Essays in the Economics of Costly Misrepresentation

by

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Abstract

This dissertation studies strategic interaction between informed parties and uninformed parties when misrepresenting private information is costly, and optimal mechanisms for the uninformed parties.

Chapters 1 and 2 analyze a screening problem in which an agent incurs a fixed cost of lying when she misrepresents her private information. In this environment, local incentive constraints are not binding in the optimal mechanism, and standard techniques for solving screening problems are not applicable. Chapter 1 establishes general properties of the optimal mechanism. Chapter 2 develops a new methodology to tackle the problem with non-local binding constraints, characterizes the optimal mechanism and computes it in special cases. The method involves a procedure that jointly solves for the binding non-local incentive constraints and the optimal allocation. The optimal mechanism has a number of novel qualitative properties, such as lack of exclusion and first-best efficient allocation at high- and low- ends of the spectrum of types. Also, bunching never occurs, as the optimal quantity allocation is always increasing in type irrespectively of type distribution.

Chapter 3 analyzes strategic interactions between lying and lie-detection, and studies the optimal design for costly lie-detection and its effectiveness. An informed sender wants to persuade an uninformed receiver to take high actions but the receiver wants to match the action with the true state. The sender makes a claim about the true state and the receiver decides whether to incur a cost to inspect the truthfulness of the claim. I show that the receiver-optimal equilibrium has a three-interval structure: types in the top interval make precise and truthful claims about the state, which are mimicked by types in the bottom interval and randomly inspected, while types in the middle interval make a truthful but vague claim that is never inspected. Compared to state verification, lie-detection is shown to be more beneficial to the receiver because it provides incentives for moderate and high types to be truthful. This suggests that fact-checking of politicians' claims is effective in holding them countable and deterring them from lying.

Lay Summary

Lying is often costly, either due to physiological and moral barriers or fear of being caught lying and the subsequent punishments. This dissertation studies how an uninformed party can make use of the potential costs of lying to elicit information from an informed party. Chapters 1 and 2 study the optimal screening mechanism when costs of lying is exogenous and independent of the size of the lie. Chapters 3 presents a model of communication in which costs of lying is endogenously derived from the possibility of lie-detection and punishment, and studies strategic interaction between lying and lie-detection.

Preface

This dissertation is original, unpublished work. Chapters 1 and 2 are based on a working paper titled "Screening Under Fixed Cost of Misrepresentation", co-authored with Professor Sergei Severinov. We equally contributed to this work at every stage of the project. Chapter 3 is my own independent work.

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To B. Wong

Introduction

While the rapid development of information technology drastically improves information accessibility, it also reduces the threshold for information providers to misrepresent information. In order to combat misinformation, it is important to understand the conflicts and strategic interaction between information providers and receivers, and study the optimal policies that resolve those conflicts. Lying and spreading misinformation often come with costs and potential consequences. They could be inherent, such as morality concerns and psychological barriers of lying, or external, such as risks of being exposed lying and subsequent consequences.

There is substantial experimental evidence supporting the hypothesis that individuals are averse to lying and incur a cost when doing so. In particular, Abeler, Becker and Falk [1] measure intrinsic cost of lying in a setup where other motives such as reputational and efficiency concerns, altruism and conditional cooperation can be ruled out, and find that lying costs are significant and widespread. Kajackaite and Gneezy [38] report experimental data indicating that intrinsic costs of lying are positive and finite. They conclude that "the evidence suggests that lying is a "normal" good for which people compare the intrinsic cost and benefit of the lie, and when the benefit from lying is higher than the intrinsic cost of lying, they lie." Abeler, Nosenzo and Raymond [2] provide a meta-analysis of 72 experimental studies with 32503 subjects and find that subjects obtain only about a quarter of the maximal payoff they could obtain by making payoff maximizing reports. They examine a range of popular explanations and conclude that the data is explained by a combination of lying cost and reputational concern.

Some evidences also suggest that the fear of being exposed lying could lead to truth-telling behaviours. Recent studies find evidence that fact-checking reduces lying behaviors of politicians (e.g. Nyhan and Reifler [47]; Lim [42]). They suggest that detecting lies could be an effective way in combating misinformation.

In light of these evidences, it is natural to wonder how would both intrinsic and external costs of lying affects information transmission between uninformed parties and informed parties, and how can uninformed parties benefits the most from the presence of these costs.

Chapters 1 and 2 of this dissertation study the optimal screening mechanism when agents incur a fixed cost to misrepresent their private information. Previous works in screening problem either assume that agents have no cost of misrepresentation (the standard model), or that the cost is increasing in the magnitude of the "lie", with zero fixed cost of lying. Fixed costs of lying appear in various forms, for example, psychological barriers and cognitive costs in committing to a lie of any size, or terminations of future interaction due to loss of "trust". This dissertation addresses the following questions:

(1) What is the optimal mechanism that maximizes the principal's payoff when facing agents with a fixed cost of lying?

(2) How does it differ from standard screening mechanism and optimal mechanism with a variable cost of lying? What are the implications of these differences?

In the optimal mechanism, allocative efficiency is improved compared with the standard screening problem without cost of lying. In particular, efficient allocations are offered to an interval of low types and high types. Allocations for the intermediate types are below the efficient level, but above the second-best level. The standard screening model and its variation with a variable cost of lying predict that individuals with low valuations or disadvantageous information positions are offered inefficient allocation, and could even be excluded from the mechanism. This socially unappealing property might raise interests of regulations and interventions. This property is not robust to the introduction of a small fixed cost of lying. On the contrary, full efficiency is achieved for low types. This result suggests that it is an optimal policy to provide a certain level of welfare programs without the need for eligibility proof. An example of such programs is food banks which target low-income households. Even though food banks normally do not require income verification from the beneficiaries, these programs are rarely abused by higher-income households because the benefits are not enough to justify the psychological and moral barriers of misrepresenting themselves as low-income households.

Methodologically, this model is substantially different from the standard screening model and screening model with a variable cost of lying. In particular, local incentive constraints are no longer binding in the optimal mechanism. Instead, agents have binding incentive constraints toward some endogenous non-local types. In this environment, standard Mirrlees' techniques for solving screening problems are not applicable. Chapters 2 of this dissertation develops a new methodology to characterize the optimal mechanism. The method involves a procedure that jointly solves for the non-locally binding incentive constraints and the optimal allocation.

Chapter 3 analyzes the optimal usage of costly lie-detecting technology and its effectiveness. An informed party (sender) communicates with an uninformed party (receiver) by making a claim about the state of world and the receiver decides whether to incur a cost to inspect the truthfulness of the claim. Costly lie-detection can be found in various real-life applications. For instance, a citizen who is skeptical about a politician's claim on policy-related issues might incur time and effort to verify the claim through trust-worthy media. Police can inspect the truthfulness of a testimony using interrogation tactics and lie-detecting machines. Previous theoretical literature in lie-detection and strategic communication assumes that the receiver is able to detect a lie from the sender with an exogenous probability. In this dissertation I make an innovation by allowing the receiver to actively construct a lie-detecting strategy contingent on sender's claim. This assumption is motivated by the observation that some claims are often more suspicious than the others and draw more attention in inspecting them. For instance, if a pharmaceutical company claims their new drug is proven to reduce any cancer risk by

99 percent, potential consumers might be skeptical and search if such claim is backed by any trust-worthy, independent studies. On the other hand, if the company makes a mild claim that the drug strengthens the immune system, consumers might just take the company's word and not bother checking it.

This Chapter addresses the following questions:
(1) Under what conditions is lie-detection technology helpful in improving the receiver's welfare?
(2) What is the optimal lie-detection equilibrium for the receiver?
(3) How do changes in the technology, in particular, the information generated by inspection, affect the receiver's welfare?

I found that lie-detection technology is helpful in improving the receiver's welfare if and only if the cost of lie-detection is sufficiently low and prior expectation of the state is not too optimistic. This result echoes a conventional belief that lie-detection is effective when the sender is suspicious. For example, the police usually conduct an interrogation only if they believe that the suspect is likely to have committed a crime. When the sender is likely to be "innocent", there is no cost-effective way to separate lies from truths. Even though full revelation is possible under lie-detection technology, it is neither credible nor optimal for the receiver for any positive cost of lie-detection. The receiver's benefit from lie-detection can be decomposed into two components. First, lie-detection generates a direct information value by distinguishing liars from truth-tellers. Second, the sender might stay honest in fear of being caught lying. Therefore, the possibility of lie-detection creates a threat that deters potential liars and facilitates information transmission. This is called the indirect deterrence effect. I show that under the optimal lie-detection policy, the direct information value of inspection is completely offset by the cost of inspection. Improvement of the receiver's ex-ante payoff is driven by the deterrence effect. This is perhaps surprising as one might expect an optimal lie-detection policy should allow the receiver to acquire as much information as possible. Such intuition turns out to be incorrect because excess amount of information acquired from inspection indicates that the policy induces sender to lie too often, and costly inspection takes place more frequently than necessary. for inspections.

The optimal lie-detection equilibrium has a three-interval structure in the state space where high types are truthful and low types lie and mimic the claims made by the high types. These claims are randomly inspected by the receiver. Moderate types refrain from exaggerating their claims in fear of being caught lying. I also study the effect of inspection technology on the receiver's welfare by comparing lie-detection with state-verification. There are substantial differences between lie-detection and state-verification. Under state-verification technology, an inspection reveals the true state of the world. There will be no uncertainty upon inspection. Under lie-detection technology, an inspection returns a binary signal on the truthfulness of the sender's claim. With a state space with more than two states, a receiver generally does not learn the true state upon lie-detection. I find that even if state-verification is feasible, lie-detection technology can yield a higher payoff to the receiver. Assuming the same unit cost of the two technologies, I show that the receiver's welfare is higher under optimal lie-detection policy compared with optimal state-verification policy.

Chapter 1

Screening Under Fixed Cost of Misrepresentation: Properties of Optimal Mechanism

1.1 Introduction

This chapter sets up the screening model under fixed cost of misrepresentation and analyzes the general structure of an optimal mechanism. An uninformed principal interacts with a privately informed agent who incurs a fixed cost of "lying" when she misrepresents her private information. The analysis of the fixed cost of lying is novel and produces qualitatively new and interesting results.

Whereas most literature on contract theories and mechanism design assume that a privately informed party is unconstrained in her ability to misrepresent and manipulate her information, several strands in this literature have explored alternative frameworks in which misrepresentation is costly. A notable direction in this research, which originated in the work of Lacker and Weinberg [41] and has been further developed by Maggi and Rodriguez-Clare [43] and Crocker and Morgan [18] considers settings in which an agent incurs a cost increasing in the size of her "lie" or type misrepresentation. Another strand of literature on honesty in mechanisms, which includes Alger and Ma [3], Alger and Renault [4] [5], and Severinov and Deneckere [54] has explored situations in which a principal has to deal with a population of agents some of whom are "honest" and are not able to misrepresent their private information, whereas a complementary fraction consists of fully "strategic" agents who behave in a standard fashion. This model differs from both of these literature in studying a setting in which the cost of misrepresentation or lying is finite and does not depend on the magnitude of a "lie."

Misrepresentation costs may exist for several reasons. First, misrepresenting the truth may require costly effort or actions either to manufacture evidence or, conversely, to hide evidence that reveals the true state of the world and conceal one's information. For example, a firm seeking a contract or an individual applying for a promotion may need to be perceived as productive, highly competent and/or creditworthy. This goal may be attained by manufacturing "evidence" exaggerating prior performance and concealing the risk of non-performance. It is plausible that the cost or the effort required to produce such favorable but inaccurate "evidence" is independent of the magnitude of misrepresentation. For instance, the cost of misrepresentation or concealment could involve the loss of future business, benefits or reputation that may have "once and for all" nature making it unrelated to the size of misrepresentation.

In common law there is a legal principle "Falsus in uno, falsus in omnibus", which translated from Latin means "false in one thing, false in everything". That is, a witness who testifies falsely about one matter is not credible to testify about any matter. Clearly, under such principle, lying is associated with a fixed cost of a reputation loss.

In a game-theoretical perspective, a fixed cost of lying could be a reduced form of the payoff difference between "good equilibrium" and "bad equilibrium" in a continuation game where any lying behaviour triggers a bad equilibrium. Consider an infinitely repeated screening environment with two long-lived agents. At the beginning of each period, each agent's private valuation is drawn from an independent and identical distribution, and a short-lived principal picks one of them to be the active agent and interacts with in a static screening model. At the end of each period, any misreporting behaviour from the active agent will be revealed. Consider a strategy profile where agents are truth-telling, and a principal switches to another agent if and only if a misreporting behaviour has been revealed last period. The principal's strategy is sequentially rational because the two agents are ex-ante identical at the beginning of each period. Theorem 1 shows that truth-telling is indeed a property of the optimal mechanism under fixed cost of lying. Given such a strategy profile, an active agent's fixed cost of lying for each period is the discounted expected informational rent for the future periods.

Second, the cost of misrepresenting the truth may have psychological or ethical nature. A moral barrier, a feeling of shame or discomfort, or stress may prevent people from lying.[1] Since being honest or not is often a binary decision, the size of a lie would not affect such psychological costs.

Third, studies in cognitive science and neuroscience indicate that lying is costly because it requires more cognitive resources (Christ et al. [14]). Therefore, if the potential benefit of lying is small, people tend not to think about it and stay honest as a default choice. On the other hand, if temptation to lie is high enough, the individuals tend to take full advantage of it regardless of the extent of a lie. For example, for a consumer pretending to be mildly interested in the product may not be easier than pretending to be not interested at all.

The misrepresentation cost in some contexts can be viewed as a psychic cost of disobedience. The agent incurs this cost when she does not follow the instructions given by the principal viewed as a figure of authority. It has been observed in the psychology literature that individuals are often reluctant to disobey the authority. Also,a deviation from those instruction may be seen as a deviation from a social norm.

While most experimental studies indicate that lying costs exist, the exact shape and nature

[1] Behavioral psychologists have studied a number of physical symptoms associated with emotional discomfort and "feeling wrong" that people experience when lying, including blushing, gaze aversion, elevated eye-blink rate, etc. See, for example, Ekman [22] [23] [24], Porter and Ten Brinke [51].

of these costs remains unclear. Gneezy, Kajackaite and Sobel [31] study the relation between the size of a lie and three different factors: payoffs, outcomes and the likelihood of being perceived as a liar. They find that while social identity (the likelihood factor) has an important impact on lying costs, the other two factors have smaller effects on lying behavior, which indicates that the distance between the report and the truth itself plays little role in the cost of lying. On the other hand, Hilbig and Hessler [36] find that willingness to lie decreases with the degree of the required distortion of the truth, which suggests that the cost of lying is increasing in the size of lie. It is likely that in reality the cost of lying includes both fixed and variable cost elements.

This model adopts the fixed cost of lying hypothesis as a working assumption. From a theoretical perspective, it is important to understand the effect of the fixed cost of lying on the optimal mechanism and pricing. As shown below, the presence of such cost reshapes the landscape of the optimal screening problem and produces qualitatively new results.

There are several properties of the optimal mechanism that are worth mentioning. First, monotonicity of the quantity allocation in type is no longer a necessary condition for implementation in our set-up. Non-monotone allocations are implementable, even though the single-crossing property (SCP) holds. In fact, when there is a fixed lying cost, the set of incentive compatible mechanisms becomes much larger, allowing quantities and transfers to be discontinuous in agent's type. However, we show that only quantity allocations increasing in type are optimal, without any additional restrictions on the parameters of the model. In the standard case, ironing (flat segments of the quantity profile) occurs if the types are drawn from a probability distribution that fails monotone hazard rate condition. Here, however, the assigned quantity is strictly increasing in the optimal mechanism, regardless of the type distribution. The reason behind this is two-fold. For one thing, an increasing quantity schedule is more profitable for the firm. Further, fixed costs imply that one can always make the quantity schedule at least slightly increasing, without violating incentive constraints.

Second, in the optimal mechanism full allocative efficiency is achieved on intervals of low and high types who are assigned the first-best quantity, while downward quantity distortion occurs for medium types. This result is in contrast to the standard "sacrifice efficiency of low types to extract more rent from the high types" logic. Given a positive fixed cost, it is not worth for any type to imitate a low type even if the latter is assigned her first-best quantity. Therefore, no distortion is needed for low types. The intuition behind the efficiency of the allocation for the high-value types is somewhat similar: it is not worth for anyone to imitate those types because, despite their high information rent, they also pay a large transfer. So, with the addition of fixed costs, the surplus from imitating those high types is negative.

Third, the efficiency of the allocation for the low types also means that there is no exclusion in the sense that every type with a positive valuation receives a positive quantity. Severinov and Deneckere [54] establish a no exclusion property when there is a positive fraction of completely honest agent. This paper shows that this property also holds when there are intermediate barriers to the agents' opportunism in the form of a fixed cost.

1.2 Model and Preliminaries

We will cast the model in the context of a relationship between a monopolistic seller, who acts as a principal, and a privately informed buyer, who acts as an agent. However, all results apply in other principal/agent settings, such as a regulator and a firm, an employer and employee.

Consider a monopolistic profit-maximizing firm facing a consumer with privately known preference parameter (value) θ distributed according to a continuously differentiable cdf $F(\theta)$ over the interval $[0,1]$ with full support and corresponding density function $f(.)$. A consumer with value θ gets utility $u(q,\theta)-t$ from consuming quantity/quality q of the good in exchange for payment t. We also adopt the following standard assumptions on $u(q,\theta)$:

Assumption 1 *(i) The function $u(q,\theta)$ is three times continuously differentiable, strictly increasing in θ when $q>0$, strictly concave in q and satisfies $u(0,\theta)=0$ for all $\theta\in[0,1]$;*
(ii) $u_q(0,\theta)>0$ for all $\theta>0$, $u_{qq}(q,\theta)<0$.
(iii) There exists q^m s.t. $u_q(q^m,\theta)<0$ for all $\theta\in[0,1]$.
(iv) There exist $\underline{K}>0$ and $\overline{K}>0$ such that $\underline{K}<u_{q\theta}<\overline{K}$ for all $q>0$ and $\theta\in[0,1]$.

Assumption 1 implies that $q^{fb}(\theta)\equiv\arg\max_q u(q,\theta)$ is well-defined, finite, strictly positive for $\theta>0$ and increasing in θ.

We also assume that the firm has zero cost of production. This is without loss of generality. Indeed, if the firm instead faced a cost of production $c(q)$, the model would be equivalent to one in which the firm's cost is identically zero, while the consumer's utility function is $u(q,\theta)-c(q)$ [2].

The firm has all the bargaining power and designs a mechanism to maximize its expected profits. The consumer can either accept or reject the mechanism offered by the firm. In the latter case she earns her reservation utility normalized to 0.

To this standard screening environment we now add a new element in the form of the fixed cost of misrepresentation or lying about one's type. Specifically, we assume that the consumer of type $\theta\in[0,1]$ incurs a cost C if, being asked to report her value to the firm, she reports some $\hat{\theta}\neq\theta$. Our goal is to characterize the firm's optimal mechanism in this environment.

It is immediate to see that, under the fixed cost of lying assumption, the Revelation Principle still applies provided that type announcement is considered to be a part of the allocation. So the mechanism designed by the firm is represented by a menu $(q(\theta),t(\theta),A(\theta))\in\mathbb{R}^+\times\mathbb{R}\times[0,1]$ where $q(\theta)$ is the quantity assigned to type θ, $t(\theta)$ is her payment to the firm and $A(\theta)$ is the type announcement recommended by the mechanism to the consumer of type θ.

Let $1(A(\theta')\neq\theta)$ denote an indicator function equal to 1 when $A(\theta')\neq\theta$ and equal to zero otherwise. Then the firm's optimal mechanism solves the following problem:

$$\max_{q(\theta),t(\theta),A(\theta)}\int_0^1 t(\theta)f(\theta)d\theta$$

[2] Note that Assumption 1 is still satisfied if the production cost function $c(q)$ incorporated into the utility function is three times continuously differentiable and convex

subject to the following incentive and individual rationality constraints:

$$u(q(\theta),\theta) - t(\theta) - C \times 1(A(\theta) \neq \theta) \geq u(q(\theta'),\theta) - t(\theta') - C \times 1(A(\theta') \neq \theta) \qquad \forall \theta, \theta' \in [0,1]$$
$$u(q(\theta),\theta) - t(\theta) - C \times 1(A(\theta) \neq \theta) \geq 0 \qquad \forall \theta \in [0,1]$$

As the next result demonstrates, we can restrict considerations to mechanisms in which there is no lying.

Theorem 1 *Consider an incentive compatible, individually rational mechanism $(q(\theta), t(\theta), A(\theta))$ such that for a set of types θ of a positive measure we have $A(\theta) \neq \theta$. Then there exists an alternative mechanism $(\hat{q}(\theta), \hat{t}(\theta), \hat{A}(\theta))$ such that $\hat{A}(\theta) = \theta$ for almost all θ and which is strictly more profitable for the principal than the original mechanism.*

Significantly, Theorem 1 implies that the firm's problem can be stated as follows:

$$\max_{q(\theta)\geq 0, t(\theta)} \int_0^1 t(\theta) f(\theta) d\theta \tag{1.1}$$

subject to

$$u(q(\theta),\theta) - t(\theta) \geq u(q(\theta'),\theta) - t(\theta') - C \qquad \forall \theta, \theta' \in [0,1] \quad (IC) \tag{1.2}$$
$$u(q(\theta),\theta) - t(\theta) \geq 0 \qquad \forall \theta \in [0,1] \quad (IR) \tag{1.3}$$

We call $(q(\theta), t(\theta))$ an optimal mechanism if it solves the principal's maximization problem (1.1) subject to (1.2) and (1.3). We now have:

Theorem 2 *An optimal mechanism exists. It is unique if $u_{\theta qq}(q,\theta) \geq 0$ for all (q,θ).*

1.3 General Structure of the Optimal Mechanism

In this section we will establish a number of important general properties of an optimal mechanism. By Theorem 1 we can from now on denote the mechanism by a tuple $(q(.), t(.))$. Given an incentive compatible individually rational mechanism $(q(.), t(.))$, let $V(\theta) = u(q(\theta), \theta) - t(\theta)$ denote the associated net payoff of the agent-type θ in this mechanism.

The next Theorem establishes important properties of an optimal mechanism.

Theorem 3 *There exists an optimal mechanism $(q(.), t(.))$ such that for all $\theta \in [0,1]$:*

1. *$V(\theta)$ is Lipschitz continuous*[3]*, $q(\theta)$ and $t(\theta)$ are continuous, with $t(\theta) \geq 0$, for all $\theta \in [0,1]$.*

[3] With Lipschitz constant $L = \max_{\theta'' \in [0,1]} u_\theta(q^m, \theta'')$

2. *$V(\theta)$ is non-decreasing;*

3. *$q(\theta)$ is strictly increasing;*

4. *$0 < q(\theta) \leq q^{fb}(\theta)$ for all $\theta > 0$;*

The continuity and monotonicity results of Theorem 3 are standard in screening models without lying costs. In particular, the continuity and monotonicity of $V(.)$ and the monotonicity of $q(.)$ are a direct consequences of the assumption that $u(.)$ is increasing in θ and are necessary for implementability.

Yet, the nature and significance of the monotonicity and continuity results in Theorem 3 are different. Particularly, the presence of fixed costs creates a positive gap between the payoffs that the agent gets by reporting her true type and by imitating a close-by type, which makes it possible to implement non-monotone and discontinuous quantity schedule $q(.)$ and payoff function $V(.)$. To see this, suppose first that $q(.)$ and $V(.)$ are continuous and monotone. Then if type θ imitated type $\theta + \epsilon$ for some small, positive or negative, ϵ, she would get a payoff equal to $V(\theta + \epsilon) + u(q(\theta + \epsilon), \theta) - u(q(\theta + \epsilon), \theta + \epsilon) - C$ which is strictly less than her payoff $V(\theta)$. So, local incentive constraints are not binding for any type θ, and we can change $q(.)$ and $V(.)$ slightly in each neighborhood and, in particular, choose them to be decreasing and/or discontinuous without violating global incentive constraints either.

So, instead of relying on incentive and individual rationality constraints, the proof of Theorem 3 uses optimality arguments and shows that the principal can strictly improve her profits by modifying a mechanism in which $V(.)$ and $q(.)$ are non-monotone and/or discontinuous.

Note that the no-exclusion property $q(\theta) > 0$ is also due to the presence of fixed cost. Indeed, for every $\theta > 0$, there exists a sufficiently small $\underline{q}(\theta) > 0$ such that $u(\underline{q}(\theta), 1) - u(\underline{q}(\theta), \theta) < C$. Then assigning $\underline{q}(\theta)$ to an excluded type θ in exchange for transfer $u(\underline{q}(\theta), \theta)$ increases the seller's expected profit without violating any other type's incentive constraint.

Although the continuity of $V(.)$ and $q(.)$ imply that local incentive constraints are not binding for any type, yet, some incentive constraints must be binding when the fixed cost is not too large, for otherwise the optimal mechanism would involve first-best quantities and full surplus extraction by the principal. Thus, identifying and characterizing the set of binding incentive constraints is an important part in our analysis, and it is especially challenging since such constraints are non-local. We deal with this issue by, at first, establishing general properties of the binding incentive constraint correspondence in the following Theorem.

First let us define the targeted type correspondence $\tau(\theta)$ in the mechanism $(q(.), t(.))$ as follows:

$$\tau(\theta) = \left\{\theta' | u(q(\theta), \theta) - t(\theta) = u(q(\theta'), \theta) - t(\theta') - C\right\} \tag{1.4}$$

In words, $\tau(\theta)$ is the set of all such types θ' that incentive constraint $IC(\theta, \theta')$ of type θ is binding. Note that $\tau(\theta)$ may be empty. If $\tau(\theta)$ is non-empty which, as we show below, is true when θ is sufficiently large, then we will call the types in $\tau(\theta)$ "targeted types" of type θ.

Also, with a slight abuse of notation, for any set $\Theta \subseteq [0,1]$, we let $\tau(\Theta) = \cup_{\theta\in\Theta}\tau(\theta)$. The following Theorem provides key properties of the correspondence $\tau(.)$.

Theorem 4 *In an optimal mechanism,*

1. *For any fixed cost C, $0 < C < \overline{C} \equiv \max_{\theta\in[0,1]} u(q^{fb}(\theta),1) - u(q^{fb}(\theta),\theta)$ there exists $\hat{\theta} \in (0,1)$ such that $\tau(\theta) \neq \emptyset$ iff $\theta \in [\hat{\theta},1]$.*

2. *The correspondence $\tau(\theta)$ is strictly increasing, upper hemicontinous and compact-valued on $[\hat{\theta},1]$, with $\max\tau(\theta) < \theta$ and $\min\tau(\theta) > 0$.* [4]

3. *For all $\theta \in [0, \max\tau(\hat{\theta})] \cup [\min\tau(1),1]$, we have $q(\theta) = q^{fb}(\theta)$.*

4. *If $\theta_1, \theta_2 \in \tau(\theta)$ for some θ and $\theta_1 < \theta_2$, then $q(\theta') = q^{fb}(\theta')$ for all $\theta' \in [\theta_1,\theta_2]$.*

5. *$V(\theta) = 0$ for all $\theta \in [0,\hat{\theta}]$, $V(\theta) > 0$ for all $\theta \in (\hat{\theta},1]$.*

By Theorem 4 the screening problem is non-trivial iff

$$C < \overline{C} = \max_{\theta\in[0,1]} u(q^{fb}(\theta),1) - u(q^{fb}(\theta),\theta).$$

For each positive fixed cost below $\overline{C}$ only sufficiently high types have binding incentive constraints pointing to some strictly lower types, and earn positive surpluses. Moreover, only intermediate types are "targeted." This is intuitive, since imitating a low type gives any other type too little surplus that is not sufficient to cover the fixed cost C. Likewise, imitating a high type does not give enough surplus for another type to cover the fixed cost because high types are paying sufficiently high transfers in exchange for a high quantity. Figure 1.1 illustrates the relationship between targeted type correspondence τ and quantity q in the optimal mechanism. Since types below $\tau(\hat{\theta})$ and above $\tau(1)$ are not targeted by any type, there is no reason for principal to distort allocation of those types. As a result, they receive the first-best quantities.

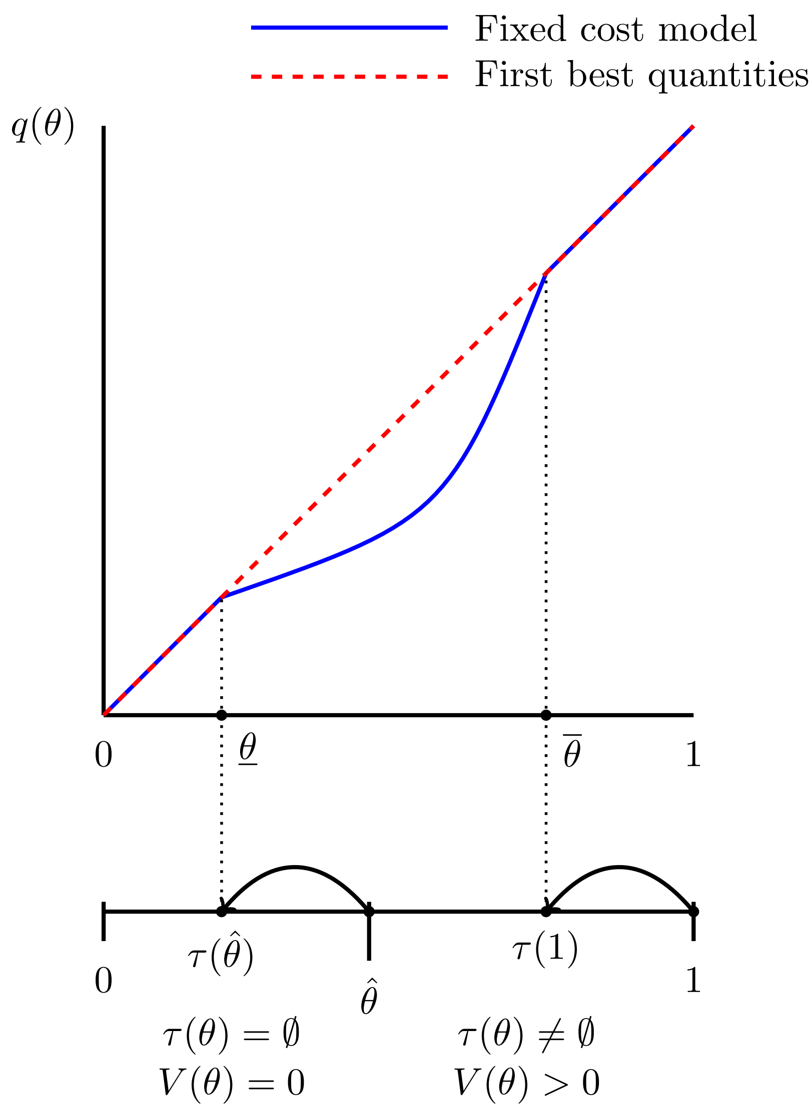
Significantly, Theorem 4 shows that targeted type correspondence is strictly increasing and compact-valued. As we will see below, the former property has particularly significant implications for the optimal mechanism.

The next Theorem provides comparative statics in C. For the purposes of this Lemma we slightly modify the notation and let $q(\theta|C)$ and $V(\theta|C)$ be the quantity and the net payoff of the type θ, respectively, and let $M(C)$ be the maximal length of a chain of targeted types in the unique optimal mechanism under fixed cost C.[5] Also, let $q^{sb}(\theta)$ and $V^{sb}(\theta)$ be the optimal

[4] By definition, τ is strictly increasing when the following is true: If $\theta > \theta'$, $t \in \tau(\theta)$ and $t' \in \tau(\theta')$, then $t > t'$.

[5] Note that for any given $C > 0$, the length $\theta - \tau(\theta)$ is bounded below for any θ because quantity and transfer are continuous in θ in the optimal mechanism. Therefore, the number of elements in the partition M is bounded above.

Figure 1.1: General Structure of Targeted Types and Quantities in the Optimal Mechanism



quantity and the net payoff of type θ, respectively, in the solution to the standard screening problem with zero cost of misrepresentation.

Theorem 5 *We have* $\lim_{C\downarrow 0} q(\theta|C) = q^{sb}(\theta)$, $\lim_{C\downarrow 0} V(\theta|C) = V^{sb}(\theta)$ *for all* $\theta \in [0,1]$, *and* $\lim_{C\downarrow 0} M(C) = \infty$.

1.4 Comparison with Related Literature

In this section, I compare this model with Maggi and Rodriguez-Clare [43]. In their paper, they also study a principal-agent screening problem in which there is a cost for the agent to misrepresent his type. The main difference between their setup and mine is that they assume that the cost of misrepresentation is a convex function of the distance between agent's reported type and true type, with zero fixed cost. I assume that cost of misrepresentation is a fixed cost. This twist in the assumption leads to several significant differences in the optimal mechanism.

First, while in my model there is no lying behaviour in the optimal mechanism, lying occurs in their optimal mechanism. Under a convex lying cost function, the principal benefits from inducing the agent to engage in some degree of lying. Mimicking a type who is already lying to the opposite direction becomes increasingly more costly because it requires even larger exaggeration. As a result, the principal can extract more informational rent from the agent by inducing lying behaviour. Under a fixed lying cost function, lying by a type does not make it more costly for another type to mimic, so the principal does not induce lying in the optimal mechanism.

Second, in their model, local incentive constraints are binding in the sense that a type's payoff function is continuous around his reported type under the optimal mechanism. In my model, there is a discontinuous jump, with the size of the fixed cost, around the agent's reported type (his true type). This discontinuity in payoff leads to non-local binding incentive constraints. In order to characterize those constraints, a different set of methodology is developed.

Third, in both their model and mine, allocative efficiency improves compared with the the standard screening model with no cost of lying. It is because lying costs relax agent's incentive constraints and thus the principal can implement more efficient allocation without deviation. However, it is worth-noting that in their model, the efficiency improvement is marginal across each type and first-best allocation is not achieved for any type. In my model, for any positive fixed cost, first-best allocation is achieved for intervals of low types and high types. Such qualitative distinction of optimal mechanisms derived from the two models might be useful in identification of lying cost structures.

1.5 Concluding Remarks

The general properties of optimal mechanism established in this chapter allow us to develop a methodology for characterizing the optimal mechanism, to formulate our problem as a dynamic

Figure 1.2: Comparison with Maggi and Rodriguez-Clare [43]

	This Model	Maggi and Rodriguez-Clare
Lying cost structure	Fixed cost invariant to true type	Convex costs dependent on the true type and the lie
Lying in optimal mechanism	No	Yes
Binding incentive constraints	Non-local	Local
Improvement in allocative efficiency	First-best allocation for low types and high types	Marginal improvement from second-best allocation

optimization one and to derive the necessary and sufficient conditions describing the optimal mechanism for general utility function and type distribution. They will be presented in chapter 2.

Chapter 2

Screening Under Fixed Cost of Misrepresentation: Characterization

2.1 Introduction

A significant difference between our model and the standard screening setting is that local incentive constraints are no longer binding when there is a fixed cost of lying. Indeed, imitating a close-by type invariably yields a lower payoff than telling the truth. Therefore, we can no longer use standard Mirrlees' method to derive the agent's surplus from the first-order condition and use it to replace incentive constraints.

Instead, we need to identify non-local incentive constraints that are binding at the optimum. To describe them, we introduced a concept of a "targeted type" $\tau(\theta)$ - a type or a set of types to which type θ has a binding incentive constraint. Significantly, $\tau(\theta)$ is endogenous, and its choice is one of the elements of the optimal design.

Further, targeted types form "chains." Specifically, if type θ targets some type θ' i.e., $\tau(\theta) = \theta'$, and type θ' targets some type θ'' i.e., $\tau(\theta') = \theta''$, then the types $\theta, \theta', \theta''$ are part of a single chain. The optimal quantity allocation of any type in a chain is then determined jointly with all other types in this chain.

In this chapter we derive a set of first-order conditions describing the optimal mechanism. These first-order conditions take the form of ordinary differential equations for the optimal quantity $q(\theta)$ and the targeted type $\tau(\theta)$. With quadratic utility function and uniform type distribution we are able to derive a closed form solution and exhibit the optimal mechanism explicitly in some cases.

The overall structure of the optimal mechanism involves an endogenous partition of the type space into intervals such that any type in an interval targets some type in the adjacent lower interval. As the fixed cost of lying decreases, the number of intervals in this partition increases, binding incentive constraints converge to the local ones i.e., $\tau(\theta) \to \theta$, and the optimal quantity allocation profile and transfers converge to the standard second-best. Conversely, the number of intervals decreases as the fixed cost becomes large. In particular, for a range of costs this partition contains only two elements. As the fixed costs increases further, binding incentive constraint disappear and the quantity allocation becomes the first-best. While not being particularly surprising, this limiting result provides an insight that second-best and first-best can be viewed as the two extreme cases as lying costs vary. Our model provides a generalization which is compatible with both cases, and also allows us to make predictions under intermediate costs of lying.

Thus the contribution of this model is two-fold. First, we characterize the optimal screening mechanism offered by a principal to an agent who incurs a fixed cost of lying, and highlight important qualitative properties of this mechanism.

The second contribution is methodological and involves developing new techniques to solve a class of principle-agent problems in which local incentive constraints are not binding and which, in contrast to standard ones, cannot be dichotomized into two parts, an implementability part which involves an envelope condition and the requirement that the allocation be monotone, and the second part involving an optimization under those constraints. We believe that the key elements of our approach, such as the characterization of binding non-local incentive constraints and the "targeted types," as well as the techniques of solving for them, could also be useful for solving other problems with binding non-local incentive constraints, potentially providing an important analytical instrument for various applications.

2.2 Deriving the Optimal Mechanism

In this section we reformulate our mechanism design problem (1.1) - (1.3) of choosing the quantity/transfer profile $(q(.), t(.))$ as a problem of choosing an optimal profile $(q(.), \tau(.), \hat{\theta})$, where $\tau(\theta)$ is a "targeted type" of θ, and where $\hat{\theta}$ is the lowest type for which $\tau(.)$ is non-empty, so that $V(\theta) > 0$ iff $\theta > \hat{\theta}$.

By Theorem 3, we may without loss of generality assume that $q(.)$, $t(.)$ and $V(.)$ are increasing and continuous, and hence almost everywhere differentiable. By Theorem 4, the correspondence τ is strictly increasing with a range $[0, 1]$. It implies that $\tau(\theta)$ is single-valued and differentiable almost everywhere. Incentive compatibility for type θ implies $\tau(\theta) = argmax_{\theta'} u(q(\theta'), \theta) - t(\theta') - C$. Since Theorem 4 implies that $0 < \tau(\theta) < 1$ for any $\theta \in [0, 1]$, $\tau(\theta)$ satisfies the following first-order condition for $\theta \in [\hat{\theta}, 1]$ almost everywhere:

$$u_q(q(\tau(\theta)), \theta)\dot{q}(\tau(\theta)) - \dot{t}(\tau(\theta)) = 0 \tag{2.1}$$

Then differentiating $V(\theta) = u(q(\tau(\theta)), \theta) - t(\tau(\theta)) - C$ at $\theta \in [\hat{\theta}, 1]$ and using (2.1) yields:

$$\dot{V}(\theta) = u_\theta(q(\tau(\theta)), \theta) + \dot{\tau}(\theta)[u_q(q(\tau(\theta)), \theta)\dot{q}(\tau(\theta)) - \dot{t}(\tau(\theta))] = u_\theta(q(\tau(\theta)), \theta) \tag{2.2}$$

Note that equation (2.2) holds almost everywhere at [0,1]. Since $V(.)$ is Lipschitz continuous and non-decreasing, it is an integral of its a.e. derivative $\dot{V}(\theta)$. So, given that $V(\theta) = 0$ for $\theta \in [0, \hat{\theta}]$, we have

$$V(\theta) = \int_{\hat{\theta}}^{\max\{\theta, \hat{\theta}\}} u_\theta(q(\tau(s)), s)ds \tag{2.3}$$

Equation (2.3) is the current model's analogue of the well-known envelope condition. The difference is that in our case the argument of $u_\theta(.)$ under the integration sign is $q(\tau(s))$, not $q(s)$, because type s has a binding incentive constraint to $\tau(s)$, not a local one. This implies that we cannot used the standard technique of substituting (2.3) into the objective and solving for the optimal profile $q(.)$. However, using (2.3) and the first-order condition (2.1) we can derive the law of motion of $q(.)$ which will allow us to characterize the optimal mechanism. First, note that

$$t(\theta) = u(q(\theta),\theta) - V(\theta) = u(q(\theta),\theta) - \int_{\hat{\theta}}^{\max\{\theta,\hat{\theta}\}} u_\theta(q(\tau(s)),s)ds. \tag{2.4}$$

Now differentiate (2.4) to get:

$$\dot{t}(\theta) = u_q(q(\theta),\theta)\dot{q}(\theta) + u_\theta(q(\theta),\theta) - 1(\theta \geq \hat{\theta})u_\theta(q(\tau(\theta)),\theta) \tag{2.5}$$

Combining (2.1) and (2.5) we obtain the following "law of motion" for all $\theta \in [\hat{\theta}, 1]$:

$$[u_q(q(\tau(\theta)),\theta) - u_q(q(\tau(\theta)),\tau(\theta))]\dot{q}(\tau(\theta))u_\theta(q(\tau(\theta)),\tau(\theta)) + 1(\tau(\theta) \geq \hat{\theta})u_\theta(q(\tau(\tau(\theta))),\tau(\theta)) = 0 \tag{2.6}$$

Next, we reformulate the objective of our problem. First, using (2.4) and integrating by parts yields the following expression for the seller's expected profits:

$$\int_0^1 [u(q(\theta),\theta) - \int_{\hat{\theta}}^{\max\{\theta,\hat{\theta}\}} u_\theta(q(\tau(s)),s)ds]f(\theta)d\theta = \int_0^1 u(q(\theta),\theta)f(\theta) - \int_{\hat{\theta}}^1 (1-F(\theta))u_\theta(q(\tau(\theta)),\theta)d\theta \tag{2.7}$$

Since $q(\theta) = q^{fb}(\theta)$ for all $\theta \in [0, \min\tau(\hat{\theta})] \cup [\max\tau(1), 1]$ we can further rewrite (2.7) as follows:

$$\int_{\theta\in[0,\min\tau(\hat{\theta})]\cup[\max\tau(1),1]} u(q^{fb}(\theta),\theta)f(\theta)d\theta + \int_{\min\tau(\hat{\theta})}^{\max\tau(1)} u(q(\theta),\theta)f(\theta)d\theta$$
$$- \int_{\hat{\theta}}^1 (1-F(\theta))u_\theta(q(\max\tau(\theta)),\theta)d\theta \tag{2.8}$$

For the sake of tractability, we will work with the problem with a single-valued τ function. The following assumptions and Lemma 1 state conditions under which $\tau(\theta)$ is single-valued.

Assumption 2 $G(\theta,\theta') \equiv u(q^{fb}(\theta'),\theta) - u(q^{fb}(\theta'),\theta')$ *is strictly quasi-concave in* θ'.

Assumption 2 holds for many commonly specified utility functions, for example, a linear quadratic one, $\theta q - \frac{q^2}{2}$.

Assumption 3 *For all* $\theta \in [0,1], q \in [0,\infty)$, $f'(\theta) \geq 0$, $u_{qqq}(q,\theta) \leq 0$, $u_{\theta qq}(q,\theta) = 0$, $u_{\theta\theta q}(q,\theta) \leq 0$.

Assumption 3 also holds under linear quadratic utility function and uniform distribution.

Lemma 1 *(i) Suppose that Assumption 2 holds and* $V(\theta) = 0$ *for any* $\theta \in \tau([0,1])$. *Then* $\tau(\theta)$ *is either single-valued or empty;*
(ii) Suppose that Assumptions 2 and 3 hold. Then $\tau(\theta)$ *is is either single-valued or empty.*

We will henceforth assume that the condition of Lemma 1 holds and hence $\tau(.)$ is a strictly increasing, single-valued function defined on $[\hat{\theta}, 1]$. Thus, objective (2.8) can be rewritten as:

$$\int_{\theta \in [0,\tau(\hat{\theta})] \cup [\tau(1),1]} u(q^{fb}(\theta),\theta) f(\theta) d\theta$$
$$+ \int_{\hat{\theta}}^{1} u(q(\tau(\theta)), \tau(\theta)) f(\tau(\theta)) \dot{\tau}(\theta) - (1 - F(\theta)) u_{\theta}(q(\tau(\theta)), \theta) d\theta \tag{2.9}$$

where the equality is obtained by making a change of variables in the second integral of objective (2.8) and combining it with the third integral.

Note that, provided that $\tau(.)$ and $q(.)$ are increasing functions, incentive constraints $IC(\theta, \theta')$ hold for all $(\theta, \theta') \in [0,1] \times [\tau(\hat{\theta}, 1]$.

Finally, recall that the following boundary conditions must hold:

$$q(\tau(1)) = q^{fb}(\tau(1)) \tag{2.10}$$
$$q(\tau(\hat{\theta})) = q^{fb}(\tau(\hat{\theta})) \tag{2.11}$$
$$V(\hat{\theta}) = u(q(\tau(\hat{\theta})), \hat{\theta}) - u(q(\tau(\hat{\theta})), \tau(\hat{\theta})) - C = 0 \tag{2.12}$$

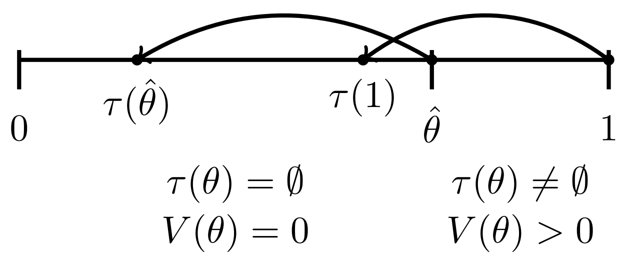
We will refer to the problem of maximizing (2.9) with respect to choice variables $(q(\theta), \tau(\theta), \hat{\theta})$ subject to the "law of motion" (2.6) and the boundary conditions (2.10)-(2.12) as a relaxed program. It is a relaxed program, because we have not imposed the incentive constraints that no type wishes to imitate a type $\theta' \in [0, \tau(\hat{\theta})] \cup [\tau(1), 1]$. Neither have we required $q(.)$ and $\tau(.)$ to be increasing, which must be the case in the optimal mechanism. At the same time, the individual rationality of the solution to the relaxed program follows directly from (2.4).

In the sequel we will solve the relaxed program and then establish that it satisfies the omitted constraints and its solution uniquely defines a solution $(q(.), t(.))$ to our original problem.

2.3 Solving for the Optimal Mechanisms: Intermediate Costs

In this section we solve the relaxed program formulated in the previous section and characterize the optimal mechanism for a range of intermediate values of the fixed cost C under which the optimal mechanism has a particularly simple structure, as shown in the next Theorem.

Figure 2.1: Structure of Targeted Types and Informational Rents Under Intermediate Costs of Lying



Theorem 6 *There exists* $\underline{C} \in (0, \overline{C})$ *such that if* $C \in (\underline{C}, \overline{C})$*, then in the optimal mechanism* $\tau(1) < \hat{\theta}$*.*

According to this Theorem, when $C \in (\underline{C}, \overline{C})$, then $\tau([0,1]) \subseteq [0, \hat{\theta}]$. Since $\tau(\theta) = \emptyset$ and $V(\theta) = 0$ for all $\theta < \hat{\theta}$, it follows that $\tau(\tau(\theta)) = \emptyset$ for all θ. Thus, the maximal length of the chain of targeted types is 1, as illustrated in Figure 2.1, and the last term in (2.6) is zero. Moreover, all types within the image of τ get zero net payoff, i.e. $V(\theta) = 0$ for all $\theta \in [\tau(\hat{\theta}), \tau(1)]$. Therefore, Lemma 1 ensures that $\tau(.)$ is a single-valued function given Assumption 2.

Our next step is to derive a solution to the relaxed program- maximizing (2.9) subject to (2.6) and (2.10)-(2.12)- via optimal control method. To this end, we will first make a change of variables. Specifically, let $Q(\theta) = q(\tau(\theta))$ be the quantity assigned to the targeted type of θ. Note that finding a solution $(q(\theta), \tau(\theta), \hat{\theta})$ to the relaxed program is equivalent to finding a solution $(Q(\theta), \tau(\theta), \hat{\theta})$. In particular, since $\dot{Q}(\theta) = \dot{q}(\tau(\theta))\dot{\tau}(\theta)$, we can rewrite (2.6) as follows:

$$\dot{Q}(\theta) = \frac{u_\theta(Q(\theta), \tau(\theta))}{u_q(Q(\theta), \theta) - u_q(Q(\theta), \tau(\theta))}\dot{\tau}(\theta) \text{ for all } \theta \in [\hat{\theta}, 1] \tag{2.13}$$

Next, let us define scrap values $S_0(\hat{\theta}, \tau(\hat{\theta}))$ and $S_1(\tau(1))$:

$$S_0(\hat{\theta}, \tau(\hat{\theta})) = \int_0^{\tau(\hat{\theta})} u(q^{fb}(\theta), \theta) f(\theta) d\theta \tag{2.14}$$

$$S_1(\tau(1)) = \int_{\tau(1)}^1 u(q^{fb}(\theta), \theta) f(\theta) d\theta \tag{2.15}$$

Now we can rewrite our relaxed program as follows:

$$\max_{Q(\theta),\tau(\theta),\hat{\theta}} \int_{\hat{\theta}}^{1} u(Q(\theta),\tau(\theta))f(\tau(\theta))\dot{\tau}(\theta) - (1-F(\theta))u_\theta(Q(\theta),\theta)]d\theta + S_0(\hat{\theta},\tau(\hat{\theta})) + S_1(\tau(1)) \tag{2.16}$$

subject to (2.13) and the boundary conditions:

$$R_1 \equiv Q(1) - q^{fb}(\tau(1)) = 0, \tag{2.17}$$

$$R_2 \equiv Q(\hat{\theta}) - q^{fb}(\tau(\hat{\theta})) = 0, \tag{2.18}$$

$$R_3 \equiv u(Q(\hat{\theta}),\hat{\theta}) - u(Q(\hat{\theta}),\tau(\hat{\theta})) - C = 0. \tag{2.19}$$

This problem is amenable to optimal control approach, with state variables $Q(.)$ and $\tau(.)$, control variable α satisfying $\dot{\tau}(\theta) = \alpha$, and a free boundary $\hat{\theta}$. Introducing the notation $h(\theta,Q,\tau) = \frac{u_\theta(Q,\tau)}{u_q(Q,\theta)-u_q(Q,\tau)}$. Equation (2.13) can be rewritten into the following laws of motion in the optimal control problem:

$$\dot{Q}(\theta) = h(\theta,Q,\tau)\alpha \tag{2.20}$$

$$\dot{\tau}(\theta) = \alpha \tag{2.21}$$

Thus, the Hamiltonian of this optimal control problem is given by:

$$H = u(Q,\tau)f(\tau)\alpha - (1-F(\theta))u_\theta(Q,\theta) + \lambda_Q h(\theta,Q,\tau)\alpha + \lambda_\tau\alpha \tag{2.22}$$

The linearity of the Hamiltonian (2.22) in the control variable α creates certain technical difficulties, as it implies that α cannot be solved for directly from the standard first-order conditions. However, Pontryagin's Maximum principle still applies and requires that the optimal control α maximizes the Hamiltonian (2.22).

Particularly, let us introduce the following *switching function*:

$$J(\theta,Q(\theta),\tau(\theta),\lambda_Q(\theta),\lambda_\tau(\theta)) = u(Q,\tau)f(\tau) + \lambda_Q h(\theta,Q,\tau) + \lambda_\tau \tag{2.23}$$

Note that the switching function J can never be strictly positive, since then the optimal value of α is infinity and, correspondingly, the value of the objective would be infinite. Optimality requires the following "switching conditions" to hold:

$$\begin{aligned} J(\theta,Q,\tau,\lambda_Q,\lambda_\tau) &< 0 \Rightarrow \alpha = 0 \\ J(\theta,Q,\tau,\lambda_Q,\lambda_\tau) &= 0 \Rightarrow \alpha \geq 0 \end{aligned}$$

An interval of θ on which $J < 0$ is called a *nonsingular arc*. The optimal solution involves setting $\alpha(\theta) = 0$ for all θ on a non-singular arc.

An interval of θ on which J vanishes ($J = 0$) is called a *singular arc*. On a singular arc, the optimality conditions do not pin down the value of the optimal control α. As a consequence, such problems of singular control are quite difficult to solve. Only a few solutions have been developed up to now, most notably Merton [45]'s celebrated portfolio choice problem in finance, and trajectory optimization in aeronautics (see e.g. Bryson and Ho [12] Ch. 8). The approach we follow here is to recover the optimal control α along a singular arc by differentiating the identity $J = 0$ with respect to θ until the control variable appears in a non-trivial way, and then solve for it. Significantly, our solution is simplified by finding that the whole domain in our case constitutes a singular arc, so that we do not have to characterize the juncture points between singular and non-singular arcs.

In addition, by Pontryagin's Maximum principle the solution has to satisfy the following costate equations:

$$\begin{aligned}
-\dot{\lambda_Q} = \frac{\partial H}{\partial Q} &= u_q(Q,\tau)f(\tau)\alpha - [1-F(\theta)]u_{\theta q}(Q,\theta) + \lambda_Q \frac{\partial h}{\partial Q}\alpha \\
&= u_q(Q,\tau)f(\tau)\alpha - [1-F(\theta)]u_{\theta q}(Q,\theta) \\
&+ \lambda_Q \frac{u_{\theta q}(Q,\tau)[u_q(Q,\theta)-u_q(Q,\tau)] - u_\theta(Q,\tau)[u_{qq}(Q,\theta)-u_{qq}(Q,\tau)]}{[u_q(Q,\theta)-u_q(Q,\tau)]^2}\alpha
\end{aligned} \tag{2.24}$$

$$\begin{aligned}
-\dot{\lambda_\tau} = \frac{\partial H}{\partial \tau} &= u_\theta(Q,\tau)f(\tau)\alpha + u(Q,\tau)f'(\tau)\alpha + \lambda_Q \frac{\partial h}{\partial \tau}\alpha \\
&= u_\theta(Q,\tau)f(\tau)\alpha + u(Q,\tau)f'(\tau)\alpha \\
&+ \lambda_Q \frac{u_{\theta\theta}(Q,\tau)[u_q(Q,\theta)-u_q(Q,\tau)] + u_\theta(Q,\tau)u_{\theta q}(Q,\tau)}{[u_q(Q,\theta)-u_q(Q,\tau)]^2}\alpha
\end{aligned} \tag{2.25}$$

In addition, the following transversality conditions have to hold for some $\gamma_1, \gamma_2, \gamma_3$:

$$\lambda_Q(1) = \gamma_1 \frac{\partial R_1}{\partial Q(1)} = \gamma_1 \tag{2.26}$$

$$\lambda_\tau(1) = \gamma_1 \frac{\partial R_1}{\partial \tau(1)} + \frac{\partial S_1}{\partial \tau(1)} = -\gamma_1 \dot{q}^{fb}(\tau(1)) - u(Q(1), \tau(1)f(\tau(1)) \tag{2.27}$$

$$-\lambda_Q(\hat{\theta}) = \gamma_2 \frac{\partial R_2}{\partial Q(\hat{\theta})} + \gamma_3 \frac{\partial R_3}{\partial Q(\hat{\theta})} = \gamma_2 + \gamma_3[u_q(Q(\hat{\theta}),\hat{\theta}) - u_q(Q(\hat{\theta}),\tau(\hat{\theta})] \tag{2.28}$$

$$\begin{aligned}
-\lambda_\tau(\hat{\theta}) &= \gamma_2 \frac{\partial R_2}{\partial \tau(\hat{\theta})} + \gamma_3 \frac{\partial R_3}{\partial \tau(\hat{\theta})} + \frac{\partial S_0}{\partial \tau(\hat{\theta})} \\
&= -\gamma_2 \dot{q}^{fb}(\tau(\hat{\theta})) - \gamma_3 u_\theta(Q(\hat{\theta}),\tau(\hat{\theta})) + u(Q(\hat{\theta}),\tau(\hat{\theta}))f(\tau(\hat{\theta}))
\end{aligned} \tag{2.29}$$

$$H(\hat{\theta}) = \gamma_3 \frac{\partial R_3}{\partial \hat{\theta}} = \gamma_3 u_\theta(Q(\hat{\theta}),\hat{\theta}) \tag{2.30}$$

Now, consider a singular arc where we have $J = u(Q,\tau)f(\tau) + \lambda_Q h(\theta,Q,\tau) + \lambda_\tau = 0$. Then by (2.22) $H(\hat{\theta}) = J(\hat{\theta})\alpha(\hat{\theta}) - [1-F(\hat{\theta})]u_\theta(Q(\hat{\theta}),\hat{\theta}) = -[1-F(\hat{\theta})]u_\theta(Q(\hat{\theta}),\hat{\theta})$. It can be easily verified that with $\gamma_3 = F(\hat{\theta}) - 1$ and $\gamma_1 = \gamma_2 = 0$, transversality conditions (2.26)-(2.30) are satisfied.

Differentiating the switching function J on a singular arc we get:

$$\begin{aligned}\frac{dJ}{d\theta} =& \dot{\lambda_Q} h(\theta, Q, \tau) + \lambda_Q \left(\frac{\partial h}{\partial \theta} + \frac{\partial h}{\partial Q}\dot{Q} + \frac{\partial h}{\partial \tau}\dot{\tau} \right) + \dot{\lambda}_\tau \\ &+ u_q(Q,\tau) f(\tau)\dot{Q} + u_\theta(Q,\tau) f(\tau)\dot{\tau} + u(Q,\tau) f'(\tau)\dot{\tau} = 0 \end{aligned} \tag{2.31}$$

Totally differentiating (2.31) yields:

$$\begin{aligned}-\dot{\lambda}_\tau =& \dot{\lambda}_Q h + \lambda_Q \left(\frac{\partial h}{\partial Q}\dot{Q} + \frac{\partial h}{\partial \tau}\dot{\tau} + \frac{\partial h}{\partial \theta} \right) \\ &+ u_q(Q,\tau) f(\tau)\dot{Q} + u_\theta(Q,\tau) f(\tau)\dot{\tau} + u(Q,\tau) f'(\tau)\dot{\tau} \end{aligned} \tag{2.32}$$

From (2.24), (2.32) and $\dot{\tau} = \alpha$,

$$\begin{aligned}-\dot{\lambda}_\tau =& - u_q(Q,\tau) f(\tau)\dot{\tau} h + (1 - F(\theta)) u_{\theta q}(Q,\theta) h - \lambda_Q \frac{\partial h}{\partial Q}\dot{\tau} h \\ &+ \lambda_Q \left(\frac{\partial h}{\partial Q}\dot{Q} + \frac{\partial h}{\partial \tau}\dot{\tau} + \frac{\partial h}{\partial \theta} \right) + u_q(Q,\tau) f(\tau)\dot{Q} + u_\theta(Q,\tau) f(\tau)\dot{\tau} + u(Q,\tau) f'(\tau)\dot{\tau} \\ =& (1 - F(\theta)) u_{\theta q}(Q,\theta) h + \lambda_Q \left(\frac{\partial h}{\partial \tau}\dot{\tau} + \frac{\partial h}{\partial \theta} \right) + u_\theta(Q,\tau) f(\tau)\dot{\tau} + u(Q,\tau) f'(\tau)\dot{\tau} \end{aligned} \tag{2.33}$$

Substituting (2.25) for $\dot{\lambda}_\tau$ on the left-hand side of (2.33) and using $\dot{\tau} = \alpha$ yields:

$$\begin{aligned}& u_\theta(Q,\tau) f(\tau)\dot{\tau} + u(Q,\tau) f'(\tau)\dot{\tau} + \lambda_Q \frac{\partial h}{\partial \tau}\dot{\tau} \\ =& (1 - F(\theta)) u_{\theta q}(Q,\theta) h + \lambda_Q \left(\frac{\partial h}{\partial \tau}\dot{\tau} + \frac{\partial h}{\partial \theta} \right) + u_\theta(Q,\tau) f(\tau)\dot{\tau} + u(Q,\tau) f'(\tau)\dot{\tau} \end{aligned}$$

which, after collecting terms and using $\dot{Q} = h\dot{\tau}$, simplifies to:

$$\lambda_Q \frac{\partial h}{\partial \theta} = -(1 - F(\theta)) u_{\theta q}(Q,\theta) h \tag{2.34}$$

Using $\frac{\partial h}{\partial \theta} = \frac{-u_{\theta q}(Q,\theta)}{u_q(Q,\theta) - u_q(Q,\tau)} h$ in (2.34) yields:

$$\lambda_Q = (1 - F(\theta))(u_q(Q,\theta) - u_q(Q,\tau)) \tag{2.35}$$

Next totally differentiate (2.35) to obtain:

$$\dot{\lambda}_Q = (1 - F(\theta))[u_{qq}(Q,\theta)\dot{Q} - u_{qq}(Q,\tau)\dot{Q} + u_{\theta q}(Q,\theta) - u_{\theta q}(Q,\tau)\dot{\tau}] - f(\theta)[u_q(Q,\theta) - u_q(Q,\tau)] \tag{2.36}$$

Now we can substitute (2.24) for $\dot{\lambda}_Q$ in (2.36) to obtain:

$$\begin{aligned}&u_q(Q,\tau)f(\tau)\dot{\tau} - [1-F(\theta)]u_{\theta q}(Q,\theta) + \lambda_Q \frac{\partial h}{\partial Q}\dot{\tau}\\ =& -(1-F(\theta))[u_{qq}(Q,\theta)\dot{Q} - u_{qq}(Q,\tau)\dot{Q} + u_{\theta q}(Q,\theta) - u_{\theta q}(Q,\tau)\dot{\tau}] + f(\theta)[u_q(Q,\theta) - u_q(Q,\tau)]\end{aligned} \tag{2.37}$$

Using (2.35), $\frac{\partial h}{\partial Q} = \frac{u_{\theta q}(Q,\tau) - [u_{qq}(Q,\theta) - u_{qq}(Q,\tau)]h}{u_q(Q,\theta) - u_q(Q,\tau)}$ and $\dot{Q} = h\dot{\tau}$ and cancelling terms in the previous equation yields the following differential equation:

$$\dot{\tau} = \frac{f(\theta)(u_q(Q,\theta) - u_q(Q,\tau))}{f(\tau)u_q(Q,\tau)} \tag{2.38}$$

Finally, using $\dot{Q} = h\dot{\tau} = \frac{u_\theta(Q,\tau)}{u_q(Q,\theta) - u_q(Q,\tau)}\dot{\tau}$ we obtain:

$$\dot{Q} = \frac{f(\theta)u_\theta(Q,\tau)}{f(\tau)u_q(Q,\tau)} \tag{2.39}$$

The system of ordinary differential equations (2.38) and (2.39) describes the dynamics of Q and τ in the optimal mechanism. The following Theorem shows that (2.38) and (2.39) with boundary conditions (2.17)-(2.19) uniquely characterize the optimal mechanism.

Theorem 7 *Suppose that $u_{\theta qq}(q,\theta) \geq 0$ for all $(q,\theta) \in \mathbf{R}_+ \times [0,1]$. For any $C \in (\underline{C}, \overline{C})$, there is a unique triple $(\tau(\theta), Q(\theta), \hat{\theta})$ such that $(\tau(\theta), Q(\theta))$ is an increasing solution to the system of ordinary differential equations (2.38) and (2.39) with boundary conditions (2.17)-(2.19), where in particular $\tau(\hat{\theta})$ is the smallest solution to (2.19).*

This triple $(\tau(\theta), Q(\theta), \hat{\theta})$ uniquely defines the optimal mechanism $(q(.), t(.))$ as follows: $q(\theta) = q^{fb}(\theta)$ for all $\theta \in [0, \tau(\hat{\theta})] \cup [\tau(1), 1]$, $q(\theta) = Q(\tau^{-1}(\theta))$ for all $\theta \in [\tau(\hat{\theta}), \tau(1)]$, and $t(.)$ is given by (2.4).

The next Theorem shows comparative statics of the optimal mechanism. Further discussion on comparative statics are given in the next subsection.

Theorem 8 *Suppose that $u_{\theta qq}(q,\theta) \geq 0$ for all $(q,\theta) \in \mathbf{R}_+ \times [0,1]$. Given any $C_i \in (\underline{C}, \overline{C})$, $i \in 1, 2$, let $(q_i(\theta), t_i(\theta))$ be the optimal mechanism and $(\tau_i(\theta), Q_i(\theta), \hat{\theta}_i)$ be the corresponding triple. If $C_2 > C_1$, then:*
(1) $\hat{\theta}_2 > \hat{\theta}_1$; (2) $\tau_2(\hat{\theta}_2) > \tau_1(\hat{\theta}_1)$;
(3) $\tau_2(\theta) < \tau_1(\theta)$ for $\theta \in [\hat{\theta}_2, 1]$; (4) $q_2(\theta) > q_1(\theta)$ for $\theta \in [\tau_2(\hat{\theta}_2), \tau_2(1)]$.

2.3.1 Quadratic-uniform Example

In this subsection we derive a closed-form solution to our problem for the case when $u(q,\theta) = \theta q - \frac{q^2}{2}$, θ is uniformly distributed on $[0,1]$, and length of chain $M = 1$.

