Configurations and Decoupling: A Few Problems in Euclidean Harmonic Analysis

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B.Sc., The Chinese University of Hong Kong, 2015
M.Phil., The Chinese University of Hong Kong, 2017

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

in

The Faculty of Graduate and Postdoctoral Studies

(Mathematics)

THE UNIVERSITY OF BRITISH COLUMBIA

(Vancouver)

April 2021

The following individuals certify that they have read, and recommend to the Faculty of Graduate and Postdoctoral Studies for acceptance, the thesis entitled:

Configurations and Decoupling: A Few Problems in Euclidean Harmonic Analysis

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Abstract

In this thesis, we study two topics in Euclidean harmonic analysis. The first one is the configurations contained in fractal-like sets in the Euclidean space. The other is decoupling for various geometric objects in the Euclidean space.

In the study of Euclidean configurations, we first discuss the background, address their subtleties and do a simple survey on this subject. Then we proceed to the proof of my main result, which demonstrates the topological property of a set containing a similar copy of sequences converging to zero.

In the study of decoupling, we first formulate a general decoupling inequality and discuss some general upper and lower bound estimates Then we move on to decoupling for manifolds in Euclidean space, and in particular curves in the plane. We then state a classical result by Bourgain and Demeter and use it to prove a decoupling inequality that works uniformly for all polynomials up to a certain degree, generalising an earlier result of Biswas et al. in the plane.

Lay Summary

This thesis studies two separate topics in harmonic analysis, namely, fractal geometry and decoupling theory.

A fractal is a complex pattern that typically takes a highly irregular shape. Yet many fractal-like patterns occur in nature, such as snowflakes, ferns, and trees. As one zooms in, fractals often exhibit self similarity. Here we study the relation between the size of fractals and the geometric figures contained within them.

Decoupling theory is a branch of classical harmonic analysis, which deals with functions formed from the superpositions of sinusoidal waves made up by different frequencies. Decoupling studies the behaviour of functions when their frequency is localized to a small neighborhood of a geometrically significant shape.

Preface

This dissertation is an original intellectual product of the author, Tongou Yang.

Chapters 4 and 5 are based on the author's publication [72]. Chapter 7 is based on the author's publication [73]. All of the topics above were directly or indirectly suggested by my advisor Malabika Pramanik.

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Acknowledgements

I express gratitude to my advisor, Professor Malabika Pramanik, for her guidance with incredible patience in the past four years. She not only provided intriguing insights in mathematics but greatly improved my presentation in mathematics. Without her I would not have produced my publications and other academic articles.

I would also like to thank my master advisor Professor Po Lam Yung for hosting and supporting me a few times in CUHK after graduation. I also thank Professor Joshua Zahl, Professor Izabella Łaba, Professor Juncheng Wei and other department members for their help in the past four years.

Finally, I thank my parents for their firm support for my PhD career in Canada.

Chapter 1

Introduction

1.1 Euclidean configurations

Fix any subset $B \subseteq \mathbb{R}^n$, called a configuration (or pattern). Given a subset $A \subseteq \mathbb{R}^n$, we say A contains a *similar*[1] *copy* of B if there is a translation $t \in \mathbb{R}^n$ and a uniform dilation $\delta \neq 0$ such that $t + \delta B \subseteq A$. For example, if A contains an open ball, then it contains a similar copy of every bounded set B. In fact, the classical Lebesgue density theorem implies the following nontrivial result: if A has positive Lebesgue measure, then it contains a similar copy of every finite configuration B.

In 1955, Erdős and Kakutani [21] first generalised the Lebesgue density theorem by constructing a perfect set $A \subseteq [0, 1]$ with Lebesgue measure 0 and Hausdorff dimension 1, such that A still contains a similar copy of every finite configuration. Since then, a lot of new results have been established, either by weakening the assumption or strengthening the conclusion of previously proved theorems. In Chapter 3 we will present a survey of many theorems and conjectures in this area. Most of the results are related to constructions of Cantor-like sets.

Part of my doctoral work is based on the study of the topological properties of sets $A \subseteq \mathbb{R}$ containing similar copies of patterns B given by *zero sequences*, that is, sequences strictly decreasing to 0.

My main result is as follows.

Theorem 1.1.1 (Theorem 1.1 of [72])**.** *If $A \subseteq \mathbb{R}$ contains a similar copy of any zero sequence, then the closure of A contains an interval.*

[1]In this thesis we do not allow rotations or reflections.

Theorem 1.1.2 (Theorem 1.2 of [72]). *Let η_m be a zero sequence. Then there is a closed and nowhere dense set $A \subseteq [0,1]$, depending on η_m, that contains a similar copy of any sequence $\alpha_m \to 0$ with $\sup_m \alpha_m/\eta_m < \infty$.*

The proof of the theorems will be presented in Chapters 4 and 5, respectively.

1.2 Decoupling

Roughly speaking, decoupling studies the superposition of waves in physical space with their frequencies lying in disjoint sets in frequency space. Mathematically, given $1 \leq p, q \leq \infty$ and finitely many disjoint sets $\mathcal{A} = \{A_i\}$ in $\mathbb{R}^n$, what is the smallest constant $D_{p,q}(\mathcal{A})$ such that for all functions f_i each with Fourier transform $\hat{f}_i$ supported on A_i, we have the inequality[2]

$$\left\| \sum_i f_i \right\|_{L^p(\mathbb{R}^n)} \leq D_{p,q}(\mathcal{A}) \left(\sum_i \|f_i\|_{L^p(\mathbb{R}^n)}^q \right)^{\frac{1}{q}} ?$$

For any finite collection $\mathcal{A}$, not necessarily disjoint, by the triangle inequality and Hölder's, it is easy to see that $D_{p,q}(\mathcal{A}) \leq (\#\mathcal{A})^{1-1/q}$.

Decoupling studies the following fundamental question. If $\mathcal{A}$ is chosen to be disjoint, then how can we give an estimate on $D_{p,q}(A)$ for a pair of exponents (p, q)? If yes, are the estimates sharp?

When $p = q = 2$, by Plancherel's identity and the disjointness of A_i it is easy to see that $D_{2,2}(\mathcal{A}) = 1$. It can be viewed as the simplest decoupling inequality. Hence, we mainly seek to find sharp estimates of $D_{p,q}(A)$ for other pairs (p, q).

Decoupling theory originated from a Fourier analytic tool developed by Wolff [71] in the study of local smoothing estimates for the cone. The subsequent work of Łaba–Wolff [52], Łaba–Pramanik [50], Pramanik–Seeger [60], and Garrigós–Seeger [27, 28] provided useful insights prior to the systematic study of decoupling. In the breakthrough work in 2015 of Bourgain and

[2]with the obvious modification for $q = \infty$.

Demeter [5], they proved a sharp decoupling estimate for a compact subset of the standard elliptic paraboloid in $\mathbb{R}^n$, namely, the graph of the function $\xi \mapsto |\xi|^2$ for $\xi \in [-1,1]^{n-1}$. In particular, for $n = 2$, what they established is the following.

Theorem 1.2.1 (Bourgain-Demeter [5], $n = 2$)**.** *Let $\phi(s) = s^2$. For $\delta \in \mathbb{N}^{-2}$, let $I_i = [(i-1)\delta^{1/2}, i\delta^{1/2}]$ be the partition of $[0,1]$ into intervals of length $\delta^{1/2}$. Let A_i be given by the δ-neighbourhood of the graph of ϕ over I_i.*

Assume $2 \le p \le 6$. Then for any $\varepsilon > 0$, there is a constant $C_{\varepsilon,p}$ depending on ε, p only, such that for all functions f_i each with Fourier support on A_i, we have

$$\left\| \sum_i f_i \right\|_{L^p(\mathbb{R}^2)} \le C_{\varepsilon,p} \delta^{-\varepsilon} \left(\sum_i \|f_i\|_{L^p(\mathbb{R}^2)}^2 \right)^{\frac{1}{2}}. \tag{1.1}$$

See Figure 1.1 for a picture of this theorem. We remark that the constant $C_{\varepsilon,p}$ here is independent of δ and all f_i, and this will be the feature of all decoupling inequalities. Also, the range of exponents $2 \le p \le 6$ is the largest we can expect if we want the growth with respect to $\delta \to 0$ to be $\delta^{-\varepsilon}$ for any $\varepsilon > 0$. This can be seen from an exponential sum estimate; see [75].

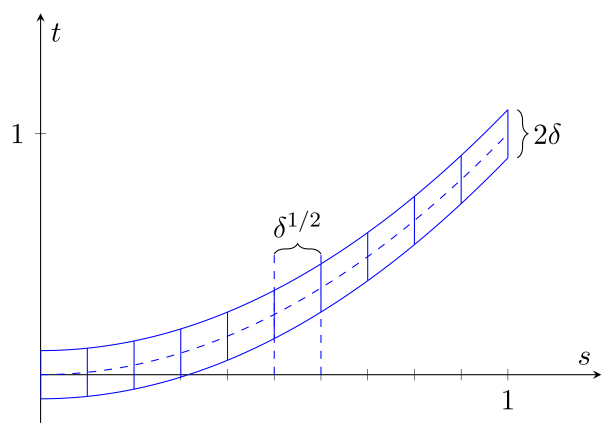

Figure 1.1: δ-neighbourhood of $t = s^2$

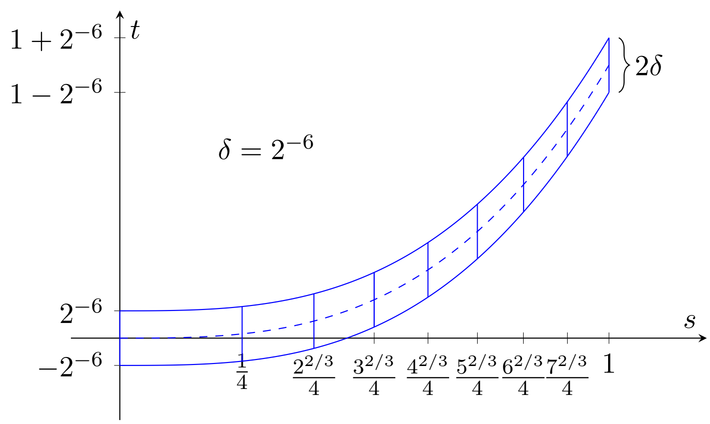

Figure 1.2: δ-neighbourhood of $t = s^3$

Part of my doctoral work in [73] is a generalisation of Theorem 1.2.1 in $n = 2$. I derive there an inequality of the form (1.1) for all polynomials up to a fixed degree.

Theorem 1.2.2 (Theorem 1.4 of [73])**.** *For any $2 \leq p \leq 6$, $d \geq 1$ and $\varepsilon > 0$, there is a constant $C_\varepsilon = C_{d,\varepsilon,p}$ such that the following is true.*

For any $0 < \delta \leq 1$, any polynomial ϕ of degree at most d, any "admissible partition" $\mathcal{P} = \{I_i\}$ of $[0,1]$ for ϕ at the scale δ and any f_i each Fourier supported on the δ-neighbourhood of the graph of ϕ over I_i, we have

$$\left\| \sum_i f_i \right\|_{L^p(\mathbb{R}^2)} \leq C_\varepsilon \delta^{-\varepsilon} \left(\sum_i \|f_i\|^2_{L^p(\mathbb{R}^2)} \right)^{\frac{1}{2}}. \tag{1.2}$$

Here, an "admissible partition" for ϕ at the scale δ is "the coarsest" partition $\mathcal{P}$ of $[0,1]$ such that the δ-neighbourhood of the graph of ϕ over each $I \in \mathcal{P}$ is an almost rectangle. The formal definition will be given in Section 7.1.

We refer to this as a "uniform decoupling inequality" since the constant C_ε is independent of ϕ and $\mathcal{P}$ in addition to δ and f_i.

In Figure 1.2 above, we also showcase an admissible partition of $[0,1]$ for

$\phi(s) = s^3$ at the scale $\delta = 2^{-6}$, given by

$$[(j-1)^{2/3}\delta^{1/3}, j^{2/3}\delta^{1/3}], \quad 1 \leq j \leq \delta^{-1/2}.$$

1.3 Outline of the thesis

The main body of the thesis is divided into two parts. The first part deals with Euclidean configurations. We will start with some preliminaries in Chapter 2. Chapter 3 is a detailed literature review. Chapters 4 and 5 are devoted to the proof of Theorems 1.1.1 and 1.1.2, respectively.

The second part deals with decoupling theory. In Chapter 6 we deal with some preliminaries of decoupling, and state some known results in the current literature. In Chapter 7 we prove Theorem 1.2.2.

Chapter 2

Configurations: Background

2.1 Definition and notation

In this chapter, we are only interested in Borel sets in Euclidean space, unless otherwise stated.

2.1.1 A little set theory

Let A, B be Borel sets in $\mathbb{R}^n$. For $\delta \in \mathbb{R}$, the notation δA is defined as the set $\{\delta a : a \in A\}$. The notation $A + B$ will be used to denote the Minkowski sum of A and B, namely, $A + B = \{a + b : a \in A, b \in B\}$. In particular, if A (and similarly B) is a singleton, by $a + B$ or $B + a$ we mean $\{a\} + B$. We also write $-A$ to mean $(-1)A$ and $B - A$ to mean $B + (-A)$.

Let $A \subseteq \mathbb{R}^n$, let $\{A_i \subseteq \mathbb{R}^n : i \in I\}$ where I is any index set, and let $t \in \mathbb{R}^n$. The following relations show that translations commute with basic set operations. The proof is immediate.

$$(A + t)^c = A^c + t, \tag{2.1}$$

$$\bigcup_{i \in I}(A_i + t) = \left(\bigcup_{i \in I} A_i\right) + t, \tag{2.2}$$

$$\bigcap_{i \in I}(A_i + t) = \left(\bigcap_{i \in I} A_i\right) + t. \tag{2.3}$$

For a Borel set $A \subseteq \mathbb{R}^n$, we use $\mathcal{L}^n(A)$ to denote its n-dimensional Lebesgue measure. The cardinality of a set will always be denoted as $\#A$.

The notation $|\cdot|$ will be reserved for the Euclidean norm of a vector in $\mathbb{R}^n$. We use $\operatorname{diam}(A)$ to denote the diameter of a set $A \subseteq \mathbb{R}^n$, defined as $\operatorname{diam}(A) = \sup\{|x-y| : x, y \in A\}$. The distance between two sets $A, B \subseteq \mathbb{R}^n$, denoted $\operatorname{dist}(A, B)$, is defined as $\inf\{|x-y| : x \in A, y \in B\}$. The notation $\operatorname{dist}(x, B)$ means $\operatorname{dist}(\{x\}, B)$.

2.1.2 Notation in topology

We use the notation $B^n(x, r)$ (or simply $B(x, r)$ when the ambient dimension is clear from the context) to denote the open ball in $\mathbb{R}^n$ of radius r centred at x. All balls (open or closed) in this thesis are assumed to be non-degenerate, that is, they have positive and finite radius. The interior and the closure of a set A in $\mathbb{R}^n$ (with respect to the standard topology, unless otherwise specified) will be denoted $\operatorname{Int}(A)$ and $\overline{A}$, respectively.

We also make the following definitions about density of sets. If $K \subseteq \mathbb{R}^n$ is a closed set, we say a set $A \subseteq \mathbb{R}^n$ is *dense* in K if for each open ball $I \subseteq K$ we have $I \cap A \neq \varnothing$. It is an easy exercise to show that $A \subseteq \mathbb{R}^n$ is dense in K if and only if $\overline{A \cap K} = K$. We say a set $A \subseteq \mathbb{R}^n$ is *nowhere dense* if $\operatorname{Int}(\overline{A}) = \varnothing$. We say a set is *somewhere dense* if it is not nowhere dense.

The following elementary lemma will be useful. The proof is elementary, but we include it here for completeness.

Lemma 2.1.1. *$A \subseteq \mathbb{R}^n$ is nowhere dense if and only if for each closed ball $K \subseteq \mathbb{R}^n$ there is an open ball $I \subseteq K$ such that $I \subseteq A^c$. As a corollary, if A and B are nowhere dense, then so is $A \cup B$.*

Proof. For the forward direction, assume towards contradiction that there is a closed ball $K \subseteq \mathbb{R}^n$ such that for all open balls $I \subseteq K$, we have $I \cap A \neq \varnothing$. Then by definition, A is dense in K. Equivalently, $\overline{A \cap K} = K$. Hence

$$\operatorname{Int}(\overline{A}) \supseteq \operatorname{Int}(\overline{A \cap K}) = \operatorname{Int}(K) \neq \varnothing,$$

which is a contradiction.

For the reverse direction, suppose towards contradiction that A is somewhere dense. Then $\overline{A}$ contains an open ball which in turn contains some closed ball K. By assumption, there is some open ball $I \subseteq K$ such that

$I \subseteq A^c$. Taking interiors on both sides, we see

$$I = \mathrm{Int}(I) \subseteq \mathrm{Int}(A^c) = \overline{A}^c \subseteq K^c.$$

But this is a contradiction to $I \subseteq K$ as I is nonempty.

Now we prove the corollary, namely, if A and B are nowhere dense, then so is $A \cup B$. Let A and B be nowhere dense. By what we have just proved, it suffices to show that for any closed ball $K \subseteq \mathbb{R}^n$ there is an open ball $I \subseteq K$ such that $I \subseteq (A \cup B)^c$. Let $K \subseteq \mathbb{R}$ be a closed ball. Since A is nowhere dense, by what we just proved, there is an open ball $I' \subseteq K$ such that $I' \subseteq A^c$. But I' contains some closed ball K', and since B is nowhere dense, by what we just proved, there is an open ball $I \subseteq K'$ such that $I \subseteq B^c$. But $K' \subseteq I' \subseteq A^c$, so $I \subseteq A^c$. Hence $I \subseteq K$ is an open ball such that $I \subseteq A^c \cap B^c = (A \cup B)^c$, so $A \cup B$ is nowhere dense. □

Remark. All density in this article will always refer to topological density as defined above, not to be confused with other notions of density such as asymptotic density or Banach density, etc. These are interesting topics to study, but they are not the main point of concern in this thesis.

2.2 Hausdorff measure and Hausdorff dimension

2.2.1 Definition and notation

We follow the formulation in Chapter 4 of [56].

Definition 2.2.1. *Given any set $A \subseteq \mathbb{R}^n$. For $0 < \delta \leq \infty$ and $s \geq 0$, define*

$$\mathcal{H}^s_\delta(A) := \inf \left\{ \sum_{i=1}^{\infty} \mathrm{diam}(A_i)^s : \bigcup_{i=1}^{\infty} A_i \supseteq A, \mathrm{diam}(A_i) < \delta \right\} \in [0, \infty].$$

Here, we adopt the convention that $\mathrm{diam}(\varnothing) = 0$ *and that* $0^0 = 1$ *(when* $s = 0$*).*

With this definition, $\mathcal{H}^s_\delta(A)$ is a decreasing function in δ. We then define

$$\mathcal{H}^s(A) := \sup_{0<\delta\le\infty} \mathcal{H}^s_\delta(A) = \lim_{\delta\to 0^+} \mathcal{H}^s_\delta(A) \in [0,\infty]. \tag{2.4}$$

This is called the s-dimensional *Hausdorff measure* of A.

Remark. $\mathcal{H}^s$ defined in this way is only an outer measure on $\mathbb{R}^n$, but using Theorem 1.7 of [56], we can show that all Borel sets are $\mathcal{H}^s$-measurable. (A short note: the "measure" in Definition 1.1 of [56] is more commonly known as an outer measure.) Since we are only interested in Borel sets in this thesis, we may simply treat $\mathcal{H}^s$ as a measure.

Lemma 2.2.2 (Theorem 4.7 of [56]). *Let $A \subseteq \mathbb{R}^n$. Then $\mathcal{H}^s(A)$ is a decreasing function in $s \ge 0$. Moreover, there is a unique real number $\alpha \in [0,n]$ such that $\mathcal{H}^s(A) = 0$ for all $s > \alpha$ and $\mathcal{H}^s(A) = \infty$ for all $0 \le s < \alpha$.*

With this, we can finally make the following definition:

Definition 2.2.3 (Hausdorff dimension). *Let $A \subseteq \mathbb{R}^n$. With the above notation, the number α is called the Hausdorff dimension of A, denoted $\dim_H(A) = \alpha$.*

2.2.2 Elementary properties and examples

When s is an integer, the Hausdorff measure $\mathcal{H}^s$ has some interesting behaviours.

Theorem 2.2.4. *For $A \subseteq \mathbb{R}^n$, the following are true.*

1. *$\mathcal{H}^0$ is just the counting measure, that is, $\mathcal{H}^0(A) = \#A$.*
2. *There is a constant $c(n) \in (0,\infty)$ such that $\mathcal{H}^n(A) = c(n)\mathcal{L}^n(A)$. As a corollary, if $\mathcal{L}^n(A) > 0$, then $\dim_H(A) = n$.*
3. *For any $s > n$, we have $\mathcal{H}^s(A) = 0$. Thus $\dim_H(A) \le n$.*
4. *Let $m \in [0,n]\cap\mathbb{N}$ and suppose A is an m-dimensional smooth manifold embedded in $\mathbb{R}^n$. Then $\mathcal{H}^m(A) = c(m)\sigma(A)$ where σ is the surface measure of A. As a result, $\dim_H(A) = m$.*

Proof. The first two assertions are proved in Section 2.2 of [22], where $\mathcal{H}^s$ is already normalised by some $c(n)$ defined in Definition 2.1. The last assertion is illustrated in Section 3.3.4 of [22]. We will only prove the third assertion here.

Let $M > 0$ be arbitrary, and let $A_M = A \cap B^n(0, M)$. Since $\mathcal{H}^s$ is an outer measure, it suffices to show that $\mathcal{H}^s(A_M) = 0$. Let $\delta > 0$ be arbitrary, and let $r < \delta$. Cover $B^n(0, M)$ by $CM^n r^{-n}$ many open balls of radius r where $C = C(n)$ is an absolute constant. By definition, we have $\mathcal{H}^s_\delta(A_M) \leq CM^n r^{-n} = CM^n r^{s-n}$, for any $r < \delta$. Letting $r \to 0$, we see that $\mathcal{H}^s_\delta(A_M) = 0$. But this holds for any $\delta > 0$. Thus $\mathcal{H}^s(A_M) = 0$ and our proof is complete. $\square$

An application of the Hausdorff dimension is to measure fractals such as the middle-third Cantor set $C \subseteq \mathbb{R}$. It follows from the case $\lambda = 1/3$ of Section 4.10 of [56] that $\dim_H(C) = \ln 2/\ln 3$. On the other hand, the fat Cantor set $F \subseteq \mathbb{R}$ (Smith–Volterra–Cantor set) has Hausdorff dimension 1, which follows from Part 2 of Theorem 2.2.4 and the fact that $\mathcal{L}^1(F) = 1/2 > 0$.

Lastly, we remark that the Hausdorff dimension is countably stable. That is, we have the following

Proposition 2.2.5. *Let $A_i, i \geq 1$ be subsets of $\mathbb{R}^n$. Then*

$$\dim_H \left(\bigcup_{i=1}^\infty A_i \right) = \sup_i \dim_H(A_i).$$

As a result, any countable set has Hausdorff dimension 0*.*

Proof. The "$\geq$" side is trivial. For the "$\leq$" side, by definition, we will show that for any $\varepsilon > 0$, $\mathcal{H}^{s+\varepsilon}(\cup_i A_i) = 0$ where $s = \sup_i \dim_H(A_i)$ ($s \leq n$ by 3 of Theorem 2.2.4). Thus for each i, $s \geq \dim_H(A_i)$ and thus $\mathcal{H}^{s+\varepsilon}(A_i) = 0$. Since $\mathcal{H}^{s+\varepsilon}$ is an outer measure, we have

$$\mathcal{H}^{s+\varepsilon} \left(\bigcup_{i=1}^\infty A_i \right) \leq \sum_{i=1}^\infty \mathcal{H}^{s+\varepsilon}(A_i) = 0.$$

It is easy to see that any singleton has zero Hausdorff dimension. Thus, the equality implies that this is true for any countable set. $\square$

2.2.3 Generalised Hausdorff measures

In Definition 2.2.1, we can replace the term $\operatorname{diam}(A_i)^s$ by $\phi(A_i)$ where ϕ is a suitable increasing function. The precise formulation, following Section 3.6 of [25], is as follows.

Definition 2.2.6 (Dimension function)**.** *Let $h : [0,\infty) \to [0,\infty)$. We say it is a dimension function if it is right-continuous, increasing, $h(0) = 0$ and $h(t) > 0$ for $t > 0$.*

For example, for $s > 0$, the functions $x \mapsto x^s$ is a dimension function.

Definition 2.2.7. *Let h be a dimension function. For a set $A \subseteq \mathbb{R}^n$ and $0 < \delta \leq \infty$, we define*

$$\mathcal{H}^h_\delta(A) = \inf\left\{\sum_i h\left(\operatorname{diam} A_i\right) : \bigcup_i A_i \supseteq A, \operatorname{diam}(A_i) < \delta\right\},$$

which is a decreasing function of δ. We then define the h-Hausdorff measure of A by

$$\mathcal{H}^h(A) := \sup_{0<\delta\leq\infty} \mathcal{H}^h_\delta(A) = \lim_{\delta\to 0^+} \mathcal{H}^h_\delta(A),$$

which is also a Borel outer measure on $\mathbb{R}^n$.

Lemma 2.2.8. *Let h be a dimension function and $A \subseteq \mathbb{R}^n$. Then $\mathcal{H}^h(A) = 0$ if and only if $\mathcal{H}^h_\infty(A) = 0$.*

Proof. By monotonicity it suffices to prove the "if" side. Let $\delta > 0$ be arbitrary and it suffices to show $\mathcal{H}^h_\delta(A) = 0$.

Since $\mathcal{H}^h_\infty(A) = 0$, for every $0 < \varepsilon < h(\delta)$ there are sets A_i with $A \subseteq \cup_i A_i$ and $\sum_i h(\operatorname{diam}(A_i)) < \varepsilon$. In particular, for each i we have $h(\operatorname{diam}(A_i)) < \varepsilon$, which implies $\operatorname{diam}(A_i) < \delta$ since $\varepsilon < h(\delta)$ and h is increasing. Thus we have $\mathcal{H}^h_\delta(A) < \varepsilon$. Letting $\varepsilon \to 0$ proves the claim. $\square$

Given a dimension function h, we say it is a *zero dimension function* or *zero dimensional* if for every $\varepsilon > 0$ there is $c_\varepsilon > 0$ such that $h(t) > c_\varepsilon t^\varepsilon$ for $0 < t < 1$. For example, the following function

$$h(x) := \begin{cases} 0, & \text{if } x = 0 \\ -\frac{1}{\ln x}, & \text{if } 0 < x \leq \frac{1}{e} \\ 1, & \text{if } x > \frac{1}{e} \end{cases} \tag{2.5}$$

is a zero dimension function.

