contradiction since L_Σ is a positive operator under either of the conditions $Ric_N > 0$ and $A_{\partial N} \geq 0$ or $Ric_N \geq 0$ and $A_{\partial N} > 0$.

This completes the proof.

□

Chapter 4

Higher Dimensional Surgery and Steklov Eigenvalues

4.1 Introduction

In this chapter we study questions related to shape optimization for the Steklov eigenvalue problem on compact Riemannian manifolds of dimension at least three. The classical theorem of R. Weinstock [57] states that among all simply connected domains in $\mathbb{R}^2$ with fixed boundary length 2π, the unit disc uniquely maximizes the first Steklov eigenvalue. It was shown in [13] that the Weinstock inequality holds in any dimension, provided one restricts to the class of convex domains. Namely, for every bounded convex domain $\Omega \subset \mathbb{R}^n$ we have $\bar{\sigma}_1(\Omega) \leq \bar{\sigma}_1(\mathbb{B}^n)$, where $\bar{\sigma}_j(\Omega) = \sigma_j(\Omega)|\partial\Omega|^{1/(n-1)}$ denotes the normalized eigenvalues. However, in the wider class consisting of contractible domains, the higher dimensional analogue of Weinstock's theorem fails since there exist contractible domains $\mathbb{B}^n_{\varepsilon,\delta}$ (see Figure 4.1) with $\bar{\sigma}_1(\mathbb{B}^n_{\varepsilon,\delta}) > \bar{\sigma}_1(\mathbb{B}^n)$ when $n \geq 3$ ([31]). The proof involves a 1-dimensional surgery in which a small tubular neighbourhood of a curve connecting boundary components is removed from an annular domain. The construction in [31] shows more generally that in dimensions greater than or equal to three the number of boundary components does not affect the supremum of the normalized first Steklov eigenvalue. This is in contrast to the situation in dimension two, where by adding an extra boundary component to a surface the normalized first Steklov eigenvalue can be made strictly larger ([30], [42]).

In this chapter we consider the effect of higher dimensional surgeries on the Steklov spectrum of compact Riemannian manifolds with boundary. Specifically, we show that one can perform surgeries of codimension two or higher while keeping the normalized Steklov eigenvalues nearly unchanged. Given a compact n-dimensional Riemannian manifold Ω with boundary, a compact properly embedded m-dimensional submanifold Σ of Ω, and $\delta > 0$ small, we let $\Omega_{\Sigma,\delta}$ denote the Lipschitz domain obtained by removing the δ tubular neighbourhood of Σ from Ω. We call this procedure a *surgery of codimension* $n-m$ (see Section 4.2 for more precise details). Our main result is the following.

Theorem 4.1.1. *The Steklov spectrum of a compact Riemannian manifold with boundary changes continuously under surgeries of codimension at least two, in the sense that, if* $n-m \geq 2$ *then* $\lim_{\delta\to 0^+} \bar{\sigma}_j(\Omega_{\Sigma,\delta}) = \bar{\sigma}_j(\Omega)$ *for* $j = 0, 1, 2, \ldots$.

One purpose of such surgeries is to simplify the topology of the manifold. When $m = 1$ and $j = 1$ this theorem was proved in [31], in which case the surgery can be applied to construct a manifold with connected boundary from any given manifold with boundary, while keeping the first normalized Steklov eigenvalue nearly unchanged. Theorem 1.1 implies more generally that the supremum of the j-th normalized eigenvalue among all manifolds is the same as the supremum among manifolds having relatively simple topology; that is, among manifolds which can be obtained by performing surgeries up to codimension two. Specifically, our result implies that given any compact Riemannian manifold Ω of dimension $n \geq 3$, and given any $\varepsilon > 0$ and $k \in \mathbb{N}$, there exists a smooth subdomain $\tilde{\Omega}$ of Ω such that the homomorphism of the m-th homology groups $i_* : H_m(\partial\tilde{\Omega}) \to H_m(\tilde{\Omega})$ induced by the inclusion $i : \partial\tilde{\Omega} \hookrightarrow \tilde{\Omega}$ is injective for $m = 0, 1, \ldots \min\{n-3, \lfloor \frac{n}{2} \rfloor\}$, and such that $|\bar{\sigma}_j(\tilde{\Omega}) - \bar{\sigma}_j(\Omega)| < \varepsilon$ for $j = 1, \ldots, k$. This conclusion involves some standard but rather involved algebraic topology, and we omit the proof since our results here are analytic in nature.

An immediate consequence of Theorem 4.1.1 is that for $n \geq 3$, balls do not maximize higher normalized Steklov eigenvalues among contractible domains in $\mathbb{R}^n$.

Corollary 4.1.2. *Let $n \geq 3$. For any $k \geq 1$, there exists a contractible domain Ω^* in $\mathbb{R}^n$ such that*

$$\bar{\sigma}_j(\Omega^*) > \bar{\sigma}_j(\mathbb{B}^n)$$

for $j = 1, \cdots, k$.

This is not surprising, especially in light of the fact that a disc does not maximize higher Steklov eigenvalues among simply connected domains in $\mathbb{R}^2$. The classical result [38] gives the upper bound $\bar{\sigma}_j(\Omega) \leq 2\pi j$ for any simply connected domain Ω in $\mathbb{R}^2$ and all $j \geq 0$. When $j = 1$, equality is characterized in Weinstock's theorem ([57]), but for $j \geq 2$ the inequality is strict ([34], [32]). However, it was shown in [34] that the inequality is sharp and is achieved in the limit by a sequence of simply connected domains degenerating to j identical discs. By analogy with the $n = 2$ case, it is natural to ask whether a similar result is true in higher dimensions. Using ideas from Theorem 4.1.1 we show that this is not the case.

Theorem 4.1.3. *For $n \geq 3$ and $j \geq 2$, the supremum of the j-th normalized Steklov eigenvalue among contractible domains in $\mathbb{R}^n$ is not achieved in the limit by a sequence of contractible domains degenerating to the disjoint union of j identical round balls.*

This is a direct consequence of a more general theorem on the convergence of Steklov eigenvalues of overlapping domains as they are pulled apart, Theorem 4.3.2, discussed in Section 4.3.

The chapter is organized as follows. In section 4.2 we recall basic knowledge about the Steklov eigenvalue problem and prove our main result, Theorem 4.1.1, on higher dimensional surgeries and the Steklov spectrum. In section 4.3 we discuss applications and related results in connection with questions about shape optimization for the Steklov problem for domains in $\mathbb{R}^n$, and prove Corollary 4.1.2, Theorem 4.3.2, and Theorem 4.1.3.

4.2 Continuity of Steklov Eigenvalues under Codimension 2 Surgeries

In this section, we recall some basic facts about the Steklov eigenvalue problem, and prove our main theorem on the continuity of eigenvalues under certain higher dimensional surgeries of at least codimension 2. Let (Ω, g) be a compact, connected n-dimensional smooth Riemannian manifold with nonempty boundary. The Steklov eigenvalue problem is given by

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = \sigma u & \text{on } \partial\Omega \end{cases}$$

where ν is the outer unit normal to Ω. When the trace operator $T : H^1(\Omega) \to L^2(\partial\Omega)$ is compact, in particular, if the domain has Lipschitz boundary, the spectrum of the Steklov eigenvalue problem is discrete

$$0 = \sigma_0 < \sigma_1(\Omega) \leq \sigma_2(\Omega) \leq \cdots$$

and the eigenvalues have a standard Rayleigh quotient characterization

$$\sigma_j(\Omega) = \inf \left\{ \frac{\int_\Omega |\nabla u|^2}{\int_{\partial\Omega} u^2} \ : \ 0 \neq u \in H^1(\Omega), \ \int_{\partial\Omega} u\phi_i = 0 \text{ for } i = 0, \ldots, j-1 \right\} \tag{4.2.1}$$

where $\{\phi_0, \phi_1, \phi_2, \ldots\}$ is a complete orthonormal basis of $L^2(\partial\Omega)$ such that ϕ_i is an eigenfunction with eigenvalue $\sigma_i(\Omega)$ for each $i = 1, 2, \ldots$. Alternatively,

$$\sigma_j(\Omega) = \inf_{E_{j+1}} \sup_{u \in E_{j+1} \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2}{\int_{\partial\Omega} u^2} \tag{4.2.2}$$

where the infimum is taken over all $(j+1)$-dimensional subspaces E_{j+1} of the Sobolev space $H^1(\Omega)$.