Given the quadratic-uniform assumptions and $M = 1$, differential equations (2.38)- (2.39) and boundary conditions (2.17)-(2.19) imply that the optimal $\tau(\theta)$ and $Q(\theta)$ satisfy the following differential equations:

$$\dot{\tau} = \frac{\theta - \tau}{\tau - Q} \tag{2.40}$$

$$\dot{Q} = \frac{Q}{\tau - Q} \tag{2.41}$$

with boundary conditions

$$Q(1) = \tau(1) \tag{2.42}$$

$$Q(\hat{\theta}) = \tau(\hat{\theta}) \tag{2.43}$$

$$Q(\hat{\theta})[\hat{\theta} - \tau(\hat{\theta})] = C \tag{2.44}$$

Ordinary differential equation system (2.40)-(2.44) has the following parametric solution defined for $t \in [\hat{t}, 1]$:

$$\theta(t) = b_1 \left(t - \frac{1 + 3\sqrt{\frac{1}{5}}}{2} t^{\frac{\sqrt{5}-1}{2}} + \frac{3\sqrt{\frac{1}{5}} - 1}{2} t^{-\frac{\sqrt{5}+1}{2}} \right) + \frac{\sqrt{5}+1}{2\sqrt{5}} t^{\frac{\sqrt{5}-1}{2}} + \frac{\sqrt{5}-1}{2\sqrt{5}} t^{-\frac{\sqrt{5}+1}{2}} \tag{2.45}$$

$$Q(t) = -\frac{b_1}{2} t \tag{2.46}$$

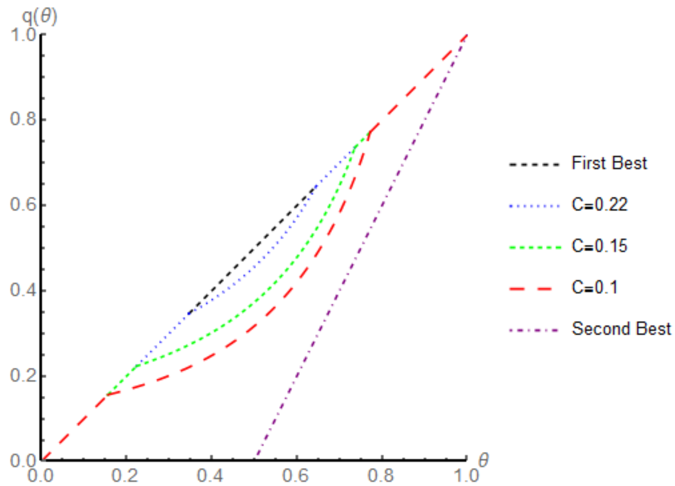
$$\tau(t) = b_1 \left(\frac{t}{2} - \frac{1 + \sqrt{\frac{1}{5}}}{2} t^{\frac{\sqrt{5}-1}{2}} - \frac{1 - \sqrt{\frac{1}{5}}}{2} t^{-\frac{\sqrt{5}+1}{2}} \right) + \frac{1}{\sqrt{5}} t^{\frac{\sqrt{5}-1}{2}} - \frac{1}{\sqrt{5}} t^{-\frac{\sqrt{5}+1}{2}} \tag{2.47}$$

$$b_1 = -\frac{\frac{1}{\sqrt{5}} \hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{1}{\sqrt{5}} \hat{t}^{-\frac{\sqrt{5}+1}{2}}}{\hat{t} - \frac{1+\sqrt{\frac{1}{5}}}{2} \hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{1-\sqrt{\frac{1}{5}}}{2} \hat{t}^{-\frac{\sqrt{5}+1}{2}}} \tag{2.48}$$

$$C = -\frac{b_1}{2} \left(b_1 \left(\frac{\hat{t}^2}{2} - \frac{1}{\sqrt{5}} \hat{t}^{\frac{\sqrt{5}+1}{2}} + \frac{1}{\sqrt{5}} \hat{t}^{-\frac{\sqrt{5}-1}{2}} \right) + \frac{\sqrt{5}-1}{2\sqrt{5}} \hat{t}^{\frac{\sqrt{5}+1}{2}} + \frac{\sqrt{5}+1}{2\sqrt{5}} \hat{t}^{-\frac{\sqrt{5}-1}{2}} \right) \tag{2.49}$$

where $(\theta(t), Q(t), \tau(t))$ characterize the implicit functions $(Q(\theta), \tau(\theta))$ defined on $\theta \in [\hat{\theta}, 1]$, with $\theta(\hat{t}) = \hat{\theta}$ and $\theta(1) = 1$. The optimal quantity $q(\theta)$ for $\theta \in [\tau(\hat{\theta}), \tau(1)]$ can be computed via the following relation: $q(\tau(t)) = Q(t)$.

Figure 2.2: Optimal Quantities, Quadratic-uniform Case



The admissible cost range for this example (where $\tau(1) < \hat{\theta}$) is $(\underline{C}, \overline{C})$, where $\overline{C} = 0.25$ and $\underline{C} \approx 0.09$. For C in this range, $(Q(\theta), \tau(\theta), q(\theta))$ and scalars $(b_1, \hat{t})$ are uniquely determined by (2.45)-(2.49).

This solution exhibits several properties. The optimal quantity $q(\theta)$ is strictly increasing in θ, which is consistent with the general property given by Theorem 3. In this particular example, $q(\theta)$ is also strictly convex for $\theta \in [\tau(\hat{\theta}), \tau(1)]$.

For comparative statics, an increase in cost of lying create potential slackness of incentive compatibility, which is filled by two forces to generate extra profit. First, principal generate higher revenue by improving efficiency of the mechanism. As illustrated in Figure 2.2, optimal quantities increase for the medium types. The interval of types with distorted quantity $[\tau(\hat{\theta}), \tau(1)]$, becomes narrower, and quantities converge to first best level as the cost goes to $\overline{C}$. Second, principal extracts more surplus from the agent. Note from Figure 2.3 that for a higher C, the targeted type $\tau(\theta)$ is lower for any given type. It reduces agent's surplus $V(\theta) = \int_{\hat{\theta}}^{\theta} u_\theta(q(\tau(\theta')), \theta')d\theta'$ in intensive margin. In addition, the cutoff type $\hat{\theta}$ is increasing in C, which reduces agent's surplus in extensive margin. As C goes to $\overline{C}$, $\hat{\theta}$ converges to 1 and all surplus is extracted.

Finally, this solution applies under the condition that any type in the image of τ gets zero surplus in the optimal mechanism, i.e. $\tau(1) \leq \hat{\theta}$. The lower bound of cost of lying that satisfies this condition is $\underline{C} \approx 0.09$. Figure 2.4 shows that $\hat{\theta}$ and $\tau(1)$ converge to each other as C approaches the lower bound.

Figure 2.3: Optimal Targeted Types, Quadratic-uniform Case

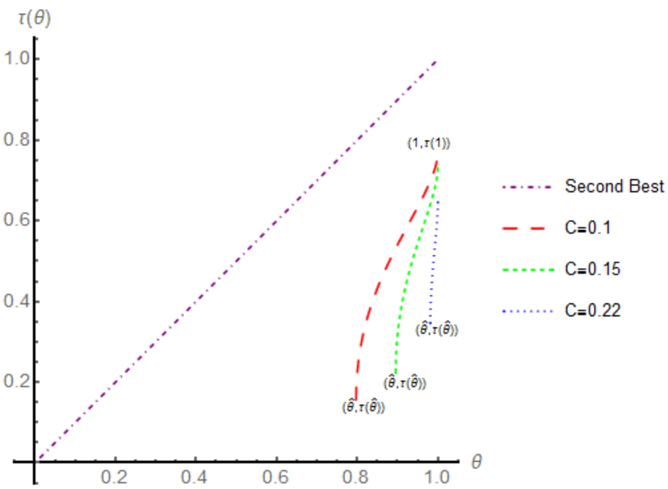
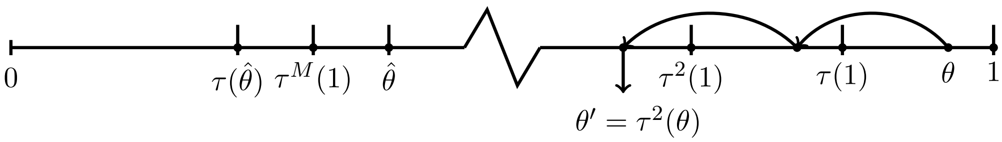


Figure 2.4: Optimal Values of $\hat{\theta}, \tau(1)$ and $\tau(\hat{\theta})$, Quadratic-uniform Case

C	$\hat{\theta}$	$\tau(1)$	$\tau(\hat{\theta})$
0.09	0.78	0.78	0.15
0.1	0.80	0.77	0.16
0.15	0.90	0.74	0.23
0.22	0.98	0.65	0.35
0.25	1	0.5	0.5

Figure 2.5: Chains of Incentive Constraint



2.4 Optimal Mechanisms: Low Costs

The goal of this section is to provide necessary and sufficient conditions for the optimal mechanism when cost of lying is below the threshold $\underline{C}$ specified in Theorem 6. First, we will use the targeted type correspondence to define a partition of the interval $[\hat{\theta}, 1]$ that we will use below to solve our problem. For this, let us define a higher-order targeted type correspondence recursively as follow: for any θ let $\tau^0(\theta) = \theta$, and for any integer $k \geq 1$ let $\tau^k(\theta) = \tau(\tau^{k-1}(\theta))$. Then $(\tau(\theta), ..., \tau^k(\theta), ...)$ is a chain of targeted types originating from θ. Let $M = \max\{k : \tau^k(1) \neq \emptyset\}$ be the length of such chain originating from the highest type $\theta = 1$. Since by Theorem 4 $\tau(.)$ is continuous and increasing, it maps the interval $[\tau^k(1), \tau^{k-1}(1)]$ onto the adjacent interval $[\tau^{k+1}(1), \tau^k(1)]$ or, equivalently, $\tau^{k-1}(.)$ maps the interval $[\tau(1), 1]$ onto the interval $[\tau^k(1), \tau^{k-1}(1)]$ for all $k \in \{1, ..., M\}$. So $\{[\tau^k(1), \tau^{k-1}(1)\}]\}_{k=1}^M$ is a parition of the interval $[\tau^M(1), 1]$ which is non-overlapping except at the boundary points.

Since, by definition $\hat{\theta} = \min\{\theta | \tau(\theta) \neq \emptyset\}$ and $M = \max\{k : \tau^k(1) \neq \emptyset\}$, we have $\tau^{M-1}(1) \geq \hat{\theta} > \tau^M(1)$. So, $[\hat{\theta}, 1] = \cup_{k \in \{1,...,M-1}[\tau^k(1), \tau^{k-1}(1)] \cup [\hat{\theta}, \tau^{M-1}(1)]$. This partition has the property that for every $k \in \{1, ..., M-1\}$ and $\theta \in [\tau^k(1), \tau^{k-1}(1)]$, we have $\tau(\theta) \in [\tau^{k+1}(1), \tau^k(1)]$, and the residual interval $[\hat{\theta}, \tau^{M-1}(1)]$ is such that $\tau(\theta) \neq \emptyset$ for $\theta \in [\hat{\theta}, \tau^{M-1}(1)]$, while $\tau(\theta) = \emptyset$ for $\theta \in [\tau^M(1), \hat{\theta})$. Then there exists $\hat{\theta}_M \in (\tau(1), 1]$ such that $\tau^{M-1}(\hat{\theta}_M) = \hat{\theta}$, and for every $\theta \in [\hat{\theta}_M, 1]$ we have $\max\{k : \tau^k(\theta) \neq \emptyset\} = M$, while for every $\theta \in [\tau(1), \hat{\theta}_M)$, $\max\{k : \tau^k(\theta) \neq \emptyset\} = M - 1$. In other words, letting $M(\theta)$ be the length of the chain of targeted types originating at $\theta \in [\tau(1), 1]$, we have:

$$M(\theta) = \begin{cases} M & \text{if } \theta \in [\hat{\theta}_M, 1] \\ M - 1 & \text{if } \theta \in [\tau(1), \hat{\theta}_M) \end{cases}$$

Therefore, we can rewrite $\int_{\hat{\theta}}^1 u(q(\tau(\theta)), \tau(\theta)) f(\tau(\theta)) \dot{\tau}(\theta) - (1 - F(\theta)) u_\theta(q(\tau(\theta)), \theta) d\theta$, the second term in the objective on the right-hand side of (2.9) as a sum of integrals over non-overlapping collection of intervals $[\tau^k(1), \tau^{k-1}(1)]$, $k \in \{1, ..., M-1\}$ and $[\hat{\theta}, \tau^{M-1}(1)]$, and then "fold it"' by making a change of variables on the interval $[\tau^k(1), \tau^{k-1}(1)]$ using the functions $\tau^{k-1}(.)$, $k \in \{1, ..., M-1\}$, and using the function $\tau^{M-1}(.)$ on the interval $[\hat{\theta}, \tau^{M-1}(1)]$. This procedure yields:

$$
\begin{aligned}
&\int_{\hat{\theta}}^{1} u(q(\tau(\theta)),\tau(\theta))f(\tau(\theta))\dot{\tau}(\theta)-(1-F(\theta))u_\theta(q(\tau(\theta)),\theta)d\theta \\
&=\sum_{k=1}^{M-1}\int_{\tau^k(1)}^{\tau^{k-1}(1)} u(q(\tau(\theta)),\tau(\theta))f(\tau(\theta))\dot{\tau}(\theta)-(1-F(\theta))u_\theta(q(\tau(\theta)),\theta)d\theta \\
&\quad+\int_{\hat{\theta}}^{\tau^{M-1}(1)} u(q(\tau(\theta)),\tau(\theta))f(\tau(\theta))\dot{\tau}(\theta)-(1-F(\theta))u_\theta(q(\tau(\theta)),\theta)d\theta \\
&=\sum_{k=1}^{M-1}\int_{\tau(1)}^{1} u(q(\tau^k(\theta)),\tau^k(\theta))f(\tau^k(\theta))\dot{\tau}^k(\theta)-(1-F(\tau^{k-1}(\theta)))u_\theta(q(\tau^k(\theta)),\tau^{k-1}(\theta))\dot{\tau}^{k-1}(\theta)d\theta \\
&\quad+\int_{\hat{\theta}_M}^{1} u(q(\tau^M(\theta)),\tau^M(\theta))f(\tau^M(\theta))\dot{\tau}^M(\theta)-(1-F(\tau^{M-1}(\theta)))u_\theta(q(\tau^M(\theta)),\tau^{M-1}(\theta))\dot{\tau}^{M-1}(\theta)d\theta
\end{aligned}
\tag{2.50}
$$

Next, let $Q^k(\theta)=q(\tau^k(\theta))$ for $k=1,...,M(\theta)$. Substituting this into (2.50) and using the result in (2.9) yields the following reformulated objective if our problem.

$$
\begin{aligned}
&\int_{\tau(1)}^{1}\sum_{k=1}^{M-1} u(Q^k(\theta),\tau^k(\theta))f(\tau^k(\theta))\dot{\tau}^k(\theta)-(1-F(\tau^{k-1}(\theta)))u_\theta(Q^k(\theta),\tau^{k-1}(\theta))\dot{\tau}^{k-1}(\theta)d\theta \\
&+\int_{\hat{\theta}^M}^{1} u(Q^M(\theta),\tau^M(\theta))f(\tau^M(\theta))\dot{\tau}^M(\theta)-(1-F(\tau^{M-1}(\theta)))u_\theta(Q^M(\theta),\tau^{M-1}(\theta))\dot{\tau}^{M-1}(\theta)d\theta \\
&+S_0(\tau^M(\hat{\theta}_M))+S_1(\tau(1))
\end{aligned}
\tag{2.51}
$$

where $S_0(\tau^M(\hat{\theta}_M))$ and $S_1(\tau(1))$ are the scrap values of our problem given by:

$$S_0(\tau^M(\hat{\theta}_M))=\int_0^{\tau^M(\hat{\theta}_M)} u(q^{fb}(\theta),\theta)f(\theta)d\theta \tag{2.52}$$

$$S_1(\tau(1))=\int_{\tau(1)}^{1} u(q^{fb}(\theta),\theta)f(\theta)d\theta \tag{2.53}$$

Next, differentiating $Q^k(\theta)=q(\tau^k(\theta))$ and using (2.6) yields:

$$\dot{Q}^k(\theta)=\frac{u_\theta(Q^k(\theta)),\tau^k(\theta))-1(\tau^k(\theta)\geq\hat{\theta})u_\theta(Q^{k+1}(\theta),\tau^k(\theta))}{u_q(Q^k(\theta),\tau^{k-1}(\theta))-u_q(Q^k(\theta),\tau^k(\theta))}\dot{\tau}^k(\theta) \tag{2.54}$$

Thus, we obtain a maximization problem with objective (2.51), $2M$ choice variables $Q^1,...,Q^M$,

$\tau^1, ..., \tau^M$, the "law of motion" (2.54), and boundary conditions

$$\tau^{k+1}(1) = \tau^k(\tau(1)) \qquad \text{for } k = 0, ..., M-1 \tag{2.55}$$

$$Q^{k+1}(1) = Q^k(\tau(1)) \qquad \text{for } k = 1, ..., M-1 \tag{2.56}$$

$$Q^1(1) = q^{fb}(\tau(1)) \tag{2.57}$$

$$Q^M(\hat{\theta}_M) = q^{fb}(\tau^M(\hat{\theta}_M)) \tag{2.58}$$

$$u(Q^M(\hat{\theta}_M), \tau^{M-1}(\hat{\theta}_M)) - u(Q^M(\hat{\theta}_M), \tau^M(\hat{\theta}_M)) - C = 0 \tag{2.59}$$

The boundary conditions (2.55) and (2.56) connect the values of (τ^{k+1}, Q^{k+1}) at 1 and the values of (τ^k, Q^k) at $\tau(1)$ ensuring the continuity of the solution. Conditions (2.57) and (2.58) ensure that the optimal first-best quantities are assigned at the lower end of the type interval $[\tau(1), 1]$ and at the upper end of the interval $[0, \tau^M(\hat{\theta}^M)]$, respectively. This is optimal by continuity as all types in $(\tau(1), 1]$ and $[0, \tau^M(\hat{\theta}^M))$ do not have any binding constraints pointing to them and are therefore assigned first-best quantities. Finally, condition (2.59) ensures that type $\hat{\theta}$ receives zero surplus.

This problem can be solved using the optimal control method. For this, one has to consider two problems. The first one - on the interval $[1, \hat{\theta}^M]$, and the second one - on the interval $[\hat{\theta}^M, \tau(1)]$. Alternatively, it can be solved using the perturbation method. We follow the latter solution strategy. (At the same time we exhibit optimal control solution in a technical Appendix at the end of the paper). The result is provided in the following Theorem:

Theorem 9 *The solution to the mazimization problem (2.51) with boundary conditions (2.55) - (2.59) satisfies the following system of differential equations of* $\tau^1, ..., \tau^M; Q^1, ..., Q^M$*:*

$$\dot{\tau}^k = \frac{f(\theta)[u_q(Q^k, \tau^{k-1}) - u_q(Q^k, \tau^k)]}{f(\tau^k) u_q(Q^k, \tau^k)} \prod_{s=1}^{k-1} \frac{u_q(Q^s, \tau^{s-1})}{u_q(Q^s, \tau^s)} \tag{2.60}$$

$$\dot{Q}^k = \begin{cases} \frac{f(\theta)[u_\theta(Q^k, \tau^k) - u_\theta(Q^{k+1}, \tau^k)]}{f(\tau^k) u_q(Q^k, \tau^k)} \prod_{s=1}^{k-1} \frac{u_q(Q^s, \tau^{s-1})}{u_q(Q^s, \tau^s)} & \textit{if } k < M(\theta) \\ \frac{f(\theta) u_\theta(Q^k, \tau^k)}{f(\tau^k) u_q(Q^k, \tau^k)} \prod_{s=1}^{k-1} \frac{u_q(Q^s, \tau^{s-1})}{u_q(Q^s, \tau^s)} & \textit{if } k = M(\theta). \end{cases} \tag{2.61}$$

Theorem 9 provides a system of $2M$ first-order differential equations (2.60) and (2.61) in $2M$ variables, with $2M + 2$ boundary conditions from (2.55)-(2.59), along with two free boundaries θ_M and $\tau(1)$. Generically, this system has a unique solution.

The following intermediate step of deriving (2.60) and (2.61) provides some intuition behind these optimality conditions.

$$u_q(Q^k, \tau^k) f(\tau^k) \dot{\tau}^k = [u_q(Q^k, \tau^{k-1}) - u_q(Q^k, \tau^k)] \sum_{s=1}^{k} f(\tau^{k-s}) \dot{\tau}^{k-s} \tag{2.62}$$

The left hand side of (2.62) is the marginal gain on efficiency for increasing $Q^k(\theta)$ given the relative density of $\tau^k(\theta)$. The right hand side of (2.62) is the marginal cost on informational rent for increasing $Q^k(\theta)$. It reflects the nature of the trade-off: for an additional unit of quantity assigned to $\tau^k(\theta)$, informational rent has to be given to the type who targets $\tau^k(\theta)$, i.e. $\tau^{k-1}(\theta)$, in order to prevent $\tau^{k-1}(\theta)$ from imitating $\tau^k(\theta)$. Furthermore, increase in informational rent makes the contract of $\tau^{k-1}(\theta)$ more attractive, so the same amount of informational rent has to be given to $\tau^{k-2}(\theta)$, and thus every preceding types in the chain, to prevent imitation.

Note that the number of elements of the partition M is endogenous. To find it, start with assuming $M = 1$ and solve it deriving the optimal θ_M for this case. If $\tau^M(1) \leq \tau^{M-1}(\theta_M)$, then M is the optimal partition size. Otherwise, repeat the same process with $M + 1$. Iterate this procedure until we have found the optimal M.

2.5 Concluding Remarks

This model sheds light on the role of fixed cost of lying in screening frameworks. The introduction of a fixed cost of lying reshape the screening problem into a new class of principle-agent problem with non-locally binding incentive constraints. We develop a method to represent the problem as an optimal control, in which the binding non-local constraints, "targeted types", and the physical allocations are jointly solved. We derive the optimality condition of the problem, which can be interpreted as an endogenous discretization of the standard optimal screening. The model produces several qualitatively novel results. We show that the standard exclusion property is not robust to a small fixed cost of lying. On the contrary, full efficiency is achieved for low types. We provide an example for the optimal mechanism given linear-quadratic utility under uniform type distribution.

While this dissertation only characterizes the optimal screening mechanism given type-independent fixed cost of lying, it is likely that the important properties of our methodological approach, such as the characterization of binding non-local incentive constraints and the targeted type concept, also apply under more general cost of lying with non-zero fixed cost.

Chapter 3

Lying and Lie-detection

3.1 Introduction

In many situations, an informed party is tempted to misrepresent information when communicating with an uninformed party. Politicians hope to impress voters by exaggerating their past achievements and future policy goals; companies want to convince consumers to believe that their products are better than what they actually are; suspects who have committed crimes often refuse to plead guilty. Such tension is widely studied in the literature with various resolutions.

Contract theory assumes the uninformed receiver can commit to a menu of outcomes corresponding to the informed sender's report of his private information. The cheap talk literature assumes no such commitment but the sender and the receiver share a degree of common interest in how they would want to respond to the sender's private information. In both cases, lying by the sender about his private information does not occur in equilibrium.

This chapter introduces a theory of lying in cheap talk communication where the receiver has access to a costly lie-detecting technology. Claims made by the sender are statements regarding the state. Some of them emerge in equilibrium as lies and others as truth[6]. Lie-detection is a broadly used technology in our daily lives. For instance, skeptical voters might go online to fact-check a politician's claim. The innovation in this dissertation is that lies emerge as equilibrium claims about the state in the face of costly lie-detection. To my knowledge, this is the first paper that develops a framework of endogenous lie-detection. It allows strategic interaction between lying and lie-detecting, and analyzes tensions between the sender's incentive of lying and the receiver's incentive of inspection. This assumption is motivated by the observation that not all claims are treated equally by an uninformed party. Oftentimes some claims are more suspicious than the others and draw more attention in inspecting them. For instance, if a health product company claims their new drug is proven to reduce any cancer risk by 99 percent, potential consumers might be skeptical and search if such claim is backed by any trustworthy, independent studies; but if the company makes a mild claim that the drug strengthens immune system, consumers might just take the company's word and do not bother checking it.

This chapter analyzes a framework with endogenous lying and lie-detection. An informed

[6]Sobel [56] points out that lying depends on the existence of accepted meanings for messages. In this paper, a sender's message has the following accepted meaning: "The true state is within Θ", where Θ is some subset of the state space. Naturally, the sender is lying if the true state is not one of the states in Θ.

sender prefers a higher action by an uninformed receiver to a lower action independent of the true state, while the receiver has a quadratic loss function and wants to match the action with the true state. The sender communicates with the receiver by making a claim modeled as a subset of possible states; a claim is truthful if it includes the true state and is a lie otherwise. Before taking the payoff relevant action, the receiver has the option to incur a cost to inspect the truthfulness of the claim. The list of claims made in an equilibrium that maximizes receiver's expected payoff is an optimal design for costly lie-detection.

Lie-detection technology improves the receiver's welfare only if lying occurs in equilibrium. If the sender never lies, the receiver has no incentive to inspect the sender's claims; If there is no inspection, babbling is the only equilibrium as the sender and the receiver share no common interest. I show that there exists equilibrium where lying and lie-detection occur if and only if inspection cost is sufficiently low and prior expectation of the state is not too high. The threshold for prior expectation increases as inspection cost decreases and converges to the upper bound of the state space as inspection cost goes to zero. Intuitively, this result comes from the conflict between the sender's incentive to lie and the receiver's incentive to inspect. Liars aim to convince the receiver that they are better than the average (prior expectation) when they get away with the lie. If the prior expectation is too high, this can happen only when a small number of liars mimic a large number of truth-tellers, but then the claim is not worth inspecting because the sender is too likely to be truthful. This result echoes a common perception that lie-detection is effective when the sender is suspicious, in the sense that there is a substantial difference between the receiver's prior belief and the belief preferred by the sender. For example, the police usually conduct an interrogation only if they believe that the suspect is likely to have committed a crime. When the sender is likely to be "innocent", there is no cost-effective way to separate lies from truths.

Even though full revelation is achievable using lie-detection technology, it is neither sequentially rational nor ex-ante optimal for the receiver because some claims are not worth inspecting. The receiver's benefit from lie-detection can be decomposed into two components. First, lie-detection generates a direct information value by distinguishing liars from truth-tellers which generally provides information about the true state. Second, the sender might stay honest in fear of being caught lying. Therefore, the possibility of lie-detection creates a threat that deters potential liars and facilitates information transmission. This is called the indirect deterrent effect. I show that under the optimal lie-detection policy, the direct information value of inspection is completely offset by the cost of inspection. Improvement of the receiver's ex-ante payoff is driven by the deterrent effect: the receiver is able to elicit information from the sender due to the credible threats of lie-detection. This is perhaps surprising as one might expect an optimal lie-detection design should allow the receiver to acquire as much information as possible. Such intuition turns out to be incorrect because excess amount of information acquired from inspection indicates that the design induces sender to lie too often, and costly inspection takes place more frequently than necessary. This suggests that lie-detection technology better serves as a mean of deterrence than a mean of information acquisition.

The receiver's ex-ante payoff depends on the degree of information transmission facilitated by the indirect deterrence, which is affected by the set of claims available to the sender. Therefore, an optimal lie-detection design involves not only a contingent plan of inspection, but also a set of admissible claims. I show that the optimal design is characterized by three intervals which partition the state space. The sender makes truthful claims when the true state is in the high interval (good types), lies and mimics the high claims when the true state is in the low interval (bad types). These high claims are randomly inspected. In fear of being caught lying and perceived as bad types, the sender in the intermediate interval (moderate types) is deterred from mimicking the high claims. These moderate types pool at a vague yet truthful claim which is not inspected by the receiver. It is optimal for the receiver to give the sender an option of being vague because precise claims require inspections to sustain, while moderate types are not distant enough with each other to justify the cost of inspection. Technically speaking, it is always optimal to pool an interval of types to a single claim and leave it uninspected because the conditional variance of a small enough interval is smaller than the inspection cost.

If the density of the bottom half of the prior distribution is concentrated toward the center, then the optimal design corresponds to a so-called decreasing mimicking mechanism, where the inspected high claims are precise, meaning that the optimal set of admissible claims consists of a vague claim that indicates moderate types and a continuum of precise high claims. The mimicking is decreasing in the sense that liars with lower types mimic truth-tellers with higher types. Inspection probabilities of those claims are chosen so that liars are indifferent between all equilibrium lies. Since liars who make higher claims will be punished by worse posterior beliefs upon lie-detection, the inspection probability need not be increasing in the level of claim. In practice, remaining silent can be interpreted as the vague claim which induces a moderate belief from the receiver. Any claims in attempts to induce better beliefs are required to be precise and will be inspected stochastically.

I study the effect of inspection technology on the receiver's welfare by comparing lie-detection with state-verification. There are substantial differences between lie-detection and state-verification. Under state-verification technology, an inspection reveals the true state of the world. There will be no uncertainty upon inspection. Under lie-detection technology, an inspection returns a binary signal on the truthfulness of the sender's claim. Information learned from an inspection is endogenously determined by the claim made by the sender. Practically, state-verification is a hard skill which requires the receiver to be able to acquire knowledge about the true state, which might not be feasible in some situation. For example, there might not be any objective evidence in the crime scene that provides further information about whether a suspect has committed to the crime. On the other hand, lie-detection can be a soft skill. A competent detective might be able to spot a lie told by the suspect using various interrogation tactics. Studies in psychology and cognitive science have shown possibilities of detecting lies using methods such as asking questions that raise cognitive load (Vrij et al. [61]), measuring brain activities (Christ et al. [14]) and reading micro-expressions (Porter and Ten Brinke [51]), with nearly 70 percent accuracy (Hartwig and Bond [34]) and 85 percent accuracy for trained interviewers (Hartwig et al. [35]).

Even if state-verification is feasible, lie-detection technology can yield a higher benefit to the receiver. Assuming the same unit cost of the two technologies, I show that the receiver's welfare is higher under optimal lie-detection design compared with optimal state-verification design. This is because revealing the true state upon inspection removes any strategic uncertainty that can serve as a threat of punishment to potential deviators. Since state-verification leads to an accurate assessment of the true state, there is no credible punishment for the liar thus sender always has the incentive to exaggerate the state to "try his luck". As a result, the deterrence effect is eliminated under state-verification technology and there will not be any informative communication. This result sheds light on the optimal approaches of fact-checking as a tool to combat misinformation in politics. The internet has enabled the public to more easily verify politicians' claims using fact-checking websites such as *FactCheck.org* and *PolitiFact*. A question regarding the socially desired mission of these organizations is whether they should focus on presenting verdicts on politicians' statements (lie-detection) or educating the public about policy-related issues (state-verification). The latter is more informative as verdicts on politicians' statements can be derived from knowledge in policy-related issues. An argument for the former is that simple verdicts cost less time to read and are easier to comprehend, compared with the complex policy-related issues. Another argument for the former is that targeting politicians' statements hold them accountable and deter them from lying. Some studies find evidence that fact-checking reduces lying behaviors of politicians (e.g. Nyhan and Reifler [47]; Lim [42]). This paper provides a theoretical ground for the deterrence argument and shows that the public can be better off under lie-detection in spite of the ignorance of details in policy-related issues.

My work presented in this chapter is mostly related to the literature of strategic communication with lie-detection. Balbuzanov [6] analyzes a cheap talk model akin to the setup in Crawford and Sobel [17], with the addition that a lie of the sender will be detected with an exogenous probability. He shows that given intermediate probability of lie-detection and sufficiently small bias, fully revealing equilibria exist. Dziuda and Salas [21] study a pure persuasion game with the same lie-detection technology as Balbuzanov [6] and show that certain refinement criteria lead to a unique equilibrium where moderate types and high types stay honest and low types lie to imitate high types. My findings in optimal mechanisms echo findings from Dziuda and Salas [21] that moderate types do not exaggerate their types to avoid being mistaken as the low type liars. The key difference between this paper and previous literature is that this paper models lie-detection as a decision of the receiver, where the probability of lie-detection can be chosen conditional on the sender's claim. This allows an analysis of tensions between the sender's incentive of lying and the receiver's incentive of inspection. Under different sets of permissible claims, the resolutions of such tensions result in different degrees of inspection and information revelation, and hence different payoffs. Therefore, this leads to a non-trivial design problem on the optimal set of permissible claims. Jehiel [37] analyzes an interesting multi-round cheap talk environment where lie can be spotted from the inconsistent messages of a forgetful liar who cannot remember the content of the lie he has told.

Another strand of literature study strategic communication with lying cost for the sender. Kartik, Ottaviani and Squintani [39] study a strategic communication environment where the sender is upwardly biased and there is an exogenous cost of misrepresentation. They show that when the state space is unbound above, fully separating equilibrium exists in which sender lies and uses inflated language. Kartik [40] studies a similar environment and shows that when state space is bound above, there is some pooling on the highest message and the degree of information revelation depends on the intensity of lying cost. While lying behaviors arise in equilibrium in both their models and my model, the natures and interpretations of lies are quite different. In my model, lies serve as disguises to confuse the receiver. Liars try to mimic the types they claim to be and the receiver cannot tell them apart without an inspection. In their models, lies serve as inflated languages. The sender tells lie to avoid being mistaken as a worse type and a strategic receiver does not confuse a liar with the type he claims to be. An alternatives interpretation of the models in Kartik, Ottaviani and Squintani [39], alongside other related works (e.g. Ottaviani and Squintani [49]; Chen [13]) is that a proportion of receivers naively believes sender's message. The coexistence of strategic and naive receivers imposes an endogenous cost for the sender to overly exaggerate the state since the naive receivers will take it at face value, which is not preferred by a sender whose bias is not too large. In the equilibria of their models, lies are chosen by the sender to balance the induced beliefs of two groups of receivers who interpret messages differently. In my model, lies are chosen to mimic the corresponding truthful senders and confuse the receiver.

For a broader discussion on the role of lying in strategic interactions, Sobel [56] establishes a general framework of lying with various applications. My model adopts the same definition of lying as in Sobel. His framework does not incorporate the possibilities of lie-detection.

3.2 The Lie-detection Model

There are a decision-maker (DM) and a sender. DM has to make a decision, but only the sender has the relevant information. Sender privately observes the state of the world, θ, which is distributed according to a continuously differentiable c.d.f. F over the normalized state space $\Theta \equiv [0,1]$, with associated density f. θ is also referred to as the sender's type. For example, θ might represent the quality of the advertised product or the severity of crimes committed by a suspect.

Message: The sender sends a message $m \in \mathcal{M}$ to DM, where $\mathcal{M}$ is the set of all measurable subsets of the state space Θ. A message sent by the sender is interpreted as a statement regarding his type. To provide a few examples, a message $m = [0.3, 0.4]$ is interpreted as the following statement: "my type lies somewhere in between 0.3 and 0.4"; a message $m = \{0.5\} \cup \{0.7\}$ is interpreted as "my type is either 0.5 or 0.7"; a message $m = \Theta$ can be interpreted as to remain silent because it essentially means "Anything is possible".

Costly inspection: DM, after observing m, chooses whether or not to inspect the message with a cost $c > 0$. An inspection reveals the truthfulness of the statement. Formally, if an inspection takes place, DM will receive a binary signal

$$s(m,\theta) = \begin{cases} t & \text{if } \theta \in m \\ l & \text{otherwise} \end{cases} \tag{3.1}$$

If DM chooses not to inspect, she receives an uninformative signal $s(m,\theta) = u$. The signal t indicates the sender's claim is inspected and confirmed to be truthful; l indicates the sender's claim is inspected and confirmed to be a lie; u indicates the sender's claim is uninspected.

Action: After observing both the message m and the inspection signal s, DM chooses a payoff relevant action $x \in [0,1]$.

Preference: DM has a quadratic loss function $-(x-\theta)^2 - c\mathbf{1}_I$, where $\mathbf{1}_I = 1$ if an inspection took place, $\mathbf{1}_I = 0$ otherwise. The sender has a von Neumann-Morgenstern utility $u(x)$ which is strictly increasing in x. In other words, there is no common interest between DM and the sender. DM wants to take an action that matches the true state, while the sender always prefers a higher action, independent of the state.

The design problem: An mechanism (q, P, X) consists of a message rule $q : \Theta \to \Delta_{\mathcal{M}}$, where $q(.|\theta)$ is type θ's probability distribution over the message space $\mathcal{M}$; an inspection rule $P : \mathcal{M} \to [0,1]$, where $P(m)$ is the probability of inspecting message m; and an action rule $X : \mathcal{M} \times \{t,l,u\} \to [0,1]$, where $X(m,s)$ is the action taken following message m and inspection signal $s \in \{t,l,u\}$.

For expositional clarity, I confine attention to pure message rule in this chapter, i.e. each type of sender θ sends a message $m_q(\theta)$ with probability 1. In the appendix, I show that for any incentive compatible mechanism with mixed message rule, there is an equivalent incentive compatible pure message mechanism which generates the same outcome distribution. Therefore, all results can be generalized to allow mixed message rules.

Given a pure message rule m_q, let $\mathcal{M}_q = m_q(\Theta)$ be the set of all on-path messages [7]. For any on-path message $m \in \mathcal{M}_q$, let

$$\Theta_q^t(m) = \{\theta \in \Theta : m_q(\theta) = m \text{ and } \theta \in m\} \tag{3.2}$$

$$\Theta_q^l(m) = \{\theta \in \Theta : m_q(\theta) = m \text{ and } \theta \notin m\} \tag{3.3}$$

$$\Theta_q^u(m) = \Theta_q^t(m) \cup \Theta_q^l(m) \tag{3.4}$$

be the sets of truthful senders, lying senders and senders of m. DM cannot commit to an inspection rule and/or an action rule. They have to be sequentially rational based on a Bayesian updated belief.

[7] Throughout this Chapter, I follow the convention and refer to $g(X)$ as $\{y : \exists x \in X \text{ such that } y \in g(x)\}$ for any function or correspondence g and set X within the domain of g.

Sequentially rational action: Since DM's utility is quadratic, her optimal action equal conditional expectation of the sender's type given the posterior belief, so an action rule X is **sequentially rational** given q if for any $m \in \mathcal{M}_q$ and $s \in \{t, l, u\}$,

$$X(m,s) = E[\Theta_q^s(m)] \tag{3.5}$$

where $E[\Theta'] \equiv \frac{\int_{\Theta'} \theta dF(\theta)}{Pr(\Theta')}$ denotes the conditional expected type given a set of type $\Theta' \subseteq \Theta$, and $Pr(\Theta') \equiv \int_{\Theta'} dF(\theta)$ denotes the probability of Θ' [8].

After observing the on-path message m and inspection signal s, DM chooses an action to match the conditional expected type of senders who send m and lead to inspection signal s given the message rule q. Instead of blindly taking a message at its face value, a Bayesian, sequentially rational DM updates her belief given the set of equilibrium senders who would pass/fail an inspection, and reacts optimally. When there is no inspection, DM remains aware of the possibility of lying and chooses an action that matches the weighted average type of the equilibrium truth-tellers and liars.

Information value of inspection: Given a message rule q and a sequentially rational action rule X, DM's expected continuation payoff if she inspects an on-path message $m \in \mathcal{M}_q$ is:

$$-w_q(m)Var(\Theta_q^l(m)) - (1 - w_q(m))Var(\Theta_q^t(m)) \tag{3.6}$$

where $w_q(m) = \frac{Pr(\Theta_q^l(m))}{Pr(\Theta_q^u(m))}$ is the conditional probability of sender being a liar given that he sends m, $Var(\Theta') \equiv \frac{\int_{\Theta'}(\theta - E[\Theta'])^2 d\theta}{Pr(\Theta')}$ denotes the conditional variance given Θ'[9]. Upon inspection, DM's expected loss from action imprecision for a message m is the weighted average conditional variance of equilibrium truth-tellers and liars of m. DM's expected continuation payoff if she does not inspect m is:

$$-Var(\Theta_q^u(m)) \tag{3.7}$$

which is the variance of the sender's type conditional on him sending m. By the law of total variance,

$$\begin{aligned} Var(\Theta_q^u(m)) =& w_q(m)Var(\Theta_q^l(m)) + (1 - w_q(m))Var(\Theta_q^t(m)) \\ &+ w_q(m)(1 - w_q(m))(E[\Theta_q^l(m)]^2 + E[\Theta_q^t(m)]^2 - 2E[\Theta_q^l(m)]E[\Theta_q^t(m)]) \end{aligned}$$

[8]It is possible that $Pr(\Theta_q^s(m)) = 0$ even if the message m is on-path, if m is sent by a set of types with zero measure but positive density. Therefore, a more precise version of condition (3.5) is that for any subset of on-path messages $M \subseteq \mathcal{M}_q$, $\int_M \int_{\Theta_q^s(m)} X(m,s) dF(\theta) dm = \int_{\Theta_q^s(M)} \theta dF(\theta)$ for $s \in \{t, l, u\}$. This ensures that DM's action rule is sequentially rational given q almost surely.

[9]Similarly, a more precise condition for $w_q(m)$ is that for any $M \subseteq \mathcal{M}_q$, $\int_M w_q(m) \int_{\Theta^l_q(m)} dF(\theta) dm = Pr(\Theta_q^l(M))$.

The information value of inspecting m is the reduction in conditional variance from the binary signal, which is the difference between (3.6) and (3.7):

$$\begin{aligned} V_q(m) =& Var(\Theta_q^u(m)) - [w_q(m)Var(\Theta_q^l(m)) + (1-w_q(m))Var(\Theta_q^t(m))] \\ =& w_q(m)(1-w_q(m))(E[\Theta_q^l(m)] - E[\Theta_q^t(m)])^2 \end{aligned} \tag{3.8}$$

An inspection allows DM to make a better inference on the sender's type and chooses more precise action accordingly. If there is a large difference between the expected type of truth-tellers and liars who send m, the value of differentiating these two groups is high. Besides, an inspection is more informative when the liar to truth-teller ratio is less extreme. If the sender of m is very likely to be on one side, not much information is revealed from an inspection. An inspection rule P is **sequentially rational** given q if for any $m \in \mathcal{M}_q$,

$$P(m) \in \begin{cases} \{0\} & \text{if } c > V_q(m) \\ [0,1] & \text{if } c = V_q(m) \\ \{1\} & \text{if } c < V_q(m) \end{cases}. \tag{3.9}$$

i.e. inspecting a message is credible only if information value of inspection is no less than cost of inspection.

Sender's optimality: Given inspection rule P and action rule X, type θ sender's expected utility from sending a message m is:

$$EU_{X,P}(m|\theta) = \begin{cases} P(m)u(X(m,t)) + (1-P(m))u(X(m,u)) & \text{if } \theta \in m \\ P(m)u(X(m,l)) + (1-P(m))u(X(m,u)) & \text{if } \theta \notin m \end{cases} \tag{3.10}$$

A pure message rule q is **optimal** given P and X if for any $\theta \in \Theta$ and $m' \in \mathcal{M}_q$ [10],

$$EU_{X,P}(m_q(\theta)|\theta) \geq EU_{X,P}(m'|\theta) \tag{3.11}$$

3.3 Incentive Compatible Mechanisms

This Section defines an incentive compatible mechanism and establishes the necessary and sufficient conditions for the existence of an incentive compatible mechanism where inspections take place with positive probability.

Definition 1 *An mechanism $\Omega \equiv (q, P, X)$ is incentive compatible if P and X are sequentially rational given q and q is optimal given P and X.*

[10] Incentive constraints over off-path messages are omitted because sequential rationality put no restriction on the inspections and actions following those messages, so we can without loss of generality let $X(m', s) = 0$ for any off-path message m', and sender will have no incentive to deviate to those messages.

Since the decision-maker has no commitment power, an incentive compatible mechanism requires that DM has no incentive to deviate after any history, so it corresponds to a Perfect Bayesian equilibrium in a game-theoretic approach. The two concepts are interchangeable under this framework. Given an incentive compatible mechanism Ω, DM's ex-ante expected payoff is:

$$EU_{DM}(\Omega) = -\int_{\mathcal{M}_q} \sum_{s=t,l} \int_{\Theta^s(m)} [(1 - P(m))(X(m,u) - \theta)^2 + P(m)[(X(m,s) - \theta)^2 + c]]dF(\theta)dm \tag{3.12}$$

Define

$$G_\Omega(x) = \int_{\mathcal{M}_q} \sum_{s=t,l} \int_{\Theta^s(m)} [(1 - P(m))\mathbf{1}(X(m,u) \leq x) + P(m)\mathbf{1}(X(m,s) \leq x)]dF(\theta)dm \tag{3.13}$$

be the distribution of induced actions under Ω, and

$$p_\Omega = \int_{\mathcal{M}_q} P(m) \int_{\Theta^u_q(m)} dF(\theta)dm \tag{3.14}$$

be the ex-ante probability that a sender is inspected under Ω. Sequential rationality of the action rule X implies that

$$EU_{DM}(\Omega) = \int_{[0,1]} x^2 dG_\Omega(x) - cp_\Omega - E[\theta^2] \tag{3.15}$$

where $E[\theta^2] \equiv \int_\Theta \theta^2 dF(\theta)$. Sender's ex-ante expected payoff is:

$$EU_S(\Omega) = \int_{[0,1]} u(x)dG_\Omega(x) \tag{3.16}$$

I refers to the pair (G_Ω, p_Ω) as the **induced outcome distribution** of an mechanism. I say two mechanisms Ω and Ω' are **distribution equivalent** if they have the same induced outcome distribution. Since (G_Ω, p_Ω) uniquely determine payoffs in an incentive compatible mechanism, two incentive compatible, distribution equivalent mechanisms induce the same expected payoffs for DM and every type of sender.

Since DM cannot commit to a sub-optimal action rule, the expected value of induced actions must equal the expected value of the state. In fact, the distribution of induced actions G is a mean-preserving contraction of F. A more dispersed G implies a more precise match between the induced actions and the states, and thus a higher expected payoff for DM.

Lemma 2 *For any incentive compatible mechanism* $\Omega = (q, P, X)$ *there exists a distribution equivalent mechanism* $(\hat{q}, \hat{P}, \hat{X})$ *such that for any* $m \in \mathcal{M}_{\hat{q}}$*:*
(i) $\hat{X}(m,t) \geq \hat{X}(m,l)$*, and*
(ii) $m = \Theta^t_{\hat{q}}(m)$*.*

Condition (i) of Lemma 2 provides a natural interpretation of the mechanism: liars pretend to be truth-tellers in the hope of inducing higher actions [11]. Condition (ii) comes from the fact that condensing the statement of a message to include only equilibrium truth-tellers is the most effective design in maintaining incentive compatibility. Under such design, any type who deviates from his equilibrium message to any other on-path message will be identified as a liar, which according to (i), gets a lower expected payoff than if he is identified as a truth-teller. Lemma 2 is useful in analyzing the set of implementable outcome because an outcome distribution is implementable if and only if it can be induced by an incentive compatible mechanism that satisfies the above properties [12]. Unless otherwise stated, any mechanism discussed henceforth satisfies conditions (i) - (ii) of Lemma 2.

Let $\mathcal{M}^0_q = \{m \in \mathcal{M}_q : P(m) = 0\}$ be the set of on-path uninspected messages. Sender's optimality requires that any messages in $\mathcal{M}^0_q$ must induce the same action, for otherwise senders who receive a lower uninspected action will deviate to a higher one. Therefore, we can without loss assume that there is at most one such message, m^0_q, and all senders of that message are truthful, i.e. $m^0_q = \Theta^u_q(m) = \Theta^t_q(m) \equiv \Theta^0_q$, where Θ^0_q is the set of types who are never inspected in equilibrium.Sequential rationality of X requires $X(m^0_q, u) = E[\Theta^0_q]$. Let $\mathcal{M}^+_q = \{m \in \mathcal{M}_q : P(m) > 0\}$ be the set of messages that are inspected with positive probability. $\mathcal{M}^+_q$ is simply referred to as the **set of inspected messages**. For $\theta \in \Theta$, I say θ is **truthful** if $\theta \in \Theta^t_q(\mathcal{M}^+_q)$; θ is **lying** if $\theta \in \Theta^l_q(\mathcal{M}^+_q)$; θ is **uninspected** if $\theta \in \Theta^0_q$.

I say Ω is a **mechanism with inspection** if $p_\Omega > 0$, i.e, some on-path messages are inspected with positive probability. The following assumption and Theorem establish the necessary and sufficient conditions for the existence of incentive compatible mechanism with inspection.

Assumption 4 ***(Low cost)*** $c < \frac{1}{4}$ *and* ***(Pessimism)*** $E[\Theta] \equiv \int_0^1 \theta dF(\theta) < \frac{1}{2} + \sqrt{\frac{1}{4} - c}$*.*

[11] In a model where sender can make a truthful claim and tricks the lie-detector to identify him as a liar (for example, by acting nervous or intentionally failing a test), then condition (i) must holds in any incentive compatible mechanism for any inspected message m, for otherwise equilibrium truth-tellers who act normally and get $X(m,t)$ will deviate to act nervously and get $X(m,l)$.

[12] Note however that oftentimes an implementable outcome distribution can also be induced by other incentive compatible mechanisms. For example, if there exists an on-path message m' which is never inspected, and Θ' is the set of senders of m', then an incentive compatible mechanism that satisfies (ii) requires the statement m' to be a subset of Θ'. However, the mechanism will still be incentive compatible if senders of m' simply " remain silent ", i.e. $m' = \Theta$. By definition, it means every type becomes truth-teller of m', but it has no effect on the sender's incentive because being truthful and lying makes no difference to the outcome when m' is never inspected.

Theorem 10 *There exists an incentive compatible mechanism with inspection if and only if Assumption 4 is satisfied.*

The credibility of inspections relies on the existence of both liars and truth-tellers. Upon receiving a message, if DM's interim expectation on the sender's type is extreme (either too high or too low), the information value of an inspection is low because the sender is either very likely to be truth-telling or very likely to be lying, and inspection is non-credible. Now consider an uninspected message m and a randomly inspected message m'. In order to incentivize the liars who send m' to take the risk of being caught, DM's interim expectation on sender's type upon receiving m' must be higher than the interim expectation upon receiving m, so that if liars of m' get away with the lie, they receive a higher payoff than those who send m. Since DM is Bayesian, her interim expectation upon receiving some inspected messages must be higher than the prior expectation, so if the prior expectation is too optimistic, the interim expectation of those messages will be too optimistic for inspection to be credible. It is worth noting that the condition is not symmetric. Even if prior expectation on the sender's type is pessimistic, it is possible to design a mechanism with pessimistic interim belief for the uninspected message and moderate interim beliefs for the inspected messages so that liars of the inspected message are incentivized and inspections are credible. Therefore, the lie-detection technology is useful when the prior expectation is moderate or pessimistic, but not when the prior expectation is optimistic. Figure 3.1 depicts the region of parameter values in which an incentive compatible mechanism with a positive probability of inspection exists. The threshold of prior expectation such that inspection is incentive compatible is decreasing in cost of inspection, meaning that when the cost is smaller, inspection is incentive compatible for a larger range of optimistic beliefs. When the cost of inspection is small, inspection is credible even if conditional expectations given the inspected statements are optimistic and information values of inspection are small. Inspection can therefore facilitate information transmission. As cost goes to 0, the lie-detection technology is useful for almost any prior distribution.

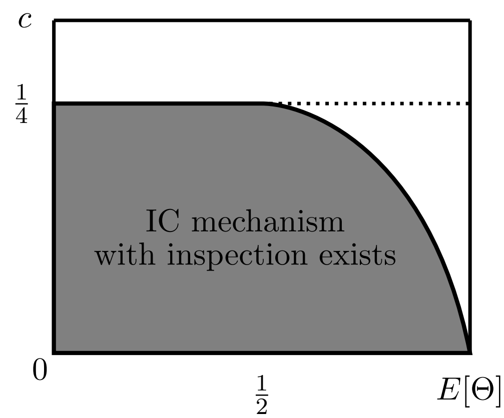
3.4 Optimal Mechanisms

This section defines optimal mechanism and establishes the properties of optimal mechanisms.

Definition 2 *An mechanism Ω is optimal if it is incentive compatible, and for any incentive compatible mechanism Ω', $EU_{DM}(\Omega) \geq EU_{DM}(\Omega')$.*

An optimal mechanism induces the highest expected payoff to DM among all incentive compatible mechanisms. I focus on analyzing the best mechanism for DM because oftentimes DM's welfare reflects the public interest, for instance consumers and voters who have to make decisions under incomplete information. The optimal mechanism indicates an upper bound to the welfare of the public under lie-detection technology. Besides, the optimal mechanism

Figure 3.1: Existence of Incentive Compatible Mechanism with Inspection



Note: Horizontal axis depicts the expectation on type under prior distribution F; Vertical axis depicts the cost of inspection.

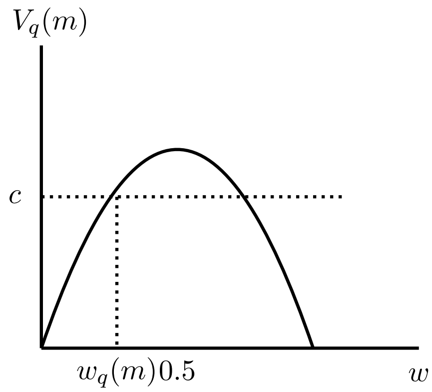
minimizes weighted average objective of inference error and inspection cost. Therefore, it can be interpreted as the most efficient way of combating misinformation using lie-detection technology. On the other hand, the sender's welfare is sensitive to his risk attitude. However, it is worth-noting that if the sender is risk neutral, he will get the same ex-ante payoff in any incentive compatible mechanism because the mean of the induced action distribution must equal the prior expectation of the state. In such case, the outcome induced by an optimal mechanism is also Pareto-efficient. When sender is risk averse, he prefers a more concentrated action distribution. In such case, the ex-ante interests of sender and receiver conflict with each other.

Now we derive some properties of an optimal mechanism. For any set of messages $M \in \mathcal{M}$, let $Pr_q(M) = Pr(\Theta_q^u(M))$ be the ex-ante probability of the senders of M under message rule q. We say a property holds **almost everywhere** for a set of messages M if it holds for a subset of messages $M' \subseteq M$ such that $Pr_q(M') = Pr_q(M)$.

Theorem 11 ***(No direct benefit of inspection)***
In an optimal mechanism Ω, $V_q(m) = c$ almost everywhere for $m \in \mathcal{M}_q^+$.

The value of lie-detection technology to DM is composed of two parts: direct information value and indirect deterrence effect. Theorem 11 says that direct information value of inspection is offset by the cost of inspection in an optimal mechanism, and the net benefit of inspection comes from its effect on the sender's incentive: some types of sender refrain from making a

Figure 3.2: The Optimal Proportion of Liars



Note: Horizontal axis depicts the proportion of liars in an inspected message m; Vertical axis depicts value of inspecting m. An optimal mechanism minimizes the proportion liars subject to the constraint that $V(m) \geq c$. The expression of $w_q(m)$ is given in (3.18).

higher claim because of the possible lie-detection. As a result, some information is transmitted through the messages in the sense that interim expectations of the sender's type upon receiving different messages are different, so DM is able to make a better inference on the sender's type even when inspection does not take place ex-post.

Theorem 12 ***(Liars are minority)***
In an optimal mechanism Ω, $w_q(m) \leq 0.5$ almost everywhere for $m \in \mathcal{M}_q^+$.

Theorem 12 says that for any inspected message in an optimal mechanism, liars are a minority compared with truth-tellers. It is because the role of liars is to sustain moderate liar to truth-teller ratios so that information values are high enough for credible inspections. Such ratios can be achieved by either a minority of liars or a majority of liars. Compared with a mechanism with a majority of liars, a mechanism with a minority of liars means that the expected types of the sender of inspected messages are higher. That creates larger differences between conditional expectations given inspected messages and the uninspected message, which means more information is transmitted through messages under a mechanism with a minority of liars.

Theorem 13 ***(Three-interval structure)***
In an optimal mechanism Ω, there exists $\underline{\theta}_\Omega$ and $\overline{\theta}_\Omega$ such that $0 \leq \underline{\theta}_\Omega < \overline{\theta}_\Omega \leq 1$ and for almost every $\theta \in \Theta$, θ is lying if $\theta < \underline{\theta}_\Omega$; truthful if $\theta > \overline{\theta}_\Omega$; uninspected if $\theta \in [\underline{\theta}_\Omega, \overline{\theta}_\Omega]$.

An optimal mechanism has a three-interval configuration such that when the state is above the cutoff $\overline{\theta}_\Omega$, sender is truthful; When the state is below the cutoff $\underline{\theta}_\Omega$, sender lies and claims that the state is somewhere above $\overline{\theta}_\Omega$, such claims are inspected with positive probabilities; When the state is intermediate, sender makes the claim in which DM does not inspect. Such configuration induces disperse inferences upon inspection, which benefit DM the most.

Under the optimal mechanism, low type senders are incentivized to lie in order to justify inspections of the truthful statements made by high type senders. Such inspections prevent moderate type senders from exaggerating their types in fear of getting caught lying and perceived as low types.

3.4.1 Optimal Mechanism: Decreasing Mimicking with Precise Statements

This subsection defines the decreasing mimicking mechanism and establishes the conditions under which such a mechanism is optimal.

For $d \in [2\sqrt{c}, 1]$, define

$$w^-(d) = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{c}{d^2}} \tag{3.17}$$

which is the smaller root of the equation $w(1-w)d^2 = c$. Given that d is the distance between the conditional expected type of truth-tellers and liars in a message m, $w^-(d)$ is the minimum proportion of liars such that information value of inspecting m is no less than c. This minimum proportion is decreasing in d, meaning that the credibility of inspection can be sustained for a smaller proportion of liars when the distance of conditional expectations is larger. Note that $2\sqrt{c}$ is the minimum required distance such that an inspection can be made credible, and $w^-(2\sqrt{c}) = \frac{1}{2}$. Theorem 11 and Theorem 12 imply for any inspected message $m \in \mathcal{M}_q^+$ in an optimal mechanism,

$$w_q(m) = w^-(X(m,t) - X(m,l)) \tag{3.18}$$

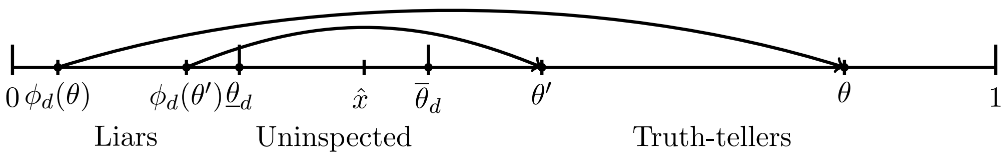
so for any inspected message in an optimal mechanism, proportion of liars is uniquely determined by the distance between expected types of truth-tellers and liars. For $x_l \in [0, 1-2\sqrt{c}]$ and $x_t \in [x_l + 2\sqrt{c}, 1]$, define

$$X_u^*(x_t, x_l) = w^-(x_t - x_l)x_l + (1 - w^-(x_t - x_l))x_t \tag{3.19}$$

which is the expected type of senders of a message m where x_t is the expected type of truth-tellers, x_l is the expected type of liars, and proportion of liars is minimized subject to DM's incentive constraint of inspection. Since DM is sequentially rational, for any $m \in \mathcal{M}_q^+$ in an optimal mechanism,

$$X(m,u) = X_u^*(X(m,t), X(m,l)) \tag{3.20}$$

Figure 3.3: The Structure of Decreasing Mimicking Mechanism Ω_d



Note: In the decreasing mimicking mechanism, types above $\overline{\theta}_d$ make truthful and precise claims, which are mimicked by liars below $\underline{\theta}_d$ according to a decreasing mimicking function. These claims are randomly inspected. Types in between $\underline{\theta}_d$ and $\overline{\theta}_d$ pool at a single claim which is never inspected. $\hat{x}$ denotes the mean of the interval $(\underline{\theta}_d, \overline{\theta}_d)$. If $\hat{x}$ is above the mid-point of this interval, the decreasing mimicking mechanism is optimal.

so the induced action when the inspection does not take place is uniquely determined by the expected type of truth-tellers and liars.

Now we define the decreasing mimicking mechanism. Define a pair of cutoffs $\underline{\theta}_d$, $\overline{\theta}_d$ and matching function $\phi_d : [\overline{\theta}_d, 1] \to [0, \underline{\theta}_d]$ as a solution of the following system of differential equation and boundary conditions:

$$\dot{\phi}_d(\theta) = -\frac{w^-(\theta - \phi_d(\theta))}{1 - w^-(\theta - \phi_d(\theta))}\frac{f(\theta)}{f(\phi_d(\theta))} \tag{3.21}$$

$$\phi_d(1) = 0 \tag{3.22}$$

$$\phi_d(\overline{\theta}_d) = \underline{\theta}_d \tag{3.23}$$

To determine the boundaries $\underline{\theta}_d$ and $\overline{\theta}_d$, first define $\hat{\theta}$ such that

$$\hat{\theta} - \phi_d(\hat{\theta}) = 2\sqrt{c} \tag{3.24}$$

Lemma 3 *If Assumption 4 is satisfied, then there exists a unique solution $(\hat{\theta}, \phi_d)$ that satisfies conditions (3.21), (3.22) and (3.24). Furthermore, there exists a unique $\overline{\theta}_d \in [\hat{\theta}, 1]$ such that such that for any $\theta \in [\hat{\theta}, 1]$, $\theta < \overline{\theta}_d$ implies $X_u^*(\theta, \phi_d(\theta)) < E[\phi_d(\theta), \theta]$; $\theta > \overline{\theta}_d$ implies $X_u^*(\theta, \phi_d(\theta)) > E[\phi_d(\theta), \theta]$.*

$\phi_d(.)$ represents a decreasing matching function from the interval of truthful types to the the interval of lying types which specifies the lying pattern in the decreasing mimicking mechanism. Lemma 3 pins down the unique upper bound of the first interval and lower bound of the second interval $(\underline{\theta}_d, \overline{\theta}_d)$. The decreasing matching function $\phi_d(.)$ is constructed by starting with matching type 1 with type 0, then the matching process goes inward. The law of motion is determined by the property that the relative weights in each matched liar-truthteller pair provides just enough incentive for DM to inspect the message. The boundary pair $(\underline{\theta}_d, \overline{\theta}_d)$ is determined so that either (i) the boundary types are closer enough that no relative weight could justify an inspection, or (ii) under the required weight, the weighted mean of the two boundary types goes below the mean of the whole interval in between the two types. In case (i), the matching process stops because there is no way to sustain inspection anymore. In case (ii), the matching process stops because the liar of a message with weighted mean below the mean of the middle interval would deviate to pool in the uninspected middle interval.

Formally define the **decreasing mimicking mechanism** Ω_d which is characterized by $(\underline{\theta}_d, \overline{\theta}_d, \phi^d)$ defined in conditions (3.21) - (3.23) and Lemma 3 such that:
(i) **Intermediate types - Uninspected vague claim**: There is an uninspected message $m_q^0 = [\underline{\theta}_d, \overline{\theta}_d]$ sent by $\theta \in [\underline{\theta}_d, \overline{\theta}_d]$ and $P(m_q^0) = 0$;
(ii) **High types - Randomly inspected, precise claims**: There is a continuum of randomly inspected messages $\mathcal{M}_q^+ = \{m = \{\theta\} : \theta \in (\overline{\theta}_d, 1]\}$, each $m \in \mathcal{M}_q^+$ sent by the truthful type $\theta = m$ and $P(m) \in (0, 1)$;
(iii) **Low types - Liars of the high claims**: Each $m \in \mathcal{M}_q^+$ is sent by a liar $\phi_d(m)$.
The action rule X is determined by sequential rationality. For $m \in \mathcal{M}_q^+$,

$$\begin{aligned} X(m,t) &= m \\ X(m,l) &= \phi_d(m) \\ X(m,u) &= X_u^*(m, \phi_d(m)) \end{aligned} \tag{3.25}$$

and

$$X(m_q^0, u) = E[\underline{\theta}_d, \overline{\theta}_d] \tag{3.26}$$

The inspection rule P for $m \in \mathcal{M}_q^+$ is determined by the incentive compatibility conditions of the liars:

$$P(m) = \frac{u(X(m,u)) - u(X(m_q^0, u))}{u(X(m,u)) - u(X(m,l))} \tag{3.27}$$

Figure 3.3 depicts the structure of the decreasing mimicking mechanism. Under Ω_d, each truthful type θ makes the precise claim "My type is θ", and each of such claim is mimicked by exactly one type of liar $\phi_d(\theta)$, where $\phi_d(.)$ is decreasing so worse liars tell bigger lies. Upon receiving each of these messages, DM is indifferent between inspecting and not inspecting. The inspection probability is chosen so that liars are indifferent between telling such lies and making the uninspected claim. The optimal mechanism specifies a list of permissible claims the sender is allowed to make: a vague claim that represents moderate states, and a continuum of precise, high claims. Requiring a precise statement for high claims helps make more precise decisions upon inspection. Random inspections of those claims are justified because each of them is made by a low type and a high type. A vague moderate claim pools the moderate types which are not distant enough to be worth inspecting [13].

Lemma 4 *If Assumption 4 is satisfied, the decreasing mimicking mechanism* Ω_d *is incentive compatible with* $0 < \underline{\theta}_d < \overline{\theta}_d < 1$, *and* $EU_{DM}(\Omega_d) > -\min\{Var(\Theta), c\}$.

Lemma 4 means that Ω_d is incentive compatible whenever there exists an incentive compatible mechanism with inspection. I say two mechanisms $\Omega = (q, X, P)$ and $\Omega' = (q', X', P')$ are **equal almost everywhere** if for almost every $\theta \in \Theta$ and $s = \{t, l, u\}$, $m_q(\theta) = m_{q'}(\theta)$, $X(m_q(\theta), s) = X'(m_{q'}(\theta), s)$ and $P(m_q(\theta)) = P'(m_{q'}(\theta))$, i.e. sender sends the same messages, induces the same actions and inspected with the same probabilities in the two mechanisms almost surely.

Theorem 14 ***(Optimality of the decreasing mimicking mechanism)*** *Suppose in an optimal mechanism* Ω, $E[\underline{\theta}_\Omega, \overline{\theta}_\Omega] > \frac{\underline{\theta}_\Omega + \overline{\theta}_\Omega}{2}$, *then* Ω *and* Ω_d *are equal almost everywhere.*

When the mean of the uninspected interval is skewed towards its boundary to the truthful interval $\overline{\theta}_\Omega$, the value of inspecting the marginal truthful type around $\overline{\theta}_\Omega$ is small, so extending the uninspected interval to the right would be beneficial. Decreasing matching minimizes the truth-teller to liar ratios and allows the uninspected interval to extend farthest to the right. The condition of Theorem 14 is satisfied under a broad class of prior distributions. Two examples are symmetric single peaked distributions and distributions with increasing density.

Remark 1 *Suppose Assumption 4 is satisfied, and either:*
(1) F is symmetric and single peaked, or
(2) $f'(\theta) > 0$ *for any* $\theta \in [0, 1]$,
then in the optimal mechanism Ω, $E[\underline{\theta}_\Omega, \overline{\theta}_\Omega] > \frac{\underline{\theta}_\Omega + \overline{\theta}_\Omega}{2}$.