The following proposition explains the term "zero dimension function".

Proposition 2.2.9. *Let h be a zero dimension function and $A \subseteq \mathbb{R}^n$ be such that $\mathcal{H}^h_\infty(A) = 0$. Then $\dim_H(A) = 0$.*

Proof. By Lemma 2.2.8 we have $\mathcal{H}^h_\delta(A) = 0$ for every $0 < \delta \leq \infty$. Let $s > 0$ be arbitrarily small. Since h is zero dimensional, there is $c > 0$ such that $h(t) > c_s t^s$ for any $0 < t < 1$. Thus for every $0 < \delta < 1$,

$$\begin{aligned} \mathcal{H}^s_\delta(A) &= \inf\left\{\sum_i (\operatorname{diam}(B_i))^s : \bigcup_i B_i \supseteq A, \operatorname{diam}(B_i) < \delta\right\} \\ &\leq \inf\left\{\sum_i c_s^{-1} h(\operatorname{diam}(B_i)) : \bigcup_i B_i \supseteq A, \operatorname{diam}(B_i) < \delta\right\} \\ &= c_s^{-1}\mathcal{H}^h_\delta(A) = 0. \end{aligned}$$

Hence $\mathcal{H}^s(A) = 0$ for any $s > 0$, and thus $\dim_H(A) = 0$. □

2.2.4 Frostman's Lemma and energy integrals

Following Section 2.5 of [57], we give a few more sophisticated characterisations of the Hausdorff dimensions for subsets of $\mathbb{R}^n$. We start with the following standard notation.

Given a Borel measure μ on $\mathbb{R}^n$, we define the support of μ, denoted $\operatorname{supp}(\mu)$, to be the intersection of all closed sets K for which $\mu(K^c) = 0$. Also, for any set $A \subseteq \mathbb{R}^n$, we say μ is supported on A if $\operatorname{supp}(\mu) \subseteq A$.

We now state Frostman's lemma.

Theorem 2.2.10 (Theorem 2.7 of [57])**.** *Let $A \subseteq \mathbb{R}^n$ be a Borel set. Then $\dim_H(A)$ is equal to the supremum of the real numbers s for which there exists a Borel probability measure μ supported on A and a constant $C > 0$ such that*

$$\mu(B^n(x, r)) \leq Cr^s, \quad \text{for all } r > 0,\ x \in A. \tag{2.6}$$

Definition 2.2.11 (Energy integral)**.** *Let μ be a positive Borel measure in $\mathbb{R}^n$. The s-energy of μ, denoted $I_s(\mu)$, is defined by*

$$I_s(\mu) = \iint |x - y|^{-s} d\mu(x) d\mu(y). \tag{2.7}$$

Using this, we obtain another characterisation of the Hausdorff dimension:

Theorem 2.2.12 (Theorem 2.8 of [57])**.** *Let $A \subseteq \mathbb{R}^n$ be a Borel set. Then $\dim_H(A)$ is equal to the supremum of all real numbers s for which there exists a Borel probability measure μ supported on A such that $I_s(\mu) < \infty$.*

Using the Fourier transform, we can give yet another characterisation of the Hausdorff dimension.

Definition 2.2.13. *If μ is a finite Borel measure, then the Fourier transform of μ, denoted $\widehat{\mu}$, is a function defined on $\mathbb{R}^n$ by*

$$\widehat{\mu}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} d\mu(x).$$

Theorem 2.2.14 (Theorem 3.10 of [57])**.** *If $0 < s < n$ and μ is a positive Borel measure on $\mathbb{R}^n$, then*

$$I_s(\mu) = c_{s,n} \int_{\mathbb{R}^n} |\hat{\mu}(\xi)|^2 |\xi|^{s-n} d\xi < \infty, \tag{2.8}$$

where $c(s, n) \in (0, \infty)$ is a constant.

Combining with Theorem 2.2.12, we see that for a Borel $A \subseteq \mathbb{R}^n$, its Hausdorff dimension is equal to the supremum of all real numbers $0 < s < n$

for which there exists a Borel probability measure μ supported on A such that

$$\int_{\mathbb{R}^n} |\hat{\mu}(\xi)|^2 |\xi|^{s-n} d\xi < \infty.$$

2.3 Other dimensions

Apart from the Hausdorff dimension, there are also a few other notions of dimensions.

2.3.1 Fourier dimension

In connection to the last section, we first define the Fourier dimension.

Definition 2.3.1 (Fourier dimension)**.** *Let $A \subseteq \mathbb{R}^n$ be a Borel set. Then the Fourier dimension of A, denoted* $\dim_F(A)$*, is defined by the supremum of the real numbers $0 < s < n$ for which there exists a Borel probability measure μ supported on A and a constant $C > 0$ such that*

$$|\hat{\mu}(\xi)| \leq C(1 + |\xi|)^{-\frac{s}{2}} \text{ for all } \xi \neq 0. \tag{2.9}$$

Combining Theorems 2.2.12, 2.2.14 and Definition 2.3.1, we have

$$\dim_F(A) \leq \dim_H(A) \tag{2.10}$$

for any A. Strict inequality may occur, as in the case of the standard middle-third Cantor set C which has $\dim_H(C) = \ln 2 / \ln 3$, while it follows from Theorem 8.1 of [57] that $\dim_F(C) = 0$.

In some cases, the Fourier dimension is also countably stable as in Proposition 2.2.5; see [17], which in particular proves the following finite stability of the Fourier dimension:

Proposition 2.3.2. *Let A, B be compact disjoint sets. Then* $\dim_F(A \cup B) = \max\{\dim_F(A), \dim_F(B)\}$.

A Borel set $A \subseteq \mathbb{R}^n$ is called a *Salem* set if $\dim_F(A) = \dim_H(A)$. Since $\dim_F(A) \leq \dim_H(A)$, we see that A is Salem if $\dim_H(A) = 0$. The standard middle-third Cantor set C is not Salem. However, it is surprisingly tricky to construct a Salem set with positive Hausdorff dimension, and most constructions have to go through probabilistic arguments (see [62] by Salem himself and [40, 41] by Kahane), which in particular implies the following theorem.

Theorem 2.3.3. *For every $0 \leq s \leq n$, there exists a Salem set $A \subseteq \mathbb{R}^n$ of Hausdorff dimension s.*

For deterministic constructions of a Salem set, the reader may refer to [35, 46].

Another interesting example is concerned with lower dimensional manifolds in $\mathbb{R}^n$. It can be shown (see p. 348 and 350 of [65]) that any compact hypersurface with nonzero Gaussian curvature is Salem. On the contrary, if $1 \leq k \leq n-1$, then a k-plane in $\mathbb{R}^n$ will always have Fourier dimension 0 and thus is not Salem.

We end this subsection with a trivial observation that any set A in $\mathbb{R}^n$ containing an interior point is Salem. Indeed, $\dim_H(A) = n$, and so it suffices to establish $\dim_F(A) = n$. But we may just take a nonnegative bump function $\phi \in C_c^\infty(\mathbb{R}^n)$ supported on an open ball contained in A with mass 1, and let $\mu = \phi(x)dx$. Since $\hat{\phi}$ is Schwartz, we have $\dim_F(A) = n$.

2.3.2 Minkowski dimension

We now introduce Minkowski dimensions, which are defined for bounded sets.

Definition 2.3.4. *Let $A \subseteq \mathbb{R}^n$ be a bounded set. For $\delta > 0$, let $N_\delta(A)$ be the smallest number of open balls of radius δ so that their union covers A.*

We then define the upper and lower Minkowski dimensions as follows:

$$\overline{\dim}_B(A) = \varlimsup_{\delta \to 0} \frac{\ln N_\delta(A)}{-\ln \delta}, \tag{2.11}$$

$$\underline{\dim}_B(A) = \varliminf_{\delta \to 0} \frac{\ln N_\delta(A)}{-\ln \delta}, \tag{2.12}$$

respectively. If $\overline{\dim}_B(A) = \underline{\dim}_B(A)$*, then we define the Minkowski dimension of A as*

$$\dim_B(A) = \lim_{\delta\to 0} \frac{\ln N_\delta(A)}{-\ln \delta}. \tag{2.13}$$

The subscript B here stands for the box-counting dimension, another name of the Minkowski dimension. There are also many other equivalent formulations; see Section 2.1 of [25]. The following proposition is very useful in computations.

Proposition 2.3.5 (Proposition 2.4 of [25])**.** *For* $A \subseteq \mathbb{R}^n$ *and* $\delta > 0$*, let* $\mathcal{N}_\delta(A)$ *denote the* δ*-neighbourhood of* A *in* $\mathbb{R}^n$*, i.e.*

$$\mathcal{N}_\delta(A) := \{x \in \mathbb{R} : \mathrm{dist}(x, A) < \delta\}. \tag{2.14}$$

Then we have the following

$$\overline{\dim}_B(A) = n - \underline{\lim}_{\delta\to 0} \frac{-\ln \mathcal{L}^n(\mathcal{N}_\delta(A))}{-\ln \delta}, \tag{2.15}$$

$$\underline{\dim}_B(A) = n - \overline{\lim}_{\delta\to 0} \frac{-\ln \mathcal{L}^n(\mathcal{N}_\delta(A))}{-\ln \delta}. \tag{2.16}$$

As a result, we have

$$\dim_B(A) = n - \lim_{\delta\to 0} \frac{-\ln \mathcal{L}^n(\mathcal{N}_\delta(A))}{-\ln \delta} \tag{2.17}$$

whenever the limit on the right hand side exists.

We observe that the Minkowski dimension is defined in an easier way than the Hausdorff dimension, and it seems to be more intuitive. Thus, one may ask why we prefer the Hausdorff dimension. One reason is to avoid the case when upper and lower Minkowski dimensions do not agree. But more importantly, Minkowski dimension is not countably stable as in Proposition 2.2.5. Indeed, it is not necessarily true that $\dim_B(\cup_i A_i) = \sup_i \dim_B(A_i)$, as can be seen from the simple example where $A = \{a_i\}$ is an enumeration of all rational numbers in $[0, 1]$ and $A_i = \{a_i\}$. In this example, it is easy to see that $\dim_B(A) = 1$ because of the density of rational numbers, while

$\dim_B(A_i) = 0$ for each i. Nevertheless, the upper Minkowski dimension is finitely stable, that is, the relation in Proposition 2.2.5 holds when the index set is finite.

Proposition 2.3.6. *For bounded sets $A, B \subseteq \mathbb{R}^n$, we have*

$$\overline{\dim}_B(A \cup B) = \max\{\overline{\dim}_B(A), \overline{\dim}_B(B)\}.$$

If $\operatorname{dist}(A, B) > 0$*, then we also have*

$$\underline{\dim}_B(A \cup B) = \max\{\underline{\dim}_B(A), \underline{\dim}_B(B)\}.$$

Proof. The first result follows from the observation that

$$\max\{N_\delta(A), N_\delta(B)\} \leq N_\delta(A \cup B) \leq N_\delta(A) + N_\delta(B).$$

For the second result, note that for δ small, the second inequality above becomes an equality by the positive separation of A, B. □

From the definition of the upper and lower Minkowski dimensions, we have the following relation.

Proposition 2.3.7. *Let $A \subseteq \mathbb{R}^n$ be bounded. Then*

$$\dim_H(A) \leq \underline{\dim}_B(A) \leq \overline{\dim}_B(A) \leq n. \tag{2.18}$$

Proof. The second inequality is trivial. The last inequality can be proved in exactly the same way as in the proof of Part 3 of Theorem 2.2.4. It remains to prove the first inequality.

Let $s = \underline{\dim}_B(A)$. Let $\varepsilon > 0$. We will show that $\mathcal{H}^{s+\varepsilon}(A) = 0$. Let $\delta > 0$, and let $U_i, 1 \leq i \leq j$ be a cover of A with open balls of diameter $r < \delta$, such that $j = N_r(A)$. Then by definition, we have

$$\mathcal{H}_\delta^{s+\varepsilon}(A) \leq N_r(A) r^{s+\varepsilon},$$

for all $r < \delta$. Since $s = \underline{\dim}_B(A)$, there is a sequence $\delta \geq r_k \searrow 0$ such that

$$\frac{\ln N_{r_k}(A)}{-\ln r_k} \leq s + \frac{\varepsilon}{2}.$$

Combining the last two inequalities shows that

$$\mathcal{H}^{s+\varepsilon}_\delta(A) \leq r_k^{\frac{\varepsilon}{2}}.$$

Letting $k \to \infty$, we have $\mathcal{H}^{s+\varepsilon}_\delta(A) = 0$. Letting $\delta \to 0$, we have $\mathcal{H}^{s+\varepsilon}(A) = 0$. $\square$

Unlike the Hausdorff dimension, a countable set may have either zero or positive Minkowski dimension. The set given by the geometric sequence $\{r^n : n \geq 1\}$ where $|r| < 1$ has zero Minkowski dimension, but it is easy to see that $A := \mathbb{Q} \cap [0, 1]$ has Minkowski dimension 1. It is also possible to have a sequence with Minkowksi dimension strictly between 0 and 1, as can be seen from the example $\dim_B(\{n^{-p} : n \geq 1\}) = (p + 1)^{-1}$ for any $p > 0$. We give a proof of the last assertion.

Proposition 2.3.8. *Let* $A = \{n^{-p} : n \geq 1\}$. *Then* $\dim_B(A) = (p + 1)^{-1}$.

Proof. Using Proposition 2.3.5, it suffices to show that

$$\lim_{\delta \to 0} \frac{-\ln \mathcal{L}^1(\mathcal{N}_\delta(A))}{-\ln \delta} = \frac{p}{p+1}.$$

Let $0 < \delta \leq (1 - 2^{-p})/2$. Since the sequence $d_k := k^{-p} - (k + 1)^{-p}, k \geq 1$ is also strictly decreasing, there is a unique $k \geq 2$ such that $d_k < 2\delta \leq d_{k-1}$. Thus $\mathcal{N}_\delta(A)$ can be expressed as

$$\mathcal{N}_\delta(A) = (-\delta, k^{-p} + \delta) \cup \left(\bigcup_{n=1}^{k-1} (n^{-p} - \delta, n^{-p} + \delta) \right),$$

which has measure $k^{-p} + 2k\delta$. Using the mean value theorem, we see that

$\delta \sim k^{-p-1}$. Letting $\delta \to 0$ shows that

$$\lim_{\delta \to 0} \frac{-\ln \mathcal{L}^1(\mathcal{N}_\delta(A))}{-\ln \delta} = \frac{p}{p+1}.$$

$\square$

Remark. In the proof, we see that $\mathcal{N}_\delta(A)$ is comprised of two parts: a single interval on the left and a disjoint union of intervals of the same length on the right. A similar structure will be useful in Chapter 4 when we prove Theorem 1.1 of [72].

Lastly, for the standard middle-third Cantor set C, we have $\dim_B(C) = \ln 2/\ln 3$, the same as $\dim_H(C)$. The proof is simple (see Example 2.2 of [25]).

We remark that there are other notions of dimension, such as the topological dimension, the Assouad dimension (see Part I of [61], the modified box-counting dimension (see Section 2.3 of [25]) and the packing dimension (see Section 3.5 of [25]).

Chapter 3

Euclidean Configurations

Given two sets $A, B \subseteq \mathbb{R}^n$, we say that A contains a similar copy of B if there exist $\delta \neq 0$ and $t \in \mathbb{R}^n$ such that $t + \delta B \subseteq A$. Intuitively speaking, we expect large sets to contain many configurations (patterns) and small sets to avoid many patterns. In the last chapter, we defined various notions of size; those notions are not all compatible, as it can often happen that sets which are large in one sense are small in another. Depending on which definition of size we choose in the last chapter, many historical results can be roughly classified into one of the four following directions.

(i) If a set is large, then it should contain many patterns.

(ii) It is possible to construct certain large sets that avoid many prescribed patterns.

(iii) If a set contains many patterns, then this set must be large.

(iv) It is possible to construct certain small sets that contain many prescribed patterns.

Some of them seem to be contradictory with others, but they are all true once we specify in what sense the set is large and how many prescribed patterns it is to contain. We will elaborate each of the rough statements above through a detailed literature review. Proofs of some facts of moderate length are postponed to the last section of this chapter.

3.1 Large sets tend to contain many patterns

We first discuss direction (i).

3.1.1 A few known results

We say a configuration $B \subseteq \mathbb{R}^n$ is *universal* if every set of positive Lebesgue measure contains a similar copy of B. As pointed out in the introduction, the Lebesgue density theorem shows that all finite sets are universal.

If we suitably weaken the assumption of the Lebesgue density Theorem, we may still obtain interesting conclusions. For example, Łaba and Pramanik [51] showed that if a compact set in $\mathbb{R}$ satisfies a certain Frostman's condition as in (2.6) and a Fourier decay condition as in (2.9), then A must contain a 3-term arithmetic progression, that is, a similar copy of $\{-1, 0, 1\}$. We also refer the reader to other results in this direction, including [1, 32, 38, 39, 54], all in higher dimensions. For example, in [38] it is proved that when $n \geq 4$, if a compact set $A \subseteq \mathbb{R}^n$ has sufficiently large Hausdorff dimension, then it contains vertices of an equilateral triangle, that is, there are distinct $x, y, z \in A$ such that $|x - y| = |x - z| = |y - z|$.

3.1.2 Distance set conjectures

For $A \subseteq \mathbb{R}^n$, we consider its *difference set* defined by $A - A := \{a - b : a, b \in A\} \subseteq \mathbb{R}^n$. We also consider its *distance set* $\Delta(A) := \{|a - b| : a, b \in A\} \subseteq [0, \infty)$.

The Erdős distance set conjecture is about the minimum cardinality of $\Delta(A)$ given a finite set $A \subseteq \mathbb{R}^n$. In symbols, this conjecture studies the function $f : \mathbb{N} \to \mathbb{N}$ defined by

$$f(m) := \min\{\#\Delta(A) : \#A = m\}.$$

In Erdős' paper [19], he proved for $n = 2$ the easier upper bound $f(m) \lesssim m(\log m)^{-1/2}$ and conjectured this should also be the sharp lower bound (up to an absolute constant). This is still an open problem, and the best bound is currently due to Guth and Katz [34], who proved that $f(m) \gtrsim m/\log m$. See also [69] for a survey which also discusses an intimately related unsolved problem, namely, the Erdős unit distance problem.

In the continuous setting, a classical result by Steinhaus [67] states that if $\mathcal{L}^1(A) > 0$, then $A - A$ and hence $\Delta(A)$ contains an open ball centred at the origin. But if we weaken the assumption to $\dim_H(A) > \alpha$ where $\alpha \in [0, n)$, then things get trickier. Falconer [24] proved that for $n \geq 2$, if $\dim_H(A) > (n+1)/2$, then $\Delta(A)$ has positive Lebesgue measure. Based on this, he formulated the long-standing conjecture that for $n \geq 2$, if $\dim_H(A) > n/2$, then $\Delta(A)$ has positive Lebesgue measure. The best result for general n is due to Erdoğan [18], who proved that if $n \geq 3$ and $A \subseteq \mathbb{R}^n$ is a compact set with $\dim_H(A) > n/2 + 1/3$, then $\mathcal{L}^1(\Delta(A)) > 0$. In the case $n = 2$, the best bound is due to Guth, Iosevich, Ou and Wang [33], who proved that the condition $\dim_H(A) > 4/3$ is sufficient. See also [29–32, 38, 54] for related results.

3.2 Large sets avoiding prescribed patterns

Now we elaborate direction (ii).

3.2.1 A few known results

In view of the Lebesgue density Theorem, one naturally asks if we can weaken the assumption while still arriving at the same conclusion. For example, on $\mathbb{R}$, from 2 of Theorem 2.2.4, we know that $\mathcal{L}^1(A) > 0$ implies $\dim_H(A) = 1$. Hence, we may ask the following question: if $A \subseteq \mathbb{R}$ with $\dim_H(A) = 1$ and $B \subseteq \mathbb{R}$ is finite, must A contain a similar copy of B? Unfortunately, the answer is negative. In fact, Keleti [47] proved that if $B \subseteq \mathbb{R}$ is a set with 3 elements, then there exists a compact set $A \subseteq \mathbb{R}$ of Hausdorff dimension 1 that does not contain a similar copy of B.

Moreover, if we make the modest restriction to 3-term arithmetic progressions, namely, $B = \{-1, 0, 1\}$, then we have an even stronger result by Shmerkin [64], who proved the existence of a compact set $A \subseteq \mathbb{R}$ of Fourier dimension 1 that does not contain a similar copy of B. (Thus A is Salem by (2.10).)

Recently, Liang and Pramanik [53] generalised the above result by proving that there is a large family of systems of linear equations, so that for any such a system, there exists a compact set $A \subseteq \mathbb{R}$ of Fourier dimension 1 that avoids all solutions to this system. We remark that it is still open whether given any $B \subseteq \mathbb{R}$ of 3 elements, there is a set A of Fourier dimension 1 that avoids any similar copy of B.

In higher dimensions, Fraser and Pramanik [26] showed that there is a much larger family of systems of equations, so that for any such a system, there exists a set with full Minkowski dimension and large Hausdorff dimension that avoids all solutions to this system.

3.2.2 Erdős similarity conjecture

Apart from the above positive results, there is a still longstanding conjecture in this direction, namely, the Erdős similarity conjecture, raised in [20].

We return to the Lebesgue density Theorem again. Another attempt to generalise this could be to retain the assumption to see if a stronger conclusion can be drawn. For example, we may ask the following question: if $A \subseteq \mathbb{R}$ is such that $\mathcal{L}^1(A) > 0$ and $B \subseteq \mathbb{R}$ is infinite, must A contain a similar copy of B? Unfortunately, the answer is negative again, even for B being a real sequence strictly decreasing to 0 (which we will refer to as a *zero sequence* from now on), as Eigen [16] and Falconer [23] independently proved that any zero sequence $\{x_n\}_{n=1}^{\infty}$ such that $\liminf_{n\to\infty} x_{n+1}/x_n = 1$ is not universal. Examples include sequences n^{-p} for any $0 < p < \infty$ and R^{-n^p} where $R > 1$, $0 < p < 1$. Inspired by this, we may now state the famous Erdős similarity conjecture.

Conjecture 3.2.1 (Erdős)**.** *There is no infinite universal set in $\mathbb{R}^n$ for any $n \geq 1$.*

We remark here that the hardest case is when $n = 1$, by the following simple proposition.

Proposition 3.2.2. *If there is an infinite universal set in $\mathbb{R}^n$ for some $n \geq 2$, then there is an infinite universal set in $\mathbb{R}$.*

Proof. Suppose there is an infinite universal set B in $\mathbb{R}^n$. Then there is at least one unit vector $u \subseteq \mathbb{R}^n$ such that $\pi(B) \subseteq \mathbb{R}$ is infinite, where $\pi(b) := b \cdot u$. We claim that $\pi(B)$ is universal. Indeed, let $A \subseteq \mathbb{R}$ be such that $\mathcal{L}^1(A) > 0$. Then $\mathcal{L}^n(\pi^{-1}(A)) > 0$. Since B is universal, there is $\delta \neq 0$ and $t \in \mathbb{R}^n$ such that $t + \delta B \subseteq \pi^{-1}(A)$. Applying π on both sides gives

$$\pi(t) + \delta\pi(B) \subseteq \pi(\pi^{-1}(A)) = A,$$

and thus $\pi(B)$ is universal. $\square$

We list some other known partial results towards this conjecture. Bourgain [3] proved that if $B_1, B_2, B_3 \subseteq \mathbb{R}$ are infinite sets, then their Minkowski sum $B_1 + B_2 + B_3$ is not universal. See also [32] for a generalisation of this result. In terms of zero sequences, one of the best results up to now is due to Kolountzakis [48], which generalises Falconer's classical result [23]. See also [49] for a measure-theoretic result, and [68] gives a nice survey of partial results in this field.

However, combining all the partial results above, it is still open whether $B = \{2^{-n} : n \geq 1\}$ is universal.

We also remark that if A is a set with full measure, then A indeed contains a similar copy of any zero sequence. In fact, more is true.

Theorem 3.2.3. *If $A \subseteq [0,1]$ has measure 1 and $B \subseteq (0,\infty)$ is a bounded countable set, then there is $\delta > 0$ such that $\delta B \subseteq A$.*

This result actually aligns with direction (i). It shows that the candidate positive measure sets used to attack the Erdős similarity conjecture cannot have full measure in any interval.

Proof. Write $B = \{\alpha_m\}_{m=1}^{\infty}$. The conclusion is equivalent to the statement that $\bigcap_{m=1}^{\infty} \alpha_m^{-1} A \cap (0,\infty) \neq \varnothing$. Since $\alpha_m^{-1} A \subseteq [0,\infty)$, it suffices to prove that $\bigcap_{m=1}^{\infty} \alpha_m^{-1} A$ has positive Lebesgue measure.

Let $b = \sup B$. Then for any m, we have $\alpha_m^{-1} A \cap [0, \alpha_m^{-1}]$ differs from $[0, \alpha_m^{-1}]$ by a set of measure 0. Thus $\bigcap_{m=1}^{\infty} \alpha_m^{-1} A$ differs from $\bigcap_{m=1}^{\infty} [0, \alpha_m^{-1}]$

by a set of measure 0, by of the following general set relation:

$$\left(\bigcap_m (A_m \sqcup N_m)\right) \cap \left(\bigcup_m A_m^c\right) \subseteq \bigcup_m N_m,$$

where $\sqcup$ denotes disjoint union. But $\bigcap_{m=1}^{\infty}[0, \alpha_m^{-1}]$ is just $[0, b^{-1}]$ or $[0, b^{-1})$, which has positive Lebesgue measure. Thus $\bigcap_{m=1}^{\infty} \alpha_m^{-1} A$ also has positive Lebesgue measure, and the result follows. $\square$

To conclude, the results in direction (ii) do not contradict those in direction (i). The idea is that if the assumptions in the theorems in direction (i) fail, then one may construct specific examples to turn this into a negative result as in direction (ii). Combining the two sides, we see that the results are the sharpest if there is a single threshold on the largeness of sets, such that if a set is larger than that, then it must contain specific configurations; meanwhile, there exist corresponding counterexamples of sets just barely smaller than the threshold but do not contain prescribed patterns.

3.3 Sets with many patterns tend to be large

We now come to direction (iii).

The famous longstanding Kakeya conjecture is in this direction. We say a compact set $E \subseteq \mathbb{R}^n$ is a *Kakeya (or Besicovitch)* set if it contains a unit line segment in every direction. That is, for every $u \in \mathbb{S}^{n-1}$, there is some $x \in \mathbb{R}^n$ such that $x + tu \in E$ for every $t \in [0, 1]$. It was shown by Besicovitch [2] that a Kakeya set could have Lebesgue measure 0.

