ON THE BARTNIK MASS OF TWO-SPHERES WITH NON-NEGATIVE CONSTANT MEAN CURVATURE

by

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The following individuals certify that they have read, and recommend to the Faculty of Graduate and Postdoctoral Studies for acceptance, the thesis entitled:

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Abstract

We establish three new upper bounds on the Bartnik quasi-local mass of triples $(\mathbb{S}^2, g, H)$ where $\mathbb{S}^2$ is a topological two sphere, g is a Riemannian metric on $\mathbb{S}^2$, and $H \geq 0$ is a specified (constant) value for the initial mean curvature. We use the initial data set approach under the additional assumptions of time-symmetry (TS) and the dominant energy condition (DEC) in which one first constructs a collar with initial boundary sphere isometric to $(\mathbb{S}^2, g)$ and then extends to an asymptotically flat (AF) 3-manifold with non-negative scalar curvature (which is the DEC under the TS setting).

The first bound extends the main result in [**13**] to include the boundary case. Precisely, we show that any metric g with non-negative first eigenvalue of the operator $-\Delta_g + K_g$ appears as an apparent horizon (in the TS/DEC/AF setting) and that its Bartnik mass is precisely the corresponding Hawking mass.

The second bound establishes that the Bartnik mass of the triple $(\mathbb{S}^2, g, H)$ is bounded above by $r/2$ whenever g has non-negative Gaussian curvature K_g and $H > 0$. This result was known when K_g is assumed to be strictly positive (see [**18**]) though the methods used there do not apply when $\min K_g = 0$.

For the last bound, given any metric g with $K_g \geq 0$ and any $H > 0$, we give an explicit constant C (depending only on g and H) such that the Bartnik mass of the triple $(\mathbb{S}^2, g, H)$ is bounded above by a quantity involving C which approaches the Hawking mass as $C \to 0$, which happens as either $H \to 0$ or as g becomes round. Moreover, C remains bounded if $H \to \infty$ or $r^2 \min K_g \to 0$. This result can be extended to arbitrary metrics (that do not necessarily satisfy $K_g \geq 0$) although the resulting bound in this case is only finite if H is sufficiently large depending on g.

Lay Summary

Mathematical general relativity, roughly speaking, is the study of mathematical models for our universe which capture how matter bends spacetime. One key feature of general relativity is that there is not a well-defined way of measuring mass since any notion of energy density is observer dependent. Despite this, estimating the mass of a spacetime (or a portion thereof) remains important in physics. There have therefore been many suggested notions of mass, one of which is due to Robert Bartnik. Though it is recognized as being very physically accurate, Bartnik mass has proven to be quite elusive as most examples in the literature provide upper bounds which are not very explicit or use strong curvature conditions. We establish three new upper bounds on the Bartnik mass of a sphere which are easily understood and weaken the curvature assumptions.

Preface

Chapter 1 is mostly expository.

The remainder of this thesis (chapters 2 and 3) is collaborative research between myself and Albert Chau. Chapter 2 consists of preliminary work that is needed in for the proofs of our main results and is, unless otherwise stated, original intellectual property of myself.

The main results, which are presented in chapter 3, are mostly from two papers that were written by myself and Albert Chau which have both been submitted for publication. The original idea for section §3.2 is due to Albert Chau and the work in this section is an adaptation of [**7**]. The original ideas for sections §3.3 and §3.4 were my own and the work in these sections are an adaptation of [**8**].

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Acknowledgements

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Dedication

To Charlotte

CHAPTER 1

Background

In this chapter, we first present the notation that will be used throughout (section §1.1), then we delve into the necessary background information that is required to understand the main results and their framework in the wider setting of general relativity. Section §1.2 summarizes the necessary definitions and theorems of Riemannian geometry. A primary reference of which is John Lee's *Introduction to Riemannian Manifolds* [**11**]. Similarly, section §1.3 summarizes key items from general relativity as well as their mathematical interpretations. While most of the basic definitions can be found in Robert Wald's *General Relativity* [**22**], many of the notions we will be discussing are more recent and their sources will be individually referenced.

1.1. Notation and Conventions

Notation	**Explanation/Definition**
$\mathbb{N}$	Natural numbers $\{1,2,3,\dots\}$
$\mathbb{R}$	Real numbers
$\mathbb{R}_{>0}, \mathbb{R}_{\geq 0}$	Positive, non-negative real numbers
$\mathbb{S}^n$	Topological n-sphere
$\mathbb{D}_r, \mathbb{D}$	Open disk of radius r, 1. That is $\{x \in \mathbb{R}^2 : \lvert x\rvert < r\}$, $\{x \in \mathbb{R}^2 : \lvert x\rvert < 1\}$
$\equiv$	Identically equal (reserved for functions)
$\approx$	Diffeomorphism, diffeomorphic manifolds
$\cong$	Isometry, isometric manifolds
LHS, RHS	Left hand side, right hand side
$:=$	Definition. LHS is given by RHS
$=:$	Definition. RHS is given by LHS
$\mathfrak{X}(M)$	Space of smooth vector fields on a manifold M
C^k	Banach space of k-times continuously differentiable functions
C^k_c	Vector space of compactly supported C^k functions
L^p	Banach space of functions f with $\int \lvert f\rvert^p < \infty$
$W^{k,p}$	Banach space of functions f with $f, \nabla f, \dots, \nabla^k f \in L^p$ ("Sobolev space")
$\nabla^{M,g}$	Levi-Civita connection on (M,g). §1.2.2
$\Delta_{M,g}$	Laplace-Beltrami operator on (M,g). §1.2.3
Rm, Ric, R, K	Riemannian, Ricci, scalar, Gaussian curvature. §1.2.3
L_g	Linear elliptic operator $-\Delta_g + K_g$. §1.3.4 and §2.2
$\lambda_1(g)$	First eigenvalue of L_g. §1.3.4 and §2.2
dA_g, dV_g	Area, volume form of a metric g. §1.2.1
$\mathrm{area}(M,g)$	Area of (M,g). That is $\int_M dA_g$. §1.2.1
$\mathrm{Met}(M)$	Bundle of Riemannian metrics on a smooth manifold M. §1.2.1
$\mathscr{M}_{>0}, \mathscr{M}_{\geq 0}, \mathscr{M}_{=0}$	$g \in \mathrm{Met}(\mathbb{S}^2)$ with $\lambda_1(g) > 0$, $\lambda_1(g) \geq 0$, $\lambda_1(g) = 0$. §1.3.4 and §2.2
$\mathcal{K}_{>0}, \mathcal{K}_{\geq 0}$	$g \in \mathrm{Met}(\mathbb{S}^2)$ with $K_g > 0$, $K_g \geq 0$ on $\mathbb{S}^2$. §1.2.3
H_Σ	(Scalar) mean curvature of hypersurface Σ. §1.2.4

All functions are real valued unless otherwise specified.

The authors of [**13**] and [**7**] use the notations $\mathscr{M}_+$ and $\overline{\mathscr{M}_+}$ instead of $\mathscr{M}_{>0}$, $\mathscr{M}_{\geq 0}$ which are favoured here. This change of notation is to maintain consistency with the other notation of $\mathbb{R}_{>0}$, $\mathcal{K}_{>0}$.

If it is clear from context, we simplify notation by dropping subscripts and superscripts. For example, $\Delta_{M,g}$ may be simplified to Δ_g if the manifold is understood but the metric is not or simply Δ if both the manifold and metric are understood.

1.2. Riemannian Geometry

In this section, we recall some basic definitions and results from Riemannian geometry. The only proofs we present are those of Lemma 1.2.13 which will be needed in the proofs of our main results. This section is not meant to be a complete overview of Riemannian geometry and not everything will be re-defined here. For a more thorough treatment, we refer the reader to [**11**]. We henceforth assume familiarity with the theory of smooth manifolds.

1.2.1. Metrics and Conformal Equivalence.

Definition 1.2.1.

(*i*) A *pseudo-Riemannian metric* g on a smooth manifold M is a symmetric 2-tensor field on M that is non-degenerate and constant *signature* on M. We say $g(p)$ has (k, l)-signature if given any local coordinates x^i, the matrix $\big(g(p)\big)_{ij}$ has k positive and l negative eigenvalues (counting multiplicity). If g is positive definite, it is called a *Riemannian metric*. We will often write $\langle X, Y\rangle_g$ or even $\langle X, Y\rangle$ for $g(X, Y)$. The space of all Riemannian metrics on a manifold M is denoted $\mathrm{Met}(M)$.

(*ii*) A *pseudo-Riemannian (Riemannian) manifold* is a pair (M^n, g) where M^n is a n-dimensional smooth manifold and g is a pseudo-Riemannian (Riemannian) metric on M. In the case $n = 2$, (M, g) is sometimes called a pseudo-Riemannian (Riemannian) surface.

(*iii*) The *volume form* of an oriented Riemannian manifold (M^n, g) is a smooth top dimensional form given by $dV_g := \sqrt{\det g}\, d\mathbf{x}$. Here $\det g = \det\left((g)_{ij}\right)$ as a matrix in local coordinates x^i. Existence of a volume form is equivalent to orientability of M. If $n = 2$, the volume form is naturally called the *area form* and will be denoted dA_g.

(*iv*) If (M, g) and (N, h) are Riemannian manifolds with $f : M \to N$ a diffeomorphism, then f is called an *isometry* if $g = f^*h$ (or even $h = f(g)$) which is shorthand for saying that

$$g_p(u, v) = h_{f(p)}(df_p(u), df_p(v)) \;\text{ for all } p \in M \text{ and } u, v \in T_pM.$$

If there is such an isometry, we say that (M, g) and (N, h) are isometric and we write $(M, g) \cong (N, h)$ or even $M \cong N$ if the metrics are understood.

(*v*) Suppose $g \in \mathrm{Met}(\mathbb{S}^2)$. We say that $(\mathbb{S}^2, g)$ is a *round sphere* (or that g is *round*) if there is a natural inclusion $\iota : (\mathbb{S}^2, g) \hookrightarrow (\mathbb{R}^3, \delta)$ in which $\iota(\mathbb{S}^2)$ is a submanifold of $\mathbb{R}^3$ consisting of a set of points of constant distance from a center, and $\iota(g)$ is the natural metric coming from the ambient Euclidean metric δ. Spheres that are not round are sometimes called *squashed*.

Later, we will be interested in comparing several different metrics on the same manifold that are related. The following definition makes this notion more precise.

Definition 1.2.2. Let g, h be two Riemannian metrics on a smooth manifold M. We say that g and h are *conformally equivalent* if there exists a smooth $w : M \to \mathbb{R}$ such that $h = e^{2w}g$ everywhere on M. By a slight abuse of notation, we say that the Riemannian manifolds (M, g) and (M, h) are *conformally equivalent* if the metrics g and h are.

It is a basic exercise to show that conformal equivalence is an equivalence relation on the set $\mathrm{Met}(M)$. A natural question is: What do the resulting equivalence classes look like? At least in the case $n = 2$, the answer to this question is completely understood and is captured by the following classical theorem (which we omit the proof of).

THEOREM 1.2.3 (Uniformization Theorem). *Every simply connected Riemannian surface is conformally equivalent to one of three Riemann surfaces: the Poincaré disc, the complex plane with the Euclidean metric, or a round sphere.*

An immediate corollary of the uniformization theorem is that every metric $g \in \mathrm{Met}(\mathbb{S}^2)$ is conformally equivalent to a round metric g_*. By further scaling by a constant factor, we may assume that $\mathrm{area}(\mathbb{S}^2, g_*) = 4\pi$. This fact will be used extensively throughout.

1.2.2. Connections.

One of the main goals of Riemannian geometry is to establish a notion of curvature which relates local properties of a manifold to its global topological properties. In order to define curvature however, we first need the notion of a *connection* on a manifold. Essentially, a connection is a coordinate independent way of taking derivatives of vector fields.

DEFINITION 1.2.4. An *connection* on a Riemannian manifold (M, g) is a map $\nabla : \mathfrak{X}(M)\times\mathfrak{X}(M) \to \mathfrak{X}(M)$, written $(X, Y) \mapsto \nabla_X Y$ satisfying

(*i*) ∇ is linear over $C^\infty(M)$ in X: For all $f_1, f_2 \in C^\infty(M)$ and $X_1, X_2, Y \in \mathfrak{X}(M)$,

$$\nabla_{f_1X_1+f_2X_2}Y = f_1\nabla_{X_1}Y + f_2\nabla_{X_2}Y.$$

(*ii*) ∇ is linear over $\mathbb{R}$ in Y: For all $a_1, a_2 \in \mathbb{R}$ and $X, Y_1, Y_2 \in \mathfrak{X}(M)$,

$$\nabla_X a_1Y_1 + a_2Y_2 = a_1\nabla_X Y_1 + a_2\nabla_X Y_2.$$

(*iii*) ∇ satisfies the *product rule for connections*: For all $f \in C^\infty(M)$ and $X, Y \in \mathfrak{X}(M)$,

$$\nabla_X(fY) = f\nabla_X Y_1 + (Xf)Y.$$

Some authors (e.g., [**11**]) use a more general definition of a connection and reserve Definition 1.2.4 to mean an *affine connection*. As it turns out, given any Riemannian manifold (M, g), there is a cannonical choice (as described in Definition 1.2.5) of connection which is affine. Therefore, whenever we write ∇, it will henceforth only refer to this natural connection.

DEFINITION 1.2.5. Given any Riemannian manifold (M, g) the *Levi-Civita connection* $\nabla^{M,g}$ (or the *gradient*) is the unique connection that is symmetric and compatible with g. That is to say that $\nabla^{M,g} = \nabla$ satisfies

$$\nabla_X Y - \nabla_Y X \equiv [X, Y] \;\text{ for all } X, Y \in \mathcal{X}(M) \qquad \text{(Symmetry)}$$
$$\nabla_X\langle Y, Z\rangle = \langle\nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle \;\text{ for all } X, Y, Z \in \mathcal{X}(M) \qquad \text{(Compatibility)}$$

The existence and uniqueness of the Levi-Cevita connection is often refered to as the *Fundamental Theorem of Riemannian Geometry.* Given a coordinate system x^i, the *Christoffel symbols* Γ^m_{ij} are the components of $\nabla_{\partial_i}\partial_j$. That is, $\nabla_{\partial_i}\partial_j = \Gamma^m_{ij}\partial_m$.

By a slight abuse of notation, we also use ∇ to denote a differential operator of functions on a Riemannian manifold (and not just vector fields).

Definition 1.2.6. Given a Riemannian manifold (M, g) and $f : M \to \mathbb{R}$ is a smooth function, the *gradient* of f (denoted ∇f) is the vector field obtained from df by *raising an index* (i.e., $\nabla f = (df)^{\#}$). That is to say that

$$df_p(w) = \langle \nabla f|_p, w\rangle \quad \text{for all } p \in M,\ w \in T_pM$$

or that

$$\nabla f = g^{ij}\partial_i f\ \partial_j$$

in local coordinates x^i.

Another differential operator that we will be using throughout is the generalization of the Laplacian to a Riemannian manifold (M, g).

Definition 1.2.7. Given a Riemannian manifold (M, g), the *Laplace-Beltrami operator* (or simply the Laplacian) is a map $\Delta_{M,g} : C^\infty(M) \to C^\infty(M),\ f \mapsto \operatorname{div}(\nabla f)$ which is given in local coordinates x^i by the formula

$$\Delta_{M,g} f := \frac{1}{\sqrt{\det g}}\frac{\partial}{\partial x^j}\left(\sqrt{\det g}\, g^{ij}\frac{\partial}{\partial x^i}\right) f. \tag{1}$$

In the case of $(M, g) \cong (\mathbb{R}^n, \delta)$ (here δ is the Euclidean metric), equation (1) simplifies to the well-known Laplacian

$$\Delta_{\mathbb{R}^n,\delta} = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}.$$

The Laplace-Beltrami operator can be defined in terms of covariant derivatives, though formula (1) is sufficient for our purposes.

Remark 1.2.8. If $M \approx \mathbb{S}^2$, which is a special case of interest for us, it is evident using Green's identity that for any $g \in \operatorname{Met}(\mathbb{S}^2)$ and any $u \in C^\infty(\mathbb{S}^2)$, we have

$$\int_{\mathbb{S}^2} \Delta_g u\ dA_g = 0$$

and $\Delta_g u \equiv 0$ if and only if $u \equiv c$ is identically constant. These facts will be used in Proposition 2.2.6.

1.2.3. Curvature.

Since the Levi-Cevita connection is a cannonical property of the metric g, it allows us to define natural notions of the curvature of a Riemannian manifold. Each of the following definitions have physical interpretations but we will not delve into those here.

Definition 1.2.9. Let (M, g) be a Riemannian manifold with its Levi-Cevita connection ∇.

(i) The *Riemannian curvature tensor* (or simply the *curvature tensor*) is a $(0, 4)$-tensor field Rm given by its action on vector fields X, Y, Z, W as

$$\operatorname{Rm}(X, Y, Z, W) = \langle \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, W\rangle$$

or in coordinates as

$$\operatorname{Rm}_{ijkl} = g_{lm}\left(\partial_i\Gamma^m_{jk} - \partial_j\Gamma^m_{ik} + \Gamma^p_{jk}\Gamma^m_{jk} - \Gamma^m_{ij}\Gamma^m_{jp}\right).$$

(*ii*) The *Ricci curvature tensor* is a $(0,2)$-tensor field Ric obtained by tracing the curvature tensor. It is given by its action on vector fields as

$$\mathrm{Ric}(X,Y) = \mathrm{tr}\ \mathrm{Rm}(\cdot, X, Y, \cdot)$$

or in coordinates as

$$\mathrm{Ric}_{ij} = g^{km}\ \mathrm{Rm}_{kijm}.$$

(*iii*) The *scalar curvature* is a function R obtained by tracing the Ricci curvature tensor:

$$R = \mathrm{tr}\ \mathrm{Ric}(\cdot,\cdot) = g^{ij}\ \mathrm{Ric}_{ij}.$$

(*iv*) The *Gaussian curvature* of a Riemannian 2-manifold (M^2, g) is given by

$$K = \frac{\langle (\nabla_{\partial_2}\nabla_{\partial_1} - \nabla_{\partial_1}\nabla_{\partial_2})\partial_1, \partial_2}{\det g}$$

where ∂_1, ∂_2 is a local coordinate frame. A well known formula for the Gaussian curvature in an orthogonal parametrization of g (that is $g = E(u,v)du^2 + G(u,v)dv^2$) is

$$K = -\frac{1}{2\sqrt{EG}}\left(\frac{\partial}{\partial u}\frac{G_u}{\sqrt{EG}} + \frac{\partial}{\partial v}\frac{E_v}{\sqrt{EG}}\right). \tag{2}$$

The existence of orthogonal parametrizations for any 2-manifold follows from Theorem 1.2.3 and the fact that the Poincaré disc metric (in the standard disc coodinates) and the round sphere metric (in stereopgraphic projection coordinates) are both *locally conformal flat*. That is to say that around every $p \in M$, there is a neighborhood U of p and a smooth $w : U \to \mathbb{R}$ such that $(U, e^{2w}g)$ is *flat* (i.e., the Riemannian curvature tensor Rm vanishes). The class of metrics $g \in \mathrm{Met}(\mathbb{S}^2)$ with non-negative (positive) Gaussian curvature is denoted by $\mathcal{K}_{\geq 0}$ ($\mathcal{K}_{>0}$).

REMARK 1.2.10. Though we will not need it for this thesis, Gaussian curvature can be defined for manifolds of dimension $n \geq 3$ (see [**11**]). It is a fact that when $n = 2$, the Ricci and scalar curvature (Ric and R respectively) are related to the Gaussian curvature K by

$$\mathrm{Ric} = Kg,\ R = 2K. \tag{3}$$

Though curvature is a local property of the metric, it was long theorized that if one understands local properties of a manifold, then one can say something about a global property of the manifold. This next theorem is, in all likelihood, the most well-known result of that flavour. It relates the Gaussian curvature (a local property) to the Euler characteristic $\chi(M)$ (a global topological property).