3.5 State-verification and Lie-detection

In this section, I compare state-verification technology to lie-detection technology, in particular, DM's welfare under the two technologies. Instead of revealing a binary signal as in (3.1), consider now the true state is revealed upon inspection, so by paying cost c to inspect the message m, DM receives the precise signal

$$s(m, \theta) = \theta \tag{3.28}$$

[13] It is worth noting that despite having a list of permissible claims, exogenous enforcement on the sender's obedience is not necessary. It is because there is always a perfect Bayesian equilibrium where any off-path claim is regarded as a signal of the worst state and punished maximally so that the sender will never deviate to any claim out of the list.

If DM chooses not to inspect, she receives an uninformative signal $s(m,\theta) = u$. Under state-verification technology, the sequentially rational action rule for DM is

$$X(m,\theta) = \theta; X(m,u) = E[\Theta_q^u(m)] \tag{3.29}$$

where $\Theta_q^u(m)$ is the set of senders who send m, and value of verifying m is the conditional variance of the sender's type:

$$V_q(m) = Var(\Theta_q^u(m)) \tag{3.30}$$

and the sequentially rational inspection rule for DM is

$$P(m) \in \begin{cases} \{0\} & \text{if } c > V_q(m) \\ [0,1] & \text{if } c = V_q(m) \\ \{1\} & \text{if } c < V_q(m) \end{cases}. \tag{3.31}$$

Type θ sender's expected utility from sending a message m is

$$EU_{X,P}(m|\theta) = P(m)u(\theta) + (1 - P(m))u(X(m,u)) \tag{3.32}$$

and sender's optimality implies that for any on-path message $m' \in \mathcal{M}_q$,

$$P(m_q(\theta))u(\theta) + (1 - P(m_q(\theta)))u(X(m_q(\theta),u) \geq P(m')u(\theta) + (1 - P(m'))u(X(m',u) \tag{3.33}$$

where $m_q(\theta)$ is the message sent by θ under the mechanism. I will show that there are only two kinds of incentive compatible mechanism under costly state-verification,
Uninformative mechanism: $P(m) = 0$ and $X(m,u) = E[\Theta]$ for any $m \in \mathcal{M}_q$, and
State-verifying mechanism: $P(m) = 1$ and $X(m,\theta) = \theta$ for any $m \in \mathcal{M}_q$.

Theorem 15 ***(No informative communication under state-verification.)***
Under costly state-verification technology, if $c > Var(\Theta)$, only the uninformative mechanism is incentive compatible; if $c < Var(\Theta)$, only the state-verifying mechanism is incentive compatible.

The ability to reveal the state precisely upon an inspection completely eliminates any incentive for the sender to transmit information. Gain from state-verification technology comes solely from the direct information value. It is contrary to the lie-detection technology, which benefits DM by manipulating the sender's incentive to transmit information. Such manipulation is possible because the nature of lie-detection creates a strategic uncertainty to DM: even if she spots a lie, she does not reveal the true type of the liar and has to decide the action base on equilibrium inference. This could benefit DM in an ex-ante sense because the sender might be deterred from deviation in fear of being mistaken as a worse type than what he actually is, and such a deterrence effect facilitates informative. However, if DM reveals the true state from an inspection, this deterrence will not be credible, and there will be no reason for the sender to stay honest. As a result, revealing more information from inspection eliminates voluntary information transmission from the sender. The following Proposition shows that learning more from inspection reduces DM's payoff. With Theorem 15, DM's ex-ante payoff under costly state-verification technology is

$$EU^s_{DM} = -\min\{Var(\Theta), c\} \tag{3.34}$$

Theorem 16 ***(DM is better off under lie-detection technology than state-verification.)*** *Let Ω^* be the optimal mechanism under lie-detection technology. Then under any inspection cost and distribution, $EU_{DM}(\Omega^*) \geq EU^s_{DM}$. Furthermore, if $c < Var(\Theta)$, then $EU_{DM}(\Omega^*) > EU^s_{DM}$.*

This result provides a theoretical foundation for the emphasis on expert's integrity, instead of the objective information. By neglecting further information about the truth (other than the information that determines whether the sender is lying), the decision-maker is able to impose a credible threat that whoever being caught lying will be perceived poorly, regardless of the sender's true type. Therefore, even though there is no common interest between experts and decision-makers, some types of experts refrain from making higher claims in fear of being perceived as a worse type than they actually are.

3.6 Comparison with Related Literature

In this section, I compare this model with two related papers in the literature, Dziuda and Salas [21] and Balbuzanov [6].

The first and perhaps most important difference that separates this model from the two papers is lie-detection technology. While they both assume a lie of the sender is detected with an exogenous probability, I model lie-detection as a costly technology and the receiver choose whether or not to use it, contingent on the sender's message. Therefore, lying and lie-detection interact and emerge as equilibrium properties in this model. In fact, the equilibrium probabilities of lie-detection are indeed contingent on the sender's messages.

Second, in Dziuda and Salas, the sender's message space contains singletons of the type space, which is interpreted as "My true type is exactly θ". An extension in Balbuzanov allows sender to make sub-interval claims ("My true type is in between x and y"). This model generalizes the message space to include all measurable subsets of the type space, thus the sender is provided with a richer set of language in communication.

Third, Dziuda and Salas and this model assume sender's payoff is state-independent and increasing in receiver's action, whereas Balbuzanov adopts the setting of biased sender as in Crawford and Sobel [17]. This leads to different welfare implications. In Balbuzanov, ex-ante preferences of sender and receiver are aligned, thus full information revelation implies Pareto optimality. This is not the case in Dziuda and Salas and this model. Whether the sender prefers information revelation depends on the shape of the state-independent utility function (risk attitude). Also note that in this model, the receiver's welfare depends both on information revelation and frequency of inspection.

In terms of equilibrium selection, Dziuda and Salas select equilibria in which any off-equilibrium message is trusted by the receiver, and the receiver responds to the detection of any lie with a same action. Balbuzanov confines attention to equilibria with full revelation. In this model I focus on the receiver-optimal equilibrium. As mentioned in Section 3.4, this allows us to identify the most efficient way of combating misinformation using lie-detection technology, taken its cost into account.

Although both selected equilibria in Dziuda and Salas and this model have a three-interval structure, there are two worth-noting differences. First, in Dziuda and Salas, types in the middle interval are truth-telling without pooling. This can be sustained due to the exogenous lie-detection. In this model, types in the middle interval are not inspected by the receiver. Therefore, any separation within the interval is unsustainable, thus they all pool into a single vague message. Second, in Dziuda and Salas, all liars from the bottom interval randomize over all messages sent by the truth-tellers from the top interval. In this model, under certain conditions each type of liar has a particular lie to tell (as shown in Section 3.4.1).

In terms of information revelation, Balbuzanov shows that fully revealing equilibria exist for a set of intermediate detection probability p and when $p = 1$. In Dziuda and Salas, as detection probability goes to 1, the lengths of the bottom and top intervals converges to 0, so the selected equilibrium approaches full revelation. In this model, as cost of inspection goes to zero, the lengths of the bottom and middle intervals converges to 0, so the receiver-optimal equilibrium approaches full revelation [14].

3.7 Concluding Remarks

Lying and lie-detection emerge from the opportunism of informed parties and the skepticism of uninformed parties. I establish a framework that allows analyses on the strategic interaction between lying and lie-detection, and characterize the optimal lie-detection policy. The results suggest that optimal lie-detection works as a credible deterrence tool. Low types are induced to lie so that inspections are justified, which deter higher types from lying. Under certain conditions, such optimal equilibrium can be achieved by allowing the sender to choose among a vague moderate claim and a continuum of precise high claims. This provides a direction for efficient allocation of resources in combating misinformation in various aspects such as politics

[14] This result is an implication of Lemma 4, which states that the receiver's payoff is bounded below by $-c$ when c is small. As $c \to 0$, the receiver's payoff converges to 0, which can only be achieved by approaching full separation with the top interval extending to the whole type space.

Figure 3.4: Comparison with Dziuda and Salas [21] and Balbuzanov [6]

	This Model	Dziuda and Salas	Balbuzanov
Lie-detection technology	Costly and endogenous	Exogenous probability	Exogenous probability
Message space	Subsets of type space	Singletons of type space	Singletons and subintervals of type space
Preference of the sender	State-independent and Monotone	State-independent and Monotone	State-dependent (biased sender)
Equilibrium selection	Receiver-optimality	Base on off-equilibrium beliefs and receiver's response upon lie-detection	Full revelation (if possible)
Information revelation in selected equilibrium	Full revelation convergence as detection cost goes to 0	Full revelation convergence as detection probability goes to 1	Full revelation achievable for some ranges of detection probability
Structure of selected equilibrium	Three-interval structure with pooling for middle interval	Three-interval structure without pooling for middle interval	Truth-telling

and product advertising.

Several potential extensions are worth mentioning. As the first attempt in the literature to study endogenous lying and costly lie-detection, I restrict attention to the setting of single round communication and lie-detection. In some applications, the sender and the receiver can conduct multiple rounds of communication and inspect lie-detection, before a final decision is made by the receiver. For instance, the police can ask the suspect multiple questions and conduct lie-detection for each claim made by the suspect. Dziuda and Salas [21] show that the receiver prefers to commit to a single round communication when the probability of lie-detection is exogenously high, because anticipating the second chance of communication makes the sender more likely to lie. It might appear that this effect is strengthened when lie-detection is costly as the receiver has to pay the cost of inspection in each round. A formal analysis is required for such an argument. Another potential extension is to allow a certain degree of common interest between the sender and the receiver, such as biased sender as in Crawford and Sobel [17]. It is not clear whether having a sender with a smaller bias would benefit the receiver when lie-detection is possible. On one hand, sender with smaller bias is willing to reveal more precise information, as suggested by the standard cheap talk model. On the other hand, when bias is small, there is no way to induce the sender to tell big lies. This hinders the formation of credible inspection. Without inspection, the sender might be tempted to tell small lies, which impede informative communication. The analysis of these opposing effects may present interesting avenues for future research.

Conclusion

This dissertation studies an important aspect in information economics: how costly misrepresentation affects information transmission and optimal mechanism designs.

Chapter 1 and 2 analyze optimal screening when the informed party can incur a fixed cost to misrepresent private information. We establish the general structure of the optimal mechanism, which exhibits several qualitatively new proprieties, such as non-locally binding incentive constraints, efficient allocations achieved for intervals of low types and high types. We develop a new methodology jointly solve for the non-local incentive constraints and optimal allocations. The characterization of binding non-local incentive constraints and the "targeted types," as well as the techniques of solving for them, could be useful for solving other problems with binding non-local incentive constraints, potentially providing an important analytical instrument for various applications.

Chapter 3 studies information transmission when misrepresentation costs is derived from the possibility of endogenous lie-detection. I show that lie-detecting technology is useful in improving welfare of the uninformed party if and only if cost of the technology is low enough and prior expectation on the state is not too optimistic. The receiver-optimal design leads to a three-interval equilibrium: low types mimic high types which are randomly inspected, while moderate types stay honest and never being inspected. Compared with state-verification, the uninformed party can achieve a higher payoff under lie-detection technology because some types of senders refrain from lying in fear of being mistaken as a worst type upon lie-detection.

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Appendix A

Appendix for Chapter 1

A.1 Proof of Theorem 1

Let $(q(\theta), t(\theta), A(\theta))$ be an incentive compatible, individually rational mechanism which satisfies $A(\theta) \neq \theta$ for all $\theta \in \Theta^l$, where the set Θ^l has a positive measure. Now consider an alternative mechanism $(\hat{q}(\theta), \hat{t}(\theta), \hat{A}(\theta))$ such that $(\hat{q}(\theta), \hat{t}(\theta), \hat{A}(\theta)) = (q(\theta), t(\theta), A(\theta))$ for all θ such that $A(\theta) = \theta$ and $(\hat{q}(\theta), \hat{t}(\theta), \hat{A}(\theta)) = (q(\theta), t(\theta) + C, \theta)$ for θ such that $A(\theta) \neq \theta$. Clearly, $(\hat{q}(\theta), \hat{t}(\theta), \hat{A}(\theta))$ is strictly more profitable for the firm, provided that it is incentive compatible and individually rational. The individual rationality of the new mechanism follows immediately from the individual rationality of the original mechanism. So we only need to show that the new mechanism is incentive compatible. Indeed, for all $\theta, \theta' \in [0, 1]$ we have:

$$
\begin{aligned}
&u(\hat{q}(\theta), \theta) - \hat{t}(\theta) - C \times 1(\hat{A}(\theta) \neq \theta) = u(\hat{q}(\theta), \theta) - \hat{t}(\theta) = u(q(\theta), \theta) - t(\theta) - C \times 1(A(\theta) \neq \theta) \\
&\geq u(q(\theta'), \theta) - t(\theta') - C \times 1(A(\theta') \neq \theta) \geq u(\hat{q}(\theta'), \theta) - \hat{t}(\theta') - C \times 1(\hat{A}(\theta') \neq \theta)
\end{aligned}
$$

where the first equality holds because $\hat{A}(\theta) = \theta$ for all $\theta \in [0, 1]$, the second equality holds by definition of $(\hat{q}(\theta), \hat{t}(\theta), \hat{A}(\theta))$, the first inequality holds because $(q(\theta), t(\theta), A(\theta))$ is incentive compatible, the second inequality holds because $\hat{q}(\theta') = q(\theta')$, $\hat{t}(\theta') \geq t(\theta')$ and $\hat{A}(\theta') = \theta \neq \theta$ for $\theta' \neq \theta$. *Q.E.D.*

A.2 Preliminary Lemmas

This section provides a series of Lemmas that facilitate the proofs of the subsequent theorems.

The first Lemma shows that the payment t is non-negative for almost every type.

Lemma 5 *Without loss of generality, we can restrict consideration to mechanisms* $(q(.), t(.))$ *such that* $t(\theta) \geq 0$ *for all* $\theta \in [0, 1]$.

Proof of lemma 5: Suppose that mechanism $(q(.), t(.))$ is incentive compatible and individually rational and $t(\theta) < 0$ iff $\theta \in \Theta^-$, where Θ^- is a non-empty subset of $[0, 1]$.

Let $(\tilde{q}(\theta), \tilde{t}(\theta)) = (q(\theta), t(\theta))$ for any $\theta \notin \Theta^-$, and $(\tilde{q}(\theta), \tilde{t}(\theta)) = (0, 0)$ for any $\theta \in \Theta^-$. So $\tilde{t}(\theta) \geq 0$ for all $\theta \in [0, 1]$. Then mechanism $(\tilde{q}(.), \tilde{t}(.))$ is individually rational for all $\theta \in [0, 1]$ and is incentive compatible for all $\theta \notin \Theta^-$. If $(\tilde{q}(.), \tilde{t}(.))$ is not incentive compatible for some $\theta \in \Theta^-$, then still θ makes a non-positive transfer instead of a negative transfer in the mechanism $(q(.), t(.))$. So $(\tilde{q}(.), \tilde{t}(.))$ is strictly more profitable for the principal than $(q(.), t(.))$ if Θ^- has a positive measure and is weakly more profitable if Θ^- has zero measure. *Q.E.D*

Lemma 6 *There exists an optimal mechanism solving the principal's maximization problem (1.1) subject to (1.2) and (1.3).*

Proof of Lemma 6: By Lemma 5 we restrict consideration to mechanisms $(q(\theta), t(\theta))$ s.t. $t(\theta) \geq 0$. Therefore, $q(\theta) \in [0, \bar{Q}]$ where $\bar{Q} = \max\{Q | u(Q, 1) \geq 0\}$ (by Assumption 1(iii) $\bar{Q} < \infty$). Indeed, if some type θ is assigned an allocation $(q(\theta), t(\theta))$ such that $q(\theta) > \bar{Q}$, then $t(\theta) < 0$ by individual rationality. Also, individual rationality requires that $t(\theta) \leq u(q^{fb}(1), 1)$ where $q^{fb}(1) = \arg\max_q u(q, 1)$.

So, our space of mechanisms is a set of bounded measurable, and hence integrable, functions $(t(\theta), q(\theta)) : [0, 1]^2 \mapsto [0, u(q^{fb}(1), 1)] \times [0, \bar{Q}]$. Endowed with pointwise convergence topology, this space is compact by Tychonoff Theorem. Note that the objective (1.1) is continuous on this space. Furthermore, the subset of this space satisfying the constraints (1.2) and (1.3) is compact and non-empty. In particular, it includes all increasing $q(.)$ coupled with transfer functions that implement such $q(.)$ in the case with no fixed costs. So by Weierstrass Theorem, there exists a solution $(q^*(.), t^*(.))$ maximizing (1.1) subject to (1.2) and (1.3). *Q.E.D.*

The next Lemma establishes continuity of $V(.)$, $t(.)$ and $q(.)$. in an optimal mechanism.

Lemma 7 *There exists an optimal mechanism $(q(.), t(.))$ such that $V(.)$ is nondecreasing, and $V(.)$, $q(.)$ and $t(.)$ are continuous at any $\theta \in [0, 1]$.*

Proof of Lemma 7: Suppose that $((q(.), t(.))$ is an optimal incentive compatible individually rational mechanism.

$V(.)$ **is increasing.** First, for any θ s.t. $V(\theta) > 0$ there exists a sequence θ_n s.t. $V(\theta) = \lim_{n\to\infty} u(q(\theta_n), \theta) - t(\theta_n) - C$. For, suppose otherwise i.e., there exists $\epsilon > 0$ s.t. $V(\theta) > u(q(\theta'), \theta) - t(\theta') - C + \epsilon$ for all $\theta' \in [0, 1]$. Then the principal can increase her profits by modifying the allocation $(q(\theta), t(\theta))$ by raising $t(\theta)$ by $\frac{\epsilon}{2}$. This modified mechanism is clearly incentive compatible and individually rational.

Now consider some θ' s.t. $\theta' > \theta$. We have $V(\theta') \geq \lim_{n\to\infty} u(q(\theta_n), \theta') - t(\theta_n) > \lim_{n\to\infty} u(q(\theta_n), \theta) - t(\theta_n) = V(\theta)$. So $V(.)$ is increasing in an optimal mechanism.

The Lipschitz continuouity of $V(.)$: There exists L, $0 < L < \infty$ s.t. $|V(\theta) - V(\theta')| \leq L|\theta - \theta'|$.

Since $V(.)$ is increasing, we have to prove this Claim only for the case $\theta > \theta'$. As shown in the previous part, there exists a sequence θ_n s.t. $V(\theta) = \lim_{n\to\infty} u(q(\theta_n), \theta) - t(\theta_n) - C$. Obviously, $V(\theta') \geq \lim_{n\to\infty} u(q(\theta_n), \theta') - t(\theta_n) - C$, and hence $V(\theta) - V(\theta') \leq \lim_{n\to\infty}(u(q(\theta_n), \theta) - u(q(\theta_n), \theta') \leq u(q^m, \theta) - u(q^m, \theta') \leq \max_{\theta''\in[0,1]} u_\theta(q^m, \theta'')(\theta - \theta')$. Taking $L = \max_{\theta''\in[0,1]} u_\theta(q^m, \theta'')$ establishes this Claim.

The continuity of $t(.)$**.** Now, using the continuity of $V(.)$ let us show that $t(.)$ is continuous in an optimal mechanism. Again, the proof is by contradiction. So suppose that there exists $\theta' \in (0,1]$, a sequence θ_n s.t. $\lim_{n\to\infty}\theta_n = \theta'$ and $t^* = \lim_{n\to\infty} t(\theta_n)$ s.t. $|t(\theta') - t^*| > \beta$. Suppose first that $t^* > t(\theta') + \beta$. By continuity of $V(.)$, it follows that $V(\theta') = u(q^*, \theta') - t^*$ where $q^* = \lim_{n\to\infty} q(\theta_n)$ (The latter limit exists because, as shown earlier, $q(\theta)$ is bounded in an optimal mechanism). Now, suppose that the principal assigns the allocation (t^*, q^*) to type θ' instead of the allocation $(t(\theta'), q(\theta'))$. This modification weakly increases the principal's profits because $t^* > t(\theta')$. Moreover, the modified mechanism is still incentive compatible and individually rational. The latter is true because $V(\theta)$ remains unchanged for all $\theta \in [0,1]$. By the same reason, $IC(\theta', \theta)$ continue to hold for all $\theta \in [0,1]$.

It remains to show that $IC(\theta, \theta')$ still hold in the modified mechanism. The proof is by contradiction, so suppose that $IC(\theta, \theta')$ now fails for some θ i.e., $V(\theta) < u(q^*, \theta) - t^* - C$. But since $(q^*, t^*) = \lim_{n\to\infty}(q(\theta_n, t(\theta_n))$, there exists θ_n for n large enough that $V(\theta) < u(q_n, \theta) - t_n - C$. So the original mechanism is not incentive compatible. Contradiction.

Now, let us consider the case $t(\theta') > t^* + \beta$. By continuity of $V(.)$, it follows that $u(q^*, \theta') = \lim_{n\to\infty} u(q(\theta_n), \theta_n) < u(q(\theta'), \theta')$

Next, fix $\tilde{q} = \frac{q^* + q(\theta')}{2}$. and define a new mechanism $(\tilde{q}(.), \tilde{t}(.))$ which differs from the original mechanism $(q(.), t(.))$ only at θ_n for $n \geq N$ where N is sufficiently large that $u(q(\theta_n), \theta_n) < u(\tilde{q}, \theta_n)$. For such n set $\tilde{t}(\theta_n) = u(\tilde{q}, \theta_n) - V(\theta_n) > t(\theta_n)$ and $\tilde{q}(\theta_n) = \tilde{q}$. So the new mechanism $(\tilde{q}(.), \tilde{t}(.))$ is more profitable for the seller than $(q(.), t(.))$.

We need to check that the new mechanism $(\tilde{q}(.), \tilde{t}(.))$ is individually rational and incentive compatible. First, the net payoff of any type $\theta \in [0,1]$ in the new mechanism, $\tilde{V}(\theta)$, satisfies $\tilde{V}(\theta) = V(\theta)$. So, $IR(\theta)$ and $IC(\theta, \theta'')$ hold for all $\theta \in [0,1]$ and $\theta'' \neq \theta_n$, $n \geq N$, because the mechanism $(\tilde{q}(.), \tilde{t}(.))$ differs from $(q(.), t(.))$ only for θ_n, $n \geq N$.

It remains to consider $IC(\theta, \theta_n)$, $n \geq N$. Since $IC(\theta, \theta')$ holds for all $\theta \in [0,1]$ in both mechanisms, we have $V(\theta) \geq u(q(\theta'), \theta) - t(\theta') - C = u(q(\theta'), \theta) - u(q(\theta'), \theta') + V(\theta') - C$. Also, since $IC(\theta, \theta_n)$ holds for all $\theta, \theta_n \in [0,1]$ in the original mechanism, $\lim_{n\to\infty}(t(\theta_n), q(\theta_n)) = (t^*, q^*)$, and $u(.)$ is continuous, it follows that for any θ, $V(\theta) \geq u(q^*, \theta) - t^* - C = u(q^*, \theta) - u(q^*, \theta') + V(\theta') - C$. So, $V(\theta) \geq \max\{u(q(\theta'), \theta) - u(q(\theta'), \theta') + V(\theta') - C, u(q^*, \theta) - u(q^*, \theta') + V(\theta') - C\} > u(\tilde{q}, \theta) - u(\tilde{q}, \theta') + V(\theta') - C$ where the inequality holds because $u_{q\theta}(q, \theta) > 0$ and $\tilde{q} \in (\min\{q^*, q(\theta')\}, \max\{q^*, q(\theta')\})$.

Finally, $u(\tilde{q}, \theta') - V(\theta') \approx u(\tilde{q}, \theta_n) - V(\theta_n) = \tilde{t}(\theta_n)$ where the approximate equality holds by continuity of $V(.)$ and $u(.)$ and because $\lim_{n\to\infty}\theta_n = \theta'$ and the equality holds by definition. It follows that $V(\theta) > u(\tilde{q}, \theta) - \tilde{t}(\theta_n) - C$ for $n \geq N$ when N is sufficiently large. Therefore, $IC(\theta, \theta_n)$ hold for all $\theta \in [0,1]$, θ_n, $n \geq N$ in the mechanism $(\tilde{q}(.), \tilde{t}(.))$.

The continuity of $q(.)$ follows from the continuity of $V(.)$ and $t(.)$. *Q.E.D*

Lemma 8 *The correspondence* $\tau(\theta)$ *is upper hemicontinuous and compact-valued in an optimal mechanism.*

Proof of Lemma 8: To establish the upper-hemicontintuity of $\tau(.)$, let (θ_n, θ'_n) be a sequence of type pairs such that $\theta'_n \in \tau(\theta_n)$ for all $n = 1, 2, ..., \infty$ and $\lim_{n\to\infty}(\theta_n, \theta'_n) = (\tilde{\theta}, \tilde{\theta}')$. We need to show that $\tilde{\theta}' \in \tau(\tilde{\theta})$. Define $\Delta U(\theta, \theta') = V(\theta) - u(q(\theta'), \theta) + t(\theta') + C$. Since $\theta'_n \in \tau(\theta_n)$, $\Delta U(\theta_n, \theta'_n) = 0$ for all n. Assumption 1 and Lemma 7 imply that $\Delta U(.)$ is continuous. Therefore we have $\Delta U(\tilde{\theta}, \tilde{\theta}') = \lim_{n\to\infty} \Delta U(\theta_n, \theta'_n) = 0$, implying that $\tilde{\theta}' \in \tau(\tilde{\theta})$.

The compact-valuedness of $\tau(.)$ follows because $\theta'' \in \tau(\theta)$ iff

$$\theta'' \in \arg\max u(q(\theta''), \theta) - t(\theta'') - C.$$

The set of such maximizers is compact by Berge's Maximum Theorem because $q(.)$ and $t(.)$ are continuous functions by Lemma 7. *Q.E.D.*

Lemma 9 shows that for any positive cost of lying, there is a positive threshold $\hat{\theta}$ such that all types below $\hat{\theta}$ do not have any binding incentive constraints and get zero surplus; all types above $\hat{\theta}$ have binding incentive constraints and get a positive surplus.

Lemma 9 *For any $C > 0$, there exists $\hat{\theta} > 0$ s.t. $\tau(\theta) = \emptyset$ iff $\theta \in [0, \hat{\theta})$ and $V(\theta) = 0$ iff $\theta \in [0, \hat{\theta}]$.*

Proof of Lemma 9: Since $u(q, 0) = 0$ for all q, we must have $t(0) = 0$ and $V(0) = 0$ in an optimal mechanism. Then, since $V(.)$ is continuous and non-decreasing by Lemma 7, it follows that there exists $\hat{\theta} \in [0, 1]$ such that $V(\theta) = 0\ \forall\theta \leq \hat{\theta}$ and $V(\theta) > 0\ \forall\theta > \hat{\theta}$ (If $V(\theta) = 0$ for all $\theta \in [0, 1]$, then $\hat{\theta} = 1$).

Now suppose there exists $\theta < \hat{\theta}$ and θ' such that $\theta' \in \tau(\theta)$, so $V(\theta) = u(q(\theta'), \theta) - t(\theta') - C \geq 0$. But then $V(\hat{\theta}) \geq u(q(\theta'), \hat{\theta}) - t(\theta') - C > 0$ because $u_\theta > 0$, which contradicts $V(\hat{\theta}) = 0$. Therefore $\tau(\theta) = \emptyset\ \forall\theta < \hat{\theta}$.

Now suppose that there exists $\theta > \hat{\theta}$ such that $\tau(\theta) = \emptyset$. Then the continuity of $V(.)$, $q(.)$ and $t(.)$ established in Lemma 7 and $\tau(\theta) = \emptyset$ imply that there exists $\epsilon > 0$ such that $V(\theta) > u(q(\theta'), \theta) - t(\theta') - C + \epsilon$ for all $\theta' \in [0, 1]$. Since $V(\theta) > 0$, the seller can increase her profit by raising $t(\theta)$ by $\min\{\epsilon, V(\theta)\}$. This modification clearly does not violate any IR or IC constraints. Therefore $\tau(\theta) \neq \emptyset\ \forall\theta > \hat{\theta}$. In addition, the upper hemicontinuity of $\tau(.)$ established in Lemma 8 implies that $\tau(\hat{\theta}) \neq \emptyset$.
Finally, for $\theta \in \tau(\hat{\theta})$, $V(\hat{\theta}) = u(q(\theta), \hat{\theta}) - t(\theta) - C = 0$. Since $C > 0$ and $t(\theta) \geq 0$, it must be the case that $\hat{\theta} > 0$.

Q.E.D

Lemma 10 shows that higher types have binding incentive constraints to types who are assigned higher quantities.

Lemma 10 *In an incentive compatible mechanism, suppose that $\theta_1 > \theta_2$, $\theta'_1 \in \tau(\theta_1)$ and $\theta'_2 \in \tau(\theta_2)$. Then $q(\theta'_1) \geq q(\theta'_2)$.*

Proof of Lemma 10: Since $\theta_1' \in \tau(\theta_1)$, $V(\theta_1) = u(q(\theta_1'),\theta_1) - t(\theta_1') - C \geq u(q(\theta_2'),\theta_1) - t(\theta_2') - C$. Similarly, $V(\theta_2) = u(q(\theta_2'),\theta_2) - t(\theta_2') - C \geq u(q(\theta_1'),\theta_2) - t(\theta_1') - C$. Combining these two inequalities yields: $u(q(\theta_1'),\theta_1) - u(q(\theta_2'),\theta_1) \geq t(\theta_1') - t(\theta_2') \geq u(q(\theta_1'),\theta_2) - u(q(\theta_2'),\theta_2)$. Since $\theta_1 > \theta_2$ and $u_{q\theta} > 0$, it must be that $q(\theta_1') \geq q(\theta_2')$. *Q.E.D*

Lemma 11 shows that optimal quantities never exceed the first-best level, only downward incentive constraints can be binding, and some incentive constraints must be binding towards types with below-first-best quantities.

Lemma 11 *In an optimal mechanism for any* $\theta \in [0,1]$, $q(\theta) \leq q^{fb}(\theta)$. *If* $\tau^{-1}(\theta)$ *is non-empty, then* $\tau^{-1}(\theta) \subseteq (\theta, 1]$. *If* $\tau^{-1}(\theta)$ *is empty, then* $q(\theta) = q^{fb}(\theta)$.

Proof of Lemma 11:

Claim 1: For any $\theta \in [0,1]$, if $\tau^{-1}(\theta)$ is non-empty, then either $\tau^{-1}(\theta) \subseteq [0,\theta)$ or $\tau^{-1}(\theta) \subseteq (\theta,1]$.

From the definition of $\tau(.)$ in (1.4) and the fact that $C > 0$ it follows that $\theta \notin \tau^{-1}(\theta)$. Now suppose that contrary to the Claim, there exists $\theta, \theta_1, \theta_2 \in [0,1]$ such that $\theta_1 < \theta < \theta_2$ and $\theta_1, \theta_2 \in \tau^{-1}(\theta)$. Since $\tau(\theta_1) \neq \emptyset$ and $\theta > \theta_1$, Lemma 9 implies there exists $\theta' \in \tau(\theta)$. By Lemma 10 we have both $q(\theta') \geq q(\theta)$ and $q(\theta') \leq q(\theta)$, so $q(\theta') = q(\theta)$. Since $IC(\theta,\theta')$ is binding, we have $u(q(\theta),\theta) - t(\theta) = u(q(\theta'),\theta) - t(\theta') - C$. Since $q(\theta) = q(\theta')$, it follows that $t(\theta') = t(\theta) - C$. But then $IC(\theta_1,\theta)$ and $IC(\theta_2,\theta)$ cannot be binding because types θ_1 and θ_2 get strictly higher payoff by imitating θ' rather than θ.

Claim 2: If $q(\theta) < q^{fb}(\theta)$, then $\tau^{-1}(\theta)$ is a non-empty subset of $(\theta,1]$; If $q(\theta) > q^{fb}(\theta)$, then $\tau^{-1}(\theta)$ is a non-empty subset of $[0,\theta)$.

Suppose that contrary to the first part of the claim, $q(\theta) < q^{fb}(\theta)$ but $\theta \notin \tau(\theta')$ for any $\theta' \in (\theta,1]$, then we have $V(\theta') > u(q(\theta),\theta') - t(\theta') - C$ for all $\theta' \in [\theta,1]$ (as $\theta \notin \tau(\theta)$). Since $[\theta,1]$ is compact, there exists $\delta > 0$ such that $V(\theta') > u(q(\theta),\theta) - t(\theta) - C + \delta$ for all $\theta' \in [\theta,1]$.

Now let $\tilde{q}(\theta)$ be the solution to $u(\tilde{q}(\theta),1) - u(q(\theta),1) = \delta$ if such exists and satisfies $\tilde{q}(\theta) \leq q^{fb}(\theta)$ and otherwise let $\tilde{q}(\theta) = q^{fb}(\theta)$.

Then the seller can increase its profits by offering an alternative mechanism $(\tilde{q}(.),\tilde{t}(.))$ in which the allocation of type θ is given by $\tilde{q}(\theta)$, $\tilde{t}(\theta) = t(\theta) + u(\tilde{q}(\theta),\theta) - u(q(\theta),\theta) > t(\theta)$ and all other elements remain the same as in the original mechanism $(q(.),t(.))$.

This modification does not affect the net payoff $V(\theta)$ of any type, so $IR(\theta)$ still hold for all θ. Also, $IC(\theta',\theta)$ hold for any $\theta' \in [0,\theta)$ in the mechanism $(\tilde{q}(.),\tilde{t}(.))$ because $u(\tilde{q}(\theta),\theta') - \tilde{t}(\theta) = u(\tilde{q}(\theta),\theta') - u(\tilde{q}(\theta),\theta) + u(q(\theta),\theta) - t(\theta) < u(q(\theta),\theta') - t(\theta)$. The last inequality holds since $\tilde{q}(\theta) > q(\theta)$, $\theta' < \theta$ and $u_{q\theta} > 0$. For $\theta' > \theta$, $V(\theta') > u(q(\theta),\theta') - t(\theta) - C + \delta \geq u(q(\theta),\theta') - t(\theta) - C + u(\tilde{q}(\theta),\theta') - u(q(\theta),\theta') > u(\tilde{q}(\theta),\theta') - \tilde{t}(\theta) - C$, which implies that $IC(\theta',\theta)$ still holds with a slack for $\theta' > \theta$.

A symmetric argument establishes the second part of the claim.

Claim 3: For any $\theta \in [0,1]$, $q(\theta) \leq q^{fb}(\theta)$.

Suppose for some θ_1, $q(\theta_1) > q^{fb}(\theta_1)$. Then by Claim 2, $\theta_1 \in \tau(\theta_0)$ for some $\theta_0 \in [0, \theta_1)$. Therefore,

$$V(\theta_0) = u(q(\theta_1), \theta_0) - t(\theta_1) - C$$

Combining this with $V(\theta_1) = u(q(\theta_1), \theta_1) - t(\theta_1)$ yields:

$$V(\theta_1) = V(\theta_0) + u(q(\theta_1), \theta_1) - u(q(\theta_1), \theta_0) + C > C.$$

Next we will show that there exists an infinite sequence $\{\theta_n\}_{n=0}^{\infty}$ such that for any $n \geq 1$, $\theta_n \in \tau(\theta_{n-1})$, $\theta_n > \theta_{n-1}$, $q(\theta_n) \geq q^{fb}(\theta_n)$ and $V(\theta_n) \geq nC$. We have established this for $n = 1$, so it suffices to establish the following inductive step: if for some fixed $k \geq 1$ these exists θ_k satisfying these conditions, then there exists θ_{k+1} for which these conditions also hold.

Indeed, since $V(\theta_k) \geq kC$, Lemma 9 implies that there exists some $\theta_{k+1} \in \tau(\theta_k)$. Since $\theta_k \in \tau(\theta_{k-1})$ and $\theta_k > \theta_{k-1}$, Lemma 10 then implies that $q(\theta_{k+1}) \geq q(\theta_k)$. If $\theta_{k+1} < \theta_k$, then $q(\theta_{k+1}) \geq q(\theta_k) > q^{fb}(\theta_k) > q^{fb}(\theta_{k+1})$, which contradicts Claim 2. Therefore $\theta_{k+1} > \theta_k$. Then $q(\theta_{k+1}) \geq q^{fb}(\theta_{k+1})$ by Claim 2.

Since $\theta_{k+1} \in \tau(\theta_k)$, we have $V(\theta_k) = u(q(\theta_{k+1}), \theta_k) - t(\theta_{k+1}) - C$. Combining this with $V(\theta_{k+1}) = u(q(\theta_{k+1}), \theta_{k+1}) - t(\theta_{k+1})$, we get:

$$V(\theta_{k+1}) = V(\theta_k) + u(q(\theta_{k+1}), \theta_{k+1}) - u(q(\theta_{k+1}), \theta_k) + C > V(\theta_k) + C > (k+1)C.$$

This completes the proof of the existence of the sequence $\{\theta_n\}_{n=0}^{\infty}$.

However, $u(q(\theta^n), \theta^n)$ is bounded from above, and so $t(\theta^n) < 0$ for sufficiently large n, contradicting Lemma 5.

Claim 4: If $q(\theta) = q^{fb}(\theta)$, then $\nexists \theta' \in (0, \theta)$ s.t. $\theta \in \tau(\theta')$

Suppose there exists some θ such that $q(\theta) = q^{fb}(\theta)$ and $\theta \in \tau(\theta')$ for some $\theta' \in [0, \theta)$. Then the same argument as in Claim 3 can be used to establish a contradiction.

Combining Claims 1-4 yields the statement of the Lemma.

Q.E.D

Relying on Lemma 11 we can now establish the uniqueness of the optimal mechanism.

Lemma 12 *Suppose that $u_{\theta qq}(q, \theta) \geq 0$ for all (q, θ). Then the optimal mechanism is unique.*

Proof of Lemma 12: By Lemma 11 only downwards incentive constrains may be binding. So it is sufficient to establish the uniqueness of the solution to the relaxed problem in which the objective (1.1) is maximized subject to the individual rationality constraints (1.3) and downwards incentive constraints i.e., (1.2) holding for all $\theta, \theta' \in [0, 1]$ s.t. $\theta \geq \theta'$. The proof is by contradiction. So suppose that there exist two solutions to this problem, $(q_1(.), t_1(.))$ and $(q_2(.), t_2(.))$. Then let $V_i(\theta) \equiv u(q_i(\theta), \theta) - t_i(\theta)$ be the agent's net payoff function in the solution $i \in \{1, 2\}$.

Next, fix some $\lambda \in (0, 1)$ and consider an allocation function $\lambda q_1(.) + (1 - \lambda) q_2(.)$ and a net payoff function $\lambda V_1(\theta) + (1 - \lambda) V_2(\theta)$. Let us demonstrate that this allocation and payoff functions define a mechanism which is associated with a strictly higher payoff for the principal and which satisfies (1.2) for all $\theta, \theta' \in [0,1]$ s.t. $\theta \geq \theta'$. The individual rationality of every type θ in (1.3) is trivially satisfied since $V_i(\theta) \geq 0$ for all θ and $i \in \{1,2\}$. Further, the transfer of type θ in this mechanism, $t^\lambda(\theta)$, is equal to

$$\begin{aligned} t^\lambda(\theta) &= u(\lambda q_1(\theta) + (1-\lambda)q_2(\theta), \theta) - (\lambda V_1(\theta) + (1-\lambda)V_2(\theta)) \\ &> \lambda u(q_1(\theta), \theta) + (1-\lambda)u(q_2(\theta), \theta) - (\lambda V_1(\theta) + (1-\lambda)V_2(\theta) = \lambda t_1(\theta) + (1-\lambda)t_2(\theta) \end{aligned}$$

Since this inequality holds for all $\theta \in [0,1]$, the principal gets a strictly higher payoff in this mechanism.

Incentive compatibility constraint in this mechanism is

$$\begin{aligned} &V_1(\theta) + (1-\lambda)V_2(\theta) \\ \geq &u(\lambda q_1(\theta') + (1-\lambda)q_2(\theta'), \theta) - u(\lambda q_1(\theta') + (1-\lambda)q_2(\theta'), \theta') + (\lambda V_1(\theta') + (1-\lambda)V_2(\theta')) - C \end{aligned} \tag{A.1}$$

Now, note that

$$u(\lambda q_1(\theta') + (1-\lambda)q_2(\theta'), \theta) - u(\lambda q_1(\theta') + (1-\lambda)q_2(\theta'), \theta') = \int_{\theta'}^{\theta} u_\theta(\lambda q_1(\theta') + (1-\lambda)q_2(\theta'), t)dt \leq$$
$$\int_{\theta'}^{\theta} \lambda u_\theta(q_1(\theta'), t) + (1-\lambda)u_\theta(q_2(\theta'), t)dt = \lambda(u(q_1(\theta'), \theta) - u(q_1(\theta'), \theta')) + (1-\lambda)(u(q_2(\theta'), \theta) - u(q_2(\theta'), \theta'))$$

where the equalities hold by integration and the inequality holds because $u_{\theta qq} \geq 0$. Combining the above inequality with the fact that incentive constraints (1.2) hold in mechanisms $(q_1(.), t_1(.))$ and $(q_2(.), t_2(.))$ implies that the incentive constraints (A.1) also hold for all $\theta, \theta' \in [0,1]$. *Q.E.D.*

The next Lemma shows that $q(\theta)$ must be strictly increasing and establishes a lower bound of slope for $q(\theta)$.

Lemma 13 *In an optimal mechanism, for any $\theta_2 > \theta_1$, $q(\theta_2) - q(\theta_1) \geq \delta_q(\theta_2 - \theta_1)$, where $\delta_q \equiv \min\{\min_{\theta\in[0,1]} \dot{q}^{fb}(\theta), \frac{\min_{\theta,q} u_\theta(q,\theta)}{\overline{K}}, \frac{\underline{K}}{\overline{K}^2}C\} > 0$, $\overline{K} = \max_{\theta\in[0,1], q(\theta)\in[0,q^{fb}(\theta)]} u_{\theta q}(q(\theta), \theta)$ and $\underline{K} = \min_{\theta\in[0,1], q(\theta)\in[0,q^{fb}(\theta)]} u_{\theta q}(q(\theta), \theta)$.*

Proof of Lemma 13: From Lemma 11 $q(\theta) \leq q^{fb}(\theta)$ for any θ. If $q(\theta_2) = q^{fb}(\theta_2)$, then $q(\theta_2) - q(\theta_1) \geq q^{fb}(\theta_2) - q^{fb}(\theta_1) \geq \min_{\theta\in[0,1]} \dot{q}^{fb}(\theta)(\theta_2 - \theta_1)$. If $q(\theta_2) < q^{fb}(\theta_2)$, then Lemma 11 implies that there exists $\tilde{\theta} > \theta_2$ such that $\theta_2 \in \tau(\tilde{\theta})$. Then by incentive compatibility of $\tilde{\theta}$, $u(q(\theta_2), \tilde{\theta}) - u(q(\theta_2), \theta_2) + V(\theta_2) - C \geq u(q(\theta_1), \tilde{\theta}) - u(q(\theta_1), \theta_1) + V(\theta_1) - C$, which implies

$$\begin{aligned} V(\theta_2) - V(\theta_1) \geq &u(q(\theta_2), \theta_2) - u(q(\theta_2), \theta_1) - [u(q(\theta_2), \tilde{\theta}) - u(q(\theta_2), \theta_1) - u(q(\theta_1), \tilde{\theta}) + u(q(\theta_1), \theta_1] \\ \geq &u(q(\theta_2), \theta_2) - u(q(\theta_2), \theta_1) - (\tilde{\theta} - \theta_1) \max\{(q(\theta_2) - q(\theta_1))\overline{K}, (q(\theta_2) - q(\theta_1))\underline{K}\} \end{aligned} \tag{A.2}$$

Suppose $V(\theta_2) = V(\theta_1) = 0$, then by (A.2) $q(\theta_2)-q(\theta_1) \geq \frac{u(q(\theta_2),\theta_2)-u(q(\theta_2),\theta_1)}{(\tilde{\theta}-\theta_1)\overline{K}} \geq \frac{\min_{\theta,q} u_\theta(q,\theta)}{\overline{K}}(\theta_2-\theta_1)$. Suppose $V(\theta_2) > 0$, then by Lemma 9 there exists $\theta_2' \in \tau(\theta_2)$. Since $\theta_2' \in \tau(\theta_2)$, $u(q(\theta_2),\theta_2)-t(\theta_2) \geq u(q(\theta_2'),\theta_2)-t(\theta_2')-C$. By incentive compatibility of $\tilde{\theta}$, $u(q(\theta_2),\tilde{\theta})-t(\theta_2) \geq u(q(\theta_2'),\tilde{\theta}) - t(\theta_2')$. These two conditions implies

$$\begin{aligned} C \leq& u(q(\theta_2),\tilde{\theta}) - u(q(\theta_2'),\tilde{\theta}) - u(q(\theta_2),\theta_2) + u(q(\theta_2'),\theta_2) \\ \leq& (\tilde{\theta}-\theta_2)(q(\theta_2)-q(\theta_2'))\overline{K} \end{aligned} \tag{A.3}$$

Furthermore, by incentive compatibility of θ_2 and θ_1, $V(\theta_2)-V(\theta_1) \leq u(q(\theta_2'),\theta_2)-u(q(\theta_2'),\theta_1)$. This inequality and (A.2) implies that

$$\begin{aligned} (\theta_2-\theta_1)(q(\theta_2)-q(\theta_2'))\underline{K} &\leq u(q(\theta_2),\theta_2) - u(q(\theta_2),\theta_1) - u(q(\theta_2'),\theta_2) + u(q(\theta_2'),\theta_1) \\ &\leq (\tilde{\theta}-\theta_1)\max\{(q(\theta_2)-q(\theta_1))\overline{K}, (q(\theta_2)-q(\theta_1))\underline{K}\} \\ &= (\tilde{\theta}-\theta_1)(q(\theta_2)-q(\theta_1))\overline{K} \end{aligned} \tag{A.4}$$

where the last inequality holds because (A.3) implies that LHS of (A.4) is positive. (A.3) and (A.4) then imply $q(\theta_2)-q(\theta_1) \geq \frac{\underline{K}}{\overline{K}^2(\tilde{\theta}-\theta_2)(\tilde{\theta}-\theta_1)}C(\theta_2-\theta_1) \geq \frac{\underline{K}}{\overline{K}^2}C(\theta_2-\theta_1)$. *Q.E.D.*

Lemmas 10 and 13 imply that binding IC correspondence is non-decreasing, i.e. suppose in an optimal mechanism $\theta_1 > \theta_2$ and $\theta_1' \in \tau(\theta_1)$, $\theta_2' \in \tau(\theta_2)$, then $\theta_1' \geq \theta_2'$.

For any $\theta' \in [0,1]$, define $\tau^{-1}(\theta') = \{\theta \in [0,1] : \theta' \in \tau(\theta)\}$.

Lemma 14 *In an optimal mechanism, $\tau^{-1}(\theta)$ is either empty or a singleton for any θ. Moreover, there exists $\delta_\tau > 0$ such that for any $\theta'' \geq \theta'$ such that $\tau^{-1}(\theta')$ and $\tau^{-1}(\theta'')$ are non-empty, $\theta''-\theta' \geq \delta_\tau[\tau^{-1}(\theta'')-\tau^{-1}(\theta')]$.*

Proof of Lemma 14: We will show an equivalent statement that there exists $\delta_\tau > 0$ such that for any $\theta'' \geq \theta'$ such that $\tau^{-1}(\theta')$ and $\tau^{-1}(\theta'')$ are non-empty, $\theta''-\theta' \geq \delta_\tau[\max\tau^{-1}(\theta'') - \min\tau^{-1}(\theta')]$. The proof is by contradiction, so suppose such δ_τ does not exist, then there exists a sequence $(\epsilon_n, \theta_n^\dagger, \theta_{2,n}, \theta_{1,n})$ such that $\theta_{2,n} \equiv \max\tau^{-1}(\theta_n^\dagger+\epsilon_n)$, $\theta_{1,n} \equiv \max\tau^{-1}(\theta_n^\dagger-\epsilon_n)$, $\epsilon_n \to 0$ and $\lim_{\epsilon_n\to 0}\theta_{2,n}-\theta_{1,n} > 0$.

Let $(\epsilon, \theta^\dagger, \theta_2, \theta_1)$ be a generic element of the sequence. Define $\hat{\delta} = \frac{\theta_2-\theta_1}{3}$, $\hat{\theta}_1 = \theta_1 + \hat{\delta}$ and $\hat{\theta}_2 = \theta_2 - \hat{\delta}$.

Consider an alternative mechanism $(\tilde{q},\tilde{t})$ such that for $\theta' \in [\theta^\dagger-2\epsilon, \theta^\dagger+2\epsilon]$, $\tilde{q}(\theta') = q(\theta')-\epsilon^2$ and $\tilde{t}(\theta') = t(\theta') - u(q(\theta'),\theta') + u(q(\theta')-\epsilon^2,\theta')$; for $\theta \in [\hat{\theta}_1,\hat{\theta}_2]$, $\tilde{t}(\theta) = t(\theta)+\Delta(\theta,\epsilon)$, where $\Delta(\theta,\epsilon) = \min_{\theta'\in[\theta^\dagger-2\epsilon,\theta^\dagger+2\epsilon]} u(q(\theta'),\theta) - u(q(\theta')-\epsilon^2,\theta) - u(q(\theta'),\theta') + u(q(\theta')-\epsilon^2,\theta')$. Note that $q(\theta^\dagger) > 0$ since $IC(\theta_1,\theta^\dagger)$ binds, so $\tilde{q}(\theta') > 0$ is well defined for small enough ϵ. We will show that all IC and IR are satisfied in the new contract.

For $\theta' \in [\theta^\dagger-2\epsilon, \theta^\dagger+2\epsilon]$, $\tilde{V}(\theta') = V(\theta')$, therefore IR are satisfied. For $\theta \in [\hat{\theta}_1,\hat{\theta}_2]$, since $\tau(\theta_1)$ is non-empty and $\lim_{\epsilon\to 0}\theta-\tau(\theta_1) \geq \hat{\delta} > 0$, Lemma 9 implies that $\lim_{\epsilon\to 0}V(\theta) > 0$ and $\lim_{\epsilon\to 0}\Delta(\theta,\epsilon) = 0$, therefore $\tilde{V}(\theta) = V(\theta)-\Delta(\theta,\epsilon) > 0$ for small enough ϵ, and IR are satisfied.

Now we have to check that $\tilde{IC}(\theta,\theta')$ are satisfied for $\theta \in [0,1]$ and $\theta' \in [\theta^\dagger - 2\epsilon, \theta^\dagger + 2\epsilon]$. For $\theta \leq \theta'$, $IC(\theta,\theta')$ is slack by Lemma 11, and by continuity $\tilde{IC}(\theta,\theta')$ is still slack for small enough ϵ. For $\theta > \theta'$ and $\theta \notin [\hat{\theta}_1, \hat{\theta}_2]$, $\tilde{IC}(\theta,\theta')$ improves. For $\theta \in [\hat{\theta}_1, \hat{\theta}_2]$, $\tilde{IC}(\theta,\theta')$ holds because:

$$\begin{aligned}
&\tilde{V}(\theta) = V(\theta) - \Delta(\theta,\epsilon)\\
&\geq V(\theta) - [u(q(\theta'),\theta) - u(q(\theta') - \epsilon^2,\theta) - u(q(\theta'),\theta') + u(q(\theta') - \epsilon^2,\theta')]\\
&\geq u(q(\theta'),\theta) - t(\theta') - C - [u(q(\theta'),\theta) - u(q(\theta') - \epsilon^2,\theta) - u(q(\theta'),\theta') + u(q(\theta') - \epsilon^2,\theta')]\\
&= u(q(\theta'),\theta) - \tilde{t}(\theta') - C - u(q(\theta'),\theta) + u(q(\theta') - \epsilon^2,\theta)\\
&= u(\tilde{q}(\theta'),\theta) - \tilde{t}(\theta') - C
\end{aligned}$$

where the first equality holds because $\tilde{t}(\theta) = t(\theta) + \Delta(\theta,\epsilon)$, the first inequality holds by definition of $\Delta(\theta,\epsilon)$, the second inequality holds by $IC(\theta,\theta')$, the second equality holds by definition of $\tilde{t}(\theta')$, the last equality holds by definition of $\tilde{q}(\theta')$.

Let $U(\theta'|\theta) = u(q(\theta'),\theta) - t(\theta') - C$ for any θ,θ'.

Claim 1: In an optimal mechanism, suppose there exist θ_2, θ_1, $\theta_1' \in \tau(\theta_1)$ and θ'' such that $(\theta_2 - \theta_1)(\theta_1' - \theta'') > 0$, then $V(\theta_2) - U(\theta''|\theta_2) \geq (\theta_2 - \theta_1)(\theta_1' - \theta'')\delta_V$, where $\delta_V \equiv \delta_q \underline{K} > 0$.

Proof: Incentive compatibility of θ_2 implies

$$V(\theta_2) - U(\theta''|\theta_2) \geq U(\theta_1'|\theta_2) - U(\theta''|\theta_2) = u(q(\theta_1'),\theta_2) - u(q(\theta''),\theta_2) - [t(\theta_1') - t(\theta'')] \quad \text{(A.5)}$$

Incentive compatibility of θ_2 implies that $V(theta_1) \equiv U(\theta_1'|\theta_1) \geq U(\theta''|\theta_1)$, which means $u(q(\theta_1'),\theta_1) - u(q(\theta''),\theta_1) - [t(\theta_1') - t(\theta'')] \geq 0$. This inequality and (A.5) imply

$$\begin{aligned}
V(\theta_2) - U(\theta''|\theta_2) \geq u(q(\theta_1'),\theta_2) - u(q(\theta''),\theta_2) - u(q(\theta_1'),\theta_1) + u(q(\theta''),\theta_1) &\geq (\theta_2 - \theta_1)(q(\theta_1') - q(\theta''))\underline{K}\\
&\geq (\theta_2 - \theta_1)(\theta_1' - \theta'')\delta_q \underline{K}
\end{aligned} \quad \text{(A.6)}$$

where the last inequality holds because of Lemma 13.

For $\theta \in [\hat{\theta}_1, \hat{\theta}_2]$ and $\theta' < \theta^\dagger - 2\epsilon$, since $\theta^\dagger - \epsilon \in \tau(\theta_1)$, Claim 1 implies $V(\theta) - U(\theta'|\theta) \geq (\theta - \theta_1)(\theta^\dagger - \epsilon - \theta')\delta_V > \hat{\delta}\epsilon\delta_V$. This inequality also holds for $\theta \in [\hat{\theta}_1, \hat{\theta}_2]$ and $\theta' > \theta^\dagger + 2\epsilon$ by a symmetrical argument. Therefore, $\frac{\tilde{V}(\theta) - U(\theta'|\theta)}{\epsilon} = \frac{V(\theta) - \Delta(\theta,\epsilon) - U(\theta'|\theta)}{\epsilon} \geq \frac{\hat{\delta}\epsilon\delta_V - \Delta(\theta,\epsilon)}{\epsilon}$. Since $\lim_{\epsilon \to 0} \frac{\hat{\delta}\epsilon\delta_V}{\epsilon} > 0$ and $\lim_{\epsilon \to 0} \frac{\Delta(\theta,\epsilon)}{\epsilon} = 0$, we have $\tilde{V}(\theta) - U(\theta'|\theta) > 0$ for small enough ϵ, and thus $\tilde{IC}(\theta,\theta')$ are satisfied.

The change in seller's profits from switching to the new mechanism is equal to $\int_{\hat{\theta}_1}^{\hat{\theta}_2} \Delta(\theta,\epsilon)dF(\theta) -$

$\int_{\theta^\dagger-2\epsilon}^{\theta^\dagger+2\epsilon} u(q(\theta),\theta') - u(q(\theta)-\epsilon^2,\theta)dF(\theta)$. Since

$$\begin{aligned}
&\lim_{\epsilon\to 0}\left[\frac{\int_{\hat\theta_1}^{\hat\theta_2}\Delta(\theta,\epsilon)dF(\theta)}{\epsilon^2} - \frac{\int_{\theta^\dagger-2\epsilon}^{\theta^\dagger+2\epsilon} u(q(\theta),\theta') - u(q(\theta)-\epsilon^2,\theta)dF(\theta)}{\epsilon^2}\right] \\
&= \int_{\hat\theta_1}^{\hat\theta_2} u_q(q(\theta^\dagger),\theta) - u_q(q(\theta^\dagger),\theta^\dagger)dF(\theta) - \lim_{\epsilon\to 0}\int_{\theta^\dagger-2\epsilon}^{\theta^\dagger+2\epsilon} u_q(q(\theta^\dagger),\theta^\dagger)dF(\theta) \\
&= \int_{\hat\theta_1}^{\hat\theta_2} u_q(q(\theta^\dagger),\theta) - u_q(q(\theta^\dagger),\theta^\dagger)dF(\theta) > 0
\end{aligned}$$

where the last inequality holds because $\hat\theta_1 > \theta_1 > \theta^\dagger - \epsilon$ due to Lemma (11), and thus $\hat\theta_1 > \theta^\dagger$ for small enough ϵ. Therefore, the alternative mechanism generates higher profit while satisfying all IC and IR for enough enough ϵ, contradiction.

Q.E.D.

Lemma 14 implies that for any θ, $\tau^{-1}(\theta)$ is either empty or a singleton. It also implies that the correspondence $\tau(.)$ is strictly increasing.

Corollary 1 *Let $\theta_1 > \theta_2$. Suppose $\theta_1' \in \tau(\theta_1)$, $\theta_2' \in \tau(\theta_2)$, then $\theta_1' > \theta_2'$.*

The next Lemma establishes a lower bound of loss if a type chooses to imitate other types.

Lemma 15 *Let $U(\theta'|\theta) = u(q(\theta'),\theta) - t(\theta') - C$ for any θ,θ'. In an optimal mechanism, for any θ_2,θ_1 and $\theta_1' \in \tau(\theta_1)$,*

$$V(\theta_2) - U(\theta_1'|\theta_2) \geq \begin{cases} \delta_V\delta_\tau\frac{(\theta_2-\theta_1)^2}{4} & \text{if } \frac{\theta_1+\theta_2}{2} \geq \hat\theta \\ \frac{\theta_1-\theta_2}{2}\min_\theta u_\theta(q(\theta_1'),\theta) & \text{if } \frac{\theta_1+\theta_2}{2} < \hat\theta \end{cases}.$$

Proof of Lemma 15: Let $\tilde\theta = \frac{\theta_1+\theta_2}{2}$. If $\tilde\theta \geq \hat\theta$, then there exists $\tilde\theta' \in \tau(\tilde\theta)$. Then $V(\theta_2) - U(\theta_1'|\theta_2) \geq \delta_V(\theta_2-\tilde\theta)(\tilde\theta'-\theta_1') \geq \delta_V\delta_\tau(\theta_2-\tilde\theta)(\tilde\theta-\theta_1)$, where the first and second inequalities hold by Lemma 14 and Claim 1 of Lemma 14. If $\tilde\theta < \hat\theta$, then since $\tau(\theta_1)$ is non-empty, it must be the case that $\theta_2 < \tilde\theta < \hat\theta \leq \theta_1$, thus Lemma 9 implies that $V(\tilde\theta) = V(\theta_2) = 0$. Incentive compatibility of $\tilde\theta$ then implies $U(\theta_1'|\tilde\theta) \leq 0$, and thus $V(\theta_2) - U(\theta_1'|\theta_2) = 0 - U(\theta_1'|\theta_2) = 0 - U(\theta_1'|\tilde\theta) + u(q(\theta_1'),\tilde\theta) - u(q(\theta_1'),\theta_2) \geq (\tilde\theta-\theta_2)\min_\theta u_\theta(q(\theta_1'),\theta)$. *Q.E.D.*

Lemma 16 *Define $\tau^{-k}(.) = \tau^{-1}(\tau^{-(k-1)}(.))$ for $k = 1,2,....$ In an optimal mechanism, there exists $\bar K < \infty$ such that for any θ, $\tau^{-k}(\theta) = \emptyset$ for some $k \leq \bar K$.*

Proof of Lemma 16: Since $\tau(.)$ is increasing, it is sufficient to establish the claim of the Lemma for $\theta_1 = \tau(\hat\theta)$. We argue by contradiction, so suppose that the claim of the Lemma is not true for θ_1. Then there exists a sequence θ_k, $k = 1,...,\infty$ such that $\theta_{k+1} = \tau^{-1}(\theta_k)$ for all $k \geq 1$ i.e., $u(q(\theta_{k+1}),\theta_{k+1}) - t(\theta_{k+1}) = u(q(\theta_k),\theta_{k+1}) - t(\theta_k) - C$. Lemma 11 implies that $\theta_k < \theta_{k+1}$. Since $\theta_k \in [0,1]$ for all k, it follows that $\lim_{k\to\infty} \theta_k - \theta_{k+1} = 0$. But then by continuity of q and t, $\lim_{k\to\infty}[u(q(\theta_{k+1}), \theta_{k+1}) - t(\theta_{k+1})] - [u(q(\theta_k), \theta_{k+1}) - t(\theta_k)] = 0 > -C$, a contradiction. *Q.E.D.*

Lemma 17 *In an optimal mechanism, if $\theta'_1, \theta'_2 \in \tau(\breve{\theta}^0)$ for some $\breve{\theta}^0$, with $\theta'_1 < \theta'_2$, then $q(\theta') = q^{fb}(\theta')$ for any $\theta' \in [\theta'_1, \theta'_2]$.*

Proof of Lemma 17:

Suppose to the contrary that $q(\theta) < q^{fb}(\theta)$ for some $\theta \in [\theta'_1, \theta'_2]$. Then by continuity of $q(.)$ there exist $\breve{\theta}'_1, \breve{\theta}'_2$ such that $\theta'_1 < \breve{\theta}'_1 < \breve{\theta}'_2 < \theta'_2$ and for any $\theta' \in [\breve{\theta}'_1, \breve{\theta}'_2]$, $q(\theta') < q^{fb}(\theta')$. Then by Lemma 11 and Corollary 1, $\tau^{-1}(\theta) = \{\breve{\theta}^0\}$ for all $\theta \in [\breve{\theta}'_1, \breve{\theta}'_2]$.

Recall that $\tau^{-k}(.) = \tau^{-1}(\tau^{-(k-1)}(.))$ where k is a positive integer k. By Lemma 16 there exists $M \geq 0$ such that $\tau^{-k}(\breve{\theta}^0)$ is singleton for $k \leq M$ and empty for $k > M$. So, if $M \geq 1$, then for $k \in \{1, ..., M\}$ let us define $\breve{\theta}^k = \tau^{-k}(\breve{\theta}^0)$.

For any $\epsilon > 0$ and $k = 0, ..., M$, let $\Theta_k(\epsilon) = [\breve{\theta}^k - (\frac{1}{\delta_\tau} + 1)^k \epsilon, \breve{\theta}^k + (\frac{1}{\delta_\tau} + 1)^k \epsilon]$.

Now consider an alternative mechanism $(\tilde{q}(.), \tilde{t}(.))$ which differs from the original mechanism $(q(.), t(.)))$ only as follows: for $\theta \in [\breve{\theta}'_1, \breve{\theta}'_2]$, $\tilde{q}(\theta) = q(\theta) + \epsilon^3$ and $\tilde{t}(\theta) = t(\theta) + u(q(\theta) + \epsilon^3, \theta) - u(q(\theta), \theta)$, and for $\theta \in \cup_{k=0}^M \Theta_k(\epsilon)$, $\tilde{t}(\theta) = t(\theta) - \Delta(\epsilon)$, where $\Delta(\epsilon) \equiv \max_{\theta' \in [\breve{\theta}'_1, \breve{\theta}'_2]} u(q(\theta') + \epsilon^3, 1) - u(q(\theta'), 1) - u(q(\theta') + \epsilon^3, \theta') + u(q(\theta'), \theta')$. We will show that all IC and IR are satisfied in the new contract.

First, IR constraints hold in $(\tilde{q}(.), \tilde{t}(.))$ because $\tilde{V}(\theta) > V(\theta)$ for $\theta \in \cup_{k=0}^M \Theta_k(\epsilon)$, and $\tilde{V}(\theta) = V(\theta)$ for all other types θ.

Now let us consider incentive constraints. For $\theta \notin \cup_{k=0}^M \Theta_k(\epsilon)$ and $\theta' \in [\breve{\theta}'_1, \breve{\theta}'_2]$ and small enough ϵ,

$$\begin{aligned}
&\tilde{V}(\theta) = V(\theta) \geq u(q(\theta'), \theta) - t(\theta') - C + \delta_V \delta_\tau \frac{\epsilon^2}{4} \\
&= u(\tilde{q}(\theta'), \theta) - \tilde{t}(\theta') - C + \delta_V \delta_\tau \frac{\epsilon^2}{2} - [u(q(\theta') + \epsilon^3, \theta) - u(q(\theta'), \theta) - u(q(\theta') + \epsilon^3, \theta') - u(q(\theta'), \theta')] \\
&> u(\tilde{q}(\theta'), \theta) - \tilde{t}(\theta') - C
\end{aligned}$$

where the first inequality holds because $\theta' \in \tau(\breve{\theta}^0)$ and $|\theta - \breve{\theta}^0| \geq \epsilon$, so Lemma 15 implies $V(\theta) - U(\theta'|\theta) \geq \delta_V \delta_\tau \frac{\epsilon^2}{4}$ for small enough ϵ; the second equality holds by the definitions of $\tilde{q}(\theta')$ and $\tilde{t}(\theta')$; the last inequality holds for small enough ϵ.

For $\theta \in \cup_{k=0}^M \Theta_k(\epsilon)$ and $\theta' \in [\breve{\theta}'_1, \breve{\theta}'_2]$,

$$\begin{aligned}
&\tilde{V}(\theta) = V(\theta) + \Delta(\epsilon) \\
&\geq u(q(\theta'), \theta) - t(\theta') - C + [u(q(\theta') + \epsilon^3, \theta) - u(q(\theta'), \theta) - u(q(\theta') + \epsilon^3, \theta') + u(q(\theta'), \theta')] \\
&= u(\tilde{q}(\theta'), \theta) - \tilde{t}(\theta') - C
\end{aligned}$$

where the first equality holds by definition of $\tilde{t}(\theta)$; the first inequality holds by $IC(\theta,\theta')$, definition of $\Delta(\theta,\epsilon)$ and $u_{\theta q}>0$; the last equality holds by definitions of $\tilde{q}(\theta')$ and $\tilde{t}(\theta')$.

