Double Diffusive Convection and Steady 2D Salt Finger Solutions in Porous Media

by

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Abstract

Double diffusive convection is a naturally occurring phenomenon playing important roles in geophysical, astrophysical, and oceanographic events alike. Herein, it is the transfer of heat by fluid movement driven by the differing rates of diffusion of temperature and salinity, developing into one of 2 regimes: diffusive convection or salt fingering. We consider this problem in a porous medium, relevant in situations regarding permafrost, magma, and soils amongst others.

We begin by performing and comparing linear and nonlinear stability analyses near the onset of instability, as in existing work. We ultimately find that the two methods result in the same bounds for the onset of instability for salt fingering and steady diffusive convection, and so we conclude there are no subcritical cases. This is further confirmed in the third section, wherein we conduct a weakly nonlinear stability analysis using asymptotic expansions. In both the diffusive convection and the salt fingering cases, the amplitude equations obtained indicate that supercritical instabilities occur. In the case of oscillatory diffusive convection, the regime of criticality depends on the relative size of the density ratio to the Lewis number.

Extending previous work by considering a porous medium, we consider modes creating the fastest growing fingers, resulting in fingers with a smaller horizontal/vertical aspect ratio. These fingers are studied first in a vertically unbounded domain, then in a bounded one. We find evolution equations for both cases, and plot the resulting steady-state solution of the temperature amplitude of the latter. We find that in the zero limit of the horizontal/vertical aspect ratio, the temperature amplitudes of the steady solutions converge in both the salt-heat and sugar-salt configurations.

Lay Summary

Convection is the movement within a fluid, caused by warmer (less dense) fluid rising and cooler (more dense) fluid sinking. Double diffusive convection (DDC) is an extension of this buoyancy-driven process incorporating a second component, mass, into the overall solution density. In this work DDC is the transfer of heat and the mass component salt, and the mixing process can be classified as one of two types: diffusive convection (DC), or salt fingering (SF). We will be considering DDC in porous media, that is to say some material containing voids through which fluid may flow. This medium is of interest, as it is relevant to many geophysical situations regarding materials such as permafrost, magma, and soils. Important parameter values permitting mixing will be determined, as well as solutions the system settles to over time.

Preface

Chapters two and three contain original, unpublished work done by the author K. Willcott with guidance from their advisor N. Balmforth, as well as a more detailed exposition of existing results. Chapters four and five are an extension into a porous medium analog of work done by M. Stern and T. Radko [29, 35].

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Acknowledgments

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Chapter 1

Introduction

Until the observation by Stommel, Arons, and Blanchard in 1956 and Melvin Stern's further investigation in 1960, the phenomenon of double diffusive convection (DDC) existed silently [34, 37]. Beginning in an oceanographical setting, the effects of double diffusion have now been found in a variety of applications - spanning from the evolution of stars and planets to layer formation in magma [27, 34]. While most laboratory experiments use sugar and salt as their components, this thesis will be considering an oceanographic setting, wherein the two components are salt and heat, the latter diffusing at a rate about 100 times faster than the former [27, 34].

Double diffusion is a buoyancy driven flow caused by differences in diffusivity, and can commonly be thought of as an extension of the Rayleigh-Bénard convection, wherein the fluid is set between two rigid plates [11, 13, 20, 27, 30, 34]. Two regimes are possible - diffusive convection and salt fingering. Both require a globally stable density stratification, the difference being whether it is salt or heat acting as the stabilizing component [27]. For salt fingering to occur, the faster diffusing component (heat) is stabilizing, and one begins with warm, salty water overlying cool, fresh water. Diffusive convection requires the opposite - a stabilizing solutal component, with cool, fresh water overlying warm, salty. Within the diffusive convection regime, there are two more branches of convection - steady and oscillatory. Oscillatory convection can also be referred to as overstability

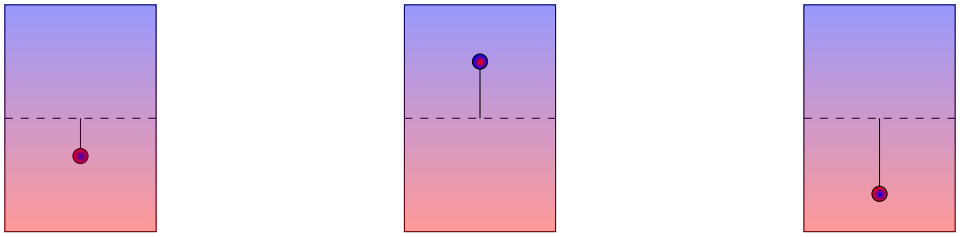


Figure 1.1: A sketch of oscillatory convection, with the ball representing some parcel of fluid, and the vertical line representing its trajectory. Here, blue represents cool, fresh water and red represents warm, salty water.

- given a parcel of fluid moving downwards, it absorbs heat from the lower water more quickly than it does salt and so it becomes relatively lighter and shoots back upwards. As this parcel cools once more, it becomes the same temperature as the surrounding fluid, but is now saltier and so it descends again, and the cycle continues [27, 33, 43]. Figure 1.1 outlines this movement.

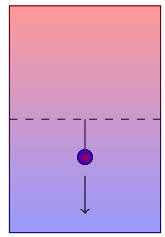


Figure 1.2: A sketch of salt fingering, with the ball representing some parcel of fluid, and the vertical line representing its trajectory. Here, blue represents cool, fresh water and red represents warm, salty water.

Salt fingering does not result in the displaced parcel moving up and down. A perturbation of the initial state causes a parcel of warm, salty fluid to move downwards, and this parcel loses heat to the surrounding fluid, becomes heavier than its surroundings and simply continues to descend. This is outlined in figure 1.2 [27, 30, 34]. In terms of the density ratio, which can also be thought of as the stability ratio, we find that the stabilizing component will contribute more [20]. We will define this ratio to be $R_\rho = \frac{\alpha \Delta S}{\beta \Delta T}$; whether the value exceeds unity or not depends on which regime we find ourselves in.

DDC has been widely studied using numerical, experimental, and analytical methods. It has been found analytically and confirmed numerically that the fastest growing finger will dominate the field and dictate the speed of convection [29, 31, 32, 35]. Since the two components used are typically sugar and salt when studied in laboratories, when numeric analyses are executed a lower Lewis number, i.e. the ratio of mass diffusivity to heat diffusivity, is used in order to create a clear comparison in results [28, 29, 35].

The onset of convection in a porous medium was studied early on by Lapwood, who determined the critical Rayleigh number is $4\pi^2$. This was further studied and confirmed by Elder [9, 19]. The finite amplitude of salt fingers has also been studied in porous media. Green examined double diffusion in a groundwater setting, and found that fingers do form, however their vertical coherence is limited by their horizontal dispersion [12]. The onset of both steady and oscillatory double-diffusive convections have been studied in porous media using a variety of boundary conditions, both analytically and numerically [7, 23, 27]. In particular, Chen (1993) found that for a saturated porous medium, the numerics were in good agreement with existing results, though computationally expensive once moving away from steady convection [7] . It has also been suggested that changes in density due to solutal concentration give rise to instabilities more effectively than changes due to temperature [12].

This thesis looks specifically at the porous media case. By definition, a porous medium is one which is permeable by water or air. Further, porosity, commonly denoted ϕ, is defined as the ratio or percent of a material occupied by void space, and is valued between 0 and 1 and so the ratio or percent of space occupied by a solid is denoted by $1 - \phi$ [23]. In order to proceed with analyses, averaging over a porous area is necessary. Herein, we will be following a spatial averaging approach, in which a representative elementary volume, or a sample of the porous media containing both solid and void components, which is a good representation of the media as a whole [2, 23].

Working in this media alters some assumptions and includes some alternative approximations. However, one approximation that is retained is the Boussinesq approximation. Here, variations in density are neglected unless the terms are multiplied with gravity - that is to say its effects on buoyancy terms dominates those on inertial terms [44]. An important alteration to the momentum equation in a porous medium is Darcy's law, in which we see a relation between the flow rate and pressure [23]:

$$\nabla p = -\frac{\mu}{k}\mathbf{u}$$

where $\mathbf{u}$ is velocity, p is pressure, μ is the dynamic viscosity of the fluid, and k is permeability which is the ease with which fluid may pass through the medium. Coupling Darcy's law with the Boussinesq approximations leaves us with the following set of equations, outlining heat and mass transfer in porous media:

$$\mathbf{u} = -\frac{k}{\mu}\left(\nabla p + \rho g \hat{\mathbf{z}}\right) \tag{1.0.1}$$

$$\Phi\frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T = \kappa_T \nabla^2 T \tag{1.0.2}$$

$$\phi\frac{\partial S}{\partial t} + \mathbf{u}\cdot\nabla S = \phi\kappa_S \nabla^2 S, \tag{1.0.3}$$

where $\rho = \rho_0(1 + \alpha(S - S_0) - \beta(T - T_0))$. Here, κ_S, κ_T are salt and heat diffusivity respectively, α, β are the solutal and thermal coefficients of expansion, Φ is defined as $\frac{\phi\rho_l c_l + (1-\phi)\rho_s c_s}{\rho_l c_s}$ where $\rho_l c_l$ and $\rho_s c_s$ are the liquid and solid volumetric heat capacities, and ρ_0 is a reference density [13, 23, 40]. We write the Lewis number (Le) as $\frac{\phi\kappa_S}{\kappa_T}$, with τ representing its inverse, $\frac{1}{Le}$. As we are working with an incompressible Newtonian fluid, our two-dimensional mass continuity is given by

$$\nabla\cdot\mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0.$$

In double diffusion, we encounter secular instabilities which occur on the timescale in which the loss of temperature causes instability [16]. And so to remove the secular terms arising in our weakly nonlinear analyses, we also employ the method of multiple scales, defining both a fast time t and a slow time $\mathcal{T} = \epsilon^2 t$ [16, 22, 24, 25].

When conducting weakly nonlinear analyses, we employ the Landau equation to determine if

the systems are sub or supercritical [1, 26]. It is an expression of the amplitude growth over time and can be written in one of two ways:

$$\frac{d|A(t)|^2}{dt} = 2\sigma_r|A(t)|^2 - l_r|A(t)|^4$$

or

$$\frac{d|A(t)|}{dt} = \sigma_r|A(t)| - \frac{l_r}{2}|A(t)|^2,$$

where σ_r is a function of some bifurcation parameter, and l_r is some constant. In this thesis, the bifurcation parameter in question will be the Rayleigh number. If l_r is 0, then the linear equation is regained. The sign of l_r dictates how the effect of σ_r on the system's stability will be interpreted. More specifically, the signs of σ_r and l_r will give insight into whether the system is sub or supercritically stable, that is to say, whether there exists some other critical value resulting in a steady, non-stationary solution. In the event of a positive l_r, as σ_r increases, the system becomes increasingly unstable, and consequently we have supercritical stability. Conversely, Landau found that if l_r is negative, stability is achieved with a negative σ_r, and so there should be another critical value lower than R_c determined; systems of this form are then classified as being subcritical.

Weakly nonlinear analysis is helpful in examining cases near the onset of instability under the influence of a small perturbation. By using asymptotic expansions and evaluating the governing equations at each order of epsilon, we will ultimately be able to find a Landau equation and be able to determine how higher order contributions of the bifurcating parameter, herein the Rayleigh number, will lead to instability.

1.1 Our Problem

In this work we will be considering two-dimensional double diffusion in a porous medium. In order to simplify the governing equations, we consider the case where porosity $\phi = 1$ as would be found in a Hele-Shaw cell [13, 42]. We will be considering dimensionless equations, and tank depth will be taken to be $d = 1$ in chapters two and three. In chapter four we consider some unbounded fluid, and in chapter five we consider the dimensionless height of the finger, denoted H. We are studying an incompressible fluid bound by impermeable surfaces. The boundary conditions are then [23]

$$w = 0 \qquad \text{at } z = 0, d.$$

For diffusive convection the solutal and thermal boundary conditions are:

$$\begin{aligned} T &= T_0 + \Delta T, \quad S = S_0 + \Delta S \qquad && \text{at } z = 0 \\ T &= T_0, \quad S = S_0 && \text{at } z = d \end{aligned}$$

and for salt fingering:

$$
\begin{aligned}
&T = T_0, \;\; S = S_0 && \text{at } z = 0 \\
&T = T_0 + \Delta T, \;\; S = S_0 + \Delta S && \text{at } z = d.
\end{aligned}
$$

In both cases, perturbations are zero on the boundaries. In the first two chapters, we consider a tank of incompressible fluid, half filled with warm, salty water and half with cool, fresh water. The determination of top and bottommost solutions will depend on the regime we are studying. In our analyses, we take the thermal Rayleigh number R to be our bifurcation parameter. We define the density ratio R_ρ in terms of the solutal and thermal Rayleigh numbers, $R_\rho = \frac{R_S}{R}$, and so all three are intrinsically linked. Increasing the solutal Rayleigh number, or decreasing the thermal Rayleigh number effectively increases the influence of the solutal field - depending on the regime in which we are in, this increases either the stabilizing or destabilizing component's contribution. We follow the linear and nonlinear analyses of the problem outlined by Hewitt [13], and the methodology of a weakly nonlinear analysis as seen in Veronis [21].

The purpose of this work is first to carefully determine the onset and type of stability of both regimes, and then to look more closely at long, thin salt fingers and determine evolutionary equations for temperature, all in a porous medium. We first consider the linear stability problem, in order to determine the critical Rayleigh number at the onset of purely thermal convection. Linear stability analyses ultimately give some sufficient condition for instability, that is to say it gives the value above which the system is unstable [14]. However, instability may also occur below this threshold, arising from the nonlinear terms, and so it is important to employ the complementary nonlinear stability analysis. In doing so, we find that for both steady diffusive convection and salt fingering, the energy bounds match those of the linear stability. This tells us that we should not expect any subcritical instabilities in the system, as the conditions for stability and instability coincide [14]. In the case of oscillatory convection, we find that there are some regions in which the nonlinear bound falls below the linear, indicating that there may be some unstable region not captured by the linearized equations.

For the third portion of the first two chapters, as others have done, we conduct a weakly nonlinear stability analysis in order to arrive at the aforementioned Landau equations, describing the slow timescale amplitude evolution over time in response to small perturbations [21, 25, 27, 29]. In order to proceed, we use asymptotic analysis in order to study equations at increasingly small orders. In

particular, we will use the following:

$$\begin{aligned}
\psi &= \psi_0 + \epsilon\psi_1 + \epsilon^2\psi_2 + \epsilon^3\psi_3 \\
T &= T_0 + \epsilon T_1 + \epsilon^2 T_2 + \epsilon^3 T_3 \\
S &= S_0 + \epsilon S_1 + \epsilon^2 S_2 + \epsilon^3 S_3 \\
R &= R_0 + \epsilon R_1 + \epsilon^2 R_2 + \epsilon^3 R_3.
\end{aligned}$$

For our purposes, we need only to expand to $O(\epsilon^3)$, as it is at this order which we will determine the evolution equation. In these chapters, the aspect ratio will be of $O(1)$, i.e. the fingers considered will be rather square in shape.

As the fastest growing finger dominates the field, in the final two chapters we move away from the state of marginal stability and look to the fastest growing finger mode [18, 29, 31, 32, 35, 41]. These fingers will have a smaller aspect ratio and so will be longer than they are wide. We consider the porous medium analog of Stern and Radko's 1998 and 2000 papers respectively and ultimately find the main difference being the resulting temperature amplitudes [29, 35]. In chapter four, we consider some vertically unbounded domain and study salt fingers of a smaller aspect ratio by considering the mode resulting in the fastest growth rate, rather than the fingers near the onset of instability. In chapter five we keep our focus on the salt fingering regime, but in a bounded domain. We ultimately determine an equation describing the amplitude of temperature with respect to height.

Chapter 2

Diffusive Convection

2.1 Assumptions and Set-Up

Before beginning the calculations pertaining to stability in the diffusive regime, we first need to address the assumptions made - both about the problem's set-up and the parameter values. The set-up we are considering is then a two-dimensional box of height d with hot, salty water underlying cool, fresh water [23, 27]. Upon nondimensionalizing[1], we take d to be 1, with $T = S = 1$ on the bottom and $T = S = 0$ on top with T and S representing dimensionless thermal and solutal concentrations respectively.

In the case of diffusive convection, the temperature field acts as the destabilizing component, and salinity stabilizes the system. For diffusive convection to occur, the stabilizing component must contribute more than the destabilizing, so in the case of $R_\rho = \frac{\alpha\Delta S}{\beta\Delta T}$, the density ratio will exceed unity [13, 17, 27]. Further, the density ρ is defined to be $\rho_0\,(1 + \alpha\Delta S - \beta\Delta T)$. In order for to satisfy double diffusion's globally stable density gradient requirement, we need a bottom-heavy configuration. Since we have declared the top and bottom values of temperature and salinity to be 0 and 1 respectively, we must satisfy the following equation:

$$\begin{aligned} \rho_{top} - \rho_{bottom} &= -\alpha\Delta S + \beta\Delta T < 0 \\ \implies \frac{\alpha\Delta S}{\beta\Delta T} &> 1. \end{aligned}$$

Within the diffusive convection regime, either steady or oscillatory convection may occur [13, 27]. We will see in the proceeding chapters that the conditions and critical Rayleigh numbers required will differ between the two regimes. We also note that in order to proceed with the oscillatory convection analysis, the system must be globally unstable (i.e. $R_\rho < 1$) [23]. And so the oscillatory convection analyses below will correspond to a density ratio R_ρ of less than one, which consequently

[1] See Appendix A for nondimensionalizing calculations

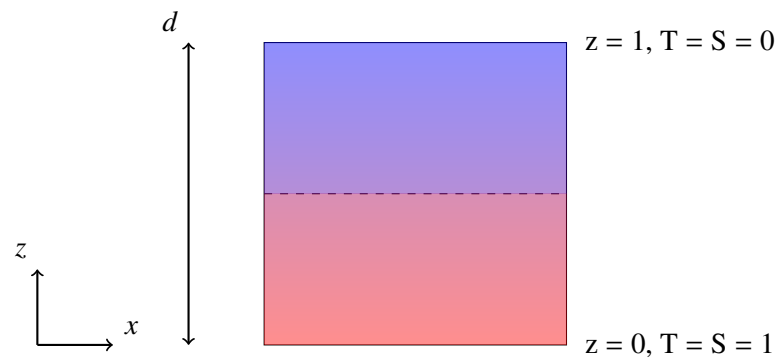


Figure 2.1: For the diffusive convection regime, a tank of depth d, with red representing hot, salty water and blue as cool, fresh water. 0 and 1 represent the dimensionless values of depth (z), temperature (T), and salinity (S).

causes a small change of sign in the governing equation in order to prevent obtaining a negative Rayleigh number.

2.2 Linear Stability Analysis

To conduct the linear stability analysis, we begin by assuming the following solution forms, comprised of the base states and perturbations satisfying the boundary conditions [13, 23, 25, 40]:

$$\begin{aligned}
\psi &= \psi^* e^{ikx+\sigma t}\sin(m\pi z) + \text{c.c.} \\
T &= 1 - z + T^* e^{ikx+\sigma t}\sin(m\pi z) + \text{c.c.} \\
S &= 1 - z + S^* e^{ikx+\sigma t}\sin(m\pi z) + \text{c.c.},
\end{aligned}$$

Where k and m are the horizontal wave numbers, and σ is the growth rate. The base states $T = S = 1 - z$ and $\psi = 0$ represent pure thermal diffusion, and the asterisk represents perturbed states. The dimensionless governing equations are:

$$\nabla^2\psi = R\left(T_x - R_\rho S_x\right) \tag{2.2.1}$$

$$T_t + \mathbf{u}\cdot\nabla T = \nabla^2 T \tag{2.2.2}$$

$$S_t + \mathbf{u}\cdot\nabla S = \frac{1}{Le}\nabla^2 S. \tag{2.2.3}$$

Since we want to linearize the equations, we are concerned with terms containing first-order perturbations (i.e. the perturbed states ψ^*, T^*, S^*). Looking at equations (2.2.2) - (2.2.3) we see there is an advection term in the latter two. We look at this term more closely before writing out the full, substituted equations. Beginning with $\mathbf{u}\cdot\nabla T$, we replace $\mathbf{u} = (u, w)$ with the streamfunction,

$\mathbf{u} = (\psi_z, -\psi_x)$ and let $\tilde{\psi} = \psi^* e^{ikx+\sigma t}\sin(m\pi z)$, $\tilde{T} = T^* e^{ikx+\sigma t}\sin(m\pi z)$ and $\tilde{S} = S^* e^{ikx+\sigma t}\sin(m\pi z)$. Substituting the solution forms into this give the following:

$$(\tilde{\psi}_z, -\tilde{\psi}_x) \cdot \left((1 - z + \tilde{T})_x, (1 - z + \tilde{T})_z\right).$$

We eliminate any second-order perturbations (i.e. any product of $\tilde{\cdot}$ terms) and retain only $\tilde{\psi}_x$. The same will be produced from $\mathbf{u} \cdot \nabla S$, and so the linearized equations is as follows:

$$\begin{aligned}
\nabla^2\psi &= R\left(T_x - R_\rho S_x\right) \\
T_t + \psi_x &= \nabla^2 T \\
S_t + \psi_x &= \frac{1}{Le}\nabla^2 S.
\end{aligned}$$

Substituting the solution forms into the above equations, and dividing through by the common term $e^{ikx+\sigma t}\sin(m\pi z)$ we obtain the following:

$$(-k^2 - m^2\pi^2)\psi^* = R(ikT^* - ikR_\rho S^*) \tag{2.2.4}$$

$$\sigma T^* + ik\psi^* = (-k^2 - m^2\pi^2)T^* \tag{2.2.5}$$

$$\sigma S^* + ik\psi^* = \frac{1}{Le}(-k^2 - m^2\pi^2)S^*. \tag{2.2.6}$$

Substituting (2.2.5) and (2.2.6) into (2.2.4), and dividing through by ψ^* leaves an equation in terms of σ. Organizing by order of σ gives:

$$-Le\sigma^2 + \sigma\left(\frac{Rk^2 Le}{k^2 + m^2\pi^2}(R_\rho - 1) - (k^2 + m^2\pi^2)(1 + Le)\right) + \left(Rk^2(R_\rho Le - 1) - (k^2 + m^2\pi^2)^2\right) = 0. \tag{2.2.7}$$

We proceed to the case of marginal stability where $\sigma = 0$. Rearranging the remaining terms, and isolating R gives:

$$R = \frac{(k^2 + m^2\pi^2)^2}{k^2(R_\rho Le - 1)}.$$

The non-zero value of m providing the strictest restriction on the critical Rayleigh number is 1. When considering the case of pure thermal convection, we take the solutal Rayleigh number to be 0 [13]. With the density ratio defined as $R_\rho = RR_S$, for pure thermal convection we are left with $R = \frac{(k^2+\pi^2)^2}{k^2}$. Minimizing over k, we find that $k = \pi$ and so the system becomes linearly unstable when $R = 4\pi^2$.

For oscillatory convection, we write $\sigma = i\omega + \sigma_r$. Again, in a porous medium we require $R_\rho < 1$ for the oscillatory case, and in order to avoid a negative Rayleigh number we change the signs in

the governing equations slightly[2]. Using a slightly altered version of (2.2.7), and setting $\sigma_r = 0$, as we are only concerned with the imaginary portion of the growth rate, leaves

$$i\omega\left(\frac{Rk^2Le}{k^2+\pi^2}(1-R_\rho)-(k^2+\pi^2)(1+Le)\right).$$

Similarly to above, we now determine the point at which this term is zero, as we are assuming a non-zero imaginary growth rate ω. After some minor algebra we arrive at the following:

$$R_{0O} = \frac{(k^2+\pi^2)^2(1+Le)}{k^2Le(1-R_\rho)}.$$

Minimizing with respect to k as above and setting $R_S = 0$, we find that the critical value once again occurs at $k = \pi$, and so oscillatory convection sets in at $R_{0C} = \frac{4\pi^2(1+Le)}{Le}$.

2.3 Nonlinear Stability Analysis

We next consider nonlinear, or energy, stability. By taking power integrals and using Euler-Lagrange equations, we will determine optimal energy stability bounds and consequently compare them to the linear stability bounds [15, 39].

We take the base steady state as $1-z$ and thermal and solutal perturbations as θ, σ respectively, writing temperature and salinity fields are written as $T = 1-z+\theta$ and $S = 1-z+\sigma$. The perturbations satisfy the homogeneous boundary conditions [4, 13]. We begin with the case of steady convection and follow it with oscillatory convection.

2.3.1 Steady Convection

Substituting the above into the dimensionless Boussinesq equations leaves

$$\mathbf{u} = -\nabla p - R\theta\hat{\mathbf{z}} + RR_\rho\sigma\hat{\mathbf{z}} \tag{2.3.1}$$

$$\theta_t - w + \mathbf{u}\cdot\nabla\theta = \nabla^2\theta \tag{2.3.2}$$

$$\sigma_t - w + \mathbf{u}\cdot\nabla\sigma = \frac{1}{Le}\nabla^2\sigma. \tag{2.3.3}$$

Taking the dot product of (2.3.1) with the perturbation velocity $\mathbf{u}$ and multiplying (2.3.2), (2.3.3) by temperature θ, and salinity σ perturbations respectively, and integrating over the domain gives the

[2]These changes and the corresponding calculations can be found in Appendix B.

power integrals [4, 13]:

$$\langle|\mathbf{u}^2|\rangle = -R\langle w\theta\rangle + RR_\rho\langle w\sigma\rangle$$
$$\frac{1}{2}\langle\theta^2\rangle_t - \langle w\theta\rangle = -\langle|\nabla\theta|^2\rangle$$
$$\frac{1}{2}\langle\sigma^2\rangle_t - \langle\sigma w\rangle = -\frac{1}{Le}\langle|\nabla\sigma|^2\rangle.$$

We also multiply (2.3.2) by σ and (2.3.3) by θ and add to obtain a fourth equation

$$\langle\theta\sigma\rangle_t - \langle\theta w\rangle - \langle\sigma w\rangle = -\left(1 + \frac{1}{Le}\right)\langle\nabla\theta\cdot\nabla\sigma\rangle.$$

Next, following previous work, we create a linear combination of the above equations [4, 13]. The combination suggested by Hewitt [13], with a slight modification reflecting the change of sign in R, is as follows:

$$\begin{aligned}
&E_t = I - D\\
&I = \left(1 + A - C^2\right)\langle w\theta\rangle + \left(A + A^2 + B^2 + C^2R_\rho\right)\langle w\sigma\rangle\\
&D = \langle|\nabla\theta|^2\rangle + \left(A^2 + B^2\right)\tau\langle|\nabla\sigma|^2\rangle + A\,(1+\tau)\,\langle\nabla\theta\cdot\nabla\sigma\rangle + \frac{C^2}{R}\langle|\mathbf{u}|^2\rangle.
\end{aligned}$$

Ultimately, we want to reduce the equations such that they no longer contain θ or σ. Again, following Hewitt, we set some $f = \theta + \frac{A(1+\tau)\sigma}{2}$. Using this relation and substituting it into I and D results in[3]:

$$\begin{aligned}
&I = \left(1 + A - C^2\right)\langle wf\rangle + \left(A + A^2 + B^2 + C^2R_\rho - \frac{A}{2}(1+\tau)\left(1 + A - C^2\right)\right)\langle w\sigma\rangle\\
&D = \langle|\nabla f|^2\rangle + \left(B^2\tau - \frac{A^2}{4}(1-\tau)^2\right)\langle|\nabla\sigma|^2\rangle + \frac{C^2}{R}\langle|\mathbf{u}|^2\rangle,
\end{aligned}$$

where E is energy, and A, B, and C are some constants which are yet to be determined. The next step is to eliminate σ terms. To accomplish this, we set the coefficients of $\langle w\sigma\rangle$ and $\langle|\nabla\sigma|^2\rangle$ equal to 0. Based on the above equations, it is then required that $A + A^2 + B^2 + C^2R_\rho - \frac{A}{2}(1+\tau)\left(1 + A - C^2\right)$ and $B^2\tau - \frac{A^2}{4}(1-\tau)^2$ equal 0. Using algebra once more we find that

$$B^2 = \frac{A^2(1-\tau)^2}{4\tau} \qquad \text{and} \qquad C^2 = -\frac{A(1-\tau)\,(2\tau + A(1+\tau))}{2\tau\left(2R_\rho + A(1+\tau)\right)}.$$

[3] Algebraic details for this section can be found in appendix B

and are left with the following:

$$\frac{\partial E}{\partial t} = I - D = -D\left(1 - \frac{I}{D}\right)$$
$$I = \left(1 + A - C^2\right)\langle wf\rangle$$
$$D = \langle|\nabla f|^2\rangle + \frac{C^2}{R}\langle|\mathbf{u}|^2\rangle.$$

For energy stability, we require a decay in energy over time. In other words, we need $\frac{\partial E}{\partial t} < 0$. For a positive D, this implies that the maximum of $\frac{I}{D}$ must be less than 1. Since incompressibility must hold at every point, we impose a point wise constraint of $\langle(\nabla \cdot \mathbf{u})\lambda\rangle$ [15, 38].