THEOREM 1.2.11 (Gauss-Bonnet). *Suppose M is a compact Riemannian 2-manifold. Then*

$$\int_M K\ dA_g = 2\pi\chi(M).$$

COROLLARY 1.2.12. If g is a round metric on $\mathbb{S}^2$, then $K_g \equiv \frac{4\pi}{\mathrm{area}(g)}$.

PROOF. Let g be a round metric on $\mathbb{S}^2$ so that K is identically constant. By Theorem 1.2.11,

$$\mathrm{area}(g)K = \int_{\mathbb{S}^2} K\ dA_g = 2\pi\chi(\mathbb{S}^2) = 4\pi$$

and therefore $K \equiv \frac{4\pi}{\mathrm{area}(g)}$. $\square$

One of the main themes of this thesis is utilizing the relationships of various quantities of conformally related metrics. The following lemma concerns those quantities that will be of utmost importance in this thesis.

LEMMA 1.2.13. Let $h = e^{2w}g$ be a conformal equivalence between Riemannian metrics g and h on a 2-manifold M^2. Then

(i) $dA_h = e^{2w}dA_g$,
(ii) $\Delta_h = e^{-2w}\Delta_g$,
(iii) $K_h = e^{-2w}(K_g - \Delta_g w)$, and
(iv) $|\nabla^h f|_h^2 = e^{-2w}|\nabla^g f|_g^2$ for any $f \in C^\infty(M)$.

PROOF. Items (i) and (iv) are a straightforward application of their definitions:

$$dA_h = \sqrt{\det h}\, d\mathbf{x} = \sqrt{\det(e^{2w}g)}\, d\mathbf{x} = \sqrt{e^{4w}\det g}\, d\mathbf{x} = e^{2w}\sqrt{\det g}\, d\mathbf{x} = e^{2w}dA_g,$$

and

$$|\nabla^h f|_h^2 = h^{ij}f_i f_j = (e^{2w}g)^{ij}f_i f_j = e^{-2w}g^{ij}f_i f_j = e^{-2w}|\nabla^g f|_g^2.$$

To prove (ii), we simply plug in $h = e^{2w}g$ in equation (1):

$$\begin{aligned}
\Delta_h &= -\frac{1}{\sqrt{\det h}}\frac{\partial}{\partial x^j}\left(\sqrt{\det h}\, h^{ij}\frac{\partial}{\partial x^i}\right)\\
&= -\frac{1}{\sqrt{\det(e^{2w}g)}}\frac{\partial}{\partial x^j}\left(\sqrt{\det(e^{2w}g)}\,(e^{2w}g)^{ij}\frac{\partial}{\partial x^i}\right)\\
&= -e^{-2w}\frac{1}{\sqrt{\det g}}\frac{\partial}{\partial x^j}\left(e^{2w}\sqrt{\det g}\, e^{-2w}g^{ij}\frac{\partial}{\partial x^i}\right)\\
&= -e^{-2w}\frac{1}{\sqrt{\det g}}\frac{\partial}{\partial x^j}\left(\sqrt{\det g}\, g^{ij}\frac{\partial}{\partial x^i}\right) = e^{-2w}\Delta_g.
\end{aligned}$$

To prove (iii), we use orthogonal local coordinates around $p \in M$ so that $g = E(u,v)du^2 + G(u,v)dv^2$ and $h = (e^{2w}E)du^2 + (e^{2w}G)dv^2$. Using equation (2), we have

$$\begin{aligned}
K_h &= -\frac{1}{2\sqrt{(e^{2w}E)(e^{2w}G)}}\left(\frac{\partial}{\partial u}\frac{(e^{2w}G)_u}{\sqrt{(e^{2w}E)(e^{2w}G)}} + \frac{\partial}{\partial v}\frac{(e^{2w}E)_v}{\sqrt{(e^{2w}E)(e^{2w}G)}}\right)\\
&= -e^{-2w}\frac{1}{2\sqrt{EG}}\left(\frac{\partial}{\partial u}\frac{2w_uG + G_u}{\sqrt{EG}} + \frac{\partial}{\partial v}\frac{2w_vE + E_v}{\sqrt{EG}}\right)\\
&= e^{-2w}K_g - e^{-2w}\frac{1}{\sqrt{EG}}\left(\frac{\partial}{\partial u}\frac{w_uG}{\sqrt{EG}} + \frac{\partial}{\partial v}\frac{w_vE}{\sqrt{EG}}\right). \qquad (4)
\end{aligned}$$

Expanding the first term in the bracket yields

$$\begin{aligned}
\frac{\partial}{\partial u}\frac{w_uG}{\sqrt{EG}} &= \frac{\sqrt{EG}(w_{uu}G + w_uG_u) - w_uG\frac{1}{2\sqrt{EG}}(EG)_u}{EG}\\
&= \frac{w_{uu}G}{\sqrt{EG}} + \frac{w_uG_u}{\sqrt{EG}} - \frac{w_uG_u}{2\sqrt{EG}} - \frac{w_uE_uG}{2E\sqrt{EG}}\\
&= \frac{w_{uu}G}{\sqrt{EG}} + \frac{w_uG_u}{2\sqrt{EG}} - \frac{w_uE_uG}{2E\sqrt{EG}}.
\end{aligned}$$

The other term is symmetric. So

$$\begin{aligned}
\frac{1}{\sqrt{EG}}\left(\frac{\partial}{\partial u}\frac{2w_uG}{\sqrt{EG}}+\frac{\partial}{\partial v}\frac{2w_vE}{\sqrt{EG}}\right) &= \frac{w_{uu}}{E}+\frac{w_uG_u}{2EG}-\frac{w_uE_u}{2E^2}+\frac{w_{vv}}{G}+\frac{w_vE_v}{2EG}-\frac{w_vG_v}{2G^2}\\
&= g^{uu}w_{uu}-g^{vv}\Gamma^u_{vv}w_u-g^{uu}\Gamma^u_{uu}w_u+g^{vv}w_{vv}-g^{uu}\Gamma^v_{uu}w_v-g^{vv}\Gamma^v_{vv}w_v\\
&= g^{ij}w_{ij}-g^{ij}\Gamma^k_{ij}\partial_k w=\mathrm{tr}_g(\nabla^2 w)=\Delta w.
\end{aligned}$$

Plugging this back into equation (4), we have $K_h = e^{-2w}(K_g - \Delta_g w)$ as desired.

□

1.2.4. Extrinsic Geometry.

We have discussed several different methods of measuring the curvature of a manifold but until now, they all measure the curvature of a manifold as an intrinsic property. If we have a natural inclusion of Riemannian manifolds $\iota : \Sigma^m \hookrightarrow M^n$, one may also ask the question of how Σ curves "relative to M". We will make this notion more precise in the special case where Σ is a *hypersurface* in M (that is, $m = n-1$). Henceforth, given any $p \in \Sigma$, we identify p with $\iota(p)$ and $T_p\Sigma$ as a subspace of $T_{\iota(p)}M$ via the pushforward $d\iota_p$.

DEFINITION 1.2.14. Let (M^n, g) be a Riemannian manifold and Σ^{n-1} a hypersurface in M with the induced metric h from M. Let ν be a choice of unit normal on Σ.

(i) The *second fundamental form* of Σ is a a map $\mathfrak{X}(\Sigma)\times\mathfrak{X}(\Sigma)\to\Gamma(NM)$ given by

$$II(X,Y) = (\nabla^M_X Y)^\perp$$

where $Z^\perp$ gives the perpendicular portion of Z to Σ.

(ii) The *scalar second fundamental form* with respect to ν is a symmetric covariant 2-tensor field given by its action on $X, Y \in \mathfrak{X}(\Sigma)$ as

$$\rho = \langle \nu, II(X,Y)\rangle.$$

Some authors refer to the scalar second fundamental form simply as the second fundamental form but we use this notation to remain consistent with [**11**]. One should also note that if ν is replaced by $-\nu$, then ρ changes signs but it otherwise independent of all choices.

(iii) The *mean curvature* of Σ with respect to ν is a function on Σ defined by

$$H = \mathrm{tr}_h\rho.$$

Again, if ν is replaced by $-\nu$, then H changes signs as well but is otherwise independent of all choices. The definition of mean curvature is an unfortunate source of inconsistency. Lee [**11**] defines the mean curvature with an extra factor of $\frac{1}{n-1}$ while many papers in the field (see [**6**],[**13**], [**17**]) use the definition given above. We chose this definition so our results are more easily compared to those in the literature.

1.3. General Relativity

In this section, we summarize the main ideas of mathematical general relativity which pertain directly to the contents of this thesis. As in section §1.2, this is by no means intented to be a complete overview of the subject matter. A more complete treatment can be found in [**22**] and [**21**] and the references therein.

1.3.1. Spacetime and Einstein's Field Equation.

A *Lorentzian manifold* (L^n, h) is a n-dimensional pseudo-Riemannian manifold with $(n-1,1)$ *signature*. That is to say, for any $p \in L$, there is some choice of coordinate x^i in which the metric h evaluated at p is given by the matrix

$$h(p)_{ij} = \begin{cases} 1 & i = j < n \\ -1 & i = j = n \\ 0 & \text{else.} \end{cases} .$$

An immediate consequence of h not being positive definite is that not all non-zero vectors at a particular point have positive "length". It is natural then to decompose the tangent space as

$$T_pL = \{X \in T_pL : \langle X, X\rangle_h > 0\} \,\cup\, \{X \in T_pL : \langle X, X\rangle_h = 0\} \,\cup\, \{X \in T_pL : \langle X, X\rangle_h < 0\}.$$

Vectors satisfying $\langle X, X\rangle_h = 0$ are said to be *null* and form a cone called a "lightcone". Vectors satisfying $\langle X, X\rangle_h < 0$ are those which are contained inside the lightcone (called *time-like*). On the other hand, those vectors satisfying $\langle X, X\rangle_h > 0$ are those which are not contained inside the lightcone (called *space-like*). Submanifolds $M \subset L$ with the induced metric are said to be null, timelike, or spacelike if the tangent space of any point $p \in M$ is composed of null, timelike, or spacelike vectors respectively.

Lorentzian manifolds arise naturally in Einstein's general theory of relativity in which spacetime is a 4-dimensional Lorentzian manifold (L^4, h). In this setting, photons travel along null curves and massive particles travel along timelike curves (i.e. slower than light). It is sometimes useful to think of a spacelike hypersurfaces (or spacelike *slices*) as being time-slices for a particular observer though we warn the reader that this need not always be the case. One of Einstein's assertions when he discovered this field is that matter curves spacetime according to the equation

$$\mathrm{Ric}_L - \frac{1}{2} R_L h = 8\pi T \tag{5}$$

which is known as Einstein's field equation(s). Here T is the stress-energy 2-tensor representing the matter field. The field of mathematical general relativity is the study of this non-linear wave equation and its solutions.

1.3.2. Initial Data Sets and Assumptions.

Full solutions to equation (5) are very difficult to understand. Such a solution (L^4, h) would characterize an entire (theoretical) universe and not simply a "snapshot" of what it looks like. The task of finding such solutions, is therefore extremely daunting. It is much simpler to split spacetime into a 3-dimensional spacelike hypersurface (M^3, γ) and whatever extra information is required to tell how M "evolves in time" persae. It turns out that the only extra information that is required is the scalar second fundamental form of M inside of L (which we denote by p). Of course, not all tuples (M^3, γ, p) will result in a solution (L, h) to equation (5). In order for this to happen, we further require (M^3, γ, p) to satisfy the *Einstein constraint equations*:

$$\begin{cases} 16\pi\mu & = R_M - |p|_M^2 + (\mathrm{tr}_M p)^2 \\ 8\pi J & = \mathrm{div}_M p - \mathrm{tr}_M p. \end{cases} \tag{6}$$

Here η is a unit normal vector to M, $\iota : M \hookrightarrow L$ is the inclusion map, $\mu = T(\eta, \eta)$ is called the *energy density*, and $J = \iota^*(T(\eta, \cdot))$ is called the *energy flux density*. With this in mind, we have the following definition.

DEFINITION 1.3.1 (Initial data sets, cf. Schoen-Yau [**21**]). An *initial data set* is a tuple (M^3, γ, p) which satisfies the Einstein constraint equations (6).

The study of 3-dimenional space-like hypersurfaces (M^3, γ) is often focused on their intrinsic Riemannian geometry, while little emphasis is placed on the ambient spacetime beyond the general relativistic context it provides. For this reason, it is often simply assumed that $p \equiv 0$ in (6), which is the so called time symmetric setting. We will also operate under this assumption for this thesis, and will henceforth ignore p.

Even with the assumption of time-symmetry (i.e., $p \equiv 0$), many initial data sets (M^3, γ) are not considered very physically accurate. It is for this reason that when studying initial data sets, one often imposes what is called an *energy condition* which attempts to capture the notion that "energy should be positive", despite the fact that this notion is not well defined (see section 1.3.3). In this thesis, we will adopt the *dominant energy condition* (DEC) on our initial data sets (M, γ) which is the requirement that $-T(\eta, \cdot)^{\#}$ is either future-pointing or null. Physically speaking, DEC is the condition that mass/energy cannot be observed to flow faster than light. While this assumption satisfies our physical intuition, it is seemingly quite puzzling from a mathematical perspective. Despite this, one can show that under the TS assumption, the DEC is equivalent with (M, γ) having non-negative scalar curvature (i.e., $R_\gamma \geq 0$).

In addition to the TS and DEC assmptions, we will make the assumption that our initial data sets (M, γ) are *asymptotically flat* (Definition 1.3.2). This captures the physical idea that (M, γ) is a slice of spacetime which models an isolated system.

DEFINITION 1.3.2 (Asymptotically flat, cf. Wald [**22**]). A Riemannian 3-manifold (M, γ) is said to be *asymptotically flat* (AF) if it satisfies the following:

(*i*) There exists a compact $K \subset M$ such that $M \setminus K \approx \mathbb{R}^3 \setminus \{|x| \leq 1\}$,

(*ii*) in the coordinates coming from the diffeomorphism in (*i*), the metric γ satisfies

$$|\gamma - \delta| + r|\partial\gamma| + r^2|\partial^2\gamma| \leq Cr^{-1}$$

where $C > 0$ and δ is the standard flat metric on $\mathbb{R}^3 \setminus \{|x| \leq 1\}$, and

(*iii*) $R_\gamma \in L^1(M)$.

Remark 1.3.3. In part (i) of Definition 1.3.2, one usually allows

$$M \setminus K \approx \sqcup_{j=1}^{m} E_j$$

where $E_j = \mathbb{R}^3 \setminus \{|x| \leq 1\}$ for all j (called the *ends* of M). We chose to omit this from Definition 1.3.2 as the only initial data sets we will be considering have a single end.

One trivial example of an asymptotically flat manifold is $(\mathbb{R}^3, \delta)$ which appears as initial datum for the Minkowski space $(\mathbb{R}^3 \times \mathbb{R}, \delta - dt^2)$, with $T \equiv 0$. We end this section with a less trivial, but more useful example of an asymptotically flat initial data set.

Example 1.3.4 (mass-m Schwarzschild). Let g_* denote a round metric on $\mathbb{S}^2$ with area 4π. The *mass-m Schwarzschild* Riemannian manifold

$$\mathcal{S}_m := \left(\mathbb{S}^2 \times (2m, \infty), r^2 g_* + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 \right)$$

is asymptotically flat (with one end) and scalar flat (i.e., $R \equiv 0$). This manifold is aptly named "mass-m" as the ADM mass (see Definition 8) is precisely equal to m. For any $r_0 > 2m$, we call the restriction $\mathcal{S}_m|_{[r_0,\infty)}$ an *outer Schwarzschild region.* The mass-m Schwarzschild manifold appears as a time symmetric initial data set for the *Lorentzian mass-m Schwarzschild* manifold

$$\left(\mathbb{S}^2 \times (2m, \infty) \times \mathbb{R}, r^2 g_* + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - \left(1 - \frac{2m}{r}\right) dt^2 \right)$$

which is a non-trivial static solution to equation (5) with $T \equiv 0$.

1.3.3. Measuring Mass.

One of the ways which general relativity differs from Newtonian mechanics is in the way that mass is measured. In general relativity, energy density and energy flux density are not well defined in the sense that they depend on the observers velocity. Because of this, there are many competing notions of mass/energy. We will introduce several of these notions of mass here while discussing how they compare. For the remainder of this section, (M^3, γ) is always AF with one end satisfying the DEC and TS assumptions. We warn the reader that under these conditions, some of the definitions seen here are somewhat simplified but will be sufficient for our purposes.

Definition 1.3.5 (Hawking mass, cf. Hawking [**9**]). For a closed 2-submanifold (Σ, g) of (M^3, γ), the *Hawking mass* of Σ with mean curvature H is

$$\mathfrak{m}_H(\Sigma, g, H) := \sqrt{\frac{\text{area}(\Sigma, g)}{16\pi}} \left(1 - \frac{1}{16\pi} \int_\Sigma H^2 \, dA_g \right). \tag{7}$$

Hawking mass is an example of *quasi-local* mass in the sense that it measures quantities defined within a finite region of space.

The case of particular interest for us is when $\Sigma \approx \mathbb{S}^2$ and H is constant. In which case, we may write r_g to denote the radius of g (that is $r_g = \sqrt{\frac{\text{area}(\mathbb{S}^2, g)}{4\pi}}$) and so equation (7) simplifies to

$$\mathfrak{m}_H(\mathbb{S}^2, g, H) = \frac{r_g}{2} \left(1 - \frac{H^2 r_g^2}{4}\right).$$

One physical interpretation of the Hawking mass of a 2-sphere is the measurement of the degree to which ingoing and outgoing lightrays are bent by the mass enclosed. We remark here that, contrary to physical intuition, the Hawking mass is not necessarily positive.

The next notion of mass we will be considering was originally formulated by Arnowitt, Deser, and Misner in 1959.

Definition 1.3.6 (ADM mass, cf. Arnowitt-Deser-Misner [**1**]). Let $K \subset M$ be as in Definition 1.3.2. That is to say that K is compact and $M \setminus K \approx \mathbb{R}^3 \setminus \{|x| \leq 1\}$. Let x^i be coordinates on M coming from this diffeomorphism. The *ADM mass* of (M, γ) is

$$\mathfrak{m}_{ADM}(M, \gamma) := \frac{1}{16\pi} \lim_{R \to \infty} \int_{|x|=R} (\partial_j \gamma_{ij} - \partial_i \gamma_{jj}) \frac{x^j}{R}\, d\sigma_R. \tag{8}$$

Here $d\sigma_R$ is the area form induced by the Euclidean metric on $\{|x| = R\} \subset \mathbb{R}^3$.

Assuming appropriate decay conditions (such as our AF assumption), Robert Bartnik showed (Lemma 2.1 in [**2**]) that the ADM mass is well-defined in the sense that the integral in (8) is independent of the choice of coordinate. Further assuming a completeness condition, Schoen and Yau proved that $\mathfrak{m}_{ADM}(M, \gamma)$ is non-negative.

Theorem 1.3.7 (Positive mass theorem, Shoen-Yau [**20, 21**]). *Let (M^3, γ) be asymptotically flat with non-negative scalar curvature. If (M^3, γ) is complete without boundary, or with boundary consisting of closed surfaces whose mean curvature vector does not point outside M, then $\mathfrak{m}_{ADM}(M, \gamma) \geq 0$ with equality if and only if $(M^3, \gamma) \cong (\mathbb{R}^3, \delta)$.*

In this thesis, we will be concerned with the special case $M \approx \mathbb{S}^2 \times [0, \infty)$. The TS/DEC conditions equate to (M, γ) having non-negative scalar curvature. If (M, γ) is asymptotically flat and satisfies a non-degeneracy condition (see Definition 1.3.8), we can relate the ADM mass of M to the Hawking mass of $\partial M \cong (\mathbb{S}^2, g)$ by the Riemannian Penrose inequality (Theorem 1.3.9).