Definition 4.2.1. *Given a smooth m-dimensional submanifold Σ embedded in the manifold Ω, with boundary $\partial\Sigma$ embedded in $\partial\Omega$, and meeting $\partial\Omega$ orthogonally along $\partial\Sigma$. Let*

$$L_\delta = \{x \in \Omega : d(x, \Sigma) < \delta\} \ \textit{and} \ T_\delta = \{x \in \Omega : d(x, \Sigma) = \delta\}.$$

Denote

$$\Omega_\delta = \Omega \setminus L_\delta.$$

Taking out the tubular neighborhood L_δ of Σ gives a domain Ω_δ. Throughout this chapter, we call this procedure surgery of dimension m or surgery of codimension n-m.

We will apply the following “no concentration" lemma to show that if $1 \leq m \leq n-2$, then the interior and boundary L^2 norms of a sequence of eigenfunctions don't concentrate near the neck T_δ as $\delta \to 0^+$. When $m = 1$, that is when Σ is a smooth curve, this phenomenon was verified in [31, Lemma 4.2]. We prove here that it is in fact still true provided $2 \leq m \leq n-2$.

Lemma 4.2.2. *Suppose $1 \leq m \leq n-2$. There exists a constant $r_0 > 0$ such that if $u_\delta \in W^{1,2}(\Omega_\delta)$ and*

$$\int_{\Omega_{r_0/2} \setminus \Omega_{r_0}} u_\delta^2 + \int_{\Omega_\delta \setminus \Omega_{r_0}} |\nabla u_\delta|^2 \leq C$$

for $\delta \in (0, r_0/2)$, with C independent of δ. Then

$$\lim_{\delta \to 0} \|u_\delta\|_{L^2(T_\delta)} = 0, \tag{4.2.3}$$

$$\lim_{s \to 0} \|u_\delta\|_{L^2(\Omega_\delta \setminus \Omega_s)} = 0, \tag{4.2.4}$$

$$\lim_{s \to 0} \|u_\delta\|_{L^2((L_s \setminus L_\delta) \cap \partial\Omega)} = 0 \tag{4.2.5}$$

for any $0 < \delta < s < r_0/2$.

Proof. For simplicity of presentation we will assume that the normal bundle of Σ is trivial; otherwise, we can consider local trivializations and add up all of the corresponding estimates. Choose $r_0 > 0$ sufficiently small such that the exponential map of the normal bundle of Σ is a diffeomorphism from the r_0 neighbourhood of the zero section onto its image, and such that the metric on the r_0 tubular neighbourhood L_{r_0} of Σ in M is uniformly equivalent to the product metric $\tilde{g} + dr^2 + r^2 g_{S^{n-m-1}}$ on $\Sigma \times D_{r_0}$, where $\tilde{g}$ is the metric on Σ induced from g, $g_{S^{n-m-1}}$ is the standard metric on the sphere S^{n-m-1} and D_t is the ball of radius t centred at the origin in $\mathbb{R}^{n-m}$. Since Σ intersects $\partial\Omega$ orthogonally along $\partial\Sigma$, we may further choose r_0 sufficiently small such that the metric on $L_{r_0} \cap \partial\Omega$ is uniformly equivalent to the product metric $g_{\partial\Sigma} + dr^2 + r^2 g_{S^{n-m-1}}$.

We first localize the support of u_δ to lie near Σ. Choose a smooth radial cut-off function such that

$$\varphi(x,r,\theta) = \begin{cases} 1 & r \leq \frac{r_0}{2} \\ 0 & r \geq r_0 \end{cases}$$

and define $v_\delta = \varphi u_\delta$. It follows from the Cauchy-Schwarz and arithmetic geometric mean inequalities that

$$|\nabla v_\delta|^2 \leq 2(\varphi^2 |\nabla u_\delta|^2 + u_\delta^2 |\nabla\varphi|^2).$$

Since $|\nabla\varphi|$ can be bounded in terms of r_0 which is a fixed number, we have

$$\int_{\Omega_\delta} |\nabla v_\delta|^2 \leq 2\int_{\Omega_\delta \setminus \Omega_{r_0}} |\nabla u_\delta|^2 + 2C_1 \int_{\Omega_{r_0/2} \setminus \Omega_{r_0}} u_\delta^2 \leq C_2 \tag{4.2.6}$$

where C_1, C_2 are constants depending only on C and r_0. We will choose δ much smaller than r_0. From the construction we have that $u_\delta = v_\delta$ on T_δ, and thus to prove equation (4.2.3) in the lemma it suffices to prove that for any $\varepsilon > 0$

$$\int_{T_\delta} u_\delta^2 = \int_{T_\delta} v_\delta^2 \leq \varepsilon \int_{\Omega_\delta} |\nabla v_\delta|^2$$

for sufficiently small δ. Since the metric on Ω_δ is uniformly equivalent to the product metric on the support of v_δ, it suffices to prove the estimate for the product metric.

For a fixed point p in Σ we denote the restriction of $v_\delta(x,r,\theta)$ to the annulus $D_{r_0} \setminus D_\delta$ in $\mathbb{R}^{n-m}$ at this point p by $v(r,\theta)$. Choose a harmonic function h on $D_{r_0} \setminus D_\delta$ as follows

$$\begin{cases} \Delta h = 0 & D_{r_0} \setminus D_\delta \\ h = v = 0 & \partial D_{r_0} \\ h = v & \partial D_\delta. \end{cases}$$

Since harmonic functions minimize the Dirichlet energy we have

$$\int_{D_{r_0}\setminus D_\delta} |\nabla h|^2 \le \int_{D_{r_0}\setminus D_\delta} |\nabla v|^2. \tag{4.2.7}$$

For any σ with $\delta \le \sigma \le r_0$ we have

$$\begin{aligned}
\int_{D_{r_0}\setminus D_\sigma} \Delta h^2 &= \int_{\partial D_{r_0}} \frac{\partial h^2}{\partial r} - \int_{\partial D_\sigma} \frac{\partial h^2}{\partial \sigma} \\
&= -\int_{\partial D_\sigma} \frac{\partial h^2}{\partial \sigma} \\
&= -\sigma^{n-m-1} \frac{d}{d\sigma}\left[\sigma^{-n+m+1} \int_{\partial D_\sigma} h^2\right]
\end{aligned}$$

where last equality follows since the volume measure on ∂D_σ is σ^{n-m-1} times that on the unit sphere ∂D_1. Since $\Delta h = 0$, we have

$$\int_{D_{r_0}\setminus D_\sigma} \Delta h^2 = 2\int_{D_{r_0}\setminus D_\sigma} |\nabla h|^2,$$

which together with (4.2.7) implies

$$\begin{aligned}
-\sigma^{n-m-1} \frac{d}{d\sigma}\left[\sigma^{-n+m+1} \int_{\partial D_\sigma} h^2\right] &= 2\int_{D_{r_0}\setminus D_\sigma} |\nabla h|^2 \\
&\le 2\int_{D_{r_0}\setminus D_\delta} |\nabla h|^2 \\
&\le 2\int_{D_{r_0}\setminus D_\delta} |\nabla v|^2.
\end{aligned}$$

Now dividing both sides by σ^{n-m-1} and integrating with respect to σ over the interval $[\delta, r_0]$ we obtain

$$\delta^{-n+m+1} \int_{\partial D_\delta} v^2 \le 2\left(\int_\delta^{r_0} \sigma^{-n+m+1}\, d\sigma\right) \int_{D_{r_0}\setminus D_\delta} |\nabla v|^2.$$

If $1 \le m \le n-2$, this implies that

$$\int_{\partial D_\delta} v^2 \le \varepsilon_n(\delta) \int_{D_{r_0}\setminus D_\delta} |\nabla v|^2$$

where $\varepsilon_{m+2}(\delta) = 2\delta \ln(\frac{r_0}{\delta})$ and $\varepsilon_n(\delta) = \frac{2\delta}{n-m-2}$ for $n \geq m+3$. Integrating the above inequality over Σ we obtain

$$\int_{T_\delta} v_\delta^2 \leq \varepsilon_n(\delta) \int_{\Sigma^m} \int_{D_{r_0} \setminus D_\delta} |\nabla v|^2 \leq \varepsilon_n(\delta) \int_{\Omega_\delta} |\nabla v_\delta|^2, \tag{4.2.8}$$

since $|\nabla v| \leq |\nabla v_\delta|$ because ∇v is the Euclidean gradient on the slice $D_{r_0} \setminus D_\delta$ at a point on Σ. Since $\lim_{\delta \to 0} \varepsilon_n(\delta) = 0$ for $1 \leq m \leq n-2$, this completes the proof of (4.2.3).