For $\theta \notin \cup_{k=0}^{M}\Theta_k(\epsilon)$, $\theta' \in \cup_{k=0}^{M}\Theta_k(\epsilon)$ and small enough ϵ,

$$\begin{aligned}\tilde{V}(\theta) = V(\theta) &> u(q(\theta'),\theta) - t(\theta') - C + \delta_V\delta_\tau\frac{\epsilon^2}{4}\\ &= u(\tilde{q}(\theta'),\theta) - \tilde{t}(\theta') - C + \delta_V\delta_\tau\frac{\epsilon^2}{2} - \Delta(\epsilon)\\ &> u(\tilde{q}(\theta'),\theta) - \tilde{t}(\theta') - C\end{aligned}$$

where the first inequality holds because $\theta' \in [\breve{\theta}^k - (\frac{1}{\delta_\tau}+1)^k\epsilon, \breve{\theta}^k + (\frac{1}{\delta_\tau}+1)^k\epsilon]$ for some k, so Lemma 14 implies $\tau^{-1}(\theta') \in [\breve{\theta}^{k+1} - \frac{1}{\delta_\tau}(\frac{1}{\delta_\tau}+1)^k\epsilon, \breve{\theta}^{k+1} + \frac{1}{\delta_\tau}(\frac{1}{\delta_\tau}+1)^k\epsilon]$, and since $|\theta - \tau^{-1}(\theta')| \geq (\frac{1}{\delta_\tau}+1)^k\epsilon \geq \epsilon$, Lemma 15 implies that $V(\theta) - U(\theta'|\theta) \geq \delta_V\delta_\tau\frac{\epsilon^2}{4}$ for small enough ϵ; the second equality holds by the definitions of $\tilde{q}(\theta')$ and $\tilde{t}(\theta')$; the last inequality holds for small enough ϵ.

For $\theta \in \cup_{k=0}^{M}\Theta_k(\epsilon)$ and $\theta' \in \cup_{k=0}^{M}\Theta_k(\epsilon)$,

$$\tilde{V}(\theta) = V(\theta) + \Delta(\epsilon) \geq u(q(\theta'),\theta) - t(\theta') - C + \Delta(\epsilon) = u(\tilde{q}(\theta'),\theta) - \tilde{t}(\theta') - C$$

where the first equality holds by definition of $\tilde{t}(\theta)$, the first inequality holds by $IC(\theta,\theta')$, the last equality holds by definition of $\tilde{t}(\theta')$. Thus, all $\tilde{IC}(\theta,\theta')$ are satisfied for small enough ϵ.

The change in seller's profits from switching to the new mechanism is equal to

$$\int_{\breve{\theta}'_1}^{\breve{\theta}'_2}[u(q(\theta')+\epsilon^3,\theta') - u(q(\theta'),\theta')]f(\theta')d\theta' - F(\cup_{k=0}^{M}[\breve{\theta}^k - (\frac{1}{\delta_\tau}+1)^k\epsilon, \breve{\theta}^k + (\frac{1}{\delta_\tau}+1)^k\epsilon])\Delta(\epsilon)$$

Since $\lim_{\epsilon\to 0}\frac{\int_{\breve{\theta}'_1}^{\breve{\theta}'_2}[u(q(\theta')+\epsilon^3,\theta') - u(q(\theta'),\theta')]f(\theta')d\theta'}{\epsilon^3} \in (0,\infty)$, $\lim_{\epsilon\to 0}\frac{\Delta(\epsilon)}{\epsilon^3} \in (0,\infty)$ and $\lim_{\epsilon\to 0} F(\cup_{k=0}^{M}[\breve{\theta}^k - (\frac{1}{\delta_\tau}+1)^k\epsilon, \breve{\theta}^k + (\frac{1}{\delta_\tau}+1)^k\epsilon]) = 0$, we conclude that when ϵ is sufficiently small, this alternative mechanism generates a higher profit while satisfying all IC and IR, contradiction.

Q.E.D.

The next Lemma establishes that the set of binding incentive constraints is non-empty when C is not too high, and there are thresholds $\underline{\theta}, \overline{\theta}$ such that for all types outside $[\underline{\theta}, \overline{\theta}]$, no type has binding incentive towards them. To this end, define

$$G(\theta,\theta') = u(q^{fb}(\theta'),\theta) - u(q^{fb}(\theta'),\theta'), \tag{A.7}$$

We have:

Lemma 18 *Let*

$$\overline{C} = \max_{\theta,\theta'\in[0,1]} G(\theta,\theta') = \max_{\theta'\in[0,1]} G(1,\theta') \tag{A.8}$$

Then: (i) $\tau([0,1]) \neq \emptyset$ *if* $C < \overline{C}$*;*
(ii) $\tau([0,1]) = \emptyset$ *if* $C > \overline{C}$*.*
(iii) For any $C > 0$*, there exists* $\underline{\theta}, \overline{\theta}$*,* $0 < \underline{\theta} \leq \overline{\theta} < 1$*, such that* $\tau(\theta) = \emptyset$ *for all* $\theta \in [0, \underline{\theta}) \cup (\overline{\theta}, 1]$*.*

Proof of Lemma 18:

(i) To prove the first claim of the Lemma we argue by contradiction. So suppose that $\tau([0,1]) = \emptyset$. Then for all $\theta \in [0,1]$ $V(\theta) = 0$ by Lemma 9, and $q(\theta) = q^{fb}(\theta)$ by Lemma 11. But then $IC(1,\theta)$ fails for some θ because $C < \overline{C} = \max_{\theta,\theta'} u(q^{fb}(\theta'),\theta) - u(q^{fb}(\theta'),\theta')$.

(ii) The proof that $\tau([0,1]) = \emptyset$ if $C > \overline{C}$ is straightforward and is therefore omitted.

(iii) For the upper bound $\overline{\theta}$, if $\tau(1) = \emptyset$, then Lemma 9 implies $\tau([0,1]) = \emptyset$, so $\overline{\theta} = \underline{\theta}$. If $\tau(1) \neq \emptyset$, then let $\overline{\theta} = \max\{\theta' : \theta' \in \tau(1)\} < 1$, where the inequality holds by Lemma 11. But then by Corollary 1 $\theta' \leq \overline{\theta}$ for any $\theta' \in \tau([0,1])$.

Finally, since $u(q^{fb}(\theta),1)$ is continuous in θ and $u(q^{fb}(0),1) = 0$ because $q^{fb}(0) = 0$, there exists $\underline{\theta} > 0$ such that $u(q^{fb}(\theta),1) - C < 0$ for all $\theta \in [0,\underline{\theta}]$. By Lemma 11 $q(\theta) \leq q^{fb}(\theta)$ and by Lemma 5 $t(\theta) > 0$, so $u(q(\theta),\theta') - t(\theta) - C \leq u(q(\theta),1) - t(\theta) - C < 0$ for any $\theta' \in [0,1]$ and $\theta \in [0,\underline{\theta}]$, which implies $\theta \notin \tau(\theta')$.

Q.E.D.

Lemma 19 shows that for a range of C, any type $\theta \in \tau([0,1])$ gets zero surplus.

Lemma 19 *There exists* $\underline{C} \in (0,\overline{C})$*, such that in the optimal mechanism for any* $C \in [\underline{C},\overline{C}]$ *we have: if* $\theta' \in \tau([0,1])$ *then* $V(\theta') = 0$*.*

Proof of Lemma 19: Recall from (A.7) that $G(\theta,\theta') = u(q^{fb}(\theta'),\theta) - u(q^{fb}(\theta'),\theta')$, and $\overline{C} = \max_{\theta,\theta'} G(\theta,\theta') = \max_{\theta'} G(1,\theta')$. Now define $G^*(\theta) = \max_{\theta'} G(\theta,\theta')$. Since $G(.,.)$ is continuous in both arguments and $u_\theta > 0$, $G^*(\theta)$ is continuous and strictly increasing.

Define

$$\hat{\Theta}(C) = \{\theta \in [0,1] : G^*(\theta) - C \geq 0\}$$

Then for any $C \in (0,\overline{C})$ the set $\hat{\Theta}(C)$ is non-empty. Furthermore, since $G^*(\theta)$ is continuous and strictly increasing in θ, there exists $\theta^C \in (0,1)$ such that $\hat{\Theta}(C) = [\theta^C,1]$, with $\lim_{C\to\overline{C}} \theta^C \to 1$.

Next, let us show that there exists $\underline{C} \in (0,\overline{C})$ such that whenever $C \in (\underline{C},\overline{C})$:

(i) $V(\theta) = 0$ for all $\theta \notin \hat{\Theta}(C)$,

(ii) $\hat{\Theta}(C) \cap \tau(\hat{\Theta}(C)) = \emptyset$.

To establish (i), suppose that $V(\theta) > 0$ for some $\theta \notin \hat{\Theta}(C)$. Then consider an alternative mechanism $(\tilde{q}(.),\tilde{t}(.))$ which differs from the original mechanism $(q(.),t(.))$ only in transfers. Particularly, $\tilde{t}(\theta) = u(q(\theta),\theta)$ for $\theta \notin \hat{\Theta}(C)$ and $\tilde{t}(\theta) = \max\{u(q(\theta),\theta^C) - C, t(\theta)\}$ for $\theta \in \hat{\Theta}(C)$.

The mechanism $(\tilde{q}(.),\tilde{t}(.))$ is (weakly) more profitable for the seller then $(q(.),t(.))$. So we only need to verify that $(\tilde{q}(.),\tilde{t}(.))$ satisfies IR and IC constraints. For $\theta \notin \hat{\Theta}(C)$, $IR(\theta)$ is binding by construction. If $\theta \in \hat{\Theta}(C)$ and $\tilde{t}(\theta) = t(\theta)$ then $IR(\theta)$ holds in $(\tilde{q}(.),\tilde{t}(.))$ because it holds in $(q(.),t(.))$. If $\tilde{t}(\theta) = u(q(\theta),\theta^C) - C > t(\theta)$, then θ gets a payoff $u(q(\theta),\theta) - u(q(\theta),\theta^C) + C > 0$. So all IR constraints hold.

Now consider IC constraints. Fix any pair $(\theta, \theta') \in ([0,1] \setminus \hat{\Theta}(C) \times \hat{\Theta}(C)$. $IC(\theta, \theta')$ holds in the mechanism $(\tilde{q}(.), \tilde{t}(.))$ because by reporting her type truthfully type θ gets zero payoff. At the same time, $\tilde{t}(\theta') \geq u(q(\theta'), \theta^C) - C$, and so type θ's payoff from imitating θ' does not exceed $u(q(\theta'), \theta) - u(q(\theta'), \theta^C) \leq 0$.

Now fix any pair $(\theta, \theta') \in \hat{\Theta}(C) \times ([0,1] \setminus \hat{\Theta}(C))$. If $\tilde{t}(\theta) = t(\theta)$, then $IC(\theta, \theta')$ holds in $(\tilde{q}(.), \tilde{t}(.))$ because it holds in $(q(.), t(.))$ and $\tilde{t}(\theta') \geq t(\theta')$. Now suppose that $\tilde{t}(\theta) = u(q(\theta), \theta^C) - C > t(\theta)$. Then $IC(\theta, \theta')$ holds iff $u(q(\theta), \theta) - u(q(\theta), \theta^C) + C \geq u(q(\theta'), \theta) - u(q(\theta'), \theta') - C$. Note that $q(\theta') \leq q^{fb}(\theta')$. So,this inequality holds if $C \geq \frac{\overline{C}}{2}$.

Next, $IC(\theta, \theta')$ holds for any pair $(\theta, \theta') \in ([0,1] \setminus \hat{\Theta}(C) \times ([0,1] \setminus \hat{\Theta}(C))$, because $q(\theta') \leq q^{fb}(\theta')$ and so, by definition of C

$$u(q(\theta'), \theta) - \tilde{t}(\theta') - C = u(q(\theta'), \theta) - u(q(\theta'), \theta') - C \leq u(q^{fb}(\theta'), \theta) - u(q^{fb}(\theta'), \theta') - C \leq 0$$

Finally, consider a pair $(\theta, \theta') \in \hat{\Theta}(C) \times \hat{\Theta}(C)$. If $\tilde{t}(\theta) = t(\theta)$, then $IC(\theta, \theta')$ holds in $(\tilde{q}(.), \tilde{t}(.))$ because it holds in $(q(.), t(.))$ and $\tilde{t}(\theta') \geq t(\theta')$.

Now suppose that $\tilde{t}(\theta) = u(q(\theta), \theta^C) - C > t(\theta)$. Since $\tilde{t}(\theta') \geq u(q(\theta'), \theta^C) - C$, $IC(\theta, \theta')$ holds if $u(q(\theta), \theta) - u(q(\theta), \theta^C) + C \geq u(q(\theta'), \theta) - u(q(\theta'), \theta^C)$. This inequality clearly holds if $q(\theta) \geq q(\theta')$. Now, if $q(\theta) < q(\theta')$, let us rewrite the last inequality as follows:

$$C \geq u(q(\theta'), \theta) - u(q(\theta), \theta) - (u(q(\theta'), \theta^C) - u(q(\theta), \theta^C)) \tag{A.9}$$

Since $q(.)$ is continuous in θ by Lemma 7, the right-hand side of inequality (A.9) converges to zero as C increase to $\bar{C}$. So the inequality (A.9) holds strictly when C is sufficiently close to $\bar{C}$. This completes the proof of incentive compatibility of the mechanism $(\tilde{q}(.), \tilde{t}(.))$.

Next, let us establish claim (ii): $\hat{\Theta}(C) \cap \tau(\hat{\Theta}(C)) = \emptyset$. The proof is by contradiction, so suppose there exists $\theta' \in \hat{\Theta}(C) \cap \tau(\hat{\Theta}(C))$. Then there exists $\theta \in \hat{\Theta}(C)$ such that $IC(\theta, \theta')$ is binding. But as we have shown above, this is not true in the optimal mechanism when C is close to $\overline{C}$. In particular, in this case (A.9) holds strictly. A contradiction. *Q.E.D.*

A.3 Proof of Theorem 2

Theorem 2 follows from Lemmas 6 and 12.

A.4 Proof of Theorem 3

Theorem 3 follows from Lemmas 5, 7, 11 and 13.

A.5 Proof of Theorem 4

Theorem 4 follows from Lemmas 8, 9, 11, 18, 17 and Corollary 1.

A.6 Proof of Theorem 5

First, let us rewrite the problem (1.1)-(1.3) as the following equivalent problem using the net payoff function $V(\theta) = u(q(\theta), \theta) - t(\theta)$:

$$\max_{q(\theta), V(\theta)} \int_0^1 [u(q(\theta), \theta) - V(\theta)] f(\theta) d\theta \tag{A.10}$$

subject to:

$$V(\theta) - V(\theta') \geq u(q(\theta'), \theta) - u(q(\theta'), \theta') - C \qquad \forall\, \theta, \theta' \in [0,1] \tag{A.11}$$

$$V(\theta) \geq 0 \qquad \forall\, \theta \in [0,1] \tag{A.12}$$

$$q(\theta) \geq 0 \qquad \forall\, \theta \in [0,1] \tag{A.13}$$

By Lemmas 5, 7 and 11, we can without loss of generality restrict $q(.)$ to belong to the space of continuous functions from $[0,1]$ to $[0, q^{fb}(1)]$ and $V(.)$ to belong to the space of continuous functions from $[0,1]$ to $[0, u(q^{fb}(1), 1)]$. Let $K = \max\{q^{fb}(\theta), u(q^{fb}(1), 1)\}$ and let $C([0,1])^{[0,K]}$ be the space of continuous functions from $[0,1]$ to $[0, K]$.

Endow $C([0,1])^{[0,K]}$ with weak-* topology.[15] By Alaoglu Theorem the space $C([0,1])^{[0,K]}$ is compact in the weak* topology, and by Tychonoff's Theorem the product $C([0,1])^{[0,K]} \times C([0,1])^{[0,K]}$ is compact in the product topology generated by the weak* topology. Further, for every value of the fixed cost C, the set of functions $(q(.), V(.)) \in C([0,1])^{[0,K]} \times C([0,1])^{[0,K]}$ that satisfy the constraints (A.11)-(A.13) is a closed subset of $C([0,1])^{[0,K]} \times C([0,1])^{[0,K]}$, and is therefore compact in the product topology generated by the weak* topology. Also, this set varies continuously with the fixed costs C. Thus, the correspondence $\{(q(.), V(.)) \in C([0,1])^{[0,K]} \times C([0,1])^{[0,K]} : (q(.), V(.))$ satisfy (A.11)-(A.13)$\}$ specifying the set of admissible quantity and surplus functions for fixed cost C is continuous in C and compact valued.

Let $(q(.|C)), V(.|C)))$ be the solution to problem (A.10)-(A.13). By Theorem 2 the solution exists and is unique. Since the objective function (A.10) is continuous in $q(.)$, $V(.)$ and C, by Berge's Maximum Theorem $(q(.|C), V(.|C))$ is upper hemicontinuous in C. This implies that $\lim_{C \downarrow 0}(q(\theta|C), V(\theta|C)) = (q(\theta|0), V(\theta|0)) \equiv (q^{sb}(\theta), V^{sb}(\theta))$ for all $\theta \in [0,1]$.

Further, $(q^{sb}(\theta), V^{sb}(\theta)) = (q(\theta|0)), V(\theta|0))$ is the standard second-best solution to our problem for $C = 0$. Note that $q^{sb}(\theta)$ is continuous and $q^{sb}(0) = 0 < q^{sb}(1) = q^{fb}(1)$. Therefore, there exist $\underline{\theta}, \overline{\theta} \in [0,1]$, $\underline{\theta} < \overline{\theta}$, such that $q^{sb}(\theta)$ is strictly increasing and $V^{sb}(\theta) > 0$ on $[\underline{\theta}, \overline{\theta}]$.

[15] A sequence $x_n(\theta)$ converges to $x(\theta)$ in the weak* topology iff $\int_0^1 x_n(\theta) y(\theta) dF(\theta) \to \int_0^1 x(\theta) y(\theta) dF(\theta)$ for all $y \in L^2(F)$.

Since $\lim_{C\downarrow 0} V(\theta|C) = V^{sb}(\theta) > 0$ for $\theta \in [\underline{\theta}, \overline{\theta}]$, Lemma 9 implies that there exists $\hat{C} > 0$ such that $\tau(\theta|C) \neq \emptyset$ for all $C \in (0, \hat{C})$ and $\theta \in [\underline{\theta}, \overline{\theta}]$.

Now to show that $\lim_{C\downarrow 0} M(C) = \infty$, fix any pair (θ, θ') s.t. $\theta \in (\underline{\theta}, \overline{\theta}]$ and $\theta' < \theta$, and consider the corresponding incentive constraint (A.11). Putting all terms on one side and taking the limit as $C \to 0$ we get:

$$\lim_{C\downarrow 0} \big(V(\theta|C) - V(\theta'|C) + C - u(q(\theta'|C), \theta) + u(q(\theta'|C), \theta') \big) =$$

$$V^{sb}(\theta) - V^{sb}(\theta') - u(q^{sb}(\theta'), \theta) - u(q^{sb}(\theta'), \theta') = \int_{\theta'}^{\theta} u_\theta(q^{sb}(s), s) ds - u(q^{sb}(\theta'), \theta) + u(q^{sb}(\theta'), \theta') > 0$$

where the last inequality holds because $q^{sb}(.)$ is increasing, strictly on $(\underline{\theta}, \overline{\theta})$. So, for any $\theta \in (\underline{\theta}, \overline{\theta}]$ and $\theta' < \theta$ we have $\tau(\theta|C) > \theta'$ when C is sufficiently small. Hence, $\lim_{C\downarrow 0} \tau(\theta|C) = \theta$ for $\theta \in [\underline{\theta}, \overline{\theta}]$.

Finally, fix some integer $M > 0$ and let $\epsilon_M = \frac{\overline{\theta} - \underline{\theta}}{M}$. Since $\lim_{C\downarrow 0} \tau(\theta|C) = \theta$ for $\theta \in [\underline{\theta}, \overline{\theta}]$, there exists $C_M > 0$ such that $\tau^{k-1}(\overline{\theta}|C) - \tau^k(\overline{\theta}|C) \leq \epsilon_M$ for any $k = 1, ..., M$ and hence $\tau^M(\overline{\theta}|C) \geq \underline{\theta}$ for all $C \in (0, C_M]$. By Corollary 1, $\tau^M(1|C) > \tau^M(\overline{\theta}|C) \geq \underline{\theta}$ for $C \in (0, C_M]$. Since M was chosen arbitrarily, it follows that $\tau^M(1|C) \neq \emptyset$ for any $M < \infty$ when C is sufficiently small i.e., $\lim_{C\downarrow 0} M(C) = \infty$.

Q.E.D.

Appendix B

Appendix for Chapter 2

In this Appendix we provide proof to Lemma 1 and Theorems 6, 7, 8, and 9.

B.1 Proof of Lemma 1

Part (i): Suppose on the contrary, there exists θ such that $\tau(\theta)$ is multi-valued. Let $\theta_2 = \max \tau(\theta) > \theta_1 = \min \tau(\theta)$. θ_1 and θ_2 exist by Lemma 8. By assumption $V(\theta_2) = V(\theta_1) = 0$, and Lemma 7 implies $V(\theta') = 0 \ \forall \theta' \in [\theta_1, \theta_2]$. By Lemma 17, $q(\theta') = q^{fb}(\theta')$ for all $\theta' \in [\theta_1, \theta_2]$. These imply that $u(q(\theta'), \theta) - t(\theta') = G(\theta, \theta')$ for all $\theta' \in [\theta_1, \theta_2]$. By Assumption 2, $G(\theta, \theta') > \min\{G(\theta, \theta_1), G(\theta, \theta_2)\}$ for all $\theta' \in (\theta_1, \theta_2)$. So, $u(q^{fb}(\theta'), \theta) - t(\theta') > \min\{u(q^{fb}(\theta_1), \theta) - t(\theta_1), u(q^{fb}(\theta_2), \theta) - t(\theta_2)\}$, contradicting that $\theta_1, \theta_2 \in \tau(\theta)$.

Part (ii): Suppose on the contrary, there exists θ such that $\tau(\theta)$ is multi-valued. If $V(\max \tau(\theta)) = 0$, then the same argument as part (i) establishes a contradiction, so it must be that $V(\max \tau(\theta)) > 0$. Let $\Theta_m = \{\theta : \tau(\theta) \text{ is multi-valued}\}$ be the set of types with multi-valued τ and $\underline{\theta}_m = \inf \Theta_m$. Fix any $\tilde{\theta} \in \Theta_m$ such that $\tilde{\theta} - \underline{\theta}_m < \frac{C}{\overline{K} q^{fb}(1)}$ where $\overline{K} = \max_{\theta \in [0,1], q(\theta) \in [0, q^{fb}(\theta)]} u_{\theta q}(q(\theta), \theta)$.

Let us show that $\theta' < \underline{\theta}_m$ for any $\theta' \in \tau(\tilde{\theta})$. By definition of $\tau(.)$, we have

$$u(q(\theta'), \tilde{\theta}) - u(q(\theta'), \tilde{\theta}) = C + V(\tilde{\theta}) - V(\theta')$$

Using $V(\theta) = \int_{\hat{\theta}}^{\theta} u_\theta(q(\tau(s)), s) ds$ in the above equation and rearranging yields:

$$\int_{\theta'}^{\tilde{\theta}} u_\theta(q(\theta'), s) ds - \int_{\theta'}^{\tilde{\theta}} u_\theta(q(\tau(s)), s) ds = \int_{\theta'}^{\tilde{\theta}} \int_{q(\tau(s))}^{q(\theta')} u_{\theta q}(q, s) dq ds = C$$

Since $q(\theta) \leq q^{fb}(1)$ for all θ and $u_{\theta q} \leq \overline{K}$, the previous equation implies that $\tilde{\theta} - \theta' \geq \frac{C}{\overline{K} q^{fb}(1)} > \tilde{\theta} - \underline{\theta}_m$.

Let $\theta_2 = \max \tau(\tilde{\theta})$ and $\theta_1 = \min \tau(\tilde{\theta})$. Since $\tau(\tilde{\theta})$ is multi-valued, we have $V(\theta_2) > 0$, and Lemma 9 implies that $\tau(\theta_2)$ is non-empty. Also since $\theta_2 < \underline{\theta}_m$, $\tau(\theta_2)$ is single-valued.

Let $\tilde{G}(\theta, \theta') = G(\theta, \theta') + V(\theta') = u(q^{fb}(\theta'), \theta) - u(q^{fb}(\theta'), \theta') + \int_{\hat{\theta}}^{\max\{\theta', \hat{\theta}\}} u_\theta(q(\tau(s)), s) ds$, where the second equality holds by definition of G and equation (2.3). By Lemma 17, $q(\theta') =$

$q^{fb}(\theta')$ for all $\theta' \in [\theta_1, \theta_2]$, which implies that $u(q(\theta'), \tilde{\theta}) - t(\theta') = \tilde{G}(\tilde{\theta}, \theta')$ for all $\theta' \in [\theta_1, \theta_2]$. We must have the following:

$$\frac{\partial \tilde{G}(\tilde{\theta}, \theta_2)}{\partial \theta_2} = u_q(q^{fb}(\theta_2), \tilde{\theta})\dot{q}^{fb}(\theta_2) - u_\theta(q^{fb}(\theta_2), \theta_2) + u_\theta(q(\tau(\theta_2)), \theta_2) \geq 0, \tag{B.1}$$

since otherwise $\tilde{G}(\tilde{\theta}, \theta) > \tilde{G}(\tilde{\theta}, \theta_2)$ for $\theta < \theta_2$ close enough to θ_2, and $IC(\tilde{\theta}, \theta)$ fails. Rearranging (B.1) gives:

$$u_q(q^{fb}(\theta_2), \tilde{\theta}) \geq [u_\theta(q^{fb}(\theta_2), \theta_2) - u_\theta(q(\tau(\theta_2)), \theta_2)]\frac{1}{\dot{q}^{fb}(\theta_2)} \tag{B.2}$$

Therefore,

$$
\begin{aligned}
&(\tilde{\theta} - \theta_2)u_{\theta q}(q^{fb}(\theta_2), \tilde{\theta}) - \int_{\theta_2}^{\tilde{\theta}} (s - \theta_2)u_{\theta\theta q}(q^{fb}(\theta_2), s)ds \\
&= \int_{\theta_2}^{\tilde{\theta}} u_{\theta q}(q^{fb}(\theta_2), \tilde{\theta})ds - \int_{\theta_2}^{\tilde{\theta}}\int_s^{\tilde{\theta}} u_{\theta\theta q}(q^{fb}(\theta_2), r)drds = \int_{\theta_2}^{\tilde{\theta}} u_{\theta q}(q^{fb}(\theta_2), s)ds \\
&\geq \int_{\tau(\theta_2)}^{\theta_2} u_{\theta q}(q^{fb}(s), \theta_2)\frac{\dot{q}^{fb}(s)}{\dot{q}^{fb}(\theta_2)}ds + \int_{q(\tau(\theta_2))}^{q^{fb}(\tau(\theta_2))} u_{\theta q}(q, \theta_2)\frac{1}{\dot{q}^{fb}(\theta_2)}dq \\
&= \int_{\tau(\theta_2)}^{\theta_2} u_{\theta q}(q^{fb}(s), \theta_2)\frac{u_{\theta q}(q^{fb}(s), s)u_{qq}(q^{fb}(\theta_2), \theta_2)}{u_{\theta q}(q^{fb}(\theta_2), \theta_2)u_{qq}(q^{fb}(s), s)}ds + \int_{q(\tau(\theta_2))}^{q^{fb}(\tau(\theta_2))} u_{\theta q}(q, \theta_2)\frac{-u_{qq}(q^{fb}(\theta_2), \theta_2)}{u_{\theta q}(q^{fb}(\theta_2), \theta_2)}dq \\
&\geq \int_{\tau(\theta_2)}^{\theta_2} u_{\theta q}(q^{fb}(s), s)ds + \int_{q(\tau(\theta_2))}^{q^{fb}(\tau(\theta_2))} [-u_{qq}(q^{fb}(\theta_2), \theta_2)]dq \\
&\geq \int_{\tau(\theta_2)}^{\theta_2} [u_{\theta q}(q^{fb}(\theta_2), \tilde{\theta})ds - \int_{\tau(\theta_2)}^{\theta_2}\int_s^{\tilde{\theta}} u_{\theta\theta q}(q^{fb}(\theta_2), r)drds + \int_{q(\tau(\theta_2))}^{q^{fb}(\tau(\theta_2))} [-u_{qq}(q^{fb}(\tau(\theta_2)), \theta_2)]dq \\
&= (\theta_2 - \tau(\theta_2))u_{\theta q}(q^{fb}(\theta_2), \tilde{\theta}) - [\int_{\theta_2}^{\tilde{\theta}} (\theta_2 - \tau(\theta_2))u_{\theta\theta q}(q^{fb}(\theta_2), s)ds + \int_{\tau(\theta_2)}^{\theta_2} (s - \tau(\theta_2))u_{\theta\theta q}(q^{fb}(\theta_2), s)ds] \\
&+ [q^{fb}(\tau(\theta_2)) - q(\tau(\theta_2))][-u_{qq}(q^{fb}(\tau(\theta_2)), \theta_2)]
\end{aligned} \tag{B.3}
$$

Where the first equality holds by integration by parts; the first inequality holds because of (B.2) and $u_q(q^{fb}(\theta_2), \theta_2) = 0$; the third equality holds becasue $\dot{q}^{fb}(\theta) = \frac{-u_{\theta q}(q^{fb}(\theta), \theta)}{u_{qq}(q^{fb}(\theta), \theta)}$; the second inequality holds because $u_{\theta qq} = 0$ and $u_{qqq} \leq 0$, so $u_{qq}(q^{fb}(\theta_2), \theta_2) \leq u_{qq}(q^{fb}(\theta'), \theta') < 0$ for $\theta_2 > \theta'$ and $u_{\theta q}(q, \theta_2) = u_{\theta q}(q', \theta_2)$ for any q, q'; the last inequality holds by integration by parts.

By Theorem 9, the following condition holds for $\theta' \in [\hat{\theta}, 1]$ almost everywhere:

$$\dot{\tau}(\theta') = \frac{\dot{\tau}^k(\tau^{-(k-1)}(\theta'))}{\dot{\tau}^{k-1}(\tau^{-(k-1)}(\theta'))} \tag{B.4}$$

This implies that for $\theta' \in [\theta_2, \tilde{\theta})$ almost everywhere:

$$\begin{aligned}\dot{\tau}(\theta') &\geq \frac{f(\theta')[u_q(q(\tau(\theta')), \theta') - u_q(q(\tau(\theta')), \tau(\theta'))]}{f(\tau(\theta'))u_q(q(\tau(\theta')), \tau(\theta'))} \\ &\geq \frac{u_q(q(\tau(\theta')), \theta') - u_q(q(\tau(\theta')), \tau(\theta'))}{u_q(q(\tau(\theta')), \tau(\theta'))} \\ &> \frac{u_q(q(\tau(\theta')), \theta') - u_q(q(\tau(\theta')), \theta_2)}{u_q(q(\tau(\theta_2)), \tau(\theta'))} \end{aligned} \quad \text{(B.5)}$$

where the second inequality holds because $\theta' > \tau(\theta')$ from theorem 4 and $f' \geq 0$, the last inequality holds because $\theta_2 > \theta_1 \in \tau(\tilde{\theta})) \geq \tau(\theta')$ and $q(\tau(\theta_2)) \leq q(\tau(\theta'))$.

Now we have

$$\begin{aligned}&[\theta_2 - \tau(\theta_2)][q^{fb}(\tau(\theta_2)) - q(\tau(\theta_2))][-u_{qq}(q^{fb}(\tau(\theta_2)), \theta_2)] + \frac{(\theta_2 - \tau(\theta_2))^2}{2} u_{\theta q}(q(\tau(\theta_2)), \tilde{\theta}) \\ &- [\int_{\theta_2}^{\tilde{\theta}} \frac{\theta_2 - \tau(\theta_2)^2}{2} u_{\theta\theta q}(q^{fb}(\theta_2), s)ds + \int_{\tau(\theta_2)}^{\theta_2} (\theta_2 - \frac{s + \tau(\theta_2)}{2})(s - \tau(\theta_2))u_{\theta\theta q}(q^{fb}(\theta_2), s)ds] \\ \geq &\int_{\tau(\theta_2)}^{\theta_2} [u_q(q(\tau(\theta_2)), \tau(\theta_2))d\theta' + \int_{\tau(\theta_2)}^{\theta_2} \int_{\tau(\theta_2)}^{\theta'} u_{\theta q}(q(\tau(\theta_2)), \tilde{\theta})dsd\theta' - \int_{\tau(\theta_2)}^{\theta_2} \int_{\tau(\theta_2)}^{\theta'} \int_s^{\tilde{\theta}} u_{\theta q}(q(\tau(\theta_2)), r)drdsd\theta' \\ = &\int_{\tau(\theta_2)}^{\theta_2} u_q(q(\tau(\theta_2)), \theta')d\theta' \geq \int_{\tau(\theta_2)}^{\theta_1} u_q(q(\tau(\theta_2)), \theta')d\theta' \\ \geq &\int_{\theta_2}^{\tilde{\theta}} u_q(q(\tau(\theta_2)), \tau(\theta'))\dot{\tau}(\theta')d\theta' > \int_{\theta_2}^{\tilde{\theta}} [u_q(q(\tau(\theta')), \theta') - u_q(q(\tau(\theta')), \theta_2)]d\theta' \\ = &\int_{\theta_2}^{\tilde{\theta}} \int_{\theta_2}^{\theta'} u_{\theta q}(q(\tau(\theta')), \tilde{\theta})dsd\theta' - \int_{\theta_2}^{\tilde{\theta}} \int_{\theta_2}^{\theta'} \int_s^{\tilde{\theta}} u_{\theta\theta q}(q(\tau(\theta')), r)drdsd\theta' \\ = &\frac{(\tilde{\theta} - \theta_2)^2}{2} u_{\theta q}(q^{fb}(\theta_2), \theta_2) - [\int_{\theta_2}^{\tilde{\theta}} (\tilde{\theta} - \frac{s + \theta_2}{2})(s - \theta_2)u_{\theta\theta q}(q^{fb}(\theta_2), s)ds] \end{aligned} \quad \text{(B.6)}$$

where the first inequality holds because $u_q(q^{fb}(\tau(\theta_2)), \tau(\theta_2)) = 0$, $u_{qq}(q^{fb}(\tau(\theta_2)), \tau(\theta_2)) \leq u_{qq}(q, \tau(\theta_2))$ for $q \in [q(\tau(\theta_2)), q^{fb}(\tau(\theta_2)]$ $u_{\theta qq} = 0$ and integration by parts. The third inequality holds by change of variable and $\tau((\theta_2, \tilde{\theta})) \subseteq (\tau(\theta_2), \theta_1)$. The fourth inequality holds by (B.5). The last equality holds because $u_{\theta qq} = 0$ and integration by parts.

Let $A = \tilde{\theta} - \theta_2$, $B = \theta_2 - \tau(\theta_2)$ and $C = [q^{fb}(\tau(\theta_2)) - q(\tau(\theta_2))][-u_{qq}(q^{fb}(\tau(\theta_2)), \theta_2)]$. Then from (B.3),

$$(A - B)u_{\theta q}(q(\tau(\theta_2)), \tilde{\theta}) + [\int_{\theta_2}^{\tilde{\theta}} (2\theta_2 - \tau(\theta_2) - s)u_{\theta\theta q}(q^{fb}(\theta_2), s)ds + \int_{\tau(\theta_2)}^{\theta_2} (s - \tau(\theta_2))u_{\theta\theta q}(q^{fb}(\theta_2), s)ds] \geq C \quad \text{(B.7)}$$

From (B.6),

$$BC > \frac{A^2-B^2}{2}u_{\theta q}(q^{fb}(\theta_2),\tilde{\theta}) - [\int_{\theta_2}^{\tilde{\theta}}(\tilde{\theta}-\frac{s+\theta_2}{2})(s-\theta_2) - \frac{\theta_2-\tau(\theta_2)^2}{2}]u_{\theta\theta q}(q^{fb}(\theta_2),s)ds$$
$$-\int_{\tau(\theta_2)}^{\theta 2}(\theta_2-\frac{s+\tau(\theta_2)}{2})(s-\tau(\theta_2))u_{\theta\theta q}(q^{fb}(\theta_2),s)ds \quad \text{(B.8)}$$

Combining (B.7) and (B.8),

$$\int_{\theta_2}^{\tilde{\theta}} k_1(s)u_{\theta\theta q}(q^{fb}(\theta_2),s)ds + \int_{\tau(\theta_2)}^{\theta_2} k_2(s)u_{\theta\theta q}(q^{fb}(\theta_2),s)ds > \frac{(A-B)^2}{2}u_{\theta q}(q^{fb}(\theta_2),\tilde{\theta}) \geq 0 \quad \text{(B.9)}$$

where $k_1(s) = \tilde{\theta}s - \theta_2 s + \tau(\theta_2)s - \frac{s^2}{2} - \tilde{\theta}\theta_2 - 2\theta_2\tau(\theta_2) + 2\theta_2^2 + \frac{\tau(\theta_2)^2}{2}$ and $k_2(s) = 2\theta_2 s - \tau(\theta_2)s - \frac{s^2}{2} - 2\theta_2\tau(\theta_2) + \frac{3\tau(\theta_2)^2)}{2}$. Since $k_2'(s) = 2\theta_2 - s - \tau(\theta_2) > 0$ for all $s \in [\tau(\theta_2),\theta_2]$ and $k_2(\theta_2) = 0$, so $k_2(s) \geq 0$ for any $s \in [\tau(\theta_2),\theta_2]$. For $k_1(.)$, since $k_1''(s) = -1 < 0$, so for any $s \in [\theta_2,\tilde{\theta}]$, $k(s) \geq \min\{k_1(\theta_2),k_1(\tilde{\theta})\} = \min\{\frac{B^2}{2},\frac{(A-B)^2}{2}\} \geq 0$. Since $u_{\theta\theta q} \leq 0$, $\int_{\theta_2}^{\tilde{\theta}} k_1(s)u_{\theta\theta q}(q^{fb}(\theta_2),s)ds + \int_{\tau(\theta_2)}^{\theta_2} k_2(s)u_{\theta\theta q}(q^{fb}(\theta_2),s)ds \leq 0$, contradicts to (B.9).Therefore, τ must be single-valued under the assumptions. *Q.E.D.*

B.2 Proof of Theorem 6

Theorem 6 follows from Lemmas 18 and 19.

B.3 Proof of Theorem 7

Let $(q(\theta),t(\theta))$ be an optimal mechanism, which exists and is unique by Theorem 2. Consider the triple $(\tau(\theta),Q(\theta),\hat{\theta})$ where $\tau(\theta)$ is defined by (1.4), $\hat{\theta} = \max\{\theta : V(\theta) = 0\}$ and $Q(\theta) = q(\tau(\theta))$ for $\theta \in [\hat{\theta},1]$. Let us show that the triple $(\tau(\theta),Q(\theta),\hat{\theta})$ is an increasing solution to the relaxed program.

Since the optimal mechanism is unique, $\tau(\theta)$ must be strictly increasing by Theorem 4, and $q(\theta)$ must be strictly increasing by Theorem 3, and so $Q(\theta) = q(\tau(\theta))$ is also strictly increasing. Since $C_i \in (\underline{C},\overline{C})$, Theorem 6 implies that $\tau(\theta) < \hat{\theta}$ for all $\theta \in [\hat{\theta},1]$, and so $\dot{Q}(\theta) = \frac{u_\theta(Q(\theta),\tau(\theta))}{u_q(Q(\theta),\theta)-u_q(Q(\theta),\tau(\theta))}\dot{\tau}(\theta)$ for all $\theta \in [\hat{\theta},1]$, which is equivalent to (2.6).

Consider any $\theta \in [\hat{\theta},1]$. Then $\tau(\theta)$ is single-valued by Assumption 2. Also, $\tau^{-1}(\theta) = \emptyset$ by Theorem 6, and so $[\tau(\hat{\theta}),\tau(1)] \cup [\hat{\theta},1] = \emptyset$. Therefore, Theorem 9 with $M = 1$ implies that for all $\theta \in [\hat{\theta},1]$, $u_q(q(\tau(\theta)),\tau(\theta))f(\tau(\theta))\dot{\tau}(\theta) = [u_q(q(\tau(\theta)),\theta) - u_q(q(\tau(\theta)),\tau(\theta))]f(\theta)$, which is equivalent to equation (2.38). In combination with $\dot{Q}(\theta) = \frac{u_\theta(Q(\theta),\tau(\theta))}{u_q(Q(\theta),\theta)-u_q(Q(\theta),\tau(\theta))}\dot{\tau}(\theta)$ this yields (2.39).

Boundary conditions (2.17) and (2.18) hold because by Theorem 4, $q(\theta) = q^{fb}(\theta)$ for $\theta \in [0,\tau(\hat{\theta})] \cup [\tau(1),1]$. Boundary condition (2.19) holds because by Theorem 4 $V(\theta) = 0$ for $\theta \in [0,\hat{\theta}]$ and $\tau(\hat{\theta}) < \hat{\theta}$, and therefore $V(\hat{\theta}) = u(q(\tau(\hat{\theta})),\hat{\theta}) - t(\tau(\hat{\theta})) - C = u(q(\tau(\hat{\theta})),\hat{\theta}) - u(q(\tau(\hat{\theta})),\tau(\hat{\theta})) - C = 0$.

Finally, let us show that $\tau(\hat{\theta})$ must be the smallest solution to (2.19). Conditions (2.18) and (2.19) imply $G(\hat{\theta},\tau(\hat{\theta})) \equiv u(q^{fb}(\tau(\hat{\theta})),\hat{\theta})) - u(q^{fb}(\tau(\hat{\theta})),\tau(\hat{\theta})) = C$. By Assumption 2 there are at most two solutions to this equation. If we set $\tau(\hat{\theta})$ to be equal to the larger solution, then $G_2(\hat{\theta},\tau(\hat{\theta})) < 0$. Therefore, there exists $\theta' < \tau(\hat{\theta})$, s.t. if we set $q(\theta') = q^{fb}(\theta')$ and $V(\theta') = 0$ it follows that $u(q(\theta'),\hat{\theta}) - t(\theta') = G(\hat{\theta},\theta') - C > G(\hat{\theta},\tau(\hat{\theta})) - C = 0$, violating $IC(\hat{\theta},\theta')$. Therefore, $\tau(\hat{\theta})$ must be the smaller solution of (2.19).

To summarize the above, we have shown that the optimal mechanism induces a triple $(\tau(\theta), Q(\theta), \hat{\theta})$ which constitutes an increasing solution to the relaxed program. Thus, to complete the proof it is sufficient to show that an increasing solution to the relaxed program is unique. We establish this below via a sequence of Claims.

First, fix some $\hat{\theta}_i$ and C_j where $i,j \in \{1,2\}$ and let Let $\Gamma(\hat{\theta}_i, C_j) = \{\theta' : G(\hat{\theta}_i,\theta') \equiv u(q^{fb}(\theta'),\hat{\theta}) - u(q^{fb}(\theta'),\theta') = C_j\}$ i.e., $\Gamma(\hat{\theta}_i, C_j)$ is the set of types satisfying the boundary condition (2.19). Suppose that $\Gamma(\hat{\theta}_i, C_j) \neq \emptyset$ for $i,j \in \{1,2\}$. Note that $\Gamma(\hat{\theta}_i, C_j)$ contains at most two elements because $G(\hat{\theta}_i,\theta') \equiv u(q^{fb}(\theta'),\hat{\theta}) - u(q^{fb}(\theta'),\theta')$ is strictly quasi-concave in θ' by Assumption 2,

Claim 1: *If $\hat{\theta}_1 > \hat{\theta}_2$ and $C_1 < C_2$, then $min\Gamma(\hat{\theta}_1, C_j) < min\Gamma(\hat{\theta}_2, C_j)$ and $min\Gamma(\hat{\theta}_i, C_1) < min\Gamma(\hat{\theta}_i, C_2)$.*

Proof of Claim 1:
Since $G(.)$ is strictly quasi-concave and $G(\hat{\theta}_i, min\Gamma_i) > G(\hat{\theta}_i, 0) = 0$, it follows that $G_2(\hat{\theta}_i, min\Gamma_i) \geq 0$. On the other hand, we have $G_1(\hat{\theta}_i, min\Gamma_i) = u_\theta(q^{fb}(min\Gamma_i),\hat{\theta}_i) > 0$. The last two inequalities together imply Claim 1. ■

Claim 2: *Suppose that there exist $(\hat{\theta}_1,\hat{\tau}_1)$ and $(\hat{\theta}_2,\hat{\tau}_2)$ such that for $i = 1,2$, $(Q_i(\theta)), \tau_i(\theta))$ is an increasing solution to the system of differential equations (2.38) and (2.39) on $[\hat{\theta}_i, 1]$ that satisfies boundary conditions $\tau_i(\hat{\theta}_i) = \hat{\tau}_i$, $Q_i(\hat{\theta}_i) = q^{fb}(\hat{\tau}_i)$ and $Q_i(1) = q^{fb}(\tau_i(1))$. Let $q_i(\theta) = Q_i(\tau_i^{-1}(\theta))$ for $\theta \in [\hat{\tau}_i, \tau_i(1)]$.*

Then the following "no-crossing" property holds:
If there exists $\theta^\dagger \in [max\{\hat{\tau}_1,\hat{\tau}_2\}, min\{\tau_1(1),\tau_2(1)\}]$ such that $q_2(\theta^\dagger) < q_1(\theta^\dagger)$, then $q_2(\theta) < q_1(\theta)$ for all $\theta \in [max\{\hat{\tau}_1,\hat{\tau}_2\}, min\{\tau_1(1),\tau_2(1)\}]$.

Proof of Claim 2:
The proof is by contradiction, so suppose that there exists $\theta' \in [max\{\hat{\tau}_1,\hat{\tau}_2\}, min\{\tau_1(1),\tau_2(1)\}]$ such that $q_2(\theta') = q_1(\theta') \equiv q'$ and $\dot{q}_2(\theta') \neq \dot{q}_1(\theta')$. Without loss of generality we can assume $\dot{q}_2(\theta') > \dot{q}_1(\theta')$. Differential equations (2.38) and (2.39) and $\dot{q}_i = \frac{\dot{Q}_i}{\dot{\tau}_i}$ imply $\frac{u_\theta(q',\theta')}{u_q(q',\tau_2^{-1}(\theta'))-u_q(q',\theta')} >$

$\frac{u_\theta(q',\theta')}{u_q(q',\tau_1^{-1}(\theta'))-u_q(q',\theta')}$. Since $u_{\theta q} > 0$, $\tau_1^{-1}(\theta') > \tau_2^{-1}(\theta')$. Let $\tilde{\theta}' = \tau_1^{-1}(\theta')$.

Next we consider the following two cases:

Case 1: $q_2(\theta) > q_1(\theta)$ for $\theta \in (\theta', min\{\tau_1(1), \tau_2(1)\}]$.

First note that (2.39) i.e., $\dot{Q}_i(\theta) = \frac{f(\theta)u_\theta(Q_i,\tau)}{f(\tau)u_q(Q_i,\tau)}$ and $\dot{Q}_i(\theta) > 0$ in combination imply that $q_i(\theta) \leq q^{fb}(\theta)$ for all $\theta \in (\hat{\tau}_i, \tau_i(1))$. It follows that $\tau_1(1) > \tau_2(1)$, for otherwise $q_2(\tau_1(1)) > q_1(\tau_1(1)) = q^{fb}(\tau_1(1))$, where the inequality hold by case assumption, and the equality holds by boundary condition (2.17), violating $q_2(.) \leq q^{fb}(.)$.

While $\tau_1(1) > \tau_2(1)$, we also have $\tau_1(\tilde{\theta}') = \theta' < \tau_2(\tilde{\theta}')$ since $\tilde{\theta}' = \tau_1^{-1}(\theta') > \tau_2^{-1}(\theta')$. Therefore there exists $\tilde{\theta}'' \in (\tilde{\theta}', 1)$ such that $\tau_1(\tilde{\theta}'') = \tau_2(\tilde{\theta}'') \equiv \theta''$ and $\dot{\tau}_1(\tilde{\theta}'') > \dot{\tau}_2(\tilde{\theta}'')$. By (2.38) the latter is equivalent to $\frac{f(\tilde{\theta}'')(u_q(Q_1(\tilde{\theta}''),\tilde{\theta}'')-u_q(Q_1(\tilde{\theta}''),\theta''))}{f(\tau_1(\theta''))u_q(Q_1(\tilde{\theta}''),\theta'')} > \frac{f(\tilde{\theta}'')(u_q(Q_2(\tilde{\theta}''),\tilde{\theta}'')-u_q(Q_2(\tilde{\theta}''),\theta''))}{f(\tau_2(\theta''))u_q(Q_2(\tilde{\theta}''),\theta'')}$. Then from $u_{qq} < 0$ and $u_{\theta qq} \geq 0$ it follows that $Q_1(\tilde{\theta}'') > Q_2(\tilde{\theta}'')$, or equivalently $q_1(\theta'') > q_2(\theta'')$. However, this contradicts the case assumption since $\theta'' \in (\theta', \tau_2(1))$.

Case 2: There exists $\theta'' \in (\theta', min\{\tau_1(1), \tau_2(1)\}]$ such that $q_2(\theta) > q_1(\theta)$ for $\theta \in (\theta', \theta'')$, $q_2(\theta'') = q_1(\theta'') \equiv q''$ and $\dot{q}_2(\theta'') < \dot{q}_1(\theta'')$.

Given $\dot{q}_2(\theta'') < \dot{q}_1(\theta'')$, a similar argument to that in Case 1 yields that $\tau_1^{-1}(\theta'') < \tau_2^{-1}(\theta'')$, and $\tau_2(\tau_1^{-1}(\theta'')) < \tau_2(\tau_2^{-1}(\theta'')) = \theta'' = \tau_1(\tau_1^{-1}(\theta''))$. Let $\tilde{\theta}'' = \tau_1^{-1}(\theta'')$. Note that $\tilde{\theta}'' > \tilde{\theta}'$ as $\theta'' > \theta'$. Since $\tau_1(\tilde{\theta}') < \tau_2(\tilde{\theta}')$ and $\tau_2(\tilde{\theta}'') < \tau_1(\tilde{\theta}'')$, there exists $\tilde{\theta}''' \in [\tilde{\theta}', \tilde{\theta}'']$ such that $\tau_1(\tilde{\theta}''') = \tau_2(\tilde{\theta}''') \equiv \theta'''$ and $\dot{\tau}_1(\tilde{\theta}''') > \dot{\tau}_2(\tilde{\theta}''')$. A similar argument to the in Case 1 yields $Q_1(\tilde{\theta}''') > Q_2(\tilde{\theta}''')$, or equivalently $q_1(\theta''') > q_2(\theta''')$. But by the case assumption $q_1(\theta''') < q_2(\theta''')$. Contradiction.

Claim 3: *If there exists $\tilde{\theta}' \in [max\{\hat{\theta}_1, \hat{\theta}_2\}, 1]$ such that $\tau_2(\tilde{\theta}') < \tau_1(\tilde{\theta}')$, then $\tau_2(\theta) < \tau_1(\theta)$ for all $\theta \in [max\{\hat{\theta}_1, \hat{\theta}_2\}, 1]$.*

Proof of Claim 3:

The proof is by contradiction, so suppose the Claim is not true. Then there exists a "crossing point" $\tilde{\theta}' \in [max\{\hat{\theta}_1, \hat{\theta}_2\}, 1]$ such that $\tau_2(\tilde{\theta}') = \tau_1(\tilde{\theta}') \equiv \theta'$ and $\dot{\tau}_1(\tilde{\theta}') \neq \dot{\tau}_2(\tilde{\theta}')$. Without loss of generality we can assume $\dot{\tau}_1(\tilde{\theta}') > \dot{\tau}_2(\tilde{\theta}')$. Then from the differential equation (2.38) it follows that $Q_1(\tilde{\theta}') > Q_2(\tilde{\theta}')$, or equivalently $q_1(\theta') > q_2(\theta')$.

Note that $\tilde{\theta}' < 1$ for otherwise we would have $\theta' = \tau_1(1) = \tau_2(1)$ and $q^{fb}(\theta') = q_1(\theta') = q_2(\theta')$ which contradicts $q_1(\theta') > q_2(\theta')$.

Now consider the following two cases:

Case 1: $\tau_1(\theta) > \tau_2(\theta)$ for $\theta \in (\tilde{\theta}', 1]$.

Since $\tau_1(1) > \tau_2(1)$, we have $q_1(\tau_2(1)) \leq q^{fb}(\tau_2(1)) = q_2(\tau_2(1))$, which combined with $q_1(\theta') > q_2(\theta')$ violates Claim 2, the no-crossing property of q.

Case 2: There exists $\tilde{\theta}'' \in (\tilde{\theta}', 1]$ such that $\tau_1(\theta) > \tau_2(\theta)$ for $\theta \in (\tilde{\theta}', \tilde{\theta}'')$, $\tau_1(\tilde{\theta}'') = \tau_2(\tilde{\theta}'') \equiv \theta''$ and $\dot{\tau}_1(\tilde{\theta}'') < \dot{\tau}_2(\tilde{\theta}'')$.

Using $\dot{\tau}_1(\tilde{\theta}'') < \dot{\tau}_2(\tilde{\theta}'')$ and $\tau_1(\tilde{\theta}'') = \tau_2(\tilde{\theta}'')$ in differential equation (2.38) yields $Q_1(\tilde{\theta}'') < Q_2(\tilde{\theta}'')$, or equivalently $q_1(\theta'') < q_2(\theta'')$, which combined with $q_1(\theta') > q_2(\theta')$ violates Claim 2, the no-crossing property of q.

Claim 4: *Suppose there exist $(\hat{\theta}_1, \hat{\tau}_1) \neq (\hat{\theta}_2, \hat{\tau}_2)$ such that for $i = 1, 2$, $(Q_i(.), \tau_i(.))$ is an increasing solution to differential equations (2.38)-(2.39) with boundary conditions (2.18)-(2.19). Then $\hat{\theta}_2 > \hat{\theta}_1$ if and only if $\hat{\tau}_2 > \hat{\tau}_1$.*

Proof of Claim 4:
Suppose not, then without loss of generality we have $\hat{\theta}_2 \geq \hat{\theta}_1$ and $\hat{\tau}_1 \geq \hat{\tau}_2$ with at least one strict inequality. Then $\tau_1(\hat{\theta}_2) \geq \tau_1(\hat{\theta}_1)$ and $\tau_1(\hat{\theta}_1) \geq \tau_2(\hat{\theta}_2)$ with at least one strict inequality, from which it immediately follows that $\tau_1(\hat{\theta}_2) > \tau_2(\hat{\theta}_2)$, and so $q_1(\tau_1(\hat{\theta}_1)) = q^{fb}(\tau_1(\hat{\theta}_1)) \geq q_2(\tau_1(\hat{\theta}_1))$. By Claim 3 (the "no-crossing" property of τ), $\tau_1(1) > \tau_2(1)$, and therefore $q_2(\tau_2(1)) = q^{fb}(\tau_2(1)) > q_1(\tau_2(1))$. The last inequality in combination with $q_1(\tau_1(\hat{\theta}_1)) \geq q_2(\tau_1(\hat{\theta}_1))$ contradict the no-crossing property of q in Claim 2.

Uniqueness. Now we can establish the uniqueness of the solution to the relaxed program relying on Claims 1-4. Again the proof is by contradiction, so suppose the solution is not unique. Then there exist $\hat{\theta}_1$ and $\hat{\theta}_2$, $\hat{\theta}_1 \neq \hat{\theta}_2$ s.t. $(Q_1(\theta), \tau_1(\theta))$ and $(Q_2(\theta), \tau_2(\theta)))$ solve the system of differential equations (2.38) and (2.39) with corresponding boundary conditions (2.17)-(2.19) where $\hat{\tau}_i \equiv \tau_i(\hat{\theta}_i) = \min \Gamma_i$. Without loss of generality suppose that $\hat{\theta}_1 > \hat{\theta}_2$. Claim 4 implies that $\hat{\tau}_1 > \hat{\tau}_2$. However, this contradicts Claim 1. *Q.E.D.*

B.4 Proof of Theorem 8

Part (1) and (2):
Claim 3 in Proof of Theorem 7 implies that $\hat{\theta}_2 \leq \hat{\theta}_1$ if and only if $\tau_2(\hat{\theta}_2) \leq \tau_1(\hat{\theta}_1)$, but that contradicts to Claim 1 in Proof of Theorem 7 given $C_2 > C_1$. Therefore, it must be the case that both $\hat{\theta}_2 > \hat{\theta}_1$ and $\tau_2(\hat{\theta}_2) > \tau_1(\hat{\theta}_1)$.

Part (3) and (4):
Part (2) and boundary condition (2.18) implies $q_2(\tau_2(\hat{\theta}_2)) = q^{fb}(\tau_2(\hat{\theta}_2)) > q_1(\tau_2(\hat{\theta}_2))$, therefore part (4) follows from the no-crossing property of q from Claim 2 in Proof of Theorem 7. Now since $q_1(\tau_2(1)) < q_2(\tau_2(1)) = q^{fb}(\tau_2(1))$, it must be the case that $\tau_1(1) > \tau_2(1)$, and part (3) follows from the no-crossing property of τ from Claim 2 in Proof of Theorem 7.

Q.E.D.

B.5 Proof of Theorem 9

The equation (2.61) is obtained by combining (2.54) and (2.60). So we only need to establish that (2.60) holds.

Note that since $\tau(.)$ is strictly increasing, upper hemicontinuous and is single-valued by assumption, $\tau^s(.)$ must also be a strictly increasing, continuous function differentiable almost everywhere on $[\underline{\theta}, \overline{\theta}]$ for $s \in \{1, ..., M\}$.

To establish (2.60), let us first assume that (2.62) holds for all $k = 1, ...M$. Let $A(k) = f(\theta) + \sum_{s=1}^{k} f(\tau^s)\dot{\tau}^s$. Then from (2.62) we have:

$$\dot{\tau}^k = \frac{u_q(Q^k, \tau^{k-1}) - u_q(Q^k, \tau^k)}{f(\tau^k) u_q(Q^k, \tau^k)} A(k-1) \tag{B.10}$$

and

$$\begin{aligned} A(k) =& A(k-1) + f(\tau^k)\dot{\tau}^k \\ =& A(k-1) \frac{u_q(Q^k, \tau^{k-1})}{u_q(Q^k, \tau^k)} \\ =& f(\theta) \prod_{s=1}^{k} \frac{u_q(Q^s, \tau^{s-1})}{u_q(Q^s, \tau^s)} \end{aligned} \tag{B.11}$$

recursively. From (B.10) and (B.11),

$$\dot{\tau}^k = \frac{f(\theta)[u_q(Q^k, \tau^{k-1}) - u_q(Q^k, \tau^k)]}{f(\tau^k) u_q(Q^k, \tau^k)} \prod_{s=1}^{k-1} \frac{u_q(Q^s, \tau^{s-1})}{u_q(Q^s, \tau^s)}$$

which is equation (2.60).

Now to establish (2.62) for all $k = 1, ...M$, we argue by contradiction. In particular, suppose that for some $\tilde{\theta} \in (\underline{\theta}, \overline{\theta})$ and $s \in \{1, ..., M\}$:

$$u_q(q(\tau^s(\tilde{\theta}))), \tau^s(\tilde{\theta})) f(\tau^s(\tilde{\theta}))\dot{\tau}^s(\tilde{\theta}) > [u_q(q(\tau^s(\tilde{\theta})), \tau^{s-1}(\tilde{\theta})) - u_q(q(\tau^s(\tilde{\theta})), \tau^s(\tilde{\theta}))] \sum_{k=1}^{s} f(\tau^{s-k}(\tilde{\theta}))\dot{\tau}^{s-k}(\tilde{\theta}). \tag{B.12}$$

(The proof in the case when the opposite inequality holds is similar and will therefore be omitted.)

Note that the left hand side of (B.12) is the marginal efficiency gain of raising q on a neighborhood around $\tau^s(\tilde{\theta})$, while its right hand side is the marginal increases in rent that the principal needs to provide to the types in the neighborhoods around every predecessor of $\tau^s(\tilde{\theta})$ in the chain of targeted types, $\tau^{s-k}(\tilde{\theta})$ for $k = 1, ..., s$. The multiplier $f(\tau^{s-k}(\tilde{\theta}))\dot{\tau}^{s-k}(\tilde{\theta})$ for $k = 0, ..., s$, reflects the relative probability weight of the neighborhood around $\tau^{s-k}(\tilde{\theta})$. So, when (B.12) holds, the principal could get higher profits by increasing the quantities assigned to the types around $\tau^s(\tilde{\theta})$ and collecting the additional revenue generated thereby, while providing increased rents required by types around $\tau^{s-k}(\tilde{\theta})$, $k = 1, ..., s$.

The rest of the proof formalizes this intuition. We proceed through three steps. In Step 1, we construct an alternative mechanism $(\tilde{q}(.), \tilde{t}(.))$ reflecting the aforementioned modification. In Steps 2 and 3 we show, respectively, that this mechanism is incentive compatible and is more proftiable for the principal than the original one, when the quantity changes for the types around $\tau^s(\tilde{\theta})$ are sufficiently small.

Step 1. Constructing an Alternative Mechanism.

First, (B.12) implies that there exists $\mu > 0$ such that

$$u_q(q(\tau^s(\tilde{\theta})), \tau^s(\tilde{\theta}))f(\tau^s(\tilde{\theta}))\dot{\tau}^s(\tilde{\theta}) - [u_q(q(\tau^s(\tilde{\theta})), \tau^{s-1}(\tilde{\theta})) - u_q(q(\tau^s(\tilde{\theta})), \tau^s(\tilde{\theta}))] \sum_{k=1}^{s} f(\tau^{s-k}(\tilde{\theta}))\dot{\tau}^{s-k}(\tilde{\theta}) - \mu > 0 \tag{B.13}$$

Note that $q(\tau^s(\tilde{\theta}) < q^{fb}(\tau^s(\tilde{\theta}))$, for otherwise the first term in (B.13) is zero while its second term is positive.

Next, for any $\epsilon > 0$ and $k = 0, ..., s$, let $\Theta_k(\epsilon) = [\tau^k(\tilde{\theta} - \epsilon) - (\frac{\delta_\tau}{2})^k \epsilon^2, \tau^k(\tilde{\theta} + \epsilon) + (\frac{\delta_\tau}{2})^k \epsilon^2]$.

Now consider an alternative mechanism $(\tilde{q}(.), \tilde{t}(.))$ which differs from the original mechanism $(q(.), t(.)))$ only as follows: for $\theta \in \Theta_s(\epsilon)$, $\tilde{q}(\theta) = q(\theta) + \epsilon^5$ and $\tilde{t}(\theta) = t(\theta) + u(q(\theta) + \epsilon^5, \theta) - u(q(\theta), \theta)$, and for $\theta \in \cup_{k=0}^{s-1}\Theta_k(\epsilon)$, $\tilde{t}(\theta) = t(\theta) - \Delta(\epsilon)$, where $\Delta(\epsilon) \equiv \max_{\theta' \in \Theta_s(\epsilon)} u(q(\theta') + \epsilon^5, \overline{\theta}_{s-1}) - u(q(\theta'), \overline{\theta}_{s-1}) - u(q(\theta') + \epsilon^5, \theta') + u(q(\theta'), \theta')$ and $\overline{\theta}_{s-1} = \max \Theta_{s-1}(\epsilon)$. We will show that all IC and IR are satisfied in the new contract.

Step 2. Establishing incentive compatibility of the alteranative mechanism.

IR constraints hold in $(\tilde{q}(.), \tilde{t}(.))$ because $\tilde{V}(\theta) > V(\theta)$ for $\theta \in \cup_{k=0}^{s-1}\Theta_k(\epsilon)$, and $\tilde{V}(\theta) = V(\theta)$ for all other types θ.

We will show that incentive constraints in the mechanism $(\tilde{q}(.), \tilde{t}(.))$, denoted by $\tilde{IC}(\theta, \theta')$, hold for all $(\theta, \theta') \in [0, 1]^2$. The argument is given separately for several subsets of $[0, 1]^2$.

First, if $\theta \in [0, 1]$ and $\theta' \notin \cup_{k=0}^{s}\Theta_k(\epsilon)$, then $\tilde{IC}(\theta, \theta')$ holds because $\tilde{V}(\theta) \geq V(\theta)$, $\tilde{q}(\theta') = q(\theta')$, $\tilde{t}(\theta') = t(\theta')$ and $IC(\theta, \theta')$ holds.

Second, if $\theta \in [0, 1]$ and $\theta' \in \Theta_0(\epsilon)$, then for small enough ϵ, $\tau^{-1}(\theta') = \emptyset$ since $\Theta_0(\epsilon) \subset (\underline{\theta}, \overline{\theta}) \subset [\tau(1), 1]$. So, in the original mechanism incentive constraints $IC(\theta, \theta')$ are slack on this set of types, with minimal slack $\delta > 0$ over all $\theta \in [0, 1]$ and all $\theta' \in \Theta_0(\epsilon)$. In the mechanism, $(\tilde{q}_j(.), \tilde{t}_j(.))$, $\tilde{V}(\theta) \geq V(\theta)$ for all $\theta \in [0, 1]$, and $\tilde{V}(\theta') = V(\theta') + \Delta(\epsilon)$ for $\theta' \in \Theta_0(\epsilon)$. Therefore, $\tilde{IC}(\theta, \theta')$ holds for sufficiently small ϵ i.e., when $\Delta(\epsilon) \leq \delta$.

Third, consider $IC(\theta, \theta')$ where $\theta \in \Theta_{s-1}(\epsilon)$ and $\theta' \in \Theta_s(\epsilon)$. Recall that $U(\theta, \theta') = u(q(\theta'), \theta) - t(\theta') - C$. So we have:

$$\begin{aligned}
&\tilde{V}(\theta) = V(\theta) + \Delta(\epsilon) \geq U(\theta, \theta') + \Delta(\epsilon) \\
&\geq u(q(\theta'), \theta) - t(\theta') - C + [u(q(\theta') + \epsilon^5, \theta) - u(q(\theta') + \epsilon^5, \theta')] - [u(q(\theta'), \theta) - u(q(\theta'), \theta')] \\
&= u(\tilde{q}(\theta'), \theta) - \tilde{t}(\theta') - C
\end{aligned}$$

where the first equality holds by definition of $\tilde{t}(\theta)$; the first inequality holds by incentive compatibility of the original mechanism; the second inequality holds by definition of $\Delta(\epsilon)$, $\theta \leq \overline{\theta}_{s-1}$ and $u_{\theta q} > 0$; the last equality holds by definitions of $\tilde{q}(\theta')$ and $\tilde{t}(\theta')$.