Definition 1.3.8 (Outerminimizing, cf. Bray [**5**]). Let (M^3, γ) be a Riemannian manifold with $M \approx \mathbb{S}^2 \times [0, \infty)$. We say that $\partial M = \mathbb{S}^2 \times \{0\}$ is *outerminimizing* if there exists no surface $\Sigma \subset M$ separating ∂M from infinity with less area.

Theorem 1.3.9 (Riemannian Penrose inequality, Bray [**5**] and Huisken-Ilmanen [**10**]). *Let $(M^3 \approx \mathbb{S}^2 \times [0, \infty), \gamma)$ be asymptotically flat with non-negative scalar curvature such that $\partial M \cong (\mathbb{S}^2, g)$ is outerminimizing with mean curvature H with respect to the inward pointing unit normal. Then*

$$\mathfrak{m}_{ADM}(M^3, \gamma) \geq \mathfrak{m}_H(\mathbb{S}^2, g, H) = \sqrt{\frac{area(\mathbb{S}^2, g)}{16\pi}} \left(1 - \frac{1}{16\pi} \int_\Sigma H^2\, dA_g\right).$$

Remark 1.3.10. Theorem 1.3.9 was originally proved in a much more general context: Bray [**5**] proved a version in which ∂M could be a disconnected surface but only allows $H = 0$, while Huisken-Ilmanen [**10**] proved a version for general H but requires ∂M to be connected. Since we will be always assuming that $M^3 \approx \mathbb{S}^2 \times [0, \infty)$, we only state this simplified version that will suffice for our purposes.

It is natural to wonder: given any $g \in \mathrm{Met}(\mathbb{S}^2)$ and a smooth function H on $\mathbb{S}^2$, is it possible to construct a manifold $(M \approx \mathbb{S}^2 \times [0, \infty), \gamma)$ which satisfies the hypothesis of Theorem 1.3.9? This is the motivation for the following definition.

Definition 1.3.11 (Admissible extension, cf. Cabrera *et al.* [**6**]). Let $g \in \mathrm{Met}(\mathbb{S}^2)$ and let H be (smooth) function on $\mathbb{S}^2$. An *admissible extension* of g is any asymptotically flat $(M^3 \approx \mathbb{S}^2 \times [0,\infty), \gamma)$ such that

(*i*) (M, γ) has non-negative scalar curvature,
(*ii*) $\gamma|_{\{t=0\}} \equiv g$,
(*iii*) ∂M has mean curvature H relative to the inward pointing unit normal, and
(*iv*) ∂M is outerminimizing.

It is highly non-obvious whether or not admissible extensions exist for arbitrary $g \in \mathrm{Met}(\mathbb{S}^2)$ and $H \in C^\infty(\mathbb{S}^2)$. If (M^3, γ) is such an extension, Theorem 1.3.9 guarantees that $\mathfrak{m}_{ADM}(M^3, \gamma) \geq \mathfrak{m}_H(\mathbb{S}^2, g, H)$. In this case, it is natural to ask the question: "how close can $\mathfrak{m}_{ADM}(M^3, \gamma)$ get to its known lower bound of $\mathfrak{m}_H(\mathbb{S}^2, g, H)$?". This question is the motivation for our last notion of mass that will be discussed here.

Definition 1.3.12 (Barnik mass, Bartnik [**3**]). Let $g \in \mathrm{Met}(\mathbb{S}^2)$ and $H \in C^\infty(\mathbb{S}^2)$. If admissible extensions of g exist, then the *Barnik mass* of the triple $(\mathbb{S}^2, g, H)$ is

$$\mathfrak{m}_B(\mathbb{S}^2, g, H) := \inf\{\mathfrak{m}_{ADM}(M^3, \gamma) : (M^3, \gamma) \text{ is an admissible extension}\}.$$

An immediate corrolary to Theorem 1.3.9 is that $\mathfrak{m}_B(\mathbb{S}^2, g, H) \geq \mathfrak{m}_H(\mathbb{S}^2, g, H)$ but despite its simple definition, finding upper bounds for the Bartnik mass has proven to be quite elusive. Not only are admissible extensions often quite involved to construct, but the ADM mass of an arbitrary admissible extension can be very difficult to calculate, nevermind the infemum of all such quantities. Despite this, there have been some progress made in recent years. Mantoulidis and Schoen [**13**] introduced a groundbreaking method in which they first construct a collar extension (similar to our generic notion of a collar that is introduced in section §2.1), and then "glue" on an outer Scwarzschild region (a more simplified version of this is gluing process is represented by Lemma 2.1.7 presented in section §2.1 as well). Using this method, they were able to prove the following result.

Theorem 1.3.13 (Theorem 2.1 in [**13**]). *If $g \in Met(\mathbb{S}^2)$ with the property that the first eigenvalue of the operator $-\Delta_g + K_g$ is positive, then*

$$\mathfrak{m}_B(\mathbb{S}^2, g, 0) = \mathfrak{m}_H(\mathbb{S}^2, g, 0).$$

Theorem 3.1.1 in this thesis is an extension of this result to boundary case of when g is a metric with the property that the first eigenvalue of the operator $-\Delta_g + K_g$ is equal to zero. The significance of these spaces of metrics in the general relativity context will be discussed in the next subsection.

1.3.4. Horizons.

General relativity is, roughly speaking, the study of how objects with mass bend spacetime itself. When an objects mass is large enough compared to the objects size (commonly refered to as its *Schwarzschild radius*), a black hole forms (at least theoretically). These are aptly called "black" because around them, a boundary forms beyond which events cannot affect an observer. This boundary is called the *event horizon* of a black hole and have been an object of great interest in physics and pop science.

Since an initial data set (M^3, γ, p) is essentially a "cross section" of a 4-dimensional spacetime (L^4, h), a natural object for us to study is the relative cross section of an event event horizon of a black hole. Such cross sections are typical examples of *apparent horizons* which are outermost closed surfaces in the initial data set whose future outgoing null normal field have zero divergence in the ambient spacetime. Under the DEC, Hawking's Theorem on the topology of black holes [**9**] states that an apparent horizons in an asymptotically flat (M, γ) is a topological 2-sphere. In this case, write g for the induced metric from γ on the apparent horizon $\Sigma \approx \mathbb{S}^2 \subset M$. Under the further assumption of TS, Huisken and Ilmanen [**10**] showed that an apparent horizon $\Sigma \subset M$ can be characterized as a stable minimal 2-sphere. Here *stability* is the condition that

$$(9) \qquad \int_{\mathbb{S}^2} \varphi L_g \varphi \, dA_g = \int_{\mathbb{S}^2} \varphi(-\Delta_g + K_g)\varphi \, dA_g \geq \frac{1}{2}\int_{\mathbb{S}^2} (R_\gamma + |\rho|^2)\varphi^2 dA_g \;\text{ for all } \varphi \in C^\infty(\mathbb{S}^2)$$

where ρ is the scalar second fundamental form on $(\mathbb{S}^2, g)$ in (M, γ). Since the DEC and TS assumptions imply that $R_\gamma \geq 0$, we find that the operator $L_g := -\Delta_g + K_g$ is non-negative on $(\mathbb{S}^2, g) \cong \partial M$. In particular, the first eigenvalue of L_g (denoted $\lambda_1(g)$) is non-negative on any apparent horizon $(\mathbb{S}^2, g)$. Therefore, we have the inclusion

$$AH \subset \{g \in \mathrm{Met}(\mathbb{S}^2) : \lambda_1(g) \geq 0\} =: \mathscr{M}_{\geq 0}$$

where

$$AH := \{g \in \mathrm{Met}(\mathbb{S}^2) \text{ coming from an apparent horizon in DEC/TS/AF setting}\}.$$

While the space of apparent horizons AH is defined somewhat extrinsicly in terms of an ambient manifold M, the space $\mathscr{M}_{\geq 0}$ is defined purely instrinsicly on $\mathbb{S}^2$. In this sense, the opposite inclusion would provide a complete instrinsic characterization of AH, though this inclusion is far from obvious. The proof of Theorem 1.3.13 (seen in [**13**]) actually proves the (partial) reverse inclusion

$$AH \supset \{g \in \mathrm{Met}(\mathbb{S}^2) : \lambda_1(g) > 0\} =: \mathscr{M}_{> 0}$$

since in the admissible extensions they construct, $\partial M \cong (\mathbb{S}^2, g)$ satisfies the stability condition given in (9). Thus, whenever $g \in \mathscr{M}_{>0}$, Mantoulidis and Schoen [**13**] proved that $(\mathbb{S}^2, g)$ appears as an apparent horizon with Bartnik mass precisely equal to the Hawking mass (in the AF/TS/DEC setting). One of the main contributions of this thesis is to extend Mantoulidis' and Schoen's result to include the "boundary case" of $g \in \mathscr{M}_{=0}$. We therefore establish the following.

THEOREM (Theorem 3.1.1). *For any $g \in \mathscr{M}_{=0}$, $g \in AH$ and*

$$\mathfrak{m}_B(\mathbb{S}^2, g, 0) = m_H(\mathbb{S}^2, g, 0).$$

Thus, when the results of Mantoulidis and Schoen are combined with Theorem 3.1.1, we have the following.

THEOREM 1.3.14. *Apparent horizon in the AF/TS/DEC setting are precisely given by the class of metrics $\mathscr{M}_{\geq 0}$. Moreover, the Bartnik mass of such a horizon is*

$$\mathfrak{m}_B(\mathbb{S}^2, g, 0) = \mathfrak{m}_H(\mathbb{S}^2, g, 0) = \sqrt{\frac{area(\mathbb{S}^2, g)}{16\pi}}.$$

CHAPTER 2

Preliminaries

Since our main theorems are bounds on the Barnik mass for specific classes of metrics, their proofs require constructing admissible extensions of metrics g on $\mathbb{S}^2$. To construct such an extension, we use a technique first introduced by Mantoulidis and Schoen in which we construct "collar extensions" of a given metric, then we "glue" on an outer Schwarzschild region (as illustrated below). Section §2.1 is devoted to the construction and the general properties of collar extensions. At the end of section §2.1, we make precise the notion of "gluing" a collar to an outer Schwarzschild region.

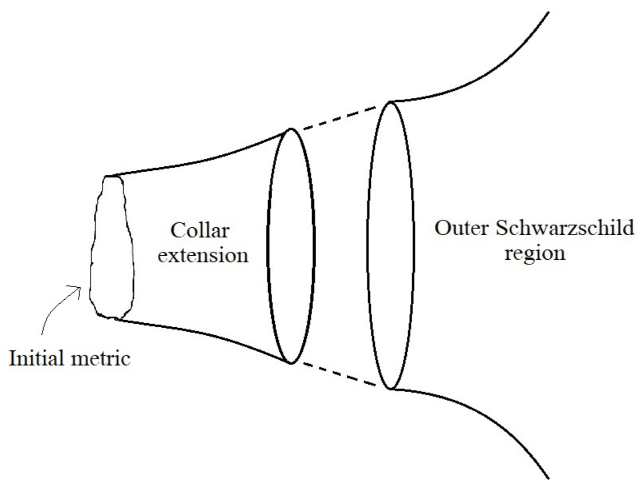

Figure 1. Gluing an outer Schwarzschild region to a g-collar

We have already seen in section §1.3.4 that the class of metrics $\mathscr{M}_{\geq 0}$ (that is the class of metrics $g \in \mathrm{Met}(\mathbb{S}^2)$ with non-negative first eigenvalue of the operator $L_g = -\Delta_g + K_g$) arises naturally in the study of apparent horizons in the TS/DEC/AF setting. Section §2.2 is devoted to better understanding the operator L_g and its eigenvalues. Proposition 2.2.6 proves the existence of metrics $g \in \mathrm{Met}(\mathbb{S}^2)$ satisfying $\lambda_1 = 0$, which in particular shows that Theorem 3.1.1 is not vacuous.

In section §2.3 in this chapter, we discuss various properties of the classes of metrics $\mathscr{M}_{\geq 0}$ and $\mathcal{K}_{\geq 0}$. Of most importance, given a metric $g \in \mathscr{M}_{\geq 0}$ or $\mathcal{K}_{\geq 0}$ we construct g-admissible paths $t \mapsto g(t)$ (see Definition 2.1.1) satisfying

$$\lambda_1(g(t)) \geq c\,t \quad \text{or} \quad \min_{\mathbb{S}^2 \times \{t\}} K_{g(t)} \geq c\,t$$

respectively. These estimates play an important role in the proofs of Theorems 3.1.1 and 3.1.2.

2.1. Collar Extensions

Definition 2.1.1. Given $g \in \mathrm{Met}(\mathbb{S}^2)$, a path of metrics $t \mapsto g(t)$ for $t \in [0,1]$ is called *g-admissible* if

(*i*) $t \mapsto g(t)$ is smooth,
(*ii*) $g(0) = g$,
(*iii*) $g(1)$ is round, and
(*iv*) $\frac{d}{dt} dA_{g(t)} \equiv 0$.

Condition (*iv*) states that $g(t)$ has pointwise constant area form which is equivalent with the path $t \mapsto g(t)$ satisfying $\mathrm{tr}_{g(t)} g'(t) \equiv 0$ (which we henceforth write simply as $\mathrm{tr}_g g' \equiv 0$). To see why this is true, note that

$$\begin{aligned} \mathrm{tr}_{g(t)} g'(t) = g(t)^{ij} g'(t)_{ij} &= \frac{1}{\det(g(t))} \left[g(t)_{11} g'(t)_{22} + g(t)_{22} g'(t)_{11} - 2 g(t)_{12} g'(t)_{12} \right] \\ &= \frac{\frac{d}{dt} \det(g(t))}{\det(g(t))} = 2 \frac{\frac{d}{dt} dA_{g(t)}}{\sqrt{\det(g(t))}}. \end{aligned}$$

So $\mathrm{tr}_g g' \equiv 0$ if and only if $\frac{d}{dt} dA_{g(t)} \equiv 0$.

If condition (*iv*) were excluded from Definition 2.1.1, it would be clear that such paths always exist. Indeed, given any $g \in \mathrm{Met}(\mathbb{S}^2)$ we can write $g = e^{2w} g_*$ where g_* is round by the uniformization theorem (Theorem 1.2.3) and the simple path $e^{2(1-t)w} g_*$ would satisfy conditions (*i*)-(*iii*). Condition (*iv*) however is significantly more complicated to resolve and the existence of such paths is due to Mantoulidis and Schoen. We include their proof here for ease of reference later.

Proposition 2.1.2 (Lemma 1.2 in [**13**]). For any $g \in \mathrm{Met}(\mathbb{S}^2)$, g-admissible paths exist.

Proof. Fix some $g \in \mathrm{Met}(\mathbb{S}^2)$. By uniformization (Theorem 1.2.3) we may write $g = e^{2w} g_*$ for some round metric g_* with area 4π. Fix a smooth decreasing function $\zeta : [0,1] \to [0,1]$ with $\zeta(0) = 1$ and $\zeta(1) = 0$. Then $t \mapsto e^{2\zeta(t) w(x)} g_*$ is a smooth path from g to the round metric g_*. Define an auxiliary function $a : [0,1] \to \mathbb{R}$ as the solution to the initial value problem

$$\begin{cases} a'(t) &= -\zeta'(t) \int_{\mathbb{S}^2} w(x) dA_{e^{2\zeta w} g_*} \\ a(0) &= 0. \end{cases}$$

That is to say that the path $t \mapsto h_t := e^{2\zeta(t) w(x) + 2a(t)} g_*$ satisfies $h_0 = g$ and

$$\frac{d}{dt} \mathrm{area}(\mathbb{S}^2, e^{2\zeta(t) w(x) + 2a(t)} g_*) = 0.$$

Now having fixed the metrics h_t, for each t, let X_t be such that $\mathrm{div}_{h_t} X_t = -2(\zeta'(t) w + a'(t))$. Take ϕ_t to be the integral flow along X_t and consider $g(t) = \phi_t^* h(t)$. Property (*iv*) follows from

$$\begin{aligned} \frac{d}{dt} dA_{g(t)} = \frac{d}{dt} \phi_t^* dA_{h(t)} &= \phi_t^* \left[\frac{d}{dt} dA_{h(t)} + \mathscr{L}_{\dot{\phi}_t} dA_{h(t)} \right] \\ &= \phi_t^* \left[\frac{1}{2} \mathrm{tr}_{h(t)} \dot{h}(t) dA_{h(t)} + \mathrm{div}_{h(t)} \dot{\phi}_t dA_{h(t)} \right] \\ &= \phi_t^* \left[2(a'(t) + \zeta'(t) w) + \mathrm{div}_{h(t)} \dot{\phi}_t \right] dA_{h(t)} = 0. \end{aligned}$$

∎

REMARK 2.1.3. In the preceeding proof, if we further require ζ to be identically constant on an interval, then $a(t)$ would also be constant there by its definition and consequently, the entire path $t \mapsto g(t)$ would be constant on this interval. We will utilize this in particular when the interval in question is a neighborhood of $t = 0$ or $t = 1$.

Associated to any g-admissible path, are two quanitities which have been discussed regularly in the literature (e.g., [**6**], [**14**],[**17**]). Although our results will only use them as a method of notation (or bookkeeping if you will), we include their definitions here for context.

DEFINITION 2.1.4. Associated to any g-admissible path are constants α and β which are defined as

$$\alpha = \frac{1}{4} \max_{\mathbb{S}^2 \times [0,1]} |g'|_g^2, \quad \text{and } \beta = r^2 \min_{\mathbb{S}^2 \times [0,1]} K_{g(t)}.$$

Here $r_{g(t)} = \sqrt{\frac{\text{area}(g(t))}{4\pi}}$ is the *radius* of the metric $g(t)$ (which is necessarily constant since $\text{tr}_g g' \equiv 0$ is equivalent with $dA_{g(t)}$ being identically constant). It is clear that $\alpha \geq 0$ with equality if and only if $t \mapsto g(t)$ is a constant path. Moreover, the Gauss-Bonnet theorem (Theorem 1.2.11) gives

$$\int_{\mathbb{S}^2} r^2 K_{g(t)}\, dA_{g(t)} = 4\pi r^2 = \int_{\mathbb{S}^2} 1\, dA_{g(t)}.$$

So $0 < \beta \leq 1$ with $\beta = 1$ if and only if $g(t)$ is round for all t.

DEFINITION 2.1.5 (Collar Extension). Given $g \in \text{Met}(\mathbb{S}^2)$, a *collar extension of* g (or a g-*collar* or simply a *collar*) is the manifold $M = \mathbb{S}^2 \times [0, 1]$ equipped with any Riemannian metric γ of the form

$$\gamma = E(t)g(t) + v(x,t)^2 dt^2$$

satisfying the following properties:

(*i*) $E : [0,1] \to \mathbb{R}_{>0}$ and $v : M \to \mathbb{R}_{>0}$ are smooth,
(*ii*) $t \mapsto g(t)$ is a g-admissible path,
(*iii*) $(\mathbb{S}^2 \times \{0\}, \gamma|_{\{t=0\}})$ is isometric to $(\mathbb{S}^2, g)$, and
(*iv*) (M, γ) has non-negative scalar curvature R_γ.

We remind the reader that conditions (*iii*) and (*iv*) above are exactly conditions (*i*) and (*ii*) in the definition of an admissible extension (Definition 1.3.11)

REMARK 2.1.6. If we further require that $E' > 0$ in Definition 2.1.5, then $(\mathbb{S}^2 \times \{0\}, \gamma|_{\{t=0\}}) \cong (\mathbb{S}^2, g)$ is "outerminimizing in M" in the sense that there are no other surfaces of lesser area seperating it from the "right" endpoint of $\{t = 1\}$. Lemma 2.1.7 will make it clear how this condition relates to the actual definition of outerminimizing given in Definition 1.3.8.