In what follows we shall prove (4.2.4) and (4.2.5) of the lemma. By the same argument used to prove (4.2.8), we have that for any $\delta \leq t < r_0/2$

$$\int_{T_t} v_\delta^2 \leq \varepsilon_n(t) \int_{\Omega_\delta} |\nabla v_\delta|^2 \leq C_2 \varepsilon_n(t)$$

where $\varepsilon_n(t)$ is defined as above, and C_2 is as in (4.2.6) and is independent of δ. Integrating with respect to t over $[\delta, s]$ for $s < r_0/2$ yields

$$\int_{\Omega_\delta \setminus \Omega_s} v_\delta^2 = \int_{L_s \setminus L_\delta} v_\delta^2 \leq \begin{cases} C_2 \left(s^2 \ln(\frac{r_0}{s}) - \delta^2 \ln(\frac{r_0}{\delta}) + \frac{s^2 - \delta^2}{2} \right), & n = m+2 \\ C_2 \, (s^2 - \delta^2)/(n-m-2), & n \geq m+3. \end{cases} \tag{4.2.9}$$

Since the right hand sides tend to zero as $s \to 0$ for all $\delta < s$, this concludes the proof of (4.2.4).

Denote $\Sigma_t = \{x \in \Sigma : dist_\Sigma(x, \partial\Sigma) \leq t\}$ and denote $\partial\Sigma_t = \{x \in \Sigma : dist_\Sigma(x, \partial\Sigma) = t\}$ (note this is only part of the topological boundary of Σ_t). We choose t sufficiently small such that $H_{\partial\Sigma_\tau} \leq C$ for all $\tau \leq t$, where $H_{\partial\Sigma_\tau}$ is the mean curvature of $\partial\Sigma_\tau$ with respect to outer unit normal. By the coarea formula,

$$\int_0^t \int_{\partial\Sigma_\tau} \left(\int_{D_s \setminus D_\delta(x)} v_\delta^2 \right) d\tau = \int_{\Sigma_t} \left(\int_{D_s \setminus D_\delta(x)} v_\delta^2 \right) \leq \int_\Sigma \int_{D_s \setminus D_\delta(x)} v_\delta^2 = \int_{L_s \setminus L_\delta} v_\delta^2.$$

Denote

$$F_{s,\delta}(\tau) := \int_{\partial\Sigma_\tau} \left(\int_{D_s \setminus D_\delta(x)} v_\delta^2 \right).$$

Then (4.2.9) yields

$$\lim_{s\to 0}\int_0^t F_{s,\delta}(\tau)\, d\tau = 0.$$

Thus there exists $\tau_0 \in (0,t)$ such that

$$\lim_{s\to 0} F_{s,\delta}(\tau_0) = 0.$$

We differentiate $F_{s,\delta}(\tau)$ to get

$$\frac{d}{d\tau}F_{s,\delta}(\tau) = \int_{\partial\Sigma_\tau}\left(\int_{D_s\setminus D_\delta(x)} 2v_\delta \frac{dv_\delta}{d\tau}\right) - \int_{\partial\Sigma_\tau}\left(\int_{D_s\setminus D_\delta(x)} v_\delta^2\right) H_{\partial\Sigma_\tau}.$$

Integrating over $[0,\tau_0]$, applying the coarea formula, and using Cauchy-Schwarz inequality and the arithmetic-geometric mean inequality result in that for any $0 < \varepsilon < 1$

$$\begin{aligned}
|F_{s,\delta}(\tau_0) - F_{s,\delta}(0)| &= \left|\int_{\Sigma_{\tau_0}}\left(\int_{D_s\setminus D_\delta(x)} 2v_\delta \frac{dv_\delta}{d\tau}\right) - \int_0^{\tau_0}\int_{\partial\Sigma_\tau}\left(\int_{D_s\setminus D_\delta(x)} v_\delta^2\right) H_{\partial\Sigma_\tau}\, d\tau\right| \\
&\leq \varepsilon \int_{\Sigma_{\tau_0}}\left(\int_{D_s\setminus D_\delta(x)} |\nabla v_\delta|^2\right) + (1/\varepsilon + C)\int_{\Sigma_{\tau_0}}\left(\int_{D_s\setminus D_\delta(x)} v_\delta^2\right) \\
&\leq C_2\varepsilon + (1/\varepsilon + C)\int_{L_s\setminus L_\delta} v_\delta^2.
\end{aligned}$$

Taking $\varepsilon = s$ and letting s tend to zero, from (4.2.9) it follows that

$$F_{s,\delta}(0) \to 0 \text{ as } s \to 0.$$

Hence we obtain the equation (4.2.5). □

Lemma 4.2.3. *Consider the following logarithmic cut-off function along the submanifold Σ^m, where $1 \leq m \leq n-2$,*

$$\varphi_\delta = \begin{cases} 0 & r \leq \delta^2 \\ \frac{2\ln\delta - \ln r}{\ln\delta} & \delta^2 \leq r \leq \delta \\ 1 & \delta \leq r. \end{cases}$$

Then $\int_\Omega |\nabla\varphi_\delta|^2 \to 0$ *as* $\delta \to 0^+$.

Proof. Integrating with respect to product metric gives

$$\int_\Omega |\nabla\varphi_\delta|^2 \le \int_\Sigma \int_{D_\delta \setminus D_{\delta^2}} |\nabla\varphi_\delta|^2 = \frac{c(n)|\Sigma|}{(\ln\delta)^2} \int_{\delta^2}^{\delta} r^{n-m-3}\, dr = c(n)|\Sigma|\,\delta(n)$$

where $\delta(n) = -1/\ln\delta$ when $n = m+2$ and $\delta(n) = (\delta^{n-m-2} - \delta^{2n-2m-4})/(n-m-2)(\ln\delta)^2$ when $n > m+2$. Since $\delta(n) \to 0$ as $\delta \to 0$, this completes the proof. $\square$

We now prove our main theorem, that given a compact Riemannian manifold with boundary (Ω^n, g), surgeries of dimension m (see Definition 4.2.1) can be performed while keeping the Steklov eigenvalues nearly unchanged, provided $1 \le m \le n-2$.

Theorem 4.1.1. *If* $1 \le m \le n-2$, *then* $\lim_{\delta\to 0^+} \sigma_j(\Omega_\delta) = \sigma_j(\Omega)$ *for* $j = 0, 1, 2, \ldots$.

Proof. Let $u_\delta^0, u_\delta^1, u_\delta^2, \ldots$ be $L^2(\partial\Omega_\delta)$-orthonormal Steklov eigenfunctions such that u_δ^j is a Steklov eigenfunction of Ω_δ with eigenvalue $\sigma_j(\Omega_\delta)$,

$$\begin{cases} \Delta u_\delta^j = 0 & \text{in } \Omega_\delta \\ \frac{\partial u_\delta^j}{\partial\eta} = \sigma_j(\Omega_\delta) u_\delta^j & \text{on } \partial\Omega_\delta. \end{cases}$$

Claim 1. *For any* $j \in \mathbb{N}$, $\sigma_j(\Omega_\delta)$ *is uniformly bounded from above for small* δ.

Proof. Let $\{f_1, \ldots, f_{j+1}\} \subset H^1(\Omega)$ be $j+1$ functions on Ω with support in Ω_{r_0} (where r_0 is as in Lemma 4.2.2), that are linearly independent on $\partial\Omega$. Let $E = \mathrm{span}\{f_1, \cdots, f_{j+1}\}$. Then if $\delta < r_0$, any function in E is a valid test function for the min-max variational characterization (4.2.2) of $\sigma_j(\Omega_\delta)$, and so

$$\sigma_j(\Omega_\delta) \le \sup_{u\in E\setminus\{0\}} \frac{\int_{\Omega_\delta} |\nabla u|^2}{\int_{\partial\Omega_\delta} u^2} = \sup_{u\in E\setminus\{0\}} \frac{\int_{\Omega_{r_0}} |\nabla u|^2}{\int_{\partial\Omega_{r_0}} u^2} \le \Lambda_j$$

where Λ_j is independent of δ. $\square$

Since u_δ^j is a Steklov eigenfunction of Ω_δ with eigenvalue $\sigma_j(\Omega_\delta)$,

$$\int_{\Omega_\delta} |\nabla u_\delta^j|^2 = \sigma_j(\Omega_\delta) \int_{\Omega_\delta} (u_\delta^j)^2 = \sigma_j(\Omega_\delta) \le \Lambda_j. \tag{4.2.10}$$

By (4.2.10) and since $\|u^j_\delta\|_{L^2(\partial\Omega_\delta)} = 1$, by standard theory,

$$\|u^j_\delta\|_{L^2(K)} \le C(\Lambda_j, K) \tag{4.2.11}$$

for any compact subset K of $\Omega \setminus \Sigma$.