We now wish to extremize $F = \frac{I}{D} + \langle(\nabla \cdot \mathbf{u})\lambda\rangle$ and do so by taking the functional derivatives $\delta f, \delta u, \delta w$ and then combining [10]. The functionals are as follows:

$$\frac{\delta F}{\delta f} : \frac{\left(1 + A - C^2\right)w}{D} + \frac{2I\nabla^2 f}{D^2} = 0 \tag{2.3.4}$$
$$\frac{\delta F}{\delta u} : -\frac{2C^2 Iu}{D^2 R} = \lambda_x \tag{2.3.5}$$
$$\frac{\delta F}{\delta w} : \frac{\left(1 + A - C^2\right)f}{D} - \frac{2C^2 Iw}{D^2 R} = \lambda_z. \tag{2.3.6}$$

Taking the z-derivative of (2.3.5), and the x-derivatives of (2.3.4) and (2.3.6) gives:

$$\left(\frac{\delta F}{\delta f}\right)_x : \frac{\left(1 + A - C^2\right)w_x}{D} + \frac{2I\nabla^2 f_x}{D^2} = 0$$
$$\left(\frac{\delta F}{\delta u}\right)_z : -\frac{2C^2 Iu_z}{D^2 R} = \lambda_{xz}$$
$$\left(\frac{\delta F}{\delta w}\right)_x : \frac{\left(1 + A - C^2\right)f_x}{D} - \frac{2C^2 Iw_x}{D^2 R} = \lambda_{zx}.$$

Rearranging, replacing (u, w) with $(\psi_z, -\psi_x)$, and combining the above equations yields

$$\left(\frac{I}{D}\right)^2 \left(\frac{4C^2}{R\left(1 + A - C^2\right)^2}\right)\nabla^4\psi = -\psi_{xx}.$$

Substituting the solution form $\psi \sim e^{ikx+\sigma t}\sin(\pi z)$ from the preceding section:

$$\left(\frac{I}{D}\right)^2 \left(\frac{4C^2}{R\left(1 + A - C^2\right)^2}\right)(k^2 + \pi^2)^2 = k^2 \implies \left(\frac{I}{D}\right)^2 = \frac{k^2 R\left(1 + A - C^2\right)^2}{(k^2 + \pi^2)^2 4C^2}.$$

As we require $\frac{I}{D} < 1$ for energy stability the following must be true:

$$\left(\frac{k^2 R\left(1+A-C^2\right)^2}{(k^2+\pi^2)^2 4C^2}\right)^{\frac{1}{2}} < 1$$
$$\implies \frac{kR^{\frac{1}{2}}\left(1+A-C^2\right)}{(k^2+\pi^2)2C} < 1$$
$$\implies R^{\frac{1}{2}} < \frac{(k^2+\pi^2)2C}{\left(1+A-C^2\right)k}.$$

We note that $\frac{k^2+\pi^2}{k} = R_{crit}^{\frac{1}{2}}$, is the critical Rayleigh number for purely thermal conduction, and so we write the following:

$$\left(\frac{R}{R_{crit}}\right)^{\frac{1}{2}} < \frac{2C}{(1+A-C^2)}.$$

We are now left to maximize $\frac{2C}{(1+A-C^2)}$, or equivalently minimizing $\frac{(1+A-C^2)}{2C}$ over A. Substituting back the aforementioned C^2 value in terms of A, differentiating with respect to A, and setting equal to 0 yields the following possible solutions:

$$A_a = \frac{\pm 2\sqrt{R_\rho \tau}}{\tau+1}, \quad A_b = \frac{\pm 2\sqrt{(\tau^3 - R_\rho)(\tau - R_\rho)} - 2(\tau^2 + R_\rho)}{(\tau+1)^2},$$

which are the same as the first two sets of solutions in [13]. The third set of solutions does not appear here as we were not working with a sixth order polynomial. To avoid imaginary numbers, we require the solutions of A to produce positive C^2 values. We also require a real valued A, but since R_ρ is greater than 1 in the steady convection regime, and τ less than 1, all solutions will be real. The solutions producing positive C^2 come from either the positive or negative root of A_b. Both roots result in the following, noting $R_\rho > 1 > \tau$:

$$\frac{R}{R_{crit}} < \frac{\tau}{R_\rho - \tau}$$

Solving for just R, which is the critical Rayleigh number for energy stability, we arrive at

$$R < \frac{(k^2+\pi^2)^2}{k^2(R_\rho Le - 1)},$$

which is identical to the bound achieved in the linear stability analysis. Based on this, we do not expect any subcritical instabilities. The resulting curves are plotted in Matlab.

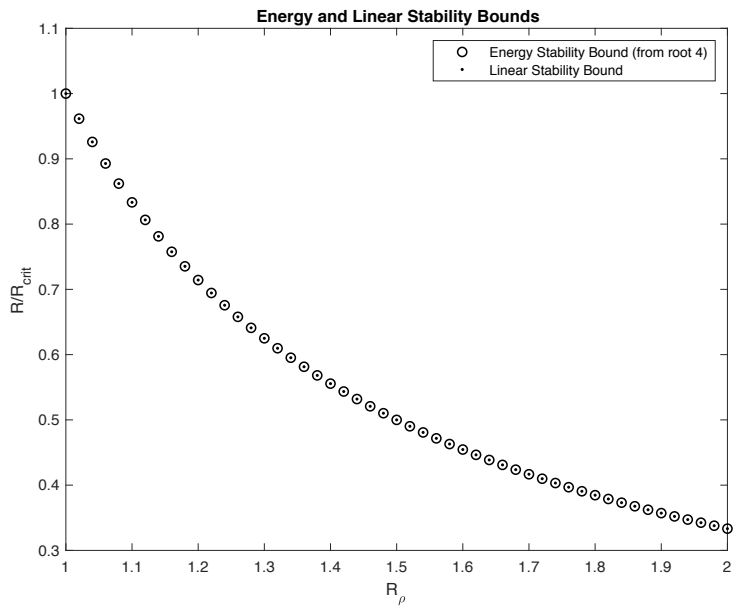


Figure 2.2: We see that the energy stability boundary for steady convection matches the linear stability boundary exactly, confirming the supercritical stability. Here, R_{crit} is the critical Rayleigh number for pure thermal convection. Roots 3 and 4 are outlined above. We have taken $\tau = 0.5$ here.

2.3.2 Oscillatory Convection

The same method as above is employed, but with a slight change of sign to take into account the new assumption that $R_\rho < 1$. Using the same base state and perturbation assumptions as previously, we begin with the following set of equations:

$$\mathbf{u} = -\nabla p + R\theta\hat{\mathbf{z}} - RR_\rho\sigma\hat{\mathbf{z}} \tag{2.3.7}$$

$$\theta_t - w + \mathbf{u}\cdot\nabla\theta = \nabla^2\theta \tag{2.3.8}$$

$$\sigma_t - w + \mathbf{u}\cdot\nabla\sigma = \frac{1}{Le}\nabla^2\sigma. \tag{2.3.9}$$

Repeating the same process as above, but accounting for the sign changes reflected in (2.3.7), we arrive at:

$$\begin{aligned}
\langle|\mathbf{u}^2|\rangle &= R\langle w\theta\rangle - RR_\rho\langle w\sigma\rangle \\
\frac{1}{2}\langle\theta^2\rangle_t - \langle w\theta\rangle &= -\langle|\nabla\theta|^2\rangle \\
\frac{1}{2}\langle\sigma^2\rangle_t - \langle\sigma w\rangle &= -\frac{1}{Le}\langle|\nabla\sigma|^2\rangle \\
\langle\theta\sigma\rangle_t - \langle\theta w\rangle - \langle\sigma w\rangle &= -\left(1+\frac{1}{Le}\right)\langle\nabla\theta\cdot\nabla\sigma\rangle.
\end{aligned}$$

We use the following linear combination to define energy:

$$
\begin{aligned}
&E_t = I - D \\
&I = \left(1 + A + C^2\right)\langle w\theta\rangle + \left(A + A^2 + B^2 - C^2 R_\rho\right)\langle w\sigma\rangle \\
&D = \langle|\nabla\theta|^2\rangle + \left(A^2 + B^2\right)\tau\langle|\nabla\sigma|^2\rangle + A\,(1+\tau)\,\langle\nabla\theta\cdot\nabla\sigma\rangle + \frac{C^2}{R}\langle|\mathbf{u}|^2\rangle.
\end{aligned}
$$

Eliminating θ and σ terms as in the preceding section, we are left with

$$
\begin{aligned}
&\frac{\partial E}{\partial t} = I - D = -D\left(1 - \frac{I}{D}\right) \\
&I = \left(1 + A + C^2\right)\langle wf\rangle \\
&D = \langle|\nabla f|^2\rangle + \frac{C^2}{R}\langle|\mathbf{u}|^2\rangle,
\end{aligned}
$$

and $C^2 = \frac{A(1-\tau)[2\tau + A(1+\tau)]}{2\tau\left[2R_\rho + A(1+\tau)\right]}$. Next, functional derivatives are determined, and yield the same results as in the preceding section, but with $(1 + A + C^2)$ as opposed to $(1 + A - C^2)$. By rearranging in a similar manner to before, we arrive at

$$
\left(\frac{I}{D}\right)^2 = \frac{k^2 R(1 + A + C^2)^2}{4(k^2 + \pi^2)^2 C^2}.
$$

We again require $\frac{I}{D}$ to be less than 1 for energy stability, and so we write

$$
\frac{kR^{1/2}(1 + A + C^2)}{2(k^2 + \pi^2)C} < 1.
$$

Noting that $\left(\frac{R_{crit-osc}}{1+\tau}\right)^{1/2} = \frac{k^2+\pi^2}{k}$ and rearranging such that the Rayleigh numbers are on the left-hand side we arrive at

$$
\left(\frac{R}{R_{crit-osc}}\right)^{1/2} < \frac{2C}{(1 + A + C^2)(1+\tau)^{1/2}}.
$$

Solving the right-hand side with respect to A gives us the following solutions

$$
A_a = \pm\frac{2\sqrt{R_\rho \tau}}{\tau + 1}, \quad A_b = \frac{\pm 2\sqrt{(\tau^3 - R_\rho)(\tau - R_\rho)} - 2(\tau^2 + R_\rho)}{(\tau + 1)^2}.
$$

We require a real A and a positive C^2, and so solutions A_b are only admissible if $R_\rho > \tau$ or $R_\rho < \tau^3 < \tau$. And so, by examining the three regions, $0 < R_\rho < \tau^3$, $\tau^3 < R_\rho < \tau$, $\tau < R_\rho < 1$, we find the following: Using the appropriate solutions outlined in Table 2.1, we find the following bounds:

Region	$0 < R_\rho < \tau^3$	$\tau^3 < R_\rho < \tau$	$\tau < R_\rho < 1$
Real A	$A_{a+,a-}, A_{b+,b-}$	$A_{a+,a-}$	$A_{a+,a-}, A_{b+,b-}$
Positive C^2	$A_{a+,a-}, A_{b+,b-}$	$A_{a+,a-}$	$A_{a+,a-}$
Maximized $\frac{2C}{(1+A+C^2)(1+\tau)^{1/2}}$	A_{b+}	A_{a-}	A_{a-}

Table 2.1: Above we see the relevant roots for each region of R_ρ. The roots outlined in the final row appear in Figure 2.3.

$$\frac{R}{R_{crit}} < \begin{cases} \frac{\tau}{(\tau+1)(\tau-R_\rho)} & R_\rho < \tau^3 \\ \frac{1-\tau}{(1-\sqrt{R_\rho}\tau)^2} & R_\rho \geq \tau^3 \end{cases}$$

Solving for the critical energy Rayleigh numbers, we arrive at

$$R < \begin{cases} \frac{(k^2+\pi^2)^2}{k^2(1-R_\rho Le)} & R_\rho < \tau^3 \\ \frac{(1-\tau)^2(k^2+\pi^2)^2}{(1-\sqrt{R_\rho}\tau)^2 k^2} & R_\rho \geq \tau^3, \end{cases}$$

neither of which matches the bound achieved in the linear stability analysis exactly. However, upon plotting the roots resulting in the maximal value of $\frac{2C}{(1+A+C^2)(1+\tau)^{1/2}}$, we find that there are regions in which the energy bound is lower than the boundary outlined by linear theory, indicating that the system is open to subcritical instabilities. Similarly to the previous section there are also areas wherein the Rayleigh numbers from energy and linear theories meet, indicating regions of supercritical instability.

2.4 Weakly Nonlinear Stability Analysis

We have divided the weakly nonlinear stability analysis into two parts - steady and oscillatory convections. In either case, we use the following asymptotic expansions, up to O(ϵ^3) so as to obtain the amplitude equations:

$$\begin{aligned} \psi &= 0 + \epsilon\psi_1 + \epsilon^2\psi_2 + \epsilon^3\psi_3 \\ T &= T_0 + \epsilon T_1 + \epsilon^2 T_2 + \epsilon^3 T_3 \\ S &= S_0 + \epsilon S_1 + \epsilon^2 S_2 + \epsilon^3 S_3 \\ R &= R_0 + \epsilon R_1 + \epsilon^2 R_2 + \epsilon^3 R_3. \end{aligned}$$

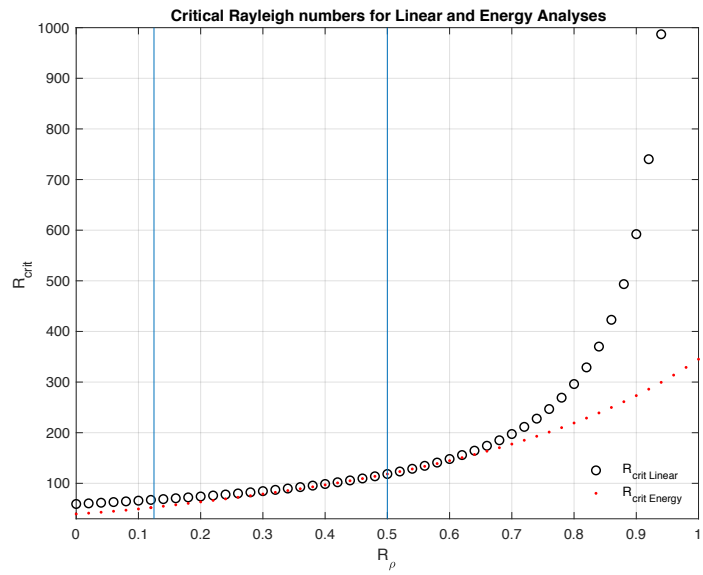


Figure 2.3: We see that the critical linear Rayleigh number is either above or overlapping the critical energy Rayleigh number, indicating that there may be both super and subcritical instabilities. Here, R_{crit} is the critical Rayleigh number for pure thermal convection. We have taken $\tau = 0.5$ here. The blue vertical lines indicate τ^3 and τ.

2.4.1 Steady Convection

We first consider the steady convection case, in which we have a real growth rate. In this case, the only influence of time in our field expressions will be found in a slow-time dependent constant $\mathcal{T}$. As with the linear analysis, we take T_0 and S_0 to be 1 - z. To proceed, we will substitute these expansions into the dimensionless equations (2.2.1) - (2.2.3) and extract the elements by order $O(\epsilon)$. To satisfy boundary conditions, we again assume solutions of the form:

$$\psi_1 \sim A(\mathcal{T})e^{ikx}\sin(\pi z) + \text{c.c.} \tag{2.4.1}$$

$$T_1 \sim B(\mathcal{T})e^{ikx}\sin(\pi z) + \text{c.c.} \tag{2.4.2}$$

$$S_1 \sim C(\mathcal{T})e^{ikx}\sin(\pi z) + \text{c.c.}, \tag{2.4.3}$$

where c.c. is the complex conjugate and $\mathcal{T}$ represents the slow time scale with $\mathcal{T} = \epsilon^2 t$. Because we only have the slow-time scale, no time dependent terms will appear until $O(\epsilon^3)$ which is the order at which we will obtain our evolution equation.

2.4.1.1 $O(\epsilon)$ Equations

At $O(\epsilon)$, the relevant equations are

$$\nabla^2\psi_1 = R_0\left(T_{1x} - R_\rho S_{1x}\right) \tag{2.4.4}$$

$$\psi_{1x} = \nabla^2 T_1 \tag{2.4.5}$$

$$\psi_{1x} = \frac{1}{Le}\nabla^2 S_1. \tag{2.4.6}$$

To reduce the number of unknowns, the coefficients of temperature and salinity are put in terms of the amplitude of velocity. Taking the x-derivative of (2.4.5) and (2.4.6) and the Laplacian of (2.4.4) results in the following set of equations

$$\nabla^4\psi_1 = R_0\left(\nabla^2 T_{1x} - R_\rho \nabla^2 S_{1x}\right) \tag{2.4.7}$$

$$\psi_{1xx} = \nabla^2 T_{1x} \tag{2.4.8}$$

$$\psi_{1xx} = \frac{1}{Le}\nabla^2 S_{1x}. \tag{2.4.9}$$

We are now able to substitute (2.4.8), (2.4.9) into (2.4.7) such that we are left with an equation all in terms of ψ_1. Substituting in the assumed solution (2.4.1) and taking the appropriate derivatives in the updated (2.4.7) leaves

$$\left(k^2 + \pi^2\right)^2 \psi_1 = R_0\left(-k^2\psi_1 + R_\rho k^2 Le\psi_1\right).$$

As expected, dividing the above by the common factor of ψ_1 and solving for R_0 gives the same critical Rayleigh number as in the linear case. The next part of the calculations will be to determine the amplitudes of $O(\epsilon)$ temperature and salinity fields in terms of velocity amplitudes $A(\mathcal{T})$. To do so, we expand equations (2.4.5) and (2.4.6) using the assumed solution forms, and solve for $B(\mathcal{T}), C(\mathcal{T})$ respectively. The resulting terms[4] are:

$$\begin{aligned}
R_0 &= \frac{(k^2+\pi^2)^2}{k^2(R_\rho Le - 1)} \\
T_1 &= \frac{-A(\mathcal{T})ik}{k^2+\pi^2} e^{ikx}\sin(\pi z) + c.c. \\
S_1 &= \frac{-A(\mathcal{T})ikLe}{k^2+\pi^2} e^{ikx}\sin(\pi z) + c.c..
\end{aligned}$$

[4]For detailed calculations at this order, as well as following orders, see Appendix B

2.4.1.2 $O(\epsilon^2)$ Equations

At the next order we acquire some new terms in our equations, though the methodology is similar to that of $O(\epsilon)$. Again, extracting terms of $O(\epsilon^2)$ we obtain:

$$\nabla^2\psi_2 = R_0\left(T_{2x} - R_\rho S_{2x}\right) + R_1\left(T_{1x} - R_\rho S_{1x}\right) \tag{2.4.10}$$

$$\psi_{1z}T_{1x} - \psi_{1x}T_{1z} + \psi_{2x} = \nabla^2 T_2 \tag{2.4.11}$$

$$\psi_{1z}S_{1x} - \psi_{1x}S_{1z} + \psi_{2x} = \frac{1}{Le}\nabla^2 S_2. \tag{2.4.12}$$

The first step is to use the information about T_1 and S_1 gained at $O(\epsilon)$. We are now able to solve for the terms arising from the advection terms, $\mathbf{u}\cdot\nabla T$ and $\mathbf{u}\cdot\nabla S$. We find that

$$\psi_{1z}T_{1x} - \psi_{1x}T_{1z} = \frac{2A(\mathcal{T})\bar{A}(\mathcal{T})k^2\pi}{k^2+\pi^2}\sin(2\pi z)$$

$$\psi_{1z}S_{1x} - \psi_{1x}S_{1z} = \frac{2LeA(\mathcal{T})\bar{A}(\mathcal{T})k^2\pi}{k^2+\pi^2}\sin(2\pi z),$$

indicating that both ψ_{2x}, $\nabla^2 T_2$, and $\nabla^2 S_2$ will contain $\sin(2\pi z)$ terms. As before, we take the Laplacian of (2.4.10) and the x-derivative of (2.4.11) and (2.4.12). Since the $\mathbf{u}\cdot\nabla T$ and $\mathbf{u}\cdot\nabla S$ terms are independent of x, this leaves:

$$\nabla^4\psi_2 = R_0\left(\nabla^2 T_{2x} - R_\rho\nabla^2 S_{2x}\right) + R_1\left(\nabla^2 T_{1x} - R_\rho\nabla^2 S_{1x}\right) \tag{2.4.13}$$

$$\psi_{2xx} = \nabla^2 T_{2x} \tag{2.4.14}$$

$$\psi_{2xx} = \frac{1}{Le}\nabla^2 S_{2x}. \tag{2.4.15}$$

We substitute (2.4.8), (2.4.9), (2.4.14) and (2.4.15) into (2.4.13) yielding:

$$\nabla^4\psi_2 = R_0\left(\psi_{2xx} - R_\rho Le\psi_{2xx}\right) + R_1\left(\psi_{1xx} - R_\rho Le\psi_{1xx}\right). \tag{2.4.16}$$

To go further, we impose the following solvability condition in order to avoid secular growth:

$$\int_0^1\int_0^{\frac{2\pi}{k}}(2.4.16)e^{ikx}\sin(\pi z)dxdz = 0.$$

The $\sin(2\pi z)$ terms in ψ_2 go to 0 due to orthogonality, leaving

$$\int_0^1\int_0^{\frac{2\pi}{k}} R_1\left(\psi_{1xx} - R_\rho Le\psi_{1xx}\right)e^{ikx}\sin(\pi z)dxdz = 0.$$

Since we know the form of ψ_1 we can evaluate the integral. Doing so gives:

$$\frac{R_1 A(\mathcal{T})\left(R_\rho Le - 1\right)k}{\pi} = 0,$$

implying R_1 must be zero, as the other terms are nonzero. Including this information in (2.4.16) leaves

$$\nabla^4\psi_2 = R_0\left(\psi_{2xx} - R_\rho Le\psi_{2xx}\right).$$

However, given that ψ_2 is only dependent on z, the only solution to this equation is the trivial one, and so we conclude that $\psi_2 = 0$. This information, combined with equations (2.4.14) and (2.4.15) shows that T_2 and S_2 are independent of x, and we arrive at the following equations:

$$\frac{2|A(\mathcal{T})|^2k^2\pi}{k^2+\pi^2}\sin(2\pi z) = T_{2zz}$$
$$\frac{2Le|A(\mathcal{T})|^2k^2\pi}{k^2+\pi^2}\sin(2\pi z) = \frac{1}{Le}S_{2zz}.$$

Integrating twice with respect to z gives:

$$T_2 = \frac{-|A(\mathcal{T})|^2k^2}{2\pi(k^2+\pi^2)}\sin(2\pi z)$$
$$S_2 = \frac{-Le^2|A(\mathcal{T})|^2k^2}{2\pi(k^2+\pi^2)}\sin(2\pi z),$$

and so, at $O(\epsilon^2)$ we have obtained the following information:

$$\psi_2 = 0$$
$$R_1 = 0$$
$$T_2 = \frac{-|A(\mathcal{T})|^2k^2}{2\pi(k^2+\pi^2)}\sin(2\pi z)$$
$$S_2 = \frac{-Le^2|A(\mathcal{T})|^2k^2}{2\pi(k^2+\pi^2)}\sin(2\pi z).$$

2.4.1.3 $O(\epsilon^3)$ Equations

Next, we substitute in $\psi_2 = R_1 = 0$, and peel off the following terms at $O(\epsilon^3)$:

$$\nabla^2\psi_3 = R_0\left(T_{3x} - R_\rho S_{3x}\right) + R_2\left(T_{1x} - R_\rho S_{1x}\right) \tag{2.4.17}$$

$$T_{1t} + \psi_{1z}T_{2x} - \psi_{1x}T_{2z} + \psi_{3x} = \nabla^2 T_3 \tag{2.4.18}$$

$$S_{1t} + \psi_{1z}S_{2x} - \psi_{1x}S_{2z} + \psi_{3x} = \frac{1}{Le}\nabla^2 S_3. \tag{2.4.19}$$

As in the previous two orders, to ultimately obtain an equation in terms of ψ_3 we must first take the x-derivatives of (2.4.18) and (2.4.19). Substituting in known solution forms, we get the following:

$$\frac{A'(\mathcal{T})k^2}{k^2+\pi^2}e^{ikx}\sin(\pi z) - \frac{A(\mathcal{T})|A(\mathcal{T})|^2k^4}{k^2+\pi^2}\left(e^{ikx}\cos(2\pi z)\sin(\pi z)\right) + \psi_{3xx} = \nabla^2 T_{3x}.$$

Applying a trigonometric identity to the cosine-sine product, and grouping by sine term gives:

$$\left(\frac{A'(\mathcal{T})k^2}{k^2+\pi^2} + \frac{A(\mathcal{T})|A(\mathcal{T})|^2k^4}{2(k^2+\pi^2)}\right)e^{ikx}\sin(\pi z) - \frac{A(\mathcal{T})|A(\mathcal{T})|^2k^4}{2(k^2+\pi^2)}e^{ikx}\sin(3\pi z) + \psi_{3xx} = \nabla^2 T_{3x}. \quad (2.4.20)$$

A similar calculation yields the following for the solutal equation:

$$\left(\frac{LeA'(\mathcal{T})k^2}{k^2+\pi^2} + \frac{Le^2A(\mathcal{T})|A(\mathcal{T})|^2k^4}{2(k^2+\pi^2)}\right)e^{ikx}\sin(\pi z) - \frac{Le^2A(\mathcal{T})|A(\mathcal{T})|^2k^4}{2(k^2+\pi^2)}e^{ikx}\sin(3\pi z) + \psi_{3xx} = \frac{1}{Le}\nabla^2 S_{3x}. \quad (2.4.21)$$

From these, we see that ψ_3 will contain some combination of terms with $e^{ikx}\sin(\pi z)$, and $e^{ikx}\sin(3\pi z)$ so write the following:

$$\psi_3 = H(\mathcal{T})e^{ikx}\sin(\pi z) + J(\mathcal{T})e^{ikx}\sin(3\pi z),$$

where $H(\mathcal{T}), J(\mathcal{T})$ are some slow-time dependent terms. Next, we take the Laplacian of (2.4.17) and substitute in the above expressions for $\nabla^2 T_{3x}, \nabla^2 S_{3x}, \nabla^2 T_{1x}$, and $\nabla^2 S_{1x}$, as well as the form of ψ_3. We proceed to group by $\sin(\pi z)$ and $\sin(3\pi z)$ terms. Grouping and moving the $H(\mathcal{T})\sin(\pi z)$ terms to the left-hand side, and substituting in the known R_0 value we find the following:

$$H(\mathcal{T})\left(\frac{(k^2+\pi^2)^2(R_\rho Le-1) + (k^2+\pi^2)^2 - R_\rho Le(k^2+\pi^2)^2}{R_\rho Le - 1}\right) = 0.$$

In order for the equation to hold, we need the coefficients of the remaining $\sin(\pi z)$ terms on the right-hand side to equal 0 as well. We are then left with the following expression:

$$0 = \frac{(k^2+\pi^2)(2A'(\mathcal{T}) + A(\mathcal{T})|A(\mathcal{T})|^2k^2)}{2(R_\rho Le-1)} + R_2A(\mathcal{T})k^2(R_\rho Le-1) - \frac{(k^2+\pi^2)R_\rho Le(2LeA'(\mathcal{T}) + Le^2A(\mathcal{T})|A(\mathcal{T})|^2k^2)}{2(R_\rho Le-1)}.$$

After some algebraic manipulation, we arrive at the Landau equation:

$$A'(\mathcal{T}) = \frac{A(\mathcal{T})R_2k^2(R_\rho Le-1)^2}{(k^2+\pi^2)(R_\rho Le^2-1)} - \frac{A(\mathcal{T})|A(\mathcal{T})|^2k^2(R_\rho Le^3-1)}{2(R_\rho Le^2-1)}.$$

Since both the Lewis number and the density ratio are greater than unity in the diffusive convection regime, both $R_\rho Le^2, R_\rho Le^3$ will also exceed 1, making $(R_\rho Le^2 - 1), (R_\rho Le^3 - 1)$ positive. Taking this into account, we find a Landau equation of the form:

$$A'(\mathcal{T}) = \lambda R_2 A(\mathcal{T}) - \Gamma A(\mathcal{T})|A(\mathcal{T})|^2,$$

where λ and Γ are positive coefficients. This form is representative of a supercritical regime, as for instability we need $R_2 > 0$. This compliments what we found in the energy stability analysis, as we were not anticipating any subcritical stabilities to arise. The supercritical stability tells us that at some Rayleigh number exceeding the critical value of $4\pi^2$, there exists some secondary Rayleigh number which allows for some steady, non-stationary motion [1]. This value is determined by taking $\left(\frac{\lambda}{\Gamma}\right)^{1/2}$.

$$|A(\mathcal{T})| = \left(\frac{2R_2(R_\rho Le - 1)^2}{(k^2 + \pi^2)(R_\rho Le^3 - 1)}\right)^{1/2}.$$

2.4.2 Oscillatory Convection

In the case of oscillatory convection, both the real and imaginary part of the growth rate, $\sigma = \sigma_r + i\omega$, are retained. As in the steady convection case, we find the real component of growth rate in the $A(\mathcal{T})$ term, leaving the imaginary component in an exponential. As stated above, despite doubly-diffusive requirement of a globally stable density gradient, it is found that oscillatory convection occurs in porous media when $R_\rho < 1$, or when we have a globally instability [8, 23]. And so we proceed with $R_\rho < 1$, and with governing equations modified to avoid a negative Rayleigh number.

The same asymptotic expansions and base states as in the preceding case are utilized, however solution forms at first order will differ slightly from the steady convection case, due to the nonzero imaginary part. Satisfying boundary conditions, we obtain the following:

$$\begin{aligned}
\psi_1 &\sim A(\mathcal{T})e^{ikx+i\omega t}\sin(\pi z) + \text{c.c.} \\
T_1 &\sim B(\mathcal{T})e^{ikx+i\omega t}\sin(\pi z) + \text{c.c.} \\
S_1 &\sim C(\mathcal{T})e^{ikx+i\omega t}\sin(\pi z) + \text{c.c.},
\end{aligned}$$

where t is the fast timescale and $\mathcal{T}$ is the slow timescale i.e. $\mathcal{T} \sim \epsilon^2 t$, and c.c. is the complex conjugate. As before, we are able to extract relevant terms at each order of ϵ by substituting in the asymptotic expansions.