However, is it possible that $\dim_H(E) < n$? In the case $n = 1$ this answer is trivially negative. However, for $n \geq 2$, this becomes a very interesting problem. Davies [11] proved that we must have $\dim_H(E) = 2$ for Kakeya sets E in $\mathbb{R}^2$, but the question remains open in dimensions $n \geq 3$. The best bound so far for all $n \neq 3, 4, 6$ is due to Hickman, Rogers and Zhang [36] and Zahl [76], who independently proved that for Kakeya sets $E \subseteq \mathbb{R}^n$, we have

$$\dim_H(E) \geq \max_{2\leq k\leq n} \min\left\{n-k+2, \frac{n^2+k^2+n-k}{2n}\right\}.$$

The best lower bounds for $n = 3, 4, 6$ are respectively $5/2 + \varepsilon$ due to Katz and Zahl [44], roughly 3.059 due to Katz and Zahl [45] and 4 due Wolff [70]. See also [4, 42, 43] and Section 23 of [57] for results that used to be optimal.

Note that Theorem 1.1.1 is also in this direction.

3.4 Small sets containing many patterns

We finally come to direction (iv). In the discussion of direction (ii), we proved that even $\dim_F(A) = 1$ does not generally guarantee the existence of a 3 term arithmetic progression in A. However, does there exist some particular set A with $\mathcal{L}^1(A) = 0$ that contains many prescribed patterns? The answer is yes, and we can do much better.

An early result by Erdős and Kakutani [21] says that there exists a perfect set $A \subseteq [0, 1]$ with $\mathcal{L}^1(A) = 0$ and $\dim_H(A) = 1$ such that it contains a similar copy of every finite set. More surprisingly, one can construct even smaller sets that contain a copy of any finite set. Máthé [55] proved implicitly that there is a compact set $A \subseteq [0, 1]$ with Hausdorff dimension 0 that contains a similar copy of every finite pattern.

Molter and Yavicoli proved another surprising result in [58]. This article included many results on small sets containing patterns in $\mathbb{R}^n$ prescribed by a large family of functions. In particular, they are able to construct an F_σ set $A \subseteq \mathbb{R}$ with Hausdorff dimension 0 that contains a translated copy of any countable set. A proof of this special case can be found in the appendix of [72].

This seems to be a much stronger result than [55], but it should be noted the above set A is not closed. Indeed, it cannot be closed, by the following simple proposition.

Proposition 3.4.1. *Suppose $A \subseteq \mathbb{R}$ contains a similar copy of any bounded sequence. Then $\overline{A}$ contains an interval. As a result, the set A mentioned above cannot be closed.*

Proof. Let $\{\alpha_m\}$ be an enumeration of all the rationals in $[0, 1]$. By assumption, there is some $t \in \mathbb{R}$ and $\delta \neq 0$ such that $t + \delta\alpha_m \in A$ for all m. Taking closure on both sides shows that $\overline{A}$ contains an interval.

If A were closed, then A would contain a interval, a contradiction to the fact that $\dim_H(A) = 0$. $\square$

The above example suggests the subtlety of this subject to some extent. Indeed, in terms of dimensionality, A is extremely small as $\dim_H(A) = 0$. However, it is quite large in the sense of topology, as it is somewhere dense.

We now take a closer look at Proposition 3.4.1. The key for this result is that a bounded sequence may have lots of accumulation points. This motivates the following question:

Question 3.4.2. *If $A \subseteq \mathbb{R}$ contains a similar copy of any zero sequence, can A be nowhere dense?*

The answer is still negative, and has been spelt out in Theorem 1.1.1 above. On the other hand, if we restrict the rate of decay of the zero sequence, then we have Theorem 1.2 of [72], which aligns with this direction.

Chapter 4

Proof of Theorem 1.1.1

In this chapter we prove Theorem 1.1.1.

4.1 Preliminaries of the proof

4.1.1 A preliminary reduction

From the statement of Theorem 1.1.1, given any $\alpha_m \searrow 0$, there is $t \in \mathbb{R}$ and $\delta \neq 0$ such that $t + \delta\alpha_m \in E$ for all m. However, δ can be either positive or negative. In this subsection, we shall show that without loss of generality, it suffices to prove the case when $\delta > 0$. More precisely, we will show that the following Proposition 4.1.1 implies Theorem 1.1.1. Once this is established, it suffices to prove Proposition 4.1.1.

Proposition 4.1.1. *Let $B \subseteq \mathbb{R}$. Suppose that for every zero sequence α_m there is some $t' \in \mathbb{R}$ and $\delta' > 0$ such that $t' + \delta'\alpha_m \in E$ for all m. Then $\overline{B}$ contains an interval.*

Remark: To avoid excessive use of extra terminology, from now on we will not be referring to Proposition 4.1.1 itself in the subsequent argument. Instead, we will assume without loss of generality that $\delta > 0$ in the assumption of Theorem 1.1.1.

Proof that Proposition 4.1.1 Implies Theorem 1.1.1

Proof. Suppose, towards contradiction, that the closure of E contains no interval, that is, E is nowhere dense. Hence $-E$ is also nowhere dense. Let $B = E \cup (-E)$. Then by Lemma 2.1.1 for $n = 1$, B is nowhere dense.

We claim that B satisfies the assumptions of Proposition 4.1.1. Let $\alpha_m \searrow 0$ strictly. By the assumption on E, there is $\delta \neq 0$ and $t \in \mathbb{R}$ such that $t + \delta\alpha_m \in E$ for all m. If $\delta > 0$, then $t + \delta\alpha_m \in E \subseteq B$; if $\delta < 0$, then $-t + (-\delta)\alpha_m \in -E \subseteq B$.

By Proposition 4.1.1, $\overline{B}$ contains an interval, which is a contradiction as we showed above that B is nowhere dense. $\square$

4.1.2 Some notation

For our future use, it is convenient to introduce the following notation. We also remark that since $n = 1$, we may spare the subscript n in the remaining parts of this chapter.

- If I is an interval with endpoints $-\infty < a < b < \infty$, we define $I^* := [a, b)$. If O is a union of intervals I_n with endpoints $-\infty < a_n < b_n < \infty$ such that $\overline{I}_n \cap \overline{I}_{n'} = \varnothing$ for $n \neq n'$, we further define $O^* := \bigcup_n I_n^* = \bigcup_n [a_n, b_n)$.
- For any set $S \subseteq \mathbb{R}$ and any $r > 0$, we write $B_-(S, r)$ for the left r-neighbourhood of the set S: $B_-(S, r) := \{x - t : x \in S,\ 0 \leq t < r\}$.

We list here some elementary properties we shall use.

Proposition 4.1.2. *(i) If $S = (a, b)$, then for each $r > 0$, $B_-(S, r) = (a - r, b)$. In particular, $B_-(S, r) \supseteq [a, b) = S^*$.*

(ii) For any index set I and any $r > 0$, $\bigcup_{i \in I} B_-(S_i, r) = B_-(\bigcup_{i \in I} S_i, r)$.

(iii) If S is a (countable or finite) union of bounded open intervals with disjoint closures, then for any $r > 0$, $B_-(S, r) \supseteq S^*$.

(iv) If $S_2 \supseteq S_1$, then for any $r > 0$, $B_-(S_2, r) \supseteq B_-(S_1, r)$.

(v) If $r < s$, then for any set S, $B_-(S, r) \subseteq B_-(S, s)$.

Proof. (i) Let $S = (a, b)$ and $r > 0$. If $y \in B_-(S, r)$, then there is $x \in S = (a, b)$ and $0 \leq t < r$ such that $y = x - t$, so $y \in (a - t, b - t) \subseteq (a - r, b - 0) = (a - r, b)$. Hence $B_-(S, r) \subseteq (a - r, b)$.

On the other hand, if $y \in (a-r, b)$, then we have two cases:

If $a < y < b$, then letting $x = y \in (a, b)$ and $t = 0$ shows that $y \in B_-(S, r)$.

If $a - r < y \leq a$, then we let $\delta = a - y \in [0, r)$, and let $0 < \epsilon < \min\{b-a, r-\delta\}$. Then we let $x = a + \epsilon \in (a, b) = S$ and $t = x - y$. Note that $x - y > a - y \geq 0$ and $x - y = a + \epsilon - y = \delta + \epsilon < \delta + r - \delta = r$. Thus $t \in [0, r)$ and so $y = x - t \in B_-(S, r)$.

Thus $B_-(S, r) \supseteq (a-r, b)$. Combining two arguments gives $B_-(S, r) = (a-r, b)$.

Since $a - r < a$ for all $r > 0$, we have $B_-(S, r) = (a-r, b) \supseteq [a, b)$.

(ii) Let $\{S_i\}_{i \in I}$ and $r > 0$. If $y \in \bigcup_{i \in I} B_-(S_i, r)$, then there is $i \in I$ such that $y \in B_-(S_i, r)$, that is, there is $x \in S_i$ and $0 \leq t < r$ such that $y = x - t$. But $S_i \subseteq \bigcup_{i \in I} S_i$, so $x \in \bigcup_{i \in I} S_i$, and thus $y \in B_-(\bigcup_{i \in I} S_i, r)$. Hence $\bigcup_{i \in I} B_-(S_i, r) \subseteq B_-(\bigcup_{i \in I} S_i, r)$.

On the other hand, if $y \in B_-(\bigcup_{i \in I} S_i, r)$, then there is $x \in \bigcup_{i \in I} S_i$ and $0 \leq t < r$ such that $y = x - t$. Since $x \in \bigcup_{i \in I} S_i$, there is $i \in I$ such that $x \in S_i$. Hence $y = x - t \in B_-(S_i, r) \subseteq \bigcup_{i \in I} B_-(S_i, r)$. Hence $\bigcup_{i \in I} B_-(S_i, r) \supseteq B_-(\bigcup_{i \in I} S_i, r)$.

(iii) Write $S = \bigcup_n (a_n, b_n)$. Then for each $r > 0$,

$$B_-(S, r) \overset{(ii)}{=} \bigcup_n B_-((a_n, b_n), r) \overset{(i)}{=} \bigcup_n (a_n - r, b_n) \overset{(i)}{\supseteq} \bigcup_n [a_n, b_n) = S^*.$$

(iv) Since $S_2 \supseteq S_1$ we can write $S_2 = (S_2 \backslash S_1) \cup S_1$. By (ii) we have $B_-(S_2, r) = B_-(S_2 \backslash S_1, r) \cup B_-(S_1, r) \supseteq B_-(S_1, r)$.

(v) Let $r < s$, and let $y \in B_-(S, r)$. Then there is $x \in S$ and $0 \leq t < r$ such that $y = x - t$. But then $0 \leq t < s$, so $y \in B_-(S, s)$. Hence $B_-(S, r) \subseteq B_-(S, s)$.

$\square$

4.2 A Cantor-like construction

The main idea behind the proof of Theorem 1.1.1 is by contradiction. To achieve the contradiction, we will assume that E is nowhere dense, and construct a Cantor-like set containing E. At each level of construction of the Cantor set, we are removing intervals with specific lengths from the middle thirds of the remaining intervals. We then construct a slowly decreasing sequence $\{\alpha_m\}$, with rate of decrease depending on the lengths of the removed intervals, such that E contains no similar copy of $\{\alpha_m\}$. This construction will be the key to our proof of Theorem 1.1.1.

We will use the following standard notations and definitions:

4.2.1 The main construction

One of the main steps in the proof of Theorem 1.1.1 is the following Cantor-type construction.

Proposition 4.2.1. *Let $A \subseteq [0,1]$ be nowhere dense. Then there is a countable collection of open sets $\{O_n : n \geq 1\}$ and a countable collection of closed intervals $\{K_{n,j} : n \geq 1, 1 \leq j \leq 2^n\}$, with the following properties:*

(a) $A \subseteq [0,1] \backslash (\bigcup_{i=1}^{n} O_i)$ *for each* $n \geq 1$.

(b) $\overline{O}_n \cap \overline{O}_{n'} = \varnothing$ *for all* $n \neq n'$.

(c) Each O_n is of the form

$$O_n = \bigcup_{j=1}^{2^{n-1}} I_{n,j}, \tag{4.1}$$

where for each n, $\{I_{n,j} : 1 \leq j \leq 2^{n-1}\}$ is a collection of open intervals of the same length (denoted by l_n) with disjoint closures. Without loss of generality, l_n can be chosen to be decreasing to 0 such that $l_n^{-1} \in \mathbb{N}$.

(d) For each n, $[0,1] \backslash \bigcup_{i=1}^{n} O_i$ is a disjoint union of 2^n closed intervals, which we denote as $\{K_{n,j} : 1 \leq j \leq 2^n\}$ from left to right. They obey the relation $[0,1] \backslash \bigcup_{i=1}^{n} O_i = \bigcup_{j=1}^{2^n} K_{n,j}$, or equivalently $[0,1] \backslash \bigcup_{i=1}^{n} \overline{O}_i =$

$\bigcup_{j=1}^{2^n} \mathrm{Int}(K_{n,j})$. *In addition,* $\mathcal{L}^1(K_{n,j}) < (2/3)^n$ *for each* n *and each* $1 \leq j \leq 2^n$.

As a consequence,

$$A \subseteq [0,1] \backslash \left(\bigcup_{n=1}^{\infty} O_n \right) = \bigcap_{n=1}^{\infty} \bigcup_{j=1}^{2^n} K_{n,j}. \tag{4.2}$$

Proof. We construct O_n inductively. In the first step, by Lemma 2.1.1 applied to A with $K = [1/3, 2/3]$, we can find an open interval $I_{1,1} \subseteq [1/3, 2/3]$ which lies in A^c. Let the length of $I_{1,1}$ be l_1 (since we can always take a shorter interval within $I_{1,1}$, we may assume $l_1^{-1} \in \mathbb{N}$), and let $O_1 := I_{1,1}$. Note that $[0,1] \backslash O_1$, which contains A, has 2 closed connected components, which we denote as $K_{1,1}$ and $K_{1,2}$ from left to right (See Figure 4.1). By construction, $[0, 1/3] \subseteq K_{1,1} \subseteq [0, 2/3)$, so $1/3 \leq \mathcal{L}^1(K_{1,1}) < 2/3$; similarly we also have $1/3 \leq \mathcal{L}^1(K_{1,2}) < 2/3$. Hence all (a)-(d) are satisfied for $n = 1$ ((b) being null here).

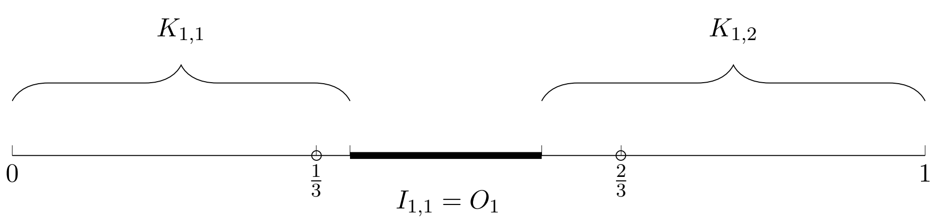

Figure 4.1: Removing an interval $I_{1,1}$ from the middle third of $[0, 1]$.

In general, at the end of the n-th step, we have obtained O_n and hence $I_{n,j}$ and $K_{n,j}$ obeying the requirements (a)-(d). In the $(n+1)$-th step, we apply Lemma 2.1.1 to A for each $1 \leq j \leq 2^n$ with $I = K_{n,j}$ and find an open sub-interval $I_{n+1,j}$ of the closed middle third of $K_{n,j}$ contained in A^c. A priori the intervals $I_{n+1,j}$ may have varying lengths. If $l > 0$ with $l^{-1} \in \mathbb{N}$ and $l \leq \min\{l_n/2, \mathcal{L}^1(I_{n+1,1}), \ldots, \mathcal{L}^1(I_{n+1,2^n})\}$, we replace each $I_{n+1,j}$, $1 \leq j \leq 2^n$ by a subinterval of length l, and we define $l_{n+1} = l$. By a slight abuse of notation we continue to call these smallest subintervals

$I_{n+1,j}$. Thus all $I_{n+1,j}$'s now have the same lengths $l_{n+1} \leq l_n/2$, such that $l_{n+1}^{-1} \in \mathbb{N}$ and that $l_n \to 0$.

(Refer to Figure 4.2, which demonstrates for a fixed $K_{n,j}$ two subsequent iterations. We remark here that the two solid dots denote the trisection points of $K_{n,j} = [a, b]$. Similarly, the four empty dots denote the trisection points of $K_{n+1,2j-1}$ and $K_{n+1,2j}$, respectively.)

Since for each $1 \leq j \leq 2^n$, $\overline{I}_{n+1,j}$ lies in the closed middle third $\tilde{K}_{n,j}$ of the closed interval $K_{n,j}$, and $\{K_{n,j} : 1 \leq j \leq 2^n\}$ are disjoint by (d) in the n-th step, we see that $\{\overline{I}_{n+1,j} : 1 \leq j \leq 2^n\}$ are disjoint. Furthermore, $\bigcup_{j=1}^{2^n} \overline{I}_{n+1,j}$ is disjoint from $\bigcup_{i=1}^{n} \overline{O}_i$ since by the n-th step we have

$$\bigcup_{i=1}^{n} \overline{O}_i = [0,1] \backslash \bigcup_{j=1}^{2^n} \mathrm{Int}(K_{n,j}) \subseteq [0,1] \backslash \bigcup_{j=1}^{2^n} \tilde{K}_{n,j} \subseteq [0,1] \backslash \bigcup_{j=1}^{2^n} \overline{I}_{n+1,j}.$$

Let $O_{n+1} := \bigcup_{j=1}^{2^n} I_{n+1,j}$ be the disjoint union of these open intervals, and by disjointness we also have $\overline{O}_{n+1} := \bigcup_{j=1}^{2^n} \overline{I}_{n+1,j}$. Then we have just showed that

$$\overline{O}_{n+1} \cap \overline{O}_i = \varnothing, \tag{4.3}$$

for all $1 \leq i \leq n$.

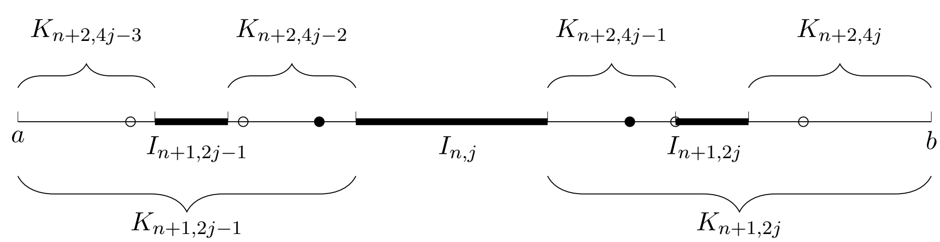

Figure 4.2: Two further iterations applied to $K_{n,j} = [a, b]$ (trisection points indicated).

We now proceed to verify conditions (a)-(d). We start with (a). Since $A \subseteq [0,1] \backslash (\bigcup_{i=1}^{n} O_i)$ by induction hypothesis, it suffices to show that

$$A \subseteq [0,1] \backslash O_{n+1}. \tag{4.4}$$

However, O_{n+1} was chosen as the union of intervals $I_{n+1,j}$, all of which are disjoint from A. Hence (4.4) follows.

We proceed to (b). In view of the induction hypothesis, this would follow if we show that $\overline{O}_{n+1} \cap \overline{O}_i = \varnothing$ for $i = 1, \ldots, n$. But this is (4.3) that we have already proved.

Part (c) follows by definition of O_{n+1} and disjointness of $\{\overline{I}_{n+1,j} : 1 \leq j \leq 2^n\}$.

For (d), since up to the n-th step we have 2^n intervals $K_{n,j}$, and given $1 \leq j \leq 2^n$, each $K_{n,j} \backslash I_{n,j}$ is a union of 2 disjoint closed intervals, we see $[0,1] \backslash \bigcup_{i=1}^{n+1} O_i$ is a disjoint union of 2^{n+1} closed intervals, which we denote as $K_{n+1,j}, 1 \leq j \leq 2^{n+1}$ from left to right.

With our choice of indices, we have $K_{n,j} \backslash I_{n,j} = K_{n+1,2j-1} \cup K_{n+1,2j}$. We write $K_{n,j} = [a,b]$, $I_{n,j} = (c,d)$, then $K_{n+1,2j-1} = [a,c]$. Since $I_{n,j}$ is a subinterval of the middle third of $K_{n,j}$, we have

$$\mathcal{L}^1(K_{n+1,2j-1}) = c - a < \frac{2}{3}(b-a) = \frac{2}{3}\mathcal{L}^1(K_{n,j}).$$

By the induction hypothesis, we have $\mathcal{L}^1(K_{n,j}) < (2/3)^n$, and so we have $\mathcal{L}^1(K_{n+1,2j-1}) < (2/3)^{n+1}$. Similarly we can show that $\mathcal{L}^1(K_{n+1,2j}) < (2/3)\mathcal{L}^1(K_{n,j}) < (2/3)^{n+1}$. As this holds for all $1 \leq j \leq 2^n$, we see that $\mathcal{L}^1(K_{n+1,j}) < (2/3)^{n+1}$ for all $1 \leq j \leq 2^{n+1}$.

Hence the induction follows. Lastly, letting $n \to \infty$ shows that

$$\begin{aligned} A \subseteq [0,1] \backslash \left(\bigcup_{n=1}^{\infty} O_n \right) &= [0,1] \cap \left(\bigcap_{n=1}^{\infty} O_n^c \right) \\ &= \bigcap_{n=1}^{\infty} ([0,1] \cap O_n^c) = \bigcap_{n=1}^{\infty} \bigcup_{j=1}^{2^n} K_{n,j}. \end{aligned}$$

□

The proof of Proposition 4.2.1 shows that any interval $K_{n,j}$ from the n-th step of the construction yields exactly two intervals $K_{n+1,2j-1}$ and $K_{n+1,2j}$

at the n-th step, i.e.

$$K_{n+1,r} \subseteq K_{n,j} \quad \text{if and only if} \quad r \in \{2j-1.2j\}.$$

Moreover, if $K_{n,j} = [a,b]$, then $a \in K_{n+1,2j-1}$, $b \in K_{n+1,2j}$.

Each interval $K_{n,j}$ generates exactly 2^k descendants after k subsequent steps. The rightmost of these intervals is $K_{n+k,2^kj}$. For fixed n and j, as k increases, the closed and bounded intervals $\{K_{n+k,2^kj} : k \geq 1\}$ form a decreasing nested sequence such that each $K_{n+k,2^kj}$, $k \geq 1$ contains the right endpoint of $K_{n,j}$, namely, $\sup K_{n,j}$. Additionally, in view of (d), we have $\mathcal{L}^1(K_{n+k,2^kj}) < (2/3)^{n+k} \to 0$. Hence the nested interval property leads to the following lemma:

Lemma 4.2.2. *Fix $n \geq 1$, $1 \leq j \leq 2^n$. Then*

$$\sup_{k\geq 1}(\inf K_{n+k,2^kj}) = \lim_{k\to\infty}(\inf K_{n+k,2^kj}) = \sup K_{n,j}.$$

4.2.2 Distribution of the deleted open sets

The following set relation will be used in the last part of the proof of Lemma 4.3.2 which leads to the main theorem. Recall the left neighbourhood B_- and the I^* notation introduced in Section 4.1.2.

Proposition 4.2.3. *The sets $\{O_n : n \geq 1\}$ constructed in the proof of Proposition 4.2.1 obey the following property: for $N \geq 1$,*

$$\bigcup_{n=N+1}^{\infty} B_-\left(O_n, \left(\frac{2}{3}\right)^n\right) \supseteq [0,1)\setminus\left(\bigcup_{n=1}^{N} O_n^*\right) = \bigcup_{j=1}^{2^N} K_{N,j}^*. \tag{4.5}$$

In other words, the intervals $\{I_{n,j}\}$ are densely distributed; if some x is not covered by any of the O_n^*'s up to stage N, then there is some $n \geq N+1$ and some j so that x will be within the left $(2/3)^n$-neighbourhood of $I_{n,j}$.

The proof of this proposition is based on the following simple observation.

Lemma 4.2.4. *Let K be a closed interval, and let $\tilde{K}$ denote its closed middle third. Then for each open interval $I \subseteq \tilde{K}$, we have*

$$B_{-}\left(I, \tfrac{2}{3}\mathcal{L}^1(K)\right) \supseteq [\inf K, \sup I).$$

(The illustration of this lemma and the proof is shown in Figure 4.3.)

Proof. Let $K = [a, b]$ and $I = (c, d)$. By (i) of Proposition 4.1.2, we have

$$B_{-}\left(I, \tfrac{2}{3}\mathcal{L}^1(K)\right) = \left(c - \tfrac{2}{3}\mathcal{L}^1(K), d\right).$$

Since $I \subseteq \tilde{K}$, we have $c < a + 2(b - a)/3$. Hence

$$c - \tfrac{2}{3}\mathcal{L}^1(K) < a + \tfrac{2}{3}(b - a) - \tfrac{2}{3}(b - a) = a.$$

Thus we have $B_{-}\left(I, \tfrac{2}{3}\mathcal{L}^1(K)\right) \supseteq [a, d) = [\inf K, \sup I)$. $\square$

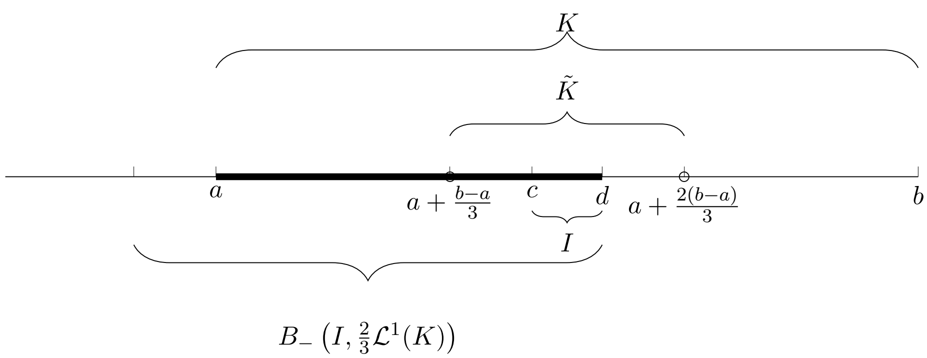

Figure 4.3: Illustration of Lemma 4.2.4, with $[a, d) = [\inf K, \sup I)$ shaded

Now we can give a proof of Proposition 4.2.3.

