Presenting Higher Categories and Weak Functors via Multi-Opetopic Nerves and Terminal Coalgebras

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Abstract

In this thesis, we construct a convenient presentation of weak n-categories for $0 \leq n \leq \omega$ whose corresponding weak n-functors in particular do not strictly preserve units. The approach constructs the finite-dimensional higher categories inductively using multi-opetopic sets, analogous to the construction of Segal categories, and multi-simplicial nerves of Tamsamani. We prove that our construction is correct in dimension two by providing a fully faithful and weakly essentially surjective functor from the category of bicategories and pseudofunctors to our weak 2-categories. For the infinite-dimensional case, we realise the weak ω-categories as formal limits of their finite-dimensional truncations, using coalgebras over an appropriate endofunctor. We also prove that these weak ω-categories admit an equivalent characterisation as infinitary opetopic sets subject to constraints analogous to the finite-dimensional case. Finally, we specialise our construction to ∞-groupoids and prove that the coalgebraic structure induces a canonical functor from nice topological spaces that defines the Poincaré ∞-groupoid construction. We show that the Poincaré ∞-groupoid has the correct higher morphisms in all dimensions and retains the information about all homotopy groups of the space. Moreover, we show that this construction preserves and reflects weak equivalences. We conclude by proposing a construction that likely recovers a space from its Poincaré ∞-groupoid which conjecturally establishes a version of the Homotopy Hypothesis.

Lay Summary

Many mathematical structures can be studied holistically through the language of categories. In fact, even categories themselves can be collected together to form an even larger category, but doing so loses much of the nuance of category theory. This is resolved by instead collecting categories together into what is called a 2-category, which adds an extra dimension of structure. In a similar way, 2-categories can also be collected together, and this collection naturally carries the structure of a 3-category. Repeating this process indefinitely leads to an overarching structure of all higher-dimensional categories, called an ∞-category. As ∞-categories are difficult to define directly, the goal of this thesis is to provide an easy step-by-step way of constructing n-categories for all n, and show how to construct ∞-categories by taking the limit as $n \to \infty$.

Preface

This thesis is original, unpublished, independent work by the author, Z. Goldthorpe. Chapter 1 provides a non-technical summary of the context for the problems addressed in the thesis, as well as a brief overview of the general approach, highlighting the main theorems of the thesis. Chapter 2 and Chapter 3 are expository, and Chapter 4 reviews all of the mathematics necessary for the main results. Chapter 5 contains the original results of the author, which may be published at a later date. Chapter 6 compares the contributions of this thesis with other works in the literature.

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Chapter 1

Introduction

The theory of categories was first developed by Eilenberg and Mac Lane in 1945, and served as a convenient general framework for algebraic or set-theoretic mathematical objects to be studied holistically, in terms of diagrams of arrows between points. These arrows—called *morphisms*—are subject to very few rules:

- Arrows that share points can be connected tail-to-tip to uniquely produce other arrows, and this composition is associative.
- There is a unit arrow at each point, and any other arrow does not change when composed with one of the unit arrows.

The theory has since witnessed rapid and diverse development, providing a convenient language for studying several abstract areas of mathematics, such as those in algebraic geometry and algebraic topology. However, with mathematics becoming more refined with more subtle notions of equivalence between mathematical objects, ordinary category theory faced similar refinements, leading to generalisations such as enriched and higher category theory.

Higher category theory generalises the points-and-arrows language of ordinary categories to allow for higher-dimensional arrows between arrows. Arrows of dimension k in a higher category are referred to as *k-morphisms*, and allow for the rules of category theory to be relaxed: arrows can still be connected tail-to-tip, but the composite arrow is no longer uniquely determined. Rather, any two candidates for the composite arrow are connected by a reversible higher-dimensional arrow.

Reversibility—or more precisely, *invertibility*—of arrows in higher dimensions serve as a more general method of studying the weaker notions of equivalence necessary in mathematics: for example, higher categories are more capable of studying spaces that are considered equivalent to their continuous deformations, such as in homotopy theory. While the intentions for higher category theory were clear, producing a precise axiomatisation of higher categories requires exponentially more rules as the dimension increases, and the ways of composing higher-dimensional morphisms also quickly become unwieldy. For this reason, explicit theory for higher categories only exist for low dimensions: Bénabou established the definition for dimension two with *bicategories* in [7], and this was extended to *tricategories* in dimension three by Gordon, Power, and Street in [17]. For higher dimensions, the standard

approach was instead to present higher categories indirectly using other mathematical objects that were easier to describe, this resulting in several candidate presentations for higher categories.

The most popular models are based on simplicial sets, such as the quasicategories of Boardman and Vogt in [9]. Simplicial sets are abstract structures built from gluing higher-dimensional triangles together, and these triangular shapes can encode how the one-dimensional arrows compose when composition is no longer strictly unique. However, their triangular shape limits their effectiveness in expressing higher-dimensional arrows, so quasicategories are presentations only of a special case of higher categories called $(\infty,1)$-*categories*: those higher categories wherein the k-morphisms are invertible for all $k > 1$. While $(\infty,1)$-categories are sufficient for handling algebraic or set-theoretic objects that are coherent up to higher homotopy, they cannot fully express homotopy theories themselves, as this requires non-invertible 2-morphisms and thus $(\infty,2)$-categories.

Even with an established presentation of $(\infty,2)$-categories, this would surely lead to the need for a theory of $(\infty,3)$-categories—for instance, showing that two different presentations of $(\infty,2)$-categories are equivalent requires comparing them in an ambient $(\infty,3)$-category. There should be a conceptual limit to these iterative abstractions, and these would be the fully weak (∞,∞)-categories—also called *weak ω-categories*—where all higher-dimensional arrows need not be invertible. Barring set-theoretic complications, the theory of weak ω-categories is entirely self-contained. However, weak ω-categories are also the most difficult to axiomatise, as they are the most structurally delicate versions of higher categories. There are a handful of proposed presentations, many of which try to generalise the ideas behind quasicategories: examples include the weak complicial sets of Verity in [47], and the opetopic sets of Baez and Dolan in [3].

Despite the technical convenience of working with simplicial objects, an artefact of using them for presentations of higher categories is that they inherently cause an obstruction in the presentation of higher functors. In ordinary category theory, the functors describe a rule of translating points and arrows of one category into those of another, in a compatible way such that composition of arrows is preserved, in the sense that the composite of the image of two arrows should coincide with the image of the composite of the same two arrows. When generalising this to higher categories, the compatibility of composition also must be relaxed. Since composition is no longer uniquely determined, the composite of the image of two arrows should be connected to the image of the composite of the same two arrows by an invertible 2-morphism, and similarly for the higher dimensions. This should also be true for the action of the unit arrows, but simplicial models of higher categories force the corresponding weak functors to preserve these units exactly, rather than up to higher invertible morphism. Such functors are called *strictly unital*, and the cause for this strict unity is that simplicial models of higher categories encode unit arrows via *degeneracies*, and the simplicial structure forces degeneracies in general to be preserved exactly.

The main goal of this thesis is to develop a theory of higher categories that is capable of presenting weak functors that are not necessarily strictly unital. As this excludes any simplicial model, we approach this problem using *opetopes* similar to those of Baez and Dolan. These generalise simplices so that the faces are no longer forced to be only triangular, but also allow for the other polygonal shapes.

This corresponds to composing strings of several arrows together in a higher category simultaneously. This generalisation removes the necessity for degeneracies, as units can now be presented by monogons (one-sided polygons) correspond to composing an empty string of arrows together. Unfortunately, the drawback of opetopes is that they require more care when defining precisely, though they are straightforward to reason with intuitively.

As we also want the theory of higher categories to be convenient to work with, we develop our higher categories inductively: there should be a uniform way of producing $(n+1)$-dimensional categories given an established theory of n-dimensional categories. For this purpose, we adapt the construction of higher nerves by Tamsamani in [46] by replacing his multi-simplicial sets with multi-opetopic sets. The additional benefit of this approach is that the resulting objects retain the categorical structure of the higher categories they present directly, such as having a straightforward way of extracting the n-morphisms for any $n \geq 0$. This is not as easy for instance in the quasicategorical model, as the higher morphisms are only encoded implicitly by the simplicial structure.

Contributions of thesis. We introduce a modification of Cheng's category of opetopes in [11] to eliminate the invertible structure maps that permute the inpute facets of opetopes. We achieve this by using Leinster's definition in [31] of opetopes via nonsymmetric operads, and produce a category $\mathbf{O}$ of opetopes by introducing opetopic analogues of the simplicial coface maps as the morphisms. This ensures that there are no nontrivial isomorphisms in $\mathbf{O}$.

We define weak n-categories as certain presheaves on $\mathbf{O}^n$, with the corresponding weak n-functors being precisely the natural transformations. The conditions on weak n-categories are defined inductively: after defining $\mathbf{wk}n\mathbf{Cat}$ and a corresponding notion of equivalence of weak n-categories, the weak $(n+1)$-categories are then defined as those functors $\mathcal{A} : \mathbf{O}^{\mathrm{op}} \to \mathbf{wk}n\mathbf{Cat}$ such that

- $\mathcal{A}_0$ is a discrete n-category,
- the target face map $t : \mathcal{A}_{2[1]} \to \mathcal{A}_1$ is an equivalence of n-categories,
- for every opetope γ, the canonical map $\mathcal{A}_\gamma \to \mathcal{A}_1^{\times_{\mathcal{A}_0} |\gamma|_1}$ is a surjective equivalence of n-categories.

with the weak $(n+1)$-functors $\Phi : \mathcal{A} \to \mathcal{B}$ defined to be the natural transformations from $\mathcal{A}$ to $\mathcal{B}$. The functor Φ is an equivalence of $(n+1)$-categories if and only if it is essentially surjective on objects and fully faithful in the appropriate sense. As there is an explicit definition by Bénabou of higher categories and weak functors in dimension two, we show that our construction of $\mathbf{wk2Cat}$ recovers the category $\mathbf{Bicat}$ of bicategories and pseudofunctors in Theorem 5.16:

Theorem A. *There is a fully faithful functor* $\mathrm{N} : \mathbf{Bicat} \to \mathrm{Func}(\mathbf{O}^{\mathrm{op}}, \mathbf{Cat})$ *that is weakly essentially surjective on the full subcategory* $\mathbf{wk2Cat}$ *of weak* 2*-categories in* $\mathrm{Func}(\mathbf{O}^{\mathrm{op}}, \mathbf{Cat})$.

In particular, this shows that our theory of weak n-functors is not strictly unital. Moreover, the proof of the above theorem implies a general proof of correctness of $\mathbf{wk}n\mathbf{Cat}$ for all $n \geq 0$: given an explicit definition of higher categories in some dimension n analogous to those of bicategories, with an

appropriate analogue of categorical limits, there should be a natural analogue of our proof of the above theorem to realise that theory as equivalent to **wk$n$Cat**. Therefore, our inductive construction provides a good presentation of all higher categories of finite dimension.

We then consider the problem of extending the construction to the infinite-dimensional case. Similar to an approach by Cheng and Leinster in [13], we define our weak ω-categories to be formal limits of their finite-dimensional truncations, but our approach differs from theirs in the choice of n-truncation: they eliminate all higher arrows above a certain dimension, leaving the composition of n-morphisms somewhat incoherent, whereas we identify the n-morphisms up to equivalence before removing the higher arrows so that the result of the truncation is always a weak n-category. This allows us to use our theory of finite-dimensional higher categories directly in studying weak ω-categories.

We make precise sense of the above approach using the theory of coalgebras over an appropriate endofunctor to obtain a category **wkωCat** of weak ω-categories and weak ω-functors. Coalgebras provide an abstract framework that allows us to define objects *coinductively*; that is, with "bottomless" recursion. We then show that our weak ω-categories admit a characterisation that strongly resembles that of our finite-dimensional higher categories in Theorem 5.37:

Theorem B. *The category* **wkωCat** *is isomorphic to the category of functors* $\mathcal{A} : \mathbf{O}^{\mathrm{op}} \to$ **wkωCat** *such that*

- $\mathcal{A}_0$ *is a discrete* ω*-category, corresponding to the set of objects of* $\mathcal{A}$*,*
- *the target face map* $t : \mathcal{A}_{2[1]} \to \mathcal{A}_1$ *is an equivalence of weak* ω*-categories,*
- *for every opetope* γ*, the canonical map* $\mathcal{A}_\gamma \to \mathcal{A}_1^{\times_{\mathcal{A}_0}|\gamma|_1}$ *is a surjective equivalence of weak* ω*-categories.*

Moreover, the equivalences of weak ω*-categories correspond to the natural transformations that are essentially surjective and fully faithful under this isomorphism.*

Moreover, **wkωCat** *is universal with the above characterisation. Let* $\mathscr{V}$ *be a category equipped with a suitable notion of weak equivalence. If* $\mathscr{V}$ *admits an inclusion* $\mathscr{V} \to \mathrm{Func}(\mathbf{O}^{\mathrm{op}}, \mathscr{V})$ *which realises its objects as certain functors* $\mathbf{O}^{\mathrm{op}} \to \mathscr{V}$ *satisfying analogous properties to those listed above, and the weak equivalences of* $\mathscr{V}$ *become fully faithful and essentially surjective under this realisation, then there is a unique functor* $\mathscr{V} \to$ **wkωCat** *which respects this realisation and the weak equivalences.*

We then briefly explore the subcategory ∞**Grpd** $\subset$ **wkωCat** of higher *groupoids*: those higher categories whose arrows are invertible in all dimensions. We show in Theorem 5.42 that ∞-groupoids also admit a coinductive characterisation as those weak ω-categories $\mathcal{A}$ such that $\mathcal{A}_\gamma$ is an ∞-groupoid for every opetope γ, and the various maps $\mathcal{A}_{2[p]} \to \mathcal{A}_1^{\times_{\mathcal{A}_0} p}$ are also equivalences for all $p \geq 0$.

The invertibility of arrows in ∞-groupoids make them more suitable in particular for studying homotopy theory, as homotopies of a space are always reversible. Grothendieck's *Homotopy Hypothesis* stipulates that the theory of higher groupoids should be equivalent to homotopy theory of spaces. Intuitively, the ∞-groupoid associated to a topological space X should be a higher-categorical generalisation

of its fundamental groupoid $\Pi_1 X$; thus, the k-morphisms of the ∞-groupoid should correspond to the k-dimensional homotopies in X.

Note that X can be realised as a groupoid weakly enriched in spaces: between any points $x, y \in X$, we have a space of continuous paths $x \to y$. This allows us to construct the ∞-groupoid corresponding to X coinductively, and we obtain a significant portion of the Homotopy Hypothesis:

Theorem C. *Given a compactly generated and weakly Hausdorff space X, there is a canonical functorial construction of its Poincaré ∞-groupoid $\Pi_\infty X$ in ∞**Grpd** $\subset$ **wkωCat** such that*

(i) the n-morphisms $f \to g$ in $\Pi_\infty X$ are the boundary-preserving homotopies from f to g in X for every $n \geq 1$; in particular, the 0-truncation of $\Pi_\infty X$ is the set of path-connected components of X, and the 1-truncation of $\Pi_\infty X$ is the fundamental groupoid of X,

(ii) for any $x \in X$, the homotopy group $\pi_n(X, x)$ is isomorphic to the group of equivalence classes of n-morphisms over x in $\Pi_\infty X$,

(iii) if the homotopy groups of X are trivial for all levels above n, then $\Pi_\infty X$ will be an n-groupoid.

Additionally, a continuous map $X \to Y$ is a weak homotopy equivalence if and only if the induced weak ω-functor $\Pi_\infty X \to \Pi_\infty Y$ is an equivalence of ∞-groupoids.

This is Theorem 5.47. We then propose a candidate space to any ∞-groupoid $\mathcal{G}$ whose Poincaré ∞-groupoid should be weakly equivalent to $\mathcal{G}$ which, if correct, would validate the Homotopy Hypothesis for our construction.

Organisation of thesis. Chapter 2 gives a more detailed exposition which outlines the main motivation for studying higher category theory and establishes the basic definitions. Chapter 3 surveys existing simplicial models of higher category theory, with the focus on highlighting the benefits and drawbacks of these models. None of the work in the first three chapters of this thesis are original; the only contributions of the author are in the organisation and presentation of the material. Chapter 4 motivates and develops the mathematics directly necessary for the main results of the thesis. In particular, the author provides a terse development of Tamsamani's construction of multi-simplicial nerves, and the definition of the category of opetopes is an adaptation of Cheng's construction applied to Leinster's definition of opetopes. Chapter 5 is entirely independent and original work of the author, and contains all of the main results of the thesis. Chapter 6 then provides a summary of the contributions of the thesis in the context of other existing work, and describes various avenues for future research.

Chapter 2

Background and Motivation

Historically, much of mathematics had been developed using set-theoretical language, with the primary objects of study being sets of elements endowed with some form of additional structure. For instance, groups are sets of symmetries subject to a rule describing how these symmetries can be combined, while topological spaces are sets of points equipped with a configuration that describes an overarching shape. The theory of any such flavour of mathematical object is then described by the functions of their underlying sets that preserve this specified structure. For the aforementioned examples: group theory is studied through group homomorphisms that map between symmetries in a way compatible with how they are combined, and topology is studied through continuous functions that locally respect the configurations of points. Broadly speaking, studying these theories in general can be achieved to some extent through the notion of a category, which reduces mathematical objects to abstract points, and shifts focus to the maps between them:

Definition 2.1. A *category* $\mathscr{C}$ consists of

- a collection $\mathscr{C}_0$ of *objects*,
- for $x, y \in \mathscr{C}_0$, a collection $\mathrm{Hom}_{\mathscr{C}}(x, y)$ of *morphisms* $f : x \to y$,
- for $x, y, z \in \mathscr{C}_0$, a map $\circ : \mathrm{Hom}_{\mathscr{C}}(y, z) \times \mathrm{Hom}_{\mathscr{C}}(x, y) \to \mathrm{Hom}_{\mathscr{C}}(x, z)$ called *composition*,
- for $x \in \mathscr{C}$, a distinct morphism $\mathrm{id}_x : x \to x$ called the *identity* on x

such that composition is

- associative: $(f \circ g) \circ h = f \circ (g \circ h)$ for all $f : y \to z$, $g : x \to y$, $h : w \to x$, and $w, x, y, z \in \mathcal{C}_0$
- unital: $f \circ \mathrm{id}_x = f$ and $\mathrm{id}_y \circ f = f$ for all $f : x \to y$ and $x, y \in \mathscr{C}_0$.

For brevity, composition of morphisms may also be denoted by juxtaposition. The category $\mathscr{C}$ is called *locally small* if $\mathrm{Hom}_{\mathscr{C}}(x, y)$ is a set for all $x, y \in \mathscr{C}_0$, and is moreover called *small* if it is locally small and $\mathscr{C}_0$ is also a set.

Refer to [33] or [40] for the basics of category theory. The prototypical example of a category is **Set**: the category whose objects are sets, and whose morphisms are functions. In practice—as is the case for the category **Grp** of groups and group homomorphisms, or **Top** of topological spaces and continuous functions—categories often lie over **Set** in the sense that the category $\mathscr{C}$ has a canonical functor $U : \mathscr{C} \to \mathbf{Set}$ sending objects $x \in \mathscr{C}_0$ to their underlying set $Ux \in \mathbf{Set}_0$. The functor is typically faithful, as the set $\mathrm{Hom}_{\mathscr{C}}(x,y)$ of morphisms $x \to y$ is precisely the subset of those functions of sets $Ux \to Uy$ which preserve the additional structure of these objects of $\mathscr{C}$.

In fact, this perspective can be modified to apply to a general (locally small) category $\mathscr{C}$ as well. Fix an object $x \in \mathscr{C}_0$, then for any other object $s \in \mathscr{C}_0$, we can think of the set of morphisms $s \to x$ as the set of "s-shaped elements" of x, allowing us to see x as a $\mathscr{C}_0$-indexed family of sets. Now, morphisms $x \to y$ in $\mathscr{C}$ become $\mathscr{C}_0$-indexed families of functions, which map s-shaped elements of x to s-shaped elements of y for any given $s \in \mathscr{C}_0$. The structure on these families of sets describing any object $x \in \mathscr{C}_0$ is given by how we are allowed to move between elements of different shapes in x (through morphisms), and the morphisms in $\mathscr{C}$ will then be exactly those families of functions which preserve this structure. Working this interpretation out formally recovers the fully faithful Yoneda embedding, which establishes an equivalence between any given category $\mathscr{C}$ and the full subcategory of representable presheaves in the category $\mathbf{PSh}\mathscr{C} := \mathrm{Func}(\mathscr{C}^{\mathrm{op}}, \mathbf{Set})$ of presheaves on $\mathscr{C}$ and natural transformations. This makes precise the idea that categories correspond to theories of structured sets.

This also reveals a limitation of ordinary categories: while they provide a convenient general framework to study the set theory of mathematical objects, they provide little aid in studying any deeper structure these objects may have. For example, suppose R is a commutative ring, then we can certainly build the category $R\mathbf{Mod}$ of R-modules and module homomorphisms. However, the language of categories is insufficient for homology, which require for instance good notions of kernels and images of morphisms. These require observing that R-modules are more than just structured sets, but are structured *abelian groups*, meaning that we need to enrich the theory of R-modules with the language from the theory of abelian groups. Remarkably, this action of enriching the theory can be reflected by endowing the hom-sets $\mathrm{Hom}_{R\mathbf{Mod}}(M,N)$ of module homomorphisms $M \to N$ with the structure of an abelian group via pointwise addition. Categories whose hom-sets are abelian groups then serve as the foundation for the classical theory of *abelian categories*, which provides a reasonable general framework for homological algebra as described for instance in [15].

A similar deficiency can even be seen from studying categories directly. As (small) categories are sets of objects equipped with additional structure, they too can be collected into a category **Cat** of small categories and functors. However, set-theoretic language becomes inadequate for appropriately comparing categories. For example, set-theoretic objects are considered the same if one can be obtained from the other via a structure-preserving relabelling of its elements, this being called an *isomorphism*. Isomorphisms of categories, however, are generally too strict to reflect when categories should be viewed as the same: categories should be instead viewed as equivalent if their internal set theories are essentially the same. We obtain the necessary language for making sense of this form of equivalence by enriching the category **Cat** over itself: the set of functors $\mathscr{C} \to \mathscr{D}$ can be made into a category $\mathrm{Func}(\mathscr{C}, \mathscr{D})$ by

taking its morphisms to be the natural transformations. This is described in more detail in Section 2.2.

2.1 Monoidal Structure and Enrichment

A category $\mathscr{C}$ *enriched* in another category $\mathscr{V}$ consists of a class $\mathscr{C}_0$ of objects as in the case of ordinary categories, but the morphisms $x \to y$ for $x, y \in \mathscr{C}_0$ collect to form an object $\mathscr{C}(x,y)$ of $\mathscr{V}$. Making this precise thus requires additional structure from $\mathscr{V}$. If the theory of $\mathscr{V}$ is rich enough, we can appeal to an enriched analogue of the Yoneda embedding as shown by Kelly in [27, §2.4] to conclude that this notion of enrichment is sufficient in allowing us to view objects of $\mathscr{C}$ as structured objects of $\mathscr{V}$. This is the case when $\mathscr{V} = \mathbf{Ab}$ is the category of abelian groups and group homomorphisms, as well as when $\mathscr{V} = \mathbf{Cat}$. Note that this is also the case when $\mathscr{V} = \mathbf{Set}$, where we recover ordinary (locally small) categories.

If $\mathscr{C}$ is a $\mathscr{V}$-enriched category, then to each object $x \in \mathscr{C}_0$, we need an analogue of an identity endomorphism $\mathrm{id}_x : x \to x$. If $\mathscr{V} = \mathbf{Set}$, then morphisms $x \to y$ of $\mathscr{C}$ are elements of $\mathrm{Hom}_{\mathscr{C}}(x,y)$, so by the set-theoretic Yoneda embedding, the morphisms of $\mathscr{C}$ for a general $\mathscr{V}$ are generalised elements of $\mathscr{C}(x,y)$ of a certain fixed shape $\mathbb{1} \in \mathscr{V}_0$. In particular, the identity on $x \in \mathscr{C}_0$ is a choice of morphism $1_x : \mathbb{1} \to \mathscr{C}(x,x)$ in $\mathscr{V}$.

More importantly, we also need a composition on $\mathscr{C}$. If $\mathscr{V} = \mathbf{Set}$, then composition is defined by a family of functions

$$\circ : \mathrm{Hom}_{\mathscr{C}}(y,z) \times \mathrm{Hom}_{\mathscr{C}}(x,y) \to \mathrm{Hom}_{\mathscr{C}}(x,z)$$

for $x, y, z \in \mathscr{C}_0$. Translating this to enrichment in a general $\mathscr{V}$ requires being able to pair objects of $\mathscr{V}$ together, which is done through a bifunctor $\otimes : \mathscr{V} \times \mathscr{V} \to \mathscr{V}$. Then, composition on $\mathscr{C}$ is a family of choices of morphisms $\circ : \mathscr{C}(y,z) \otimes \mathscr{C}(x,y) \to \mathscr{C}(x,z)$ in $\mathscr{V}$ for $x, y, z \in \mathscr{C}_0$. To assert the usual axioms of category theory on this structure for $\mathscr{C}$, we need $\mathbb{1}$ and $\otimes$ in $\mathscr{V}$ to satisfy axioms of their own:

- The unit axiom on $\mathscr{C}$ should reflect that composition with identity endomorphisms preserves morphisms, which means for example that the two morphisms

$$\begin{tikzcd}
\mathbb{1} \otimes \mathscr{C}(x,y) \arrow[r, "1_y \otimes \mathscr{C}(x,y)"] & \mathscr{C}(y,y) \otimes \mathscr{C}(x,y) \arrow[r, "\circ"] & \mathscr{C}(x,y) \\
\mathscr{C}(x,y) \arrow[rr, "\mathrm{id}_{\mathscr{C}(x,y)}"'] & & \mathscr{C}(x,y)
\end{tikzcd}$$

  should be comparable in $\mathscr{V}$ for all $x, y, z \in \mathscr{C}_0$. This is impossible unless we can identify their domains, which suggests that $\mathbb{1} \otimes v$ should be naturally isomorphic to v for all $v \in \mathscr{V}_0$. Composing identities in $\mathscr{C}$ on the right implies an analogous isomorphism between $v \otimes \mathbb{1}$ and v.

- The associativity axiom on $\mathscr{C}$ should reflect that the two ways of performing a three-fold com-

posite of morphisms in $\mathscr{C}$ should be equal, meaning that the two morphisms

$$\big(\mathscr{C}(y,z)\otimes\mathscr{C}(x,y)\big)\otimes\mathscr{C}(w,x)\xrightarrow{(\circ)\otimes\mathscr{C}(w,x)}\mathscr{C}(x,z)\otimes\mathscr{C}(w,x)\xrightarrow{\circ}\mathscr{C}(w,z)$$

$$\mathscr{C}(y,z)\otimes\big(\mathscr{C}(x,y)\otimes\mathscr{C}(w,x)\big)\xrightarrow[\mathscr{C}(y,z)\otimes(\circ)]{}\mathscr{C}(y,z)\otimes\mathscr{C}(w,y)\xrightarrow[\circ]{}\mathscr{C}(w,z)$$

should also be comparable in $\mathscr{V}$ for all $w,x,y,z\in\mathscr{C}_0$. This again requires identifying domains, which suggests that $(v\otimes v')\otimes v''$ should be isomorphic to $v\otimes(v'\otimes v'')$ naturally in $v,v',v''\in\mathscr{V}_0$.

To ensure that there is no ambiguity in how to rebracket iterated applications of $\otimes$, the aforementioned natural isomorphisms should also satisfy some coherence axioms. This ultimately implies that $\mathscr{V}$ should at least be equipped with the structure of a *monoidal category* in order to discuss enrichment.

Definition 2.2. A monoidal category $(\mathscr{V},\otimes,\mathbb{1})$ is a category $\mathscr{V}$ equipped with a fixed object $\mathbb{1}\in\mathscr{V}_0$ called its *tensor unit*, a bifunctor $\otimes:\mathscr{V}\times\mathscr{V}\to\mathscr{V}$ called its *tensor product*, families of isomorphisms $\lambda_v:\mathbb{1}\otimes v\xrightarrow{\sim}v$ and $\rho_v:v\otimes\mathbb{1}\xrightarrow{\sim}v$ natural in $v\in\mathscr{V}_0$ called *unitors*, and a family of isomorphisms $\alpha_{u,v,w}:(u\otimes v)\otimes w\xrightarrow{\sim}u\otimes(v\otimes w)$ natural in $u,v,w\in\mathscr{V}_0$ called *associators*. This structure is subject to coherence axioms given by commutativity of the pentagon

$$\begin{tikzcd}
((u\otimes v)\otimes w)\otimes x \arrow[r, "\alpha_{u\otimes v,w,x}"] \arrow[d, "\alpha_{u,v,w}\otimes x"'] & (u\otimes v)\otimes(w\otimes x) \arrow[r, "\alpha_{u,v,w\otimes x}"] & u\otimes(v\otimes(w\otimes x)) \\
(u\otimes(v\otimes w))\otimes x \arrow[rr, "\alpha_{u,v\otimes w,x}"'] & & u\otimes((v\otimes w)\otimes x) \arrow[u, "u\otimes\alpha_{v,w,x}"']
\end{tikzcd}$$

for all $u,v,w,x\in\mathscr{V}_0$, and the triangle

$$\begin{tikzcd}
(u\otimes\mathbb{1})\otimes v \arrow[rr, "\alpha_{u,\mathbb{1},v}"] \arrow[dr, "\rho_u\otimes v"'] & & u\otimes(\mathbb{1}\otimes v) \arrow[dl, "u\otimes\lambda_v"] \\
& u\otimes v &
\end{tikzcd}$$

for all $u,v\in\mathscr{V}_0$.

If $\mathscr{V}$ has finite products, then it automatically admits a monoidal structure with the tensor product being given by the categorical product $(\times)$, and the tensor unit by the terminal object (pt). The unitors and associators are induced by the universal properties of these limits, and their uniqueness ensures that the coherence axioms are already satisfied. The triple $(\mathscr{V},\times,\mathrm{pt})$ is then called a *cartesian monoidal category*. Note that while cartesian monoidal categories are easy to find and construct, the greater generality is necessary to cover all of our basic examples of enrichment. For instance, enrichment over abelian groups is not done through its cartesian structure $(\mathbf{Ab},\oplus,0)$: the category of abelian groups has a zero object, meaning there is a unique morphism $0\to G$ for any abelian group G, and this map picks out the neutral element of G. This is problematic since the identity endomorphism in $\mathrm{Hom}_{R\mathbf{Mod}}(M,M)$ is never the zero map on M unless $M=0$. Instead, the usual monoidal structure on $\mathbf{Ab}$ is given by the

tensor product of abelian groups $(\mathbf{Ab}, \otimes_{\mathbb{Z}}, \mathbb{Z})$, which is the prototypical example of a monoidal category. In this case, set-theoretic elements of an abelian group G correspond to group homomorphisms $\mathbb{Z} \to G$.

Since the category of abelian groups is precisely the category of $\mathbb{Z}$-modules, we can enrich **Ab** over itself by endowing its hom-sets with pointwise addition. Similarly, **Cat** is enriched over itself, as we have a category $\mathrm{Func}(\mathscr{C}, \mathscr{D})$ of functors between any two categories $\mathscr{C}$ and $\mathscr{D}$. This generalises the tautological enrichment of **Set** over itself. Notice that in the case **Set**, its cartesian monoidal structure and self-enrichment are connected by *currying*: functions $f : X \times Y \to Z$ of sets naturally correspond to maps $f : X \to \mathrm{Hom}_{\mathbf{Set}}(Y, Z)$ by partial evaluation. This correspondence extends to **Ab** and **Cat** as well, which motivates the following definition:

Definition 2.3. A monoidal category $(\mathscr{V}, \otimes, \mathbb{1})$ is called (right) *closed* if for every $v \in \mathscr{V}_0$, the functor $v \otimes (-)$ admits a right adjoint $[v, -]$.

Let $(\mathscr{V}, \otimes, \mathbb{1})$ be closed monoidal. Through the adjunction $v \otimes (-) \dashv [v, -]$, generalised elements $\mathbb{1} \to [v, w]$ correspond to morphisms $v \cong \mathbb{1} \otimes v \to w$ naturally in $w \in \mathscr{V}_0$. In other words, $[v, w]$ realises $\mathrm{Hom}_{\mathscr{V}}(v, w)$ as an object of $\mathscr{V}$ for all $v, w \in \mathscr{V}_0$. For this reason, $[-, -]$ is called the *internal hom* of $\mathscr{V}$, and it defines a canonical enrichment of $\mathscr{V}$ over itself. The adjunction counit provides morphisms $\mathrm{ev}_w : [v, w] \otimes v \to w$ internalising the notion of evaluating a morphism $v \to w$ at an element of v to yield an element of w. The identity endomorphisms $\mathbb{1} \to [v, v]$ are then the adjuncts of the left unitors $\mathbb{1} \otimes v \cong v$, and the composition rule $[v, w] \otimes [u, v] \to [u, w]$ is the adjunct of the composite

$$\big([v, w] \otimes [u, v]\big) \otimes u \xrightarrow{\alpha_{[v,w],[u,v],u}} [v, w] \otimes \big([u, v] \otimes u\big) \xrightarrow{[v,w] \otimes \mathrm{ev}_v} [v, w] \otimes v \xrightarrow{\mathrm{ev}_w} w$$

2.2 Higher Category Theory

Given any monoidal category $\mathscr{V}$, we can define a category $\mathscr{V}\mathbf{Cat}$ of (small) $\mathscr{V}$-enriched categories and $\mathscr{V}$-enriched functors. If the monoidal structure is symmetric (cf. Definition 2.6 below), then given $\mathscr{V}$-enriched categories $\mathscr{C}$ and $\mathscr{D}$, we can define their tensor product $\mathscr{C} \otimes \mathscr{D}$ by taking the class of objects to be $(\mathscr{C} \otimes \mathscr{D})_0 := \mathscr{C}_0 \times \mathscr{D}_0$, and for $c, c' \in \mathscr{C}_0$ and $d, d' \in \mathscr{D}_0$ taking the hom-objects to be $(\mathscr{C} \otimes \mathscr{D})((c, d), (c', d')) := \mathscr{C}(c, c') \otimes \mathscr{D}(d, d')$. Note that the symmetry is necessary for a well-defined composition on this construction. With $\mathbf{B}\mathbb{1}$ where $(\mathbf{B}\mathbb{1})_0 := \{*\}$ and $\mathbf{B}\mathbb{1}(*, *) := \mathbb{1}$, this data altogether forms another symmetric monoidal category $(\mathscr{V}\mathbf{Cat}, \otimes, \mathbf{B}\mathbb{1})$. Kelly discusses this in detail in [27, §1.4]. The cartesian monoidal structure $(\mathscr{V}, \times, \mathrm{pt})$ is symmetric, for example, and by this construction we can see that the induced monoidal structure on $\mathscr{V}$-enriched categories is cartesian also.

If $(\mathscr{V}, \otimes, \mathbb{1})$ is moreover complete and closed monoidal, then the same will hold for $(\mathscr{V}\mathbf{Cat}, \otimes, \mathbf{B}\mathbb{1})$. Completeness is straightforward: given a functor $F : \mathscr{J} \to \mathscr{V}\mathbf{Cat}$, then $\varprojlim F$ has for its set of objects $(\varprojlim F)_0 = \varprojlim_{j \in \mathscr{J}_0} F(j)_0$ and for its hom-objects $(\varprojlim F)(x, y) = \varprojlim_{j \in \mathscr{J}_0} F(j)(\pi_j x, \pi_j y)$, where the maps $\pi_j : (\varprojlim F)_0 \to F(j)_0$ are the canonical projections produced by the limit. The categorical structure on $\varprojlim F$ is then induced by the universal property of its hom-objects. Kelly formalises the closed monoidal structure in [27, Chapter 2] by constructing enriched functor categories via ends.

Recall that categories enriched in the cartesian closed and complete category $(\mathbf{Set}, \times, \mathrm{pt})$ are the locally small categories. In particular, this means $(\mathbf{Set})\mathbf{Cat} = \mathbf{Cat}$, which is cartesian closed and complete. Repeating this, a **Cat**-enriched category $\mathscr{C}$ consists of objects as before, but between any two such objects $x, y \in \mathscr{C}_0$ we now have a hom-category $\mathscr{C}(x,y)$. Thus, the enrichment provides us with morphisms $x \to y$ via the objects of $\mathscr{C}(x,y)$, and also with higher morphisms between these morphisms, which are called 2-*morphisms*. This additional structure allows us to define within $\mathscr{C}$ various category-theoretic constructions, such as adjunctions (via their unit and counit formulation) and equivalences. **Cat**-enriched categories are called *strict* 2-*categories*, and we denote the category of (small) strict 2-categories by $2\mathbf{Cat} := (\mathbf{Cat})\mathbf{Cat}$. Again, this is a cartesian closed and complete category, so the process can be repeated.

If we take small 0-categories to be synonymous with sets, then we obtain an inductive definition of strict n-categories by setting $0\mathbf{Cat} := \mathbf{Set}$ and $(n+1)\mathbf{Cat} := (n\mathbf{Cat})\mathbf{Cat}$ for all $n \geq 0$. Moreover, if 0-morphisms are synonymous with objects, then we can inductively define $(k+1)$-morphisms of an $(n+1)$-category to be the k-morphisms of its hom-n-categories. We have a fully faithful embedding $\mathrm{disc} : \mathbf{Set} \hookrightarrow \mathbf{Cat}$ which views sets X as discrete categories $\mathrm{disc}\, X$ whose objects are the elements of X, and whose hom-sets are given by

$$\mathrm{Hom}_{\mathrm{disc}\, X}(x,y) = \begin{cases} \mathrm{pt}, & x = y \\ \varnothing, & \text{otherwise} \end{cases}$$

for all $x, y \in X$. This induces a similar embedding $\mathbf{Cat} \hookrightarrow 2\mathbf{Cat}$ which views categories as *locally discrete* 2-categories in that its only 2-morphisms are identities. Inductively, we obtain a chain of inclusions

$$0\mathbf{Cat} \hookrightarrow 1\mathbf{Cat} \hookrightarrow 2\mathbf{Cat} \hookrightarrow 3\mathbf{Cat} \hookrightarrow \cdots \hookrightarrow n\mathbf{Cat} \hookrightarrow \cdots$$

where in general, we think of an n-category as a higher category whose k-morphisms are trivial (i.e., identities) for all $k > n$. This alludes to a general notion of a higher category for which k-morphisms are allowed to be nontrivial for arbitrary $k > 0$. Such a higher category can in fact be defined, and admits a surprisingly compact definition via globular sets that can be found in [44]. Such objects are called *strict* ω-*categories*, with ω referring to the first infinite ordinal, being the colimit of the ordinal inclusions $0 \hookrightarrow 1 \hookrightarrow 2 \hookrightarrow \cdots$, and they collect into another category $\omega\mathbf{Cat}$ which contains $n\mathbf{Cat}$ for all $n \geq 0$. Moreover, Street shows in [44, Theorem 1.5] that this is cannot be pushed further, as there is an equivalence of categories $(\omega\mathbf{Cat})\mathbf{Cat} \simeq \omega\mathbf{Cat}$.

Cheng and Leinster later show in [13, Theorem 3.6] that $\omega\mathbf{Cat}$ is in fact a limit of a tower of forgetful functors

$$\cdots \to n\mathbf{Cat} \to \cdots \to 3\mathbf{Cat} \to 2\mathbf{Cat} \to 1\mathbf{Cat} \to 0\mathbf{Cat}$$

where the map $(n+1)\mathbf{Cat} \to n\mathbf{Cat}$ sends a strict $(n+1)$-category to the underlying strict n-category obtained by discarding the $(n+1)$-morphisms. The fact that $(\omega\mathbf{Cat})\mathbf{Cat} \cong \omega\mathbf{Cat}$ is even an isomorphism of categories then follows from general abstract nonsense. Briefly, the inclusions $n\mathbf{Cat} \hookrightarrow (n+1)\mathbf{Cat}$

endow n**Cat** with the structure of a *coalgebra* over the enrichment endofunctor $\mathscr{V} \mapsto \mathscr{V}$**Cat** on cartesian monoidal categories, and the limit of the above tower constructs ω**Cat** as the terminal coalgebra over this endofunctor.

However, this is not the end of the story. The n-categories obtained by iterated enrichment are called *strict* because the enrichment only uses the 1-categorical (i.e., set-theoretic) structure of the cartesian monoidal category n**Cat**. In particular, objects of n**Cat** are compared up to strict isomorphism when performing the enrichment, and so the resulting $(n+1)$-categories have a composition rule that is associative and unital on-the-nose rather than up to a weaker notion of n-equivalence. This restriction hampers the applicability of these higher categories to other fields of mathematics.

2.2.1 Weak Higher Categories

The notion of equivalence between objects in an n-category, which may be called an *n-equivalence*, is defined inductively starting with 0-equivalence meaning genuine equality. For $n \geq 1$, an equivalence of objects x, y in an n-category $\mathscr{C}$ is given by a pair of 1-morphisms $f : x \to y$ and $g : y \to x$ such that $g \circ f$ is $(n-1)$-equivalent to id_x in $\mathscr{C}(x,x)$ and $f \circ g$ is $(n-1)$-equivalent to id_y in $\mathscr{C}(y,y)$. The cartesian closed structure of n**Cat** canonically makes it into an $(n+1)$-category, from which the $(n+1)$-equivalences become the appropriate notion of *equivalence of n-categories*.

This weaker notion of equivalence resembles homotopy equivalences of spaces. In fact, higher category theory can be thought of as a directed analogue of homotopy theory: objects correspond to points in space, morphisms to directed paths, 2-morphisms to directed homotopies between paths, and so on. From this perspective, one would expect higher category theory to subsume homotopy theory by restricting to higher groupoids, where all the higher morphisms are invertible. This is the content of Grothendieck's *Homotopy Hypothesis*, which stipulates that (small) n-groupoids should correspond to homotopy n-types, up to equivalence. However, this is not achieved by strict n-categories.

Strict n-groupoids can be defined inductively: the small 0-groupoids are precisely the sets, and the strict $(n+1)$-groupoids for any $n \geq 0$ are those strict $(n+1)$-categories whose 1-morphisms are all equivalences, and whose hom-n-categories are all n-groupoids. On the other hand, (weak) homotopy n-types are topological spaces considered modulo continuous functions which induce isomorphisms on homotopy groups π_k for all $0 \leq k \leq n$. The Homotopy Hypothesis clearly holds when $n = 0$, since homotopy 0-types can be represented by discrete spaces, and discrete spaces correspond to sets and thus to 0-groupoids.

The hypothesis is also satisfied when $n = 1$. Consider the functor $\Pi_1 : \mathbf{Top} \to \mathbf{Cat}$ which sends a topological space X to its *fundamental groupoid* $\Pi_1 X$, whose objects are the points of X and whose morphisms $x \to y$ are the continuous paths from x to y in X considered up to endpoint-preserving homotopy. Just as for the fundamental groups of a space, the quotient by homotopy ensures that concatenation of paths produces a well-defined associative and unital composition on $\Pi_1 X$. Moreover, two spaces have the same homotopy 1-type if and only if their fundamental groupoids are equivalent as categories. Eilenberg and Mac Lane have shown that we can also move in the other direction: given a groupoid $\mathscr{G}$, there exists a CW complex $K(\mathscr{G}, 1)$ such that $\Pi_1 K(\mathscr{G}, 1) = \mathscr{G}$. An explicit construction is described for

instance in [4, §1.4].

It turns out that strict 2-groupoids can also encode homotopy 2-types, but this is coincidental. The main obstacle for strict n-groupoids to model homotopy n-types in general is the existence of nontrivial Whitehead brackets, which reflect nontrivial commutation relations between higher homotopies of a space. More concretely, Simpson proves in [43, Theorem 2.7.2] that any reasonable construction analogous to $K(\mathscr{G},1)$ for strict 3-groupoids will fail to obtain the homotopy 3-type of the sphere $\mathbb{S}^2$.

Recall for a pointed topological space X that its *loop space* ΩX is the space of pointed continuous functions $\mathbb{S}^1 \to X$ (with the compact-open topology). We have natural isomorphisms $\pi_k(\Omega X) \cong \pi_{k+1}(X)$ for all $k \geq 0$, so ΩX is a homotopy $(n-1)$-type if X is a homotopy n-type for any $n \geq 1$. Undoing the action of producing loop spaces would thus provide a means of creating and studying higher homotopy types. However, not all spaces X can be realised as the loop space ΩY for some Y. Loop spaces carry a natural weak algebraic structure from the fact that loops can be concatenated, and so if X carries analogous structure, we define its *delooping* $\mathbf{B}X$ to be a path-connected pointed space with $X \simeq \Omega\mathbf{B}X$. In particular, this ensures that $\pi_{k+1}(\mathbf{B}X) \cong \pi_k(X)$ for all $k \geq 0$, so $\mathbf{B}X$ is a homotopy $(n+1)$-type for all homotopy n-types X.

A notion of delooping also exists in higher category theory, and the algebraic structure necessary for delooping to be possible here is for the category to be *monoidal*. For example, if M is a monoid, then we can reinterpret it as a category $\mathbf{B}M$ where $(\mathbf{B}M)_0 := \{*\}$ and $\mathrm{Hom}_{\mathbf{B}M}(*,*) := M$. The monoid unit and multiplication on M become the identity and composition, respectively, on $\mathbf{B}M$, and this defines the categorical delooping of M. If $(\mathscr{V},\otimes,\mathbb{1})$ is a monoidal category, then we would expect that it can also be delooped into a one-object 2-category, but strict 2-categories require a strictly associative composition, unlike the tensor product $\otimes$ that is associative only up to coherent isomorphism. However, by Mac Lane's Coherence Theorem (cf. [33, §VII.2]), we have that any monoidal category is equivalent to one that is strictly associative and monoidal, and the equivalence is strongly monoidal in the sense that the monoidal structure is preserved up to coherent isomorphism (cf. Remark 2.9). This explains why homotopy 2-types can be successfully presented by strict 2-groupoids. In higher dimensions, we no longer have access to such a strong coherence theorem as we face the following obstruction:

Proposition 2.4 (Eckmann-Hilton argument)**.** *Let $\mathscr{C}$ be a strict 2-category, and fix an object $x \in \mathscr{C}_0$. Set $M := \mathrm{Hom}_{\mathscr{C}(x,x)}(\mathrm{id}_x,\mathrm{id}_x)$, then M carries two strict monoidal structures: the first is* vertical composition $\circ$ *induced by the composition of 2-morphisms on the category $\mathrm{Hom}_{\mathscr{C}}(x,x)$, and the second is* horizontal composition $\otimes$ *induced by the* **Cat***-enriched composition of 1-morphisms on the 2-category $\mathscr{C}$. These define the same monoidal structure on M, and this structure is moreover commutative.*

Proof. The functoriality of $\otimes : \mathscr{C}(x,x) \times \mathscr{C}(x,x) \to \mathscr{C}(x,x)$ implies the *exchange law* for $a,b,c,d \in M$:

$$(a \circ b) \otimes (c \circ d) = (a \otimes c) \circ (b \otimes d)$$

We can explicitly check that the identity 2-cell on id_x is the unit with respect to either of the above products, but this also follows from the above identity and the fact that each product has a unit: let $1_\otimes$

and $1_\circ$ be the units for $\otimes$ and $\circ$, respectively, then

$$\begin{aligned} 1_\otimes = 1_\otimes \otimes 1_\otimes &= (1_\otimes \circ 1_\circ) \otimes (1_\circ \circ 1_\otimes) \\ &= (1_\otimes \otimes 1_\circ) \circ (1_\circ \otimes 1_\otimes) && \text{(exchange law)} \\ &= 1_\circ \circ 1_\circ = 1_\circ \end{aligned}$$

Now, denote this unit by 1, then for $a, b \in M$ we have

$$a \otimes b = (a \circ 1) \otimes (1 \circ b) = (a \otimes 1) \circ (1 \otimes b) = a \circ b$$

showing that the two multiplications coincide. Finally,

$$a \circ b = a \otimes b = (1 \circ a) \otimes (b \circ 1) = (1 \otimes b) \circ (a \otimes 1) = b \circ a$$

shows that the multiplication on M is commutative. □

Corollary 2.5. *If $\mathscr{C}$ is a strict 3-category, then $\mathscr{C}(x,x)(\mathrm{id}_x, \mathrm{id}_x)$ is a strict commutative monoidal category.*

Therefore, the monoidal categories $\mathscr{V}$ that are twice-deloopable into a strict 3-category are those that are equivalent to a strict commutative one. On the other hand, one would expect that a monoidal category equipped with an additional compatible monoidal structure would also be sufficient for being twice-deloopable, as the two tensor products could stand in for the horizontal and vertical composition. Tracing through the Eckmann-Hilton argument with two weak monoidal structures on $\mathscr{V}$, we obtain a single monoidal category equipped with a *braiding* rather than commutativity:

Definition 2.6. A *braided monoidal category* is a monoidal category $(\mathscr{V}, \otimes, \mathbb{1})$ equipped with an isomorphism $c_{u,v} : u \otimes v \xrightarrow{\sim} v \otimes u$ natural in $u, v \in \mathscr{V}_0$, called a *braiding*, that is coherent in the sense that the hexagons

$$\begin{tikzcd} (u \otimes v) \otimes w \arrow[r, "c_{u,v} \otimes w"] \arrow[d, "\alpha_{u,v,w}"'] & (v \otimes u) \otimes w \arrow[r, "\alpha_{v,u,w}"] & v \otimes (u \otimes w) \arrow[d, "v \otimes c_{u,w}"] \\ u \otimes (v \otimes w) \arrow[r, "c_{u,v \otimes w}"'] & (v \otimes w) \otimes u \arrow[r, "\alpha_{v,w,u}"'] & v \otimes (w \otimes u) \end{tikzcd}$$

and

$$\begin{tikzcd} u \otimes (v \otimes w) \arrow[r, "u \otimes c_{v,w}"] \arrow[d, "\alpha^{-1}_{u,v,w}"'] & u \otimes (w \otimes v) \arrow[r, "\alpha^{-1}_{u,w,v}"] & (u \otimes w) \otimes v \arrow[d, "c_{u,w} \otimes v"] \\ (u \otimes v) \otimes w \arrow[r, "c_{u \otimes v,w}"'] & w \otimes (u \otimes v) \arrow[r, "w \otimes c_{u,v}"'] & w \otimes (v \otimes u) \end{tikzcd}$$

commute for all $u, v, w \in \mathscr{V}_0$.

In particular, the double braid $c_{v,u} c_{u,v}$—called the *monodromy*—may be a nontrivial automorphism of $u \otimes v$. The prototypical example of a braided monoidal category is the *braid category*, constructed

by Joyal and Street in [25, Example 2.1] by taking the coproduct of the delooping of the Artin braid groups, where the tensor product is given by concatenating braids, and so in particular has nontrivial monodromy. From the Eckmann-Hilton argument, a monoidal category is twice-deloopable into a strict 3-category only if it is braided and the braiding has trivial monodromy; that is, only if $\mathscr{V}$ is *symmetric monoidal*.

Indeed, suppose F is a braided strongly monoidal equivalence from a braided monoidal category $\mathscr{V}$ to a strict commutative one. F being braided means that the braiding on $\mathscr{V}$ commutes with the strong monoidal coherence isomorphisms of F, which is stated more precisely in [25, Definition 2.3], and this reflects that F encodes an equivalence of 3-categories (so that this is the correct type of equivalence to discuss two-fold delooping). Consider the square

$$\begin{tikzcd} Fu \otimes Fv \arrow[r, "\cong"] \arrow[d, "\mathrm{id}_{Fu\otimes Fv}"'] & F(u\otimes v) \arrow[d, dashed] \\ Fu \otimes Fv \arrow[r, "\cong"] & F(u\otimes v) \end{tikzcd}$$

where the horizontal isomorphisms are the same and given by the strongly monoidal structure of F. This can be made commutative by taking the dashed arrow to be $F(\mathrm{id}_{u\otimes v})$ by the functoriality of F, but it can also be made commutative by taking the dashed arrow to be $F(c_{v,u}c_{u,v})$ by the fact that F is braided. As all other arrows are isomorphisms, this implies that $F(c_{v,u}c_{u,v}) = F(\mathrm{id}_{u\otimes v})$. As F is an equivalence, it is fully faithful, and thus $c_{v,u}c_{u,v} = \mathrm{id}_{u\otimes v}$, meaning $\mathscr{V}$ must have already had trivial monodromy and was thus symmetric.

The nontrivial structure of braided monoidal categories makes them useful in three-dimensional topology, and also provide a convenient language to study three-dimensional topological quantum field theories as discussed in [6, Chapter 4]. To extend these applications to higher dimensions, we need *weak* n-categories.

2.2.2 Bicategories

Bénabou took the first step towards establishing algebraic definitions of weak higher categories, providing an axiomatisation of weak 2-categories called *bicategories* that can be found in [7].

Definition 2.7. A bicategory $\mathscr{B}$ consists of a class $\mathscr{B}_0$ of objects and a category $\mathrm{Hom}_{\mathscr{B}}(x,y)$ of morphisms and 2-morphisms for all $x,y \in \mathscr{B}_0$ such that every $x \in \mathscr{B}_0$ admits an identity endomorphism $\mathrm{id}_x \in \mathrm{Hom}_{\mathscr{B}}(x,x)$, and such that there is a *horizontal composition functor*

$$\otimes : \mathrm{Hom}_{\mathscr{B}}(y,z) \times \mathrm{Hom}_{\mathscr{B}}(x,y) \to \mathrm{Hom}_{\mathscr{B}}(x,z)$$

for all $x,y,z \in \mathscr{B}_0$. Denote morphisms $f \in \mathrm{Hom}_{\mathscr{B}}(x,y)$ by arrows $f : x \to y$ as before, and 2-morphisms $\theta \in \mathrm{Hom}_{\mathscr{B}}(x,y)(f,g)$ by double arrows $\theta : f \Rightarrow g$.

These data are subject to unity and associativity axioms expressed respectively by the existence of *unitors* $\lambda_{x,y}(f) : \mathrm{id}_y \otimes f \overset{\sim}{\Rightarrow} f$ and $\rho_{x,y}(f) : f \otimes \mathrm{id}_x \overset{\sim}{\Rightarrow} f$ both natural in $f : x \to y$ for all $x,y \in \mathscr{B}_0$, and

associators $\alpha_{w,x,y,z}(f,g,h) : (f \otimes g) \otimes h \overset{\sim}{\Rightarrow} f \otimes (g \otimes h)$ natural in $f : y \to z$, $g : x \to y$, $h : w \to x$ for all $w,x,y,z \in \mathscr{B}_0$. These data are moreover subject to coherence given by commutativity of the pentagon

$$\begin{tikzcd}
((e \otimes f) \otimes g) \otimes h \arrow[r, Rightarrow, "\alpha(e\otimes f,g,h)"] \arrow[d, Rightarrow, "\alpha(e,f,g)\otimes h"'] & (e \otimes f) \otimes (g \otimes h) \arrow[r, Rightarrow, "\alpha(e,f,g\otimes h)"] & e \otimes (f \otimes (g \otimes h)) \\
(e \otimes (f \otimes g)) \otimes h \arrow[rr, Rightarrow, "\alpha(e,f\otimes g,h)"'] & & e \otimes ((f \otimes g) \otimes h) \arrow[u, Rightarrow, "e\otimes\alpha(f,g,h)"']
\end{tikzcd}$$

for all (horizontally) composable morphisms e, f, g, h, and the triangle

$$\begin{tikzcd}
(f \otimes \mathrm{id}_x) \otimes g \arrow[rr, Rightarrow, "\alpha(f,\mathrm{id}_x,g)"] \arrow[dr, Rightarrow, "\rho(f)\otimes g"'] & & f \otimes (\mathrm{id}_x \otimes g) \arrow[dl, Rightarrow, "f\otimes\lambda(g)"] \\
& f \otimes g &
\end{tikzcd}$$

for all $f : x \to y$ and $g : w \to x$ with $w,x,y \in \mathscr{B}_0$. When the intention is clear, horizontal composition of 1-morphisms may also be denoted by juxtaposition, as will vertical composition of 2-morphisms. In particular, horizontal composition of 2-morphisms will always be denoted by $\otimes$.