2.4.2.1 $O(\epsilon)$ Equations

At first order $O(\epsilon)$, the equations are as follows:

$$\begin{aligned}
&\nabla^2\psi_1 = R_0(-T_{1x} + R_\rho S_{1x}) + R_1(-T_{0x} + R_\rho S_{0x})\\
&i\omega T_1 + \psi_{1z}T_{0x} - \psi_{1x}T_{0z} = \nabla^2 T_1\\
&i\omega S_1 + \psi_{1z}S_{0x} - \psi_{1x}S_{0z} = \frac{1}{Le}\nabla^2 S_1.
\end{aligned}$$

Since the base states for temperature and salinity are $T_0 = S_0 = 1 - z$, it follows that $T_{0x} = S_{0x} = 0$ and $T_{0z} = S_{0z} = -1$. Substituting in these values gives

$$\nabla^2\psi_1 = R_0(-T_{1x} + R_\rho S_{1x}) \tag{2.4.22}$$

$$i\omega T_1 + \psi_{1x} = \nabla^2 T_1 \tag{2.4.23}$$

$$i\omega S_1 + \psi_{1x} = \frac{1}{Le}\nabla^2 S_1. \tag{2.4.24}$$

The goal is again to obtain $B(\mathcal{T})$ and $C(\mathcal{T})$ in terms of $A(\mathcal{T})$. Using the assumed solution forms above, (2.4.23) becomes

$$\begin{aligned}
&i\omega B(\mathcal{T})e^{ikx+i\omega t}\sin(\pi z) + ikA(\mathcal{T})e^{ikx+i\omega t}\sin(\pi z) = (-k^2 - \pi^2)B(\mathcal{T})e^{ikx+i\omega t}\sin(\pi z)\\
&\implies i\omega B(\mathcal{T}) + ikA(\mathcal{T}) = (-k^2 - \pi^2)B(\mathcal{T})\\
&\implies B(\mathcal{T}) = \frac{-ikA(\mathcal{T})}{i\omega + k^2 + \pi^2}.
\end{aligned}$$

Similarly, (2.4.24) becomes

$$\begin{aligned}
&i\omega C(\mathcal{T})e^{ikx+i\omega t}\sin(\pi z) + ikA(\mathcal{T})e^{ikx+i\omega t}\sin(\pi z) = \frac{(-k^2 - \pi^2)}{Le}C(\mathcal{T})e^{ikx+i\omega t}\sin(\pi z)\\
&\implies i\omega C(\mathcal{T}) + ikA(\mathcal{T}) = \frac{(-k^2 - \pi^2)}{Le}C(\mathcal{T})\\
&\implies C(\mathcal{T}) = \frac{-LeikA(\mathcal{T})}{Lei\omega + k^2 + \pi^2}.
\end{aligned}$$

Since the slow-time dependent coefficients of thermal and solutal fields can now be written in terms of $A(\mathcal{T})$, at $O(\epsilon)$ we arrive at:

$$\begin{aligned}
&\psi_1 \sim A(\mathcal{T})e^{ikx+i\omega t}\sin(\pi z) + \text{c.c.}\\
&T_1 \sim \frac{-ikA(\mathcal{T})}{i\omega + k^2 + \pi^2}e^{ikx+i\omega t}\sin(\pi z) + \text{c.c.}\\
&S_1 \sim \frac{-LeikA(\mathcal{T})}{Lei\omega + k^2 + \pi^2}e^{ikx+i\omega t}\sin(\pi z) + \text{c.c.}.
\end{aligned}$$

Noting that T_1 can be written as $\frac{-ik}{i\omega+k^2+\pi^2}\psi_1$ and S_1 as $\frac{-Leik}{Lei\omega+k^2+\pi^2}\psi_1$, we are able to proceed to the determination of R_0. Substituting these solutions into (2.4.22) results in an equation entirely in terms of ψ_1:

$$\begin{aligned}
\nabla^2\psi_1 &= R_0(-ikT_1 + R_\rho ikS_1) \\
\implies \left(-k^2-\pi^2\right)\psi_1 &= R_0\left(\frac{-k^2}{i\omega+k^2+\pi^2} + R_\rho\frac{Lek^2}{Lei\omega+k^2+\pi^2}\right)\psi_1.
\end{aligned}$$

Canceling ψ_1, multiplying through by $\left(i\omega+k^2+\pi^2\right)\cdot\left(Lei\omega+k^2+\pi^2\right)$ and solving for R_0 gives the following:

$$R_0 = \frac{\left(k^2+\pi^2\right)\left(Lei\omega+k^2+\pi^2\right)\left(i\omega+k^2+\pi^2\right)}{k^2\left(Lei\omega+k^2+\pi^2-R_\rho Le\left(i\omega+k^2+\pi^2\right)\right)}.$$

Expanding the products, retaining only the imaginary components, and simplifying the remaining terms gives our critical Rayleigh number at first order:

$$R_0 = \frac{\left(k^2+\pi^2\right)^2(1+Le)}{k^2Le\left(1-R_\rho\right)},$$

which is, as expected, identical to that achieved in the linear stability analysis.

2.4.2.2 $O(\epsilon^2)$ Equations

Using the same approach as above, at $O(\epsilon^2)$ we obtain the following, noting that the time derivatives present at this order are with respect to the fast time t :

$$\nabla^2\psi_2 = R_0\left(-T_{2x}+R_\rho S_{2x}\right) + R_1\left(-T_{1x}+R_\rho S_{1x}\right) \tag{2.4.25}$$

$$T_{2t} + \psi_{1z}T_{1x} - \psi_{1x}T_{1z} + \psi_{2x} = \nabla^2 T_2 \tag{2.4.26}$$

$$S_{2t} + \psi_{1z}S_{1x} - \psi_{1x}S_{1z} + \psi_{2x} = \frac{1}{Le}\nabla^2 S_2. \tag{2.4.27}$$

We begin by solving $\psi_{1z}T_{1x} - \psi_{1x}T_{1z}$ and $\psi_{1z}S_{1x} - \psi_{1x}S_{1z}$[5]. As in the previous section, we begin with temperature:

$$\overbrace{\left(A(\mathcal{T})\pi e^{ikx+i\omega t}\cos(\pi z) + c.c\right)}^{\psi_{1z}}\overbrace{\left(\frac{k^2 A(\mathcal{T})e^{ikx+i\omega t}\sin(\pi z)}{i\omega + k^2 + \pi^2} + c.c.\right)}^{T_{1x}}$$
$$- \underbrace{\left(A(\mathcal{T})ike^{ikx+i\omega t}\sin(\pi z) + c.c.\right)}_{\psi_{1x}}\underbrace{\left(\frac{-ik\pi A(\mathcal{T})e^{ikx+i\omega t}\cos(\pi z)}{i\omega + k^2 + \pi^2} + c.c.\right)}_{T_{1z}}.$$

Again, we find that the only terms remaining after expanding are ones containing only $\sin(2\pi z)$:

$$\psi_{1z}T_{1x} - \psi_{1x}T_{1z} = \frac{2\left(k^2 + \pi^2\right)|A(\mathcal{T})|^2 k^2 \pi \sin(2\pi z)}{\left(k^2 + \pi^2\right)^2 + \omega^2}.$$

Doing the same for salt gives the same result, differing only in the appearance of the Lewis number:

$$\psi_{1z}S_{1x} - \psi_{1x}S_{1z} = \frac{2Le\left(k^2 + \pi^2\right)|A(\mathcal{T})|^2 k^2 \pi \sin(2\pi z)}{\left(k^2 + \pi^2\right)^2 + \omega^2 Le^2}.$$

This shows us that S_2, T_2, and ψ_{2x} (and consequently ψ_2) contain $\sin(2\pi z)$ terms. We note that there are no fast time t or x components. Then, $S_{2x} = S_{2t} = T_{2x} = T_{2t} = \psi_{2xx} = 0$. Using this information and writing $\psi_2 \sim D(\mathcal{T})\sin(2\pi z)$, we rewrite (2.4.25) as

$$\begin{aligned} &\psi_{2zz} = R_1(-T_{1x} + R_\rho S_{1x}) \\ \implies &-4\pi^2\psi_2 = R_1\left(\frac{-k^2}{i\omega + k^2 + \pi^2} + R_\rho \frac{Lek^2}{Lei\omega + k^2 + \pi^2}\right)\psi_1. \end{aligned} \tag{2.4.28}$$

We now impose the following solvability condition on (2.4.28); we multiply by $e^{-ikx-i\omega t}\sin(\pi z)$, integrate over the domain, and set equal to 0.

$$\int_0^1 \int_0^{\frac{2\pi}{k}} \int_0^{\frac{2\pi}{\omega}} (2.4.28) e^{-ikx-i\omega t}\sin(\pi z) dt dx dz = 0.$$

[5]More detailed calculations can be found in Appendix B.

By orthogonality, the left-hand side of (2.4.28) will be 0 once evaluated, so the integral is reduced to

$$\int_0^1 \int_0^{\frac{2\pi}{k}} \int_0^{\frac{2\pi}{\omega}} R_1 \left(\frac{-k^2}{i\omega + k^2 + \pi^2} + R_\rho \frac{Lek^2}{Lei\omega + k^2 + \pi^2} \right) \psi_1 e^{-ikx - i\omega t} \sin(\pi z) dt dx dz = 0$$
$$\implies R_1 \left(\frac{-k^2}{i\omega + k^2 + \pi^2} + R_\rho \frac{Lek^2}{Lei\omega + k^2 + \pi^2} \right) \cdot \frac{2\pi A(\mathcal{T})}{k\omega} = 0.$$

Since the rest of the terms are non-zero, it remains that R_1 must be 0. Applying this information to (2.4.28), we get

$$-4\pi^2 \psi_2 = 0 \implies \psi_2 = 0.$$

We are now able to solve for the slow-time coefficients of the temperature and salinity fields where $T_2 \sim D_1(\mathcal{T})\sin(2\pi z)$ and $S_2 \sim D_2(\mathcal{T})\sin(2\pi z)$, and $D_{1,2}(\mathcal{T})$ are some coefficients yet to be determined. Beginning with (2.4.26) and using known values we write

$$\begin{aligned}
&\psi_{1z} T_{1x} - \psi_{1x} T_{1z} = T_{2zz} \\
\implies &\frac{2\left(k^2 + \pi^2\right) |A(\mathcal{T})|^2 k^2 \pi \sin(2\pi z)}{\left(k^2 + \pi^2\right)^2 + \omega^2} = -4\pi^2 D_1(\mathcal{T}) \sin(2\pi z) \\
\implies &D_1(\mathcal{T}) = \frac{-\left(k^2 + \pi^2\right) |A(\mathcal{T})|^2 k^2}{2\pi \left(\left(k^2 + \pi^2\right)^2 + \omega^2 \right)}.
\end{aligned}$$

Doing the same for (2.4.27) gives

$$D_2(\mathcal{T}) = \frac{-Le^2 \left(k^2 + \pi^2\right) |A(\mathcal{T})|^2 k^2}{2\pi \left(\left(k^2 + \pi^2\right)^2 + Le^2 \omega^2 \right)},$$

and so at $O(\epsilon^2)$ we are able to write everything in terms of $A(\mathcal{T})$ and have:

$$\begin{aligned}
T_2 &= \frac{-\left(k^2 + \pi^2\right) |A(\mathcal{T})|^2 k^2}{2\pi \left(\left(k^2 + \pi^2\right)^2 + \omega^2 \right)} \sin(2\pi z) \\
S_2 &= \frac{-Le^2 \left(k^2 + \pi^2\right) |A(\mathcal{T})|^2 k^2}{2\pi \left(\left(k^2 + \pi^2\right)^2 + Le^2 \omega^2 \right)} \sin(2\pi z) \\
\psi_2 &= 0 \\
R_1 &= 0.
\end{aligned}$$

2.4.2.3 $O(\epsilon^3)$ Equations

The relevant terms at $O(\epsilon^3)$ are:

$$\nabla^2\psi_3 = R_0\left(-T_{3x} + R_\rho S_{3x}\right) + R_2\left(-T_{1x} + R_\rho S_{1x}\right) \tag{2.4.29}$$

$$T_{1\mathcal{T}} + i\omega T_3 - \psi_{1x}T_{2z} + \psi_{3x} = \nabla^2 T_3 \tag{2.4.30}$$

$$S_{1\mathcal{T}} + i\omega S_3 - \psi_{1x}S_{2z} + \psi_{3x} = \frac{1}{Le}\nabla^2 S_3. \tag{2.4.31}$$

At this order, the goal is not to obtain solutions for each T_3, S_3, ψ_3, and R_2 as before, but rather to obtain an evolutionary (Landau) equation in terms of $A(\mathcal{T})$ to determine the criticality of the system based on the sign of R_2 which leads to instability.

Substituting the known forms of $T_{1,2}, S_{1,2}, \psi_{1,2}$ into equations (2.4.30) and (2.4.31), we obtain equations which will indicate the forms of T_3, S_3, and ψ_3. We begin with the equation pertaining to temperature:

$$\left(i\omega - \nabla^2\right)T_3 + \psi_{3x} - \overbrace{\frac{ikA'(\mathcal{T})}{i\omega + k^2 + \pi^2}e^{ikx+i\omega t}\sin(\pi z)}^{T_{1\mathcal{T}}} - \underbrace{\left(ikA(\mathcal{T})e^{ikx+i\omega t}\sin(\pi z)\right)}_{\psi_{1x}}\underbrace{\left(\frac{-\left(k^2+\pi^2\right)|A(\mathcal{T})|^2k^2}{\left(\left(k^2+\pi^2\right)^2+\omega^2\right)}\cos(2\pi z)\right)}_{T_{2z}} = 0.$$

Expanding, and applying a trigonometric identity to the $\sin(\pi z)\cos(\pi z)$ term, we arrive at

$$\left(i\omega - \nabla^2\right)T_3 + \psi_{3x} - \frac{ikA'(\mathcal{T})}{i\omega + k^2 + \pi^2}e^{ikx+i\omega t}\sin(\pi z) + \frac{ik^3A(\mathcal{T})|A(\mathcal{T})|^2(k^2+\pi^2)e^{ikx+i\omega t}}{(k^2+\pi^2)^2+\omega^2}\left(\frac{\sin(3\pi z) - \sin(\pi z)}{2}\right) = 0. \tag{2.4.32}$$

Performing identical calculations for the solutal equation (2.4.31), we obtain

$$\left(i\omega - \frac{1}{Le}\nabla^2\right)S_3 + \psi_{3x} - \frac{LeikA'(\mathcal{T})}{Lei\omega + k^2 + \pi^2}e^{ikx+i\omega t}\sin(\pi z) + \frac{Le^2ik^3A(\mathcal{T})|A(\mathcal{T})|^2(k^2+\pi^2)e^{ikx+i\omega t}}{(k^2+\pi^2)^2+Le^2\omega^2}\left(\frac{\sin(3\pi z) - \sin(\pi z)}{2}\right) = 0. \tag{2.4.33}$$

In order for the equation to hold, we see that S_3, T_3, ψ_{3x} must all contain some combination of $\sin(\pi z)$ and $\sin(3\pi z)$ terms; we write $T_3, S_3, \psi_{3x} \sim H(\mathcal{T})_{T,S,\psi}e^{ikx+i\omega t}\sin(\pi z) + J(\mathcal{T})_{T,S,\psi}e^{ikx+i\omega t}\sin(3\pi z)$, where $H(\mathcal{T})_{T,S,\psi}$ and $J(\mathcal{T})_{T,S,\psi}$ are slow-time dependent coefficients for thermal (T), solutal (S), and velocity (ψ) fields. Now that we know the general form of the solutions at $O(\epsilon^3)$, we will

solve for the slow-time dependent coefficients of T_3, S_3 in terms of $H(\mathcal{T})_\psi$ and $A(\mathcal{T})$, then combine (2.4.29)-(2.4.31) and impose solvability conditions and finally employ Matlab to obtain an evolution equation for $A(\mathcal{T})$.

After substituting in (2.4.32) and (2.4.33) into (2.4.29), we multiply by $e^{ikx+i\omega t}\sin(\pi z)$ and integrate over the domain and time. Due to orthogonality, only terms containing $\sin(\pi z)$ will remain. To solve for the last set of coefficients, we then consider only the $\sin(\pi z)$ component of the solutions. We first solve for the slow-time dependent coefficient for T_3, $H(\mathcal{T})_T$. Beginning with (2.4.32), we discard $\sin(3\pi z)$ terms and substitute in the solution forms for T_3 and ψ_3:

$$\left(i\omega + k^2 + \pi^2\right)\left(H(\mathcal{T})_T e^{ikx+i\omega t}\sin(\pi z)\right) + ikH(\mathcal{T})_\psi e^{ikx+i\omega t}\sin(\pi z) - \\ \frac{ikA'(\mathcal{T})}{i\omega + k^2 + \pi^2} e^{ikx+i\omega t}\sin(\pi z) - \frac{ik^3 A(\mathcal{T})|A(\mathcal{T})|^2(k^2+\pi^2)e^{ikx+i\omega t}}{(k^2+\pi^2)^2 + \omega^2}\left(\frac{\sin(\pi z)}{2}\right) = 0.$$

Dividing through by $e^{ikx+i\omega t}\sin(\pi z)$ and rearranging, we obtain the following expression for $H(\mathcal{T})_T$:

$$H(\mathcal{T})_T = \left(\frac{ik^3 A(\mathcal{T})|A(\mathcal{T})|^2(k^2+\pi^2)}{2\left((k^2+\pi^2)^2+\omega^2\right)} + \frac{ikA'(\mathcal{T})}{i\omega + k^2 + \pi^2} - ikH(\mathcal{T})_\psi\right)\frac{1}{i\omega + k^2 + \pi^2}.$$

An identical procedure for (2.4.33) yields

$$H(\mathcal{T})_S = \left(\frac{Le^2 ik^3 A(\mathcal{T})|A(\mathcal{T})|^2(k^2+\pi^2)}{2\left((k^2+\pi^2)^2+Le^2\omega^2\right)} + \frac{LeikA'(\mathcal{T})}{Lei\omega + k^2 + \pi^2} - ikH(\mathcal{T})_\psi\right)\frac{Le}{Lei\omega + k^2 + \pi^2}.$$

Now, substituting the solution and coefficient forms into (2.4.29), we proceed to determining the coefficients of the evolution equation. Since we are considering only $\sin(\pi z)$ terms, once applying the Laplacian operator to ψ_3 and taking the x-derivatives of $T_{1,3}, S_{1,3}$ we are able to group terms and remove the common factor $e^{ikx+i\omega t}\sin(\pi z)$ leaving

$$-(k^2+\pi^2)H(\mathcal{T})_\psi = R_0\left(-ikH(\mathcal{T})_T + R_\rho ikH(\mathcal{T})_S\right) + R_2\left(\frac{-k^2 A(\mathcal{T})}{i\omega + k^2 + \pi^2} + R_\rho\frac{k^2 LeA(\mathcal{T})}{Lei\omega + k^2 + \pi^2}\right).$$

Substituting the terms $H(\mathcal{T})_T$ and $H(\mathcal{T})_S$, we arrive at an equation completely in terms of $H(\mathcal{T})_\psi$. We proceed by moving all terms containing $H(\mathcal{T})_\psi$ to the left hand side and substituting in the known form of R_0. After some algebraic manipulations[6] we find that the coefficients of $H(\mathcal{T})_\psi$ go to zero, and so in order the have balance the remaining terms on the right hand side must also equal zero. With the help of a symbolic solver[7], the Landau equation is found to be

$$A(\mathcal{T})' = \frac{R_2 k^2(1-R_\rho)}{2(k^2+\pi^2)}A(\mathcal{T}) - \frac{k^2 Le(1-R_\rho)}{8(R_\rho Le - 1)}A(\mathcal{T})^3.$$

[6]See Appendix B.
[7]See Appendix E for the code.

Since we required a stability ratio less than unity, the sign of the $A(\mathcal{T})^3$ coefficient will depend on the relative size of R_ρ to τ. If $R_\rho > \tau$ we can write the Landau equation in the following form:

$$A(\mathcal{T})' = \lambda R_2 A(\mathcal{T}) - \Gamma A(\mathcal{T})^3,$$

where both λ and Γ are positive. Since the only way to achieve instability is by having R_2 positive, we have a supercritical instability. Doing the same as in the previous section, we find that as $\mathcal{T} \to \infty$, the amplitude approaches the following:

$$|A(\mathcal{T})| = \frac{4R_2(R_\rho Le - 1)}{Le(k^2 + \pi^2)}.$$

If $R_\rho < \tau$ then we write the equation as

$$A(\mathcal{T})' = \lambda R_2 A(\mathcal{T}) + \Gamma A(\mathcal{T})^3,$$

where λ and Γ are again positive. With this change of sign in the second coefficient, the equation now describes a regime of subcritical instability. This is unsurprising, as the nonlinear stability analysis indicated that the system was open to both sub and supercritical instabilities.

2.5 Discussion

In this chapter we have seen complementary results from linear, nonlinear, and weakly nonlinear stability analyses. Section 1 concluded that the linear equations give an upper bound for the critical Rayleigh number. That is to say, while values below the critical Rayleigh number could result in instability, for steady convection any $R > R_{crit} = 4\pi^2$ will definitively result in an unstable system [23, 39]. The nonlinear analysis resulted in a lower bound for stability, giving the lowest possible Rayleigh number resulting in an unstable system. As the two bounds were identical, we could conclude that there are no critical Rayleigh numbers below $4\pi^2$ - the nonlinear analysis did not capture any new regions of instability which could have potentially been excluded from the linear analysis. Because of this, we went forward expecting to see only supercritical stability in the system, which was confirmed by the weakly nonlinear analysis. Once the amplitude equation was put into the form of the Landau equation, we were able to see that in order for instabilities to arise, the higher order Rayleigh number terms (R_2) were required to be positive. Any negative R_2 would cause R to fall below the threshold of instability, and no convection would ensue. In the steady convection regime, we were then able to determine the amplitude the system would settle to over long time.

In the case of oscillatory convection, we found that depending on the Lewis number's relative size to the density ratio, the system fell either into sub or supercritical regimes. Subcritical instabilities could arise in regions where the energy bound was lower than the linear, as the nonlinear

analysis may have captured information missed by the linearized equations. This is consistent with the nonlinear stability results, as we found the system would be open to both regimes. We do note, however, that since a globally unstable density ratio was required to proceed with the oscillatory analysis, it technically does not fall under the double diffusion category.

Chapter 3

Salt Fingering Convection

3.1 Assumptions and Set-Up

Salt fingering occurs when the faster diffusing field is stabilizing, and the slower destabilizing so we take temperature and salinity to be stabilizing and destabilizing respectively [23, 27, 34]. We effectively flip the preceding tank corresponding to diffusive convection, by putting warm, salty water on top of cool, fresh water. The new tank set-up is pictured in figure 3.1

The density ratio R_ρ remains $\frac{\alpha\Delta S}{\beta\Delta T}$, but due to the new stabilizing and destabilizing roles of temperature and salinity, salt fingering will occur only when $R_\rho < 1$. A density ratio of less than unity indicates that density increases as we approach the lower boundary by water getting relatively cooler and saltier, and so we have the required globally stable density stratification [27, 34]. Using the same approach as in the previous section, but noting that the top and bottom boundary values have been exchanged, we arrive at the following relation:

$$\begin{aligned} \rho_{top} - \rho_{bottom} &= \alpha\Delta S - \beta\Delta T < 0 \\ \implies \frac{\alpha\Delta S}{\beta\Delta T} &< 1. \end{aligned}$$

The dimensionless Lewis number $Le = \frac{\kappa_T}{\phi\kappa_S}$ remains greater than or equal to unity as the thermal diffusivity is greater than solutal diffusivity. We again take on impermeable surfaces:

3.2 Linear Stability Analysis

In the case of salt fingering, the methodology for linear stability analysis is identical to that of diffusive convection in the preceding chapter. However, the base state and the governing equations are slightly different. Since the tank in question has effectively been flipped, the base state and

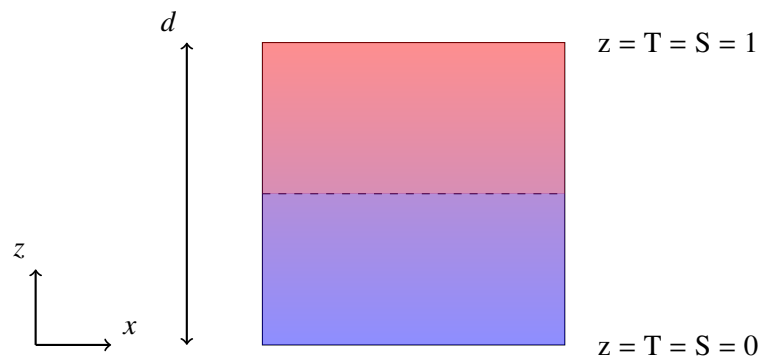


Figure 3.1: For the salt fingering regime, a tank of depth d, with red representing hot, salty water and blue as cool, fresh water. 0 and 1 represent the dimensionless values of depth (z), temperature (T), and salinity (S).

perturbations are now written as:

$$\begin{aligned}
\psi &= \psi^* e^{ikx+\sigma t}\sin(m\pi z) + \text{c.c.} \\
T &= z + T^* e^{ikx+\sigma t}\sin(m\pi z) + \text{c.c.} \\
S &= z + S^* e^{ikx+\sigma t}\sin(m\pi z) + \text{c.c.},
\end{aligned}$$

where k and m are wave numbers, and σ is the complex growth rate. The base steady states, representing pure thermal diffusion, are given as $T_0 = z$, $S_0 = z$, $\psi_0 = 0$, and variables with an asterisk (*) represent the perturbations. The governing equations differ by those in the diffusive convection regime only in signs:

$$\begin{aligned}
\nabla^2\psi &= R\left(-T_x + R_\rho S_x\right) \\
T_t - \psi_x &= \nabla^2 T \\
S_t - \psi_x &= \frac{1}{Le}\nabla^2 S.
\end{aligned}$$

We also note that the density ratio R_ρ is required to be smaller than unity in the salt fingering regime. Since the method does not change between regimes, we jump ahead to the resulting critical values[1]. We ultimately find that marginal stability occurs when $R = \frac{(k^2+\pi^2)^2}{k^2(R_\rho Le-1)}$. As in the case of diffusive convection, we wish to determine the value of the Rayleigh number at the onset of convection. We again set $R_S = 0$, and so $R_\rho = RR_S = 0$, leaving $R = \frac{(k^2+\pi^2)^2}{k^2}$. Minimizing over k, we again find that $k = \pi$ and critical Rayleigh number $R_c = 4\pi^2$.

It is at this value of the Rayleigh number where convection will begin to occur. In section 3.3.4, we will determine whether the system is in a subcritical or supercritical regime, by considering the

[1]Work done can be found in Appendix C.

values of higher-order R leading to instability. Given the Landau equation $A'(\mathcal{T}) = \lambda A(\mathcal{T}) - \Gamma A(\mathcal{T})^3$, we expect λ to be the same as the σ derived above, as if $\Gamma = 0$, we regain the linear solution [26].

An analog of Radko's work [27] shows that to have a positive Rayleigh number, we must have $R_\rho Le > 1$. Beginning with the dimensionless linearized equation we set the z wavenumber to 0, as salt fingers are typically longer than they are wide [27, 29, 35]. Using the same solution form as above, and repeating the substitution method leads to the following:

$$-k^2 = R\left(\frac{k^2}{\sigma + k^2} - R_\rho Le \frac{k^2}{Le\sigma + k^2}\right).$$

Setting σ equal to zero once more to evaluate at the marginal stability leaves:

$$\frac{k^2}{R} = \left(R_\rho Le - 1\right).$$

As k^2 will be a positive value, in order for the left and right hand sides to be of the same sign, if $R > 0$ then we must have $R_\rho Le > 1$, and consequently $R_\rho > \frac{1}{Le} = \tau$.

3.3 Nonlinear Stability Analysis

After changing the base states from $1 - z$ to z and altering signs in the governing equations, the methodology for nonlinear (energy) stability analysis in the salt fingering regime is identical to that of the preceding chapter. The new equations are as follows:

$$\begin{aligned}
&\mathbf{u} = -\nabla p + R\theta\hat{\mathbf{z}} - RR_\rho\sigma\hat{\mathbf{z}} \\
&\theta_t + w + \mathbf{u}\cdot\nabla\theta = \nabla^2\theta \\
&\sigma_t + w + \mathbf{u}\cdot\nabla\sigma = \frac{1}{Le}\nabla^2\sigma.
\end{aligned}$$

Because of the differing governing equations, the linear combinations and consequently the solutions of the coefficient A will be changed slightly from the diffusive convection case. Of the four solutions obtained[2] we ultimately find the following to be admissible:

$$A = \frac{\pm 2\sqrt{(\tau^3 - R_\rho)(\tau - R_\rho)} - 2(\tau^2 + R_\rho)}{(\tau + 1)^2}.$$

We find that while both roots give positive C^2 and real A, only the negative root provides a positive bound for $\frac{R}{R_{crit}}$. We are then left with

$$\frac{R}{R_{crit}} < \frac{\tau}{R_\rho - \tau}.$$

[2]See Appendix C for all solutions.

Replacing R_{crit} with $\frac{(k^2+\pi^2)^2}{k^2}$, we find that the critical Rayleigh number for energy stability matches that obtained in the linear stability analysis. Based on this, we do not expect any subcritical instabilities in the region $R_\rho > \tau$, as there will not be any region of instability below that indicated by the linear bound. These curves are plotted in Matlab and pictured below.