Proof. Fix N. Recall that (d) of Proposition 4.2.1 gives that for each N, $[0, 1)\backslash\left(\bigcup_{n=1}^{N} O_n\right) = \bigcup_{j=1}^{2^N} K_{N,j}$. Since $\{K_{N,j} : 1 \leq j \leq 2^N\}$ are disjoint, using our definition of I^* for each interval I introduced above, we also have $[0, 1)\backslash\left(\bigcup_{n=1}^{N} O_n^*\right) = \bigcup_{j=1}^{2^N} K_{N,j}^*$.

Fix N, j and consider a single $K_{N,j}$ (See Figure 4.2 again). For $k \geq 1$, since the middle third of $K_{N+k-1,2^{k-1}j}$ contains $I_{N+k,2^{k-1}j}$, by Lemma 4.2.4 applied to $K_{N+k-1,2^{k-1}j}$, we have

$$B_{-}\left(I_{N+k,2^{k-1}j}, \tfrac{2}{3}\mathcal{L}^1(K_{N+k-1,2^{k-1}j})\right) \supseteq [\inf K_{N+k-1,2^{k-1}j}, \sup I_{N+k,2^{k-1}j}). \tag{4.6}$$

Again, since $I_{N+k,2^{k-1}j}$ is deleted from $K_{N+k-1,2^{k-1}j}$ whose "child" on the right is $K_{N+k,2^kj}$, we have

$$\sup I_{N+k,2^{k-1}j} = \inf K_{N+k,2^kj}. \tag{4.7}$$

Taking the union over $k \geq 1$ on both sides in (4.6), we have

$$\begin{aligned}
&\bigcup_{k=1}^{\infty} B_{-}\left(I_{N+k,2^{k-1}j}, \frac{2}{3}\mathcal{L}^1(K_{N+k-1,2^{k-1}j})\right) \\
&\supseteq \bigcup_{k=1}^{\infty} [\inf K_{N+k-1,2^{k-1}j}, \sup I_{N+k,2^{k-1}j}) \\
\text{(by (4.7))} &= \bigcup_{k=1}^{\infty} [\inf K_{N+k-1,2^{k-1}j}, \inf K_{N+k,2^kj}).
\end{aligned}$$

We observe that for each k, the k-th interval above is adjacent to the $(k+1)$-th one. As a result, the union is a single interval given by

$$[\inf K_{N,j}, \sup_{k\geq 1}(\inf K_{N+k,2^kj})).$$

But by Lemma 4.2.2, $\sup_{k\geq 1}(\inf K_{N+k,2^kj}) = \sup K_{N,j}$, so

$$[\inf K_{N,j}, \sup_{k\geq 1}(\inf K_{N+k,2^kj})) = [\inf K_{N,j}, \sup K_{N,j}) = K^*_{N,j}.$$

What we have just shown is then

$$\bigcup_{k=1}^{\infty} B_{-}\left(I_{N+k,2^{k-1}j}, \frac{2}{3}\mathcal{L}^1(K_{N+k-1,2^{k-1}j})\right) \supseteq K^*_{N,j}. \tag{4.8}$$

Thus the left hand side of (4.5) is equal to:

$$
\begin{aligned}
&\bigcup_{n=N+1}^{\infty} B_{-}\left(O_n, \left(\frac{2}{3}\right)^n\right) \\
&= \bigcup_{k=1}^{\infty} B_{-}\left(O_{N+k}, \left(\frac{2}{3}\right)^{N+k}\right) \\
\text{(by (c) of Prop. 4.2.1)} &= \bigcup_{k=1}^{\infty} B_{-}\left(\bigcup_{l=1}^{2^{N+k-1}} I_{N+k,l}, \left(\frac{2}{3}\right)^{N+k}\right) \\
\text{(by (iv) of Prop. 4.1.2)} &\supseteq \bigcup_{k=1}^{\infty} B_{-}\left(\bigcup_{j=1}^{2^N} I_{N+k,2^{k-1}j}, \left(\frac{2}{3}\right)^{N+k}\right) \\
\text{(by (ii) of Prop. 4.1.2)} &= \bigcup_{j=1}^{2^N} \bigcup_{k=1}^{\infty} B_{-}\left(I_{N+k,2^{k-1}j}, \left(\frac{2}{3}\right)^{N+k}\right) \\
&\supseteq \bigcup_{j=1}^{2^N} \bigcup_{k=1}^{\infty} B_{-}\left(I_{N+k,2^{k-1}j}, \frac{2}{3}\mathcal{L}^1(K_{N+k-1,2^{k-1}j})\right) \\
\text{(by (4.8))} &\supseteq \bigcup_{j=1}^{2^N} K^*_{N,j},
\end{aligned}
$$

where in the second to last line we have used (d) of Proposition 4.2.1 and (v) of Proposition 4.1.2. □

4.3 Proof of Theorem 1.1.1

We will prove Theorem 1.1.1 by contradiction. Suppose E is nowhere dense. For $k \in \mathbb{Z}$, write

$$E_k = E \cap [k, k+1). \tag{4.9}$$

Then for each $k \in \mathbb{Z}$, $E_k - k \subseteq [0,1]$ is nowhere dense, so we can use Proposition 4.2.1 with $A = E_k - k \subseteq [0,1]$ to find $O_n^{(k)} \subseteq [k, k+1]$ and $I_{n,j}^{(k)} \subseteq [k, k+1]$ with lengths $l_n^{(k)}$ as specified by (c) of Proposition 4.2.1.

4.3.1 Constructing a slowly decreasing sequence $\{\alpha_m\}$

With the countable collection of sequences $\{l_n^{(k)}\}_{n=1}^\infty$ indexed by k, we are going to pick an extremely slowly decreasing sequence $\alpha_m \searrow 0$ depending on $\{l_n^{(k)}\}$, such that E does not contain any similar copy of $\{\alpha_m\}$.

Note that for each k, $\{l_n^{(k)}\}$ is a sequence in n that decreases to 0, but the rate may vary for different k. By the following lemma, we are going to construct a strictly decreasing sequence $\{\mu_n\}$ which decreases more rapidly than $\{l_n^{(k)}\}$ for any k.

Lemma 4.3.1. *For each $k \in \mathbb{Z}$, let $\{l_n^{(k)}\}_{n=1}^\infty$ with $(l_n^{(k)})^{-1} \in \mathbb{N}$ be strictly decreasing to 0. Then there is a sequence $\{\mu_n\}$ with $\mu_n^{-1} \in \mathbb{N}$ which also decreases strictly to 0, such that for any $k \in \mathbb{Z}$ and any $n \geq |k|$ we have $\mu_n \leq l_n^{(k)}$.*

Proof. Let $\mu_n = \min\{l_n^{(k)} : |k| \leq n\}$. Then $\mu_n > 0$ for all n since $l_n^{(k)} > 0$ for all k and n. Also, $\mu_n^{-1} \in \mathbb{N}$.

We prove that $\{\mu_n\}$ is strictly decreasing. Indeed, let $n \geq 2$, then

$$\begin{aligned}
\mu_n &= \min\{l_n^{(k)} : |k| \leq n\} \\
&\leq \min\{l_n^{(k)} : |k| \leq n-1\} \\
&< \min\{l_{n-1}^{(k)} : |k| \leq n-1\} = \mu_{n-1},
\end{aligned}$$

where the strict inequality follows since for each k, $\{l_n^{(k)}\}$ is strictly decreasing with respect to n. Lastly, fix $k \geq 1$. By definition, if $n \geq |k|$, then $\mu_n = \min\{l_n^{(k)} : |k| \leq n\} \leq l_n^{(k)}$. □

Now we start to construct $\{\alpha_m\}$. We set $N_0 := 0$ and $N_n := \mu_n^{-1} + N_{n-1}$ for $n \geq 1$, so $N_n \in \mathbb{N}$ and increases strictly to ∞.

We then define $\{\alpha_m\}_{m=1}^\infty$ as follows:

$$\alpha_m = \frac{1}{n} - \left(\frac{1}{n} - \frac{1}{n+1}\right) \frac{m - N_{n-1} - 1}{N_n - N_{n-1}}, \quad m = N_{n-1} + 1, \ldots, N_n. \quad (4.10)$$

That is, we set

$$\alpha_1 = \alpha_{N_0+1} = 1, \quad \alpha_{N_1+1} = \frac{1}{2}, \quad \alpha_{N_2+1} = \frac{1}{3}, \quad \dots \quad \alpha_{N_n+1} = \frac{1}{n+1}, \quad \dots, \tag{4.11}$$

and the choice of α_m for intermediate values of m is made by linearly interpolating between the two closest values, namely, $N_{n-1}+1 < m < N_n+1$.

Thus

$$\alpha_m - \alpha_{m+1} = \frac{1}{n(n+1)(N_n - N_{n-1})}, \quad N_{n-1} + 1 \leq m \leq N_n.$$

Since $N_n - N_{n-1} = \mu_n^{-1}$ is increasing, it follows that $\alpha_m - \alpha_{m+1}$ is decreasing. Since $\alpha_m - \alpha_{m+1} > 0$, we see that $\{\alpha_m\}$ is strictly decreasing.

Refer to Figure 4.3.1, which shows the sequence in the case $N_1 = 4$ and $N_2 = 8$. For example, α_m decreases from 1 to 1/2 in $N_1 = 4$ steps of equal size $1/2 \times 1/4 = 1/8$. It then decreases from 1/2 to 1/3 in $N_2 - N_1 = 4$ steps of equal size $1/6 \times 1/4 = 1/24$.

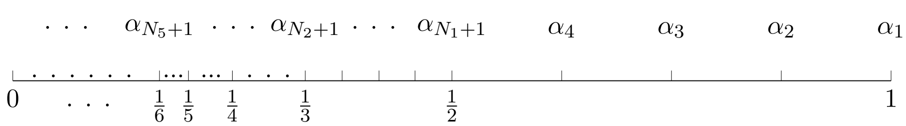

Figure 4.4: $\{\alpha_m\}$ when $N_1 = 4$, $N_2 = 8$

We will then prove the following lemma.

Lemma 4.3.2. *Let $\{\alpha_m : m \geq 1\}$ be the sequence defined in* (4.10). *For every $k \geq 1$, E_k denotes the set in* (4.9). *Then for every $\delta > 0$ and $m_0 \geq 1$, we have*

$$[0, 1) \cap \left(\bigcap_{m=m_0}^{\infty} (E_k - k) - \delta\alpha_m \right) = \varnothing. \tag{4.12}$$

The lemma will be proved in Section 4.4.

4.3.2 Proof of Theorem 1.1.1 assuming Lemma 4.3.2

Recall that at the beginning of this section, we have assumed towards contradiction that E is nowhere dense and from this constructed each E_k and a slowly decreasing $\{\alpha_m\}$. We will show that E contains no similar copy of $\{\alpha_m\}$, which contradicts the assumption of Theorem 1.1.1 and thus finishes our proof.

Suppose, towards contradiction, that there is $t \in \mathbb{R}$ and $\delta \neq 0$ such that $t + \delta\alpha_m \in E$ for all m. Recalling the preliminary reduction in Section 4.1.1, we may assume without loss of generality that $\delta > 0$.

Thus there is $k \in \mathbb{Z}$ such that E_k contains all but finitely many terms of $t + \delta\alpha_m$. Indeed, there is a unique $k \in \mathbb{Z}$ with $t \in [k, k+1)$. Since $t + \delta\alpha_m \searrow t$, there is $m_0 = m_0(\{\alpha_m\}, E)$ such that $t + \delta\alpha_m < k + 1$ for all $m \geq m_0$, so $t + \delta\alpha_m \in E_k = E \cap [k, k+1]$ for all $m \geq m_0$. Equivalently, $t - k + \delta\alpha_m \in E_k - k \subseteq [0, 1]$ for $m \geq m_0$. Letting $m \to \infty$ also shows that $t - k \subseteq [0, 1)$. Rewriting this in set notation, we have

$$t - k \in [0, 1) \cap \left(\bigcap_{m=m_0}^{\infty} (E_k - k) - \delta\alpha_m \right),$$

which is a contradiction to Lemma 4.3.2. This proves Theorem 1.1.1.

4.4 Translation of an interval

In this section, we will prove Lemma 4.3.2. The main ingredients of this proof are two structural results concerning the union of translations of an interval. These results are contained in Lemma 4.4.1 and 4.4.2 below. The proof of Lemma 4.3.2 assuming these results appear in Section 4.4.4.

Before stating the lemma, we point out a minor simplification of notation. We will temporarily drop the dependence of every term on k until it becomes necessary. This helps us get rid of using excessively cumbersome notation.

To be more precise, for each $k \geq 1$, let us write $A := E_k - k \subseteq [0, 1]$, and unless otherwise specified, $O_n^{(k)}$, $I_{n,j}^{(k)}$ and $l_n^{(k)}$ (defined at the beginning of this section) will be denoted by O_n, $I_{n,j}$ and l_n, respectively.

In the new notation, (4.12) in Lemma 4.3.2 reads

$$[0,1) \cap \left(\bigcap_{m=m_0}^{\infty} A - \delta\alpha_m \right) = \varnothing. \tag{4.13}$$

4.4.1 Structure of union of translates of an interval

Fix n and we examine carefully $\bigcup_{m=1}^{\infty} O_n - \delta\alpha_m$ for a large n. Let us recall that $O_n = \bigcup_{j=1}^{2^{n-1}} I_{n,j}$ from (4.1) of Proposition 4.2.1, and fix one connected component $I_{n,j}$ of O_n.

Let

$$M(n) = M(n, m_0, \delta) = \min\{m \geq m_0 : \delta(\alpha_m - \alpha_{m+1}) < l_n\}. \tag{4.14}$$

We note that $M(n)$ is finite since $\alpha_m - \alpha_{m+1} \searrow 0$. By the monotonicity of $\alpha_m - \alpha_{m+1}$, for all $m \geq M(n)$, we have $\delta(\alpha_m - \alpha_{m+1}) < l_n$. It is worth noting that $M(n)$ depends δ and m_0, but this dependence is suppressed because the subsequent argument does not rely on the specified value of δ and m_0.

Lemma 4.4.1. *Let $\{\alpha_m\}_{m=1}^{\infty}$ be a sequence strictly decreasing to 0 such that $\alpha_m - \alpha_{m+1}$ is also decreasing. Then for any $m_0 \geq 1$ and $M(n)$ as in (4.14), we can decompose the countable union of intervals $\bigcup_{m=m_0}^{\infty} I_{n,j} - \delta\alpha_m$ into a disjoint union of U_1 and U_2, where*

$$U_1 = U_1(j) = \bigcup_{m=m_0}^{M(n)-1} I_{n,j} - \delta\alpha_m$$

is a disjoint union of open intervals of the same length l_n, and

$$U_2 = U_2(j) = \bigcup_{m=M(n)}^{\infty} I_{n,j} - \delta\alpha_m$$

is a single open interval with length $l_n + \delta\alpha_{M(n)}$ and the same right endpoint *as $I_{n,j}$. Using our B_- notation, this can be written as*

$$U_2 = B_-(I_{n,j}, \delta\alpha_m). \tag{4.15}$$

This lemma is illustrated in Figure 4.5. In this figure, we first fix an interval $I = I_{n,j}$ and show the relative positions of $I - \delta\alpha_m$ for different choices of $m \geq m_0$. To showcase the threshold for the overlapping phenomenon, we draw these intervals indexed by m along the vertical axis.

We also remark that U_1 and U_2 again depend on n, j (as well as δ and m_0), but we suppress the dependence for the moment since for now we will be only considering one single $I_{n,j}$. Another crucial observation is that our $M(n)$ is independent of the choice of j, so it works for all intervals $\{I_{n,j}, 1 \leq j \leq 2^{n-1}\}$ in the n-th iteration of the construction in the proof of Proposition 4.2.1. In the future, we call U_1 the disjoint part and U_2 the overlapping part.

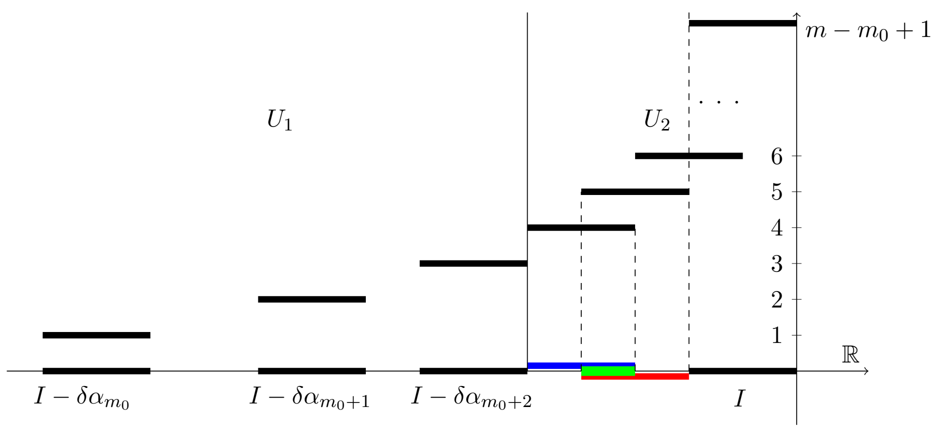

Figure 4.5: Structure of $\bigcup_{m=m_0}^{\infty} I_{n,j} - \delta\alpha_m$ when $M(n) = m_0 + 3$

Proof of Lemma 4.4.1. As all $I_{n,j} - \delta\alpha_m$ are open intervals and α_m is strictly decreasing, U_1 is a disjoint union if and only if for each $m_0 \leq m \leq M(n) - 2$, we have $\sup I_{n,j} - \delta\alpha_m \leq \inf I_{n,j} - \delta\alpha_{m+1}$. This is true if and only if $\delta(\alpha_m - \alpha_{m+1}) \geq \sup I_{n,j} - \inf I_{n,j} = l_n$ for all $1 \leq m \leq M(n) - 2$, which follows from the definition (4.14) of $M(n)$. Since $\{I_{n,j} - \delta\alpha_m : 1 \leq m \leq M(n) - 1\}$ are translates of the interval $I_{n,j}$, they have the same length l_n.

Since $\delta\alpha_m$ is strictly decreasing, U_1 and U_2 are disjoint if and only if $I_{n,j} - \delta\alpha_{M(n)-1}$ and $I_{n,j} - \delta\alpha_{M(n)}$ are disjoint. This is true if and only if $\delta(\alpha_{M(n)-1} - \alpha_{M(n)}) \geq l_n$, which holds by (4.14).

The infinite union U_2 is a single open interval if and only if for each $m \geq M(n)$, we have $\sup I_{n,j} - \delta\alpha_m > \inf I_{n,j} - \delta\alpha_{m+1}$. This is true if and only if $\delta(\alpha_m - \alpha_{m+1}) < \sup I_{n,j} - \inf I_{n,j} = l_n$ for all $m \geq M(n)$, which follows from (4.14).

Lastly, since α_m decreases strictly to 0, $\sup I_{n,j} - \delta\alpha_m$ increases strictly to $\sup I_{n,j}$ as $m \to \infty$. Since we have shown that U_2 is an open interval, we have $U_2 = (\inf I_{n,j} - \delta\alpha_{M(n)}, \sup I_{n,j})$. By Part (i) of Proposition 4.1.2, we have $U_2 = B_-(I_{n,j}, \delta\alpha_{M(n)})$, which is (4.15). □

4.4.2 Slow Decay of $\{\alpha_m\}$

In this subsection, we prove the following lemma, which is a result of the slow decay of $\{\alpha_m\}$.

Lemma 4.4.2. *Let $k \geq 1$. Then there is $n_0 = n_0(k, \delta, m_0)$ such that*

$$\alpha_{M(n)} \geq (n+1)^{-1}, \quad \textit{for all } n \geq n_0. \tag{4.16}$$

Recall that $M(n)$ depends implicitly on k.

We first prove that there is $n_0 = n_0(k, \delta, m_0)$ such that $M(n) \leq N_n$ for all $n \geq n_0$. (Recall N_n was defined in the construction of $\{\alpha_m\}$ at the end of Section 4.3.1, and does not depend on k.) Indeed, by definition of $M(n)$, this is true if and only if

$$\delta(\alpha_{N_n} - \alpha_{N_n+1}) < l_n \tag{4.17}$$

for all large n such that $N_n \geq m_0$. But by construction of the sequence α_m, we have

$$\delta(\alpha_{N_n} - \alpha_{N_n+1}) = \frac{\delta}{n(n+1)(N_n - N_{n-1})} = \frac{\delta}{n(n+1)\mu_n^{-1}},$$

which will be strictly less than μ_n if $n > \delta^{-1}$. But by Lemma 4.3.1, $\mu_n \leq l_n := l_n^{(k)}$ for all $n \geq |k|$. Hence (4.17) holds if $n \geq \max\{\delta^{-1}, |k|\}$.

Since $N_n \to \infty$, there is n_1 such that $N_n \geq m_0$ for all $n \geq n_1$. Hence we may choose $n_0 > \max\{\delta^{-1}, |k|, n_1\}$ so that $M(n) \leq N_n$ for all $n \geq n_0$. By monotonicity of α_m and recalling (4.11), we have

$$\alpha_{M(n)} \geq \alpha_{N_n} > \alpha_{N_n+1} = (n+1)^{-1}, \quad \text{for all } n \geq n_0,$$

which is (4.16).

4.4.3 A corollary of Lemma 4.4.1 and Lemma 4.4.2

In this subsection, we prove the following set relation:

$$\bigcup_{m=m_0}^{\infty} \bigcup_{n=1}^{\infty} O_n - \delta\alpha_m \supseteq [0, 1). \tag{4.18}$$

For the proof of (4.18), we will be only interested in the overlapping part. For each n and j, we have

$$\bigcup_{m=m_0}^{\infty} I_{n,j} - \delta\alpha_m \supseteq U_2(j) \overset{(4.15)}{=} B_-(I_{n,j}, \delta\alpha_{M(n)}). \tag{4.19}$$

Recall that $M(n)$ is independent of j. Thus we can take the union over $1 \leq j \leq 2^{n-1}$ on both sides of (4.19) and obtain

$$\bigcup_{j=1}^{2^{n-1}} \bigcup_{m=m_0}^{\infty} I_{n,j} - \delta\alpha_m \supseteq \bigcup_{j=1}^{2^{n-1}} B_-(I_{n,j}, \delta\alpha_{M(n)}). \tag{4.20}$$

Swapping the unions on the left hand side of (4.20) and by (4.1) and the relation (2.1), we see it is equal to $\bigcup_{m=m_0}^{\infty} O_n - \delta\alpha_m$. By (4.1) and (ii) of Proposition 4.1.2, the right hand side of (4.20) is equal to $B_-(O_n, \delta\alpha_{M(n)})$. We have thus showed

$$\bigcup_{m=m_0}^{\infty} O_n - \delta\alpha_m \supseteq B_-(O_n, \delta\alpha_{M(n)}). \tag{4.21}$$

Now we invoke Lemma 4.4.2 to find an n_0 such that $\alpha_{M(n)} \geq (n+1)^{-1}$ for all $n \geq n_0$. We then choose an integer $N \geq n_0$ such that for all $n \geq N$, we have $\delta/(n+1) \geq (2/3)^n$. This implies

$$\delta\alpha_{M(n)} \geq (2/3)^n, \text{ for all } n \geq N. \tag{4.22}$$

Taking union over n on both sides of (4.21), we have

$$\begin{aligned}
&\bigcup_{n=1}^{\infty} \bigcup_{m=m_0}^{\infty} O_n - \delta\alpha_m \\
&\supseteq \bigcup_{n=1}^{\infty} B_-(O_n, \delta\alpha_{M(n)}) \\
&= \left(\bigcup_{n=1}^{N} B_-(O_n, \delta\alpha_{M(n)}) \right) \cup \left(\bigcup_{n=N+1}^{\infty} B_-(O_n, \delta\alpha_{M(n)}) \right) \\
&\supseteq \left(\bigcup_{n=1}^{N} B_-(O_n, \delta\alpha_{M(n)}) \right) \cup \left(\bigcup_{n=N+1}^{\infty} B_- \left(O_n, \left(\frac{2}{3} \right)^n \right) \right) \\
&\supseteq \left(\bigcup_{n=1}^{N} B_-(O_n, \delta\alpha_{M(n)}) \right) \cup \left([0,1) \backslash \left(\bigcup_{n=1}^{N} O_n^* \right) \right) \\
&\supseteq \left(\bigcup_{n=1}^{N} O_n^* \right) \cup \left([0,1) \backslash \left(\bigcup_{n=1}^{N} O_n^* \right) \right) \supseteq [0,1),
\end{aligned}$$

where in the fourth line we have used (v) of Proposition 4.1.2 and (4.22), in the fifth line we have used (4.5) in Proposition 4.2.3, and in the sixth line we have used (iii) of Proposition 4.1.2. Hence (4.18) follows.

4.4.4 Proof of Lemma 4.3.2

Using (4.18) we can now prove Lemma 4.3.2, which is expressed in the form (4.13). We will use the relations (2.1) through (2.3) here.

By the inclusion relation (4.2) in Proposition 4.2.1, for any $\delta > 0$,

$$\begin{aligned}
\bigcap_{m=m_0}^{\infty} A - \delta\alpha_m &= \bigcap_{m=m_0}^{\infty} \left([0,1] \backslash \left(\bigcup_{n=1}^{\infty} O_n \right) - \delta\alpha_m \right) \\
&= \bigcap_{m=m_0}^{\infty} \left([0,1] \cap \left(\bigcap_{n=1}^{\infty} O_n^c \right) - \delta\alpha_m \right) \\
&\subseteq \bigcap_{m=m_0}^{\infty} \left(\bigcap_{n=1}^{\infty} O_n^c - \delta\alpha_m \right) \\
&= \bigcap_{m=m_0}^{\infty} \bigcap_{n=1}^{\infty} (O_n^c - \delta\alpha_m).
\end{aligned}$$

Now we take complements in $[0,1)$ on both sides of (4.18) which was obtained in the previous section. This gives

$$\begin{aligned}
\varnothing &\supseteq [0,1) \cap \left(\bigcap_{m=m_0}^{\infty} \bigcap_{n=1}^{\infty} (O_n - \delta\alpha_m)^c \right) \\
&= [0,1) \cap \left(\bigcap_{m=m_0}^{\infty} \bigcap_{n=1}^{\infty} O_n^c - \delta\alpha_m \right) \\
&\supseteq [0,1) \cap \left(\bigcap_{m=m_0}^{\infty} A - \delta\alpha_m \right),
\end{aligned}$$

which is (4.13). This finishes the proof of Lemma 4.3.2 and thus Theorem 1.1.1.

Chapter 5

Proof of Theorem 1.1.2

In this chapter we prove Theorem 1.1.2.