Certainly any strict 2-category can be viewed as a bicategory in the obvious way. From the definition, we can also see immediately how any monoidal category $(\mathscr{V}, \otimes, \mathbb{1})$ admits a delooping into a one-object bicategory $\mathbf{B}\mathscr{V}$. In fact, this delooping process can be extended to weak monoidal functors in the same way to produce *lax functors*:

Definition 2.8. Let $\mathscr{B}, \mathscr{B}'$ be bicategories, then a lax functor $F : \mathscr{B} \to \mathscr{B}'$ consists of a map $F : \mathscr{B}_0 \to \mathscr{B}'_0$ of objects and *local functors* $F_{x,y} : \mathrm{Hom}_{\mathscr{B}}(x,y) \to \mathrm{Hom}_{\mathscr{B}'}(Fx,Fy)$ equipped with a *lax unity constraint* $F^0_x : \mathrm{id}_{Fx} \Rightarrow F_{x,x}(\mathrm{id}_x)$ for $x \in \mathscr{B}_0$ and a *lax functoriality constraint* $F^2_{f,g} : F_{y,z}f \otimes F_{x,y}g \Rightarrow F_{x,z}(f \otimes g)$ natural in $f : y \to z$ and $g : x \to y$ for all $x,y,z \in \mathscr{B}_0$. Diagrammatically, these constraints are given by

$$\begin{tikzcd}
& \mathrm{Hom}_{\mathscr{B}}(x,x) \arrow[dr, "F_{x,x}"] \arrow[d, phantom, "F^0_x \Uparrow"] & \\
\mathbf{1} \arrow[ur, "\underline{\mathrm{id}_x}"] \arrow[rr, "\underline{\mathrm{id}_{Fx}}"'] & & \mathrm{Hom}_{\mathscr{B}'}(Fx,Fx)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Hom}_{\mathscr{B}}(y,z) \times \mathrm{Hom}_{\mathscr{B}}(x,y) \arrow[r, "\otimes"] \arrow[d, "F_{y,z}\times F_{x,y}"'] \arrow[dr, phantom, "F^2 \Nearrow"] & \mathrm{Hom}_{\mathscr{B}}(x,z) \arrow[d, "F_{x,z}"] \\
\mathrm{Hom}_{\mathscr{B}'}(Fy,Fz) \times \mathrm{Hom}_{\mathscr{C}}(Fx,Fy) \arrow[r, "\otimes"'] & \mathrm{Hom}_{\mathscr{B}'}(Fx,Fz)
\end{tikzcd}$$

where $\mathbf{1}$ is the terminal object in $\mathbf{Cat}$, and for instance $\underline{\mathrm{id}_x} : \mathbf{1} \to \mathrm{Hom}_{\mathscr{B}}(x,x)$ corresponds to the functor which picks out $\mathrm{id}_x \in \mathrm{Hom}_{\mathscr{B}}(x,x)_0$. These constraints are then subject to compatibility with associators

via commutativity of

$$\begin{tikzcd}
(Ff \otimes Fg) \otimes Fh \arrow[r, Rightarrow, "\alpha(Ff,Fg,Fh)"] \arrow[d, Rightarrow, "F^2_{f,g} \otimes \mathrm{id}_{Fh}"'] & Ff \otimes (Fg \otimes Fh) \arrow[d, Rightarrow, "\mathrm{id}_{Ff} \otimes F^2_{g,h}"] \\
F(f \otimes g) \otimes Fh \arrow[d, Rightarrow, "F^2_{f \otimes g,h}"'] & Ff \otimes F(g \otimes h) \arrow[d, Rightarrow, "F^2_{f,g \otimes h}"] \\
F(f \otimes (g \otimes h)) \arrow[r, Rightarrow, "F(\alpha(f,g,h))"'] & F((f \otimes g) \otimes h)
\end{tikzcd}$$

for all composable morphisms f, g, h, and also compatibility with unitors via commutativity of

$$\begin{tikzcd}
\mathrm{id}_{Fy} \otimes Ff \arrow[r, Rightarrow, "\lambda(Ff)"] \arrow[d, Rightarrow, "F^0_y \otimes \mathrm{id}_{Ff}"'] & Ff \\
F(\mathrm{id}_y) \otimes Ff \arrow[r, Rightarrow, "F^2_{\mathrm{id}_y,f}"'] & F(\mathrm{id}_y \otimes f) \arrow[u, Rightarrow, "F(\lambda(f))"']
\end{tikzcd}
\qquad
\begin{tikzcd}
Ff \otimes \mathrm{id}_x \arrow[r, Rightarrow, "\rho(Ff)"] \arrow[d, Rightarrow, "\mathrm{id}_{Ff} \otimes F^0_x"'] & Ff \\
Ff \otimes F(\mathrm{id}_x) \arrow[r, Rightarrow, "F^2_{f,\mathrm{id}_x}"'] & F(f \otimes \mathrm{id}_x) \arrow[u, Rightarrow, "F(\rho(f))"']
\end{tikzcd}$$

for all $f : x \to y$ with $x, y \in \mathscr{B}_0$.

If the components of F^0 and F^2 are all isomorphisms, then F is called a *pseudofunctor*. If they are moreover identities, then F is called a *strict* 2*-functor*.

Remark 2.9. It follows from the above definitions that a functor $F : \mathscr{V} \to \mathscr{W}$ between monoidal categories is weak monoidal exactly when it corresponds to a lax functor $\mathbf{B}\mathscr{V} \to \mathbf{B}\mathscr{W}$, and is likewise strongly monoidal exactly when it corresponds to a pseudofunctor.

As these three classes of functors have well-defined composition and evident identity maps, we can collect (small) bicategories to form categories $\mathbf{Bicat}_{\mathrm{str}}$, $\mathbf{Bicat}$, and $\mathbf{Bicat}_{\mathrm{Lax}}$, where the morphisms are given by strict 2-functors, pseudofunctors, and lax functors, respectively. However, just as the category $\mathbf{Cat}$ of small categories is more appropriately two-dimensional, $\mathbf{Bicat}$ naturally carries the structure of a weak 3-category (called a *tricategory*). In fact, the 1-categorical structure of $\mathbf{Bicat}$ and $\mathbf{Bicat}_{\mathrm{Lax}}$ is much less well-behaved than $2\mathbf{Cat}$ and even $\mathbf{Bicat}_{\mathrm{str}}$: for instance, $\mathbf{Bicat}$ and $\mathbf{Bicat}_{\mathrm{Lax}}$ are not finitely complete, as shown in [29, Example 4.5]. To obtain the complete picture for $\mathbf{Bicat}$, we provide it with appropriate notions of 2- and 3-morphisms.

Definition 2.10. Suppose $\mathscr{B}, \mathscr{B}'$ are bicategories with pseudofunctors $F, G : \mathscr{B} \to \mathscr{B}'$. A *lax natural transformation* $\theta : F \Rightarrow G$ consists of a morphism $\theta_x : Fx \to Gx$ in $\mathscr{B}'$ for every $x \in \mathscr{B}_0$ and 2-morphisms $\theta_f : (Gf) \otimes \theta_x \Rightarrow \theta_y \otimes (Ff)$ natural in $f : x \to y$ in $\mathscr{B}$ as per the diagram

$$\begin{tikzcd}
Fx \arrow[r, "Ff"] \arrow[d, "\theta_x"'] \arrow[dr, phantom, "\Nearrow \theta_f"] & Fy \arrow[d, "\theta_y"] \\
Gx \arrow[r, "Gf"'] & Gy
\end{tikzcd}$$

and the data is subject to lax unity as per the pasting diagram identity

$$
\begin{tikzcd}
Fx \arrow[r, bend left, "F(\mathrm{id}_x)"] \arrow[d, "\theta_x"'] \arrow[dr, phantom, "\overset{\theta_{\mathrm{id}_x}}{\Rrightarrow}"] & Fx \arrow[d, "\theta_x"] \\
Gx \arrow[r, bend left, "G(\mathrm{id}_x)"] \arrow[r, bend right, "\mathrm{id}_{Gx}"'] \arrow[r, phantom, "G^0 \Uparrow"] & Gx
\end{tikzcd}
=
\begin{tikzcd}
Fx \arrow[r, bend left, "F(\mathrm{id}_x)"] \arrow[r, bend right, "\mathrm{id}_{Fx}"'] \arrow[r, phantom, "F^0 \Uparrow"] \arrow[d, "\theta_x"'] \arrow[dr, bend left, "\theta_x"] \arrow[dr, phantom, "\overset{\lambda}{\Rrightarrow}"] & Fx \arrow[d, "\theta_x"] \arrow[dl, phantom, "\overset{\rho^{-1}}{\Rrightarrow}"] \\
Gx \arrow[r, bend right, "\mathrm{id}_{Gx}"'] & Gx
\end{tikzcd}
$$

as well as lax naturality as per the pasting diagram identity

$$
\begin{tikzcd}
Fx \arrow[rr, bend left, "F(gf)"] \arrow[d, "\theta_x"'] \arrow[rr, phantom, "\overset{\theta_{gf}}{\Rrightarrow}"] & & Fz \arrow[d, "\theta_z"] \\
Gx \arrow[rr, bend left, "G(gf)"] \arrow[dr, "Gf"'] & & Gz \\
& Gy \arrow[ur, "Gg"'] \arrow[u, phantom, "G^2 \Uparrow"] &
\end{tikzcd}
=
\begin{tikzcd}
Fx \arrow[rr, bend left, "F(gf)"] \arrow[dr, "Ff"'] \arrow[ddd, "\theta_x"'] & & Fz \arrow[ddd, "\theta_z"] \\
& Fy \arrow[ur, "Fg"'] \arrow[u, phantom, "F^2 \Uparrow"] \arrow[dd, "\theta_y"] \arrow[dl, phantom, "\overset{\theta_f}{\Rrightarrow}"] \arrow[dr, phantom, "\overset{\theta_g}{\Rrightarrow}"] & \\
& & \\
Gx \arrow[dr, "Gf"'] & & Gz \\
& Gy \arrow[ur, "Gg"'] &
\end{tikzcd}
$$

If all θ_f are invertible, then θ is called a *pseudonatural transformation*.

Given two lax natural transformations $\theta, \theta' : F \Rightarrow G$, a *modification* $\Gamma : \theta \Rrightarrow \theta'$ consists of a 2-morphism $\Gamma_x : \theta_x \Rightarrow \theta'_x$ in $\mathscr{B}'$ for every $x \in \mathscr{B}_0$ such that we have commutativity of the square

$$
\begin{tikzcd}
(Gf) \otimes \theta_x \arrow[r, Rightarrow, "Gf \otimes \Gamma_x"] \arrow[d, Rightarrow, "\theta_f"'] & (Gf) \otimes \theta'_x \arrow[d, Rightarrow, "\theta'_f"] \\
\theta_y \otimes (Ff) \arrow[r, Rightarrow, "\Gamma_y \otimes Ff"'] & \theta_y \otimes (Ff)
\end{tikzcd}
$$

Fix two bicategories $\mathscr{B}, \mathscr{B}'$. From [23, Theorem 4.4.11] and its corollaries, we can construct a bicategory $\mathrm{Lax}(\mathscr{B}, \mathscr{B}')$ of lax functors, lax natural transformations, and modifications, as well as a bicategory $\mathrm{Func}^{\mathrm{ps}}(\mathscr{B}, \mathscr{B}')$ of pseudofunctors, pseudonatural transformations, and modifications. These can be used to think of **Bicat** or $\mathbf{Bicat}_{\mathrm{Lax}}$ more as tricategories than ordinary 1-categories. We can also use these constructions to produce an analogue of the Yoneda Lemma and hence a Yoneda embedding $\mathscr{B} \hookrightarrow \mathrm{Func}^{\mathrm{ps}}(\mathscr{B}^{\mathrm{op}}, \mathbf{Cat})$, as is done in [23, Lemma 8.3.12]. This shows that the pseudofunctors form the most natural framework for studying the general theory of bicategories. In particular, the representable 2-presheaves of bicategories are not strict: nontrivial associators in the bicategory yield nontrivial functoriality constraints in the representable 2-presheaves, and likewise for the unity constraints from nontrivial unitors.

Since **Cat** is a strict 2-category, so is $\mathrm{Func}^{\mathrm{ps}}(\mathscr{B}^{\mathrm{op}}, \mathbf{Cat})$ for any bicategory $\mathscr{B}$, as pseudonatural trans-

formations are composed levelwise in **Cat**. Therefore, the Yoneda embedding also induces an equivalence between any bicategory $\mathscr{B}$ and a sub-2-category of $\mathrm{Func}^{\mathrm{ps}}(\mathscr{B}^{\mathrm{op}}, \mathbf{Cat})$. In particular, this shows that any bicategory is equivalent to a strict one, generalising the Coherence Theorem from monoidal categories to arbitrary bicategories. This is formalised in [23, Theorem 8.4.1].

2.2.3 Monads and Enriched Categories

Lax functors also offer a terse definition of monoids. Given a monoidal category $(\mathscr{V}, \otimes, \mathbb{1})$, a *monoid* in $\mathscr{V}$ is an object $M \in \mathscr{V}_0$ equipped with a *neutral element* $e : \mathbb{1} \to M$ and a *multiplication* $\mu : M \otimes M \to M$ that is associative and unital in the sense that

$$\begin{tikzcd}
(M \otimes M) \otimes M \arrow[r, "\alpha_{M,M,M}"] \arrow[d, "\mu \otimes M"'] & M \otimes (M \otimes M) \arrow[d, "M \otimes \mu"] \\
M \otimes M \arrow[r, "\mu"'] & M & M \otimes M \arrow[l, "\mu"]
\end{tikzcd}$$

as well as the diagrams

$$\begin{tikzcd}
\mathbb{1} \otimes M \arrow[rr, "\lambda_M"] \arrow[dr, "e \otimes M"'] & & M \\
& M \otimes M \arrow[ur, "\mu"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
M \otimes \mathbb{1} \arrow[rr, "\rho_M"] \arrow[dr, "M \otimes e"'] & & M \\
& M \otimes M \arrow[ur, "\mu"'] &
\end{tikzcd}$$

commute, respectively. Accordingly, a *monoid homomorphism* in $\mathscr{V}$ is a morphism $\varphi : M \to M'$ that respects the respective monoidal structures in the sense that the diagrams

$$\begin{tikzcd}
M \otimes M \arrow[r, "\mu"] \arrow[d, "\varphi \otimes \varphi"'] & M \arrow[d, "\varphi"] \\
M' \otimes M' \arrow[r, "\mu'"'] & M'
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{1} \arrow[r, "e"] \arrow[dr, "e'"'] & M \arrow[d, "\varphi"] \\
& M'
\end{tikzcd}$$

commute. As the identity morphisms are always monoid homomorphisms, and monoid homomorphisms are closed under the usual composition in $\mathscr{V}$, we obtain a category $\mathbf{Mon}(\mathscr{V})$ of monoids in $\mathscr{V}$. For example, $\mathbf{Mon}(\mathbf{Set}) = \mathbf{Mon}$ is the usual category of monoids, and $\mathbf{Mon}(\mathbf{Ab}) = \mathbf{Ring}$ is the category of (non-commutative unital) rings.

With the language of bicategories, a monoid in $\mathscr{V}$ reduces to a lax functor $\widetilde{M} : \mathbf{1} \to \mathbf{B}\mathscr{V}$, where we view the category $\mathbf{1}$ as a locally discrete bicategory, in the sense that its hom-set is viewed as a discrete category. Indeed, with this interpretation, the underlying object of the monoid is given by $M := \widetilde{M}(\mathrm{id}_*)$, and the neutral element and multiplication on M are given by the lax functoriality and unity constraints $e := \widetilde{M}^0_*$ and $\mu := \widetilde{M}^2_{\mathrm{id}_*, \mathrm{id}_*}$, respectively. The monoid homomorphisms $\varphi : M \to M'$ correspond to *oplax natural transformations* (lax natural transformations whose lax coherence 2-morphisms are reversed) $\widetilde{\varphi} : \widetilde{M} \Rightarrow \widetilde{M}'$ as per [23, Proposition 4.3.11]. In particular, the realisation of $\mathbf{Mon}(\mathscr{V})$ as a category of lax functors $\mathbf{1} \to \mathbf{B}\mathscr{V}$ immediately implies—at least on objects—that any weak monoidal functor

$F : \mathscr{V} \to \mathscr{W}$ induces a change-of-base functor $F_* : \mathbf{Mon}(\mathscr{V}) \to \mathbf{Mon}(\mathscr{W})$. Explicitly, F_* is given by post-composition of $\mathbf{B}F$ on lax functors $\mathbf{1} \to \mathbf{B}\mathscr{V}$. Presenting monoids in this way, we can readily generalise them to the context of bicategories, giving the theory of *monads*:

Definition 2.11. The category of monads in a bicategory $\mathscr{B}$ is the category $\mathbf{Mnd}(\mathscr{B}) := \mathrm{Lax}(\mathbf{1}, \mathscr{B})$ of lax functors and lax natural transformations.

If (T, μ, η) is a monad on some object $x \in \mathscr{B}_0$, then this means T is an endomorphism $x \to x$ equipped with a unit $\eta : \mathrm{id}_x \Rightarrow T$ and multiplication $\mu : TT \Rightarrow T$. A *left T-module* (coming from an object $y \in \mathscr{B}_0$) is a 1-morphism $A : y \to x$ of $\mathscr{B}$ equipped with a 2-morphism $\nu : TA \Rightarrow A$ called the *action* of T such that the diagrams

$$\begin{tikzcd} (TT)A \arrow[r, Rightarrow, "\alpha(T,T,A)"] \arrow[d, Rightarrow, "\mu A"'] & T(TA) \arrow[r, Rightarrow, "T\nu"] & TA \arrow[d, Rightarrow, "\nu"] \\ TA \arrow[rr, Rightarrow, "\nu"'] & & A \end{tikzcd} \qquad \begin{tikzcd} \mathrm{id}_x A \arrow[r, Rightarrow, "\eta A"] & TA \arrow[d, Rightarrow, "\nu"] \\ A \arrow[u, Rightarrow, "\lambda(A)"] \arrow[r, Rightarrow, "\mathrm{id}_A"'] & A \end{tikzcd}$$

commute. Accordingly, a left T-module homomorphism $\phi : (A, \nu) \to (A', \nu')$ is a 2-morphism $\phi : A \Rightarrow A'$ (which forces that the domains of A and A' are the same) which respects the actions in the sense that

$$\begin{tikzcd} TA \arrow[r, Rightarrow, "T\phi"] \arrow[d, Rightarrow, "\nu"'] & TA' \arrow[d, Rightarrow, "\nu'"] \\ A \arrow[r, Rightarrow, "\phi"'] & A' \end{tikzcd}$$

commutes. It is then clear that left T-modules coming from the same object y with T-module homomorphisms collect to form a category $T\mathbf{Mod}_{y/}$.

Monads in the strict 2-category **Cat** (which are precisely the monads of ordinary category theory) are particularly useful, appearing in various fields of mathematics as well as computer science such as in universal algebra and functional programming, as discussed for example in [10, §4] and [36]. Given a category $\mathscr{C}$ and a monad $T : \mathscr{C} \to \mathscr{C}$, the left T-modules of particular interest are the *T-algebras*, which are the left T-modules (A, ν) with $A : \mathbf{1} \to \mathbf{Cat}$. With their corresponding homomorphisms, T-algebras collect to form the *Eilenberg-Moore category* $\mathscr{C}^T := T\mathbf{Mod}_{\mathbf{1}/}$. Note that by mapping from the terminal category, a T-algebra is equivalently given by an object $A \in \mathscr{C}_0$ equipped with an action given by a morphism $\nu : TA \to A$ in $\mathscr{C}$. Likewise, T-algebra homomorphisms reduce to action-preserving morphisms in $\mathscr{C}$, and so we have an evident forgetful functor $U : \mathscr{C}^T \to \mathscr{C}$. Thinking of T as encoding abstract semantics for an algebraic theory, then T-algebras correspond to algebras in the sense of universal algebra that model the semantics with concrete functions over a set.

The above monad T induces a left adjoint $T \dashv U$ which sends an object $x \in \mathscr{C}_0$ to the free T-algebra on x, whose underlying object is Tx and whose action is given by $\mu_x : TTx \to Tx$. The composite UT then recovers the original monad on $\mathscr{C}$. In fact, this universally characterises the Eilenberg-Moore category up to isomorphism, which is stated more precisely for example in [40, Proposition 5.2.12].

Given an arbitrary adjunction $L : \mathscr{C} \rightleftarrows \mathscr{D} : R$, we have a canonical monad structure on RL over $\mathscr{C}$, where the unit is given by the adjunction unit $\eta : \mathrm{id}_{\mathscr{C}} \Rightarrow RL$, and the multiplication by the counit as $R\varepsilon L : RLRL \Rightarrow RL$, with the monad axioms following by the zigzag identities of the adjunction. From this adjunction, we have an induced functor $R : \mathscr{D} \to \mathscr{C}^{RL}$ into the Eilenberg-Moore category given by sending an object $x \in \mathscr{D}_0$ to the RL-algebra whose underlying object is Rx and whose action is given by $\varepsilon_{Rx} : RLRx \to Rx$. If this defines an equivalence $\mathscr{D} \simeq \mathscr{C}^{RL}$, then the functor $R : \mathscr{D} \to \mathscr{C}$ is said to be *monadic*, and Riehl gives some instances of this phenomenon in [40, Example 5.2.6]. We will briefly discuss the monadicity of monoids over sets in Section 4.3, but the main motivating example of that section is that **Cat** as a 1-category is monadic over the category of directed graphs.

Enriched categories. The definition of monoid objects in a monoidal category by lax functors also invites a generalisation on the domain side. The delooping of monoids allows us to think of categories as many-object variants of monoids. Likewise, given a monoidal category $\mathscr{V}$, categories enriched in $\mathscr{V}$ are many-object generalisations of monoids in $\mathscr{V}$. This interpretation can be formalised using lax functors: if a monoid in $\mathscr{V}$ is a lax functor $\mathbf{1} \to \mathbf{B}\mathscr{V}$, then a $\mathscr{V}$-enriched category with a set S of objects should be a lax functor from S to $\mathbf{B}\mathscr{V}$ as well, where S is endowed with the appropriate bicategorical structure. This is achieved by taking the *codiscrete category* $\operatorname{codisc} S$, where $(\operatorname{codisc} S)_0 := S$, and $\mathrm{Hom}_{\operatorname{codisc} S}(x,y) := \mathbf{1}$ contains a unique morphism $x \to y$ for all $x, y \in S$.

Now, a lax functor $\mathscr{C} : \operatorname{codisc} S \to \mathbf{B}\mathscr{V}$ is uniquely determined on objects since $(\mathbf{B}\mathscr{V})_0 = \{*\}$, and is locally given by functors $\mathbf{1} = \mathrm{Hom}_{\operatorname{codisc} S}(x,y) \to \mathrm{Hom}_{\mathbf{B}\mathscr{V}}(*,*) = \mathscr{V}$, meaning that it picks out an object $\mathscr{C}(x,y) \in \mathscr{V}_0$ for all $x, y \in S$. Similar to the case for monoid objects, the lax unity constraint picks for every $x \in S$ an identity $1_x := \mathscr{C}^0_x : \mathbb{1} \to \mathscr{C}(x,x)$ while the lax functoriality constraint produces for every $x, y, z \in S$ a composition $\circ := \mathscr{C}^2_{!,!} : \mathscr{C}(y,z) \otimes \mathscr{C}(x,y) \to \mathscr{C}(x,y)$. The coherence constraints of $\mathscr{C}$ correspond precisely to the usual axioms for $\mathscr{C}$ to be a category enriched in $\mathscr{V}$ in the sense defined in [27, §1.2].

We can then define for any set S a category $\mathscr{V}\mathbf{Cat}_S$ of $\mathscr{V}$-enriched categories over S, whose objects are lax functors $\operatorname{codisc} S \to \mathscr{V}$ and whose morphisms are oplax natural transformations where the components are all given by the tensor unit[1] $\mathbb{1}$. Note that $\mathscr{V}\mathbf{Cat}_{\mathrm{pt}} = \mathbf{Mon}(\mathscr{V})$, as expected. We can then reconstruct the entire category $\mathscr{V}\mathbf{Cat}$ via the *Grothendieck construction*, which is worked out in more detail in Appendix A. In particular, this makes it readily apparent that when given a weak monoidal functor $F : \mathscr{V} \to \mathscr{W}$, we obtain a change-of-base functor $F_* : \mathscr{V}\mathbf{Cat} \to \mathscr{W}\mathbf{Cat}$. Explicitly, the action is given on a $\mathscr{V}$-enriched category $\mathscr{C}$ by post-composition to give $F_*\mathscr{C} : \operatorname{codisc} \mathscr{C}_0 \xrightarrow{\mathscr{C}} \mathbf{B}\mathscr{V} \xrightarrow{\mathbf{B}F} \mathbf{B}\mathscr{W}$, meaning that the objects remain the same as before, and the hom-objects are given by $(F_*\mathscr{C})(x,y) := F(\mathscr{C}(x,y))$ for $x, y \in \mathscr{C}_0$.

[1] This restriction on the oplax natural transformations corresponds to the fact that $\mathscr{V}$-enriched functors in $\mathscr{V}\mathbf{Cat}_S$ should act as identities on the fixed set S of objects.

Chapter 3

Geometric Models for Higher Categories

Unlike in the strict setting, the definition of a weak 2-category via bicategories is already large even after establishing ordinary category theory. Gurski's PhD thesis [18] extends the construction one step further, refining the weak 3-categories called *tricategories* first introduced in [17] and working out the fundamental theory of these objects. Here, on top of having associators and unitors as in the case of bicategories, the pentagon and triangle axioms relax to invertible modifications which are themselves subject to further coherence axioms. This axiomatisation grows exponentially as the dimension increases, though arguably algorithmically: the coherence diagrams always take the shape of *associahedra* (also called *Stasheff polytopes*). For instance, Trimble has explicitly written down the coherence axioms for weak 4-categories in response to Street challenging him to do so, and the 51-page feat can be found on Baez's homepage [5]. While the axiomatisation of his "tetracategories" is impressive to look at, such a definition is far too unwieldy to work with in practice.

Therefore, higher categories are instead studied through geometric presentations, which avoid the need for an explicit essentially algebraic axiomatisation of the theory. This is inspired by the Homotopy Hypothesis: if k-morphisms are weakly invertible for every $k \geq 1$, then we can encode the weak ω-category with a topological space considered up to homotopy. This circumvents the need to define higher coherence constraints, and this is mainly because concatenation of paths and higher homotopies in a topological space are not explicitly defined, but are rather just characterised up to even higher homotopy. This is similar to the situation with a cartesian monoidal category: formally, the cartesian products are only characterised up to unique isomorphism, so realising a cartesian monoidal category properly requires making a choice of representative for every product, and the uniqueness constraint on the universal property ensures that the coherence axioms of a monoidal category then hold.

It is convenient when building models for higher categories from partial answers—such as spaces for ∞-groupoids and our usual definitions for categories or bicategories—to refine the hierarchy of higher categories to speak of (n,r)-*categories* in the sense of [4, Definition 8], where $n,r \leq \infty$. In the presence of an ambient notion of a weak ω-category, an (n,r)-category is a weak higher category whose k-morphisms all exist and are unique for $k > n+1$, and whose k-morphisms are all weakly invertible for $k > r$. For example, (n,n)-categories are precisely the weak n-categories and $(n,0)$-categories are the n-groupoids for all $n \leq \infty$.

From this perspective and through the Homotopy Hypothesis, it follows that **Top** with its homotopy theory forms an $(\infty,1)$-category: namely, the $(\infty,1)$-category ∞**Grpd** of ∞-groupoids. More generally, thinking of $(\infty,1)$-categories as being weakly enriched in ∞-groupoids, $(\infty,1)$-categories encode *homotopy-coherent* category theory in the sense that they provide general frameworks to do homotopy theory, and all of the categorical axioms that define them are themselves only described up to coherent homotopy.

3.1 Model Categories

An $(\infty,1)$-category whose composition and units are strict on the level of 1-morphisms, such as **Top**, can make sense of strict 1-categorical notions such as limits and colimits. However, the correct notion of equivalence between objects in an $(\infty,1)$-category is not by isomorphism, but rather by morphisms with homotopy inverses (similar to the situation in n-categories), and this greatly affects the desiderata of categorical constructions. In particular, limits and colimits should be modified appropriately so that they are invariant under these equivalence, rather than just up to isomorphism. For instance, consider the strict pullback square in **Top**:

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \{0\} \arrow[d] \\ \{1\} \arrow[r] & {[0,1]} \end{tikzcd}$$

Considering this up to homotopy, $[0,1] \simeq \{*\}$ is contractible, so the pullback should be equivalent to the pullback

$$\begin{tikzcd} \{\star\} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \{0\} \arrow[d] \\ \{1\} \arrow[r] & \{*\} \end{tikzcd}$$

yet the induced map $\varnothing \to \{\star\}$ cannot be made into a homotopy equivalence. This means that the strict pullbacks in **Top** do not present pullbacks in the corresponding $(\infty,1)$-category ∞**Grpd**.

One way to work in the underlying 1-category of such an $(\infty,1)$-category is to remember which of its 1-morphisms are equivalences. This results in a *category with weak equivalences*, which is a pair $(\mathscr{C},\mathcal{W})$ where $\mathscr{C}$ is an ordinary category, and $\mathcal{W} \subseteq \mathscr{C}_1$ is a class of morphisms called weak equivalences such that

- $\mathcal{W}$ contains the isomorphisms of $\mathscr{C}$
- weak equivalences satisfy *two-out-of-three*: if any two of f, g, $g \circ f$ are weak equivalences, then so is the third, for any composable pair of morphisms f,g in $\mathscr{C}$

The challenge is then to redefine limits and colimits in $\mathscr{C}$ so that they are characterised up to these weak equivalences, but this turns out to be quite difficult in general. Thus, abstract homotopy theory often uses additional scaffolding to make this more feasible, this structure given by the notion of a *model*

category studied extensively for example in Hovey's book [22]. For brevity, we will present here the more concise definition described by Riehl in [39].

Definition 3.1. Consider commutative squares of the form

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "\ell"'] & \bullet \arrow[d, "r"] \\ \bullet \arrow[r] \arrow[ur, dashed, "k"] & \bullet \end{tikzcd} \tag{Eq. 3.1}$$

Given a class $\mathcal{R}$ of morphisms, denote by $\mathrm{llp}(\mathcal{R})$ the class of those morphisms ℓ satisfying the *left lifting property* agianst $\mathcal{R}$; that is, those ℓ for which a lift k exists for any $r \in \mathcal{R}$ in (Eq. 3.1). Dually, given a class $\mathcal{L}$ of morphisms, denote by $\mathrm{rlp}(\mathcal{L})$ the class of those morphisms r which satisfy the *right lifting property* against $\mathcal{L}$; that is, those r for which a lift k exists for any $\ell \in \mathcal{L}$ in (Eq. 3.1).

A *weak factorisation system* in a category $\mathscr{C}$ is then a pair $(\mathcal{L}, \mathcal{R})$ of classes of morphisms in $\mathscr{C}$ such that

- every morphism f in $\mathscr{C}$ admits a factorisation $f = r \circ \ell$ where $\ell \in \mathcal{L}$ and $r \in \mathcal{R}$
- $\mathcal{R} = \mathrm{rlp}(\mathcal{L})$ and $\mathcal{L} = \mathrm{llp}(\mathcal{R})$

Note that invertibility implies that $\mathcal{L}$ and $\mathcal{R}$ will both contain the isomorphisms of $\mathscr{C}$.

Finally, a *model category* is a quadruple $(\mathscr{C}, \mathrm{cof}, \mathrm{fib}, \mathcal{W})$ where $\mathscr{C}$ is a complete and cocomplete category, and $\mathrm{cof}, \mathrm{fib}, \mathcal{W}$ are classes of morphisms in $\mathscr{C}$ such that

- $\mathcal{W}$ satisfies two-out-of-three
- $(\mathrm{cof} \cap \mathcal{W}, \mathrm{fib})$ and $(\mathrm{cof}, \mathrm{fib} \cap \mathcal{W})$ form weak factorisation systems in $\mathscr{C}$

Morphisms in $\mathcal{W}$ are called *weak equivalences* and are denoted by decorating the arrow with $x \xrightarrow{\sim} y$. Morphisms in fib are called *fibrations* and denoted $x \twoheadrightarrow y$, while morphisms in cof are called *cofibrations* and denoted $x \rightarrowtail y$. Fibrations and cofibrations which are also weak equivalences are called *acyclic*.

From the definition, any two classes in $(\mathrm{cof}, \mathrm{fib}, \mathcal{W})$ uniquely determine the third. For instance, the prototypical example is the Quillen model structure on **Top**, which is uniquely defined by its weak equivalences and fibrations. Namely:

- The weak equivalences are the weak homotopy equivalences, just as in ∞**Grpd** as per the Homotopy Hypothesis.
- The fibrations are the *Serre fibrations*, which are those continuous maps which satisfy the right lifting property against the inclusions $\mathbb{D}^n \times \{0\} \hookrightarrow \mathbb{D}^n \times [0,1]$ for $n \geq 0$, where $\mathbb{D}^n$ is the n-dimensional disk.

That this induces a model category is proven in [22, §2.4]. Therefore, we can say that $\mathbf{Top}_{\mathrm{Quillen}}$ presents the $(\infty, 1)$-category of spaces and thus of ∞-groupoids. However, comparing the weak equivalences here to the correct notion of equivalence in an $(\infty, 1)$-category, there is an obvious discrepancy since

weak homotopy equivalences do not necessarily have homotopy inverses. There is in fact another model structure on **Top** worked out by Strøm in [45] where the weak equivalences are the genuine homotopy equivalences, but this phenomenon does not happen in a general model category.

The structure of any model category $(\mathscr{C}, \mathrm{cof}, \mathrm{fib}, \mathcal{W})$ induces two dual notions of homotopy between morphisms. Let $y \in \mathscr{C}_0$, then we can construct a *path object* $y^{\mathbb{I}}$ by factoring the diagonal map into $\Delta_y : y \overset{\sim}{\rightarrowtail} y^{\mathbb{I}} \twoheadrightarrow y \times y$, which is then well-defined up to weak equivalence. With the two projections $d_0, d_1 : y^{\mathbb{I}} \rightrightarrows y$ corresponding to taking the endpoints of any given path in $y^{\mathbb{I}}$, we can then define a *right homotopy* between morphisms $f, g : x \to y$ to be a morphism $h : x \to y^{\mathbb{I}}$ such that $d_0 h = f$ and $d_1 h = g$. Dually, we can construct for $x \in \mathscr{C}_0$ a *cylinder object* $\mathrm{cyl}\, x$ by factoring the fold map into $\nabla_x : x \amalg x \rightarrowtail \mathrm{cyl}\, x \overset{\sim}{\twoheadrightarrow} x$, which morally corresponds to taking the product of x with an interval. With the two coprojections $s_0, s_1 : x \rightrightarrows \mathrm{cyl}\, x$ which pick out each endpoint face of the cylinder, we can likewise define a *left homotopy* between morphisms $f, g : x \to y$ to be a morphism $h : \mathrm{cyl}\, x \to y$ such that $hs_0 = f$ and $hs_1 = g$. These two notions of homotopy are generally incomparable unless the objects involved are particularly well-behaved.

Recall that Whitehead's Theorem tells us all weak homotopy equivalences between CW complexes are genuine homotopy equivalences. As every topological space admits a CW approximation, we can thus adjust spaces in $\mathbf{Top}_{\mathrm{Quillen}}$ up to weak equivalence so that the notion of weak equivalence is correct from a $(\infty, 1)$-categorical perspective. This phenomenon generalises to any model category $(\mathscr{C}, \mathrm{cof}, \mathrm{fib}, \mathcal{W})$: call an object $x \in \mathscr{C}_0$ *cofibrant* if the unique map $\emptyset \to x$ is a cofibration, and dually call it *fibrant* if the unique map $x \to \mathrm{pt}$ is a fibration. If an object is both cofibrant and fibrant, say that it is *bifibrant*. By the factorisation properties, any object $x \in \mathscr{C}_0$ admits a *cofibrant resolution* $\emptyset \rightarrowtail \mathsf{Q}x \overset{\sim}{\twoheadrightarrow} x$ as well as a *fibrant resolution* $x \overset{\sim}{\rightarrowtail} \mathsf{R}x \twoheadrightarrow \mathrm{pt}$, both of which are well-defined up to weak equivalence. In fact, applying both resolutions yields a bifibrant object $\mathsf{RQ}x$ that is related to x by a zigzag of weak equivalences, as can be seen by the diagram

$$\begin{tikzcd} \emptyset \arrow[r, tail] & \mathsf{Q}x \arrow[r, two heads, "\sim"] \arrow[d, tail, "\wr"] & x \\ & \mathsf{RQ}x \arrow[r, two heads] & \mathrm{pt} \end{tikzcd}$$

with cofibrancy of $\mathsf{RQ}x$ following as cofibrations are closed under composition. Restricting to bifibrant objects, we obtain the following generalisation of Whitehead's Theorem:

Proposition 3.2. *Let $\mathscr{C}$ be a model category and suppose $x, y \in \mathscr{C}_0$ with x cofibrant and y fibrant. Then, there exists a left homotopy between two morphisms $x \to y$ if and only if there exists a right homotopy between them, and this defines an equivalence relation $(\sim)$ on $\mathrm{Hom}_{\mathscr{C}}(x, y)$. Moreover, if x and y are bifibrant, then a morphism $x \to y$ is a weak equivalence in $\mathscr{C}$ if and only if it is a homotopy equivalence with respect to $(\sim)$.*

Proof. This follows from [22, Corollary 1.2.7 and Proposition 1.2.8]. □

This suggests that a model category more accurately presents an $(\infty, 1)$-category whose class of objects is given (up to equivalence) by the bifibrant objects, and whose 2-morphisms are encoded by

(left or right) homotopies. In particular, this can be seen by considering the 1-truncation of the $(\infty,1)$-category, which is the 1-category obtained by taking the 1-morphisms of the $(\infty,1)$-category modulo higher equivalence, and then forgetting the higher morphisms. This procedure reduces 1-morphisms that were originally equivalences into isomorphisms in the 1-truncation, meaning that if the $(\infty,1)$-category were presented by a category with weak equivalences $(\mathscr{C},\mathcal{W})$, then its 1-truncation should be the localisation $\mathscr{C}[\mathcal{W}^{-1}]$ obtained by formally inverting the morphisms in $\mathcal{W}$.

In general, this category can be quite difficult to study, as its morphisms are given by zigzags of morphisms of $\mathscr{C}$, where the arrows pointing backwards come from $\mathcal{W}$, and the zigzags are subject to the relations that express them as inverse to their forward-facing counterparts. Fortunately, if $\mathscr{C}$ is a model category, then the inclusion $\mathscr{C}^\circ \hookrightarrow \mathscr{C}$ of the full subcategory on bifibrant objects induces an equivalence of categories $\mathscr{C}^\circ/(\sim) \simeq \mathscr{C}[\mathcal{W}^{-1}]$, where the former is given by $\mathscr{C}^\circ$ with its morphisms taken modulo the homotopy relation $(\sim)$ of Proposition 3.2.

All topological spaces are fibrant in $\mathbf{Top}_{\text{Quillen}}$, and we will see briefly in the next section that CW complexes are among the cofibrant objects. The above discussion thus implies that the ∞-groupoids are presented more accurately by CW complexes and their homotopy theory. This is perhaps more satisfactory, since CW complexes are more combinatorial in nature, and higher categories should be purely combinatorial objects. In fact, this suggests that we might be able to filter away the topological structure of CW complexes altogether and produce an alternative presentation of ∞-groupoids that is truly combinatorial.

3.2 Simplicial Sets

The acyclic cofibrations in $\mathbf{Top}_{\text{Quillen}}$ are determined by having the left lifting property against Serre fibrations, which implies in particular that the inclusions $\mathbb{D}^n \times \{0\} \hookrightarrow \mathbb{D}^n \times [0,1]$ are acyclic cofibrations for all $n \geq 0$, where $\mathbb{D}^n$ denotes the n-dimensional disk. On the other hand, there is also a simple family of non-acyclic cofibrations, namely the boundary inclusions of disks $\partial\mathbb{D}^n \hookrightarrow \mathbb{D}^n$ for $n \geq 0$. Given these, we can generate further examples of cofibrations via closure properties induced on this class of morphisms by virtue of the fact that it is characterised by a left lifting property.

More precisely, suppose $\mathcal{R}$ is any class of morphisms. Then, $\text{llp}(\mathcal{R})$ is closed under pushout in the sense that the pushout of a morphism in $\text{llp}(\mathcal{R})$ along any other morphism will remain in $\text{llp}(\mathcal{R})$. This is immediate from the universal property of a pushout as can be seen in the diagram below:

$$\begin{tikzcd}
u \arrow[r] \arrow[d, "i\in\text{llp}(\mathcal{R})"'] & w \arrow[r] \arrow[d, "i'"'] & x \arrow[d, "\in\mathcal{R}"] \\
v \arrow[r] \arrow[ur, dashed, "k"] & v \amalg_u w \arrow[r] \arrow[ur, dotted, "k'"'] & y
\end{tikzcd}$$

If i' is the pushout of some $i \in \text{llp}(\mathcal{R})$, then given any lifting problem for i' against some morphism in $\mathcal{R}$, we can form the above diagram. The morphism k then exists by the definition of i, and then the universal property of the pushout produces the morphism k' which serves as a solution to the original lifting property for i'. The class $\text{llp}(\mathcal{R})$ is also closed under transfinite composition: given a family

$(u_\xi)_{\xi<\lambda}$ of objects indexed by some ordinal λ and morphisms $i_\xi : u_\xi \to u_{\xi+1}$ which all lie in $\mathrm{llp}(\mathcal{R})$, then the coprojection $i : u_0 \to \varinjlim_{\xi<\lambda} u_\xi =: u_\lambda$ will also fall in $\mathrm{llp}(\mathcal{R})$. This follows by transfinite induction on λ, as can be seen in the diagram below:

$$\begin{tikzcd}
u_0 \arrow[r] \arrow[d, dashed] & x \arrow[dd, "\in\mathcal{R}"] \\
u_\xi \arrow[ur, dashed, "k"] \arrow[d, "i_\xi \in \mathrm{llp}(\mathcal{R})"'] & \\
u_{\xi+1} \arrow[uur, dotted, "k'"] \arrow[d, dashed] & \\
u_\lambda \arrow[r] & y
\end{tikzcd}$$

If the partial composite $u_0 \to u_\xi$ admits a lift k, then we can use this to form a lifting property of i_ξ against $\mathcal{R}$ which produces a lift k' that also serves as a lift for the composite $u_0 \to u_{\xi+1}$. If ξ is a limit ordinal and all partial composites $u_0 \to u_{\xi'}$ for $\xi' < \xi$ admit a lift $u_{\xi'} \to x$, then the universal property of the composite $u_0 \to \varinjlim_{\xi'<\xi} u_{\xi'} = u_\xi$ induces from each of these lifts a unique map $k : u_\xi \to x$.

Therefore, given a class $\mathcal{I}$ of morphisms, define the class $\mathrm{cell}(\mathcal{I})$ of *relative $\mathcal{I}$-cell complexes* to be the class of morphisms obtained by taking transfinite composites of pushouts of morphisms in $\mathcal{I}$. It follows from the above discussion that if $\mathcal{I} \subseteq \mathrm{llp}(\mathcal{R})$, then $\mathrm{cell}(\mathcal{I}) \subseteq \mathrm{llp}(\mathcal{R})$ also. Note that $\mathrm{cell}\{\partial\mathbb{D}^n \hookrightarrow \mathbb{D}^n \mid n \geq 0\}$ is the usual class of relative cell complexes in **Top**, meaning in particular that cell complexes (and hence CW complexes) are cofibrant in $\mathbf{Top}_{\mathrm{Quillen}}$.

Classes of morphisms defined by a left lifting property are also preserved under taking retracts in the arrow category, meaning that if $i : u \to v$ is in $\mathrm{llp}(\mathcal{R})$, then so is any morphism i' fitting in the diagram

$$\begin{tikzcd}
w \arrow[r] \arrow[d, "i'"'] \arrow[rr, bend left, "\mathrm{id}_w"] & u \arrow[r] \arrow[d, "i"'] & w \arrow[d, "i'"] \\
z \arrow[r] \arrow[rr, bend right, "\mathrm{id}_z"'] & v \arrow[r] & z
\end{tikzcd}$$

also lies in $\mathrm{llp}(\mathcal{R})$. Indeed, any lifting problem for i' against some morphism in $\mathcal{R}$ induces through the retraction a lifting problem for i which produces a lift k. Composing with the section then yields the desired lift k', as illustrated in the diagram

$$\begin{tikzcd}
w \arrow[r] \arrow[d, "i'"'] \arrow[rr, bend left, "\mathrm{id}_w"] & u \arrow[r] \arrow[d, "i"'] & w \arrow[r] \arrow[d, "i'"] & x \arrow[d, "\in\mathcal{R}"] \\
z \arrow[r, "s"'] \arrow[urr, dotted, "ks"] \arrow[rr, bend right, "\mathrm{id}_z"'] & v \arrow[r] \arrow[urr, dashed, "k"] & z \arrow[r] & y
\end{tikzcd}$$

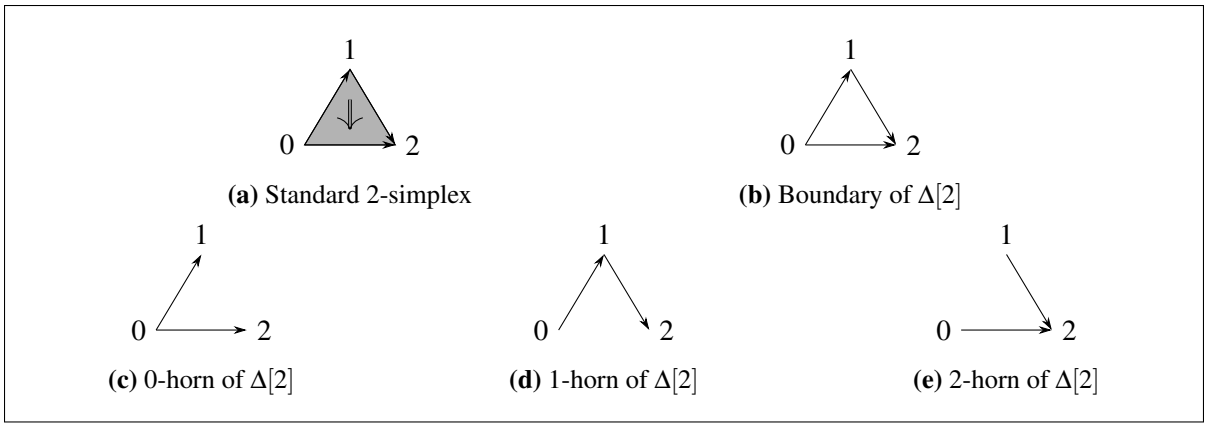
Accordingly, given a class $\mathcal{I}$ of morphisms, define the class $\mathrm{cof}(\mathcal{I})$ to be the class of morphisms obtained by taking retracts of relative $\mathcal{I}$-cell complexes. It then follows from the above discussion that if $\mathcal{I} \subseteq \mathrm{llp}(\mathcal{R})$, then $\mathrm{cof}(\mathcal{I}) \subseteq \mathrm{llp}(\mathcal{R})$ also. In fact, the class $\mathrm{cof}\{\partial\mathbb{D}^n \hookrightarrow \mathbb{D}^n \mid n \geq 0\}$ is precisely the class of cofibrations in $\mathbf{Top}_{\mathrm{Quillen}}$, meaning that the cofibrations are generated by the boundary inclusions of disks. Similarly, $\mathrm{cof}\{\mathbb{D}^n \times \{0\} \hookrightarrow \mathbb{D}^n \times [0,1] \mid n \geq 0\}$ is precisely the class of acyclic cofibrations on $\mathbf{Top}_{\mathrm{Quillen}}$. These generating families of morphisms will also uniquely determine the entire model structure of $\mathbf{Top}_{\mathrm{Quillen}}$, as (acyclic) fibrations are determined by the right lifting property against these classes, and then by the factorisation properties it follows that the weak equivalences are precisely compositions of acyclic fibrations followed by acyclic cofibrations. Therefore, to extract the combinatorial nature of the homotopy theory of topological spaces, it is enough to do so for the generating (acyclic) cofibrations. By triangulating the disks, this leads to the study of *simplicial sets*.

Let Δ be the full subcategory of **Cat** on the categories $[n] := \{0 \to 1 \to \cdots \to n\}$ for $n \geq 0$. This is to say that Δ is the category whose objects are the nonempty finite sets $\{0,\ldots,n\}$ equipped with the standard ordering on numbers, whose morphisms are given by monotone maps. This category can also be described by generators and relations, being the category whose morphisms are generated by *coface maps* $d^i : [n-1] \to [n]$ for $0 \leq i \leq n$ (which are explicitly the inclusions $[n-1] \hookrightarrow [n]$ whose images do not contain i, and thus $d^i(j) = j$ for $j < i$ and $d^i(j) = j+1$ for $j \geq i$) and *codegeneracy maps* $s^i : [n+1] \to [n]$ for $0 \leq i \leq n$ (which are the surjections $[n+1] \to [n]$ that combine i with $i+1$, meaning that $s^i(j) = j$ for $j \leq i$ and $s^i(j) = j-1$ for $j > i$), subject to the *simplicial identities*

$$\begin{aligned} d^j d^i &= d^i d^{j-1}, && (i < j) \\ s^j s^i &= s^{i-1} s^j, && (i > j) \end{aligned} \qquad\qquad s^j d^i = \begin{cases} d^i s^{j-1}, & (i < j) \\ \mathrm{id}, & (j \leq i \leq j+1) \\ d^{i-1} s^j, & (i > j+1) \end{cases}$$

The category of simplicial sets is then defined to be the category $\mathbf{sSet} := \mathbf{PSh}\,\Delta$ of presheaves on Δ. Given a simplicial set $A : \Delta^{\mathrm{op}} \to \mathbf{Set}$, we thus have a set $A_n := A([n])$ of n-cells for all $n \geq 0$ as well as face maps $d_i := A(d^i) : A_n \to A_{n-1}$ and degeneracy maps $s_i := A(s^i) : A_n \to A_{n+1}$ which are subject to the dual identities to those listed above. An n-cell is then said to be degenerate if it lies in the image of one of the degeneracy maps, and non-degenerate otherwise. The geometric intuition is that a simplicial set is a complex of simplices of various dimensions glued together, analogous to how CW complexes are constructed from gluing cells of various dimensions together: A_n is the set of n-dimensional simplices of A, and the face maps indicate which $(n-1)$-dimensional simplices in A_{n-1} correspond to the boundaries of these. The degenerate elements are then those simplices which are completely flattened by these identifications.

For example, consider the *standard n-simplex* for any $n \geq 0$, which is defined as the representable presheaf $\Delta[n] := \mathrm{Hom}_\Delta(-,[n])$. The 0-cells of $\Delta[n]$ are the maps $[0] \to [n]$ and thus correspond to the elements of $[n] = \{0,\ldots,n\}$. More generally, the nondegenerate k-cells of $\Delta[n]$ are the monotone injections $[k] \hookrightarrow [n]$ and thus correspond to size-$(k+1)$ subsets of $[n]$. Geometrically, these are the k-dimensional simplices spanned by the chosen $k+1$ vertices; in particular, there is a unique nondegenerate n-cell,



(a) Standard 2-simplex **(b)** Boundary of $\Delta[2]$

(c) 0-horn of $\Delta[2]$ **(d)** 1-horn of $\Delta[2]$ **(e)** 2-horn of $\Delta[2]$

Figure 3.1: Visualisations of $\Delta[2]$, $\partial\Delta[2]$, and $\Lambda^i[2]$

and all k-cells are degenerate for $k > n$. Thus, as the name suggests, $\Delta[n]$ represents an abstract n-dimensional simplex. Supporting our intuition for a general simplicial set A, the Yoneda lemma ensures that the elements of A_n naturally correspond to simplicial maps $\Delta[n] \to A$, meaning that A_n indeed can be thought of as the set of n-dimensional simplices in A.

From the standard n-simplex, we can construct its boundary $\partial\Delta[n] \subset \Delta[n]$ as the simplicial set obtained by discarding the unique nondegenerate n-cell given by the identity $[n] \to [n]$ in $\Delta[n]_n$. For $0 \leq i \leq n$, we can further define the i-horn $\Lambda^i[n]$ of $\Delta[n]$ by taking $\partial\Delta[n]$ and discarding the unique nondegenerate $(n-1)$-cell excluding vertex i; that is, by discarding the simplicial map represented by the coface $d^i : \Delta[n-1] \to \Delta[n]$. Visualisations of these three families of simplicial sets in dimension two are given in Figure 3.1.

To be more precise about how simplicial sets encode the combinatorial data of the homotopy types of topological spaces, first note that the standard n-simplices have natural incarnations as topological spaces: for instance, we can realise them as subspaces of Euclidean space via

$$|\Delta[n]| := \left\{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1}_{\geq 0} \middle| x_0 + \cdots + x_n = 1\right\} \subset \mathbb{R}^{n+1}$$

For a general simplicial set A, we can construct a topological space $|A|$ inductively as follows. Take for every n-cell $a \in A_n$ a disjoint copy of $|\Delta[n]|$. If $n > 0$, then glue the faces of this space to to the topological $(n-1)$-dimensional simplices already constructed corresponding to the faces $d_0 a, \ldots, d_n a$. This construction readily extends to simplicial maps and produces the geometric realisation functor $|-| : \mathbf{sSet} \to \mathbf{Top}$ which can be seen to map simplicial sets to CW complexes. Conversely, to extract combinatorial information from a general topological space, we proceed through the path of least resistance by taking for any topological space X its singular simplicial complex or *nerve* $\mathrm{N}X$, whose n-cells are just the continuous maps $|\Delta[n]| \to X$. This defines an adjunction $|-| \dashv \mathrm{N}$, which is described more formally and in greater generality in Section 3.3.

We wish for **sSet** to preserve the homotopy theory of $\mathbf{Top}_{\text{Quillen}}$ through the above adjunction. This leads to the Quillen model structure on **sSet**, which is developed in detail in [22, Chapter 3]. The weak

equivalences in $\mathbf{sSet}_{\text{Quillen}}$ are those simplicial maps whose geometric realisation is a (weak) homotopy equivalence in **Top**. The fibrations are a combinatorial analogue of those maps satisfying the homotopy lifting property called *Kan fibrations*, which are those simplicial maps which satisfy the right lifting property against the horn inclusions $\Lambda^i[n] \hookrightarrow \Delta[n]$ for all $0 \leq i \leq n$. As before, this uniquely characterises the cofibrations: the acyclic cofibrations are then generated by the horn inclusions, being precisely the class $\text{cof}\{\Lambda^i[n] \hookrightarrow \Delta[n] \mid 0 \leq i \leq n\}$, and the cofibrations in general turn out to be the monomorphisms in **sSet**. In particular, all objects in $\mathbf{sSet}_{\text{Quillen}}$ are cofibrant, and the fibrant objects are called *Kan complexes*. That the previously established adjunction preserves the homotopy theories of these model categories is then expressed more formally by saying that $|-| \dashv \mathrm{N}$ is a *Quillen equivalence*, which is proven for instance in [22, Theorems 2.4.23 and Theorem 3.6.7].

Definition 3.3. Adjoint functors $L : \mathscr{C} \rightleftarrows \mathscr{D} : R$ between model categories are called left and right Quillen functors if they satisfy any of the equivalent properties:

- L preserves cofibrations and acyclic cofibrations;
- R preserves fibrations and acyclic fibrations;
- L preserves cofibrations and R preserves fibrations.

These properties are seen to be equivalent by translating lifting problems through the natural isomorphism $\text{Hom}_{\mathscr{D}}(L(-),-) \cong \text{Hom}_{\mathscr{C}}(-,R(-))$.

In this situation, $L \dashv R$ establishes a Quillen equivalence if the *derived units* $x \to RLx \to R(\mathsf{R}Lx)$ for all cofibrant $x \in \mathscr{C}_0$ are weak equivalences in $\mathscr{C}$, and the *derived counits* $L(\mathsf{Q}Ry) \to LRy \to y$ for all fibrant $y \in \mathscr{D}_0$ are weak equivalences in $\mathscr{D}$. This definition is independent of the choice of fibrant and cofibrant resolutions by Ken Brown's Lemma (cf. [22, Lemma 1.1.12]).

Quillen adjunctions form the appropriate homomorphisms of model categories, and correspond to adjunctions of the $(\infty,1)$-categories they present, which is shown concretely in [35, Theorem 2.1]. This means that Quillen equivalences present more appropriate equivalences of $(\infty,1)$-categories, so in particular the Quillen equivalence $|-| : \mathbf{sSet}_{\text{Quillen}} \rightleftarrows \mathbf{Top}_{\text{Quillen}} : \mathrm{N}$ implies that ∞-groupoids can be equivalently described by the more combinatorial Kan complexes.