Fourth, if $\theta \in \cup_{k=0}^{s-1}\Theta_k(\epsilon)$ and $\theta' \in \cup_{k=1}^{s-1}\Theta_k(\epsilon)$, then $\tilde{V}(\theta) = V(\theta) + \Delta(\epsilon)$ and $\tilde{V}(\theta') = V(\theta') + \Delta(\epsilon)$ since both θ and θ' get the same quantity as in the original mechanism but their transfer is decreased by $\Delta(\epsilon)$ in $(\tilde{q}(.), \tilde{t}(.))$. Therefore, $\tilde{IC}(\theta, \theta')$ holds because $IC(\theta, \theta')$ holds.

Fifth, if $\theta \notin \Theta_{s-1}(\epsilon)$ and $\theta' \in \Theta_s(\epsilon)$, when ϵ is sufficiently small we have

$$\begin{aligned}
&\tilde{V}(\theta) = V(\theta) > u(q(\theta'), \theta) - t(\theta') - C + \delta_V \delta_\tau (\frac{\delta_\tau}{2})^{2(s-1)} \frac{\epsilon^4}{16} \\
&= u(\tilde{q}(\theta'), \theta) - \tilde{t}(\theta') - C + \delta_V \delta_\tau (\frac{\delta_\tau}{2})^{2(s-1)} \frac{\epsilon^4}{16} - [u(q(\theta') + \epsilon^5, \theta) - u(q(\theta'), \theta) - u(q(\theta') + \epsilon^5, \theta') + u(q(\theta'), \theta')] \\
&> u(\tilde{q}(\theta'), \theta) - \tilde{t}(\theta') - C
\end{aligned}$$

where the first inequality holds because $\theta' \in [\tau^s(\tilde{\theta} - \epsilon) - (\frac{\delta_\tau}{2})^s \epsilon^2, \tau^s(\tilde{\theta} + \epsilon) + (\frac{\delta_\tau}{2})^s \epsilon^2]$, so Lemma 14 implies $\tau^{-1}(\theta') \in [\tau^{s-1}(\tilde{\theta} - \epsilon) - \frac{1}{2}(\frac{\delta_\tau}{2})^{s-1}\epsilon^2, \tau^{s-1}(\tilde{\theta} + \epsilon) + \frac{1}{2}(\frac{\delta_\tau}{2})^{s-1}\epsilon^2]$, and since $|\theta - \tau^{-1}(\theta')| \geq \frac{1}{2}(\frac{\delta_\tau}{2})^{s-1}\epsilon^2$, Lemma 15 implies that $V(\theta) - U(\theta'|\theta) \geq \delta_V \delta_\tau (\frac{\delta_\tau}{2})^{2(s-1)} \frac{\epsilon^4}{16}$ for small enough ϵ; the second equality holds by the definitions of $\tilde{q}(\theta')$ and $\tilde{t}(\theta')$; the last inequality holds for small enough ϵ.

Sixth, if $\theta \notin \cup_{k=0}^{s-1}\Theta_k(\epsilon)$ and $\theta' \in \Theta_r(\epsilon)$ for $r = 1, ..., s-1$, when ϵ is sufficiently small we have

$$\begin{aligned}
&\tilde{V}(\theta) = V(\theta) > u(q(\theta'), \theta) - t(\theta') - C + \delta_V \delta_\tau (\frac{\delta_\tau}{2})^{2(r-1)} \frac{\epsilon^4}{16} \\
&= u(\tilde{q}(\theta'), \theta) - \tilde{t}(\theta') - C + \delta_V \delta_\tau (\frac{\delta_\tau}{2})^{2(r-1)} \frac{\epsilon^4}{16} - \Delta(\epsilon) \\
&> u(\tilde{q}(\theta'), \theta) - \tilde{t}(\theta') - C
\end{aligned}$$

where the first inequality holds because $\theta' \in [\tau^r(\tilde{\theta} - \epsilon) - (\frac{\delta_\tau}{2})^r \epsilon^2, \tau^r(\tilde{\theta} + \epsilon) + (\frac{\delta_\tau}{2})^r \epsilon^2]$, so Lemma 14 implies $\tau^{-1}(\theta') \in [\tau^{r-1}(\tilde{\theta} - \epsilon) - \frac{1}{2}(\frac{\delta_\tau}{2})^{r-1}\epsilon^2, \tau^{r-1}(\tilde{\theta} + \epsilon) + \frac{1}{2}(\frac{\delta_\tau}{2})^{r-1}\epsilon^2]$, and since $|\theta - \tau^{-1}(\theta')| \geq \frac{1}{2}(\frac{\delta_\tau}{2})^{r-1}\epsilon^2$, Lemma 15 implies that $V(\theta) - U(\theta'|\theta) \geq \delta_V \delta_\tau (\frac{\delta_\tau}{2})^{2(r-1)} \frac{\epsilon^4}{16}$ for small enough ϵ; the second equality holds by the definitions of $\tilde{q}(\theta')$ and $\tilde{t}(\theta')$; the last inequality holds for small enough ϵ.

Step 3. Establishing that the the mechanism $(\tilde{q}(.), \tilde{t}(.))$ is more profitable for the principal than the original mechanism.

The change in seller's profits from switching to the new mechanism is equal to

$$\Pi(\epsilon) = \int_{\Theta_s(\epsilon)} [u(q(\theta) + \epsilon^5, \theta) - u(q(\theta), \theta)] f(\theta) d\theta - \Delta(\epsilon) \sum_{k=0}^{s-1} \int_{\Theta_k(\epsilon)} f(\theta) d\theta$$

and thus

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\frac{\Pi(\epsilon)}{\epsilon^6} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}[\int_{\Theta_s(\epsilon)} u_q(q(\theta),\theta)f(\theta)d\theta - \max_{\theta'\in\Theta_s(\epsilon)}[u_q(q(\theta'),\overline{\theta}_{s-1}) - u_q(q(\theta'),\theta')]\sum_{k=0}^{s-1}\int_{\Theta_k(\epsilon)} f(\theta)d\theta]\\
&=\lim_{\epsilon\to 0}\frac{1}{\epsilon}[\int_{\tau^s(\tilde{\theta}-\epsilon)}^{\tau^s(\tilde{\theta}+\epsilon)} u_q(q(\theta),\theta)f(\theta)d\theta - \max_{\theta'\in\Theta_s(\epsilon)}[u_q(q(\theta'),\overline{\theta}_{s-1}) - u_q(q(\theta'),\theta')]\sum_{k=0}^{s-1}\int_{\tau^k(\tilde{\theta}-\epsilon)}^{\tau^k(\tilde{\theta}+\epsilon)} f(\theta)d\theta]\\
&=\lim_{\epsilon\to 0}\frac{1}{\epsilon}[\int_{\tilde{\theta}-\epsilon}^{\tilde{\theta}+\epsilon} u_q(q(\tau^s(\theta)),\tau^s(\theta))f(\tau^s(\theta))\dot{\tau}^s(\theta) - \max_{\theta'\in\Theta_s(\epsilon)}[u_q(q(\theta'),\overline{\theta}_{s-1}) - u_q(q(\theta'),\theta')]\sum_{k=0}^{s-1}\dot{\tau}^k(\theta)f(\tau^k(\theta))d\theta]\\
&=2[u_q(q(\tau^s(\tilde{\theta}),\tau^s(\tilde{\theta}))f(\tau^s(\tilde{\theta}))\dot{\tau}^s(\tilde{\theta}) - [u_q(q(\tau^s(\tilde{\theta})),\tau^{s-1}(\tilde{\theta})) - u_q(q(\tau^s(\tilde{\theta})),\tau^s(\tilde{\theta}))]\sum_{k=0}^{s-1}\dot{\tau}^k(\tilde{\theta})f(\tau^k(\tilde{\theta}))]\\
&>2\mu > 0
\end{aligned}
$$

where the first equality holds by definition of Δ; the second equality holds by eliminating the second order residuals in $\cup_{k=0}^{s}\Theta_k(\epsilon)$; the third equality is derived using change of variables; the fourth equality holds since $\overline{\theta}_{s-1} \to \tau^{s-1}(\tilde{\theta})$ and $\Theta_s(\epsilon) \to \{\tau^s(\tilde{\theta})\}$ as $\epsilon \to 0$; and the first equality holds by (B.13).

Therefore, change of profit is positive for small enough ϵ, which contradicts the optimality of the original mechanism.

Q.E.D.

Appendix C

Appendix for Chapter 3

C.1 Proof of Lemma 2

Fix an incentive compatible mechanism Ω. Let $M_1 = \{m \in \mathcal{M}_q : X(m,l) > X(m,t)\}$ be the set of on-path messages such that the induced action of liars is higher than the induced action of truth-tellers. Define a mortified mechanism $\hat{\Omega}$ such that for each $m \in \mathcal{M}_q/M_1$, the set of senders remain unchanged but they now send the transformed message $T(m) = \Theta_q^t(m)$. For each $m \in M_1$, the set of senders remain unchanged but they now send the transformed message $T(m) = \Theta_q^l(m)$. For inspection probabilities, let $\hat{P}(T(m)) = P(m)$ for each $m \in \mathcal{M}_q$.

The set of on-path messages of the modified mechanism $\hat{\Omega}$ is $\mathcal{M}_{\hat{q}} = T(\mathcal{M}_q)$. The sequentially rational actions for $\hat{\Omega}$ are $\hat{X}(T(m), s) = X(m, s)$ for $m \in \mathcal{M}_q/M_1$ and $s = t, l, u$; $\hat{X}(T(m), t) = X(m, l)$, $\hat{X}(T(m), l) = X(m, t)$ and $\hat{X}(T(m), u) = X(m, u)$ for $m \in M_1$. It is straight-forward that for all $m \in \mathcal{M}_q$ $\hat{X}(T(m), t) \geq \hat{X}(T(m), l)$ and $T(m) = \Theta_q^t(T(m))$. Therefore, condition (i) and (ii) are satisfied in the mortified mechanism $\hat{\Omega}$. Furthermore, since the induced actions remain unchanged for every type of sender, so $\hat{\Omega}$ and Ω are distribution equivalent.

To see that $\hat{\Omega}$ is incentive compatible, note that for $m \in \mathcal{M}_q/M_1$, $w_q(m) = w_(\hat{q})(T(m))$, and for $m \in M_1$, $w_q(m) = 1 - w_(\hat{q})(T(m))$. Therefore, for any $m \in \mathcal{M}_q$, $V_q(m) = w_q(m)(1 - w_q(m))(X(m,t) - X(m,l))^2 = w_(\hat{q})(T(m))(1 - w_(\hat{q})(T(m)))(\hat{X}(T(m),t) - \hat{X}(T(m),l))^2 = V_{\hat{q}}(T(m))$, thus (3.9) remains satisfied in $\hat{\Omega}$. To check incentive constraints (3.11), note that the equilibrium payoff of each type of sender remain unchanged, i.e. $EU_{\hat{X},\hat{P}}(m_{\hat{q}}(\theta)|\theta) = EU_{X,P}(m_q(\theta)|\theta)$. By the definition of the mortified set of message, any type θ would be identified as a liar of any on-path message other than its equilibrium message, i.e. $\theta \notin m'$ for any $m' \in T(\mathcal{M}_q)$ and $m' \neq m_{\hat{q}}(\theta)$. This combined with the fact that $\hat{X}(T(m), t) \geq \hat{X}(T(m), l)$ imply $EU_{\hat{X},\hat{P}}(T(m')|\theta) \leq EU_{X,P}(m'|\theta)$ for any $m' \in \mathcal{M}_q$. Therefore, $EU_{\hat{X},\hat{P}}(m_{\hat{q}}(\theta)|\theta) = EU_{X,P}(m_q(\theta)|\theta) \geq EU_{X,P}(m'|\theta) \geq EU_{\hat{X},\hat{P}}(T(m')|\theta)$ for any $\theta \in \Theta$ and $m' \in \mathcal{M}_q$, where the first inequality holds by incentive compatibility of the original mechanism, thus (3.11) is satisfied in the modified mechanism. Therefore, we conclude that $\hat{\Omega}$ is incentive compatible.

Q.E.D.

C.2 Preliminary Lemmas

This section provides a series of Lemmas that facilitate the proofs of the subsequent theorems.

Below are some definitions and terminologies that are useful for proving the results. For $d \in [2\sqrt{c}, 1]$, let

$$h(d) = \frac{w^-(d)}{1 - w^-(d)} \tag{C.1}$$

be the required liar to truth-teller ratio to maintain incentive for DM to inspect a message. It can be verified that $h(.)$ is a strictly decreasing and strictly convex function, with $\lim_{d\to 2\sqrt{c}} h'(d) = -\infty$ and $\lim_{d\to 2\sqrt{c}} h''(d) = +\infty$.

We say θ is **essentially revealed upon inspection** in Ω if $x^d_\Omega(\theta) = \theta$.

The following Proposition establishes the necessary and sufficient conditions of an incentive compatible mechanism.

Lemma 20 *Let q and X be a pair of message and action rule that satisfies (i) - (ii) of Lemma 2, X satisfies DM's sequential rationality (3.5) given q, and let m^0_q be the (potentially non-exist) uninspected message. Then there exists an inspection rule $P(.)$ on the set of inspected messages M^+_q such that (q, P, X) is incentive compatible if and only if for any $m, m' \in \mathcal{M}^+_q$:*
(a) $X(m,l) \le X(m^0_q, u) < X(m', u)$ if m^0_q exists; $X(m,l) < X(m',u)$ otherwise;
(b) $w_q(m)(1-w_q(m))(X(m,t)-X(m,l))^2 \begin{cases} = c & \text{if } X(m,l) < X(m^0_q,u) \\ \ge c & \text{if } X(m,l) = X(m^0_q,u) \end{cases}$; If m^0_q does not exists, replace $X(m^0_q, u)$ with $\sup_{m'\in\mathcal{M}^+_q} X(m', l)$.
In particular, $P(m) = \frac{u(X(m,u)) - u(x^0_q)}{u(X(m,u)) - u(X(m,l))}$, where $x^0_q = X(m^0_q, u)$ if m^0_q exists;
$x^0_q \in [\sup_{m'\in\mathcal{M}^+_q} X(m', l), \inf_{m''\in\mathcal{M}^+_q} X(m'', u)]$ if m^0_q does not exist and $v_q(m) = c$ for all $m \in \mathcal{M}^+_q$; $x^0_q = \max_{m'\in\mathcal{M}^+_q} X(m', l)$ otherwise.

Proof of Lemma 20: Given (i) of Lemma 2 we have $X(m,t) \ge X(m,l)$, and for any $m \in \mathcal{M}^+_q$, incentive compatibility requires $X(m,t) > X(m,l)$, for otherwise value of inspection $V_q(m) = w_q(m)(1 - w_q(m))(X(m,t) - X(m,l))^2 = 0$, violating DM's sequential rationality. (3.5) then implies that $X(m,t) > X(m,u) > X(m,l)$ for any $m \in \mathcal{M}^+_q$. (i) of Proposition 2 then imply that if m^0_q exists, $P(m)u(X(m,l)) + (1 - P(m))u(X(m,u)) = P(m^0_q)u(X(m^0_q, l)) + (1 - P(m^0_q))u(X(m^0_q, u)) = u(X(m^0_q, u))$, which means $P(m) = \frac{u(X(m,u)) - u(X(m^0_q,u))}{u(X(m,u)) - u(X(m,l))}$. Since sender's utility $u(.)$ is strictly increasing, there exists such $P(m) \in (0, 1]$ if and only if $X(m,l) \le X(m^0_q, u) < X(m,u)$, which holds for all $m, m' \in \mathcal{M}^+_q$, so $X(m,l) \le X(m^0_q, u) < X(m', u)$. If $X(m,l) < X(m^0_q, u)$, it must be $P(m) \in (0,1)$, so sequentially rational inspection requires $V_q(m) = c$; If $X(m,l) = X(m^0_q, u)$, $P(m) = 1$, sequentially rational inspection requires $V_q(m) \ge c$.

If m_q^0 does not exist, then for any $m, m' \in \mathcal{M}_q^+$, $P(m)u(X(m,l)) + (1-P(m))u(X(m,u)) = P(m')u(X(m',l)) + (1-P(m'))u(X(m',u)) = u(X(m',u))$, which can be achieved with some strictly positive $P(.)$ if and only if $\sup_{m' \in \mathcal{M}_q^+} X(m',l) < \inf_{m'' \in \mathcal{M}_q^+} X(m'',u)$. Sequentially rational inspection requires $V_q(m) = c$ for any m such that $P < 1$, which must be the case when $X(m,l) < \sup_{m' \in \mathcal{M}_q^+} X(m',l)$. $V_q(m) \geq c$ and $P(m) = 1$ is allowed if and only if $X(m,l) = \max_{m' \in \mathcal{M}_q^+} X(m',l)$. *Q.E.D.*

Given an mechanism Ω, DM's expected payoff of Ω when every messages are ex-post uninspected and actions $X(m,u)$ are induced is

$$\begin{aligned} EU_{DM}^U(\Omega) &= \int_{\mathcal{M}_q^+} Pr(\Theta_q^u(m))X(m,u)^2 dm + Pr(\Theta_q^u(m_q^0))X(m_q^0,u)^2 - E[\theta^2] \\ &= \int_{[0,1]} x^2 dG_\Omega^u(x) - E[\theta^2] \end{aligned} \tag{C.2}$$

Where

$$G_\Omega^u(x) = \int_{\mathcal{M}_q} \int_{\Theta_q^u(m)} \mathbf{1}(X(m,u) \leq x) dF(\theta) dm \tag{C.3}$$

is the distribution of induced actions when messages are ex-post uninspected; DM's expected payoff when every messages in $\mathcal{M}_q^+$ are ex-post inspected and actions $X(m,t)$ ($X(m,l)$) are induced when sender is truthful (lying) is

$$\begin{aligned} EU_{DM}^I(\Omega) = &\int_{\mathcal{M}_q^+} Pr(\Theta_q^u(m))[w_q(m)X(m,l)^2 + (1-w_q(m))X(m,t)^2 - c]dm \\ &+ Pr(\Theta_q^u(m_q^0))X(m_q^0,u)^2 - E[\theta^2] \\ =&(1 - Pr(\Theta_q^u(m_q^0))) \int_{[0,1]} (x^2 - c) dG_\Omega^i(x) + Pr(\Theta_q^u(m_q^0))X(m_q^0,u)^2 - E[\theta^2] \end{aligned} \tag{C.4}$$

where

$$G_\Omega^i(x) = \frac{1}{1 - Pr(\Theta_q^0)} \int_{\mathcal{M}_q^+} \sum_{s=t,l} \int_{\Theta_q^s(m)} \mathbf{1}(X(m,s) \leq x)] dF(\theta) dm \tag{C.5}$$

is the distribution of actions induced by messages with positive probability of inspection, when those messages are ex-post inspected.

Lemma 21 *Let Ω be an mechanism such that X satisfies (3.5) given q, and $V_q(m) = c$ almost everywhere for $m \in \mathcal{M}_q^+$, then $EU_{DM}(\Omega) = EU_{DM}^U(\Omega) = EU_{DM}^I(\Omega)$.*

Proof of Lemma 21: From equation (3.12),

$$
\begin{aligned}
&EU_{DM}(\Omega)\\
=&-\int_{\Theta}\int_{\mathcal{M}_q} q(m|\theta)[(1-P(m))(X(m,u)-\theta)^2+P(m)\sum_{s=t,l}\mathbf{1}(\theta\in\Theta_s(m))[(X(m,s)-\theta)^2+c]]dmdF(\theta)\\
=&-\int_{\mathcal{M}_q^+}[(1-P(m))\int_{\Theta_q^u(m)}(X(m,u)-\theta)^2dF(\theta)+P(m)\sum_{s=t,l}\int_{\Theta_q^s(m)}[(X(m,s)-\theta)^2+c]dF(\theta)]dm\\
&-\int_{\Theta_q^u(m_q^0)}(X(m_q^0,u)-\theta)^2dF(\theta)\\
=&\int_{\mathcal{M}_q^+}[(1-P(m))\int_{\Theta_q^u(m)}dF(\theta)X(m,u)^2+P(m)\sum_{s=t,l}\int_{\Theta_q^s(m)}dF(\theta)[X(m,s)^2-c]]dm\\
&+\int_{\Theta_q^u(m_q^0)}dF(\theta)X(m_q^0,u)^2-E[\theta^2]\\
=&\int_{\mathcal{M}_q^+}Pr(\Theta_q^u(m))[(1-P(m))X(m,u)^2+P(m)[w_q(m)X(m,l)^2+(1-w_q(m))X(m,t)^2-c]]dm\\
&+Pr(\Theta_q^u(m_q^0))X(m_q^0,u)^2-E[\theta^2] \qquad \text{(C.6)}
\end{aligned}
$$

where the second equality holds because of (3.2) - (3.4), the third equality holds because (3.5) implies $-\int_{\Theta_q^s(m)}(X(m,s)-\theta)^2dF(\theta)=\int_{\Theta_q^s(m)}X(m,s)^2dF(\theta)-\int_{\Theta_q^s(m)}\theta^2dF(\theta)$ for each $(m,s)\in\mathcal{M}_q\times\{t,l,u\}$; the forth equality holds because $w_q(m)Pr(\Theta_q^u(m))=Pr(\Theta_q^l(m))$ and $(1-w_q(m))Pr(\Theta_q^u(m))=Pr(\Theta_q^t(m))$.

Since $X(m,u)=w_q(m)X(m,l)+(1-w_q(m))X(m,t)$, so for any $m\in\mathcal{M}_q^+$, $w_q(m)X(m,l)^2+(1-w_q(m))X(m,t)^2-X(m,u)^2=w_q(m)(1-w_q(m))(X(m,t)-X(m,l))^2\equiv V_q(m)$. Therefore, $V_q(m)=c$ implies that

$$X(m,u)^2=w_q(m)X(m,l)^2+(1-w_q(m))X(m,t)^2-c \qquad \text{(C.7)}$$

holds almost everywhere for $m\in\mathcal{M}_q^+$, thus

$$
\begin{aligned}
EU_{DM}(\Omega)=&\int_{\mathcal{M}_q^+}Pr(\Theta_q^u(m))X(m,u)^2dm+Pr(\Theta_q^u(m_q^0))X(m_q^0,u)^2-E[\theta^2]\\
=&EU_{DM}^U(\Omega)
\end{aligned}
$$

and

$$
\begin{aligned}
EU_{DM}(\Omega)=&\int_{\mathcal{M}_q^+}Pr(\Theta_q^u(m))[w_q(m)X(m,l)^2+(1-w_q(m))X(m,t)^2-c]dm\\
&+Pr(\Theta_q^u(m_q^0))X(m_q^0,u)^2-E[\theta^2]\\
=&EU_{DM}^I(\Omega)
\end{aligned}
$$

Q.E.D.

Lemma 22 *For any $\epsilon > 0$, suppose there is a mechanism Ω such that X satisfies (3.5) given q, $w_q(m) = w^-(X(m,t) - X(m,l))$ almost everywhere for $m \in \mathcal{M}_q^+$, $\sup_{m \in \mathcal{M}_q^+} X(m,l) < X(m_q^0, u) + \epsilon$ and $\inf_{m \in \mathcal{M}_q^+} X(m,u) > X(m_q^0, u) - \epsilon$, then there exists an incentive compatible mechanism $\hat{\Omega}$ such that $EU_{DM}(\hat{\Omega}) > EU_{DM}(\Omega) - 4\epsilon^2$.*

Proof of Lemma 22: Let $M^u = \{m \in \mathcal{M}_q^+ : X(m_q^0, u) - \epsilon < X(m,u) \leq X(m_q^0, u)\}$ be the set of inspected messages that violate $X(m_q^0, u) < X(m,u)$. Define $\hat{x}$ that solves

$$Pr(\Theta_q^0 \cup \Theta_q^u(M^u))(\hat{x} - E[\Theta_q^0 \cup \Theta_q^u(M^u)]) = \int_{m \in \mathcal{M}_q^+ : X(m,l) \geq \hat{x}} Pr(\Theta_q^l(m))(X(m,l) - \hat{x}) dm \tag{C.8}$$

If $\sup_{m \in \mathcal{M}_q^+} X(m,l) > E[\Theta_q^0 \cup \Theta_q^u(M^u)]$, and $\hat{x} = E[\Theta_q^0 \cup \Theta_q^u(M^u)]$ otherwise. Let $M^l = \{m \in \mathcal{M}_q^+ : X(m,l) \geq \hat{x}\}$. Since X satisfies (3.5), we have

$$E[\Theta_q^0 \cup \Theta_q^u(M^u) \cup \Theta_q^l(M^l)] = \hat{x} \in (X(m_q^0, u) - \epsilon, X(m_q^0, u) + \epsilon) \tag{C.9}$$

and

$$X(m,u) > \hat{x} > X(m,l) \text{ for any } m \in M_q^+ / (M^u \cup M^l) \tag{C.10}$$

Now for each $m \in M^l$ define $\hat{w} = w^-(X(m,t) - \hat{x})$ be the new required weight of liars given that the liar-induced action of m is $\hat{x}$. Since $w^-(.)$ is a decreasing function, $w_q(m) = w^-(X(m,t) - X(m,l))$ and $X(m,l) \geq \tilde{x}$ for $m \in M^l$, so $\hat{w}(m) \leq w_q(m)$. Let $\hat{p} = \int_{M^l} \frac{\hat{w}(m)}{1-\hat{w}(m)} \int_{\Theta_q^t(m)} dF(\theta) dm$ be the total mass of liars required to be pooled with truth-tellers in M^l, given that liar-induced actions are $\hat{x}$. $\hat{w}(m) \leq w_q(m)$ implies $\hat{p} \leq Pr(\Theta_q^l(M^l)) = \int_{M^l} \frac{w_q(m)}{1-w_q(m)} \int_{\Theta_q^t(m)} dF(\theta) dm$.

Let $\hat{\Theta} = \Theta_q^0 \cup \Theta_q^u(M^u) \cup \Theta_q^l(M^l)$ be the pool of modifying types. Let $\underline{z} = \frac{Pr(\hat{\Theta}) - \hat{p}}{Pr(\hat{\Theta})}$. Assign an arbitrary strict ranking $r : M^l \to \mathbb{R}$ to the message set M^l. Then for any $m \in M^l$, let

$$z^-(m) = \underline{z} + \frac{1}{Pr(\hat{\Theta})} \int_{m' \in M^l : r(m') < r(m)} \frac{\hat{w}(m')}{1 - \hat{w}(m')} \int_{\Theta_q^t(m')} dF(\theta) dm' \tag{C.11}$$

$$z^+(m) = z^-(m) + \frac{1}{Pr(\hat{\Theta})} \frac{\hat{w}(m)}{1 - \hat{w}(m)} \int_{\Theta_q^t(m)} dF(\theta) \tag{C.12}$$

Define an modified messaging and action rules $\hat{q}, \hat{X}$ where other things remain unchanged, except the set of messages M^u is off-path; the uninspected message is modified to $m_{\hat{q}}^0 = \hat{\Theta}(\underline{z})$

with the set of sender identical to the statement, where $\hat{\Theta}(\underline{z})$ is a mean preserving division of $\hat{\Theta}$ so that $E[\hat{\Theta}(\underline{z}] = E[\hat{\Theta}] = \hat{x}$ and $Pr(\hat{\Theta}(\underline{z})) = \underline{z}Pr(\hat{\Theta}) = Pr(\hat{\Theta}) - \hat{p}$; For any $m \in M^l$, the set of truth-tellers remain unchanged, while the set of liars is modified to $\Theta^l_{\hat{q}}(m) = \hat{\Theta}(z^+(m))/int(\hat{\Theta}(z^-(m)))$, a mean preserving division of $\hat{\Theta}$ where $int(X)$ is the interior of set X, so that $E[\Theta^l_{\hat{q}}(m)] = \hat{x}$ and the set has measure $\frac{\hat{w}(m)}{1-\hat{w}(m)} \int_{\Theta^t_q(m)} dF(\theta)$.

The sequentially rational actions for the modified uninspected messages $m^0_{\hat{q}}$ is

$$\hat{X}(m^0_{\hat{q}}, u) = E[\hat{\Theta}(\underline{z})] = \hat{x} \tag{C.13}$$

and for $m \in M^l$,

$$\begin{aligned} \hat{X}(m,t) &= X(m,t) \\ \hat{X}(m,l) &= E[\Theta^l_{\hat{q}}(m)] = \hat{x} \\ \hat{X}(m,u) &= \hat{w}(m)\hat{x} + (1-\hat{w}(m))X(m,t) \end{aligned} \tag{C.14}$$

where $X(m,t) > X(m,l) \geq \hat{x}$. (C.10), (C.13) and (C.14) imply that

$$\hat{X}(m,u) > \hat{X}(m^0_{\hat{q}}, u) \leq \hat{X}(m,l) \text{ for any } m \in M^+_{\hat{q}} \tag{C.15}$$

so $(\hat{q}, \hat{X})$ satisfies (a) of Lemma 20. Furthermore, by the definition of $\Theta^l_{\hat{q}}(m)$ for $m \in M^l$, we have

$$w_{\hat{q}}(m) = \hat{w}(m) = w^-(\hat{X}(m,t) - \hat{X}(m,l)) \tag{C.16}$$

and thus

$$V_{\hat{q}}(m) = c \tag{C.17}$$

so $(\hat{q}, \hat{X})$ satisfies (b) of Lemma 20. Therefore, there exists $\hat{P}$ such that $\hat{\Omega} = (\hat{q}, \hat{P}, \hat{X})$ is incentive compatible.

To compare DM's ex-ante payoffs, since $V_q(m) = V_{\hat{q}}(m) = c$, (C.7) holds for both mechanisms, thus (C.6) implies

$$\begin{aligned} EU_{DM}(\Omega) = &\int_{\mathcal{M}^+_q/M^l} Pr(\Theta^u_q(m))X(m,u)^2 dm + Pr(\Theta^0_q)X(m^0_q,u)^2 - E[\theta^2] \\ &+ \int_{M^l} [Pr(\Theta^l_q(m))X(m,l)^2 + Pr(\Theta^t_q(m))X(m,t)^2 - Pr(\Theta^u_q(m))c] dm \end{aligned} \tag{C.18}$$

and

$$\begin{aligned} EU_{DM}(\hat{\Omega}) = &\int_{\mathcal{M}^+_q/(M^l \cup M^u)} Pr(\Theta^u_q(m))X(m,u)^2 dm + Pr(\hat{\Theta}(\underline{z}))\hat{x}^2 - E[\theta^2] \\ &+ \int_{M^l} [Pr(\hat{\Theta}^l_q(m))\hat{x}^2 + Pr(\Theta^t_q(m))X(m,t)^2 - Pr(\hat{\Theta}^u_q(m))c] dm \end{aligned} \tag{C.19}$$

so

$$
\begin{aligned}
&EU_{DM}(\Omega) - EU_{DM}(\hat{\Omega}) \\
=&\int_{M^u} Pr(\Theta_q^u(m))X(m,u)^2 dm + Pr(\Theta_q^0)X(m_q^0,u)^2 + \int_{M^l} Pr(\Theta_q^l(m))(X(m,l)^2 - c)dm \\
&- Pr(\hat{\Theta}(\underline{z}))\hat{x}^2 - \int_{M^l} Pr(\hat{\Theta}_q^l(m))(\hat{x}^2 - c)dm \\
=&\int_{M^u} Pr(\Theta_q^u(m))X(m,u)^2 dm + Pr(\Theta_q^0)X(m_q^0,u)^2 + \int_{M^l} Pr(\Theta_q^l(m))X(m,l)^2 dm \\
&- Pr(\hat{\Theta})\hat{x}^2 + c\int_{M^l} [Pr(\hat{\Theta}_q^l(m)) - Pr(\Theta_q^l(m))]dm \\
\leq&\int_{M^u} Pr(\Theta_q^u(m))X(m,u)^2 dm + Pr(\Theta_q^0)X(m_q^0,u)^2 + \int_{M^l} Pr(\Theta_q^l(m))X(m,l)^2 dm - Pr(\hat{\Theta})\hat{x}^2 \\
=&\int_{M^u} Pr(\Theta_q^u(m))[X(m,u)^2 - \hat{x}^2]dm + \int_{M^l} Pr(\Theta_q^l(m))[X(m,l)^2 - \hat{x}^2]dm \\
&+ Pr(\Theta_q^0)[X(m_q^0,u)^2 - \hat{x}^2] \\
=&\int_{M^u} Pr(\Theta_q^u(m))[X(m,u) - \hat{x}]^2 dm + \int_{M^l} Pr(\Theta_q^l(m))[X(m,l) - \hat{x}]^2 dm \\
&+ Pr(\Theta_q^0)[X(m_q^0,u) - \hat{x}]^2 \\
<&4Pr(\hat{\Theta})\epsilon^2 \leq 4\epsilon^2
\end{aligned} \tag{C.20}
$$

where the first inequality holds because $\int_{M^l} Pr(\hat{\Theta}_q^l(m)) - Pr(\Theta_q^l(m))dm = \hat{p} - Pr(\Theta_q^l(M^l)) \leq 0$; the third equality holds because $\hat{\Theta} = \Theta_q^0 \cup \Theta_q^u(M^u) \cup \Theta_q^l(M^l)$; the fourth equality holds because (3.5) implies that $\int_{M^u} Pr(\Theta_q^u(m))X(m,u)dm + \int_{M^l} Pr(\Theta_q^l(m))X(m,l)dm + Pr(\Theta_q^0)X(m_q^0,u) = E[\hat{\Theta}] = \hat{x}$; the second inequality holds because $|X(m,s) - X(m_q^0)| < \epsilon$ for $m \in M^s$, $s \in \{u,l\}$ by definitions of M^l, M^u and $|\hat{x} - X(m_q^0)| < \epsilon$ by (C.9); the last inequality holds because $Pr(\hat{\Theta}) \leq 1$. *Q.E.D.*

Lemma 23 $\frac{dx_u^*(x_t,x_l)}{dx_t} > 0$ *and* $\frac{dx_u^*(x_t,x_l)}{dx_l} < 0$.

Proof of Lemma 23:

$$
\begin{aligned}
\frac{dx_u^*(x_t,x_l)}{dx_t} =& 1 - w^-(x_t - x_l) - \frac{dw^-(x_t - x_l)}{dx_t}[x_t - x_l] \\
=&\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{c}{(x_t - x_l)^2}} + (\frac{1}{4} - \frac{c}{(x_t - x_l)^2})^{-0.5}\frac{c}{(x_t - x_l)^2} \\
>&0
\end{aligned}
$$

$$
\begin{aligned}
\frac{dx_u^*(x_t,x_l)}{dx_l} =& w^-(x_t-x_l) - \frac{dw^-(x_t-x_l)}{dx_l}[x_t-x_l] \\
=& \frac{1}{2} - \sqrt{\frac{1}{4}-\frac{c}{(x_t-x_l)^2}} - (\frac{1}{4}-\frac{c}{(x_t-x_l)^2})^{-0.5}\frac{c}{(x_t-x_l)^2} \\
=& (\frac{1}{4}-\frac{c}{(x_t-x_l)^2})^{-0.5}[\frac{1}{2}\sqrt{\frac{1}{4}-\frac{c}{(x_t-x_l)^2}} - (\frac{1}{4}-\frac{c}{(x_t-x_l)^2}) - \frac{c}{(x_t-x_l)^2}] \\
=& (\frac{1}{4}-\frac{c}{(x_t-x_l)^2})^{-0.5}[\frac{1}{2}\sqrt{\frac{1}{4}-\frac{c}{(x_t-x_l)^2}} - \frac{1}{4}] \\
<& 0
\end{aligned}
$$

where the last inequality holds because $\sqrt{\frac{1}{4}-\frac{c}{(x_t-x_l)^2}} < \frac{1}{2}$. *Q.E.D.*

Lemma 24 *Let* $(x_0, x_1, x_2, \hat{x}, \delta) \in [0,1]^5$. *If* $x_1 > \hat{x}$, $\hat{x} > \frac{x_1+x_2}{2}$ *and* $(x_1-x_0)(x_0-x_2)-(x_1-\hat{x})(\hat{x}-x_2) > \delta > 0$, *then* $\hat{x} > x_0 + \delta$.

Proof of Lemma 24: First we have

$$
(\hat{x}-x_0)(\hat{x}+x_0-\mu_1-\mu_2) = (x_1-x_0)(x_0-x_2)-(x_1-\hat{x})(\hat{x}-x_2) > \delta \tag{C.21}
$$

It must be the case that $\hat{x} > x_0$, for otherwise $x_0 \geq \hat{x} > \frac{x_1+x_2}{2}$ would implies that LHS of (C.21) is non-positive. $x_1 > \hat{x}$ implies that $\hat{x}+x_0-\mu_1-\mu_2)1$, therefore $\hat{x}-x_0 > \delta$. *Q.E.D.*

For an Ω, let $\overline{\theta}^0 = inf\{\theta : Pr([0,\theta]\cap\Theta_q^0) = Pr(\Theta_q^0)\}$ and $\underline{\theta}^0 = sup\{\theta : Pr([\theta,1]\cap\Theta_q^0) = Pr(\Theta_q^0)\}$ be the probabilistic upper bound and lower bound of the set of uninspected types, $\mu = E[\Theta_q^0]$ be the mean of Θ_q^0.

Lemma 25 *In an optimal mechanism* Ω, $(\overline{\theta}^0-\mu)(\mu-\underline{\theta}^0) \leq c$.

Proof of Lemma 25: Suppose contrary to the claim, $(\overline{\theta}^0-\mu)(\mu-\underline{\theta}^0) > c$, then there exists $\delta > 0$ such that $(\overline{\theta}^0-\mu)(\mu-\underline{\theta}^0)-c > \delta$, and for any small enough $\epsilon > 0$ there exist $\Theta_1, \Theta_2 \subseteq \Theta_q^0$ such that $Pr(\Theta_1) = Pr(\Theta_2) = \epsilon$ and

$$
(\mu_1-\mu)(\mu-\mu_2)-c > \delta \tag{C.22}
$$

where $\mu_1 \equiv E[\Theta_1]$ and $\mu_2 \equiv E[\Theta_2]$. Let $\hat{w} = w^-(\mu_1-\mu_2)$, where $w^-(.)$ is the minimum liar weight function defined in (3.17), which is well defined because (C.22) implies that $\mu_1 - \mu_2 > 2\sqrt{c}$. Create a mean-preserving division of Θ_2, $\Theta_2(\frac{\hat{w}}{1-\hat{w}})$, so $E[\Theta_2(\frac{\hat{w}}{1-\hat{w}})] = E[\Theta_2]$ and $Pr(\Theta_2(\frac{\hat{w}}{1-\hat{w}})) = \frac{\hat{w}}{1-\hat{w}}Pr(\Theta_2)$.

Now define an modified mechanism $\hat{\Omega} = (\hat{q}, \hat{P}, \hat{X})$ where other things remain unchanged, except the uninspected message is modified to $m^0_{\hat{q}} = \Theta^0_{\hat{q}} \equiv \Theta^0_q/(\Theta_1 \cup \Theta_2(\frac{\hat{w}}{1-\hat{w}}))$, with the set of sender identical to the statement $\Theta^0_{\hat{q}}$. An inspected message $\hat{m} = \Theta_1$ is added, with truthful senders $\Theta^t_{\hat{q}}(\hat{m}) = \Theta_1$ and lying senders $\Theta^l_{\hat{q}}(\hat{m}) = \Theta_2(\frac{\hat{w}}{1-\hat{w}})$.

The sequentially rational actions for the modified message $\hat{m}$ are

$$\hat{X}(\hat{m}, t) = E[\Theta_1] = \mu_1$$
$$\hat{X}(\hat{m}, l) = E[\Theta_2(\frac{\hat{w}}{1-\hat{w}})] = \mu_2$$
$$\hat{X}(\hat{m}, u) = E[\Theta_1 \cup \Theta_2(\frac{\hat{w}}{1-\hat{w}})] = \hat{w}\mu_2 + (1-\hat{w})\mu_1$$

the last equality holds because $\frac{Pr(\Theta_2(\frac{\hat{w}}{1-\hat{w}}))}{Pr(\Theta_2(\frac{\hat{w}}{1-\hat{w}}))+Pr(\Theta_1)} = \frac{Pr(\Theta_2(\frac{\hat{w}}{1-\hat{w}}))}{Pr(\Theta_2(\frac{\hat{w}}{1-\hat{w}}))+Pr(\Theta_2)} = \hat{w}$. The sequentially rational action for the modified message $m^0_{\hat{q}}$ is

$$\hat{X}(m^0_{\hat{q}}, u) = E[\Theta^0_q/(\Theta_1 \cup \Theta_2(\frac{\hat{w}}{1-\hat{w}}))]$$

The definition of $\hat{w}$ implies that $V_q(\hat{m}) = \hat{w}(1-\hat{w})(\mu_1-\mu_2)^2 = c$, so (b) of Lemma 20 is satisfied for $\hat{m}$. The above equality also implies

$$(\mu_1 - \hat{X}(\hat{m}, u))(\hat{X}(\hat{m}, u) - \mu_2) = c \qquad (C.23)$$

and since $\hat{w} < \frac{1}{2}$, we have $\hat{X}(\hat{m}, u) > \frac{\mu_1+\mu_2}{2}$, so Lemma 24, (C.22) and (C.23) imply

$$\hat{X}(\hat{m}, u) > \mu + \delta \qquad (C.24)$$

By sequential rationality,

$$Pr(\Theta_1 \cup \Theta_2(\frac{\hat{w}}{1-\hat{w}}))\hat{X}(\hat{m}, u) + Pr(\Theta^0_q/(\Theta_1 \cup \Theta_2(\frac{\hat{w}}{1-\hat{w}})))\hat{X}(m^0_{\hat{q}}, u) = Pr(\Theta^0_q)\mu \qquad (C.25)$$

where $Pr(\Theta_1 \cup \Theta_2(\frac{\hat{w}}{1-\hat{w}})) = \epsilon + \frac{\hat{w}}{1-\hat{w}}\epsilon = \frac{\epsilon}{1-\hat{w}}$. Rearranging (C.25) yields

$$\begin{aligned}\hat{X}(m^0_{\hat{q}}, u) =& \mu_0 - \frac{\epsilon}{(1-\hat{w})(Pr(\Theta^0_q) - \frac{\epsilon}{1-\hat{w}})}(\hat{X}(\hat{m}, u) - \mu) \\ >& \mu_0 - \frac{2\epsilon}{Pr(\Theta^0_q) - 2\epsilon}\end{aligned} \qquad (C.26)$$

because $\hat{w} < 0.5$ and $\hat{X}(\hat{m}, u) - \mu < 1$. Now we have

$$\hat{X}(m^0_{\hat{q}}, u) < \mu < \hat{X}(\hat{m}, u) \qquad (C.27)$$

and for any unmodified on-path message $m \in \mathcal{M}_q^+$, incentive compatibility of the original mechanism means $\hat{X}(m,u) = X(m,u) > \mu > \hat{X}(m_{\hat{q}}^0, u)$, so

$$\inf_{m \in M_{\hat{q}}^+} \hat{X}(m,u) > \hat{X}(m_{\hat{q}}^0, u) \tag{C.28}$$

Since $\hat{X}(\hat{m}, l) = \mu_2 < \hat{w}\mu_2 + (1-\hat{w})\mu_1 = \hat{X}(m_{\hat{q}}^0, u)$ and $\sup_{m \in \mathcal{M}_q^+} X(m,l) \leq \mu$ by incentive compatibility of the original mechanism, so (C.26) implies

$$\sup_{m \in M_{\hat{q}}^+} X(m,l) < \hat{X}(m_{\hat{q}}^0, u) + \frac{2\epsilon}{Pr(\Theta_q^0) - 2\epsilon} \tag{C.29}$$

Since the original mechanism is optimal, $w_q(m) = w^-(X(m,t) - X(m,l))$ hold almost everywhere for unmodified message $m \in \mathcal{M}_q^+$. For the modified message, $w_{\hat{q}}(\hat{m}) = w^-(\mu_1 - \mu_2) = w^-(\hat{X}(\hat{m},t) - \hat{X}(\hat{m},l))$, so we have

$$w_{\hat{q}}(m) = w^-(\hat{X}(m,t) - \hat{X}(m,l)) \tag{C.30}$$

hold almost everywhere at $\mathcal{M}_{\hat{q}}^+$.

To compare DM's payoffs, since $V_q(m) = V_{\hat{q}}(m) = c$, by Lemma 21, $EU_{DM}(\Omega) = EU_{DM}^U(\Omega)$ and $EU_{DM}(\hat{\Omega}) = EU_{DM}^U(\hat{\Omega})$, so

$$\begin{aligned}
&EU_{DM}(\hat{\Omega}) - EU_{DM}(\Omega) \\
=&\frac{\epsilon}{1-\hat{w}}\hat{X}(\hat{m},u)^2 + [Pr(\Theta_q^0) - \frac{\epsilon}{1-\hat{w}}]\hat{X}(m_{\hat{q}}^0,u)^2 - Pr(\Theta_q^0)(\mu)^2 \\
=&\frac{\epsilon}{1-\hat{w}}[\hat{X}(\hat{m},u) - \mu]^2 + [Pr(\Theta_q^0) - \frac{\epsilon}{1-\hat{w}}][\hat{X}(m_{\hat{q}}^0,u) - \mu]^2 \\
>&\frac{\epsilon}{1-\hat{w}}\delta^2 > 2\delta^2\epsilon
\end{aligned} \tag{C.31}$$

where the second equality holds because $\frac{\epsilon}{1-\hat{w}}\hat{X}(\hat{m},u) + [Pr(\Theta_q^0) - \frac{\epsilon}{1-\hat{w}}]\hat{X}(m_{\hat{q}}^0,u) - Pr(\Theta_q^0)(\mu) = 0$ by (C.25); the first inequality holds because of (C.24), and the last inequality hold because $\hat{w} < 0.5$. Finally, given (C.28) - (C.30), Lemma 22 implies that there exist an incentive compatible mechanism $\tilde{\Omega}$ such that $EU_{DM}(\tilde{\Omega}) > EU_{DM}(\hat{\Omega}) - 4(\frac{2\epsilon}{Pr(\Theta_q^0)-2\epsilon})^2$, then by (C.31) $EU_{DM}(\tilde{\Omega}) - EU_{DM}(\Omega) > 2\delta^2\epsilon - 4(\frac{2\epsilon}{Pr(\Theta_q^0)-2\epsilon})^2 > 0$ for small enough ϵ, but it contradicts that Ω is an optimal mechanism. Therefore, we conclude that $(\overline{\theta}^0 - \mu)(\mu - \underline{\theta}^0) \leq c$ in an optimal mechanism. *Q.E.D.*

For an mechanism Ω such that $Pr(\Theta_q^0) > 0$, let $\underline{\theta}^t = \sup\{\theta : Pr([\theta, 1] \cap \Theta_q^t(\mathcal{M}_q^+)) = Pr(\Theta_q^t(\mathcal{M}_q^+)\}$ and $\overline{\theta}_q^l = sup\{\theta : Pr([0,\theta] \cap \Theta^l(\mathcal{M}_q^+)) = Pr(\Theta_q^l(\mathcal{M}_q^+))\}$ be the probabilistic lower bound of the set of inspected truthful types and probabilistic upper bound of the set of inspected lying types.

Lemma 26 *In an optimal mechanism Ω, $\underline{\theta}^t \geq \overline{\theta}^0$.*

Proof of Lemma 26: Suppose contrary to the claim, $\underline{\theta}^t < \overline{\theta}^0$, then there exists $\delta > 0$ such that $\underline{\theta}^t < \overline{\theta}^0 - \delta$. Let $M_t = \{m \in \mathcal{M}_q^+ : \exists \theta \in \Theta_q^t(m)$ such that $\theta < \overline{\theta}^0 - \delta\}$ be the set of messages containing truthful types below $\overline{\theta}^0 - \delta$, and $\Theta_t = \{\theta : \exists m \in M_t$ such that $\theta \in \Theta_q^t(m), \theta < \overline{\theta}^0 - \delta$ and $\theta \leq X(m,t)\}$ be the set of such truthful types with a weaker higher induced action. We have $Pr(\Theta_t) > 0$ and there exist positive measure subset of uninspected types $\Theta_0 \subseteq \Theta_q^0$ such that for any $\theta_t \in \Theta_t$ and $\overline{\mu} \equiv E[\Theta_0]$,

$$\overline{\mu} - \theta_t > \delta \tag{C.32}$$

Denote $a = \frac{Pr(\Theta_t)}{Pr(\Theta_q^t(M_t))} > 0$, and let $M_t^a = \cap M \subseteq M_t : \frac{Pr(\Theta_t \cap \Theta_q^t(M))}{Pr(\Theta_q^t(M))} \geq a$ be the subset containing every message in M_t where proportion of truthful types within Θ_t is no less than a. Since $Pr(M_t^a) > 0$, for any $\epsilon > 0$ there is positive measure subset $M' \subseteq M_t^a$ such that for any $m, m' \in M'$ and $s \in \{t, l, u\}$,

$$|X(m,s) - X(m',s)| < \epsilon \tag{C.33}$$

and $Pr(M') \leq \epsilon$ (If every $m \in M_t^a$ is such that $Pr(m) > \epsilon$, then we can take any of such message m and divide it into two messages m_1, m_2 with exact same induced actions and $Pr(m_1) = \epsilon$, $Pr(m_2) = Pr(m) - \epsilon$ and take $M' = \{m_1\}$). Therefore, M' satisfies $Pr(M') \in (0, \epsilon)$, $\frac{Pr(\Theta' \cap \Theta_q^t(M'))}{Pr(\Theta_q^t(M'))} \geq a$.
Let $\Theta' = \Theta_q^t(M') \cap \Theta_t$ be the set of truthful types in M' that satisfies (C.32) and $\theta < X(m,t)$ for $\theta \in \Theta_q^t(m) \cap \Theta'$. Since $M' \subseteq M_t^a$, we have

$$Pr(\Theta') \geq a Pr(\Theta_q^t(M')) \tag{C.34}$$

Let $\Theta^l = \Theta_q^l(M')$, $\Theta^t = \Theta_q^t(M')$ and $\Theta^u = \Theta^l \cup \Theta^t$ be the aggregate set of truth-tellers, liars and senders of M' ;$E^u = E[\Theta^u]$, $E^t = E[\Theta^t]$, $E^l = E[\Theta^l]$, $E' = E[\Theta']$ be their corresponding expected values, and $z^t = \frac{Pr(\Theta^t)}{Pr(\Theta^u)}$, $z^l = \frac{Pr(\Theta^l)}{Pr(\Theta^u)} = 1 - z^t$, $z' = \frac{Pr(\Theta')}{Pr(\Theta^u)}$ be their corresponding ratios of measure to set of senders Θ^u.Since $\Theta' \subseteq \Theta_t$,

$$E' < \overline{\mu} - \delta \tag{C.35}$$

and

$$E' \leq E^t \tag{C.36}$$

Since the original mechanism is optimal, we have $(X(m,t) - X(m,u))(X(m,u) - X(m,l)) = c$ and $w_q(m) = w^-(X(m,t) - X(m,l)) \leq 0.5$ almost everywhere at M', so $|E^s - X(m,s)| < \epsilon$ for any $m \in M'$ and $s = t, l, u$ imply

$$|(1-z^t)z^t(E^t-E^l)^2 - c| \equiv |(E^t-E^u)(E^u-E^l) - c| < 4\epsilon^2 \tag{C.37}$$

$$z^t \geq 0.5 \tag{C.38}$$

For any $z \in [0,\infty)$, let $E^t(z) = \frac{z^tE^t - z'E' + z\overline{\mu}}{z^t - z' + z}$ be the expected value of the set $(\Theta_q^t(M')/\Theta') \cup \Theta_z$, where Θ_z is a set with expected value $\overline{\mu}$ and measure $zPr(\Theta^u)$, and $E^u(z) = \frac{z^tE^t - z'E' + z\overline{\mu} + (1-z^t)E^l}{1-z'+z}$ be the expected values of $(\Theta_q^u(M')/\Theta') \cup \Theta_z$. Define $\hat{z}$ that solves

$$\frac{(1-z^t)(z^t-z'+z)}{(1-z'+z)^2}(E^t(z)-E^l)^2 \equiv (E^t(z)-E^u(z))(E^u(z)-E^l) = c \tag{C.39}$$

$$z^t + z - z' \geq 1 - z^t \tag{C.40}$$

By definition of $E^t(z)$,

$$\begin{aligned} E^t(z) - E^t =& \frac{z}{z^t}(\overline{\mu} - E') - \frac{z-z'}{z^t}(E^t(z) - E') \\ \geq& \frac{z}{z^t}\delta - \frac{z-z'}{z^t}(E^t(z) - E') \\ \geq& a\delta - \frac{z-z'}{z^t}(E^t(z) - E') \end{aligned} \tag{C.41}$$

where the first inequality holds by (C.35)., the second inequality holds by (C.34). When $z = z'$, $E^t(z) - E^t \geq a\delta$, and $(1-z^t)z^t = \frac{1-z^t)(z^t-z'+z)}{(1-z'+z)^2}$, so for small enough ϵ, $\frac{(1-z^t)(z^t-z'+z)}{(1-z'+z)^2}(E^t(z) - E^l)^2 > (1-z^t)z^t(E^t - E^l + a\delta)^2 > (1-z^t)z^t(E^t-E^l)^2 - 4\epsilon^2 > c$; On the other hand, $\lim_{z\to\infty} \frac{(1-z^t)(z^t-z'+z)}{(1-z'+z)^2}(E^t(z) - E^l)^2 \to 0 < c$. Therefore, for small enough ϵ there exists $\hat{z} \in (z', \infty)$ that is solution to (C.39) and (C.60).

We claim that for small enough ϵ, $E^t(\hat{z}) - E^t \geq \epsilon^{\frac{1}{3}}$. Suppose $E^t(\hat{z}) - E^t < \epsilon^{\frac{1}{3}}$, then (C.41) implies $a\delta - \frac{\hat{z}-z'}{z^t}(E^t(\hat{z}) - E') < \epsilon^{\frac{1}{3}}$, and since $E^t(\hat{z}) - E' \leq 1$, we have $\hat{z} - z' > z^t(a\delta - \epsilon^{\frac{1}{3}})$. Now $(1-z^t)z^t - \frac{(1-z^t)(z^t-z'+\hat{z})}{(1-z'+\hat{z})^2} = \frac{(1-z^t)(\hat{z}-z')[(\hat{z}-z')z^t+2z^t-1]}{(1-z'+\hat{z})^2} > \frac{(1-z^t)z^t}{(1-z'+z)^2}(\hat{z}-z')^2 > \frac{(1-z^t)z^t}{(1+z^t(a\delta-\epsilon^{\frac{1}{3}}))^2}(z^t(a\delta-\epsilon^{\frac{1}{3}}))^2$; and $(E^t(\hat{z}) - E^l)^2 - (E^t - E^l)^2 < (E^t - E^l + \epsilon^{\frac{1}{3}})^2 - (E^t - E^l)^2 = 2\epsilon^{\frac{1}{3}}(E^t - E^l) - \epsilon^{\frac{2}{3}}$. Therefore, for small enough ϵ, $\frac{(1-z^t)(z^t-z'+z)}{(1-z'+z)^2}(E^t(z) - E^l)^2 < [(1-z^t)z^t - \frac{(1-z^t)z^t}{(1+z^t(a\delta-\epsilon^{\frac{1}{3}}))^2}(z^t(a\delta-\epsilon^{\frac{1}{3}}))^2][(E^t-E^l)^2 + 2\epsilon^{\frac{1}{3}}(E^t-E^l) - \epsilon^{\frac{2}{3}}] < (1-z^t)z^t(E^t-E^l)^2 - 4\epsilon^2 < c$, but it contradicts to (C.39), so for small enough ϵ, we have

$$E^t(\hat{z}) - E^t \geq \epsilon^{\frac{1}{3}} \tag{C.42}$$

and by definition of $E^u(z)$ and E^u, we have $E^u(\hat{z}) - E^u = \hat{z}(\overline{\mu} - E') - (\hat{z} - z')(E^u(z) - E') = z^t(E^t(\hat{z}) - E^t) + (\hat{z} - z')(E^t(\hat{z}) - E^u(\hat{z})) > z^t(E^t(\hat{z}) - E^t)$, then (C.38) and (C.42 imply

$$E^u(\hat{z}) - E^u \geq \frac{1}{2}\epsilon^{\frac{1}{3}} \tag{C.43}$$

Define a modified truth-tellers set $\hat{\Theta}^t = (\Theta^t_q(M')/\Theta') \cup \Theta_0(\frac{Pr(\Theta^u)}{Pr(\Theta_0)}\hat{z})$, where $\Theta_0(\frac{Pr(\Theta^u)}{Pr(\Theta_0)}\hat{z})$ is a mean-preserving division of Θ_0 so that $E[\Theta_0(\frac{Pr(\Theta^u)}{Pr(\Theta_0)}\hat{z})] = \overline{\mu}$ and $Pr(\Theta_0(\frac{Pr(\Theta^u)}{Pr(\Theta_0)}\hat{z})) = \hat{z}Pr(\Theta^u)$. Let $\hat{\Theta}^u = \hat{\Theta}^t \cup \Theta^l$ be the modified set of senders. As a results, we have

$$E[\hat{\Theta}^t] = \frac{z^t Pr(\Theta^u)E^t - z' Pr(\Theta^u)E' + \hat{z}Pr(\Theta^u)\overline{\mu}}{(z^t - z' + \hat{z})Pr(\Theta^u)} = E^t(\hat{z}) \tag{C.44}$$

$$Pr(\hat{\Theta}^t) = (z^t + \hat{z} - z')Pr(\Theta^u) = Pr(\Theta^t) + (\hat{z} - z'))Pr(\Theta^u) \tag{C.45}$$

$$E[\hat{\Theta}^u] = E^u(\hat{z}) \tag{C.46}$$

$$Pr(\hat{\Theta}^u) = Pr(\Theta^u) + (\hat{z} - z'))Pr(\Theta^u) \tag{C.47}$$

Define the modified uninspected set $\Theta^0_{\hat{q}} = (\Theta^0_q / \cup \Theta')/\Theta_0(\frac{Pr(\Theta^u)}{Pr(\Theta_0)}\hat{z})$, so that

$$Pr(\Theta^0_{\hat{q}}) = Pr(\Theta^0_q) - (\hat{z} - z'))Pr(\Theta^u) \tag{C.48}$$

and since $\Theta^0_{\hat{q}} \cup \hat{\Theta}^u = \Theta^0_q \cup \Theta^u$, we have

$$Pr(\Theta^0_{\hat{q}})E[\Theta^0_{\hat{q}}] + Pr(\hat{\Theta}^u)E^u(\hat{z}) = Pr(\Theta^0_q)E[\Theta^0_q] + Pr(\Theta^u)E^u \tag{C.49}$$

and thus

$$\begin{aligned} E[\Theta^0_{\hat{q}}] - E[\Theta^0_q] &= \frac{Pr(\Theta^u)}{Pr(\Theta^0_q)}(E^u(\hat{z}) - E^u) + \frac{(\hat{z} - z')Pr(\Theta^u)}{Pr(\Theta^0_q)}(E^u(\hat{z}) - E[\Theta^0_{\hat{q}}]) \\ &\leq \frac{1 + z^{max} - z'}{Pr(\Theta^0_q)}Pr(\Theta^u) \end{aligned} \tag{C.50}$$

where the inequality holds for z^{max} equals the larger root of $\frac{(1-z^t)(z^t - z' + z)}{(1 - z' + z)^2} = c$, so (C.39) implies $\hat{z} \leq z^{max}$.

Now define an modified message and action rules $\hat{q}, \hat{X}$ where other things remain unchanged, except the set of messages M' is off-path; The uninspected message is modified to $m^0_{\hat{q}} = \Theta^0_{\hat{q}}$ with the set of senders identical to the statement, and an message $\hat{m} = \hat{\Theta}^t$ is added with the set of truthful senders $\Theta^t_{\hat{q}}(\hat{m}) = \hat{\Theta}^t$, and the set of lying senders $\Theta^l_{\hat{q}}(\hat{m}) = \Theta^l$.

The sequentially rational actions for the modified message $\hat{m}$ are

$$\hat{X}(\hat{m}, t) = E[\hat{\Theta}^t] = E^t(\hat{z})$$

$$\hat{X}(\hat{m}, l) = E[\Theta^l] = E^l$$

$$\hat{X}(\hat{m}, u) = E[\hat{\Theta}^t \cup \Theta^l) = E^u(\hat{z})$$

The sequentially rational action for the modified uninspected message $m^0_{\hat{q}}$ is

$$\hat{X}(m^0_{\hat{q}}, u) = E[\Theta^0_{\hat{q}}]$$

For small enough ϵ, $\hat{X}(\hat{m}, u) \geq E^t + \epsilon^{\frac{1}{3}} > \inf_{m \in M'} X(m, u) - \epsilon + \epsilon^{\frac{1}{3}} > \inf_{m \in M'} X(m, u)$, and for any unmodified on-path message $m \in \mathcal{M}^+_q$, incentive compatibility of the original mechanism means $\hat{X}(m, u) = X(m, u) > \mu > \hat{X}(m^0_{\hat{q}}, u)$, so

$$\inf_{m \in M^+_{\hat{q}}} \hat{X}(m, u) > \hat{X}(m^0_{\hat{q}}, u) \tag{C.51}$$

Since $\hat{X}(\hat{m}, l) = E^l \leq \sup_{m \in M'} X(m, l)$ and $\sup_{m \in \mathcal{M}^+_q} X(m, l) \leq \mu$ by incentive compatibility of the original mechanism, so (C.50) implies

$$\sup_{m \in M^+_{\hat{q}}} X(m, u) < \hat{X}(m^0_{\hat{q}}, u) + \frac{1 + z^{max} - z'}{Pr(\Theta^0_q)} Pr(\Theta^u) \tag{C.52}$$

Since the original mechanism is optimal, $w_q(m) = w^-(X(m, t) - X(m, l))$ hold almost everywhere for unmodified message $m \in \mathcal{M}^+_q / M'$. For the modified message, (C.39) and (C.60) imply $w_{\hat{q}}(\hat{m}) = w^-(E^t(\hat{z}) - E^l) = w^-(\hat{X}(\hat{m}, t) - \hat{X}(\hat{m}, l))$, so we have

$$w_{\hat{q}}(m) = w^-(\hat{X}(m, t) - \hat{X}(m, l)) \tag{C.53}$$

hold almost everywhere at $\mathcal{M}^+_{\hat{q}}$.

To compare DM's payoffs, since $V_q(m) = V_{\hat{q}}(m) = c$, by Lemma 21, $EU_{DM}(\Omega) = EU^U_{DM}(\Omega)$ and $EU_{DM}(\hat{\Omega}) = EU^U_{DM}(\hat{\Omega})$, so

$$\begin{aligned}
&EU_{DM}(\hat{\Omega}) - EU_{DM}(\Omega) \\
=&Pr(\Theta^0_{\hat{q}})E[\Theta^0_{\hat{q}}]^2 + Pr(\hat{\Theta}^u)E^u(\hat{z})^2 - Pr(\Theta^0_q)E[\Theta^0_q]^2 - Pr(\Theta^u)(E^u)^2 \\
=&- Pr(\Theta^0_q)[(E[\Theta^0_q] - E[\Theta^0_{\hat{q}}])^2 + 2E[\Theta^0_{\hat{q}}](E[\Theta^0_q] - E[\Theta^0_{\hat{q}}])] \\
&+ Pr(\Theta^u)[(E^u(\hat{z}) - E^u)^2 + 2E^u(E^u(\hat{z}) - E^u)] \\
&+ (\hat{z} - z')Pr(\Theta^u)[(E^u(\hat{z}) - E[\Theta^0_{\hat{q}}])^2 + 2E[\Theta^0_{\hat{q}}](E^u(\hat{z}) - E[\Theta^0_{\hat{q}}])] \\
=&- Pr(\Theta^0_q)(E[\Theta^0_q] - E[\Theta^0_{\hat{q}}])^2 + Pr(\Theta^u)(E^u(\hat{z}) - E^u)^2 + (\hat{z} - z')Pr(\Theta^u)(E^t(\hat{z}) - E[\Theta^0_{\hat{q}}])^2 \\
>&- Pr(\Theta^0_q)(\frac{1 + z^{max} - z'}{Pr(\Theta^0_q)} Pr(\Theta^u))^2 + Pr(\Theta^u)\frac{1}{4}\epsilon^{\frac{2}{3}}
\end{aligned} \tag{C.54}$$

Where the second equality holds by (C.47) and (C.48); the third equality holds by (C.49) and $E^u(\hat{z}) > E^u > E[\Theta^0_{\hat{q}}]$; the inequality holds by (C.43) and (C.50) for small enough ϵ.

Finally, given (C.51) - (C.53), Lemma 22 implies that there exists an incentive compatible mechanism $\tilde{\Omega}$ such that $EU_{DM}(\tilde{\Omega}) > EU_{DM}(\hat{\Omega}) - 4(\frac{1+z^{max}-z'}{Pr(\Theta_q^0)}Pr(\Theta^u))^2$, then by (C.54), for small enough ϵ,

$$\begin{aligned}
&EU_{DM}(\tilde{\Omega}) - EU_{DM}(\Omega)\\
>&Pr(\Theta^u)[\frac{1}{4}\epsilon^{\frac{2}{3}} - Pr(\Theta_q^0)(\frac{1+z^{max}-z'}{Pr(\Theta_q^0)})^2 Pr(\Theta^u) - 4(\frac{1+z^{max}-z'}{Pr(\Theta_q^0)})^2 Pr(\Theta^u)]\\
\geq&Pr(\Theta^u)[\frac{1}{4}\epsilon^{\frac{2}{3}} - Pr(\Theta_q^0)(\frac{1+z^{max}-z'}{Pr(\Theta_q^0)})^2\epsilon - 4(\frac{1+z^{max}-z'}{Pr(\Theta_q^0)})^2\epsilon] > 0
\end{aligned}$$

where the second inequality holds because $Pr(\Theta_q^0) = Pr(M') \leq \epsilon$, but it contradicts that Ω is an optimal mechanism. Therefore, we conclude that $\underline{\theta}^t) \geq \overline{\theta}^0$ in an optimal mechanism. *Q.E.D.*

Lemma 27 *In an optimal mechanism Ω, $w_q(m) < 0.5$ almost everywhere for $m \in \mathcal{M}_q^+$.*

Proof of Lemma 27: Suppose Contrary to the claim, there exist a positive measure set of messages $M \subseteq \mathcal{M}_q^+$ such that for any $m \in M$, $w_q(m) = 0.5$, i.e. $X(m,u) = \frac{X(m,t)+X(m,l)}{2}$, then $V_q(m) = (X(m,t) - X(m,u))(X(m,u) - X(m,l)) = c$ implies that $X(m,t) - X(m,u) = X(m,u) - X(m,l) = \sqrt{c}$. By Lemma 20, $X(m,u) > X(m_q^0, u) \equiv \mu$, so $X(m,l) > \mu - \sqrt{c}$. Therefore, there exists $\delta > 0$ and a positive measure set of message $M' \subseteq \mathcal{M}_q^+$ such that for any $m \in M'$,

$$(\mu - X(m,l))^2 < c - \delta \tag{C.55}$$

Now we consider two cases.