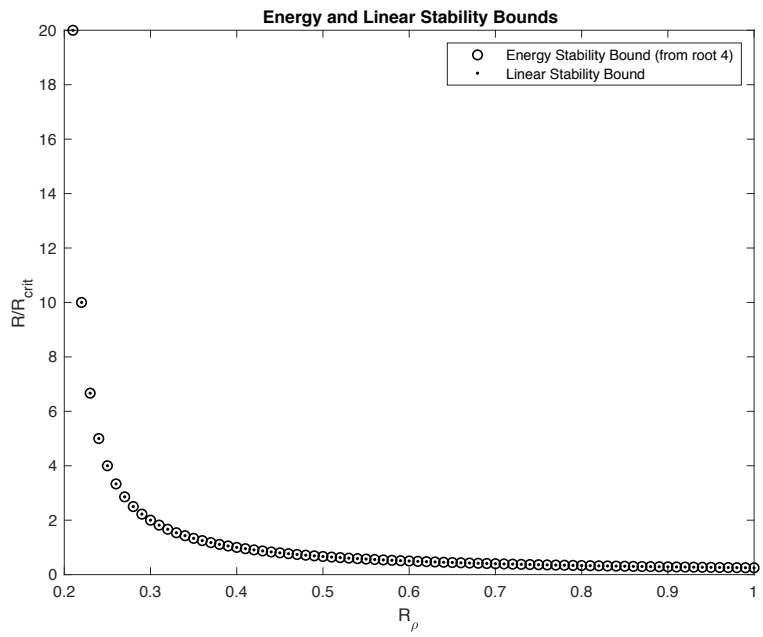


Figure 3.2: We see that the energy stability boundary matches the linear stability boundary, confirming the supercritical stability. Here, R_{crit} is the critical Rayleigh number for pure thermal convection. $\tau = 0.2$

3.4 Weakly Nonlinear Stability Analysis

Keeping with the theme of preceding sections, the weakly nonlinear analysis is carried out in an identical manner as for diffusive convection. The base states of salinity and temperature will again be z, rather than $1 - z$ as in the diffusive convection case, and the governing equations will differ by signs:

$$\nabla^2\psi = R\left(-T_x + R_\rho S_x\right)$$
$$T_t - \psi_x = \nabla^2 T$$
$$S_t - \psi_x = \frac{1}{Le}\nabla^2 S.$$

The solution forms will maintain the following form:

$$\psi_1 \sim A(\mathcal{T})e^{ikx}\sin(\pi z)+ \text{ c.c.} \tag{3.4.1}$$

$$T_1 \sim B(\mathcal{T})e^{ikx}\sin(\pi z)+ \text{ c.c.} \tag{3.4.2}$$

$$S_1 \sim C(\mathcal{T})e^{ikx}\sin(\pi z)+ \text{ c.c.}, \tag{3.4.3}$$

where c.c. represents the complex conjugate, and $A(\mathcal{T}), B(\mathcal{T}), C(\mathcal{T})$ are the slow time coefficients. We repeat the algebra done in section 2.4, and arrive at the following Landau equation[3]:

$$A'(\mathcal{T}) = \frac{A(\mathcal{T})R_2k^2(1-R_\rho Le)^2}{(k^2+\pi^2)(R_\rho Le^2-1)} - \frac{A(\mathcal{T})|A(\mathcal{T})|^2k^2(R_\rho Le^3-1)}{2(R_\rho Le^2-1)}.$$

Since we have $Le > 1$ and $R_\rho < 1$ with the relation $R_\rho > \frac{1}{Le} > \frac{1}{Le^2} > \frac{1}{Le^3}$, the coefficients of both $A(\mathcal{T})$ and $A(\mathcal{T})|A(\mathcal{T})|^2$ will be positive, and the Landau equation can be written as follows:

$$A'(\mathcal{T}) = \lambda R_2 A(\mathcal{T}) - \Gamma A(\mathcal{T})|A(\mathcal{T})|^2.$$

Instability may only be achieved for $R_2 > 0$, so we have a case of supercritical stability, as predicted by the energy stability analysis in the preceding section. As in the preceding chapter, we can find the steady solution as $\mathcal{T} \to \infty$ and find it to be:

$$|A(\mathcal{T})| = \left(\frac{2R_2(R_\rho Le-1)^2}{(R_\rho Le^3-1)(k^2+\pi^2)}\right)^{1/2}.$$

3.5 Discussion

As in the previous chapter's steady convection discussion, we find the linear and nonlinear stability analyses to be complementary. We again find that at $R = 4\pi^2$, linear analysis tells us that there will be no stability above this value, and nonlinear analysis tells us there will be no instability below it. We again conclude that no subcritical stability should exist in this regime, as there are no regions in which the nonlinear bound is below the linear. The weakly nonlinear analysis confirms this, by presenting us with a Landau equation describing supercritical stability and no possibility of a subcritical regime due to our choices of R_ρ with respect to τ. We also find the amplitude $A(\mathcal{T})$ to which the system settles over long time.

Since this and the preceding chapter's weakly nonlinear analyses take place near the onset of instability, we find that the fingers will be more square in shape. In fact, both the horizontal and vertical wavenumbers described in the critical Rayleigh number are π, and so they will have a 1:1

[3]Details of the calculations can be found in Appendix C, though they remain largely the same as in the case of steady diffusive convection.

aspect ratio meaning that their horizontal and vertical scales will be the same. One can picture fingers near the onset of instability as being rather short and square, as it is the beginning of the growth of the fingers; they have not yet had time to extend vertically resulting in their 1:1 aspect ratio. Since the fingers lengthen with time, in an unbounded domain we expect to see a smaller aspect ratio in moving away from the initial onset of instability [30, 31, 32, 36]. And so, in oceanographic settings fingers will be longer than they are wide. In the following section, we will determine the wavenumber for the fastest growing finger and find an evolutionary equation based on it.

Chapter 4

2D Salt Fingers in a Vertically Unbounded Fluid

In the preceding chapters, we found solutions for salt fingers with similarly large horizontal and vertical wavenumbers. In oceanic settings, salt fingers tend to have a high aspect ratio and so it is of interest to consider the case where the vertical wavenumber is much smaller than the horizontal [35] .

Similarly to above, we begin with the dimensionless Boussinesq equations with a warm salty water overlying cool fresh, again taking $(u, w) = (\psi_z, -\psi_x)$. We take temperature to be the stabilizing field, and so $R_\rho = \frac{\alpha\Delta S}{\beta\Delta T} < 1$, as temperature must contribute more to the stability ratio in order to obtain global stability. We also have $\tau = \frac{\phi\kappa_S}{\kappa_T} < 1$, as the solutal diffusivity is smaller than thermal diffusivity. Our system of equations is then:

$$\nabla^2\psi = R\left(-T_x + R_\rho S_x\right) \tag{4.0.1}$$

$$T_t + \mathbf{u}\cdot\nabla T = \nabla^2 T \tag{4.0.2}$$

$$S_t + \mathbf{u}\cdot\nabla S = \tau\nabla^2 S. \tag{4.0.3}$$

We also have the restrictions $R_\rho > \tau$, and so we will take $\frac{R_\rho}{\tau} - 1 = \epsilon > 0$. We write the perturbations as:

$$T = z + T'(x, z, t) + \theta(z, t) \qquad \text{and} \qquad S = z + S'(x, z, t) + \sigma(z, t),$$

with the horizontal averages $\overline{T'} = \overline{S'} = \overline{w} = 0$ and vertical averages of $\theta, \sigma = 0$ and continue, following the methodology of Stern and Radko [35]. Beginning with equations (4.0.2) and (4.0.3),

we substitute in the perturbed solutal and thermal fields. The simplified equations are as follows:

$$\nabla^2\psi = R\left(-T'(x,z,t)_x + R_\rho S'(x,z,t)_x\right) \tag{4.0.4}$$

$$\left(T'(x,z,t) + \theta(z,t)\right)_t + w + w\theta_z + \mathbf{u}\cdot\nabla T'(x,z,t) = \nabla^2 T'(x,z,t) + \theta_{zz} \tag{4.0.5}$$

$$\left(S'(x,z,t) + \sigma(z,t)\right)_t + w + w\sigma_z + \mathbf{u}\cdot\nabla S'(x,z,t) = \tau\nabla^2 S'(x,z,t) + \tau\sigma_{zz}. \tag{4.0.6}$$

Taking the horizontal mean of the preceding equations gives the mean field equations

$$\theta_t + \left(\overline{T'w}\right)_z = \theta_{zz} \tag{4.0.7}$$

$$\sigma_t + \left(\overline{S'w}\right)_z = \tau\sigma_{zz}, \tag{4.0.8}$$

and subtracting these from equations (4.0.5) and (4.0.6) leaves

$$T'_t + w + \nabla\cdot(\mathbf{u}T') + w\theta_z - \left(\overline{T'w}\right)_z = \nabla^2 T' \tag{4.0.9}$$

$$S'_t + w + \nabla\cdot(\mathbf{u}S') + w\sigma_z - \left(\overline{S'w}\right)_z = \tau\nabla^2 S'. \tag{4.0.10}$$

Rearranging the above equations such that we have T' and S' on the left-hand side gives

$$\left(\frac{\partial}{\partial t} - \nabla^2\right)T' = \left(\overline{T'w}\right)_z - w - \nabla\cdot(\mathbf{u}T') - w\theta_z \tag{4.0.11}$$

$$\left(\frac{\partial}{\partial t} - \tau\nabla^2\right)S' = \left(\overline{S'w}\right)_z - w - \nabla\cdot(\mathbf{u}S') - w\sigma_z. \tag{4.0.12}$$

Next we substitute the rearranged equations into (4.0.4), and multiply through by $\left(\frac{\partial}{\partial t} - \nabla^2\right)\left(\frac{\partial}{\partial t} - \tau\nabla^2\right)$, resulting in the following expression:

$$\left(\frac{\partial}{\partial t} - \nabla^2\right)\left(\frac{\partial}{\partial t} - \tau\nabla^2\right)\nabla^2\psi = -R\left(\left(\overline{T'w}\right)_z - w - \nabla\cdot(\mathbf{u}T') - w\theta_z\right)_x\left(\frac{\partial}{\partial t} - \tau\nabla^2\right)$$
$$+ RR_\rho\left(\left(\overline{S'w}\right)_z - w - \nabla\cdot(\mathbf{u}S') - w\sigma_z\right)_x\left(\frac{\partial}{\partial t} - \nabla^2\right).$$

We note that (u, w) can be written in terms of the streamfunction $(\psi_z, -\psi_x)$. And so we rearrange the equation again, moving the linear terms to the left-hand side, and leaving the nonlinear terms of the right-hand side.

$$LHS : \left(\frac{\partial}{\partial t} - \nabla^2\right)\left(\frac{\partial}{\partial t} - \tau\nabla^2\right)\nabla^2\psi + R\left(\frac{\partial}{\partial t} - \tau\nabla^2\right)\psi_{xx} - RR_\rho\left(\frac{\partial}{\partial t} - \nabla^2\right)\psi_{xx} \tag{4.0.13}$$

$$RHS : -R\left(\frac{\partial}{\partial t} - \tau\nabla^2\right)\left(\left(\overline{T'w}\right)_z - \nabla\cdot(\mathbf{u}T') - w\theta_z\right)_x + \\ RR_\rho\left(\frac{\partial}{\partial t} - \nabla^2\right)\left(\left(\overline{S'w}\right)_z - \nabla\cdot(\mathbf{u}S') - w\sigma_z\right)_x. \quad (4.0.14)$$

The terms on the right-hand side containing θ and σ come from the mean field and triad terms give $\nabla\cdot(\mathbf{u}T')$ and $\nabla\cdot(\mathbf{u}S')$.

We begin by solving the linear problem, and so we take the solution form $\psi \sim e^{\lambda t}\sin(kx)\cos(mz)$. Substituting this into (4.0.13) and applying the appropriate operators we get the following:

$$-\left(\lambda^2 + \tau\lambda(k^2+m^2) + \lambda(k^2+m^2) + \tau(k^2+m^2)^2\right)(k^2+m^2) \\ -Rk^2\left(\lambda + \tau(k^2+m^2)\right) + RR_\rho k^2\left(\lambda + (k^2+m^2)\right) = 0.$$

Grouping by order of λ, dividing through by $\tau(k^2+m^2)$, and using the relation $\epsilon = \frac{R_\rho}{\tau} - 1$:

$$-\frac{\lambda^2}{\tau} + \lambda\left(-(k^2+m^2)\left(1+\frac{1}{\tau}\right) - \frac{Rk^2(1-R_\rho)}{\tau(k^2+m^2)}\right) + \left(-(k^2+m^2)^2 - Rk^2 + \frac{RR_\rho k^2}{\tau}\right) = 0 \\ \implies -\frac{\lambda^2}{\tau} + \lambda\left(-(k^2+m^2)\left(1+\frac{1}{\tau}\right) - \frac{Rk^2(1-R_\rho)}{\tau(k^2+m^2)}\right) + \left(-(k^2+m^2)^2 + \epsilon Rk^2\right) = 0.$$

Following Stern and Radko, we consider the $\epsilon \to 0$ for small λ [35]. As in the previous linear analysis, due to the small λ we discard the λ^2 term. Rearranging the remaining terms leaves

$$\lambda = \frac{(k^2+m^2)^2 - \epsilon Rk^2}{-(k^2+m^2)\left(1+\frac{1}{\tau}\right) - \frac{Rk^2(1-R_\rho)}{\tau(k^2+m^2)}}.$$

Next, we must determine the spatial scaling. In order to do so, we let $(k,m) = \epsilon^q(k_0,m_0)$. Substituting this in to the previous equation we get

$$\lambda = \frac{\epsilon^{4q}(k_0^2+m_0^2)^2 - \epsilon^{2q+1}Rk_0^2}{-\epsilon^{2q}(k_0^2+m_0^2)\left(1+\frac{1}{\tau}\right) - \frac{Rk_0^2(1-R_\rho)}{\tau(k_0^2+m_0^2)}}.$$

In order to balance terms in the numerator, we need $4q = 2q+1$ to hold true, and so we find that q must be $\frac{1}{2}$. Using the scaling $(k,m) = \epsilon^{\frac{1}{2}}(k_0,m_0)$ in the preceding equation and Taylor expanding about $\epsilon = 0$, the leading order term is

$$\lambda = \frac{\epsilon^2\tau(k_0^2+m_0^2)\left(Rk_0^2 - (k_0^2+m_0^2)^2\right)}{R(1-R_\rho)k_0^2}. \quad (4.0.15)$$

In order to maximize the growth rate λ, we see that $m_0 = 0$. By applying this and taking the

derivative with respect to k_0^2 and setting equal to 0, we are able to find the horizontal wavenumber producing the maximum growth rate:

$$\frac{\epsilon^2\tau(R-2k_0^2)}{R(1-R_\rho)} = 0 \implies k_0 = \left(\frac{R}{2}\right)^{\frac{1}{2}}. \tag{4.0.16}$$

Substituting k_0 into (4.0.15) gives the following maximum growth rate:

$$\lambda_{max} = \frac{R\tau\epsilon^2}{4(1-R_\rho)}. \tag{4.0.17}$$

Now that we have information regarding the fastest growing finger, we move on to obtaining a density equation. This is executed by taking the linear combination $\rho' = -T' + R_\rho S'$, with $T' =$ (4.0.11) and $S' =$ (4.0.12). This will allow us to consider the terms with respect to their effects on density.

$$\left(\frac{R_\rho}{\tau}S' - T'\right)_t + \epsilon w + \left(\frac{R_\rho}{\tau}\nabla\cdot(\mathbf{u}S') - \nabla\cdot(\mathbf{u}T')\right) + \left((\overline{wT'})_z - \frac{R_\rho}{\tau}(\overline{wS'})_z\right) + w\left(\frac{R_\rho}{\tau}\sigma_z - \theta_z\right) = \nabla^2\rho'. \tag{4.0.18}$$

With these equations in hand, we proceed to determine the variable scalings. We know that the horizontal scale x is $\epsilon^{-\frac{1}{2}}$ from (4.0.16) and that the timescale is $t \sim \epsilon^{-2}$ from (4.0.17). Since the nonlinear terms arising from the mean field (σ_z, θ_z) must be balanced by the linear terms (ϵw) we find that $w\sigma_z \sim \epsilon w \sim w\theta_z$, so $\theta_z \sim \sigma_z \sim \epsilon$. Next, considering the linear terms in equation (4.0.9), we expect $w \sim \nabla^2 T'$. However, since we are considering long, thin fingers the z-scale will result in higher order ϵ terms, and will disappear in the $\epsilon \to 0$ limit. We then look only to the x-derivatives, and arrive at $w \sim T'_{xx} \sim \epsilon T'$. The same relation can be found for salt and so we also have $T' \sim S'$. Equations (4.0.7) and (4.0.8) give us $\overline{T'w} \sim \theta_z \sim \epsilon$ and $\overline{S'w} \sim \sigma_z \sim \epsilon$.

Since we also expect the nonlinear terms in (4.0.18) to balance, we have $w\epsilon \sim (\overline{wT'})_z$. However, using above relations we write the following: $w\epsilon \sim \epsilon\frac{\partial}{\partial z}$. This tells us that $w \sim \frac{\partial}{\partial z}$. We can now write an equivalence for σ so that we can ultimately find the vertical velocity scaling. Using $\sigma_z \sim \epsilon$ and $w \sim \frac{\partial}{\partial z}$, we write $\sigma \sim \frac{\epsilon}{w}$. We return to the mean field salinity equation and require the following balance of terms $\sigma_t \sim (\overline{S'w})_z$. Substituting gives $\frac{\epsilon}{w}\cdot\epsilon^2 \sim \epsilon w$ so we have $w \sim \epsilon$. Using the density equation once more, we find that $\rho \sim w\epsilon$. Again, taking only the x-derivatives of the Laplacian, we have $\rho \sim \epsilon$. We also find that $S' \sim \epsilon^0 \sim T'$, from the fact that $w \sim \epsilon T'$. Finally, we employ the continuity equation to find the horizontal velocity scaling and arrive at $u \sim \epsilon^{\frac{3}{2}}$. A summary of the

resulting scalings are outlined below.

$$t = \epsilon^{-2} t_0 \qquad x = \epsilon^{-\frac{1}{2}} x_0 \qquad z = \epsilon^{-1} z_0 \qquad (T', S') = \epsilon^0 (T_0, S_0)$$
$$u = \epsilon^{-\frac{3}{2}} u_0 \qquad w = \epsilon w_0 \qquad (\theta_z, \sigma_z) = \epsilon(\beta_T, \beta_s) \qquad \rho' = \epsilon \rho_0$$

Applying these scalings to (4.0.4), (4.0.11), and (4.0.12) and taking the limit as $\epsilon \to 0$ gives the following expressions relating velocity to temperature, salinity, and density[1]:

$$w_0 = \frac{\partial^2}{\partial x_0^2} \tau S_0 \qquad w_0 = \frac{\partial^2}{\partial x_0^2} T_0 \qquad w_0 = -\rho_0 R \tag{4.0.19}$$

We can now use these relations to work towards obtaining an equation entirely in terms of temperature (T_0). We begin by applying the scalings to (4.0.18) and taking the limit as $\epsilon \to 0$. We then introduce the preceding relations (4.0.19), noting that we can relate temperature and salinity by $S_{0xx} = \frac{1}{\tau} T_{0xx}$:

$$\left(\frac{R_\rho}{\tau^2} - 1\right) \frac{\partial}{\partial t_0} T_0 + \left(\frac{R_\rho}{\tau^2} - 1\right) \nabla \cdot (\mathbf{u}_0 T_0) - \left(\frac{R_\rho}{\tau^2} - 1\right) \frac{\partial}{\partial z_0} (\overline{w_0 T_0}) + \frac{\partial^2}{\partial x_0^2} T_0 \left(1 + \frac{R_\rho}{\tau} \beta_S - \beta_T\right) = -\frac{\partial^4}{\partial x_0^4} \frac{1}{R} T_0.$$

Multiplying through by R, and noting that in the $\epsilon \to 0$ limit, $\frac{R_\rho}{\tau} \approx 1$ we can simplify to the following evolutionary equation:

$$R\left(\frac{1}{\tau} - 1\right)\left(\frac{\partial}{\partial t_0} T_0 + \nabla \cdot (\mathbf{u}_0 T_0) - \frac{\partial}{\partial z_0} (\overline{w_0 T_0})\right) + R \frac{\partial^2}{\partial x_0^2} T_0 \left(1 + \beta_S - \beta_T\right) = -\frac{\partial^4}{\partial x_0^4} T_0. \tag{4.0.20}$$

Following Stern and Radko [35], we use the mean field equations to find expressions for β_T, β_S in terms of the temperature field so as to obtain an evolutionary equation entirely in terms of T_0. Since we have $(\theta_z, \sigma_z) = \epsilon(\beta_T, \beta_S)$, we first differentiate the mean field equations (4.0.7) and (4.0.8), then apply the scalings and take the limit as $\epsilon \to 0$. Doing so gives the following:

$$\left(\frac{\partial}{\partial t_0} - \frac{\partial^2}{\partial z_0^2}\right) \beta_T = -\frac{\partial^2}{\partial z_0^2} \left(\overline{T_0 \frac{\partial^2}{\partial x_0^2} T_0}\right)$$
$$\left(\frac{\partial}{\partial t_0} - \tau \frac{\partial^2}{\partial z_0^2}\right) \beta_S = -\frac{1}{\tau} \frac{\partial^2}{\partial z_0^2} \left(\overline{T_0 \frac{\partial^2}{\partial x_0^2} T_0}\right).$$

In order to completely express the evolutionary equation is terms of temperature alone, we must also relate velocities to the temperature field. We write the now-scaled velocity in terms of the

[1] Details can be found in Appendix D

streamfunction i.e. $(u_0, w_0) = (\frac{\partial}{\partial z_0}\psi_0, -\frac{\partial}{\partial x_0}\psi_0)$ in order to satisfy the continuity equation, and also note that $w_0 = \frac{\partial^2}{\partial x_0^2}T_0$. Using this information, we obtain the following:

$$w_0 = -\frac{\partial}{\partial x_0}\psi_0 \implies \frac{\partial^2}{\partial x_0^2}T_0 = -\frac{\partial}{\partial x_0}\psi_0 \implies \psi_0 = -\frac{\partial}{\partial x_0}T_0,$$

and consequently find that

$$(u_0, w_0) = (\frac{\partial}{\partial z_0}\psi_0, -\frac{\partial}{\partial x_0}\psi_0) \implies (u_0, w_0) = \left(-\frac{\partial^2}{\partial x_0 \partial z_0}T_0, \frac{\partial^2}{\partial x_0^2}T_0\right).$$

Now that the evolutionary equation is in terms of only temperature, we take the vertical average over an infinite vertical length such that we capture the case of $m = 0$. Doing so will eliminate any z derivatives in (4.0.20) and will allow us to consider the effects on the mode (x, z) with wavenumbers $(k, 0)$. Letting $\langle T_0 \rangle$ denote said vertical average, (4.0.20) becomes:

$$R\left(\frac{1}{\tau} - 1\right)\left(\frac{\partial}{\partial t_0}\langle T_0 \rangle + \frac{\partial}{\partial x_0}\langle uT_0 \rangle\right) + \frac{\partial^2}{\partial x_0^2}\langle T_0 R\,(1 + \beta_S - \beta_T)\rangle + \frac{\partial^4}{\partial x_0^4}\langle T_0 \rangle = 0.$$

4.1 Discussion

We conduct a similar analysis to that of Stern and Radko [35], noting that since the following terms are taken from the vertically averaged density equation, we will be considering them in terms of the $m = 0$ mode. We see that the $\frac{\partial^2}{\partial x_0^2}T_0$ and $\frac{\partial^4}{\partial x_0^4}T_0$ terms describe the linear growth of the system, and are balanced by the mean field terms β_S and β_T, providing a method of controlling this growth. The term $\frac{\partial}{\partial x_0}\langle uT_0 \rangle$ can interact with other terms by exchanging energy through non-resonant triads, which helps to stabilize the system [5, 28, 35]. We also found that the horizontal wavenumber resulting in the fastest growing finger is significantly smaller in a porous medium; $k_{0_{porous}} = \left(\frac{R}{2}\right)^{1/2}$ compared to $k_{0_{non-porous}} = \left(\frac{1}{3}\right)^{1/4}$. This indicates that in a porous medium, the horizontal wavelength of the fastest growing finger will be much smaller than that of a finger in a non-porous medium. We also note that when compared to the wavenumbers at the onset of instability (π) and the fastest growing finger $\left(\left(\frac{R}{2}\right)^{1/2}\right)$, the latter is much larger and so we are anticipating a shorter horizontal wavelength and thinner finger structure.

This analytical solution is only relevant for small ϵ, and although it was not analyzed numerically, it is useful in showing that linear growth is balanced by both triad and mean field terms. The time and horizontal spatial scalings derived in this chapter will also by applied in the following section.

Chapter 5

Steady Solutions of 2D Salt Fingers in a Vertically Bounded Layer

Extending the preceding chapter, we now look to solutions in a vertically bounded layer, and include thin boundary layers in our calculations. We proceed with the porous analog of Radko and Stern [29] and as usual, begin with the dimensionless Boussinesq equations:

$$\nabla^2\psi = R\left(-T_x + R_\rho S_x\right) \tag{5.0.1}$$

$$T_t + \mathbf{u}\cdot\nabla T + w = \nabla^2 T \tag{5.0.2}$$

$$S_t + \mathbf{u}\cdot\nabla S + w = \tau\nabla^2 S, \tag{5.0.3}$$

where $(T, S) = [T(x, z, t), S(x, z, t)]$ are the perturbed quantities and the lone w comes from the linear base state $T = S = z$. Following the calculations from chapter 4, we use the scalings below, noting that since we are no longer in an unbounded vertical domain, we restrict the z-scaling to that of the horizontal spatial scaling:

$$(x, z) = \epsilon^{-\frac{1}{2}}(x_0, z_0) \qquad\qquad t = \epsilon^{-2}t_0$$

We scale temperature $T \sim \epsilon^q T_0$. Using the above scalings, we balance the linear terms of the advection-diffusion equation i.e.

$$w \sim \nabla^2 T \implies w \sim \epsilon^{q+1}\nabla^2 T_0 \implies w \sim \epsilon^{q+1} w_0.$$

Similarly for equation (5.0.3), taking $S \sim \epsilon^j S_0$,

$$w \sim \tau\nabla^2 S \implies \epsilon^{q+1} w_0 \sim \epsilon^{1+j}\tau\nabla^2 S_0 \implies q = j \implies S \sim \epsilon^q S_0.$$

And so we see that S, T are both scaled by ϵ^q. From the preceding chapter, we found the horizontal wavenumber producing the maximum growth rate was $k_0 = \left(\frac{R}{2}\right)^{\frac{1}{2}}$. We then write temperature, salinity, and vertical velocity in terms of harmonic functions of x with wavenumber k_0:

$$T = \epsilon^q T_0 \sin(k_0 x_0) + ... \qquad S = \epsilon^q S_0 \sin(k_0 x_0) + ... \qquad w = \epsilon^{q+1} w_0 \sin(k_0 x_0) + ... \quad (5.0.4)$$

with higher order terms being determined below. Now, substituting (5.0.4) into (5.0.2) and (5.0.3) and considering the leading order terms by taking the limit as $\epsilon \to 0$, we are left only with the linear terms

$$w_0 \sin(k_0 x_0) = \nabla^2 T_0 \sin(k_0 x_0) \implies w_0 = \left(\frac{\partial^2}{\partial z_0^2} - k_0^2\right) T_0.$$

Doing the same for the solutal equation we arrive at

$$w_0 \sin(k_0 x_0) = \tau \nabla^2 S_0 \sin(k_0 x_0) \implies w_0 = \tau \left(\frac{\partial^2}{\partial z_0^2} - k_0^2\right) S_0,$$

and find that $T_0 = \tau S_0$. Now that we have an expression for the vertical velocity, we also need one for the horizontal velocity u. We do so by noting that the incompressibility condition,$\nabla \cdot \mathbf{u} = u_x + w_z = 0$ must be satisfied. Since we have made the vertical and horizontal spatial scalings the same, u must be scaled the same as w, i.e. $u \sim \epsilon^{q+1} u_0$. We also note that since w (and consequently, w_z) contains $\sin(k_0 x_0)$, then u_x must have a $\sin(k_0 x_0)$ term, and so $u \sim \epsilon^{q+1} u_0 \cos(k_0 x_0)$. Using these forms, we are able to solve for u_0:

$$u_x + w_z = 0 \implies -k_0 u_0 \sin(k_0 x_0) + \frac{\partial}{\partial z_0} w_0 \sin(k_0 x_0) = 0 \implies u_0 = \frac{1}{k_0} \frac{\partial}{\partial z_0} w_0.$$

Inserting the known form of w_0 gives

$$u_0 = \left(\frac{1}{k_0} \frac{\partial^3}{\partial z_0^3} - \frac{\partial}{\partial z_0} k_0\right) T_0.$$

We now consider both the mean field and triad terms, but will ultimately find that triad terms may be neglected. The leading-order mean field approximation leaves the following terms from equations (5.0.2) and (5.0.3):

$$\left(\overline{wT}\right)_z = \nabla^2 \theta \quad (5.0.5)$$

$$\left(\overline{wS}\right)_z = \tau \nabla^2 \sigma. \quad (5.0.6)$$

Using the scalings above for (5.0.5) we find that $(\theta, \sigma) \sim \epsilon^{2q+\frac{1}{2}} (\theta_0, \sigma_0)$. Substituting in the determined forms of u, w, T, and S and taking the horizontal average of wT and wS gives the following

expressions in terms of T_0, θ_0, and σ_0:

$$\frac{1}{2}\frac{\partial}{\partial z_0}\left[\frac{\partial^2}{\partial z_0^2}T_0 - k_0^2 T_0\right]\cdot T_0 = \frac{\partial^2}{\partial z_0^2}\theta_0 \tag{5.0.7}$$

$$\frac{1}{2}\frac{\partial}{\partial z_0}\left[\frac{\partial^2}{\partial z_0^2}T_0 - k_0^2 T_0\right]\cdot T_0 = \tau^2\frac{\partial^2}{\partial z_0^2}\sigma_0. \tag{5.0.8}$$

For the triad terms, we use the equations $\nabla\cdot(\mathbf{u}T) = \nabla^2 T$ and $\nabla\cdot(\mathbf{u}S) = \tau\nabla^2 S$. We begin with the temperature equation. Noting that the terms $(uT)_x$ and $(wT)_z$ will produce $\cos(2k_0x_0)$ terms, on the left-hand side we will take $T = \epsilon^{2q+\frac{1}{2}}T_{tr}\cos(2k_0x_0)$. Inserting this information in, we obtain the following[1]:

$$\begin{aligned}
&(u_0\cos(k_0x_0)T_0\sin(k_0x_0))_x + (w_0\sin(k_0x_0)T_0\sin(k_0x_0))_z = \left(\frac{\partial}{\partial z_0^2} - 4k_0^2\right)T_{tr}\cos(2k_0x_0)\\
\implies &\left(\partial z_0^3 T_0 - k_0\partial z_0 T_0\right)\cdot T_0 - \frac{1}{2}\frac{\partial}{\partial z_0}\left(\partial z_0^2 T_0 - k_0^2 T_0\right)\cdot T_0 -\\
&\qquad\qquad \frac{1}{2}\left(\partial z_0^2 T_0 - k_0^2 T_0\right)\cdot\frac{\partial}{\partial z_0}T_0 = \left(\frac{\partial^2}{\partial z_0^2} - 4k_0^2\right)T_{tr}\\
\implies &\frac{1}{2}\left(\partial z_0^3 T_0\cdot T_0 - \partial z_0^2 T_0\cdot\frac{\partial}{\partial z_0}T_0\right) = \left(\frac{\partial^2}{\partial z_0^2} - 4k_0^2\right)T_{tr}.
\end{aligned} \tag{5.0.9}$$

After completing an analogous calculation and using the relation $S_0 = \frac{T_0}{\tau}$, the equation for the solutal triad mode is determined to be

$$\frac{1}{2}\left(\partial_{z_0}^3 T_0\cdot T_0 - \partial z_0^2 T_0\cdot\partial z_0 T_0\right) = \tau^2\left(\frac{\partial^2}{\partial z_0^2} - 4k_0^2\right)S_{tr}.$$

As in the non-porous case, we find that we are able to neglect the triad terms in favour of the mean field terms, due to their relative size following the analysis of Stern and Radko (2000) [29]. In the interior we find that $\frac{\partial}{\partial z_0} \sim \frac{1}{H} \sim k_0\mu$, where H is the vertical height and μ is the aspect ratio. Since we are considering long, thin fingers the aspect ratio will be quite small[2], and so $k_0\mu$ will be much smaller than k_0. Using this, we find that (5.0.7) gives $\theta \sim k_0^2 H T_0^2 \sim \frac{k_0}{\mu}T_0^2$ and (5.0.9) gives $k_0\mu^3 T_0^2 \sim T_{tr}$. We now compare the sizes of the triad and mean field terms by taking $\frac{T_{tr}}{\theta} \sim \mu^4$, indicating that the mean field terms are significantly larger than the triad terms. We must now check the relative sizes at the boundary layers. Continuity of temperature allows us to make the following relations $\frac{\partial}{\partial z_0}T_{tr} \sim \frac{1}{d_0}T_{tr}|_{z_0=d_0}$ and $\frac{\partial}{\partial z_0}\theta \sim \frac{1}{d_0}\theta|_{z_0=d_0}$ where d_0 represents the boundary layer height. Since the boundary layer terms will be of the same magnitude as those in the interior, we can neglect

[1]Here, $\partial_{z_0}^i = \frac{\partial^i}{\partial z_0^i}$.