On top of being a combinatorial model of ∞-groupoids, $\mathbf{sSet}_{\text{Quillen}}$ also celebrates nicer general structure compared to its topological counterpart, which makes it more convenient to work in. Most notably, by being a category of presheaves, **sSet** is cartesian closed, and its internal hom $\text{Map}(A,B)$ is defined by $\text{Map}(A,B)_n = \text{Hom}_{\mathbf{sSet}}(A \times \Delta[n], B)$. In fact, the internal hom—also called a *mapping space*—is compatible with the model structure, in the sense that it makes $\mathbf{sSet}_{\text{Quillen}}$ into a *cartesian model category*, as proven in [22, Proposition 4.2.8].

Definition 3.4. A *monoidal model category* is a closed symmetric monoidal category $(\mathscr{C}, \otimes, \mathbb{1})$ where $\mathscr{C}$ is equipped also with a model structure that is subject to the following:

- Given cofibrations $i : w \rightarrowtail x$ and $j : y \rightarrowtail z$, the *pushout product* $i \,\widehat{\otimes}\, j$ defined as the uniquely existing morphism in

$$\begin{tikzcd}
w \otimes y \arrow[r, "i\otimes y"] \arrow[d, "w\otimes j"'] \arrow[dr, phantom, "\ulcorner"] & x \otimes y \arrow[d] \arrow[ddr, bend left, "x\otimes j"] & \\
w \otimes z \arrow[r] \arrow[drr, bend right, "i\otimes z"'] & (x\otimes y) \amalg_{w\otimes y} (w \otimes z) \arrow[dr, dashed, "i\widehat{\otimes} j", "\exists!"'] & \\
& & x \otimes z
\end{tikzcd}$$

  is also a cofibration, and is moreover acyclic if either of i or j is.

- The induced map $(\mathsf{Q}\mathbb{1}) \otimes x \to \mathbb{1} \otimes x$ is a weak equivalence for every cofibrant $x \in \mathscr{C}_0$.

If the monoidal structure on $\mathscr{C}$ is cartesian, then $\mathscr{C}$ is called a cartesian model category.

If $(\mathscr{C}, \otimes, \mathbb{1})$ is a monoidal model category, then the pushout product axiom implies in particular that whenever $x \in \mathscr{C}_0$ is cofibrant, then $(-) \otimes x \dashv [x, -]$ defines a Quillen adjunction. Indeed, the pushout product of any cofibration $j : y \rightarrowtail z$ with the cofibration $\emptyset \rightarrowtail x$ yields the morphism $j \otimes x$ and thus shows it to be a cofibration which is even acyclic once j is. Therefore, $(-) \otimes x$ preserves cofibrations and acyclic cofibrations and thus is a left Quillen functor. As a result, the internal hom $[x, -]$ is a right Quillen functor and hence in particular preserves fibrant objects as well as weak equivalences between fibrant objects by Ken Brown's Lemma. Specialising to the cartesian model category $\mathbf{sSet}_{\text{Quillen}}$, this means that $\mathrm{Map}(A, B)$ is a Kan complex once B is, providing a convenient way of seeing $\infty\mathbf{Grpd}$ as weakly enriched in itself.

Unfortunately, while using Kan complexes and derivative models of higher categories through simplicial sets will certainly present the higher categories properly, they are inherently limited with regards to higher functors. More specifically, the functors of these simplicial models of higher categories will always be strictly unital, which will be illustrated in the next section. The reason that the Homotopy Hypothesis cannot remedy this is because of the following. For a topological space X, the components of its fundamental ∞-groupoid $\Pi_\infty X$ are straightforward to define: the objects are points in X, the morphisms are paths in X, the 2-morphisms are endpoint-preserving homotopies in X, the 3-morphisms are endpoint-path-preserving higher homotopies in X, and so on. The delicacy is in formalising the canonical higher categorical structure that these components carry; in particular, there is no unique choice for how to compose higher homotopies in X, and accordingly the associators and unitors at each level are not unique either. However, $\Pi_\infty X$ does carry canonical units: given any k-morphism ϕ in $\Pi_\infty X$, we can take the identity $(k+1)$-morphism id_ϕ to be the constant higher homotopy on ϕ. Continuous maps $f : X \to Y$ induce higher functors $\Pi_\infty X \to \Pi_\infty Y$ in the obvious way, by sending a k-morphism ϕ in $\Pi_\infty X$ to the post-composite $f\phi$ in $\Pi_\infty Y$, and this will always preserve the canonical units on the nose. Therefore, even the Homotopy Hypothesis itself can only see strictly unital higher functors.

3.3 Nerve and Realisation

To see explicitly why simplicial maps between Kan complexes yield strictly unital higher functors, we need to better understand how the combinatorial structure of Kan complexes represents higher categories. We will draw intuition from the 1-truncated situation and look at the *Grothendieck nerve* of 1-categories. Both the Grothendieck nerve and the singular simplicial complex are special cases of a general construction of nerves, which is originally attributed to Kan's work in [26, §3].

Let $\mathscr{S}$ be a small category such as the category Δ of simplices, where the objects are thought of as abstract shapes and the morphisms as structure maps between these shapes. The category $\mathbf{PSh}\,\mathscr{S}$ of presheaves is then intuitively the category of complexes of the shapes in $\mathscr{S}$. To realise these $\mathscr{S}$-shaped complexes concretely as objects in some cocomplete category $\mathscr{C}$, Kan shows that it is enough to first realise the elementary shapes—that is, the objects of $\mathscr{S}$—and build the complexes from these. More precisely, the abstract shapes of $\mathscr{S}$ appear as elementary complexes in $\mathbf{PSh}\,\mathscr{S}$ via the Yoneda embedding, and we can construct a realisation functor $\mathbf{PSh}\,\mathscr{S} \to \mathscr{C}$ by first defining how the elementary shapes behave via a covariant functor $S : \mathscr{S} \to \mathscr{C}$. This is analogous to defining a linear transformation between vector spaces by linearly extending from a choice of actions on basis elements; indeed, this is because the Yoneda embedding realises $\mathbf{PSh}\,\mathscr{S}$ as the *free cocompletion* of $\mathscr{S}$, and thus we can extend $S : \mathscr{S} \to \mathscr{C}$ cocontinuously to yield the *realisation functor* $|-| : \mathbf{PSh}\,\mathscr{S} \to \mathscr{C}$. Explicitly, the realisation is defined as the left Kan extension of S along the Yoneda embedding, making it a coend: for $A : \mathscr{S}^{\mathrm{op}} \to \mathbf{Set}$, we have

$$|A| := \left(\operatorname*{Lan}_{\mathscr{S} \hookrightarrow \mathbf{PSh}\,\mathscr{S}}\right)(A) = \int^{s \in \mathscr{S}_0} \coprod_{A(s)} S(s)$$

By definition, realisation is left adjoint to the *nerve functor* $\mathrm{N} : \mathscr{C} \to \mathbf{PSh}\,\mathscr{S}$ that sends an object $C \in \mathscr{C}_0$ to the presheaf $\mathrm{N}C := \mathrm{Hom}_{\mathscr{C}}(S(-), C) : \mathscr{S}^{\mathrm{op}} \to \mathbf{Set}$. Indeed, this ensures that the realisation functor does extend S: using the adjunction and the Yoneda Lemma, we have

$$\mathrm{Hom}_{\mathscr{C}}(|\mathrm{Hom}_{\mathscr{S}}(-,s)|, C) \cong \mathrm{Nat}(\mathrm{Hom}_{\mathscr{S}}(-,s), \mathrm{N}C) \cong (\mathrm{N}C)_s =: \mathrm{Hom}_{\mathscr{C}}(S(s), \mathscr{C})$$

naturally in $s \in \mathscr{S}_0$ and $C \in \mathscr{C}_0$. By the fully faithfulness of the Yoneda embedding, this implies $|\mathrm{Hom}_{\mathscr{S}}(-,s)| \cong S(s)$ naturally in $s \in \mathscr{S}_0$ as desired.

For example, the geometric realisation functor $\mathbf{sSet} \to \mathbf{Top}$ is realised by taking $S : \Delta \to \mathbf{Top}$ to be $S[n] := |\Delta[n]|$. The coface maps $d^i : [n-1] \to [n]$ are realised in $\mathbf{Top}$ by inserting zeroes in the ith coordinates as

$$|\Delta[n-1]| \to |\Delta[n]|, \quad (x_0, \ldots, x_{n-1}) \mapsto (x_0, \ldots, 0, \ldots, x_{n-1})$$

which appropriately realises $|\Delta[n-1]|$ as the ith face of $|\Delta[n]|$. On the other hand the codegeneracy maps $s^i : [n+1] \to [n]$ are realised by adding the ith and $(i+1)$st coordinates as

$$|\Delta[n+1]| \to |\Delta[n]|, \quad (x_0, \ldots, x_{n+1}) \mapsto (x_0, \ldots, x_i + x_{i+1}, \ldots, x_{n+1})$$

which projects $|\Delta[n+1]|$ onto its ith face. Spelling out the coend for an arbitrary simplicial set recovers

the geometric realisation described in Section 3.2.

Looking at 1-categories, we already have a canonical realisation of standard simplices as categories by definition: we can realise Δ as a full subcategory of **Cat**, and thus we can take $S : \Delta \to \mathbf{Cat}$ to be the corresponding inclusion functor. The induced nerve $\mathrm{N} : \mathbf{Cat} \to \mathbf{sSet}$ by definition sends a (small) category $\mathscr{C}$ to the simplicial set $\mathrm{N}\mathscr{C}$ called the *Grothendieck nerve* of $\mathscr{C}$, whose n-cells are the functors $[n] \to \mathscr{C}$. This is equivalent to taking $(\mathrm{N}\mathscr{C})_n$ to be the set of length-n strings of composable morphisms in $\mathscr{C}$; in particular, $(\mathrm{N}\mathscr{C})_0 = \mathscr{C}_0$ and $(\mathrm{N}\mathscr{C})_1 = \mathscr{C}_1$. From this perspective, we find that the face maps $d_i : (\mathrm{N}\mathscr{C})_n \to (\mathrm{N}\mathscr{C})_{n-1}$ act by composing the ith and $(i-1)$st morphisms when $0 < i < n$, and the degeneracy maps $s_i : (\mathrm{N}\mathscr{C})_n \to (\mathrm{N}\mathscr{C})_{n+1}$ act by inserting an identity morphism at the ith position. Therefore, the Grothendieck nerve completely preserves the structure of its underlying category.

We can also characterise the essential image of the Grothendieck nerve: a simplicial set is isomorphic to the nerve of a small category if and only if it admits a unique right lift against any *inner* horn inclusion, which is an inclusion of simplicial sets of the form $\Lambda^i[n] \hookrightarrow \Delta[n]$ with $0 < i < n$. This is proven rigorously in [32, Proposition 1.1.2.2], but intuitively this is because having unique lifts against inner horn inclusions means that we can use the inner face maps to produce a well-defined composition rule for the 1-cells, and the identity 1-cells for this composition rule will come from the degeneracy maps. For instance, the unique lifts against the inclusion $\Lambda^1[2] \hookrightarrow \Delta[2]$ correspond to finding unique dashed arrows in the diagram

$$\begin{tikzcd} & x_1 \arrow[dr, "f_{1,2}"] & \\ x_0 \arrow[ur, "f_{0,1}"] \arrow[rr, dashed] & & x_2 \end{tikzcd}$$

of 1-cells. Viewing this as a commutative diagram means that the dashed arrow corresponds to the composite $f_{1,2} \circ f_{0,1}$, and this is indeed the case when the simplicial set is given by the Grothendieck nerve of a category.

Note however that nerves of categories are not necessarily Kan complexes. In fact, the nerve of a category is a Kan complex if and only if the category is a groupoid. Indeed, suppose $\mathrm{N}\mathscr{C}$ is a Kan complex and let $f : x \to y$ be any morphism in $\mathscr{C}$. Then, lifts of outer horn inclusions would imply in particular that we can find morphisms which fit as the dashed arrows in the commutative diagrams

$$\begin{tikzcd} & y \arrow[dr, dashed] & \\ x \arrow[ur, "f"] \arrow[rr, "\mathrm{id}_x"'] & & x \end{tikzcd} \qquad \begin{tikzcd} & x \arrow[dr, "f"] & \\ y \arrow[ur, dashed] \arrow[rr, "\mathrm{id}_y"'] & & y \end{tikzcd}$$

coming from horn inclusions $\Lambda^0[2] \hookrightarrow \Delta[2]$ and $\Lambda^2[2] \hookrightarrow \Delta[2]$, respectively. These produce retractions and sections of f, respectively, and thus f will be an isomorphism. Moreover, f being an isomorphism makes all retractions and sections equal to f^{-1}, so this shows that the essential image of the Grothendieck nerve when restricted to groupoids is spanned by those simplicial sets who admit unique right lifts against all horn inclusions.

Relaxing the uniqueness constraint on lifts against horn inclusions in the nerve of a groupoid when

moving to general Kan complexes reflects relaxing the uniqueness for how to compose morphisms, which frees composition to be defined only up to coherent homotopy. As mentioned in the beginning of the chapter, this eliminates the need for higher unitors and associators, albeit at the cost of no longer having a purely algebraic theory of higher categories. This point of view also motivates one of the most popular presentations of $(\infty,1)$-categories in practice called weak Kan complexes or *quasicategories*. Based on the characterisation of nerves of 1-categories, a quasicategory is a simplicial set which satisfies the right lifting property against all inner horn inclusions. This incarnation of $(\infty,1)$-category theory translates much of ordinary category theory into a homotopy-coherent setting, as is explored at length for instance in [32].

However, being simplicial presentations of higher categories, Kan complexes and quasicategories have some drawbacks from a categorical perspective. As already mentioned, the corresponding higher functors given by simplicial maps are necessarily strictly unital by the need to preserve degeneracies, but we also face some inconveniences when trying to define even higher categories as simplicial sets. Namely, while simplicial sets readily present the objects and morphisms of a higher category with its 0-cells and 1-cells respectively, the higher morphisms cannot be extracted without padding with degeneracies due to the non-globular shape of simplices. This also leads to it being less clear how to define the appropriate hom-objects: for example, refer to [32, §1.2.2] for various (homotopy equivalent) ways of constructing the hom-∞-groupoids of a quasicategory.

3.3.1 The Duskin Nerve

To be more explicit about the shortcomings of simplicial presentations of higher categories, we will consider a natural generalisation of the Grothendieck nerve for bicategories. This construction was first studied by Duskin in [14], but a more refined account can be found in [23, §5.4]. Briefly, the Duskin nerve is what would have been the right adjoint to the realisation induced by the cosimplicial bicategory $\Delta \hookrightarrow \mathbf{Cat} \hookrightarrow \mathbf{Bicat}_{\mathrm{ULax}}$. This means that the Duskin nerve of a bicategory $\mathscr{B}$ is the simplicial set $\mathrm{N}\mathscr{B}$ where $(\mathrm{N}\mathscr{B})_n := \mathrm{ULax}([n],\mathscr{B})$ is the set of strictly unital lax functors $[n] \to \mathscr{B}$. Note, however, that we do not actually obtain a corresponding realisation functor, as $\mathbf{Bicat}_{\mathrm{ULax}}$ is not cocomplete.

As **Cat** is a full subcategory of $\mathbf{Bicat}_{\mathrm{ULax}}$, we recover the Grothendieck nerve when restricting the Duskin nerve to 1-categories. The reason for already restricting the 2-functors to be strictly unital is to ensure that the n-cells have the expected diagrammatic shape in the bicategory. More specifically, we want $(\mathrm{N}\mathscr{B})_0$ to correspond to the set of objects of $\mathscr{B}$, which is only possible via 2-functors $[0] \to \mathscr{B}$ if we restrict to those that are strictly unital; otherwise, we would have the set of monads over $\mathscr{B}$. With the Duskin nerve, $(\mathrm{N}\mathscr{B})_0$ and $(\mathrm{N}\mathscr{B})_1$ are the sets of objects and morphisms in $\mathscr{B}$, respectively, as before. One dimension higher, we find that $(\mathrm{N}\mathscr{B})_2$ consists of tuples $(f,g,h;\theta)$ where f,g,h are morphisms and $\theta : gf \Rightarrow h$ is a 2-morphism in $\mathscr{B}$, which fits into a diagram that is analogous to the standard 2-simplex of Figure 3.1:

$$\begin{tikzcd} & y \arrow[dr, "g"] \arrow[d, phantom, "\Downarrow\theta"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & {} & z \end{tikzcd}$$

Note that this means $(\mathrm{N}\mathscr{B})_2$ cannot encode the 2-morphisms as easily as the 1-morphisms and objects were encoded. More generally, n-cells of the Duskin nerve correspond to n-simplex-shaped pasting diagrams in $\mathscr{B}$; that is, the elements of $(\mathrm{N}\mathscr{B})_n$ consist of objects x_i for $0 \leq i \leq n$, morphisms $f_{i,j} : x_i \to x_j$ for $0 \leq i < j \leq n$, and 2-morphisms $\theta_{i,j,k} : f_{j,k} f_{i,j} \Rightarrow f_{i,k}$ for $0 \leq i < j < k \leq n$ (from the lax functoriality constraints) subject to the identity

$$\begin{tikzcd}
 & x_j \arrow[r, "f_{j,k}"] \arrow[dr, "f_{j,l}"] \arrow[d, phantom, "\Swarrow \theta_{j,k,l}"] & x_k \arrow[d, "f_{k,l}"] \\
x_i \arrow[ur, "f_{i,j}"] \arrow[rr, "f_{i,l}"'] \arrow[urr, phantom, "\Searrow \theta_{i,j,l}"] & & x_l
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
x_j \arrow[r, "f_{j,k}"] \arrow[dr, phantom, "\Searrow \theta_{i,j,k}"] & x_k \arrow[dr, "f_{k,l}"] \arrow[dl, phantom, "\Swarrow \theta_{i,k,l}"] & \\
x_i \arrow[u, "f_{i,j}"] \arrow[ur, "f_{i,k}"] \arrow[rr, "f_{i,l}"'] & & x_l
\end{tikzcd}$$

for all $0 \leq i < j < k < l \leq n$. Note that pasting diagrams make it clear how the above identity relates to the shape of a 3-simplex, but suppress the associator $\alpha(f_{k,l}, f_{j,k}, f_{i,j})$ that is necessary to make the perimeters compatible; the explicit commutative diagram of 2-morphisms is given in [23, Equation (5.4.13)]. The horizontal composition rules are now given in the higher cells implicitly, as the face maps $d_\ell : (\mathrm{N}\mathscr{B})_n \to (\mathrm{N}\mathscr{B})_{n-1}$ send such a family $(x_i, f_{i,j}, \theta_{i,j,k})$ to the subfamily where none of the indices i, j, k are equal to ℓ. This allows for composition to be defined only up to coherent 2-isomorphism. On the other hand, the degeneracy maps $s_\ell : (\mathrm{N}\mathscr{B})_n \to (\mathrm{N}\mathscr{B})_{n+1}$ sends a family $(x_i, f_{i,j}, \theta_{i,j,k})$ to the family which includes another copy of x_ℓ, inserts the identity morphism $\mathrm{id}_{x_\ell} : x_\ell \to x_\ell$, and the unitors $\lambda(f_{i,\ell})$ for $i < \ell$ and $\rho(f_{\ell,j})$ for $j > l$.

Duskin described the essential image of this nerve construction in [14, Theorem 8.6], and moreover characterised the nerves of $(2,1)$-categories (that is, 2-truncated $(\infty,1)$-categories) and 2-groupoids. He proved that a simplicial set is isomorphic to the nerve of a $(2,1)$-category if and only if it is a quasi-category whose lifts against inner horn inclusions of dimension at least two are unique, and likewise is isomorphic to the nerve of a 2-groupoid iff it is a Kan complex whose lifts against any horn inclusion of dimension at least two are unique. Therefore, the Duskin nerve is the correct simplicial nerve for bicategories, and we may use it to reveal the insufficiency of simplicial sets for presenting higher functors. More specifically, we have the following well-known result:

Proposition 3.5. *The Duskin nerve is a fully faithful functor* $\mathrm{N} : \mathbf{Bicat}_{\mathrm{ULax}} \to \mathbf{sSet}$.

Proof. To see that it is faithful, suppose $F, G : \mathscr{B} \to \mathscr{B}'$ are normal lax functors such that $\mathrm{N}F = \mathrm{N}G$. As $(\mathrm{N}\mathscr{B})_0$ and $(\mathrm{N}\mathscr{B})_1$ are precisely the sets of objects and morphisms of $\mathscr{B}$, respectively, and analogously for $\mathscr{B}'$, we have that F and G must necessarily act on objects and morphisms in the same way. As for 2-morphisms, note that $(\mathrm{N}F)_2$ maps

$$\begin{tikzcd}
 & y \arrow[dr, "g"] \arrow[d, phantom, "\Downarrow \theta"] & \\
x \arrow[ur, "f"] \arrow[rr, "h"'] & & z
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
 & Fy \arrow[dr, "Fg"] \arrow[d, phantom, "\Downarrow F^2_{g,f}"] & \\
Fx \arrow[ur, "Ff"] \arrow[rr, "F(gf)"] \arrow[rr, bend right, "Fh"'] \arrow[rr, phantom, bend right=40, "\Downarrow F\theta"] & & Fz
\end{tikzcd}$$

and analogously for $(\mathrm{N}G)_2$. By considering the case where $h = gf$ and $\theta = \mathrm{id}_{gf}$, this shows that F and G must have the same lax functoriality constraint, as they must strictly preserve identity 2-morphisms from being local functors. Finally, since F and G are strictly unital, F^0 and G^0 are identities, and thus $F^2_{\mathrm{id}_y,f}$ and $G^2_{\mathrm{id}_y,f}$ must be invertible for all $f : x \to y$ by the compatibility constraints of F with the left unitor described in Definition 2.8, and so considering the case where $g = \mathrm{id}_y$, the invertibility allows us to conclude that F and G must have also agreed on 2-morphisms, showing that $F = G$.

To see that the nerve functor is full, suppose we have a simplicial map $\Phi : \mathrm{N}\mathscr{B} \to \mathrm{N}\mathscr{B}'$. For this to be the nerve of a strictly unital lax functor $F : \mathscr{B} \to \mathscr{B}'$, the action of Φ and F on objects and morphisms must agree exactly. Moreover, we can extract the lax functoriality constraint as before by considering the image of id_{gf} under Φ and taking the interior 2-simplex. By considering thew case where $g = \mathrm{id}_y$, $h = f$, and $\theta = \lambda_{x,y}(f)$, its image under Φ reveals how F must act on the unitor. With this, we can extract how F acts on an arbitrary 2-morphism $\gamma : f \Rightarrow h$ by considering the case where $g = \mathrm{id}_y$ and $\theta = \gamma \otimes \lambda_{x,y}(f)$, taking its image under Φ, and composing with $(F\lambda_{x,y}(f))^{-1}$. That this construction produces a strictly unital lax functor is elementary but technical, and is spelled out in [19, Proof of 3.17], where he takes advantage of a natural stratification on the nerves of bicategories.

It remains to check that $\mathrm{N}F = \Phi$. Indeed, it already agrees on objects and morphisms by definition. To see that its action is consistent on the 2-cells, note that Φ sends the 3-cell

$$\begin{tikzcd}
 & y \arrow[r, "g"] \arrow[dr, "g"] \arrow[r, phantom, shift right=4ex, "\lambda_{y,z}(g)"] & z \arrow[d, "\mathrm{id}_z"] \\
x \arrow[ur, "f"] \arrow[rr, "h"'] \arrow[ur, phantom, shift right=4ex, "\theta"] & & z
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
y \arrow[r, "g"] \arrow[dr, phantom, shift right=4ex, "\mathrm{id}_{gf}"] & z \arrow[dr, "\mathrm{id}_z"] \arrow[dl, phantom, "\theta\otimes\lambda"] & \\
x \arrow[u, "f"] \arrow[ur, "gf"] \arrow[rr, "h"'] & & z
\end{tikzcd}$$

by the definition of F to

$$\begin{tikzcd}
 & Fy \arrow[r, "Fg"] \arrow[dr, "Fg"] \arrow[r, phantom, shift right=4ex, "F(\lambda_{y,z}(g))"] & Fz \arrow[d, "\mathrm{id}_{Fz}"] \\
Fx \arrow[ur, "Ff"] \arrow[rr, "Fh"'] \arrow[ur, phantom, shift right=4ex, "\Phi\theta"] & & Fz
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
Fy \arrow[r, "Fg"] \arrow[dr, phantom, shift right=4ex, "F^2_{g,f}"] & Fz \arrow[dr, "\mathrm{id}_{Fz}"] \arrow[dl, phantom, "F\theta\otimes\lambda"] & \\
Fx \arrow[u, "Ff"] \arrow[ur, "F(gf)"] \arrow[rr, "Fh"'] & & Fz
\end{tikzcd}$$

for any $\theta : gf \Rightarrow h$. As F is a lax functor, the unitors cancel by the corresponding compatibility constraint on F, and we are left with $\Phi\theta$ on the left and the definition of $(\mathrm{N}F)(\theta)$ on the right, as desired. The actions on higher simplices coincide by definition, so this shows $\mathrm{N}F = \Phi$. $\square$

On top of establishing that simplicial maps are precisely the strictly unital maps, we can see from the above proof also how awkward it is to translate between how the lax functors act on 2-morphisms and how the induced simplicial maps act on 2-cells. The goal in the remaining chapters will be to work around both of these obstacles.

Chapter 4

Towards an Unbiased Presentation

We have seen in Section 3.3 how relaxing the definition of composition of 1-morphisms up to higher coherence significantly simplifies the axiomatisation of various higher categories; however, we have also seen how simplicial sets complicate the study of higher morphisms, which is apparent for instance in the proof of Proposition 3.5. On the other hand, the original approach to (strict) higher categories via iterated enrichment allowed higher morphisms and their composition to be handled by induction while the challenge was addressing 1-morphisms. This inspires trying to combine these two approaches so that we may have both well-defined weak composition, as well as readily available higher hom-categories.

Suppose $(\mathscr{V}, \otimes, \mathbb{1})$ is a monoidal category and $\mathscr{C}$ is a (strictly) $\mathscr{V}$-enriched category, then an enriched analogue of the Grothendieck nerve for $\mathscr{C}$ should be a *simplicial object* in $\mathscr{V}$; that is, it should be a functor $\mathrm{N}\mathscr{C} : \Delta^{\mathrm{op}} \to \mathscr{V}$. Recall that the n-cells of the nerve of an ordinary category are given by length-n strings of composable morphisms, so assuming $\mathscr{V}$ has finite coproducts, a natural start point would be to take

$$(\mathrm{N}\mathscr{C})_n := \coprod_{x_0,\ldots,x_n \in \mathscr{C}_0} \left(\bigotimes_{i=1}^{n} \mathscr{C}(x_{i-1}, x_i) \right)$$

In particular, this would mean $(\mathrm{N}\mathscr{C})_1 = \coprod_{x,y \in \mathscr{C}_0} \mathscr{C}(x,y)$ and $(\mathrm{N}\mathscr{C})_0 = \coprod_{\mathscr{C}_0} \mathbb{1}$. The degeneracy maps would correspond to inserting identity morphisms, and the inner face maps $d_i : (\mathrm{N}\mathscr{C})_n \to (\mathrm{N}\mathscr{C})_{n-1}$ for $0 < i < n$ can still be induced by the composition rules.

Unfortunately, such a construction cannot be completed to form a simplicial object in general, with the main obstacle being to define the remaining face maps $d_0, d_n : (\mathrm{N}\mathscr{C})_n \to (\mathrm{N}\mathscr{C})_{n-1}$. In the original Grothendieck nerve, these remove the first and last morphisms in the length-n string, and thus would require the tensor product to have projections. Moreover, when $n = 1$, these face maps must define domain and codomain morphisms $\coprod_{x,y \in \mathscr{C}_0} \mathscr{C}(x,y) \rightrightarrows \coprod_{\mathscr{C}_0} \mathbb{1}$, which would require canonical maps into the tensor unit. Effectively, this forces $(\mathscr{V}, \otimes, \mathbb{1})$ to be cartesian monoidal. This direction is studied extensively in Pellisier's PhD Thesis [37], where he focuses the case where $\mathscr{V}$ is a "catégorie discrétisante," which is a cartesian monoidal category with a reasonable notion of set-like objects. However, this suggests that the nerve construction and general enrichment are likely incompatible. The issue is that the nerve of a general (small) $\mathscr{V}$-enriched category $\mathscr{C}$ cannot be defined because $\mathscr{C}_0$ is a set rather than an object of

$\mathscr{V}$. Therefore, to properly iteratively utilise nerves to define higher categories, we would have to have a way of defining $\mathscr{C}$ solely using the internal language of $\mathscr{V}$.

4.1 Internal Categories

The process of translating set-theoretic structures into more categorical language so that they make sense in more general categories is called *internalisation*, and capitalises on how categories are meant to serve as a general framework for doing set theory. One example that we have already seen is that of monoid objects in Section 2.2.3, which internalise monoids into any monoidal category. A more ubiquitous example is with groups, whose internalisation into a cartesian monoidal category is given by monoid objects further equipped with an inversion endomorphism subject to the axioms of group theory. Group objects produce several familiar examples of structured groups: topological groups in **Top**, algebraic groups in the category of algebraic varieties, Lie groups in the category of smooth manifolds, and Hopf algebras in the opposite of the category of commutative rings.

We can likewise internalise categories themselves into any category with finite limits, and this can in fact be used as an alternative means to produce (strict) higher categories. While less prevalent, internal categories can serve as a means of categorifying set-theoretic structures to allow for more nuanced study of the structure's theory. For example, categories internal to the category of vector spaces (over a fixed field) yield 2-*vector spaces* in the sense of [2], wherein the objects, being vectors, formalise directions in space as is normally the case for vectors, but the morphisms provide a means of formalising *infinitesimal* directions as well. This is useful for instance in higher Lie theory.

Definition 4.1. Let $\mathscr{E}$ be a finitely complete category, then a category $\mathscr{C}$ *internal to* $\mathscr{E}$ consists of a pair of objects $\mathscr{C}_0, \mathscr{C}_1 \in \mathscr{E}_0$ equipped with source and target morphisms $s, t : \mathscr{C}_1 \rightrightarrows \mathscr{C}_0$, an identity-assigning morphism $e : \mathscr{C}_0 \to \mathscr{C}_1$, and a composition morphism $c : \mathscr{C}_1 \times_{\mathscr{C}_0} \mathscr{C}_1 \to \mathscr{C}_1$ mapping out of the pullback in the diagram

$$\begin{tikzcd} \mathscr{C}_1 \times_{\mathscr{C}_0} \mathscr{C}_1 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathscr{C}_1 \arrow[d, "s"] \\ \mathscr{C}_1 \arrow[r, "t"'] & \mathscr{C}_0 \end{tikzcd}$$

The structure is subject to type compatibility, asserting that e is a section of both s and t, and asserting commutativity of the squares

$$\begin{tikzcd} \mathscr{C}_1 \times_{\mathscr{C}_0} \mathscr{C}_1 \arrow[r, "c"] \arrow[d, "\pi_1"'] & \mathscr{C}_1 \arrow[d, "s"] \\ \mathscr{C}_1 \arrow[r, "s"'] & \mathscr{C}_0 \end{tikzcd} \qquad \begin{tikzcd} \mathscr{C}_1 \times_{\mathscr{C}_0} \mathscr{C}_1 \arrow[r, "c"] \arrow[d, "\pi_2"'] & \mathscr{C}_1 \arrow[d, "t"] \\ \mathscr{C}_1 \arrow[r, "t"'] & \mathscr{C}_1 \end{tikzcd}$$

These ensure that the (co)domains of identities and composites are as expected of data for a category. Finally, composition is subject to the axioms of category theory, which is expressed by asserting com-

mutativity of the diagrams

$$\begin{tikzcd}
\mathscr{C}_1 \times_{\mathscr{C}_0} \mathscr{C}_1 \times_{\mathscr{C}_0} \mathscr{C}_1 \arrow[r, "c \times_{\mathscr{C}_0} \mathscr{C}_1"] \arrow[d, "\mathscr{C}_1 \times_{\mathscr{C}_0} c"'] & \mathscr{C}_1 \times_{\mathscr{C}_0} \mathscr{C}_1 \arrow[d, "c"] \\
\mathscr{C}_1 \times_{\mathscr{C}_0} \mathscr{C}_1 \arrow[r, "c"'] & \mathscr{C}_1
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{C}_0 \times_{\mathscr{C}_0} \mathscr{C}_1 \arrow[r, "e \times_{\mathscr{C}_0} \mathscr{C}_1"] \arrow[dr, "\pi_2"'] & \mathscr{C}_1 \times_{\mathscr{C}_0} \mathscr{C}_1 \arrow[d, "c"] & \mathscr{C}_1 \times_{\mathscr{C}_0} \mathscr{C}_0 \arrow[l, "\mathscr{C}_1 \times_{\mathscr{C}_0} e"'] \arrow[dl, "\pi_1"] \\
& \mathscr{C}_1 &
\end{tikzcd}$$

asserting associativity and unity, respectively.

Given two categories $\mathscr{C}$ and $\mathscr{D}$ internal to $\mathscr{E}$, an *internal functor* $F : \mathscr{C} \to \mathscr{D}$ consists of morphisms $F_0 : \mathscr{C}_0 \to \mathscr{D}_0$ and $F_1 : \mathscr{C}_1 \to \mathscr{D}_1$ in $\mathscr{E}$ such that they commute with the structure morphisms s, t, e, c of $\mathscr{C}$ and $\mathscr{D}$. As functors defined this way are closed under composition, and $\mathrm{id}_{\mathscr{C}} = (\mathrm{id}_{\mathscr{C}_0}, \mathrm{id}_{\mathscr{C}_1})$ provides an identity functor on any internal category $\mathscr{C}$, we have a category $\mathbf{Cat}(\mathscr{E})$ of categories and functors internal to $\mathscr{E}$.

By design, $\mathbf{Cat}(\mathbf{Set}) = \mathbf{Cat}$, meaning categories internal to **Set** are precisely the small categories. For any finitely complete category $\mathscr{E}$, the category $\mathbf{Cat}(\mathscr{E})$ will also be finitely complete. Indeed, given a functor $F : \mathscr{J} \to \mathbf{Cat}(\mathscr{E})$ where $\mathscr{J}$ is finite, then the internal category $\varprojlim F$ can be constructed by taking $(\varprojlim F)_0 = \varprojlim_{j \in \mathscr{J}_0} F(j)_0$ and $(\varprojlim F)_1 = \varprojlim_{j \in \mathscr{J}_0} F(j)_1$, where the structure morphisms are then induced by the universal properties of these limits over the structure morphisms on each of the internal categories $F(j)$. Therefore, we are allowed to iteratively internalise categories as a way of producing strict higher categories. However, unlike enrichment, this construction generally provides too much structural flexibility for the class of objects.

We see this immediately with categories internal to **Cat**: for any $\mathscr{C} \in \mathbf{Cat}(\mathbf{Cat})_0$, the morphisms of the category $\mathscr{C}_1$ of morphisms serve as 2-morphisms for $\mathscr{C}$, and the objects of the category $\mathscr{C}_0$ of objects serve as the objects for $\mathscr{C}$, but now we have two independent sets of morphisms which both act as 1-morphisms for $\mathscr{C}$. By the generic construction of internalisation, we obtain our usual 1-morphisms as the objects of $\mathscr{C}_1$—called *horizontal* morphisms—but since $\mathscr{C}_0$ is now a category, we also have 1-morphisms coming from the morphisms of $\mathscr{C}_0$—then called *vertical* morphisms. Therefore, a generic 2-morphism of $\mathscr{C}$ takes the shape of

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow"] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet
\end{tikzcd}$$

and these can be composed both horizontally and vertically. Due to the orthogonality of the two categorical structures $\mathscr{C}$ ends up carrying, categories internal to **Cat** are instead called *double categories*.

Fortunately, we can recover strict 2-categories from double categories by taking the category of objects to be discrete. More precisely, we have a fully faithful inclusion $2\mathbf{Cat} \hookrightarrow \mathbf{Cat}(\mathbf{Cat})$ which sends a strict 2-category $\mathscr{C}$ to the double category whose category of objects is given by $\mathrm{disc}(\mathscr{C}_0)$ and whose category of morphisms is $\coprod_{x,y \in \mathscr{C}_0} \mathscr{C}(x,y)$. The essential image of this functor consists of those double categories whose vertical morphisms are trivial.

More generally, the category $(n+1)\mathbf{Cat}$ is equivalently the full subcategory of $\mathbf{Cat}(n\mathbf{Cat})$ on those internal categories whose n-category of objects is discrete in the sense that it lies in the image of the

inclusion $\mathbf{Set} \hookrightarrow n\mathbf{Cat}$. Therefore, internalisation provides a reasonable alternative for producing higher categories, and this approach allows more readily for an analogue of the Grothendieck nerve construction.

4.1.1 The Segal Condition

Let $\mathscr{E}$ be a finitely complete category, then unlike in the case for enrichment, we can readily generalise the Grothendieck nerve construction to categories internal to $\mathscr{E}$ to produce simplicial objects in $\mathscr{E}$. Fix an internal category $\mathscr{C} \in \mathbf{Cat}(\mathscr{E})_0$, then just as the n-cells of the Grothendieck nerve of an ordinary category were the length-n composable strings of morphisms, we can define $(\mathrm{N}\mathscr{C})_n \in \mathscr{E}_0$ to be the iterated pullback

$$(\mathrm{N}\mathscr{C})_n := \mathscr{C}_1^{\times_{\mathscr{C}_0} n} = \underbrace{\mathscr{C}_1 \times_{\mathscr{C}_0} \mathscr{C}_1 \times_{\mathscr{C}_0} \cdots \times_{\mathscr{C}_0} \mathscr{C}_1}_{n \text{ times}}$$

which is shorthand for the limit of the diagram

$$\begin{tikzcd}
\mathscr{C}_1 \arrow[dr, "s"'] & & \mathscr{C}_1 \arrow[dl, "t"] \arrow[dr, "s"'] & \cdots & \mathscr{C}_1 \arrow[dl, "t"] \arrow[dr, "s"'] & & \mathscr{C}_1 \arrow[dl, "t"] \\
& \mathscr{C}_0 & & \cdots & & \mathscr{C}_0 &
\end{tikzcd}$$

In particular, $(\mathrm{N}\mathscr{C})_0 = \mathscr{C}_0$ and $(\mathrm{N}\mathscr{C})_1 = \mathscr{C}_1$ as expected. To produce a simplicial object $\mathrm{N}\mathscr{C} : \Delta^{\mathrm{op}} \to \mathscr{E}$, define its face and degeneracy morphisms as follows. The first face $d_0 : (\mathrm{N}\mathscr{C})_n \to (\mathrm{N}\mathscr{C})_{n-1}$ internalises the act of taking the domain of the first morphism and is thus given by the morphism

$$(\mathrm{N}\mathscr{C})_n = \mathscr{C}_1 \times_{\mathscr{C}_0} \mathscr{C}_1^{\times_{\mathscr{C}_0}(n-1)} \xrightarrow{s \times_{\mathscr{C}_0} \mathrm{id}} \mathscr{C}_0 \times_{\mathscr{C}_0} \mathscr{C}_1^{\times_{\mathscr{C}_0}(n-1)} \cong \mathscr{C}_1^{\times_{\mathscr{C}_0}(n-1)} = (\mathrm{N}\mathscr{C})_{n-1}$$

Similarly, the last face d_n takes the codomain of the last internal morphism. The intermediate faces $d_i : (\mathrm{N}\mathscr{C})_n \to (\mathrm{N}\mathscr{C})_{n-1}$ for $0 < i < n$ correspond to composing the ith and $(i+1)$st morphisms, and are thus given by

$$\underbrace{\mathscr{C}_1^{\times_{\mathscr{C}_0}(i-1)} \times_{\mathscr{C}_0} \mathscr{C}_1 \times_{\mathscr{C}_0} \mathscr{C}_1 \times_{\mathscr{C}_0} \mathscr{C}_1^{\times_{\mathscr{C}_0}(n-i-1)}}_{(\mathrm{N}\mathscr{C})_n} \xrightarrow{\mathrm{id} \times_{\mathscr{C}_0} c \times_{\mathscr{C}_0} \mathrm{id}} \underbrace{\mathscr{C}_1^{\times_{\mathscr{C}_0}(i-1)} \times_{\mathscr{C}_0} \mathscr{C}_1 \times_{\mathscr{C}_0} \mathscr{C}_1^{\times_{\mathscr{C}_0}(n-i-1)}}_{(\mathrm{N}\mathscr{C})_{n-1}}$$

The degeneracy maps $s_i : (\mathrm{N}\mathscr{C})_n \to (\mathrm{N}\mathscr{C})_{n+1}$ for $0 \leq i \leq n$ on the other hand correspond to inserting an identity morphism at index i and thus are given by

$$\underbrace{\mathscr{C}_1^{\times_{\mathscr{C}_0} i} \times_{\mathscr{C}_0} \mathscr{C}_1^{\times_{\mathscr{C}_0}(n-i)}}_{(\mathrm{N}\mathscr{C})_n} \cong \mathscr{C}_1^{\times_{\mathscr{C}_0} i} \times_{\mathscr{C}_0} \mathscr{C}_0 \times_{\mathscr{C}_0} \mathscr{C}_1^{\times_{\mathscr{C}_0}(n-i)} \xrightarrow{\mathrm{id} \times_{\mathscr{C}_0} e \times_{\mathscr{C}_0} \mathrm{id}} \underbrace{\mathscr{C}_1^{\times_{\mathscr{C}_0} i} \times_{\mathscr{C}_0} \mathscr{C}_1 \times_{\mathscr{C}_0} \mathscr{C}_1^{\times_{\mathscr{C}_0}(n-i)}}_{(\mathrm{N}\mathscr{C})_{n+1}}$$

That these satisfy the simplicial identities follows precisely from the fact that the composition in $\mathscr{C}$ is associative and unital. Moreover, this general nerve construction canonically extends to a functor $\mathrm{N} : \mathbf{Cat}(\mathscr{E}) \to \mathrm{Func}(\Delta^{\mathrm{op}}, \mathscr{E})$ by taking an internal functor $F : \mathscr{C} \to \mathscr{D}$ to the simplicial morphism $\mathrm{N}F$

whose action on n-cells is induced by the universal property of the limits defining $(\mathrm{N}\mathscr{C})_n$ and $(\mathrm{N}\mathscr{D})_n$, which may be expressed symbolically by saying $(\mathrm{N}F)_n = F_1^{\times_{F_0} n}$.

In Section 3.3, the essential image of the Grothendieck nerve for ordinary categories was characterised in terms of lifting properties against horn inclusions. However, it can also be characterised by the fact that the nerve of an ordinary category provides no additional information after dimension two, which says in particular that nerves of categories are 2-*coskeletal* as simplicial sets. Of course, not all 2-coskeletal simplicial sets are nerves of categories, but the characterisation of the essential image will then only propagate from the fact that the 2-cells must reflect the compositional structure of a category: any composable pair of morphisms must have a unique composite. Being 2-coskeletal then handles the fact that the composition is associative, and the degeneracies ensure that the composition is unital.

Given a (2-coskeletal) simplicial set A, the composable pairs of morphisms are given by the pullback $A_1 \times_{A_0} A_1$ of $d_1, d_0 : A_1 \rightrightarrows A_0$. Note that every 2-cell $\theta \in A_2$ induces a composable pair $(d_2\theta, d_0\theta)$ of 1-cells by the simplicial identities, meaning that d_2 and d_0 induce a unique map $A_2 \to A_1 \times_{A_0} A_1$. The corresponding composite for such a pair is then given by $d_1\theta$, so to say that every composable pair of morphisms in A has a composite means asserting that this canonical map $A_2 \to A_1 \times_{A_0} A_1$ is surjective. To ensure that the choice of composite is unique, we just ensure that the choice of 2-cell for any composable pair which may provide a composite through d_1 is necessarily unique, meaning that $A_2 \to A_1 \times_{A_0} A_1$ must also be injective.

Therefore, a simplicial set A is the nerve of a category if and only if it is 2-coskeletal and the canonical map $A_2 \to A_1 \times_{A_0} A_1$ is bijective. Unwinding what it means to be 2-coskeletal, we obtain the (strict) *Segal condition*. For any $0 \leq i < n$, call the inclusion $\{i, i+1\} \hookrightarrow [n]$ in Δ an *inert morphism*, then a simplicial set X is said to satisfy the Segal condition if the canonical maps $A_n \to A_1^{\times_{A_0} n}$ induced by the inert morphisms into $[n]$ are all isomorphisms. This recovers the well-known characterisation of the essential image of the Grothendieck nerve, and in fact generalises readily to arbitrary internal categories:

Lemma 4.2. *Let $\mathscr{E}$ be a finitely complete category. A simplicial object $A : \Delta^{\mathrm{op}} \to \mathscr{E}$ is isomorphic to the nerve of a category internal to $\mathscr{E}$ if and only if it satisfies the Segal condition.*

Proof. The nerves of categories internal to $\mathscr{E}$ satisfy the Segal condition by definition. Suppose A is a simplicial object such that the inert morphisms into $[n]$ induce an isomorphism $A_n \xrightarrow{\sim} A_1^{\times_{A_0} n}$ for all $n \geq 0$. Consider the internal category $\mathscr{C} \in \mathbf{Cat}(\mathscr{E})_0$ where $\mathscr{C}_0 := A_0$ and $\mathscr{C}_1 := A_1$, the source and target maps are given by $s := d_0 : A_1 \to A_0$ and $t := d_1 : A_0 \to A_1$, respectively, the identity-assigning morphism is taken to be $e := s_0 : A_0 \to A_1$, and the composition map is given by $c : A_1 \times_{A_0} A_1 \cong A_2 \xrightarrow{d_1} A_1$ by the Segal conditions. The simplicial identities ensure that the type compatibility constraints on $\mathscr{C}$ are satisfied.

To see that this composition rule is associative, consider first the diagram

$$\begin{tikzcd}
A_3 \arrow[rr, "d_1"] \arrow[dr, "(d_3,d_0d_0)"] \arrow[dd, "(d_2d_3,d_0d_3,d_0d_0)"'] & & A_2 \arrow[dd, "(d_2,d_0)"] \\
& A_2 \times_{A_0} A_1 \arrow[dl, "(d_2,d_0)\times_{A_0}A_1"'] \arrow[dr, "d_1\times_{A_0}A_1"] & \\
A_1 \times_{A_0} A_1 \times_{A_0} A_1 \arrow[rr, "c\times_{A_0}A_1"'] & & A_1 \times_{A_0} A_1
\end{tikzcd}$$

where the tuples indicate which morphisms induce by universal property the maps into the pullbacks. Commutativity of the lower triangle follows by the definition of c, and the rest by the simplicial identities $d_i d_j = d_{j-1} d_i$ for $i < j$. Now, consider the diagram

$$\begin{tikzcd}
A_3 \arrow[rr, "d_1"] \arrow[dr, "\sim"] \arrow[dd, "d_2"'] & & A_2 \arrow[d, "\wr"] \arrow[ddd, bend left, "d_1"] \\
& A_1 \times_{A_0} A_1 \times_{A_0} A_1 \arrow[r, "c\times_{A_0}A_1"'] \arrow[d, "A_1\times_{A_0}c"'] & A_1 \times_{A_0} A_1 \arrow[d, "c"] \\
A_2 \arrow[r, "\sim"] \arrow[rr, bend right, "d_1"'] & A_1 \times_{A_0} A_1 \arrow[r, "c"] & A_1
\end{tikzcd}$$

We have just shown that the topmost square commutes, and an analogous argument shows the same for the square on the left. That the right and lower triangles commute is precisely the definition of c. Since the indicated morphisms are invertible by the Segal conditions, it follows that the inner square commutes, which establishes associativity of c.

To see that c is left unital, consider the diagram

$$\begin{tikzcd}
A_1 \arrow[r, "s_0"] \arrow[d, equal, "(d_1,\mathrm{id})"'] \arrow[dr, "(s_0d_1,\mathrm{id})"] & A_2 \arrow[d, "\wr"', "(d_2,d_0)"] \arrow[ddd, bend left, "d_1"] \\
A_0 \times_{A_0} A_1 \arrow[r, "e\times_{A_0}A_1"'] \arrow[dr, "\pi_2"'] & A_1 \times_{A_0} A_1 \arrow[d, "c"] \\
& A_1
\end{tikzcd}$$

That the rightmost triangle commutes is the definition of c, and the leftmost triangle commutes because $e = s_0$. The simplicial identities $d_i s_j = s_j d_{i-1}$ for $i > j+1$ and $d_i s_j = \mathrm{id}$ for $j \leq i \leq j+1$ imply commutativity of the topmost triangle as well as the perimeter. Since the indicated morphism is invertible, this implies that the bottom left triangle commutes, which shows that c is left unital. The proof for being right unital is analogous.

Therefore, $\mathscr{C}$ indeed defines a category object in $\mathscr{E}$, and the Segal condition ensures that the induced maps $A_n \to A_1^{\times_{A_0} n} = (\mathrm{N}\mathscr{C})_n$ are isomorphisms, and these collect to define a simplicial isomorphism

$A \cong \mathrm{N}\mathscr{C}$ as desired. $\square$

The Segal condition ensures that all simplicial data in the nerve of a category is uniquely determined by the 0- and 1-cells, and this extends to the simplicial morphisms between nerves: the simplicial identities force any $\Phi : \mathrm{N}\mathscr{C} \to \mathrm{N}\mathscr{D}$ to be completely determined by its action on 0- and 1-cells, meaning more precisely that $\Phi_n = \Phi_1^{\times_{\Phi_0} n}$ is uniquely determined by the universal property of the fibre products over inert morphisms for all $n \geq 0$. As Φ_0, Φ_1 necessarily provide the data for a functor $\mathscr{C} \to \mathscr{D}$ (whose nerve is then necessarily equal to Φ), this shows that $\mathrm{N} : \mathbf{Cat}(\mathscr{E}) \to \mathrm{Func}(\Delta^{\mathrm{op}}, \mathscr{E})$ is fully faithful and thus establishes an equivalence of categories between the categories internal to $\mathscr{E}$ and the simplicial objects of $\mathscr{E}$ that satisfy the Segal condition.

4.2 Tamsamani n-Nerves

If we view strict 2-categories as double categories, then the nerve construction of Section 4.1.1 restricts to a fully faithful functor $\mathrm{N} : 2\mathbf{Cat} \to \mathrm{Func}(\Delta^{\mathrm{op}}, \mathbf{Cat})$, whose essential image consists of those simplicial categories $A : \Delta^{\mathrm{op}} \to \mathbf{Cat}$ with A_0 discrete that satisfy the strict Segal condition. This implies in particular that the functor $A_2 \to A_1 \times_{A_0} A_1$ is an isomorphism in $\mathbf{Cat}$, meaning that it is fully faithful and bijective on objects. In other words, there is a one-to-one correspondence between composable pairs of morphisms f and g, and commutative triangles of the form

$$\begin{tikzcd} \bullet \arrow[rr, dashed] \arrow[dr, "f"'] & & \bullet \\ & \bullet \arrow[ur, "g"'] & \end{tikzcd}$$

which is functorial with respect to 2-morphisms. As discussed in the previous section, this correspondence provides a composition rule for morphisms (with functoriality giving horizontal composition for 2-morphisms), while the simplicial structure of A handles associativity and unity.

The advantage of this point of view is that it suggests a relaxation of the Segal condition to account for bicategories. Isomorphisms are generally too restrictive as a way of comparing categories, with the more reasonable means of comparison being through equivalences of categories. If we relax the Segal condition on a simplicial category A to assert only that the canonical functors $A_n \to A_1^{\times_{A_0} n}$ are equivalences of categories, then A presents a 2-category whose structure is specified only up to unique 2-isomorphism. Indeed, the composition rule on A is induced by weakly inverting $A_2 \to A_1 \times_{A_0} A_1$, implying that a choice of composite for any composable pair of morphisms is unique only up to unique 2-isomorphism. Analogously, associativity of composition requires a choice of weak inverse for the Segal map $A_3 \to A_1 \times_{A_0} A_1 \times_{A_0} A_1$, and this choice reflects choosing a coherent natural associator for the composition in A. Unitors follow similarly, which suggests that simplicial categories with a discrete category of 0-cells that satisfy the Segal condition up to equivalence correspond to the bicategories of Section 2.2.2.

Tamsamani formalises this observation in his PhD thesis [46], motivating an inductive definition of weak n-categories similar to the iterated internalisation for the strict case done in the beginning of

Section 4.1. The construction requires an appropriate notion of weak equivalence at each step of the induction in order to make sense of the Segal condition, so he defines these weak equivalences inductively using a well-behaved 0-truncation functor that sends weak n-categories to their sets of equivalence classes of objects. The idea is that equivalences of $(n+1)$-categories should be essentially surjective on objects and locally equivalences of hom-n-categories. [46, Définition 1.3.2] unwinds the inductive nature of the construction of his higher categories—which he calls *n-nerves*—so we will follow a more streamlined approach that is similar in spirit to that of [21, §2].

Definition 4.3. Define the category n**PNerve** of *n-prenerves* with an inclusion of discrete objects $\mathrm{disc} : n\mathbf{Set} \hookrightarrow n\mathbf{PNerve}$ as follows. Start with $0\mathbf{PNerve} := \mathbf{Set}$ with the identity endofunctor as the inclusion of discrete objects. Inductively, define $(n+1)\mathbf{PNerve}$ to be the full subcategory of $\mathrm{Func}(\Delta^{\mathrm{op}}, n\mathbf{PNerve}$ on those simplicial n-prenerves $\mathcal{A}$ for which $\mathcal{A}_0$ is discrete. Then, take the corresponding inclusion $\mathrm{disc} : \mathbf{Set} \hookrightarrow (n+1)\mathbf{PNerve}$ to be the functor which sends a set S to the constant functor on $\mathrm{disc}\, S \in n\mathbf{PNerve}_0$.

Unpacking the inductive definition as done in the beginning of [21, §2], we find that n**PNerve** is the full subcategory of $\mathbf{PSh}(\Delta^n)$ on those presheaves whose value on $([m_1], \ldots, [m_n])$ does not depend on any m_j for $j < i$ once $m_i = 0$:

Lemma 4.4. *Define Θ^n to be the quotient of Δ^n modulo the equivalence generated by identifying $([m_1], \ldots, [m_n]) \sim ([m'_1], \ldots, [m'_n])$ whenever there exists some $1 \leq i \leq n$ where $m_i = m'_i = 0$ and $m_j = m'_j$ for all $j < i$. Then, we have an isomorphism of categories $n\mathbf{PNerve} \cong \mathbf{PSh}(\Theta^n)$, under which the inclusion* $\mathrm{disc} : \mathbf{Set} \hookrightarrow n\mathbf{PNerve}$ *corresponds to the inclusion* $\mathbf{Set} \hookrightarrow \mathbf{PSh}(\Theta^n)$ *mapping a set S to the constant presheaf on S.*

In particular, n**PNerve** is complete, and its limits are computed levelwise. From here, we define the category of n-nerves to be a full subcategory $n\mathbf{Nerve} \subseteq n\mathbf{PNerve}$ equipped with a class of weak equivalences (containing the isomorphisms) as well as a truncation functor $\tau_{\leq 0} : n\mathbf{Nerve} \to \mathbf{Set}$ such that

(N1) n**Nerve** has fibre products over discrete objects,

(N2) weak equivalences are stable under pullback over discrete objects in the sense that the dashed arrow in

$$\begin{tikzcd}
\mathcal{A}' \times_{\mathrm{disc}\, S} \mathcal{B}' \arrow[r] \arrow[d] \arrow[dr, dashed] & \mathcal{B}' \arrow[dr, "g"] & \\
\mathcal{A}' \arrow[dr, "f"'] & \mathcal{A} \times_{\mathrm{disc}\, S} \mathcal{B} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{B} \arrow[d] \\
& \mathcal{A} \arrow[r] & \mathrm{disc}\, S
\end{tikzcd}$$

is a weak equivalence once f and g are,

(N3) $\tau_{\leq 0}$ preserves fibre products over discrete objects and sends weak equivalences to bijections.

The above desiderata are clearly satisfied when $n = 0$ by taking $0\mathbf{Nerve} := 0\mathbf{PNerve} = \mathbf{Set}$, declaring the weak equivalences to be the bijections, and taking $\tau_{\leq 0}$ to be the identity endofunctor.