Elliptic boundary estimates ([33, Theorem 6.29]) give bounds

$$\|u^j_\delta\|_{C^{2,\alpha}(K)} \le C\|u^j_\delta\|_{C^0(K)}$$

for any compact subset K of $\Omega\setminus\Sigma$ for all sufficiently small δ, where $C = C(j, \alpha, \Lambda_j, K)$. By Sobolev embedding and interpolation inequalities ([1, Theorem 5.2], [33, (7.10)]),

$$\|u^j_\delta\|_{C^0(K)} \le C\left(\varepsilon\|u^j_\delta\|_{C^2(K)} + \varepsilon^{-\mu}\|u^j_\delta\|_{L^2(K)}\right)$$

where $\varepsilon > 0$ can be taken arbitrarily small, $\mu > 0$ depends on n, and C depends on K. Hence $\|u^j_\delta\|_{C^{2,\alpha}(K)} \le C$ with C independent of δ. By the Arzela-Ascoli theorem and a diagonal sequence argument, there exists a sequence $\delta_i \to 0$ such that for all j, $u^j_{\delta_i}$ converges in $C^2(K)$ on compact subsets $K \subset \Omega\setminus\Sigma$ to a harmonic function u^j on $\Omega\setminus\Sigma$, satisfying

$$\frac{\partial u^j}{\partial \eta} = \sigma^j u^j \quad \text{on} \quad \partial\Omega\setminus\partial\Sigma$$

with $\sigma^j = \lim_{i\to\infty}\sigma_j(\Omega_{\delta_i})$.

Claim 2. *For each $j \ge 1 \in \mathbb{N}$, u^j can be extended to a Steklov eigenfunction of Ω with eigenvalue σ^j.*

Proof. First observe that $u^j \in H^1(\Omega\setminus\Sigma)$. Fix the compact subset Ω_{δ_N} for some large N. Then $\|u^j_{\delta_i}\|^2_{L^2(\Omega_{\delta_i})} = \|u^j_{\delta_i}\|^2_{L^2(\Omega_{\delta_N})} + \|u^j_{\delta_i}\|^2_{L^2(\Omega_{\delta_i}\setminus\Omega_{\delta_N})}$ is uniformly bounded independent of i. The first term is bounded by (4.2.11) and the second term is bounded by (4.2.4) of Lemma 4.2.2. This together with (4.2.10) shows that $\|u^j_{\delta_i}\|_{H^1(\Omega_{\delta_i})}$ is uniformly bounded. Thus $u^j \in H^1(\Omega\setminus\Sigma)$.

For any function $\psi \in W^{1,2}\cap L^\infty(\Omega)$, define $\psi_\delta = \psi\varphi_\delta$ where φ_δ is defined in Lemma 4.2.3. Since u^j is a harmonic function on $\Omega\setminus\Sigma$ and satisfies $\frac{\partial u^j}{\partial\eta} = \sigma^j u^j$

on $\partial\Omega \setminus \partial\Sigma$, and ψ_δ vanishes near Σ, we have

$$\int_{\Omega\setminus\Sigma} \nabla u^j \cdot \nabla \psi_\delta = \sigma^j \int_{\partial\Omega\setminus\partial\Sigma} u^j \psi_\delta,$$

and so

$$\int_\Omega \psi \nabla u^j \cdot \nabla \varphi_\delta + \varphi_\delta \nabla u^j \cdot \nabla \psi = \sigma^j \int_{\partial\Omega} u^j \psi_\delta. \tag{4.2.12}$$

From Hölder's inequality it follows that

$$\left| \int_\Omega \psi \nabla u^j \cdot \nabla \varphi_\delta \right| \le \int_\Omega |\psi \nabla u^j| |\nabla \varphi_\delta| \le \left(\int_\Omega |\psi \nabla u^j|^2 \right)^{\frac{1}{2}} \left(\int_\Omega |\nabla \varphi_\delta|^2 \right)^{\frac{1}{2}}.$$

Therefore, by Lemma 4.2.3

$$\int_\Omega \psi \nabla u^j \cdot \nabla \varphi_\delta \to 0 \ \text{ as } \ \delta \to 0.$$

Since $|\psi_\delta| \le |\psi| \in L^\infty(\Omega)$ and $u^j \in H^1(\Omega \setminus \Sigma)$, taking limits both sides of (4.2.12) and applying the dominated convergence theorem, we obtain

$$\int_\Omega \nabla u^j \cdot \nabla \psi = \sigma^j \int_{\partial\Omega} u^j \psi$$

which implies the claim. □

Claim 3. *The set of functions $\{u^j\}$ is orthonormal on the boundary of Ω, i.e.,*

$$\int_{\partial\Omega} u^j u^l = \delta_{jl} \qquad \textit{for } \ j,l \ge 0.$$

Proof. Let $\varepsilon > 0$. Since u^j are Steklov eigenfunctions by Claim 2, and therefore smooth by the regularity theory, there exists sufficiently small $s > 0$ such that $|\int_{L_s\cap\partial\Omega} u^j u^l| < \varepsilon/4$. According to Hölder's inequality and (4.2.3) and (4.2.5) of Lemma 2.2, we may furthermore assume that s is chosen sufficiently small such that $|\int_{T_{\delta_i}} u^j_{\delta_i} u^l_{\delta_i}| < \varepsilon/4$ and $|\int_{(L_s\setminus L_{\delta_i})\cap\partial\Omega} u^j_{\delta_i} u^l_{\delta_i}| < \varepsilon/4$ hold for i large enough. The fact that $u^j_{\delta_i}$ uniformly converges to u^j in any compact subset implies that

$|\int_{\partial\Omega\setminus L_s} u^j_{\delta_i} u^l_{\delta_i} - \int_{\partial\Omega\setminus L_s} u^j u^l| < \varepsilon/4$ for i large enough. Hence it follows that

$$\begin{aligned}
\left|\int_{\partial\Omega} u^j u^l - \delta_{jl}\right| &< \left|\int_{\partial\Omega\setminus L_s} u^j u^l - \delta_{jl}\right| + \varepsilon/4 \\
&< \left|\int_{\partial\Omega\setminus L_s} u^j_{\delta_i} u^l_{\delta_i} - \delta_{jl}\right| + \varepsilon/2 \\
&= \left|\int_{\partial\Omega_{\delta_i}} u^j_{\delta_i} u^l_{\delta_i} - \int_{(L_s\setminus L_{\delta_i})\cap\partial\Omega} u^j_{\delta_i} u^l_{\delta_i} - \int_{T_{\delta_i}} u^j_{\delta_i} u^l_{\delta_i} - \delta_{jl}\right| + \varepsilon/2 \\
&< \left|\int_{(L_s\setminus L_{\delta_i})\cap\partial\Omega} u^j_{\delta_i} u^l_{\delta_i}\right| + \left|\int_{T_{\delta_i}} u^j_{\delta_i} u^l_{\delta_i}\right| + \varepsilon/2 \\
&< \varepsilon/4 + \varepsilon/4 + \varepsilon/2 = \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary this yields the desired result. □

Theorem 4.1.1 is proved if we can show that $\sigma^j = \sigma_j(\Omega)$. This is what we shall do in the following, by applying induction. It is easy to see that $\lim_{\delta\to 0}\sigma_0(\Omega_\delta) = \sigma_0(\Omega)$ since they are always zero. Hereafter, we fix j and assume that for each $0 \le i \le j-1$, u^i is a Steklov eigenfunction of Ω with eigenvalue $\sigma_i(\Omega)$. First, we show that $\sigma_j(\Omega) \le \sigma^j$. To see this, from Claim 3 it follows that

$$\int_{\partial\Omega} u^j = \int_{\partial\Omega} u^1 u^j = \cdots = \int_{\partial\Omega} u^{j-1} u^j = 0,$$

which guarantees that u^j is an admissible test function for $\sigma_j(\Omega)$, and thus $\sigma^j \ge \sigma_j(\Omega)$. To prove the theorem, we need to prove the reverse inequality.