[2]In the case of our numerical computations, we take μ to be 0.1.

the triad terms in favour of the mean field terms through the entire finger zone [29].

To find higher order T, S terms, we consider the interaction between the mean field terms, θ, σ and the fastest growing mode, k_0. And so, beginning with $T_2\epsilon^j, S_2\epsilon^l$, we write:

$$\epsilon^{3q+2}\frac{\partial}{\partial z_0}(w_0\theta_0) = \epsilon^{j+1}\nabla^2 T_2 \implies j = 3q+1$$
$$\epsilon^{3q+2}\frac{\partial}{\partial z_0}(w_0\sigma_0) = \epsilon^{l+1}\tau\nabla^2 S_2 \implies l = 3q+1.$$

To find higher order w terms, we ensure that $\nabla^2\psi = R\left(-T_x + R_\rho S_x\right)$ is satisfied. At leading order, noting that $\nabla^2\psi$ has the same scaling as w_x, we have the following balance:

$$\epsilon^{q+\frac{3}{2}}\nabla^2\psi_0 = \epsilon^{j+\frac{1}{2}}\frac{\partial}{\partial x_0}R(-T + R_\rho S) \implies j = q+1.$$

And so, similarly to in chapter 4 the difference between T and $R_\rho S$ is smaller than T or S independently; in this case, it is an order ϵ smaller. Gong on to the next order, with $\epsilon^j w_2$:

$$\epsilon^{j+\frac{1}{2}}\nabla^2\psi_0 = \epsilon^{3q+\frac{5}{2}}\frac{\partial}{\partial x_0}R(-T_2 + R_\rho S_2) \implies j = 3q+2.$$

All together, up to order ϵ^{3q+2} we have:

$$T = \epsilon^q T_0\sin(k_0x_0) + \epsilon^{2q+\frac{1}{2}}\theta_0 + \epsilon^{3q+1}T_2\sin(k_0x_0) + ...$$
$$S = \epsilon^q S_0\sin(k_0x_0) + \epsilon^{2q+\frac{1}{2}}\sigma_0 + \epsilon^{3q+1}S_2\sin(k_0x_0) + ...$$
$$w = \epsilon^{q+1}w_0\sin(k_0x_0) + \epsilon^{3q+2}w_2\sin(k_0x_0) + ...$$

To determine the value of q, we ensure that the nonlinear terms equilibrate, and so we want $w\theta_z \sim \frac{\partial}{\partial t}T$. Putting in our scalings, we have $3q+2 = 2+q \implies q = 0$. Now, noting that we can write $\frac{1}{\tau} = \frac{(\epsilon+1)}{R_\rho}$, at $O(\epsilon^{3q+2}) = O(\epsilon^2)$, equations (5.0.2) and (5.0.3) give the following:

$$\frac{\partial}{\partial t_0}T_0 + w_0\frac{\partial}{\partial z_0}\theta_0 + w_2 = \nabla^2 T_2 \tag{5.0.10}$$
$$\frac{\partial}{\partial t_0}T_0 + \tau w_0\frac{\partial}{\partial z_0}\sigma_0 + \frac{\tau^2}{R_\rho}w_2 + \frac{\tau^2}{R_\rho}w_0 = \tau^2\nabla^2 S_2. \tag{5.0.11}$$

where the second equation occurs once more from the substitution $S_0 = \frac{T_0}{\tau}$. We now take the leading order $O\left(\epsilon^{\frac{3}{2}}\right)$ of equation (5.0.1), and using known forms of w_0, u_0 and the relation $S_0 = \frac{T_0}{\tau}$

obtain:

$$
\begin{aligned}
&-\frac{\partial}{\partial x_0} w_0 \sin(k_0 x_0) + \frac{\partial}{\partial z_0} u_0 \cos(k_0 x_0) = R\left(-\frac{\partial}{\partial x_0} T_2 \sin(k_0 x_0) + R_\rho \frac{\partial}{\partial x_0} S_2 \sin(k_0 x_0)\right) \\
\implies &-k_0 w_0 + \frac{\partial}{\partial z_0} u_0 = R\left(-k_0 T_2 + R_\rho k_0 S_2\right) \\
\implies &-k_0\left(\frac{\partial^2}{\partial z_0^2} - k_0^2\right) T_0 + \frac{\partial}{\partial z_0}\left(\frac{1}{k_0}\frac{\partial^3}{\partial z_0^3} - k_0 \frac{\partial}{\partial z_0}\right) T_0 = R\left(-k_0 T_2 + R_\rho k_0 S_2\right) \\
\implies &\left(\frac{\partial^2}{\partial z_0^2} - k_0^2\right)^2 T_0 = k_0^2 R\left(-T_2 + R_\rho S_2\right).
\end{aligned} \tag{5.0.12}
$$

Combining (5.0.10) and (5.0.11), we are able to eliminate w_2 and are left with:

$$
-\left(\frac{\partial^2}{\partial z_0^2} - k_0^2\right)\left(-T_2 + R_\rho S_2\right) + \left(\frac{R_\rho}{\tau^2} - 1\right)\frac{\partial}{\partial t_0} T_0 + \frac{R_\rho}{\tau} w_0 \frac{\partial}{\partial z_0}\left(\sigma_0 - \theta_0\right) + w_0 = 0. \tag{5.0.13}
$$

We are considering the $\epsilon \to 0$ limit, so $\tau = R_\rho$ and we can use (5.0.12) to eliminate the T_2, S_2 terms. Equations (5.0.7) and (5.0.8) give us the relation $\sigma = \frac{\theta}{\tau^2}$. Using these substitutions, we find a sixth order equation from writing (5.0.13) in terms of T_0 and θ_0:

$$
\begin{aligned}
\frac{1}{2}\left(\frac{1}{\tau^2} - 1\right)\left(\frac{\partial^2}{\partial z_0^2} - k_0^2\right) T_0 \left[\left(\partial z_0^2 T_0 - k_0^2 T_0\right)\cdot T_0 - \left\langle\left(\partial z_0^2 T_0 - k_0^2 T_0\right)\cdot T_0\right\rangle\right] \\
- \frac{1}{R k_0^2}\left(\frac{\partial^2}{\partial z_0^2} - k_0^2\right)^3 T_0 + \left(\frac{1}{\tau} - 1\right)\frac{\partial}{\partial t_0} T_0 + \left(\frac{\partial^2}{\partial z_0^2} - k_0^2\right) T_0 = 0,
\end{aligned}
$$

where $\langle ... \rangle$ denotes the vertical average over the finger domain. This equation describes the behaviour of T_0 over the length of the fastest growing finger and we are able to use it to solve for steady solutions numerically. Since our wavenumber k_0 is dependent on the Rayleigh number, the wave lengths will be much shorter than those of Radko and Stern [29], and so we expect the height of our fingers to be much shorter as well, in order to maintain an aspect ratio large enough to retain stability.

5.1 Weakly Nonlinear Theory

In order to acquire a good guess for the numerical solution, we begin by writing out some asymptotic expansions to employ a weakly nonlinear analysis:

$$\begin{aligned} T_0 &= \epsilon f_1 + \epsilon^3 f_3 + ... \\ R &= R_0 + \epsilon^2 R_2. \\ C &= \epsilon^2 C_2 + ..., \end{aligned}$$

where C is the constant of integration from θ_0, i.e. $\left\langle \left(\partial z_0^2 T_0 - k_0^2 T_0\right) \cdot T_0 \right\rangle$. We effectively rescale, letting $k_0 = 1$ and instead perturb the Rayleigh number R. Beginning at $O(\epsilon)$, we obtain the following:

$$\left(f_1^{(2)} - f_1\right) - \frac{1}{R_0}\left(f_1^{(6)} - 3f_1^{(4)} + 3f_1^{(2)} -^6 f_1\right) = 0.$$

Letting $f_1 = A\sin(\mu z_0)$, we substitute and then promptly divide through by like the terms $A\sin(\mu z_0)$. This allows us to solve for R_0:

$$\begin{aligned} & -(\mu^2+1) + \frac{1}{R_0}\left(\mu^6 + 3\mu^4 + 3\mu^2 + 1\right) = 0 \\ \implies & -(\mu^2+1) + \frac{1}{R_0}\left(\mu^2+1\right)^3 = 0 \\ \implies & -1 + \frac{1}{R_0}\left(\mu^2+1\right)^2 = 0 \\ \implies & R_0 = (\mu^2+1)^2. \end{aligned}$$

At $O(\epsilon^3)$ we repeat the process and obtain:

$$\begin{aligned} R_0 \frac{1}{2}\left(\frac{1}{\tau^2} - 1\right)\left((f_1^{(2)})^2 f_1 - 2f_1^{(2)} f_1^2 + f_1^3 - C_2 f_1^{(2)} + C_2 f_1\right) + R_0\left(f_3^{(2)} - f_3\right) \\ + R_2\left(f_1^{(2)} - f_1\right) - \left(f_3^{(6)} - 3f_3^{(4)} + 3f_3^{(2)} -^6 f_3\right). \end{aligned}$$

Since the f_1 terms will produce $\sin(\mu z_0)$ and $\sin(3\mu z_0)$ terms, we assume the form $f_3 \sim B\sin(\mu z_0) + D\sin(3\mu z_0)$. As in previous chapters, we collect like terms. In particular, we pay attention to the coefficients of the $B\sin(\mu z)$ terms. Collecting these we see that they end up being 0:

$$\begin{aligned} & B\left(\mu^6 + 3\mu^4 + (3 - R_0)\mu^2 - R_0 + 1\right)\sin(\mu z_0) \\ \implies & B\left((\mu^2+1)^3 - R_0(\mu^2+1)\right)\sin(\mu z_0) \\ \implies & B\left((\mu^2+1)^3 - (\mu^2+1)^2(\mu^2+1)\right)\sin(\mu z_0) \\ & = 0. \end{aligned}$$

Since these $\sin(\mu z_0)$ terms go to zero, we know that the remaining $\sin(\mu z_0)$ terms must do the same. The remaining $\sin(\mu z_0)$ coefficients, all produced by f_1 are:

$$
\begin{aligned}
&R_0 \frac{1}{2}\left(\frac{1}{\tau^2} - 1\right)\left(\frac{3A^3}{4}(\mu^2+1)^2 + C_2 A(\mu^2+1)\right) - R_2 A(\mu^2+1) = 0 \\
\implies &R_0 \frac{1}{2}\left(\frac{1}{\tau^2} - 1\right)\left(\frac{3A^3}{4}(\mu^2+1) + C_2 A\right) - R_2 A = 0. \qquad (5.1.1)
\end{aligned}
$$

Before proceeding, we must determine the value of C_2. Vertically integrating over $f_1^{(2)} f_1 - f_1^2$, and letting the vertical height H equal $\frac{\pi}{\mu}$ in order to satisfy boundary conditions, we arrive at:

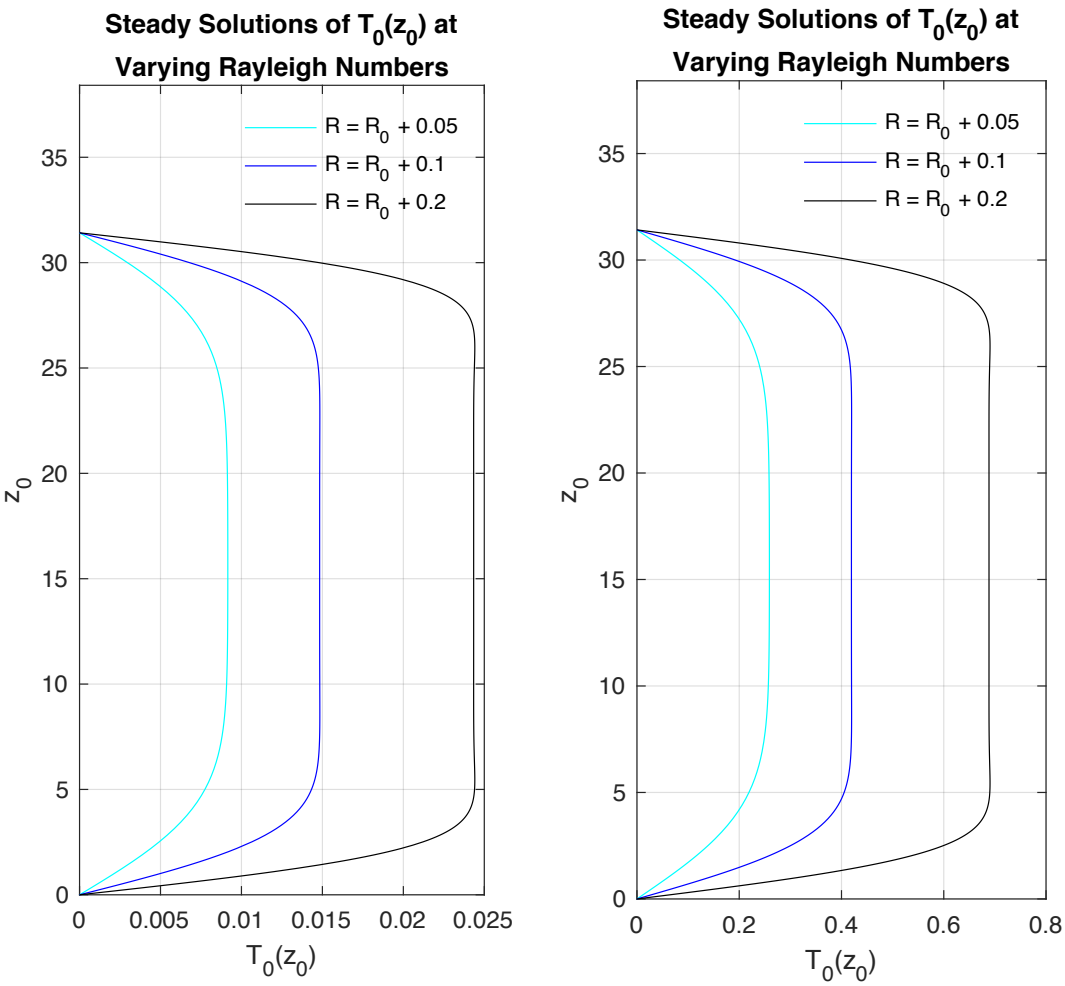
$$
\begin{aligned}
C_2 &= \int_0^H f_1^{(2)} f_1 - f_1^2 dz_0 \\
&= \int_0^{\frac{\pi}{\mu}} -A^2\mu^2 \sin(\mu z_0)^2 - A^2 \sin(\mu z_0)^2 dz \\
&= -\frac{A^2 \pi(\mu^2+1)}{2\mu}. \qquad (5.1.2)
\end{aligned}
$$

Dividing (5.1.2) by H to obtain an average, and substituting it into (5.1.1) we are able to solve for the amplitude A.

$$
\begin{aligned}
&R_0 \frac{1}{2}\left(\frac{1}{\tau^2} - 1\right)\left(\frac{3A^3(\mu^2+1)}{4} - \frac{A^3(\mu^2+1)}{2}\right) - R_2 A = 0 \\
\implies &A^3 R_0 \frac{(\mu^2+1)}{8}\left(\frac{1}{\tau^2} - 1\right) - R_2 A = 0 \\
\implies &A^2 = \frac{R_2 8\tau^2}{R_0(\mu^2+1)(1-\tau^2)} \\
\implies &A = \sqrt{\frac{R_2 8\tau^2}{(\mu^2+1)^3(1-\tau^2)}}.
\end{aligned}
$$

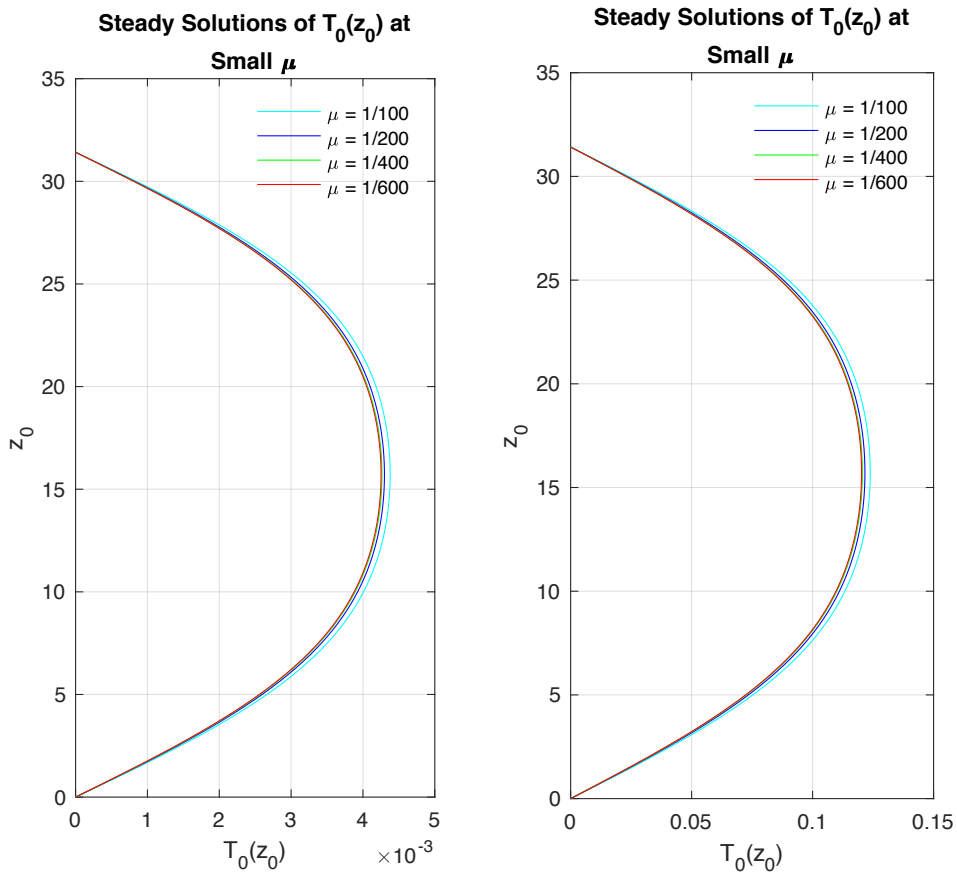
In order to arrive at a real valued amplitude, we require R_2 to be positive. Using the above values as an initial guess, we are able to plot the steady solutions in Matlab[3]. The results are displayed in Figure 5.1. Figure 5.2 shows the amplitudes to which the numeric solutions converge in the $\mu \to 0$ limit, and Figure 5.3 compares the numeric and analytic solutions in this same limit.

[3]Code for the plot can be found in Appendix E.



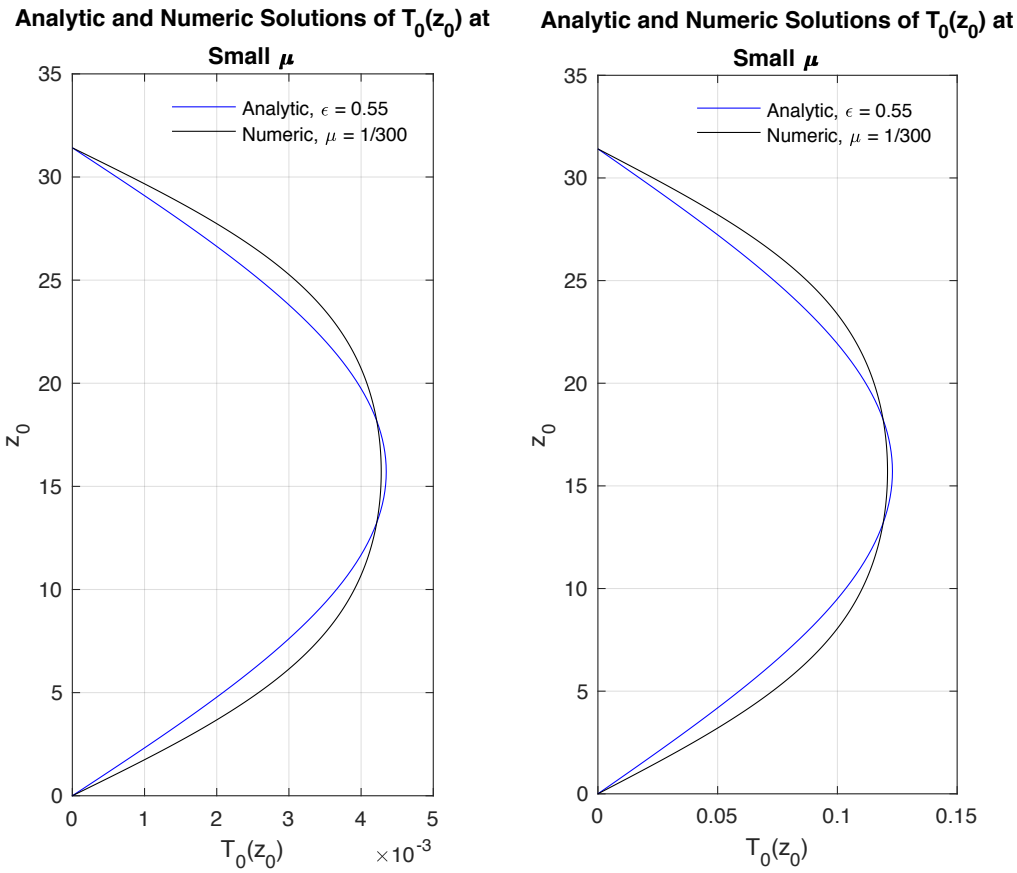
(a) (b)

Figure 5.1: Steady solutions of the salt fingers in an enclosed porous medium, over varying Rayleigh numbers. In (a) we use the salt-heat configuration and thus take $\tau = \frac{1}{80}$. In (b) we follow the sugar-salt values and use $\tau = \frac{1}{3}$. Other parameter values remain the same between the two - $\mu = 0.1, R_0 = (\mu^2 + 1)^2$.



(a) (b)

Figure 5.2: Solutions of the salt fingers in an enclosed porous medium, over varying μ. In (a) we use the salt-heat configuration and thus take $\tau = \frac{1}{80}$. In (b) we follow the sugar-salt values and use $\tau = \frac{1}{3}$. In both cases we use $R_0 + 0.05 = (\mu^2 + 1)^2 + 0.05$.



(a) (b)

Figure 5.3: Above are the amplitudes predicted by the numerical and analytical analyses. R_0+0.05 = $(\mu^2 + 1)^2 + 0.05$ for both, with $\mu = \frac{1}{80}, \frac{1}{3}$ in (a) and (b) respectively.

5.2 Discussion

From Figures 5.1a and 5.1b we see that the fingers have a uniform temperature, going to 0 at the top and bottom as per our boundary conditions, much like the non-porous results from Stern and Radko [29]. We see that changing the Lewis number (from 3 to 80) so that it's appropriate for oceanographic settings changes the maximum temperature attained, but not so much the general shape of the temperature distribution. As we get increasingly far from the critical Rayleigh number, we see that the temperature amplitude increases. We also note that at a lower Lewis number (higher τ, indicating a more dominant thermal diffusivity), with all other parameters unchanged, the temperature amplitude is higher. The steady solution in the porous media is thus similar to that of the non-porous results, but with a smaller maximal amplitude. As in Stern and Radko's paper, we find that the amplitude is almost independent of z outside of the boundary layers [29].

Stern and Radko found that in order to achieve a regular steady solution, the horizontal/vertical aspect ratio μ must not be smaller than $1 : 200$ [29]. In our case, it appears as though we are able to find reasonable amplitude solutions even at incredibly small values of μ[4]. It appears as though as μ approaches the zero limit, the temperature amplitude converges to about 0.0042 for the salt-heat case and 0.12 for the sugar-salt case.

For the analytic solution of T_0, we took the asymptotic expansion $\epsilon f_1 + \epsilon^3 f_3 + ...$, where f_1 was decided to be $A\sin(\mu z_0)$. Since only A was computed, we plot only the solution using f_1. Using the computed value of A in the $\mu \to 0$ limit, we see that the agreement between numeric and analytic solutions depends on the range of ϵ. For all values of ϵ, the analytic and numeric solutions have similar shapes but as ϵ approaches a value of about 0.55 the two begin to agree quite nicely. It should be noted that while Figures 5.3a and 5.3b are both done using $\mu = \frac{1}{300}$, results were the same for $\mu = \frac{1}{100}, \frac{1}{200}$, up to and past $\mu = \frac{1}{600}$.

[4]Code for Figure 5.2 is identical to that used for 5.1, but with varying mu values, instead of varying Rt. Figure 5.3 uses this same code as well, with the addition of $\epsilon A \sin(\mu z_0)$ being plotted for the analytical curve.

Chapter 6

Conclusion

In the stability analyses of chapters two and three, we have seen that in both the (steady) diffusive convection and the salt fingering regimes there is agreement between the linear and nonlinear stability analyses. In fact, we saw that the two methods gave the same critical Rayleigh numbers for the onset of instability, confirming that there are no other critical values below the calculated threshold. Weakly nonlinear analysis further confirmed this by providing us with a Landau equation representing supercritical stability and allowed us to determine the amplitudes the systems tend towards in the limit $\mathcal{T} \to \infty$. In the case of oscillatory diffusive convection, we saw that the system was open to both sub and supercritical instabilities, depending on the value of R_ρ relative to τ. Weakly nonlinear analysis indicated that when R_ρ was greater than τ, the system would enter a regime of supercriticality; a subcritical regime could be achieved when R_ρ was less than τ.