Definition 4.5. Assuming $n\mathbf{Nerve}$ is equipped with a class of weak equivalences and a truncation functor satisfying (N1), (N2), (N3), define $(n+1)\mathbf{Nerve}$ to be the full subcategory of $(n+1)\mathbf{PNerve}$ on those $(n+1)$-prenerves $\mathcal{A}$ such that for every $p \geq 0$,

- $\mathcal{A}_p \in n\mathbf{Nerve}_0$,
- (Segal condition) the Segal map $\mathcal{A}_p \to \mathcal{A}_1^{\times_{\mathcal{A}_0} p}$ induced by inert morphisms is a weak equivalence of n-nerves

Proposition 4.6. $(n+1)\mathbf{Nerve}$ *as in Definition 4.5 satisfies (N1).*

Proof. Consider a cospan $\mathcal{A} \to \operatorname{disc} S \leftarrow \mathcal{B}$ of $(n+1)$-nerves. The pullback in $(n+1)\mathbf{PNerve}$ is computed levelwise, so it is enough to show that this defines an $(n+1)$-nerve. By induction, $\mathcal{A}_p \times_{\operatorname{disc} S} \mathcal{B}_p$ will be an n-nerve for all $p \geq 0$, so we are left to check the Segal condition. Since $\mathcal{A}$ and $\mathcal{B}$ are assumed to be n-nerves, they satisfy the Segal condition, so since weak equivalences are stable under pullback over discrete objects, we obtain a weak equivalence $\mathcal{A}_p \times_{\operatorname{disc} S} \mathcal{B}_p \to \mathcal{A}_1^{\times_{\mathcal{A}_0} p} \times_{\operatorname{disc} S} \mathcal{B}_1^{\times_{\mathcal{B}_0} p}$. Finally, since limits commute with limits, the codomain is isomorphic to $(\mathcal{A}_1 \times_{\operatorname{disc} S} \mathcal{B}_1)^{\times_{(\mathcal{A}_0 \times_{\operatorname{disc} S} \mathcal{B}_0)} p}$, showing that $\mathcal{A} \times_{\operatorname{disc} S} \mathcal{B}$ is an $(n+1)$-nerve, as desired. $\square$

Define $\tau_{\leq 1} := (\tau_{\leq 0})_* : \operatorname{Func}(\Delta^{\mathrm{op}}, n\mathbf{Nerve}) \to \operatorname{Func}(\Delta^{\mathrm{op}}, \mathbf{Set}) = \mathbf{sSet}$, then this functor preserves fibre products over discrete objects since these are computed levelwise in both categories. We have that $\tau_{\leq 1}\mathcal{A}$ is (the Grothendieck nerve of) a category for any $(n+1)$-nerve $\mathcal{A}$. Indeed, we have by definition that $(\tau_{\leq 1}\mathcal{A})_p = \tau_{\leq 0}(\mathcal{A}_p)$, so $\tau_{\leq 0}$ will send the weak equivalence $\mathcal{A}_p \to \mathcal{A}_1^{\times_{\mathcal{A}_0} p}$ to the bijection $\tau_{\leq 0}(\mathcal{A}_p) \to \tau_{\leq 0}\left(\mathcal{A}_1^{\times_{\mathcal{A}_0} p}\right) \cong \tau_{\leq 0}(\mathcal{A}_1)^{\times_{\tau_{\leq 0}(\mathcal{A}_0)} p}$ for every $p \geq 0$, putting $\tau_{\leq 1}\mathcal{A}$ in the essential image of the Grothendieck nerve by Lemma 4.2. We can thus compose $\tau_{\leq 1}$ with the functor $\tau_0 : \mathbf{Cat} \to \mathbf{Set}$ which sends small categories to their sets of isomorphism classes of objects:

Definition 4.7. With $(n+1)\mathbf{Nerve}$ as in Definition 4.5, define its truncation functor to be the composite $\tau_{\leq 0} : (n+1)\mathbf{Nerve} \xrightarrow{\tau_{\leq 1}} \mathbf{Cat} \xrightarrow{\tau_0} \mathbf{Set}$.

Remark 4.8. Since τ_0 and $\tau_{\leq 1}$ both preserve fibre products over discrete objects, the same holds for $\tau_{\leq 0}$.

The truncation functor allows us to define an analogue of *essential surjectivity* of a morphism Φ to mean its truncation $\tau_{\leq 0}\Phi$ is a surjection of sets, which establishes half of the definition of a weak equivalence in $(n+1)\mathbf{Nerve}$. Fully faithfulness is defined by Φ being a weak equivalence on hom-n-nerves:

Definition 4.9. For $\mathcal{A} \in (n+1)\mathbf{Nerve}_0$, identify the discrete n-nerve $\mathcal{A}_0$ with its underlying set. The two coface maps $d^0, d^1 : [0] \rightrightarrows [1]$ induce a map $\mathcal{A}_1 \to \mathcal{A}_0 \times \mathcal{A}_0$, from which we define for any pair of

objects $x, y \in \mathcal{A}_0$ the *hom-n-nerve* $\mathcal{A}(x,y)$ to be the fibre

$$\begin{tikzcd} \mathcal{A}(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{A}_1 \arrow[d] \\ \mathrm{pt} \arrow[r, "(x,y)"'] & \mathcal{A}_0 \times \mathcal{A}_0 \end{tikzcd}$$

As $\mathcal{A}_0 \times \mathcal{A}_0$ and pt are discrete, $\mathcal{A}(x,y)$ is indeed an n-nerve. Now, define a morphism $\Phi : \mathcal{A} \to \mathcal{B}$ of $(n+1)$-nerves to be *fully faithful* if the induced maps $\Phi_{x,y} : \mathcal{A}(x,y) \to \mathcal{B}(\Phi x, \Phi y)$ of n-nerves are weak equivalences for every $x, y \in \mathcal{A}_0$. If Φ is fully faithful and $\tau_{\leq 0}\Phi$ is surjective, then say that Φ is a *weak equivalence*.

Lemma 4.10. $(n+1)$**Nerve** *with the weak equivalences and truncation defined above satisfies (N2) and (N3).*

Proof. If $\Phi : \mathcal{A} \to \mathcal{B}$ is a weak equivalence of $(n+1)$-nerves, then $\tau_{\leq 1}\Phi$ will be an equivalence of ordinary categories. Indeed, $(\tau_{\leq 1}\Phi)_{x,y} = \tau_{\leq 0}(\Phi_{x,y})$ is bijective for every $x, y \in \mathcal{A}_0$, and $\tau_0\tau_{\leq 1}\Phi$ is surjective. Therefore, $\tau_{\leq 0}\Phi = \tau_0\tau_{\leq 1}\Phi$ is a bijection, proving (N3).

Note that fully faithfulness of maps in $(n+1)$**Nerve** is preserved by pullbacks over discrete objects by (N2) on n**Nerve**. On the other hand, essential surjectivity are seen to be preserved by pullbacks over discrete objects by taking $\tau_{\leq 0}$ of the pullback diagram. As the weak equivalences are the fully faithful and essentially surjective maps, this proves (N2). □

This completes the inductive construction of Tamsamani n-nerves for all finite $n \geq 0$.

4.2.1 Double Nerves

Tamsamani 0-nerves are precisely the sets by definition, and Lemma 4.2 establishes an equivalence of categories **Cat** $\simeq$ 1**Nerve**. In dimension two, Tamsamani proves in [46, §1.4] that his 2-nerves recover the bicategories of Bénabou discussed in Section 2.2.2 by describing how to construct a 2-nerve from a bicategory and vice versa. Note that since 1-nerves are already equivalent to small categories, this reduces to comparing bicategories with simplicial categories $\mathcal{A} : \Delta^{\mathrm{op}} \to$ **Cat** where $\mathcal{A}_0$ is discrete and the Segal maps $\mathcal{A}_p \to \mathcal{A}_1^{\times_{\mathcal{A}_0} p}$ are equivalences of categories for all $p \geq 0$.

Given a bicategory $\mathscr{B}$, Tamsamani constructs a *double nerve* $\mathrm{N}\mathscr{B} : \Delta^{\mathrm{op}} \to$ **Cat** as follows. The objects of $(\mathrm{N}\mathscr{B})_n$ are families $(x_\bullet, f_{\bullet\bullet}, \theta_{\bullet\bullet\bullet})$ where $x_i \in \mathscr{B}_0$ for $0 \leq i \leq n$, $f_{i,j} : x_i \to x_j$ for $0 \leq i < j \leq n$, and $\theta_{i,j,k} : f_{j,k}f_{i,j} \overset{\sim}{\Longrightarrow} f_{i,k}$ for $0 \leq i < j < k \leq n$. Intuitively, this means $(\mathrm{N}\mathscr{B})_{n,0}$ consists of strings of n composable morphisms $f_{i-1,i}$ for $1 \leq i \leq n$ along with weak composites at all stages up to a single overall weak composite $f_{0,n}$. The morphisms $(x_\bullet, f_{\bullet\bullet}, \theta_{\bullet\bullet\bullet}) \to (x_\bullet, f'_{\bullet\bullet}, \theta'_{\bullet\bullet\bullet})$ necessarily map between families with the same tuple of objects of $\mathscr{B}$, and are given by families of 2-morphisms $\xi_{i,j} : f_{i,j} \Rightarrow f'_{i,j}$ for all $0 \leq i < j \leq n$ such that $\xi_{i,k} \circ \theta_{i,j,k} = \theta'_{i,j,k} \circ (\xi_{j,k} \otimes \xi_{i,j})$ for every $0 \leq i < j < k \leq n$; that is, the morphisms of $(\mathrm{N}\mathscr{B})_n$ are given by families of commutative triangular prisms between weak composites.

The face maps $d_i : (\mathrm{N}\mathscr{B})_n \to (\mathrm{N}\mathscr{B})_{n-1}$ act by dropping all elements of the tuples in $(x_\bullet, f_{\bullet\bullet}, \theta_{\bullet\bullet\bullet})$ with an index specified by i, which intuitively corresponds to taking the chosen weak composite of the

morphisms $f_{i-1,i}$ and $f_{i,i+1}$ when $0 < i < n$. On the other hand, the degeneracies $s_i : (\mathrm{N}\mathscr{B})_n \to (\mathrm{N}\mathscr{B})_{n+1}$ act by inserting an additional copy of x_i into $x_\bullet$, inserting id_{x_i} into $f_{\bullet\bullet}$, and inserting the unitors $\lambda_{x_j,x_i}(f_{j,i})$ for $j < i$ and $\rho_{x_i,x_j}(f_{i,j})$ for $j > i$ into $\theta_{\bullet\bullet\bullet}$ where appropriate, which corresponds to inserting an identity at the ith position and choosing all weak composites with it to just be determined by the unitors of $\mathscr{B}$.

By definition, we find $(\mathrm{N}\mathscr{B})_0 = \mathrm{disc}\,\mathscr{B}_0$, and $(\mathrm{N}\mathscr{B})_1 = \coprod_{x,y\in\mathscr{B}_0} \mathrm{Hom}_{\mathscr{B}}(x,y)$ as categories. Moreover, if $\mathscr{B}$ is locally discrete and hence an ordinary category, its double nerve is levelwise discrete and recover its Grothendieck nerve. Consider the Segal maps $(\mathrm{N}\mathscr{B})_n \to (\mathrm{N}\mathscr{B})_1^{\times(\mathrm{N}\mathscr{B})_0{}^n}$, which send an n-cell $(x_\bullet, f_{\bullet\bullet}, \theta_{\bullet\bullet\bullet})$ to the string $(f_{0,1},\ldots,f_{n-1,n})$. This is genuinely surjective, as any length-n composable string of morphisms can be obtained from the n-cell given by taking all weak composites to be those chosen by the horizontal composition rule of $\mathscr{B}$. The map is also full for the same reason, as length-n horizontally composable strings of 2-morphisms can be obtained in the same way. The functor is moreover faithful because the $\theta_{\bullet\bullet\bullet}$ are chosen to be invertible 2-morphisms, and so any morphism $\xi_{\bullet\bullet}$ of n-cells is uniquely determined by the 2-morphisms $\xi_{0,1},\ldots,\xi_{n-1,n}$ from the identity $\xi_{i,k}\circ\theta_{i,j,k} = \theta'_{i,j,k}\circ(\xi_{j,k}\otimes\xi_{i,j})$ for all $0 \le i < j < k \le n$. Therefore, the Segal maps are equivalences of categories, showing that $\mathrm{N}\mathscr{B}$ is indeed a 2-nerve.

Conversely, any 2-nerve $\mathcal{A} : \Delta^{\mathrm{op}} \to \mathbf{Cat}$ is levelwise equivalent to the double nerve of some bicategory. The construction of $\mathscr{B}$ is similar in spirit to the proof of Lemma 4.2, and is done explicitly in [46, Théorème 1.4.2]: take $\mathscr{B}_0 := \mathcal{A}_{0,0}$, and for $x,y \in \mathscr{B}_0$ define $\mathrm{Hom}_{\mathscr{B}}(x,y) := \mathcal{A}(x,y)$. To define horizontal composition, we need the axiom of choice to form a weak inverse of the equivalence $\mathcal{A}_2 \to \mathcal{A}_1 \times_{\mathcal{A}_0} \mathcal{A}_1$, after which $\otimes$ can be pulled via fibres from the composite $\mathcal{A}_1 \times_{\mathcal{A}_0} \mathcal{A}_1 \to \mathcal{A}_2 \xrightarrow{d_1} \mathcal{A}_1$. Associators for $\otimes$ are likewise obtained from weakly inverting $\mathcal{A}_3 \to \mathcal{A}_1^{\times_{\mathcal{A}_0}3}$. The reliance on the axiom of choice in defining composition on $\mathscr{B}$ reflects how n-nerves do not make any explicit choices for composites and instead only leaves them specified up to equivalence. This also means that maps of n-nerves will only preserve composition up to equivalence by design, implying that n-nerves also serve to present a class of pseudofunctors rather than strict 2-functors.

However, the identity endomorphisms of $\mathscr{B}$ are given by the degeneracy functor $s_0 : \mathcal{A}_0 \to \mathcal{A}_1$, with their corresponding unitors coming from the simplicial identities of s_0 with d_0 and d_2. In particular, this means that the identities play a critical role in the structure of $\mathcal{A}$ as a simplicial category; moreover, as simplicial functors must strictly preserve simplicial structure, this means that the pseudofunctors presented by maps of 2-nerves are necessarily strictly unital. Lack and Paoli prove this explicitly in [30, Theorem 3.7]. The similarity to the restrictions imposed by the simplicial maps between Duskin nerves of bicategories seen in Section 3.3.1 suggests that the main obstacle to relaxing strict unity is degeneracy in Δ.

4.3 Unbiased Operations

To overcome the obstacle posed by simplicial degeneracy maps in presenting weakly unital functors, Kock offers an alternative in [28] with *fair n-categories*. Here, the simplex category is replaced with the *fat delta*, which consists of stratified simplices without codegeneracy maps. The idea is that the stratification replaces the degeneracies in indicating which morphisms serve as candidates for identities.

Kock originally proposed fair categories in order to provide a general framework to address an open conjecture [42, Conjecture 2] of Simpson, which states that all weak higher categories are equivalent to one with a composition that is strictly associative but weakly unital.

However, the distinction made between composition and units in fair n-categories is arguably artificial. Associative and unital binary operations are often just a practical presentation of *unbiased* multiary operations. This is evidently the case for set-theoretic operations such as binary union and intersection, but is true also for more algebraic operations such as addition and multiplication. While unbiased operations require more work to write down formally, they can streamline other definitions and arguments so that they are more uniform in shape, which can make the objects easier to reason with conceptually. For example, primality of an integer p is often defined in a biased fashion, stating that p is non-invertible, and $p \mid ab$ implies $p \mid a$ or $p \mid b$. On top of commonly being confusing to the public as to why 1 is excluded from being prime, the first condition is also consequently an edge case that must be handled separately in proofs. Conversely, the equivalent unbiased definition of primality for p is that $p \mid \prod_{i=1}^{n} a_i$ implies $p \mid a_i$ for some $1 \leq i \leq n$. The usual primality definition is recovered by taking the above definition with $n = 0$ and $n = 2$, respectively, and now we have a more uniform characterisation of primes.

The unbiased presentation of monoidal binary operations is also more natural from a categorical perspective. More precisely, the monadicity of monoids over sets follows from the equivalence between the usual (biased) definition of monoidal structure and associative families of multiary operations. Explicitly, the forgetful functor $U : \mathbf{Mon} \to \mathbf{Set}$ admits a left adjoint given by the Kleene star operator $(-)^* : \mathbf{Set} \to \mathbf{Mon}$, which sends a set X to the collection $X^* := \bigcup_{p \geq 0} X^p$ of finite-length lists of elements of X. This has a canonical monoid structure given by list concatenation, and the monoidal unit is the empty list. The algebras of the induced monad $T := U(-)^*$ on $\mathbf{Set}$ consist of a ground set A and an action $\nu : TA \to A$ that commutes with the monad unit and multiplication of T. Specifically, the action assigns an element $\nu(a_1, \ldots, a_n) \in A$ to every list of elements $(a_1, \ldots, a_p) \in A^*$ such that $\nu(a) = a$ for singleton lists, and $\nu(\nu(a_1^1, \ldots, a_{q_1}^1), \ldots, \nu(a_1^p, \ldots, a_{q_p}^p)) = \nu(a_1^1, \ldots, a_{q_p}^p)$. This is precisely the axiomatisation of an unbiased monoidal structure on A, and the biased structure is recovered by taking the monoidal unit to be $e := \nu()$ on the empty list, and taking the product to be $a \cdot b := \nu(a, b)$. The higher arity operations then ensure that the associativity and unit axioms of a monoid are met: in particular, $a \cdot (b \cdot c) = \nu(a, \nu(b, c)) = \nu(a, b, c) = \nu(\nu(a, b), c) = (a \cdot b) \cdot c$ proves associativity, and right unity follows because $a \cdot e = \nu(a, \nu()) = \nu(a) = a$ with left unity following analogously.

Geometric presentations of higher categories by multisimplicial nerves such as the one of Tamsamani are already mostly unbiased. As we have already seen, binary composition on an n-nerve $\mathcal{A}$ comes from inverting the Segal map $\mathcal{A}_2 \to \mathcal{A}_1 \times_{\mathcal{A}_0} \mathcal{A}_1$, and then we obtain associators for this composition from choosing an inverse for $\mathcal{A}_3 \to \mathcal{A}_1^{\times_{\mathcal{A}_0} 3}$; however, the way we obtain these associators for the binary composition is via two *unbiased* associators that relate the binary composition to its ternary analogue. Informally, the equivalence $(f \otimes g) \otimes h \simeq f \otimes (g \otimes h)$ of stacked binary composites is given by the composite of equivalences $(f \otimes g) \otimes h \simeq (f \otimes g \otimes h) \simeq f \otimes (g \otimes h)$. Higher coherence equivalences between associators then arise similarly from quaternary composition rules, and so on. In general, we

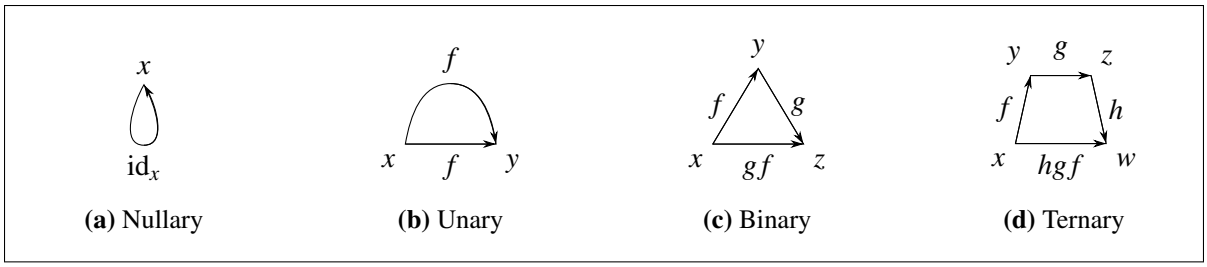


Figure 4.1: Multiary composition diagrams

can obtain p-ary composition for the n-nerve $\mathcal{A}$ by choosing an inverse of the p-ary Segal map and then taking fibres of the composite $\mathcal{A}_1^{\times_{\mathcal{A}_0} p} \to \mathcal{A}_p \xrightarrow{d_1 \ldots d_{p-1}} \mathcal{A}_1$. However, this definition only makes sense when $p \geq 1$, so we cannot recover units in this way. This problem is avoided in Section 4.2.1 by taking $d_{-1} : \mathcal{A}_0 \to \mathcal{A}_1$ to be the degeneracy map s_0.

The reason this works for arity $p \geq 1$ is as follows. By viewing p-ary composition as a commutative diagram, we obtain a directed graph in the shape of a $(p+1)$-gon as in Figure 4.1. When $p \geq 2$, this diagram arises as a subgraph of the complete graph on $p+1$ vertices. Orienting the edges of the complete graph consistently with the orientation of the $(p+1)$-gon, we recover the skeleton of a standard p-simplex. The triangles then correspond to intermediate partial composites: for example, in Figure 4.1d, the graph would embed into the standard 3-simplex, whose triangular faces correspond to taking the composites $g \circ f$, $h \circ g$, $h \circ (gf)$, and $(hg) \circ f$. This is no longer precisely the case when $p = 1$, but unary composition is itself degenerate in the sense that the unary composite of any morphism should just be itself, making the commutative digon in Figure 4.1b more appropriately just a single directed edge and thus precisely a 1-simplex. We cannot recover the case when $p = 0$ with simplices since the 0-simplex by definition has no nondegenerate edges, whereas the nullary composition in Figure 4.1a has one which constitutes the weakly degenerate identity endomorphism.

The conceptual benefit mentioned earlier for using unbiased operations is especially highlighted when axiomatising associativity up to coherent equivalence. While the usual biased definition is very terse, requiring only one type of associator and two types of unitors on the level of 2-morphisms, the coherence constraints described by higher invertible morphisms corresponding to the pentagon and triangle identities are not self-evident to non-experts; moreover, it is difficult to see that these axioms alone are sufficient in describing a fully coherent associative composition. On the other hand, the associators relating the multiary family of composition rules are far more natural, giving the n-ary operation one associator for every partitioning of n, corresponding to every possible way of bracketing an n-fold composition. The coherence constraint is then that any two ways of iteratively subdividing a string of composable morphisms yields the same associator. While this means that the unbiased definition has far more axioms, the constraints are much more natural, which allow for them to be more easily generalised.

4.3.1 Unbiased Bicategories

The observation about monoids being monadic over sets can be generalised: small categories are directed graphs equipped with a meaningful composition rule, and the corresponding forgetful functor turns out to also be monadic. The left adjoint is given by the *free category monad* of [31, Example 6.5.3], which sends a directed graph to the category whose objects are the vertices of the graph, and whose morphisms are given by finite-length paths of edges. That the algebras of this monad are precisely the small categories is proven entirely analogously to the case for monoids, and in particular describes the unbiased axiomatisation of category theory. This monadicity will be revisited in greater generality in Section 4.3.2, but here we will categorify this phenomenon to fit the context of bicategories.

The categorification of monads and their algebras up one dimension yields 2-monads and their pseudoalgebras as defined in [8], where the algebraic axioms are relaxed to only hold up to specified coherent isomorphism. It turns out that bicategories are 2-monadic over a categorification of directed graphs called **Cat**-graphs, where between any two vertices is a small category of edges rather than a mere set. The 2-monadicity is proven explicitly and in more generality in [41, §5], and the pseudoalgebras of the resulting free strict 2-category 2-monad are precisely the bicategories, presented with unbiased composition rules akin to those of [31, Definitions 3.4.1, 3.4.3]:

Definition 4.11. An *unbiased bicategory* $\mathscr{B}$ consists of a class $\mathscr{B}_0$ of objects and a category $\mathrm{Hom}_{\mathscr{B}}(x,y)$ of morphisms and 2-morphisms for all $x,y \in \mathscr{B}_0$ as in the biased case, then equipped for every integer $p \geq 0$ and $x_0,\dots,x_p \in \mathscr{B}_0$ with a (horizontal) p-fold composition functor

$$\otimes^p : \mathrm{Hom}_{\mathscr{B}}(x_{p-1},x_p) \times \cdots \times \mathrm{Hom}_{\mathscr{B}}(x_0,x_1) \to \mathrm{Hom}_{\mathscr{B}}(x_0,x_p)$$

For brevity, we will write $\bigotimes_{i=1}^p f_i := \otimes^p(\vec{f})$ where $\vec{f} = (f_1,\dots,f_p)$. Associativity of these composition rules is expressed by a chosen family of 2-isomorphisms for $p \geq 0$ and $q_1,\dots,q_p \geq 0$

$$\alpha^p_{\vec{f}^1,\dots,\vec{f}^p} : \bigotimes_{i=1}^{p} \bigotimes_{j=1}^{q_i} f^i_j \overset{\sim}{\Longrightarrow} \otimes^{\Sigma_{i=1}^p q_i}(f^1_1,\dots,f^p_{q_p})$$

natural in all composable f^i_j, called (unbiased) *associators*, as well as 2-isomorphisms

$$\iota_f : f \overset{\sim}{\Longrightarrow} \otimes^1(f)$$

natural in f called *inserters*. The data are then subject to commutativity of the coherence square

$$\begin{tikzcd}
\bigotimes_{i=1}^{p} \bigotimes_{j=1}^{q_i} \bigotimes_{k=1}^{r_{i,j}} f^{i,j}_k \arrow[r, Rightarrow, "\alpha^p"] \arrow[d, Rightarrow, "\otimes_i \alpha^{q_i}"'] & \otimes^{\Sigma_i q_i}\left(\otimes^{r_{1,1}}(\vec{f}^{1,1}),\dots,\otimes^{r_{p,q_p}}(\vec{f}^{p,q_p})\right) \arrow[d, Rightarrow, "\alpha^{\Sigma_i q_i}"] \\
\bigotimes_{i=1}^{p}\left(\otimes^{\Sigma_j r_{i,j}}(f^{i,1}_1,\dots,f^{i,q_i}_{r_{i,q_i}})\right) \arrow[r, Rightarrow, "\alpha^p"'] & \otimes^{\Sigma_i \Sigma_j r_{i,j}}(f^{1,1}_1,\dots,f^{p,q_p}_{r_{p,q_p}})
\end{tikzcd}$$

for all morphisms $f_k^{i,j}$ and integers $p, q_i, r_{i,j} \geq 0$, and also commutativity of the coherence triangles

$$\begin{tikzcd}
\bigotimes_{i=1}^{p} f_i \arrow[r, Rightarrow, "\otimes_i \iota_{f_i}"] \arrow[dr, Rightarrow, "\mathrm{id}"'] & \bigotimes_{i=1}^{p} \left(\otimes^1(f_i)\right) \arrow[d, Rightarrow, "\alpha^p"] \\
& \bigotimes_{i=1}^{p} f_i
\end{tikzcd}
\qquad
\begin{tikzcd}
\otimes^1\left(\bigotimes_{i=1}^{p} f_i\right) \arrow[d, Rightarrow, "\alpha^1"'] & \bigotimes_{i=1}^{p} f_i \arrow[l, Rightarrow, "\iota"'] \arrow[dl, Rightarrow, "\mathrm{id}"] \\
\bigotimes_{i=1}^{p} f_i &
\end{tikzcd}$$

for all $\vec{f}$. Denote the unique morphisms picked by the nullary composites $\otimes^0 : \mathbf{1} \to \mathrm{Hom}_{\mathscr{B}}(x,x)$ by id_x, and use infix notation for binary composites so that $f \otimes g := \otimes^2(f,g)$.

Given two unbiased bicategories $\mathscr{B}, \mathscr{B}'$, a *lax functor* $F : \mathscr{B} \to \mathscr{B}'$ consists of a map of objects $F_0 : \mathscr{B}_0 \to \mathscr{B}'_0$, a local functor $F_{x,y} : \mathrm{Hom}_{\mathscr{B}}(x,y) \to \mathrm{Hom}_{\mathscr{B}'}(Fx, Fy)$ for all $x, y \in \mathscr{B}_0$, and a *lax (unbiased) functoriality constraint* $F^p_{f_1,\ldots,f_p} : \bigotimes_{i=1}^p F f_i \Rightarrow F\left(\bigotimes_{i=1}^p f_i\right)$ natural in the morphisms f_i in $\mathscr{B}$ for every $p \geq 0$ such that they satisfy commutativity of the compatibility pentagon

$$\begin{tikzcd}
\bigotimes_{i=1}^{p} \bigotimes_{j=1}^{q_i} F f_j^i \arrow[r, Rightarrow, "\otimes_i F^{q_i}"] \arrow[d, Rightarrow, "\alpha^p"'] & \bigotimes_{i=1}^{p} F\left(\bigotimes_{j=1}^{q_i} f_j^i\right) \arrow[r, Rightarrow, "F^p"] & F\left(\bigotimes_{i=1}^{p} \bigotimes_{j=1}^{q_i} f_j^i\right) \arrow[d, Rightarrow, "F(\alpha^p)"] \\
\otimes^{\Sigma_i q_i}(F f_1^1, \ldots, F f_{q_p}^p) \arrow[rr, Rightarrow, "F^{\Sigma_i q_i}"'] & & F\left(\otimes^{\Sigma_i q_i}(f_1^1, \ldots, f_{q_p}^p)\right)
\end{tikzcd}$$

and compatibility triangle

$$\begin{tikzcd}
& Ff \arrow[dl, Rightarrow, "\iota"'] \arrow[dr, Rightarrow, "F\iota"] & \\
\otimes^1(Ff) \arrow[rr, Rightarrow, "F^1"'] & & F\left(\otimes^1 f\right)
\end{tikzcd}$$

If F^p is invertible for every $p \geq 0$, then F is called a *pseudofunctor* as before. Unbiased bicategories and their lax or pseudofunctors collect to form (1-)categories $\mathbf{UBicat}_{\mathrm{Lax}}$ and $\mathbf{UBicat}$, respectively.

While shrouded in technicalities, the axioms of an unbiased bicategory convey more directly that computing a p-fold composition should be unique up to unique coherent isomorphism, as it extracts the key properties that induce coherence for objects defined by universal property. In particular, there are some cases where a weakly associative composition is more easily realised in its unbiased form. For instance, the p-fold products are more readily available in a cartesian monoidal category, and this is evident when proving that the cartesian product is indeed monoidal: the (biased) associators are induced from the fact that both $(x \times y) \times z$ and $x \times (y \times z)$ satisfy the universal property of the three-fold product $x \times y \times z$ and are thus uniquely isomorphic; likewise, the pentagon axiom follows from the uniqueness property when noting that all of the various ways of using binary products to combine objects w, x, y, z will satisfy the universal property of the four-fold product $w \times x \times y \times z$.

The unbiased associators arise from respecting the multiplication of the free strict 2-category 2-

monad up to coherent isomorphism, and the inserters likewise from respecting the 2-monad unit. The reason for the two different types of associators (and consequently two different types of coherence axioms) is that the definition of 2-monad used is itself biased.Intuitively, the coherence square asserts that any two ways of unbracketing composites yield the same coherence 2-isomorphism, while the coherence triangles assert that any two ways of rebracketing composites also yield the same coherence 2-isomorphism. If we were to use unbiased 2-monads instead, we would obtain an equivalent characterisation of unbiased bicategories with a far more technical family of associators:

Lemma 4.12. *The following data is equivalent to providing an unbiased bicategory as in Definition 4.11:* $\mathscr{B}_0$, $\mathrm{Hom}_{\mathscr{B}}$, *and* $\otimes^p$ *for* $p \geq 0$ *as before, but now with* 2*-isomorphisms*

$$\beta^{N;\vec{p}}_{\vec{f}} : \bigotimes_{i_1=1}^{p^1} \bigotimes_{i_2=1}^{p^2_{i_1}} \cdots \bigotimes_{i_N=1}^{p^N_{i_1,\ldots,i_{N-1}}} f_{i_1,\ldots,i_N} \xRightarrow{\sim} \bigotimes_{\vec{i}} f_{\vec{i}}$$

for all $N \geq 0$ *and* $p^1, \ldots, p^N_{p^1,\ldots,p^{N-1}} \geq 0$ *and natural over composable morphisms* $f_{i_1,\ldots,i_N}$*, subject to the identities*

$$\mathrm{id}_{\otimes^p(\vec{f})} = \beta^{1;p}_{\vec{f}}; \qquad \left(\bigotimes_{i_1=1}^{p^1} \cdots \bigotimes_{i_N=1}^{p^N_{i_1,\ldots,i_{N-1}}} \beta^{N_{i_1,\ldots,i_N}} \right) \beta^{N;\vec{p}} = \beta^{\Sigma_{i_1} \cdots \Sigma_{i_N} N_{i_1,\ldots,i_N}}$$

Proof. This ultimately amounts to unpacking definitions. Given the above 2-isomorphisms, we recover associators and inserters by taking $\alpha^\bullet := \beta^{2;\bullet}$ and $\iota := \beta^0$, respectively. On the other hand, given an unbiased bicategory, we can produce the 2-isomorphisms β^N inductively on N, starting naturally with $\beta^0 := \iota$, and then taking

$$\beta^{N+1;\vec{n}}_{\vec{f}} : \bigotimes_{i_0=1}^{p^0} \left(\bigotimes_{i_1=1}^{p^1_{i_0}} \bigotimes_{i_2=1}^{p^2_{i_0,i_1}} \cdots \bigotimes_{i_N=1}^{p^N_{i_0,i_1,\ldots,i_{N-1}}} f_{i_0 i_1,\ldots,i_N} \right) \xRightarrow{\otimes_{i_0} \beta^{N;p}} \bigotimes_{i_0=1}^{p^0} \bigotimes_{\vec{i}} f_{i_0,\vec{i}} \xRightarrow{\alpha} \bigotimes_{i_0,\vec{i}} f_{i_0,\vec{i}}$$

The fact that $\mathrm{id} = \beta^1$ then follows from the left coherence triangle for inserters, and the other identity follows inductively from the coherence square for associators. □

Remark 4.13. We can see that there is still a shadow of bias in the above characterisation, as the β^N are subject to two different identities. We could make this further unbiased by combining these identities into a much larger family of identities of the form "$(\otimes_\bullet \beta) \ldots \beta = \beta$" with appropriate indexing, but this would only be useful in the context of 3-categories.

Given an unbiased bicategory as per Definition 4.11, we recover an ordinary bicategory from taking the nullary and binary composites, as suggested by the special notation for these arities. The (biased) associator $(f \otimes g) \otimes h \xRightarrow{\sim} f \otimes (g \otimes h)$ is obtained from the composite

$$(f \otimes g) \otimes h \xRightarrow[\sim]{(f \otimes g) \otimes \iota_h} (f \otimes g) \otimes (h) \xRightarrow[\sim]{\alpha^2_{(f,g),(h)}} \otimes^3(f,g,h) \xRightarrow[\sim]{(\alpha^2_{(f),(g,h)})^{-1}} (f) \otimes (g \otimes h) \xRightarrow[\sim]{\iota_f^{-1} \otimes (g \otimes h)} f \otimes (g \otimes h)$$

while the left unitor $\mathrm{id}\otimes f \overset{\sim}{\Longrightarrow} f$ is obtained from

$$\mathrm{id}\otimes f \overset{\mathrm{id}\otimes\iota_f}{\overset{\sim}{\Longrightarrow}} \mathrm{id}\otimes(f) \overset{\alpha^2_{(),(f)}}{\overset{\sim}{\Longrightarrow}} (f) \overset{\iota_f^{-1}}{\overset{\sim}{\Longrightarrow}} f$$

with the right unitors defined analogously.

From a biased bicategory, there are several ways to resolve its bias: for instance, we can define its multiary compositions with a left-associative bias by taking $\otimes^0 := \mathrm{id}_x : \mathbf{1} \to \mathrm{Hom}(x,x)$ and then proceeding inductively with $\bigotimes_{i=1}^{p+1} f_i := \left(\bigotimes_{i=1}^{p} f_i\right) \otimes f_{p+1}$. The inserters would be given by identity 2-isomorphisms, and the pentagon and triangle axioms ensure that any path of rebracketing using ordinary associators and unitors can be used to produce the unbiased associators for these p-fold compositions. These actions of introducing and resolving biases for bicategories extend naturally to lax and pseudofunctors, and induce equivalences of categories $\mathbf{Bicat} \simeq \mathbf{UBicat}$ and $\mathbf{Bicat}_{\mathrm{Lax}} \simeq \mathbf{UBicat}_{\mathrm{Lax}}$ by the Irrelevance of Signature Theorem in [31, §3.4].

4.3.2 Generalised Operads

While the Segal condition yields a nearly unbiased composition, the triangular shape of simplices indicate that simplicial nerves are still biased: triangles can only encode binary operations, and the n-cells of the nerve of a higher category really just provide a string of n composable morphisms with every possible intermediate way of composing morphisms together through these binary operations. In order to resolve this bias, the conceptually simplest approach is to modify the category of shapes from the simplex category Δ to one which allows for other directed polygons than just the triangle in dimension two. This will also remove the necessity of degeneracy, as nullary composition can then be obtained from the polygon in Figure 4.1a.

Fleshing this out results in *opetopes*,[1] which are unfortunately much more technical to construct than simplices, but fortunately yield a much simpler category of shapes due to the absense of codegeneracy maps. Opetopes were first introduced by Baez and Dolan in [3] for the same purpose of constructing presentations of higher categories, where they sought to generalise the simplicial model of quasicategories to weak ω-categories as opetopic sets subject to analogous lifting properties. However, they defined opetopic sets directly, rather than via a shape category, and moreover their original definition suffered from redundancy due to input facets having labels. For example, there would be 3! incarnations of the 2-opetope corresponding to the ternary composition shape of Figure 4.1d.

Cheng resolves this in [11] by generalising the construction to *symmetric multicategories*, and further defines a category of opetopes whose presheaves recover the desired opetopic sets. In this category, opetopes corresponding to the same shape but with possibly different labels for the input facets become isomorphic. However, to ensure that our category of opetopes is discrete at each dimension in the sense that the only nontrivial maps are face maps, we will follow Leinster's approach in [31, §7] via *nonsymmetric operads* that drops the labelling altogether. The category of opetopes can then be constructed

[1] The authors of [3] and [31] emphasise that the pronunciation of "opetope" is /ˌɒpəˈtəʊp/, as it is a portmanteau of "operation" and "polytope" for its role as a geometric presentation of multiary operations.

analogously to Cheng, and she proves in [12] that these categories of opetopes are equivalent.

The following definitions can be found in [31, §§4–7].

Definition 4.14. Let $\mathscr{E}$ be a category and T a monad on $\mathscr{E}$. Say that the pair $(\mathscr{E},T)$ is *cartesian* if $\mathscr{E}$ is finitely complete, T preserves pullbacks, and the naturality squares for the unit and multiplication transformations on T are pullback squares. If $(\mathscr{E},T)$ is cartesian, call it *suitable* if moreover the following hold:

- $\mathscr{E}$ has disjoint finite coproducts, meaning that it has an initial object $\emptyset$, the coproduct $x \amalg y$ exists for all $x,y \in \mathscr{E}_0$ with monic coprojections $x,y \hookrightarrow x \amalg y$, and $x \times_{x \amalg y} y = \emptyset$.
- Finite coproducts in $\mathscr{E}$ are pullback-stable, meaning for any morphism $z \to x \amalg y$ in $\mathscr{E}$ that we have $(x \times_{x \amalg y} z) \amalg (y \times_{x \amalg y} z) = z$.
- Given a functor $\omega \to \mathscr{E}$, where ω is the first infinite ordinal viewed as a posetal category, such that all morphisms in ω are mapped to monomorphisms (such a functor is called a *nested sequence*), then its colimit exists, and the canonical coprojections are again monic.
- Colimits of nested sequences are pullback-stable, meaning that given a functor $F : \Lambda^2[2] \times \omega \to \mathscr{E}$ such that $F^i : \omega \to \mathscr{E}$ is a nested sequence for every $i \in \Lambda^2[2]_0$ (where $\Lambda^2[2] = \{\bullet \to \bullet \leftarrow \bullet\}$ is the walking cospan category, whose nerve is the simplicial 2-horn of the same name), then $\varinjlim_n \varprojlim_i F^i_n = \varprojlim_i \varinjlim_n F^i_n$.
- T preserves colimits of nested sequences.

Definition 4.15. The bicategory $\mathbf{Span}(\mathscr{E},T)$ of T-spans in $\mathscr{E}$ for a cartesian pair $(\mathscr{E},T)$ is given as follows. The objects are the same as those of $\mathscr{E}$, the morphisms $f : a \nrightarrow a'$ are given by spans of the form $Ta \xleftarrow{f_0} b \xrightarrow{f_1} a'$ in $\mathscr{E}$ for $b \in \mathscr{E}_0$, and the corresponding 2-morphisms $\theta : f \Rightarrow g : a \nrightarrow a'$ are given by morphisms $\theta : b \to b'$ in $\mathscr{E}$ such that

$$\begin{tikzcd}
 & b \arrow[dl, "f_0"'] \arrow[dr, "f_1"] \arrow[dd, "\theta"] & \\
Ta & & a' \\
 & b' \arrow[ul, "g_0"] \arrow[ur, "g_1"'] &
\end{tikzcd}$$

commutes. Composition of spans $f : a \nrightarrow a'$ and $f' : a' \nrightarrow a''$ is given by choosing a fixed pullback in

the diagram below and taking the encompassing span:

$$\begin{tikzcd}
Tb \times_{Ta} b' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & b' \arrow[r, "f_1'"] \arrow[d, "f_0'"'] & a'' \\
Tb \arrow[r, "Tf_1"'] \arrow[d, "Tf_0"'] & Ta' & \\
TTa \arrow[d, "\mu_a^T"'] & & \\
Ta & &
\end{tikzcd}$$

where μ^T is the multiplication transformation on T. The natural choice of an identity $a \nrightarrow a$ is then just the span $Ta \xleftarrow{\eta_a^T} a \xrightarrow{\mathrm{id}_a} a$, and the unitors and associators are induced by the universal property of the pullback in the above diagram, using the fact that T is cartesian.

Definition 4.16. Let $(\mathscr{E},T)$ be cartesian. For $V \in \mathscr{E}_0$, define the category of $(\mathscr{E},T)$-*graphs* with V the object of vertices to be the monoidal endomorphism category $(\mathscr{E},T)\mathbf{Graph}_V := \mathrm{Hom}_{\mathbf{Span}(\mathscr{E},T)}(V,V)$. This construction is functorial in V: given a morphism $V \to V'$ in $\mathscr{E}$, we obtain a corresponding functor $(\mathscr{E},T)\mathbf{Graph}_V \to (\mathscr{E},T)\mathbf{Graph}_{V'}$ that sends a graph $TV \leftarrow E \to V$ to the graph

$$TV' \leftarrow TV \leftarrow E \to V \to V'$$

extending to graph morphisms in the obvious way. Therefore, the covariant Grothendieck construction (cf. Appendix A) produces the category

$$(\mathscr{E},T)\mathbf{Graph} := \mathscr{E} \int (\mathscr{E},T)\mathbf{Graph}_{(-)}$$

of all $(\mathscr{E},T)$-graphs.

Unpacking the definition, the category of $(\mathscr{E},T)$-graphs consists of spans $T\mathscr{M}_0 \xleftarrow{s} \mathscr{M}_1 \xrightarrow{t} \mathscr{M}_0$ representing graphs $\mathscr{M}$ with object of vertices $\mathscr{M}_0$ and object of edges $\mathscr{M}_1$. The source and target vertices of any edge are intuitively chosen by s and t, respectively. Note that the source of an edge is generalised from being a single vertex as in usual graphs to allow for a family of vertices, depending on the structure of the monad T. Graph homomorphisms $\Phi : \mathscr{M} \to \mathscr{M}'$ are commutative rectangles

$$\begin{tikzcd}
T\mathscr{M}_0 \arrow[d, "T\Phi_0"'] & \mathscr{M}_1 \arrow[l, "s"'] \arrow[r, "t"] \arrow[d, "\Phi_1"] & \mathscr{M}_0 \arrow[d, "\Phi_0"] \\
T\mathscr{M}_0' & \mathscr{M}_1' \arrow[l, "s'"] \arrow[r, "t'"'] & \mathscr{M}_0'
\end{tikzcd}$$

which are intuitively maps of vertices and edges that respect sources and targets.

Definition 4.17. Let $(\mathscr{E},T)$ be cartesian. For $V \in \mathscr{E}_0$, the category of $(\mathscr{E},T)$-*multicategories* on V is the category of monoids $(\mathscr{E},T)\mathbf{Multicat}_V := \mathbf{Mon}(\mathscr{E},T)\mathbf{Graph}_V$. This construction is also functorial

in V: given a morphism $V \to V'$ in $\mathscr{E}$, the induced functor $(\mathscr{E},T)\mathbf{Graph}_V \to (\mathscr{E},T)\mathbf{Graph}_{V'}$ of graphs is weak monoidal by the universal property of the pullbacks involved, and thus further induces a functor of monoid categories. Therefore, we have by the Grothendieck construction a category

$$(\mathscr{E},T)\mathbf{Multicat} := \mathscr{E}\int (\mathscr{E},T)\mathbf{Multicat}_{(-)}$$

of $(\mathscr{E},T)$-multicategories and $(\mathscr{E},T)$-multifunctors.

Moreover, the canonical forgetful functor $U_V : (\mathscr{E},T)\mathbf{Multicat}_V \to (\mathscr{E},T)\mathbf{Graph}_V$ is natural in V and thus lifts to a forgetful functor $U : (\mathscr{E},T)\mathbf{Multicat} \to (\mathscr{E},T)\mathbf{Graph}$ that lies over $\mathscr{E}$.

An $(\mathscr{E},T)$-multicategory is given by a $(\mathscr{E},T)$-graph $\mathscr{M}$ equipped with an identity-assigning morphism $e : \mathscr{M}_0 \to \mathscr{M}_1$ and a composition morphism $c : T\mathscr{M}_1 \times_{T\mathscr{M}_0} \mathscr{M}_1 \to \mathscr{M}_1$ fitting in the commutative diagrams

$$\begin{tikzcd}
 & \mathscr{M}_0 \arrow[dl, "\eta^T_{\mathscr{M}_0}"'] \arrow[dr, "\mathrm{id}"] \arrow[dd, "e"] & \\
T\mathscr{M}_0 & & \mathscr{M}_0 \\
 & \mathscr{M}_1 \arrow[ul, "s"] \arrow[ur, "t"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & T\mathscr{M}_1 \times_{T\mathscr{M}_0} \mathscr{M}_1 \arrow[dl, "\mu^T_{\mathscr{M}_0} \circ Ts"'] \arrow[dr, "t"] \arrow[dd, "c"] & \\
T\mathscr{M}_0 & & \mathscr{M}_0 \\
 & \mathscr{M}_1 \arrow[ul, "s"] \arrow[ur, "t"'] &
\end{tikzcd}$$

where the span on top of the right hand diagram is the horizontal composite $\mathscr{M} \otimes \mathscr{M}$ in $\mathbf{Span}(\mathscr{E},T)$. The monad axioms on $\mathscr{M}$ assert that c is associative with unit e in the same way as for a category, and a multifunctor $\Phi : \mathscr{M} \to \mathscr{M}'$ is a morphism of the underlying graphs that respects e and c in the sense that the diagrams

$$\begin{tikzcd}
\mathscr{M}_0 \arrow[r, "e"] \arrow[d, "\Phi_0"'] & \mathscr{M}_1 \arrow[d, "\Phi_1"] \\
\mathscr{M}'_0 \arrow[r, "e'"'] & \mathscr{M}'_1
\end{tikzcd}
\qquad
\begin{tikzcd}
T\mathscr{M}_1 \times_{T\mathscr{M}_0} \mathscr{M}_1 \arrow[r, "c"] \arrow[d, "T\Phi_1 \times_{T\Phi_0} \Phi_1"'] & \mathscr{M}_1 \arrow[d, "\Phi_1"] \\
T\mathscr{M}'_1 \times_{T\mathscr{M}'_0} \mathscr{M}'_1 \arrow[r, "c'"'] & \mathscr{M}'_1
\end{tikzcd}$$

both commute. With this explicit formulation, the forgetful functor $U : (\mathscr{E},T)\mathbf{Multicat} \to (\mathscr{E},T)\mathbf{Graph}$ becomes obvious, and can be seen to lie over the projection from these categories onto $\mathscr{E}$ that retain only the underlying object of vertices.

Example 4.18. The pair $(\mathbf{Set}, \mathrm{id})$ is suitable. Indeed, **Set** is complete and cocomplete, and the coproducts are readily seen to be disjoint and pullback-stable just by checking the identities element-wise. Pullback stability of colimits of nested sequences follows from the fact that finite limits commute with filtered colimits in **Set**. As for the suitability of the identity monad, there is nothing to check.

More generally, if $\mathscr{E}$ is a finitely complete category, then $(\mathscr{E}, \mathrm{id})$ will at least be cartesian. This is enough structure to follow through the constructions in Definition 4.17, and we can see that $(\mathscr{E}, \mathrm{id})$-graphs produce graph objects in $\mathscr{E}$, and more importantly $(\mathscr{E}, \mathrm{id})\mathbf{Multicat} = \mathbf{Cat}(\mathscr{E})$ recovers the categories internal to $\mathscr{E}$.

Example 4.19. The pair $(\mathbf{Set}, (-)^*)$ is also suitable, where $(-)^*$ is the Kleene star, and this can be easily checked elementwise. This example highlights the role of the monad in generalising the internal categories of the above example, as $(\mathbf{Set}, (-)^*)\mathbf{Multicat} = \mathbf{Multicat}$ recovers the category of

ordinary small multicategories, which are categories where the morphisms have several domains and one codomain. In more detail, a (small) multicategory $\mathscr{M}$ consists of a set $\mathscr{M}_0$ of objects, and hom-sets $\mathrm{Hom}_{\mathscr{M}}(x_1,\dots,x_p;y)$ of morphisms $(x_1,\dots,x_p)\to y$ for all $x_1,\dots,x_p,y\in\mathscr{M}_0$ and $p\geq 0$. Composition is then given by an associative family of maps

$$\mathrm{Hom}_{\mathscr{M}}(y_1,\dots,y_p;z)\times\prod_{i=1}^{p}\mathrm{Hom}_{\mathscr{M}}(x_1^i,\dots,x_{q_i}^i;y_i)\to\mathrm{Hom}_{\mathscr{M}}(x_1^1,\dots,x_{q_p}^p;z)$$

with identity morphisms $\mathrm{id}_x:(x)\to x$ as before.

For example, any monoidal category $\mathscr{V}$ (with an unbiased tensor product) has the natural structure of a multicategory $\mathscr{V}^{\otimes}$ where $\mathrm{Hom}_{\mathscr{V}^{\otimes}}(x_1,\dots,x_p;y):=\mathrm{Hom}_{\mathscr{V}}(x_1\otimes\cdots\otimes x_p,y)$.

The main benefit here of establishing the above generalisation of the monadicity phenomenon of categories over graphs is that under suitable initial conditions, the construction can be iterated and provide a streamlined means for producing intricate unbiased higher-dimensional structures.

Lemma 4.20. *Let $(\mathscr{E},T)$ be suitable. Then, $(\mathscr{E},T)$**Multicat** is monadic over $(\mathscr{E},T)$**Graph**, meaning that the canonical forgetful functor $U:(\mathscr{E},T)$**Multicat** $\to(\mathscr{E},T)$**Graph** admits a left adjoint $F\dashv U$, and the $(\mathscr{E},T)$-multicategories are precisely the algebras for the induced monad UF. Moreover, the resulting pair $((\mathscr{E},T)$**Graph**$,UF)$ is suitable.*

*This is also true on fibres: if $V\in\mathscr{E}_0$, then $(\mathscr{E},T)$**Multicat**$_V$ is monadic over $(\mathscr{E},T)$**Graph**$_V$, and the induced monad forms a suitable pair with $(\mathscr{E},T)$**Graph**$_V$.*

Proof. The full proof can be found in [31, Theorems 6.5.2, 6.5.4]. Here, we will only reconstruct the "free multicategory" left adjoint F, as it will be needed when discussing opetopes in the next section.

Let $\mathscr{M}$ be an $(\mathscr{E},T)$-graph with vertex object $V\in\mathscr{E}_0$, then the free $(\mathscr{E},T)$-multicategory on $\mathscr{M}$ is the colimit of a nested sequence of graphs $\mathscr{M}^0\hookrightarrow\mathscr{M}^1\hookrightarrow\mathscr{M}^2\hookrightarrow\dots$ on V. This generalises the construction of the free monoid on a set X, which is done by taking a colimit of the sets $X^{\leq p}:=\bigcup_{k\leq p}X^k$ of finite strings of length at most p. Indeed, $\mathscr{M}^p$ consists of paths on the graph $\mathscr{M}$ of length at most p. At $p=0$, this makes $\mathscr{M}^0$ the identity span $TV\xleftarrow{\eta_V^T}V\xrightarrow{\mathrm{id}}V$, as this yields the discrete graph on V. Given $\mathscr{M}^p$, we then take $\mathscr{M}^{p+1}:=\mathscr{M}^0\amalg(\mathscr{M}\otimes\mathscr{M}^p)$, which provides new null paths and appends to all existing paths a new edge from $\mathscr{M}$.

The chain of inclusions $i^p:\mathscr{M}^p\hookrightarrow\mathscr{M}^{p+1}$ are defined inductively with $i^0:\mathscr{M}^0\hookrightarrow\mathscr{M}^0\amalg(\mathscr{M}\otimes\mathscr{M}^0)$ being given by the first coprojection, and $i^{p+1}:=\mathrm{id}_{\mathscr{M}^0}\amalg(\mathrm{id}_{\mathscr{M}}\otimes i^p)$. In general, a morphism $i:x\to y$ is monic if and only if its kernel pair is trivial in the sense that

$$\begin{tikzcd} x \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & x \arrow[d, "i"] \\ x \arrow[r, "i"'] & y \end{tikzcd}$$

is a pullback square. Since coproducts in $\mathscr{E}$ are pullback-stable with monic coprojections, this means that the i^p are monic for all $p\geq 0$ and thus the nested sequence of edge objects $\mathscr{M}_1^0\hookrightarrow\mathscr{M}_1^1\hookrightarrow\dots$

admits a colimit in $\mathscr{E}$. The universal property of the colimit induces canonical morphisms onto TV and V from those of the $\mathscr{M}^p$, making this colimit a $(\mathscr{E},T)$-graph and thus a colimit $F_V\mathscr{M} := \varinjlim_p \mathscr{M}^p$ in $(\mathscr{E},T)\mathbf{Graph}_V$ also.

In words, $F_V\mathscr{M}$ is the graph of paths on $\mathscr{M}$, and it carries a canonical $(\mathscr{E},T)$-multicategorical structure, where composition is induced by path concatenation, and identities are given by null paths. Precisely, the identity-assigning map is given by the canonical coprojection $e : \mathscr{M}^0 \hookrightarrow F_V\mathscr{M}$. As for composition, we have maps $\mathscr{M}^p \otimes \mathscr{M}^q \to \mathscr{M}^{p+q}$ obtained from noting that pullback stability of coproducts in $\mathscr{E}$ implies $\mathscr{M}^p = \coprod_{k\leq p} \mathscr{M}^{\otimes k}$, and thus $\mathscr{M}^p \otimes \mathscr{M}^q = \coprod_{k\leq p}\coprod_{k'\leq q} \mathscr{M}^{\otimes(k+k')}$, which canonically embeds into $\mathscr{M}^{p+q} = \coprod_{k\leq p+q} \mathscr{M}^{\otimes k}$. Taking colimits as $p,q \to \infty$ and recalling that colimits of nested sequences are also pullback-stable, these produce the desired composition map $c : F_V\mathscr{M} \otimes F_V\mathscr{M} \to F_V\mathscr{M}$.

This construction extends to a functor $F_V : (\mathscr{E},T)\mathbf{Graph}_V \to (\mathscr{E},T)\mathbf{Multicat}_V$ on each fibre naturally in V. Therefore, we can combine them via the (covariant) Grothendieck construction to produce the overall left adjoint $F \dashv U$. □

Definition 4.21. If $(\mathscr{E},T)$ is suitable, then denote by $(\mathscr{E}^+,T^+)$ the pair where $\mathscr{E}^+ := (\mathscr{E},T)\mathbf{Graph}$ and T^+ is the monad on $(\mathscr{E},T)\mathbf{Graph}$ whose algebras are the $(\mathscr{E},T)$-multicategories. Similarly, if $V \in \mathscr{E}_0$, denote by $(\mathscr{E}_V^+,T_V^+)$ the pair where $\mathscr{E}_V^+ := (\mathscr{E},T)\mathbf{Graph}_V$ and T_V^+ is the monad on $(\mathscr{E},T)\mathbf{Graph}_V$ whose algebras are the $(\mathscr{E},T)$-multicategories over V. By Lemma 4.20, the pairs $(\mathscr{E}^+,T^+)$ and $(\mathscr{E}_V^+,T_V^+)$ always exist and are suitable.