Denote the j-th Steklov eigenfunction of Ω by v, and define

$$f = v - \sum_{i=0}^{j-1}\left(\int_{\partial\Omega_\delta} v u^i_\delta\right) u^i_\delta.$$

In fact, f is the projection of v to the orthogonal complement of first j eigenspaces of Ω_δ. Therefore,

$$\int_{\partial\Omega_\delta} f u^i_\delta = 0, \text{ for } i = 0, \cdots, j-1.$$

Thus f is an admissible test function for $\sigma_j(\Omega_\delta)$.

On the one hand, let's estimate the denominator of the Rayleigh quotient,

$$\int_{\partial\Omega_\delta} f^2 = \int_{\partial\Omega_\delta} v^2 - \sum_{i=0}^{j-1}\left(\int_{\partial\Omega_\delta} vu_\delta^i\right)^2. \tag{4.2.13}$$

It's clear that

$$\lim_{\delta\to 0}\int_{\partial\Omega_\delta} v^2 = \int_{\partial\Omega} v^2$$

and

$$\lim_{\delta\to 0}\int_{\partial\Omega_\delta} vu_\delta^i = \int_{\partial\Omega} vu^i = 0 \;\text{ for } i = 0,\cdots,j-1.$$

On the other hand, we may estimate the numerator of the Rayleigh quotient as follows

$$\int_{\Omega_\delta}|\nabla f|^2 = \int_{\Omega_\delta}|\nabla v|^2 + \sum_{i=1}^{j-1}\left(\int_{\partial\Omega_\delta} vu_\delta^i\right)^2\int_{\Omega_\delta}|\nabla u_\delta^i|^2 - 2\sum_{i=1}^{j-1}\left(\int_{\partial\Omega_\delta} vu_\delta^i\right)\int_{\Omega_\delta}\langle\nabla v,\nabla u_\delta^i\rangle. \tag{4.2.14}$$

Similarly,

$$\int_{\Omega_\delta}|\nabla u_\delta^i|^2 = \sigma_1(\Omega_\delta) < C_1$$

and

$$\int_{\Omega_\delta}\langle\nabla v,\nabla u_\delta^i\rangle \le C_2\left(\int_{\Omega_\delta}|\nabla u_\delta^i|^2\right)^{\frac{1}{2}} \le C_3,$$

where all the constant C_1, C_2, C_3 are independent of δ for small δ.

In the end, we combine the estimates (4.2.13), (4.2.14) in the characterization (4.2.1) and take the limit to get

$$\limsup_{\delta\to 0}\sigma_j(\Omega_\delta) \le \lim_{\delta\to 0}\frac{\int_{\Omega_\delta}|\nabla f|^2}{\int_{\partial\Omega_\delta} f^2} = \frac{\int_\Omega|\nabla v|^2}{\int_{\partial\Omega} v^2} = \sigma_j(\Omega).$$

This completes the proof of the theorem. □

4.3 Some Results on Shape Optimization for Steklov Eigenvalues in $\mathbb{R}^n$

For any bounded domain $\Omega \subset \mathbb{R}^n$ there are upper bounds on all normalized Steklov eigenvalues, $\sigma_j(\Omega)|\partial\Omega|^{\frac{1}{n-1}} \leq C(n) j^{\frac{2}{n}}$ ([21]). An explicit upper bound for the first normalized Steklov eigenvalue was given in [31, Proposition 2.1], however the upper bound is not expected to be sharp. It is an open question to determine the sharp upper bound.

Question 4.3.1 ([35])**.** *On which domain (or in the limit of which sequence of domains) is the supremum of $\bar{\sigma}_j(\Omega)$ over all bounded domains $\Omega \subset \mathbb{R}^n$ realized?*

A consequence of Theorem 4.1.1 is that the supremum of the j-th normalized eigenvalue among all domains is the same as the supremum among domains having relatively simple topology; that is, among domains which can be obtained by performing surgeries up to codimension two (as discussed in Section 4.2).

Another immediate consequence of our surgery result is that when $n \geq 3$ the unit ball is not the maximizer for $\bar{\sigma}_j(\Omega)$, even among contractible domains. This was proved for $j = 1$ in [31, Theorem 1.1].

Corollary 4.1.2. *Let $n \geq 3$. For any $k \geq 1$, there exist a contractible domain Ω^* in $\mathbb{R}^n$ such that*

$$\bar{\sigma}_j(\Omega^*) > \bar{\sigma}_j(\mathbb{B}^n)$$

for $j = 1, \cdots, k$.

Proof. Let $\mathbb{B}^n_\varepsilon$ denote the ball of radius ε centred at the origin in $\mathbb{R}^n$. It was shown in [31, Proposition 3.1] that for $\varepsilon > 0$ sufficiently small, $\bar{\sigma}_j(\mathbb{B}^n \setminus \mathbb{B}^n_\varepsilon) > \bar{\sigma}_j(\mathbb{B}^n)$. We may then perform a 1-dimensional surgery on $\mathbb{B}^n \setminus \mathbb{B}^n_\varepsilon$ to construct a contractible domain, while changing the Steklov eigenvalues by an arbitrarily small amount. For example, let $\mathbb{B}^n_{\varepsilon,\delta}$ denote the domain obtained by removing a δ-tube around a radial segment (see Figure 4.1). Then since $n \geq 3$, by Theorem 4.1.1, for sufficiently small $\varepsilon > 0$ we have

$$\lim_{\delta \to 0} \bar{\sigma}_j(\mathbb{B}^n_{\varepsilon,\delta}) = \bar{\sigma}_j(\mathbb{B}^n \setminus \mathbb{B}^n_\varepsilon) > \bar{\sigma}_j(\mathbb{B}^n).$$

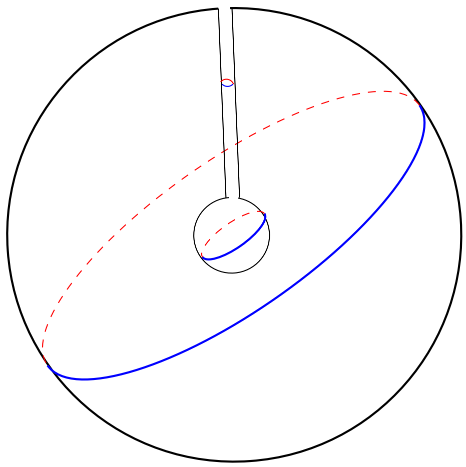

Figure 4.1: The construction of the contractible domain $\mathbb{B}^n_{\varepsilon,\delta}$ in $\mathbb{R}^n$ for $n \geq 3$: δ is the radius of the neck and ε is the radius of the inner ball, and we delete the neck and inner ball from the unit ball.

Therefore, for any fixed k, we can find a small $\delta > 0$ such that $\bar{\sigma}_j(\mathbb{B}^n_{\varepsilon,\delta}) > \bar{\sigma}_j(\mathbb{B}^n)$ for $j = 1, \cdots, k$. We let $\Omega^* = \mathbb{B}^n_{\varepsilon,\delta}$. $\square$

As discussed in the introduction, this result is not surprising, especially in light of [34] which shows that the supremum of the j-th normalized eigenvalue among simply connected domains in $\mathbb{R}^2$ is achieved in the limit by a sequence of simply connected domains degenerating to the disjoint union of j identical discs. We now consider contractible domains of this type in higher dimensions.

Consider the necklace-like contractible domain $\Omega^n_{\varepsilon,l}$ in $\mathbb{R}^n$ which is the union of l n-dimensional unit balls centred along a common axis and positioned such that adjacent balls overlap by a distance ε^2 along the axis; see Figure 4.2. As ε tends to zero, $\Omega^n_{\varepsilon,l}$ converges to l identical unit balls touching tangentially at $l-1$ points, which we denote by $p_1, \cdots, p_{l-1}$ (see Figure 4.3). We denote l disjoint unit balls by $\sqcup_l \mathbb{B}^n$.