Case 1: there is a message $m' \in M'$ such that $Pr(m') \equiv a > 0$.
Let $\Theta^l = \Theta_q^l(m')$, $\Theta^t = \Theta_q^t(m')$ and $\Theta^u = \Theta^l \cup \Theta^t$ be the set of truth-tellers, liars and senders of m' ;$E^u = E[\Theta^u]$, $E^t = E[\Theta^t]$, $E^l = E[\Theta^l]$, be their corresponding expected values. Then we have

$$Pr(\Theta^l) = Pr(\Theta^t) = \frac{a}{2} \tag{C.56}$$

$$E^t - E^u = E^u - E^l = \sqrt{c} \tag{C.57}$$

$$(\mu - E^l)^2 < c - \delta \tag{C.58}$$

For $\epsilon \in (0,1]$ and $s \in \{t,l\}$, let $\Theta_\epsilon^s = \Theta^s/\Theta^s(1-\epsilon)$ the outer ring mean-preserving division of Θ^s so that $E[\Theta_\epsilon^s] = E^s$, $Pr(\Theta_\epsilon^s) = \frac{a}{2}\epsilon$. Since Θ_ϵ^s and $\Theta^s/\Theta_\epsilon^s$ induce the same actions $X(m',s)$ with the same weight of truthful and lying types, we can without loss separate them into two messages m_ϵ and $m_{1-\epsilon}$. Now for $z \in [0,1]$, let $\theta_\epsilon^l(z) = \inf\theta : Pr(\Theta_\epsilon^l \cap [\theta,1]) = zPr(\Theta^l)$

be the $1-z$ percentile type in Θ^l_ϵ. Let $E^l_\epsilon(z) = E[\Theta^l_\epsilon \cap [0, \theta^l_\epsilon(z))]$ be is the expected value for the bottom $1-z$ percentile types in Θ^l_ϵ. Note that $E^l_\epsilon(z) = \frac{\int_0^z \theta^l_\epsilon(z')dz'}{1-z}$, $E^l_\epsilon(0) = E^l$ and $\frac{dE^l_\epsilon(z)}{dz} = E^l_\epsilon(z) - \theta^l_\epsilon(z)$. Now define $\hat{z}_\epsilon$ that solves

$$\frac{1-z}{(2-z)^2}(E^t - E^l_\epsilon(z))^2 = c \tag{C.59}$$

$$z \in (0,1) \tag{C.60}$$

To show that there exists such solution, When $z=1$, $\frac{1-z}{(2-z)^2}(E^t - E^l_\epsilon(z))^2 = 0 < c$; when $z=0$, $\frac{1-z}{(2-z)^2}(E^t - E^l_\epsilon(z))^2 = \frac{1}{4}(E^t - E^l_\epsilon(0))^2 = \frac{1}{4}(E^t - E^l)^2 = c$, where the last equality holds because of (C.57). Since

$$\begin{aligned}
\lim_{z\to 0^+} d[\frac{1-z}{(2-z)^2}(E^t - E^l_\epsilon(z))^2]/dz &= -\frac{1}{4}2(E^t - E^l)\lim_{z\to 0^+}\frac{dE^l_\epsilon(z)}{dz} \\
&= \sqrt{c}(\theta^l_\epsilon(0) - E^l) \\
&= \sqrt{c}(\sup \Theta^l - E^l) > 0,
\end{aligned} \tag{C.61}$$

such $\hat{z}_\epsilon \in (0,1)$ exists for any $\epsilon \in (0,1]$. Now Define a modified inspected liar set $\hat{\Theta}^l = \Theta^l_\epsilon \cap [0, \theta^l_\epsilon(\hat{z}_\epsilon))$ so that

$$Pr(\hat{\Theta}^l) = \frac{a}{2}(1-\hat{z}_\epsilon)\epsilon \tag{C.62}$$

$$E[\hat{\Theta}^l] = E^l_\epsilon(\hat{z}_\epsilon) \tag{C.63}$$

$$\frac{Pr(\hat{\Theta}^l)Pr(\Theta^t_\epsilon)}{(Pr(\hat{\Theta}^l) + Pr(\Theta^t_\epsilon))^2}(E^t - E[\hat{\Theta}^l])^2 = \frac{1-\hat{z}_\epsilon}{(2-\hat{z}_\epsilon)^2}(E^t - E^l_\epsilon(\hat{z}_\epsilon))^2 = c \tag{C.64}$$

Define the modified uninspected set $\Theta^0_{\hat{q}} = \Theta^0_q \cup (\Theta^l_\epsilon \cap [\theta^l_\epsilon(\hat{z}_\epsilon), 1])$, so that

$$Pr(\Theta^0_{\hat{q}}) = Pr(\Theta^0_q) + \frac{a}{2}\hat{z}_\epsilon\epsilon \tag{C.65}$$

$$E[\Theta^0_{\hat{q}}] = \frac{Pr(\Theta^0_q)\mu + \frac{a}{2}\hat{z}_\epsilon\epsilon\overline{E}^l}{Pr(\Theta^0_q) + \frac{a}{2}\hat{z}_\epsilon\epsilon} \tag{C.66}$$

where $\overline{E}^l \equiv E[\Theta^l_\epsilon \cap [\theta^l_\epsilon(\hat{z}_\epsilon), 1]] > E^l$ is the expected value for the top z percentile types in Θ^l_ϵ.

Now define the modified message and action rules $\hat{q}, \hat{X}$ where other things remain unchanged, except the messages m_ϵ is modified to message $\hat{m} = \Theta^t$ with the set of truthful senders $\Theta^t_{\hat{q}}(\hat{m}) = \Theta^t$, and the set of lying senders $\Theta^l_{\hat{q}}(\hat{m}) = \hat{\Theta}^l$; The uninspected message is modified to $m^0_{\hat{q}} = \Theta^0_{\hat{q}}$ with the set of senders identical to the statement.

The sequentially rational actions for the modified message $\hat{m}$ are

$$\begin{aligned}\hat{X}(\hat{m},t) &= E[\Theta^t] = E^t\\ \hat{X}(\hat{m},l) &= E[\hat{\Theta}^l] = E^l_\epsilon(\hat{z}_\epsilon)\\ \hat{X}(\hat{m},u) &= E[\hat{\Theta}^t\cup\Theta^l)] = x^*_u(E^t, E^l_\epsilon(\hat{z}_\epsilon)) > x^*_u(E^t,E^l) = X(m',u)\end{aligned} \tag{C.67}$$

where the second equality of (C.67) holds by (C.64); the inequality holds by Lemma 23 and $E^l_\epsilon(\hat{z}_\epsilon) < E^l$; the last equality holds by optimality of Ω.
The sequentially rational action for the modified uninspected message $m^0_{\hat{q}}$ is

$$\hat{X}(m^0_{\hat{q}},u) = E[\Theta^0_{\hat{q}}] = X(m^0_q,u) - \frac{a\hat{z}_\epsilon}{2Pr(\Theta^0_q)}(\hat{X}(m^0_{\hat{q}},u) - \overline{E}^l)\epsilon \tag{C.68}$$

Since $\Theta^0_{\hat{q}}\cup\hat{\Theta}^l = \Theta^0_q\cup\Theta^l_\epsilon$, we have $Pr(\Theta^0_{\hat{q}}) - Pr(\Theta^0_q) = Pr(\Theta^l_\epsilon) - Pr(\hat{\Theta}^l) = \frac{a}{2}\hat{z}_\epsilon\epsilon$ and

$$Pr(\Theta^0_q)(\mu - E[\Theta^0_{\hat{q}}]) + Pr(\hat{\Theta}^l)(E^l - E^l_\epsilon(\hat{z}_\epsilon)) - \frac{a}{2}\hat{z}_\epsilon\epsilon(E[\Theta^0_{\hat{q}}] - E^l) = 0 \tag{C.69}$$

Since the original mechanism is optimal, $w_q(m) = w^-(X(m,t) - X(m,l))$ hold almost everywhere for unmodified message $m\in\mathcal{M}^+_q/m_\epsilon$. For the modified message, (C.64) implies $w_{\hat{q}}(\hat{m}) = w^-(E^t - E^l(\hat{z}_\epsilon)) = w^-(\hat{X}(\hat{m},t) - \hat{X}(\hat{m},l))$, so we have

$$w_{\hat{q}}(m) = w^-(\hat{X}(m,t) - \hat{X}(m,l)) \tag{C.70}$$

hold almost everywhere at $\mathcal{M}^+_{\hat{q}}$.

To compare DM's payoffs, since $V_q(m) = V_{\hat{q}}(m) = c$, by Lemma 21, $EU_{DM}(\Omega) = EU^I_{DM}(\Omega)$ and $EU_{DM}(\hat{\Omega}) = EU^I_{DM}(\hat{\Omega})$, so

$$\begin{aligned}
&EU_{DM}(\hat{\Omega}) - EU_{DM}(\Omega)\\
=&Pr(\Theta^0_{\hat{q}})E[\Theta^0_{\hat{q}}]^2 + Pr(\hat{\Theta}^l)(E^l_\epsilon(\hat{z}_\epsilon)^2 - c) - Pr(\Theta^0_q)\mu^2 - Pr(\Theta^u)((E^l)^2 - c)\\
=&-Pr(\Theta^0_q)(\mu^2 - E[\Theta^0_{\hat{q}}]^2) - Pr(\hat{\Theta}^l)((E^l)^2 - E^l_\epsilon(\hat{z}_\epsilon)^2) + \frac{a}{2}\hat{z}_\epsilon\epsilon(E[\Theta^0_{\hat{q}}]^2 - (E^l)^2) + \frac{a}{2}\hat{z}_\epsilon\epsilon c\\
=&-Pr(\Theta^0_q)(\mu^2 - E[\Theta^0_{\hat{q}}]^2 - 2E[\Theta^0_{\hat{q}}](\mu - E[\Theta^0_{\hat{q}}])) - Pr(\hat{\Theta}^l)((E^l)^2 - E^l_\epsilon(\hat{z}_\epsilon)^2 - 2E[\Theta^0_{\hat{q}}]((E^l - E^l_\epsilon(\hat{z}_\epsilon)))\\
&+\frac{a}{2}\hat{z}_\epsilon\epsilon(E[\Theta^0_{\hat{q}}]^2 - (E^l)^2 - 2E[\Theta^0_{\hat{q}}](E[\Theta^0_{\hat{q}}] - E^l)) + \frac{a}{2}\hat{z}_\epsilon\epsilon c\\
=&-Pr(\Theta^0_q)(\mu - E[\Theta^0_{\hat{q}}])^2 - Pr(\hat{\Theta}^l)((E^l - E^l_\epsilon(\hat{z}_\epsilon))(E^l + E^l_\epsilon(\hat{z}_\epsilon) - 2E[\Theta^0_{\hat{q}}]))\\
&+\frac{a}{2}\hat{z}_\epsilon\epsilon(c - (E[\Theta^0_{\hat{q}}] - E^l)^2)\\
>&-Pr(\Theta^0_q)(\frac{a\hat{z}_\epsilon}{2Pr(\Theta^0_q)}(\hat{X}(m^0_{\hat{q}},u) - \overline{E}^l)\epsilon)^2\epsilon^2 + \frac{a}{2}\hat{z}_\epsilon\epsilon\delta
\end{aligned} \tag{C.71}$$

where the third equality holds by (C.69); the inequality holds because of (C.58), (C.68) and $E[\Theta^0_{\hat{q}}] \approx \mu > E^l > E^l_\epsilon(\hat{z}_\epsilon)$ for small enough ϵ.

Case 2: $Pr(m) = 0$ **for all** $m \in M'$**:**

If almost every $m, m' \in M'$ induces the same actions $X(m, s) = X(m', s) = \overline{X}^s$, then it is without loss to pool them into a same message with positive measure, and case 1 applies. Now if there exist $\delta' > 0$ and two positive measure subsets of messages $M'_1 \subseteq M'$ and $M'_2 \subseteq M'$ such that for any $(m_1, m_2) \in M'_1 \times M'_2$ and $s \in t, l, u$,

$$X(m_1, s) - X(m_2, s) > \delta' \tag{C.72}$$

then for any $\epsilon > 0$ there exist two positive measure subsets $M''_1 \subseteq M'_1$ and $M''_2 \subseteq M'_2$ such that for any $i = 1, 2$, $m, m' \in M''_i$ and $s \in t, l, u$,

$$|X(m, s) - X(m', s)| < \epsilon \tag{C.73}$$

and $Pr(M''_1) = zPr(M''_2) \leq \epsilon$ for any $z \in (0, \infty)$. For $i = 1, 2$ and $s \in t, l, u$, let $\Theta^s_i = \Theta^s_q(M''_i)$ be the aggregate sets of truthful senders, lying senders and senders of M''_i, and $E^s_i = E[\Theta^s_i]$ be their corresponding expected value. Since $w_q(m) = 0.5$ for any $m \in M_i$, we have

$$Pr(\Theta^l_i) = Pr(\Theta^t_i) \tag{C.74}$$

$$E^t_i - E^u_i = E^u_i - E^l_i = \sqrt{c} \tag{C.75}$$

$$(\mu - E^l_i)^2 < c - \delta \tag{C.76}$$

and thus

$$E^t_1 - E^t_2 = E^l_1 - E^l_2 > \delta' \tag{C.77}$$

For $z \in (0, \infty)$, let $\hat{\Theta}^t(z) = \Theta^t_1 \cup \Theta^t_2$ be the aggregate set of truthful type, and $E^t(z) = E[\hat{\Theta}^t(z)] = \Theta^t_2 + \frac{z}{1+z}(\Theta^t_1 - \Theta^t_2)$. Define $\hat{z}$ that solves

$$\frac{1+z}{(2+z)^2}(E^t(z) - E^l_2)^2 = c \tag{C.78}$$

$$z \in (0, \infty) \tag{C.79}$$

To show that there exists such solution, When $z \to \infty$, $\frac{1+z}{(2+z)^2}(E^t(z) - E^l_2)^2 \to 0 < c$; when $z = 0$, $\frac{1+z}{(2+z)^2}(E^t(z) - E^l_2)^2 = \frac{1}{4}(E^t_2 - E^l_2)^2 = \frac{1}{4}(E^t - E^l)^2 = c$.Since

$$\begin{aligned}\lim_{z \to 0^+} d[\frac{1+z}{(2+z)^2}(E^t(z) - E^l_2)^2]/dz &= -\frac{1}{4}2(E^t_2 - E^l_2) \lim_{z \to 0^+} \frac{dE^t(z)}{dz} \\ &= \sqrt{c}(E^t_1 - E^t_2) \\ &> \sqrt{c}\delta' > 0, \end{aligned} \tag{C.80}$$

such $\hat{z}_\epsilon \in (0,1)$ exists. Now take a set of messages M_1'' such that $Pr(M_1'') = \hat{\Theta}^t Pr(M_2'') \leq \epsilon$. Denote $b = Pr(M_2'')$ and let $\hat{\Theta}^t(\hat{z})$ be the modified inspected truth-teller set so that

$$Pr(\hat{\Theta}^t) = (1+\hat{z})\frac{b}{2} \tag{C.81}$$

$$E[\hat{\Theta}^t] = E^t(\hat{z}) = \Theta_2^t + \frac{\hat{z}}{1+\hat{z}}(\Theta_1^t - \Theta_2^t) \tag{C.82}$$

$$\frac{Pr(\Theta_2^l)Pr(\hat{\Theta}^t)}{(Pr(\Theta_2^l) + Pr(\hat{\Theta}^t)^2}(E[\hat{\Theta}^t] - E_2^l)^2 = \frac{1+\hat{z}}{(2+\hat{z})^2}(E^t(z) - E_2^l)^2 = c \tag{C.83}$$

Define the modified uninspected set $\Theta_{\hat{q}}^0 = \Theta_q^0 \cup \Theta_1^l$, so that

$$Pr(\Theta_{\hat{q}}^0) = Pr(\Theta_q^0) + \hat{z}\frac{Pr(M_2'')}{2} \tag{C.84}$$

$$E[\Theta_{\hat{q}}^0] = \frac{Pr(\Theta_q^0)\mu + \hat{z}\frac{Pr(M_2'')}{2}E_1^l}{Pr(\Theta_q^0) + \hat{z}\frac{b}{2}} \tag{C.85}$$

Now define the modified message and action rules $\hat{q}, \hat{X}$ where other things remain unchanged, except the set of messages $M_1'' \cup M_2''$ is off-path; an inspected message $\hat{m} = \hat{\Theta}^t(\hat{z})$ is added with the set of truthful senders $\Theta_{\hat{q}}^t(\hat{m}) = \hat{\Theta}^t(\hat{z})$, and the set of lying senders $\Theta_{\hat{q}}^l(\hat{m}) = \Theta_2^l$; The uninspected message is modified to $m_{\hat{q}}^0 = \Theta_{\hat{q}}^0$ with the set of senders identical to the statement.

The sequentially rational actions for the modified message $\hat{m}$ are

$$\begin{aligned}
&\hat{X}(\hat{m}, t) = E[\hat{\Theta}^t(\hat{z})] = E^t(\hat{z}) \\
&\hat{X}(\hat{m}, l) = E[\hat{\Theta}^l] = E_2^l \\
&\hat{X}(\hat{m}, u) = E[\hat{\Theta}^t \cup \Theta^l)] = x_u^*(E^t(z), E_2^l > x_u^*(E_2^t, E_2^l) = E_2^u
\end{aligned} \tag{C.86}$$

where the second equality of (C.67) holds by (C.83); the inequality holds by Lemma 23 and $E^t(\hat{z}) > E_2^t$; the last equality holds by optimality of Ω.
The sequentially rational action for the modified uninspected message $m_{\hat{q}}^0$ is

$$\hat{X}(m_{\hat{q}}^0, u) = E[\Theta_{\hat{q}}^0] = X(m_q^0, u) - \frac{\hat{z}b}{2Pr(\Theta_q^0)}(\hat{X}(m_{\hat{q}}^0, u) - E_1^l) \tag{C.87}$$

Since $\Theta_{\hat{q}}^0 = \Theta_q^0 \cup \Theta_1^l$, $\hat{\Theta}^t(\hat{z}) = \Theta_1^t \cup \Theta_2^t$ and $Pr(\Theta_{\hat{q}}^0) - Pr(\Theta_q^0) = Pr(\hat{\Theta}^t) - Pr(\Theta_2^t) = Pr(\Theta_1^t) =$

$Pr(\Theta_1^l) = \frac{\hat{z}}{2}b,$

$$Pr(\Theta_q^0)(\mu - E[\Theta_{\hat{q}}^0]) - \frac{\hat{z}}{2}b(E[\Theta_{\hat{q}}^0] - E_1^l) = 0 \tag{C.88}$$

$$\int_{m\in M_i''}(E_i^s - X(m,s))\int_{\Theta_q^s(m)} dF(\theta)dm = 0 \text{ for } i = 1,2; s = t,l \tag{C.89}$$

$$(1+\hat{z})E^t(\hat{z}) - E_2^t - \hat{z}E_1^t = 0 \tag{C.90}$$

Since the original mechanism is optimal, $w_q(m) = w^-(X(m,t) - X(m,l))$ hold almost everywhere for unmodified message $m \in \mathcal{M}_q^+/m_\epsilon$. For the modified message, (C.83) implies $w_{\hat{q}}(\hat{m}) = w^-(E^t - E^l(\hat{z}_\epsilon)) = w^-(\hat{X}(\hat{m},t) - \hat{X}(\hat{m},l))$, so

$$w_{\hat{q}}(m) = w^-(\hat{X}(m,t) - \hat{X}(m,l)) \tag{C.91}$$

hold almost everywhere at $\mathcal{M}_{\hat{q}}^+$.

To compare DM's payoffs, since $V_q(m) = V_{\hat{q}}(m) = c$, by Lemma 21, $EU_{DM}(\Omega) = EU_{DM}^I(\Omega)$ and $EU_{DM}(\hat{\Omega}) = EU_{DM}^I(\hat{\Omega})$, so

$$\begin{aligned}
&EU_{DM}(\hat{\Omega}) - EU_{DM}(\Omega)\\
=&Pr(\Theta_{\hat{q}}^0)E[\Theta_{\hat{q}}^0]^2 + Pr(\hat{\Theta}^t)(E^t(\hat{z})^2 - c) + Pr(\Theta_2^l)((E_2^l)^2 - c)\\
&- Pr(\Theta_q^0)\mu^2 - \sum_{i=1,2}\sum_{s=t,l}\int_{M_i''}(X(m,s)^2 - c)\int_{\Theta_q^s(m)} dF(\theta)dm\\
=&-Pr(\Theta_q^0)(\mu^2 - E[\Theta_{\hat{q}}^0]^2) - \frac{b}{2}\hat{z}((E_1^l)^2 - E[\Theta_{\hat{q}}^0]^2) - \frac{b}{2}((1+\hat{z})E^t(\hat{z})^2 - (E_2^t)^2 - \hat{z}(E_1^t)^2)\\
&- \sum_{i=1,2}\sum_{s=t,l}\int_{M_i''}(X(m,s)^2 - (E_i^s)^2)\int_{\Theta_q^s(m)} dF(\theta)dm + \frac{b}{2}\hat{z}c\\
=&-Pr(\Theta_q^0)(\mu - E[\Theta_{\hat{q}}^0])^2 - \frac{b}{2}\hat{z}(E_1^l - E[\Theta_{\hat{q}}^0])^2 - \frac{b}{2}(\frac{\hat{z}}{1+\hat{z}}(E_1^t - E_2^t)^2)\\
&- \sum_{i=1,2}\sum_{s=t,l}\int_{M_i''}(X(m,s) - E_i^s)^2\int_{\Theta_q^s(m)} dF(\theta)dm + \frac{b}{2}\hat{z}c\\
\geq&-Pr(\Theta_q^0)(\frac{\hat{z}b}{2Pr(\Theta_q^0)}(\hat{X}(m_{\hat{q}}^0,u) - E_1^l))^2 - b(1+\hat{z})\frac{1}{4}\epsilon^2 + \frac{b}{2}\hat{z}[c - \frac{1}{1+\hat{z}}(E_1^t - E_2^t)^2 - (E[\Theta_{\hat{q}}^0] - E_1^l)^2]\\
>&-Pr(\Theta_q^0)(\frac{\hat{z}}{2Pr(\Theta_q^0)}(\hat{X}(m_{\hat{q}}^0,u) - E_1^l))^2b^2 - b(1+\hat{z})\frac{1}{4}\epsilon^2 + \frac{b}{2}\hat{z}[c - (\mu - E_2^l)^2]\\
>&-Pr(\Theta_q^0)(\frac{\hat{z}}{2Pr(\Theta_q^0)}(\hat{X}(m_{\hat{q}}^0,u) - E_1^l))^2b^2 - b(1+\hat{z})\frac{1}{4}\epsilon^2 + \frac{b}{2}\hat{z}\delta
\end{aligned} \tag{C.92}$$

where the third equality holds by (C.88)-(C.90); the first inequality holds because of (C.87), (C.73) and Popoviciu's inequality; the second inequality holds for small ϵ because $E_1^t - E_2^t = E_1^l - E_2^l$ and $\mu \approx E[\Theta_{\hat{q}}^0] > E_1^l > E_2^l$; the last inequality holds by (C.76).

Finally for Case $j = 1, 2$, (C.68) and (C.87) imply that $X(m^0_q, u) - \hat{X}(m^0_{\hat{q}}, u) = K_j(\epsilon)$ where

$$K_1(\epsilon) = \frac{a\hat{z}_\epsilon}{2Pr(\Theta^0_q)}(\hat{X}(m^0_{\hat{q}}, u) - \overline{E}^l)\epsilon$$

$$K_2(\epsilon) = \frac{\hat{z}}{2Pr(\Theta^0_q)}(\hat{X}(m^0_{\hat{q}}, u) - E^l_1)b \leq \frac{\hat{z}}{2Pr(\Theta^0_q)}(\hat{X}(m^0_{\hat{q}}, u) - E^l_1)\epsilon$$

so Lemma 22 implies that there exists an incentive compatible mechanism $\tilde{\Omega}$ such that $EU_{DM}(\tilde{\Omega}) > EU_{DM}(\hat{\Omega}) - 4K_j(\epsilon)^2$, then by (C.71) and (C.92) $EU_{DM}(\tilde{\Omega}) - EU_{DM}(\Omega) > \frac{a}{2}\hat{z}_\epsilon\epsilon\delta - [4 + Pr(\Theta^0_q)](\frac{a\hat{z}_\epsilon}{2Pr(\Theta^0_q)}(\hat{X}(m^0_{\hat{q}}, u) - \overline{E}^l)\epsilon)^2$ for case 1 and $EU_{DM}(\tilde{\Omega}) - EU_{DM}(\Omega) > \frac{b}{2}\hat{z}\delta - b(1+\hat{z})\frac{1}{4}\frac{1}{4}\epsilon^2 - [4 + Pr(\Theta^0_q)][(\frac{\hat{z}}{2Pr(\Theta^0_q)}(\hat{X}(m^0_{\hat{q}}, u) - E^l_1))]^2 b^2$ for case 2, where $b \leq \epsilon$ goes to 0 as $\epsilon \to 0$, so in both cases $EU_{DM}(\tilde{\Omega}) - EU_{DM}(\Omega) > 0$ for small enough ϵ, but it contradicts that Ω is an optimal mechanism. Therefore, we conclude that $w_q(m) < 0.5$ almost everywhere for $m \in \mathcal{M}^+_q$ in an optimal mechanism. *Q.E.D.*

Lemma 28 *Suppose for an optimal mechanism Ω, $Pr(\Theta^0_q) > 0$ and $\mu - \overline{\theta}^l > \underline{\theta}^t - \mu$, then $\underline{\theta}^t \in (\mu, \mu + \sqrt{c})$ and for almost every $\theta \in (\underline{\theta}^t, \mu + \sqrt{c}) \cap \Theta^t_q(\mathcal{M}^+_q)$, θ is essentially revealed upon inspection in Ω, and $\Theta^t_q(m_q(\theta)) = \{\theta\}$.*

Proof of Lemma 28: First We claim that $\underline{\theta}^t = \overline{\theta}^0 > \mu$, where the inequality holds by definition and $Pr(\Theta^0_q) > 0$. Suppose $\underline{\theta}^t \neq \overline{\theta}^0$, then by Lemma 26 $\underline{\theta}^t > \overline{\theta}^0$, and since $\Theta^t_q(\mathcal{M}^+_q) \cup \Theta^t_q(\mathcal{M}^+_q) \cup \Theta^0_q = [0, 1]$, we have $\underline{\theta}^t = \overline{\theta}^l$, but that contradicts $\mu - \overline{\theta}^l > \underline{\theta}^t - \mu > 0$. Now $\underline{\theta}^t = \overline{\theta}^0$ and $\Theta^t_q(\mathcal{M}^+_q) \cup \Theta^t_q(\mathcal{M}^+_q) \cup \Theta^0_q = [0, 1]$ imply $\overline{\theta}^l \geq \underline{\theta}^0$. Then $\underline{\theta}^t = \overline{\theta}^0$, $\overline{\theta}^l \geq \underline{\theta}^0$, $\mu - \overline{\theta}^l > \underline{\theta}^t - \mu > 0$ and Lemma 25 imply that $\underline{\theta}^t - \mu < \sqrt{c}$.

Now we show that for almost every $\theta \in (\underline{\theta}^t, \mu + \sqrt{c}) \cap \Theta^t_q(\mathcal{M}^+_q)$, θ is essentially revealed upon inspection in Ω. Suppose on the contrary, there exist a positive measure set $\Theta_t \subseteq (\underline{\theta}^t, \mu + \sqrt{c}) \cap \Theta^t_q(\mathcal{M}^+_q)$ such that for any $\theta \in \Theta_t$, $x^d_\Omega(\theta) = X(m_q(\theta), t) \neq \theta$. Let $\Theta'_t = \{\theta \in \Theta_t : X(m_q(\theta), t) < \theta\}$. Since $X(.)$ satisfies (3.5), Θ'_t must have positive measure. For any $m \in m_q(\Theta'_t)$, Lemma 20 implies that $X(m, u) > \mu$, and Lemma 27 implies $w_q(m) < 0.5$, so there exist $\delta > 0$ and a positive measure subset $\Theta''_t \subseteq \Theta'_t$ such that for any $\theta \in \Theta''_t$ and $m \in m_q(\Theta''_t)$,

$$(\theta - \mu)^2 < c - \delta \tag{C.93}$$

$$\theta < x^d_\Omega(\theta) - \delta \tag{C.94}$$

$$X(m, u) > \mu + \delta \tag{C.95}$$

$$w_q(m) < 0.5 - \delta \tag{C.96}$$

Denote $a = \frac{Pr(\Theta_t'')}{Pr(\Theta_q^t(m_q(\Theta_t'')))} > 0$, and let $M_t^a = \cap M \subseteq m_q(\Theta_t'') : \frac{Pr(\Theta_t'' \cap \Theta_q^t(M))}{Pr(\Theta_q^t(M))} \geq a$ be the subset containing every message in $m_q(\Theta_t'')$ where proportion of truthful types within Θ_t'' is no less than a. Since $Pr(M_t^a) > 0$, for any $\epsilon > 0$ there is positive measure subset $M' \subseteq M_t^a$ such that for any $m, m' \in M'$ and $s \in \{t, l, u\}$,

$$|X(m,s) - X(m',s)| < \epsilon \tag{C.97}$$

and $Pr(M') \leq \epsilon$ Denote $b = Pr(M') \equiv Pr(\Theta_q^u(M'))$. Let $\Theta' = \Theta_q^t(M') \cap \Theta_t''$ be the set of truthful types in M' that satisfies (C.93) and(C.94). Since $M' \subseteq M_t^a$, we have

$$Pr(\Theta') \geq aPr(\Theta_q^t(M')) \tag{C.98}$$

Let $\Theta^l = \Theta_q^l(M')$, $\Theta^t = \Theta_q^t(M')$ and $\Theta^u = \Theta^l \cup \Theta^t$ be the aggregate set of truth-tellers, liars and senders of M'; $\Theta_{ex} = \Theta_q^t(M')/\Theta'$ be the set of truth-tellers excluding those in Θ'; $E^u = E[\Theta^u]$, $E^t = E[\Theta^t]$, $E^l = E[\Theta^l]$, $E' = E[\Theta']$ and $E_{ex} = E[\Theta_{ex}]$ be their corresponding expected values, so that

$$E^t = \frac{z'E' + z_{ex}E_{ex}}{z' + z_{ex}} \tag{C.99}$$

and $z^t = \frac{Pr(\Theta^t)}{b}$, $z^l = \frac{Pr(\Theta^l)}{b} = 1 - z^t$, $z' = \frac{Pr(\Theta')}{b}$ and $z_{ex} = \frac{Pr(\Theta_{ex})}{b}$ be their corresponding ratios of measure to set of senders Θ^u. Since $\Theta' \subseteq \Theta_t''$ and $M' \in M_t^a$,

$$(E' - \mu)^2 < c - \delta \tag{C.100}$$

$$E' + \delta < E^t < E_{ex} \tag{C.101}$$

$$z' \geq az^t \tag{C.102}$$

Since the original mechanism is optimal, we have $(X(m,t) - X(m,u))(X(m,u) - X(m,l)) = c$ and $w_q(m) = w^-(X(m,t) - X(m,l)) \leq 0.5$ almost everywhere at M', so $|E^s - X(m,s)| < \epsilon$ for any $m \in M'$ and $s = t, l, u$ and (C.96) imply

$$|(1 - z^t)z^t(E^t - E^l)^2 - c| \equiv |(E^t - E^u)(E^u - E^l) - c| < 4\epsilon^2 \tag{C.103}$$

$$z^t > 0.5 + \delta \tag{C.104}$$

Therefore $z^l = 1 - z^t \in (\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{c+4\epsilon^2}{(E^t - E^l)^2}}, \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{c-4\epsilon^2}{(E^t - E^l)^2}})$, so

$$\frac{z^l}{z^t} = \frac{w^-(E^t - E^l)}{1 - w^-(E^t - E^l)} + k_1(\epsilon)\epsilon^2 = h(E^t - E^l) + k_1(\epsilon)\epsilon^2 \tag{C.105}$$

where $k_1(\epsilon)$ is a bounded function.
For any $r \in [0,1]$, let $E^t(r) = \frac{z^tE^t - rz'E' - \epsilon z_{ex}E_{ex}}{z^t - rz' - \epsilon z_{ex}}$ be the expected value of the set $(\Theta_q^t(M')/(\Theta_r') \cup$

$\Theta_{ex,\epsilon}$), where Θ'_r is a set with expected value E' and measure $rPr(Theta')$, and $\Theta_{ex,\epsilon}$ is a set with expected value E_{ex} and measure $\epsilon Pr(\Theta_{ex})$. Define $\hat{r} \in (0,1)$ that solves

$$h(E_{ex} - E^l)\epsilon z_{ex} + h(E^t(r) - E^l)(z^t - rz' - \epsilon z_{ex}) = z^l \tag{C.106}$$

To show that such solution exists for small enough ϵ, $E^t(1) = E_{ex} > E^t$, and $h(E_{ex} - E^l)\epsilon z_{ex} + h(E^t(r) - E^l)(z^t - rz' - \epsilon z_{ex}) = h(E_{ex} - E^l)(z^t - z') < h(E^t - E^l)z^t \to z^l$, where the first inequality holds as $h(.)$ is a decreasing function, and (C.105) implies $\lim_{\epsilon\to 0} h(E^t - E^l)z^t - z^l = 0$, so when $r = 1$, LHS of (C.106) is smaller than z^l for small ϵ; On the other end, $E^t(0) = E^t - \frac{\epsilon z_{ex}}{z^t - \epsilon z_{ex}}(E_{ex} - E^t) < E^t < E_{ex}$ and $E^t = \frac{z_{ex}}{z^t}\epsilon E^{ex} + (1 - \frac{z_{ex}}{z^t}\epsilon)E^t(0)$, so by second order taylor expansion, $h(E^t - E^l) = h(E^t(0) - E^l) + h'(E^t(0) - E^l)\frac{z_{ex}}{z^t}(E_{ex} - E^t(0))\epsilon + h''(\tilde{E} - E^l)(\frac{z_{ex}}{z^t}(E_{ex} - E^t(0))\epsilon)^2$, where $\tilde{E} \in [E^t(0), E_{ex}]$. Then

$$\begin{aligned}
&(1 - \frac{z_{ex}}{z^t}\epsilon)h(E^t(0) - E^l) + \frac{z_{ex}}{z^t}\epsilon h(E_{ex} - E^l) - h(E^t - E^l) \\
=&h(E^t(0) - E^l) + \frac{z_{ex}}{z^t}\epsilon[h(E_{ex} - E^l) - h(E^t(0) - E^l)] - h(E^t - E^l) \\
=&\frac{z_{ex}}{z^t}\epsilon[h(E_{ex} - E^l) - h(E^t(0) - E^l) - (E_{ex} - E^t(0))h'(E^t(0) - E^l)] \\
&- h''(\tilde{E} - E^l)(\frac{z_{ex}}{z^t}(E_{ex} - E^t(0))\epsilon)^2 \\
\equiv&k_2(\epsilon)\epsilon
\end{aligned} \tag{C.107}$$

where $\lim_{\epsilon\to 0} k_2(\epsilon) > 0$ because $h(.)$ is strictly convex and $E_{ex} > E^t \approx E^t(0)$, so $h(E_{ex} - E^l) - h(E^t(0) - E^l) - (E_{ex} - E^t(0))h'(E^t(0) - E^l)$ is bound away from 0. Therefore, $h(E_{ex} - E^l)\epsilon z_{ex} + h(E^t(r) - E^l)(z^t - rz' - \epsilon z_{ex}) = z^t[(1 - \frac{z_{ex}}{z^t}\epsilon)h(E^t(0) - E^l) + \frac{z_{ex}}{z^t}\epsilon h(E_{ex} - E^l)] = z^t[h(E^t - E^l) + k_2(\epsilon)\epsilon] = z^l + k_1(\epsilon)\epsilon^2 + z^t k_2(\epsilon)\epsilon > z^l$ for small enough ϵ, so when $r = 0$, LHS of (C.106) is larger than z^l for small ϵ, thus there exists $\hat{r} \in (0,1)$ such that (C.106) is satisfied. Furthermore, we have $\lim_{\epsilon\to 0} \hat{r} = 0$, for otherwise $\lim_{\epsilon\to 0} E^t(\hat{r}) > E^t$, and $\lim_{\epsilon\to 0} h(E_{ex} - E^l)\epsilon z_{ex} + h(E^t(\hat{r}) - E^l)(z^t - \hat{r}z' - \epsilon z_{ex}) = \lim_{\epsilon\to 0} h(E^t(\hat{r}) - E^l)(z^t - \hat{r}z') < \lim_{\epsilon\to 0} h(E^t - E^l)z^t = z^l$, contradicting (C.106).

Now let $\Theta_{ex,\epsilon} = \Theta_{ex}(\epsilon)$ the modified upper set of inspected truth-teller, where $\Theta_{ex}(\epsilon)$ is a mean-preserving division of Θ_{ex} so that

$$Pr(\Theta_{ex,\epsilon}) = \epsilon Pr(\Theta_{ex}) = \epsilon z_{ex} b \tag{C.108}$$

$$E[\Theta_{ex,\epsilon}] = E_{ex} \tag{C.109}$$

Let $\Theta^0_{\hat{q}} = \Theta^0_q \cup \Theta'(\hat{r})$ be the modified uninspected set, where $\Theta'(\hat{r})$ is a mean-preserving division of Θ' so that

$$Pr(\Theta'(\hat{r})) = \hat{r}Pr(\Theta') = \hat{r}z'b \tag{C.110}$$

$$E[\Theta'(\hat{r})] = E' \tag{C.111}$$

and thus

$$Pr(\Theta^0_{\hat{q}}) = Pr(\Theta^0_q) + \hat{r}z'b \tag{C.112}$$

$$E[\Theta^0_{\hat{q}}] = \frac{Pr(\Theta^0_q)\mu + \hat{r}z'bE'}{Pr(\Theta^0_q) + \hat{r}z'b} \tag{C.113}$$

Let $\hat{\Theta} = (\Theta'/\Theta'(\hat{r})) \cup (\Theta_{ex}/\Theta_{ex,\epsilon})$ be the modified lower set of inspected truth-teller, so that

$$Pr(\hat{\Theta}) = (1-\epsilon)Pr(\Theta_{ex}) + (1-\hat{r})Pr(\Theta') = [z^t - \epsilon z_{ex} - \hat{r}z')]b \tag{C.114}$$

$$E[\hat{\Theta}] = \frac{z^t E^t - rz'E' - \epsilon z_{ex}E_{ex}}{z^t - rz' - \epsilon z_{ex}} = E^t(r) \tag{C.115}$$

Let $\overline{z}^l = h(E_{ex} - E^l)\epsilon z_{ex}$ be the required share of liars for the modified upper set of inspected truth-teller, and $\underline{z}^l = h(E^t(\hat{r}) - E^l)(z^t - \hat{r}z' - \epsilon z_{ex})$ be the the required share of liars for the modified lower set of inspected truth-teller. (C.106) implies that $\overline{z}^l + \underline{z}^l = z^l$. Let $\overline{\Theta}^l = \Theta^l(\frac{\overline{z}^l}{z^l})$ and $\underline{\Theta}^l = \Theta^l/\overline{\Theta}^l$ be the mean-preserving divisions of Θ^l so that $E[\overline{\Theta}^l] = E[\underline{\Theta}^l] = E^l$, $Pr(\overline{\Theta}^l) = h(E_{ex} - E^l)\epsilon z_{ex}b$ and $Pr(\underline{\Theta}^l) = h(E^t(\hat{r}) - E^l)(z^t - \hat{r}z' - \epsilon z_{ex})b$. By the definition of $h(.)$,

$$\frac{Pr(\overline{\Theta}^l)Pr(\Theta_{ex,\epsilon})}{(Pr(\overline{\Theta}^l) + Pr(\Theta_{ex,\epsilon}))^2}(E_{ex} - E^l)^2 = c \tag{C.116}$$

$$\frac{Pr(\underline{\Theta}^l)Pr(\hat{\Theta})}{(Pr(\underline{\Theta}^l) + Pr(\hat{\Theta}))^2}(E^t(\hat{r}) - E^l)^2 = c \tag{C.117}$$

Now define the modified message and action rules $\hat{q}, \hat{X}$ where other things remain unchanged, except the set of messages M' is off-path; an upper inspected message $m_{ex} = \Theta_{ex,\epsilon}$ is added with the set of truthful senders $\Theta^t_{\hat{q}}(m_{ex}) = \Theta_{ex,\epsilon}$, and the set of lying senders $\Theta^l_{\hat{q}}(m_{ex}) = \overline{\Theta}^l$; an lower inspected message $\hat{m} = \hat{\Theta}$ is added with the set of truthful senders $\Theta^t_{\hat{q}}(\hat{m}) = \hat{\Theta}$, and the set of lying senders $\Theta^l_{\hat{q}}(\hat{m}) = \underline{\Theta}^l$; The uninspected message is modified to $m^0_{\hat{q}} = \Theta^0_{\hat{q}}$ with the set of senders identical to the statement.

The sequentially rational actions for the upper modified message m_{ex} are

$$\begin{aligned} \hat{X}(m_{ex}, t) &= E_{ex} \\ \hat{X}(m_{ex}, l) &= E^l \\ \hat{X}(m_{ex}, u) &= x^*_u(E_{ex}, E^l) > x^*_u(E^t, E^l) > \mu \end{aligned} \tag{C.118}$$

The sequentially rational actions for the lower modified message $\hat{m}$ are

$$\begin{aligned} \hat{X}(\hat{m}, t) &= E^t(\hat{r}) \\ \hat{X}(\hat{m}, l) &= E^l \\ \hat{X}(\hat{m}, u) &= x^*_u(E^t(\hat{r}), E^l_2) \approx x^*_u(E^t, E^l) > \mu \end{aligned} \tag{C.119}$$

where the approximation of (C.119) holds as $\epsilon \to 0$, and the inequality holds by (C.95). The sequentially rational action for the modified uninspected message $m^0_{\hat{q}}$ is

$$\hat{X}(m^0_{\hat{q}}, u) = X(m^0_q, u) + \frac{\hat{r}z'b}{Pr(\Theta^0_q) + \hat{r}z'b}(E' - X(m^0_q, u)) \tag{C.120}$$

Since the original mechanism is optimal, $w_q(m) = w^-(X(m,t) - X(m,l))$ hold almost everywhere for unmodified message $m \in \mathcal{M}^+_q / m_\epsilon$. For the modified message, (C.116) and (C.117) imply $w_{\hat{q}}(m_{ex}) == w^-(E_{ex} - E^l)$ and $w_{\hat{q}}(\hat{m}) = w^-(E^t(\hat{r}) - E^l)$, so

$$w_{\hat{q}}(m) = w^-(\hat{X}(m,t) - \hat{X}(m,l)) \tag{C.121}$$

hold almost everywhere at $\mathcal{M}^+_{\hat{q}}$.

To compare DM's payoffs, since $V_q(m) = V_{\hat{q}}(m) = c$, by Lemma 21, $EU_{DM}(\Omega) = EU^I_{DM}(\Omega)$ and $EU_{DM}(\hat{\Omega}) = EU^I_{DM}(\hat{\Omega})$, so

$$\begin{aligned}
&EU_{DM}(\hat{\Omega}) - EU_{DM}(\Omega)\\
=&(Pr(\Theta^0_q) + \hat{r}z'b)E[\Theta^0_{\hat{q}}]^2 + \epsilon z_{ex}b(E^2_{ex} - c) + (z^t - \epsilon z_{ex} - \hat{r}z')b(E^t(\hat{r})^2 - c) + z^l b((E^l)^2 - c)\\
&- Pr(\Theta^0_q)\mu^2 - \sum_{s=t,l}\int_{M'}(X(m,s)^2 - c)\int_{\Theta^s_q(m)} dF(\theta)dm\\
=&-[Pr(\Theta^0_q)\mu^2 + \hat{r}z'b(E')^2 - (Pr(\Theta^0_q) + \hat{r}z'b)E[\Theta^0_{\hat{q}}]^2] + \hat{r}z'bc\\
&+ b[\epsilon z_{ex}E^2_{ex} + \hat{r}z'(E')^2 + (z^t - \epsilon z_{ex} - \hat{r}z')E^t(\hat{r})^2 - z^t(E^t)^2]\\
&- \sum_{s=t,l}\int_{M'}(X(m,s)^2 - (E^s)^2)\int_{\Theta^s_q(m)} dF(\theta)dm\\
=&-Pr(\Theta^0_q)(E[\Theta^0_{\hat{q}}] - \mu)^2 + \hat{r}z'b[c - (E' - E[\Theta^0_{\hat{q}}])^2]\\
&+ b[\epsilon z_{ex}(E_{ex} - E^t)^2 + \hat{r}z'(E^t - E')^2 + (z^t - \epsilon z_{ex} - \hat{r}z')(E^t - E^t(\hat{r}))^2]\\
&- \sum_{s=t,l}\int_{M'}(X(m,s) - (E^s))^2\int_{\Theta^s_q(m)} dF(\theta)dm\\
>&-Pr(\Theta^0_q)(\frac{\hat{r}z'b}{Pr(\Theta^0_q) + \hat{r}z'b}(E' - X(m^0_q, u)))^2 + \hat{r}z'b\delta + \frac{1}{4z_{ex}}a^2\delta^2 b\epsilon - \frac{1}{4}b\epsilon^2
\end{aligned} \tag{C.122}$$

where the third equality holds by sequential rational actions ; the inequality holds because of (C.120), $(E' - E[\Theta^0_{\hat{q}}])^2 < (E' - \mu)^2 < c - \delta$, $b\epsilon z_{ex}(E_{ex} - E^t)^2 = b\epsilon z_{ex}[\frac{z'}{z_{ex}}(E^t - E')]^2 > b\epsilon z_{ex}[\frac{az^t}{z_{ex}}\delta]^2 > b\epsilon z_{ex}[\frac{0.5a}{z_{ex}}\delta]^2$, (C.73) and Popoviciu's inequality.
(C.120) implies that $\hat{X}(m^0_{\hat{q}}, u) - X(m^0_q, u) = \frac{z'}{Pr(\Theta^0_q) + \hat{r}z'b}(E' - X(m^0_q, u))\hat{r}b$, so Lemma 22 implies that there exists an incentive compatible mechanism $\tilde{\Omega}$ such that $EU_{DM}(\tilde{\Omega}) > EU_{DM}(\hat{\Omega}) -$

$4(\frac{z'}{Pr(\Theta_q^0)+\hat{r}z'b}(E'-X(m_q^0,u))\hat{r}b)^2$, then by (C.122) $EU_{DM}(\tilde{\Omega})-EU_{DM}(\Omega) > \hat{r}z'b\delta+\frac{1}{4z_{ex}}a^2\delta^2b\epsilon-\frac{1}{4}b\epsilon^2-[4+Pr(\Theta_q^0)](\frac{z'}{Pr(\Theta_q^0)+\hat{r}z'b}(E'-X(m_q^0,u))\hat{r}b)^2$, where $b \leq \epsilon$ goes to 0 as $\epsilon \to 0$, so $EU_{DM}(\tilde{\Omega})-EU_{DM}(\Omega) > 0$ for small enough ϵ, but it contradicts that Ω is an optimal mechanism. Therefore, we have shown that for almost every $\theta \in (\underline{\theta}^t, \mu+\sqrt{c}) \cap \Theta_q^t(\mathcal{M}_q^+)$, θ is essentially revealed upon inspection in an optimal mechanism Ω.

Finally, suppose there is a positive measure set $\Theta_t \in (\underline{\theta}^t, \mu+\sqrt{c}) \cap \Theta_q^t(\mathcal{M}_q^+)$, such that for $\theta \in \Theta_t$, $x_\Omega^d(\theta) = \theta$ but $\Theta_q^t(m_q(\theta)) \neq \{\theta\}$, then there is a positive measure set of truthful types $\Theta' = \Theta_q^t(m_q(\Theta_t))/\Theta_t$ who pool with Θ_t but not in Θ'. Then since $x_\Omega^d(\theta) = \theta$ for $\theta \in \Theta_t$, sequential rationality of X means there exists positive measure set $\Theta'_- = \{\theta \in \Theta' : \theta < X_\Omega^d(\theta)\}$, but it means that for $\theta \in \Theta'_-$, $\theta < X_\Omega^d(\theta) < \mu+\sqrt{c}$ is not essentially revealed upon inspection, contradicts to the first statement. *Q.E.D.*

Lemma 29 *Suppose there is a mechanism Ω such that $Pr(\Theta_q^0) > 0$, $\underline{\theta_t} \in (\mu, \mu+\sqrt{c})$, X is sequentially rational given q, and there is a message $\hat{m} \in \mathcal{M}_q^+$ such that conditions (a) and (b) of Lemma 20 and $w_q(m) = w^-(X(m,t)-X(m,l))$ are satisfied almost everywhere for $m \in \mathcal{M}_q^+/\hat{m}$ and Properties stated in Lemma 28 are satisfied for $\Theta_q^t(\mathcal{M}_q^+\hat{m})$, except that $Pr(\hat{m}) > 0$, $X(\hat{m},t) > \inf_{m\in\mathcal{M}+_q} X(m,t)$, $X_u^*(X(\hat{m},t), X(\hat{m},l)) > \mu$ and $h_q(\hat{m}) \equiv \frac{Pr(\Theta_q^l(\hat{m}))}{Pr(\Theta_q^t(\hat{m}))} < h(X(\hat{m},t)-X(\hat{m},l))$.*
Let $\hat{b} = (1-\frac{h_q(\hat{m})}{h(X(\hat{m},t)-X(\hat{m},l))})Pr(\Theta_q^t(\hat{m}))$. Then for any small enough $\hat{b}$ there exists an incentive compatible mechanism $\hat{\Omega}$ such that $EU_{DM}^I(\hat{\Omega}) > EU_{DM}^I(\Omega) + k(\hat{b})\hat{b}$, where $\lim_{\hat{b}\to 0} k(\hat{b}) > 0$.

Proof of Lemma 29: Let $z^t = \frac{Pr(\Theta_q^t(\hat{m}))}{Pr(\hat{m})}$ and $z^l = \frac{Pr(\Theta_q^l(\hat{m}))}{Pr(\hat{m})}$, be the share of truthful types and lying types in $\hat{m}$, $z_r^t = \frac{z^l}{h(X(\hat{m},t)-X(\hat{m},l))}$ and $z_e^t = z^t - z_r^t$ be the required share of truthful types and excess share of truthful types. We have

$$z_e^t = z^t(1-\frac{h_q(\hat{m})}{h(X(\hat{m},t)-X(\hat{m},l))}) > 0 \tag{C.123}$$

Let $\Theta^l = \Theta_q^l(\hat{m})$ be the set of liars in $\hat{m}$, $\Theta_r^t = \Theta_q^t(\hat{m})(\frac{z_r^t}{z^t})$ and $\Theta_e^t = \Theta_q^t(\hat{m})/\Theta_q^t(\hat{m})(\frac{z_r^t}{z^t})$ be the mean-preserving divisions of $\Theta_q^t(\hat{m})$ so that

$$E[\Theta_r^t] = E[\Theta_e^t] = E[\Theta_q^t(\hat{m})] = X(\hat{m},t) \tag{C.124}$$

$$Pr(\Theta_r^t) = \frac{z_r^t}{z^t}Pr(\Theta_q^t(\hat{m})) = \frac{Pr(\Theta_q^l(\hat{m}))}{h(X(\hat{m},t)-X(\hat{m},l))} \tag{C.125}$$

$$Pr(\Theta_e^t) = \frac{z_e^t}{z^t}Pr(\Theta_q^t(\hat{m})) = (1-\frac{h_q(\hat{m})}{h(X(\hat{m},t)-X(\hat{m},l))})Pr(\Theta_q^t(\hat{m})) = \hat{b} \tag{C.126}$$

Let $\underline{\Theta}_t = \{\theta \in \Theta^t_q(\mathcal{M}^+_q/\hat{m}) : \theta < X(\hat{m},t)\}$. $\underline{\Theta}_t$ has positive measure because $X(\hat{m},t) > \inf_{m\in\mathcal{M}+_q} X(m,t)$. Then by Lemma 28, there exists $\delta > 0$ and a positive measure subset $\underline{\Theta}'_t \subseteq \underline{\Theta}_t$ such that for any $\theta \in \underline{\Theta}'_t$,

$$(\theta-\mu)^2 < c-\delta \tag{C.127}$$

$$x^d_\Omega(\theta) = \theta \tag{C.128}$$

$$\Theta^t_q(m_q(\theta)) = \{\theta\} \tag{C.129}$$

Let $M_t = m_q(\underline{\Theta}'_t)$ be the set of messages sent by truthful types $\underline{\Theta}'_t$. (C.129) and sequential rationality of X imply that for any $m \in M_t$, $\Theta^t_q(m) = \{X(m,t)\}$. By definition of M_t, for any $m \in M_t$, $X(\hat{m},t) > X(m,t)$, and since $h(X(m,t)-X(m,l))$ is well-defined in Ω, $h(X(\hat{m},t)-X(m,l))$ is also well-defined with $h(X(\hat{m},t)-X(m,l)) < h(X(m,t)-X(m,l))$. Since $Pr(M_t) > 0$, for small enough $\hat{b}$, we have $\int_{m\in M_t} \frac{1}{h(X(\hat{m},t)-X(m,l))}\int_{\Theta^l_q(m)} dF(\theta)dm \geq \hat{b}$, and because M_t is a collection of zero measure messages, so there exists $M'_t \subseteq M_t$ such that

$$\int_{m\in M'_t} \frac{1}{h(X(\hat{m},t)-X(m,l))}\int_{\Theta^l_q(m)} dF(\theta)dm = \hat{b} \tag{C.130}$$

Assign an arbitrary strict ranking $r : M'_t \to \mathbb{R}$ to the message set M'_t. Then for any $m \in M'_t$, let

$$z(m) = \frac{1}{\hat{b}}\int_{m'\in M'_t : r(m')\leq r(m)} \frac{1}{h(X(\hat{m},t)-X(m',l))}\int_{\Theta^t_q(m')} dF(\theta)dm' \tag{C.131}$$

be the cumulative required share of truthful types in Θ^t_e to pair with the liars in M'_t .

Define an modified messaging and action rules $\hat{q}, \hat{X}$ where other things remain unchanged, except the message $\hat{(m)}$ is modified to $T(\hat{m}$, the set of truthful senders $\Theta^t_{\hat{q}}(T(\hat{m})) = \Theta^t_r$ and the set of lying senders $\Theta^l_{\hat{q}}(T(\hat{m})) = \Theta^l$; for each $m \in M'_t$, m is modified to $T(m)$ with the set of truthful senders $\Theta^t_{\hat{q}}(T(m)) = \Theta^t_e(z(m))\ int(\Theta^t_e(z(m)))$ and the set of lying senders $\Theta^l_{\hat{q}}(T(m)) = \Theta^l_q(m)$, where $\Theta^t_e(z(m))\ int(\Theta^t_e(z(m)))$ is the boundary set of a mean preserving division of $\Theta^t_e(z(m))$ so that $E[\Theta^t_{\hat{q}}(T(m))] = E[\Theta^t_e] = X(\hat{m},t)$ and the set has measure $\frac{1}{h(X(\hat{m},t)-X(m,l))}\int_{\Theta^t_q(m)} dF(\theta)$; The uninspected message is modified to $m^0_{\hat{q}}$ with the set of senders $\Theta^0_q \cup \Theta^t_q(M'_t)$.

The sequentially rational actions for $T(\hat{m})$ are

$$\begin{aligned}\hat{X}(T(\hat{m}),t) &= X(\hat{m},t)\\ \hat{X}(T(\hat{m}),l) &= X(\hat{m},l)\\ \hat{X}(T(\hat{m}),u) &= x^*_u(X(\hat{m},t),X(\hat{m},l)) > \mu\end{aligned} \tag{C.132}$$

For each $m \in M'_t$, The sequentially rational actions for $T(m)$ are

$$\hat{X}(T(m),t) = X(\hat{m},t)$$
$$\hat{X}(T(m),l) = X(m,l)$$
$$\hat{X}(T(m),u) = x_u^*(X(\hat{m},t),X(m,l)) > x_u^*(X(m,t),X(m,l)) > \mu \tag{C.133}$$

where the first inequality of (C.162) holds because $X(\hat{m},t) > X(m,t)$. The sequentially rational action for the modified uninspected message $m_{\hat{q}}^0$ is

$$\hat{X}(m_{\hat{q}}^0,u) = X(m_q^0,u) + \frac{1}{Pr(\Theta_q^0) + Pr(\Theta_q^t(M'_t))}\int_{M'_t}(X(m,t) - X(m_q^0,u))\int_{\Theta_q^t(m)} dF(\theta)dm$$
$$< X(m_q^0,u) + \frac{Pr(\Theta_q^t(M'_t))}{Pr(\Theta_q^0) + Pr(\Theta_q^t(M'_t))} \tag{C.134}$$

where the inequality holds because $X(m,t) - X(m_q^0,u) \in (0,1)$. In the original mechanism, $w_q(m) = w^-(X(m,t) - X(m,l))$ hold almost everywhere for unmodified message $m \in \mathcal{M}_q^+/(M'_t \cup \hat{m})$. For the modified message, (C.125) and (C.131) imply $w_{\hat{q}}(T(\hat{m})) == w^-(X(\hat{m},t) - X(\hat{m},l))$ and $w_{\hat{q}}(T(m)) = w^-(X(\hat{m},t) - X(m,t))$, so

$$w_{\hat{q}}(m) = w^-(\hat{X}(m,t) - \hat{X}(m,l)) \tag{C.135}$$

hold almost everywhere at $\mathcal{M}_{\hat{q}}^+$. $w_q(m) = w^-(X(m,t)-X(m,l))$ almost everywhere for $m \in M'_t$ implies that $Pr(\Theta_q^t(M'_t)) = \int_{m\in M'_t}\frac{1}{h(X(m,t)-X(m,l))}\int_{\Theta_q^l(m)} dF(\theta)dm$, combined with (C.130) means

$$Pr(\Theta_q^t(M'_t)) = \frac{\int_{m\in M'_t}\frac{1}{h(X(m,t)-X(m,l))}\int_{\Theta_q^l(m)} dF(\theta)dm}{\int_{m\in M'_t}\frac{1}{h(X(\hat{m},t)-X(m,l))}\int_{\Theta_q^l(m)} dF(\theta)dm}\hat{b} \in (h(1)\hat{b},\hat{b}) \tag{C.136}$$

where the upper bound holds because $X(\hat{m},t) > X(m,t)$ and $h(.)$ is a decreasing function, the lower bound holds because $w_q(m) > 0.5$ means $h(X(m,t) - X(m,l)) < 1$ and $h(X(\hat{m},t) - X(m,l)) > h(1)$.

To compare DM's payoffs,

$$EU_{DM}^I(\hat{\Omega}) - EU_{DM}^I(\Omega)$$
$$=(Pr(\Theta_q^0) + Pr(\Theta_q^t(M'_t)))E[\Theta_{\hat{q}}^0]^2$$
$$- Pr(\Theta_q^0)\mu^2 - \int_{M'_t}(X(m,t)^2 - c)\int_{\Theta_q^t(m)} dF(\theta)dm$$
$$= - Pr(\Theta_q^0)(E[\Theta_{\hat{q}}^0] - \mu)^2 + \int_{M'_t} c - (X(m,t) - E[\Theta_{\hat{q}}^0])^2\int_{\Theta_q^t(m)} dF(\theta)dm$$
$$> - Pr(\Theta_q^0)(\frac{Pr(\Theta_q^t(M'_t))}{Pr(\Theta_q^0) + Pr(\Theta_q^t(M'_t))})^2 + Pr(\Theta_q^t(M'_t))\delta \tag{C.137}$$

where the second equality holds by sequential rational actions ; the inequality holds because of (C.134), (C.127) and (C.128). Since $V_q(m) = V_{\hat{q}}(m) = c$, by Lemma 21, $EU_{DM}(\hat{\Omega}) = EU^U_{DM}(\hat{\Omega}) = EU^I_{DM}(\hat{\Omega})$, and (C.134) implies that $\hat{X}(m^0_{\hat{q}}, u) - X(m^0_q, u) < \frac{Pr(\Theta^t_q(M'_t))}{Pr(\Theta^0_q)+Pr(\Theta^t_q(M'_t))}$, so Lemma 22 implies that there exists an incentive compatible mechanism $\tilde{\Omega}$ such that $EU^I_{DM}(\tilde{\Omega}) > EU^I_{DM}(\hat{\Omega}) - 4(\frac{Pr(\Theta^t_q(M'_t))}{Pr(\Theta^0_q)+Pr(\Theta^t_q(M'_t))})^2$, then by (C.163) $EU^I_{DM}(\tilde{\Omega}) - EU^I_{DM}(\Omega) > Pr(\Theta^t_q(M'_t))\delta - [4 + Pr(\Theta^0_q)](\frac{Pr(\Theta^t_q(M'_t))}{Pr(\Theta^0_q)+Pr(\Theta^t_q(M'_t))})^2$, and by (C.136), $EU^I_{DM}(\tilde{\Omega}) - EU^I_{DM}(\Omega) = k(\hat{b})\hat{b}$, where $\lim_{\hat{b}\to 0} k(\hat{b}) > h(1)\delta > 0$. *Q.E.D.*

Lemma 30 *Suppose for an optimal mechanism Ω, $Pr(\Theta^0_q) > 0$ and $\mu - \overline{\theta}^l > \underline{\theta}^t - \mu$, then for almost every $\theta \in \Theta^u_q(\mathcal{M}^+_q)$, θ is essentially revealed upon inspection in Ω, and for almost every $m \in \mathcal{M}^+_q$ and $s = t, l$, $\Theta^s_q(m) = \{X(m,s)\}$.*

Proof of Lemma 30: We will show that for almost every lying types $\theta \in \Theta^l_q(\mathcal{M}^+_q)$, θ is essentially revealed upon inspection in Ω, the proof for truthful types is symmetrical and omitted.

Suppose on the contrary, there exist a positive measure set $\Theta_l \subseteq \Theta^l_q(\mathcal{M}^+_q)$ such that for any $\theta \in \Theta_l$, $x^d_\Omega(\theta) = X(m_q(\theta), l) \neq \theta$. Let $\Theta'_l = \{\theta \in \Theta_t : X(m_q(\theta), l) > \theta\}$. Since $X(.)$ satisfies (3.5), Θ'_l must have positive measure. For any $m \in m_q(\Theta'_l)$,Lemma 27 implies $w_q(m) < 0.5$, so there exist $\delta > 0$ and a positive measure subset $\Theta''_l \subseteq \Theta'_t$ such that for any $\theta \in \Theta''_l$ and $m \in m_q(\Theta''_l)$,

$$X(m,t) > \inf_{m' \in \mathcal{M}^+_q} X(m', t) \tag{C.138}$$

$$\theta < x^d_\Omega(\theta) - \delta \tag{C.139}$$

$$X(m,u) > \mu + \delta \tag{C.140}$$

$$w_q(m) < 0.5 - \delta \tag{C.141}$$

Denote $a = \frac{Pr(\Theta''_l)}{Pr(\Theta^l_q(m_q(\Theta''_l)))} > 0$, and let $M^a_l = \cap M \subseteq m_q(\Theta''_l) : \frac{Pr(\Theta''_l \cap \Theta^l_q(M))}{Pr(\Theta^l_q(M))} \geq a$ be the subset containing every message in $m_q(\Theta''_t)$ where proportion of truthful types within Θ''_l is no less than a. Since $Pr(M^a_l) > 0$, for any $\epsilon > 0$ there is positive measure subset $M' \subseteq M^a_l$ such that for any $m, m' \in M'$ and $s \in \{t, l, u\}$,

$$|X(m,s) - X(m',s)| < \epsilon \tag{C.142}$$

and $Pr(M') \leq \epsilon$ Denote $b = Pr(M') \equiv Pr(\Theta^u_q(M'))$. Let $\Theta' = \Theta^l_q(M') \cap \Theta''_l$ be the set of lying types in M' that satisfies (C.139). Since $M' \subseteq M^a_l$, we have

$$Pr(\Theta') \geq aPr(\Theta^l_q(M')) \tag{C.143}$$

Let $\Theta^l = \Theta^l_q(M')$, $\Theta^t = \Theta^t_q(M')$ and $\Theta^u = \Theta^l \cup \Theta^t$ be the aggregate set of truth-tellers, liars and senders of M'; $\Theta_{ex} = \Theta^l_q(M')/\Theta'$ be the set of liars excluding those in Θ'; $E^u = E[\Theta^u]$, $E^t = E[\Theta^t]$, $E^l = E[\Theta^l]$, $E' = E[\Theta']$ and $E_{ex} = E[\Theta_{ex}]$ be their corresponding expected values, so that

$$E^l = \frac{z'E' + z_{ex}E_{ex}}{z' + z_{ex}} \tag{C.144}$$

and $z^t = \frac{Pr(\Theta^t)}{b}$, $z^l = \frac{Pr(\Theta^l)}{b} = 1 - z^t$, $z' = \frac{Pr(\Theta')}{b}$ and $z_{ex} = \frac{Pr(\Theta_{ex})}{b}$ be their corresponding ratios of measure to set of senders Θ^u. Since $\Theta' \subseteq \Theta''_l$ and $M' \in M^a_l$,

$$E^t > \inf_{m' \in \mathcal{M}^+_q} X(m', t) \tag{C.145}$$

$$E' + \delta > E^l > E_{ex} \tag{C.146}$$

$$z' \geq az^l \tag{C.147}$$

Since the original mechanism is optimal, we have $(X(m,t) - X(m,u))(X(m,u) - X(m,l)) = c$ and $w_q(m) = w^-(X(m,t) - X(m,l)) \leq 0.5$ almost everywhere at M', so $|E^s - X(m,s)| < \epsilon$ for any $m \in M'$ and $s = t, l, u$ and (C.141) imply

$$|(1 - z^t)z^t(E^t - E^l)^2 - c| \equiv |(E^t - E^u)(E^u - E^l) - c| < 4\epsilon^2 \tag{C.148}$$

$$z^t > 0.5 + \delta \tag{C.149}$$

Therefore $z^l = 1 - z^t \in (\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{c+4\epsilon^2}{(E^t-E^l)^2}}, \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{c-4\epsilon^2}{(E^t-E^l)^2}})$, so

$$\frac{z^t}{z^l} = \frac{1 - w^-(E^t - E^l)}{w^-(E^t - E^l)} + k_1(\epsilon)\epsilon^2 = J(E^t - E^l) + k_1(\epsilon)\epsilon^2 \tag{C.150}$$

where $J(.) = \frac{1}{h(.)}$ is the required truth-teller to liar ratio, and $k_1(\epsilon)$ is a bounded function.