Baez and Dolan had originally defined opetopes via *symmetric operads*. Loosely speaking, an operad is a system of abstract operators of various arities that can have their outputs plugged into other operators to form larger ones. The operad is symmetric if the abstract operators are insensitive to input permutation up to isomorphism. The issue with this original approach was that the "slice operation" they used to build operads of higher dimension would forget the symmetries of the lower-dimensional operads they were built from, meaning many higher operads that should have been equivalent were no longer isomorphic. With Leinster's approach, we avoid this altogether by working with (generalised) nonsymmetric operads and picking a representative for every isomorphism class of symmetric operads.

Note that abstract operators in an operad have no restrictions on how they can be wired together, so long as output nodes are only connected to input nodes of other operads such that no feedback loops are formed. We can generalise the abstract operators so that their input and output nodes have associated colours, and then further restrict composability to be between nodes that share the same colour. Formalising this idea would recover the ordinary multicategories of Example 4.19, which is why multicategories are also referred to as coloured operads: the colours make up the multicategory objects, and the abstract operators the multimorphisms. In particular, ordinary operads are just multicategories on one object.

This also means that operads can be internalised: if $(\mathscr{E},T)$ is cartesian, and $\mathrm{pt} \in \mathscr{E}_0$ is a terminal object, then the corresponding category of generalised operads can be taken to be $(\mathscr{E},T)\mathbf{Multicat}_{\mathrm{pt}}$. If $(\mathscr{E},T)$ is suitable, then Definition 4.21 provides a procedure $(\mathscr{E},T) \mapsto (\mathscr{E}_{\mathrm{pt}}^+,T_{\mathrm{pt}}^+)$ which serves as the nonsymmetric analogue of Baez and Dolan's slice operation. Note that the T_{pt}^+-algebras will be precisely the $(\mathscr{E},T)$-operads.

4.4 The Category of Opetopes

Here, we will finally define the shape category **O** that will replace the simplex category for nerves of higher categories. We will construct opetopes as in [31, §7.1] and define structure maps according to [11, §1.4].

Given any cartesian pair $(\mathscr{E}, T)$, the category $\mathscr{E}_V^+$ of $(\mathscr{E}, T)$-graphs on V is equivalent to the slice category $(\mathscr{E} \downarrow TV \times V)$ since the presence of finite products in $\mathscr{E}$ allow us to identify spans $TV \leftarrow E \rightarrow V$ with morphisms $E \rightarrow TV \times V$, and graph morphisms on V are likewise identifiable with morphisms of edge objects in $\mathscr{E}$ that lie over $TV \times V$. Taking $V = \mathrm{pt}$ to be terminal in $\mathscr{E}$, this means in particular that $\mathscr{E}_{\mathrm{pt}}^+ \cong (\mathscr{E} \downarrow T(\mathrm{pt}))$. From Example 4.18, we have seen that the pair $(\mathscr{E}^0, T^0) := (\mathbf{Set}, \mathrm{id})$ is suitable, and this will serve as the base of our inductive definition of suitable pairs $(\mathscr{E}^n, T^n)$ as iterative slices. More precisely, define $\mathscr{E}^{n+1} := (\mathscr{E}^n)_{\mathrm{pt}}^+ \cong (\mathscr{E}^n \downarrow T^n(\mathrm{pt}))$ and $T^{n+1} := (T^n)_{\mathrm{pt}}^+$.

Proposition 4.22. *For every $n \geq 0$, there is a unique set O_n, up to bijection, such that $\mathscr{E}^n \cong (\mathbf{Set} \downarrow O_n)$. Call O_n the set of n-*opetopes.

Proof. This is proven in [31, §7.1], but we will recount the proof here to help describe the elements of O_n later. Note that uniqueness follows from objects being determined up to isomorphism by the isomorphism class of their slice category.

The proposition is certainly true when $n = 0$ with $O_0 := \mathrm{pt}$. Inductively, suppose $\mathscr{E}^n \cong (\mathbf{Set} \downarrow O_n)$ for some $n \geq 0$ and consider $\mathscr{E}^{n+1} \cong (\mathscr{E}^n \downarrow T^n(\mathrm{pt}))$. By being a slice category, $\mathscr{E}^n$ has a terminal object given by $\mathrm{id}_{O_n} : O_n \rightarrow O_n$, so consider the object $t := T^n(\mathrm{pt}) = T^n(\mathrm{id}_{O_n})$ in $(\mathbf{Set} \downarrow O_n)$ and define O_{n+1} to be the domain of t as a function in **Set**. Then, $\mathscr{E}^{n+1} \cong (\mathscr{E}^n \downarrow t) \cong ((\mathbf{Set} \downarrow O_n) \downarrow t)$ is isomorphic to the category whose objects are commutative triangles

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & O_{n+1} \arrow[dl, "t"] \\ & O_n & \end{tikzcd}$$

for X some set. The morphisms of this category are functions $X \rightarrow X'$ which lie over t in the above triangle. Commutativity forces the map $X \rightarrow O_n$ in the triangle to be uniquely determined as the composite of the map $X \rightarrow O_{n+1}$ and t, which means that the forgetful functor $((\mathbf{Set} \downarrow O_n) \downarrow t) \rightarrow (\mathbf{Set} \downarrow O_{n+1})$ is in fact an isomorphism of categories, establishing that $\mathscr{E}^{n+1} \cong (\mathbf{Set} \downarrow O_{n+1})$, as desired. $\square$

The above proof provides an explicit induction for constructing n-opetopes via $O_{n+1} := T^n(O_n)$, as we can retrace through the details of the proof of Lemma 4.20 to understand how the monad T^n acts. Additionally, the above proof provides a canonical map $t : O_{n+1} \rightarrow O_n$ for all $n \geq 0$.

At dimension zero, we have $\mathscr{E}^0 = \mathbf{Set}$, $T^0 = \mathrm{id}$, and $O_0 = \mathrm{pt}$ by definition, meaning we have only one 0-opetope. This also means $O_1 = T^0(O_0) = \mathrm{pt}$, which in turn implies that $\mathscr{E}^1 \cong (\mathbf{Set} \downarrow \mathrm{pt}) \cong \mathbf{Set}$ is the usual cartesian monoidal category of sets. The monoid objects in $\mathscr{E}^1$ are just the usual set-theoretic monoids, so $T^1 = (-)^*$ recovers the Kleene star or "free monoid" monad.

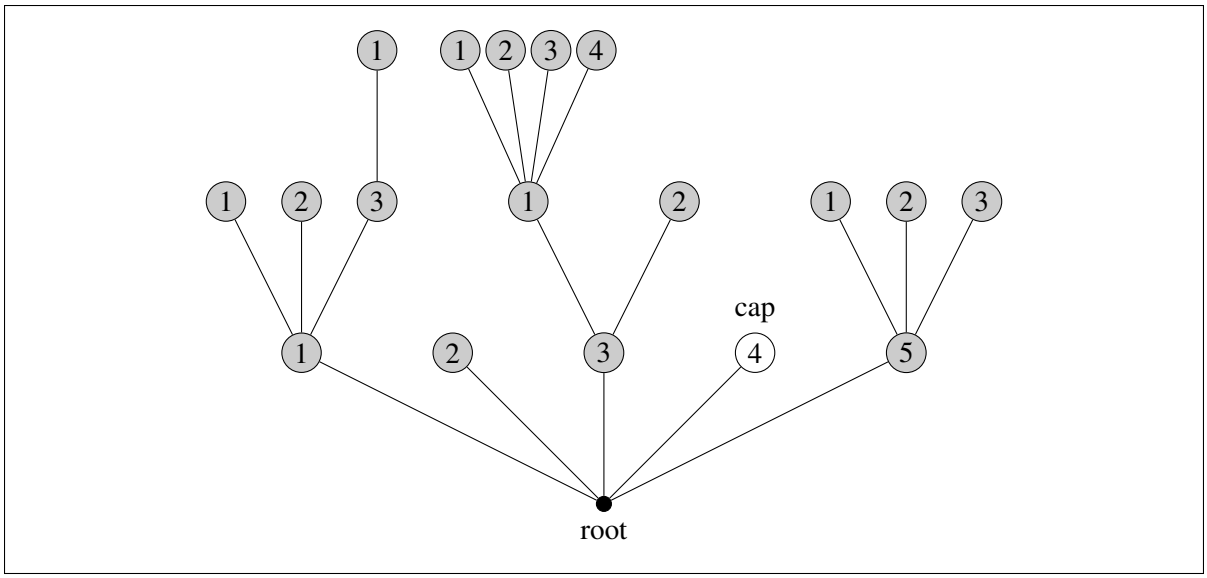


Figure 4.2: Example of an oriented capped rooted tree

Moving to dimension two, $O_2 := T^1(O_1) = \mathrm{pt}^* \cong \mathbb{N}_0$ is the monoid of nonnegative integers under addition, and therefore $\mathscr{E}^2 = (\mathbf{Set} \downarrow \mathbb{N}_0)$ is the category of graded sets. The monoidal product on $\mathscr{E}^2$ is given on graded sets $X = (X_p)_{p\geq 0}$ and $Y = (Y_p)_{p\geq 0}$ by

$$(X \otimes Y)_p := \bigcup_{\substack{q_1+\cdots+q_k=p \\ k\geq 0}} X_{q_1} \times \cdots \times X_{q_k} \times Y_k$$

We can interpret the degree-p elements of the graded set X as circuit gates with p inputs and one output, then $X \otimes Y$ consists of gates $y \in Y_k$ for some $k \geq 0$ with each of their k input nodes wired to the output node of some gate in X, and the degree is determined by the resulting number of (open) input nodes of this circuit. As in Example 4.19, the monoids in $\mathscr{E}^2$ are the plain operads. This interpretation realises the gates as abstract operators, and wiring gates together yields other operators such that this process is associative and unital. Therefore, T^2 is the "free operad" monad that takes a graded set X and produces the operad whose operators are freely generated by the gates of X by wiring them together.

Under this interpretation, $O_2 \cong \mathbb{N}_0$ is the graded set with a unique gate of each arity, and so by thinking of each such gate as a vertex, $O_3 = T^2(O_2)$ can be identified with the set of *oriented capped rooted trees*, with the root being the unique output node and the leaves being input nodes. The caps on a rooted trees are just labels on a subset of their leaves to indicate that a rooted tree cannot be connected there, which corresponds to the nullary gate $0 \in \mathbb{N}_0 \cong O_2$. The orientation is given by a fixed enumeration of the child nodes for each node of the tree, which corresponds to fixing an order of the inputs for the corresponding gate that the node represents. The canonical map $t : O_3 \to O_2$ maps an oriented capped rooted tree to its number of uncapped leaves. Figure 4.2 gives an example of an oriented rooted tree with a single cap and twelve uncapped leaves. The root is placed at the bottom of the tree to suggest that this is the output, while the leaves are the inputs, corresponding to the fact that

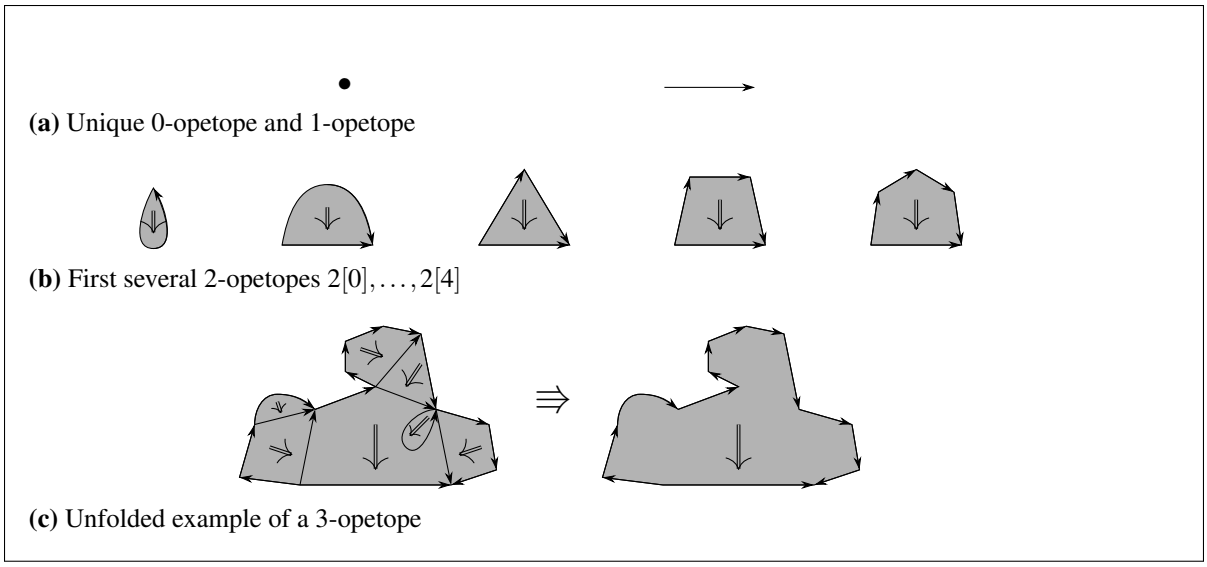


Figure 4.3: Visualisation of opetopes

its target 2-opetope is the twelve-input gate.

Alternatively, we can use dual diagrams to view the p-ary abstract operator in O_2 instead as a $(p+1)$-gon with p input edges and one output edge. Then, O_3 may be identified with the set of polyhedra with various polygonal input faces and a single output face. The benefit of this visualisation of opetopes as polytopes is that it corresponds more readily to diagrams of (higher) morphisms: for instance, O_2 corresponds to the set of multiary composition diagram shapes of Figure 4.1.

Inductively, given that an n-opetope γ for $n > 0$ is determined by its configuration of input facets $s_k\gamma \in O_{n-1}$ with k in some index set (which produces a single output facet $t\gamma \in O_{n-1}$), consider the following procedure:

- Start with a finite sequence $\gamma_1, \ldots, \gamma_m$ of n-opetopes for some integer $m \geq 0$.
- For each $1 < j \leq m$, glue the output facet $t\gamma_j$ of γ_j to an input facet $s_k\gamma_i$ of an opetope γ_i for some $i < j$ such that $s_k\gamma_i = t\gamma_j$. If this is impossible, declare that this procedure failed; otherwise, say that this procedure is successful.

Then, O_{n+1} is the collection of all configurations Γ of n-opetopes obtained from successful runs of the above procedure. Any such configuration Γ yields from the unglued input facets a configuration of $(n-1)$-opetopes, which by induction determines a unique n-opetope. This n-opetope is taken to be the output facet $t\Gamma$, while the n-opetopes in the configuration define the input facets $s_k\Gamma$, and so we are allowed to continue this procedure inductively. Note that taking the output facet defines the canonical map $t : O_{n+1} \to O_n$ of Proposition 4.22.

Just as the name suggests, this interpretation allows us to view n-opetopes as directed n-dimensional polytopes that encode abstract operators by having a prescribed output facet, and therefore as an unbiased and nondegenerate generalisation of the abstract n-simplices of Section 3.2. A visualisation of

opetopes in the first few dimensions that is similar to that for simplices is given in Figure 4.3. In particular, the example of a 3-opetope in Figure 4.3c is the dual polytope corresponding to the rooted tree in Figure 4.2.

We compile these opetopes into a category analogous to that of Cheng in [11, §1.4]:

Definition 4.23. For an n-opetope $\gamma \in O_n$ with $n > 0$, recall that it is determined by its configuration of input $(n-1)$-opetope facets $s_k\gamma$, and has a unique output $(n-1)$-opetope facet $t\gamma$. Define the *category of opetopes* $\mathbf{O}$ to be the category whose set of objects is $\coprod_{n\geq 0} O_n$ and whose morphisms are generated by

- *target coface maps* $t : t\gamma \to \gamma$ for all $\gamma \in O_n$ with $n > 0$, and
- *source coface maps* $s^\delta : \delta \to \gamma$ for every input opetope $\delta = s_k\gamma$ of $\gamma \in O_n$ with $n > 0$

subject to the following *opetopic identities*: for any two input $(n-1)$-opetopes δ, δ' of γ wherein the output facet $\varepsilon := t\delta$ of δ is glued to the input facet $s^\varepsilon : \varepsilon \to \delta'$ of δ', we have commutativity of the square

$$\begin{tikzcd} \varepsilon \arrow[r, "s^\varepsilon"] \arrow[d, "t"'] & \delta' \arrow[d, "s^{\delta'}"] \\ \delta \arrow[r, "s^\delta"'] & \gamma \end{tikzcd}$$

Write $\dim\gamma := n$ if $\gamma \in O_n$, then the coface maps all decrease dimension by exactly one.

Note that the coface maps in Cheng's original category of opetopes defined in [11, §1.4] are moreover subject to several other compatibility constraints on top of the opetopic identities described above. This is because her definition of opetopes via generalised symmetric operads requires the coface maps to reflect input insensitivities to permutation up to isomorphism. As our generalised operads are nonsymmetric, this is unnecessary in our case. The generating coface maps of $\mathbf{O}$ correspond to the simplicial coface maps $d^i : [n] \to [n+1]$ of Δ, and the asserted opetopic identities likewise correspond to the simplicial identities of the form $d^j d^i = d^i d^{j-1}$ for $i < j$.

We can also realise the visual intuition for opetopes provided in Figure 4.3 formally with the shape functor $J : \mathbf{O} \to \mathbf{Top}$ from [31, §7.4]:

Definition 4.24. Let $\mathbf{O}_{\leq n}$ be the full subcategory of $\mathbf{O}$ on opetopes γ with $\dim\gamma \leq n$. Define the functors $J_n : \mathbf{O}_{\leq n} \to \mathbf{Top}$ inductively as follows. Set $J_0 : \mathbf{O}_{\leq 0} = \mathbf{1} \to \mathbf{Top}$ to map to the singleton space. Given $J_n : \mathbf{O}_{\leq n} \to \mathbf{Top}$ for some $n \geq 0$, we have from Section 3.3 an induced nerve-realisation adjunction

$$R_n : \mathbf{PSh}(\mathbf{O}_{\leq n}) \rightleftarrows \mathbf{Top} : \mathrm{Hom}_{\mathbf{O}_{\leq n}}(J_n, -)$$

where in particular $R_n(\mathrm{Hom}_{\mathbf{O}_{\leq n}}(-,\delta)) \cong J_n(\delta)$ for every δ in $\mathbf{O}_{\leq n}$. Extend J_n to $\mathbf{O}_{\leq n+1}$ to define a functor $J_{n+1} : \mathbf{O}_{\leq n+1} \to \mathbf{Top}$ by setting

$$J_{n+1}(\gamma) := \mathrm{Cone}\left(R_n(\mathrm{Hom}_{\mathbf{O}}(-,\gamma)|_{\mathbf{O}_{\leq n}})\right)$$

for $\gamma \in O_{n+1}$, where $\mathrm{Cone}(X)$ of a space X is the contractible space $\frac{X\times[0,1]}{X\times\{1\}}$. For a face map $d:\delta\to\gamma$ in $\mathbf{O}_{\leq n+1}$ with $\dim\delta = n$ and $\dim\gamma = n+1$, define $J_{n+1}(d)$ to be the composite

$$J_{n+1}(\delta) \xrightarrow{\sim} R_n\left(\mathrm{Hom}_{\mathbf{O}}(-,\delta)|_{\mathbf{O}_{\leq n}}\right) \xrightarrow{R_n(d_*)} R_n\left(\mathrm{Hom}_{\mathbf{O}}(-,\gamma)|_{\mathbf{O}_{\leq n}}\right) \hookrightarrow J_{n+1}(\gamma)$$

Note that $\mathbf{O} = \varinjlim_n \mathbf{O}_{\leq n}$ is the colimit of the inclusions $\mathbf{O}_{\leq n} \subset \mathbf{O}_{\leq n+1}$. Therefore, the above family of functors $J_n : \mathbf{O}_{\leq n} \to \mathbf{Top}$ induces a unique co-opetopic space $J : \mathbf{O} \to \mathbf{Top}$.

Proposition 4.25. *$J(\gamma)$ is a contractible CW complex for every opetope γ. Moreover, if $\delta\to\gamma$ is a map of opetopes with $\dim\delta < \dim\gamma$, then $J(\delta)\to J(\gamma)$ is an inclusion of $J(\delta)$ as a subcomplex of $\partial J(\gamma)$.*

Proof. It is clear from the definition that $J(\gamma)$ is contractible for all opetopes γ. We prove the remainder of the proposition by induction on $J_n : \mathbf{O}_{\leq n} \to \mathbf{Top}$. This is trivial for $n = 0$, so suppose the proposition is true for some $n \geq 0$. For $\gamma \in O_{n+1}$, we have that

$$B := R_n\left(\mathrm{Hom}_{\mathbf{O}}(-,\gamma)|_{\mathbf{O}_{\leq n}}\right) = \int^{\delta\in(\mathbf{O}_{\leq n})_0} \mathrm{Hom}_{\mathbf{O}}(\delta,\gamma)\times J_n(\delta)$$

where the hom-set is given the discrete topology. B is thus a quotient of the disjoint union of CW complexes $\{d\}\times J_n(\delta)$ with $\delta\in(\mathbf{O}_{\leq n})_0$ and $d:\delta\to\gamma$.

For any map $e:\varepsilon\to\varepsilon'$ in $\mathbf{O}_{\leq n}$, we have by induction that $J_n(\varepsilon)$ is a subcomplex of $\partial J_n(\varepsilon')$ via e. The quotient B then identifies for any $d:\varepsilon'\to\gamma$ the complex $\{de\}\times J_n(\varepsilon)$ of B with the subcomplex $\{d\}\times J_n(\varepsilon)\subset\partial(\{d\}\times J_n(\varepsilon'))$. Therefore, B is the result of gluing CW complexes along boundary subcomplexes, and is thus a CW complex also. This shows that $J_{n+1}(\gamma) := \mathrm{Cone}(B)$ is indeed a CW complex.

For a face map $d:\delta\to\gamma$ of γ with $\dim\delta = n$, the map $J_{n+1}(d)$ is the composite of the subcomplex inclusions $J_n(\delta)\cong\{d\}\times J_n(\delta)\subset B$ and the inclusion $B\hookrightarrow\mathrm{Cone}(B)$ which factors through $\partial(\mathrm{Cone}(B))$. Therefore, the proposition holds for J_{n+1} as well. By induction, and taking the colimit, this proves the proposition for $J:\mathbf{O}\to\mathbf{Top}$. □

As opetopes are far more structurally involved than their simplicial analogue, they are also difficult to label. As we will primarily be dealing with opetopes of dimension at most three, we will establish some notation for symbolically referring to them. The notation for 3-dimensional opetopes will be cumbersome, but we will largely only refer to a special subfamily of 3-opetopes which are of depth two when viewed as capped rooted trees.

There is a unique 0-opetope and 1-opetope, so denote these by 0 and 1, respectively. In dimension two, each 2-opetope corresponds to a natural number p, so denote by $2[p]$ the 2-opetope with p input edges. For 3-opetopes, view them as oriented capped rooted trees and produce the name by traversing the tree with a depth-first search as follows. Do not label uncapped leaf nodes, and then inductively label a node by its number of input edges followed by a comma-separated list of labels of its child subtrees (in the order given by their orientation) wrapped in parentheses. The array $\vec{p}$ of numbers obtained for the overall 3-opetope is then given by the label for its root, and we then denote the 3-opetope by $3[\vec{p}]$. If

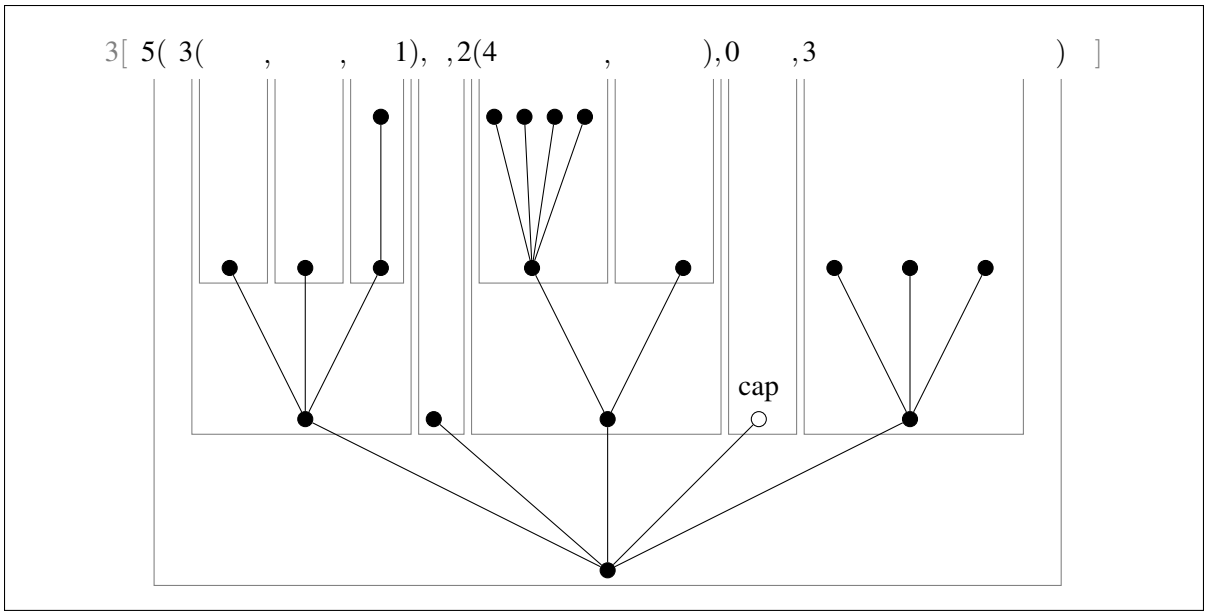


Figure 4.4: Example of labelling a capped rooted tree

all input subtrees of a vertex are given by leaf nodes, then the vertex may simply be labelled by its arity without traversing further. This process is illustrated in Figure 4.4, where the orientation of the vertices is the same as in Figure 4.2.

Chapter 5

Opetopic Nerves

Now that we have defined the shape category $\mathbf{O}$ of opetopes, we can replace the simplex category Δ for the nerve-based definition of higher categories. We will see that much of the construction of Section 4.2 readily translates to the opetopic case, using an analogue of the simplicial Segal condition. However, due to the deliberate removal of degeneracies, the Segal condition alone will be insufficient, so we will also need another constraint on the structure morphisms to characterise opetopic higher categories.

We will start by establishing an opetopic analogue of the Grothendieck nerve, and prove a similar result to Lemma 4.2, which we can define via a co-opetopic category $S : \mathbf{O} \to \mathbf{Cat}$ that forgets all but the 1-dimensional structure of the opetopes.

Definition 5.1. For an opetope γ, define the set $\{\gamma\}_1$ inductively on $\dim \gamma$ as follows.

- Set $\{0\}_1 := \varnothing$ and $\{1\}_1 := \mathrm{Hom}_{\mathbf{O}}(1,1) = \mathrm{pt}$.
- For the 2-opetope $2[p]$, set $\{2[p]\}_1 := \mathrm{Hom}_{\mathbf{O}}(1, 2[p]) \setminus \{t\}$ to be its set of source 1-opetopes.
- If $\dim \gamma > 2$, then set $\{\gamma\}_0 := \{t\gamma\}_0$ and $\{\gamma\}_1 := \{t\gamma\}_1$.

Call$\{\gamma\}_1$ the set of *input* 1-*opetopes* of γ, and denote by $|\gamma|_1$ the cardinality of $\{\gamma\}_1$.

Proposition 5.2. *The input* 1-*opetopes induce a linear order on the* 0-*opetopes* $0 \to \gamma$ *of* γ *by asserting that* $0 \xrightarrow{s} 1 \to e$ *to be less than* $0 \xrightarrow{t} 1 \to e$ *for any* $e \in \{\gamma\}_1$. *Moreover, for a map* $\delta \to \gamma$, *the induced map* $\mathrm{Hom}_{\mathbf{O}}(0,\delta) \to \mathrm{Hom}_{\mathbf{O}}(0,\gamma)$ *of* 0-*opetopes is monotone.*

Proof. The linear order is trivial for 0 and 1. For 2-opetopes, the linear order follows by construction, as $2[p]$ is constructed from gluing p many 1-opetopes source-to-target (cf. Figure 5.1). That maps $\delta \to \gamma$ preserve the above ordering on 0-opetopes when $\dim \gamma \leq 2$ is then clear by design. In particular, the ordering on 0-opetopes is preserved for target coface maps $t : 1 \to 2[p]$.

If $\dim \gamma \geq 2$, note that a 1-opetope $1 \to \gamma$ is either an input 1-opetope, or the target of some coface $2[p] \to \gamma$. This is clear if $\dim \gamma = 2$. If $\dim \gamma > 2$ and an input 1-opetope of some source facet of γ is not glued to the target 1-opetope of another source facet, then it is a 1-opetope of $t\gamma$ by the definition of $t\gamma$. By induction this 1-opetope $1 \to t\gamma$ is either an input 1-opetope of $t\gamma$ (and thus of γ), or is the target 1-opetope of some coface $2[p] \to t\gamma \to \gamma$.

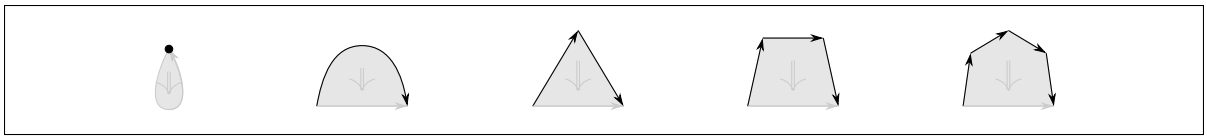


Figure 5.1: Chains of input 1-opetopes in the first several 2-opetopes

Suppose $\delta \to \gamma$ and let $e \in \{\delta\}_1$. Let $x : 0 \xrightarrow{s} 1 \xrightarrow{e} \delta$ and $y : 0 \xrightarrow{t} 1 \xrightarrow{e} \delta$, then $x < y$ in δ. If e is an input 1-opetope of γ, then $x < y$ in γ by definition. Otherwise, by the previous paragraph, e is the target 1-opetope of some coface $2[p] \to \gamma$. In this case, note that $x < y$ in $2[p]$ as well, because the target map $t : 1 \to 2[p]$ induces a monotone map on 0-opetopes. If the source 1-opetopes of $2[p] \to \gamma$ are input 1-opetopes of γ, then this implies $x < y$ in γ. Otherwise, we may recurse on the 1-opetopes of $2[p] \to \gamma$ that are not input 1-opetopes of γ, as they are target 1-opetopes of other 2-dimensional cofaces of γ. As γ consists of only finitely many opetopes of smaller dimension, this recursion must eventually terminate at input 1-opetopes, after which we find that $x < y$ in γ. $\square$

Definition 5.3. Define the functor $S : \mathbf{O} \to \mathbf{Cat}$ with $S(\gamma) := [|\gamma|_1]$ set as the chain $\{0 < \cdots < |\gamma|_1\}$ of 0-opetopes of γ, and $S(\delta) \to S(\gamma)$ the corresponding inclusion of chains for any map $\delta \to \gamma$. Using the general construction in Section 3.3, S induces an adjunction $|-| : \mathbf{PSh}(\mathbf{O}) \rightleftarrows \mathbf{Cat} : \mathrm{N}$ between small categories and opetopic sets. The right adjoint is the *opetopic nerve*, which is given on a category $\mathscr{C}$ by $(\mathrm{N}\mathscr{C})_\gamma := \mathrm{Func}(S(\gamma), \mathscr{C})$.

As in the case of the Grothendieck nerve, we have for the opetopic nerve of a category $\mathscr{C}$ that $(\mathrm{N}\mathscr{C})_\gamma = \mathscr{C}_1^{\times_{\mathscr{C}_0} |\gamma|_1}$, meaning that the γ-cells of the opetopic nerve of $\mathscr{C}$ are the length-$|\gamma|_1$ strings of composable morphisms in $\mathscr{C}$. In particular, $(\mathrm{N}\mathscr{C})_0 = \mathscr{C}_0$ and $(\mathrm{N}\mathscr{C})_1 = \mathscr{C}_1$ as before.

We can also describe the opetopic nerve of $\mathscr{C}$ more explicitly: for any opetope γ, consider its 1-skeleton, which is the directed graph obtained by remembering only the vertices (0-opetopes) and edges (1-opetopes) of γ, then define $(\mathrm{N}\mathscr{C})_\gamma$ to be the set of commutative diagrams in $\mathscr{C}$ whose shape is given by the 1-skeleton of γ. The coface maps $\delta \to \gamma$ induce inclusions of 1-skeleta and thus provide a natural function $(\mathrm{N}\mathscr{C})_\gamma \to (\mathrm{N}\mathscr{C})_\delta$ by sending such a commutative diagram to the subdiagram corresponding to the sub-1-skeleton of δ in γ. Due to the commutativity constraint on the cells in $(\mathrm{N}\mathscr{C})_\gamma$, each diagram is completely determined by the morphisms chosen for the source 1-opetopes of γ, and so we recover the same definition $(\mathrm{N}\mathscr{C})_\gamma = \mathscr{C}_1^{\times_{\mathscr{C}_0} |\gamma|_1}$ as before. This has an obvious generalisation to internal categories, and leads to an analogue of the strict Segal condition of Lemma 4.2.

However, a strict opetopic Segal condition is insufficient to characterise the essential image of the above nerve construction. For example, fix a monoid M (in **Set**) and consider the set $\widetilde{M} := \{0,1\} \times M$ equipped with the unbiased product $\prod_{i=1}^p (b_i, m_i) := \left(0, \prod_{i=1}^p m_i\right)$. This almost defines an associative and unital multiplication on $\widetilde{M}$, except $\prod^1 (1, m) = (0, m)$ for all $m \in M$ shows that the unary product is not well-behaved. However, we can still follow the above nerve construction and define a functor $A : \mathbf{O}^{\mathrm{op}} \to \mathbf{Set}$ by taking A_γ to be the set of length-$|\gamma|_1$ strings of elements of $\widetilde{M}$ and the face maps on A to be given by multiplying the appropriate elements of a string together. We can see by construction

then that $A_0 = \mathrm{pt}$ and $A_1 = \widetilde{M}$, and so $A_\gamma = \widetilde{M}^{|\gamma|_1} = A_1^{\times_{A_0}|\gamma|_1}$ despite not A being isomorphic to the nerve of a category.

The reason for the insufficiency of the Segal condition is the lack of codegeneracy maps in $\mathbf{O}$. Recall that the codegeneracy maps in Δ were necessary to provide units for nerves (cf. Lemma 4.2). The problem with this incarnation of units is that the resulting units are preserved too strictly in general. As identity morphisms should instead arise from nullary composition, $\mathbf{O}$ is already capable of providing them from the Segal condition on the opetope $2[0]$; however, there are no restrictions on the unary operations obtained from the Segal condition on the opetope $2[1]$ to act essentially trivially. This issue can be resolved by simply asserting that every morphism arises as a unary composite of some other morphism: for instance, we can see that $\prod^1$ on $\widetilde{M}$ in the previous example is not surjective. In the simplicial case, this surjectivity is automatic by the simplicial identity $d_i s_i = \mathrm{id}$, which ensures that the face maps of a simplicial set are already split epic.

Lemma 5.4. *An opetopic set $A : \mathbf{O}^{\mathrm{op}} \to \mathbf{Set}$ is isomorphic to the opetopic nerve of a category if and only if it satisfies the following:*

- *(Segal condition) The canonical map $A_\gamma \to A_1^{\times_{A_0}|\gamma|_1}$ (which we will refer to as a* Segal map*) is an isomorphism for all opetopes γ.*
- *(Unary condition) The target face map $t : A_{2[1]} \to A_1$ is an isomorphism.*

Proof. That the opetopic nerve of a category satisfies the Segal and unary conditions follows by definition. Suppose $A : \mathbf{O}^{\mathrm{op}} \to \mathbf{Set}$ satisfies the above conditions, then we will show that $A \cong \mathrm{N}\mathscr{C}$ for some category $\mathscr{C}$. Define $\mathscr{C}_0 := A_0$ and $\mathscr{C}_1 := A_1$ with the source and target maps defined by the corresponding source and target cofaces $0 \rightrightarrows 1$ in $\mathbf{O}$. Define composition in an unbiased manner: for any $p \geq 0$, define p-ary composition on $\mathscr{C}$ by $c^p : A_1^{\times_{A_0} p} \leftarrow A_{2[p]} \xrightarrow{t} A_1$, inverting the left-facing arrow by the Segal condition.

To see that this composition is associative in the unbiased sense, note that the opetopic identities ensure that the following diagram commutes for all $p \geq 0$ and $q_1, \ldots, q_p \geq 0$:

$$\begin{tikzcd}
\prod_{i=1}^{p} A_0 A_1^{\times_{A_0} q_i} & & A_{2[\sum_i q_i]} \arrow[ll, "\sim"'] \arrow[ddr, "t"] & \\
\prod_{i=1}^{p} A_0 A_{2[q_i]} \arrow[u, "\wr"] \arrow[d, "t^{\times_{A_0} p}"'] & & A_{3[p(\vec{q})]} \arrow[ll, "(s^{2[q_i]})_i"] \arrow[ul, "\sim"] \arrow[u, "t"] \arrow[d, "s^{2[p]}"'] & A_1 \\
A_1^{\times_{A_0} p} & & A_{2[p]} \arrow[ll, "\sim"] \arrow[ur, "t"'] &
\end{tikzcd}$$

The indicated isomorphisms are given by the Segal maps, so we can invert these and obtain the commu-

tative diagram

$$\begin{tikzcd}
\prod_{i=1}^{p} {}_{A_0}A_1^{\times_{A_0} q_i} \arrow[r, "\sim"] \arrow[d, "\wr"'] \arrow[dr, "\sim"] & A_{2[\sum_i q_i]} \arrow[dr, "t"] & \\
\prod_{i=1}^{p} {}_{A_0}A_{2[q_i]} \arrow[d, "t"'] & A_{3[p(\vec{q})]} \arrow[u, "t"] \arrow[d, "s"'] & A_1 \\
A_1^{\times_{A_0} p} \arrow[r, "\sim"'] & A_{2[p]} \arrow[ur, "t"'] &
\end{tikzcd} \tag{Eq. 5.1}$$

The lower path $\prod_{i=1}^{p} {}_{A_0}A_1^{\times_{A_0} q_i} \to A_1$ composes morphisms of $\mathscr{C}$ via $c^p(c^{q_1}, \ldots, c^{q_p})$ while the upper path composes the morphisms all at once with $c^{\sum_i q_i}$, so commutativity here proves that the family of compositions on $\mathscr{C}$ is associative.

It remains to check that unary composition $c^1 : A_1 \xrightarrow{s^{-1}} A_{2[1]} \xrightarrow{t} A_1$ acts by the identity. This follows from $t : A_{2[1]} \to A_1$ being epic: by the associativity proven above with $p = q_1 = 1$, we have $c^1 c^1 = c^1$; that is, $ts^{-1}ts^{-1} = ts^{-1}$, which implies $ts^{-1}t = t$. As t is an epimorphism, this means $s^{-1}t = \mathrm{id}$, from which the invertibility of s^{-1} gives $c^1 = ts^{-1} = \mathrm{id}$. We can then recover $\mathscr{C}$ as a (biased) category in $\mathscr{E}$ by taking $c := c^2$ and $e := c^0$, and the Segal maps provide the natural isomorphisms $A_\gamma \to (\mathrm{N}\mathscr{C})_\gamma$, as desired. □

Remark 5.5. The proof reveals that the unary condition can be relaxed to just asserting that the target face map is an epimorphism.

With functors being determined by their action on objects and morphisms, the above nerve construction naturally extends to a faithful functor $\mathrm{N} : \mathbf{Cat} \to \mathrm{Func}(\mathbf{O}^{\mathrm{op}}, \mathbf{Set})$. As opetopic maps between nerves must respect the structure maps, they will respect the Segal maps, and thus also their inverses, meaning that opetopic maps between nerves are automatically functorial as well. Therefore, Lemma 5.4 describes **Cat** as a full subcategory of $\mathrm{Func}(\mathbf{O}^{\mathrm{op}}, \mathbf{Set})$.

Remark 5.6. We can define an analogue of opetopic nerves of categories internal to a finitely complete category $\mathscr{E}$ in a similar way as in Section 4.1. Given $\mathscr{C} \in \mathbf{Cat}(\mathscr{E})_0$, define $(\mathrm{N}\mathscr{C})_\gamma := \mathscr{C}_1^{\times_{\mathscr{C}_0} |\gamma|_1}$, and the maps $(\mathrm{N}\mathscr{C})_\gamma \to (\mathrm{N}\mathscr{C})_\delta$ correspond to composition in $\mathscr{C}$ for all $\delta \to \gamma$. Then, Lemma 5.4 generalises to realise $\mathbf{Cat}(\mathscr{E})$ as a full subcategory of $\mathrm{Func}(\mathbf{O}^{\mathrm{op}}, \mathscr{E})$. In particular, by taking $\mathscr{E} := n\mathbf{Cat}$, we find that strict $(n+1)$-categories correspond to functors $A : \mathbf{O}^{\mathrm{op}} \to n\mathbf{Cat}$ such that

- A_0 is a discrete n-category.
- $A_\gamma \to A_1^{\times_{A_0} |\gamma|_1}$ is an isomorphism for all opetopes γ.
- $t : A_{2[1]} \to A_1$ is an isomorphism.

5.1 Unbiased Double Nerve

We can generalise the idea of taking cells of $\mathrm{N}\mathscr{C}$ as commutative diagrams in $\mathscr{C}$ to the context of unbiased bicategories as well to yield an analogue of Section 4.2.1. Let $\mathscr{B}$ be an unbiased bicategory, then its nerve should be a functor $\mathrm{N}\mathscr{B} : \mathbf{O}^{\mathrm{op}} \to \mathbf{Cat}$ (functors of this type may be called *opetopic categories*) whose γ-cells for an opetope γ are diagrams that commute up to specified 2-isomorphisms, where the shape of these diagrams of morphisms and 2-isomorphisms correspond to the 2-skeleton of γ.

Definition 5.7. Define the unbiased double nerve functor $\mathrm{N} : \mathbf{UBicat} \to \mathrm{Func}(\mathbf{O}^{\mathrm{op}}, \mathbf{Cat})$ as follows.

Consider families in $\mathscr{B}$ of the form $(x_\bullet, f_\bullet, \theta_\bullet)$ with objects $x_i \in \mathscr{B}_0$ for every vertex $i : 0 \to \gamma$, morphisms $f_j : x_{sj} \to x_{tj}$ for every edge $j : 1 \to \gamma$, and 2-isomorphisms $\theta_k : \bigotimes_{i=1}^p f_{s^i k} \overset{\sim}{\Rightarrow} f_{tk}$ for every face $k : 2[p] \to \gamma$. For every 3-opetope $\delta \to \gamma$, the above family produces a diagram of 2-isomorphisms in $\mathscr{B}$ via the various faces $k : 2[p] \to \delta \to \gamma$, so take the objects of $(\mathrm{N}\mathscr{B})_\gamma$ to be those families $(x_\bullet, f_\bullet, \theta_\bullet)$ where every diagram of 2-isomorphisms induced by some 3-opetope $\delta \to \gamma$ commutes. Just as with the simplicial double nerve, take the morphisms $(x_\bullet, f_\bullet, \theta_\bullet) \to (x_\bullet, f'_\bullet, \theta'_\bullet)$ in $\mathrm{N}\mathscr{B}$ to be families of 2-morphisms $\xi_j : f_j \Rightarrow f'_j$ such that $\xi_{tk} \circ \theta_k = \theta'_k \circ \bigotimes_{i=1}^p \xi_{s^i k}$ for every face $k : 2[p] \to \gamma$. With this construction, the face maps become apparent, and produce the desired opetopic category $\mathrm{N}\mathscr{B}$.

Given bicateogires $\mathscr{B}, \mathscr{C}$ and a pseudofunctor $F : \mathscr{B} \to \mathscr{C}$, we can define its nerve $\mathrm{N}F : \mathrm{N}\mathscr{B} \to \mathrm{N}\mathscr{C}$ to be the opetopic functor that sends a γ-cell $(x_\bullet, f_\bullet, \theta_\bullet)$ in $\mathscr{B}$ to the γ-cell $(Fx_\bullet, Ff_\bullet, \vartheta_\bullet)$, where the 2-isomorphisms are given by the composite $\vartheta_k : \bigotimes_{i=1}^p Ff_{s^i k} \overset{F^p}{\Rightarrow} F\left(\bigotimes_{i=1}^p f_{s^i k}\right) \overset{F(\theta^p)}{\Longrightarrow} Ff_{tk}$. The latter is a well-defined γ-cell in $\mathscr{C}$ by the pseudofunctoriality of F.

As before, it follows from the definition that $(\mathrm{N}\mathscr{B})_0 = \mathrm{disc}\,\mathscr{B}_0$ and $(\mathrm{N}\mathscr{B})_1 = \coprod_{x,y \in \mathscr{B}_0} \mathrm{Hom}_{\mathscr{B}}(x,y)$, and we recover the opetopic nerve of the previous section if $\mathscr{B}$ were in fact a 1-category. The Segal maps $(\mathrm{N}\mathscr{B})_\gamma \to (\mathrm{N}\mathscr{B})_1^{\times_{(\mathrm{N}\mathscr{B})_0} |\gamma|_1}$ send a γ-cell $(x_\bullet, f_\bullet, \theta_\bullet)$ to the composable string of morphisms f_j corresponding to the source 1-opetopes j of γ. Reasoning exactly as with the simplicial double nerve, the Segal maps will be genuinely surjective and fully faithful, and thus in particular will be equivalences of categories. Moreover, since $(\mathrm{N}\mathscr{B})_{2[1]}$ is the category of 2-isomorphisms and commutative squares of 2-morphisms in $\mathscr{B}$, the map $t : (\mathrm{N}\mathscr{B})_{2[1]} \to (\mathrm{N}\mathscr{B})_1$ that sends a 2-isomorphism to its codomain can be easily seen to be strictly surjective. Again by the invertibility of the 2-morphisms in $2[1]$-cells, the map is also fully faithful and thus an equivalence of categories as well.

The goal now is to understand how to reverse engineer this process; that is, recognise which opetopic categories are levelwise equivalent to the unbiased double nerve of a bicategory, and recover the underlying bicategory. Refer back to Lemma 5.4: the Segal maps there were asserted to be isomorphisms into a limit. As limits are defined up to unique isomorphism, this allows us to reinterpret the strict Segal condition as asserting that the object A_γ with its structure morphisms into the input 1-faces and vertices forms a universal cone; that is, A_γ *is* the limit $A_1^{\times_{A_0} |\gamma|_1}$. This perspective makes the commutativity of the

diagram

$$\begin{tikzcd}
\prod_{i=1}^{p} {}_{A_0}A_1^{\times_{A_0} q_i} \arrow[r, "\sim"] \arrow[d, "\wr"'] \arrow[dr, "\sim"] & A_{2[\sum_i q_i]} \arrow[dr, "t"] & \\
\prod_{i=1}^{p} {}_{A_0}A_{2[q_i]} \arrow[d, "t"'] & A_{3[p(\vec{q})]} \arrow[u, "t"] \arrow[d, "s"'] & A_1 \\
A_1^{\times_{A_0} p} \arrow[r, "\sim"'] & A_{2[p]} \arrow[ur, "t"'] &
\end{tikzcd}$$

from (Eq. 5.1) more intuitively clear: the pentagon on the lower left commutes by the universal property of $A_{2[p]}$ as a limit via the Segal condition, as it makes the two paths mapping into it from $\prod_{i=1}^{p} {}_{A_0}A_1^{\times_{A_0} q_i}$ necessarily equal; likewise, the universal property of $A_{2[\sum_i q_i]}$ makes the upper triangle commute. Therefore, our goal is to view the Segal condition for double nerves in a similar way so that associativity can follow by universal property as well. This will make the argument more readable, and also easier to generalise to higher dimensions (though the proof will still be quite involved).

5.1.1 Higher Limits

Given an unbiased bicategory $\mathscr{B}$, we have seen that the Segal maps $(\mathrm{N}\mathscr{B})_\gamma \to (\mathrm{N}\mathscr{B})_1^{\times_{(\mathrm{N}\mathscr{B})_0} |\gamma|_1}$ of its nerve are surjective equivalences of categories. Unlike in the 1-dimensional case, this is insufficient for witnessing $(\mathrm{N}\mathscr{B})_\gamma$ as a limit, as limits are not necessarily preserved under equivalences. Therefore, we will study the double nerve with the 2-categorical analogue of a limit, which is only defined up to equivalence rather than isomorphism.

Given a category $\mathscr{C}$, recall that the limit of a diagram $F : \mathscr{J} \to \mathscr{C}$, if it exists, is a representing object for the corresponding limit of representable presheaves $\varprojlim_{j \in \mathscr{J}_0} \mathrm{Hom}_{\mathscr{C}}(-, Fj) : \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$. Note that the limit of a general diagram of sets $G : \mathscr{J} \to \mathbf{Set}$ is just the set of tuples $u_\bullet \in \prod_{j \in \mathscr{J}_0} Gj$ such that $(Gf)(u_j) = u_{j'}$ for every $f : j \to j'$ in $\mathscr{J}$, and this is equivalently the set of natural transformations $\mathrm{const}_{\mathrm{pt}} \Rightarrow G$ from the functor $\mathrm{const}_{\mathrm{pt}} : \mathscr{J} \to \mathbf{Set}$ mapping everything in $\mathscr{J}$ to a singleton. In other words, $\varprojlim G \cong \mathrm{Nat}(\mathrm{const}_{\mathrm{pt}}, G)$. Applying this pointwise to our diagram of representable presheaves for F, we find in general that the limit of a diagram $F : \mathscr{J} \to \mathscr{C}$ is an object $\varprojlim F \in \mathscr{C}_0$ equipped with isomorphisms of sets

$$\mathrm{Hom}_{\mathscr{C}}(x, \varprojlim F) \cong \varprojlim_{j \in \mathscr{J}_0} \mathrm{Hom}_{\mathscr{C}}(x, Fj) \cong \mathrm{Nat}(\mathrm{const}_{\mathrm{pt}}, \mathrm{Hom}_{\mathscr{C}}(x, F))$$

natural in $x \in \mathscr{C}_0$. We can also read off the universal property of $\varprojlim F$ from the above characterisation: $\mathrm{Nat}(\mathrm{const}_{\mathrm{pt}}, \mathrm{Hom}_{\mathscr{C}}(x, F))$ is precisely the set of cones from x to F, and we have a natural correspondence between these and morphisms $x \to \varprojlim F$. In particular, the universal cone with $x = \varprojlim F$ corresponds to the identity endomorphism on $\varprojlim F$, and naturality of this correspondence in x unwinds to say that the morphism $x \to \varprojlim F$ for any cone from x to F is precisely the unique morphism that factors this cone

through the universal cone for $\varprojlim F$.

The representability characterisation of limits generalises to the 2-dimensional setting:

Definition 5.8. Fix a bicategory $\mathscr{B}$, and let $F : \mathscr{J} \to \mathscr{B}$ be a pseudofunctor. The *conical* 2*-limit* of F is defined to be an object $\varprojlim^2 F \in \mathscr{B}_0$ equipped with a pseudonatural equivalence of categories

$$\mathrm{Hom}_{\mathscr{B}}(-, \varprojlim{}^2 F) \simeq \underbrace{\mathrm{Nat}^{\mathrm{ps}}(\mathrm{const}_{\mathbf{1}}, \mathrm{Hom}_{\mathscr{B}}(-,F))}_{\mathrm{Hom}_{\mathrm{Func}^{\mathrm{ps}}(\mathscr{J},\mathbf{Cat})}(\mathrm{const}_{\mathbf{1}},\mathrm{Hom}_{\mathscr{B}}(-,F))} : \mathscr{B}^{\mathrm{op}} \to \mathbf{Cat}$$

By the bicategorical Yoneda Lemma (cf. [23, Lemma 8.3.12]), this uniquely characterises the 2-limit up to equivalence in $\mathscr{B}$, as desired. These are well-known objects, and can be found for instance in [23, §5.1].

We can extract from this definition the 2-universal property of $\varprojlim^2 F$. The objects of the category $\mathrm{Nat}^{\mathrm{ps}}(\mathrm{const}_{\mathbf{1}}, \mathrm{Hom}_{\mathscr{C}}(x,F))$ are the cones from x to F that commute up to coherent specified 2-isomorphism, and the morphisms are families of 2-morphisms between the cone morphisms that commute with the coherence 2-isomorphisms. The equivalence with the category of morphisms $x \to \varprojlim^2 F$ means that every such cone from x is in the essential image of some morphism $x \to \varprojlim^2 F$ that is unique up to unique 2-isomorphism. The image of the identity endomorphism on $\varprojlim^2 F$ gives the 2-universal cone from $\varprojlim^2 F$ to F, and pseudonaturality in x tells us that any cone from x to F factors up to coherent 2-isomorphism through the 2-universal cone via some $x \to \varprojlim^2 F$, and any two such factoring morphisms are uniquely 2-isomorphic from the pseudonatural transformation being a levelwise equivalence.

The 2-universal property will be important when proving coherence for the underlying unbiased bicategories of opetopic 2-nerves, but we can also compute 2-limits much more concretely in **Cat**: suppose $G : \mathscr{J} \to \mathbf{Cat}$ is a pseudofunctor, then $\varprojlim^2 G \simeq \mathrm{Nat}^{\mathrm{ps}}(\mathrm{const}_{\mathbf{1}}, G)$. Indeed, a pseudofunctor $\Phi : \mathscr{X} \to \mathrm{Nat}^{\mathrm{ps}}(\mathrm{const}_{\mathbf{1}}, G)$ is equivalently a pseudonatural transformation $\widetilde{\Phi} : \mathrm{const}_{\mathbf{1}} \Rightarrow \mathrm{Func}^{\mathrm{ps}}(\mathscr{X}, G)$ of pseudofunctors $\mathscr{J} \to \mathbf{Cat}$ by taking its components to be $\widetilde{\Phi}_j := \Phi(-)j : \mathscr{X} \to Gj$ for $j \in \mathscr{J}_0$ (recalling that a pseudofunctor $\mathbf{1} \to \mathrm{Func}^{\mathrm{ps}}(\mathscr{X}, Gj)$ is up to equivalence a pseudofunctor $\mathscr{X} \to Gj$) and similarly for morphisms. This identification extends readily to pseudonatural transformations $\Phi \Rightarrow \Psi$ by identifying them with modifications $\widetilde{\Phi} \Rrightarrow \widetilde{\Psi}$. Reversing this identification is obvious, establishing the desired equivalence

$$\mathrm{Func}^{\mathrm{ps}}(\mathscr{X}, \mathrm{Nat}^{\mathrm{ps}}(\mathrm{const}_{\mathbf{1}}, G)) \simeq \mathrm{Nat}^{\mathrm{ps}}(\mathrm{const}_{\mathbf{1}}, \mathrm{Func}^{\mathrm{ps}}(\mathscr{X}, G))$$

which is moreover pseudonatural in $\mathscr{X}$.

Example 5.9 (2-pullback)**.** Given a cospan $\mathscr{X} \xrightarrow{F} \mathscr{Z} \xleftarrow{G} \mathscr{Y}$ of categories, a 2-cone from $\mathbf{1}$ is equivalently a choice of objects $x \in \mathscr{X}_0$, $y \in \mathscr{Y}_0$, $z \in \mathscr{Z}_0$ and specified isomorphisms $Fx \cong z$ and $Gy \cong z$. Any such cone is isomorphic to one where the isomorphism $Fx \cong z$ is given by an identity, so we can simplify our canonical 2-limit $\mathscr{X} \times^2_{\mathscr{Z}} \mathscr{Y}$ of the cospan to be the category whose objects are triples (x, y, ϕ) with $x \in \mathscr{X}_0$, $y \in \mathscr{Y}_0$, and $\phi : Fx \cong Gy$ in $\mathscr{Z}$, and whose morphisms $(x, y, \phi) \to (x', y', \phi')$ are given by pairs of morphisms $f : x \to x'$ and $g : y \to y'$ such that $Ff \circ \phi' = \phi \circ Gg$.

Note that the strict pullback $\mathscr{X} \times_{\mathscr{Z}} \mathscr{Y}$ by definition forms a cone with this cospan and thus admits an essentially unique functor into $\mathscr{X} \times^2_{\mathscr{Z}} \mathscr{Y}$ by its 2-universal property. In fact, $\mathscr{X} \times_{\mathscr{Z}} \mathscr{Y}$ is the full subcategory of the 2-pullback on those triples (x, y, ϕ) where ϕ is an identity, and the essentially unique functor $\mathscr{X} \times_{\mathscr{Z}} \mathscr{Y} \to \mathscr{X} \times^2_{\mathscr{Z}} \mathscr{Y}$ can be taken to be the corresponding inclusion. In particular, if $\mathscr{Z}$ is discrete, then $\mathscr{X} \times^2_{\mathscr{Z}} \mathscr{Y}$ recovers the strict pullback of categories.