Using similar ideas as in the proof of Theorem 4.1.1, we show that the Steklov spectrum of $\Omega^n_{\varepsilon,l}$ converges to that of $\sqcup_l \mathbb{B}^n$ as $\varepsilon \to 0$.

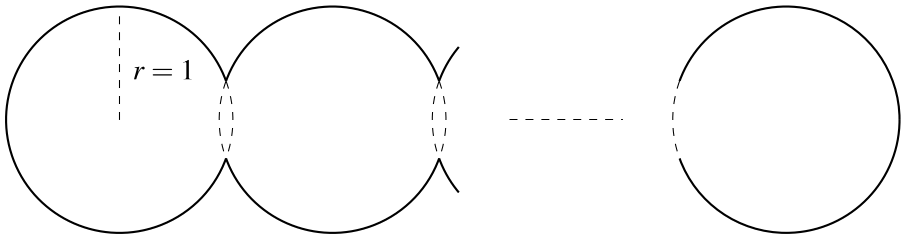

Figure 4.2: The construction of $\Omega^n_{\varepsilon,l}$ in $\mathbb{R}^n$, $n \geq 2$: l overlapping unit balls.

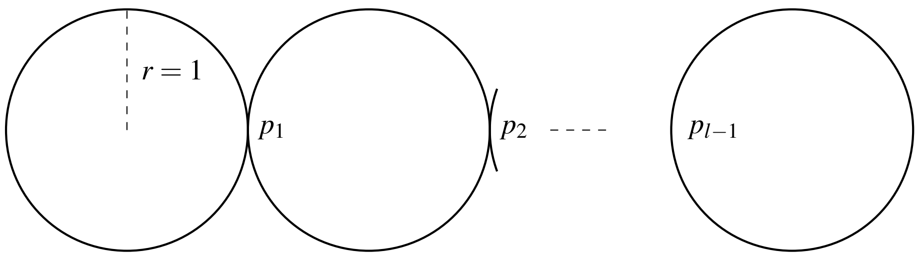

Figure 4.3: Limit of $\Omega^n_{\varepsilon,l}$ in $\mathbb{R}^n$, $n \geq 2$, as $\varepsilon \to 0$.

Theorem 4.3.2. *For every $j \geq 0$, the j-th normalized Steklov eigenvalue of $\Omega^n_{\varepsilon,l}$, where $n \geq 2$, converges to the j-th normalized Steklov eigenvalue of the disjoint union of l unit balls as ε tends to zero, i.e.,*

$$\lim_{\varepsilon \to 0^+} \bar{\sigma}_j(\Omega^n_{\varepsilon,l}) = \sigma_j(\sqcup_l \mathbb{B}^n) \cdot (l \cdot |\mathbb{S}^{n-1}|)^{\frac{1}{n-1}}. \tag{4.3.1}$$

Remark 4.3.3. *(1) The above theorem holds for more general overlapping domains. For example, in the construction we could replace the unit ball by the domain $\mathbb{B}^n_{\varepsilon,\delta}$ from the proof of Corollary 4.1.2 (Figure* 4.1*). More generally, the same proof works for domains with smooth boundary overlapping at elliptic points on the boundary, and such that other intersections do not occur; (2) Comparing* (4.3.1) *with Theorem 1.3.1 in [34], the index of the Steklov eigenvalue doesn't have to be same as the number l of balls we overlap. (3) This result was proved in [14, Example 3]. However, we use a different approach here.*

Proof of Theorem 4.3.2. Since $|\partial\Omega^n_{\varepsilon,l}|$ converges to $l|\mathbb{S}^{n-1}|$, it suffices to prove that

$$\lim_{\varepsilon\to 0^+} \sigma_j(\Omega^n_{\varepsilon,l}) = \sigma_j(\sqcup_l \mathbb{B}^n). \tag{4.3.2}$$

We fix $l \geq 2$ and $n \geq 2$, and without ambiguity, we simply use the notation Ω_ε for $\Omega^n_{\varepsilon,l}$ in the proof. We split the proof into four steps.

Step 1: Uniform energy bound and uniform convergence in compact subsets. We consider the Steklov eigenvalue problem on Ω_ε. Let $\{u^j_\varepsilon\}_{j=0}^\infty$ be $L^2(\partial\Omega_\varepsilon)$-orthonormal Steklov eigenfunctions of Ω_ε such that u^j_ε is an eigenfunction with eigenvalue $\sigma_j(\Omega_\varepsilon)$. By an argument similar to the proof of Claim 1, for each fixed $j \in \mathbb{N}$, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$, $\sigma_j(\Omega_\varepsilon)$, and hence the energy, is uniformly bounded independent of ε. As in the proof of Theorem 4.1.1, elliptic boundary estimates give uniform bounds on u^j_ε and its derivatives up to $\partial\Omega_\varepsilon$, and there exists a sequence $\varepsilon_i \to 0$ such that for each j, $u^j_{\varepsilon_i}$ converges in $C^2(K)$ as $i \to \infty$ on any compact subset K of $\sqcup_l \overline{\mathbb{B}^n} \setminus \{p_1, \ldots, p_{l-1}\}$ to a harmonic function u^j on $\sqcup_l \mathbb{B}^n$ satisfying

$$\frac{\partial u^j}{\partial \eta} = \sigma^j u^j \text{ on } \sqcup_l \partial\mathbb{B}^n \setminus \{p_1, \ldots, p_{l-1}\}$$

with $\sigma^j = \lim_{i\to\infty} \sigma_j(\Omega_{\varepsilon_i})$.

Step 2: Nonconcentration of L^2 norms of eigenfunctions around $p_1, \cdots, p_{l-1}$. For a point p on the boundary of the unit ball $\mathbb{B}^n$, we consider the geodesic ball $\mathscr{B}_s(p)$ in $\mathbb{B}^n$ centred at p with radius s. Notice that if $s \geq \varepsilon$ and p is chosen along the axis of the domain Ω_ε then $\mathscr{B}_s(p)$ will contain the intersection of the adjacent overlapping balls. The boundary of $\mathscr{B}_s(p)$ consists of two parts: $\Gamma^1_s(p) = \partial\mathscr{B}_s(p) \cap \mathbb{B}$ and $\Gamma^2_s(p) = \partial\mathscr{B}_s(p) \setminus \Gamma^2_s(p)$. In the following we choose r_0 small such that $\mathscr{B}_s(p)$ is uniformly equivalent to a Euclidean half ball of radius s for $s < r_0$.

We claim that for any smooth function u on $\mathscr{B}_s(p)$ with $u = 0$ on $\Gamma^1_s(p)$,

$$\int_{\mathscr{B}_s(p)} u^2 \leq C_1(n)\, s^2 \int_{\mathscr{B}_s(p)} |\nabla u|^2 \tag{4.3.3}$$

and

$$\int_{\Gamma^2_s(p)} u^2 \leq C_2(n)\, s \int_{\mathscr{B}_s(p)} |\nabla u|^2. \tag{4.3.4}$$

Here $C_1(n), C_2(n)$ are constants depending only on the dimension n. The above two Poincare inequalities follow from estimates of the first Dirichlet-Neumann eigenvalue and the first Dirichlet-Steklov eigenvalue of a Euclidean half ball (or see [32, Lemma 4.5 & Lemma 4.6]).