For any $\epsilon \in [0, 1]$, let $E^l(\epsilon) = \frac{z^l tE^l - \epsilon z_{ex}E_{ex}}{z^l - \epsilon z_{ex}}$ be the expected value of the set $(\Theta^l_q(M')/\Theta_{ex,\epsilon})$, where $\Theta_{ex,\epsilon}$ is a set with expected value E_{ex} and measure $\epsilon Pr(\Theta_{ex})$.
Since $E^l = \frac{z_{ex}}{z^l}\epsilon E^{ex} + (1 - \frac{z_{ex}}{z^l}\epsilon)E^l(\epsilon)$, by second order taylor expansion, $J(E^t - E^l) = J(E^t - E^l(\epsilon)) + J'(E^t - E^l(\epsilon))\frac{z_{ex}}{z^l}(E^l(\epsilon) - E_{ex})\epsilon + J''(E^t - \tilde{E})(\frac{z_{ex}}{z^l}(E^l(\epsilon) - E_{ex})\epsilon)^2$, where $\tilde{E} \in [E_{ex}, E^l(\epsilon)]$. Then

$$\begin{aligned}
&(1 - \frac{z_{ex}}{z^l}\epsilon)J(E^t - E^l(\epsilon)) + \frac{z_{ex}}{z^t}\epsilon J(E^t - E_{ex}) - J(E^t - E^l) \\
=&J(E^t - E^l(\epsilon)) + \frac{z_{ex}}{z^l}\epsilon[J(E^t - E_{ex}) - J(E^t - E^l(\epsilon))] - J(E^t - E^l) \\
=&\frac{z_{ex}}{z^t}\epsilon[J(E^t - E_{ex}) - J(E^t - E^l(\epsilon)) - (E^l(\epsilon) - E_{ex})J'(E^t - E^l(\epsilon))] \\
&- J''(E^t - \tilde{E})(\frac{z_{ex}}{z^t}(E_{ex}) - E^l(\epsilon))\epsilon)^2 \\
\equiv& - k_2(\epsilon)\epsilon
\end{aligned} \tag{C.151}$$

where $\lim_{\epsilon\to 0} k_2(\epsilon) > 0$ because $J(.)$ is strictly concave and $E_{ex} < E^l \approx E^l(\epsilon))$, so $J(E^t - E_{ex}) - J(E^t - E^l(\epsilon)) - (E_{ex}) - E^l(\epsilon))J'(E^t - E^l(\epsilon))$ is negative and bound away from 0. Therefore, (C.150) and (C.151) imply

$$\begin{aligned}(z^l - z_{ex}\epsilon)J(E^t - E^l(\epsilon)) &= z^l J(E^t - E^l) - z_{ex}\epsilon J(E^t - E_{ex}) - z^l k_2(\epsilon)\epsilon \\ &= z^t - z_{ex}\epsilon J(E^t - E_{ex}) - z^l\epsilon[k_2(\epsilon) - k_1(\epsilon)\epsilon] \end{aligned} \tag{C.152}$$

Now let $\Theta^l_{ex,\epsilon} = \Theta_{ex}(\epsilon)$ be the set of liars for the modified lower message, where $\Theta_{ex}(\epsilon)$ is a mean-preserving division of Θ_{ex} so that

$$Pr(\Theta^l_{ex,\epsilon}) = \epsilon Pr(\Theta_{ex}) = \epsilon z_{ex} b \tag{C.153}$$

$$E[\Theta^l_{ex,\epsilon}] = E_{ex} \tag{C.154}$$

Let $z^t_{ex} = J(E^t - E_{ex})\epsilon z_{ex}$ be the required share of truth-tellers for the modified lower message, and $\Theta^t_{ex,\epsilon} = \Theta^t(\frac{z^t_{ex}}{z^t})$ be the set of truth-tellers for the modified lower message, where $\Theta^t(\frac{z^t_{ex}}{z^t})$ is a mean-preserving division of Θ^t so that

$$Pr(\Theta^t_{ex,\epsilon}) = \frac{z^t_{ex}}{z^t} Pr(\Theta^t) = J(E^t - E_{ex})\epsilon z_{ex} b \tag{C.155}$$

$$E[\Theta^t_{ex,\epsilon}] = E^t \tag{C.156}$$

Let $\hat{\Theta}^l = (\Theta^l/(\Theta_{ex}/\Theta_{ex,\epsilon})$ be the set of liars for the modified upper message, so that

$$Pr(\hat{\Theta}^l) = (1-\epsilon)Pr(\Theta_{ex}) + Pr(\Theta') = [z^l - \epsilon z_{ex})]b \tag{C.157}$$

$$E[\hat{\Theta}] = \frac{z^l E^l - \epsilon z_{ex} E_{ex}}{z^l - \epsilon z_{ex}} = E^l(\epsilon) \tag{C.158}$$

Let $\hat{\Theta}^t = \Theta^t/\Theta^t_{ex,\epsilon}$ be the set of truth-tellers for the modified upper message, which is the residual from $\Theta^t_{ex,\epsilon}$ so that

$$Pr(\hat{\Theta}^t) = (z^t - J(E^t - E_{ex})\epsilon z_{ex})b \tag{C.159}$$

$$E[\Theta^t_{ex,\epsilon}] = E^t \tag{C.160}$$

Now define the modified message and action rules $\hat{q}, \hat{X}$ where other things remain unchanged, except the set of messages M' is off-path; an lower inspected message m_{ex} is added with the set of truthful senders $\Theta^t_{\hat{q}}(m_{ex}) = \Theta^t_{ex,\epsilon}$, and the set of lying senders $\Theta^l_{\hat{q}}(m_{ex}) = \Theta^l_{ex,\epsilon}$; an upper inspected message $\hat{m}$ is added with the set of truthful senders $\Theta^t_{\hat{q}}(\hat{m}) = \hat{\Theta}^t$, and the set of lying senders $\Theta^l_{\hat{q}}(\hat{m}) = \hat{\Theta}^l$.

The sequentially rational actions for the message m_{ex} are

$$\begin{aligned}\hat{X}(m_{ex},t) &= E^t \\ \hat{X}(m_{ex},l) &= E_{ex} \\ \hat{X}(m_{ex},u) &= x_u^*(E^t,E_{ex}) > x_u^*(E^t,E^l) > \mu \end{aligned} \tag{C.161}$$

The sequentially rational actions for the message $\hat{m}$ are

$$\begin{aligned}\hat{X}(\hat{m},t) &= E^t \\ \hat{X}(\hat{m},l) &= E^l(\epsilon) \\ x_u^*(E^t,E^l(\epsilon)) &\approx x_u^*(E^t,E^l) > \mu \end{aligned} \tag{C.162}$$

where the approximation of (C.162) holds as $\epsilon \to 0$, and the inequality holds by (C.140).

To compare DM's payoffs,

$$\begin{aligned}& EU^I_{DM}(\hat{\Omega}) - EU^I_{DM}(\Omega) \\ =& \epsilon z_{ex} b(E_{ex}^2 - c) + (z^l - \epsilon z_{ex}) b(E^l(\epsilon)^2 - c) + z^t b((E^t)^2 - c) \\ & - \sum_{s=t,l} \int_{M'} X(m,s)^2 - c \int_{\Theta_q^s(m)} dF(\theta) dm \\ =& \epsilon z_{ex} b(E_{ex} - E^l)^2 + (z^l - \epsilon z_{ex}) b(E^l - E^l(\epsilon))^2 \\ & - \sum_{s=t,l} \int_{M'} (X(m,s) - E^s)^2 \int_{\Theta_q^s(m)} dF(\theta) dm \\ >& \epsilon z_{ex} b(E_{ex} - E^l)^2 + (z^l - \epsilon z_{ex}) b(E^l - E^l(\epsilon))^2 - \frac{1}{4}\epsilon^2 b \end{aligned} \tag{C.163}$$

where the second equality holds by sequential rational actions ; the inequality holds because of (C.142) and Popoviciu's inequality. By definition of z_{ex}^t and $J(.)$, $w_{\hat{q}}(m_{ex}) = w^-(E^t - E_{ex})$. Now For $\hat{m}$,

$$\begin{aligned} h_{\hat{q}}(\hat{m}) = \frac{Pr(\hat{\Theta}^l)}{Pr(\hat{\Theta}^t)} &= \frac{z^l - z_{ex}\epsilon}{z^t - J(E^t - E_{ex}) z_{ex}\epsilon} \\ &= \frac{z^l - z_{ex}\epsilon}{(z^l - z_{ex}\epsilon) J(E^t - E^l(\epsilon)) + z^l \epsilon [k_2(\epsilon) - k_1(\epsilon)\epsilon]} \end{aligned} \tag{C.164}$$

and since $h(E^t - E^l(\epsilon)) - \frac{1}{J(E^t - E^l(\epsilon))}$,

$$\begin{aligned} \hat{b} \equiv (1 - \frac{h_{\hat{q}}(\hat{m}))}{h(E^t - E^l(\epsilon))}) Pr(\hat{\Theta}^t) &= h_q(\hat{m}) \frac{z^l}{z^l - z_{ex}\epsilon} (z^t - J(E^t - E^l(\epsilon)) z_{ex}\epsilon) b (k_2(\epsilon) - k_1(\epsilon)\epsilon)\epsilon \\ &\approx h(E^t - E^l) z^t k_2(\epsilon) \epsilon b \end{aligned} \tag{C.165}$$

for small ϵ. Since $\lim_{\epsilon\to 0}\hat{b}=0$, so Lemma 29, (C.145) and (C.162) imply that for small enough ϵ there exists an incentive compatible mechanism $\tilde{\Omega}$ such that $EU^I_{DM}(\tilde{\Omega}) > EU^I_{DM}(\hat{\Omega}) > k_{\hat{b}}\hat{b} > 0$, and thus for $EU_{DM}(\tilde{\Omega}) - EU_{DM}(\Omega) > 0$ for small enough ϵ, but it contradicts that Ω is an optimal mechanism. Therefore, we have shown that for almost every $\theta \in \Theta^u_q(\mathcal{M}^+_q)$, θ is essentially revealed upon inspection in an optimal mechanism Ω.

Finally, suppose there is a positive measure set $M' \subseteq \mathcal{M}^+_q)$ and $s = t, l$ such that for $m \in M'$, $\Theta^s_q(m) \neq \{X(m,s)\}$, then there is a positive measure set Θ' such that $m_q(\theta) \neq \theta$, which contradicts that θ is essentially revealed upon inspection. *Q.E.D.*

Lemma 30 implies in an optimal mechanism where $Pr(\Theta^0_q) > 0$ and $\mu - \overline{\theta}^l > \underline{\theta}^t - \mu$, every inspected type is separated upon inspection, so there exists an matching function from the set of truth-tellers to the set of liars $\phi_q : \Theta^t_q(\mathcal{M}^+_q) \to \Theta^l_q(\mathcal{M}^+_q)$ such that for $m_q(\theta) = m_q(\theta')$ if and only if $\phi_q(\theta) = \theta'$.

Lemma 31 *Suppose for an optimal mechanism Ω, $Pr(\Theta^0_q) > 0$ and $\mu - \overline{\theta}^l > \underline{\theta}^t - \mu$, then $(\underline{\theta}^t - \mu)(\mu - \overline{\theta}^l) \geq c$ and for any $m_1, m_2 \in \mathcal{M}^+_q$, $X(m_1,t) > X(m_2,t)$ if and only if $X(m_1,l) < X(m_2,l)$.*

Proof of Lemma 31: We consider two cases that contrary to the claim.

Case 1: $(\underline{\theta}^t - \mu)(\mu - \overline{\theta}^l) \geq c$ and there exists positive measure sets $M_1, M_2 \subseteq \mathcal{M}^+_q$ such that $M_1 \cap M_2 = \emptyset$ and for all $(m_1,m_2) \in (M_1,M_2)$, $X(m_1,t) \geq X(m_2,t)$ and $X(m_1,l) \geq X(m_2,l)$:

By Lemma 30 every inspected types is separating, then for $m_1 \neq m_2$ and $s = t, l$, $X(m_1,s) \neq X(m_2,s)$, so it must be the case that for any $(m_1,m_2) \in (M_1,M_2)$ $X(m_1,t) > X(m_2,t)$ and $X(m_1,l) > X(m_2,l)$. Therefore, there exist $\delta > 0$ such that for any $\epsilon > 0$ there are positive measure subsets $M'_1 \subseteq M_1$ and $M'_2 \subseteq M_2$ such that for any $m_i \in M'_i$, $i = 1,2$ and $s = t, l$,

$$X(m_1,s) > X(m_2,s) > \delta \text{ for } (m_1,m_2) \in (M_1,M_2) \tag{C.166}$$

$$|X(m,s) - X(m',s)| < \epsilon \text{ for } m, m' \in M_i, i = 1,2 \tag{C.167}$$

$$X(m_2,t) > \inf_{m' \in \mathcal{M}^+_q} X(m',t) \tag{C.168}$$

and $\frac{1}{r}Pr(M'_2) = Pr(M'_1) \leq \epsilon$ for any $r \in (0,1)$. Denote $b = Pr(M'_1)$. For $i \in 1,2$ and $s \in t,l,u$, let $\Theta^s_i = \Theta^s_q(M'_i)$ be the aggregate sets of truthful senders, lying senders and senders of M''_i, and $E^s_i = E[\Theta^s_i]$ be their corresponding expected value and $z^s_i = \frac{Pr(\Theta^s_i)}{Pr(\Theta^u_i)}$ be their corresponding ratios of measure to set of senders Θ^u_i.

Since the original mechanism is optimal, we have $(X(m,t) - X(m,u))(X(m,u) - X(m,l)) = c$ and $w_q(m) = w^-(X(m,t) - X(m,l)) \leq 0.5$ almost everywhere at M'_i, so $|E^s - X(m,s)| < \epsilon$ for any $m \in M'$ and $s = t, l$ and implies

$$\frac{z_i^l}{z_i^t} = \frac{w^-(E_i^t - E_i^l)}{1 - w^-(E_i^t - E_i^l)} + k_1(\epsilon)\epsilon^2 = h(E_i^t - E_i^l) + k_i(\epsilon)\epsilon^2 \tag{C.169}$$

where $k_i(\epsilon)$ is a bounded function. Define $\hat{r} = \frac{z_1^t}{z_2^l} h(E_1^t - E_2^l)$, and take the sets M_1, M_2 such that $Pr(M_2') = \hat{r} Pr(M_1')$, so that

$$\frac{Pr(\Theta_2^l)}{Pr(\Theta_1^t)} = \frac{\hat{r} z_2^l}{z_1^t} = h(E_1^t - E_2^l) \tag{C.170}$$

Now define the modified message and action rules $\hat{q}, \hat{X}$ where other things remain unchanged, except the set of messages $M_1' \cup M_2'$ is off-path; an inspected message m_{ex} is added with the set of truthful senders $\Theta_{\hat{q}}^t(m_{ex}) = \Theta_1^t$, and the set of lying senders $\Theta_{\hat{q}}^l(m_{ex}) = \Theta_2'$; an inspected message $\hat{m}$ is added with the set of truthful senders $\Theta_{\hat{q}}^t(\hat{m}) = \Theta_2^t$, and the set of lying senders $\Theta_{\hat{q}}^l(\hat{m}) = \Theta_1^t$.

The sequentially rational actions for the lower modified message m_{ex} are

$$\begin{aligned} \hat{X}(m_{ex}, t) &= E_1^t \\ \hat{X}(m_{ex}, l) &= E_2^l \\ \hat{X}(m_{ex}, u) &= x_u^*(E_1^t, E_2^l) > x_u^*(E_1^t, E_1^l) > \mu \end{aligned} \tag{C.171}$$

The sequentially rational actions for the upper modified message $\hat{m}$ are

$$\begin{aligned} \hat{X}(\hat{m}, t) &= E_2^t \\ \hat{X}(\hat{m}, l) &= E_1^l \\ x_u^*(E^t, E^l(\epsilon)) &= x_u^*(E_2^t, E_1^l) > \mu \end{aligned} \tag{C.172}$$

where the inequality holds because $(E_2^t - \mu)(\mu - E_1^l) > (\underline{\theta}^t - \mu)(\mu - \overline{\theta}^l) \geq c$.

To compare DM's payoffs,

$$\begin{aligned} & EU_{DM}^I(\hat{\Omega}) - EU_{DM}^I(\Omega) \\ =& b \sum_{s=t,l} [(z_1^s)^2 - c] + \hat{r} b \sum_{s=t,l} [(z_2^s)^2 - c] \\ & - \sum_{i=1,2} \sum_{s=t,l} \int_{M_i'} [X(m,s)^2 - c] \int_{\Theta_q^s(m)} dF(\theta) dm \\ =& - \sum_{i=1,2} \sum_{s=t,l} \int_{M'} (X(m,s) - E_i^s)^2 \int_{\Theta_q^s(m)} dF(\theta) dm \\ >& - (1 + \hat{r}) \frac{1}{4} \epsilon^2 b \end{aligned} \tag{C.173}$$

where the second equality holds by sequential rational actions ; the inequality holds because of (C.167) and Popoviciu's inequality. By (C.170), $w_{\hat{q}}(m_{ex}) = w^-(E_1^t - E_2^l)$. Now For $\hat{m}$,

$$\begin{aligned} h_{\hat{q}}(\hat{m}) = \frac{Pr(\Theta_1^l)}{Pr(\Theta_2^t)} &= \frac{z_1^l}{\hat{r} z_2^t} \\ &= \frac{z_1^l z_2^l}{z_1^t z_2^t} \frac{1}{h(E_1^t - E_2^l)} \\ &= \frac{h(E_1^t - E_1^l) h(E_2^t - E_2^l)}{h(E_1^t - E_2^l)} + g(\epsilon)\epsilon^2 \end{aligned} \tag{C.174}$$

where the second equality holds by definition of $\hat{r}$, and the last equality holds by (C.169), where $g(\epsilon)$ is a bounded function. Since $h(.)$ is a strictly convex function and $E_1^t - E_2^l - \delta > \max_i E_i^t - E_i^l \geq \min_i E_i^t - E_i^l > E_2^t - E_1^l + \delta$,

$$\begin{aligned} \hat{b} &\equiv (1 - \frac{h_{\hat{q}}(\hat{m})}{h(E_2^t - E_1^l)}) Pr(\Theta_2^t) \\ &\approx (1 - \frac{h(E_1^t - E_1^l) h(E_2^t - E_2^l)}{h(E_1^t - E_2^l) h(E_2^t - E_1^l)}) \hat{r} z_2^t b \end{aligned} \tag{C.175}$$

for small ϵ. Since $\lim_{\epsilon \to 0} b = 0$, so Lemma 29, (C.168) and (C.172) imply that for small enough ϵ there exists an incentive compatible mechanism $\tilde{\Omega}$ such that $EU_{DM}^I(\tilde{\Omega}) > EU_{DM}^I(\hat{\Omega}) > k_{\hat{b}}\hat{b} > 0$, and thus for $EU_{DM}(\tilde{\Omega}) - EU_{DM}(\Omega) > 0$ for small enough ϵ, but it contradicts that Ω is an optimal mechanism.

Case 2: $(\underline{\theta}^t - \mu)(\mu - \overline{\theta}^l) < c$:

Then there exists positive measure $\Theta^l \subseteq \Theta_q^l(\mathcal{M}_q^+)$ such that for any $\theta \in \Theta_l$, $(\underline{\theta}^t - \mu)(\mu - \theta) < c$. Incentive compatibility implies $(\phi^{-1}(\theta) - \mu)(\mu - \theta) > c$, where $\phi^{-1}(\theta)$ is the truth-teller who match with θ, then there exists $\delta_1 > 0$ and positive measure subset $\Theta_i^l \subseteq \Theta^l$ such that for all $\theta \in \Theta_i^l$, $(\phi^{-1}(\theta) - \delta_1 - \mu)(\mu - \theta) > c$. Now for each $\theta \in \Theta_i^l$, define $R(\theta)$ be such that $(R(\theta) - \mu)(\mu - \theta) = c$, So we have

$$\underline{\theta}^t < R(\theta) < \phi^{-1}(\theta) - \delta_1 \tag{C.176}$$

By Lemma 26, $R(\theta) > \underline{\theta}^t \geq \overline{\theta}^0$, so $R(\theta) \in \Theta_q^t(\mathcal{M}_q^+)$. Let $\Theta_{ii}^l = \{\theta \in \Theta_i^l : R(\theta) \notin \phi^{-1}(\Theta_l')\}$ to remove any liars type with $R(\theta)$ duplicate with $\phi^{-1}(\theta')$ for other types in Θ_i^l. Second inequality of (C.176) implies that Θ_{ii}^l has positive surplus. Incentive compatibility implies $(R(\theta) - \mu)(\mu - \phi(R(\theta))) > c$, so there exists $\delta_2 > 0$ and positive measure subset $\Theta_{iii}^l \subseteq \Theta_{ii}^l$ such that for all $\theta \in \Theta_{iii}^l$, $(R(\theta) - \mu)(\mu - \phi(R(\theta)) + \delta_2) > c$, So we have

$$\phi(R(\theta)) < \theta - \delta_2 \tag{C.177}$$

let $\delta = min\{\delta_1, \delta_2\}$, and pick a positive measure subset $\Theta^l_{iv} \subseteq \Theta^l_{iii}$ such that for any $\theta, \theta' \in \Theta^l_{iv}$, $\max\{|\theta - \theta'|, |R(\theta) - R(\theta')|, |\phi^{-1}(\Theta_l) - \phi^{-1}(\Theta'_l)|, |\phi(R(\theta)) - \phi(R(\theta'))|\} < \epsilon\frac{\delta}{4}$, so that by (C.176) and (C.177), for any $\theta, \theta' \in \Theta^l_{iv}$,

$$R(\theta') < \phi^{-1}(\theta) - \frac{\delta}{2} \tag{C.178}$$

$$\phi(R(\theta')) < \theta - \frac{\delta}{2} \tag{C.179}$$

Now let $M_1 = m_q(\Theta^l_{iv})$ and $M_2 = m_q(R(\Theta^l_{iv}))$, so for every $(m_1, m_2) \in (M_1, M_2)$,

$$X(m_2, t) < X(m_1, t) - \frac{\delta}{2} \tag{C.180}$$

$$X(m_2, l) < X(m_1, l) - \frac{\delta}{2} \tag{C.181}$$

and for $\theta \in \Theta^l_q(M_1)$ there exists $\theta' \in \Theta^t_q(M_2)$ such that

$$(\theta' - \mu)(\mu - \theta) = c \tag{C.182}$$

Now for $i = 1, 2$, $s = t, l$, let $\Theta^s_i = \Theta^s_q(M_i)$ be the truthful and lying sets, and for $\epsilon \in (0, 1)$, $\theta^s_i(\epsilon) = \{\theta \in \Theta^s_i : Pr(\Theta^s_i \cap [0, \theta]) = \epsilon Pr(\Theta^s_i)\}$ be the $\epsilon - th$ percentile of Θ^s_i. Define $M_1(\epsilon) = m_q(\Theta^l_1 \cap [0, \theta^l_1(\epsilon^2)])$ be the subset of messages of M_1 where the liars are on the bottom $\epsilon^2 - th$ percentile, and $M_2(\epsilon) = m_q(\Theta^t_2 \cap [0, \theta^l_1(1 - \epsilon^2)])$ be the subset of messages of M_2 where the truth-tellers are on the top $\epsilon^2 - th$ percentile, so that for small enough ϵ,

$$|X(m, s) - X(m', s)| < \epsilon \text{ for } m, m' \in M_i(\epsilon), i = 1, 2 \tag{C.183}$$

$$X(m_2, t) > \inf_{m' \in \mathcal{M}^+_q} X(m', t) \tag{C.184}$$

$$(E^t_2(\epsilon) - \mu)(\mu - E^l_1(\epsilon)) > c \tag{C.185}$$

where $E^s_i(\epsilon) = E[\Theta^s_q(M_i(\epsilon))]$, which implies $X *_u (E^t_2(\epsilon), E^l_1(\epsilon)) > \mu$. Then we can derive a contradiction using the same argument as Case 1. *Q.E.D.*

Lemma 32 *Suppose for an optimal mechanism* Ω*,* $Pr(\Theta^0_q) > 0$ *and* $\underline{\theta}^t - \mu \geq \mu - \overline{\theta}^l$*, then* $\overline{\theta}^l \leq \underline{\theta}^0$*.*

Proof of Lemma 32: Suppose contrary to the claim, $\overline{\theta}^l > \underline{\theta}^0$, then it must be the case that $\mu - \overline{\theta}^l \leq \sqrt{c}$, for otherwise Lemma 26 and $\Theta^t_q(\mathcal{M}^+_q) \cup \Theta^t_q(\mathcal{M}^+_q) \cup \Theta^0_q = [0, 1]$ imply that $\underline{\theta}^t = \overline{\theta}^0$, thus $(\overline{\theta}^0 - \mu)(\mu - \underline{\theta}^0) = (\underline{\theta}^t - \mu)(\mu - \underline{\theta}^0) > (\underline{\theta}^t - \mu)(\mu - \overline{\theta}^l) > c$, which contradicts Lemma 25.

Now since $\overline{\theta}^l > \underline{\theta}^0$, there exist $\delta > 0$, a positive measure subset of uninspected types $\Theta_0 \subseteq \Theta_q^0$ and a positive set of lying types $\Theta_l \subseteq \Theta_q^l(\mathcal{M}_q^+)$ such that for any $\theta \in \Theta_l$, $\underline{\mu} \equiv E[\Theta_0]$ and $m \in m_q(\Theta_l)$

$$\theta - \underline{\mu} > \delta \tag{C.186}$$

$$\theta \geq x_\Omega^d(\theta_l) \tag{C.187}$$

$$X(m,u) > \mu + \delta \tag{C.188}$$

$$X(m,l) < \mu - \delta \tag{C.189}$$

$$w_q(m) < 0.5 - \delta \tag{C.190}$$

Let $M_l = m_q(\Theta_l)$ be the set of messages sent by those lying types, $a = \frac{Pr(\Theta_l)}{Pr(\Theta_q^l(M_l))} > 0$, and $M_l^a = \cap M \subseteq M_l : \frac{Pr(\Theta_l \cap \Theta_q^l(M))}{Pr(\Theta_q^l(M))} \geq a$ be the subset containing every message in M_l where proportion of lying types within Θ_l is no less than a. Since $Pr(M_t^a) > 0$, for any $\epsilon > 0$ there is positive measure subset $M' \subseteq M_t^a$ such that for any $m, m' \in M'$ and $s \in \{t, l, u\}$ and $\theta \in \Theta_l \cap \Theta_q^l(M_l^a)$,

$$|X(m,s) - X(m',s)| < \epsilon \tag{C.191}$$

$$\theta > \mu - \sqrt{c} - \epsilon \tag{C.192}$$

and $Pr(M') \equiv b \leq \epsilon$.
For $s \in \{t, l, u\}$, let $\Theta^s = \Theta_q^s(M')$, $E^s[\Theta^s]$ and $z^s = \frac{Pr(\Theta^s)}{b}$ be the aggregate set of truth-tellers, liars and senders of M', their corresponding expected values and ratios of measure to set of senders Θ^u. Let $\Theta' = \Theta_q^l(M') \cap \Theta_l$ be the set of lying types in M' that satisfies (C.186) and (C.187), and $\Theta_{ex} = \Theta^l/\Theta'$ be the set of lying types who send M' but not in Θ'; $E' = E[\Theta']$, $E_{ex} = E[\Theta_{ex}]$, $z' = \frac{Pr(\Theta')}{b}$ and $z_{ex} = \frac{Pr(\Theta_{ex})}{b}$ be their corresponding expected values and ratios of measure to set of senders Θ^u. By (C.186), (C.187), (C.192) and $M' \subseteq M_l^a$,

$$E' > \underline{\mu} + \delta \tag{C.193}$$

$$E' \geq E^l \geq E_{ex} \tag{C.194}$$

$$E' \geq \mu - \sqrt{c} - \epsilon \tag{C.195}$$

$$z' \geq a z^l \tag{C.196}$$

Since the original mechanism is optimal, we have $(X(m,t) - X(m,u))(X(m,u) - X(m,l)) = c$ and $w_q(m) = w^-(X(m,t) - X(m,l)) \leq 0.5$ almost everywhere at M', so $|E^s - X(m,s)| < \epsilon$ for any $m \in M'$ and $s = t, l, u$ imply

$$\frac{z^l}{z^t} = \frac{w^-(E^t - E^l)}{1 - w^-(E^t - E^l)} + k_1(\epsilon)\epsilon^2 = h(E^t - E^l) + k_1(\epsilon)\epsilon^2 \tag{C.197}$$

$$\frac{z^t}{z^l} = \frac{1 - w^-(E^t - E^l)}{w^-(E^t - E^l)} + k_1(\epsilon)\epsilon^2 = J(E^t - E^l) + k_2(\epsilon)\epsilon^2 \tag{C.198}$$

where $k_1(\epsilon)$ and $k_2(\epsilon)$ are bounded functions.

Now we consider two cases.

Cases 1: $\lim_{\epsilon\to 0} E' - E^l = 0$**:**

Let $\hat{r} = h(E^t - \underline{\mu})\frac{z^t}{z^l}$. Since $lim_{\epsilon\to 0}E^l - \underline{\mu} = lim_{\epsilon\to 0}E' - \underline{\mu} > \delta > 0$ and $h'(.) < 0$, (C.197) imply that

$$lim_{\epsilon\to 0}\hat{r} = lim_{\epsilon\to 0}\frac{h(E^t - \underline{\mu})}{h(E^t - E^l)} \in (0,1) \tag{C.199}$$

Let $\hat{\Theta}^l = \Theta_0(\hat{r}z^l b\frac{1}{Pr(\Theta_0)})$ be the modified set of liars, where $\Theta_0(\hat{r}z^l b\frac{1}{Pr(\Theta_0)})$ is a mean-preserving division of Θ_0 so that

$$Pr(\hat{\Theta}^l) = \hat{r}z^l b = \hat{r}Pr(\Theta^l) \tag{C.200}$$

$$E[\hat{\Theta}^l] = \underline{\mu} \tag{C.201}$$

Let $\Theta^0_{\hat{q}} = (\Theta^0_q/\hat{\Theta}^l) \cup \Theta^l$ be the modified uninspected set.

Now define the modified message ans action rules $\hat{q}, \hat{X}$ where other things remain unchanged, except the set of messages M' is off-path; an inspected message $\hat{m}$ is added with the set of truthful senders $\Theta^t_{\hat{q}}(\hat{m}) = \Theta^t$ and the set of lying senders $\Theta^l_{\hat{q}}(\hat{m}) = \hat{\Theta}^l$; The uninspected message is modified to $m^0_{\hat{q}}$ with the set of senders $\Theta^0_{\hat{q}}$.

The sequentially rational actions for $\hat{m}$ are

$$\begin{aligned}
&\hat{X}(\hat{m},t) = E^t \\
&\hat{X}(\hat{m},l) = \underline{\mu} \\
&\hat{X}(\hat{m},t) = x^*_u(E^t,\underline{\mu}) > x^*_u(E^t,E^l) > \mu
\end{aligned} \tag{C.202}$$

where the first inequality holds for small enough ϵ because $\lim_{\epsilon\to 0} E^l - \underline{\mu} > 0$, and the second inequality holds by optimality of Ω.

The sequentially rational action for the modified uninspected message $m^0_{\hat{q}}$ is

$$\begin{aligned}
\hat{X}(m^0_{\hat{q}},u) &= \mu + \frac{\hat{r}z^l b}{Pr(\Theta^0_q) + (1-\hat{r})z^l b}(\mu - \underline{\mu}) - \frac{z^l b}{Pr(\Theta^0_q) + (1-\hat{r})z^l b}(\mu - E^l) \\
&= \mu + k_3(\epsilon)b
\end{aligned} \tag{C.203}$$

where $k_3(\epsilon)$ is a bounded function.

By (C.200) and the definition of $\hat{r}$, $Pr(\hat{\Theta}^l) = h(E^t - \underline{\mu})Pr(\Theta^t)$, so $w_{\hat{q}}(\hat{m}) = w^-(E^t - \underline{\mu}) = w^-(\hat{X}(\hat{m},t) - \hat{X}(\hat{m},l))$, so

$$w_{\hat{q}}(m) = w^-(\hat{X}(m,t) - \hat{X}(m,l)) \tag{C.204}$$

hold almost everywhere at $\mathcal{M}_{\hat{q}}^{+}$.

To compare DM's payoffs, since $V_q(m) = V_{\hat{q}}(m) = c$, by Lemma 21, $EU_{DM}(\Omega) = EU_{DM}^{I}(\Omega)$ and $EU_{DM}(\hat{\Omega}) = EU_{DM}^{I}(\hat{\Omega})$, so

$$\begin{aligned}
&EU_{DM}(\hat{\Omega}) - EU_{DM}(\Omega)\\
=&(Pr(\Theta_q^0) + (1-\hat{r})z^l b)E[\Theta_{\hat{q}}^0]^2 + \hat{r}z^l b[\underline{\mu}^2 - c] + z^t b[(E^t)^2 - c]\\
&- Pr(\Theta_q^0)\mu^2 - \sum_{s=t,l}\int_{M'}(X(m,s)^2 - c)\int_{\Theta_q^s(m)} dF(\theta)dm\\
=&- Pr(\Theta_q^0)(\mu - E[\Theta_{\hat{q}}^0])^2 + \hat{r}z^l b[(E[\Theta_{\hat{q}}^0] - \underline{\mu})^2 - c] - z^l b[(E[\Theta_{\hat{q}}^0] - E^l)^2 - c]\\
&- \sum_{s=t,l}\int_{M'}(X(m,s) - E^s)^2\int_{\Theta_q^s(m)} dF(\theta)dm\\
>&- Pr(\Theta_q^0)k_3(\epsilon)^2 b^2 - (1-\hat{r})z^l b[(E[\Theta_{\hat{q}}^0] - E^l)^2 - c]\\
&+ \hat{r}z^l b[(E[\Theta_{\hat{q}}^0] - \underline{\mu})^2 - (E[\Theta_{\hat{q}}^0] - E^l)^2] - \frac{1}{4}\epsilon^2 b\\
>&- Pr(\Theta_q^0)k_3(\epsilon)^2 b^2 - (1-\hat{r})z^l b[(E[\Theta_{\hat{q}}^0] - E^l)^2 - c]\\
&+ \hat{r}z^l b(E^l - \underline{\mu})^2 - \frac{1}{4}\epsilon^2 b\\
>&\hat{r}z^l b\delta^2 - k_4(\epsilon)\epsilon b
\end{aligned} \tag{C.205}$$

where the second equality holds by sequential rational actions ; the first inequality holds because of (C.203), (C.191) and Popoviciu's inequality; the last inequality holds for small enough ϵ and a bounded function $k_4(\epsilon)$ because $b \le \epsilon$, so $Pr(\Theta_q^0)k_3(\epsilon)^2 b^2 \le Pr(\Theta_q^0)k_3(\epsilon)^2\epsilon b$; (C.195) implies $\lim_{\epsilon\to 0}(E[\Theta_{\hat{q}}^0] - E^l)^2 - c = \lim_{\epsilon\to 0}(\mu - E')^2 - c \le 0$, and (C.193) implies $\lim_{\epsilon\to 0}(E^l - \underline{\mu})^2 = \lim_{\epsilon\to 0}(E' - \underline{\mu})^2 > \delta^2$.

Since $V_q(m) = V_{\hat{q}}(m) = c$, by Lemma 21, $EU_{DM}(\hat{\Omega}) = EU_{DM}^{U}(\hat{\Omega}) = EU_{DM}^{I}(\hat{\Omega})$, and (C.203) implies that $\hat{X}(m_{\hat{q}}^0, u) - X(m_q^0, u) < k_3(\epsilon)b$, so Lemma 22 implies that there exists an incentive compatible mechanism $\tilde{\Omega}$ such that $EU_{DM}(\tilde{\Omega}) > EU_{DM}(\hat{\Omega}) - 4(k_3(\epsilon)b)^2$, then by (C.205) $EU_{DM}(\tilde{\Omega}) - EU_{DM}(\Omega) > \hat{r}z^l b\delta^2 - k_4(\epsilon)\epsilon b - 4(k_3(\epsilon)b)^2 > 0$ for small enough ϵ, but it contradicts that Ω is an optimal mechanism.

Case 2:$\lim_{\epsilon\to 0} E' - E^l > 0$:

For $r \in [0,1]$, define $E_1^l(r) = E^l + r\epsilon^{\frac{1}{3}}(E' - E_{ex})$ and $E_2^l(r) = E^l - r\epsilon^{\frac{1}{3}}(E' - E_{ex}) - \epsilon(E' - \underline{\mu})$. Define $\hat{r} \in (0,1)$ that solves

$$J(E^t - E_1^l(r)) + J(E^t - E_2^l(r)) = 2\frac{z^t}{z^l} \tag{C.206}$$

To show that such solution exists for small enough ϵ, $E_1^l(0) = E^l$ and $E_2^l(0) = E^l - \epsilon(E' - \underline{\mu}) <$

E^l, by (C.193) and (C.198),

$$\begin{aligned}
&\lim_{\epsilon\to 0}\frac{1}{\epsilon}[J(E^t-E_1^l(0))+J(E^t-E_2^l(0))-2\frac{z^t}{z^l}]\\
=&\lim_{\epsilon\to 0}\frac{1}{\epsilon}[J(E^t-E^l)+J(E^t-E^l+\epsilon(E'-\underline{\mu}))-2(J(E^t-E^l)+k_2(\epsilon)\epsilon^2)]\\
=&J'(E^t-E^l)(E'-\underline{\mu})>J'(E^t-E^l)\delta>0 \qquad \text{(C.207)}
\end{aligned}$$

On the other side, $J(E^t-E_1^l(1))=J(E^t-E^l)-J'(E^t-E^l)(\epsilon^{\frac{1}{3}}(E'-E_{ex}))+J''(E^t-\tilde{E}_1)(\epsilon^{\frac{1}{3}}(E'-E_{ex}))^2$ and $J(E^t-E_1^2(1))=J(E^t-E^l)+J'(E^t-E^l)(\epsilon^{\frac{1}{3}}(E'-E_{ex})+\epsilon(E'-\epsilon))+J''(E^t-\tilde{E}_2)(\epsilon^{\frac{1}{3}}(E'-E_{ex})+\epsilon(E'-\epsilon))^2$, where $E_1^l(1)>\tilde{E}_1>E^l>\tilde{E}_2>E_1^2(1)$. Therefore, $J(E^t-E_1^l(1))+J(E^t-E_2^l(1))=2J(E^t-E^l)+J'(E^t-E^l)\epsilon(E'-\epsilon)+J''(E^t-\tilde{E}_1)(\epsilon^{\frac{1}{3}}(E'-E_{ex}))^2+J''(E^t-\tilde{E}_2)(\epsilon^{\frac{1}{3}}(E'-E_{ex})+\epsilon(E'-\epsilon))^2$, thus

$$\begin{aligned}
&\lim_{\epsilon\to 0}\frac{1}{\epsilon^{\frac{2}{3}}}[J(E^t-E_1^l(1))+J(E^t-E_2^l(1))-2\frac{z^t}{z^l}]\\
=&\lim_{\epsilon\to 0}\frac{1}{\epsilon^{\frac{2}{3}}}[J(E^t-E_1^l(1))+J(E^t-E_2^l(1))-2(J(E^t-E^l)+k_2(\epsilon)\epsilon^2)]\\
=&h''(E^t-E^l)(E'-E_{ex})^2>0 \qquad \text{(C.208)}
\end{aligned}$$

where the inequality holds because $J(.)$ is strictly concave and $E'-E_{ex}>E'-E^l>0$, thus for small enough $\epsilon>0$, there exists $\hat{r}\in(0,1)$ such that (C.206) is satisfied.

Divide Θ_{ex} into two mean-preserving divisions $\Theta_{ex,1}$ and $\Theta_{ex,2}$ so that $E[\Theta_{ex,1}]=E[\Theta_{ex,2}]=E_{ex}$, $Pr(\Theta_{ex,1})=\frac{1}{2}[Pr(\Theta_{ex})-z^lb\hat{r}\epsilon^{\frac{1}{3}}]$ and $Pr(\Theta_{ex,2})=\frac{1}{2}[Pr(\Theta_{ex})+z^lb\hat{r}\epsilon^{\frac{1}{3}}]$; Divide Θ' into three mean-preserving divisions Θ'_0, Θ'_1 and Θ'_2 so that $E[\Theta'_0]=E[\Theta'_1]=E[\Theta'_2]=E'$, $Pr(\Theta'_0)=\frac{1}{2}z^lb\epsilon$, $Pr(\Theta'_1)=\frac{1}{2}[Pr(\Theta')+z^lb\hat{r}\epsilon^{\frac{1}{3}}]$ and $Pr(\Theta'_2)=\frac{1}{2}[Pr(\Theta')-z^lb\hat{r}\epsilon^{\frac{1}{3}}-z^lb\epsilon]$; Divide Θ^t into two mean-preserving divisions $\hat{\Theta}_1^t$ and $\hat{\Theta}_2^t$ so that $E[\hat{\Theta}_1^t]=E[\hat{\Theta}_2^t]=E^t$, $Pr(\hat{\Theta}_1^t)=\frac{1}{2}J(E^t-E_1^l(\hat{r}))Pr(\Theta^l)$ and $Pr(\hat{\Theta}_2^t)=\frac{1}{2}J(E^t-E_2^l(\hat{r}))Pr(\Theta^l)$; Divide a mean-preserving set $\Theta_{0,2}$ from Θ_0 so that $E[\Theta_{0,2}]=\underline{\mu}$ and $Pr(\Theta_{0,2})=\frac{1}{2}z^lb\epsilon$.

Let $\hat{\Theta}_1^l=\Theta_{ex,1}\cup\Theta'_1$ be the set of liars for the modified upper message, so that

$$\begin{aligned}
Pr(\hat{\Theta}_1^l)&=\frac{1}{2}[Pr(\Theta_{ex})-z^lb\hat{r}\epsilon^{\frac{1}{3}}]+\frac{1}{2}[Pr(\Theta')+z^lb\hat{r}\epsilon^{\frac{1}{3}}]\\
&=\frac{1}{2}z^lb \qquad \text{(C.209)}\\
E[\hat{\Theta}_1^l]&=\frac{E_{ex}[Pr(\Theta_{ex})-z^lb\hat{r}\epsilon^{\frac{1}{3}}]+E'[Pr(\Theta')+z^lb\hat{r}\epsilon^{\frac{1}{3}}]}{z^lb}\\
&=E^l+\hat{r}\epsilon^{\frac{1}{3}}(E'-E_{ex})=E_1^l(\hat{r}) \qquad \text{(C.210)}
\end{aligned}$$

Let $\hat{\Theta}^l_2 = \Theta_{ex,2} \cup \Theta'_2 \cup \Theta_{0,2}$ be the set of liars for the modified lower message, so that

$$Pr(\hat{\Theta}^l_2) = \frac{1}{2}[Pr(\Theta_{ex}) + z^l b\hat{r}\epsilon^{\frac{1}{3}}] + \frac{1}{2}[Pr(\Theta') - z^l b\hat{r}\epsilon^{\frac{1}{3}} - z^l b\epsilon] + \frac{1}{2}z^l b\epsilon$$
$$= \frac{1}{2}z^l b \tag{C.211}$$
$$E[\hat{\Theta}^l_1] = \frac{E_{ex}[Pr(\Theta_{ex}) + z^l b\hat{r}\epsilon^{\frac{1}{3}}] + E'[Pr(\Theta') - z^l b\hat{r}\epsilon^{\frac{1}{3}} - z^l b\epsilon] + \underline{\mu} z^l b\epsilon}{z^l b}$$
$$= E^l - \hat{r}\epsilon^{\frac{1}{3}}(E' - E_{ex}) - \epsilon(E' - \underline{\mu}) = E^l_2(\hat{r}) \tag{C.212}$$

Let $\Theta^0_{\hat{q}} = (\Theta^0_q / \Theta_{0,2}) \cup \Theta'_0$ be the modified uninspected set.

Now define the modified message ans action rules $\hat{q}, \hat{X}$ where other things remain unchanged, except the set of messages M' is off-path; an upper inspected message m_1 is added with the set of truthful senders $\Theta^t_{\hat{q}}(m_1) = \hat{\Theta}^t_1$ and the set of lying senders $\Theta^l_{\hat{q}}(m_1) = \hat{\Theta}^l_1$; an lower inspected message m_2 is added with the set of truthful senders $\Theta^t_{\hat{q}}(m_2) = \hat{\Theta}^t_2$ and the set of lying senders $\Theta^l_{\hat{q}}(m_2) = \hat{\Theta}^l_2$; The uninspected message is modified to $m^0_{\hat{q}}$ with the set of senders $\Theta^0_{\hat{q}}$

The sequentially rational actions for m_1 are

$$\hat{X}(m_1, t) = E^t$$
$$\hat{X}(m_1, l) = E^l_1(\hat{r})$$
$$\hat{X}(m_1, t) = x^*_u(E^t, E^l_1(\hat{r})) \approx x^*_u(E^t, E^l) > \mu \tag{C.213}$$

where the approximation holds for small enough ϵ because $\lim_{\epsilon \to 0} E^l_1(\hat{r}) - E^l = 0$, and the second inequality holds by optimality of Ω.

The sequentially rational actions for m_2 are

$$\hat{X}(m_2, t) = E^t$$
$$\hat{X}(m_2, l) = E^l_2(\hat{r})$$
$$\hat{X}(m_2, t) = x^*_u(E^t, E^l_2(\hat{r})) > x^*_u(E^t, E^l) > \mu \tag{C.214}$$

The sequentially rational action for the modified uninspected message $m^0_{\hat{q}}$ is

$$\hat{X}(m^0_{\hat{q}}, u) = \mu + \frac{\epsilon z^l b}{Pr(\Theta^0_q)}(E' - \underline{\mu}) = k_5(\epsilon) b\epsilon \tag{C.215}$$

where $k_5(\epsilon)$ is a bounded function. Since for $i = 1, 2$, $Pr(\hat{\Theta}^t_i) = J(E^t - E^l_i(\hat{r}))Pr(\hat{\Theta}^l_i)$, so $w_{\hat{q}}(m_i) = w^-(E^t - E^l_i(\hat{r})) = w^-(\hat{X}(m_i, t) - \hat{X}(m_i, l))$, so

$$w_{\hat{q}}(m) = w^-(\hat{X}(m, t) - \hat{X}(m, l)) \tag{C.216}$$

hold almost everywhere at $\mathcal{M}^+_{\hat{q}}$.

To compare DM's payoffs, since $V_q(m) = V_{\hat{q}}(m) = c$, by Lemma 21, $EU_{DM}(\Omega) = EU^I_{DM}(\Omega)$ and $EU_{DM}(\hat{\Omega}) = EU^I_{DM}(\hat{\Omega})$, so

$$\begin{aligned}
&EU_{DM}(\hat{\Omega}) - EU_{DM}(\Omega)\\
=&Pr(\Theta^0_q)E[\Theta^0_{\hat{q}}]^2 + \frac{1}{2}z^l b \sum_{i=1,2}[E^l_i(\hat{r})^2 - c] + z^t b[(E^t)^2 - c]\\
&- Pr(\Theta^0_q)\mu^2 - \sum_{s=t,l}\int_{M'}(X(m,s)^2 - c)\int_{\Theta^s_q(m)} dF(\theta)dm\\
=&- Pr(\Theta^0_q)(\mu - E[\Theta^0_{\hat{q}}])^2 - \frac{1}{2}z^l b\epsilon(E' - E[\Theta^0_{\hat{q}}])^2 + \frac{1}{2}z^l b\epsilon(\underline{\mu} - E[\Theta^0_{\hat{q}}])^2\\
&+ \frac{1}{2}z^l b\hat{r}\epsilon^{\frac{1}{3}}\sum_{i=1,2}(E^l_i(\hat{r}) - E^l)^2 + \frac{1}{2}z^l b\epsilon(E' - E^l)^2 - \frac{1}{2}z^l b\epsilon(\underline{\mu} - E^l)^2\\
&- \sum_{s=t,l}\int_{M'}(X(m,s) - E^s)^2\int_{\Theta^s_q(m)} dF(\theta)dm\\
>&- Pr(\Theta^0_q)[k_5(\epsilon)b\epsilon]^2 - \frac{1}{2}z^l b\epsilon(E' - E[\Theta^0_{\hat{q}}])^2 + \frac{1}{2}z^l b\epsilon(\underline{\mu} - E[\Theta^0_{\hat{q}}])^2\\
&+ \frac{1}{2}z^l b\hat{r}\epsilon^{\frac{1}{3}}\sum_{i=1,2}(E^l_i(\hat{r}) - E^l)^2 + \frac{1}{2}z^l b\epsilon(E' - E^l)^2 - \frac{1}{2}z^l b\epsilon(\underline{\mu} - E^l)^2 - \frac{1}{4}\epsilon^2 b\\
>&- Pr(\Theta^0_q)[k_5(\epsilon)b\epsilon]^2 + z^l b\epsilon(E[\Theta^0_{\hat{q}}] - E^l)(E' - \underline{\mu}) - \frac{1}{4}\epsilon^2 b\\
>&- Pr(\Theta^0_q)[k_5(\epsilon)b\epsilon]^2 + z^l b\epsilon\delta^2 - \frac{1}{4}\epsilon^2 b\\
>&z^l b\epsilon\delta^2 - k_6(\epsilon)b\epsilon^2
\end{aligned} \tag{C.217}$$

where the second equality holds by sequential rational actions ; the first inequality holds because of (C.215), (C.191) and Popoviciu's inequality; the third inequality holds because (C.189) and (C.193) imply that $E[\Theta^0_{\hat{q}}] - E^l > \mu - E^l > \delta$ and $E' - \underline{\mu} > \delta$; the last inequality holds for a bounded function $k_6(\epsilon)$.

Since $V_q(m) = V_{\hat{q}}(m) = c$, by Lemma 21, $EU_{DM}(\hat{\Omega}) = EU^U_{DM}(\hat{\Omega}) = EU^I_{DM}(\hat{\Omega})$, and (C.215) implies that $\hat{X}(m^0_{\hat{q}}, u) - X(m^0_q, u) < k_5(\epsilon)b\epsilon$, so Lemma 22 implies that there exists an incentive compatible mechanism $\tilde{\Omega}$ such that $EU_{DM}(\tilde{\Omega}) > EU_{DM}(\hat{\Omega}) - 4(k_5(\epsilon)b\epsilon)^2$, then by (C.205) $EU_{DM}(\tilde{\Omega}) - EU_{DM}(\Omega) > z^l b\epsilon\delta^2 - k_6(\epsilon)b\epsilon^2 - 4(k_5(\epsilon)b\epsilon)^2 > 0$ for small enough ϵ, but it contradicts that Ω is an optimal mechanism. *Q.E.D.*

Lemma 33 *In an optimal mechanism Ω with $Pr(\Theta^0_q) > 0$, $\overline{\theta}^l = \underline{\theta}^0$ and $\underline{\theta}^t = \overline{\theta}^0$. Furthermore, if $\underline{\theta}^t - \mu < \mu - \overline{\theta}^l$, then $(\underline{\theta}^t - \mu)(\mu - \overline{\theta}^l) = c$.*

Proof of Lemma 33: For $\underline{\theta}^t - \mu \geq \mu - \overline{\theta}^l$, Lemma 26 and Lemma 32 imply $\underline{\theta}^t \geq \overline{\theta}^0 > \underline{\theta}^0 \geq \overline{\theta}^l$. Since $\Theta_q^t(\mathcal{M}_q^+) \cup \Theta_q^t(\mathcal{M}_q^+) \cup \Theta_q^0 = [0,1]$, we have $\underline{\theta}^t = \overline{\theta}^0 > \underline{\theta}^0 = \overline{\theta}^l$.

For $\underline{\theta}^t - \mu < \mu - \overline{\theta}^l$, by Lemma 26 $\mu - \overline{\theta}^l > \underline{\theta}^t - \mu \geq \overline{\theta}^0 - \mu > 0$, so $\underline{\theta}^t = \overline{\theta}^0$ and $\overline{\theta}^l \geq \underline{\theta}^0$. Then Lemma 25 and Lemma 31 imply $(\underline{\theta}^t - \mu)(\mu - \overline{\theta}^l) \geq c \geq (\overline{\theta}^0 - \mu)(\mu - \underline{\theta}^0) = (\underline{\theta}^t - \mu)(\mu - \underline{\theta}^0)$, so $\overline{\theta}^l = \underline{\theta}^0$ and $(\underline{\theta}^t - \mu)(\mu - \overline{\theta}^l) = c$. *Q.E.D.*

C.3 Proof of Lemma 3

Since $w^-(d) \in (0, \frac{1}{2}]$ is well-defined and positive for any $d \geq 2\sqrt{c}$, and $c \leq \frac{1}{4}$ means $2\sqrt{c} \leq 1$, so $\dot{\phi}_d(1) = -\frac{w^-(1)}{1-w^-(1)}\frac{f(1)}{f(0)}$ is well-defined, and for any $\theta \leq 1$ in which $\theta - \phi_d(\theta) \geq 2\sqrt{c}$, $\dot{\phi}_d(\theta)$ is well-defined and negative, which means $\frac{d(\theta-\phi_d(\theta))}{d\theta} > 1$. Therefore, there exists unique solutions ϕ_d and $\hat{\theta} \in (0,1]$ that satisfy (3.21), (3.22) and (3.24).

To show the second part of the statement, for any $\theta \in [\hat{\theta}, 1]$,

$$\frac{dX_u^*(\theta, \phi_d(\theta))}{d\theta} = \frac{\partial X_u^*(\theta, \phi_d(\theta))}{\partial\theta} + \frac{\partial X_u^*(\theta, \phi_d(\theta))}{\partial\phi_d(\theta)}\dot{\phi}_d(\theta) > 0 \tag{C.218}$$

where the inequality holds by Lemma 23 and $\dot{\phi}_d(\theta) < 0$. Also,

$$\begin{aligned}
&\frac{dE[\phi_d(\theta), \theta]}{d\theta} \\
=&\frac{\partial E[\phi_d(\theta), \theta]}{\partial\theta} + \frac{\partial E[\phi_d(\theta), \theta]}{\partial\phi_d(\theta)}\dot{\phi}_d(\theta) \\
=&\frac{f(\theta)(\theta - E[\phi_d(\theta), \theta])}{Pr([\phi_d(\theta), \theta])} - \frac{f(\phi_d(\theta))(\phi_d(\theta) - E[\phi_d(\theta), \theta])}{Pr([\phi_d(\theta), \theta])}\dot{\phi}_d(\theta) \\
=&\frac{f(\theta)}{Pr([\phi_d(\theta), \theta])(1 - w^-(\theta - \phi_d(\theta)))}[(1 - w^-(\theta - \phi_d(\theta))\theta + w^-(\theta - \phi_d(\theta))\phi_d(\theta) - E[\phi_d(\theta), \theta]] \\
=&\frac{f(\theta)}{Pr([\phi_d(\theta), \theta])(1 - w^-(\theta - \phi_d(\theta)))}[X_u^*(\theta, \phi_d(\theta)) - E[\phi_d(\theta), \theta]]
\end{aligned} \tag{C.219}$$

Where the third equality holds by (3.21), the fourth equality holds by (C.1), and the last equality holds by (3.20). Since (C.218) and (C.219) means when $X_u^*(\theta, \phi_d(\theta)) \leq E[\phi_d(\theta), \theta]$, $X_u^*(\theta, \phi_d(\theta)) - E[\phi_d(\theta), \theta]$ is strictly increasing in θ, therefore for any $\theta' < \theta$,

$$X_u^*(\theta, \phi_d(\theta)) \leq E[\phi_d(\theta), \theta] \Rightarrow X_u^*(\theta', \phi_d(\theta')) < E[\phi_d(\theta'), \theta'] \tag{C.220}$$

By assumption of the Lemma, $X_u^*(1, \phi_d(1)) = X_u^*(1, 0) = \frac{1}{2} + \sqrt{\frac{1}{4} - c} > E[\Theta] \equiv E[0,1] = E[\phi_d(1), 1]$. Now we consider two cases.

Case 1: $X_u^*(\hat{\theta}, \phi_d(\hat{\theta})) > E[\phi_d(\hat{\theta}), \hat{\theta}]$**:** then (C.220) and $X_u^*(1, \phi_d(1)) > E[\phi_d(1), 1]$ imply that $X_u^*(\theta, \phi_d(\theta)) > E[\phi_d(\theta), \theta]$ for any $\theta \in [\hat{\theta}, 1]$, then $\overline{\theta}_d = \hat{\theta}$.

Case 2: $X_u^*(\hat{\theta}, \phi_d(\hat{\theta})) \leq E[\phi_d(\hat{\theta}), \hat{\theta}]$**:** then since $X_u^*(1, \phi_d(1)) > E[\phi_d(1), 1]$, there exists $\overline{\theta}_d \in [\hat{\theta}, 1)$ such that $X_u^*(\overline{\theta}_d, \phi_d(\overline{\theta}_d)) = E[\phi_d(\overline{\theta}_d), \overline{\theta}_d]$. (C.220) implies that $X_u^*(\theta, \phi_d(\theta)) < E[\phi_d(\theta), \theta]$ for $\theta < \overline{\theta}_d$ and $X_u^*(\theta, \phi_d(\theta)) > E[\phi_d(\theta), \theta]$ for $\theta > \overline{\theta}_d$. *Q.E.D.*

C.4 Proof of Lemma 4

To show that Ω_d is incentive compatible, for $m \in \mathcal{M}_q^+$,

$$\begin{aligned}\frac{w_q(m)}{1 - w_q(m)} &= \lim_{\epsilon} \frac{Pr([\phi_d(m+\epsilon), \phi_d(m-\epsilon)])}{Pr([m-\epsilon, m+\epsilon])} \\ &= \frac{-\dot{\phi}_d(m) f(\phi_d(m))}{f(m)} = \frac{w^-(m - \phi_d(m))}{1 - w^-(m - \phi_d(m))} \end{aligned} \tag{C.221}$$

where the first equality holds because of the continuously decreasing message rule, and the third equality holds by (3.21). Therefore, $w_q(m) = w^-(m - \phi_d(m))$ and $V_q(m) = c$, thus condition (b) of Lemma 20 is satisfied.

$X(m_q^0, u)$, $X(m, t)$ and $X(m, l)$ for $m \in \mathcal{M}_q^+$ are clearly sequentially rational given the message rule. For $m \in \mathcal{M}_q^+$,

$$\begin{aligned}X(m, u) = X_u^*(m, \phi_d(m)) &= w^-(m - \phi_d(m))\phi_d(m) + (1 - w^-(m - \phi_d(m)))m \\ &= w_q(m) X(m, l) + (1 - w_q(m)) X(m, t)\end{aligned}$$

is also sequentially rational. For any $m \in \mathcal{M}_q^+ = (\overline{\theta}_d, 1]$,

$$\begin{aligned}X(m, u) = X_u^*(m, \phi_d(m)) > E[\phi_d(m), m] &> E[\phi(\overline{\theta}_d), \overline{\theta}_d] \\ &= X(m_q^0, u) > \underline{\theta}_d > \phi_d(m) = X(m, l)\end{aligned}$$

where the first and second inequalities holds because by the definition of $\overline{\theta}_d$, $X_u^*(\theta, \phi_d(\theta)) > E[\phi_d(\theta), \theta]$ for any $\theta \in (\overline{\theta}_d, 1]$, then by (C.219) $E[\phi_d(\theta), \theta]$ is strictly increasing for $\theta \in (\overline{\theta}_d, 1]$. Therefore, condition (a) of Lemma 20 is satisfied, and thus Ω_d is incentive compatible with the inspection rule specified by (3.27). Since $c < \frac{1}{4}$ implies $2\sqrt{c} < 1 = 1 - \phi_d(1)$, so $\hat{\theta} < 1$, and $E[\Theta] < \frac{1}{2} + \sqrt{\frac{1}{4} - c}$ implies $X_u^*(1, \phi_d(1)) > E[0, 1]$, so $\overline{\theta}_d < 1$. Then $\underline{\theta}_d = \phi_d(\overline{\theta}_d) > 0$ because ϕ_d is strictly decreasing.

Since $V_q(m) = c$ for any $m \in \mathcal{M}_q^+$, Lemma 21 implies $EU_{DM}(\Omega_d) = EU^U_{DM}(\Omega_d) = EU^I_{DM}(\Omega_d)$. To show that $EU_{DM}(\Omega_d) > -c$,

$$\begin{aligned}
&EU_{DM}(\Omega_d) = EU^I_{DM}(\Omega_d) \\
=&Pr([\underline{\theta}_d, \overline{\theta}_d])E[\underline{\theta}_d, \overline{\theta}_d]^2 + \int_{[0,\underline{\theta}_d)\cup(\overline{\theta}_d,1]} (\theta^2 - c)dF(\theta) - E[\theta^2] \\
=& - Pr([\underline{\theta}_d, \overline{\theta}_d])Var([\underline{\theta}_d, \overline{\theta}_d]) - (1 - Pr([\underline{\theta}_d, \overline{\theta}_d]))c \\
=& - c + Pr([\underline{\theta}_d, \overline{\theta}_d])[c - Var([\underline{\theta}_d, \overline{\theta}_d])] \\
>& - c + Pr([\underline{\theta}_d, \overline{\theta}_d])[c - (\overline{\theta}_d - E[\underline{\theta}_d, \overline{\theta}_d])(E[\underline{\theta}_d, \overline{\theta}_d] - \underline{\theta}_d)] \\
\geq& - c
\end{aligned}$$

where the first inequality holds because of Bhatia–Davis inequality, and the second inequality holds because definition of $\overline{\theta}_d$ implies either $X^*_u(\overline{\theta}_d, \underline{\theta}_d) = E[\underline{\theta}_d, \overline{\theta}_d]$, which means $(\overline{\theta}_d - E[\underline{\theta}_d, \overline{\theta}_d])(E[\underline{\theta}_d, \overline{\theta}_d] - \underline{\theta}_d) = c$, or $\overline{\theta}_d - \underline{\theta}_d = 2\sqrt{c}$, which means $(\overline{\theta}_d - E[\underline{\theta}_d, \overline{\theta}_d])(E[\underline{\theta}_d, \overline{\theta}_d] - \underline{\theta}_d) \leq c$. To show that $EU_{DM}(\Omega_d) > -Var(\Theta)$,

$$\begin{aligned}
&EU_{DM}(\Omega_d) = EU^U_{DM}(\Omega_d) \\
=&Pr([\underline{\theta}_d, \overline{\theta}_d])E[\underline{\theta}_d, \overline{\theta}_d]^2 + \int_{(\overline{\theta}_d}^1 X(\{\theta\}, u)^2(f(\theta) + \dot{\phi}_d(\theta)f(\phi_d(\theta)))d\theta - E[\theta^2] \\
=&Pr([\underline{\theta}_d, \overline{\theta}_d])E[\underline{\theta}_d, \overline{\theta}_d]^2 + \int_{\overline{\theta}_d}^1 X(\{\theta\}, u)^2(f(\theta) + \dot{\phi}_d(\theta)f(\phi_d(\theta)))d\theta - (Var(\Theta) + E[\Theta]^2) \\
=&Pr([\underline{\theta}_d, \overline{\theta}_d])(E[\underline{\theta}_d, \overline{\theta}_d] - E[\Theta])^2 + \int_{\overline{\theta}_d}^1 (X(\{\theta\}, u) - E[\Theta])^2(f(\theta) + \dot{\phi}_d(\theta)f(\phi_d(\theta)))d\theta - Var(\Theta) \\
>& - Var(\Theta)
\end{aligned}$$

where the last equality holds because of sequential rationality of X, and the inequality holds because $c < \frac{1}{4}$ implies $2\sqrt{c} < 1 = 1 - \phi_d(1)$ and $E[\Theta] < \frac{1}{2} + \sqrt{\frac{1}{4} - c}$ implies $X^*_u(1, \phi_d(1)) > E[0, 1]$, so $\overline{\theta}_d < 1$. *Q.E.D.*

C.5 Proof of Theorem 10

"Only if":

Let $\Omega \equiv (q, P, X)$ be an incentive compatible mechanism where $p_\Omega > 0$, then the set of inspected messages $\mathcal{M}_q^+$ has positive measure, so (3.5) implies that for almost every $m \in \mathcal{M}_q^+$, $X(m, t) < 1$ and $X(m, l) > 0$, so $V_q(m) = w_q(m)(1 - w_q(m))(X(m, t) - X(m, l))^2 < \frac{1}{4}$. Since (3.8) and (3.9) imply $V_q(m) \geq c$, it must be the case that $c \leq V_q(m) < \frac{1}{4}$.