With the convention $h(t) = E(t)g(t)$, it is well known that the scalar curvature of a collar extension (M, γ) is given by

$$R_\gamma = 2K_h - 2v^{-1}\Delta_h v + v^{-2}\left[-\text{tr}_h h'' - \frac{1}{4}(\text{tr}_h h')^2 + \frac{3}{4}|h'|_h^2 + \frac{\partial_t v}{v}\text{tr}_h h'\right]. \tag{10}$$

It will be useful for us later to have this expression in terms of $E(t)$ and $g(t)$ directly instead of $h(t)$. So to that end, we compute

$$\text{tr}_h h' = E^{-1}\text{tr}_g[E'g + Eg'] = E^{-1}E'\,\text{tr}_g g + \text{tr}_g g' = 2E^{-1}E'$$

and

$$|h'|_h^2 = E^{-2}|E'g + Eg'|_g^2 = E^{-2}[2(E')^2 + E^2|g'|_g^2].$$

Using this, we have

$$-\frac{1}{4}(\mathrm{tr}_h h')^2 + \frac{3}{4}|h'|_h^2 = -E^{-2}(E')^2 + \frac{3}{2}E^{-2}(E')^2 + \frac{3}{4}|g'|_g^2 = \frac{1}{2}E^{-2}(E')^2 + \frac{3}{4}|g'|_g^2.$$

Proposition 1.2.13 (*iii*) yields $K_{h(t)} = E(t)^{-1}\left(K_{g(t)} - \frac{1}{2}\Delta_{g(t)}\log(E(t))\right) = E(t)^{-1}K_{g(t)}$ since E is a function only of t. We also have

$$\mathrm{tr}_h h'' = E^{-1}\mathrm{tr}_g[E''g + 2E'g' + Eg''] = 2E^{-1}E'' + \mathrm{tr}_g g''$$

and

$$0 = [(\mathrm{tr}_g g')]' = \mathrm{tr}_g g'' - |g'|_g^2 \implies \mathrm{tr}_g g'' = |g'|_g^2.$$

Using these simplifications in equation (10) yields

$$R_\gamma = 2E^{-1}K_{g(t)} + v^{-2}\left[-2E^{-1}E'' - \frac{1}{4}|g'|_g^2 + \frac{1}{2}E^{-2}(E')^2 + 2E^{-1}E'\frac{\partial_t v}{v}\right]. \tag{11}$$

Given a collar extension (M, γ), consider the submanifold $\Sigma_t := \mathbb{S}^2 \times \{t\}$ with unit normal vector

$$\nu = -\frac{1}{v(x,t)}\frac{\partial}{\partial t}.$$

We refer to Σ_t as the *foliating sphere at time t* and ν as the *inward pointing* unit normal. The mean curvature of Σ_t as a submanifold of (M, γ) with respect ν is calculated as

$$\begin{aligned}\rho(t)_{ij} = \langle \nu, II(\partial_i, \partial_j)\rangle_\gamma &= \gamma_{ab}\nu^a((\nabla^M_{\partial_i}\partial_j)^\perp)^b = \gamma_{tt}\nu^t((\nabla^M_{\partial_i}\partial_j)^\perp)^t \\ &= \gamma_{tt}\frac{1}{-v(x,t)}(\Gamma^M)^t_{ij} = \gamma_{tt}\frac{1}{-v(x,t)}(-\frac{1}{2}\gamma^{tt}\gamma_{ij;t}) = \frac{1}{2v(x,t)}h_{ij;t}.\end{aligned}$$

Since $h(t) = E(t)g(t)$, we have

$$h' = E'(t)g(t) + E(t)\dot{g}(t) \implies \mathrm{tr}_h h' = E^{-1}\mathrm{tr}_g h' = 2E'(t)E^{-1}(t).$$

Note that we used $\mathrm{tr}_g\dot{g} \equiv 0$ in the above calculation. Therefore

$$H_t = \mathrm{tr}_{h(t)}\rho(t) = \frac{1}{2v(x,t)}\mathrm{tr}_h(\dot{h}) = \frac{E'(t)E^{-1}(t)}{v(x,t)}.$$

To prove our main theorems, we will take a specified $g \in \mathrm{Met}(\mathbb{S}^2)$ and a constant $H \geq 0$ and we will construct a g-collar $(\mathbb{S}^2 \times [0,1], \gamma)$ such that the mean curvature of Σ_0 satisfies $H_0 \equiv H$ (with respect to the inward pointing normal). Once we have such a collar, we would like to "glue" an outer Schwarzschild region (with controlled ADM mass) onto the right ($t = 1$) end of the collar. This "gluing" idea was first introduced by Mantoulidis and Schoen [**13**] in 2015 who used it to prove Theorem 1.3.13. Since that time, their method of "gluing" has been somewhat refined. The following lemma of Cabrera *et al.*, is one of those refinements and is the tool that we will use to construct admissible extensions from a g-collar.

Lemma 2.1.7 (Proposition 2.1 in [**6**]). Consider a collar

$$(M = \mathbb{S}^2 \times [0,1], \gamma = E(t)g(t) + v(x,t)^2dt^2)$$

as in Definition 2.1.5. Suppose there exists a constant $0 < a < 1$ such that

(1) $E'(t) > 0$ for all $t \in [0,1]$,

(2) v is identically constant on $\mathbb{S}^2 \times [a,1]$,

(3) $R_\gamma \geq 0$ with $R_\gamma > 0$ on $\mathbb{S}^2 \times [a,1]$,

(4) $g(t) \equiv g(1)$ which is round on $\mathbb{S}^2 \times [a,1]$,

(5) $H_1 > 0$, and

(6) $\mathfrak{m}_H(\Sigma_1, H_1) \geq 0$.

Then for any $\epsilon > 0$, there exists a smooth, rotationally symmetric, asymptotically flat Riemannian 3-manifold $(\widetilde{M} \approx \mathbb{S}^2 \times [0,\infty), \tilde{\gamma})$ satisfying the following properties:

(i) $R_{\tilde{\gamma}} \geq 0$,

(i) $\widetilde{M}$, outside a compact set, is isometric to an outer Schwarzschild region with ADM mass $m := \mathfrak{m}_H(\Sigma_1, H_1) + \epsilon$,

(iii) $\tilde{\gamma} \equiv \gamma$ on the region $\mathbb{S}^2 \times [0, \frac{a+1}{2}]$, and

(iv) $\partial\widetilde{M} \cong (\mathbb{S}^2, g)$ is outerminimizing.

Given a metric $g \in \mathrm{Met}(\mathbb{S}^2)$ and a constant $H \geq 0$, if we can construct a g-collar $(\mathbb{S}^2 \times [0,1], \gamma)$ satisfying conditions (1)-(6) above, then we immediately have that

$$\mathfrak{m}_B(\mathbb{S}^2, g, H) \leq \mathfrak{m}_H(\Sigma_1, H_1)$$

since the resulting manifolds that Lemma 2.1.7 constructs are admissible extensions of g with ADM mass arbitrarily close to $\mathfrak{m}_H(\Sigma_1, H_1)$. The proofs of our main theorems will therefore consist primarily of said collar constructions in such a way that $\mathfrak{m}_H(\Sigma_1, H_1)$ is arbitrarily close to the given bound.

2.2. The Operator L and its Eigenvalues

On a Riemannian manifold (Σ, g), we define an elliptic differential operator

$$L_g := -\Delta_g + K_g$$

so that $L_g u = -\Delta_g u + K_g u$. Even though L is well-defined on any Riemannian manifold, we will henceforth restrict our attention to the special case $\Sigma \approx \mathbb{S}^2$. Throughout this section, we equip $\mathbb{S}^2$ with an arbitrary Riemannian metric g so we will drop it from notation. Note that for any $u, v \in C^\infty(\mathbb{S}^2)$, we may apply Green's identity to see that

$$\int_{\mathbb{S}^2} vLu = \int_{\mathbb{S}^2} -v\Delta u + Kuv \overset{\text{Green's}}{=} \int_{\mathbb{S}^2} \langle \nabla u, \nabla v\rangle + Kuv.$$

Therefore, $B : C^\infty(\mathbb{S}^2) \times C^\infty(\mathbb{S}^2) \to C^\infty(\mathbb{S}^2)$, $(u, v) \mapsto \langle \nabla u, \nabla v\rangle + Kuv$ is called the *relevent bilinear form* of L. The *index form* of L is $I(u) = \int_{\mathbb{S}^2} B(u, u)$. A natural question one may ask is: Are there minimizers of the index form I (after normalization of course) and if so, what form do they take? The answer to this question is yes but it is not immediately obvious as to why this should be true. It is obvious that in order to even make sense of $I(u)$, we need to at least require $u \in W^{1,2}(\mathbb{S}^2)$. With this in mind, we have the following definition.

Definition 2.2.1. The *first eigenvalue* of L_g is

$$\lambda_1(L_g) := \inf_{\substack{u \in W^{1,2}(\mathbb{S}^2)\\ u\not\equiv 0}} \frac{\int_{\mathbb{S}^2} B(u,u)}{\int_{\mathbb{S}^2} u^2}.$$

Since we are working with a fixed operator L, we will denote $\lambda_1(L_g)$ as $\lambda_1(g)$ or even just λ_1 if the metric g is understood. The following propositions (2.2.2, 2.2.3 and 2.2.4) are basic results from the study of elliptic operators but we include their statements and (brief) proofs for completeness.

Proposition 2.2.2. Fix some metric $g \in \mathrm{Met}(\mathbb{S}^2)$ with L and λ_1 as defined above. Then

(i) the infemum in Definition 2.2.1 is achieved by some minimizer $u \in W^{1,2}(\mathbb{S}^2)$,
(ii) the minimizer u is actually soooth, i.e., $u \in C^\infty(\mathbb{S}^2)$, and
(iii) the minimizer u satisfies $Lu = \lambda_1 u$ on $\mathbb{S}^2$.

Proof. Let $\{u_i\}_{i\in\mathbb{N}} \subset W^{1,2}(\mathbb{S}^2)$ be a (normalized) sequence such that $\int_{\mathbb{S}^2} u_i^2 = 1$ for all $i \in \mathbb{N}$ and $I(u_i) \searrow \lambda_1$ as $i \to \infty$. Since $W^{1,2}(\mathbb{S}^2) = W^{1,2}_0(\mathbb{S}^2) \hookrightarrow L^2(\mathbb{S}^2)$ compactly embeds, there is some subsequence (still calling it $\{u_i\}_{i\in\mathbb{N}}$) and some $u \in L^2(\mathbb{S}^2)$ such that $u_i \to u$ in L^2 (and consequently $\int_{\mathbb{S}^2} u^2 = 1$). Then we have

$$I(u_i - u_j) + \lambda_1 \int_{\mathbb{S}^2} (u_i + u_j)^2 \le I(u_i - u_j) + I(u_i + u_j) = 2I(u_i) + 2I(u_j).$$

But as $i, j \to \infty$, $2I(u_i) + 2I(u_j) \to 4\lambda_1$ and

$$\lambda_1 \int_{\mathbb{S}^2} (u_i + u_j)^2 \to \lambda_1 \int_{\mathbb{S}^2} (2u)^2 = 4\lambda_1 \int_{\mathbb{S}^2} u^2 = 4\lambda_1.$$

Therefore $I(u_i - u_j) \to 0$ as $i, j \to \infty$ and so $\{u_i\}_{i\in\mathbb{N}}$ is Cauchy in $W^{1,2}(\mathbb{S}^2)$. By the completeness of this space, $u \in W^{1,2}(\mathbb{S}^2)$ and $I(u) = \lambda_1$ which proves (i). Fix some $\eta \in W^{1,2}(\mathbb{S}^2)$ so that

$$\left.\frac{d}{dt}\right|_{t=0} \frac{I(u + t\eta)}{\int_{\mathbb{S}^2}(u + t\eta)^2} = 0$$

since u minimizes this quantity. Expanding out the derivative and setting $t = 0$ then yields

$$\left(\int_{\mathbb{S}^2} 2\langle \nabla u, \nabla \eta \rangle + 2Ku\eta\right)\left(\int_{\mathbb{S}^2} u^2\right) - I(u)\int_{\mathbb{S}^2} 2u\eta = 0$$

and so

$$\int_{\mathbb{S}^2} \eta Lu = \int_{\mathbb{S}^2} (-\Delta u + Ku)\eta = \int_{\mathbb{S}^2} \langle \nabla u, \nabla \eta \rangle + Ku\eta = \lambda_1 \int_{\mathbb{S}^2} u\eta.$$

Thus u is a weak solution of $Lu = \lambda_1 u$. Elliptic regularity implies that $u \in C^\infty(\mathbb{S}^2)$ and that u is in fact a strong solution to $Lu = \lambda_1 u$. This proves *(ii)* and *(iii)*. $\square$

Now given $\lambda_1, \ldots, \lambda_{k-1}$ (and relevant eigenfunctions $\{u_i\}$), we define

$$\lambda_k(L) := \inf\left\{\frac{I(u,u)}{\int_{\mathbb{S}^2} u^2} \;:\; u \not\equiv 0, \int_{\mathbb{S}^2} uu_j = 0 \;\text{ whenever } u_j \text{ is an } j\text{'th eigenfunction for all } 1 \le j < k\right\}.$$

Of course, Proposition 2.2.2. can be repeated for each k to show that the infemum is achieved for some $u \in C^\infty(\mathbb{S}^2)$ and that $Lu = \lambda_k u$. We call such a u a *k'th eigenfunction* and λ_k the *k'th eigenvalue*. Standard theory shows that $(\lambda_k)_{k\in\mathbb{N}}$ is a discrete sequence (with possible repetitions) that tends to infinity as $k \to \infty$. The *multiplicity* of λ_k is the number of independent eigenfunctions associated to the eigenvalue λ_k and the *k'th eigenspace* is the span of the k'th eigenfunctions. There is more we can say about the first eigenspace.

Proposition 2.2.3. λ_1 has multiplicity 1 and first eigenfunctions do not change sign.

Proof. Let u be a first eigenfunction (which is smooth by Proposition 2.2.2) so that $\lambda_1 = I(u)/\int_{\mathbb{S}} u^2$. After observing that $I(|u|) = I(u)$, we have that $|u|$ is also a first eigenfunction. That is, $L|u| + \lambda_1|u| = 0$. Now Harnack inequality ($\max u \le C \min u$) implies that $\min |u| > 0$ since $|u| \ge 0$ and $u \not\equiv 0$. Since u is smooth, it is continuous and so it must not change signs (otherwise it would have a zero by the intermediate value theorem). Since first eigenfunctions are either positive or negative, it is impossible to have

$$\int_{\mathbb{S}^2} uv = 0$$

for any two such functions. Therefore, λ_1 has multiplicity 1. $\square$

This next proposition will be critical in the proof of Theorem 3.1.1 §3.2.

Proposition 2.2.4. Given a smooth path $[0,1] \to \mathrm{Met}(\mathbb{S}^2)$, $t \mapsto g(t)$, there exists $u : [0,1]\times\mathbb{S}^2 \to \mathbb{R}_{>0}$ such that

- *(i)* u is smooth and positive on $\mathbb{S}^2 \times [0,1]$,
- *(ii)* $u_t(\cdot) := u(\cdot, t) : \mathbb{S}^2 \to \mathbb{R}_{>0}$ is a first eigenfunction of $L_{g(t)}$, and
- *(ii)* u_t has unit L^2 norm with respect to the area form $dA_{g(t)}$ for all $t \in [0,1]$.

Proof. Lemma A.1 in [**13**] yields a smooth $v : \mathbb{S}^2 \times [0,1]$ such that $v_t := v(\cdot, t)$ is a first eigenfunction of $g(t)$. We know by Proposition 2.2.3 that each v_t is never equal to 0 on $\mathbb{S}^2$ for all t. Therefore, v is never 0 on $\mathbb{S}^2 \times [0,1]$. So by possibly replacing v with $-v$, we have that $v > 0$ on $\mathbb{S}^2 \times [0,1]$. Finally, we can normalize

$$u(x,t) = \frac{v(x,t)}{\left(\int_{\mathbb{S}^2} v(x,t)^2 \, dA_{g(t)}\right)^{1/2}}$$

The resulting function is clearly positive and satisfies *(ii)* and *(iii)*. u is smooth since both v and the path $t \mapsto g(t)$ are smooth. $\square$

Until now, eigenvalues and eigenfunctions of L seem somewhat mysterious. The following proposition shows that, at least for round spheres, the first eigenvalue and associated eigenfunctions are well understood.

PROPOSITION 2.2.5. If g_* is a round metric on $\mathbb{S}^2$, then $\lambda_1(g_*) = \frac{4\pi}{\text{area}(g)}$ with associated eigenfunction(s) identically constant(s).

PROOF. For any $u \in W^{1,2}(\mathbb{S}^2)$ (not identically zero), we have

$$\frac{I(u)}{\int_{\mathbb{S}^2} u^2} = \frac{\int_{\mathbb{S}^2} |\nabla u|_g^2 + K_g u^2}{\int_{\mathbb{S}^2} u^2} = \frac{\int_{\mathbb{S}^2} |\nabla u|_g^2}{\int_{\mathbb{S}^2} u^2} + \frac{4\pi}{\text{area}(g_*)} \geq \frac{4\pi}{\text{area}(g_*)}.$$

On the other hand, $u \equiv c \in \mathbb{R}$ has $I(u)/\int_{\mathbb{S}^2} u^2 = \frac{4\pi}{\text{area}(g_*)}$. Therefore $\lambda_1(g) = \frac{4\pi}{\text{area}(g_*)}$. $\square$

In section §3.2, we will be constructing admissible extensions of metrics in $\mathscr{M}_{=0}$ which is the class of metrics $g \in \text{Met}(\mathbb{S}^2)$ with $\lambda_1(g) = 0$. Though a priori, it is not immediately obvious that such metrics even exist. The following proposition addresses this concern.

PROPOSITION 2.2.6. $\mathscr{M}_{=0}$ is non-empty.

Although the statement is very simple, the proof is somewhat technical. We begin with a lemma.

LEMMA 2.2.7. For any $p \in \mathbb{S}^2$, there exists a coodinate chart (U, ϕ) containing p and $v \in W^{1,2}(\mathbb{S}^2)$ such that

(i) $v > 0$ on U and $v = 0$ on $\mathbb{S}^2 \setminus U$,
(ii) v is smooth on U, and
(iii) there exists some open $V \subset U$ such that $\overline{V} \supset \partial U$ and $\Delta_* v > 0$ on V.