Chapters four and five provided us with evolutionary equations for long, thin salt fingers in terms of the temperature, with only time and z-dependence. While only the chapter five equation was evaluated numerically, this reduced model allowed for fast computations. The resulting figure was of a similar form to that obtained in Stern and Radko (2000), but with a slightly shorter length scale, and significantly smaller temperature amplitude. We were also able to find reasonable solutions at extremely small μ, checking up to $\mu = \frac{1}{600}$ and saw that solutions converged to amplitudes of 0.0042 and 0.12 for salt-heat and sugar-salt configurations respectively. Chapter four's evolution equation was not evaluated numerically, but we were still able to determine that the growth caused by the linear terms was balanced by both the mean field and triad terms. Based on the wavenumber of the fastest growing finger, we found that in a porous medium these fingers would have a smaller width than in non-porous media.

Only the two-dimensional cases were studied in this thesis, and so further work can be done to extend the analyses to three-dimensional fingers. Further, as this thesis contained purely analytical and numerical results, the case of the fastest growing salt fingers in a porous medium could be studied experimentally for further understanding.

Bibliography

[1] On the problem of turbulence. In D. Ter Haar, editor, *Collected Papers of L.D. Landau*, pages 387 – 391. Pergamon, 1965. ISBN 978-0-08-010586-4.

[2] Y. Bachmat and J. Bear. Macroscopic modelling of transport phenomena in porous media. 1: The continuum approach. *Transport in Porous Media*, 1(3):213–240, 1986.

[3] P. G. Baines and A. E. Gill. On thermohaline convection with linear gradients. *Journal of fluid mechanics*, 37(2):289–306, 1969.

[4] N. J. Balmforth, S. A. Ghadge, A. Kettapun, and S. D. Mandre. Bounds on double-diffusive convection. *Journal of Fluid Mechanics*, 569:29–50, 2006.

[5] M. D. Bustamante and B. Quinn. Robust energy transfer mechanism and critically balanced turbulence via non-resonant triads in nonlinear wave systems. 2013.

[6] E. Castillo, A. Luceno, and P. Pedregal. Composition functionals in calculus of variations. application to products and quotients. *Mathematical Models and Methods in Applied Sciences - M3AS*, 18, 01 2008. doi: 10.1142/S0218202508002607.

[7] F. Chen and C. F. Chen. Double-diffusive fingering convection in a porous medium. *International Journal of Heat and Mass Transfer*, 36(3):793–807, 1993.

[8] C. A. Cooper, R. J. Glass, and S. W. Tyler. Experimental investigation of the stability boundary for double-diffusive finger convection in a hele-shaw cell. *Water resources research*, 33(4): 517–526, 1997.

[9] J. W. Elder. Steady free convection in a porous medium heated from below. *Journal of fluid mechanics*, 27(1):29–48, 1967.

[10] E. Engel and R. M. Dreizler. *Density functional theory: an advanced course*. Springer, New York;Heidelberg [Germany], 2011. ISBN 3642140890;9783642140891.

[11] L. R. O. F.R.S. Lix. on convection currents in a horizontal layer of fluid, when the higher temperature is on the under side. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 32(192):529–546, 1916. doi: 10.1080/14786441608635602.

[12] T. Green. Scales for double-diffusive fingering in porous media. *Water Resources Research*, 20(9):1225–1229, 1984.

[13] D. R. Hewitt. Double diffusive convection in a porous medium. 10 2015.

[14] D. D. Joseph. *Stability of Fluid Motions I*, volume 27. Springer Berlin Heidelberg, 1976. ISBN 3642809936.

[15] D. D. Joseph. *Stability of Fluid Motions II*, volume 28. Springer Berlin Heidelberg, 1976. ISBN 9783642809965.

[16] R. Kippenhahn and A. Weigert. *Stellar structure and evolution*. Springer-Verlag, New York;Berlin, 1st, corr. 3rd printing edition, 1994.

[17] M. D. T. J. Knobloch, E. and N. Weiss. Transitions to chaos in two-dimensional double-diffusive convection. *Journal of Fluid Mechanics*, 166:409–448, 1986. doi: 10.1017/S0022112086000216.

[18] E. Kunze. Limits on growing, finite-length salt fingers: A richardson number constraint. *Journal of marine research*, 45(3):533–556, 1987.

[19] E. R. Lapwood. Convection of a fluid in a porous medium. *Mathematical Proceedings of the Cambridge Philosophical Society*, 44(4):508–521, 1948. doi: 10.1017/S030500410002452X.

[20] e. a. Lerman, Abraham. *Physics and Chemistry of Lakes*. Springer Berlin Heidelberg, 1995. doi: 10.1007/978-3-642-85132-2.

[21] W. V. R. Malkus and G. Veronis. Finite amplitude cellular convection. *Journal of fluid mechanics*, 4(3):225–260, 2006.

[22] A. C. Newell and J. A. Whitehead. Finite bandwidth, finite amplitude convection. *Journal of fluid mechanics*, 38(2):279–303, 2006.

[23] D. A. Nield and A. Bejan. *Convection in Porous Media*. Springer, 1992. ISBN 0-387-97651-5.

[24] R. E. O'Malley and D. B. Williams. Deriving amplitude equations for weakly-nonlinear oscillators and their generalizations. *Journal of Computational and Applied Mathematics*, 190(1): 3–21, 2006.

[25] M. R. E. Proctor and J. Y. Holyer. Planform selection in salt fingers. *Journal of fluid mechanics*, 168(1):241–253, 1986.

[26] M. Provansal, C. Mathis, and L. Boyer. Bénard-von kármán instability: transient and forced regimes. *Journal of fluid mechanics*, 182(1):1–22, 1987.

[27] T. Radko. *Double-Diffusive Convection*. Cambridge: Cambridge University Press, 2013. doi: 10.1017/CBO9781139034173.

[28] T. Radko and M. E. Stern. Salt fingers in three dimensions. *Journal of marine research*, 57(3): 471–502, 1999.

[29] T. Radko and M. E. Stern. Finite-amplitude salt fingers in a vertically bounded layer. *Journal of fluid mechanics*, 425:133–160, 2000.

[30] R. Schmitt. Double diffusion in oceanography. *Annual review of fluid mechanics*, 26(1):255–285, 1994.

[31] R. W. Schmitt. The growth rate of super-critical salt fingers. *Deep-sea research. Part A. Oceanographic research papers*, 26(1):23–40, 1979.

[32] C. Y. Shen. The evolution of the double-diffusive instability: Salt fingers. *Physics of Fluids A: Fluid Dynamics*, 1(5):829–844, 1989.

[33] V. Steinberg and H. Brand. Convective instabilities of binary mixtures with fast chemical reaction in a porous medium. *The Journal of chemical physics*, 78(5):2655–2660, 1983.

[34] M. E. Stern. The "salt-fountain" and thermohaline convection. *Tellus*, 12(2):172–175, 5 1960. ISSN 2153-3490. doi: 10.1111/j.2153-3490.1960.tb01295.x.

[35] M. E. Stern and T. Radko. The salt finger amplitude in unbounded t-s gradient layers. *Journal of Marine Research*, 56(1):157–196, 1998. doi: 10.1357/002224098321836154.

[36] M. E. Stern and J. Simeonov. Amplitude equilibration of sugar-salt fingers. *Journal of fluid mechanics*, 508:265–286, 2004.

[37] H. Stommel, A. B. Arons, and D. Blanchard. An oceanographical curiosity: the perpetual salt fountain. *Deep-sea research (1953)*, 3(2):152–153, 1956.

[38] B. Straughan. *The Energy Method, Stability, and Nonlinear Convection*, volume 91. Springer New York, New York, NY, 1992. ISBN 1475721943.

[39] B. Straughan. *Stability and wave motion in porous media*, volume 165;165.;. Springer, New York, 1. aufl. edition, 2008. ISBN 9780387765419;0387765417;.

[40] J. W. Taunton, E. N. Lightfoot, and T. Green. Thermohaline instability and salt fingers in a porous medium. *The Physics of Fluids*, 15(5):748–753, 1972. doi: 10.1063/1.1693979.

[41] J. Taylor. Laboratory experiments on the formation of salt fingers after the decay of turbulence. *Journal of Geophysical Research: Oceans*, 96(C7):12497–12510, 1991.

[42] P. M. J. Trevelyan, C. Almarcha, and A. De Wit. Buoyancy-driven instabilities of miscible two-layer stratifications in porous media and hele-shaw cells. *Journal of fluid mechanics*, 670: 38–65, 2011.

[43] J.-H. Xie, B. Miquel, K. Julien, and E. Knobloch. A reduced model for salt-finger convection in the small diffusivity ratio limit. *Fluids (Basel)*, 2(1):6, 2017;2016;.

[44] R. Zeytounian. *Asymptotic Modeling of Atmospheric Flows*. Springer Berlin Heidelberg, Berlin, Heidelberg, 1990. ISBN 9783642738029.

Appendix A

Nondimensionalization

As in previous works, we wish to procure the dimensionless governing equations [4, 13, 23]. We begin with the Boussinesq equations:

$$\rho = \rho_0[1 - \beta(T - T_0) + \alpha(S - S_0)] \tag{A.0.1}$$

$$\mathbf{u} = -\frac{k}{\mu}(\nabla p + \rho g \hat{\mathbf{z}}) \tag{A.0.2}$$

$$\Phi T_t + \mathbf{u} \cdot \nabla T = \kappa_T \nabla^2 T \tag{A.0.3}$$

$$\phi S_t + \mathbf{u} \cdot \nabla S = \phi \kappa_S \nabla^2 S. \tag{A.0.4}$$

First we will determine the scalings for time and velocity. We begin with (A.0.3), in which we scale temperature with the temperature gradient ΔT and lengths by d [13, 23]. We write these re-scalings as

$$T = \Delta T\, T^*, \quad \frac{\partial}{\partial z} = \frac{\partial}{\partial z^* d}, \quad \frac{\partial}{\partial x} = \frac{\partial}{\partial x^* d}, \quad \mathbf{u} = \square_1 \mathbf{u}^*, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial t^* \square_2},$$

where $(...)^*$ denotes the dimensionless groups, and $\square_{1,2}$ denote scalings which are yet to be determined. Inserting these new definitions into (A.0.3) gives

$$\begin{aligned}
&\Phi \frac{\partial \Delta T T^*}{\partial t^* \square_2} + \square_1 \mathbf{u}^* \cdot \left(\frac{\partial \Delta T T^*}{\partial x^* d}, \frac{\partial \Delta T T^*}{\partial z^* d} \right) = \kappa_T \frac{\Delta T T^*}{d^2} \\
\implies &\square_1 = \frac{\kappa_T}{d}, \text{ and } \square_2 = \frac{d^2 \Phi}{\kappa_T} \\
\implies &T^*_{t^*} + \mathbf{u}^* \cdot \nabla T^* = \nabla^2 T^*,
\end{aligned}$$

where velocity and time scalings were determined by matching terms. Using the newfound velocity and time scalings $\square_{1,2}$, and using the salinity gradient ΔS to scale salinity in equation (A.0.4), the

following is obtained:

$$\phi\frac{\partial\Delta S\,S^*\kappa_T}{\partial t^* d^2\Phi} + \frac{\kappa_T}{d}\mathbf{u}^*\cdot\left(\frac{\partial\Delta S\,S^*}{\partial x^* d},\frac{\partial\Delta S\,S^*}{\partial z^* d}\right) = \phi\kappa_S\,\frac{\Delta S\,S^*}{d^2}$$
$$\implies \frac{\phi}{\Phi}S^*_{t^*} + \mathbf{u}^*\cdot\nabla S^* = \frac{\phi\kappa_S}{\kappa_T}\nabla^2 S^*.$$

Dropping the $(...)^*$ decoration results in the following dimensionless equations:

$$T_t + \mathbf{u}\cdot\nabla T = \nabla^2 T$$
$$\frac{\phi}{\Phi}S_t + \mathbf{u}\cdot\nabla S = \frac{\phi\kappa_S}{\kappa_T}\nabla^2 S.$$

Now we return to the dimensional equations and substitute (A.0.1) into (A.0.2). After some work, this will give us the third and final equation used to describe the double-diffusive convection.

$$\mathbf{u} = -\frac{k}{\mu}\left(\underbrace{\nabla p}_{1} + \underbrace{\rho_0[1-\beta(T-T_0)+\alpha(S-S_0)]g\hat{\mathbf{z}}}_{2}\right). \tag{A.0.5}$$

Taking the curl of (A.0.5), on the left-hand side of the equation we now have the negative vorticity, i.e. $-\nabla^2\psi$. On the right had side, since the curl of a gradient is 0, term 1 will also be 0. Consider now term 2: since there are only z-components, we will be left with the x-derivatives of these terms. This leaves $\frac{-k\rho_0 g}{\mu}(\alpha S_x - \beta T_x)$. And so, together we have the following:

$$-\nabla^2\psi = \frac{-k\rho_0 g}{\mu}(\alpha S_x - \beta T_x).$$

To nondimensionalize, we use the same scalings as above, noting that $\nabla\times\mathbf{u} = -\nabla^2\psi$ will have the dimensions of velocity over length.

$$\frac{\partial^2\kappa_T\psi^*}{\partial x^{*2}d^2} + \frac{\partial^2\kappa_T\psi^*}{\partial z^{*2}d^2} = \frac{k\rho_0 g}{\mu}\left(\alpha\frac{\partial\Delta S\,S^*}{\partial x^* d} - \beta\frac{\partial\Delta T\,T^*}{\partial x^* d}\right)$$
$$\implies \frac{\partial^2\psi^*}{\partial x^{*2}} + \frac{\partial^2\psi^*}{\partial z^{*2}} = \frac{k\rho_0 g d}{\mu\kappa_T}(\alpha\Delta S\,S^*_{x^*} - \beta\Delta T\,T^*_{x^*})$$
$$\implies \nabla^2\psi^* = \frac{k\rho_0 g d}{\mu\kappa_T}(\alpha\Delta S\,S^*_{x^*} - \beta\Delta T\,T^*_{x^*}).$$

Once again dropping the $(...)^*$ decoration, as well as taking the Rayleigh number R to be $\frac{dk\beta\rho_0\Delta T g}{\mu\kappa_T}$ and density ratio $R_\rho = \frac{\alpha\Delta S}{\beta\Delta T}$ as per [13, 23] we get

$$\implies \nabla^2\psi = R\left(R_\rho S_x - T_x\right),$$

and so the dimensionless governing equations are as follows:

$$\nabla^2\psi = R\left(R_\rho S_x - T_x\right)$$
$$T_t + \mathbf{u}\cdot\nabla T = \nabla^2 T$$
$$\frac{\phi}{\Phi}S_t + \mathbf{u}\cdot\nabla S = \frac{\phi\kappa_S}{\kappa_T}\nabla^2 S.$$

In the case of diffusive convection, we let R take on a negative, so that we arrive at a positive Rayleigh number in our calculations. This makes the equations for diffusive convection

$$\nabla^2\psi = R\left(-R_\rho S_x + T_x\right)$$
$$T_t + \mathbf{u}\cdot\nabla T = \nabla^2 T$$
$$\frac{\phi}{\Phi}S_t + \mathbf{u}\cdot\nabla S = \frac{\phi\kappa_S}{\kappa_T}\nabla^2 S.$$

Appendix B

Full Calculations of the Diffusive Convection Case

B.1 Linear - Oscillatory Case

We modify the governing equations by taking the negative of the right-hand side of the velocity equation

$$\begin{aligned}
\nabla^2\psi &= R\left(-T_x + R_\rho S_x\right)\\
T_t + \psi_x &= \nabla^2 T\\
S_t + \psi_x &= \frac{1}{Le}\nabla^2 S,
\end{aligned}$$

keeping thermal and solutal equations the same as for steady convection. Following the same steps as for steady convection, we arrive at the following equation in terms of σ:

$$-Le\sigma^2 + \sigma\left(\frac{Rk^2Le}{k^2+m^2\pi^2}(1-R_\rho) - (k^2+m^2\pi^2)(1+Le)\right) + \left(Rk^2(1-R_\rho Le) - (k^2+m^2\pi^2)^2\right) = 0.$$

Setting $\sigma = i\omega + \sigma_r$, where ω and σ_r are real and retaining only the imaginary parts gives

$$i\omega\left(\frac{Rk^2Le}{k^2+m^2\pi^2}(1-R_\rho) - (k^2+m^2\pi^2)(1+Le)\right) = 0.$$

B.2 Nonlinear

B.2.1 Power Integrals

We begin with the perturbed governing equations from section, noting that the perturbations θ, σ satisfy homogeneous boundary conditions 3.3.

$$\mathbf{u} = -\nabla p - R\theta\hat{\mathbf{z}} + RR_\rho\sigma\hat{\mathbf{z}} \tag{B.2.1}$$

$$\theta_t - w + \mathbf{u}\cdot\nabla\theta = \nabla^2\theta \tag{B.2.2}$$

$$\sigma_t - w + \mathbf{u}\cdot\nabla\sigma = \frac{1}{Le}\nabla^2\sigma. \tag{B.2.3}$$

Multiplying the above equations by $\mathbf{u}$, θ, and σ respectively, and integrating over the domain will yield power integrals. Doing so and applying boundary conditions and the divergence theorem, we obtain the following:

$$\int_0^{\frac{2\pi}{m}}\int_0^{\frac{2\pi}{k}} \mathbf{u}\cdot\mathbf{u}\,dxdz = \int_0^{\frac{2\pi}{m}}\int_0^{\frac{2\pi}{k}} -\mathbf{u}\cdot\nabla p + RR_\rho\sigma\hat{\mathbf{z}}\cdot\mathbf{u} - R\theta\hat{\mathbf{z}}\cdot\mathbf{u}\,dxdz$$
$$\implies \langle|\mathbf{u}|^2\rangle = -RR_\rho\langle w\sigma\rangle + R\langle w\theta\rangle.$$

Similarly for temperature and salt:

$$\int_0^{\frac{2\pi}{m}}\int_0^{\frac{2\pi}{k}} \theta_t\theta + (\mathbf{u}\cdot\nabla\theta)\,\theta - w\theta\,dxdz = \int_0^{\frac{2\pi}{m}}\int_0^{\frac{2\pi}{k}} \theta\nabla^2\theta\,dxdz$$
$$\implies \frac{1}{2}\langle\theta^2\rangle_t - \langle w\theta\rangle = -\langle|\nabla\theta|^2\rangle,$$

and

$$\int_0^{\frac{2\pi}{m}}\int_0^{\frac{2\pi}{k}} \sigma_t\sigma + (\mathbf{u}\cdot\nabla\sigma)\,\sigma - w\sigma\,dxdz = \int_0^{\frac{2\pi}{m}}\int_0^{\frac{2\pi}{k}} \frac{1}{Le}\sigma\nabla^2\sigma\,dxdz$$
$$\implies \frac{1}{2}\langle\sigma^2\rangle_t - \langle w\sigma\rangle = -\frac{1}{Le}\langle|\nabla\sigma|^2\rangle,$$

where the negative signs in the gradient terms come from the divergence theorem. As θ_t^2 and σ_t^2 produce $2\theta_t\theta$ and $2\sigma_t\sigma$, the $\frac{1}{2}$ is added in to counteract it. Note that the angle brackets $\langle ...\rangle$ represent integration over the domain.

The final equation required for the stability analysis is obtained by multiplying (B.2.2) by σ,

(B.2.3) by θ, summing together, and integrating over the domain:

$$\theta_t\sigma - w\sigma + (\mathbf{u}\cdot\nabla\theta)\,\sigma = \sigma\nabla^2\theta$$
$$+$$
$$\sigma_t\theta - w\theta + (\mathbf{u}\cdot\nabla\sigma)\,\theta = \frac{\theta}{Le}\nabla^2\sigma$$

$$\begin{aligned}
&\implies \int_0^{\frac{2\pi}{m}}\int_0^{\frac{2\pi}{k}} \theta_t\sigma - w\sigma + (\mathbf{u}\cdot\nabla\theta)\,\sigma + \sigma_t\theta - w\theta + (\mathbf{u}\cdot\nabla\sigma)\,\theta\; dxdz \\
&\implies \int_0^{\frac{2\pi}{m}}\int_0^{\frac{2\pi}{k}} \sigma\nabla^2\theta + \frac{\theta}{Le}\nabla^2\sigma\; dxdz \\
&\implies \langle\theta\sigma\rangle_t - \langle w\sigma\rangle - \langle w\theta\rangle = -(1+\tau)\langle\nabla\sigma\cdot\nabla\theta\rangle,
\end{aligned}$$

with the negative associated with the last term once again arising from the divergence theorem.

B.2.2 Eliminating θ

With the equations laid out, we move onto eliminating σ, θ terms from the linear combination outlined in the main body. Using the substitution $f = \theta + \frac{A(1+\tau)\sigma}{2}$, we begin with the combination for I:

$$\begin{aligned}
I &= \left(1 + A - C^2\right)\langle w\theta\rangle + \left(A + A^2 + B^2 + C^2R_\rho\right)\langle w\sigma\rangle \\
&\implies \left(1 + A - C^2\right)\langle w\left(f - \frac{A(1+\tau)\sigma}{2}\right)\rangle + \left(A + A^2 + B^2 + C^2R_\rho\right)\langle w\sigma\rangle \\
&\implies \left(1 + A - C^2\right)\langle wf\rangle - \left(1 + A - C^2\right)\left(\frac{A(1+\tau)}{2}\right)\langle w\sigma\rangle + \left(A + A^2 + B^2 + C^2R_\rho\right)\langle w\sigma\rangle \\
&\implies \left(1 + A - C^2\right)\langle wf\rangle + \left(A + A^2 + B^2 + C^2R_\rho - \frac{A}{2}(1+\tau)\left(1 + A - C^2\right)\right)\langle w\sigma\rangle.
\end{aligned}$$

Doing the same for D:

$$
\begin{aligned}
D &= \langle|\nabla\theta|^2\rangle + \left(A^2 + B^2\right)\tau\langle|\nabla\sigma|^2\rangle + A\,(1+\tau)\,\langle\nabla\theta\cdot\nabla\sigma\rangle + \frac{C^2}{R}\langle|\mathbf{u}|^2\rangle \\
&\implies \langle|\nabla\left(f - \frac{A(1+\tau)\sigma}{2}\right)|^2\rangle + \left(A^2 + B^2\right)\tau\langle|\nabla\sigma|^2\rangle + A\,(1+\tau)\,\langle\nabla\left(f - \frac{A(1+\tau)\sigma}{2}\right)\cdot\nabla\sigma\rangle + \frac{C^2}{R}\langle|\mathbf{u}|^2\rangle \\
&\implies \langle|\nabla f|^2\rangle - A(1+\tau)\langle\nabla f\cdot\nabla\sigma\rangle + \frac{A^2(1+\tau)^2}{4}\langle|\nabla\sigma|^2\rangle + \left(A^2 + B^2\right)\tau\langle|\nabla\sigma|^2\rangle \\
&\qquad + \frac{C^2}{R}\langle|\mathbf{u}|^2\rangle + A(1+\tau)\left(\langle\nabla f\cdot\nabla\sigma\rangle - \frac{A(1+\tau)}{2}\langle|\nabla\sigma|^2\rangle\right) \\
&\implies \langle|\nabla f|^2\rangle + \left(B^2\tau - \frac{A^2}{4}(1-\tau)^2\right)\langle|\nabla\sigma|^2\rangle + \frac{C^2}{R}\langle|\mathbf{u}|^2\rangle.
\end{aligned}
$$

B.2.3 Functional Derivatives

As described in the main body, we eliminate terms involving σ by carefully choosing B^2, C^2. Including a point wise constraint due to incompressibility, we are then left to extremize

$$
F = \frac{I}{D} + \langle(\nabla\cdot\mathbf{u})\lambda\rangle = \frac{\alpha\langle wf\rangle}{\langle|\nabla f|^2\rangle + \frac{C^2}{R}\langle|\mathbf{u}|^2\rangle} + \langle(\nabla\cdot\mathbf{u})\lambda\rangle,
$$

and do so by taking functional derivatives $\delta f, \delta u$, and δw. Writing $f = f + \delta f, u = u + \delta u$, and $w = w + \delta w$, we begin with the f functional, letting α represent $\left(1 + A - C^2\right)$ in order to keep things a bit tidier. Using quotient rule and retaining terms of $O(\delta f)$, noting that the incompressibility constraint does not contain f, and goes to zero here we arrive at the following [6]:

$$
\begin{aligned}
F =& \frac{\alpha\langle(w+\delta w)(f+\delta f)\rangle}{\langle|\nabla f + \nabla\delta f|^2\rangle + \frac{C^2}{R}\langle|(u+\delta u, w+\delta w)|^2\rangle} + \langle\nabla\cdot(u+\delta u, w+\delta w)\lambda\rangle \\
\implies& \frac{\delta F}{\delta f} : \frac{\alpha w}{D} + \frac{2I\nabla^2 f}{D^2} = 0 \\
\implies& \frac{\delta F}{\delta f} : \frac{-\alpha w D}{2I} = \nabla^2 f \\
\implies& \frac{\delta F}{\delta f} : \frac{\alpha\psi_x D}{2I} = \nabla^2 f,
\end{aligned}
$$

where the gradient term produces the following due to the divergence theorem:

$$
\begin{aligned}
\langle|\nabla f + \nabla\delta f|^2\rangle \implies& \int_0^{\frac{2\pi}{m}}\int_0^{\frac{2\pi}{k}} |\nabla f|^2 + 2\nabla\delta f\cdot\nabla f + |\nabla\delta f|^2 dxdz \\
\implies& \langle|\nabla f|^2\rangle - 2\langle\delta f\nabla^2 f\rangle + \langle|\nabla\delta f|^2\rangle.
\end{aligned}
$$

Doing the same for δu and δw. We note that there are no δu terms in the numerator, and so the calculation is simplified slightly:

$$
\begin{aligned}
F =& \frac{\alpha\langle (w+\delta w)(f+\delta f)\rangle}{\langle |\nabla f + \nabla \delta f|^2\rangle + \frac{C^2}{R}\langle |(u+\delta u, w+\delta w)|^2\rangle} + \langle \nabla\cdot(u+\delta u, w+\delta w)\lambda\rangle \\
\implies \frac{\delta F}{\delta u} :& -\frac{I}{D^2}\frac{2C^2}{R} = \lambda_x.
\end{aligned}
$$

Now for w, once again applying quotient rule and retaining $O(\delta w)$ terms we obtain

$$
\frac{\delta F}{\delta w} : \frac{\alpha f}{D^2} - \frac{I}{D^2}\frac{2wC^2}{R} = \lambda_z.
$$

B.2.4 Checking validity of A solutions

Given the solutions of A

$$
A_a = \pm\frac{2\sqrt{R_\rho \tau}}{\tau+1}, \quad A_b = \frac{\pm 2\sqrt{(\tau^3 - R_\rho)(\tau - R_\rho)} - 2(\tau^2 + R_\rho)}{(\tau+1)^2},
$$

we must determine which are admissible, by checking which result in positive C^2 values. We begin with the negative root A_a, followed by the positive root:

$$
\begin{aligned}
C^2_{A_a-} =& \frac{(1-\tau)\sqrt{R_\rho\tau}(\tau - \sqrt{R_\rho\tau})}{\tau(\tau+1)(R_\rho - \sqrt{R_\rho\tau})} \\
\implies& \frac{(1-\tau)(\sqrt{\tau} - \sqrt{R_\rho})}{(\tau+1)(\sqrt{R_\rho} - \sqrt{\tau})} \\
\implies& \frac{\tau-1}{1+\tau} < 0
\end{aligned}
$$

$$
\begin{aligned}
C^2_{A_a+} =& \frac{(\tau-1)\sqrt{R_\rho\tau}(\tau + \sqrt{R_\rho\tau})}{\tau(\tau+1)(R_\rho + \sqrt{R_\rho\tau})} \\
\implies& \frac{(\tau-1)(\sqrt{\tau} + \sqrt{R_\rho})}{(\tau+1)(\sqrt{R_\rho} + \sqrt{\tau})} \\
\implies& \frac{\tau-1}{1+\tau} < 0.
\end{aligned}
$$

And so we discard both roots of the first solution. We proceed with A_b, with root 3 corresponding to the negative root, and root 4 to the positive. Since in the diffusive convection regime, we have

$R_\rho > 1 > \tau$, both roots of A_b are real. The positive root of A_b yields the following C^2:

$$C^2_{A_b+} = -\frac{2(1-\tau)\left(\sqrt{(\tau-R_\rho)(\tau^3-R_\rho)}-\tau^2-R_\rho\right)\left(\sqrt{(\tau-R_\rho)(\tau^3-R_\rho)}-\tau^2-R_\rho+\tau(\tau+1)\right)}{2\tau(\tau+1)^2\left(\sqrt{(\tau-R_\rho)(\tau^3-R_\rho)}-\tau^2-R_\rho+R_\rho(\tau+1)\right)},$$

with $C^2_{A_b-}$ being similar, but with different signs. Since the resulting C^2 expressions are quite complex, we opt to use Matlab to determine their signs and find that both roots give $C^2 > 0$.

B.2.5 Oscillatory Convection

We note that until the equation

$$\left(\frac{I}{D}\right)^2 = \frac{k^2\alpha^2 R}{4C^2(k^2+\pi^2)^2},$$

the work is identical to the steady convection case, with some differences in signs in the governing equations and consequently α, C, I, and D. We also note that $R_{crit-osc} = \frac{(k^2+\pi^2)^2(1+\tau)}{k^2}$, and so in the expression being extremized, we pick up a factor of $(1+\tau)^{-1/2}$. We arrive at the same solutions A as in the case of steady convection. However, we find that the validity of solutions now depends on the value of R_ρ relative to τ.