Remark 5.10. Recall from Section 4.2 that it was crucial for the induction that the category n**Nerve** had fibre products over discrete objects, and moreover that weak equivalences were stable under such pullbacks. That the above example showing in particular that 2-fibre products over discrete categories can be presented with a usual fibre product suggests that these assumptions on n**Nerve** really just describe the properties necessary for using 1-categorical language to work with higher limits without much loss of generality.

Let $\mathcal{A} : \mathbf{O}^{\mathrm{op}} \to \mathbf{Cat}$ be an opetopic category with $\mathcal{A}_0$ discrete, and γ an n-opetope for some $n \geq 3$ to ensure that $|\gamma|_1 = |t\gamma|_1$. Suppose the Segal maps $f : \mathcal{A}_{t\gamma} \to \mathcal{A}_1^{\times_{\mathcal{A}_0}|\gamma|_1}$ and $g : \mathcal{A}_\gamma \to \mathcal{A}_1^{\times_{\mathcal{A}_0}|\gamma|_1}$ into the 1-fibre product are surjective equivalences of categories. By the axiom of choice, we can find sections $f^{-1} : \mathcal{A}_1^{\times_{\mathcal{A}_0}|\gamma|_1} \to \mathcal{A}_{t\gamma}$ of f and $g^{-1} : \mathcal{A}_1^{\times_{\mathcal{A}_0}|\gamma|_1} \to \mathcal{A}_\gamma$ of g that are also weak inverses. Fix natural isomorphisms $\xi : \mathrm{id} \overset{\sim}{\Rightarrow} gg^{-1}$ and $\zeta : \mathrm{id} \overset{\sim}{\Rightarrow} ff^{-1}$, then for every input edge $i : 1 \to \gamma$ we obtain a natural isomorphism $\xi_{(i)} : \pi_i \overset{\sim}{\Rightarrow} ig^{-1}$ from the pasting diagram

$$\begin{tikzcd}
\mathcal{A}_1^{\times_{\mathcal{A}_0}|\gamma|_1} \arrow[dd, "\mathrm{id}"'] \arrow[dr, "g^{-1}"] \arrow[drr, phantom, "\overset{\sim}{\underset{\xi}{\Longrightarrow}}"] & & \\
& \mathcal{A}_\gamma \arrow[dl, "g"] \arrow[dr, "i"] \arrow[d, phantom, "\circlearrowleft"] & \\
\mathcal{A}_1^{\times_{\mathcal{A}_0}|\gamma|_1} \arrow[rr, "\pi_i"'] & & \mathcal{A}_1
\end{tikzcd}$$

where $\pi_i : \mathcal{A}_1^{\times_{\mathcal{A}_0}|\gamma|_1} \to \mathcal{A}_1$ is the canonical projection. Since $i : 1 \to \gamma$ is also an input edge of $t\gamma$ using the fact that $\dim \gamma \geq 3$, we similarly obtain a natural isomorphism $\zeta_{(i)} : \pi_i \overset{\sim}{\Rightarrow} if^{-1}$ from the pasting diagram

$$\begin{tikzcd}
\mathcal{A}_1^{\times_{\mathcal{A}_0}|\gamma|_1} \arrow[dd, "\mathrm{id}"'] \arrow[dr, "f^{-1}"] \arrow[drr, phantom, "\overset{\sim}{\underset{\zeta}{\Longrightarrow}}"] & & \\
& \mathcal{A}_{t\gamma} \arrow[dl, "f"] \arrow[dr, "i"] \arrow[d, phantom, "\circlearrowleft"] & \\
\mathcal{A}_1^{\times_{\mathcal{A}_0}|\gamma|_1} \arrow[rr, "\pi_i"'] & & \mathcal{A}_1
\end{tikzcd}$$

By Example 5.9, the 1-fibre product $\mathcal{A}_1^{\times_{\mathcal{A}_0}|\gamma|_1}$ being over a discrete category $\mathcal{A}_0$ also satisfies the universal property of the corresponding 2-pullback. Given that $g : \mathcal{A}_\gamma \to \mathcal{A}_1^{\times_{\mathcal{A}_0}|\gamma|_1}$ is an equivalence of categories, this means that $\mathcal{A}_\gamma$ will also satisfy this 2-universal property, which implies the following

key observation:

Lemma 5.11. *Let $\mathcal{A} : \mathbf{O}^{\mathrm{op}} \to \mathbf{Cat}$ be an opetopic category with $\mathcal{A}_0$ discrete, and γ an n-opetope for some $n \geq 3$. Suppose the Segal maps $f : \mathcal{A}_{t\gamma} \to \mathcal{A}_1^{\times_{\mathcal{A}_0}|\gamma|_1}$ and $g : \mathcal{A}_\gamma \to \mathcal{A}_1^{\times_{\mathcal{A}_0}|\gamma|_1}$ into the 1-fibre product—that is, computing $\mathcal{A}_1^{\times_{\mathcal{A}_0}|\gamma|_1}$ in* **Cat** *as a 1-category—are surjective equivalences of categories, and fix $f^{-1}, g^{-1}, \zeta_{(i)}, \xi_{(i)}$ as above. Then, there exists a unique natural isomorphism $\alpha : tg^{-1} \overset{\sim}{\Rightarrow} f^{-1}$ as in the diagram*

$$\begin{tikzcd}
\mathcal{A}_1^{\times_{\mathcal{A}_0}|\gamma|_1} \arrow[d, "g^{-1}"'] \arrow[dr, bend left, "f^{-1}"] \arrow[dr, phantom, "\exists!\alpha \overset{\sim}{\Rightarrow}"] & \\
\mathcal{A}_\gamma \arrow[r, "t"'] & \mathcal{A}_{t\gamma}
\end{tikzcd}$$

such that for every input 1-opetope $i : 1 \to \gamma$, we have $(i \otimes \alpha) \circ \xi_{(i)} = \zeta_{(i)}$ as natural isomorphisms $\pi_i \overset{\sim}{\Rightarrow} if^{-1}$.

While the above lemma also follows by a simple direct computation with the natural isomorphisms involved, the realisation of α by 2-universal property as uniquely determined by the natural isomorphisms $\xi : \mathrm{id} \overset{\sim}{\Rightarrow} gg^{-1}$ and $\zeta : \mathrm{id} \overset{\sim}{\Rightarrow} ff^{-1}$ generalises to other similar diagrams involving sections of Segal maps in an opetopic category. This will be instrumental when proving Theorem 5.12: we will fix sections g^{-1} for all Segal maps g, as well as natural isomorphisms $\mathrm{id} \overset{\sim}{\Rightarrow} gg^{-1}$, and implicitly use them to construct and argue uniqueness of various natural isomorphisms similar to α in Lemma 5.11.

5.1.2 Recovering the Underlying Algebraic Structure

With the language of 2-limits established, we can generalise the proof of Lemma 5.4 to unbiased bicategories. We will moreover confirm that this opetopic approach fully characterises the pseudofunctors as well.

Theorem 5.12. *Let $\mathcal{A} : \mathbf{O}^{\mathrm{op}} \to \mathbf{Cat}$ be an opetopic category satisfying:*

- *(Discreteness condition) $\mathcal{A}_0$ is discrete.*
- *(Unary condition) The target face map $t : \mathcal{A}_{2[1]} \to \mathcal{A}_1$ is an equivalence of categories.*
- *(Segal condition) The map $\mathcal{A}_\gamma \to \mathcal{A}_1^{\times_{\mathcal{A}_0}|\gamma|_1}$ into the 1-fibre product is a surjective equivalence of categories for all opetopes γ.*

Then, there exists an unbiased bicategory $\mathscr{B}$ with a levelwise equivalence $\mathcal{A} \to \mathrm{N}\mathscr{B}$.

Remark 5.13. As surjectivity on objects is not stable under equivalences of categories, this does not completely characterise the weak essential image of the double nerve construction. The reason for asserting 1-categorical constraints on $\mathcal{A}$ is an artefact of how hom-categories arise as 1-fibres of the overall category of morphisms and 2-morphisms in a bicategory. By making the equivalences surjective, we guarantee that the Segal maps admit weak inverses that are also sections, which is important for ensuring that composites strictly respect domains and codomains; otherwise, we will only have isomorphisms $\mathrm{dom}(g \otimes f) \cong \mathrm{dom}\, f$ for composable morphisms f, g.

Note that this was unnecessary to assert in the simplicial context because Segal maps are already split epic: the Segal maps are induced from the inert morphisms $\{i, i+1\} \hookrightarrow [n]$ which all have retractions in the other direction by mapping $k \in [n]$ to i if $k \leq i$ and otherwise to $i+1$. As retractions map to sections under a contravariant functor (such as a presheaf), these combine to form a section for the induced Segal map.

Proof of Theorem 5.12. We will construct an unbiased bicategory $\mathscr{B}$ from $\mathcal{A}$ in a similar manner to the simplicial case. Take $\mathscr{B}_0 := (\mathcal{A}_0)_0$ and for $x, y \in \mathscr{B}_0$ define $\mathrm{Hom}_{\mathscr{B}}(x,y)$ as the 1-fibre

$$\begin{tikzcd} \mathrm{Hom}_{\mathscr{B}}(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{A}_1 \arrow[d, "(s,t)"] \\ \mathbf{1} \arrow[r, "(x,y)"'] & \mathcal{A}_0 \times \mathcal{A}_0 \end{tikzcd}$$

This establishes the objects, morphisms, and 2-morphisms of the proposed bicategory $\mathscr{B}$.

To define and establish associativity of the composition on $\mathscr{B}$, fix once and for all sections g^{-1} for every Segal map $g : \mathcal{A}_\gamma \to \mathcal{A}_1^{\times_{\mathcal{A}_0} |\gamma|_1}$ as well as natural isomorphisms $\mathrm{id} \overset{\simeq}{\Rightarrow} gg^{-1}$. This is possible by the axiom of choice, since the Segal maps are assumed to be surjective equivalences of categories.

Unbiased composition. For each integer $p \geq 0$, the p-ary composition on $\mathscr{B}$ is induced by taking the fixed section of the Segal map in the span $\otimes^p : \mathcal{A}_1^{\times_{\mathcal{A}_0} p} \leftarrow \mathcal{A}_{2[p]} \xrightarrow{t} \mathcal{A}_1$. In particular, note that the unary composition when $p = 1$ is given by $\otimes^1 := ts^{-1} : \mathcal{A}_1 \leftarrow \mathcal{A}_{2[1]} \to \mathcal{A}_1$, where s^{-1} is the chosen section of the Segal map $s : \mathcal{A}_{2[1]} \to \mathcal{A}_1$. We need to check that p-ary composition $\otimes^p$ as defined above restricts to a functor

$$\otimes^p : \mathrm{Hom}_{\mathscr{B}}(x_0, x_1) \times \cdots \times \mathrm{Hom}_{\mathscr{B}}(x_{p-1}, x_p) \to \mathrm{Hom}_{\mathscr{B}}(x_0, x_p)$$

for all objects $x_0, \ldots, x_p \in \mathscr{B}_0$. This indeed holds because we take a section of the Segal map rather than an arbitrary weak inverse: the canonical inclusions of the hom-categories into $\mathcal{A}_1$ induce a map fitting as the dashed arrow in the commutative square (in fact, this is a pullback square)

$$\begin{tikzcd} \mathrm{Hom}_{\mathscr{B}}(x_0, x_1) \times \cdots \times \mathrm{Hom}_{\mathscr{B}}(x_{p-1}, x_p) \arrow[r, dashed, "\exists!"] \arrow[d] & \mathcal{A}_1^{\times_{\mathcal{A}_0} p} \arrow[d] \\ \mathbf{1} \arrow[r, "(x_0, \ldots, x_p)"'] & \mathcal{A}_0^{\times(p+1)} \end{tikzcd}$$

where the vertical map $\mathcal{A}_1^{\times_{\mathcal{A}_0} p} \to \mathcal{A}_0^{\times(p+1)}$ sends a composable string of 1-cells $(f_1, \ldots, f_p)$ to the tuple $(sf_1, sf_2, \ldots, sf_p, tf_p)$. Note that $tf_i = sf_{i+1}$ for all $1 \leq i < p$. This establishes commutativity of the

pentagon on the left of the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{B}}(x_0,x_1)\times\cdots\times\mathrm{Hom}_{\mathscr{B}}(x_{p-1},x_p) \arrow[rr] \arrow[ddd, bend right] & & \mathcal{A}_1^{\times_{\mathcal{A}_0} p} \arrow[dl, "\mathrm{id}"'] \arrow[d] \arrow[dr, "\otimes^p"] & \\
& \mathcal{A}_1^{\times_{\mathcal{A}_0} p} \arrow[d] & \mathcal{A}_{2[p]} \arrow[l, "\sim"] \arrow[r, "t"'] & \mathcal{A}_1 \arrow[d, "(s,t)"] \\
& \mathbf{1} \arrow[r, "(x_0,\ldots,x_p)"] \arrow[rr, bend right, "(x_0,x_p)"'] & \mathcal{A}_0^{\times(p+1)} \arrow[r, "(\mathrm{proj}_0,\mathrm{proj}_p)"] & \mathcal{A}_0\times\mathcal{A}_0
\end{tikzcd}$$

Commutativity of the triangle on the bottom is obvious, and commutativity of the lower right pentagon follows by opetopic identities. The upper right triangle commutes by the definition of $\otimes^p$, and the triangle to its left commutes because the map $\mathcal{A}_1^{\times_{\mathcal{A}_0} p} \to \mathcal{A}_{2[p]}$ was chosen to be a section of the Segal map. This proves commutativity of the perimeter in the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{B}}(x_0,x_1)\times\cdots\times\mathrm{Hom}_{\mathscr{B}}(x_{p-1},x_p) \arrow[rr] \arrow[dr, dashed, "\exists!\otimes^p"] \arrow[ddr, bend right] & & \mathcal{A}_1^{\times_{\mathcal{A}_0} p} \arrow[dr, "\otimes^p"] & \\
& \mathrm{Hom}_{\mathscr{B}}(x_0,x_p) \arrow[rr] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & & \mathcal{A}_1 \arrow[d] \\
& \mathbf{1} \arrow[r, "(x_0,x_p)"'] & \mathcal{A}_0\times\mathcal{A}_0 &
\end{tikzcd}$$

and thus we obtain the desired composition map as the unique dashed arrow indicated above. We will prove associativity of these composition maps by looking at the global maps $\otimes^p : \mathcal{A}_1^{\times_{\mathcal{A}_0} p} \to \mathcal{A}_1$, as this will imply associativity for the composition maps in $\mathscr{B}$ by taking fibres.

Associators and inserters. Now that we have established p-ary composition functors for $\mathscr{B}$, we will construct the unbiased associators. Recall from Lemma 5.4 that we have a (strictly) commutative diagram

$$\begin{tikzcd}
\prod_{i=1}^p \mathcal{A}_0\mathcal{A}_1^{\times_{\mathcal{A}_0} q_i} & & \mathcal{A}_{2[\sum_i q_i]} \arrow[ll, "\sim"'] \arrow[dr, "t"] & \\
\prod_{i=1}^p \mathcal{A}_0\mathcal{A}_{2[q_i]} \arrow[u, "\wr"] \arrow[d, "t^{\times_{\mathcal{A}_0} p}"'] & & \mathcal{A}_{3[p(\vec{q})]} \arrow[ll, "(s^{2[q_i]})_i"] \arrow[ul, "\sim"'] \arrow[u, "t"] \arrow[d, "s^{2[p]}"'] & \mathcal{A}_1 \\
\mathcal{A}_1^{\times_{\mathcal{A}_0} p} & & \mathcal{A}_{2[p]} \arrow[ll, "\sim"] \arrow[ur, "t"'] &
\end{tikzcd}$$

by opetopic identities. Our Segal maps are only weakly invertible, so the diagram obtained after reversing the Segal maps in the above diagram (cf. (Eq. 5.1) from Lemma 5.4) will commute only up to specified 2-isomorphisms. The resulting pasting diagram of these 2-isomorphisms will form our desired

associators.

More precisely, for $p \geq 0$ and $q_1, \ldots, q_p \geq 0$, consider the pasting diagram

$$\begin{tikzcd}
\prod_{i=1}^{p} \mathcal{A}_0 \mathcal{A}_1^{\times_{\mathcal{A}_0} q_i} \arrow[r, "\sim"] \arrow[d, "\wr"'] \arrow[dr, "\sim"'] & \mathcal{A}_{2[\sum_i q_i]} \arrow[dr, "t"] & \\
\prod_{i=1}^{p} \mathcal{A}_0 \mathcal{A}_{2[q_i]} \arrow[d, "t"'] \arrow[dr, phantom, "\Nearrow \alpha_{(s)}"] & \mathcal{A}_{3[p(\vec{q})]} \arrow[u, "t"'] \arrow[d, "s"] \arrow[r, phantom, "\circlearrowleft"] & \mathcal{A}_1 \\
\mathcal{A}_1^{\times_{\mathcal{A}_0} p} \arrow[r, "\sim"'] & \mathcal{A}_{2[p]} \arrow[ur, "t"'] & \arrow[u, phantom, "\Nearrow \alpha_{(t)}"]
\end{tikzcd} \tag{Eq. 5.2}$$

where the morphisms marked as equivalences are the chosen sections of Segal maps. The natural isomorphism $\alpha_{(t)}$ arises from the 2-universal property of $\mathcal{A}_{2[\sum_i q_i]}$ as a 2-limit by an argument analogous to that of Lemma 5.11. Note that $\alpha_{(t)}$ is implicitly uniquely determined by the fixed natural isomorphisms $\mathrm{id} \overset{\sim}{\Rightarrow} gg^{-1}$ for every Segal map g. Similarly, $\alpha_{(s)}$ exists by the 2-universal property of $\mathcal{A}_{2[p]}$ as the 2-limit $\mathcal{A}_1^{\times^2_{\mathcal{A}_0} p}$. The remaining square commutes by opetopic identities, and the overall diagram defines the corresponding associator α^p; specifically, we have

$$\alpha^p := (t \otimes \alpha_{(s)}) \circ (t \otimes \alpha_{(t)}) : \bigotimes_{i=1}^{p} \otimes^{q_i} \cong \otimes^{\sum_i q_i} : \mathcal{A}_1^{\times_{\mathcal{A}_0} \sum_i q_i} \to \mathcal{A}_1$$

Recall that unary composition is given by $\otimes^1 := ts^{-1} : \mathcal{A}_1 \leftarrow \mathcal{A}_{2[1]} \to \mathcal{A}_1$ with s^{-1} the chosen section to the Segal map $s : \mathcal{A}_{2[1]} \to \mathcal{A}_1$. Taking $p = q_1 = 1$ in (Equation Eq. 5.2) provides a natural isomorphism $ts^{-1}ts^{-1} \cong ts^{-1}$, and thus by precomposing with s a natural isomorphism $\phi : t \overset{\sim}{\Rightarrow} ts^{-1}t$. By assumption, t is essentially surjective, so the axiom of choice provides every $y \in (\mathcal{A}_1)_0$ with an object $x_y \in (\mathcal{A}_{2[1]})_0$ and an isomorphism $\psi_y : y \overset{\sim}{\Rightarrow} tx_y$. For $f : y \to z$ in $\mathcal{A}_1$, let $\widetilde{f} := \psi_z f \psi_y^{-1}$ and consider the diagram

$$\begin{tikzcd}
y \arrow[r, "\psi_y", "\cong"'] \arrow[d, "f"'] & tx_y \arrow[r, "\phi_{x_y}", "\cong"'] \arrow[d, dashed, "\widetilde{f}"'] & ts^{-1}tx_y \arrow[r, "ts^{-1}\psi_y^{-1}", "\cong"'] \arrow[d, dashed, "ts^{-1}\widetilde{f}"] & ts^{-1}y \arrow[d, "ts^{-1}f"] \\
z \arrow[r, "\cong", "\psi_z"'] & tx_z \arrow[r, "\cong", "\phi_{x_z}"'] & ts^{-1}tx_z \arrow[r, "\cong", "ts^{-1}\psi_z^{-1}"'] & ts^{-1}z
\end{tikzcd}$$

The left and right squares commute by the definition of $\widetilde{f}$, and the middle square commutes because the fullness of t allows us to write $\widetilde{f} = t(g)$ for some $g : x_y \to x_z$, and then we use the naturality of ϕ. Therefore, $(ts^{-1}\psi^{-1}) \otimes \phi_{x_\bullet} \otimes \psi$ defines a natural isomorphism $\iota : \mathrm{id} \cong \otimes^1 : \mathcal{A}_1 \to \mathcal{A}_1$ which serves as our inserter.

Coherence axioms. In order to show that $\mathscr{B}$ equipped with the established p-ary composition functors and associators indeed defines an unbiased bicategory, it remains to check that the associators

satisfy commutativity of the diagram

$$\begin{tikzcd}
\bigotimes_{i=1}^{p}\bigotimes_{j=1}^{q_i}\bigotimes^{r_{i,j}} \arrow[r, Rightarrow, "\alpha^p", "\sim"'] \arrow[d, Rightarrow, "\bigotimes_i \alpha^{q_i}"', "\wr"] & \bigotimes^{\sum_i q_i}\left(\bigotimes^{r_{1,1}},\ldots,\bigotimes^{r_{p,q_p}}\right) \arrow[d, Rightarrow, "\alpha^{\sum_i q_i}", "\wr"'] \\
\bigotimes_{i=1}^{p}\bigotimes^{\sum_j r_{i,j}} \arrow[r, Rightarrow, "\sim", "\alpha^p"'] & \bigotimes^{\sum_i\sum_j r_{i,j}}
\end{tikzcd}$$

We will do so by defining the natural isomorphism $\beta^{3;p(\vec{q}(\vec{r}))}$ from Lemma 4.12 and show that the two triangles in

$$\begin{tikzcd}
\bigotimes_{i=1}^{p}\bigotimes_{j=1}^{q_i}\bigotimes^{r_{i,j}} \arrow[r, Rightarrow, "\alpha^p", "\sim"'] \arrow[d, Rightarrow, "\bigotimes_i \alpha^{q_i}"', "\wr"] \arrow[dr, Rightarrow, "\beta^{3;p(\vec{q}(\vec{r}))}", "\sim"'] & \bigotimes^{\sum_i q_i}\left(\bigotimes^{r_{1,1}},\ldots,\bigotimes^{r_{p,q_p}}\right) \arrow[d, Rightarrow, "\alpha^{\sum_i q_i}", "\wr"'] \\
\bigotimes_{i=1}^{p}\bigotimes^{\sum_j r_{i,j}} \arrow[r, Rightarrow, "\sim", "\alpha^p"'] & \bigotimes^{\sum_i\sum_j r_{i,j}}
\end{tikzcd} \tag{Eq. 5.3}$$

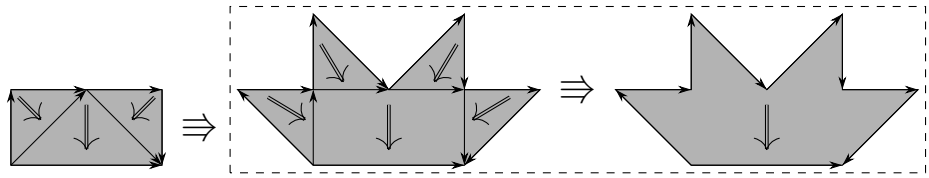
both commute.

Similar to the unbiased associators, define $\beta^{3;p(\vec{q}(\vec{r}))}$ to be the pasting diagram

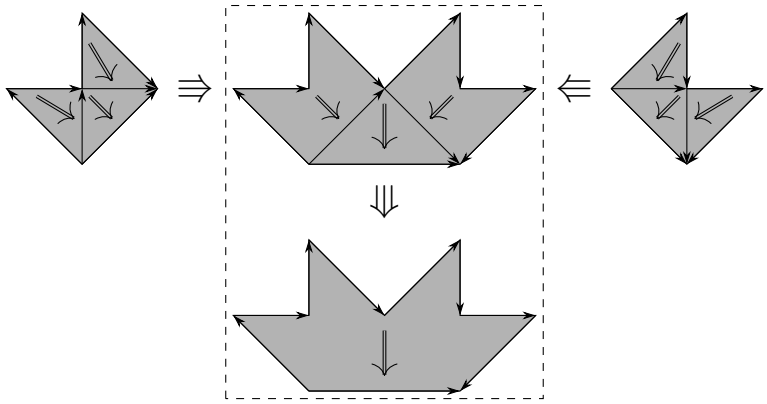
$$\begin{tikzcd}
\prod_{i=1}^{p}\mathcal{A}_0\prod_{j=1}^{q_i}\mathcal{A}_0\mathcal{A}_1^{\times_{\mathcal{A}_0} r_{i,j}} \arrow[rr, "\sim"] \arrow[d, "\wr"] \arrow[dr, "\sim"] & & \mathcal{A}_{2[\sum_i\sum_j r_{i,j}]} \arrow[dr, "t"] & \\
\prod_{i=1}^{p}\mathcal{A}_0\prod_{j=1}^{q_i}\mathcal{A}_0\mathcal{A}_{2[r_{i,j}]} \arrow[d, "t"] & & \mathcal{A}_{3[p(\vec{q}(\vec{r}))]} \arrow[u, Rightarrow, "\beta_{(t)}", "\wr"'] \arrow[dr, "s"] \arrow[r, phantom, "\circlearrowleft"] & \mathcal{A}_1 \\
\prod_{i=1}^{p}\mathcal{A}_0\mathcal{A}_1^{\times_{\mathcal{A}_0} q_i} \arrow[r, "\sim"'] \arrow[urr, Rightarrow, "\beta_{(s)}", "\sim"'] & \prod_{i=1}^{p}\mathcal{A}_0\mathcal{A}_{2[q_i]} \arrow[r, "t"'] & \mathcal{A}_1^{\times_{\mathcal{A}_0} p} \arrow[r, "\sim"'] & \mathcal{A}_{2[p]} \arrow[u, "t"']
\end{tikzcd} \tag{Eq. 5.4}$$

where $\beta_{(t)}$ comes from the 2-universal property of $\mathcal{A}_{2[\sum_i\sum_j r_{i,j}]}$, and $\beta_{(s)}$ from that of $\mathcal{A}_{2[p]}$.

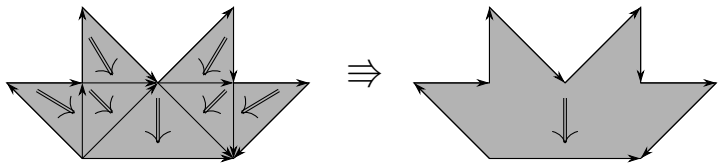
We will first show that the upper right triangle of (Eq. 5.3) commutes. Let $4[\urcorner]$ be the 4-opetope that compares these two composition complexes of natural isomorphisms: explicitly, define its complex of source 3-opetopes by pasting $3[p(\vec{q})]$ to the central input 2-opetope of $3[(\sum_i q_i)(\vec{r})]$. The corresponding target 3-opetope of this complex is $3[p(\vec{q}(\vec{r}))]$. The example where all $p, q_i, r_{i,j} = 3$ is given in Figure 5.2a.



(a) Complex of source 3-opetopes for $4[\ulcorner]$



(b) Complex of source 3-opetopes for $4[\llcorner]$



(c) Target 3-opetope for both $4[\ulcorner]$ and $4[\llcorner]$

Figure 5.2: Examples of the 4-opetopes used for associator coherence

Now, we can subdivide (Eq. 5.4) to yield the equal pasting diagram

$$
\begin{tikzcd}
\prod_{i=1}^{p} \mathcal{A}_0 \prod_{j=1}^{q_i} \mathcal{A}_0 \mathcal{A}_1^{\times_{\mathcal{A}_0} r_{i,j}} \arrow[rrr, "\sim"] \arrow[d, "\wr"'] \arrow[dr, "\sim"] \arrow[drr, "\sim"] & & & \mathcal{A}_{2[\sum_i \sum_j r_{i,j}]} \arrow[dr, "t"] & \\
\prod_{i=1}^{p} \mathcal{A}_0 \prod_{j=1}^{q_i} \mathcal{A}_0 \mathcal{A}_{2[r_{i,j}]} \arrow[r, Rightarrow, "\beta^1_{(s)}", "\sim"'] \arrow[d, "t"'] & \mathcal{A}_{4[\ulcorner]} \arrow[r, "t"'] \arrow[dl, "s"] \arrow[u, phantom, "\wr \Uparrow \beta^0_{(s)}"] \arrow[d, phantom, "\Nearrow \beta^2_{(s)}"] & \mathcal{A}_{3[p(\vec{q}(\vec{r}))]} \arrow[ur, "t"] \arrow[dr, "s"] \arrow[u, phantom, "\wr \Uparrow \beta_{(t)}"] & \circlearrowleft & \mathcal{A}_1 \\
\prod_{i=1}^{p} \mathcal{A}_0 \mathcal{A}_1^{\times_{\mathcal{A}_0} q_i} \arrow[r, "\sim"'] & \prod_{i=1}^{p} \mathcal{A}_0 \mathcal{A}_{2[q_i]} \arrow[r, "t"'] & \mathcal{A}_1^{\times_{\mathcal{A}_0} p} \arrow[r, "\sim"'] & \mathcal{A}_{2[p]} \arrow[ur, "t"'] &
\end{tikzcd}
$$

the maps $\beta^0_{(s)}$, $\beta^1_{(s)}$, $\beta^2_{(s)}$ arising from the 2-universal properties of $\mathcal{A}_{3[p(\vec{q}(\vec{r}))]}$, $\mathcal{A}_1^{\times_{\mathcal{A}_0} \sum_i q_i}$, and $\mathcal{A}_{2[p]}$, respectively. That these three natural isomorphisms paste together to recover $\beta_{(s)}$ in (Eq. 5.4) follows from

the 2-universal property of $\mathcal{A}_{2[p]}$ making $\beta_{(s)}$ unique. Composing $\beta^0_{(s)}$ and $\beta_{(t)}$ together then yields the pasting diagram

$$\begin{tikzcd}
\prod_{i=1}^{p}\mathcal{A}_0\prod_{j=1}^{q_i}\mathcal{A}_0\mathcal{A}_1^{\times_{\mathcal{A}_0} r_{i,j}} \arrow[rr, "\sim"] \arrow[d, "\wr"] \arrow[dr, "\sim"] & & \mathcal{A}_{2[\sum_i\sum_j r_{i,j}]} \arrow[dr, "t"] & \\
\prod_{i=1}^{p}\mathcal{A}_0\prod_{j=1}^{q_i}\mathcal{A}_0\mathcal{A}_{2[r_{i,j}]} \arrow[r, Rightarrow, "\beta^1_{(s)}", "\sim"'] \arrow[d, "t"] & \mathcal{A}_{4[\urcorner]} \arrow[ur, "t"] \arrow[dl, "s"] \arrow[dr, "s"] \arrow[u, phantom, "\Nearrow \sim \ \beta^0_{(t)}"] \arrow[r, phantom, "\circlearrowleft"] & & \mathcal{A}_1 \\
\prod_{i=1}^{p}\mathcal{A}_0\mathcal{A}_1^{\times_{\mathcal{A}_0} q_i} \arrow[r, "\sim"'] & \prod_{i=1}^{p}\mathcal{A}_0\mathcal{A}_{2[q_i]} \arrow[r, "t"'] \arrow[u, phantom, "\wr\Uparrow\beta^2_{(s)}"] & \mathcal{A}_1^{\times_{\mathcal{A}_0} p} \arrow[r, "\sim"'] & \mathcal{A}_{2[p]} \arrow[u, "t"']
\end{tikzcd}$$

Subdividing the commutative square based on the input 3-opetopes of $4[\urcorner]$ then gives

$$\begin{tikzcd}
\prod_{i=1}^{p}\mathcal{A}_0\prod_{j=1}^{q_i}\mathcal{A}_0\mathcal{A}_1^{\times_{\mathcal{A}_0} r_{i,j}} \arrow[rr, "\sim"] \arrow[d, "\wr"] \arrow[dr, "\sim"] \arrow[drr, "\sim"] & & \mathcal{A}_{2[\sum_i\sum_j r_{i,j}]} \arrow[dr, "t"] & \\
& \mathcal{A}_{3[(\sum_i q_i)(\vec{r})]} \arrow[ur, "t", "\wr\Uparrow\alpha^{\sum_i q_i}_{(t)}"'] \arrow[dr, "s"] \arrow[r, phantom, "\circlearrowleft"] & & \mathcal{A}_1 \\
\prod_{i=1}^{p}\mathcal{A}_0\prod_{j=1}^{q_i}\mathcal{A}_0\mathcal{A}_{2[r_{i,j}]} \arrow[r, Rightarrow, "\beta^1_{(s)}", "\sim"'] \arrow[d, "t"] & \mathcal{A}_{4[\urcorner]} \arrow[ur, "s", "\wr\Uparrow\alpha^{(0)}_{(s)}"'] \arrow[dl, "s"] \arrow[dr, "s"] \arrow[r, phantom, "\circlearrowleft"] & \mathcal{A}_{2[\sum_i q_i]} \arrow[ur, "t"] \arrow[r, "t"] & \mathcal{A}_1 \\
\prod_{i=1}^{p}\mathcal{A}_0\mathcal{A}_1^{\times_{\mathcal{A}_0} q_i} \arrow[r, "\sim"'] \arrow[urr, "\sim", "\alpha^{(0)}_{(t)}"'] & \prod_{i=1}^{p}\mathcal{A}_0\mathcal{A}_{2[q_i]} \arrow[r, "t"'] & \mathcal{A}_{3[p(\vec{q})]} \arrow[ur, "t"] \arrow[dr, "s"] \arrow[r, phantom, "\circlearrowleft"] & \mathcal{A}_1 \\
& & \mathcal{A}_1^{\times_{\mathcal{A}_0} p} \arrow[r, "\sim"'] \arrow[u, phantom, "\wr\Uparrow\alpha^{p}_{(s)}"] & \mathcal{A}_{2[p]} \arrow[uur, "t"']
\end{tikzcd}$$

with $\alpha^{\sum_i q_i}_{(t)}$, $\alpha^{(0)}_{(s)}$, $\alpha^{(0)}_{(t)}$, $\alpha^{p}_{(s)}$ arising from the 2-universal properties of $\mathcal{A}_{2[\sum_i\sum_j r_{i,j}]}$, $\mathcal{A}_{3[(\sum_i q_i)(\vec{r})]}$, $\mathcal{A}_{3[p(\vec{q})]}$, $\mathcal{A}_{2[p]}$, respectively. Equality with the previous pasting diagram follows from the 2-universal properties of $\mathcal{A}_{2[\sum_i\sum_j r_{i,j}]}$ and $\mathcal{A}_{2[p]}$ for the nontrivial natural isomorphisms, and by opetopic identities for the strictly

commutative parts. We can then finally compare this with the pasting diagram

$$
\begin{tikzcd}
\prod_{i=1}^{p} \mathcal{A}_0 \prod_{j=1}^{q_i} \mathcal{A}_0 \mathcal{A}_1^{\times_{\mathcal{A}_0} r_{i,j}} \arrow[rrr, "\sim"] \arrow[dr, "\sim"] \arrow[d, "\wr"'] & & & \mathcal{A}_{2[\sum_i \sum_j r_{i,j}]} \arrow[dr, "t"] & \\
\prod_{i=1}^{p} \mathcal{A}_0 \prod_{j=1}^{q_i} \mathcal{A}_0 \mathcal{A}_{2[r_{i,j}]} \arrow[d, "t"'] \arrow[ur, phantom, "\Uparrow \alpha_{(t)}^{\sum_i q_i}"] & & \mathcal{A}_{3[(\sum_i q_i)(\vec{r})]} \arrow[ur, "t"] \arrow[dr, "s"] \arrow[r, phantom, "\circlearrowright"] & \mathcal{A}_{2[\sum_i q_i]} \arrow[r, "t"] & \mathcal{A}_1 \\
\prod_{i=1}^{p} \mathcal{A}_0 \mathcal{A}_1^{\times_{\mathcal{A}_0} q_i} \arrow[rr, "\sim"'] \arrow[urr, bend left, "\sim"] \arrow[urr, bend right, "\sim"'] \arrow[urr, Rightarrow, "\alpha_{(s)}^{\sum_i q_i}"] & & \prod_{i=1}^{p} \mathcal{A}_0 \mathcal{A}_{2[q_i]} \arrow[r, "t"'] \arrow[ur, bend left, "\sim"] \arrow[ur, Leftrightarrow, "\alpha_{(t)}^{p}"] & \mathcal{A}_1^{\times_{\mathcal{A}_0} p} \arrow[r, "\sim"'] \arrow[ur, "t"] \arrow[r, phantom, "\circlearrowright"] & \mathcal{A}_{2[p]} \arrow[uur, "t"'] \\
& & & \mathcal{A}_{3[p(\vec{q})]} \arrow[u, phantom, "\Uparrow \alpha_{(s)}^{p}"] \arrow[ur, "s"'] &
\end{tikzcd}
\tag{Eq. 5.5}
$$

The existence of $\alpha_{(s)}^{\sum_i q_i}$ and $\alpha_{(t)}^{p}$, and the fact that they coincide with the natural isomorphisms in the previous pasting diagram all follow from the 2-universal property of $\mathcal{A}_{2[\sum_i q_i]}$. Referring to (Eq. 5.2), we find that these natural isomorphisms paste together to form the unbiased associators $\alpha^{\sum_i q_i}$ and α^{p} to give

$$
\begin{tikzcd}
\prod_{i=1}^{p} \mathcal{A}_0 \prod_{j=1}^{q_i} \mathcal{A}_0 \mathcal{A}_1^{\times_{\mathcal{A}_0} r_{i,j}} \arrow[rrr, "\sim"] \arrow[d, "\wr"'] \arrow[drrr, Rightarrow, "\sim", "\alpha^{\sum_i q_i}"'] & & & \mathcal{A}_{2[\sum_i \sum_j r_{i,j}]} \arrow[dr, "t"] & \\
\prod_{i=1}^{p} \mathcal{A}_0 \prod_{j=1}^{q_i} \mathcal{A}_0 \mathcal{A}_{2[r_{i,j}]} \arrow[d, "t"'] & & & \mathcal{A}_{2[\sum_i q_i]} \arrow[r, "t"] & \mathcal{A}_1 \\
\prod_{i=1}^{p} \mathcal{A}_0 \mathcal{A}_1^{\times_{\mathcal{A}_0} q_i} \arrow[rr, "\sim"'] \arrow[urrr, "\sim"] & & \prod_{i=1}^{p} \mathcal{A}_0 \mathcal{A}_{2[q_i]} \arrow[r, "t"'] \arrow[urr, Rightarrow, "\sim", "\alpha^{p}"'] & \mathcal{A}_1^{\times_{\mathcal{A}_0} p} \arrow[r, "\sim"'] & \mathcal{A}_{2[p]} \arrow[uur, "t"']
\end{tikzcd}
$$

This proves that the upper right triangle in (Eq. 5.3) commutes.

If we use the 4-opetope $4[\llcorner]$ whose complex of source 3-opetopes is given by pasting each $3[q_i(\vec{r}_i)]$ for $1 \leq i \leq p$ to the leaf input 2-opetopes of $3[p(\overrightarrow{\sum_j r_{-,j}})]$ (cf. Figure 5.2b) and insert this into (Eq. 5.4), we can proceed in a similar manner to show that the lower left triangle in (Eq. 5.3) commutes.

Subdivide the pasting diagram (Eq. 5.4) for $\beta^{3;p(\vec{q}(\vec{r}))}$ through our new 4-opetope $4[\llcorner]$ to obtain

$$\begin{tikzcd}
\prod_{i=1}^{p}\mathcal{A}_0\prod_{j=1}^{q_i}\mathcal{A}_0\mathcal{A}_1^{\times_{\mathcal{A}_0}r_{i,j}} \arrow[rrr, "\sim"] \arrow[d, "\wr"'] \arrow[dr, "\sim"] \arrow[drr, "\sim"] & & & \mathcal{A}_{2[\sum_i\sum_j r_{i,j}]} \arrow[dr, "t"] & \\
\prod_{i=1}^{p}\mathcal{A}_0\prod_{j=1}^{q_i}\mathcal{A}_0\mathcal{A}_{2[r_{i,j}]} \arrow[d, "t"'] \arrow[r, Rightarrow, "\beta^{1'}_{(s)}", "\sim"'] & \mathcal{A}_{4[\llcorner]} \arrow[r, "t"'] \arrow[dl, "s"] \arrow[u, phantom, "\wr\Uparrow\beta^{0'}_{(s)}"] \arrow[dr, phantom, "\wr\Uparrow\beta^{2'}_{(s)}"] & \mathcal{A}_{3[p(\vec{q}(\vec{r}))]} \arrow[ur, "t"] \arrow[dr, "s"'] \arrow[u, phantom, "\wr\Uparrow\beta_{(t)}"] \arrow[r, phantom, "\circlearrowleft"] & & \mathcal{A}_1 \\
\prod_{i=1}^{p}\mathcal{A}_0\mathcal{A}_1^{\times_{\mathcal{A}_0}q_i} \arrow[r, "\sim"'] & \prod_{i=1}^{p}\mathcal{A}_0\mathcal{A}_{2[q_i]} \arrow[r, "t"'] & \mathcal{A}_1^{\times_{\mathcal{A}_0}p} \arrow[r, "\sim"'] & \mathcal{A}_{2[p]} \arrow[ur, "t"'] &
\end{tikzcd}$$

We can compose the natural isomorphisms $\beta^{0'}_{(s)}$ and $\beta_{(t)}$ together to form

$$\begin{tikzcd}
\prod_{i=1}^{p}\mathcal{A}_0\prod_{j=1}^{q_i}\mathcal{A}_0\mathcal{A}_1^{\times_{\mathcal{A}_0}r_{i,j}} \arrow[rrr, "\sim"] \arrow[d, "\wr"'] \arrow[dr, "\sim"] & & & \mathcal{A}_{2[\sum_i\sum_j r_{i,j}]} \arrow[dr, "t"] & \\
\prod_{i=1}^{p}\mathcal{A}_0\prod_{j=1}^{q_i}\mathcal{A}_0\mathcal{A}_{2[r_{i,j}]} \arrow[d, "t"'] \arrow[r, Rightarrow, "\beta^{1'}_{(s)}", "\sim"'] & \mathcal{A}_{4[\llcorner]} \arrow[r, "t"'] \arrow[dl, "s"] \arrow[ur, phantom, "\wr\Uparrow\beta^{0'}_{(t)}"] \arrow[dr, "s"'] \arrow[r, phantom, "\circlearrowleft"] & & \mathcal{A}_1 \\
\prod_{i=1}^{p}\mathcal{A}_0\mathcal{A}_1^{\times_{\mathcal{A}_0}q_i} \arrow[r, "\sim"'] & \prod_{i=1}^{p}\mathcal{A}_0\mathcal{A}_{2[q_i]} \arrow[r, "t"'] \arrow[u, phantom, "\wr\Uparrow\beta^{2'}_{(s)}"] & \mathcal{A}_1^{\times_{\mathcal{A}_0}p} \arrow[r, "\sim"'] & \mathcal{A}_{2[p]} \arrow[ur, "t"'] &
\end{tikzcd}$$

Subdividing by the input 3-opetopes of $4[\llcorner]$, we obtain the diagram

$$\begin{tikzcd}
\prod_{i=1}^{p}\mathcal{A}_0\prod_{j=1}^{q_i}\mathcal{A}_0\mathcal{A}_1^{\times_{\mathcal{A}_0}r_{i,j}} \arrow[rrr, "\sim"] \arrow[d, "\wr"'] \arrow[dr, "\sim"] \arrow[drr, "\sim"] \arrow[drrr, "\sim"] & & & & \mathcal{A}_{2[\sum_i\sum_j r_{i,j}]} \arrow[dr, "t"] & \\
\prod_{i=1}^{p}\mathcal{A}_0\prod_{j=1}^{q_i}\mathcal{A}_0\mathcal{A}_{2[r_{i,j}]} \arrow[d, "t"'] \arrow[dr, Rightarrow, "\prod_i\mathcal{A}_0\alpha^{q_i}_{(s)}", "\sim"'] & \prod_{i=1}^{p}\mathcal{A}_0\mathcal{A}_{3[q_i(\vec{r_i})]} \arrow[d, "s"] \arrow[r, Rightarrow, "\alpha^{(1')}_{(s)}", "\sim"'] \arrow[dr, Rightarrow, "\sim", "\alpha^{(0')}_{(s)}"'] & \mathcal{A}_{4[\llcorner]} \arrow[l, "s"] \arrow[r, "s"'] \arrow[dr, Rightarrow, "\sim", "\alpha^{(0')}_{(t)}"'] & \mathcal{A}_{3[p(\sum_j r_{\vec{\sqcup},j})]} \arrow[ur, "t"] \arrow[d, "s"] \arrow[r, phantom, "\circlearrowleft"] & & \mathcal{A}_1 \\
\prod_{i=1}^{p}\mathcal{A}_0\mathcal{A}_1^{\times_{\mathcal{A}_0}q_i} \arrow[r, "\sim"'] & \prod_{i=1}^{p}\mathcal{A}_0\mathcal{A}_{2[q_i]} \arrow[r, "t"'] & \mathcal{A}_1^{\times_{\mathcal{A}_0}p} \arrow[r, "\sim"'] & \mathcal{A}_{2[p]} \arrow[ur, "t"'] & &
\end{tikzcd}$$

The natural isomorphisms $\prod_i\mathcal{A}_0\alpha^{q_i}_{(s)}$, $\alpha^{(0')}_{(t)}$, and $\alpha^{(1')}_{(s)}$ combine to form an equal pasting diagram to that of $\beta^{1'}_{(s)}$ and $\beta^{2'}_{(s)}$ by the 2-universal property of $\mathcal{A}_{2[p]}$. Similarly, $\alpha^{(0')}_{(s)}$ and $\alpha^{p}_{(t)}$ combine to form $\beta^{0'}_{(t)}$ by

the 2-universal property of $\mathcal{A}_{2[\sum_i \sum_j r_{i,j}]}$. This produces the pasting diagram

$$\begin{tikzcd}
\prod_{i=1}^{p} \mathcal{A}_0 \prod_{j=1}^{q_i} \mathcal{A}_0 \mathcal{A}_1^{\times_{\mathcal{A}_0} r_{i,j}} \arrow[rrr, "\sim"] \arrow[d, "\wr"'] \arrow[dr, "\sim"] \arrow[drr, "\sim"] \arrow[drrr, "\sim"] & & & \mathcal{A}_{2[\sum_i \sum_j r_{i,j}]} \arrow[dr, "t"] \arrow[dl, phantom, "\Leftarrow \alpha^p_{(t)}"] & \\
\prod_{i=1}^{p} \mathcal{A}_0 \prod_{j=1}^{q_i} \mathcal{A}_0 \mathcal{A}_{2[r_{i,j}]} \arrow[d, "t"'] \arrow[ur, phantom, "\Rightarrow \prod_i \mathcal{A}_0 \alpha^{q_i}_{(t)}"] & \prod_{i=1}^{p} \mathcal{A}_0 \mathcal{A}_{3[q_i(\vec{r}_i)]} \arrow[r, "t"'] \arrow[d, "s"] \arrow[dr, phantom, "\circlearrowleft"] & \prod_{i=1}^{p} \mathcal{A}_0 \mathcal{A}_{2[\sum_j r_{i,j}]} \arrow[d, "t"] \arrow[ur, phantom, "\Rightarrow \alpha^p_{(s)}"] & \mathcal{A}_{3[p(\sum_j \vec{r_{-,j}})]} \arrow[u, "t"] \arrow[d, "s"] \arrow[r, phantom, "\circlearrowleft"] & \mathcal{A}_1 \\
\prod_{i=1}^{p} \mathcal{A}_0 \mathcal{A}_1^{\times_{\mathcal{A}_0} q_i} \arrow[r, "\sim"'] \arrow[ur, phantom, "\Rightarrow \prod_i \mathcal{A}_0 \alpha^{q_i}_{(s)}"] & \prod_{i=1}^{p} \mathcal{A}_0 \mathcal{A}_{2[q_i]} \arrow[r, "t"'] & \mathcal{A}_1^{\times_{\mathcal{A}_0} p} \arrow[r, "\sim"'] & \mathcal{A}_{2[p]} \arrow[ur, "t"'] &
\end{tikzcd}$$

Note that $\prod_i \mathcal{A}_0 \alpha^{q_i}_{(t)}$ and $\alpha^p_{(s)}$ compose to the same natural isomorphism as $\alpha^{(0')}_{(t)}$, $\alpha^{(1')}_{(s)}$, and $\alpha^{(0')}_{(s)}$ by the 2-universal property of $\mathcal{A}_{2[p]}$. Referring to (Eq. 5.2), we find that these natural isomorphisms paste together to form the unbiased associators $\prod_i \mathcal{A}_0 \alpha^{q_i}$ and α^p to give

$$\begin{tikzcd}
\prod_{i=1}^{p} \mathcal{A}_0 \prod_{j=1}^{q_i} \mathcal{A}_0 \mathcal{A}_1^{\times_{\mathcal{A}_0} r_{i,j}} \arrow[rr, "\sim"] \arrow[d, "\wr"'] \arrow[drr, "\sim"] & & \mathcal{A}_{2[\sum_i \sum_j r_{i,j}]} \arrow[dr, "t"] & \\
\prod_{i=1}^{p} \mathcal{A}_0 \prod_{j=1}^{q_i} \mathcal{A}_0 \mathcal{A}_{2[r_{i,j}]} \arrow[d, "t"'] & \prod_{i=1}^{p} \mathcal{A}_0 \mathcal{A}_{2[\sum_j r_{i,j}]} \arrow[r, Rightarrow, "\sim", "\alpha^p"'] \arrow[d, "t"] & & \mathcal{A}_1 \\
\prod_{i=1}^{p} \mathcal{A}_0 \mathcal{A}_1^{\times_{\mathcal{A}_0} q_i} \arrow[r, "\sim"'] \arrow[ur, Rightarrow, "\sim", "\prod_i \mathcal{A}_0 \alpha^{q_i}"'] & \prod_{i=1}^{p} \mathcal{A}_0 \mathcal{A}_{2[q_i]} \arrow[r, "t"'] & \mathcal{A}_1^{\times_{\mathcal{A}_0} p} \arrow[r, "\sim"'] & \mathcal{A}_{2[p]} \arrow[ur, "t"']
\end{tikzcd}$$

which is precisely the pasting diagram for the lower left path in (Eq. 5.3).

Therefore, the lower left and upper right triangles of (Eq. 5.3) commute, showing coherence for the unbiased associators of $\mathscr{B}$. The coherence constraints for inserters can be proven entirely analogously, invoking 2-universal properties to compare the pasting diagrams for rebracketing and unbracketing to the identity pasting diagram. This completes the proof that $\mathscr{B}$ is indeed an unbiased bicategory. By construction, the Segal maps for $\mathcal{A}$ then define a levelwise equivalence $\mathcal{A} \to \mathrm{N}\mathscr{B}$, as desired. □

Remark 5.14. Similar to the observation made in Remark 5.5, Theorem 5.12 remains true if we relax the unary condition to just assert that $t : \mathcal{A}_{2[1]} \to \mathcal{A}_1$ is essentially surjective and full.

Note that just as in the simplicial case, reconstructing the unbiased bicategory from its nerve requires several invocations of the axiom of choice to weakly invert the Segal maps. This is because the nerve provides a means of describing coherence for weakly associative operators without explicitly specifying coherence isomorphisms for every weak identity the operators carry.

Lemma 5.15. *The unbiased double nerve functor* $\mathrm{N} : \mathbf{UBicat} \to \mathrm{Func}(\mathbf{O}^{\mathrm{op}}, \mathbf{Cat})$ *is fully faithful.*

Proof. Any pseudofunctor $F : \mathscr{B} \to \mathscr{B}'$ has its action on objects, morphisms, and 2-morphisms determined uniquely by how its nerve acts on the categories $(\mathrm{N}\mathscr{B})_0$ and $(\mathrm{N}\mathscr{B})_1$; as for the functoriality constraint, these can be extracted from the fact that the nerve sends the $2[p]$-cell $(x_\bullet, f_\bullet, \alpha^p_{\vec{f}})$ to $(Fx_\bullet, Ff_\bullet, F(\alpha^p_{\vec{f}}) \circ F^p)$, which uniquely determines F^p from the invertibility of $F(\alpha^p_{\vec{f}})$. This establishes that the unbiased double nerve is faithful.

It remains to show that this functor is full. Let $\mathcal{A} = \mathrm{N}\mathscr{B}$ and $\mathcal{B} = \mathrm{N}\mathscr{B}'$ be double nerves of unbiased bicategories and suppose $\Phi : \mathcal{A} \to \mathcal{B}$ is an opetopic functor, then the goal is to show that $\Phi = \mathrm{N}F$ for some pseudofunctor $F : \mathscr{B} \to \mathscr{B}'$. Define $F_0 := (\Phi_0)_0$ on objects, and $F_{x,y}$ as the unique functor induced by the universal property of the 1-fibre in

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{B}}(x,y) \arrow[r] \arrow[dr, dashed, "\exists! F_{x,y}"] \arrow[ddr, bend right, "!"'] & \mathcal{A}_1 \arrow[dr, "\Phi_1"] & \\
& \mathrm{Hom}_{\mathscr{B}'}(\Phi_0 x, \Phi_0 y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{B}_1 \arrow[d, "(s,t)"] \\
& \mathbf{1} \arrow[r, "(\Phi_0 x, \Phi_0 y)"'] & \mathcal{B}_0 \times \mathcal{B}_0
\end{tikzcd}$$

The functoriality constraints $F^p : \otimes^p \Phi_1 \Rightarrow \Phi_1 \otimes^p$ are induced by the pasting diagrams

$$\begin{tikzcd}
\mathcal{A}_1^{\times_{\mathcal{A}_0} p} \arrow[r, "\sim"] \arrow[d, "\Phi_1^{\times_{\Phi_0} p}"'] \arrow[dr, phantom, "\Nearrow \Phi^p"] & \mathcal{A}_{2[p]} \arrow[r, "t"] \arrow[d, "\Phi_{2[p]}"] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{A}_1 \arrow[d, "\Phi_1"] \\
\mathcal{B}_1^{\times_{\mathcal{B}_0} p} \arrow[r, "\sim"'] & \mathcal{B}_{2[p]} \arrow[r, "t"'] & \mathcal{B}_1
\end{tikzcd}$$

in the sense that F^p is obtained as the fibres of $t \otimes \Phi^p$. Note that the inverted Segal maps are defined explicitly via the composition rules of $\mathscr{B}$ and $\mathscr{B}'$: for instance, the map $\mathcal{A}_1^{\times_{\mathcal{A}_0} p} \to \mathcal{A}_{2[p]}$ is given by sending a length-p string of composable morphisms $(f_1, \ldots, f_p)$ in $\mathscr{B}$ to the $2[p]$-cell whose 2-isomorphism is given by the identity on $\bigotimes_{i=1}^p f_i$. The natural isomorphism Φ^p then comes from the 2-universal property of $\mathcal{B}_{2[p]}$ as the 2-limit $\mathcal{B}_1^{\times^2_{\mathcal{B}_0} p}$ as before with a similar argument to Lemma 5.11.