We start with a brief sketch of the proof. First, we introduce the definition of threshold sequences, and then prove Proposition 5.1.2 which is just Theorem 1.1.2 with an additional assumption that the prescribed $\{\beta_m\}$ can be replaced by a threshold sequence (definition right below) $\{\eta_m\}$. After that, we will show that Proposition 5.1.2 and Lemma 5.1.3 together imply Theorem 1.1.2. Lastly we give a proof of Lemma 5.1.3.

5.1 Threshold sequences

Definition 5.1.1 (Threshold Sequence)**.** *Let η_m be a zero sequence. We say η_m is a threshold sequence if $\eta_m - \eta_{m+1}$ is non-increasing.*

Proposition 5.1.2. *Let $\{\eta_m\}_{m=1}^{\infty}$ be a threshold sequence. Then there is a closed and nowhere dense set $A \subseteq [0,1]$, depending on $\{\eta_m\}$, such that for any sequence $\alpha_m \to 0$ with $\sup_m |\alpha_m|/\eta_m < \infty$, there is $\delta > 0$ and $t \in \mathbb{R}$ such that $t + \delta\alpha_m \in A$ for all m.*

For the demonstration to be more clear, we give a proof of Proposition 5.1.2 in the next section.

Lemma 5.1.3. *Let $\{\beta_m\}$ be a zero sequence. Then there is a threshold sequence $\{\eta_m\}$ such that $\beta_m \leq \eta_m$ for all m.*

5.1.1 Proof of Theorem 1.1.2

Assuming that Proposition 5.1.2 and Lemma 5.1.3 holds, we now prove Theorem 1.1.2

Proof. Let $\{\beta_m\}$ be given as in Theorem 1.1.2. By Lemma 5.1.3, find a threshold sequence $\{\eta_m\}$ such that $\beta_m \le \eta_m$. By Proposition 5.1.2 applied to $\{\eta_m\}$, we can find a closed and nowhere dense $A \subseteq [0,1]$, depending on $\{\eta_m\}$, such that for all $\sup_m |\alpha_m|/\eta_m < \infty$, there is $\delta > 0$ and $t \in \mathbb{R}$ such that $t + \delta\alpha_m \in A$ for all m. But $\beta_m \le \eta_m$, and hence $\sup_m |\alpha_m|/\beta_m < \infty$ implies $\sup_m |\alpha_m|/\eta_m < \infty$. Lastly, since $\{\eta_m\}$ depends on $\{\beta_m\}$ only, A also depends on $\{\beta_m\}$ only. $\square$

5.2 Proof of Proposition 5.1.2

Now we prove Proposition 5.1.2.

5.2.1 Construction of the compact set

We start with any countable collection of open intervals V_n that forms a countable base for the standard topology on $(0,1)$. For example, we can choose $\{V_n\}$ to be the countable collection of all open intervals in $(0,1)$ with rational centres and rational radii. Our set A will be of the form

$$A = [0,1] \setminus \bigcup_{n=1}^{\infty} J_n \tag{5.1}$$

for a carefully chosen collection of open intervals $J_n \subseteq V_n$ whose lengths λ_n are to be specified (See (5.10)). With this definition, $A \subseteq [0,1]$ is automatically closed and nowhere dense.

5.2.2 A measure-theoretic argument

We will figure out what conditions can be imposed on λ_n so that the set A we defined satisfies the affine containment property as stated in Proposition 5.1.2.

Let α_m with $\sup_m |\alpha_m|/\eta_m < \infty$. Assuming λ_n has been chosen, we are going to find $\delta > 0$ and $t \in \mathbb{R}$ such that $t + \delta\alpha_m \in A$ for all m. In contrast to (4.13), we show that there is $0 < \delta < 1$ such that the following set relation holds:

$$\bigcap_{m=1}^{\infty} A - \delta\alpha_m \neq \varnothing, \tag{5.2}$$

which is true if, in particular,

$$\mathcal{L}^1\left(\bigcap_{m=1}^{\infty} A - \delta\alpha_m\right) > 0. \tag{5.3}$$

But using (5.1) and the set relation (2.2), we can compute

$$\bigcap_{m=1}^{\infty} A - \delta\alpha_m = [0,1] \cap \left(\bigcap_{m=1}^{\infty}\bigcap_{n=1}^{\infty} J_n^c - \delta\alpha_m\right).$$

Thus (5.3) holds if and only if

$$1 > \mathcal{L}^1\left([0,1]\backslash \bigcap_{m=1}^{\infty}\bigcap_{n=1}^{\infty} J_n^c - \delta\alpha_m\right) = \mathcal{L}^1\left([0,1] \cap \bigcup_{m=1}^{\infty}\bigcup_{n=1}^{\infty} J_n - \delta\alpha_m\right).$$

Hence it suffices to show that there is $\delta > 0$ such that

$$1 > \mathcal{L}^1\left(\bigcup_{m=1}^{\infty}\bigcup_{n=1}^{\infty} J_n - \delta\alpha_m\right) = \mathcal{L}^1\left(\bigcup_{n=1}^{\infty}\left(\bigcup_{m=1}^{\infty} J_n - \delta\alpha_m\right)\right).$$

It further suffices to show there is $\delta > 0$ such that

$$\sum_{n=1}^{\infty} \mathcal{L}^1\left(\bigcup_{m=1}^{\infty} J_n - \delta\alpha_m\right) < 1. \tag{5.4}$$

The following proposition will imply (5.4):

Proposition 5.2.1. *1. For any $\delta > 0$ and any $n \geq 1$,*

$$\lim_{\delta\to 0^+} \mathcal{L}^1\left(\bigcup_{m=1}^{\infty} J_n - \delta\alpha_m\right) = \lambda_n.$$

2. Let $\delta_0 > 0$ be a fixed constant such that $|\alpha_m| \leq \frac{\eta_m}{2\delta_0}$ for all $m \geq 1$. (Such δ_0 exists since $\sup_m |\alpha_m|/\eta_m < \infty$, and note that δ_0 does not *depend on m, n.) Then for any $0 < \delta < \delta_0$ and any $n \geq 1$,*

$$\mathcal{L}^1\left(\bigcup_{m=1}^{\infty} J_n - \delta\alpha_m\right) \leq \mathcal{L}^1\left(\bigcup_{m=1}^{\infty} J_n - \eta_m\right). \tag{5.5}$$

3.

$$\sum_{n=1}^{\infty} \mathcal{L}^1\left(\bigcup_{m=1}^{\infty} J_n - \eta_m\right) < \infty.$$

Indeed, if all of the above are true, then by the Dominated Convergence Theorem applied to $f_\delta(n) = \mathcal{L}^1(\cup_{m=1}^{\infty} J_n - \delta\alpha_m)$ with the measure space being the counting measure on $\mathbb{N}$, we get

$$\lim_{\delta\to 0^+} \sum_{n=1}^{\infty} \mathcal{L}^1\left(\bigcup_{m=1}^{\infty} J_n - \delta\alpha_m\right) = \sum_{n=1}^{\infty} \lambda_n.$$

Thus (5.4) holds since $\sum_{n=1}^{\infty} \lambda_n < 1$ by (5.10).

5.2.3 Proof of Proposition 5.2.1

We first prove Part (1). Let $\delta > 0$ and $n \geq 1$. Denote $J_n := (a, b)$. Since $\alpha_m \to 0$, it is bounded. Let $c = \inf\{\alpha_m : m \geq 1\}$ and $d = \sup\{\alpha_m : m \geq 1\}$. Then we have $\inf(J_n - \delta\alpha_m) = a - \delta\alpha_m \geq a - \delta d$, and $\sup(J_n - \delta\alpha_m) = b - \delta\alpha_m \leq b - \delta c$. Hence $\bigcup_{m=1}^{\infty} J_n - \delta\alpha_m \subseteq (a - \delta d, b - \delta c)$, so

$$\mathcal{L}^1\left(\bigcup_{m=1}^{\infty} J_n - \delta\alpha_m\right) \leq b - a + \delta(d - c) = \lambda_n + \delta(d - c).$$

On the other hand, $\bigcup_{m=1}^{\infty} J_n - \delta\alpha_m \supseteq J_n - \delta\alpha_1 = (a - \delta\alpha_1, b - \delta\alpha_1)$, so $\mathcal{L}^1\left(\bigcup_{m=1}^{\infty} J_n - \delta\alpha_m\right) \geq b - a = \lambda_n$. Hence the squeeze law implies that $\mathcal{L}^1\left(\bigcup_{m=1}^{\infty} J_n - \delta\alpha_m\right)$ converges to λ_n as $\delta \to 0^+$.

Now we come to Part (2). Define, similar to (4.14),

$$T(n) := \min\{m : \eta_m - \eta_{m+1} < \lambda_n\}. \tag{5.6}$$

Since η_m is a threshold sequence (see Definition 5.1.1), it decreases strictly to 0 and $\eta_m - \eta_{m+1}$ is also decreasing. Thus we have $\eta_m - \eta_{m+1} < \lambda_n$ if and only if $m \geq T(n)$.

By Lemma 4.4.1, we have that $U_1 := \bigcup_{m=1}^{T(n)-1} J_n - \eta_m$ is a disjoint union of open intervals of length λ_n, that $U_2 := \bigcup_{m=T(n)}^{\infty} J_n - \eta_m$ is a single open interval of length $\eta_{T(n)} + \lambda_n$, and that $\bigcup_{m=1}^{T(n)-1} J_n - \eta_m$ and $\bigcup_{m=T(n)}^{\infty} J_n - \eta_m$ are disjoint. Thus the right hand side of (5.5) can be computed as:

$$\mathcal{L}^1\left(\bigcup_{m=1}^{\infty} J_n - \eta_m\right) = (T(n)-1)\lambda_n + \eta_{T(n)} + \lambda_n = T(n)\lambda_n + \eta_{T(n)}. \quad (5.7)$$

Now we come to the left hand side of (5.5). Regardless of the positions of the intervals $\{J_n - \delta\alpha_m\}_{m=1}^{T(n)-1}$, we always have

$$\mathcal{L}^1\left(\bigcup_{m=1}^{T(n)-1} J_n - \delta\alpha_m\right) \leq \sum_{m=1}^{T(n)-1} \mathcal{L}^1\left(J_n - \delta\alpha_m\right) = (T(n)-1)\lambda_n.$$

On the other hand, by Part 2 of Proposition 5.2.1, for all $0 < \delta < \delta_0$ and for all $m \geq 1$, we have $\delta|\alpha_m| \leq \frac{\eta_m}{2}$. Denote $J_n = (a, b)$. Then for all $m \geq T(n)$, we have

$$\sup(J_n - \delta\alpha_m) = b - \delta\alpha_m \leq b + \frac{\eta_m}{2} \leq b + \frac{\eta_{T(n)}}{2}.$$

Similarly, for all $m \geq T(n)$, we have $\inf(J_n - \delta\alpha_m) \geq a - \frac{\eta_{T(n)}}{2}$. This implies $\bigcup_{m=T(n)}^{\infty} J_n - \delta\alpha_m \subseteq \left(a - \frac{\eta_{T(n)}}{2}, b + \frac{\eta_{T(n)}}{2}\right)$, and so

$$\mathcal{L}^1\left(\bigcup_{m=T(n)}^{\infty} J_n - \delta\alpha_m\right) \leq \eta_{T(n)} + b - a = \eta_{T(n)} + \lambda_n.$$

Thus

$$\begin{aligned}
\mathcal{L}^1\left(\bigcup_{m=1}^{\infty} J_n - \delta\alpha_m\right) &\leq \mathcal{L}^1\left(\bigcup_{m=1}^{T(n)-1} J_n - \delta\alpha_m\right) + \mathcal{L}^1\left(\bigcup_{m=T(n)}^{\infty} J_n - \delta\alpha_m\right)\\
&\leq (T(n)-1)\lambda_n + \eta_{T(n)} + \lambda_n\\
\text{(by (5.7))} &= \mathcal{L}^1\left(\bigcup_{m=1}^{\infty} J_n - \eta_m\right).
\end{aligned}$$

This finishes the proof of Part (2) of the proposition.

It remains to prove Part (3). By (5.7) this is equivalent to

$$\sum_{n=1}^{\infty} T(n)\lambda_n + \eta_{T(n)} < \infty. \tag{5.8}$$

To this end, we need to specify our choice of λ_n.

Define $K(n) := 2\min\{m : \eta_m < n^{-2}\}$. $K(n)$ is well defined since $\eta_m \searrow 0$, and in particular, we have

$$K(n) \text{ is even} \qquad \text{and} \qquad \eta_{\frac{K_n}{2}} < n^{-2}. \tag{5.9}$$

Recall that V_n's are open intervals that form a topological base for $(0,1)$ and that J_n are chosen to be subintervals of V_n for each n.

Then we define:

$$\lambda_n = \min\left\{|V_n|, 2^{-n}, \eta_{K(n)} - \eta_{K(n)+1}\right\} > 0. \tag{5.10}$$

Note that $\lambda_n \leq \eta_{K(n)} - \eta_{K(n)+1}$, so $T(n) > K(n)$ by definition of $T(n)$ in (5.6). By monotonicity of $\{\eta_m\}$ and (5.9), we have

$$\sum_{n=1}^{\infty} \eta_{\left\lfloor \frac{T(n)}{2} \right\rfloor} \leq \sum_{n=1}^{\infty} \eta_{\left\lfloor \frac{K(n)}{2} \right\rfloor} = \sum_{n=1}^{\infty} \eta_{\frac{K(n)}{2}} < \sum_{n=1}^{\infty} n^{-2} < \infty. \tag{5.11}$$

Also note that since η_m is decreasing, $\eta_{T(n)} \leq \eta_{\lfloor T(n)/2 \rfloor}$ is also summable by (5.11).

The definition of $T(n)$ (5.6) implies that for all $m < T(n)$ we have

$\eta_m - \eta_{m+1} \geq \lambda_n$. Hence we can bound $T(n)\lambda_n$ from above by:

$$\begin{aligned} T(n)\lambda_n = 2\frac{T(n)}{2}\lambda_n &\leq 2\left(T(n) - \left\lfloor \frac{T(n)}{2} \right\rfloor\right)\lambda_n \\ &\leq 2\left(\eta_{\left\lfloor \frac{T(n)}{2} \right\rfloor} - \eta_{\left\lfloor \frac{T(n)}{2} \right\rfloor+1} + \cdots + \eta_{T(n)-1} - \eta_{T(n)}\right) \\ &= 2\eta_{\left\lfloor \frac{T(n)}{2} \right\rfloor} - 2\eta_{T(n)}, \end{aligned}$$

which is summable by (5.11) and the note following it. This proves (5.8), and thus Part (3) of Proposition 5.2.1.

5.2.4 Proof of Lemma 5.1.3

Let $\beta_m \searrow 0$ be given. Let $\eta_1 = \beta_1$ and $\eta_2 = \beta_2$. For $m \geq 3$, we define

$$\eta_m = \max\{\beta_m, 2\eta_{m-1} - \eta_{m-2}\}.$$

By this definition, we have $\eta_m \geq \beta_m$ for all $m \geq 1$ as well as $\eta_{m-1} - \eta_m \leq \eta_{m-2} - \eta_{m-1}$ for all $m \geq 3$. It remains to show that η_m strictly decreases to 0.

We first show by induction that η_m is strictly decreasing. First, $\eta_2 = \beta_2 < \beta_1 = \eta_1$. Assuming $\eta_{m-1} < \eta_{m-2}$ for all $m \geq m_0$ where $m_0 \geq 3$, we will show that $\eta_m < \eta_{m-1}$. We have 2 cases:

- If $\beta_m = \max\{\beta_m, 2\eta_{m-1} - \eta_{m-2}\}$, then $\eta_m = \beta_m < \beta_{m-1} \leq \eta_{m-1}$ as β_m is assumed to be strictly decreasing.
- If $2\eta_{m-1} - \eta_{m-2} = \max\{\beta_m, 2\eta_{m-1} - \eta_{m-2}\}$, then $\eta_m = 2\eta_{m-1} - \eta_{m-2} < \eta_{m-1}$, since the last inequality equivalent to $\eta_{m-1} < \eta_{m-2}$ which is our induction assumption.

Next we show that η_m converges to 0. We have two cases:

- If there is $N \geq 3$ such that for all $m \geq N$, $\beta_m \leq 2\eta_{m-1} - \eta_{m-2}$, then $\eta_m = 2\eta_{m-1} - \eta_{m-2}$ for all $m \geq N$. Thus $\{\eta_m : m \geq N-2\}$ is an infinite arithmetic progression of common difference $\eta_{N-1} - \eta_{N-2} < 0$

marching to the left. Hence if $m \geq N - 2 + \frac{\eta_{N-2}}{\eta_{N-2} - \eta_{N-1}}$, then $\eta_m \leq 0$, which is a contradiction since by definition, $\eta_m \geq \beta_m > 0$ for all m.

- Otherwise, $\beta_m > 2\eta_{m-1} - \eta_{m-2}$ infinitely often, so there is a subsequence $\eta_{m_k} = \beta_{m_k}$ for all k. Since $\beta_m \to 0$, we have $\eta_{m_k} \to 0$. But $\{\eta_m\}$ is a strictly decreasing sequence, so $\{\eta_m\}$ itself also converges to 0.

Chapter 6

Introduction to Decoupling

In this chapter, we come to the second main topic of this thesis, namely, decoupling theory.

Throughout the thesis we will write

$$e(z) := \exp(2\pi i z).$$

The Fourier transform of a Schwartz function $f : \mathbb{R}^n \to \mathbb{C}$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e(-x \cdot \xi) dx.$$

Here we recall that a Schwartz function $f : \mathbb{R}^n \to \mathbb{C}$ is a function satisfying the following assumption:

$$\sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta f(x)| < \infty, \quad \text{for any multi-indices } \alpha, \beta.$$

We have the Fourier inversion formula

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e(x \cdot \xi) d\xi.$$

We remark that there are various formulations of decoupling. For example, in [5] decoupling is formulated using neighbourhoods of a hypersurface, and in [8] decoupling is formulated using the extension operator, but in Section 5.1 there they also study the relation between the neighbourhood version and the extension operator version. In this thesis we use the neighbourhood version of decoupling, which we formulate in a moment.

Notation. Throughout this rest of the thesis, we use the standard notation

$$A \lesssim_{p_1,p_2,\ldots,p_k} B$$

to mean that for some constant C' depending on the parameters $p_1, \ldots, p_k$ only, we have $A \leq C'B$. We also just write $A \lesssim B$ if the constant C' is absolute, or the dependence on the parameters is of no importance.

6.1 General decoupling

We shall first formulate decoupling in a very general way.

6.1.1 Formulation of decoupling

We make the following definition of decoupling, which is more general than the version in the introduction since we do not require the disjointness of $\mathcal{A}$ at this moment.

Definition 6.1.1 (General decoupling)**.** *Let $1 \leq p, q \leq \infty$. Let $\mathcal{A} = \{A_i\}$ be a finite collection of bounded open sets of $\mathbb{R}^n$. Define $D_{p,q}(\mathcal{A})$ to be the smallest constant such that for all functions $f_i \in L^p(\mathbb{R}^n)$ each of which has its Fourier transform supported on A_i, we have*

$$\left\| \sum_i f_i \right\|_{L^p(\mathbb{R}^n)} \leq D_{p,q}(\mathcal{A}) \left\| \|f_i\|_{L^p(\mathbb{R}^n)} \right\|_{l^q(i)}. \tag{6.1}$$

We refer to (6.1) as an $l^q(L^p)$-decoupling.

Remark. In most cases we are interested in, the sets A_i will be taken to be an almost disjoint (i.e. their intersection has zero n-dimensional Lebesugue measure) partition of a neighbourhood of a compact subset of some manifold in $\mathbb{R}^n$, as in the case of Theorem 1.2.2.

6.1.2 General estimates

In this subsection we prove some general bounds of global decoupling. We say they are general because they hold regardless of the choice of the collection $\mathcal{A}$.

Proposition 6.1.2 (General decoupling)**.** *With the above setting, for all* $1 \le p, q \le \infty$ *and any* $\mathcal{A}$ *we have the trivial bound*[3]

$$1 \le D_{p,q}(\mathcal{A}) \le (\#\mathcal{A})^{1-1/q}. \tag{6.2}$$

Taking $q = 1$, we have the following corollary.

Corollary 6.1.3 (Sharp decoupling at $q = 1$)**.** *For any* $1 \le p \le \infty$ *we have* $D_{p,1}(\mathcal{A}) = 1$*.*

Proof of Proposition 6.1.2. The inequality $1 \le D_{p,q}$ follows by taking f to be a nonzero function Fourier supported on a single A_i. The upper bound $D_{p,q}(\mathcal{A}) \le (\#\mathcal{A})^{1/q}$ follows from the triangle and Hölder's inequalities. Indeed, let f_i be Fourier supported on A_i. Then

$$\left\|\sum_i f_i\right\|_{L^p(\mathbb{R}^n)} \le \sum_i \|f_i\|_{L^p(\mathbb{R}^n)} \le (\#\mathcal{A})^{1-\frac{1}{q}} \left\| \|f_i\|_{L^p(\mathbb{R}^n)} \right\|_{l^q(i)}.$$

Since this works for an arbitrary functions $f_i \in L^p(\mathbb{R}^n)$ Fourier supported on A_i, our claim follows. $\square$

Proposition 6.1.4 (Lower bound for $p \le q$)**.** *If* $1 \le p \le q \le \infty$*, then for any* $\mathcal{A}$ *we have* $D_{p,q}(\mathcal{A}) \ge (\#\mathcal{A})^{1/p-1/q}$*.*

For $p = 1$, combined with the trivial upper bound in Proposition 6.1.2, we have the following corollary.

Corollary 6.1.5 (Sharp decoupling at $p = 1$)**.** *For any* $1 \le q \le \infty$ *we have* $D_{1,q}(\mathcal{A}) = (\#\mathcal{A})^{1-1/q}$*.*

Proof of Proposition 6.1.4. Fix Schwartz functions g_i each with Fourier support in A_i with $\|g_i\|_{L^p(\mathbb{R}^n)} = 1$, and let c_i be arbitrary constants. Fix a unit vector v. For each k, define a $(\#\mathcal{A})$-tuple of functions $f_{k,i}$

$$f_{k,i}(x) = c_i g_{k,i}(x) := c_i g_i(x + kiv).$$

[3] Throughout the thesis we make the usual convention $1/p = 0$ when $p = \infty$.

By definition of $D_{p,q}(\mathcal{A})$, we have

$$\left\|\sum_i f_{k,i}\right\|_{L^p(\mathbb{R}^n)} \leq D_{p,q}(\mathcal{A})\left\|c_i\|g_{k,i}\|_{L^p(\mathbb{R}^n)}\right\|_{l^q(i)}.$$

On the other hand, we have

$$\|g_{k,i}\|_{L^p(\mathbb{R}^n)} = \|g_i\|_{L^p(\mathbb{R}^n)} = 1$$

by translation invariance. For the left hand side, note that

$$\lim_{k\to\infty}\left\|\sum_i f_{k,i}\right\|_{L^p(\mathbb{R}^n)} = \left\|c_i\|g_i\|_{L^p(\mathbb{R}^n)}\right\|_{l^p(i)} = \|c_i\|_{l^p(i)},$$

where the first equality follows from a general measure-theoretic property for L^p-functions[4] since the "essential supports" of $g_{k,i}$ become "essentially disjoint" as $n \to \infty$.

Thus, we have

$$\|c_i\|_{l^p(i)} \leq D_{p,q}(\mathcal{A})\|c_i\|_{l^q(i)}.$$

Since c_i's are arbitrary constants, this forces to $D_{p,q}(\mathcal{A}) \geq (\#\mathcal{A})^{1/p-1/q}$.

□

Using Hölder's inequality for l^q norms, we trivially arrive at

Proposition 6.1.6 (Hölder's inequality)**.** *For all* $1 \leq p \leq \infty$ *and* $1 \leq q_1 \leq q_2 \leq \infty$ *we have*

$$D_{p,q_2}(\mathcal{A}) \leq (\#\mathcal{A})^{\frac{1}{q_1}-\frac{1}{q_2}} D_{p,q_1}(\mathcal{A}).$$

6.1.3 Disjointness

In most cases, decoupling estimates are formulated when $\mathcal{A}$ consists of disjoint[5] subsets A_i.

[4]when $p = \infty$ the decay of g_i is needed, which is true since g_i is Schwartz here.

[5]In the decoupling part of this thesis we say $A, B \subseteq \mathbb{R}^n$ are essentially disjoint (or just disjoint) if $\mathcal{L}^n(A \cap B) = 0$.

Remark. Starting from this point, we will always assume that $\mathcal{A}$ consists of disjoing subsets, unless otherwise specified (for example, in Section 6.1.6).

Proposition 6.1.7 (General decoupling for $q \le p = 2$)**.** *For every* $1 \le q \le 2$ *we have* $D_{2,q}(\mathcal{A}) = 1$.

Proof. Let f_i be Fourier supported on A_i. By Plancherel's identity and the disjointness of A_i, we have

$$\left\| \sum_i f_i \right\|_{L^2(\mathbb{R}^n)} = \left(\sum_i \|f_i\|^2_{L^2(\mathbb{R}^n)} \right)^{\frac{1}{2}}.$$

Thus, if $q \le 2$, using the trivial inequality $\|\cdot\|_{l^2} \le \|\cdot\|_{l^q}$, the result follows. □

Corollary 6.1.8 (General decoupling for $2 = p \le q$)**.** *For any* $2 \le q \le \infty$ *we have* $D_{2,q}(\mathcal{A}) = (\#\mathcal{A})^{1/2-1/q}$. *In particular,* $D_{2,\infty}(\mathcal{A}) = (\#\mathcal{A})^{\frac{1}{2}}$.

Proof of corollary. Applying Proposition 6.1.6 with $p = 2$, $q_1 = 2$ and $q_2 = q$, we have $D_{2,q}(\mathcal{A}) \le (\#\mathcal{A})^{1/2-1/q} D_{2,2}(\mathcal{A})$. But since $D_{2,2} = 1$ by Proposition 6.1.7, we have $D_{2,q}(\mathcal{A}) \le (\#\mathcal{A})^{1/2-1/q}$. The lower bound then follows from Proposition 6.1.4. □

So far we have seen that for a lot of pairs of Lebesgue exponents (p, q) we have obtained sharp decoupling estimates without any geometric information of $\mathcal{A}$ but the disjointness.

In the following diagram 6.1, the red and blue lines indicate the pairs (p, q) for which sharp general decoupling has been established. In particular, the blue lines indicate (p, q) for which $D_{p,q}(\mathcal{A}) = 1$.

6.1.4 Interpolation of general decoupling

We can improve the decoupling estimates shown in Figure 6.1 using the following linear interpolation theorem for mixed norms. The proof can be found in a nice survey [37] on the interpolation of mixed normed spaces.