PROOF. Let (U_0, ϕ_0) be a coodinate chart centered at p with the property that $\phi_0(U_0) = \mathbb{D}$. Now given (U_n, ϕ_n), let $U_{n+1} = \phi_n^{-1}(\mathbb{D}_{1/2})$ and let $\phi_{n+1} : U_{n+1} \to \mathbb{R}^2$ be defined as

$$\phi_{n+1}(x) = 2\phi_n(x).$$

Observe that $(U_n)_n$ is a nested sequence of sets whose interstection is exactly p and $\phi_n(U_n) = \mathbb{D}$ for all $n \in \mathbb{N}$. Define $\gamma_n := (\phi_n^{-1})^* g$ so that $\{\gamma_n\}$ are Riemannian metrics on $\mathbb{D}$. We can further require $\gamma_0(0)_{ij} = \delta_{ij}$ by possibly modifying our choice of (U_0, ϕ_0). I claim that $\lim_{n\to\infty} 2^{2n}\gamma_n = \gamma$ in the C^1 norm where γ is the standard Euclidean metric on $\mathbb{D}$. To see this, fix $n \in \mathbb{N}$, let x^i be the coordinates of (U_n, ϕ_n) and let y^a be the coordinates of (U_0, ϕ_0). By the way they're defined, we have

$$\frac{\partial y^a}{\partial x^i} = \frac{1}{2^n}\delta_i^a.$$

For simplicity, write h for γ_0. We have

$$(\gamma_n)_{ij}(z) = h_{ab}(z/2^n)\frac{\partial y^a}{\partial x^i}\frac{\partial y^b}{\partial x^i j} = h_{ab}(z/2^n)\left(\frac{1}{2^n}\delta_i^a\right)\left(\frac{1}{2^n}\delta_j^b\right) = \frac{1}{2^{2n}}h_{ij}(z/2^n),$$

and

$$(\gamma_n)^{ij}(z) = h^{ab}(z/2^n)\frac{\partial x^i}{\partial y^a}\frac{\partial x^j}{\partial y^b} = h^{ab}(z/2^n)\left(2^n\delta_a^i\right)\left(2^n\delta_b^j\right) = 2^{2n}h^{ij}(z/2^n), \tag{12}$$

and

$$\left(\frac{\partial}{\partial x^i}\gamma_n\right)(z) = \frac{1}{2^{2n}}\frac{\partial}{\partial x^i}\left[h(z/2^n)\right] = \frac{1}{2^{2n}}\left(\frac{\partial}{\partial x^i}h\right)(z/2^n)\frac{1}{2^n}. \tag{13}$$

Define

$$M := \max\left\{\sup_{z\in\mathbb{D}_{1/2}}\left(\frac{\partial}{\partial x^1}h\right)(z), \sup_{z\in\mathbb{D}_{1/2}}\left(\frac{\partial}{\partial x^2}h\right)(z)\right\} < \infty.$$

Whenever $n \geq 2$, we have $\frac{1}{2^n} < \frac{1}{2}$ and therefore

$$2^{2n}\sup_{i,z\in\mathbb{D}}\left|\left(\frac{\partial}{\partial x^i}\gamma_n\right)(z)\right| \leq \frac{1}{2^n}\sup_{i,z\in\mathbb{D}_{1/2}}\left|\left(\frac{\partial}{\partial x^i}h\right)(z)\right| = \frac{M}{2^n} \to 0 \text{ as } n\to\infty.$$

Since $h(0)_{ij} = \delta_{ij}$ and $z/2^n \to 0$ for any $z \in \mathbb{D}$, we have

$$2^{2n}(\gamma_n)_{ij}(z) = h_{ij}(z/2^n) \to \delta_{ij}.$$

This proves that $\lim_{n\to\infty} 2^{2n}\gamma_n = \gamma$ in the C^1 norm. Note that $2^{2n}\gamma_n$ actually converges to γ in any C^k norm but this is not necessary for our purposes.

Consider $u : \mathbb{D} \to \mathbb{R}$, $x \mapsto (1-|x|)^2$ which is smooth on $\mathbb{D}$. Then using polar coordinates,

$$\Delta_\gamma u = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)(1-r)^2 = 2 - \frac{2(1-r)}{r}.$$

Note that on the set $\widetilde{V} := \mathbb{D}\setminus\overline{\mathbb{D}_{2/3}} = \{z\in\mathbb{R}^2 : 2/3 < |z| < 1\}$, we have

$$\Delta_\gamma u > 2 - \frac{2\left(1-\frac{2}{3}\right)}{\frac{2}{3}} = 1$$

Now using equations (12) and (13) in the definition of Δ_{γ_n}, we have

$$\begin{aligned}\Delta_{\gamma_n}\Big|_z &= \frac{1}{\sqrt{\det\gamma_n(z)}}\frac{\partial}{\partial x^i}\Big|_z\left(\gamma_n^{ij}\sqrt{\det\gamma_n}\frac{\partial}{\partial x^j}\right)\\ &= \partial_i\gamma_n^{ij}(z)\frac{\partial}{\partial x^j} + \frac{\gamma_n^{ij}(z)\partial_i(\det\gamma_n(z))}{2\det\gamma_n(z)}\frac{\partial}{\partial x^j} + \gamma_n^{ij}(z)\frac{\partial^2}{\partial x^j\partial x^i}\\ &= 2^{2n}\left[\partial_i h^{ij}(z/2^n)\frac{\partial}{\partial x^j} + \frac{h^{ij}(z/2^n)\partial_i(\det h(z/2^n))}{2\det h(z/2^n)}\frac{\partial}{\partial x^j} + h^{ij}(z/2^n)\frac{\partial^2}{\partial x^j\partial x^i}\right].\end{aligned}$$

As discussed above, the first two terms in the brackets converge to 0 as $n\to\infty$. So we can find $N\in\mathbb{N}$ large enough so that on $\widetilde{V}$, we have

$$\partial_i h^{ij}(z/2^n)\frac{\partial u}{\partial x^j} \geq -\frac{1}{4}, \quad \frac{\gamma_n^{ij}(z)\partial_i(\det\gamma_n(z))}{2\det\gamma_n(z)}\frac{\partial u}{\partial x^j} \geq -\frac{1}{4},$$

and

$$h^{ij}(z/2^n)\frac{\partial^2 u}{\partial x^j\partial x^i} \geq \frac{1}{2}.$$

Therefore $\Delta_{\gamma_N}(u) > 0$ on $\widetilde{V}$. Now take $U = U_N$, $v : \mathbb{S}^2 \to \mathbb{R}$ defined as

$$v(x) = \begin{cases} u(\phi_N(x)) & x\in U\\ 0 & x\notin U.\end{cases}$$

Then $v \in W^{1,2}(\mathbb{S}^2)$ and is easily seen to satisfy conditions (i), (ii) and (iii) with $V = \phi_N^{-1}(\widetilde{V})$. This completes the proof of the lemma.

□

Proof of Proposition 2.2.6. Let $g_* \in \mathrm{Met}(\mathbb{S}^2)$ be a round metric with area 4π and let $h = e^{2w} g_*$ where $w : \mathbb{S}^2 \to \mathbb{R}$ is any non-constant smooth function. Combining Lemma 1.2.13 and Definition 2.2.1 yields

$$\lambda_1(h) = \inf_{\substack{u \in W^{1,2}(\mathbb{S}^2) \\ u \not\equiv 0}} \frac{\int_{\mathbb{S}^2} -u\Delta_* u + (1 - \Delta_* w)u^2 \, dA_*}{\int_{\mathbb{S}^2} u^2 \, dA_h}.$$

Now recall the two facts about Δ from Remark 1.2.8:

$$\Delta_* w \not\equiv 0 \ \text{ and } \ \int_{\mathbb{S}^2} \Delta_* w \, dA_* = 0.$$

Therefore, there is some $p \in \mathbb{S}^2$ be such that $\Delta_* w|_p > 0$. Let (U, ϕ), N and $v \in W^{1,2}(\mathbb{S}^2)$ be as in Lemma 2.2.7. That is, (U, ϕ) is a coordinate chart containing p, v is smooth and positive on U while vanishing on $\mathbb{S}^2 \setminus U$, and N is an open neighborhood of ∂U such that $\Delta_* v > 0$ on $N \cap U$. By possibly shrinking U, we may further assume that $\Delta_* w \geq c > 0$ on U. Since v is strictly positive and smooth on U, we have

$$\rho := \inf_{U \setminus N} v^2 > 0, \ \text{ and } \ \Omega := \sup_{U \setminus N} |-v\Delta_* v| < \infty$$

as the set $U \setminus N$ is compact. Let $g = e^{2Aw} g_*$ where

$$A > \max\left\{\frac{1}{c}, \frac{\Omega + \rho}{c\rho}\right\}.$$

Then

$$\begin{aligned}
\lambda_1(g) &= \inf_{\substack{u \in W^{1,2}(\mathbb{S}^2) \\ u \not\equiv 0}} \frac{\int_{\mathbb{S}^2} -u\Delta_* u + (1 - A\Delta_* w)u^2 \, dA_*}{\int_{\mathbb{S}^2} u^2 \, dA_g} \\
&\leq \frac{1}{\int_{\mathbb{S}^2} v^2 \ \ dA_g} \int_{\mathbb{S}^2} -v\Delta_* v + (1 - A\Delta_* w)v^2 \, dA_* \\
&= \frac{1}{\int_U v^2 \ \ dA_g} \int_U -v\Delta_* v + (1 - A\Delta_* w)v^2 \, dA_*.
\end{aligned}$$

By construction, we have $v > 0$ on all of U, $\Delta_* v > 0$ on $N \cap U$ and $1 - A\Delta_* w \leq 1 - Ac < 0$ on U. Therefore

$$\begin{aligned}
\lambda_1(g) &\leq \frac{1}{\int_U v^2 \ \ dA_g} \int_U -v\Delta_* v + (1 - A\Delta_* w)v^2 \, dA_* \\
&\leq \frac{1}{\int_U v^2 \ \ dA_g} \int_U -v\Delta_* v + (1 - Ac)\rho \, dA_* \\
&\leq \frac{1}{\int_U v^2 \ \ dA_g} \int_{U \setminus N} -v\Delta_* v + (1 - A\Delta_* w)v^2 \, dA_* \\
&\leq \frac{1}{\int_U v^2 \ \ dA_g} \int_{U \setminus N} \Omega + (1 - Ac)\rho \, dA_*.
\end{aligned}$$

This final quantity is negative by our choice of A. Now for brevity, replace w with Aw. Define $g_t := e^{2tw} g_*$ so that $g_0 = g_*$ and $g_1 = g$. Let $u : \mathbb{S}^2 \times [0, 1] \to \mathbb{R}_{>0}$ be smooth with the property that $u(\cdot, s) =: u_s(\cdot)$ is a first eigenfunction of L_{g_s} with unit L^2 norm with respect to the area form dA_{g_s}.

Then

$$
\begin{aligned}
|\lambda_1(g_t) - \lambda_1(g_s)| &= |I_{g_t}(u_t) - I_{g_s}(u_s)| \\
&= \left| \int_{\mathbb{S}^2} -u_t \Delta_* u_t + (1 - t\Delta_* w) u_t^2 \; dA_* - \int_{\mathbb{S}^2} -u_s \Delta_* u_s + (1 - s\Delta_* w) u_s^2 \; dA_* \right| \\
&\leq \int_{\mathbb{S}^2} |u_s \Delta_* u_s - u_t \Delta_* u_t| + |\Delta_* w| \, |s u_s^2 - t u_t^2| \; dA_*
\end{aligned}
$$

Since u varies smoothly in time, this quantity can be made arbitrarily small by taking $|s - t| \to 0$. Therefore, the map $t \mapsto \lambda_1(-\Delta_{g_t} + K_{g_t})$ is continuous. Then since $\lambda_1(g_*) = 1$ and $\lambda_1(g) < 0$, we can apply the intermediate value theorem to find some t_0 such that $\lambda_1(g_{t_0}) = 0$. This completes the proof.

□

2.3. Properties of $\mathscr{M}_{\geq 0}$ and $\mathcal{K}_{\geq 0}$

In this section, we delve into some properties of two classes of metrics: $\mathscr{M}_{\geq 0}$ and $\mathcal{K}_{\geq 0}$. We have already seen in section §1.3 that the class of metrics $\mathscr{M}_{\geq 0}$ is of critical importance in general relativity as these are exactly the metrics which appear as apparent horizons in the TS/DEC/AF setting. The other class of metrics we will discuss here is $\mathcal{K}_{\geq 0}$ which is the topic of Theorems 3.1.2 and 3.1.4. We first recall our notation.

Definition 2.3.1. With λ_1 as in section §2.2 and with K denoting the Gaussian curvature, let

$$\begin{aligned}
\mathscr{M}_{\geq 0} &:= \{g \in \mathrm{Met}(\mathbb{S}^2) : \lambda_1(g) \geq 0\}, \quad \mathcal{K}_{\geq 0} := \{g \in \mathrm{Met}(\mathbb{S}^2) : K_g \geq 0\},\\
\mathscr{M}_{> 0} &:= \{g \in \mathrm{Met}(\mathbb{S}^2) : \lambda_1(g) > 0\}, \quad \mathcal{K}_{> 0} := \{g \in \mathrm{Met}(\mathbb{S}^2) : K_g > 0\},\\
\mathscr{M}_{= 0} &:= \{g \in \mathrm{Met}(\mathbb{S}^2) : \lambda_1(g) = 0\}, \quad \mathcal{K}_{= 0} := \{g \in \mathrm{Met}(\mathbb{S}^2) : K_g = 0\}.
\end{aligned}$$

Remark 2.3.2. Aside from it being useful from a compuation viewpoint, the class of metrics $\mathcal{K}_{>0}$ has been extensively surveyed in this context (e.g., [**6**], [**16**], [**17**]). Also computationally useful is the slightly larger class of metrics $\mathcal{K}_{\geq 0}$ which, in the authors knowledge, has not been greatly studied in this context. One reason why this class is harder to work with, is that any path $t \mapsto g(t)$ starting at a metric $g \in \mathcal{K}_{\geq 0}$ may have $\beta \leq 0$ which can make it challenging to ensure that corresponding collars have non-negative scalar curvature. Our Corollary 2.3.10 is one tool which helps us get around this possible issue.

We begin with a few basic properties.

Proposition 2.3.3. All of the classes of metrics from Definition 2.3.1 are invariant under dilations.

Proof. Let $g \in \mathrm{Met}(\mathbb{S}^2)$ and let $h = e^{2t}$ for some $t \in \mathbb{R}$. By Lemma 1.2.13, we have

$$K_h = e^{-2t}(K_g - \Delta_g t) = e^{-2t} K_g$$

so that $h \in \mathcal{K}_{\geq 0}$ (or $\mathcal{K}_{>0}$ or $\mathcal{K}_{=0}$) if and only if $g \in \mathcal{K}_{\geq 0}$ (or $\mathcal{K}_{>0}$ or $\mathcal{K}_{=0}$). Similarly,

$$\begin{aligned}
\lambda_1(h) &= \inf_{\substack{u \in W^{1,2}(\mathbb{S}^2)\\ u \not\equiv 0}} \frac{\int_{\mathbb{S}^2} |\nabla^h u|_h^2 + K_h u^2 \, dA_h}{\int_{\mathbb{S}^2} u^2 \, dA_h}\\
&= e^{-2t} \inf_{\substack{u \in W^{1,2}(\mathbb{S}^2)\\ u \not\equiv 0}} \frac{\int_{\mathbb{S}^2} |\nabla^g u|_g^2 + K_g u^2 \, dA_g}{\int_{\mathbb{S}^2} u^2 \, dA_g} = e^{-2t}\lambda_1(g).
\end{aligned}$$

So $h \in \mathscr{M}_{\geq 0}$ (or $\mathscr{M}_{>0}$ or $\mathscr{M}_{=0}$) if and only if $g \in \mathscr{M}_{\geq 0}$ (or $\mathscr{M}_{>0}$ or $\mathscr{M}_{=0}$). □

Proposition 2.3.4. $\mathcal{K}_{\geq 0} \subset \mathscr{M}_{>0}$.

Proof. Let $g \in \mathcal{K}_{\geq 0}$ and let $f \in C^\infty(\mathbb{S}^2)$ be a first eigenfunction of g with unit L^2 norm (with respect to dA_g). Then

$$\lambda_1(g) = \int_{\mathbb{S}^2} |\nabla^g f|_g^2 + K_g f^2 \, dA_g \geq \int_{\mathbb{S}^2} K_g f^2 \, dA_g \geq 0.$$

But $f^2 > 0$ on $\mathbb{S}^2$ by Proposition 2.2.3 so $\lambda_1(g) > 0$ unless $K_g \equiv 0$. But this is impossible by Theorem 1.2.11 which gives

$$\int_{\mathbb{S}^2} K_g \, dA_g = 2\pi\chi(\mathbb{S}^2) = 4\pi.$$

□

As we have seen in section §1.3.4, the class of metrics $\mathscr{M}_{\geq 0}$ arises naturally in the study of apparent horizons. Historically however, the examples of metrics g arising from horizons in the literature often feature the stronger assumption that $g \in \mathcal{K}_{\geq 0}$ or even $g \in \mathcal{K}_{>0}$ (e.g., [**4**], [**12**]). The following theorem of Mantoulidis and Schoen establishes that $\mathscr{M}_{>0}$ (and therefore also $\mathscr{M}_{\geq 0}$) contains elements with arbitrarily large negative integral curvature.

THEOREM 2.3.5 (Theorem 3.1 in [**13**]). *For every $c > 0$, the subset*

$$\left\{ g \in \mathscr{M}_{>0} : \int_{\mathbb{S}^2} (K_g)_- \, dA_g \geq c \right\}$$

is C^1 dense in $\mathscr{M}_{>0}$. Here $(K_g)_- = \max\{0, -K_g\}$.

To prove theorem 3.1.1, given a $g \in \mathscr{M}_{=0}$, we will need a g-admissible path $t \mapsto g(t)$ that remains in $\mathscr{M}_{\geq 0}$. The following proposition is the first step towards this.

PROPOSITION 2.3.6. $\mathscr{M}_{\geq 0}$ is path connected.

PROOF. Let $g \in \mathscr{M}_{\geq 0}$. By uniformization we may write $g = e^{2w} g_*$ for a round metric g_* with area 4π. It suffices to show that the path $t \mapsto g_t := e^{2tw} g_*$ remains within $\mathscr{M}_{\geq 0}$. Let $f \in C^\infty(\mathbb{S}^2)$, $f \not\equiv 0$. Recall from section §2.2 that the relevant bilinear form of L_{g_t} is

$$B_{g_t}(f, f) = \int_{\mathbb{S}^2} |\nabla^{g_t} f|^2_{g_t} + K_{g_t} f^2 \, dA_{g_t}.$$

Using Lemma 1.2.13 and Corollary 1.2.12, we have

$$\begin{aligned} \int_{\mathbb{S}^2} |\nabla^{g_t} f|^2_{g_t} + K_{g_t} f^2 \, dA_{g_t} &= \int_{\mathbb{S}^2} e^{-2tw} |\nabla^{g_*} f|^2_{g_*} + e^{-2tw} (K_{g_*} - t\Delta_* w) f^2 \, (e^{2tw} dA_{g_*}) \\ &= \int_{\mathbb{S}^2} |\nabla^{g_*} f|^2_{g_*} + (K_{g_*} - t\Delta_* w) f^2 \, dA_{g_*} \\ &= \int_{\mathbb{S}^2} |\nabla^{g_*} f|^2_{g_*} + (1 - t\Delta_* w) f^2 \, dA_{g_*}. \end{aligned} \tag{14}$$

Letting $t = 0$ and using Proposition 2.2.5 we have

$$\int_{\mathbb{S}^2} |\nabla^{g_*} f|^2_{g_*} + f^2 \, dA_{g_*} \geq \lambda_1(g_*) \int_{\mathbb{S}^2} f^2 \, dA_{g_*} = \int_{\mathbb{S}^2} f^2 \, dA_{g_*}$$

and at $t = 1$,

$$\int_{\mathbb{S}^2} |\nabla^g f|^2_g + K_g f^2 \, dA_g = \lambda_1(g) \int_{\mathbb{S}^2} f^2 \, dA_g \geq 0$$

since $g_1 = g \in \mathscr{M}_{\geq 0}$ by assumption. Equation (14) is also linear in t, so it is positive for all $t \in [0, 1)$. Choosing f to be a first eigenfunction of g_t yields the desired result. $\square$

Proposition 2.3.6 not only shows that $\mathscr{M}_{\geq 0}$ is path connected, but the technique of the proof also allows us to obtain a lower bound on the first eigenvalue of a path of conformally related metrics.

LEMMA 2.3.7. Let $g \in \mathscr{M}_{\geq 0}$. By uniformization we may write $g = e^{2w} g_*$ for a round metric g_* with area 4π. Let $\zeta : [0, 1] \to [0, 1]$ be a smooth decreasing function with $\zeta(0) = 1$, $\zeta'(0) < 0$, and define $g_t := e^{2w\zeta(t)} g_*$ so that $g_0 = g$, $g_1 = g_*$. Then $\lambda_1(g_t) \geq \tilde{c} t$ for some $\tilde{c} > 0$ and all $t \in [0, 1]$.