It remains to show that this construction is coherent. We will first show that the diagram below commutes:

$$\begin{tikzcd}
\bigotimes_{i=1}^p \otimes^{q_i} \Phi_1 \arrow[r, "\otimes_i F^{q_i}", "\sim"'] \arrow[d, Rightarrow, "\alpha'^p", "\wr"'] & \bigotimes_{i=1}^p \Phi_1 \otimes^{q_i} \arrow[r, "F^p", "\sim"'] & \Phi_1 \bigotimes_{i=1}^p \otimes^{q_i} \arrow[d, Rightarrow, "\wr", "\Phi_1(\alpha^p)"'] \\
\otimes^{\Sigma_i q_i} \Phi_1 \arrow[rr, Rightarrow, "\sim", "F^{\Sigma_i q_i}"'] & & \Phi_1 \otimes^{\Sigma_i q_i}
\end{tikzcd}$$

As before, we will do so by introducing an intermediate natural isomorphism $F^{p(\vec{q})}$ along the diagonal

of the above diagram as in

$$\begin{tikzcd}
\bigotimes_{i=1}^{p} \otimes^{q_i} \Phi_1 \arrow[r, "\otimes_i F^{q_i}", "\sim"'] \arrow[d, "\alpha'^p", "\wr"'] \arrow[dr, "F^{p(\vec{q})}", "\sim"'] & \bigotimes_{i=1}^{p} \Phi_1 \otimes^{q_i} \arrow[r, "F^p", "\sim"'] & \Phi_1 \bigotimes_{i=1}^{p} \otimes^{q_i} \arrow[d, "\wr", "\Phi_1(\alpha^p)"'] \\
\otimes^{\Sigma_i q_i} \Phi_1 \arrow[rr, "\sim", "F^{\Sigma_i q_i}"'] & & \Phi_1 \otimes^{\Sigma_i q_i}
\end{tikzcd} \tag{Eq. 5.6}$$

and show that the two triangles commute separately. Define $F^{p(\vec{q})}$ to be the pasting diagram

$$\begin{tikzcd}
\prod_{i=1}^{p} {}_{\mathcal{A}_0}\mathcal{A}_1^{\times_{\mathcal{A}_0} q_i} \arrow[r, "\sim"] \arrow[d, "\prod_i {}_{\Phi_0}\Phi_1^{\times_{\Phi_0} q_i}"'] \arrow[dr, "\sim"'] & \mathcal{A}_{2[\Sigma_i q_i]} \arrow[rr, "t"] \arrow[dr, "\Phi_{2[\Sigma_i q_i]}"] \arrow[d, phantom, "\overset{\sim}{\Nearrow} \alpha^p_{(t)}"] & & \mathcal{A}_1 \arrow[dd, "\Phi_1"] \\
\prod_{i=1}^{p} {}_{\mathcal{B}_0}\mathcal{B}_1^{\times_{\mathcal{B}_0} q_i} \arrow[dd, "\wr"'] & \mathcal{A}_{3[p(\vec{q})]} \arrow[u, "t"'] \arrow[r, "\Phi_{3[p(\vec{q})]}"] \arrow[d, "s"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{B}_{2[\Sigma_i q_i]} \arrow[dr, "t"] & \\
& \mathcal{A}_{2[p]} \arrow[r, "\Phi_{2[p]}"] \arrow[d, phantom, "\overset{\sim}{\Nearrow} \phi_{(0)}"] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{B}_{3[p(\vec{q})]} \arrow[u, "t"'] \arrow[d, "s"] & \\
\prod_{i=1}^{p} {}_{\mathcal{B}_0}\mathcal{B}_{2[q_i]} \arrow[r, "t"'] & \mathcal{B}_1^{\times_{\mathcal{B}_0} p} \arrow[r, "\sim"'] & \mathcal{B}_{2[p]} \arrow[r, "t"'] & \mathcal{B}_1
\end{tikzcd} \tag{Eq. 5.7}$$

The squares commute by the naturality of Φ and the opetopic identities; the natural isomorphisms $\alpha^p_{(t)}$ and $\phi_{(0)}$ arise from the 2-universal properties of $\mathcal{A}_{2[\Sigma_i q_i]}$ and $\mathcal{B}_{2[p]}$, respectively, as in Lemma 5.11.

The upper right path in (Eq. 5.6) is given by the pasting diagram

$$\begin{tikzcd}
\prod_{i=1}^{p} {}_{\mathcal{A}_0}\mathcal{A}_1^{\times_{\mathcal{A}_0} q_i} \arrow[rr, "\sim"] \arrow[d, "\prod_i {}_{\Phi_0}\Phi_1^{\times_{\Phi_0} q_i}"'] \arrow[dr, "\sim"'] \arrow[drr, "\sim"'] & & \mathcal{A}_{2[\Sigma_i q_i]} \arrow[r, "t"] \arrow[d, phantom, "\overset{\sim}{\Nearrow} \alpha^p_{(t)}"] & \mathcal{A}_1 \arrow[dd, "\Phi_1"] \\
\prod_{i=1}^{p} {}_{\mathcal{B}_0}\mathcal{B}_1^{\times_{\mathcal{B}_0} q_i} \arrow[r, "\sim", "\prod_i {}_{\mathcal{B}_0}\Phi^{q_i}"'] \arrow[dd, "\wr"'] & \prod_{i=1}^{p} {}_{\mathcal{A}_0}\mathcal{A}_{2[q_i]} \arrow[dr, "\prod_i {}_{\Phi_0}\Phi_{2[q_i]}"'] \arrow[d, "t"] & \mathcal{A}_{3[p(\vec{q})]} \arrow[u, "t"'] \arrow[d, "s"] \arrow[dr, phantom, "\circlearrowleft"] & \\
& \mathcal{A}_1^{\times_{\mathcal{A}_0} p} \arrow[r, "\sim"] \arrow[d, "{}_{\Phi_0}\Phi_1^{\times_{\Phi_0} p}"'] \arrow[dr, phantom, "\circlearrowleft"] \arrow[r, phantom, "\overset{\sim}{\Nearrow} \alpha^p_{(s)}"] & \mathcal{A}_{2[p]} \arrow[u, phantom, "\overset{\sim}{\Nearrow} \Phi^p"] \arrow[d, "\Phi_{2[p]}"] & \\
\prod_{i=1}^{p} {}_{\mathcal{B}_0}\mathcal{B}_{2[q_i]} \arrow[r, "t"'] & \mathcal{B}_1^{\times_{\mathcal{B}_0} p} \arrow[r, "\sim"'] & \mathcal{B}_{2[p]} \arrow[r, "t"'] & \mathcal{B}_1
\end{tikzcd}$$

Indeed, $\alpha^p_{(s)}$ and $\alpha^p_{(t)}$ combine to form the associator $\alpha^p : \bigotimes_{i=1}^{p} \otimes^{q_i} \Rightarrow \otimes^{\Sigma_i q_i}$ for $\mathscr{B}$ just as in (Eq. 5.2).

That this pasting diagram agrees with (Eq. 5.7) is because $\alpha^{p}_{(s)}$, Φ^{p}, and $\prod_i {}_{\mathcal{B}_0}\Phi^{q_i}$ paste together to produce the unique natural isomorphism $\phi_{(0)}$ by the 2-universal property of $\mathcal{B}_{2[p]}$.

On the other hand, the lower left path in (Eq. 5.6) is given by the pasting diagram

$$
\begin{tikzcd}
\prod_{i=1}^{p} {}_{\mathcal{A}_0}\mathcal{A}_1^{\times_{\mathcal{A}_0} q_i} \arrow[rr, "\sim"] \arrow[d, "\prod_i {}_{\Phi_0}\Phi_1^{\times_{\Phi_0} q_i}"'] \arrow[drr, phantom, "\overset{\sim}{\Longrightarrow}\ \Phi^{\Sigma_i q_i}"] & & \mathcal{A}_{2[\Sigma_i q_i]} \arrow[r, "t"] \arrow[d, "\Phi_{2[\Sigma_i q_i]}"] & \mathcal{A}_1 \arrow[dd, "\Phi_1"] \\
\prod_{i=1}^{p} {}_{\mathcal{B}_0}\mathcal{B}_1^{\times_{\mathcal{B}_0} q_i} \arrow[rr, "\sim"] \arrow[dr, "\sim"] \arrow[dd, "\wr"] & & \mathcal{B}_{2[\Sigma_i q_i]} \arrow[dr, "t"] & \\
& \mathcal{B}_{3[p(\vec{q})]} \arrow[ur, "t"] \arrow[uu, Rightarrow, "\alpha'^{p}_{(t)}"] \arrow[dr, "s"] & & \\
\prod_{i=1}^{p} {}_{\mathcal{B}_0}\mathcal{B}_{2[q_i]} \arrow[r, "t"] \arrow[ur, phantom, "\overset{\sim}{\Longrightarrow}\ \alpha'^{p}_{(s)}"] & \mathcal{B}_1^{\times_{\mathcal{B}_0} p} \arrow[r, "\sim"] & \mathcal{B}_{2[p]} \arrow[r, "t"] & \mathcal{B}_1
\end{tikzcd}
$$

with $\alpha'^{p}_{(s)}$ and $\alpha'^{p}_{(t)}$ forming the associator α'^{p} for $\mathscr{B}'$. This will be equal to the pasting diagram

$$
\begin{tikzcd}
\prod_{i=1}^{p} {}_{\mathcal{A}_0}\mathcal{A}_1^{\times_{\mathcal{A}_0} q_i} \arrow[r, "\sim"] \arrow[d, "\prod_i {}_{\Phi_0}\Phi_1^{\times_{\Phi_0} q_i}"'] \arrow[dr, "\sim"] & \mathcal{A}_{2[\Sigma_i q_i]} \arrow[rr, "t"] \arrow[dr, "\Phi_{2[\Sigma_i q_i]}"] & & \mathcal{A}_1 \arrow[dd, "\Phi_1"] \\
\prod_{i=1}^{p} {}_{\mathcal{B}_0}\mathcal{B}_1^{\times_{\mathcal{B}_0} q_i} \arrow[r, Rightarrow, "\sim", "\phi_{(1)}"'] \arrow[dr, "\sim"] \arrow[dd, "\wr"] & \mathcal{A}_{3[p(\vec{q})]} \arrow[ur, "t"] \arrow[dr, "\Phi_{3[p(\vec{q})]}"] & \mathcal{B}_{2[\Sigma_i q_i]} \arrow[dr, "t"] & \\
& & \mathcal{B}_{3[p(\vec{q})]} \arrow[u, "t"] \arrow[d, "s"] & \\
\prod_{i=1}^{p} {}_{\mathcal{B}_0}\mathcal{B}_{2[q_i]} \arrow[r, "t"] \arrow[ur, Rightarrow, "\sim", "\alpha'^{p}_{(s)}"'] & \mathcal{B}_1^{\times_{\mathcal{B}_0} p} \arrow[r, "\sim"] & \mathcal{B}_{2[p]} \arrow[r, "t"] & \mathcal{B}_1
\end{tikzcd}
$$

with the composition of $\alpha'^{p}_{(t)}$ with $\Phi^{\Sigma_i q_i}$ coinciding with the composition of $\phi_{(1)}$ and $\alpha^{p}_{(t)}$ by the 2-universal property of $\mathcal{B}_{2[\Sigma_i q_i]}$. This pasting diagram is then equal to (Eq. 5.7) as $\alpha'^{p}_{(s)}$ and $\phi_{(1)}$ compose to recover $\phi_{(0)}$ by the 2-universal property of $\mathcal{B}_{2[p]}$.

Therefore, both triangles in(Eq. 5.6) commute, proving the associator identity for F. The inserter identity follows from a similar argument, from which it follows that F as defined indeed yields a pseudofunctor whose nerve is Φ. $\square$

By combining Theorem 5.12 and Lemma 5.15, we have proven our first main result:

Theorem 5.16. *The unbiased double nerve defines a fully faithful functor* $\mathrm{N} : \mathbf{Bicat} \to \mathrm{Func}(\mathbf{O}^{\mathrm{op}}, \mathbf{Cat})$ *that is weakly essentially surjective on those* $\mathcal{A} : \mathbf{O}^{\mathrm{op}} \to \mathbf{Cat}$ *satisfying:*

- *(Discreteness condition)* $\mathcal{A}_0$ *is discrete.*
- *(Unary condition) The target face map* $t : \mathcal{A}_{2[1]} \to \mathcal{A}_1$ *is an equivalence of categories.*
- *(Segal condition) The map* $\mathcal{A}_\gamma \to \mathcal{A}_1^{\times_{\mathcal{A}_0} |\gamma|_1}$ *into the* 1*-fibre product is a surjective equivalence of categories for all opetopes* γ*.*

5.2 Higher Opetopic Nerves

We can now generalise the work in Section 5.1 to produce an inductive definition of both weak n-categories and the pseudofunctors between them, yielding large 1-categories $\mathbf{wk}n\mathbf{Cat}$ for $n \geq 0$. The idea is to take the weak $(n+1)$-categories to be the opetopic weak n-categories $\mathcal{A} : \mathbf{O}^{\mathrm{op}} \to \mathbf{wk}n\mathbf{Cat}$ subject to direct analogues of the discreteness, unary, and Segal conditions listed in Theorem 5.16. This inductive framework for defining higher categories requires well-defined notions of weak equivalence, objectwise surjectivity, and discreteness in $\mathbf{wk}n\mathbf{Cat}$ to characterise the objects of $\mathbf{wk}(n+1)\mathbf{Cat}$. In fact, we can generalise the above construction to any sufficiently nice category $\mathscr{V}$ equipped with appropriate notions of weak equivalences, objectwise surjections, and discreteness to obtain a category $\mathbf{wk}\mathscr{V}\mathbf{Cat}$ of categories weakly internal[1] to $\mathscr{V}$.

Definition 5.17. Define the (huge) category[2] $\mathfrak{Wk}\mathfrak{Cat}$ of (large) categories of (small) weak categories as follows. The objects $\mathscr{V} \in \mathfrak{Wk}\mathfrak{Cat}_0$ are quadruples $(\mathscr{V}, W, E, \tau_{\leq 0})$ where

- $\mathscr{V}$ is a (locally small) category containing $\mathbf{Set}$ as a full subcategory of *discrete objects* via a fully faithful inclusion $\mathrm{disc} : \mathbf{Set} \to \mathscr{V}$,
- $W \subseteq \mathscr{V}_1$ is a class of morphisms in $\mathscr{V}$ called *weak equivalences*, which contains the isomorphisms,
- $E \subseteq \mathscr{V}_1$ is a class of morphisms in $\mathscr{V}$ called *objectwise surjections*, which contains the surjective maps in $\mathbf{Set}$,
- $\tau_{\leq 0} : \mathscr{V} \to \mathbf{Set}$ is a retraction of the inclusion of discrete objects,

which are subject to the additional constraints

[1] The collection of objects for any category in $\mathbf{wk}\mathscr{V}\mathbf{Cat}$ is discrete, so $\mathbf{wk}\mathscr{V}\mathbf{Cat}$ may be more appropriately interpreted as a category of categories weakly enriched in $\mathscr{V}$.

[2] The usual axioms of set theory prohibit sets from becoming "too large" in order to avoid paradoxes such as the well-known Russell's Paradox. The simplest way to avoid an explicit use of proper classes is via *Grothendieck universes*, which are sets that are large enough to model enough axioms of set theory so that we can develop mathematics comfortably within them. In this case, we have two Grothendieck universes $V \subsetneq V'$, and a set is called *small* if it lies in V, *large* if it lies in V', and *huge* otherwise.

(W1) $\mathscr{V}$ has all discrete fibre products; that is, all cospans $\mathcal{A} \to \operatorname{disc} S \leftarrow \mathcal{B}$ in $\mathscr{V}$ have a limit. Moreover, fibre products consisting only of discrete objects remain discrete,

(W2) W (resp. E) is preserved along discrete fibre products, in the sense that the dashed arrow in

$$\begin{tikzcd}
\mathcal{A}' \times_{\operatorname{disc} S} \mathcal{B}' \arrow[r] \arrow[d] \arrow[dr, dashed] & \mathcal{B}' \arrow[dr, "g"] & \\
\mathcal{A}' \arrow[dr, "f"'] & \mathcal{A} \times_{\operatorname{disc} S} \mathcal{B} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{B} \arrow[d] \\
& \mathcal{A} \arrow[r] & \operatorname{disc} S
\end{tikzcd}$$

is a weak equivalence (resp. objectwise surjection) once f and g are,

(W3) $\tau_{\leq 0}$ preserves discrete fibre products, sends weak equivalences to bijections, and objectwise surjections to surjections.

The morphisms $F : \mathscr{V} \to \mathscr{W}$ are then functors that preserve the discrete objects, weak equivalences, objectwise surjections, and discrete fibre products.

Example 5.18. The category **Set** of sets, with W the class of bijections, E the class of surjections, and $\tau_{\leq 0}^{\mathbf{Set}}$ given by the identity, forms a category of weak categories. The only possibly nontrivial fact to check is that surjectivity is preserved along fibre products, which reduces to a simple diagram chase.

Indeed, given a cospan $A \xrightarrow{p} S \xleftarrow{q} B$ of sets, its pullback is canonically given by the set

$$A \times_S B = \{(a,b) \in A \times B \mid p(a) = q(b)\}$$

Given surjections $f : A' \to A$ and $g : B' \to B$, the map $A' \times_S B' \to A \times_S B$ simply acts by sending pairs $(a',b') \mapsto (f(a'), g(b'))$. For any $(a,b) \in A \times_S B$, we have $f(a') = a$ and $g(b') = b$ for some $a' \in A'$ and $b' \in B'$, and $p(f(a')) = p(a) = q(b) = q(g(b'))$, so $(a',b') \in A' \times_S B'$ also, proving surjectivity is preserved.

In fact, **Set** is the terminal object in $\mathfrak{WkCat}$. Indeed, any functor $F : \mathscr{V} \to \mathbf{Set}$ in $\mathfrak{WkCat}$ must commute with $\tau_{\leq 0}$, meaning that $\tau_{\leq 0}^{\mathbf{Set}} \circ F = \tau_{\leq 0}^{\mathscr{V}}$, and thus $F = \tau_{\leq 0}^{\mathscr{V}}$ is necessarily unique. By the assumed structure on $\tau_{\leq 0}^{\mathscr{V}}$, this does define a morphism $\mathscr{V} \to \mathbf{Set}$ of $\mathfrak{WkCat}$, ensuring that the necessarily unique functor lies in $\mathfrak{WkCat}$, as desired. Dually, the inclusion of discrete objects in any category $\mathscr{V}$ of weak categories also serves as a unique morphism $\mathbf{Set} \to \mathscr{V}$ in $\mathfrak{WkCat}$, meaning that **Set** is actually a zero object.

Lemma 5.19. $\mathfrak{WkCat}$ *has all cofiltered limits.*

Proof. Let $\mathscr{X} : \mathscr{J} \to \mathfrak{WkCat}$ be a functor with $\mathscr{J}$ cofiltered, then consider the category $\varprojlim \mathscr{X}$ computed as a limit in the huge category **CAT** of large categories. Explicitly, objects (resp. morphisms) of $\varprojlim \mathscr{X}$ are $\mathscr{J}_0$-indexed tuples $(u_j)_{j \in \mathscr{J}_0}$ of objects (resp. morphisms) of each $\mathscr{X}_j$ such that $\mathscr{X}_\phi(u_i) = u_j$ for every $\phi : i \to j$ in $\mathscr{J}$. By its universal property as a limit in **CAT**, any cone from $\mathscr{V}$ to $\mathscr{X}$ in $\mathfrak{WkCat}$

admits a necessarily unique ordinary functor to $\varprojlim \mathscr{X}$, so it remains to endow $\varprojlim \mathscr{X}$ with the structure necessary to make this functor lie in $\mathfrak{Wk}\mathfrak{Cat}$ also.

Note that **Set** embeds fully faithfully into $\varprojlim \mathscr{X}$ via the inclusions $\mathrm{disc} : \mathbf{Set} \hookrightarrow \mathscr{X}_j$; explicitly, the discrete objects of $\varprojlim \mathscr{X}$ are the constant tuples $(\mathrm{disc}\, S)_{j \in \mathscr{J}_0}$ for $S \in \mathbf{Set}_0$. Define the class W of weak equivalences (resp. E of objectwise surjections) to be those $(f_j)_{j \in \mathscr{J}_0}$ such that f_j is a weak equivalence (resp. objectwise surjection) of $\mathscr{X}_j$ for every $j \in \mathscr{J}_0$.

Define the truncation functor $\tau_{\leq 0} : \varprojlim \mathscr{X} \to \mathbf{Set}$ to be the composite $\tau_{\leq 0} : \varprojlim \mathscr{X} \xrightarrow{\pi_j} \mathscr{X}_j \xrightarrow{\tau_{\leq 0}} \mathbf{Set}$ for some arbitrary $j \in \mathscr{J}_0$, where π_j is the canonical projection map. This definition is independent of the choice of j, and at least one such j exists, both due to the fact that $\mathscr{J}$ is cofiltered. Indeed, $\mathscr{J}$ is nonempty because the empty diagram in $\mathscr{J}$ must have a cone, so suppose $j, j' \in \mathscr{J}_0$. As $\mathscr{J}$ is cofiltered, we can find a span $j \xleftarrow{\phi} i \xrightarrow{\psi} j'$ in $\mathscr{J}_0$ and thus we obtain a commutative diagram

$$\begin{tikzcd}
 & \varprojlim \mathscr{X} \arrow[dl, "\pi_j"'] \arrow[d, "\pi_i"] \arrow[dr, "\pi_{j'}"] & \\
\mathscr{X}_j \arrow[dr, "\tau_{\leq 0}"'] & \mathscr{X}_i \arrow[l, "\mathscr{X}_\phi"'] \arrow[r, "\mathscr{X}_\psi"] \arrow[d, "\tau_{\leq 0}"] & \mathscr{X}_{j'} \arrow[dl, "\tau_{\leq 0}"] \\
 & \mathbf{Set} &
\end{tikzcd}$$

where the top triangles commute from $\varprojlim \mathscr{X}$ and its projections being a (universal) cone to $\mathscr{X}$, and the lower triangles commute because $\mathscr{X}_\phi$ and $\mathscr{X}_\psi$ lie in $\mathfrak{Wk}\mathfrak{Cat}$ and thus preserve truncation. By design, the overall truncation functor will send weak equivalences to bijections and objectwise surjections to surjections.

Limits in $\varprojlim \mathscr{X}$ are computed pointwise when they exist in each $\mathscr{X}_j$, so in particular $\varprojlim \mathscr{X}$ has all discrete fibre products. By being computed pointwise, this makes it clear that the weak equivalences and objectwise surjections are preserved along discrete fibre products, and also that $\tau_{\leq 0}$ preserves these limits. Therefore, $\varprojlim \mathscr{X} \in \mathfrak{Wk}\mathfrak{Cat}_0$, and it is a straightforward to see that its projection maps make it a universal cone to $\mathscr{X}$ in $\mathfrak{Wk}\mathfrak{Cat}$, as desired. □

5.2.1 The Weak Internalisation Endofunctor

We can now describe the construction of Section 4.2 in the opetopic setting in the full generality of $\mathfrak{Wk}\mathfrak{Cat}$, which yields an endofunctor $\mathbf{wk}(-)\mathbf{Cat} : \mathfrak{Wk}\mathfrak{Cat} \to \mathfrak{Wk}\mathfrak{Cat}$.

Definition 5.20. For $\mathscr{V} \in \mathfrak{Wk}\mathfrak{Cat}_0$, define $\mathbf{wk}\mathscr{V}\mathbf{Cat}$ to be the full subcategory of $\mathrm{Func}(\mathbf{O}^{\mathrm{op}}, \mathscr{V})$ spanned by those opetopic objects $\mathcal{A} : \mathbf{O}^{\mathrm{op}} \to \mathscr{V}$ satisfying

- (Discreteness condition) $\mathcal{A}_0$ is discrete,
- (Unary condition) $t : \mathcal{A}_{2[1]} \to \mathcal{A}_1$ is a weak equivalence,
- (Segal condition) the Segal map $\mathcal{A}_\gamma \to \mathcal{A}_1^{\times_{\mathcal{A}_0} |\gamma|_1}$ is an equivalence of weak n-categories that is surjective on objects for all opetopes γ.

Note that the constant functors $\mathbf{O}^{\mathrm{op}} \to \mathscr{V}$ that map onto discrete objects in $\mathscr{V}$ will trivially satisfy these conditions since their structure maps are all identities, and isomorphisms are always weak equivalences. These form the discrete objects in $\mathbf{wk}\mathscr{V}\mathbf{Cat}$:

Proposition 5.21. *Let $\mathscr{V} \in \mathfrak{WkCat}_0$, then* **Set** *embeds fully faithfully into* $\mathbf{wk}\mathscr{V}\mathbf{Cat}$ *by sending a set S to the constant functor* $\operatorname{disc} S : \mathbf{O}^{\mathrm{op}} \to \mathscr{V}$ *on the discrete object* $\operatorname{disc}_{\mathscr{V}} S \in \mathscr{V}_0$ *corresponding to S in $\mathscr{V}$. Define the image of this embedding to be the class of discrete weak $\mathscr{V}$-categories, then* $\mathbf{wk}\mathscr{V}\mathbf{Cat}$ *satisfies (W1): the limit of any cospan $\mathcal{A} \to \operatorname{disc} S \leftarrow \mathcal{B}$ in* $\mathbf{wk}\mathscr{V}\mathbf{Cat}$ *exists and is moreover discrete if $\mathcal{A}$ and $\mathcal{B}$ are.*

Proof. If $\mathcal{A} \to \operatorname{disc} S \leftarrow \mathcal{B}$ is a cospan in $\mathbf{wk}\mathscr{V}\mathbf{Cat}$, then its fibre product is computed levelwise in $\operatorname{Func}(\mathbf{O}^{\mathrm{op}}, \mathscr{V})$. Note that $(\mathcal{A} \times_S \mathcal{B})_0 = \mathcal{A}_0 \times_S \mathcal{B}_0$ is already discrete, and the unary and Segal conditions follow from the fact that weak equivalences and objectwise surjections in $\mathscr{V}$ are preserved along discrete fibre products. Therefore, $\mathcal{A} \times_S \mathcal{B}$ lies in $\mathbf{wk}\mathscr{V}\mathbf{Cat}$. That this limit is discrete if $\mathcal{A}$ and $\mathcal{B}$ are follows from this being true in $\mathscr{V}$ and how limits are computed levelwise in $\mathbf{wk}\mathscr{V}\mathbf{Cat}$. □

Let $\widehat{\tau}_{\leq 1} := (\tau_{\leq 0})_* : \operatorname{Func}(\mathbf{O}^{\mathrm{op}}, \mathscr{V}) \to \operatorname{Func}(\mathbf{O}^{\mathrm{op}}, \mathbf{Set})$ act by truncating opetopic objects levelwise, then its restriction to $\mathbf{wk}\mathscr{V}\mathbf{Cat}$ preserves discrete fibre products from the same holding for $\tau_{\leq 0}$ in $\mathscr{V}$. If $\mathcal{A} \in \mathbf{wk}\mathscr{V}\mathbf{Cat}_0$, then $\tau_{\leq 0}(\mathcal{A}_{2[1]}) \to \tau_{\leq 0}(\mathcal{A}_1)$ is a bijection by (W3), and similarly the Segal map $\tau_{\leq 0}(\mathcal{A}_\gamma) \to \tau_{\leq 0}(\mathcal{A}_1^{\times_{\mathcal{A}_0}|\gamma|_1}) = \tau_{\leq 0}(\mathcal{A}_1)^{\times_{\mathcal{A}_0}|\gamma|_1}$ is a bijection for every opetope γ. This shows by Lemma 5.4 that $\widehat{\tau}_{\leq 1}(\mathcal{A})$ is a small category and thus $\widehat{\tau}_{\leq 1} : \mathbf{wk}\mathscr{V}\mathbf{Cat} \to \mathbf{Cat}$. In particular, this allows us to make the following definition:

Definition 5.22. For $\mathscr{V} \in \mathfrak{WkCat}_0$, define the 0-truncation functor of weak $\mathscr{V}$-categories to be the composite $\tau_{\leq 0} : \mathbf{wk}\mathscr{V}\mathbf{Cat} \xrightarrow{\tau_{\leq 1}} \mathbf{Cat} \xrightarrow{\tau_0} \mathbf{Set}$, which sends a weak $\mathscr{V}$-category to the set of isomorphism classes of its 1-truncation.

Remark 5.23. It follows from this definition that $\tau_{\leq 0}(\operatorname{disc} S) = S$ for any set S, showing that $\tau_{\leq 0}$ is a retraction of the inclusion of discrete objects in Proposition 5.21. Note also that $\tau_{\leq 0}$ preserves discrete fibre products because the same is true for $\widehat{\tau}_{\leq 1}$ and τ_0.

In order to realise $\mathbf{wk}\mathscr{V}\mathbf{Cat}$ as a category of weak categories, it remains to define appropriate classes of weak equivalences and objectwise surjections.

Definition 5.24. For $\mathscr{V} \in \mathfrak{WkCat}_0$, let $\Phi : \mathcal{A} \to \mathcal{B}$ be a weak $\mathscr{V}$-functor (that is, a morphism in $\mathbf{wk}\mathscr{V}\mathbf{Cat}$). Say that Φ is an *objectwise surjection* if $\Phi_0 : \mathcal{A}_0 \to \mathcal{B}_0$ is a surjection of sets. Similarly, say that Φ is *essentially surjective* (on objects) if $\tau_{\leq 0}\Phi : \tau_{\leq 0}\mathcal{A} \to \tau_{\leq 0}\mathcal{B}$ is a surjection of sets.

Remark 5.25. It follows from the diagram chase in Example 5.18 for sets that objectwise surjectivity is preserved along discrete fibre products. As $\tau_{\leq 0}$ preserves discrete fibre products, the same diagram chase shows that essential surjectivity is preserved along discrete fibre products as well.

Definition 5.26. Let $\mathscr{V} \in \mathfrak{Wk}\mathfrak{Cat}_0$. For a weak $\mathscr{V}$-category $\mathcal{A}$ and $x, y \in \mathcal{A}_0$, define the hom-$\mathscr{V}$-object $\mathcal{A}(x,y)$ to be the 1-fibre

$$\begin{tikzcd} \mathcal{A}(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{A}_1 \arrow[d, "(s,t)"] \\ \mathrm{disc}(\mathrm{pt}) \arrow[r, "(x,y)"'] & \mathcal{A}_0 \times \mathcal{A}_0 \end{tikzcd}$$

of the map induced by the two coface maps $s, t : 0 \rightrightarrows 1$. This is well-defined by Proposition 5.21 as $\mathcal{A}_0 \times \mathcal{A}_0$ is discrete.

Say that a weak $\mathscr{V}$-functor $\Phi : \mathcal{A} \to \mathcal{B}$ is *fully faithful* if for all $x, y \in \mathcal{A}_0$, the induced local map of fibres $\Phi_{x,y} : \mathcal{A}(x,y) \to \mathcal{B}(\Phi x, \Phi y)$ is a weak equivalence in $\mathscr{V}$.

Remark 5.27. Fully faithfulness is preserved along discrete fibre products. Indeed, suppose we have a cospan $\mathcal{A} \to \mathrm{disc}\, S \leftarrow \mathcal{B}$ of weak $\mathscr{V}$-categories and fully faithful weak $\mathscr{V}$-functors $\Phi : \mathcal{A}' \to \mathcal{A}$ and $\Psi : \mathcal{B}' \to \mathcal{B}$. For $x, y \in (\mathcal{A}' \times_{\mathrm{disc}\, S} \mathcal{B}')_0 = \mathcal{A}'_0 \times_S \mathcal{B}'_0$, the fibre $(\Phi \times_{\mathrm{disc}\, S} \Psi)_{x,y}$ is computed by a limit and thus commutes with the discrete fibre product, meaning that $(\Phi \times_{\mathrm{disc}\, S} \Psi)_{x,y} = \Phi_{x,y} \times_{\mathrm{disc}\, S} \Psi_{x,y}$. Since weak equivalences in $\mathscr{V}$ are preserved along discrete fibre products, it follows that $\Phi \times_{\mathrm{disc}\, S} \Psi$ is fully faithful, as desired.

Definition 5.28. For $\mathscr{V} \in \mathfrak{Wk}\mathfrak{Cat}_0$, define the *weak equivalences* in $\mathbf{wk}\mathscr{V}\mathbf{Cat}$ to be those weak $\mathscr{V}$-functors that are fully faithful and essentially surjective.

Remark 5.29. Note that the weak $\mathscr{V}$-functors $\Phi : \mathcal{A} \to \mathcal{B}$ that are isomorphisms are also weak equivalences under this definition: essential surjectivity follows from Φ_0 being bijective, and fully faithfulness from the fact that $\Phi_{x,y}$ is an isomorphism in $\mathscr{V}$ for all $x, y \in \mathcal{A}_0$.

This establishes all of the data necessary to realise $\mathbf{wk}\mathscr{V}\mathbf{Cat}$ as a category of weak categories, so it remains to show that this data satisfies the constraints of Definition 5.17:

Lemma 5.30. *Let $\mathscr{V} \in \mathfrak{Wk}\mathfrak{Cat}_0$, then $\mathbf{wk}\mathscr{V}\mathbf{Cat} \in \mathfrak{Wk}\mathfrak{Cat}_0$ with the structure established above. Moreover, this construction extends to define an endofunctor $\mathbf{wk}(-)\mathbf{Cat}$ on $\mathfrak{Wk}\mathfrak{Cat}$ by sending $F : \mathscr{V} \to \mathscr{W}$ in $\mathfrak{Wk}\mathfrak{Cat}$ to the functor $F_* : \mathbf{wk}\mathscr{V}\mathbf{Cat} \to \mathbf{wk}\mathscr{W}\mathbf{Cat}$ that acts by F levelwise.*

Proof. We have already shown (W1) in Proposition 5.21. (W2) follows from Remark 5.25 and Remark 5.29, which show that objectwise and essential surjections, and fully faithfulness respectively are preserved along discrete fibre products. It remains to show (W3). Preservation of discrete fibre products by $\tau_{\leq 0}$ was observed in Remark 5.23. Let $\Phi : \mathcal{A} \to \mathcal{B}$ be a weak $\mathscr{V}$-functor, and that $\tau_{\leq 0}$ sends objectwise surjections to surjections is obvious.

Let $\Phi : \mathcal{A} \to \mathcal{B}$ be a weak $\mathscr{V}$-functor that is a weak equivalence, then $\widehat{\tau}_{\leq 1}\Phi$ is an equivalence of ordinary categories. Indeed, $\widehat{\tau}_{\leq 1}\Phi$ is essentially surjective because $\tau_0(\widehat{\tau}_{\leq 1}\Phi) = \tau_{\leq 0}\Phi$ is surjective by assumption. To see that $\widehat{\tau}_{\leq 1}\Phi$ is fully faithful, note that as Φ is fully faithful, $\Phi_{x,y}$ is a weak equivalence in $\mathscr{V}$, which shows that $\tau_{\leq 0}(\Phi_{x,y}) = (\widehat{\tau}_{\leq 1}\Phi)_{x,y}$ is a bijection of sets. Therefore, since $\widehat{\tau}_{\leq 1}\Phi$ is an equivalence of categories, it follows that $\tau_{\leq 0}\Phi = \tau_0(\widehat{\tau}_{\leq 1}\Phi)$ is a bijection of sets, showing that $\tau_{\leq 0}$ sends weak equivalences to bijections as desired.

That the construction $\mathscr{V} \mapsto \mathbf{wk}\mathscr{V}\mathbf{Cat}$ is functorial follows from the fact that functors $F : \mathscr{V} \to \mathscr{W}$ in $\mathfrak{WkCat}$ must preserve discrete objects, weak equivalences, objectwise surjections, and truncations on $\mathscr{V}$. This ensures that the induced functor $F_* : \mathrm{Func}(\mathbf{O}^{\mathrm{op}}, \mathscr{V}) \to \mathrm{Func}(\mathbf{O}^{\mathrm{op}}, \mathscr{W})$ restricts to a functor $\mathbf{wk}\mathscr{V}\mathbf{Cat} \to \mathbf{wk}\mathscr{W}\mathbf{Cat}$, and moreover that F_* continues to preserve discrete objects, weak equivalences, objectwise surjections, and truncations on $\mathbf{wk}\mathscr{V}\mathbf{Cat}$. □

5.2.2 Weak ω-Categories

Having formalised the induction step for weak higher categories, we can finally explicitly define the category of weak n-categories for every finite $n \geq 0$.

Definition 5.31. Define the category of weak 0-categories to be the terminal object $\mathbf{wk0Cat} := \mathbf{Set}$ of $\mathfrak{WkCat}$. Inductively, define $\mathbf{wk}(n+1)\mathbf{Cat} := \mathbf{wk}(\mathbf{wk}n\mathbf{Cat})\mathbf{Cat}$ for $n \geq 0$.

The work done in Lemma 5.4 and Theorem 5.16 prove that $\mathbf{wk}n\mathbf{Cat}$ presents the correct weak n-categories and n-functors for $n \leq 2$ up to equivalence, making our construction a viable candidate for presenting (unbiased) higher categories as well as their functors. In fact, by viewing the Segal condition on a weak n-category $\mathcal{A}$ as asserting that all $\mathcal{A}_\gamma$ satisfy the appropriate universal property of an n-limit, the proofs of correctness for $n \leq 2$ likely generalise readily to proving that $\mathbf{wk}n\mathbf{Cat}$ agrees with coherent algebraic definitions of weak n-categories for higher n as well (such as with the tricategories of [18] when $n = 3$), though these proofs would be shrouded in exponentially more technical details and unwieldy diagrams.

Now that we have established a notion of weak n-categories for all finite $n \geq 0$, we can consider the case where $n = \omega$ is infinite. A weak ω-category $\mathcal{A}$ should consist of possibly nontrivial k-morphisms for every $k \geq 0$, and coherence for (unbiased) composition in dimension k should be expressed by invertible $(k+1)$-morphisms, which are then subject to their own coherence by higher morphisms and so on. However, all of these higher morphisms are finite-dimensional, which means that we should recover any particular higher morphism or coherence morphism of $\mathcal{A}$ from its n-truncation $\tau_{\leq n}\mathcal{A}$ for a sufficiently large $n \geq 0$. As $\tau_{\leq n}\mathcal{A}$ should be a weak n-category, this reduces the problem of studying weak ω-categories to studying their truncations: the weak n-categories $\tau_{\leq n}\mathcal{A}$ for every $n \geq 0$ should uniquely characterise the weak ω-category $\mathcal{A}$. This motivates the following definition:

Definition 5.32. Define the *n-truncation* $\tau_{\leq n} : \mathbf{wk}(n+1)\mathbf{Cat} \to \mathbf{wk}n\mathbf{Cat}$ inductively: take $\tau_{\leq 0}$ to be the truncation functor $\mathbf{wk1Cat} \to \mathbf{Set} = \mathbf{wk0Cat}$, and then define $\tau_{\leq n+1} := (\tau_{\leq n})_*$ to be the image of $\tau_{\leq n}$ under $\mathbf{wk}(-)\mathbf{Cat}$. This produces a diagram

$$\cdots \longrightarrow \mathbf{wk3Cat} \xrightarrow{\tau_{\leq 2}} \mathbf{wk2Cat} \xrightarrow{\tau_{\leq 1}} \mathbf{wk1Cat} \xrightarrow{\tau_{\leq 0}} \mathbf{wk0Cat}$$

in $\mathfrak{WkCat}$. Define the category of weak ω-categories to be the limit $\mathbf{wk}\omega\mathbf{Cat} := \varprojlim_n \mathbf{wk}n\mathbf{Cat}$ of this diagram, and denote by $\tau_{\leq n} : \mathbf{wk}\omega\mathbf{Cat} \to \mathbf{wk}n\mathbf{Cat}$ the induced canonical projections.

By Lemma 5.19, we know that $\mathbf{wk}\omega\mathbf{Cat}$ is well-defined, and moreover that an object $\mathcal{A} \in \mathbf{wk}\omega\mathbf{Cat}_0$

corresponds to an infinite tuple $(\dots, \tau_{\leq 2}\mathcal{A}, \tau_{\leq 1}\mathcal{A}, \tau_{\leq 0}\mathcal{A})$ such that $\tau_{\leq n}(\tau_{\leq n+1}\mathcal{A}) = \tau_{\leq n}\mathcal{A}$ for all $n \geq 0$. This allows us to think of weak ω-categories as formal limits of their n-truncations as $n \to \infty$.

In some cases, however, it is more useful to view weak ω-categories as the result of a bottomless recursion: a weak ω-category $\mathcal{A}$ should be a collection $\mathcal{A}_0$ of objects such that between any $x, y \in \mathcal{A}_0$ we have a weak ω-category $\mathcal{A}(x,y)$ of higher morphisms, further equipped with a weakly coherent (unbiased) composition. This can simplify the construction of weak ω-categories in some cases. For example, regarding the Homotopy Hypothesis, we want to realise a space X as a weak ω-category $\Pi_\infty X$ whose objects are points, and whose higher morphisms are higher homotopies. With our current characterisation of weak ω-categories, we would have to first construct a weak n-category $\Pi_n X$ for every $n \geq 0$ such that $\tau_{\leq n}\Pi_{n+1}X = \Pi_n X$, and then $\Pi_\infty X$ would be their formal limit. On the other hand, $\Pi_\infty X$ should be the weak ω-category whose objects are the points of X, and whose hom-ω-categories $(\Pi_\infty X)(x,y)$ for $x, y \in X$ are the weak ω-categories corresponding to the space of paths from x to y in X. The construction of objects through a bottomless recursion is called *coinduction*, and can be formalised using terminal coalgebras over an endofunctor.

Algebras over an endofunctor generalise the construction of algebras over a monad discussed in Section 2.2.3. Recall that for a monad T, its algebras are defined to be objects A equipped with an action $TA \to A$ subject to compatibility with the monad structure on T. If T is merely an endofunctor, then the compatibility constraints no longer make sense, so an algebra over an arbitrary T is simply an object A with a morphism $TA \to A$. Coalgebras of T are the dual concept:

Definition 5.33. Given a category $\mathscr{E}$ and an endofunctor $T : \mathscr{E} \to \mathscr{E}$, a *coalgebra* over T is an object $X \in \mathscr{E}_0$ equipped with a coaction morphism $\upsilon : X \to TX$. A coalgebra homomorphism $(X, \upsilon) \to (X', \upsilon')$ is then just a morphism $\phi : X \to X'$ in $\mathscr{E}$ which respects the coaction in the sense that

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d, "\upsilon"'] & X' \arrow[d, "\upsilon'"] \\ TX \arrow[r, "T\phi"'] & TX' \end{tikzcd}$$

commutes.

A coalgebra over $\mathbf{wk}(-)\mathbf{Cat}$ requires a category $\mathscr{V} \in \mathfrak{WkCat}_0$ and a coaction $\mathscr{V} \to \mathbf{wk}\mathscr{V}\mathbf{Cat}$. This is impossible to do in general, but Example 5.18 shows that $\mathbf{Set} = \mathbf{wk}0\mathbf{Cat}$ carries a unique coalgebra structure from being an initial object in $\mathfrak{WkCat}$, and that the coaction is given by the inclusion of sets $\mathrm{disc} : \mathbf{wk}0\mathbf{Cat} \to \mathbf{wk}1\mathbf{Cat}$. Inductively applying $\mathbf{wk}(-)\mathbf{Cat}$ to this inclusion, we obtain a canonical coalgebra structure on $\mathbf{wk}n\mathbf{Cat}$ for every $n \geq 0$. Explicitly, the coaction $\mathbf{wk}n\mathbf{Cat} \to \mathbf{wk}(n+1)\mathbf{Cat}$ realises weak n-categories as weak $(n+1)$-categories whose $(n+1)$-morphisms are trivial.

As these coactions allow us to realise a weak n-category as a weak m-category for all $m \geq n$, these induce a canonical inclusion $\mathbf{wk}n\mathbf{Cat} \to \mathbf{wk}\omega\mathbf{Cat}$ for all $n \geq 0$ which realise weak n-categories as weak ω-categories whose k-morphisms are trivial for all $k > n$. A coaction on $\mathbf{wk}\omega\mathbf{Cat}$ would realise weak ω-categories as categories weakly enriched in weak ω-categories, so we should expect that in this case the coaction should be trivial; that is, an isomorphism.

From the limit definition of $\mathbf{wk}\omega\mathbf{Cat}$, we have a canonical map

$$R : \mathbf{wk}(\mathbf{wk}\omega\mathbf{Cat})\mathbf{Cat} = \mathbf{wk}(\varprojlim_n \mathbf{wk}n\mathbf{Cat})\mathbf{Cat} \to \varprojlim_n \mathbf{wk}(\mathbf{wk}n\mathbf{Cat})\mathbf{Cat} = \varprojlim_{\mathbf{wk}} (n+1)\mathbf{Cat} = \mathbf{wk}\omega\mathbf{Cat}$$

The objects of $\mathbf{wk}(\mathbf{wk}\omega\mathbf{Cat})\mathbf{Cat}$ are particular functors $\mathcal{A} : \mathbf{O}^{\mathrm{op}} \to \mathbf{wk}\omega\mathbf{Cat}$, which allows us to identify $\mathcal{A}$ with a family of formal limits $\mathcal{A}_\gamma = (\ldots, \tau_{\leq 2}(\mathcal{A}_\gamma), \tau_{\leq 1}(\mathcal{A}_\gamma), \tau_{\leq 0}(\mathcal{A}_\gamma))$ for each opetope γ. The corresponding weak ω-category $R(\mathcal{A})$ is given by setting $(\tau_{\leq n+1}R(\mathcal{A}))_\gamma := \tau_{\leq n}(\mathcal{A}_\gamma)$ for every $n \geq 0$ and opetope γ, and $\tau_{\leq 0}R(\mathcal{A}) := \tau_0(\widehat{\tau}_{\leq 1}\mathcal{A}_\bullet)$. This is the inverse of the canonical coaction on $\mathbf{wk}\omega\mathbf{Cat}$:

Lemma 5.34. *The canoncial map $R : \mathbf{wk}(\mathbf{wk}\omega\mathbf{Cat})\mathbf{Cat} \to \mathbf{wk}\omega\mathbf{Cat}$ is an isomorphism in $\mathfrak{WkCat}$.*

Proof. We will construct an inverse $R^{-1} : \mathbf{wk}\omega\mathbf{Cat} \to \mathbf{wk}(\mathbf{wk}\omega\mathbf{Cat})\mathbf{Cat}$ as follows. For a weak ω-category $\mathcal{B} = (\ldots, \tau_{\leq 1}\mathcal{B}, \tau_{\leq 0}\mathcal{B})$, define $\mathcal{A} := R^{-1}(\mathcal{B}) : \mathbf{O}^{\mathrm{op}} \to \mathbf{wk}\omega\mathbf{Cat}$ by taking $\tau_{\leq n}(\mathcal{A}_\gamma) := (\tau_{\leq n+1}\mathcal{B})_\gamma$ for every $n \geq 0$ and opetope γ, extending to morphisms in the obvious way.

We first check that indeed $\mathcal{A}$ lies in $\mathbf{wk}(\mathbf{wk}\omega\mathbf{Cat})\mathbf{Cat}$. $\mathcal{A}_0$ is discrete because $\tau_{\leq n}(\mathcal{A}_0) := (\tau_{\leq n+1}\mathcal{B})_0$ is discrete for every n and thus

$$\tau_{\leq n}(\mathcal{A}_0) := (\tau_{\leq n+1}\mathcal{B})_0 = \tau_{\leq 0}((\tau_{\leq n+1}\mathcal{B})_0) =: (\widehat{\tau}_{\leq 1}\tau_{\leq n+1}\mathcal{B})_0 = (\tau_{\leq 1}\mathcal{B})_0$$

is independent of n, making $\mathcal{A}_0$ a constant functor on discrete objects. For the unary condition, we have $\mathcal{A}_{2[1]} \to \mathcal{A}_1$ is a weak equivalence if and only if its n-truncation is for every $n \geq 0$, but its n-truncation is just $\tau_{\leq n}(\mathcal{A}_{2[1]}) = (\tau_{\leq n+1}\mathcal{B})_{2[1]} \to (\tau_{\leq n+1}\mathcal{B})_1 = \tau_{\leq n}(\mathcal{A}_1)$. This is a weak equivalence by virtue of $\tau_{\leq n+1}\mathcal{B}$ being a weak $(n+1)$-category. The Segal condition follows similarly because $\mathcal{A}_1^{\times_{\mathcal{A}_0}|\gamma|_1}$ is computed levelwise in $\mathbf{wk}\omega\mathbf{Cat}$, and $\tau_{\leq n+1}\mathcal{B}$ being a weak $(n+1)$-category as well implies that $\tau_{\leq n}(\mathcal{A}_\gamma) = (\tau_{\leq n+1}\mathcal{B})_\gamma \to (\tau_{\leq n+1}\mathcal{B})_1^{\times_{(\tau_{\leq n+1}\mathcal{B})_0}|\gamma|_1} = (\tau_{\leq n}(\mathcal{A}_1))^{\times_{\tau_{\leq n}(\mathcal{A}_0)}|\gamma|_1}$ is an objectwise surjective weak equivalence.

For any $\mathcal{A}$ in $\mathbf{wk}(\mathbf{wk}\omega\mathbf{Cat})\mathbf{Cat}$, it is clear that $R^{-1}R(\mathcal{A}) = \mathcal{A}$, since

$$\tau_{\leq n}R^{-1}R(\mathcal{A}_\gamma) = \tau_{\leq n+1}R(\mathcal{A}_\gamma) = \tau_{\leq n}(\mathcal{A}_\gamma)$$

for every $n \geq 0$. On the other hand, let $\mathcal{B} \in (\mathbf{wk}\omega\mathbf{Cat})_0$, then $\tau_{\leq n+1}RR^{-1}(\mathcal{B}) = \tau_{\leq n}R^{-1}(\mathcal{B}) = \tau_{\leq n+1}\mathcal{B}$ for every $n \geq 0$. In order to see that $RR^{-1}(\mathcal{B}) = \mathcal{B}$, we need to check 0-truncation. By definition, $\widehat{\tau}_{\leq 1}R^{-1}(\mathcal{B})$ maps $\gamma \mapsto \tau_{\leq 0}(R^{-1}(\mathcal{B})_\gamma) = \widehat{\tau}_{\leq 1}(\mathcal{B})_\gamma$, which proves $\widehat{\tau}_{\leq 1}R^{-1}(\mathcal{B}) = \widehat{\tau}_{\leq 1}\mathcal{B}$, so

$$\tau_{\leq 0}RR^{-1}(\mathcal{B}) := \tau_0(\widehat{\tau}_{\leq 1}R^{-1}(\mathcal{B})) = \tau_0\widehat{\tau}_{\leq 1}(\mathcal{B}) = \tau_{\leq 0}\mathcal{B}$$

which implies $RR^{-1}(\mathcal{B}) = \mathcal{B}$.

Therefore, we have proven that R^{-1} is the inverse of R in $\mathbf{CAT}$, so now we need to check that R^{-1} is a morphism of $\mathfrak{WkCat}$. It is clear that R^{-1} preserves discrete objects, objectwise surjectivity, and 0-truncation. As equivalences of categories preserve limits, R^{-1} also preserves discrete fibre products.

Given a weak equivalence $\Psi : \mathcal{B} \to \mathcal{B}'$ in $\mathbf{wk}\omega\mathbf{Cat}$, then we need to check that the induced map $\Phi := R^{-1}(\Psi) : \mathcal{A} \to \mathcal{A}'$ is essentially surjective and fully faithful in $\mathbf{wk}(\mathbf{wk}\omega\mathbf{Cat})\mathbf{Cat}$. To see essen-

tial surjectivity, we have that $\tau_{\leq 0}\Phi := \tau_0\widehat{\tau}_{\leq 1}\Phi = \tau_0(\tau_{\leq 1}\Psi)$ is bijective since Ψ is a weak equivalence means $\tau_{\leq 1}\Psi$ is an equivalence of categories. Note that the hom-ω-category $\mathcal{A}(x,y)$ for $x,y \in \mathcal{A}_0$ is given by $\tau_{\leq n}(\mathcal{A}(x,y)) = (\tau_{\leq n+1}\mathcal{B})(x,y)$ at each level. In particular, $\tau_{\leq n}(\Phi_{x,y}) = (\tau_{\leq n+1}\Psi)_{x,y}$ is a weak equivalence for all $n \geq 0$ because $\tau_{\leq n+1}\Psi$ is fully faithful in $\mathbf{wk}(n+1)\mathbf{Cat}$, which establishes the fully faithfulness of Φ.

Therefore, $R^{-1} : \mathbf{wk}\omega\mathbf{Cat} \to \mathbf{wk}(\mathbf{wk}\omega\mathbf{Cat})\mathbf{Cat}$ is a morphism in $\mathfrak{WkCat}$, as desired. □

Remark 5.35. The proof shows in particular that the set of objects of a weak ω-category $\mathcal{A}$ is precisely the set of objects of its 1-truncation $\tau_{\leq 1}\mathcal{A} \in \mathbf{wk1Cat}$. Applying this inductively, this means that we can extract the k-morphisms of $\mathcal{A}$ just from looking at the k-morphisms $\tau_{k+1}\mathcal{A}$. This means that we can perform compositions of k-morphisms in $\tau_{\leq k+1}\mathcal{A}$, although the associators would have to be obtained by looking them up in $\tau_{\leq k+2}\mathcal{A}$, and so on for higher coherence equivalences.

Moreover, the realisation of a weak ω-category $\mathcal{A}$ as a functor $\mathbf{O}^{\mathrm{op}} \to \mathbf{wk}\omega\mathbf{Cat}$ provides a means to define the *hom-ω-category* of morphisms $x \to y$ in $\mathcal{A}$ for any $x,y \in \mathcal{A}_0$ to be the fibre product

$$\begin{tikzcd} \mathcal{A}(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{A}_1 \arrow[d, "(s,t)"] \\ \mathrm{pt} \arrow[r, "(x,y)"'] & \mathcal{A}_0 \times \mathcal{A}_0 \end{tikzcd}$$

as in Definition 5.26. As mentioned in the above proof of Lemma 5.34, discrete fibre products in $\mathbf{wk}\omega\mathbf{Cat}$ are computed levelwise, which means that $\tau_{\leq n}(\mathcal{A}(x,y)) = (\tau_{\leq n}\mathcal{A})(x,y)$ for every $n \geq 0$.

This verifies that the canonical coaction on $\mathbf{wk}\omega\mathbf{Cat}$ is trivial, which is consistent with our coinductive intuition that weak ω-categories should be precisely the categories weakly enriched in $\mathbf{wk}\omega\mathbf{Cat}$. In fact, it now follows from general abstract nonsense that $\mathbf{wk}\omega\mathbf{Cat}$ is the terminal coalgebra over $\mathbf{wk}(-)\mathbf{Cat}$. This means that in order to realise objects of some $\mathscr{V} \in \mathfrak{WkCat}_0$ as weak ω-categories, it is enough to realise them as weak $\mathscr{V}$-categories via a coaction $\mathscr{V} \to \mathbf{wk}\mathscr{V}\mathbf{Cat}$, as coinduction then induces a unique map $\mathscr{V} \to \mathbf{wk}\omega\mathbf{Cat}$. We reproduce the general construction of terminal coalgebras below, as it also describes the construction of the unique terminal map from any other coalgebra:

Lemma 5.36 (Adámek's construction)**.** *Let $\mathscr{E}$ be a category and $T : \mathscr{E} \to \mathscr{E}$ an endofunctor. If $\mathscr{E}$ has a terminal object* pt*, and the limit of*

$$\cdots \longrightarrow T^3(\mathrm{pt}) \xrightarrow{T^2!} T^2(\mathrm{pt}) \xrightarrow{T!} T(\mathrm{pt}) \xrightarrow{!} \mathrm{pt}$$

exists and is preserved by T, then the limit defines a terminal coalgebra over T.

Proof. This result can be found in [1]. Let $X := \varprojlim_n T^n(\mathrm{pt})$ in $\mathscr{E}$, then the map $TX \to X$ induced by the univeral property of X is an isomorphism by assumption, so define the coaction $\upsilon : X \to TX$ to be its inverse.

Let (X', υ') be any coalgebra over T, then define morphisms $\phi_n : X' \to T^n(\mathrm{pt})$ inductively with $\phi_0 : X' \to \mathrm{pt}$ uniquely determined, and then taking $\phi_{n+1} : X' \xrightarrow{\upsilon'} TX' \xrightarrow{T\phi_n} T^{n+1}(\mathrm{pt})$. By construction,

we certainly get $! \circ \phi_1 = \phi_0$ by the uniqueness of the map $X' \to \mathrm{pt}$. Inductively, if $T^n! \circ \phi_{n+1} = \phi_n$, then applying T to this identity implies we have commutativity of diagram

$$\begin{tikzcd}
 & X' \arrow[d, "\upsilon'"] \arrow[ddl, "\phi_{n+1}"'] \arrow[ddr, "\phi_{n+2}"] & \\
 & TX' \arrow[dl, "T\phi_n"] \arrow[dr, "T\phi_{n+1}"'] & \\
T^n(\mathrm{pt}) & & T^{n+1}(\mathrm{pt}) \arrow[ll, "T^n!"]
\end{tikzcd}$$

showing $T^{n+1}! \circ \phi_{n+2} = \phi_{n+1}$. Inductively, this defines a cone from X' to $T^\bullet(\mathrm{pt})$, inducing a unique map $\phi : X' \to X$. The uniqueness implies commutativity of

$$\begin{tikzcd}
X' \arrow[r, "\phi"] \arrow[d, "\upsilon'"'] & X \\
TX' \arrow[r, "T\phi"'] & TX \arrow[u, "\upsilon^{-1}"']
\end{tikzcd}$$

which, after inverting υ, means that $\phi : (X', \upsilon') \to (X, \upsilon)$ is a coalgebra homomorphism. Moreover, any coalgebra homomorphism $\psi : (X', \upsilon') \to (X, \upsilon)$ will induce maps $\psi_n : X' \to X \xrightarrow{\pi_n} T^n(\mathrm{pt})$ through the canonical projections from the limit X such that we have commutativity of

$$\begin{tikzcd}
X' \arrow[r, "\psi"] \arrow[d, "\upsilon'"'] & X \arrow[r, "\pi_n"] \arrow[dr, "\pi_{n+1}"] & T^n(\mathrm{pt}) \\
TX' \arrow[r, "T\psi"'] & TX \arrow[r, "T\pi_n"'] \arrow[u, "\upsilon^{-1}"'] & T^{n+1}(\mathrm{pt}) \arrow[u, "T^n!"']
\end{tikzcd}$$

implying that $\psi_{n+1} = T\psi_n \circ \upsilon'$ with $\psi_0 : X' \to \mathrm{pt}$ uniquely determind, meaning inductively that $\psi_n = \phi_n$ for every $n \geq 0$ and thus $\psi = \phi$ after taking limits. $\square$

As **Set** is the terminal object in $\mathfrak{WkCat}$, we can see that **wkωCat** is constructed precisely as in Lemma 5.36. From Lemma 5.34, this construction is preserved by **wk**$(-)$**Cat**, and so we arrive at the universal property of the category of weak ω-categories:

Theorem 5.37. **wkωCat** *is the terminal coalgebra over* **wk**$(-)$**Cat**.