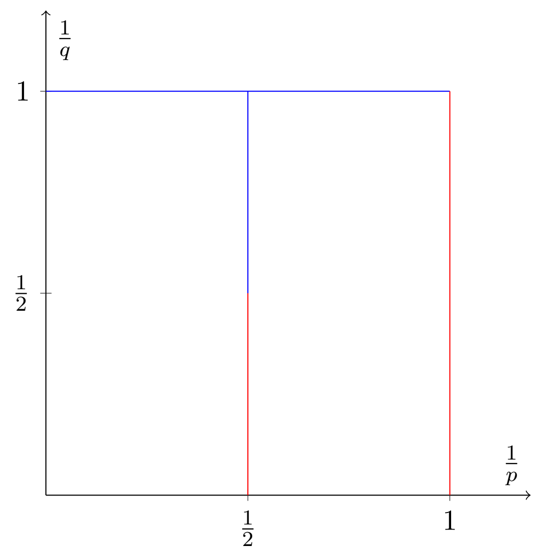

Figure 6.1: Known pairs of sharp general decoupling

Theorem 6.1.9. *Let X, Y, M, N be σ-finite measure spaces. For $k = 0, 1$, let $1 \leq a_k, b_k, c_k, d_k \leq \infty$. Let $0 < \theta < 1$ and*

$$\frac{1-\theta}{a_0} + \frac{\theta}{a_1} = \frac{1}{a}, \quad \frac{1-\theta}{b_0} + \frac{\theta}{b_1} = \frac{1}{b}, \quad \frac{1-\theta}{c_0} + \frac{\theta}{c_1} = \frac{1}{c}, \quad \frac{1-\theta}{d_0} + \frac{\theta}{d_1} = \frac{1}{d}.$$

Assume in addition $a, c < \infty$.

Assume T is a linear operator from $L^{a_k}(X, L^{c_k}(M))$ to $L^{b_k}(Y, L^{d_k}(N))$ with operator norms B_k, $k = 0, 1$. Then T also maps $L^a(X, L^c(M))$ to $L^b(Y, L^d(N))$ with operator norm bounded above by $B_0^{1-\theta} B_1^{\theta}$.

The decoupling inequality is not yet in the form of an operator bound, because of the Fourier support condition. To deal with this, inspired by the simple treatment in Section 3.2.3 of [77], we generalise a little and impose a minor technical assumption which holds under most circumstances we are interested in.

Definition 6.1.10 (Smooth cutoff condition)**.** *Let $\mathcal{A} = \{A_i\}$ be given as in Definition 6.1 and assume $\mathcal{A}$ is disjoint. We say $\mathcal{A}$ satisfies a smooth cutoff condition, if there is an absolute constant C such that for each i there exists* *a Schwartz function* ψ_i *obeying the following:*

1. $\|\psi_i\|_{L^1(\mathbb{R}^n)} \leq C$.

2. $\widehat{\psi_i} = 1$ *on* A_i.

3. $\widehat{\psi_i}$ *is supported on some* A_i' *with the following condition: for any* $1 \leq p, q \leq \infty$ *and any functions* g_i *Fourier supported on* A_i'*, we have*

$$\left\| \sum_i g_i \right\|_{L^p(\mathbb{R}^n)} \lesssim_{C,p,q,n} D_{p,q}(\mathcal{A}) \left\| \|g_i\|_{L^p(\mathbb{R}^n)} \right\|_{l^q(i)}. \tag{6.3}$$

Now we apply the interpolation theorem to get the following proposition.

Proposition 6.1.11. *Assume* $\mathcal{A}$ *satisfies the smooth cutoff condition as in Definition 6.1.10. Let* $\theta \in (0,1)$ *and assume* $1 \leq p_k, q_k, p, q \leq \infty$*,* $k = 0,1$ *satisfy*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Assume in addition that $p, q < \infty$*. Then*

$$D_{p,q}(\mathcal{A}) \lesssim_{C,p,q} D_{p_0,q_0}(\mathcal{A})^{1-\theta} D_{p_1,q_1}(\mathcal{A})^{\theta}.$$

Proof. Define the vector-input linear operator T by

$$T(f_i)_{i=1}^{\#\mathcal{A}} = \sum_{i=1}^{\#\mathcal{A}} f_i * \psi_i,$$

where f_i are arbitrary functions (with no Fourier support assumption) in $L^p(\mathbb{R}^n)$. Take

$$X = \{1, \ldots, \#\mathcal{A}\}, \quad M = N = \mathbb{R}^n, \quad Y = \{1\},$$

and the following choices of Lebesgue exponents:

$$a_k = q_k, c_k = p_k, d_k = p_k, \quad k = 0, 1, \quad \text{and} \quad a = q, c = p, d = p.$$

We will show that T maps $L^{a_k}(X, L^{c_k}(M))$ to $L^{b_k}(Y, L^{d_k}(N))$ with operator norms $B_k \lesssim D_{p_k,q_k}(\mathcal{A})$.

Given arbitrary functions $f_i \in L^p(\mathbb{R}^n)$. Since $f_i * \psi_i$ is Fourier supported on A_i', by (6.3), we have

$$\|T(f_i)_i\|_{L^{p_k}(\mathbb{R}^n)} \lesssim D_{p_k,q_k}(\mathcal{A}) \Big\| \|f_i * \psi_i\|_{L^{p_k}(\mathbb{R}^n)} \Big\|_{l^{q_k}(i)}.$$

On the other hand, for each i, using Young's convolution inequality, we have for any $1 \leq p \leq \infty$

$$\|f_i * \psi_i\|_{L^p(\mathbb{R}^n)} \leq C \|f_i\|_{L^p(\mathbb{R}^n)}.$$

Hence, we arrive at

$$\|T(f_i)_i\|_{L^{p_k}(\mathbb{R}^n)} \lesssim D_{p_k,q_k}(\mathcal{A}) \Big\| \|f_i\|_{L^{p_k}(\mathbb{R}^n)} \Big\|_{l^{q_k}(i)},$$

which is valid for all $(\#\mathcal{A})$-tuples of functions g_i. Thus, we may apply the interpolation theorem to get

$$\|T(f_i)_i\|_{L^p(\mathbb{R}^n)} \lesssim D_{p_0,q_0}(\mathcal{A})^{1-\theta} D_{p_1,q_1}(\mathcal{A})^{\theta} \Big\| \|f_i\|_{L^p(\mathbb{R}^n)} \Big\|_{l^q(i)},$$

for all $(\#\mathcal{A})$-tuples of functions f_i. In particular, this works for all functions f_i with Fourier support on A_i. Since $\widehat{\psi_i} = 1$ on A_i, we have $T(f_i)_i = \sum_i f_i$. Hence

$$\|f\|_{L^p(\mathbb{R}^n)} \lesssim D_{p_0,q_0}(\mathcal{A})^{1-\theta} D_{p_1,q_1}(\mathcal{A})^{\theta} \Big\| \|f|_{A_i}\|_{L^p(\mathbb{R}^n)} \Big\|_{l^q(i)},$$

from which the result follows. □

Using interpolation and known upper and lower bounds of general decoupling, we have

Proposition 6.1.12. *Assume $\mathcal{A}_i$ satisfies the smooth cutoff condition in Definition 6.1.10. Then for all exponents (p, q) with $(1/p, 1/q)$ lying in the* *triangle* $\mathbf{B}$ *with vertices* $(1/2, 1/2), (0, 1), (1, 1)$*, we have*

$$D_{p,q}(\mathcal{A}) \sim 1.$$

Also, for all exponents (p, q) *with* $(1/p, 1/q)$ *lying in the trapezoid* $\mathbf{R}$ *with vertices* $(1/2, 1/2), (1, 1), (1, 0), (1/2, 0)$*, we have*

$$D_{p,q}(\mathcal{A}) \sim (\#\mathcal{A})^{\frac{1}{p}-\frac{1}{q}}.$$

The implicit constants here depends on p, q *and the constant* C *in the smooth cutoff condition.*

Proof. Refer to Figure 6.1. The first part $D_{p,q}(\mathcal{A}) \sim 1$ follows directly from interpolation between the vertices $(1/2, 1/2), (0, 1), (1, 1)$. For the second part, given $(1/p, 1/q) \in \mathbf{R}$, we use Hölder's inequality 6.1.6 and $D_{p,p}(\mathcal{A}) \sim 1$ to get the required upper bound. The lower bound follows from Proposition 6.1.4. □

Assuming the smooth cutoff condition, the blue triangle $\mathbf{B}$ and the red trapezoid $\mathbf{R}$ in Figure 6.2 are the range of exponents (p, q) where we have sharp general decoupling. In particular, in $\mathbf{B}$ we even have $D_{p,q}(\mathcal{A}) \sim 1$. Therefore, it then suffices to consider the exponents (p, q) for which $(1/p, 1/q)$ lies in the trapezoid $\mathbf{W}$ defined by the vertices $(0, 0)$, $(1/2, 0)$, $(1/2, 1/2)$, $(0, 1)$, i.e. the white region in Figure 6.2.

Proposition 6.1.13. *If* $(1/p, 1/q)$ *lies in* $\mathbf{W}$*, then we have*

$$D_{p,q}(\mathcal{A}) \lesssim (\#\mathcal{A})^{1-\frac{1}{p}-\frac{1}{q}}. \tag{6.4}$$

In particular, for $2 \leq p \leq \infty$ *we have the* $l^2(L^p)$ *and* $l^p(L^p)$ *estimates*

$$D_{p,2}(\mathcal{A}) \lesssim (\#\mathcal{A})^{\frac{1}{2}-\frac{1}{p}}, \quad D_{p,p}(\mathcal{A}) \lesssim (\#\mathcal{A})^{1-\frac{2}{p}}. \tag{6.5}$$

Proof. Let $(1/p, 1/q)$ be given. Then we see $(1/p, 1-1/p)$ lies on the straight line $1/p+1/q = 1$ which is in $\mathbf{B}$. Hence $D_{p,(1-1/p)^{-1}} \sim 1$. Applying Hölder's

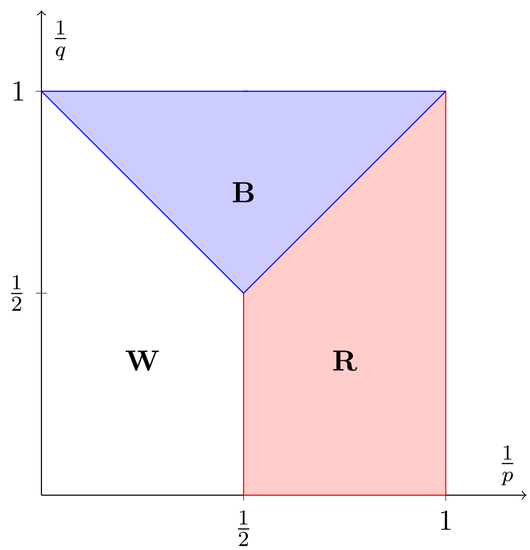

Figure 6.2: Known pairs of sharp general decoupling, with interpolation

inequality 6.1.6 we then have

$$D_{p,q}(\mathcal{A}) \lesssim (\#\mathcal{A})^{1-\frac{1}{p}-\frac{1}{q}}.$$

$\square$

Therefore, inspired by the result given by interpolation, below we are mostly interested in $p \geq 2$. In fact, the most studied pairs of exponents are $q = 2 \leq p$ and $2 \leq p = q$. The former is usually called an l^2-decoupling and the latter is usually called an l^p-decoupling.

6.1.5 Flat decoupling

So far we have studied decoupling in a very general setting. Now we shall focus on a special case in which $\mathcal{A}$ is the partition of a rectangle into identical pieces, as is the case of Proposition 9.5 of [12].

Theorem 6.1.14 (Flat decoupling)**.** *Let A be a rectangle in $\mathbb{R}^n$ and $\mathcal{A} = \{A_i\}$ be the decomposition of A into N congruent and parallel rectangles* *using $N-1$ parallel hyperplanes. Then for all $1 \leq p, q \leq \infty$ we have*

$$D_{p,q}(\mathcal{A}) \sim \begin{cases} 1, & \text{if } (1/p, 1/q) \in \mathbf{B} \\ (\#\mathcal{A})^{\frac{1}{p}-\frac{1}{q}}, & \text{if } (1/p, 1/q) \in \mathbf{R} \\ (\#\mathcal{A})^{1-\frac{1}{p}-\frac{1}{q}}, & \text{if } (1/p, 1/q) \in \mathbf{W} \end{cases}. \tag{6.6}$$

Proof of Theorem 6.1.14. The proof is divided into a few parts.

1. We first deal with simple invariance observations. By the translation-modulation and rotation invariance of the Fourier transform, it suffices to consider A to be axis-parallel and centred at 0. We now use a simple scaling argument to show that it is also dilation invariant in each coordinate, namely, for any non-isotropic dilation mapping

$$\Delta(x_1, \ldots, x_n) := (\lambda_1 x_1, \ldots, \lambda_n x_n), \quad \lambda_i > 0 \quad \forall i,$$

we have $D_{p,q}(\mathcal{A}) = D_{p,q}(\Delta\mathcal{A})$, where $\Delta\mathcal{A} := \{\Delta A_i : A_i \in \mathcal{A}\}$.

Suppose the conclusion holds for $\mathcal{B}$ which comprises of a partition of $[0,1]^n$ into N congruent and parallel rectangles B_i. Now given a general axis-parallel A, a partition $\mathcal{A}$ and f_i Fourier supported on A_i as in the assumption. Denote by l_i the length of A in the direction e_i. Then we define g_i by

$$g_i(x_1, \ldots, x_n) = l_1^{-1} \cdots l_n^{-1} f_i(l_1^{-1} x_1, \ldots, l_n^{-1} x_n), \tag{6.7}$$

so that

$$\widehat{g_i}(\xi_1, \ldots, \xi_n) = \widehat{f_i}(l_i \xi_1, \ldots, l_n \xi_n)$$

is supported on $[0,1]^n$. Applying the result for $\mathcal{B}$, we have

$$\left\| \sum_i g_i \right\|_{L^p(\mathbb{R}^n)} \leq D_{p,q}(\mathcal{B}) \left\| \|g_i\|_{L^p(\mathbb{R}^n)} \right\|_{l^q(i)}.$$

Now using (6.7), rescale back to f and cancelling the factors to get

$$\left\|\sum_i f_i\right\|_{L^p(\mathbb{R}^n)} \leq D_{p,q}(\mathcal{B})\left\|\|f_i\|_{L^p(\mathbb{R}^n)}\right\|_{l^q(i)}.$$

Thus $D_{p,q}(\mathcal{A}) \leq D_{p,q}(\mathcal{B})$. By symmetry we thus have $D_{p,q}(\mathcal{A}) = D_{p,q}(\mathcal{B})$.

With this, we then have $D_{p,q}(\mathcal{A}) = D_{p,q}(N)$ where the latter is defined to be the smallest constant such that for all f_i Fourier supported in A_i we have

$$\left\|\sum_i f_i\right\|_{L^p(\mathbb{R}^n)} \leq D_{p,q}(N)\left\|\|f_i\|_{L^p(\mathbb{R}^n)}\right\|_{l^q(i)}, \tag{6.8}$$

where we may take $A_i = [0,1]^{n-1} \times [(i-1)N^{-1}, iN^{-1}]$.

2. We now prove the upper bound for $D_{p,q}(N)$. We then check the smooth cutoff condition 6.1.10 so that we can use Proposition 6.1.12 to obtain the result in **B** and **R**.

   To this end, we simply choose A_i' to be the twice of A_i, that is, the uniform dilation of A_i by a factor of 2 with respect to the centre of A_i. Then it is easy to pick ψ_i with the first two assumptions satisfied. For (6.3), we let g_i be supported on A_i'. By our choice of A_i' (the twice of A_i), they have bounded overlap in the sense that $\sum_i 1_{A_i} \leq 2^{n-1}$. Hence, we may split the collection $\{A_i'\}$ into 2^{n-1} subcollections $\mathcal{A}_j$, each of which has cardinality less than N and is exactly an almost disjoint partition of another rectangle. See Figure 6.3, where each A_i is represented by a black rectangle and $\{A_i'\}$ is partitioned into the two families of blue and green rectangles. (The green rectangles in the figure are slightly enlarged visually to showcase the separation from the blue rectangles.)

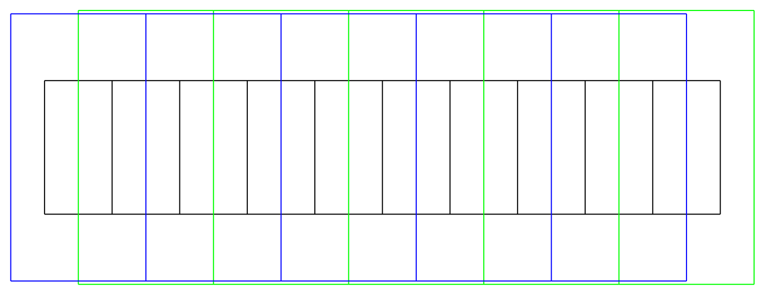

Figure 6.3: A_i and A_i' in $n = 2$

Thus, by the triangle and Hölder's inequalities we have

$$\begin{aligned}
\left\| \sum_i f_i \right\|_{L^p(\mathbb{R}^n)} &\leq \sum_j \left\| \sum_{i:A_i' \in \mathcal{A}_j} f_i \right\|_{L^p(\mathbb{R}^n)} \\
&\leq D_{p,q}(N) \sum_j \left\| \|f_i\|_{L^p(\mathbb{R}^n)} \right\|_{l^q(i:A_i' \in \mathcal{A}_j)} \\
&\lesssim_q D_{p,q}(N) \left\| \|f_i\|_{L^p(\mathbb{R}^n)} \right\|_{l^q(i)}.
\end{aligned}$$

Hence the smooth cutoff condition holds and we have the required upper bounds in **B** and **R**.

The required upper bound in **W** follows from Proposition 6.1.13.

3. Lastly we come to the lower bounds. We will show that

$$D_{p,q}(N) \gtrsim N^{1-1/p-1/q} \tag{6.9}$$

works for all $1 \leq p, q \leq \infty$, which gives the sharp lower bound for (p, q) in the range **W**. The lower bounds for **B** and **R** have been established in Proposition 6.1.12 (and are better than (6.9).

To prove (6.9), we first fix a nonzero and nonnegative smooth function η supported on $[0, 1]^n$. We then take f_i so that

$$\hat{f}_i(\xi_1, \ldots, \xi_n) = \eta\left(\xi_1, \ldots, \xi_{n-1}, N\xi_n - i + 1\right),$$

which is supported on A_i. With this choice of f_i, we substitute it into (6.8). The right hand side of (6.8) is essentially

$$N^{-1+\frac{1}{p}+\frac{1}{q}}.$$

To compute the left hand side of (6.8), we write $x = (x_1, \dots, x_n)$ and $\xi = (\xi_1, \dots, \xi_n)$. Since η is nonnegative, for any $1 \le i \le N$ we have

$$\begin{aligned}
&\mathrm{Re}(f_i)(x) \\
&= \int_{\mathbb{R}^n} \eta(\xi_1, \dots, \xi_{n-1}, N\xi_n - i + 1)\cos(2\pi x \cdot \xi) d\xi \\
&= \frac{\int_{[0,1]^n} \eta(\xi)\cos(2\pi(x_1\xi_1 + \dots + x_{n-1}\xi_{n-1} + x_n N^{-1}(\xi_n + i - 1)))d\xi}{N}.
\end{aligned}$$

Hence, if $x \in [0, (10N)^{-1}]^n$, then for $1 \le i \le N$ and $\xi \in [0,1]^n$ we have

$$|x_1\xi_1 + \dots + x_{n-1}\xi_{n-1} + x_n N^{-1}(\xi_n + i - 1)| \le \frac{1}{10},$$

from which it follows that

$$\mathrm{Re}(f_i)(x) \ge \frac{1}{2N}\|\eta\|_{L^1(\mathbb{R}^n)}$$

for all i, using the simple fact that $\cos(2\pi t) > 1/2$ for $|t| < 1/10$. Hence

$$\left\|\sum_i f_i\right\|_{L^p(\mathbb{R}^n)} \ge \left\|\sum_i f_i\right\|_{L^p([0,(10N)^{-1}]^n)} \gtrsim 1.$$

Combining the estimates on both sides gives (6.9).

$\square$

6.1.6 Bounded overlap

In practice it is common to deal with decoupling for a slightly non-disjoint family $\mathcal{A} = \{A_i\}$ with bounded overlap. This means that instead of requiring $\{A_i\}$ to be almost disjoint, we impose the slightly weaker assumption that $\sum_i 1_{A_i} = O(1)$, where the implicit constant depends on unimportant parameters only (such as the dimension). In this case, we still have $D_{2,2}(\mathcal{A}) = O(1)$. Indeed, this is true by a simple tiling argument similar to the proof of Part 2 of Theorem 6.1.14. More precisely, we split $\mathcal{A}$ into $O(1)$ many subcollections, so that each of them is disjoint, and then apply the triangle and Hölder's inequalities.

Hence, from now on the decoupling formulation also works well with a bounded overlapping collection $\mathcal{A}$.

6.2 Decoupling for manifolds

In this section we will focus on more specific cases of decoupling, namely, decoupling related to manifolds such as curves and surfaces in $\mathbb{R}^n$.

Before we introduce the formal definition, we give a brief description. In all decoupling inequalities in this thesis we work with a compact subset of a manifold M in $\mathbb{R}^n$. We are then given a tiny scale $\delta > 0$, and our test function f will be Fourier supported on some neighbourhood of M where the scale depends on δ. The collection $\mathcal{A} = \{A_i\}$ will be given by a suitable partition of the aforesaid neighbourhood.

6.2.1 Formulation of decoupling

Let M be a compact piece of a manifold of dimension $1 \leq m \leq n-1$. Thus, it can be divided into finitely many subsets, each of which can be parametrized as the graph of a function $\phi : [0,1]^m \to \mathbb{R}^{n-m}$. In below this will always be assumed.

Let M be an m-dimensional manifold in $\mathbb{R}^n$ and let $\phi : [0,1]^m \to \mathbb{R}^{n-m}$ be a local parametrization of M. For $\delta > 0$, let $\mathcal{P}_\delta = \{S_i\}$ be a partition of $[0,1]^m$ into open subsets.

Definition 6.2.1 (Neighbourhoods)**.** *The (vertical) δ-neighbourhood of the graph of ϕ over S_i, denoted $\mathcal{N}_\delta^\phi(S_i)$, is defined by*

$$\mathcal{N}_\delta^\phi(S_i) := \{(\xi', \xi'') : \xi' \in S_i, |\xi'' - \phi(\xi')| < \delta\}. \tag{6.10}$$

Remark. The advantage of using vertical neighbourhoods is that they are almost disjoint and will not exceed the domain of ϕ. In this thesis we try to avoid working with the natural neighbourhoods defined by $\{\xi : \mathrm{dist}(\xi, S) < \delta\}$, for technical reasons. Yet, under relatively general assumptions (such as $1 < p < \infty$ together with some minor regularity assumption on S_i), they can be shown to be equivalent. We omit the details.

Definition 6.2.2 (Decoupling for a manifold)**.** *Consider the family* $\mathcal{A}_\delta = \{\mathcal{N}_\delta^\phi(S_i) : S_i \in \mathcal{P}_\delta\}$ *where* $\mathcal{N}_\delta^\phi(S_i)$ *is defined as in* (6.10)*. For* $1 \leq p, q \leq \infty$*, define*

$$D_{p,q}^\phi(\mathcal{P}_\delta) = D_{p,q}(\mathcal{A}_\delta). \tag{6.11}$$

That is, $D_{p,q}^\phi(\mathcal{P}_\delta)$ *is the best constant such that for any* f_i *Fourier supported on* $\mathcal{N}_\delta^\phi(S_i)$*, we have*

$$\left\| \sum_i f_i \right\|_{L^p(\mathbb{R}^n)} \leq D_{p,q}^\phi(\mathcal{P}_\delta) \left\| \|f_i\|_{L^p(\mathbb{R}^n)} \right\|_{l^q(i)}. \tag{6.12}$$

Remark. For a generic decoupling inequality, we are always concerned with extremely small values of δ and thus $\#\mathcal{P}_\delta$ will be huge. We can usually tolerate a loss of a constant factor depending on ϕ, n, p, q but not on δ^α where $\alpha > 0$ is a fixed power. In many cases, however, we have to tolerate an ε-loss. This means that it is generally acceptable to prove an inequality of the form

$$D_{p,q}(\mathcal{P}_\delta) \leq C_\varepsilon \delta^{-\varepsilon},$$

for every small $\varepsilon > 0$, where C_ε depends on ε as well as other parameters mentioned above.

6.2.2 A literature review

With all the terminology introduced, let us do a brief survey on the most important results in decoupling theory. The most fundamental breakthrough is the paper of Bourgain and Demeter in 2015 [5] which we already mentioned in the introduction. In the newest terminology we introduced, the theorem says that if $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ is defined by $\phi(\xi) = |\xi|^2$, and if $\mathcal{P}_\delta$ is defined as the uniform partition of $[-1,1]^{n-1}$ into cubes of side length $\delta^{1/2}$, then for all $2 \leq p \leq \frac{2(n+1)}{n-1}$ we have $D^{\phi}_{p,2}(\mathcal{P}_\delta) \lesssim_\varepsilon \delta^{-\varepsilon}$. In the same paper, they also generalised this theorem to all C^3-functions $\phi : [-1,1]^{n-2}$ with positive-definite Hessian determinant.

In a later work of Bourgain and Demeter [7], they extended the result in [5] to surfaces with nonzero Gaussian curvature over a compact domain, but with the $l^2(L^p)$ inequality replaced by a corresponding weaker $l^p(L^p)$ inequality. This weakening is unavoidable as the graph of a non-convex function may contain a straight line, in which case we have flat decoupling (see Theorem 6.1.14 or Proposition 6.3.3 below). The case when the surface has zero Gaussian curvature somewhere is still unknown in general; see Conjecture 8.2.2 in the conclusion chapter of this thesis.

So far we have only discussed decoupling for hypersurfaces, namely, manifolds of codimension 1. On the other extreme, decoupling for curves in higher dimensions is also of interest. In 2016, Bougain, Demeter and Guth [9] proved a remarkable decoupling theorem for the moment curve $(t, t^2, \ldots, t^n) \in \mathbb{R}^n$ for $t \in [0,1]$, which has some profound application in number theory, particularly the proof of Vinogradov's mean value theorem. There are also decoupling results for manifolds of intermediate codimension, see, for instance [6, 13, 59].

Decoupling theory has also been applied to fully or partially solve many open problems in partial differential equations and Euclidean configurations. For instance, decoupling can be applied to prove the almost everywhere convergence of the solution to Schrödinger's equation (see [14, 15]) and a partial result (the best up to date of this thesis) on Falconer's distance set problem (see [33]).