PROOF. Since $\zeta : [0, 1] \to [0, 1]$ is smooth, we can write $\zeta(t) = 1 + \zeta'(0)t + O(t^2)$. Let $u : [0, 1] \times \mathbb{S}^2 \to \mathbb{R}_{>0}$ be as in Proposition 2.2.4. That is, u is smooth and $u_s(\cdot) := u(\cdot, s)$ is a first eigenvalue of g_s with unit L^2 norm (with respect to the area form dA_{g_s}). Using the linearity of equation (14), we have

$$\begin{aligned}
\lambda_1(g_t) = \frac{I(u_t)}{\int u_t^2 \, dA_{g_t}} &= \int |\nabla^{g_t} u_t|_{g_t}^2 + K_{g_t} u_t^2 \, dA_{g_t} \\
&= \zeta(t) \int |\nabla^{g_0} u_t|_{g_0}^2 + K_{g_0} u_t^2 \, dA_{g_0} + (1-\zeta(t)) \int |\nabla^{g_1} u_t|_{g_1}^2 + K_{g_1} u_t^2 \, dA_{g_1} \\
&= \zeta(t) \int |\nabla^{g} u_t|_{g}^2 + K_{g} u_t^2 \, dA_{g} + (1-[1+\zeta'(0)t + O(t^2)]) \int |\nabla^{*} u_t|_{*}^2 + u_t^2 \, dA_{*} \\
&\geq \zeta(t)\lambda_1(g) \int u_t^2 \, dA_g + [-\zeta'(0)t + O(t^2)])\lambda_1(g_*) \int u_t^2 \, dA_* \\
&\geq [-4\pi\zeta'(0)\lambda_1(g_*) \min_x u_t(x)^2]t + O(t^2)\lambda_1(g_*) \int u_t^2 \, dA_h.
\end{aligned}$$

Since $u : [0,1] \times \mathbb{S}^2 \to \mathbb{R}$ is smooth, $\min_x u_t(x)^2 \geq \min_x u_0(x)^2/2$ for all t small enough. We also have

$$\left| O(t^2)\lambda_1(h) \int u_t^2 \, dA_h \right| \leq [-2\pi\zeta'(0)\lambda_1(g_*) \min_x u_t(x)^2]t$$

for all $0 < t \ll 1$. Therefore

$$\lambda_1(g_t) \geq [-2\pi\zeta'(0)\lambda_1(h) \min_x u_0(x)^2]t \quad \text{for all } 0 < t \ll 1. \tag{15}$$

The bracketed term is strictly positive since $\lambda_1(g_*) = 1$ by Lemma 2.2.5, $\zeta'(0) < 0$, and $\min_x u_0^2 > 0$ since u_0 is strictly positive on a compact set. By possibly shrinking the constant in inequality (15), we can improve the interval to all of $[0,1]$ since $t \mapsto \lambda_1(g_t)$ is continuous (by the proof of Proposition 2.2.6) $\lambda_1(g_t) > 0$ on $(0,1]$ and $\lambda_1(g_1) = 1$. □

Lemma 2.3.8. For any $g \in \mathscr{M}_{\geq 0}$, there exists a g-admissible path $t \mapsto g(t)$ with the property that $g(t) \equiv g(1)$ for all $t \in [1/2, 1]$ and such that

$$\lambda_1(g(t)) \geq c\,t \quad \text{for all } t \in [0,1] \quad \text{for some } c > 0. \tag{16}$$

Remark 2.3.9. If $g \in \mathscr{M}_{>0}$, then Lemma 1.2 in [**13**] constructs a g-admissible path $t \mapsto g(t)$ remaining in $\mathscr{M}_{>0}$ satisifying

$$\min_{t\in[0,1]} \lambda_1(g(t)) > 0$$

which is a stronger condition than (16). We will therefore assume that $g \in \mathscr{M}_{=0}$ and follow a similar construction.

Proof of Lemma 2.3.8. Fix $g \in \mathscr{M}_{=0}$. Proposition 2.1.2 gives a g-admissible path

$$t \mapsto g(t) = \phi_t^* \left(e^{2\zeta(t)w(x) + 2a(t)} g_* \right).$$

By further requiring $\zeta \equiv 0$ on $[1/2, 1]$, we have $g(t) \equiv g(1)$ on $[1/2, 1]$ by Remark 2.1.3. Note that $t \mapsto e^{2\zeta(t)w(x)} g_*$ remains in the space $\mathscr{M}_{\geq 0}$ by Proposition 2.3.6. If $\zeta'(0) < 0$, then by Proposition 2.3.7, there exists some $\tilde{c} > 0$ such that

$$\lambda_1(e^{2\zeta(t)w(x)} g_*) \geq \tilde{c}\,t \quad \text{for all } t \in [0,1]$$

Since $\mathscr{M}_{\geq 0}$ is invariant under dilations it follows that $t \mapsto h(t) := e^{2\zeta(t)w(x)+2a(t)} g_* \in \mathscr{M}_{\geq 0}$ too, with

$$\lambda_1(h(t)) \geq \tilde{c}\, e^{-2a(t)} t \quad \text{for all } t \in [0,1].$$

Note that $\lambda_1(g(t)) = \lambda_1(h(t))$ since $g(t)$ is obtained from $h(t)$ by a series of isometries. Lastly, since $a : [0,1] \to \mathbb{R}$ is smooth, it achieves its supremum. Therefore, taking $c = e^{-2\max a(t)}\tilde{c}$ completes the proof. $\square$

As we saw section §1.3, the space $\mathscr{M}_{\geq 0}$ arises naturally in the context of apparent horizons. But only one of our main results (Theorem 3.1.1) pertains to this space. We therefore turn to a space of metrics with non-negative Gaussian curvature (denoted $\mathcal{K}_{\geq 0}$) that is marginally easier to understand and is the setting of interest in other main results (Theorem 3.1.2 and 3.1.4). The following corollary follows immediately from the proof of Lemma 2.3.8 but include a full proof here as it will be useful in the proof of Theorem 3.1.2.

Corollary 2.3.10. For any $g \in \mathcal{K}_{\geq 0}$, there exists a g-admissible path $t \mapsto g(t)$ such that $g(t) \equiv g(1)$ for all $t \in [1/2, 1]$ and

$$\min_{x \in \mathbb{S}^2} K_{g(t)} \geq c\,t \quad \text{for all } t \in [0,1] \quad \text{for some } c > 0.$$

Proof. Let $g \in \mathcal{K}_{\geq 0}$ so that $g \in \mathscr{M}_{>0}$ by Proposition 2.3.4. Applying Lemma 2.3.8 yields a g-admissible path

$$t \mapsto g(t) = \phi_t^* \left(e^{2\zeta(t) w(x) + 2a(t)} g_* \right)$$

satisfying $g(t) \equiv g(1)$ for all $t \in [1/2, 1]$. By the properties of ϕ_t, we have

$$\min_{x \in \mathbb{S}^2} K_{g(t)} = \min_{x \in \mathbb{S}^2} K_{h(t)}.$$

Now using Lemma 1.2.13 part (iii), we have

$$\begin{aligned} K_{e^{2\zeta(t)w(x)+2a(t)} g_*} &= e^{-2a(t)} K_{e^{2\zeta(t)w(x)} g_*} \\ &= e^{-2a(t) - 2\zeta(t)w(x)} (K_{g_*} - \zeta(t)\Delta_* w) \\ &= e^{-2a(t) - 2\zeta(t)w(x)} (1 - \zeta(t)\Delta_* w). \end{aligned}$$

Plugging in $t = 0$ and using that $g \in \mathcal{K}_{\geq 0}$ yields $\Delta_* w \leq 1$ on $\mathbb{S}^2$. Since $\zeta(t) \in [0,1]$ for all $t \in [0,1]$, we subsequently have $K_{h(t)} \geq 0$. Let $B = \min_{\mathbb{S}^2 \times [0,1]} e^{-2a(t) - 2\zeta(t)w(x)}$ so that

$$K_{g(t)} \geq B(1 - \zeta(t)) = B(1 - (1 + \zeta'(0)t + O(t^2)) \geq \frac{1}{2} B\zeta'(0)t \quad \text{for all } 0 \leq t \ll 1.$$

Combining this with the fact that, away from $t = 0$, $K_{g(t)}$ is uniformly from below by a positive constant gives

$$\min_{x \in \mathbb{S}^2} K_{g(t)} \geq c\,t \quad \text{for all } t \in [0,1] \quad \text{for some } c > 0.$$

$\square$

CHAPTER 3

Main Results

This chapter is devoted to the statements and the proofs of our main results. In section §3.1, we state the results, define the relevant constants, and discuss how the results compare to each other and to the preceeding literature. Each of our theorems takes a metric $g \in \mathrm{Met}(\mathbb{S}^2)$ satisfying a certain property, a non-negative specified constant H and produces an upper bound on the Bartnik mass of the triple $(\mathbb{S}^2, g, H)$. These theorems are summarized below for convenience.

Theorem (Theorem 3.1.1). *Let $g \in \mathscr{M}_{=0}$ and $H = 0$. Then $g \in AH$ and*

$$\mathfrak{m}_B(\mathbb{S}^2, g, H) \leq \mathfrak{m}_H(\mathbb{S}^2, g, H).$$

Theorem (Theorem 3.1.2). *Let $g \in \mathcal{K}_{\geq 0}$ and $H > 0$. Then*

$$\mathfrak{m}_B(\mathbb{S}^2, g, H) \leq \sqrt{\frac{area(g)}{16\pi}} = \frac{r_g}{2}.$$

Theorem (Theorem 3.1.4). *Let $g \in \mathcal{K}_{\geq 0}$ and $H > 0$. Then*

$$\mathfrak{m}_B(\mathbb{S}^2, g, H) \leq \max\left\{\frac{r_g\sqrt{1+C}}{2}\left(1 - \frac{r_g^2 H^2}{4(1+C)}\right), 0\right\}$$

where $C = C(g, H) \to 0$ as either $H \to 0^+$ or as g becomes round.

The proofs of these are presented in sections §3.2, §3.3, and §3.4 respectively. The final theorem we give is similar to Theorem 3.1.4 except that it does not require the given metric g to be non-negatively curved.

Theorem (Theorem 3.1.6). *Let $g \in Met(\mathbb{S}^2)$ and $H > 0$. Then*

$$\mathfrak{m}_B(\mathbb{S}^2, g, H) \leq \max\left\{\frac{r_g\sqrt{1+D}}{2}\left(1 - \frac{r_g^2 H^2}{4(1+D)}\right), 0\right\}.$$

The downside of not requiring $g \in \mathcal{K}_{\geq 0}$ is that the associated constant $D = D(g, H)$ no longer has a closed form expression and is not necessarily finite for an arbitrary pair (g, H). When it is finite however, the proof is nearly identitcal to the proof of Theorem 3.1.4 and will be discussed briefly in section §3.4.

3.1. Statement of Results

The setting and significance of our first theorem was already discussed in section §1.3.4. As mentioned there, when paired with the results of Mantoulidis and Schoen [**13**], this theorem establishes that every $g \in \mathscr{M}_{\geq 0}$ appears as an apparent horizon in the TS/DEC/AF and that the Bartnik mass of such horizons is bounded above by the Hawking mass.

Theorem 3.1.1. *Let $g \in \mathscr{M}_{=0}$ and $H = 0$. Then*

$$\mathfrak{m}_B(\mathbb{S}^2, g, H) \leq \mathfrak{m}_H(\mathbb{S}^2, g, H).$$

Moreover, in the admissible extensions we construct, $\partial M \cong (\mathbb{S}^2, g)$ satisfy the stability given in (9)*. Therefore, we also have*

$$\mathscr{M}_{=0} \subset AH$$

where AH is as defined in section §1.3.

Mantoulidis and Schoen [**13**] proved the same bound given in Theorem 3.1.1 when $g \in \mathscr{M}_{>0}$. Their proof however, relies on the fact that $\lambda_1(g) > 0$, and thus does not extend to the case $g \in \mathscr{M}_{=0}$. In either case, when this bound is coupled with the Riemannian Penrose inequality (Theorem 1.3.9), we have equality

$$\mathfrak{m}_B(\mathbb{S}^2, g, 0) = \mathfrak{m}_H(\mathbb{S}^2, g, 0)$$

for any $g \in \mathscr{M}_{\geq 0}$. Our next two results pertain to the case of when $H > 0$ is constant and g has non-negative Gaussian curvature.

Given a constant $H > 0$ and a metric $g \in \mathcal{K}_{>0}$ such that $\mathfrak{m}_H(\mathbb{S}^2, g, H) > 0$, there have been several works which give a bound of the form

$$\mathfrak{m}_B(\mathbb{S}^2, g, H) \leq \mathfrak{C}(g, H)\mathfrak{m}_H(\mathbb{S}^2, g, H)$$

where $\mathfrak{C}(g, H)$ is a constant depending only on g and H. In [**6**], this factor satisfies $\mathfrak{C}(g, H) \to 1$ as $H \to 0^+$ or as g becomes round while $\mathfrak{C}(g, H) \to \infty$ as $\min r_g^2 K_g \to 0$ or as H approaches a finite upper bound determined by g. In [**16**], estimates for $\mathfrak{C}(g, H)$ were obtained satisfying these same properties, except that H could be arbitrarily large. In this case however, $\mathfrak{C}(g, H)$ would be arbitrarily large as well. The methods in the works above are all comparable, based on an interpolation between the conformal factors of g and a round metric, a corresponding collar construction on $\mathbb{S}^2 \times [0, 1]$, and a gluing of the collar to an exterior Schwarzchild region. Using different methods an estimate for $\mathfrak{C}(g, H)$ was obtained in [**12**] using the longtime solution to the Ricci flow on $\mathbb{S}^2$ starting from g, where in particular $\mathfrak{C}(g, H) \to 1$ as g becomes round. In Theorem 3.1.2 below, we obtain an upper bound for the Bartnik mass of any pair (g, H) with $g \in \mathcal{K}_{\geq 0}$ which depends only on the area of g (and not on H). In particular, the bound does not degenerate as $\min r_g^2 K_g \to 0$ or $H \to \infty$.

Theorem 3.1.2. *Let $g \in \mathcal{K}_{\geq 0}$ and $H > 0$. Then*

$$\mathfrak{m}_B(\mathbb{S}^2, g, H) \leq \sqrt{\frac{area(g)}{16\pi}} = \frac{r_g}{2}$$

REMARK 3.1.3. In [**18**], assuming the stronger condition that $g \in \mathcal{K}_{>0}$, Miao and Xie adapted the method in [**19**] and its variation in [**15**] to show that

$$\mathfrak{m}_B(\mathbb{S}^2, g, H) \leq \frac{r_g}{2}$$

for any strictly postivive smooth function H. Their technique however, is considerably different than ours and does not include the case $g \in \mathcal{K}_{=0}$.

Although the bound given in Theorem 3.1.2 does not degenerate as $\min r_g^2 K_g \to 0^+$ or $H \to \infty$, it is inadequate in the sense that the bound is not improved when $H \to 0^+$ or as g becomes round. Our next result is comparable to Theorem 3.1.2 in the sense that it has the same hypothesis but is an improvement in the sense that the given bound approaches $\mathfrak{m}_H(\mathbb{S}^2, g, H)$ if either $H \to 0^+$ or g becomes round while still remaining bounded even if $\min r_g^2 K_g \to 0$ or $H \to \infty$.

THEOREM 3.1.4. *Let $g \in \mathcal{K}_{\geq 0}$ and $H > 0$. Then with $C = C(g, H)$ as in Definition 3.1.8, we have*

$$\mathfrak{m}_B(\mathbb{S}^2, g, H) \leq \max\left\{\frac{r_g\sqrt{1+C}}{2}\left(1 - \frac{r_g^2 H^2}{4(1+C)}\right), 0\right\}.$$

Moreover, $C \to 0^+$ as either $H \to 0^+$ or as g becomes round.

REMARK 3.1.5. As mentioned above, Theorem 3.1.2 and Theorem 3.1.4 have the same hypothesis that $g \in \mathcal{K}_{\geq 0}$ and $H > 0$. We can therefore compare the results to see which gives the better upper bound. We have

$$\frac{r_g\sqrt{1+C}}{2}\left(1 - \frac{r_g^2 H^2}{4(1+C)}\right) < \frac{r_g}{2} \iff H > \frac{2}{r_g}\sqrt{1 + C - \sqrt{1+C}}$$

Before defining the constant $C(g, H)$, we state a more general version of Theorem 3.1.4 which applies to any smooth metric on $\mathbb{S}^2$ and not just those with non-negative Gaussian curvature.

THEOREM 3.1.6. *Let $g \in Met(\mathbb{S}^2)$ (possibly with points of negative Gaussian curvature) and $H > 0$. Then with $D = D(g, H)$ as in Definition 3.1.7, we have*

$$\mathfrak{m}_B(\mathbb{S}^2, g, H) \leq \max\left\{\frac{r_g\sqrt{1+D}}{2}\left(1 - \frac{r_g^2 H^2}{4(1+D)}\right), 0\right\}.$$

Theorem 3.1.6 is intimately related to Theorem 3.1.4 in the sense that applying it to a metric $g \in \mathcal{K}_{\geq 0}$, the resulting bound is actually an improvement to the bound given in Theorem 3.1.4. We choose to state these results seperately, as for an arbitrary metric $g \in \mathrm{Met}(\mathbb{S}^2)$, the constant D is not easily understood. When $g \in \mathcal{K}_{\geq 0}$ however, we can bound the constant $D(g, H)$ from above by $C(g, H)$ which is expressed as the infemum of a definitive quantity. The constants D and C are given in Definition 3.1.7 and Definition 3.1.8 respectively.

DEFINITION 3.1.7 (The constant $D(g, H)$). Let $g \in \mathrm{Met}(\mathbb{S}^2)$ and $H > 0$. Given any g-admissible path $\xi = (g(t))$, we say that $d > 0$ is a (ξ, H)-*good constant* if

$$\min_{\mathbb{S}^2 \times [0,1]}\left[\frac{4d^2}{H^2} K_{g(t)}(1 + d\sqrt{t}) - 2t|g'|_g^2(1 + d\sqrt{t})^2 + d^2\right] > 0.$$

We then define the constant $D(g, H)$ as

$$D(g, H) := \inf_{\xi, d}\{d \text{ such that } \xi \text{ is a } g\text{-admissible path and } d \text{ is a } (\xi, H)\text{-good constant}\}$$

with the convention that $\inf \emptyset = +\infty$.

For an arbitrary $g \in \mathrm{Met}(\mathbb{S}^2)$ and $H > 0$, it need not be the case that $D(g, H) < \infty$. A necessary condition for $D(g, H)$ to be finite is clearly

$$\min_{\mathbb{S}^2} K_g > -\frac{H^2}{4}$$

though this is far from sufficient. As it turns out, given any $g \in \mathrm{Met}(\mathbb{S}^2)$, we can choose $H > 0$ sufficiently large to ensure that $D(g, H) < \infty$. For a treatment of this, we direct the reader to [**8**].

In the case that $g \in \mathcal{K}_{\geq 0}$, any constant $d > 0$ satisfying

$$\frac{4d^2}{H^2} K_{g(t)} - 2t|g'|_g^2 (1 + d\sqrt{t})^2 + d^2 \geq 0 \;\text{ on } \mathbb{S}^2 \times [0, 1] \tag{17}$$

is a (ξ, H)-good constant (except for possibly at $t = 0$ but we will see in the proof of Theorem 3.1.4, this will not matter). Condition (17) is equivalent with

$$d^2 \left(\frac{4K_{g(t)}}{H^2} + 1 \right) \geq (1 + d\sqrt{t})^2 2t|g'|_g^2 \;\text{ on } \mathbb{S}^2 \times [0, 1].$$

Taking the square root of both sides and grouping the d terms gives

$$d \left(\sqrt{\frac{4K_{g(t)} + H^2}{H^2}} - \sqrt{2t^2|g'|_g^2} \right) \geq \sqrt{2t|g'|_g^2} \;\text{ on } \mathbb{S}^2 \times [0, 1]. \tag{18}$$

If we further assume that

$$\frac{4K_{g(t)} + H^2}{H^2} > 2t^2|g'|_g^2 \;\text{ on } \mathbb{S}^2 \times [0, 1],$$

then condition (18) is equivalent with

$$d \geq \max_{\mathbb{S}^2 \times [0,1]} \frac{\sqrt{2t|g'|_g^2}}{\sqrt{\frac{4K_{g(t)} + H^2}{H^2}} - \sqrt{2t^2|g'|_g^2}}.$$

This is the motivation for the following definition.