According to [32, Lemma 4.3] there exists s_1 and a smooth cut-off function ζ, which is 0 on $\mathscr{B}_{r_0} \setminus \mathscr{B}_{s_1}$ and 1 on $\mathscr{B}_s$, such that for any smooth function u defined on $\mathscr{B}_{r_0}$,

$$\int_{\mathscr{B}_{r_1} \setminus \mathscr{B}_s} |\nabla \zeta|^2 u^2 \leq C(s) \left(\int_{\mathscr{B}_{r_0} \setminus \mathscr{B}_s} u^2 + \int_{\mathscr{B}_{r_0}} |\nabla u|^2 \right). \tag{4.3.5}$$

Here $s < s_1 = s_1(s) < r_0$ and $s_1(s) = o(1), C(s) = o(1)$. Now if u is any smooth function defined on the unit ball $\mathbb{B}^n$, applying ζu to $(4.3.3)$ we have

$$\begin{aligned} \int_{\mathscr{B}_s(p)} u^2 &\leq \int_{\mathscr{B}_{s_1}(p)} (\zeta u)^2 \\ &\leq C_1(n) s_1^2 \int_{\mathscr{B}_{s_1}(p)} |\nabla \zeta u|^2 \\ &\leq C_1(n) s_1^2 \left(\int_{\mathscr{B}_{s_1}(p)} |\nabla u|^2 + \int_{\mathscr{B}_{s_1}(p) \setminus \mathscr{B}_s(p)} |\nabla \zeta|^2 u^2 \right) \\ &\leq C_1(s) \left(\int_{\mathscr{B}_{r_0}(p)} |\nabla u|^2 + \int_{\mathscr{B}_{r_0}(p) \setminus \mathscr{B}_s(p)} u^2 \right) \end{aligned} \tag{4.3.6}$$

where $C_1(s) = o(1)$, and we used $(4.3.5)$ in the last inequality. Similarly, applying ζu to $(4.3.4)$ we obtain

$$\int_{\Gamma_s^2(p)} u^2 \leq C_2(s) \left(\int_{\mathscr{B}_{r_0}(p)} |\nabla u|^2 + \int_{\mathscr{B}_{r_0}(p) \setminus \mathscr{B}_s(p)} u^2 \right) \tag{4.3.7}$$

where $C_2(s) = o(1)$.

Now we consider the eigenfunction u_ε^j on domain Ω_ε. In what follows we shall show that interior and boundary L^2 norms of u_ε^j don't concentrate near $p_1, \ldots, p_{l-1}$ as $\varepsilon \to 0$. The common axis of the domain Ω_ε intersects the boundaries of the overlapping balls at $2l-2$ points, $p_1^{(1)}(\varepsilon), p_1^{(2)}(\varepsilon), \cdots, p_{l-1}^{(1)}(\varepsilon), p_{l-1}^{(2)}(\varepsilon)$, such that as $\varepsilon \to 0$, $p_k^{(1)}(\varepsilon)$ and $p_k^{(2)}(\varepsilon)$ converge to p_k for $k = 1, \cdots, l-1$. We apply the inequality $(4.3.6)$ at these points to the restriction of the function u_ε^j to each ball.

Summing the inequalities yields that for any $\varepsilon < s < s_1(s) < r_0$,

$$\int_{\cup_{i=1}^{2}\cup_{k=1}^{l-1}\mathscr{B}_s(p_k^{(i)}(\varepsilon))}(u_\varepsilon^j)^2 \le 2C_1(s)\left(\int_{\Omega_\varepsilon}|\nabla u_\varepsilon^j|^2+\int_{\cup_{i=1}^{2}\cup_{k=1}^{l-1}\mathscr{B}_{r_0}(p_k^{(i)}(\varepsilon))\setminus\mathscr{B}_s(p_k^{(i)}(\varepsilon))}(u_\varepsilon^j)^2\right)$$
$$\le C\cdot C_1(s)$$

where the last inequality follows from the trace inequality and the uniform energy bound. Similarly, applying (4.3.7) we obtain that for any $\varepsilon < s < s_1(s) < r_0$,

$$\begin{aligned}\int_{\cup_{i=1}^{2}\cup_{k=1}^{l-1}\Gamma_s^2(p_k^{(i)}(\varepsilon))\setminus\Gamma_\varepsilon^2(p_k^{(i)}(\varepsilon))}(u_\varepsilon^j)^2 &\le \int_{\cup_{i=1}^{2}\cup_{k=1}^{l-1}\Gamma_s^2(p_k^{(i)}(\varepsilon))}(u_\varepsilon^j)^2\\ &\le 2C_2(s)\int_{\Omega_\varepsilon}|\nabla u_\varepsilon^j|^2\\ &+\int_{\cup_{i=1}^{2}\cup_{k=1}^{l-1}\mathscr{B}_{r_0}(p_k^{(i)}(\varepsilon))\setminus\mathscr{B}_s(p_k^{(i)}(\varepsilon))}(u_\varepsilon^j)^2\\ &\le C\cdot C_2(s).\end{aligned}$$

Hence following the ideas of the proofs of Claims 2 and 3, $\|u_\varepsilon^j\|_{H^1(\Omega_\varepsilon)}$ is uniformly bounded and therefore $u^j \in H^1(\sqcup_l B^n)$, u^j extends to a Steklov eigenfunction of $\sqcup_l\mathbb{B}^n$, and $\{u^j\}$ are orthonormal on the boundary of $\sqcup_l\mathbb{B}^n$.

Step 3: Construction of cut-off function. Define the following cut-off function on $\sqcup_l\mathbb{B}^n$,

$$\varphi_\varepsilon = \begin{cases} 0, & r < \varepsilon^2 \\ \frac{\ln r - \ln\varepsilon^2}{\ln\varepsilon - \ln\varepsilon^2} & \varepsilon^2 \le r \le \varepsilon \\ 1 & r > \varepsilon \end{cases}$$

where r is the distance function to the nearest of the points $p_1,\dots,p_{l-1}$. Let $T_t = \{x \in \sqcup_l\mathbb{B}^n : r(x) \le t\}$. We have

$$\int_{\sqcup_l\mathbb{B}^n}|\nabla\varphi_\varepsilon|^2 = \frac{1}{(\ln\varepsilon)^2}\int_{T_\varepsilon\setminus T_{\varepsilon^2}}\frac{1}{r^2} \le \frac{(2l-2)C(n)}{(\ln\varepsilon)^2}\int_{\varepsilon^2}^{\varepsilon} r^{n-3}\,dr = C(n,l)\varepsilon_n(\varepsilon) \tag{4.3.8}$$

where $\varepsilon_2(\varepsilon) = -1/\ln\varepsilon$ and $\varepsilon_n(\varepsilon) = \varepsilon^{n-2}(1-\varepsilon^{n-2})/(\ln\varepsilon)^2$ for $n \ge 3$.

Step 4: Proof of the convergence of eigenvalues. We now use induction to prove that (4.3.2) holds: $\lim_{\varepsilon\to 0}\sigma_j(\Omega_\varepsilon) = \sigma_j(\sqcup_l\mathbb{B}^n)$. It is clear that $\sigma_0(\Omega_\varepsilon) = 0$ for any ε and $\sigma_0(\sqcup_l \mathbb{B}^n) = 0$, and so (4.3.2) holds for $j = 0$. For fixed $j \in \mathbb{N}$, suppose that (4.3.2) holds for $i = 0, 1, \cdots, j-1$; that is, u^i is a Steklov eigenfunction of $\sqcup_l \mathbb{B}^n$ with eigenvalue $\sigma_i(\sqcup_l \mathbb{B}^n)$ for $i = 0, 1, \cdots, j-1$. According to Step 1, we have

$$\int_{\partial(\sqcup_l \mathbb{B}^n)} u^j = \int_{\partial(\sqcup_l \mathbb{B}^n)} u^1 u^j = \cdots = \int_{\partial(\sqcup_l \mathbb{B}^n)} u^{j-1} u^j = 0.$$

Then u^j is an admissible test function for the j-th eigenvalue of $\sqcup_l \mathbb{B}^n$. Thus $\sigma_j(\sqcup_l \mathbb{B}^n) \leq \sigma^j$, and we have

$$\liminf_{\varepsilon \to 0^+} \sigma_j(\Omega_\varepsilon) \geq \sigma_j(\sqcup_l \mathbb{B}^n). \tag{4.3.9}$$

In order to prove the theorem, we need to prove the reverse inequality. We will use a j-th eigenfunction of $\sqcup_l \mathbb{B}^n$, which we denote by v, to construct an admissible test function for the j-th eigenvalue of Ω_{ε^2}, where ε^2 is the square of ε. This part of the proof is slightly different from the proof of Theorem 4.1.1, since we need to cut off and glue functions to construct the admissible test function to approximate the j-th eigenvalue of Ω_{ε^2}.