6.2.3 Linear invariance

We observe that adding to ϕ an affine transformation does not change the decoupling constant.

Proposition 6.2.3. *Let $\psi(\xi') = \phi(\xi') + L(\xi')$ where $L : \mathbb{R}^m \to \mathbb{R}^{n-m}$ is an* *affine transformation. Then for any partition $\mathcal{P}_\delta$ we have*

$$D^{\phi}_{p,q}(\mathcal{P}_\delta) = D^{\psi}_{p,q}(\mathcal{P}_\delta).$$

Proof. We will only prove the $\le$ direction and the opposite inequality follows by symmetry.

Let f_i have Fourier support in $\mathcal{N}^{\phi}_{\delta}(S_i)$. We will show that

$$\left\|\sum_i f_i\right\|_{L^p(\mathbb{R}^n)} \le L^{\psi}_{p,q}(\mathcal{P}_\delta)\Big\|\|f_i\|_{L^p(\mathbb{R}^n)}\Big\|_{l^q(i)}.$$

Let g_i be defined by the relation

$$\widehat{g_i}(\xi) = \widehat{f_i}(\xi', \xi'' - L\xi'),$$

where $\xi' = (\xi_1, \ldots, \xi_m)$ and $\xi'' = (\xi_{m+1}, \ldots, \xi_n)$. Thus g_i is Fourier supported on $\mathcal{N}^{\psi}_{\delta}(S_i)$.

Then we can apply the definition of $L^{\psi}_{p,q,E}(\mathcal{P}_\delta)$ to get

$$\left\|\sum_i g_i\right\|_{L^p(\mathbb{R}^n)} \le L^{\psi}_{p,q}(\mathcal{P}_\delta)\Big\|\|g_i\|_{L^p(\mathbb{R}^n)}\Big\|_{l^q(i)}.$$

But by the definition of g_i we have

$$g_i(x) = f_i(x' + L^t x'', x'').$$

It is then easy to see that

$$\|g_i\|_{L^p(\mathbb{R}^n)} = \|f_i\|_{L^p(\mathbb{R}^n)}, \quad \left\|\sum_i g_i\right\|_{L^p(\mathbb{R}^n)} = \left\|\sum_i f_i\right\|_{L^p(\mathbb{R}^n)},$$

from which the conclusion follows. $\square$

6.2.4 Rough cutoff

We lastly mention a rough cutoff formulation of decoupling. Let $1 < p < \infty$ and let $\mathcal{P}_\delta = \{S_i\}$ consist of disjoint rectangles in $\mathbb{R}^{n-m}$. Then by the boundedness of the Hilbert transform, the Fourier multiplier

$$(\xi', \xi'') \mapsto 1_{S_i}(\xi')$$

is bounded on $L^p(\mathbb{R}^n)$, with the operator norm depending on p, n only. That is, for $1 < p < \infty$ and rectangles $S \subseteq \mathbb{R}^m$ we always have the following inequality

$$\|f_S\|_{L^p(\mathbb{R}^n)} \lesssim_{n,p} \|f\|_{L^p(\mathbb{R}^n)}, \tag{6.13}$$

for all $f \in L^p(\mathbb{R}^n)$, where f_S denotes the Fourier restriction of f on the strip $S \times \mathbb{R}^{n-m}$, and this notation will be used from now on.

With this, we see that $D^\phi_{p,q}(\mathcal{P}_\delta)$ is also comparable to the best constant M such that for all function $f \in L^p(\mathbb{R}^n)$ Fourier supported on $\cup_i \mathcal{N}^\phi_\delta(S_i)$, we have

$$\|f\|_{L^p(\mathbb{R}^2)} \leq M(\mathcal{P}_\delta) \Big\| \|f_{S_i}\|_{L^p(\mathbb{R}^n)} \Big\|_{l^q(i)}.$$

6.3 Decoupling for curves: geometric aspects

In this section we restrict our attention to curves in $\mathbb{R}^2$, and study the most important geometric properties that will play an important role in decoupling. Many of the following results admit a natural generalisation to higher dimensions, but we do not pursue that here.

6.3.1 Flatness

We have seen in Theorem 6.1.14 that decoupling is sharp and well-understood in the "flat case", i.e when ϕ is given by a constant function. By Proposition 6.2.3 the same is true if ϕ is a linear function. More generally, this continues to hold if ϕ is "essentially linear" over a considerable portion of $[0, 1]$. To make this precise, we introduce the following terminology.

Definition 6.3.1 (δ-flatness)**.** *Let $\phi : [0,1] \to \mathbb{R}$ be C^2. We say ϕ is δ-flat over an interval $I \subseteq [0,1]$, if for any $u, v \in I$ we have*

$$|\phi(v) - \phi(u) - \phi'(u)(v-u)| \leq 2\delta. \tag{6.14}$$

Note that if ϕ is a linear function, then ϕ is δ-flat over the entire $[0,1]$. Also, by Taylor expansion, (6.14) is true if we have

$$\sup_{u \in I} |\phi''(u)| \mathcal{L}^1(I)^2 \leq 4\delta. \tag{6.15}$$

Proposition 6.3.2 (Almost rectangles)**.** *Let $\phi : [0,1] \to \mathbb{R}$ be C^2 with $|\phi'| + |\phi''| = O(1)$, and assume ϕ is δ-flat over an interval $I \subseteq [0,1]$ which has length $l \geq \delta$. Then $\mathcal{N}_\delta^\phi(I)$ is essentially a rectangle of dimensions $l \times \delta$, that is, there is an actual rectangle $T \subseteq \mathbb{R}^2$ of dimensions $l \times \delta$ and absolute constants C_1, C_2, C_3 such that*

$$C_1 T \subseteq \mathcal{N}_{C_2\delta}^\phi(I) \subseteq C_3 T,$$

where CT is the dilation of T by a factor of T with respect to its centre.

Proof. Without loss of generality, take $I = [0, l]$ and let $s = l/2 \in I$. Let T be the rectangle centred at $(s, \phi(s))$ and with sides parallel to $(1, \phi'(s))$ and $(-\phi'(s), 1)$ with lengths l and δ, respectively. We claim that T is as required.

We first prove $C_1 T \subseteq \mathcal{N}_{C_2\delta}^\phi(I)$. Let $(w, t) \in T$. Then

$$|(w - s, t - \phi(s)) \cdot (1, \phi'(s))| \lesssim C_1 l, \tag{6.16}$$

$$|(w - s, t - \phi(s)) \cdot (-\phi'(s), 1)| \lesssim C_1 \delta, \tag{6.17}$$

where the implicit constants are absolute. To show $(w, t) \in \mathcal{N}_{C_2\delta}^\phi(I)$, we will show

$$|w - s| \leq l/2, \quad |t - \phi(w)| \leq C_2 \delta.$$

The first ensures that $w \in I$ and the second ensures that $(w, t) \in \mathcal{N}_{C_2\delta}^\phi(I)$.

For the first inequality, since $(1, \phi'(s))$ and $(-\phi'(s), 1)$ form an essentially orthonormal basis for $\mathbb{R}^2$ as $|\phi'| = O(1)$, by $\delta \leq l$, (6.16) and (6.17) we have

$$|(w-s, t-\phi(s))| \lesssim C_1 l.$$

Thus, in particular, we have $|w-s| \lesssim C_1 l$. Hence, if C_1 is chosen small enough, then $|w-s| \leq l/2$.

For $|t-\phi(w)|$, we use triangle inequality

$$\begin{aligned} |t-\phi(w)| &\leq |t-\phi(s)-\phi'(s)(w-s)| + |\phi(w)-\phi(s)-\phi'(s)(w-s)| \\ &\lesssim C_1\delta + \delta, \end{aligned}$$

where in the last line we have used (6.17), (6.14) and the bound on $\|\phi''\|$. Choosing C_2 suitably, we are done.

Now we come to prove $\mathcal{N}^{\phi}_{C_2\delta}(I) \subseteq C_3 T$. Let $(u,v) \in \mathcal{N}^{\phi}_{C_2\delta}(I)$. Then $u \in I$ and $|v-\phi(u)| \leq C_2\delta$. Since $\delta \leq l$ it suffices to prove

$$\begin{aligned} &|(u-s, \phi(u)-\phi(s)) \cdot (1, \phi'(s))| \lesssim l, \\ &|(u-s, \phi(u)-\phi(s)) \cdot (-\phi'(s), 1)| \lesssim \delta. \end{aligned}$$

The former inequality follows easily since $|u-s| \leq l/2$ and ϕ' is bounded. The second inequality follows by (6.14). Choosing C_3 suitably, we are done.

$\square$

Proposition 6.3.3 (Flat decoupling)**.** *Let ϕ be a smooth function with $|\phi'| + |\phi''| = O(1)$. Let $\mathcal{P}_\delta$ be a partition on $[0,1]$ such that each $I \in \mathcal{P}_\delta$ has length at least δ. Suppose also that there is a sub-collection $\mathcal{S} \subseteq \mathcal{P}_\delta$ consisting of consecutive intervals of the same length, such that ϕ is δ-flat over their union $\cup\mathcal{S}$. Then $D^{\phi}_{p,q}(\mathcal{P}_\delta) \gtrsim \#\mathcal{S}^{1-1/p-1/q}$.*

Proof. Let $J := \cup\mathcal{S}$ and let $l := \mathcal{L}^1(J) \geq \delta$. Proposition 6.2.3 allows us to assume $\phi(s) = \phi'(s) = 0$ where s is the centre of J. Since ϕ is δ-flat over J, the proof of Proposition 6.3.2 shows that for some absolute constant C, $\mathcal{N}^{\phi}_{C\delta}(J)$ contains an axis-parallel rectangle T of dimensions $\sim l \times \delta$. Moreover, the part of T over each $I \in \mathcal{S}$ is contained in $\mathcal{N}^{\phi}_{\delta}(I)$, and the number of $I \in \mathcal{S}$ so that $I \times \mathbb{R}$ intersects T is essentially $\#\mathcal{S}$.

Thus, we can take $f_i = 1_{T\cap(I_i\times\mathbb{R})}$ where I_i runs through $\mathcal{S}$, so that $\sum_i f_i = 1_T$. Then we may apply Theorem 6.1.14 to conclude that $D^{\phi}_{p,q}(\mathcal{P}_\delta) \gtrsim \#\mathcal{S}^{1-1/p-1/q}$. $\square$

Thus, we see that ϕ being flat over a large collection of intervals leads to flat decoupling. To prevent this from happening, we would like ϕ to be "curved" in an appropriate sense. One immediate condition to impose is that $|\phi''|$ is bounded away from 0 over $[0,1]$, which is the case of Bourgain-Demeter decoupling [5] in the case $n = 2$ (see Theorem 1.2.1.) Note that in this case, since $\phi(s) = s^2$, direct computation shows that ϕ is δ-flat over each interval I of length $\delta^{1/2}$, which is also essentially the largest length of an interval on which ϕ is δ-flat.

6.4 Decoupling for curves with nonzero curvature

In this section we study Theorem 1.2.1 and prove a quantitative version of it that will be used later. For simplicity, we will restrict to $n = 2$. For higher dimensional results like decoupling for the hyperbolic paraboloid and the moment curve, the reader may refer to [5, 8–10] as well as [77] for details.

The main goal of this section is to upgrade Theorem 1.2.1 to the following Lemma for all functions with nonzero second derivative. (Note we are using the rough cutoff version now. Refer to Section 6.2.4 for the notation.)

Lemma 6.4.1. *Let $2 \le p \le 6$ and $M > 1$. Let $\phi : [0,1] \to \mathbb{R}$ be a C^3 function with*

$$\inf_{s\in[0,1]} |\phi''(s)| \ge M^{-1}, \quad \|\phi''\|_\infty \le M, \quad \|\phi'''\|_\infty \le M. \tag{6.18}$$

For $\delta \in \mathbb{N}^{-2}$, let $\mathcal{P}_\delta$ be the partition of $[0,1]$ into intervals of equal length $\delta^{1/2}$. Then for all function f Fourier supported on $\mathcal{N}^{\phi}_{M\delta}([0,1])$, we have

$$\|f\|_{L^p(\mathbb{R}^2)} \le C_{\varepsilon,M,p}\delta^{-\varepsilon}\left(\sum_{I\in\mathcal{P}_\delta}\|f_I\|^2_{L^p(\mathbb{R}^2)}\right)^{\frac{1}{2}}. \tag{6.19}$$

6.4.1 Decoupling for parabolas

To prove the lemma, we will first prove it for all parabolas.

Proposition 6.4.2. *Lemma 6.4.1 holds for all parabolas* $\phi(s) = as^2 + bs + c$ *satisfying* (6.18)*.*

Proof. By Proposition 6.2.3, it suffices to prove it for $\phi(s) = as^2$, where $|a| \geq M^{-1}/2$ by (6.18). We then use a simple scaling argument. Let f be Fourier supported on $\mathcal{N}^{\phi}_{M\delta}([0,1])$. Then let g be such that

$$\widehat{g}(s,t) = \widehat{f}(s,at),$$

and hence g is Fourier supported on $\mathcal{N}^{s^2}_{2M^2\delta}([0,1])$. We then apply Theorem 1.2.1 at the scale $2M^2\delta$ (enlarging M so that $2M^2\delta \in \mathbb{N}^{-2}$ if necessary) to get

$$\|g\|_{L^p(\mathbb{R}^2)} \leq C_{\varepsilon,p}(2M^2\delta)^{-\varepsilon} \left(\sum_{J \in \mathcal{P}_{2M\delta}} \|g_J\|^2_{L^p(\mathbb{R}^2)} \right)^{\frac{1}{2}},$$

where $\mathcal{P}_{2M^2\delta}$ is the partition of $[0,1]$ into subintervals of length $(2M^2\delta)^{1/2}$. Next, use the triangle and Cauchy-Schwarz inequalities to get

$$\|g_J\|_{L^p(\mathbb{R}^2)} \leq C_M \left(\sum_{I \in \mathcal{P}_\delta, I \subseteq J} \|g_I\|^2_{L^p(\mathbb{R}^2)} \right)^{\frac{1}{2}}. \tag{6.20}$$

Combining the above two inequalities, using $M^\varepsilon \leq 1$ and rescaling back to f, we are done. $\square$

6.4.2 Induction on scales

We now prove Lemma 6.4.1.

Proof. We basically follow the same idea of Section 7 of [5]. Let $K(\delta) = K(\delta, M, p)$ be the best constant such that under the assumptions of Lemma

6.4.1, we have

$$\|f\|_{L^p(\mathbb{R}^2)} \le K(\delta) \left(\sum_{I \in \mathcal{P}_\delta} \|f_I\|^2_{L^p(\mathbb{R}^2)} \right)^{\frac{1}{2}}.$$

Our goal is to prove that $K(\delta) \le C_{\varepsilon,M,p}\delta^{-\varepsilon}$.

Let $\delta' \in \mathbb{N}^{-2} \cap (0, \delta]$ be an intermediate scale to be determined. Since f is Fourier supported on $\mathcal{N}^{\phi}_{M\delta}([0,1])$ and $\delta' \ge \delta$, it is Fourier supported on $\mathcal{N}^{\phi}_{M\delta'}([0,1])$ as well.

Thus, if we let $\mathcal{P}_{\delta'}$ be the partition of $[0,1]$ into subintervals of length $\delta'^{1/2}$, then

$$\|f\|_{L^p(\mathbb{R}^2)} \le K(\delta') \left(\sum_{J \in \mathcal{P}_{\delta'}} \|f_J\|^2_{L^p(\mathbb{R}^2)} \right)^{\frac{1}{2}}.$$

Our hope is that the graph of ϕ over each J is approximated by a parabola with error $O(\delta)$. Fix any $s_J \in J$ for each J. We define

$$p_J(s) = \phi(s_J) + \phi'(s_J)(s - s_J) + \frac{1}{2}\phi''(s_J)(s - s_J)^2.$$

Then by Taylor's theorem, for $s \in J$, we have

$$|\phi(s) - p_J(s)| \le \frac{\|\phi'''\|_\infty}{6}\delta'^{3/2} \le \frac{M}{6}\delta'^{3/2}.$$

This suggests that we take δ' to be the smallest number in $\mathbb{N}^{-2}$ such that $\delta' \ge \delta^{2/3}$.

With this choice of δ' (so $\delta' < 4\delta^{2/3}$), since f is Fourier supported on $\mathcal{N}^{\phi}_{M\delta}([0,1])$, we have f_J is Fourier supported on $\mathcal{N}^{p_J}_{CM\delta}([0,1])$. Since p_J satisfies (6.18), we can apply Proposition 6.4.2 to get

$$\|f_J\|_{L^p(\mathbb{R}^2)} \lesssim_{\varepsilon,M,p} \delta^{-\varepsilon} \left(\sum_{I \subseteq J} \|f_I\|^2_{L^p(\mathbb{R}^2)} \right)^{\frac{1}{2}}.$$

Squaring both sides and summing over J, we obtain

$$\|f\|_{L^p(\mathbb{R}^2)} \lesssim_{\varepsilon,M,p} K(\delta')\delta^{-\varepsilon} \left(\sum_{I \in \mathcal{P}_\delta} \|f_I\|^2_{L^p(\mathbb{R}^2)} \right)^{\frac{1}{2}}.$$

This implies that

$$K(\delta) \leq C_{\varepsilon,M,p}\delta^{-\varepsilon} K(\delta'),$$

where we recall δ' is the smallest number in $\mathbb{N}^{-2}$ such that $\delta' \geq \delta^{2/3}$. For $\delta < 1/4$, we iterate this inequality n times until we get to the scale $1/4$:

$$K(\delta) \leq C^n_{\varepsilon,M,p}\delta^{-\varepsilon\left(1+\frac{2}{3}+\cdots+\left(\frac{2}{3}\right)^{n-1}\right)} K\left(\frac{1}{4}\right) \leq CC^n_{\varepsilon,M,p}\delta^{-3\varepsilon},$$

since $K(1/4) \sim 1$ by the triangle and Cauchy-Schwarz inequalities. Lastly, since $\delta' \geq \delta^{2/3}$ in each iteration and n is the first time we stop the iteration, we have $\delta^{(2/3)^{n-1}} < 1/4$. This shows that

$$n < 1 + \frac{\log\log(\delta^{-1}) - \log\log 4}{\log(3/2)} \leq C \log\log(\delta^{-1}),$$

for some suitable absolute constant C. Thus

$$C^n_{\varepsilon,M,p} \leq (\log(\delta^{-1}))^{C \log C_{\varepsilon,M,p}} \lesssim_{\varepsilon,M,p} \delta^{-\varepsilon},$$

and hence we have $K(\delta) \lesssim_{\varepsilon,M,p} \delta^{-4\varepsilon}$. □

Chapter 7

Uniform Decoupling Theorem

In this chapter we prove the uniform decoupling Theorem 1.2.2. We will first formally define the notion of an admissible partition, formulate a rigorous version of the uniform decoupling theorem, and prove it.

7.1 Admissible partitions

From the discussion in Section 6.3, we see that if we have an interval I on which ϕ is δ-flat, then further decomposition of I will likely lead to flat decoupling as in the case of Proposition 6.3.3. Hence, we would like to partition $[0,1]$ into "maximal" subintervals I on each of which ϕ is δ-flat. This motivates the following definition.

Definition 7.1.1 (Admissible partitions)**.** *Let $\phi : [0,1] \to \mathbb{R}$ be C^1 and let $\mathcal{P}$ be a partition of $[0,1]$. We say $\mathcal{P}$ is*

1. *a super-admissible partition for ϕ at the scale δ, if ϕ is δ-flat over each $I \in \mathcal{P}$.*
2. *a sub-admissible partition for ϕ at the scale δ, if for each pair of adjacent intervals $I, J \in \mathcal{P}$, ϕ is not δ-flat over $I \cup J$.*
3. *an admissible partition for ϕ at scale the δ, if it is both super-admissible and sub-admissible for ϕ at the scale δ.*

From this definition it is clear that the partition $\mathcal{P}_\delta$ of $[0,1]$ is admissible for $\phi(s) = s^2$ at the scale δ.

We first show that admissible partitions exist for every C^2 function ϕ over $[0,1]$.

Proposition 7.1.2 (Existence of admissible partitions)**.** *Let $\phi : [0,1] \to \mathbb{R}$ be C^2. Then for any $\delta > 0$, there exists an admissible partition $\mathcal{P}$ of $[0,1]$ for ϕ at the scale δ.*

Proof. If ϕ is linear, then the trivial partition is admissible. Otherwise, $\|\phi''\|_\infty > 0$ and we will construct an admissible partition $\mathcal{P}$ of I_0 for ϕ at the scale δ.

Let $a_0 = 0$. Let

$$a_1 := \max\{t \in [0,1] : \max_{s,c\in[0,t]} |\phi(s) - \phi(c) + \phi'(c)(s-c)| \leq 2\delta\}.$$

Such a_1 always exists in $[0,1]$ since ϕ is C^2. If $a_1 = 1$, then we arrive at the trivial partition which is admissible for ϕ at the scale δ.

If $a_1 < 1$, then there are $s, c \in [0, a_1]$, either s or c being a_1, such that

$$|\phi(s) - \phi(c) + \phi'(c)(s-c)| = 2\delta.$$

Taylor's theorem implies that

$$\left\|\phi''\right\|_\infty (s-c)^2 \geq 4\delta,$$

so

$$a_1 \geq |s-c| \geq \frac{2\delta^{1/2}}{\|\phi''\|_\infty^{1/2}}.$$

Let

$$a_2 := \max\{t \in [a_1,1] : \max_{s,c\in[a_1,t]} |\phi(s) - \phi(c) + \phi'(c)(s-c)| \leq 2\delta\}.$$

If $a_2 = 1$, then $\mathcal{P} := \{[0,a_1],[a_1,1]\}$ is an admissible partition of $[0,1]$ for ϕ at the scale δ. Otherwise, by the same analysis above, we have $a_2 - a_1 \geq 2\sqrt{\delta/\|\phi''\|_\infty}$. Then define a_3 in the above fashion, and repeat.

This process must stop at finite time since $a_n - a_{n-1} \geq 2\sqrt{\delta/\|\phi''\|_\infty}$ for all $n \geq 1$. $\square$

The next proposition is about the lengths of intervals constituting a sub-admissible partition for certain phase functions.

Proposition 7.1.3. *Let $\phi : [0,1] \to \mathbb{R}$ be C^2. Let $\delta > 0$ and suppose $\mathcal{P}$ is a sub-admissible partition of $[0,1]$ for ϕ at scale δ. Then there is a partition $\mathcal{P}'$ of $[0,1]$, such that each interval $I \in \mathcal{P}'$, except possibly the last one, is a union of two adjacent intervals in $\mathcal{P}$ and has length bounded below by*

$$2\sqrt{\frac{\delta}{\|\phi''\|_\infty}}.$$

As a result, the number of intervals in $\mathcal{P}$ is bounded above by $\delta^{-1/2}\|\phi''\|_\infty^{1/2} + 1$.

Proof. Denote $\mathcal{P}$ as a_j, $0 \leq j \leq n$. If $\mathcal{P}$ is the trivial partition, then we define $\mathcal{P}'$ to be trivial as well. If not, then in particular, ϕ cannot be linear, so $\|\phi''\|_\infty > 0$. By (6.15), we have

$$a_{j+2} - a_j \geq \frac{2\delta^{1/2}}{\|\phi''\|_\infty^{1/2}}.$$

Therefore, we can define $\mathcal{P}'$ as follows. If n is even, then we define

$$\mathcal{P}' = \{[a_{2k}, a_{2(k+1)}] : 0 \leq k \leq n/2\}.$$

If n is odd, then we define

$$\mathcal{P}' = \{[a_{2k}, a_{2(k+1)}] : 0 \leq k \leq (n-3)/2\} \cup \{[a_{n-1}, a_n]\}.$$

Thus $\mathcal{P}'$ is as required.

The bound on the number of intervals in $\mathcal{P}$ follows immediately. $\square$

7.2 Uniform decoupling theorem

Now we can state the l^2-uniform decoupling theorem.

Theorem 7.2.1 (Uniform l^2-decoupling for polynomials)**.** *For any $2 \le p \le 6$, $d \ge 0$ and $\varepsilon > 0$, there is a constant $C_\varepsilon = C_{d,\varepsilon,p}$ such that the following is true. For any $0 < \delta \le 1$, any polynomial $\phi : [0,1] \to \mathbb{R}$ of degree at most d with coefficients bounded by 1 in absolute value, any sub-admissible partition $\mathcal{P}_\delta$ of $[0,1]$ for ϕ at the scale δ and any $f \in L^p(\mathbb{R}^2)$ Fourier supported on $\mathcal{N}_\delta^\phi([0,1])$, we have*

$$\|f\|_{L^p(\mathbb{R}^2)} \le C_\varepsilon \delta^{-\varepsilon} \left(\sum_{I \in \mathcal{P}_\delta} \|f_I\|^2_{L^p(\mathbb{R}^2)} \right)^{\frac{1}{2}}. \tag{7.1}$$

7.3 Proof of uniform decoupling theorem

Assuming a bootstrap inequality to be stated soon, we now give a proof of the theorem.

Proof. Let $D_p^d(\delta)$, $\delta > 0$ be the best constant such that under the assumption of Theorem 7.2.1, we have

$$\|f\|_{L^p(\mathbb{R}^2)} \le D_p^d(\delta) \left(\sum_{I \in \mathcal{P}_\delta} \|f_I\|^2_{L^p(\mathbb{R}^2)} \right)^{\frac{1}{2}}. \tag{7.2}$$

Our goal is to prove that $D_p^d(\delta) \le C_\varepsilon \delta^{-\varepsilon}$ for all $\varepsilon > 0$, where C_ε depends on ε, p, d only.

Indeed, this follows once we have proved the following key inequality. It will be referred to as a "bootstrap inequality" since it will be applied in the Proof of the uniform decoupling theorem at each intermediate scale. [6]

Theorem 7.3.1 (Bootstrap inequality)**.** *Let $2 \le p \le 6$, $\varepsilon > 0$, $d \ge 0$. For any $M > 1$, there is a constant $C_{\varepsilon,M} = C(p,d,\varepsilon,M) \ge 1$ and a constant $K = K(d) \ge 1$, such that for each $0 < \delta \le 1$ we have*

$$D_p^d(\delta) \le K(C_{\varepsilon,M}\delta^{-\varepsilon} + \sup_{\delta' \ge M\delta} D_p^d(\delta')). \tag{7.3}$$

[6] In the following we will often drop the dependence on p since it will be fixed throughout the proof.

Assuming Theorem 7.3.1, we will now finish the proof of Theorem 7.2.1.