Definition 3.1.8 (The constant $C(g, H)$). A g-admissible path $t \mapsto g(t)$ is called (g, H)-*admissible* if it satisfies

$$\min_{\mathbb{S}^2 \times [0,1]} \left(\frac{4K_{g(t)} + H^2}{H^2} - 2t^2|g'|_g^2 \right) > 0.$$

In Proposition 3.4.2, we will show that (g, H)-admissible paths exist for any pair (g, H) where $g \in \mathcal{K}_{\geq 0}$ and constant $H > 0$. For any such pair (g, H), we define the constant $C(g, H)$ as

$$C(g, H) := \inf_{(g,H)\text{-admissible paths}} \left[\max_{\mathbb{S}^2 \times [0,1]} \frac{\sqrt{2t|g'|_g^2}}{\sqrt{\frac{4K_{g(t)} + H^2}{H^2}} - \sqrt{2t^2|g'|_g^2}} \right].$$

Note that $C(g, H)$ is necessarily finite and non-negative by the definition of a (g, H)-admissible path. If g is close to a round metric, one can construct (g, H)-admissible paths such that

$$\sqrt{2t|g'|_g^2}, \sqrt{2t^2|g'|_g^2} \ll 1$$

and therefore $C(g,H) \ll 1$. Furthermore, it will be clear in Proposition 3.4.2 that g-admissible paths exist satisfying

$$\max_{\mathbb{S}^2\times[0,1]} 2t^2|g'|_g^2 \le 1.$$

If $\xi = (g(t))$ is such a path, then ξ is a (g,H)-admissible path for any $H > 0$ since

$$\max_{\mathbb{S}^2\times[0,1]} 2t^2|g'|_g^2 \le 1 < \min_{\mathbb{S}^2\times[0,1]} \frac{4K_{g(t)} + H^2}{H^2}.$$

Thus, taking $H \to 0^+$ for such a path ξ, we see that

$$\max_{\mathbb{S}^2\times[0,1]} \frac{\sqrt{2t|g'|_g^2}}{\sqrt{\frac{4K_{g(t)}+H^2}{H^2}} - \sqrt{2t^2|g'|_g^2}} \to 0$$

and therefore $C(g,H) \to 0^+$. This proves the claims that $C \to 0^+$ as either $H \to 0^+$ or as g becomes round.

The proofs of each of the theorems presented in this will depend heavily on Lemma 2.1.7 which originally appeared (in a slightly different form) in Cabrera *et al.* [**6**]. This Lemma (repeated below for ease of reference) takes a collar $\mathbb{S}^2 \times [0,1]$ with certain specified properties and produces an admissible extension with ADM mass arbitrarily close to the Hawking mass of the rightmost foliating sphere of the collar. Therefore, in each of the proofs, we will construct collar extensions in which the rightmost foliating sphere has Hawking mass arbitrarily close to the given bound for $\mathfrak{m}_B(\mathbb{S}^2, g, H)$. Henceforth, given a collar $\mathbb{S}^2 \times [0,1]$, we will write Σ_t for the foliating sphere at time t (that is $\Sigma_t := \mathbb{S}^2 \times \{t\}$) and H_t for the mean curvature of Σ_t with respect to the inward pointing unit normal.

Lemma (Lemma 2.1.7). *Consider a collar*

$$(M = \mathbb{S}^2 \times [0,1], \gamma = E(t)g(t) + v(x,t)^2 dt^2)$$

as in Definition 2.1.5. Suppose there exists a constant $0 < a < 1$ *such that*

(1) $E'(t) > 0$ *for all* $t \in [0,1]$,
(2) v *is identically constant on* $\mathbb{S}^2 \times [a,1]$,
(3) $R_\gamma \ge 0$ *with* $R_\gamma > 0$ *on* $\mathbb{S}^2 \times [a,1]$,
(4) $g(t) \equiv g(1)$ *which is round on* $\mathbb{S}^2 \times [a,1]$,
(5) $H_1 > 0$, *and*
(6) $\mathfrak{m}_H(\Sigma_1, H_1) \ge 0$.

Then for any $\epsilon > 0$, *there exists a smooth, rotationally symmetric, asymptotically flat Riemannian 3-manifold* $(\widetilde{M} \approx \mathbb{S}^2 \times [0,\infty), \tilde{\gamma})$ *satisfying the following properties:*

(*i*) $R_{\tilde{\gamma}} \ge 0$,
(*i*) $\widetilde{M}$, *outside a compact set, is isometric to an outer Schwarzschild region with ADM mass* $m := \mathfrak{m}_H(\Sigma_1, H_1) + \epsilon$,
(*iii*) $\tilde{\gamma} \equiv \gamma$ *on the region* $\mathbb{S}^2 \times [0, \frac{a+1}{2}]$, *and*
(*iv*) $\partial\widetilde{M} \cong (\mathbb{S}^2, g)$ *is outerminimizing.*

3.2. Proof of Theorem 3.1.1

In this section, we prove the first of our three main theorems presented in section §3.1. We state it again here:

THEOREM (Theorem 3.1.1). *Let $g \in \mathscr{M}_{=0}$ and $H = 0$. Then*

$$\mathfrak{m}_B(\mathbb{S}^2, g, H) \le \mathfrak{m}_H(\mathbb{S}^2, g, H) = \frac{r_g}{2}.$$

We begin by fixing $g \in \mathscr{M}_{=0}$ and a path $t \mapsto g(t)$ as constructed in Lemma 2.3.8. That is to say $t \mapsto g(t)$ is a g-admissible path such that $g(t) \equiv g(1)$ on $[1/2, 1]$ and

$$\lambda_1(g(t)) \ge c\,t \text{ for all } t \in [0,1] \text{ and some } c > 0.$$

Let $u : \mathbb{S}^2 \times [0,1] \to \mathbb{R}_{>0}$ be smooth such that $u_t(\cdot) := u(\cdot, t)$ is a first eigenfunction of $g(t)$ with unit L^2 norm with respect to the area form $dA_{g(t)}$. See Proposition 2.2.4 for the existence of such a function u. To prove Theorem 3.1.1, we will construct collar extensions of g which satisfy Lemma 2.1.7 with the Hawking mass of Σ_1 arbitrarly close to $\frac{r_g}{2}$.

LEMMA 3.2.1 (Collar Construction). There exists $A_0 \gg 1$ such that for all $0 < \epsilon \le 1$, $A \ge A_0$, the topological cylinder $\mathbb{S}^2 \times (0, 1]$ endowed with the metric

$$\gamma = (1 + \epsilon t^2)g(t) + \Phi(t)^2 u(t, \cdot)^2 dt^2$$

has the following properties:

(*i*) $R_\gamma > 0$ with $R_\gamma \to 0$ uniformly as $t \to 0$,

(*ii*) $H_t > 0$ for all $t \in (0, 1]$ and $H_t \to 0$ uniformly as $t \to 0$,

(*iii*) $\mathfrak{m}_H(\Sigma_1, H_1) \to \frac{r_g\sqrt{1+\epsilon}}{2}$ as $A \to \infty$.

Here $\Phi(t) : (0, 1] \to \mathbb{R}_+$ is defined as

$$\Phi(t) = \begin{cases} \frac{A}{\sqrt{t}} & : t \in (0, \frac{1}{4}] \\ \varphi(t) & : t \in (\frac{1}{4}, \frac{1}{2}] \\ 2A - 1 & : t \in (\frac{1}{2}, 1] \end{cases}$$

Here φ is a smooth, decreasing, convex function chosen so that $\Phi \in C^\infty((0, 1])$.

REMARK 3.2.2. The presumed singularity at $t = 0$ is superficial; In the new coordinate $s = \sqrt{t}$ on $\mathbb{S}^2 \times (0, 1/4)$, we have

$$\gamma = (1 + \epsilon s^4)g(s^2) + 4A^2 u(s^2, \cdot)^2 ds^2$$

which extends smoothly to the closed manifold $\mathbb{S}^2 \times [0, 1]$. Moreover, since mean curvature is coordinate invariant and continuous along the foliating spheres in $\mathbb{S}^2 \times (0, 1]$, we obtain that the boundary sphere $\{s = 0\}$ is minimal in $\mathbb{S}^2 \times [0, 1]$ relative to the extension by part (*ii*) of the Lemma. In fact, Lemma 3.2.1 could have been stated and proved for this simpler parametrization as well, but we chose to use the parameter t in the proof for ease of reference to Lemma 1.3 in [**13**] (from which this construction is motivated) and to stay consistent with Lemma 2.2 in [**7**] (from which this construction first appeared).

PROOF OF LEMMA 3.2.1. Write $E(t) = 1 + \epsilon t^2$, $h(t) = E(t)g(t)$, and $v_t(x) := v(x, t) = \Phi(t)u(x, t)$. From section §2.1, we have

$$R_\gamma = 2K_{h(t)} - 2v^{-1}\Delta_h v + v^{-2}\left[-2E^{-1}E'' - \frac{1}{4}|g'|_g^2 + \frac{1}{2}E^{-2}(E')^2 + 2E^{-1}E'\frac{\partial_t v}{v}\right]. \tag{19}$$

Since u_t is a first eigenfunction of $g(t)$ with eigenvalue $\lambda(t)$, we have that v_t is a first eigenfunction of $h(t)$ with eigenvalue $E^{-1}\lambda_1(t)$. Using this and the definitions of E and v, we simplify equation (19) to

$$\begin{aligned} R_\gamma &= 2(1+\epsilon t^2)^{-1}\lambda_1(t) + \Phi(t)^{-2}u^{-2}\left[-4\epsilon(1+\epsilon t^2)^{-1} - \frac{1}{4}|g'|_g^2 + 2t^2\epsilon^2(1+\epsilon t^2)^{-2} + 4\epsilon(1+\epsilon t^2)^{-1}\frac{\partial_t v}{v}\right] \\ &\geq \Phi(t)^{-2}u^{-2}(1+\epsilon t^2)^{-1}\left[2\lambda_1(t)\Phi(t)^2u^2 - 4\epsilon - \frac{1}{4}|g'|_g^2 + 4\epsilon\frac{\partial_t\Phi}{\Phi} + 4\epsilon\frac{\partial_t u}{u}\right]. \end{aligned}$$

Since Φ and u are both positive functions, if we want to establish the bound $R_\gamma > 0$, it suffices to show that the bracketed quantity is positive for all $t \in (0,1]$. To that end, we note that $\min u^2 > 0$ and $\max|\partial_t \log u| < \infty$ since u is smooth and positive on the larger set $\mathbb{S}^2 \times [0,1]$ which is compact. So it suffices to prove the weaker condition

$$(*) := 2\Phi(t)^2\lambda(t)\min_{\mathbb{S}^2\times[0,1]} u^2 - 4\epsilon - \alpha - 4\epsilon\max_{\mathbb{S}^2\times[0,1]}|\partial_t \log u| - 4\epsilon t\frac{\partial_t\Phi(t)}{\Phi(t)} > 0 \quad \text{for all } t \in (0,1].$$

Here $\alpha = \max \frac{1}{4}|g'|_g^2$ as in Definition 2.1.4. Then using

$$4\epsilon t\frac{\partial_t\Phi(t)}{\Phi(t)} \geq -2\epsilon t\frac{A/t^{3/2}}{A/t^{1/2}} = -2\epsilon,$$

we have

$$(*) \geq 2A^2c\min_{\mathbb{S}^2\times[0,1]} u^2 - 6\epsilon - \alpha - 4\epsilon\max_{\mathbb{S}^2\times[0,1]}|\partial_t \log u|.$$

This final quantity is positive for all $A \geq A_0$ for some large enough A_0. The claim that $R_\gamma \to 0$ uniformly as $t \to 0$ follows from the fact that for $t \leq 1/4$, $\Phi(t)^{-2} = t/A^2 \to 0$ as $t \to 0$. This proves (i).

For (ii), recall that the mean curvature of a foliating sphere Σ_t with respect to the inward pointing unit normal is

$$H_t = \frac{2\epsilon t(1+\epsilon t^2)^{-1}}{v(t,\cdot)} = \frac{2\epsilon t(1+\epsilon t^2)^{-1}}{\Phi(t)u(t,\cdot)}. \tag{20}$$

In particular, $H_t > 0$ for all $t \in (0,1]$ and $H_t \to 0$ uniformly as $t \to 0^+$.

For (iii), we first note that since $g(t) \equiv g(1)$ for all $t \in [1/2,1]$ and $g(1)$ is round, we know by Proposition 2.2.5 that $u_t \equiv T$ for all $t \in [1/2,1]$. This constant T is easy to calculate since the path $t \mapsto g(t)$ has constant area form and u_t has unit L^2 norm. For $t \in [1/2,1]$, We have

$$1 = \int_{\mathbb{S}^2} u_t^2\, dA_{g(t)} = \text{area}(\mathbb{S}^2, g(t))T^2 = 4\pi r_g^2 T^2 \implies T = \frac{1}{2r_g\sqrt{\pi}}.$$

Therefore, the mean curvature of Σ_1 is

$$H_1 = \frac{4r_g\sqrt{\pi}(1+\epsilon)^{-1}}{2A-1}$$

and the Hawking mass of Σ_1 is

$$\frac{r_g\sqrt{1+\epsilon}}{2}\left(1 - \frac{1}{16\pi}\int_{\Sigma_1} H_1^2\right)$$

which converges to $\frac{r_g\sqrt{1+\epsilon}}{2}$ as $A \to \infty$. $\square$

The inequality part of Theorem 3.1.1 is now immediate: By taking ϵ small and A sufficiently large (depending on ϵ), we construct a collar satisfying Lemma 2.1.7 with $m_H(\Sigma_1)$ arbitrarily close to $\frac{r_g}{2}$. This then yields admissible extensions of g with ADM mass arbitrarily close to $\frac{r_g}{2}$ so we are left with the inequality

$$\mathfrak{m}_B(\mathbb{S}^2, g, 0) \leq \frac{r_g}{2}.$$

This therefore establishes that if $g \in \mathscr{M}_{=0}$, then

$$\mathfrak{m}_B(\mathbb{S}^2, g, 0) \leq \frac{r_g}{2} = \mathfrak{m}_H(\mathbb{S}^2, g, 0)$$

but we have not yet established that $g \in AH$. Recall that Σ_0 is stable if

$$\int_{\mathbb{S}^2} \varphi(-\Delta_g + K_g)\varphi \, dA_g \geq \frac{1}{2} \int_{\mathbb{S}^2} (R_\gamma + |\rho|^2)\varphi^2 dA_g \text{ for all } \varphi \in C^\infty(\mathbb{S}^2).$$

In our constructions, $R_\gamma \to 0$ uniformly as $t \to 0$ and the scalar second fundamental form of Σ_t (denoted as $\rho(x,t)$) with $t \in (0, 1/4]$ is calculated as

$$\rho(x,t) = \frac{h'(t)}{2v(x,t)} = \sqrt{t}\left(\frac{(1+\epsilon t^2)g'(t) + 2\epsilon t g(t)}{2A\, u(x,t)}\right)$$

with the bracketed term remaining bounded for all t. Therefore, simply by continuity, we have $\rho \equiv 0$ on and $R_\gamma \equiv 0$ on Σ_0. On the other hand, $g \in \mathscr{M}_{=0}$. So

$$\int_{\mathbb{S}^2} \varphi(-\Delta_g + K_g)\varphi \, dA_g \geq \lambda_1(g) \int_{\mathbb{S}^2} \varphi^2 \, dA_g = 0$$

for any test function $\varphi \in C^\infty(\mathbb{S}^2)$. Therefore, our admissible extensions satisfy the stability condition which establishes the inclusion

$$\mathscr{M}_{=0} \subset AH$$

and completes the proof.

3.3. Proof of Theorem 3.1.2

In this section, we prove the second of the three main theorems listed in section §3.1. We state it again here:

THEOREM (Theorem 3.1.2). *Let $g \in \mathcal{K}_{\geq 0}$ and $H > 0$ a constant. Then*

$$\mathfrak{m}_B(\mathbb{S}^2, g, H) \leq \frac{r_g}{2}.$$

We begin by fixing $g \in \mathcal{K}_{\geq 0}$ and a path $t \mapsto g(t)$ as in Corollary 2.3.10. We then proceed in similar fashion as section §3.2 by constructing collars satisfying Lemma 2.1.7 with $\mathfrak{m}_B(\Sigma_1)$ arbitrarily close to $\frac{r_g}{2}$. Though they appear similar, the collars constructed here vary significantly from those in the previous section.

LEMMA 3.3.1 (Collar Construction). Fix $H > 0$ and $\epsilon > 0$. There exists $A_0 \gg 1$ such that for all $A \geq A_0$, the cylinder $M \approx \mathbb{S}^2 \times [0,1]$ endowed with the metric

$$\gamma = (1 + \epsilon t) g(t) + \Phi(t)^2 dt^2$$

has the following properties:

(*i*) $R_\gamma > 0$,

(*ii*) $H_t > 0$ is constant for all $t \in [0,1]$ with $H_0 = H$, and

(*iii*) $\mathfrak{m}_H(\Sigma_1, H_1) \to \frac{r\sqrt{1+\epsilon}}{2}$ as $A \to \infty$.

Here $\Phi(t) : (0,1] \to \mathbb{R}_{>0}$ is defined as

$$\Phi(t) = \begin{cases} At + \frac{\epsilon}{H} & : t \in [0, \frac{1}{4}] \\ \varphi(t) & : t \in (\frac{1}{4}, \frac{1}{2}] \\ \frac{A}{4} + \frac{\epsilon}{H} + 1 & : t \in (\frac{1}{2}, 1] \end{cases}$$

where φ is a smooth, increasing, concave function chosen so that $\Phi \in C^\infty([0,1])$.

PROOF. Write $E(t) = 1 + \epsilon t$ and $h(t) = E(t) g(t)$. Section §2.1 gives

$$\begin{aligned} R_\gamma &= 2E^{-1} K_{g(t)} + \Phi^{-2} \left[-2E^{-1}E'' - \frac{1}{4}|g'|_g^2 + \frac{1}{2} E^{-2}(E')^2 + 2E^{-1}E' \frac{\partial_t \Phi}{\Phi} \right] \\ &= 2(1+\epsilon t)^{-1} K_{g(t)} + \Phi^{-2} \left[-\frac{1}{4}|g'|_g^2 + \frac{1}{2}\epsilon^2 (1+\epsilon t)^{-2} + 2\epsilon (1+\epsilon t)^{-1} \frac{\partial_t \Phi}{\Phi} \right] \\ &\geq 2(1+\epsilon t)^{-1} \Phi^{-2} \left[2K_{g(t)} \Phi^2 - \frac{1}{4}|g'|_g^2 + 2\epsilon \frac{\partial_t \Phi}{\Phi} \right]. \end{aligned}$$

Recall that $K_{g(t)} \geq c\,t$ by construction. In order to show $R_\gamma > 0$, it suffices establish

$$(*) := 2c\,t\Phi^2 - \frac{1}{4}|g'|_g^2 + 2\epsilon \frac{\partial_t \Phi}{\Phi} > 0 \text{ on } \mathbb{S}^2 \times [0,1].$$

As in Definition 2.1.4, write $\alpha = \max \frac{1}{4}|g'|_g^2$ and let $\delta = \min\left\{\frac{1}{4}, \frac{\epsilon}{\alpha}\right\}$. Using the facts that Φ is non-decreasing and $\delta \leq \frac{1}{4}$, we have

$$\frac{\partial_t \Phi(t)}{\Phi(t)} \geq \frac{A}{\Phi(\delta)} = \frac{A}{A\delta + \frac{\epsilon}{H}} \geq \frac{A}{A\frac{\epsilon}{\alpha} + \frac{\epsilon}{H}} = \left(\frac{\alpha}{\epsilon}\right) \frac{A}{A + \frac{\alpha}{H}} \quad \text{for all } t \in [0, \delta].$$

So whenever $A > \alpha/H$ and $t \in [0, \delta]$, we have

$$\frac{A}{A + \frac{\alpha}{H}} > \frac{\frac{A}{2} + \frac{\alpha}{2H}}{A + \frac{\alpha}{H}} = \frac{1}{2} \implies \frac{\partial_t \Phi}{\Phi} \geq \frac{\alpha}{2\epsilon} \implies 2\epsilon \frac{\partial_t \Phi}{\Phi} > \alpha.$$

So if $A > \alpha/H$, then $(*) > 0$ on $[0, \delta]$. Now if we also require $A \geq \sqrt{\frac{\alpha}{2c\delta^3}}$, then for $t \in [\delta, 1]$,

$$2ct\Phi(t)^2 \geq 2ct\Phi(\delta)^2 = 2ct\left(A\delta + \frac{\epsilon}{H}\right)^2 > 2c\delta^3 A^2 \geq \alpha.$$

So whenever A is large enough (as described), we have $(*) > 0$ on $[0, \delta] \cup [\delta, 1] = [0, 1]$ and thus $R_\gamma > 0$ on M which shows (i).