Let $f = v\varphi_\varepsilon - \sum_{i=0}^{j-1} (\int_{\partial\Omega_{\varepsilon^2}} v\varphi_\varepsilon u^i_{\varepsilon^2}) u^i_{\varepsilon^2}$, where we recall that $\{u^i_{\varepsilon^2}\}_{i=0}^{\infty}$ are orthonormal eigenfunctions of Ω_{ε^2}. By construction, f is orthogonal to $u^0_{\varepsilon^2}, \cdots, u^{j-1}_{\varepsilon^2}$ on $\partial\Omega_{\varepsilon^2}$, and so

$$\sigma_j(\Omega_{\varepsilon^2}) \leq \frac{\int_{\Omega_{\varepsilon^2}} |\nabla f|^2}{\int_{\partial\Omega_{\varepsilon^2}} f^2}.$$

A simple calculation turns the numerator to

$$\begin{aligned} \int_{\Omega_{\varepsilon^2}} |\nabla f|^2 \quad = \quad & \int_{\Omega_{\varepsilon^2}} |\nabla (v\varphi_\varepsilon)|^2 + \sum_{i=1}^{j-1} \sigma_i(\Omega_{\varepsilon^2}) \left(\int_{\partial\Omega_{\varepsilon^2}} v\varphi_\varepsilon u^i_{\varepsilon^2} \right)^2 \\ & -2 \sum_{i=1}^{j-1} \int_{\partial\Omega_{\varepsilon^2}} v\varphi_\varepsilon u^i_{\varepsilon^2} \int_{\Omega_{\varepsilon^2}} \langle \nabla u^i_{\varepsilon^2}, \nabla (v\varphi_\varepsilon) \rangle. \end{aligned} \tag{4.3.10}$$

The first term can be estimated as

$$\begin{aligned}
\int_{\Omega_{\varepsilon^2}} |\nabla(v\varphi_\varepsilon)|^2 &\leq \int_{\Omega_\varepsilon} |\nabla v|^2 + 2\int_{T_\varepsilon \setminus T_{\varepsilon^2}} \varphi_\varepsilon^2 |\nabla v|^2 + v^2 |\nabla \varphi_\varepsilon|^2 \\
&\leq \int_{\sqcup_l \mathbb{B}^n} |\nabla v|^2 + C_1 |T_\varepsilon \setminus T_{\varepsilon^2}| + C_2 \int_{T_\varepsilon \setminus T_{\varepsilon^2}} |\nabla \varphi_\varepsilon|^2 \\
&= \int_{\sqcup_l \mathbb{B}^n} |\nabla v|^2 + C_3(\varepsilon) \qquad (4.3.11)
\end{aligned}$$

with $C_3(\varepsilon) \to 0$ as $\varepsilon \to 0^+$ by (4.3.8). It's not hard to check that for $0 \leq i \leq j$,

$$\lim_{\varepsilon \to 0^+} \int_{\partial \Omega_{\varepsilon^2}} v \varphi_\varepsilon u^i_{\varepsilon^2} = \int_{\sqcup_l \mathbb{B}^n} v u^i = 0$$

and

$$\left| \int_{\Omega_{\varepsilon^2}} \langle \nabla u^i_{\varepsilon^2}, \nabla(v\varphi_\varepsilon) \rangle \right| \leq (\sigma_i(\Omega_{\varepsilon^2}))^{\frac{1}{2}} \left(\int_{\Omega_{\varepsilon^2}} |\nabla(v\varphi_\varepsilon)|^2 \right)^{\frac{1}{2}}.$$

Since $\sigma_i(\Omega_{\varepsilon^2})$ are uniformly bounded in ε, and by (4.3.11), the last two terms of the right hand side of (4.3.10) both tend to zero as $\varepsilon \to 0^+$. In all, we have an inequality for the numerator,

$$\int_{\Omega_{\varepsilon^2}} |\nabla f|^2 \leq \int_{\sqcup_l \mathbb{B}^n} |\nabla v|^2 + C_4(\varepsilon)$$

with $C_4(\varepsilon) \to 0$ as $\varepsilon \to 0^+$.

On the other hand, we expand the denominator as follows

$$\int_{\partial \Omega_{\varepsilon^2}} f^2 = \int_{\partial \Omega_{\varepsilon^2}} (v\varphi_\varepsilon)^2 - \sum_{i=0}^{j-1} \left(\int_{\partial \Omega_{\varepsilon^2}} v \varphi_\varepsilon u^i_{\varepsilon^2} \right)^2 \to \int_{\partial(\sqcup_l \mathbb{B}^n)} v^2$$

as $\varepsilon \to 0$, since the second term tends to zero as $\varepsilon \to 0$.

Now we combine all the estimates to get

$$\limsup_{\varepsilon \to 0} \sigma_j(\Omega_{\varepsilon^2}) \leq \lim_{\varepsilon \to 0} \frac{\int_{\Omega_{\varepsilon^2}} |\nabla f|^2}{\int_{\partial \Omega_{\varepsilon^2}} f^2} \leq \frac{\int_{\sqcup_l \mathbb{B}^n} |\nabla v|^2}{\int_{\partial(\sqcup_l \mathbb{B}^n)} v^2} = \sigma_j(\sqcup_l \mathbb{B}^n).$$

Therefore together with (4.3.9) we have $\lim_{\varepsilon \to 0} \sigma_j(\Omega_\varepsilon) = \sigma_j(\sqcup_l \mathbb{B}^n)$. $\square$

As a special case of Theorem 4.3.2, we obtain the following generalization of [34, Theorem 1.3.1] to higher dimensions.

Theorem 4.3.4. *For the domains $\Omega^n_{\varepsilon,j} \subset \mathbb{R}^n$ (see Figure 4.2), $n \geq 2$, we have*

$$\lim_{\varepsilon \to 0^+} \bar{\sigma}_j(\Omega^n_{\varepsilon,j}) = (j \cdot |\mathbb{S}^{n-1}|)^{\frac{1}{n-1}}.$$

Proof. When $l = j$, $\sigma_j(\sqcup_j \mathbb{B}^n) = 1$, thus $\lim_{\varepsilon \to 0^+} \sigma_j(\Omega^n_{\varepsilon,j}) = (j \cdot |\mathbb{S}^{n-1}|)^{\frac{1}{n-1}}$. □

Following Remark 4.3.3, if we take the domains that we overlap to be $\mathbb{B}^n_{\varepsilon,\delta}$, we see that when $n \geq 3$ the domains $\Omega^n_{\varepsilon,j}$ do not attain the supremum of the j-th Steklov eigenvalue in the limit as $\varepsilon \to 0$ for any j, in contrast to the case in dimension two [34].

Theorem 4.1.3. *For $n \geq 3$ the supremum of the j-th normalized Steklov eigenvalue among contractible domains in $\mathbb{R}^n$ is not achieved in the limit by a sequence of contractible domains degenerating to the disjoint union of j identical round balls.*

Proof. Let $\tilde{\Omega}^n_{\varepsilon,j}$ be the domain obtained by overlapping j copies of the domain $\mathbb{B}^n_{\varepsilon,\delta}$. Then by the proof of Theorem 4.3.2,

$$\lim_{\varepsilon \to 0^+} \bar{\sigma}_j(\tilde{\Omega}^n_{\varepsilon,j}) = j^{\frac{1}{n-1}} \cdot \sigma_j(\mathbb{B}^n_{\varepsilon,\delta}) \cdot |\partial \mathbb{B}^n_{\varepsilon,\delta}|^{\frac{1}{n-1}}.$$

By Theorem 4.3.4, Corollary 4.1.2, and Theorem 4.1.1 it follows that

$$\lim_{\varepsilon \to 0^+} \bar{\sigma}_j(\tilde{\Omega}^n_{\varepsilon,j}) > \lim_{\varepsilon \to 0^+} \bar{\sigma}_j(\Omega^n_{\varepsilon,j}).$$

□

We end this section with following corollary.

Corollary 4.3.5. *For certain j and n, if $\varepsilon > 0$ is sufficiently small, then $\bar{\sigma}_j(\Omega^n_{\varepsilon,j}) > \bar{\sigma}_j(\mathbb{B}^n)$.*

Proof. By Theorem 4.3.4, we have $\liminf_{\varepsilon \to 0^+} \bar{\sigma}_j(\Omega^n_{\varepsilon,j}) = (jn\omega_n)^{\frac{1}{n-1}} > \bar{\sigma}_j(\mathbb{B}^n)$ if $j^{1/(n-1)} > \sigma_j(\mathbb{B}^n)$. By simple calculation, this only holds in certain cases. For example, when $n = 3$, it holds for all j except when j is $4, 9, 16, \ldots$. For higher n, it holds all j less than a certain finite number depending on n. In these cases we can choose sufficiently small ε such that $\bar{\sigma}_j(\Omega^n_{\varepsilon,j}) > \bar{\sigma}_j(\mathbb{B}^n)$. $\square$

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