B.3 Weakly Nonlinear

B.3.1 Steady Convection

B.3.1.1 $O(\epsilon^2)$ Calculations

At $O(\epsilon^2)$ we retain the following terms:

$$\begin{aligned}
\nabla^2\psi_2 &= R_0\left(T_{2x} - R_\rho S_{2x}\right) + R_1\left(T_{1x} - R_\rho S_{1x}\right)\\
\psi_{1z}T_{1x} - \psi_{1x}T_{1z} + \psi_{2x} &= \nabla^2 T_2\\
\psi_{1z}S_{1x} - \psi_{1x}S_{1z} + \psi_{2x} &= \frac{1}{Le}\nabla^2 S_2.
\end{aligned}$$

Now we must calculate the terms arising from $\mathbf{u}\cdot\nabla T$ and $\mathbf{u}\cdot\nabla S$. We do so below. We start with $\psi_{1z}T_{1x} - \psi_{1x}T_{1z}$.

$$\begin{aligned}
&\left(A(\mathcal{T})\pi e^{ikx}\cos(\pi z) + \text{c.c}\right)\left(\frac{A(\mathcal{T})k^2}{k^2+\pi^2}e^{ikx}\sin(\pi z) + \text{c.c}\right)\\
&\qquad - \left(A(\mathcal{T})ike^{ikx}\sin(\pi z) + \text{c.c}\right)\left(\frac{-A(\mathcal{T})ik\pi}{k^2+\pi^2}e^{ikx}\cos(\pi z) + \text{c.c}\right).
\end{aligned}$$

Expanding the expression gives

$$A(\mathcal{T})^2k^2\pi e^{2ikx}\cos(\pi z)\sin(\pi z) + 2\frac{|A(\mathcal{T})|^2\pi k^2\cos(\pi z)\sin(\pi z)}{k^2+\pi^2} + \frac{|\bar{A}(\mathcal{T})|^2k^2\pi e^{-2ikx}\cos(\pi z)\sin(\pi z)}{k^2+\pi^2}$$
$$-\left(A(\mathcal{T})^2k^2\pi e^{2ikx}\cos(\pi z)\sin(\pi z) - 2\frac{|A(\mathcal{T})|^2\pi k^2\cos(\pi z)\sin(\pi z)}{k^2+\pi^2} + \frac{|\bar{A}(\mathcal{T})|^2k^2\pi e^{-2ikx}\cos(\pi z)\sin(\pi z)}{k^2+\pi^2}\right).$$

Now, combining like terms leaves

$$\begin{aligned}\psi_{1z}T_{1x} - \psi_{1x}T_{1z} =& \frac{4A(\mathcal{T})\bar{A}(T)k^2\pi}{k^2+\pi^2}\cos(\pi z)\sin(\pi z)\\ \Rightarrow& \frac{2|A(\mathcal{T})|^2k^2\pi}{k^2+\pi^2}\sin(2\pi z).\end{aligned}$$

Next, solving $\psi_{1z}S_{1x} - \psi_{1x}S_{1z}$ we arrive at

$$\begin{aligned}&\left(A(\mathcal{T})\pi e^{ikx}\cos(\pi z)+\text{ c.c}\right)\left(A\frac{(\mathcal{T})Lek^2}{k^2+\pi^2}e^{ikx}\sin(\pi z)+\text{ c.c}\right)\\ &\qquad - \left(A(\mathcal{T})ike^{ikx}\sin(\pi z)+\text{ c.c}\right)\left(\frac{-A(\mathcal{T})Leik\pi}{k^2+\pi^2}e^{ikx}\cos(\pi z)+\text{ c.c}\right),\end{aligned}$$

which results in an identical expression as that above, save for an additional factor of Le.

$$\psi_{1z}S_{1x} - \psi_{1x}S_{1z} = \frac{2Le|A(\mathcal{T})|^2k^2\pi}{k^2+\pi^2}\sin(2\pi z).$$

B.3.1.2 $O(\epsilon^3)$ Calculations

The governing equations at $O(\epsilon^3)$ are

$$\nabla^2\psi_3 = R_0\left(T_{3x} - R_\rho S_{3x}\right) + R_2\left(T_{1x} - R_\rho S_{1x}\right) \tag{B.3.1}$$

$$T_{1t} - \psi_{1x}T_{2z} + \psi_{3x} = \nabla^2 T_3 \tag{B.3.2}$$

$$S_{1t} - \psi_{1x}S_{2z} + \psi_{3x} = \frac{1}{Le}\nabla^2 S_3. \tag{B.3.3}$$

We then take the Laplacian of (B.3.1) and the x-derivative of (B.3.2) and (B.3.3). Substituting (2.4.20), (2.4.21), and (2.4.1.3) into the Laplacian of (B.3.1) gives

$$
\begin{aligned}
&H(\mathcal{T})\left(k^2+\pi^2\right)^2\sin(\pi z)e^{ikx}+J(\mathcal{T})\left(k^2+9\pi^2\right)^2\sin(3\pi z)e^{ikx}\\
=&R_0\Bigg[\left(\frac{A'(\mathcal{T})k^2}{k^2+\pi^2}+\frac{A(\mathcal{T})|A(\mathcal{T})|^2k^4}{2(k^2+\pi^2)}-k^2H(\mathcal{T})\right)e^{ikx}\sin(\pi z)\\
&-\left(\frac{A(\mathcal{T})|A(\mathcal{T})|^2k^4}{2(k^2+\pi^2)}+k^2J(\mathcal{T})\right)e^{ikx}\sin(3\pi z)\Bigg]\\
&-R_0R_\rho Le\Bigg[\left(\frac{LeA'(\mathcal{T})k^2}{k^2+\pi^2}+\frac{Le^2A(\mathcal{T})|A(\mathcal{T})|^2k^4}{2(k^2+\pi^2)}-k^2H(\mathcal{T})\right)e^{ikx}\sin(\pi z)\\
&-\left(\frac{Le^2A(\mathcal{T})|A(\mathcal{T})|^2k^4}{2(k^2+\pi^2)}+k^2J(\mathcal{T})\right)e^{ikx}\sin(3\pi z)\Bigg]\\
&-R_2\left(A(\mathcal{T})k^2e^{ikx}\sin(\pi z)\right)+R_2R_\rho Le\left(A(\mathcal{T})k^2e^{ikx}\sin(\pi z)\right).
\end{aligned}
$$

Next, we group by sinusoidal terms - focusing first on the $\sin(\pi z)$ terms. Collecting these, and putting terms containing $H(\mathcal{T})$ to the left-hand side gives the following equation (note that there is a common term of e^{ikx}, and so we can eliminate it):

$$
\begin{aligned}
&H(\mathcal{T})\left(k^2+\pi^2\right)^2\sin(\pi z)+R_0k^2H(\mathcal{T})\sin(\pi z)-R_0R_\rho Lek^2H(\mathcal{T})\sin(\pi z)\\
=&R_0\left[\left(\frac{A'(\mathcal{T})k^2}{k^2+\pi^2}+\frac{A(\mathcal{T})|A(\mathcal{T})|^2k^4}{2(k^2+\pi^2)}\right)\sin(\pi z)\right]\\
&-R_0R_\rho Le\left[\left(\frac{LeA'(\mathcal{T})k^2}{k^2+\pi^2}+\frac{Le^2A(\mathcal{T})|A(\mathcal{T})|^2k^4}{2(k^2+\pi^2)}\right)\sin(\pi z)\right]\\
&+R_2A(\mathcal{T})k^2\left(R_\rho Le-1\right)\sin(\pi z).
\end{aligned}
$$

Substituting in the known R_0 and simplifying, the left-hand side of the equation becomes

$$
H(\mathcal{T})\left(\frac{(k^2+\pi^2)^2(R_\rho Le-1)+(k^2+\pi^2)^2-R_\rho Le(k^2+\pi^2)^2}{R_\rho Le-1}\right)=0.
$$

As stated in the main body, we need to coefficients on the right-hand side to equal 0 as well. We are then left with the following:

$$
\begin{aligned}
0=\frac{(k^2+\pi^2)(2A'(\mathcal{T})+A(\mathcal{T})|A(\mathcal{T})|^2k^2)}{2(R_\rho Le-1)}&+R_2A(\mathcal{T})k^2(R_\rho Le-1)\\
&-\frac{(k^2+\pi^2)R_\rho Le(2LeA'(\mathcal{T})+Le^2A(\mathcal{T})|A(\mathcal{T})|^2k^2)}{2(R_\rho Le-1)}.
\end{aligned}
$$

Multiplying by $2(R_\rho Le-1)$ and grouping by $A(\mathcal{T})$ sets us up to solve for the Landau equation easily:

$$2A'(\mathcal{T})(k^2+\pi^2)(1-R_\rho Le^2)+A(\mathcal{T})|A(\mathcal{T})|^2k^2(k^2+\pi^2)(1-R_\rho Le^3) \\ +2A(\mathcal{T})R_2k^2(R_\rho Le-1)^2=0.$$

Rearranging into the form of the Landau equation, we get

$$A'(\mathcal{T})=\frac{A(\mathcal{T})R_2k^2(R_\rho Le-1)^2}{(k^2+\pi^2)(R_\rho Le^2-1)}-\frac{A(\mathcal{T})|A(\mathcal{T})|^2k^2(R_\rho Le^3-1)}{2(R_\rho Le^2-1)}.$$

B.3.2 Oscillatory Convection

B.3.2.1 $O(\epsilon^2)$ Calculations

Moving on to $O(\epsilon^2)$, the governing equations are as follows. Note that we take the fast time derivative t and not the slow time derivative T.

$$\nabla^2\psi_2=R_0\left(-T_{2x}+R_\rho S_{2x}\right)+R_1\left(-T_{1x}+R_\rho S_{1x}\right) \tag{B.3.4}$$

$$T_{2t}+\psi_{1z}T_{1x}-\psi_{1x}T_{1z}+\psi_{2x}=\nabla^2T_2 \tag{B.3.5}$$

$$S_{2t}+\psi_{1z}S_{1x}-\psi_{1x}S_{1z}+\psi_{2x}=\frac{1}{Le}\nabla^2S_2. \tag{B.3.6}$$

Breaking up the calculations, we begin by evaluating $\psi_{1z}T_{1x}-\psi_{1x}T_{1z}$:

$$\overbrace{\left(A(\mathcal{T})\pi e^{ikx+i\omega t}\cos(\pi z)+c.c\right)}^{\psi_{1z}}\overbrace{\left(\frac{k^2A(\mathcal{T})e^{ikx+i\omega t}\sin(\pi z)}{i\omega+k^2+\pi^2}+c.c.\right)}^{T_{1x}} \\ -\underbrace{\left(A(\mathcal{T})ike^{ikx+i\omega t}\sin(\pi z)+c.c.\right)}_{\psi_{1x}}\underbrace{\left(\frac{-ik\pi A(\mathcal{T})e^{ikx+i\omega t}\cos(\pi z)}{i\omega+k^2+\pi^2}+c.c.\right)}_{T_{1z}}.$$

Expanding,

$$\left[\frac{A(\mathcal{T})^2k^2\pi e^{2ikx+2i\omega t}}{i\omega+k^2+\pi^2}+\frac{|A(\mathcal{T})|^2k^2\pi}{-i\omega+k^2+\pi^2}+\frac{|A(\mathcal{T})|^2k^2\pi}{i\omega+k^2+\pi^2}+\frac{\bar{A}(\mathcal{T})^2k^2\pi e^{2ikx+2i\omega t}}{-i\omega+k^2+\pi^2}\right. \\ \left.-\left(\frac{A(\mathcal{T})^2k^2\pi e^{2ikx+2i\omega t}}{i\omega+k^2+\pi^2}-\frac{|A(\mathcal{T})|^2k^2\pi}{-i\omega+k^2+\pi^2}-\frac{|A(\mathcal{T})|^2k^2\pi}{i\omega+k^2+\pi^2}+\frac{\bar{A}(\mathcal{T})^2k^2\pi e^{2ikx+2i\omega t}}{-i\omega+k^2+\pi^2}\right)\right]\sin(\pi z)\cos(\pi z),$$

we find that the remaining terms are

$$\begin{aligned}\psi_{1z}T_{1x} - \psi_{1x}T_{1z} &= \frac{2|A(\mathcal{T})|^2k^2\pi\cos(\pi z)\sin(\pi z)}{k^2+\pi^2-i\omega} + \frac{2|A(\mathcal{T})|^2k^2\pi\cos(\pi z)\sin(\pi z)}{k^2+\pi^2+i\omega} \\ &= \frac{4\left(k^2+\pi^2\right)|A(\mathcal{T})|^2k^2\pi\cos(\pi z)\sin(\pi z)}{\left(k^2+\pi^2\right)^2+\omega^2} \\ &= \frac{2\left(k^2+\pi^2\right)|A(\mathcal{T})|^2k^2\pi\sin(2\pi z)}{\left(k^2+\pi^2\right)^2+\omega^2}.\end{aligned}$$

Doing the same for equation (B.3.6) gives the same result, differing only in the addition of the Lewis number:

$$\psi_{1z}S_{1x} - \psi_{1x}S_{1z} = \frac{2Le\left(k^2+\pi^2\right)|A(\mathcal{T})|^2k^2\pi\sin(2\pi z)}{\left(k^2+\pi^2\right)^2+\omega^2Le^2}.$$

B.3.2.2 $O(\epsilon^3)$ Calculations

At $O(\epsilon^3)$, taking into account that $\psi_2 = R_1 = 0$ the system is described by

$$\nabla^2\psi_3 = R_0\left(-T_{3x} + R_\rho S_{3x}\right) + R_2\left(-T_{1x} + R_\rho S_{1x}\right) \tag{B.3.7}$$

$$T_{1\mathcal{T}} + i\omega T_3 - \psi_{1x}T_{2z} + \psi_{3x} = \nabla^2 T_3 \tag{B.3.8}$$

$$S_{1\mathcal{T}} + i\omega S_3 - \psi_{1x}S_{2z} + \psi_{3x} = \frac{1}{Le}\nabla^2 S_3. \tag{B.3.9}$$

Continuing with the calculations for $H(\mathcal{T})_{T,S,\psi}$:

$$H(\mathcal{T})_T = \left(\frac{ik^3A(\mathcal{T})|A(\mathcal{T})|^2(k^2+\pi^2)}{2\left((k^2+\pi^2)^2+\omega^2\right)} + \frac{ikA'(\mathcal{T})}{i\omega+k^2+\pi^2} - ikH(\mathcal{T})_\psi\right)\frac{1}{i\omega+k^2+\pi^2},$$

and

$$H(\mathcal{T})_S = \left(\frac{Le^2ik^3A(\mathcal{T})|A(\mathcal{T})|^2(k^2+\pi^2)}{2\left((k^2+\pi^2)^2+Le^2\omega^2\right)} + \frac{LeikA'(\mathcal{T})}{Lei\omega+k^2+\pi^2} - ikH(\mathcal{T})_\psi\right)\frac{Le}{Lei\omega+k^2+\pi^2}.$$

Now, substituting the solution and coefficient forms into (B.3.7), we proceed to determining the coefficients of the evolution equation. Considering only $\sin(\pi z)$ terms, we apply the Laplacian operator to (B.3.7) and take the x-derivatives of (B.3.8) and (B.3.9), we are able to remove the common factor $e^{ikx+i\omega t}\sin(\pi z)$ leaving

$$-(k^2+\pi^2)H(\mathcal{T})_\psi = R_0\left(-ikH(\mathcal{T})_T + R_\rho ikH(\mathcal{T})_S\right) + R_2\left(\frac{-k^2A(\mathcal{T})}{i\omega+k^2+\pi^2} + R_\rho\frac{k^2LeA(\mathcal{T})}{Lei\omega+k^2+\pi^2}\right). \tag{B.3.10}$$

Inserting $H(\mathcal{T})_{T,S}$ from above into (B.3.10), we first collect and combine all terms containing $H(\mathcal{T})_\psi$, and move them to the left-hand side of (B.3.10):

$$H(\mathcal{T})_\psi\left(-(k^2+\pi^2)+R_0\frac{k^2}{i\omega+k^2+\pi^2}-R_0R_\rho\frac{Lek^2}{Lei\omega+k^2+\pi^2}\right)=RHS$$

$$\implies H(\mathcal{T})_\psi\left(-(k^2+\pi^2)+\frac{\left(k^2+\pi^2\right)^2(1+Le)}{Le\left(1-R_\rho\right)}\frac{1}{i\omega+k^2+\pi^2}-\frac{\left(k^2+\pi^2\right)^2(1+Le)}{Le\left(1-R_\rho\right)}R_\rho\frac{Le}{Lei\omega+k^2+\pi^2}\right)$$
$$=RHS.$$

We now put the terms over the common denominator $Le(1-R\rho)(i\omega+k^2+\pi^2)(Lei\omega+k^2+\pi^2)$ and continue with our algebra, only writing the numerator in the proceeding calculations.

$$\implies H(\mathcal{T})_\psi\Big(-(k^2+\pi^2)Le(1-R_\rho)(i\omega+k^2+\pi^2)(Lei\omega+k^2+\pi^2)$$
$$+(k^2+\pi^2)^2(1+Le)\left(Lei\omega+k^2+\pi^2-R_\rho Le(i\omega+k^2+\pi^2)\right)\Big)=RHS$$

$$\implies H(\mathcal{T})_\psi\Big(-Le(k^2+\pi^2)(1-R_\rho)\left(-Le\omega^2+(i\omega+Lei\omega)(k^2+\pi^2)+(k^2+\pi^2)^2\right)$$
$$+(k^2+\pi^2)^2(1+Le)\left(Lei\omega-R_\rho Lei\omega+(k^2+\pi^2)(1-R_\rho Le)\right)\Big)=RHS.$$

Retaining only the imaginary parts leaves

$$\implies H(\mathcal{T})_\psi\left(-Le(k^2+\pi^2)^2(1-R_\rho)(i\omega+Lei\omega)+(k^2+\pi^2)^2(1+Le)\left(Lei\omega-R_\rho Lei\omega\right)\right)=RHS$$
$$\implies H(\mathcal{T})_\psi(k^2+\pi^2)^2Le\omega\left(-(1-R_\rho)(1+Le)+(1+Le)\left(1-R_\rho\right)\right)i=RHS$$
$$\implies 0=RHS,$$

and so the right-hand side terms must equal to zero. Using the Matlab code in appendix E, we solve for the coefficients of the Landau equation and obtain the following:

$$A'(\mathcal{T})=\frac{R_2k^2(1-R_\rho)}{2(k^2+\pi^2)}A(\mathcal{T})-\frac{k^2Le(1-R_\rho)}{8(R_\rho Le-1)}A(\mathcal{T})^3.$$

Appendix C

Full Calculations of Salt Fingering Case

C.1 Linear

Beginning with the governing equations in the main body, substituting in the solution forms, and eliminating the common factor $e^{ikx}\sin(m\pi z)$, the full set of equations is:

$$-(k^2+m^2\pi^2)\psi^* = R\left(-ikT^* + R_\rho ikS^*\right) \tag{C.1.1}$$

$$\sigma T^* - ik\psi^* = (-k^2-m^2\pi^2)T^* \tag{C.1.2}$$

$$\sigma S^* - ik\psi^* = \frac{(-k^2-m^2\pi^2)}{Le}S^*. \tag{C.1.3}$$

Isolating T^* and S^* gives $T^* = \frac{ik\psi^*}{k^2+m^2\pi^2+\sigma}$ and $S^* = \frac{ikLe\psi^*}{k^2+m^2\pi^2+Le\sigma}$. Substituting these into (C.1.1) gives

$$-(k^2+m^2\pi^2)\psi^* = R\left(-ik\frac{ik\psi^*}{k^2+m^2\pi^2+\sigma} + R_\rho ik\frac{ikLe\psi^*}{k^2+m^2\pi^2+Le\sigma}\right).$$

Canceling the common factor ψ^*, multiplying through by $\left(k^2+m^2\pi^2+\sigma\right)\left(k^2+m^2\pi^2+Le\sigma\right)$ and organizing by order of σ we obtain:

$$\sigma^2 Le + \sigma\left(\frac{k^2RLe(1-R_\rho)}{k^2+m^2\pi^2} + (Le+1)(k^2+m^2\pi^2)\right) + k^2R(1-R_\rho Le) + (k^2+m^2\pi^2)^2 = 0. \tag{C.1.4}$$

To go further, we look at marginal stability where $\sigma = 0$. Doing so results in the critical Rayleigh number $R = \frac{(k^2+\pi^2)^2}{k^2(R_\rho Le-1)}$. Minimizing over m we find that the most restrictive threshold of instability occurs at $m = 1$, allowing us to rewrite (C.1.4) as

$$\sigma^2 Le + \sigma\left(\frac{k^2RLe(1-R_\rho)}{k^2+\pi^2} + (Le+1)(k^2+\pi^2)\right) + k^2R(1-R_\rho Le) + (k^2+\pi^2)^2 = 0. \tag{C.1.5}$$

Setting σ equal to 0, the critical Rayleigh number is revealed to be $R = \frac{(k^2+\pi^2)^2}{k^2(R_\rho Le-1)}$.

C.2 Nonlinear

We begin with the perturbed governing equations, noting that the perturbations θ, σ satisfy homogeneous boundary conditions:

$$\mathbf{u} = -\nabla p + R\theta\hat{\mathbf{z}} - RR_\rho\sigma\hat{\mathbf{z}} \tag{C.2.1}$$

$$\theta_t + w + \mathbf{u}\cdot\nabla\theta = \nabla^2\theta \tag{C.2.2}$$

$$\sigma_t + w + \mathbf{u}\cdot\nabla\sigma = \frac{1}{Le}\nabla^2\sigma. \tag{C.2.3}$$

C.2.1 Power Integrals

Multiplying the above equations by $\mathbf{u}$, θ, and σ respectively, and integrating over the domain will yield power integrals. Doing so and applying boundary conditions and the divergence theorem, we obtain the following:

$$\int_0^{\frac{2\pi}{m}}\int_0^{\frac{2\pi}{k}} \mathbf{u}\cdot\mathbf{u}\,dxdz = \int_0^{\frac{2\pi}{m}}\int_0^{\frac{2\pi}{k}} -\mathbf{u}\cdot\nabla p - RR_\rho\sigma\hat{\mathbf{z}}\cdot\mathbf{u} + R\theta\hat{\mathbf{z}}\cdot\mathbf{u}\,dxdz$$
$$\implies \langle|\mathbf{u}|^2\rangle = -RR_\rho\langle w\sigma\rangle + R\langle w\theta\rangle.$$

Similarly for temperature and salt:

$$\int_0^{\frac{2\pi}{m}}\int_0^{\frac{2\pi}{k}} \theta_t\theta + (\mathbf{u}\cdot\nabla\theta)\,\theta + w\theta\,dxdz = \int_0^{\frac{2\pi}{m}}\int_0^{\frac{2\pi}{k}} \theta\nabla^2\theta\,dxdz$$
$$\implies \frac{1}{2}\langle\theta^2\rangle_t + \langle w\theta\rangle = -\langle|\nabla\theta|^2\rangle,$$

and

$$\int_0^{\frac{2\pi}{m}}\int_0^{\frac{2\pi}{k}} \sigma_t\sigma + (\mathbf{u}\cdot\nabla\sigma)\,\sigma + w\sigma\,dxdz = \int_0^{\frac{2\pi}{m}}\int_0^{\frac{2\pi}{k}} \frac{1}{Le}\sigma\nabla^2\sigma\,dxdz$$
$$\implies \frac{1}{2}\langle\sigma^2\rangle_t + \langle w\sigma\rangle = -\frac{1}{Le}\langle|\nabla\sigma|^2\rangle,$$

where the negative signs in the gradient terms come from the divergence theorem. As θ_t^2 and σ_t^2 produce $2\theta_t\theta$ and $2\sigma_t\sigma$, the $\frac{1}{2}$ is added in to counteract it. Note that the angle brackets $\langle ...\rangle$ represent integration over the domain.

The final equation required for the stability analysis is obtained by multiplying (C.2.2) by σ,

(C.2.3) by θ, summing together, and integrating over the domain:

$$\theta_t\sigma + w\sigma + (\mathbf{u}\cdot\nabla\theta)\,\sigma = \sigma\nabla^2\theta$$
$$+$$
$$\sigma_t\theta + w\theta + (\mathbf{u}\cdot\nabla\sigma)\,\theta = \frac{\theta}{Le}\nabla^2\sigma$$

$$\begin{aligned}
&\implies \int_0^{\frac{2\pi}{m}}\int_0^{\frac{2\pi}{k}} \theta_t\sigma + w\sigma + (\mathbf{u}\cdot\nabla\theta)\,\sigma + \sigma_t\theta + w\theta + (\mathbf{u}\cdot\nabla\sigma)\,\theta\; dxdz\\
&\implies \int_0^{\frac{2\pi}{m}}\int_0^{\frac{2\pi}{k}} \sigma\nabla^2\theta + \frac{\theta}{Le}\nabla^2\sigma\; dxdz\\
&\implies \langle\theta\sigma\rangle_t + \langle w\sigma\rangle + \langle w\theta\rangle = -(1+\tau)\langle\nabla\sigma\cdot\nabla\theta\rangle,
\end{aligned}$$

with the negative associated with the last term once again arising from the divergence theorem.

C.2.2 Linear Combination and Eliminating θ, σ

Taking into account the changes of sign, we arrive at a similar combination to the case of diffusive convection:

$$\begin{aligned}
&E_t = I - D\\
&I = \left(-1 - A + C^2\right)\langle w\theta\rangle + \left(-A - A^2 - B^2 - C^2R_\rho\right)\langle w\sigma\rangle\\
&D = \langle|\nabla\theta|^2\rangle + \left(A^2 + B^2\right)\tau\langle|\nabla\sigma|^2\rangle + A\,(1+\tau)\,\langle\nabla\theta\cdot\nabla\sigma\rangle + \frac{C^2}{R}\langle|\mathbf{u}|^2\rangle.
\end{aligned}$$

With the equations laid out, we move onto eliminating σ, θ terms from the linear combination outlined in the main body. Using the substitution $f = \theta + \frac{A(1+\tau)\sigma}{2}$, we begin with the combination for I:

$$\begin{aligned}
I &= \left(C^2 - A - 1\right)\langle w\theta\rangle + \left(-A - A^2 - B^2 - C^2R_\rho\right)\langle w\sigma\rangle\\
&\implies \left(C^2 - A - 1\right)\langle w\left(f - \frac{A(1+\tau)\sigma}{2}\right)\rangle + \left(-A - A^2 - B^2 - C^2R_\rho\right)\langle w\sigma\rangle\\
&\implies \left(C^2 - A - 1\right)\langle wf\rangle - \left(C^2 - A - 1\right)\left(\frac{A(1+\tau)}{2}\right)\langle w\sigma\rangle + \left(-A - A^2 - B^2 - C^2R_\rho\right)\langle w\sigma\rangle\\
&\implies \left(C^2 - A - 1\right)\langle wf\rangle + \left(\frac{A}{2}(1+\tau)\left(1 + A - C^2\right) - A - A^2 - B^2 - C^2R_\rho\right)\langle w\sigma\rangle.
\end{aligned}$$

Doing the same for D:

$$
\begin{aligned}
D &= \langle|\nabla\theta|^2\rangle + \left(A^2 + B^2\right)\tau\langle|\nabla\sigma|^2\rangle + A\,(1+\tau)\,\langle\nabla\theta\cdot\nabla\sigma\rangle + \frac{C^2}{R}\langle|\mathbf{u}|^2\rangle \\
&\implies \langle|\nabla\left(f - \frac{A(1+\tau)\sigma}{2}\right)|^2\rangle + \left(A^2 + B^2\right)\tau\langle|\nabla\sigma|^2\rangle + A\,(1+\tau)\,\langle\nabla\left(f - \frac{A(1+\tau)\sigma}{2}\right)\cdot\nabla\sigma\rangle + \frac{C^2}{R}\langle|\mathbf{u}|^2\rangle \\
&\implies \langle|\nabla f|^2\rangle - A(1+\tau)\langle\nabla f\cdot\nabla\sigma\rangle + \frac{A^2(1+\tau)^2}{4}\langle|\nabla\sigma|^2\rangle + \left(A^2 + B^2\right)\tau\langle|\nabla\sigma|^2\rangle \\
&\qquad\qquad + \frac{C^2}{R}\langle|\mathbf{u}|^2\rangle + A(1+\tau)\left(\langle\nabla f\cdot\nabla\sigma\rangle - \frac{A(1+\tau)}{2}\langle|\nabla\sigma|^2\rangle\right) \\
&\implies \langle|\nabla f|^2\rangle + \left(B^2\tau - \frac{A^2}{4}(1-\tau)^2\right)\langle|\nabla\sigma|^2\rangle + \frac{C^2}{R}\langle|\mathbf{u}|^2\rangle.
\end{aligned}
$$

In order to eliminate σ terms we set

$$
B^2 = \frac{A^2(1-\tau)^2}{4\tau} \qquad \text{and} \qquad C^2 = \frac{-A(1-\tau)\,(2\tau + A(1+\tau))}{2\tau\left(2R_\rho + A(1+\tau)\right)},
$$

leaving

$$
\begin{aligned}
&\frac{\partial E}{\partial t} = I - D = -D\left(1 - \frac{I}{D}\right) \\
&I = \left(C^2 - A - 1\right)\langle wf\rangle \\
&D = \langle|\nabla f|^2\rangle + \frac{C^2}{R}\langle|\mathbf{u}|^2\rangle.
\end{aligned}
$$