For (ii), recall that the mean curvature of a foliating sphere Σ_t with respect to the inward pointing unit normal is

$$H_t = \frac{\epsilon(1+\epsilon t)^{-1}}{\Phi(t)} \tag{21}$$

which satisfies $H_t > 0$ (and constant) for all $t \in [0, 1]$ and $H_0 = \frac{\epsilon}{\Phi(0)} = \frac{\epsilon}{\epsilon/H} = H$.

For (iii), note equation (21) evaluated at $t = 1$ yields

$$H_1 = \frac{\epsilon(1+\epsilon)^{-1}}{\frac{A}{4} + \frac{\epsilon}{H} + 1}.$$

So for any fixed $\epsilon > 0$, taking $A \to \infty$ makes $H_1 \to 0$ and therefore

$$\mathfrak{m}_H(\Sigma_1) = \sqrt{\frac{|\Sigma_1|}{16\pi}}\left(1 - \frac{1}{16\pi}\int_{\Sigma_1} H_1^2\right) \longrightarrow \frac{r_g\sqrt{1+\epsilon}}{2} \quad \text{as } A \to \infty.$$

$\square$

To finish the proof of Theorem 3.1.2, let $\epsilon > 0$ and take $A \gg 1$ sufficiently large so that $\mathfrak{m}_H(\Sigma_1, H_1)$ is no larger than say $\frac{r_g\sqrt{1+2\epsilon}}{2}$. Lemma 2.1.7 then allows us to construct admissible extensions of g with ADM mass satisfying

$$\frac{r_g\sqrt{1+2\epsilon}}{2} < \mathfrak{m}_{\mathrm{ADM}} \leq \frac{r_g\sqrt{1+3\epsilon}}{2}.$$

Taking $\epsilon \to 0^+$ yields the desired bound

$$\mathfrak{m}_B(\mathbb{S}^2, g, H) \leq \frac{r_g}{2}$$

3.4. Proof of Theorem 3.1.4

In this section, we present the proof of our final main theorem. Here it is again:

THEOREM 3.4.1 (Theorem 3.1.4). *If $g \in \mathcal{K}_{\geq 0}$ and $H > 0$ then*

$$\mathfrak{m}_B(\mathbb{S}^2, g, H) \leq \max\left\{\frac{r_g\sqrt{1+C}}{2}\left(1 - \frac{r_g^2 H^2}{4(1+C)}\right), 0\right\}$$

where $C = C(g, H)$ is defined in Definition 3.1.7

Unlike in the proof of Theorems 3.1.1 and 3.1.2, where we used explicit use of the fact that we were working with paths $t \mapsto g(t)$ such that $g(t) = g(1)$ for in a neighborhood of $t = 1$, in the definition of (g, H)-admissible paths, this condition is notably absent. With that in mind, we have the following lemma.

LEMMA 3.4.2. For any $g \in \mathcal{K}_{\geq 0}$ and constant $H > 0$, there exists a (g, H)-admissible path $\xi = (g(t))$. Furthermore, for any $\epsilon > 0$, we may choose $\xi = (g(t))$ such that

$$\max_{\mathbb{S}^2 \times [0,1]} \frac{\sqrt{2t|g'|_g^2}}{\sqrt{\frac{K_{g(t)}+H^2}{H^2}} - \sqrt{2t^2|g'|_g^2}} \leq C(g, H) + \epsilon \tag{22}$$

and $g(t) \equiv g(1)$ for all $t \in [1 - \theta, 1]$ for some $\theta > 0$.

PROOF. For the remainder of the proof, we will drop the x argument for brevity. By Proposition 2.1.2 and Remark 2.1.3, there exists a g-admissible path $t \mapsto h(t)$ satisfying $h(t) = g$ for all $t > 0$ sufficiently small. Fix such a path and consider the modified path of metrics $h_c(t)$ for $t \in [0, 1]$, defined as

$$h_c(t) = \begin{cases} g & 0 \leq t \leq e^{-\frac{1}{c}} \\ h(c\log t + 1) & e^{-\frac{1}{c}} \leq \ t \leq 1 \end{cases}$$

First note that $t \mapsto h_c(t)$ is smooth for any $c > 0$ since $h(t)$ is assumed to be constant in a neighborhood of $t = 0$. By the facts that $h(t)$ itself is g-admissible, and that $h_c(t)$ is just a reparametrization of $h(t)$ we see that $h_c(t)$ is also g-admissible. We may further shrink c if necessary so that

$$0 < c < \sqrt{\frac{\min_{t\in[0,1]} 4K_{h_c(t)} + H^2}{2H^2 \max_{s\in[0,1]} |h'(s)|_{h(s)}^2}}$$

which gives

$$\max_{t\in[0,1]} 2t^2|h_c'(t)|_{h_c(t)}^2 = \max_{t\in[e^{-1/c},1]} 2t^2|h_c'(t)|_{h_c(t)}^2 = \max_{s\in[0,1]} c^2 2|h'(s)|_{h(s)}^2 < \frac{\min_{t\in[0,1]} 4K_{h_c(t)} + H^2}{H^2}$$

making $t \mapsto h_c(t)$ an (g, H)-admissible path.

Now let $\epsilon > 0$ be given and consider a (g, H)-admissible path $t \mapsto g(t)$ satisfying

$$\max_{t\in[0,1]} \frac{\sqrt{2t|g'|_g^2}}{\sqrt{\frac{4K_{g(t)}+H^2}{H^2}} - \sqrt{2t^2|g'|_g^2}} \leq C(g, H) + \epsilon/2 \tag{23}$$

We will construct a family of (g, H)-admissible paths $g_\theta(t)$ satisfying $g_\theta(t) = g_\theta(1)$ for $t \in [1 - \theta, 1]$ and also inequality (22) for sufficiently small θ. For each $\theta \in (0, 1/3)$, we consider a smooth auxilary function $\sigma_\theta : [0,1] \to [0,1]$ satisfying

$$\begin{cases} \sigma_\theta(t) = \frac{t}{1-2\theta},\ \forall\, t \in [0, 1-3\theta] \\ \sigma_\theta(t) = 1,\ \forall\, t \in [1-\theta, 1] \\ 0 \le \sigma_\theta'(t) \le \frac{1}{1-2\theta},\ \forall\, t \in [0,1]. \end{cases}$$

Such a function can be constructed by mollification as discussed in [**6**]. Then the path $t \mapsto g_\theta(t)$ given by $g_\theta(t) := g(\sigma_\theta(t))$ satisfies $g_\theta(t) = g(1)$ for all $t \in [1-\theta, 1]$. For all $t \in [0,1]$, we have

$$\begin{aligned} 2t^2|g_\theta'(t)|^2_{g_\theta(t)} &\le \frac{1}{(1-2\theta)^2} 2t^2 |g'(\sigma_\theta(t))|^2_{g_\theta(t)} \\ &\le \frac{1}{(1-2\theta)^2} 2(\sigma_\theta(t))^2 |g'(\sigma_\theta(t))|^2_{g(\sigma_\theta(t))} \\ &\le \frac{(1-c)}{(1-2\theta)^2}\left(\frac{4K_{g(\sigma_\theta(t))} + H^2}{H^2}\right) \end{aligned}$$

for some $c > 0$ using the fact that $g(t)$ is (g,H)-admissible. In particular, for all $\theta > 0$ sufficiently small we have

$$2t^2|g_\theta'(t)|^2_{g_\theta(t)} \le \left(1 - \frac{c}{2}\right)\frac{4K_{g_\theta(t)} + H^2}{H^2} \tag{24}$$

for $t \in [0,1]$, where we have used the fact that $K_{g(\sigma_\theta(t))} = K_{g_\theta(t)}$, and thus $g_\theta(t)$ is (g,H)-admissible.

On the other hand, we have

$$|g_\theta'(t)|^2_{g_\theta(t)} = \sigma_\theta'(t)^2 |g'(\sigma_\theta(t))|^2_{g(\sigma_\theta(t))},$$

and thus

$$\begin{aligned} 2t|g_\theta'(t)|^2_{g_\theta(t)} &= 2t\sigma_\theta'(t)^2|g'(\sigma_\theta(t))|^2_{g(\sigma_\theta(t))} \\ &= \frac{\sigma_\theta'(t)^2 t}{\sigma_\theta(t)}(2\sigma_\theta(t)|g'(\sigma_\theta(t))|^2_{g(\sigma_\theta(t))}) \\ &\le \frac{t}{\sigma_\theta(t)(1-2\theta)^2}(2\sigma_\theta(t)|g'(\sigma_\theta(t))|^2_{g(\sigma_\theta(t))}) \end{aligned}$$

Also, as $\frac{t/\sigma_\theta(t)}{(1-2\theta)^2} \to 1$ uniformly for $t \in [0,1]$ as $\theta \to 0$, it follows from inequality (24) that for sufficiently small θ we have

$$\left(\frac{t/\sigma_\theta(t)}{(1-2\theta)^2}\right)^2 2t^2|g_\theta'(t)|^2_{g_\theta(t)} \le \frac{4K_{g_\theta(t)} + H^2}{H^2} \quad \text{for all } t \in [0,1].$$

Combining these estimates and letting $s = \sigma_\theta(t)$, we may then estimate as follows for $\theta > 0$ sufficiently small

$$\begin{aligned}
\max_{t\in[0,1]} \frac{\sqrt{2t|g_\theta'(t)|^2_{g_\theta(t)}}}{\sqrt{\frac{4K_{g_\theta(t)}+H^2}{H^2}} - \sqrt{2\,t^2|g_\theta'(t)|^2_{g_\theta(t)}}} &\leq \max_{s\in[0,1]} \frac{\frac{t/s}{(1-2\theta)^2}\sqrt{2s\,|g'(s)|^2_{g(s)}}}{\sqrt{\frac{4K_{g(s)}+H^2}{H^2}} - \frac{t^2/s}{(1-2\theta)^2}\sqrt{2\,s^2|g'(s)|^2_{g(s)}}} \\
&\leq \max_{s\in[0,1]} \frac{\frac{t/s}{(1-2\theta)^2}\sqrt{2s\,|g'(s)|^2_{g(s)}}}{\sqrt{\frac{4K_{g(s)}+H^2}{H^2}} - \frac{t/s}{(1-2\theta)^2}\sqrt{2\,s^2|g'(s)|^2_{g(s)}}} \\
&\leq C(g,H) + \epsilon
\end{aligned}$$

where we have used the fact that $t \leq 1$ in the second last inequaltiy, and for the last inequality we have used equation (22) and the fact that $\frac{t/\sigma_\theta(t)}{(1-2\theta)^2} \to 1$ uniformly for $t \in [0,1]$ as $\theta \to 0$. □

Lemma 3.4.3 (Collar Construction). Let $g \in \mathcal{K}_{\geq 0}$, $H > 0$ a constant and let $\xi = (g(t))$ be a (g,H)-admissible path. The topological cylinder $M = \mathbb{S}^2 \times (0,1]$ endowed with the metric

$$\gamma = (1 + c\sqrt{t})g(t) + \frac{c^2}{4H^2 t}dt^2$$

has the following properties:

(i) $R_\gamma > 0$,
(ii) $H_t = \frac{H}{1+c\sqrt{t}}$ for all $t \in (0,1]$,
(iii) $\mathfrak{m}_H(\Sigma_1, H_1) = \frac{r_g\sqrt{1+c}}{2}\left(1 - \frac{r^2H^2}{4(1+c)}\right)$.

Here $c > 0$ is any constant satisfying

$$c \geq C_\xi := \max_{\mathbb{S}^2\times[0,1]} \frac{\sqrt{2t|g'|^2_g}}{\sqrt{\frac{4K_{g(t)}+H^2}{H^2}} - \sqrt{2t^2|g'|^2_g}}.$$

Remark 3.4.4. Like in the proof of Theorem 3.1.1, the presumed singularity at $t = 0$ is superficial. Changing cooridnates to $s = \sqrt{t}$ for $t \in (0,1]$ gives

$$\gamma = (1 + 2Hs)g\left(\frac{Hs^2}{2c}\right) + ds^2$$

which is no longer singular and can therefore be extended to the closed manifold $\mathbb{S}^2 \times [0,1]$. Since scalar and mean curvature are coordinate invariant quantities, we choose to work in the t-coordinates for ease of calculation and to stay consistent with [**8**] from which this lemma first appeared.

Proof of Lemma 3.4.3. Write $E(t) = 1 + c\sqrt{t}$ and $\Phi(t) = \frac{c}{2H\sqrt{t}}$. Notice that

$$-2E^{-1}E'' + 2E^{-1}E'\frac{\partial_t\Phi}{\Phi} = E^{-1}\left[-2\left(-\frac{c}{4}t^{-3/2}\right) + 2\left(\frac{1}{2}ct^{-1/2}\right)\left(\frac{-t^{-3/2}}{2t^{-1/2}}\right)\right] \equiv 0.$$

Then by our calculations in section §2.1, we have

$$\begin{aligned}
R_\gamma &= \Phi^{-2}\left[2E^{-1}K_{g(t)}\Phi(t)^2 - \frac{1}{4}|g'|^2_g + \frac{1}{2}E^{-2}(E')^2\right] \\
&= \Phi^{-2}\left[2(1+c\sqrt{t})^{-1}K_{g(t)}\left(\frac{c}{2H\sqrt{t}}\right)^2 - \frac{1}{4}|g'|^2_g + \frac{1}{2}(1+c\sqrt{t})^{-2}\left(\frac{c}{2\sqrt{t}}\right)^2\right].
\end{aligned}$$

So in order to ensure $R_\gamma > 0$ on M (and consequently $R_\gamma \geq 0$ on $\widetilde{M} = \mathbb{S}^2 \times [0,1]$ in the new s-coordinates as in Remark 3.4.4), it suffices to require the (strictly) stronger condition

$$(1+c\sqrt{t})^{-2}K_{g(t)}\frac{c^2}{2H^2t} - \frac{1}{4}|g'|_g^2 + \frac{1}{8t}(1+c\sqrt{t})^{-2}c^2 \geq 0 \text{ on } \widetilde{M}$$

or equivalently

$$\frac{4c^2}{H^2}K_{g(t)} - 2t|g'|_g^2(1+c\sqrt{t})^2 + c^2 \geq 0 \text{ on } \widetilde{M}.$$

But we saw in section §3.1 that because ξ is a (g,H)-admissible path, this is equivalent with

$$c \geq \max_{\mathbb{S}^2\times[0,1]} \frac{\sqrt{2t|g'|_g^2}}{\sqrt{\frac{4K_{g(t)}+H^2}{H^2}} - \sqrt{2t^2|g'|_g^2}}$$

which is true by the definition of C_ξ. This proves (i).

The mean curvature of Σ_t is

$$H_t = \frac{E'(t)}{\Phi(t)E(t)} = \frac{\frac{c}{2\sqrt{t}}}{(\frac{c}{2H\sqrt{t}})(1+c\sqrt{t})} = \frac{H}{1+c\sqrt{t}}$$

which gives (ii). Property (iii) is then immediate from the definition of Hawking mass. □

Proof of Theorem 3.1.4. Let $g \in \mathcal{K}_{\geq 0}$, $H > 0$ and $\epsilon > 0$. If $H < \frac{2\sqrt{1+C}}{r_g}$, then by Lemma 3.4.2, we can find a (g,H)-admissible path $\xi = (g(t))$ with C_ξ such that $C_\xi < C + \epsilon$ and $\frac{2\sqrt{1+C}}{r_g} \geq 0$. We can apply Lemma 3.4.3 to the (g,H)-admissible path ξ which satisfies the hypothesis of Lemma 2.1.7. Taking $\epsilon \to 0^+$ yields

$$\mathfrak{m}_B(\mathbb{S}^2, g, H) \leq \frac{r_g\sqrt{1+C}}{2}\left(1 - \frac{r_g^2H^2}{4(1+C)}\right).$$

Now suppose that $H \geq \frac{2\sqrt{1+C}}{r_g}$ and thus

$$c := \frac{H^2r_g^2(1+\epsilon)}{4} - 1 > C \geq 0.$$

By Lemma 3.4.2, we can find a (g,H)-admissible path $\xi = (g(t))$ such that

$$c \geq C_\xi > C \geq 0.$$

We can then apply Lemma 3.4.3 to the (g,H)-admissible path ξ and the constant c. The resulting collar again satisfies the hypothesis of Lemma 2.1.7 with

$$\mathfrak{m}_H(\Sigma_1, H_1) = \frac{r_g\sqrt{1+c}}{2}\left(1 - \frac{r_g^2H^2}{4(1+c)}\right) = \frac{r_g\epsilon\sqrt{1+c}}{2(1+\epsilon)}.$$

Assuming $\epsilon \leq 1$, this yields

$$\mathfrak{m}_B(\mathbb{S}^2, g, H) \leq \frac{r_g\epsilon\sqrt{1+c}}{2(1+\epsilon)} \leq \epsilon\frac{r_g^2H}{2\sqrt{2}}$$

Taking $\epsilon \to 0^+$ yields

$$\mathfrak{m}_B(\mathbb{S}^2, g, H) \leq 0.$$

□

Though we have been exclusively dealing with the case $g \in \mathcal{K}_{\geq 0}$, the proof featured above only needs a slight modification to prove Theorem 3.1.6. For this, we state a version of Lemma 3.4.3 for the more general case of when g is not necessarily in $\mathcal{K}_{\geq 0}$.

Lemma 3.4.5 (Collar Construction). Let $g \in \mathrm{Met}(\mathbb{S}^2)$ and let $H > 0$ be a large enough constant so that the set

$$\{d \text{ such that } \xi \text{ is a } g\text{-admissible path and } d \text{ is a } (\xi, H)\text{-good constant}\}$$

is non-empty. Let d be such a constant with $\xi = (g(t))$ the associated g-admissible path. Then the topological cylinder $M = \mathbb{S}^2 \times (0, 1]$ endowed with the metric

$$\gamma = (1 + d\sqrt{t})g(t) + \frac{d^2}{4H^2 t} dt^2$$

has the following properties:

(i) $R_\gamma > 0$,

(ii) $H_t = \frac{H}{1+d\sqrt{t}}$ for all $t \in (0, 1]$,

(iii) $\mathfrak{m}_H(\Sigma_1, H_1) = \frac{r_g\sqrt{1+d}}{2}\left(1 - \frac{r^2 H^2}{4(1+d)}\right)$.

Proof. Items (ii) and (iii) are analogous to the respective proofs in Lemma 3.4.3. To prove item (i), recall that the scalar curvature of γ is

$$R_\gamma = \frac{4H^2 t}{d^2}\left[2(1 + d\sqrt{t})^{-1} K_{g(t)} \left(\frac{d}{2H\sqrt{t}}\right)^2 - \frac{1}{4}|g'|_g^2 + \frac{1}{2}(1 + d\sqrt{t})^{-2}\left(\frac{d}{2\sqrt{t}}\right)^2\right]$$

which is necessarily positive on all of $\mathbb{S}^2 \times (0, 1]$ since d is a (ξ, H)-good constant. $\square$

The proof of Theorem 3.1.6 is now analogous to the proof of Theorem 3.1.4.

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