By (3.8) and the definition of $w^-(.)$, we have that $w^-(X(m,t)-X(m,l)) = \min\{w \in [0,1] : V_q(m) \geq c\}$. Since for any $m \in \mathcal{M}_q^+$, $V_q(m) \geq c$, so $w_q(m) \geq w^-(X(m,t) - X(m,l))$, and thus $X(m,u) = w_q(m)X(m,l)+(1-w_q(m))X(m,t) \leq X_u^*(X(m,t), X(m,l))$. Since X is sequentially rational, $X(m,t) = E[\Theta_q^t(m)] \leq 1$ and $X(m,t) = E[\Theta_q^l(m)] \geq 0$, with the inequalities hold strictly if $Pr(\Theta_q^u(m)) > 0$. Therefore, $X(m,u) \leq X_u^*(X(m,t), X(m,l)) \leq X_u^*(1,0) = \frac{1}{2} + \sqrt{\frac{1}{4} - c}$, where the second inequality holds since by Lemma 23 $X_u^*(X(m,t), X(m,l))$ is strictly increasing in $X(m,t)$ and strictly decreasing in $X(m,l)$, and it hold strictly if $Pr(\Theta_q^u(m)) > 0$.

Since $\Theta_q^0 \cup \Theta_q^u(\mathcal{M}_q^+) = \Theta$, so $Pr(\Theta_q^0) = 1 - P(\Theta_q^u(\mathcal{M}_q^+))$ and

$$(1 - P(\Theta_q^u(\mathcal{M}_q^+)))E[\Theta_q^0] + P(\Theta_q^u(\mathcal{M}_q^+))E[\Theta_q^u(\mathcal{M}_q^+)] = E[\Theta] \tag{C.222}$$

. Since X is sequentially rational,

$$\begin{aligned} P(\Theta_q^u(\mathcal{M}_q^+))E[\Theta_q^u(\mathcal{M}_q^+)] &= \int_{\mathcal{M}_q^+} X(m,u) \int_{\Theta_q^u(m)} dF(\theta)dm \\ &\leq X_u^*(1,0) \int_{\mathcal{M}_q^+} \int_{\Theta_q^u(m)} dF(\theta)dm \\ &= Pr(\Theta_q^u(\mathcal{M}_q^+))(\frac{1}{2} + \sqrt{\frac{1}{4} - c}) \end{aligned} \tag{C.223}$$

where the inequality holds strictly if $Pr(\Theta_q^u(\mathcal{M}_q^+)) > 0$. If $Pr(\Theta_q^0) = 0$, then $P(\Theta_q^u(\mathcal{M}_q^+)) = 1$ and (C.222) and (C.223) imply $E[\Theta] = E[\Theta_q^u(\mathcal{M}_q^+)] < \frac{1}{2} + \sqrt{\frac{1}{4} - c}$. If $Pr(\Theta_q^0) > 0$, then Lemma 20 implies that for any $m \in \mathcal{M}_q^+$, $E[\Theta_q^0] = X(m_q^0, u) < X(m,u)$, so by (C.223) $E[\Theta_q^0] < E[\Theta_q^u(\mathcal{M}_q^+)] \leq \frac{1}{2} + \sqrt{\frac{1}{4} - c}$, then (C.222) implies $E[\Theta] < \frac{1}{2} + \sqrt{\frac{1}{4} - c}$.

"If":

The decreasing mechanism Ω_d defined at (3.21) - (3.26) is an example, where Lemma 4 implies that if Assumption 4 is satisfied, then Ω_d is incentive compatible with $0 < \underline{\theta}_d < \overline{\theta}_d < 1$, and thus $p_{\Omega_d} = \int_{\overline{\theta}_d}^1 P(\{\theta\})[f(\theta) + \dot{\phi}_d(\theta) f(\phi_d(\theta))]d\theta > 0$. *Q.E.D.*

C.6 Proof of Theorem 11

Suppose contrary to the claim, there exists a positive measure set inspected messages $M_1 \subseteq \mathcal{M}_q^+$ such that $V_q(m) \neq c$ for all $m \in M_1$. Since $P(m) > 0$, sequential rationality (3.9) then requires that $V_q(m) > c$ and $P(m) = 1$, and (b) of Lemma 20 implies that for all $m \in M_1$, $X(m,l) = \hat{x}$, where $\hat{x} = X(m_q^0, u)$ if m_q^0 exists, $\hat{x} = \max_{m'} X(m', l)$ otherwise. Therefore, we have $E[\Theta_q^l(m)] = E[\Theta_q^l(M_1)] = \hat{x}$ for all $m \in M_1$.

Denote $\hat{\Theta} = \Theta^l_q(M_1)$ be the set of liars who send $m \in M_1$ in the original mechanism. For $m \in M_1$, denote $\hat{w}(m) = w^-(X(m,t) - \hat{x})$ be the smallest weight on liars such that value of inspection is no less than c. Since for all $m \in M_1$, $V_q(m) > c$, so $w_q(m) > \hat{w}(m)$. Now define $\hat{p} = \int_{M_1}\int_{\Theta^t_q(m)} \frac{\hat{w}(m)}{1-\hat{w}(m)} dF(\theta)dm$, which is the total minimum measure of liars required to match with truth-tellers of $m \in M_1$ such that value of inspection is no less than c. We have $\hat{p} < Pr(\hat{\Theta}) = \int_{M_1}\int_{\Theta^t_q(m)} \frac{w_q(m)}{1-w_q(m)} dF(\theta)dm$.

Assign an arbitrary strict ranking $r : M_1 \to \mathbb{R}$ to the message set M_1. Then for any $m \in M^l$, let

$$z^-(m) = \frac{1}{Pr(\Theta^l_q(M_1))} \int_{m' \in M_1 : r(m') < r(m)} \frac{\hat{w}(m')}{1-\hat{w}(m')} \int_{\Theta^t_q(m')} dF(\theta)dm' \tag{C.224}$$

$$z^+(m) = \frac{1}{Pr(\Theta^l_q(M_1))} \int_{m' \in M_1 : r(m') = r(m)} \frac{\hat{w}(m')}{1-\hat{w}(m')} \int_{\Theta^t_q(m')} dF(\theta)dm' \tag{C.225}$$

be the cumulative required fraction of liars.

For any positive measure set of types $\hat{\Theta}$, define the mean-preserving division $\hat{\Theta}(z) = \hat{\Theta} \cap [\underline{\theta}(z), \overline{\theta}(z)]$ such that $\underline{\theta}(z)$ and $\overline{\theta}(z)$ solve

$$Pr(\hat{\Theta}(z)) = zPr(\hat{\Theta}) \tag{C.226}$$

$$E[\hat{\Theta}(z)] = E[\hat{\Theta}] \tag{C.227}$$

Define an modified messaging and action rules $\hat{q}, \hat{X}$ where other things remain unchanged, except the set of messages M_1. The uninspected message is modified to $m^0_{\hat{q}} = m^0_q \cup (\hat{\Theta}/\hat{\Theta}(\frac{\hat{p}}{Pr(\hat{\Theta})}))$, where $\hat{\Theta}/\hat{\Theta}(\frac{\hat{p}}{Pr(\hat{\Theta})})$ is a mean-preserving division of $\hat{\Theta}$ with mean $\hat{x}$ and measure $Pr(\hat{\Theta}) - \hat{p}$. For $m \in M_1$, the set of truth-tellers remain unchanged, while the set of liars is modified to $\Theta^l_{\hat{q}}(m) = \hat{\Theta}(z^+(m))/int(\hat{\Theta}(z^-(m)))$, a mean preserving division of $\hat{\Theta}$ where $int(X)$ is the interior of set X, so that $E[\Theta^l_{\hat{q}}(m)] = \hat{x}$ and the set has measure $\frac{\hat{w}(m)}{1-\hat{w}(m)} \int_{\Theta^t_q(m)} dF(\theta)$.

The sequentially rational actions for the modified uninspected messages $m^0_{\hat{q}}$ is

$$\hat{X}(m^0_{\hat{q}}, u) = E[\hat{\Theta}(\underline{z})] = \hat{x} \tag{C.228}$$

and for $m \in M_1$,

$$\begin{aligned} \hat{X}(m,t) &= X(m,t) \\ \hat{X}(m,l) &= X(m,l) = \hat{x} \\ \hat{X}(m,u) &= \hat{w}(m)\hat{x} + (1-\hat{w}(m))X(m,t) \end{aligned} \tag{C.229}$$

where $\hat{X}(m,u) > \hat{x}$, so $(\hat{q}, \hat{X})$ satisfies (a) of Lemma 20. Furthermore, by the definition of $\Theta^l_{\hat{q}}(m)$ for $m \in M_l$, we have

$$w_{\hat{q}}(m) = \hat{w}(m) = w^-(\hat{X}(m,t) - \hat{X}(m,l)) \tag{C.230}$$

and thus

$$V_{\hat{q}}(m) = c \tag{C.231}$$

so $(\hat{q}, \hat{X})$ satisfies (b) of Lemma 20. Therefore, there exists $\hat{P}$ such that $\hat{\Omega} = (\hat{q}, \hat{P}, \hat{X})$ is incentive compatible.

Under the modified mechanism $\hat{\Omega}$, the sequentially rational actions remain unchanged for every type, but the ex-ante probability of inspection is reduced by $Pr(\hat{\Theta}) - \hat{p} > 0$. Therefore, $EU_{DM}(\hat{\Omega}) > EU_{DM}(\Omega)$, contradicts that Ω is an optimal mechanism. *Q.E.D.*

C.7 Proof of Theorem 12

By Theorem 11 $V_q(m) = w_q(m)(1 - w_q(m))(X(m,t) - X(m,l))^2 = c$ for $m \in \mathcal{M}_q^+$, ans since $X(m,u) = w_q(m)X(m,l) + (1 - w_q(m)X(m,t))$, we have $(X(m,t) - X(m,u))((X(m,u) - X(m,l))) = c$. Suppose contrary to the claim, $w_q(m) > 0.5$ some positive measure set of messages in $\mathcal{M}_q^+$, which implies $X(m,t) - X(m,u) > (X(m,u) - X(m,l))$, take a positive measure set of message $M^+ \in \mathcal{M}_q^+$ such that $X(m,t) - X(m,u) > (X(m,u) - X(m,l)) + \delta$ for some $\delta > 0$. Then for any $\epsilon > 0$ there exists a positive measure set of message $M_\epsilon^+ \subseteq M^+$ such that for any $m, m' \in M_\epsilon^+$ and $s = t, l, u$, $|X(m,s) - X(m',s)| < \epsilon$ and $X(m,t) - X(m,u) + \delta < X(m,u) - X(m,l)$.

Let $\Theta_\epsilon^l = \Theta_q^l(M_\epsilon^+)$ and $\Theta_\epsilon^t = \Theta_q^t(M_\epsilon^+)$ be the aggregate set of truth-tellers and liars of M_ϵ^+, and $Pr_\epsilon^l = Pr(\Theta_q^l(M_\epsilon^+))$ and $Pr_\epsilon^t = Pr(\Theta_q^t(M_\epsilon^+))$ be the measure of the two sets. Let $E_\epsilon^l = E[\Theta_\epsilon^l]$, $E_\epsilon^t = E[\Theta_\epsilon^t]$ and $E_\epsilon^u = E[\Theta_\epsilon^l \cup \Theta_\epsilon^t]$ be the corresponding expected values of the sets. Note that $|E_\epsilon^s - X(m,s)| < \epsilon$ for any $m \in M_\epsilon^+$ and $s = t, l, u$, so we have

$$E_\epsilon^t - E_\epsilon^u > E_\epsilon^u - E_\epsilon^l + \delta - 2\epsilon \tag{C.232}$$

$$|(E_\epsilon^t - E_\epsilon^u)(E_\epsilon^u - E_\epsilon^l) - c| < 4\epsilon^2 \tag{C.233}$$

Let $\hat{E}$ be the larger root of $(E_\epsilon^t - \hat{E})(\hat{E} - E_\epsilon^l) - c = 0$. (C.232) and (C.233) imply that for small enough ϵ, $E_\epsilon^t - \hat{E} < \hat{E} - E_\epsilon^l$ and $\hat{E} > E_\epsilon^u + \delta$. Fix any $m \in \mathcal{M}_q^+$ and Let $\underline{u} = P(m)u(X(m,l)) + (1 - P(m))u(X(m,u))$ be the expected payoff of the liars, Let $\hat{x} = u^{-1}(\underline{u})$ be its certainty equivalence. Note that Proposition 2 implies any liar can mimic the payoff of any other liar, so incentive compatibility means all liars receive the same payoff, and $\hat{x} = X(m_q^0, u)$ if an uninspected message m_q^0 exists. Lemma 20 implies $X(m,l) \leq \hat{x} < X(m,u)$ for any $m \in M_\epsilon^+$, so we have

$$E_\epsilon^l \leq \hat{x} < E_\epsilon^u < \hat{E} - \delta < E_\epsilon^t - \delta \tag{C.234}$$

Let z_l, z_t solve

$$z_l Pr_\epsilon^l E_\epsilon^l + z_t Pr_\epsilon^t E_\epsilon^t = (z_l Pr_\epsilon^l + z_t Pr_\epsilon^t)\hat{x} \tag{C.235}$$

$$(1-z_l) Pr_\epsilon^l E_\epsilon^l + (1-z_t) Pr_\epsilon^t E_\epsilon^t = [(1-z_l) Pr_\epsilon^l + (1-z_t) Pr_\epsilon^t]\hat{E} \tag{C.236}$$

Since $Pr_\epsilon^l E_\epsilon^l + Pr_\epsilon^t E_\epsilon^t = (Pr_\epsilon^l + Pr_\epsilon^t)E_\epsilon^u$, so (C.234) means $z_l \in (0,1)$ and $z_t \in [0,1)$

For any positive measure set of types $\hat{\Theta}$, define the mean-preserving division $\hat{\Theta}(z) = \hat{\Theta} \cap [\underline{\theta}(z), \overline{\theta}(z)]$ such that $\underline{\theta}(z)$ and $\overline{\theta}(z)$ solve

$$Pr(\hat{\Theta}(z)) = zPr(\hat{\Theta}) \tag{C.237}$$

$$E[\hat{\Theta}(z)] = E[\hat{\Theta}] \tag{C.238}$$

We divide the liar set Θ_ϵ^l into $\Theta_\epsilon^l(z_l)$ and $\Theta_\epsilon^l/\Theta_\epsilon^l(z_l)$, and truthful set Θ_ϵ^t into $\Theta_\epsilon^t(z_t)$ and $\Theta_\epsilon^t/\Theta_\epsilon^t(z_t)$. The mean-preserving divisions implies $E[\Theta_\epsilon^l(z_l)] = E[\Theta_\epsilon^l/\Theta_\epsilon^l(z_l)] = E_\epsilon^l$ and $E[\Theta_\epsilon^t(z_t)] = E[\Theta_\epsilon^t/\Theta_\epsilon^t(z_t)] = E_\epsilon^t$. From (C.235) and (C.236) we have $E[\Theta_\epsilon^l(z_l) \cup \Theta_\epsilon^t(z_t)] = \hat{x}$ and $E[\Theta_\epsilon^l/\Theta_\epsilon^l(z_l) \cup \Theta_\epsilon^t/\Theta_\epsilon^t(z_t)] = \hat{E}$.

Now define an modified mechanism $\hat{\Omega} = (\hat{q}, \hat{P}, \hat{X})$ where other things remain unchanged, except the set of messages M_ϵ^+ is off-path and an message $\hat{m} = \Theta_\epsilon^t/\Theta_\epsilon^t(z_t)$ is added with $\hat{q}(\hat{m}|\theta) = 1$ for $\theta \in \Theta_\epsilon^l/\Theta_\epsilon^l(z_l) \cup \Theta_\epsilon^t/\Theta_\epsilon^t(z_t)$. The uninspected message m_q^0 (if exists) is modified to $m_{\hat{q}}^0 = m_q^0 \cup \Theta_\epsilon^l(z_l) \cup \Theta_\epsilon^t(z_t)$ with $\hat{q}(m_{\hat{q}}^0|\theta) = 1$ for $\theta \in \Theta_\epsilon^l(z_l) \cup \Theta_\epsilon^t(z_t)$.

The sequentially rational actions for the modified messages $\hat{m}$ and $m_{\hat{q}}^0$ are $\hat{X}(\hat{m}, t) = E_\epsilon^t$, $\hat{X}(\hat{m}, l) = E_\epsilon^l$, $\hat{X}(\hat{m}, u) = \hat{E}$, $\hat{X}(m_{\hat{q}}^0, u) = \hat{x}$. By (C.234) we still have $\hat{X}(m, l) \le \hat{X}(m_{\hat{q}}^0, u) < \hat{X}(m, u)$ for all $m \in \mathcal{M}_{\hat{q}}^+$, so (a) in Lemma 20 is satisfied. For the newly added inspected message $\hat{m}$, $(\hat{X}(m_{\hat{q}}^0, t) - \hat{X}(m_{\hat{q}}^0, u))(\hat{X}(m_{\hat{q}}^0, u) - \hat{X}(m_{\hat{q}}^0, l)) = (E_\epsilon^t - \hat{E})(\hat{E} - E_\epsilon^l) = c$, so (b) in Lemma 20 is satisfied. Therefore there exists $\hat{P}$ such that $\hat{\Omega}$ is incentive compatible.

To compare DM's exante payoffs, let G_Ω^u and $G_{\hat{\Omega}}^u$ be the distribution of uninspected induced actions of the two mechanism defined in (C.3). By sequential rationality the two distributions have the same mean $\int_0^1 x dG_{\hat{\Omega}}^u(x) = \int_0^1 x dG_\Omega^u(x) = \int_\Theta \theta dF(\theta)$ and they differ only by actions induced by the set $\Theta_\epsilon^l \cup \Theta_\epsilon^t$. In the original mechanism Ω, a type in $\Theta_\epsilon^l \cup \Theta_\epsilon^t$ sends some $m \in M_\epsilon^+$ with induced action $X(m, u)$ where $|X(m, u) - E_\epsilon^u| < \epsilon$; In the modified mechanism $\hat{\Omega}$, a type in $\Theta_\epsilon^l \cup \Theta_\epsilon^t$ send either $\hat{m}$ or $m_{\hat{q}}^0$ with induced action either $\hat{X}(m_{\hat{q}}^0, u) = \hat{x}$ or $\hat{X}(\hat{m}, u) = \hat{E}$. (C.234) implies that for small enough ϵ, $X(m_{\hat{q}}^0, u) < X(m, u) < \hat{X}(\hat{m}, u)$ for any $m \in M_\epsilon^+$. Therefore, $G_{\hat{\Omega}}^u$ is a mean-preserving spread of G_Ω^u, which means $\int_{[0,1]} x^2 dG_{\hat{\Omega}}^u(x) > \int_{[0,1]} x^2 dG_\Omega^u(x)$, then (C.2) implies $EU_{DM}(\hat{\Omega}) > EU_{DM}(\Omega)$, contradicts that Ω is an optimal mechanism. *Q.E.D.*

C.8 Proof of Theorem 13

By Lemma 33 the statement is true if $Pr(\Theta_q^0) > 0$. Now suppose there is an optimal mechanism Ω in which $Pr(\Theta_q^0) = 0$, then Theorem 11 implies $V_q(m) = c$ for almost every $m \in \mathcal{M}_q^+$, then Lemma 21 implies $EU_{DM}(\Omega) = EU_{DM}^I(\Omega) \leq -c$. Since Ω is incentive compatible, Assumption 4 holds, but then Lemma 4 implies that the decreasing mechanism Ω_d is also incentive compatible, with $EU_{DM}(\Omega_d) > c \geq EU_{DM}(\Omega)$, so Ω with $Pr(\Theta_q^0) = 0$ cannot be optimal. *Q.E.D.*

C.9 Proof of Theorem 14

For an optimal mechanism Ω where $E[\underline{\theta}_\Omega, \overline{\theta}_\Omega] > \frac{\underline{\theta}_\Omega + \overline{\theta}_\Omega}{2}$, by Lemma 30 for almost every $m \in \mathcal{M}_q^+$, $\Theta_q^t(m) = \{X(m,s)\}$ and $\Theta_q^l(m) = \{X(m,l)\}$ are two singleton sets, so there exist $\overline{\Theta} \subseteq [\overline{\theta}_\Omega, 1]$ where $Pr(\overline{\Theta}) = Pr([\overline{\theta}_\Omega, 1])$, $\underline{\Theta} \subseteq [0, \underline{\theta}_\Omega]$ where $Pr(\underline{\Theta}) = Pr([0, \underline{\theta}_\Omega])$ and a bijective matching function $\phi : \overline{\Theta} \to \underline{\Theta}$ such that for $m \in \overline{\Theta}$, $\Theta_q^t(m) = \{m\}$ and $\Theta_q^l(m) = \{\phi(m)\}$. By Lemma 31,

$$\begin{aligned} m_1 > m_2 &\iff X(m_1, t) > X(m_2, t) \\ &\iff X(m_1, l) < X(m_2, l) \iff \phi(m_1) < \phi(m_2) \end{aligned}$$

so ϕ is a strictly decreasing function. Since Ω is optimal, $w_q(m) = w^-(m - \phi(m))$, so (C.221) implies that ϕ must the a solution of (3.21), so $\phi = \phi_d$ with almost identical domain.

Lemma 33 implies $(\overline{\theta}_\Omega - E[\underline{\theta}_\Omega, \overline{\theta}_\Omega])(E[\underline{\theta}_\Omega, \overline{\theta}_\Omega] - \underline{\theta}_\Omega) = c$, which combines with $E[\underline{\theta}_\Omega, \overline{\theta}_\Omega] > \frac{\underline{\theta}_\Omega + \overline{\theta}_\Omega}{2}$ implies that $X_u^*(\overline{\theta}_\Omega, \underline{\theta}_\Omega) = E[\underline{\theta}_\Omega, \overline{\theta}_\Omega]$. By Lemma 3 there exists a unique $\overline{\theta}_d$ such that $X_u^*(\overline{\theta}_d, \phi(\overline{\theta}_d)) = E[\phi_d(\overline{\theta}_d), \overline{\theta}_d]$, so $\overline{\theta}_\Omega = \overline{\theta}_d$ and $\underline{\theta}_\Omega = \phi(\overline{\theta}_d) = \underline{\theta}_d$.

Therefore, Ω and Ω_d are both characterized by the two cutoffs $\overline{\theta}_d, \underline{\theta}_d$ and matching function ϕ_d, which is defined for almost the same set of types, thus they are equal almost everywhere. *Q.E.D.*

C.10 Proof of Theorem 15

We claim that for any $m, m' \in \mathcal{M}_q$ such that $P(m) < 1$ and $P(m') < 1$, $X(m,u) = X(m',u)$ and $P(m) = P(m')$. Suppose contrary to the claim, $X(m',u) > X(m,u)$. Then since $X(m,u) = E[\Theta_q^u(m)]$, there exists $\overline{\theta} \geq X(m,u)$ such that $m_q(\overline{\theta}) = m$. Since (3.32) implies $[1 - P(m)][X(m,u) - \overline{\theta}] \geq [1 - P(m')][X(m',u) - \overline{\theta}]$, thus $P(m) > P(m')$. Similarly, there exists $\underline{\theta} \leq X(m,u)$ such that $m_q(\underline{\theta}) = m$, and (3.32) implies $[1 - P(m)][X(m,u) - \underline{\theta}] \geq [1 - P(m')][X(m',u) - \underline{\theta}]$, thus $P(m) < P(m')$, contradiction. Therefore, it must be the case that $X(m,u) = X(m',u)$, which by (3.32) implies that $P(m) = P(m')$.

We claim that if there exists $\tilde{m} \in \mathcal{M}_q$ such that $P(\tilde{m}) = 1$, then $P(m) = 1$ for almost every $m \in \mathcal{M}_q$. Suppose Contrary to the claim, there exists positive measure subset $\hat{\Theta} \subseteq \Theta$ such that $P(m_q(\theta)) < 1$ for any $\theta \in \hat{\Theta}$, then the first claim implies that $X(m_q(\theta), u) = E[\hat{\Theta}]$, and since $Pr(\hat{\Theta}) > 0$, there exists $\theta \in \hat{\Theta}$ such that $\theta > E[\hat{\Theta}] = X(m_q(\theta), u)$, but then $[1 - P(m_q(\theta)][X(m_q(\theta), u) - \theta] < 0 = [1 - P(\tilde{m})][X(\tilde{m}, u) - \theta]$, $m_q(\theta)$ is not optimal for θ, contradiction.

The first two claims implies that $P(m) = \hat{P}$ for almost every $m \in \mathcal{M}_q$, where $hatP$ is a constant, and if $hatP < 1$, $X(m, u) = E[\Theta]$ for almost every $m \in \mathcal{M}_q$. Since $P(m) = P(m')$, (3.30) and (3.31) imply that $v_q(m) = V_q(m') = Var(\Theta_q^u(m)) = Var(\Theta_q^u(m')) = Var(\Theta)$. Therefore, If $c > Var(\Theta)$, then $c > v_q(m)$ and $P(m) = 0$ for almost every $m \in \mathcal{M}_q$, and the mechanism is uninformative; if $c < Var(\Theta)$, then $c < v_q(m)$ and $P(m) = 1$ for almost every $m \in \mathcal{M}_q$, the mechanism is state-verifying. *Q.E.D.*

C.11 Proof of Theorem 16

Suppose $c \geq Var(\Theta)$, then $EU_{DM}^s = Var(\Theta)$. Consider an uninformative mechanism Ω under lie-detection technology, where $m_q(\theta) = m_q^0 = \Theta$ for any $\theta \in \Theta$ and $P(m_q^0) = 0$, $X(m_q^0, u) = E[\Theta]$. It is clear that such mechanism is incentive compatible, and $EU_{DM}(\Omega) = Var(\Theta) = EU_{DM}^s$. Therefore, $EU_{DM}(\Omega^*) \geq EU_{DM}(\Omega) = EU_{DM}^s$.//

Suppose $c < Var(\Theta)$, then $c < Var(\Theta) < (1 - E\Theta)(E(\Theta) - 0) \leq \frac{1}{4}$, where the second inequality holds by Bhatia–Davis inequality. Since $c < (1 - E\Theta)(E(\Theta) - 0)$, we have $\frac{1}{2} - \sqrt{\frac{1}{4} - c} < E(\Theta) < \frac{1}{2} + \sqrt{\frac{1}{4} - c}$, thus Assumption 4 is satisfied, and by Lemma 4, Ω_d is incentive compatible with $EU_{DM}(\Omega_d) < c$. Therefore, $EU_{DM}(\Omega^*) \geq EU_{DM}(\Omega_d) > c = EU_{DM}^s$. *Q.E.D.*

C.12 Generalization for Mixed Messaging Rules

This Appendix generalizes the model by allowing mixed message rule and shows the result that any mixed message rule has an equivalent corresponding pure message rule. Proof of Lemma 2 is also included. We consider a potentially mixed message rule q, and let $\mathcal{M}_q = \{m \in \mathcal{M} : m \in supp(q(.|\Theta))\}$ be the set of all on-path messages, where $q(.|\Theta) \equiv \int_\Theta q(.|\theta)dF(\theta)$ is the marginal message distribution. For any $m \in \mathcal{M}$, let $\Theta_t(m) = \{\theta \in m\}$ and $\Theta_l(m) = \{\theta \in \Theta/m\}$ be the sets of truthful types and liars of a statement m, and let $\Theta_u(m) = \Theta$ for notational convenience. Then for a sequentially rational action rule X, any subset of on-path message $M \subseteq \mathcal{M}_q$ and $s \in \{t, l, u\}$,

$$\int_M \int_{\Theta_s(m)} q(m|\theta)X(m, s)dF(\theta)dm = \int_M \int_{\Theta_s(m)} q(m|\theta)\theta dF(\theta)dm \tag{C.239}$$

and information value of inspection satisfies

$$V_q(m) = w_q(m)(1 - w_q(m))(X(m,t) - X(m,l))^2 \tag{C.240}$$

where for any $M \subseteq \mathcal{M}_q$, $w_q(.)$ satisfies

$$\int_M w_q(m) \int_\Theta q(m|\theta)dF(\theta)dm = \int_M \int_{\Theta_l(m)} q(m|\theta)dF(\theta)dm$$

Sequential rationality of inspection rule P follows condition (3.9) with $V_q(.)$ defined at (C.240). A message rule q is **optimal** given P and X if for any $\theta \in \Theta$, $m \in supp(q(.|\theta))$ and $m' \in supp(q(.|\Theta))$,

$$EU_{X,P}(m|\theta) \geq EU_{X,P}(m'|\theta) \tag{C.241}$$

Then definition 1 defines incentive compatible mechanism. Given an incentive compatible mechanism Ω, DM's exante expected payoff is:

$$\begin{aligned} EU_{DM}(\Omega) = -\int_\Theta \int_{\mathcal{M}_q} q(m|\theta)[(1 - P(m))(X(m,u) - \theta)^2 \\ + P(m) \sum_{s=t,l} \mathbf{1}(\theta \in \Theta_s(m))[(X(m,s) - \theta)^2 + c]]dmdF(\theta) \end{aligned} \tag{C.242}$$

Sequential rationality of the action rule X implies that

$$EU_{DM}(\Omega) = \int_{[0,1]} x^2 dG_\Omega(x) - cp_\Omega - E[\theta^2] \tag{C.243}$$

Where

$$\begin{aligned} G_\Omega(x) = \int_\Theta \int_{\mathcal{M}_q} q(m|\theta)[(1 - P(m))\mathbf{1}(X(m,u) \leq x) \\ + P(m) \sum_{s=t,l} \mathbf{1}(\theta \in \Theta_s(m))\mathbf{1}(X(m,s) \leq x)]dmdF(\theta) \end{aligned} \tag{C.244}$$

is the distribution of induced action,

$$p_\Omega = \int_{\mathcal{M}_q} P(m) \int_\Theta q(m|\theta)dF(\theta)dm \tag{C.245}$$

is the exante inspection probability, and $E[\theta^2] \equiv \int_\Theta \theta^2 dF(\theta)$.

The following proposition shows that any incentive compatible mechanism with mixed message rule has an corresponding incentive compatible mechanism with pure message rule that generate the same outcome distribution, alongside with the proof of Lemma 2.

Proposition 1 *For any incentive compatible mechanism* $\Omega = (q, P, X)$ *there exists a distribution equivalent mechanism* $(\hat{q}, \hat{P}, \hat{X})$ *such that :*
(i) $\hat{q}(.|\theta)$ *is degenerate for any* $\theta \in \Theta$*;*
(ii) $\hat{X}(m, t) \geq \hat{X}(m, l)$ *for any* $m \in \mathcal{M}_{\hat{q}}$*;*
(iii) $m = \{\theta \in \Theta : \hat{q}(m|\theta) = 1$ *and* $\theta \in m\}$ *for any* $m \in \mathcal{M}_{\hat{q}}$*.*

Proof of Proposition 1: Given an incentive compatible mechanism $\Omega = (q, P, X)$, let $\mathcal{M}_q^0 = \{m \in \mathcal{M}_q : P(m) = 0\}$ and $\mathcal{M}_q^+ = \{m \in \mathcal{M}_q : P(m) > 0\}$ be the sets of on-path uninspected messages and inspected messages respectively. For $m \in \mathcal{M}_q$, let $\Theta_q(m) = \{\theta \in \Theta : m \in supp(q(.|\theta))\}$ be the set of types who send m. For $m \in \mathcal{M}_q^0$, we can without loss assume $X(m, t) = X(m, l) = X(m, u) \equiv \underline{x}(m)$ since $X(m, t)$ and $X(m, l)$ does not affect the induced distribution. For $m \in \mathcal{M}_q^+$, it must be the case that $X(m, t) \neq X(m, l)$, otherwise equations (C.239) and (3.8) means $V_q(m) = 0$, contradicts to $P(m) > 0$. Let $\overline{x}(m) = \max\{X(m, t), X(m, l)\}$ and $\underline{x}(m) = \min\{X(m, t), X(m, l)\}$ be the larger and smaller induced action of m, $\overline{\Theta}_q(m) = \begin{cases} m \cap \Theta_q(m) & \text{if} X(m,t) > X(m,l) \\ ([0,1]/m) \cap \Theta_q(m) & \text{if} X(m,t) < X(m,l) \end{cases}$ and $\underline{\Theta}_q(m) = \Theta_q(m)/\overline{\Theta}_q(m)$ be the sets of types who send m and induce $\overline{x}(m)$ and $\underline{x}(m)$ respectively, in case of inspection.

For any $m \in \mathcal{M}_q$, denote C and $\underline{EU}(m) = P(m)u(\underline{x}(m)) + (1 - P(m))u(X(m, u))$ and $\overline{EU}(m) = P(m)u(\overline{x}(m)) + (1 - P(m))u(X(m, u))$ be the expected utilities for the senders of m who induces the smaller and larger actions, respectively. The following Lemma shows that expected utility for senders who induces the lower actions must be the same for all on-path messages.

Lemma 34 *For any* $m, m' \in \mathcal{M}_q$, $\overline{EU}(m) \geq \underline{EU}(m) = \underline{EU}(m')$*.*

Proof of Lemma 34: The first inequality holds directly from the definition of $\overline{x}$ and $\underline{x}$. Now suppose on the contrary, there exists $m, m' \in \mathcal{M}_q$ such that $\underline{EU}(m) > \underline{EU}(m')$, then by deviating to m, the sender of m' who induce $\underline{x}(m')$ will get either $\underline{EU}(m) > \underline{EU}(m')$ or $\overline{EU}(m) \geq \underline{EU}(m) > \underline{EU}(m')$, that contradicts to sender's optimality (C.241).

■

The following Lemma shows that no type can mix between a message m in which he would induce the larger action $\overline{x}(m)$ and another message m' which is either uninspected or he would induce the smaller action $\underline{x}(m')$.

Lemma 35 *For any* $\theta \in \Theta$*, there does not exist* $m, m' \in supp(q(.|\theta)$ *such that*
(1) $P(m) > 0$ *and* $\theta \in \overline{\Theta}_q(m)$*,*
(2) Either $P(m') = 0$*, or* $P(m') > 0$ *and* $\theta \in \underline{\Theta}_q(m')$*.*

Proof of Lemma 35: Suppose on the contrary, there exists a type θ and such pair of messages $m, m' \in supp(q(.|\theta)$. Then sender's optimality (C.241) requires $\underline{EU}(m') = \overline{EU}(m)$, and $P(m) >$

0 implies that value of inspecting m is positive, so $\overline{x}_(m) > \underline{x}_(m)$, but that means $\underline{EU}(m') = \overline{EU}(m) > \underline{EU}(m)$, which contradicts Lemma 34. ■

Let $\overline{\Theta}_q = \{\theta \in \Theta : \exists m \in \mathcal{M}_q^+ \text{ such that } \theta \in \overline{\Theta}_q(m)\}$ and $\underline{\Theta}_q = \{\theta \in \Theta : \exists m \in \mathcal{M}_q^+ \text{ such that } \theta \in \underline{\Theta}_q(m)\}$ be the sets of types who induce the larger action and smaller action for some on-path messages; $\Theta_q^0 = \{\theta \in \Theta : \exists m \in \mathcal{M}_q^0 \text{ such that } \theta \in \Theta_q(m)\}$ be the sets of types who send some uninspected message. It is straight-forward that $\overline{\Theta}_q \cup \underline{\Theta}_q \cup \Theta_q^0 = \Theta$. Lemma 35 implies that $\overline{\Theta}_q \cap (\underline{\Theta}_q \cup \Theta_q^0) = \emptyset$.

We refer to $\overline{\Theta}_q$ and $\underline{\Theta}_q \cup \Theta_q^0$ as the sets of superior types and inferior types, respectively. If $\overline{\Theta}_q$ and $\underline{\Theta}_q$ has zero measure, then messages are uninspected almost surely, i.e. $p_I = 0$, then sender's optimality and sequential rationality of DM imply that the induced action is a single mass point at $x = E[\Theta]$, which can be trivially achieved by pooling every $\theta \in \Theta$ at $m = \Theta$. Therefore, we consider the case where both $\overline{\Theta}_q$ and $\underline{\Theta}_q \cup \Theta_q^0$ have positive measure.

Lemma 36 *For any $\Theta' \subseteq \Theta$, $M' \subseteq \mathcal{M}$, message rule $q : \Theta' \to \Delta\mathcal{M}$ and induced action function $x(.) : M' \to [0,1]$, let*

$$F(y|\Theta', M', q(.)) = \frac{\int_{\Theta'} \int_{M'} q(m|\theta)\mathbf{1}(\theta \leq y)dmdF(\theta)}{\int_{\Theta'} \int_{M'} q(m|\theta)dmdF(\theta)}$$

be the conditional type distribution, and

$$G(y|\Theta', M', x(.), q(.)) = \frac{\int_{\Theta'} \int_{M'} q(m|\theta)\mathbf{1}(x(m) \leq y)dmdF(\theta)}{\int_{\Theta'} \int_{M'} q(m|\theta)dmdF(\theta)}$$

be the conditional induced action distribution. If

$$\int_{\Theta'} q(m'|\theta)x(m')dF(\theta) = \int_{\Theta'} q(m'|\theta)\theta dF(\theta) \text{ for any } m' \in M' \tag{C.246}$$

then $G(.|\Theta', M', x(.), q(.))$ second order stochastically dominates (S.O.S.D.) $F(.|\Theta', M', q(.))$, and the two distribution have the same mean.

Proof of Lemma 36: Let $\Delta(y) = (\int_{\Theta'} \int_{M'} q(m|\theta)dmdF(\theta)) \int_0^y [F(y'|\Theta', M', q(.)) - G(y'|\Theta', M', x(.), q(.))]dy'$. Then,

$$
\begin{aligned}
\Delta(y) = & \int_{\Theta'} \int_{M'} q(m|\theta) \int_0^y [\mathbf{1}(\theta \leq y') - \mathbf{1}(x(m) \leq y')] dy' dm dF(\theta) \\
= & \int_{\Theta' \cap [0,y]} \int_{M'} q(m|\theta)(y-\theta) dm dF(\theta) \\
& - \int_{\Theta'} \int_{m \in M' : x(m) \leq y} q(m|\theta)(y - x(m)) dm dF(\theta) \\
= & \int_{\Theta' \cap [0,y]} \int_{M'} q(m|\theta)(y-\theta) dm dF(\theta) \\
& - \int_{\Theta'} \int_{m \in M' : x(m) \leq y} q(m|\theta)(y - \theta) dm dF(\theta) \\
= & \int_{\Theta' \cap [0,y]} \int_{m \in M' : x(m) > y'} q(m|\theta)(y-\theta) dm dF(\theta) \\
& - \int_{\Theta' \cap (y,1]} \int_{m \in M' : x(m) \leq y} q(m|\theta)(y - \theta) dm dF(\theta) \\
\geq & 0
\end{aligned}
$$

where the third equality hold because of (C.246) Therefore $\int_0^y [F(y'|\Theta', M', q(.)) - G(y'|\Theta', M', x(.), q(.))] dy' \geq 0$ for any $y \in [0,1]$, $G(.|\Theta', M', x(.), q(.))$ second order stochastically dominates $F(.|\Theta', M', q(.))$. Besides, since $\int_0^1 [F(y'|\Theta', M', q(.)) - G(y'|\Theta', M', x(.), q(.))] dy' = 0$, which implies $\int_0^1 y' dF(y'|\Theta', M', q(.)) = \int_0^1 y' dG(y'|\Theta', M', x(.), q(.))$, thus the two distributions have the same mean. ■

Let $F_1(\theta) = \frac{\int_{\theta' \in \overline{\Theta}_q \cap [0,\theta]} dF(\theta')}{\int_{\theta' \in \overline{\Theta}_q} dF(\theta')}$ and $F_2(.) = \frac{\int_{\theta' \in * \underline{\Theta}_q \cup \Theta_q^0 (\cap [0,\theta]} dF(\theta')}{\int_{\theta' \in \underline{\Theta}_q \cup \Theta_q^0} dF(\theta')}$ be the conditional distributions of superior types and inferior types; Let

$$
G_1(x) = \frac{\int_{\overline{\Theta}_q} \int_{\mathcal{M}_q^+} q(m|\theta) \mathbf{1}(\overline{x}(m) \leq x) dm dF(\theta)}{\int_{\overline{\Theta}_q} dF(\theta)}
$$

be the distribution of superior induced action under inspection;

$$
G_2(x) = \frac{\int_{\underline{\Theta}_q \cup \Theta_q^0} \int_{\mathcal{M}_q} q(m|\theta) \mathbf{1}(\underline{x}(m) \leq x) dm dF(\theta)}{\int_{\underline{\Theta}_q \cup \Theta_q^0} dF(\theta)}
$$

be the distribution of inferior induced action under inspection and induced action of uninspected messages. For any $m' \in \mathcal{M}_q^+$ we have

$$
\begin{aligned}
\int_{\overline{\Theta}_q} q(m'|\theta) \overline{x}(m') dF(\theta) = & \int_{\overline{\Theta}_q} q(m'|\theta) \max\{X(m',t), X(m',l)\} dF(\theta) \\
= & \int_{\overline{\Theta}_q} q(m'|\theta) \theta dF(\theta) \qquad \text{(C.247)}
\end{aligned}
$$

Where the second equality holds because of (C.239) and $\overline{\Theta}_q \cap (\underline{\Theta}_q \cup \Theta_q^0) = \emptyset$. Similarly, For any $m' \in \mathcal{M}_q^+$

$$\int_{\underline{\Theta}_q \cup \Theta_q^0} q(m'|\theta)\underline{x}(m')dF(\theta) = \int_{\underline{\Theta}_q \cup \Theta_q^0} q(m'|\theta)\min\{X(m',t), X(m',l)\}dF(\theta)$$
$$= \int_{\underline{\Theta}_q \cup \Theta_q^0} q(m'|\theta)\theta dF(\theta) \tag{C.248}$$

and for any $m' \in \mathcal{M}_q^0$

$$\int_{\underline{\Theta}_q \cup \Theta_q^0} q(m'|\theta)\underline{x}(m')dF(\theta) = \int_{\underline{\Theta}_q \cup \Theta_q^0} q(m'|\theta)X(m',u)dF(\theta)$$
$$= \int_{\underline{\Theta}_q \cup \Theta_q^0} q(m'|\theta)\theta dF(\theta) \tag{C.249}$$

Therefore, for $i = 1, 2$ the distribution pair $(F_i(.), G_i(.))$ satisfies condition (C.246), so Lemma 36 implies that $G_i(.)$ S.O.S.D. $F_i(.)$ and the two distributions have the same mean. Denote $\Theta_1 = \overline{\Theta}_q$ and $\Theta_2 = \underline{\Theta}_q \cup \Theta_q^0$. The following Lemma shows that there exist a pure message rule on Θ_i that induces the same sequentially rational action distribution G_i.

Lemma 37 *For $i = 1, 2$, there exists a function $\Gamma_i : [0,1] \to 2^{\Theta_i}$ such that:*
(i) $\int_{\theta \in \Gamma_i([0,1])} dF_i(\theta) = 1$;
(ii) For any collection of disjoint set $\{A_j\}$, $\int_{\theta \in \Gamma_i(\cup_j A_j)} dF_i(\theta) = \sum_j \int_{\theta \in \Gamma_i(A_j)} dF_i(\theta)$ (iii) for any interval $[a, b] \in [0, 1]$, $\int_{\theta \in \Gamma_i([a,b])} dF_i(\theta) = b - a$, and $\int_{\theta \in \Gamma_i([a,b])} \theta dF_i(\theta) = \int_a^b x_i(z)dz$, where $x_i(z) = G_i^{-1}(z^+) \equiv sup\{x \in [0,1] : G_i(x) \le z\}$ is the inverse function of G_i.

Proof of Lemma 37: For $z \in [0, 1]$, let $\theta_i(z) = F_i^{-1}(z^+) \equiv sup\{\theta \in [0, 1] : F_i(\theta) \le z\}$ be the inverse function of F_i, and $\theta_i'(z^+)$ be its right derivative. We have $\theta_i'(z^+) = \frac{\int_{\Theta_i} dF(\theta)}{f(\theta_i(z))}$. If $f(\theta_i(z)) = 0$, let $\theta_i'(z^+) = +\infty$. Similarly, let $x_i'(z^+)$ be the right derivative of the inverse function of G_i. Since G_i S.O.S.D. F_i by Lemma 36, we have $\int_0^a x_i(z)dz \ge \int_0^a \theta_i(z)dz$ for all $a \in [0, 1]$.

Let $y_0 = \sup\{y \in [0, 1] : \theta_i'(F_i(y)) \le x_i'(G_i(y)) \forall y \in [0, y_0)\}$. For $a \in [0, 1]$, define $\Gamma_i(a) = \{\theta_i(\underline{z}(a))\} \cup \{\theta_i(\overline{z}(a))\}$, where $(\underline{z}(.), \overline{z}(.))$ is a pair of differential equations defined as follows:

$$\underline{z}(0) = 0 \tag{C.250}$$
$$\overline{z}(0) = F_i(y_0) \tag{C.251}$$
$$\underline{z}'(a) + \overline{z}'(a) = 1 \tag{C.252}$$
$$\underline{z}'(a)\theta_i(\underline{z}(a)) + \overline{z}'(a)\theta_i(\overline{z}(a)) = x_i(a) \tag{C.253}$$
$$\underline{z}'(a) \ge 0; \overline{z}'(a) \ge 0 \tag{C.254}$$

Now we need to ensure that $(\underline{z}(a), \overline{z}(a))$ has a valid solution for $a \in [0,1]$. Note that in order for (C.252) - (C.254) to hold, it is necessary that $\theta_i(\underline{z}(a)) \leq x_i(a) \leq \theta_i(\overline{z}(a))$ for any a. It is satisfied at $a = 0$, i.e. $\theta_i(0) \leq x_i(0) \leq \theta_i(F_i(y_0)) = y_0$, with at least one strict inequality, where $x_i(0) \leq y_0$ because $x_i'(.)$ is not defined for $y < x_i(0)$; if $x_i(0) < \theta_i(0)$, then $\int_0^a x_i(z)dz < \int_0^a \theta_i(0)dz$ for small z, contradicts to the fact that G_i S.O.S.D. F_i; if $y_0 = \theta_i(0) = x_i(0)$, then $x_i(z) < \theta_i(z)$ for small z, contradicts to the fact that G_i S.O.S.D. F_i. Now suppose for some $\hat{a} \in (0,1]$, $\underline{z}(a)$ and $\overline{z}(a)$ are well-defined for $a \in [0, \hat{a}]$, with $\underline{z}(a) \leq F_i(y_0)$. Now we consider the following potential cases:

Case 1: If $\theta_i(\underline{z}(\hat{a})) < x_i(\hat{a}) < \theta_i(\overline{z}(\hat{a}))$ and $\theta_i(\underline{z}(\hat{a})) < y_0$, then the system has a solution for a right neighbourhood $a \in (\hat{a}, \hat{a}+\epsilon]$, where $\theta_i(\underline{z}(a)) < y_0$.

Case 2: If $\theta_i(\underline{z}(\hat{a})) = x_i(\hat{a}) < y_0 \leq \theta_i(\overline{z}(\hat{a}))$, we have $\theta_i'(\hat{a})\underline{z}'(\hat{a}) \leq \theta_i'(\hat{a}) \leq x_i'(\hat{a})$ where the second inequality hold by the definition of y_0. Therefore, there is a solution for a right neighbourhood $a \in (\hat{a}, \hat{a}+\epsilon]$ such that $\theta_i(\underline{z}(a) \leq x_i(a) < y_0$.

Case 3: If $\theta_i(\underline{z}(\hat{a})) < y_0 \leq x_i(\hat{a}) = \theta_i(\overline{z}(\hat{a}))$ and $x_i'(\hat{a}) \leq \theta_i'(\overline{z}(\hat{a}))$, then for $\overline{z}'(\hat{a}) = 1$, $\theta_i'(\overline{z}(\hat{a}))\overline{z}'(\hat{a}) \geq x_i'(\hat{a})$ and so there is a solution for the right neighbourhood such that $\theta_i(\overline{z}(a) \geq x_i(a)$.

Case 4: If for some $\hat{a}_j$, $j \geq 1$, $\theta_i(\underline{z}(\hat{a}_j)) < y_0 \leq x_i(\hat{a}_j) = \theta_i(\overline{z}(\hat{a}_j))$ and $x_i'(\hat{a}_j) > \theta_i'(\overline{z}(\hat{a}_j))$.

In this case, we introduce a jump to the system. Let $y_j = \sup\{y \in [x_i(\hat{a}_j), 1] : \theta_i'(F_i(y)) \leq x_i'(G_i(y)) \forall y \in [x_i(\hat{a}_j), y_j)\}$. Note that $y_j > x_i(\hat{a}_j)$ because $x_i'(\hat{a}_j) > \theta_i'(\overline{z}(\hat{a}_j))$. Let $\overline{z}(\hat{a}_j^+) = F_i(y_j)$ be the new initial conditions for $\overline{z}(.)$ at $\hat{a}_j$. If case 4 occurs at some $\hat{a}_{j+1} > ... > \hat{a}_j$, then define $y_{j+1} = \sup\{y \in [x_i(\hat{a}_{j+1}), 1] : \theta_i'(F_i(y)) \leq x_i'(G_i(y)) \forall y \in [x_i(\hat{a}_{j+1}), y_{j+1})\}$ and let $\overline{z}(\hat{a}_{j+1}^+) = F_i(y_{j+1})$.

Case 5: If for some $\tilde{a}_k$, $k \geq 0$, $\theta_i(\underline{z}(\tilde{a}_k)) = y_k$, and if y_{k+1} has been defined at some $\hat{a}_{k+1} < \tilde{a}_k$, then introduce a jump $\underline{z}(\tilde{a}_k^+) = \overline{z}(\hat{a}_{k+1})$; If y_{k+1} has not been defined, then we have $\underline{z}([0, \tilde{a}_k]) = [0, y_0] \cup (\overline{z}(\hat{a}_1), y_1] \cup ... \cup (\overline{z}(\hat{a}_k), y_k]$; $\overline{z}([0, \tilde{a}_k]) = [y_0, \overline{z}(\hat{a}_1)] \cup (y_1, \overline{z}(\hat{a}_2)] \cup ... \cup (y_k, \overline{z}(\tilde{a}_k)]$, combined with (C.252) imply that $\overline{z}(\tilde{a}_k) = \tilde{a}_k$, then (C.253) implies that $\int_0^{\tilde{a}_k} \theta dF_i(\theta) = \int_0^{\tilde{a}_k} x_i(z)dz$. If $\tilde{a}_k = 1$, a complete solution is defined; if $\tilde{a}_k < 1$, it must be the case that $x_i'(\tilde{a}_k) \geq \theta_i(\overline{z}(\tilde{a}_k))$ because G_i S.O.S.D. F_i. Then define $y_{k+1} = \sup\{y \in [x_i(\tilde{a}_k), 1] : \theta_i'(F_i(y)) \leq x_i'(G_i(y)) \forall y \in [x_i(\tilde{a}_k), y_{k+1})\}$, and introduce jumps $\overline{z}(\tilde{a}_k^+) = F_i(y_{k+1})$ and $\underline{z}(\tilde{a}_k^+) = \overline{z}(\tilde{a}_k)$.

The above construction shows that there exists a solution of $\underline{z}(.), \overline{z}(.)$ such that $\underline{z}([0,1]) \cap \overline{z}([0,1])$ has zero measure, then (C.252) and (C.254) imply $\underline{z}([0,1]) \cup \overline{z}([0,1])$ has measure 1, so $\int_{\theta \in \Gamma_i([0,1])} dF_i(\theta) = \int_{\theta \in \theta_i(\underline{z}([0,1])) \cup \theta_i(\overline{z}([0,1]))} dF_i(\theta) = \int_{z \in \underline{z}([0,1]) \cup \overline{z}([0,1])} dz = 1$, condition (i) of the lemma is satisfied. (C.254) implies that for any collection of disjoint sets $\{A_j\}$, $\cap_j(\underline{z}(A_j) \cup \overline{z}(A_j))$ has zero measure, so $\cap_j(\Gamma_i(A_j)) = \cap_j(\theta_i(\underline{z}(A_j)) \cup \theta_i(\overline{z}(A_j)))$ has zero measure, thus condition (ii) is satisfied. (C.252) and (C.254) imply that for any $[a,b] \in [0,1]$, $\int_{\theta_i(\underline{z}([a,b])) \cup \theta_i(\underline{z}([a,b]))} dF_i(\theta) = b - a$ since $\underline{z}([a,b]) \cap \overline{z}([a,b])$; has zero measure; and (C.253) implies that $\int_{\theta_i(\underline{z}([a,b])) \cup \theta_i(\underline{z}([a,b]))} \theta dF_i(\theta) = \int_a^b x_i(z)dz$. Therefore, conditional (iii) is satisfied. ■

For $i = 1, 2$, let $\Theta_i^d = \{\theta \in \Theta_i : \exists a, b \in [0,1] \text{ s.t. } a \neq b \text{ and } \theta \in \Gamma_i(a) \cap \Gamma_i(b)\}$ be the set of duplicate type; $\Theta_i^n = \{\theta \in \Theta_i : \theta \notin \Gamma_i([0,1])\}$ be the set of non-included type. Conditions (i) and (ii) of Lemma 37 ensure that the two sets have zero measure. Now for $a \in [0,1]$, let $\hat{\Gamma}_i(a) = (\Gamma_i(a)/\Theta_i^d) \cup \{\theta \in \Theta_i^d \cup \Theta_i^n : \theta = a\}$. Since $\Theta_i^d \cup \Theta_i^n$ has zero measure, condition (iii) of Lemma 37 still applies to $\hat{\Gamma}_i$, but now $\hat{\Gamma}_i([0,1]) = \Theta_i$ and $\hat{\Gamma}_i(a) \cap \hat{\Gamma}_i(b) = \emptyset$ for any $a \neq b$. Now we can make a transformation for the original mechanism (q, P, X) to a new mechanism $(\hat{q}, \hat{P}, \hat{X})$. First assign an arbitrary strict ranking $r : \mathcal{M}_q \to \mathbb{R}$ to the original set of on-path messages. Then for any $m \in \mathcal{M}_q^+$, let

$$z_1^-(m) = G_1^-(\overline{x}(m)) + \int_{m' \in \mathcal{M}_q^+ : \overline{x}(m') = \overline{x}(m) \text{ and } r(m') < r(m)} \int_{\Theta_1} q(m'|\theta) dF_1(\theta) dm' \tag{C.255}$$

$$z_1^+(m) = z_1^-(m) + \int_{\Theta_1} q(m|\theta) dF_1(\theta) \tag{C.256}$$

where $G_1^-(x)$ is G_i evaluated at the left limit of x. Similarly, let

$$z_2^-(m) = G_2^-(\underline{x}(m)) + \int_{m' \in \mathcal{M}_q : \underline{x}(m') = \underline{x}(m) \text{ and } r(m') < r(m)} \int_{\Theta_2} q(m'|\theta) dF_2(\theta) dm' \tag{C.257}$$

$$z_2^+(m) = z_2^-(m) + \int_{\Theta_2} q(m|\theta) dF_2(\theta) \tag{C.258}$$

Note that $z_i^-(m)$ and $z_i^+(m)$ specifies the cumulative positions and measure of message m in the induced action distribution G_i, while r is an arbitrary ranking to break the tie for any messages that induce the same actions, so that each message occupies a unique position in the distribution.

For each inspected on-path message $m \in \mathcal{M}_q^+$, define the transformed message $T(m)$, set of truth-telling senders $\hat{\Theta}_t(m)$ and set of lying senders $\hat{\Theta}_l(m)$ as follows:

$$T(m) = \begin{cases} \hat{\Gamma}_1((z_1^-(m), z_1^+(m)]) & \text{if } z_1^+(m) > z_1^-(m) \\ \hat{\Gamma}_1(z_1^-(m)) & \text{if } z_1^+(m) = z_1^-(m) \end{cases} \tag{C.259}$$

$$\hat{\Theta}_t(m) = T(m) \tag{C.260}$$

$$\hat{\Theta}_l(m) = \begin{cases} \hat{\Gamma}_2((z_2^-(m), z_2^+(m)]) & \text{if } z_2^+(m) > z_2^-(m) \\ \hat{\Gamma}_2(z_2^-(m)) & \text{if } z_2^+(m) = z_2^-(m) \end{cases} \tag{C.261}$$

And for each uninspected on-path message $m \in \mathcal{M}_q^0$,

$$T(m) = \begin{cases} \hat{\Gamma}_2((z_2^-(m), z_2^+(m)]) & \text{if } z_2^+(m) > z_2^-(m) \\ \hat{\Gamma}_2(z_2^-(m)) & \text{if } z_2^+(m) = z_2^-(m) \end{cases} \tag{C.262}$$

$$\hat{\Theta}_t(m) = T(m) \tag{C.263}$$

$$\hat{\Theta}_l(m) = \emptyset \tag{C.264}$$

Corresponding to each message m in the original mechanism, the set of types $\hat{\Theta}_t(m)\cup\hat{\Theta}_l(m)$ send the transformed message $T(m)$ with probability 1 in the new mechanism $(\hat{q},\hat{P},\hat{X})$, i.e. for each $m \in \mathcal{M}_q$,

$$\hat{q}(T(m)|\theta) = \begin{cases} 1 & \text{if } \theta \in \hat{\Theta}_t(m) \cup \hat{\Theta}_l(m) \\ 0 & \text{Otherwise} \end{cases} \tag{C.265}$$

and the inspection probability and induced actions for $T(m)$ in the new mechanism are identical to those for m in the original mechanism, with the truthful types always inducing the higher action for an inspected message, i.e.

$$\hat{P}(T(m)) = P(m) \tag{C.266}$$

$$\hat{X}(T(m),u) = X(m,u) \tag{C.267}$$

$$\hat{X}(T(m),t) = \overline{x}(m) \tag{C.268}$$

$$\hat{X}(T(m),l) = \underline{x}(m) \tag{C.269}$$

Conditions (i) and (ii) of Proposition 1 are satisfied by (C.265), (C.268) and (C.269); Condition (iii) is satisfied because $T(m) \subseteq \hat{\Theta}_t(m) \cup \hat{\Theta}_l(m)$. Given any set of inspected messages $M \subseteq \mathcal{M}_q^+$,

$$\begin{aligned}
\int_{\hat{\Theta}_t(M)} dF(\theta) &= \int_{\Theta_1} dF(\theta) \int_{\hat{\Theta}_t(M)} dF_1(\theta) \\
&= \int_{\Theta_1} dF(\theta) \int_{\cup_{m\in M}\hat{\Gamma}_i([z_1^-(m),z_1^+(m)])} dF_1(\theta) \\
&= \int_{\Theta_1} dF(\theta) \int_0^1 \mathbb{1}(\exists m \in M : s \in [z_1^-(m), z_1^+(m)]) ds \\
&= \int_{\Theta_1} dF(\theta) \int_M \int_{\Theta_1} q(m|\theta) dF_1(\theta) dm \\
&= \int_M \int_{\Theta_1} q(m|\theta) dF(\theta) dm
\end{aligned} \tag{C.270}$$

where the second equality holds by definition of $\hat{\Theta}_t(.)$; the third equality holds because by (iii) of Lemma 37, $\int_{\hat{\Gamma}_i([a,b])} dF_1(\theta) = b - a$; the forth equality holds by (C.256). (C.270) implies that the exante probability of the sender sending the transformed set of messages $T(M)$ in $(\hat{q},\hat{P},\hat{X})$ and being a truthful type equal the sender exante probability of the sender sending set of messages M in (q,P,X) and being the type that induce a higher action. Similarly, we have

$$\int_{\hat{\Theta}_l(M)} dF(\theta) = \int_M \int_{\Theta_2} q(m|\theta) dF(\theta) dm \tag{C.271}$$

which means the exante probability of the sender sending $T(M)$ in $(\hat{q}, \hat{P}, \hat{X})$ and being a liar equal the exante probability of the sender sending M in (q, P, X) and being the type that induce a lower action. Combining (C.270), (C.271) and (C.265), we have

$$\int_M \int_\Theta q(T(m)|\theta)dF(\theta)dm = \int_M \int_\Theta q(m|\theta)dF(\theta)dm \tag{C.272}$$

which means the exante probability of the sender sending $T(M)$ in $(\hat{q}, \hat{P}, \hat{X})$ the exante probability of the sender sending M in (q, P, X).
For any set of uninspected message $M^0 \subseteq \mathcal{M}_q^0$,

$$\int_{\hat{\Theta}_t(M^0)} dF(\theta) = \int_{M^0} \int_{\Theta_2} q(m|\theta)dF(\theta)dm \tag{C.273}$$

Given $M \subseteq \mathcal{M}_q^+$, the exante expected type of the truthful senders who send the corresponding transformed set of messages $T(M)$ in $(\hat{q}, \hat{P}, \hat{X})$ is

$$\begin{aligned}
\int_M \int_{\theta \in T(m)} \hat{q}(T(m)|\theta)\theta dF(\theta)dm &= \int_{\hat{\Theta}_t(M)} \theta dF(\theta) \\
&= \int_{\Theta_1} dF(\theta) \int_{\hat{\Theta}_t(M)} \theta dF_1(\theta) \\
&= \int_{\Theta_1} dF(\theta) \int_{\cup_{m\in M}\hat{\Gamma}_i([z_1^-(m), z_1^+(m)])} \theta dF_1(\theta) \\
&= \int_{\Theta_1} dF(\theta) \int_0^1 \mathbb{1}(\exists m \in M : s \in [z_1^-(m), z_1^+(m)]) x_i(s)ds \\
&= \int_{\Theta_1} dF(\theta) \int_M \int_{\Theta_1} q(m|\theta)dF_1(\theta)\overline{x}(m)dm \\
&= \int_M \int_{\Theta_1} q(m|\theta)dF(\theta)\overline{x}(m)dm \\
&= \int_M \int_{\Theta_1} q(m|\theta)dF(\theta)\hat{X}(T(m), t)dm \\
&= \int_M \int_{\theta \in T(m)} \hat{q}(T(m)|\theta)dF(\theta)\hat{X}(T(m), t)dm
\end{aligned} \tag{C.274}$$

where the first equality holds by (C.260) and (C.265); the fourth equality holds by (iii) of Lemma 37, $\int_{\hat{\Gamma}_i([a,b])} \theta dF_1(\theta) = \int_a^b x_i(s)ds$; the fifth equality because $x_i(s) = G_1^{-1}(s)$, and (C.255) - (C.256) imply that $G_1^{-1}(s) = \overline{x}(m)$ for $s \in [z_1^-(m), z_1^+(m)]$; the seventh equality holds by (C.268), and the last equality holds by (C.260), (C.265) and (C.270). Similarly, we have

$$\int_M \int_{\theta \notin T(m)} \hat{q}(T(m)|\theta)\theta dF(\theta)dm = \int_M \int_{\theta \notin T(m)} \hat{q}(T(m)|\theta)dF(\theta)\hat{X}(T(m), l)dm \tag{C.275}$$

and

$$\int_M \int_\Theta \hat{q}(T(m)|\theta)\theta dF(\theta)dm = \int_M \int_{\Theta_1} q(m|\theta)dF(\theta)\overline{x}(m)dm + \int_M \int_{\Theta_2} q(m|\theta)dF(\theta)\underline{x}(m)dm$$
$$= \int_M \int_\Theta q(m|\theta)dF(\theta)X(m,u)dm$$
$$= \int_M \int_\Theta \hat{q}(T(m)|\theta)dF(\theta)\hat{X}(T(m),u)dm \tag{C.276}$$

where the first equality holds by (C.274) and (C.275); the first equality holds by incentive compatibility of (q, P, X); the last equality holds by (C.272) and (C.267).

For any set of uninspected message $M^0 \subseteq \mathcal{M}_q^0$,

$$\int_M \int_{\theta \notin T(m)} \hat{q}(T(m)|\theta)\theta dF(\theta)dm = \int_M \int_{\hat{\Theta}_l(M)} dF(\theta)\underline{x}(m)dm$$
$$= \int_M \int_{\hat{\Theta}_l(M)} dF(\theta)X(m,u)dm$$
$$= \int_M \int_\Theta \hat{q}(T(m)|\theta)dF(\theta)\hat{X}(T(m),u)dm \tag{C.277}$$

(C.274) - (C.277) imply that $\hat{X}$ is sequentially rational given $\hat{q}$. In $(\hat{q}, \hat{P}, \hat{X})$, value of inspection for any transformed set of messages $T(M)$ is given by

$$\int_M \int_{\hat{\Theta}_t(m)} dF(\theta) \int_{\hat{\Theta}_l(m)} dF(\theta)(\hat{X}(T(m),t) - \hat{X}(T(m),l))^2 dm$$
$$= \int_M \int_{\Theta_1} q(m|\theta)dF(\theta) \int_{\Theta_2} q(m|\theta)dF(\theta)(X(m,t) - X(m,l))^2 dm \tag{C.278}$$

so value of inspecting $T(M)$ in $(\hat{q}, \hat{P}, \hat{X})$ equal value of inspecting M in (q, P, X), this combined with (C.266) imply that $\hat{P}$ is sequentially rational given $\hat{q}$.

To show that $\hat{q}$ is optimal given $\hat{P}$ and $\hat{X}$, (C.260) and (C.266) - (C.269) imply for any $m \in \mathcal{M}_q^0$ and $\theta \in \hat{\Theta}_t(m)$,

$$EU_{\hat{X},\hat{P}}(T(m)|\theta) = P(m)u(\overline{x}(m)) + (1 - P(m))u(X(m,u)) = \overline{EU}(m) \tag{C.279}$$

and for $\theta \in \hat{\Theta}_l(m)$,

$$EU_{\hat{X},\hat{P}}(T(m)|\theta) = P(m)u(\underline{x}(m)) + (1 - P(m))u(X(m,u)) = \underline{EU}(m) \tag{C.280}$$

If $\theta \in \hat{\Theta}_t(m) cup \hat{\Theta}_l(m)$ deviates to any other messages $T(m') \neq T(m)$, then since $\theta \notin T(m') = \hat{\Theta}_t(m')$, θ will be a liar of $T(m')$ and

$$EU_{\hat{X},\hat{P}}(T(m')|\theta) = P(m')u(\underline{x}(m')) + (1 - P(m'))u(X(m',u)) = \underline{EU}(m') \tag{C.281}$$

Since $\overline{EU}(m) \geq \underline{EU}(m) = \underline{EU}(m')$ by Lemma 34, sender's optimality is satisfied. This concludes that $(\hat{q}, \hat{P}, \hat{X})$ is incentive compatible.

Finally, since there is an one to one transformation from $\mathcal{M}_q$ to $T(\mathcal{M}_q)$, with the same inspection probability, induced actions and measures of senders for each pair $(m, T(m))$, it is straight-forward that (q, P, X) and $(\hat{q}, \hat{P}, \hat{X})$ are distributional equivalent.

Q.E.D.