Remark 5.38. Cheng and Leinster describe in [13, Theorem 3.6] how the category of strict ω-categories arises as the terminal coalgebra for the enrichment endofunctor $\mathscr{V} \mapsto \mathscr{V}$**Cat** in the category of cartesian monoidal categories and strong monoidal functors, and they also use Adámek's construction above. However, as the terminal cartesian monoidal category is **1**, the induced maps $(n+1)$**Cat** $\to n$**Cat** send a strict $(n+1)$-category to the underlying strict n-category obtained by throwing away all of the $(n+1)$-morphisms. Indeed, this is because they identified $(\mathbf{1})$**Cat**—the category of categories enriched in the terminal category **1**, which is equivalently the category of complete graphs—with **Set** by identifying a set X with the complete graph on X.

This has little effect in the strict setting, but in order to generalise this approach to weak higher categories, Cheng and Leinster had to work with weak n-categories that lacked coherent composition in dimension n in order to build their weak ω-categories. This is necessary in their approach because forgetting the higher morphisms eliminates the higher coherence isomorphisms necessary for weak composition. The advantage of our approach is that the induced maps $\mathbf{wk}(n+1)\mathbf{Cat} \to \mathbf{wk}n\mathbf{Cat}$ will be given by reducing n-morphisms to their $(n+1)$-isomorphism classes before forgetting the $(n+1)$-morphisms, making our presentation of weak ω-categories a limit of fully coherent weak n-categories instead.

Comparing with the Tamsamani n-nerve. The construction of $\mathbf{wk}n\mathbf{Cat}$ for $0 \leq n \leq \omega$ does not rely on much special structure of $\mathbf{O}$; in particular, we can follow this construction almost verbatim with Δ to reproduce the multi-simplicial higher categories of Tamsamani. More precisely, we can construct an analogous endofunctor

$$(-)\mathbf{Nerve} : \mathfrak{Wk}\mathfrak{Cat} \to \mathfrak{Wk}\mathfrak{Cat}$$

which sends a category $\mathscr{V}$ of weak categories to the full subcategory $\mathscr{V}\mathbf{Nerve} \subseteq \mathrm{Func}(\Delta^{\mathrm{op}}, \mathscr{V})$ of those simplicial objects $\mathcal{A} : \Delta^{\mathrm{op}} \to \mathscr{V}$ such that $\mathcal{A}_0$ is discrete, and $\mathcal{A}_p \to \mathcal{A}_1^{\times_{\mathcal{A}_0} p}$ is a weak equivalence for all $p \geq 0$. Note that Remark 5.13 ensures that the simplicial Segal maps are always surjective.

The induced map $\widehat{\tau}_{\leq 1} := (\tau_{\leq 0})_* : \mathrm{Func}(\Delta^{\mathrm{op}}, \mathscr{V}) \to \mathbf{sSet}$ restricts on $\mathscr{V}$-nerves to Grothendieck nerves of ordinary categories, so we can define 0-truncation of a $\mathscr{V}$-nerve to be the composite

$$\tau_{\leq 0} : \mathscr{V}\mathbf{Nerve} \xrightarrow{\widehat{\tau}_{\leq 1}} \mathbf{Cat} \xrightarrow{\tau_0} \mathbf{Set}$$

as before. The weak equivalences of $\mathscr{V}$-nerves are then the fully faithful and essentially surjective functors, and the objectwise surjections are the surjections on 0-cells.

Beginning with $0\mathbf{Nerve} := \mathbf{Set}$ and setting $(n+1)\mathbf{Nerve} := (n\mathbf{Nerve})\mathbf{Nerve}$, we recover the Tamsamani n-nerves of Section 4.2 for every $n \geq 0$, which are moreover coalgebras over $(-)\mathbf{Nerve}$ in the same way as before. We can then adapt the proof of Theorem 5.37 to find that $\omega\mathbf{Nerve} := \varprojlim_n n\mathbf{Nerve}$ is the terminal coalgebra over $(-)\mathbf{Nerve}$, meaning that our general construction also completes the picture for Tamsamani's construction.

Moreover, we can canonically compare multi-simplicial n-nerves with our presentation of weak n-categories for all $0 \leq n \leq \omega$:

Proposition 5.39. $(-)\mathbf{Nerve}$ *is a subfunctor of* $\mathbf{wk}(-)\mathbf{Cat}$*. In particular, there is a canonical inclusion of Tamsamani n-nerves into weak n-categories for every* $0 \leq n \leq \omega$ *which preserves and reflects weak equivalences.*

Proof. Recall that we have a co-opetopic category $S : \mathbf{O} \to \mathbf{Cat}$ which sends an opetope γ to the chain $[|\gamma|_1]$. This evidently factors through the simplex category $\Delta \hookrightarrow \mathbf{Cat}$ and thus induces a map $S^* : \mathrm{Func}(\Delta^{\mathrm{op}}, \mathscr{V}) \to \mathrm{Func}(\mathbf{O}^{\mathrm{op}}, \mathscr{V})$ for any category $\mathscr{V}$, which is in fact injective on objects and morphisms.

Fix $\mathscr{V} \in \mathfrak{Wk}\mathfrak{Cat}_0$ and suppose $\mathcal{A} \in \mathscr{V}\mathbf{Nerve} \subseteq \mathrm{Func}(\Delta^{\mathrm{op}}, \mathscr{V})$. Then, $(S^*\mathcal{A})_0 = \mathcal{A}_0$ is discrete, and

$(S^*\mathcal{A})_{2[1]} = \mathcal{A}_1 = (S^*\mathcal{A})_1$ means that $t : (S^*\mathcal{A})_{2[1]} \to (S^*\mathcal{A})_1$ is an identity and thus in particular a weak equivalence. Similarly, the Segal maps $(S^*\mathcal{A}_1)^{\times_{(S^*\mathcal{A})_0}|\gamma|_1} \to (S^*\mathcal{A})_\gamma$ are just precisely the simplicial Segal maps $\mathcal{A}_1^{\times_{\mathcal{A}_0}|\gamma|_1} \to \mathcal{A}_{|\gamma|_1}$ and thus are already objectwise surjective weak equivalences. Therefore, S^* restricts to an inclusion $\mathscr{V}\mathbf{Nerve} \hookrightarrow \mathbf{wk}\mathscr{V}\mathbf{Cat}$ that is natural in $\mathscr{V}$, as desired. □

5.2.3 Towards the Homotopy Hypothesis

Now that we have established weak n-categories for all $0 \leq n \leq \omega$, we want to develop a corresponding theory of higher groupoids in order to address some form of the Homotopy Hypothesis mentioned in Section 2.2.1: namely, that homotopy n-types correspond to n-groupoids for every $0 \leq n \leq \infty$. This requires first establishing full subcategories $n\mathbf{Grpd} \subseteq \mathbf{wk}n\mathbf{Cat}$ for every $n \geq 0$, which we will obtain by defining a subfunctor $(-)\mathbf{Grpd}$ of $\mathbf{wk}(-)\mathbf{Cat}$. To motivate this construction, consider the 1-dimensional case: if $\mathscr{G}$ is a (small) groupoid, consider its opetopic nerve $\mathrm{N}\mathscr{G} \in \mathbf{wk1Cat}_0$. The p-ary composition is obtained by the Segal condition on $(\mathrm{N}\mathscr{G})_{2[p]}$, and reflects that the equation $f_1 \dots f_p = f$ can always be solved for f; however, since $\mathscr{G}$ is a groupoid, we can also solve this equation for any of the f_i for $1 \leq i \leq p$, and this can be expressed by a similar isomorphism to the Segal map.

Fix $\mathcal{A} \in \mathbf{wk1Cat}_0$. For $1 \leq i \leq p$, let $\widehat{s_i} : \mathcal{A}_{2[p]} \to \mathcal{A}_1^{\times_{\mathcal{A}_0} p}$ be the universal map induced by the source 1-opetopes $s_j : \mathcal{A}_{2[p]} \to \mathcal{A}_1$ for $j \neq i$ and the target 1-opetope $t : \mathcal{A}_{2[p]} \to \mathcal{A}_1$. Solving $f_1 \dots f_p = f$ for f_i amounts to inverting $\widehat{s_i}$ and taking the composite $\mathcal{A}_1^{\times_{\mathcal{A}_0} p} \xleftarrow{\widehat{s_i}} \mathcal{A}_{2[p]} \xrightarrow{s_i} \mathcal{A}_1$, and this is possible if and only if $\mathcal{A}$ is isomorphic to the opetopic nerve of a groupoid. Indeed, invertibility of $\widehat{s_i}$ is clear when $\mathcal{A}$ is isomorphic to the nerve of a groupoid. Conversely, $\mathcal{A}$ is always isomorphic to the nerve of some category $\mathscr{C}$, so in particular having inverses to $\widehat{s_i} : \mathcal{A}_{2[2]} \to \mathcal{A}_1 \times_{\mathcal{A}_0} \mathcal{A}_1$ means in particular that for any $f : x \to y$ in $\mathscr{C}$, we can find 2[2]-cells in $\mathcal{A}$ that correspond to the commutative diagrams

$$\begin{tikzcd} & y \arrow[dr, dashed] & \\ x \arrow[ur, "f"] \arrow[rr, "\mathrm{id}_x"'] & & x \end{tikzcd} \qquad \begin{tikzcd} & x \arrow[dr, "f"] & \\ y \arrow[ur, dashed] \arrow[rr, "\mathrm{id}_y"'] & & y \end{tikzcd}$$

thus showing that f has both left and right inverses and is thus invertible.

Definition 5.40. If $\mathscr{V} \in \mathfrak{Wk}\mathfrak{Cat}_0$, then define $\mathscr{V}\mathbf{Grpd}$ to be the full subcategory of $\mathbf{wk}\mathscr{V}\mathbf{Cat}$ on those $\mathcal{A} : \mathbf{O}^{\mathrm{op}} \to \mathscr{V}$ satisfying the *groupoid condition* that the map $\widehat{s_i} : \mathcal{A}_{2[p]} \to \mathcal{A}_1$ induced by the source 1-opetopes $s_j : \mathcal{A}_{2[p]} \to \mathcal{A}_1$ for $j \neq i$ and the target 1-opetope $t : \mathcal{A}_{2[p]} \to \mathcal{A}_1$ is an objectwise surjective weak equivalence for all $p \geq 0$ and $1 \leq i \leq p$.

$\mathscr{V}\mathbf{Grpd}$ is closed under discrete fibre products because preservation of the groupoid condition is entirely analogous to the preservation of the Segal condition. Inheriting the weak equivalences, objectwise surjections, and truncation map from $\mathbf{wk}\mathscr{V}\mathbf{Cat}$, this yields a category of weak categories $\mathscr{V}\mathbf{Grpd} \in \mathfrak{Wk}\mathfrak{Cat}_0$. As before, this extends readily to functors, so that we obtain a subfunctor $(-)\mathbf{Grpd}$ of $\mathbf{wk}(-)\mathbf{Cat}$.

Definition 5.41. Define the category of 0-groupoids to be $0\mathbf{Grpd} := \mathbf{Set}$, and inductively define the category of $(n+1)$-groupoids to be $(n+1)\mathbf{Grpd} := (n\mathbf{Grpd})\mathbf{Grpd}$ for every $n \geq 0$. We obtain n-truncation functors $\tau_{\leq n} : (n+1)\mathbf{Grpd} \to n\mathbf{Grpd}$ by inductively applying $(-)\mathbf{Grpd}$ to the truncation map $\tau_{\leq 0} : 1\mathbf{Grpd} \to \mathbf{Set} = 0\mathbf{Grpd}$. This is equivalently the restriction of the truncation maps of Definition 5.32 to higher groupoids. This produces a tower

$$\cdots \longrightarrow 3\mathbf{Grpd} \xrightarrow{\tau_{\leq 2}} 2\mathbf{Grpd} \xrightarrow{\tau_{\leq 1}} 1\mathbf{Grpd} \xrightarrow{\tau_{\leq 0}} 0\mathbf{Grpd}$$

in $\mathfrak{WkCat}$, from which we define the category of ∞-groupoids to be the limit $\infty\mathbf{Grpd} := \varprojlim_n n\mathbf{Grpd}$. Denote by $\tau_{\leq n} : \infty\mathbf{Grpd} \to n\mathbf{Grpd}$ the induced canonical projections.

As $(-)\mathbf{Grpd}$ is a subfunctor of $\mathbf{wk}(-)\mathbf{Cat}$, we obtain a canonical inclusion$\infty\mathbf{Grpd} \to \mathbf{wk}\omega\mathbf{Cat}$ which identifies ∞-groupoids with weak ω-categories whose n-truncation is an n-groupoid for every $n \geq 0$. Moreover, the similarity between the groupoid condition and the Segal condition allows us to retrace our work for $\mathbf{wk}\omega\mathbf{Cat}$ to obtain:

Theorem 5.42. $\infty\mathbf{Grpd}$ *is the terminal coalgebra over* $(-)\mathbf{Grpd}$.

Proof. *Mutatis mutandis*, the argument is just as for Theorem 5.37. □

In particular, this allows to define the Poincaré ∞-groupoid of a topological space coinductively. As our focus in this section is to make steps towards the Homotopy Hypothesis, we will restrict our attention to the full subcategory $\mathbf{CGWH} \subset \mathbf{Top}$ of compactly generated weakly Hausdorff spaces. Indeed, inheriting the cofibrations, fibrations, and weak equivalences from the Quillen model structure in $\mathbf{Top}$, we obtain a Quillen equivalent model structure on $\mathbf{CGWH}$ by [22, Theorems 2.4.23 and 2.4.25]. The benefit is that the space $\mathrm{Map}(X,Y)$ of continuous functions $X \to Y$ between compactly generated weakly Hausdorff spaces with the compact-open topology defines an internal hom for the cartesian monoidal structure on $\mathbf{CGWH}$, and [22, Proposition 4.2.11] shows that this makes $\mathbf{CGWH}_{\mathrm{Quillen}}$ a cartesian model category. The Quillen model structure on $\mathbf{CGWH}$ also makes it into a category of weak categories:

Lemma 5.43. $(\mathbf{CGWH}, W, E, \pi_0) \in \mathfrak{WkCat}_0$*, where* W *is the class of weak homotopy equivalences,* E *is the class of continuous maps that are surjective on the underlying sets, and* π_0 *sends a space to its set of path-connected components.*

Proof. $\mathbf{CGWH}$ is complete and thus in particular has discrete fibre products. Moreover, the fibre product of sets in $\mathbf{CGWH}$ will always be a set. The surjections are preserved along discrete fibre products from the same diagram chase as done for $\mathbf{Set}$ in Example 5.18. It is easy to see that π_0 preserves discrete fibre products and surjections, while it sends weak homotopy equivalences to bijections by definition.

Preservation of weak equivalences along discrete fibre products can be checked explicitly, but also follows from the fact that discrete fibre products are homotopy limits in $\mathbf{CGWH}_{\mathrm{Quillen}}$. In general, for a Reedy category $\mathscr{R}$ and any model category $\mathscr{M}$, there is a canonical model structure on $\mathrm{Func}(\mathscr{R},\mathscr{M})$ called the *Reedy model structure*, where the weak equivalences are the levelwise weak equivalences. The constant functor $\mathscr{M} \to \mathrm{Func}(\mathscr{R},\mathscr{M})_{\mathrm{Reedy}}$ will be a left Quillen functor, and so in particular its

right adjoint $\varprojlim : \mathrm{Func}(\mathscr{R}, \mathscr{M}) \to \mathscr{M}$ is a right Quillen functor. By Ken Brown's Lemma, this proves that taking limits preserves weak equivalences of Reedy fibrant diagrams. The relevant theory for this argument is discussed in more detail in [22, §5.2]. The walking cospan is especially a Reedy category, and the corresponding Reedy fibrant cospans are those whose objects are fibrant with one of the leg morphisms being a fibration. In $\mathbf{CGWH}_{\mathrm{Quillen}}$, this means all cospans of the form $A \to \mathrm{disc}\, S \leftarrow B$ are Reedy fibrant, since all maps into a discrete set are Serre fibrations, and all spaces are already fibrant. In particular, this means that levelwise weak equivalences between such cospans induce weak equivalences of limits, as desired. □

The goal now is to endow **CGWH** with coalgebraic structure over $(-)\mathbf{Grpd}$, so that the Poincaré ∞-groupoid construction arises as the unique coalgebra homomorphism $\Pi_\infty : \mathbf{CGWH} \to \infty\mathbf{Grpd}$. Intuitively, the coalgebraic structure on **CGWH** realises a space X as a groupoid on its underlying set that is weakly enriched in the $(\infty,1)$-category of spaces, taking its morphisms to be the continuous paths in X. We will formalise this with the co-opetopic space $J : \mathbf{O} \to \mathbf{CGWH}$ constructed in Definition 4.24.

As in Section 3.3, the co-opetopic space J induces a nerve $\mathrm{N} : \mathbf{CGWH} \to \mathrm{Func}(\mathbf{O}^{\mathrm{op}}, \mathbf{CGWH})$ where $(\mathrm{N}X)_\gamma := \mathrm{Map}(J(\gamma), X)$. However, $\mathrm{N}X$ will not be an object of $(\mathbf{CGWH})\mathbf{Grpd}$ for the simple reason that $(\mathrm{N}X)_0 = X$ as spaces, and is thus not discrete unless X is. For such a nerve to be effective, we would have to develop an opetopic analogue of *complete Segal spaces* as originally defined by Rezk in [38] rather than Segal categories, as Segal spaces allow the set of objects to be endowed with nontrivial topology. Therefore, we restrict the nerve as follows:

Definition 5.44. For an opetope γ, let $v_0, \ldots, v_{|\gamma|_1}$ be the 0-opetopes of γ. For any $x_0, \ldots, x_{|\gamma|_1} \in X$, let $\mathrm{Map}_{\vec{x}}(\gamma, X)$ be the subspace of continuous functions $f : J(\gamma) \to X$ such that $f(v_i) = x_i$ for all i. Then, define $(\widehat{\mathrm{N}}X)_\gamma := \coprod_{\vec{x}} \mathrm{Map}_{\vec{x}}(\gamma, X)$ to be the disjoint union of all these possible spaces. Given a coface $\delta \to \gamma$, so we obtain a continuous map $(\widehat{\mathrm{N}}X)_\gamma \to (\widehat{\mathrm{N}}X)_\delta$ simply by restricting the functions $J(\gamma) \to X$ to $J(\delta)$. This construction of $\widehat{\mathrm{N}}$ extends canonically to continuous maps by postcomposition to define the *restricted nerve* $\widehat{\mathrm{N}} : \mathbf{CGWH} \to \mathrm{Func}(\mathbf{O}^{\mathrm{op}}, \mathbf{CGWH})$.

Lemma 5.45. *The restricted nerve* $\widehat{\mathrm{N}} : \mathbf{CGWH} \to \mathrm{Func}(\mathbf{O}^{\mathrm{op}}, \mathbf{CGWH})$ *defines a coalgebraic structure on* **CGWH** *over* $(-)\mathbf{Grpd}$.

Proof. We need to first show that $\widehat{\mathrm{N}}X \in (\mathbf{CGWH})\mathbf{Grpd}_0$ for every space X. By design, $(\widehat{\mathrm{N}}X)_0 = \mathrm{disc}\, X$, so it remains to verify the unary condition, the Segal condition, and the groupoid condition, which all amount to checking that appropriate maps are surjective weak homotopy equivalences. The key is that the map $J(1) \to J(2[1])$ corresponding to the unary condition, and the maps $J(1)^{\amalg_{J(0)}|\gamma|_1} \to J(\gamma)$ corresponding to the Segal and groupoid conditions, are all acyclic cofibrations of cofibrant objects in **CGWH**. As the arguments are similar for all three conditions, we will only explicitly check the Segal condition. Fix an opetope γ.

By Proposition 4.25, the map $\iota : J(1)^{\amalg_{J(0)}|\gamma|_1} \to J(\gamma)$ is an inclusion of contractible CW complexes and is therefore an acyclic cofibration of cofibrant objects in **CGWH**. In particular, this ensures that we

have a lift

$$\begin{tikzcd} J(1)^{\amalg_{J(0)}|\gamma|_1} \arrow[r, "\mathrm{id}"] \arrow[d, tail, "\iota"'] & J(1)^{\amalg_{J(0)}|\gamma|_1} \arrow[d, two heads, "!"] \\ J(\gamma) \arrow[r, "!"'] \arrow[ur, dashed, "\exists"] & \mathrm{pt} \end{tikzcd}$$

which is a retraction of ι. The Segal map $(\widehat{\mathrm{N}}X)_\gamma \to (\widehat{\mathrm{N}}X)_1^{\times_{(\widehat{\mathrm{N}}X)_0}|\gamma|_0}$ is the coproduct over all points $x_0,\dots,x_{|\gamma|_1} \in X$ of the maps

$$\mathrm{Map}_{\vec{x}}(\gamma,X) \to \mathrm{Map}_{(x_0,x_1)}(1,X) \times_{\{x_1\}} \mathrm{Map}_{(x_1,x_2)}(1,X) \times_{\{x_2\}} \cdots \times_{\{x_{|\gamma|_1-1}\}} \mathrm{Map}_{(x_{|\gamma|_1-1},x_{|\gamma|_1})}(1,X) \quad \text{(Eq. 5.8)}$$

For a space Y with marked points $y_0,\dots,y_{|\gamma|_1} \in Y$, let $\mathrm{Map}_{\vec{x}}(Y,X)$ be the space of continuous functions $f : Y \to X$ where $f(y_i) = x_i$ for every i. Then, the codomain of the map in (Eq. 5.8) is homeomorphic to $\mathrm{Map}_{\vec{x}}(J(1)^{\amalg_{J(0)}|\gamma|_1},X)$ by concatenating paths. This can be checked explicitly, but also follows from the cartesian closed structure of **CGWH**, as this implies that the internal hom $\mathrm{Map}(-,-)$ is continuous and thus sends colimits in the first variable to limits, after which the desired homeomorphism follows by restricting to the fibres over $\vec{x}$. The map in (Eq. 5.8) is thus precisely $\mathrm{Map}_{\vec{x}}(\iota,X)$. As we have seen that ι admits a retraction, this means $\mathrm{Map}_{\vec{x}}(\iota,X)$ admits a section and is thus in particular surjective.

View the $x_0,\dots,x_{|\gamma|_1} \in X$ as marked points, and likewise the points $v_0,\dots,v_{|\gamma|_1}$ as marked points of $J(\gamma)$ and $J(1)^{\amalg_{J(0)}|\gamma|_1}$. The forgetful functor $\mathbf{CGWH}_{(|\gamma|_1+1)} \to \mathbf{CGWH}$ from the category of spaces with $|\gamma|_1+1$ marked points and their continuous functions that preserve the marking has a left adjoint which transfers the Quillen model structure of **CGWH** to the marked context, meaning that maps are cofibrations, fibrations, or weak equivalences in $\mathbf{CGWH}_{(|\gamma|_1+1)}$ if and only if they are when viewed as unmarked maps. Moreover, $\mathbf{CGWH}_{(|\gamma|_1+1)}$ will be a monoidal model category. This follows by iteratively applying [22, Proposition 4.2.9]. In particular, since all spaces are fibrant, the functor $\mathrm{Map}_{\vec{x}}(-,X)$ will preserve weak equivalences between cofibrant objects by [22, Lemma 4.2.2] and Ken Brown's Lemma. Since CW complexes are cofibrant, it follows that $\mathrm{Map}_{\vec{x}}(\gamma,X) \to \mathrm{Map}_{\vec{x}}(J(1)^{\amalg_{J(0)}|\gamma|_1},X)$ is a weak homotopy equivalence.

Taking coproducts over all markings $\vec{x}$ of X, this shows that the Segal map for γ induces an isomorphism $\pi_n((\widehat{\mathrm{N}}X)_\gamma,\theta) \to \pi_n((\widehat{\mathrm{N}}X)_1^{\times_{(\widehat{\mathrm{N}}X)_0}|\gamma|_1},\theta)$ for all basepoints θ and all $n \geq 1$. Since the fibres for each marking $\vec{x}$ of X is disjoint, the Segal map will also induce a bijection on path-connected components. Therefore, the Segal maps in $\widehat{\mathrm{N}}X$ are surjective weak homotopy equivalences, verifying the Segal condition. The unary condition and groupoidal condition can be checked in the same way, meaning that $\widehat{\mathrm{N}}X \in (\mathbf{CGWH})\mathbf{Grpd}_0$ for every space X. For this construction to define a coalgebra over $(-)\mathbf{Grpd}$, we need to ensure that $\widehat{\mathrm{N}}$ preserves all of the structure of **CGWH** as a category of weak categories.

We can readily check that $\widehat{\mathrm{N}}$ preserves discrete objects and surjections from its construction, and it preserves all limits because of the continuity of the internal hom in the second variable and how limits commute with taking fibres. Suppose $f : X \to Y$ is a weak homotopy equivalence, then we want to see that $\widehat{\mathrm{N}}f$ is essentially surjective and fully faithful. Note that the 1-truncation of $\widehat{\mathrm{N}}X$ by definition recovers the fundamental groupoid $\Pi_1 X = \widehat{\tau}_{\leq 1}(\widehat{\mathrm{N}}X)$, which means that $\tau_{\leq 0}(\widehat{\mathrm{N}}X) = \pi_0(X)$ and thus

$\tau_{\leq 0}\widehat{\mathrm{N}}f$ is bijective from f inducing a bijection on π_0. This establishes essential surjectivity. As for being fully faithful, note for $x, y \in (\widehat{\mathrm{N}}X)_0$ that the discrete fibre product defining the hom-space from Definition 5.26 gives precisely that $(\widehat{\mathrm{N}}X)(x,y) = \mathrm{Map}_{(x,y)}(1,X) = \mathrm{Map}_{(x,y)}(J(1),X)$. Since the space $J(1) \cong [0,1]$ is a CW complex and thus cofibrant in **CGWH**, it follows that $\mathrm{Map}_{(x,y)}(J(1),-)$ preserves weak equivalences of fibrant objects, and thus $f : X \to Y$ being a weak homotopy equivalence means the same for $(\widehat{\mathrm{N}}f)_{x,y}$. Therefore, $\widehat{\mathrm{N}}$ preserves weak equivalences as well. □

The now established coalgebra structure on **CGWH** allows us to employ Theorem 5.42 to coinductively define the Poincaré ∞-groupoid of a space:

Definition 5.46. Define the Poincaré ∞-groupoid construction to be the unqiue $(-)$**Grpd**-coalgebra homomorphism $\Pi_\infty : \mathbf{CGWH} \to \infty\mathbf{Grpd}$. Accordingly, define the Poincaré n-groupoid to be the n-truncation $\Pi_n := \tau_{\leq n}\Pi_\infty : \mathbf{CGWH} \to n\mathbf{Grpd}$.

Theorem 5.47. *The Poincaré ∞-groupoid of a compactly generated and weakly Hausdorff space X satisfies the following:*

(i) $\tau_{\leq 0}\Pi_\infty X = \pi_0 X$ *and* $\tau_{\leq 1}\Pi_\infty X = \Pi_1 X$*, where* $\Pi_1 X$ *here is the fundamental groupoid of* X*.*

(ii) The n-morphisms $f \to g$ in $\Pi_\infty X$ are the boundary-preserving homotopies from f to g in X for every $n \geq 1$.

(iii) For $x \in X$ and $n \geq 1$, the composition of n-morphisms in $\Pi_\infty X$ induces group structure on the hom-set

$$\Pi_n(X,x) := (\tau_{\leq n}\Pi_\infty X)\underbrace{(x,x)(x,x)\dots(x,x)}_{n \text{ times}}$$

and moreover this group is isomorphic to $\pi_n(X,x)$.

Additionally, a map $f : X \to Y$ is a weak homotopy equivalence of spaces if and only if the corresponding weak ω-functor $\Pi_\infty f : \Pi_\infty X \to \Pi_\infty Y$ is an equivalence of ∞-groupoids.

Proof. As Π_∞ must preserve 0-truncation, we have $\tau_{\leq 0}\Pi_\infty = \pi_0$. Similarly, we have $\tau_{\leq 1}\Pi_\infty = \Pi_1$ by the commutativity of the diagram

$$\begin{tikzcd}
 & \mathbf{CGWH} \arrow[d, "\widehat{\mathrm{N}}"'] \arrow[ddl, bend right, "\Pi_\infty"'] \arrow[ddr, bend left, "\Pi_1"] & \\
 & (\mathbf{CGWH})\mathbf{Grpd} \arrow[d, "(\Pi_\infty)_*"'] \arrow[dr, "\widehat{\tau}_{\leq 1}"] & \\
\infty\mathbf{Grpd} \arrow[r, "\cong"] \arrow[rr, bend right, "\tau_{\leq 1}"'] & (\infty\mathbf{Grpd})\mathbf{Grpd} \arrow[r, "\widehat{\tau}_{\leq 1}"] & \mathbf{Grpd}
\end{tikzcd}$$

where the square on the left commutes from Π_∞ being a coalgebra homomorphism, the upper triangle commutes by the construction of $\widehat{\mathrm{N}}$ as observed win Lemma 5.45, the middle triangle commutes from

the identity on 0-truncations applied levelwise, as per the definition of the $\widehat{\tau}_{\leq 1}$, and the bottom triangle commutes by the definition of $\tau_{\leq 1}$. This proves (i), which allows us to write $\Pi_n := \tau_{\leq n}\Pi_\infty$ as in Definition 5.46 without risk of ambiguity.

From Remark 5.35, the n-morphisms of $\Pi_\infty X$ can be obtained just from looking at the n-morphisms of $\Pi_{n+1}X$, which we can understand inductively through the restricted nerve $\widehat{\mathrm{N}}$. Explicitly, the proof of Adámek's construction applied to $(-)\mathbf{Grpd}$ shows that $\Pi_\infty : \mathbf{CGWH} \to \infty\mathbf{Grpd}$ is obtained from the family of morphisms $\phi_n : \mathbf{CGWH} \to n\mathbf{Grpd}$ defined inductively with $\phi_0 := \pi_0 : \mathbf{CGWH} \to \mathbf{Set}$ and ϕ_{n+1} being the composite $\mathbf{CGWH} \xrightarrow{\widehat{\mathrm{N}}} (\mathbf{CGWH})\mathbf{Grpd} \xrightarrow{(\phi_n)_*} (n+1)\mathbf{Grpd}$, where $(\phi_n)_*$ acts levelwise. Moreover, these maps ϕ_n were uniquely determined to be the composites

$$\phi_n : \mathbf{CGWH} \xrightarrow{\Pi_\infty} \infty\mathbf{Grpd} \xrightarrow{\tau_{\leq n}} n\mathbf{Grpd}$$

which are precisely the maps Π_n. Therefore, we obtain an inductive definition of Π_n with $\Pi_0 = \pi_0$ and $\Pi_{n+1} = (\Pi_n)_* \circ \widehat{\mathrm{N}}$. Unrolling this induction shows that $\Pi_n X$ is obtained by applying $\widehat{\mathrm{N}}$ levelwise n times, and then taking path-connected components of each level as in

$$\begin{aligned} &\mathbf{CGWH} \xrightarrow{\widehat{\mathrm{N}}} (\mathbf{CGWH})\mathbf{Grpd} \xrightarrow{(\widehat{\mathrm{N}})_*} ((\mathbf{CGWH})\mathbf{Grpd})\mathbf{Grpd} \xrightarrow{(\widehat{\mathrm{N}})_*} \dots \\ &\dots \xrightarrow{(\widehat{\mathrm{N}})_*} ((((\mathbf{CGWH})\mathbf{Grpd})\dots)\mathbf{Grpd})\mathbf{Grpd} \xrightarrow{(\pi_0)_*} n\mathbf{Grpd} \end{aligned}$$

As $(\widehat{\mathrm{N}}X)(x,y) = \mathrm{Map}_{(x,y)}(1,X)$ is the space of paths from x to y for any $x,y \in X$, this observation inductively proves (ii). For instance, given paths $f,g : x \to y$ in X, the 2-morphisms $f \Rightarrow g$ are the points of $\left(\widehat{\mathrm{N}}(\widehat{\mathrm{N}}(x,y))\right)(f,g) = \mathrm{Map}_{(f,g)}(1, \mathrm{Map}_{(x,y)}(1,X))$, which are precisely the endpoint-preserving homotopies $f \simeq g$.

By iteratively taking hom-groupoids of $\Pi_n X$ at a point $x \in X$, we find that

$$\Pi_n(X,x) := (\Pi_n X)\underbrace{(x,x)(\mathrm{id}_x,\mathrm{id}_x)\dots(\mathrm{id}_x,\mathrm{id}_x)}_{n \text{ times}} = \pi_0 \mathrm{Map}_{(\mathrm{id}_x,\mathrm{id}_x)}(1, \mathrm{Map}_{(\mathrm{id}_x,\mathrm{id}_x)}(1, \dots \mathrm{Map}_{(x,x)}(1,X)\dots))$$

After currying the right hand side, this says that $\Pi_n(X,x)$ is the set of homotopy classes of maps $f : J(1)^n \to X$ such that $f(p_1,\dots,p_n) = x$ once any p_i is given by a source or target vertex of $J(1)$. This is precisely the homotopy classes of pointed maps $(\mathbb{S}^n,*) \to (X,x)$, which is the underlying set of $\pi_n(X,x)$. Showing (iii) now reduces to showing that composition in $\Pi_n X$ of n-morphisms is given by path concatenation when $n \geq 1$, but this follows inductively from the fact that Π_1 recovers the fundamental groupoid of X wherein composition is precisely path concatenation.

It remains to show that Π_∞ preserves and reflects weak equivalence. Preservation follows from the fact that Π_∞ is a morphism of $\mathfrak{WkCat}$, so conversely suppose $f : X \to Y$ is a continuous map such that $\Pi_\infty f : \Pi_\infty X \to \Pi_\infty Y$ is an equivalence of ∞-groupoids. For $x \in X$, fully faithfulness of $\Pi_\infty f$ inductively implies that the induced maps $(\Pi_\infty X)(x,x)\dots(x,x) \to (\Pi_\infty Y)(fx,fx)\dots(fx,fx)$ are weak equivalences. Taking the n-truncation of these maps induce isomorphisms $\pi_n(X,x) \to \pi_n(Y,fx)$ of groups for all $n \geq 1$ from (iii). For $n = 0$, (i) implies that the 0-truncation $\Pi_0 f = \pi_0 f$ of $\Pi_\infty f$ is

a bijection as well. Therefore, if $\Pi_\infty f$ is an equivalence of ∞-groupoids, then f is already a weak homotopy equivalence, as desired. $\square$

Corollary 5.48. *A space $X \in \mathbf{CGWH}_0$ is a homotopy n-type if and only if $\Pi_\infty X$ is an n-groupoid, in the sense that it lies in the image of the canonical inclusion $n\mathbf{Grpd} \hookrightarrow \infty\mathbf{Grpd}$.*

Remark 5.49. Since continuous maps of spaces are uniquely determined by their action on elements, and the elements of a space make up the objects of its Poincaré ∞-groupoid, it follows that Π_∞ is faithful. However, the functor is likely not full, because continuous maps induce strictly unital weak functors of higher groupoids when the higher identity morphisms are all chosen to be constant homotopies.

This establishes most of the Homotopy Hypothesis, leaving out only the converse direction: given an ∞-groupoid $\mathcal{G}$, can we find a space $K(\mathcal{G})$ and a weak equivalence $\mathcal{G} \simeq \Pi_\infty K(\mathcal{G})$? Tamsamani proposed a construction for his simplicial n-nerves that modelled n-groupoids in [46, §2.5] which provided for every multisimplicial cell a product of geometric realisations of standard simplices, and then glued them together according to the structure maps of the n-nerve. This generalises the construction in [4, §1.4] of $K(\mathscr{G},1)$ for a 1-groupoid $\mathscr{G}$ alluded to in Section 2.2.1. Proceeding similarly, a reasonable guess for $K(\mathcal{G})$ given an ∞-groupoid $\mathcal{G}$ would be the quotient space

$$K(\mathcal{G}) := \coprod_{n\geq 0} \coprod_{\gamma_1,\ldots,\gamma_n} \left(\mathcal{G}_{\gamma_1,\ldots,\gamma_n,0} \times \prod_{i=1}^{n} J(\gamma_i) \right) \Big/ (\sim)$$

computed in **CGWH**, where $(\sim)$ encodes the structure of $\mathcal{G}$. More explicitly, $(\sim)$ would identify $\mathcal{G}_{\gamma_1,\ldots,\gamma_n,0} \sim \mathcal{G}_{\gamma_1,\ldots,\gamma_k,0,\ldots,0}$ for $k \leq n$ to preserve the discreteness property of $\mathcal{G}$. Given maps $j_i : \delta_i \to \gamma_i$ in $\mathbf{O}$, the relation $(\sim)$ would also identify points according to the pushout

$$\begin{tikzcd}
\mathcal{G}_{\gamma_1,\ldots,\gamma_n,0} \times \prod_{i=1}^{n} J(\delta_i) \arrow[r, "\prod_{i=1}^{n} J(j_i)"] \arrow[d, "\mathcal{G}_{j_1,\ldots,j_n,0}"'] \arrow[dr, phantom, "\ulcorner"] & \mathcal{G}_{\gamma_1,\ldots,\gamma_n,0} \times \prod_{i=1}^{n} J(\gamma_i) \arrow[d] \\
\mathcal{G}_{\delta_1,\ldots,\delta_n,0} \times \prod_{i=1}^{n} J(\delta_i) \arrow[r] & \left[\dfrac{\left(\mathcal{G}_{\gamma_1,\ldots,\gamma_n,0} \times \prod_{i=1}^{n} J(\gamma_i)\right) \amalg \left(\mathcal{G}_{\delta_1,\ldots,\delta_n,0} \times \prod_{i=1}^{n} J(\delta_i)\right)}{(\sim)} \right]
\end{tikzcd}$$

Therefore, $K(\mathcal{G})$ is the space obtained by giving each multi-opetopic cell of $\mathcal{G}$ a geometric realisation and then gluing these realisations together according to the coface maps of the opetopes involved. While one might intuitively expect the Poincaré ∞-groupoid of $K(\mathcal{G})$ to be weakly equivalent to $\mathcal{G}$ since any higher homotopy in $K(\mathcal{G})$ by construction should admit a homotopy to one corresponding to a higher morphism in $\mathcal{G}$, we are unsure of how to approach this formally and thus leave it to conjecture:

Conjecture 5.50. *Given an ∞-groupoid $\mathcal{G}$, the space $K(\mathcal{G})$ as defined above admits a weak equivalence $\mathcal{G} \to \Pi_\infty K(\mathcal{G})$.*

In conjunction with Theorem 5.47, this would establish an appropriate version of the Homotopy Hypothesis for our unbiased opetopic presentation of weak ω-categories.

Chapter 6

Conclusion

The main goal of this thesis was to produce a uniform, inductive algorithm for constructing weak higher categories and their corresponding weak functors. To this end, the thesis was very successful, producing the categories $\mathbf{wk}n\mathbf{Cat}$ of weak n-categories for every finite $n \geq 0$ through a notion of weak iterated internalisation. The approach is similar to the construction of higher nerves due to Tamsamani, but the obstacle to higher nerves modelling weakly unital functors with simplicial nerves was the presence of strictly preserved codegeneracy maps.

The idea for our construction was to replace the simplex category with a category of shapes without codegeneracy maps, encoding the units of higher categories in the same way as composition. This led to the notion of unbiased composition rules and the construction of the category of opetopes, after which Tamsamani's construction could be retraced with little additional effort. As units were replaced with nullary composition, the resulting functors were necessarily already weakly unital, and we proved this precisely for higher categories of dimension up to two. Moreover, the argument used for the proof of correctness offers a natural generalisation which would prove the same result for higher dimensions given sufficient higher categorical language; namely, given the appropriate notion of higher limits.

By describing the inductive construction of higher categories through an endofunctor $\mathbf{wk}(-)\mathbf{Cat}$, we were also able to extend the construction indefinitely to define the category $\mathbf{wk}\omega\mathbf{Cat}$ of infinite-dimensional weak ω-categories, which encompasses all higher categories obtained from the original induction. The technique used the theory of endofunctor coalgebras, from which we realised $\mathbf{wk}\omega\mathbf{Cat}$ to be the universal coalgebra over $\mathbf{wk}(-)\mathbf{Cat}$. This was similar to an earlier approach of Cheng and Leinster, but in our case the finite truncations of $\mathbf{wk}\omega\mathbf{Cat}$ recover our weak n-categories exactly, rather than partially incoherent n-categories resulting from just forgetting all higher morphisms.

We then adapted the above construction to yield an inductive construction of n-groupoids via a subfunctor $(-)\mathbf{Grpd}$ of $\mathbf{wk}(-)\mathbf{Cat}$, with the intention of testing this new presentation of higher categories against the Homotopy Hypothesis. With $\infty\mathbf{Grpd}$ being a terminal coalgebra as well, the Poincaré ∞-groupoid construction Π_∞ arises by universal property from the coalgebraic structure on the category of (nice) topological spaces. We proved that Π_∞ identifies homotopy n-types with n-groupoids for all $0 \leq n \leq \infty$, and also identifies weak homotopy equivalences of spaces with equivalences of ∞-groupoids. Moreover, we showed that the higher morphisms of the Poincaré ∞-groupoid are precisely

the higher homotopies of the corresponding space, as desired. In particular, we showed that the Poincaré ∞-groupoid contains the homotopy groups of the space as automorphism groups.

Finally, we constructed a candidate space $K(\mathcal{G})$ for any ∞-groupoid $\mathcal{G}$ whose Poincaré ∞-groupoid should be equivalent to $\mathcal{G}$. Modulo the conjectured correctness of this construction, this would prove that our proposed presentation for weak ω-categories is both capable of presenting weakly unital functors and also satisfies the Homotopy Hypothesis.

With the numerous competing definitions for weak higher categories already in literature and folklore, our construction of **wk$n$Cat** for $0 \leq n \leq \omega$ provides several advantages:

- We can readily extract the k-morphisms of a weak ω-category for any $k \geq 0$. This is similar to the multisimplicial constructions of weak n-categories by Tamsamani in [46] and (∞, n)-categories by Simpson in [43], and similar also to the complete Segal spaces of Rezk in [38]. Note that this is unlike the case for the quasicategories of Boardman and Vogt in [9] and similar stratified generalisations such as the weak complicial sets of [47], both which describe higher categories as simplicial sets with appropriate lifting properties.

- More generally, we can readily construct hom-ω-categories of a weak ω-category between any two objects as observed in Remark 5.35. This separates our construction also from even the unbiased presentations of weak ω-categories via opetopic sets as constructed by Baez and Dolan in [3], or multitopic sets as introduced by Hermida, Makkai, and Power in [20] and [34].

- The functors between weak higher categories are themselves weak; in particular, they are weakly unital, just as in the other opetopic and multitopic approaches, but the approaches using simplicial sets.

- The construction of weak higher categories of finite dimension is inductive, unlike the purely algebraic approaches or the quasicategorical model. Moreover, the weak ω-categories can be studied through their finite truncations.

- The Poincaré ∞-groupoid construction from compactly generated weakly Hausdorff spaces to ∞**Grpd** is canonical, even encoding all of the higher homotopy groups. Modulo one conjecture, our construction of higher categories satisfies the Homotopy Hypothesis, which is an important litmus test that all presentations of higher categories should pass.

6.1 Future Work

Beyond resolving Conjecture 5.50, there are important extensions to this work that need to be addressed in order to establish a solid theory of higher categories with this model. Most importantly, we lack appropriate weak ω-categories $\mathrm{Func}(\mathcal{A}, \mathcal{B})$ of functors $\mathcal{A} \to \mathcal{B}$ for $\mathcal{A}, \mathcal{B} \in$ **wkωCat**$_0$. Just as **Cat** is canonically a 2-category and **Bicat** is similarly a tricategory, **wk$n$Cat** should carry the structure of a weak $(n+1)$-category for every $n \geq 0$, where the morphisms are weak functors, 2-morphisms are weak natural transformations, 3-morphisms are weak modifications, and so on. In particular, **wkωCat** should

carry the structure of a (huge) weak ω-category. From our discussion of 2-limits in Section 5.1.1, establishing functor ω-categories in $\mathbf{wk}\omega\mathbf{Cat}$ would provide a means to define weak limits, which are essential for universal constructions in higher category theory.

Constructing weak functor ω-categories in $\mathbf{wk}\omega\mathbf{Cat}$ is also a special case of the problem of formalising *weak enrichment* in a higher category. The current state of the art for weak enrichment has been developed by Gepner and Haugseng in [16], where they worked in the context of quasicategories. The idea behind their approach generalised a correspondence between categories (strictly) enriched in a monoidal category $\mathscr{V}$ with set S of objects, and functors between the associative operad on S to the multicategory $\mathscr{V}^{\otimes}$ (cf. Example 4.19). They thus used quasicategorical analogues of operads to define enrichment in a monoidal quasicategory $\mathscr{V}$, and constructed a quasicategory of $\mathscr{V}$-enriched categories. However, by being simplicial in nature, all of the resulting enriched $(\infty,1)$-functors are necessarily strictly unital, and the construction is moreover limited to studying the $(\infty,1)$-category theory of the enrichment, unless the monoidal structure on $\mathscr{V}$ is sufficiently symmetric.

Their operadic approach to enrichment is very closely related to the observations made at the end of Section 2.2.3 that categories (strictly) enriched in a monoidal category $\mathscr{V}$ with object set S correspond to lax functors from the codiscrete category on S to the delooping of $\mathscr{V}$. In fact, the author had begun this thesis with the lofty ambition of making headway in the direction of weak enrichment, and this observation is what had originally motivated formalising a theory of higher categories with weakly unital functors. The *Delooping Hypothesis* (cf. [4, Hypothesis 22]) identifies monoidal weak n-categories with weak $(n+1)$-categories with one object for all $0 \leq n \leq \omega$. Under this hypothesis, we can define a *monoidal weak ω-category* to be a weak ω-category $\mathcal{A}$ with $\mathcal{A}_0 = \mathrm{pt}$. Then, categories with object set S that are weakly enriched in $\mathcal{A}$ should correspond to *lax ω-functors* $\operatorname{codisc} S \to \mathscr{V}$.

One way to formalise lax ω-functors would be to relax the axioms of higher opetopic nerves described in the beginning of Section 5.2 so that the nerves correspond to *lax* higher categories, where the associativity coherence constraints are expressed by higher morphisms rather than equivalences. The corresponding maps between lax nerves should then present higher lax functors. To see this, consider the 2-dimensional case, where we consider lax unbiased bicategories in the sense of [31, Definition 3.4.1]. There is a generalisation of the unbiased double nerve of Section 5.1 which makes sense also for a lax unbiased bicategory $\mathscr{B}$: for an opetope γ, take the γ-cells of the nerve $\mathrm{N}_{\mathrm{Lax}}\mathscr{B}$ to be families of the form $(x_\bullet, f_\bullet, \theta_\bullet)$ as with the unbiased double nerve, but allow the 2-morphisms $\theta_\bullet$ to be non-invertible. Following the proof of Lemma 5.15 on the fully faithfulness of the unbiased double nerve construction, we can see that for any map $F : \mathrm{N}_{\mathrm{Lax}}\mathscr{B} \to \mathrm{N}_{\mathrm{Lax}}\mathscr{B}'$ between unbiased (lax) bicategories $\mathscr{B}$ and $\mathscr{B}'$, the p-ary functoriality constraint F^p is no longer necessarily invertible, meaning that F corresponds to a lax functor $\mathscr{B} \to \mathscr{B}'$.

A major obstacle in generalising weak ω-categories to a lax coherent context is that there is no clear analogue of the Segal condition. Even for an unbiased bicategory $\mathscr{B}$ that is not necessarily lax, the Segal maps $(\mathrm{N}_{\mathrm{Lax}}\mathscr{B})_\gamma \to (\mathrm{N}_{\mathrm{Lax}}\mathscr{B})^{\times_{\mathscr{B}_0}|\gamma|_1}$ are no longer equivalences of categories. Therefore, the first challenge is to characterise the (weak) essential image of $\mathrm{N}_{\mathrm{Lax}} : \mathbf{UBicat}_{\mathrm{Lax}} \to \mathbf{Cat}$.

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Appendix A

The Grothendieck Construction

Recall from Section 2.2.3 that for any monoidal category $\mathscr{V}$ and any set S of objects, we can define a $\mathscr{V}$-enriched category as a lax functor $\operatorname{codisc} S \to \mathbf{B}\mathscr{V}$. As in the case for monoids in $\mathscr{V}$, we can extend this to define a category $\mathscr{V}\mathbf{Cat}_S$ where the morphisms between $\mathscr{V}$-enriched categories on S are the oplax natural transformations whose component 1-morphisms are given by the tensor unit $\mathbb{1}$ in $\mathscr{V}$. However, the desired category $\mathscr{V}\mathbf{Cat}$ requires allowing S to vary, and this only provides a $\mathbf{Set}_0$-indexed family of categories which appear to be fibres of $\mathscr{V}\mathbf{Cat}$.

More precisely, the appropriate category $\mathscr{V}\mathbf{Cat}$ should come with a canonical functor $\mathscr{V}\mathbf{Cat} \to \mathbf{Set}$ that sends $\mathscr{V}$-enriched categories to their sets of objects, such that $\mathscr{V}\mathbf{Cat}_S$ is the subcategory which lies over S via this projection for any set S. As the construction of $\mathscr{V}\mathbf{Cat}_S$ is contravariantly functorial over $\mathbf{Set}$, we can reverse this intuition and recover $\mathscr{V}\mathbf{Cat}$ via the *Grothendieck construction* described in [24, §B1.3].

Explicitly, given a pseudofunctor $F : \mathscr{C}^{\mathrm{op}} \to \mathbf{Cat}$ from an ordinary category $\mathscr{C}$ viewed as a locally discrete bicategory into the 2-category of small categories, the Grothendieck construction explicitly yields the category $\mathscr{C} \int F$ where

- the objects are pairs (c,x) where $c \in \mathscr{C}_0$ and $x \in (Fc)_0$
- the morphisms $(c,x) \to (d,y)$ are given by pairs (f,ϕ) where $f : c \to d$ in $\mathscr{C}$ and $\phi : x \to (Ff)(y)$ in Fc

and composition is defined as $(g,\psi) \circ (f,\phi) := (g \circ f, (Ff)(\psi) \circ \phi)$. This category has an obvious canonical projection $\mathscr{C} \int F \to \mathscr{C}$, for which the fibre at any $c \in \mathscr{C}$ recovers the category Fc. In fact, this construction extends naturally to a 2-functor $\mathscr{C} \int : \mathrm{Func}^{\mathrm{ps}}(\mathscr{C}^{\mathrm{op}}, \mathbf{Cat}) \to (\mathbf{Cat} \downarrow \mathscr{C})$ from the 2-category of pseudofunctors, pseudonatural transformations, and modifications to the 2-category of functors over $\mathscr{C}$, commutative triangles of functors, and natural transformations. In fact, the Grothendieck construction establishes an isomorphism of 2-categories between $\mathrm{Func}^{\mathrm{ps}}(\mathscr{C}^{\mathrm{op}}, \mathbf{Cat})$ and the sub-2-category of *Grothendieck fibrations* in $(\mathbf{Cat} \downarrow \mathscr{C})$. However, this fact is not necessary for the main discussion.

Returning to the pseudofunctor $\mathscr{V}\mathbf{Cat}_{(-)} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Cat}$, the Grothendieck construction will indeed construct the desired category $\mathscr{V}\mathbf{Cat}$. Certainly the objects of $\mathbf{Set} \int \mathscr{V}\mathbf{Cat}_{(-)}$ are correct, as these are

just $\mathscr{V}$-enriched categories over an arbitrary set of objects, but now a morphism $\Phi : (\mathscr{C}_0, \mathscr{C}) \to (\mathscr{D}_0, \mathscr{D})$ consists of a function $F : \mathscr{C}_0 \to \mathscr{D}_0$ of sets and a certain oplax natural transformation $\mathscr{C} \Rightarrow \Phi^* \mathscr{D}$ from $\mathscr{V}\mathbf{Cat}_{\mathscr{C}_0}$, which is ultimately[1] a family of morphisms $\Phi_{x,y} : \mathscr{C}(x,y) \to \mathscr{D}(\Phi x, \Phi y)$ in $\mathscr{V}$ such that we have commutativity of

$$\begin{tikzcd}
\mathscr{C}(y,z) \otimes \mathscr{C}(x,y) \arrow[r, "\circ"] \arrow[d, "\Phi_{y,z} \otimes \Phi_{x,y}"'] & \mathscr{C}(x,z) \arrow[d, "\Phi_{x,z}"] \\
\mathscr{D}(\Phi y, \Phi z) \otimes \mathscr{D}(\Phi x, \Phi y) \arrow[r, "\circ"'] & \mathscr{D}(\Phi x, \Phi z)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{1} \arrow[r, "1_x"] \arrow[dr, "1_{\Phi x}"'] & \mathscr{C}(x,x) \arrow[d, "\Phi_{x,x}"] \\
& \mathscr{D}(\Phi x, \Phi x)
\end{tikzcd}$$

This is precisely the definition of a $\mathscr{V}$-enriched functor as in [27].

In particular, if $F : \mathscr{V} \to \mathscr{W}$ is a lax monoidal functor between monoidal categories, then it induces an ordinary functor $F_S^* : \mathscr{V}\mathbf{Cat}_S \to \mathscr{W}\mathbf{Cat}_S$ for any set S just as in the case for monoidal categories, and this induced functor is pseudonatural in S. Thus, the Grothendieck construction provides us with an induced functor $F^* : \mathscr{V}\mathbf{Cat} \to \mathscr{W}\mathbf{Cat}$, as alluded to at the end of Section 2.2.3.

There is also a covariant analogue of the construction: given a pseudofunctor $F : \mathscr{C} \to \mathbf{Cat}$, the covariant Grothendieck construction yields the category also[2] denoted $\mathscr{C} \int F$, where

- the objects are pairs (c,x) where $c \in \mathscr{C}_0$ and $x \in (Fc)_0$
- the morphisms $(c,x) \to (d,y)$ are pairs (f,ϕ) where $f : c \to d$ in $\mathscr{C}$, but $\phi : (Ff)(x) \to y$ in Fc

with composition defined as $(g,\psi) \circ (f,\phi) := (g \circ f, \psi \circ (Fg)(\phi))$. Again, this category has an obvious projection $\mathscr{C} \int F \to \mathscr{C}$, and the construction extends to a 2-functor $\mathscr{C} \int : \mathrm{Func}^{\mathrm{ps}}(\mathscr{C}, \mathbf{Cat}) \to (\mathbf{Cat} \downarrow \mathscr{C})$.

[1] The family of morphisms is technically mapping $\mathscr{C}(x,y) \otimes \mathbb{1} \to \mathbb{1} \otimes \mathscr{D}(\Phi x, \Phi y)$, but by padding with unitors from $\mathscr{V}$, we can suppress the tensor units.

[2] While the notation may seem to conflict with its contravariant analogue, note that for a pseudofunctor $F : \mathscr{C}^{\mathrm{op}} \to \mathbf{Cat}$, the contravariant Grothendieck construction from before would yield $\mathscr{C} \int F$ lying over $\mathscr{C}$, whereas the covariant construction would give $\mathscr{C}^{\mathrm{op}} \int F$ lying over $\mathscr{C}^{\mathrm{op}}$.