Let $M > 1$ to be determined. Denote $S(r) := \sup_{\delta' \geq r} D_p^d(\delta')$. Then (7.3) says that

$$D_p^d(\delta) \leq K(C_{\varepsilon,M}\delta^{-\varepsilon} + S(M\delta)). \tag{7.4}$$

For any $\delta' \geq M\delta$ but $\delta' \leq 1$, we may apply (7.3) again to get

$$D_p^d(\delta') \leq K(C_{\varepsilon,M}\delta'^{-\varepsilon} + S(M\delta')).$$

Taking supremum over $\delta' \geq M\delta$ to the above equation, we have

$$S(M\delta) \leq K(C_{\varepsilon,M}M^{-\varepsilon}\delta^{-\varepsilon} + S(M^2\delta)) \leq K(C_{\varepsilon,M}\delta^{-\varepsilon} + S(M^2\delta)).$$

Plugging into (7.4), we have

$$D_p^d(\delta) \leq KC_{\varepsilon,M}\delta^{-\varepsilon} + K^2C_{\varepsilon,M}\delta^{-\varepsilon} + K^2S(M^2\delta).$$

In general, for each $n \geq 0$, if $M^n\delta \leq 1$, then

$$D_p^d(\delta) \leq K^nS(M^n\delta) + \sum_{m=1}^{n} C_{\varepsilon,M}\delta^{-\varepsilon}K^m \leq K^nS(M^n\delta) + nK^nC_{\varepsilon,M}\delta^{-\varepsilon}.$$

We choose $n \geq 0$ to be the smallest integer such that $M^n\delta \geq 1$, and thus $n < \frac{\log(\delta^{-1})}{\log M} + 1$.

We also have $S(M^n\delta) = 1$. Indeed, if $\delta' \geq M^n\delta \geq 1$, then all sub-admissible partitions of $[0,1]$ for ϕ at the scale δ' must be the trivial partition, in view of (6.15). Thus

$$K^nS(M^n\delta) + nK^nC_{\varepsilon,M}\delta^{-\varepsilon} \leq K^n + (2K)^nC_{\varepsilon,M}\delta^{-\varepsilon} \leq (3K)^nC_{\varepsilon,M}\delta^{-\varepsilon},$$

and thus

$$D_p^d(\delta) \leq (3K)^{\frac{\log(\delta^{-1})}{\log M}+1}C_{\varepsilon,M}\delta^{-\varepsilon} = (\delta^{-1})^{\frac{\log(3K)}{\log M}}3KC_{\varepsilon,M}\delta^{-\varepsilon}.$$

Now we may choose $M = (3K)^{1/\varepsilon}$, so M depends on d, ε only. If we choose

$C_\varepsilon = 3KC_{\varepsilon,M}$, then

$$D_p^d(\delta) \leq C_\varepsilon \delta^{-2\varepsilon},$$

which finishes the proof. $\square$

Thus, all that remains is to prove Theorem 7.3.1.

7.4 Decoupling for curves with nonzero curvature

In this section we further upgrade Lemma 6.4.1 to include the case of all sub-admissible partitions.

Theorem 7.4.1. *Let $2 \leq p \leq 6$. Let $M \geq 1$ and let $\phi : [0,1] \to \mathbb{R}$ be a C^3 function with*

$$\|\phi'''\|_\infty + \|\phi''\|_\infty \leq M \inf_{s \in [0,1]} |\phi''(s)| \quad \text{and} \quad \|\phi''\|_\infty \leq M.. \tag{7.5}$$

Then for any $\varepsilon > 0$, there is some $C_{\varepsilon,M} = C_{\varepsilon,M,p}$ such that for any $0 < \delta \leq 1$ and any sub-admissible partition $\mathcal{P}$ of $[0,1]$ for ϕ at the scale δ, we have $D^\phi_{p,2}(\mathcal{P}) \leq C_{\varepsilon,M}\delta^{-\varepsilon}$ for any $\varepsilon > 0$.

The rest of this section is devoted to the proof of this theorem. The main ingredients of the proof include Lemmas 7.1.3, 6.4.1 and the following simple tiling argument.

Proposition 7.4.2. *Let $0 < l_0 \leq 1/4$ and let $\mathcal{P}$ be a collection of disjoint subintervals of $[0,1]$ with lengths bounded above by $2l_0$ and below by l_0. Then there is $l \in 2^{-\mathbb{N}}$ with $l/l_0 \in [4,8)$ and two subcollections $\mathcal{U}_i$, $i = 1,2$ of $\mathcal{P}$, such that the following statements are true.*

1. *For each $I \in \mathcal{U}_1$, there is some $1 \leq j \leq l^{-1}$ such that $I \subseteq [(j-1)l, jl]$. Moreover, each such $[(j-1)l, jl]$ contains less than 8 intervals I.*

2. *For each $I \in \mathcal{U}_2$, there is some $1 \leq j \leq l^{-1}$ such that $I \subseteq [(j-1/2)l, (j+1/2)l] \cap [0,1]$. Moreover, each such $[(j-1/2)l, (j+1/2)l]$ contains less than 8 intervals I.*

Proof. Let $l \in 2^{-\mathbb{N}}$ be the smallest number such that $l \geq 4l_0$, so $l/l_0 \in [4, 8)$. Each interval $I \in \mathcal{P}$ has length at most $2l_0 \leq l/2$. Include I inside $\mathcal{U}_1$ if it is fully contained in a dyadic interval $[(j-1)l, jl]$ for some $1 \leq j \leq l^{-1}$. Otherwise, it has to be fully contained in $[(j-1/2)l, (j+1/2)l]$ for some $1 \leq j \leq l^{-1}$, so we can include it in the collection $\mathcal{U}_2$. The bound on the number of intervals $I \in \mathcal{P}$ contained in each dyadic interval follows from the lower bound of the lengths of the intervals I. $\square$

Now we can give a proof of Theorem 7.4.1, in a series of steps.

7.4.1 A few technical reductions

By the scaling invariance of δ-flatness we may assume $\inf_{s\in[0,1]} \phi''(s) = 1$. Let $f \in L^p(\mathbb{R}^2)$ with Fourier support on $\mathcal{N}_\delta^\phi([0,1])$ and $\mathcal{P}$ be a sub-admissible partition of $[0,1]$ for ϕ at the scale δ.

We invoke Lemma 7.1.3 to get the coarser partition $\mathcal{P}'$. Since $\|\phi''\|_\infty \leq M$, each interval $I \in \mathcal{P}'$, except possibly the last one, is a union of two adjacent intervals in $\mathcal{P}$ and has length bounded below by $2(\delta/M)^{1/2}$. As a result, the number of intervals in $\mathcal{P}$ is bounded above by $(M/\delta)^{1/2} + 1$.

By the triangle and Cauchy-Schwarz inequalities we may assume the number of intervals in $\mathcal{P}$ is even, and that each interval in $\mathcal{P}'$ has length bounded below by $2(\delta/M)^{1/2}$. Since each interval $I \in \mathcal{P}'$ is a union of two adjacent intervals in $\mathcal{P}$, by the triangle and Cauchy-Schwarz inequalities it suffices to prove that

$$\|f\|_{L^p(\mathbb{R}^2)} \lesssim_{\varepsilon,M} \delta^{-\varepsilon} \left(\sum_{I\in\mathcal{P}'} \|f_I\|_{L^p(\mathbb{R}^2)}^2 \right)^{\frac{1}{2}} .$$

We now partition the collection $\mathcal{P}'$ according to the lengths of intervals. Let I^* be an interval in $\mathcal{P}'$ with maximum length (we may of course assume $|I^*| \leq 1/2$). Let $\mathcal{P}'_1$ be the collection of intervals in $\mathcal{P}'$ with length $> |I^*|/2$. For each $k \geq 2$, let $\mathcal{P}'_k$ be the collection of intervals in $\mathcal{P}'$ with length in the range $(2^{-k}|I^*|, 2^{-k+1}|I^*|]$. Since each interval in $\mathcal{P}'$ has length bounded below by $2(\delta/M)^{1/2}$, we have only $O(\log(\delta^{-1}))$ many such collections. Since we can afford logarithmic losses, it suffices to show for each $\mathcal{P}'_k$ that

$$\left\| f_{\cup\{I: I\in\mathcal{P}'_k\}} \right\|_{L^p(\mathbb{R}^2)} \lesssim_{\varepsilon,M} \delta^{-\varepsilon} \left(\sum_{I\in\mathcal{P}'_k} \|f_I\|^2_{L^p(\mathbb{R}^2)} \right)^{\frac{1}{2}}.$$

Now fix such $k \geq 1$. We can apply Proposition 7.4.2 with $l_0 = 2^{-k}|I^*| \leq 1/4$ and $\mathcal{P} = \mathcal{P}'_k$ to get the corresponding $l = l(k)$ and $\mathcal{U}_i = \mathcal{U}_i(k)$, $i = 1, 2$. Also, note that $l \geq 8(\delta/M)^{1/2}$ since $l_0 \geq 2(\delta/M)^{1/2}$.

By the triangle and Cauchy-Schwarz inequalities again, it suffices to prove for $i = 1, 2$ that

$$\|f_{J_i}\|_{L^p(\mathbb{R}^2)} \lesssim_{\varepsilon,M} \delta^{-\varepsilon} \left(\sum_{I\in\mathcal{U}_i} \|f_I\|^2_{L^p(\mathbb{R}^2)} \right)^{\frac{1}{2}},$$

where $J_i := \cup\{I : I \in \mathcal{U}_i\}$, $i = 1, 2$.

7.4.2 Applying Lemma 6.4.1

We deal with $i = 1$ first. We have $\delta \leq l^2$ by our choice of l. By (7.5), we may apply Lemma 6.4.1 with the scale $\delta = l^2$ to get

$$\|f_{J_1}\|_{L^p(\mathbb{R}^2)} \lesssim_{\varepsilon,M} l^{-\varepsilon} \left(\sum_{j=1}^{l^{-1}} \left\| f_{[(j-1)l,jl]\cap J_1} \right\|^2_{L^p(\mathbb{R}^2)} \right)^{\frac{1}{2}}.$$

But by Proposition 7.4.2, for each j, $[(j-1)l, jl] \cap J_1$ is equal to a union of less than 8 intervals $I \in \mathcal{U}_1$. By the triangle and Cauchy-Schwarz inequalities, we have

$$\|f_{J_1}\|_{L^p(\mathbb{R}^2)} \lesssim_{\varepsilon,M} \delta^{-\varepsilon} \left(\sum_{I\in\mathcal{U}_1} \|f_I\|^2_{L^p(\mathbb{R}^2)} \right)^{\frac{1}{2}}.$$

For the case $i = 2$, we note that translating the domain of ϕ to the right by $\sigma := l/2$ is equivalent to changing $\phi(s)$ to $\tilde{\phi}(s) := \phi(s + \sigma)$, $s \in [0, 1 - \sigma]$. Since the domain of $\tilde{\phi}$ is now a subset of $[0, 1]$, the conditions in (7.5) still hold. Hence, the same argument for the case $i = 1$ works in this case.

7.5 A rescaling theorem

The following rescaling theorem resembles the parabolic rescaling theorem in [5].

Notation. From now on we denote

$$D_p^\phi(\delta) = \sup\{D_{p,2}^\phi(\mathcal{P}_\delta) : \mathcal{P}_\delta \text{ is sub-admissible for } \phi \text{ at the scale } \delta\}.$$

The subscript $q = 2$ is dropped because throughout the section we are considering l^2-decoupling.

Theorem 7.5.1 (Rescaling)**.** *Let $\phi \in C^2([0,1])$, $0 < \delta \leq 1$ and $\mathcal{P}$ be a sub-admissible partition of $[0,1]$ for ϕ at the scale δ. Then for any $J = [\alpha, \beta]$ which is a union of consecutive intervals in $\mathcal{P}$, there exists another C^2-function ψ such that for any $f \in L^p(\mathbb{R}^2)$ with Fourier support in $\mathcal{N}_\delta^\phi(J)$, we have*

$$\|f\|_{L^p(\mathbb{R}^2)} \leq D_p^\psi((\beta-\alpha)^{-1}\delta)\left(\sum_{I\in\mathcal{P}, I\subseteq J}\|f_I\|_{L^p(\mathbb{R}^2)}^2\right)^{\frac{1}{2}}. \tag{7.6}$$

In particular, if ϕ is a polynomial of degree at most d, then so is ψ. As a result

$$\|f\|_{L^p(\mathbb{R}^2)} \leq D_p^d((\beta-\alpha)^{-1}\delta)\left(\sum_{I\in\mathcal{P}, I\subseteq J}\|f_I\|_{L^p(\mathbb{R}^2)}^2\right)^{\frac{1}{2}}, \tag{7.7}$$

where $D_p^d(\delta)$ was defined at the beginning of the proof of Theorem 7.2.1.

Proof. By a change of variables, we have

$$f_{[\alpha,\beta]}(x,y) = \int_{-\delta}^{\delta}\int_\alpha^\beta \hat{f}(s, \phi(s)+t)e(xs + y(\phi(s)+t))dsdt.$$

Define $s' = (s-\alpha)/(\beta-\alpha) \in [0,1]$. Then by direct computation,

$$\begin{aligned} f_{[\alpha,\beta]}(x,y) = (\beta-\alpha) \int_{-\delta}^{\delta} e(ty) \int_0^1 & \hat{f}(s, \phi(s)+t) \\ & \cdot e(x(\beta-\alpha)s' + \alpha) \\ & \cdot e\left(y\phi(\alpha + (\beta-\alpha)s')\right) ds' dt. \end{aligned}$$

We define ψ by

$$\psi(s') = (\beta-\alpha)^{-1}\phi(\alpha + (\beta-\alpha)s'). \tag{7.8}$$

Thus $\psi(s') = (\beta-\alpha)^{-1}\phi(s)$.

Define $t' = (\beta-\alpha)^{-1}t$ and $(x', y') = (\beta-\alpha)(x,y)$. We also define another function F by the relation $\hat{F}(s', \psi(s')+t') = \hat{f}(s, \phi(s)+t)$. More explicitly, for any $(u,v) \in \mathbb{R}^2$, $\hat{F}$ is defined as

$$\hat{F}(u,v) = \hat{f}((\beta-\alpha)u + \alpha, (\beta-\alpha)v).$$

Then we see that $F \in L^p(\mathbb{R}^2)$ and is Fourier supported on $\mathcal{N}^{\psi}_{(\beta-\alpha)^{-1}\delta}([0,1])$. Thus, in the above notation, we arrive at

$$\begin{aligned} f_{[\alpha,\beta]}(x,y) &= e(\alpha x) \int_{-\frac{\delta}{(\beta-\alpha)}}^{\frac{\delta}{(\beta-\alpha)}} e(t'y') \int_0^1 \hat{F}(s', \psi(s')+t') e(x's') e(y'\psi(s')) ds' dt' \\ &= e(\alpha x) F(x', y'). \end{aligned}$$

Also, observe that the following partition of $[0,1]$

$$\mathcal{P}' := \left\{ I' = \frac{I-\alpha}{\beta-\alpha} : I \in \mathcal{P} \right\}$$

is sub-admissible for ψ at the scale $(\beta-\alpha)^{-1}\delta$. Applying the definition of $D_p^{\psi}((\beta-\alpha)^{-1}\delta)$, we have

$$\|F\|_{L^p(\mathbb{R}^2)} \leq D_p^{\psi}((\beta-\alpha)^{-1}\delta) \left(\sum_{I' \in \mathcal{P}'} \|F_{I'}\|^2_{L^p(\mathbb{R}^2)} \right)^{\frac{1}{2}}.$$

Rescaling back, we obtain

$$\|f\|_{L^p(\mathbb{R}^2)} \le D_p^{\psi}((\beta-\alpha)^{-1}\delta) \left(\sum_{I\in\mathcal{P}, I\subseteq J} \|f_I\|^2_{L^p(\mathbb{R}^2)} \right)^{\frac{1}{2}},$$

which proves (7.6). □

7.6 Proof of the bootstrap inequality

Using Theorem 7.4.1 and Theorem 7.5.1, we now prove Theorem 7.3.1. The main idea is to partition $[0,1]$ into subintervals according as whether $|\phi''|$ is bounded below. On subintervals where $|\phi''|$ is bounded below, we use decoupling for curves with nonvanishing curvature, which is Theorem 7.4.1. Otherwise, we use the rescaling Theorem 7.5.1 and the following lemma on the polynomial sub-level sets.

Lemma 7.6.1. *Let $P : \mathbb{R} \to \mathbb{R}$ be a polynomial of degree at most d. Then for any $r > 0$, the set*

$$B(P,r) := \{s \in [0,1] : |P(s)| < r\} \tag{7.9}$$

is a union of at most d intervals relatively open in $[0,1]$ and each of them have length at most

$$C_d \left(\frac{r}{\sup_{s\in[0,1]} |P(s)|} \right)^{\frac{1}{d}}. \tag{7.10}$$

The implicit constant here is independent of the choice of P.

Proof. The assertion on the number of intervals follows easily from the fundamental theorem of algebra. The assertion on the lengths of the intervals follows from Proposition 2.2 of [66] in case $n = 1$ as well as the observation that $\sup_{s\in[0,1]} |P(s)|$ is essentially the sum of all the coefficients of a polynomial P. □

We will also need the following Markov Brother's inequality.

Lemma 7.6.2. *(Inequality of A. and M. Markov [63]) For any polynomial P of degree at most d and any $\alpha < \beta$,*

$$\sup_{s\in[\alpha,\beta]} \left|P'(s)\right| \leq \frac{2d^2}{\beta-\alpha} \sup_{s\in[\alpha,\beta]} |P(s)|.$$

Proof. For $\alpha = -1$ and $\beta = 1$, this is the classical inequality of the Markov brothers. Several different proofs of this may be found in [63]. The proof for a general interval $[\alpha, \beta]$ follows by mapping it into $[-1, 1]$ by an affine transformation. $\square$

Now we can prove the bootstrap inequality.

Proof. Let ϕ be polynomial of degree at most d. For $M > 1$, we will find $K = K(d)$ and $C_{\varepsilon,M} = C_{\varepsilon,M,d,p}$ such that

$$D_p^{\phi}(\delta) \leq K(C_{\varepsilon,M}\delta^{-\varepsilon} + \sup_{\delta'\geq M\delta} D_p^d(\delta')). \tag{7.11}$$

Let $\delta > 0$ and $\mathcal{P}$ be a sub-admissible partition of $[0, 1]$ for ϕ at the scale δ. Let $f \in L^p(\mathbb{R}^2)$ with Fourier support in $\mathcal{N}_\delta^{\phi}([0, 1])$.

Since ϕ be polynomial of degree at most d, we have ϕ'' is a polynomial of degree at most $d - 2$.

Take $B = B(\phi'', M^{-d})$ as in (7.9). Split $\mathcal{P}$ into 3 subcollections

$$\begin{aligned}
\mathcal{P}_1 &:= \{I \in \mathcal{P} : I \subseteq B\} \\
\mathcal{P}_2 &:= \{I \in \mathcal{P} : I \subseteq [0, 1]\backslash B\} \\
\mathcal{P}_3 &:= \{I \in \mathcal{P} : I \cap B \neq \varnothing \text{ and } I\backslash B \neq \varnothing\}.
\end{aligned}$$

Denote $f_i := f_{\cup\{I:I\in\mathcal{P}_i\}}$, $i = 1, 2, 3$.

Since B has at most $d - 2$ connected components, $\mathcal{P}_3$ has cardinality bounded above by $O(d)$, and so by the triangle and Cauchy-Schwarz inequalities it suffices to consider $\mathcal{P}_1$ and $\mathcal{P}_2$.

Consider $\mathcal{P}_1$ first. Write $B = \cup_i^N J_i$ where $N \leq d$ and $\mathcal{L}^1(J_i) \lesssim_d M^{-1}$.

Apply (7.7) to J_i to get

$$\|(f_1)_{J_i}\|_{L^p(\mathbb{R}^2)} \le D_p^d(\mathcal{L}^1(J_i)^{-1}\delta)\left(\sum_{I\in\mathcal{P}, I\subseteq J_i}\|f_I\|^2_{L^p(\mathbb{R}^2)}\right)^{\frac{1}{2}}$$
$$\le \sup_{\delta'\ge M\delta} D_p^d(\delta')\left(\sum_{I\in\mathcal{P}, I\subseteq J_1}\|f_I\|^2_{L^p(\mathbb{R}^2)}\right)^{\frac{1}{2}}.$$

Thus

$$\|f_1\|_{L^p(\mathbb{R}^2)} \lesssim_d \sup_{\delta'\ge M\delta} D_p^d(\delta')\left(\sum_{I\in\mathcal{P}_1}\|f_I\|^2_{L^p(\mathbb{R}^2)}\right)^{\frac{1}{2}}.$$

Now we come to $\mathcal{P}_2$. Write $[0,1]\backslash B = \cup_i^{N'} J_i'$ where $N' = O(d)$. Apply (7.6) to $J_i' := [\alpha, \beta]$ to get

$$\left\|(f_2)_{J_i'}\right\|_{L^p(\mathbb{R}^2)} \le D_p^{\psi}(\mathcal{L}^1(J_i)^{-1}\delta)\left(\sum_{I\in\mathcal{P}, I\subseteq J_i'}\|f_I\|^2_{L^p(\mathbb{R}^2)}\right)^{\frac{1}{2}},$$

where $\psi(s) = (\beta-\alpha)^{-1}\phi(\alpha+(\beta-\alpha)s)$ as in (7.8).

Using Lemma 7.6.2 to ϕ'', we have ψ satisfies (7.5) with M replaced by M^d. Hence, by Theorem 7.4.1, we have

$$D_p^{\psi}(\mathcal{L}^1(J_i)^{-1}\delta) \lesssim_{\varepsilon,M,d,p} (\mathcal{L}^1(J_i)^{-1}\delta)^{-\varepsilon} \le \delta^{-\varepsilon}.$$

Thus, by the triangle and Cauchy-Schwarz inequalities,

$$\|f_2\|_{L^p(\mathbb{R}^2)} \lesssim_{\varepsilon,d,M,p} \delta^{-\varepsilon}\left(\sum_{I\in\mathcal{P}_2}\|f_I\|^2_{L^p(\mathbb{R}^2)}\right)^{\frac{1}{2}}.$$

Combining the estimates above, we have

$$\|f\|_{L^p(\mathbb{R}^2)} \le K\left(C_{\varepsilon,d,M,p}\delta^{-\varepsilon} + \sup_{\delta'\ge M\delta} D_p^d(\delta')\right)\left(\sum_{I\in\mathcal{P}}\|f_I\|^2_{L^p(\mathbb{R}^2)}\right)^{\frac{1}{2}},$$

for some absolute constant $K = K(d)$. Since f and $\mathcal{P}$ are arbitrary, we have (7.11).

This finishes the proof of the bootstrap inequality and consequently the uniform decoupling theorem. $\square$

Chapter 8

Conclusion

In this thesis we studied two topics in harmonic analysis: Euclidean configurations and decoupling. Finally we point out some future directions of research in both topics.

8.1 Euclidean configurations

The proof of Theorem 1.1.2 relies heavily on measure theory. For this reason, we are only able to obtain the set A with positive Lebesgue measure. It is natural to ask if such a set A could be constructed to have zero Lebesgue measure, when the prescribed sequence β_m decays quickly enough. As a model problem, we may ask the following question:

Problem 8.1.1. *If $A \subseteq \mathbb{R}$ is a closed set that contains a similar copy of every sequence that converges to 0 faster than $\{2^{-n}\}$, must A have positive Lebesgue measure?*

Next we consider higher dimensional generalisations.

Problem 8.1.2. *If $A \subseteq \mathbb{R}^n$ contains a similar copy of every convergent sequence, must the closure of A contain an open ball?*

The proof of Theorem 1.1.1 relies heavily on the interval structure of the real line, which cannot be trivially generalised to higher dimensions.

8.2 Decoupling

In Theorem 7.2.1 we studied a uniform l^2-decoupling theorem for polynomials of a fixed degree. It is also natural to ask if this result generalises to higher dimensions.

Problem 8.2.1. *In $\mathbb{R}^n$, given a pair of exponents (p,q) and a degree d, let $D^d_{p,q}(\delta)$ be the supremum of all decoupling constants $D^{\phi}_{p,q}(\mathcal{P}_\delta)$ as ϕ ranges through all real polynomial in $n-1$ variables of degree at most d, and as $\mathcal{P}_\delta$ ranges through all "admissible partitions" of $[0,1]^{n-1}$ for ϕ at the scale δ. Then what can we say about $D^d_{p,q}(\delta)$?*

In fact, for $n \geq 3$ it is not even obvious how one might formulate the notion of admissibility of a partition, which may consist of rectangles in different orientations. In fact, this is related to a conjecture by Bourgain, Demeter and Kemp [10], which we state below.

Conjecture 8.2.2 (Bourgain, Demeter and Kemp, [10])**.** *Let $\phi : (-2,2)^2 \to \mathbb{R}$ be a real analytic function. Then for every $\varepsilon > 0$, there is a constant C_ε, depending on ϕ and ε only, such that the following is true. For every $0 < \delta < 1$, there is a boundedly overlapping family $\mathcal{P} = \mathcal{P}_\delta$ of rectangles covering $[-1,1]^2$, such that ϕ is δ-flat (in the sense of* (6.14)*) over each $P \in \mathcal{P}$, and for any function $f : \mathbb{R}^3 \to \mathbb{C}$ with Fourier support on the δ-neighbourhood of the graph of ϕ above $[-1,1]^2$, we have the l^4-decoupling inequality:*

$$\|f\|_{L^4(\mathbb{R}^3)} \leq C_\varepsilon \delta^{-\varepsilon} \# \mathcal{P}^{\frac{1}{4}} \left(\sum_{P \in \mathcal{P}} \|f_P\|^4_{L^4(\mathbb{R}^3)} \right)^{\frac{1}{4}}. \tag{8.1}$$

In our language, this conjecture says that for each analytic function, there exists a boundedly overlapping family $\mathcal{P}_\delta$ as above, such that

$$D^{\phi}_{4,4}(\mathcal{P}_\delta) \leq C_\varepsilon \delta^{-\varepsilon} \# \mathcal{P}^{\frac{1}{4}}.$$

We remark that we cannot aim for a l^2 decoupling inequality since the function ϕ may have negative Hessian determinant; see the discussion in Section 6.2.2.

In a recent work [74] with Jianhui Li, I have made some partial progress on this conjecture. We are able to prove the conjecture for each single mixed-homogeneous polynomial ϕ (for an exact definition, see the first displayed equation in Section 1.2 of [74]). However, neither is our decoupling result uniform in all mixed-homogeneous polynomials with a bounded degree, nor does it hold for a single general polynomial ϕ which may not be mixed-homogeneous. The full conjecture still remains widely open for us.

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