C.2.3 Functional Derivatives

We can eliminate terms involving σ by carefully choosing B^2, C^2. Including a point-wise constraint due to incompressibility, we are then left to extremize

$$
F = \frac{I}{D} + \langle(\nabla\cdot\mathbf{u})\lambda\rangle = \frac{\alpha\langle wf\rangle}{\langle|\nabla f|^2\rangle + \frac{C^2}{R}\langle|\mathbf{u}|^2\rangle} + \langle(\nabla\cdot\mathbf{u})\lambda\rangle,
$$

and do so by taking functional derivatives $\delta f, \delta u$, and δw. Writing $f = f + \delta f, u = u + \delta u$, and $w = w + \delta w$, we begin with the f functional, letting α represent $\left(C^2 - A - 1\right)$ in order to keep things a bit tidier. Using quotient rule and retaining terms of $O(\delta f)$, noting that the incompressibility

constraint does not contain f, and therefore goes to zero we arrive at [6]

$$
\begin{aligned}
F =& \frac{\alpha\langle(w+\delta w)(f+\delta f)\rangle}{\langle|\nabla f+\nabla\delta f|^2\rangle+\frac{C^2}{R}\langle|(u+\delta u, w+\delta w)|^2\rangle} + \langle\nabla\cdot(u+\delta u, w+\delta w)\lambda\rangle \\
\implies& \frac{\delta F}{\delta f} : \frac{\alpha w}{D} + \frac{2I\nabla^2 f}{D^2} = 0 \\
\implies& \frac{\delta F}{\delta f} : \frac{-\alpha w D}{2I} = \nabla^2 f \\
\implies& \frac{\delta F}{\delta f} : \frac{\alpha\psi_x D}{2I} = \nabla^2 f,
\end{aligned}
$$

where the gradient term produces the following due to the divergence theorem:

$$
\begin{aligned}
\langle|\nabla f+\nabla\delta f|^2\rangle \implies& \int_0^{\frac{2\pi}{m}}\int_0^{\frac{2\pi}{k}} |\nabla f|^2 + 2\nabla\delta f\cdot\nabla f + |\nabla\delta f|^2 dxdz \\
\implies& \langle|\nabla f|^2\rangle - 2\langle\delta f\nabla^2 f\rangle + \langle|\nabla\delta f|^2\rangle.
\end{aligned}
$$

Doing the same for δu and δw. We note that there are no δu terms in the numerator, and so the calculation is simplified slightly:

$$
\begin{aligned}
F =& \frac{\alpha\langle(w+\delta w)(f+\delta f)\rangle}{\langle|\nabla f+\nabla\delta f|^2\rangle+\frac{C^2}{R}\langle|(u+\delta u, w+\delta w)|^2\rangle} + \langle\nabla\cdot(u+\delta u, w+\delta w)\lambda\rangle \\
\implies& \frac{\delta F}{\delta u} : -\frac{I}{D^2}\frac{2C^2}{R} = \lambda_x.
\end{aligned}
$$

Now for w, once again applying quotient rule and retaining $O(\delta w)$ terms we arrive at

$$
\frac{\delta F}{\delta w} : \frac{\alpha f}{D^2} - \frac{I}{D^2}\frac{2wC^2}{R} = \lambda_z.
$$

As in the previous section, in order to combine the above functionals into a useful expression, we take the x-derivatives of δw and δf, the z-derivative of δu:

$$
\begin{aligned}
\left(\frac{\delta F}{\delta f}\right)_x &: \frac{\left(C^2-A-1\right)w_x}{D} + \frac{2I\nabla^2 f_x}{D^2} = 0 \\
\left(\frac{\delta F}{\delta u}\right)_z &: -\frac{2C^2 I u_z}{D^2 R} = \lambda_{xz} \\
\left(\frac{\delta F}{\delta w}\right)_x &: \frac{\left(C^2-A-1\right)f_x}{D} - \frac{2C^2 I w_x}{D^2 R} = \lambda_{zx}.
\end{aligned}
$$

Since δu and δw have the same λ_{xz} on the right-hand side, we can set those equations equal to each other and then take the Laplacian. We also replace (u, w) with $(\psi_z, -\psi_x)$:

$$\left(\frac{I}{D}\right)^2 \left(\frac{4C^2}{R\left(C^2 - A - 1\right)^2}\right) \nabla^4 \psi = -\psi_{xx}.$$

Substituting the solution form $\psi \sim e^{ikx+\sigma t}\sin(\pi z)$ from the preceding section gives

$$\left(\frac{I}{D}\right)^2 \left(\frac{4C^2}{R\left(C^2 - A - 1\right)^2}\right)(k^2 + \pi^2)^2 = k^2 \implies \left(\frac{I}{D}\right)^2 = \frac{k^2 R\left(C^2 - A - 1\right)^2}{(k^2 + \pi^2)^2 4C^2}.$$

Since $\frac{I}{D} < 1$ is required for energy stability the following must be true:

$$R^{\frac{1}{2}} < \frac{(k^2 + \pi^2)2C}{\left(C^2 - A - 1\right)k}.$$

Noting that $\frac{k^2+\pi^2}{k} = R_{crit}^{\frac{1}{2}}$, we write the following:

$$\left(\frac{R}{R_{crit}}\right)^{\frac{1}{2}} < \frac{2C}{\left(C^2 - A - 1\right)}.$$

We now wish to maximize $\frac{2C}{(C^2-A-1)}$, or equivalently minimize $\frac{(C^2-A-1)}{2C}$ over A.

C.2.4 Checking validity of A solutions

Given the solutions of A

$$A_a = \pm\frac{2\sqrt{R_\rho \tau}}{\tau + 1}, \quad A_b = \frac{\pm 2\sqrt{(\tau^3 - R_\rho)(\tau - R_\rho)} - 2(\tau^2 + R_\rho)}{(\tau + 1)^2},$$

we must determine which are admissible, by checking which result in positive C^2 values. We find that both A_{a-} and A_{a+} result in a C^2 value of $\frac{\tau-}{\tau+1}$, which is negative as τ is less than 1 and so we discard them.

The C^2 expressions resulting from solutions A_b are unwieldy and so I used a numerical solver to determine which of the solutions give positive C^2. Both roots of A_b give positive C^2 and real A_b, and it is relevant for all R_ρ as in the salt fingering regime we require $R_\rho > \tau$.

C.3 Weakly Nonlinear

C.3.1 $O(\epsilon)$ Calculations

At $O(\epsilon)$, we find that the relevant terms are

$$\nabla^2\psi_1 = R_0\left(-T_{1x} + R_\rho S_{1x}\right) \tag{C.3.1}$$

$$-\psi_{1x} = \nabla^2 T_1 \tag{C.3.2}$$

$$-\psi_{1x} = \frac{1}{Le}\nabla^2 S_1. \tag{C.3.3}$$

Taking the Laplacian of (C.3.1), and the x-derivative of (C.3.2),(C.3.3), the following is obtained:

$$\nabla^4\psi_1 = R_0\left(-\nabla^2 T_{1x} + R_\rho \nabla^2 S_{1x}\right) \tag{C.3.4}$$

$$-\psi_{1xx} = \nabla^2 T_{1x} \tag{C.3.5}$$

$$-\psi_{1xx} = \frac{1}{Le}\nabla^2 S_{1x}. \tag{C.3.6}$$

We are now able to substitute (C.3.5), (C.3.6) into (C.3.4) such that we are left with an equation all in terms of ψ_1. Substituting in the assumed solution (3.4.1) and taking the appropriate derivatives in the updated (C.3.4) leaves

$$\left(k^2+\pi^2\right)^2\psi_1 = R_0\left(-k^2\psi_1 + R_\rho k^2 Le\psi_1\right).$$

As expected, dividing the above by the common factor of ψ_1 and solving for R_0 gives $\frac{(k^2+\pi^2)^2}{k^2(R_\rho Le-1)}$ which is the same critical Rayleigh number as derived in the linear section. The next part of the calculations will be to determine the amplitudes of $O(\epsilon)$ temperature and salinity fields in terms of velocity amplitudes ($A(\mathcal{T})$). To do so, we expand equations (C.3.2) and (C.3.3) using the assumed solution forms, and solve for $B(\mathcal{T}), C(\mathcal{T})$ respectively:

$$-A(\mathcal{T})ike^{ikx}\sin(\pi z) = \left(-k^2-\pi^2\right)B(\mathcal{T})e^{ikx}\sin(\pi z)$$

$$-A(\mathcal{T})ike^{ikx}\sin(\pi z) = \frac{\left(-k^2-\pi^2\right)}{Le}C(\mathcal{T})e^{ikx}\sin(\pi z).$$

Simplifying each of the above by dividing through by like terms (in this case, the exponentials and the sinusoidal terms), and rearranging gives the following:

$$B(\mathcal{T}) = \frac{A(\mathcal{T})ik}{k^2+\pi^2}$$

$$C(\mathcal{T}) = \frac{A(\mathcal{T})ikLe}{k^2+\pi^2}.$$

Replacing the slow-time coefficients in the thermal and solutal expressions results in the following solutions at $O(\epsilon)$:

$$
\begin{aligned}
R_0 &= \frac{(k^2+\pi^2)^2}{k^2(R_\rho Le - 1)} \\
T_1 &= \frac{A(\mathcal{T})ik}{k^2+\pi^2} e^{ikx}\sin(\pi z) + c.c. \\
S_1 &= \frac{A(\mathcal{T})ikLe}{k^2+\pi^2} e^{ikx}\sin(\pi z) + c.c..
\end{aligned}
$$

C.3.2 $O(\epsilon^2)$ Calculations

At the next order we acquire some new terms in our equations, though the methodology is similar to that of $O(\epsilon)$. Again, extracting terms of $O(\epsilon^2)$ we obtain

$$\nabla^2\psi_2 = R_0\left(-T_{2x} + R_\rho S_{2x}\right) + R_1\left(-T_{1x} + R_\rho S_{1x}\right) \tag{C.3.7}$$

$$\psi_{1z}T_{1x} - \psi_{1x}T_{1z} - \psi_{2x} = \nabla^2 T_2 \tag{C.3.8}$$

$$\psi_{1z}S_{1x} - \psi_{1x}S_{1z} - \psi_{2x} = \frac{1}{Le}\nabla^2 S_2. \tag{C.3.9}$$

Now we must calculate the terms arising from $\mathbf{u}\cdot\nabla T$ and $\mathbf{u}\cdot\nabla S$. We do so below. We start with $\psi_{1z}T_{1x} - \psi_{1x}T_{1z}$.

$$
\begin{aligned}
&\left(A(\mathcal{T})\pi e^{ikx}\cos(\pi z) + \text{c.c}\right)\left(-\frac{A(\mathcal{T})k^2}{k^2+\pi^2}e^{ikx}\sin(\pi z) + \text{c.c}\right) \\
&\qquad - \left(A(\mathcal{T})ike^{ikx}\sin(\pi z) + \text{c.c}\right)\left(\frac{A(\mathcal{T})ik\pi}{k^2+\pi^2}e^{ikx}\cos(\pi z) + \text{c.c}\right).
\end{aligned}
$$

Expanding

$$
\begin{aligned}
&\Bigg[-\frac{A(\mathcal{T})^2k^2\pi e^{2ikx}}{k^2+\pi^2} - \frac{|A(\mathcal{T})|^2k^2\pi}{k^2+\pi^2} - \frac{|A(\mathcal{T})|^2k^2\pi}{k^2+\pi^2} - \frac{\bar{A}(\mathcal{T})^2k^2\pi e^{2ikx}}{k^2+\pi^2} \\
&\qquad - \left(-\frac{A(\mathcal{T})^2k^2\pi e^{2ikx}}{k^2+\pi^2} + \frac{|A(\mathcal{T})|^2k^2\pi}{k^2+\pi^2} + \frac{|A(\mathcal{T})|^2k^2\pi}{k^2+\pi^2} - \frac{\bar{A}(\mathcal{T})^2k^2\pi e^{2ikx}}{k^2+\pi^2}\right)\Bigg]\sin(\pi z)\cos(\pi z),
\end{aligned}
$$

and combining like terms leaves

$$
\begin{aligned}
\psi_{1z}T_{1x} - \psi_{1x}T_{1z} &= -\frac{4|A(\mathcal{T})|^2k^2\pi}{k^2+\pi^2}\sin(\pi z)\cos(\pi z) \\
&\Rightarrow -\frac{2|A(\mathcal{T})|^2k^2\pi}{k^2+\pi^2}\sin(2\pi z).
\end{aligned} \tag{C.3.10}
$$

Similar calculations yield

$$\psi_{1z}S_{1x} - \psi_{1x}S_{1z} = -\frac{2Le|A(\mathcal{T})|^2k^2\pi}{k^2+\pi^2}\sin(2\pi z). \tag{C.3.11}$$

As before, we take the Laplacian of (C.3.7) and the x-derivative of (C.3.8) and (C.3.9). Since the $\mathbf{u}\cdot\nabla T$ and $\mathbf{u}\cdot\nabla S$ terms are independent of x, this leaves

$$\nabla^4\psi_2 = R_0\left(-\nabla^2 T_{2x} + R_\rho\nabla^2 S_{2x}\right) + R_1\left(-\nabla^2 T_{1x} + R_\rho\nabla^2 S_{1x}\right) \tag{C.3.12}$$

$$-\psi_{2xx} = \nabla^2 T_{2x} \tag{C.3.13}$$

$$-\psi_{2xx} = \frac{1}{Le}\nabla^2 S_{2x}. \tag{C.3.14}$$

We substitute (C.3.13), and (C.3.14) into (C.3.12) yielding

$$\nabla^4\psi_2 = R_0\left(\psi_{2xx} - R_\rho Le\psi_{2xx}\right) + R_1\left(\psi_{1xx} - R_\rho Le\psi_{1xx}\right). \tag{C.3.15}$$

To go further, we impose the following solvability condition:

$$\int_0^1\int_0^{\frac{2\pi}{k}}(C.3.15)e^{ikx}\sin(\pi z)dxdz = 0.$$

Orthogonality once more reduces the integral, so substituting in the form of ψ_1 and evaluating gives

$$\frac{R_1A(\mathcal{T})\left(R_\rho Le - 1\right)k}{\pi} = 0.$$

All other factors are non-zero, and so this implies that R_1 must be zero. including this information in (C.3.15) leaves

$$\nabla^4\psi_2 = R_0\left(\psi_{2xx} - R_\rho Le\psi_{2xx}\right),$$

however, given that ψ_2 is only dependent on z the only solution to this equation is the trivial one, and so we conclude that $\psi_2 = 0$. With $\psi_2 = 0$, (C.3.10), and (C.3.11), after some rearranging and integrating with respect to z twice, we arrive at

$$T_2 = \frac{|A(\mathcal{T})|^2k^2}{2\pi(k^2+\pi^2)}\sin(2\pi z)$$

$$S_2 = \frac{Le^2|A(\mathcal{T})|^2k^2}{2\pi(k^2+\pi^2)}\sin(2\pi z).$$

C.3.3 $O(\epsilon^3)$ Calculations

Finally, we evaluate equations at $O(\epsilon^3)$. Since $\psi_2 = R_1 = 0$, we are able to immediately reduce the equations to

$$\nabla^2\psi_3 = R_0\left(-T_{3x} + R_\rho S_{3x}\right) + R_2\left(-T_{1x} + R_\rho S_{1x}\right) \tag{C.3.16}$$

$$T_{1t} - \psi_{1x}T_{2z} - \psi_{3x} = \nabla^2 T_3 \tag{C.3.17}$$

$$S_{1t} - \psi_{1x}S_{2z} - \psi_{3x} = \frac{1}{Le}\nabla^2 S_3. \tag{C.3.18}$$

Once again, we proceed by taking the x-derivative of (C.3.17) and (C.3.18). Substituting known solution forms into the x-derivative of (C.3.17), we get the following:

$$-\frac{A'(\mathcal{T})k^2}{k^2+\pi^2}e^{ikx}\sin(\pi z) + \frac{A(\mathcal{T})|A(\mathcal{T})|^2k^4}{k^2+\pi^2}\left(e^{ikx}\cos(2\pi z)\sin(\pi z)\right) - \psi_{3xx} = \nabla^2 T_{3x}.$$

Applying a trigonometric identity to the cosine-sine product, and grouping by sine terms gives:

$$\left(-\frac{A'(\mathcal{T})k^2}{k^2+\pi^2} - \frac{A(\mathcal{T})|A(\mathcal{T})|^2k^4}{2(k^2+\pi^2)}\right)e^{ikx}\sin(\pi z) + \frac{A(\mathcal{T})|A(\mathcal{T})|^2k^4}{2(k^2+\pi^2)}e^{ikx}\sin(3\pi z) - \psi_{3xx} = \nabla^2 T_{3x}. \tag{C.3.19}$$

A similar calculation gives the following for the solutal equation:

$$\left(-\frac{LeA'(\mathcal{T})k^2}{k^2+\pi^2} - \frac{Le^2A(\mathcal{T})|A(\mathcal{T})|^2k^4}{2(k^2+\pi^2)}\right)e^{ikx}\sin(\pi z) + \frac{Le^2A(\mathcal{T})|A(\mathcal{T})|^2k^4}{2(k^2+\pi^2)}e^{ikx}\sin(3\pi z) - \psi_{3xx} = \frac{1}{Le}\nabla^2 S_{3x}. \tag{C.3.20}$$

From these, we see that ψ_3 will contain some combination of terms with e^{ikx}, $\sin(\pi z)$, and $\sin(3\pi z)$ so write the following:

$$\psi_3 = H(\mathcal{T})e^{ikx}\sin(\pi z) + J(\mathcal{T})e^{ikx}\sin(3\pi z),$$

where $H(\mathcal{T})$, $J(\mathcal{T})$ are some slow-time dependent terms. Substituting the solution forms of T_3, S_3, and ψ_3 into the Laplacian of (C.3.16) gives

$$
\begin{aligned}
&H(\mathcal{T})\left(k^2+\pi^2\right)^2\sin(\pi z)e^{ikx}+J(\mathcal{T})\left(k^2+9\pi^2\right)^2\sin(3\pi z)e^{ikx}=\\
&R_0\Bigg[\left(\frac{A'(\mathcal{T})k^2}{k^2+\pi^2}+\frac{A(\mathcal{T})|A(\mathcal{T})|^2k^4}{2(k^2+\pi^2)}-k^2H(\mathcal{T})\right)e^{ikx}\sin(\pi z)\\
&-\left(\frac{A(\mathcal{T})|A(\mathcal{T})|^2k^4}{2(k^2+\pi^2)}+k^2J(\mathcal{T})\right)e^{ikx}\sin(3\pi z)\Bigg]\\
&-R_0R_\rho Le\Bigg[\left(\frac{LeA'(\mathcal{T})k^2}{k^2+\pi^2}+\frac{Le^2A(\mathcal{T})|A(\mathcal{T})|^2k^4}{2(k^2+\pi^2)}-k^2H(\mathcal{T})\right)e^{ikx}\sin(\pi z)\\
&-\left(\frac{Le^2A(\mathcal{T})|A(\mathcal{T})|^2k^4}{2(k^2+\pi^2)}+k^2J(\mathcal{T})\right)e^{ikx}\sin(3\pi z)\Bigg]\\
&-R_2\left(A(\mathcal{T})k^2e^{ikx}\sin(\pi z)\right)+R_2R_\rho Le\left(A(\mathcal{T})k^2e^{ikx}\sin(\pi z)\right).
\end{aligned}
$$

Next, we group by sinusoidal terms - focusing first on the $\sin(\pi z)$ terms. Collecting these, and putting terms containing $H(\mathcal{T})$ to the left-hand side gives the following equation (note that there is a common term of e^{ikx}, and so we can eliminate it):

$$
\begin{aligned}
&H(\mathcal{T})\left(k^2+\pi^2\right)^2\sin(\pi z)+R_0k^2H(\mathcal{T})\sin(\pi z)-R_0R_\rho Lek^2H(\mathcal{T})\sin(\pi z)=\\
&R_0\left[\left(\frac{A'(\mathcal{T})k^2}{k^2+\pi^2}+\frac{A(\mathcal{T})|A(\mathcal{T})|^2k^4}{2(k^2+\pi^2)}\right)\sin(\pi z)\right]\\
&-R_0R_\rho Le\left[\left(\frac{LeA'(\mathcal{T})k^2}{k^2+\pi^2}+\frac{Le^2A(\mathcal{T})|A(\mathcal{T})|^2k^4}{2(k^2+\pi^2)}\right)\sin(\pi z)\right]\\
&+R_2A(\mathcal{T})k^2\left(R_\rho Le-1\right)\sin(\pi z).
\end{aligned}
$$

Inserting the known value of R_0 and eliminating the common factor $e^{ikx}\sin(\pi z)$, the left-hand side of the equation becomes:

$$
H(\mathcal{T})\left((k^2+\pi^2)^2-k^2\left(R_\rho Le-1\right)\frac{(k^2+\pi^2)^2}{k^2(R_\rho Le-1)}\right)=0.
$$

In order for the equation to hold, the terms on the right-hand side must also be equal to zero. After some algebra, the right-hand side becomes:

$$
\begin{aligned}
0=\frac{(k^2+\pi^2)(2A'(\mathcal{T})+A(\mathcal{T})|A(\mathcal{T})|^2k^2)}{2(R_\rho Le-1)}&+R_2A(\mathcal{T})k^2(R_\rho Le-1)\\
&-\frac{(k^2+\pi^2)R_\rho Le(2LeA'(\mathcal{T})+Le^2A(\mathcal{T})|A(\mathcal{T})|^2k^2)}{2(R_\rho Le-1)}.
\end{aligned}
$$

Multiplying by $2(R_\rho Le - 1)$ and grouping by $A(\mathcal{T})$ sets us up to solve for the Landau equation easily:

$$2A'(\mathcal{T})(k^2+\pi^2)(1-R_\rho Le^2) + A(\mathcal{T})|A(\mathcal{T})|^2 k^2(k^2+\pi^2)(1-R_\rho Le^3) \\ + 2A(\mathcal{T})R_2 k^2(1-R_\rho Le)^2 = 0.$$

Rearranging into the form of the Landau equation, we get

$$A'(\mathcal{T}) = \frac{A(\mathcal{T})R_2 k^2(1-R_\rho Le)^2}{(k^2+\pi^2)(R_\rho Le^2-1)} - \frac{A(\mathcal{T})|A(\mathcal{T})|^2 k^2(R_\rho Le^3-1)}{2(R_\rho Le^2-1)}.$$

Appendix D

Calculation Details for Chapter Four

To obtain the density equation (4.0.18) in chapter four, we use the expressions derived for temperature and salinity and combine them using the linear combination $R_\rho S' - T' = \rho'$, noting that we first divide the salinity equation through by τ:

$$\left(\frac{R_\rho}{\tau}\frac{\partial}{\partial t} - R_\rho \nabla^2\right) S' - \left(\frac{\partial}{\partial t} - \nabla^2\right) T' =$$
$$\frac{R_\rho}{\tau}\left(\overline{S'w}\right)_z - \frac{R_\rho}{\tau} w - \frac{R_\rho}{\tau}\nabla\cdot(\mathbf{u}S') - \frac{R_\rho}{\tau} w\sigma_z - \left(\overline{T'w}\right)_z + w + \nabla\cdot(\mathbf{u}T') + w\theta_z$$

$$\implies \frac{\partial}{\partial t}\left(\frac{R_\rho}{\tau}S' - T'\right) - \nabla^2\left(R_\rho S' - T'\right) =$$
$$\left(\frac{R_\rho}{\tau}\left(\overline{S'w}\right)_z - \left(\overline{T'w}\right)_z\right) - \left(\frac{R_\rho}{\tau} - 1\right) w - \left(\frac{R_\rho}{\tau}\nabla\cdot(\mathbf{u}S') - \nabla\cdot(\mathbf{u}T')\right) - \left(\frac{R_\rho}{\tau}\sigma_z - \theta_z\right) w.$$

Since we have defined $\epsilon = \frac{R_\rho}{\tau} - 1$, we can write $\left(\frac{R_\rho}{\tau} - 1\right) w$ as ϵw. Using the scalings described in the main body of the text, we go into detail regarding the relations between T_0, S_0, and w_0. We need to rescale the following equations and consider the $\epsilon \to 0$ limit.

$$\nabla^2\psi = R\left(-T'_x + R_\rho S'_x\right)$$
$$\left(\frac{\partial}{\partial t} - \nabla^2\right) T' = \left(\overline{T'w}\right)_z - w - \nabla\cdot(\mathbf{u}T') - w\theta_z$$
$$\left(\frac{\partial}{\partial t} - \tau\nabla^2\right) S' = \left(\overline{S'w}\right)_z - w - \nabla\cdot(\mathbf{u}S') - w\sigma_z.$$

The scaled equations prior to taking the limit are as follows.

$$
\begin{aligned}
&-\epsilon^{\frac{3}{2}}\frac{\partial}{\partial x_0}w_0+\epsilon^{\frac{5}{2}}\frac{\partial}{\partial z_0}u_0=\epsilon^{\frac{3}{2}}R\frac{\partial}{\partial x_0}\rho_0\\
&\epsilon^2\frac{\partial}{\partial t_0}T_0-\epsilon\frac{\partial^2}{\partial x_0^2}T_0-\epsilon^2\frac{\partial^2}{\partial z_0^2}T_0=\epsilon^2\frac{\partial}{\partial z_0}(\overline{T_0w_0})-\epsilon w_0-\epsilon^2\frac{\partial}{\partial x_0}(u_0T_0)-\epsilon^2\frac{\partial}{\partial z_0}(w_0T_0)-\epsilon^2w_0\beta_T\\
&\epsilon^2\frac{\partial}{\partial t_0}S_0-\epsilon\tau\frac{\partial^2}{\partial x_0^2}S_0-\epsilon^2\tau\frac{\partial^2}{\partial z_0^2}S_0=\epsilon^2\frac{\partial}{\partial z_0}(\overline{S_0w_0})-\epsilon w_0-\epsilon^2\frac{\partial}{\partial x_0}(u_0S_0)-\epsilon^2\frac{\partial}{\partial z_0}(w_0S_0)-\epsilon^2w_0\beta_S\,.
\end{aligned}
$$

Dividing through by $\epsilon^{\frac{3}{2}}$ in the first equation and by ϵ in the second and third, then taking $\epsilon\to 0$ leaves the following:

$$
\begin{aligned}
w_0&=-R\rho_0\\
\frac{\partial^2}{\partial x_0^2}T_0&=w_0\\
\tau\frac{\partial^2}{\partial x_0^2}S_0&=w_0,
\end{aligned}
$$

and so we can write w_0 and S_0 in terms of T_0:

$$
\begin{aligned}
w_0&=\frac{\partial^2}{\partial x_0^2}T_0\\
\frac{\partial^2}{\partial x_0^2}S_0&=\frac{1}{\tau}\frac{\partial^2}{\partial x_0^2}T_0.
\end{aligned}
$$

We can also reduce the mean field equation to terms of β_S, β_T, and T_0. We begin with the mean field equations and take the z derivative

$$
\begin{aligned}
\theta_{tz}+\left(\overline{T'w}\right)_{zz}&=\theta_{zzz}\\
\sigma_{tz}+\left(\overline{S'w}\right)_{zz}&=\tau\sigma_{zzz}.
\end{aligned}
$$

As above, we rescale

$$
\begin{aligned}
\epsilon^3\frac{\partial}{\partial t_0}\beta_T-\epsilon^3\frac{\partial^2}{\partial z_0^2}\beta_T&=-\epsilon^3\frac{\partial^2}{\partial z_0^2}\left(\overline{T_0w_0}\right)\\
\epsilon^3\frac{\partial}{\partial t_0}\beta_S-\epsilon^3\tau\frac{\partial^2}{\partial z_0^2}\beta_S&=-\epsilon^3\frac{\partial^2}{\partial z_0^2}\left(\overline{S_0w_0}\right).
\end{aligned}
$$

Dividing by ϵ^3 and rewriting S_0 in terms of T_0 gives

$$\left(\frac{\partial}{\partial t_0} - \frac{\partial^2}{\partial z_0^2}\right)\beta_T = -\frac{\partial^2}{\partial z_0^2}\left(\overline{T_0 \frac{\partial^2}{\partial x_0^2} T_0}\right)$$
$$\left(\frac{\partial}{\partial t_0} - \tau\frac{\partial^2}{\partial z_0^2}\right)\beta_S = -\frac{1}{\tau}\frac{\partial^2}{\partial z_0^2}\left(\overline{T_0 \frac{\partial^2}{\partial x_0^2} T_0}\right).$$

Appendix E

Matlab Code

E.1 Code for Figures in Section 2.3

Below is the code for the figure corresponding to steady convection:

```
syms a C2 A Rrho tau

k = pi;
tau = 0.5;
Le = 1/tau;
Rrho = 1:0.05:Le;
Rcrit = (k^2+pi^2)^2./(k^2.*(Rrho.*Le-1)); %magnitude of Rcrit

syms root1 root2 root3 root4 root5 root6

%testing to see which root gives an appropriately signed C^2 value.
root1 = 2*(sqrt(Rrho.*tau))./(1+tau);
root2 = -2*(sqrt(Rrho.*tau))./(1+tau);
root3 = -2*(Rrho+tau^2+sqrt(Rrho.^2+tau^4-Rrho.*(tau+tau^3)))/((1+tau)^2);
root4 = -2*(Rrho+tau^2-sqrt(Rrho.^2+tau^4-Rrho.*(tau+tau^3)))/((1+tau)^2);

%roots 1, 2, 3 and 4 give positive C^2. Can choose to plot either.
C2root3 = -root3.*(1-tau).*(2*tau+root3.*(1+tau))./((2*tau).*...
                (2.*Rrho+root3.*(1+tau)));
aroot3 = (1+root3-C2root3);
min3=(2.*(C2root3).^(1/2))./(aroot3);
```

```
C2root4 = -root4.*(1-tau).*(2*tau+root4.*(1+tau))./((2*tau).*...
                (2.*Rrho+root4.*(1+tau)));
aroot4 = (1+root4-C2root4);
min4=(2.*(C2root4).^(1/2))./(aroot4);

plot(Rrho,min4,'ok')
hold on
plot(Rrho,((Rcrit)./(4*pi^2)).^(1/2),'.k')
hold off

xlabel('R_\rho')
ylabel('(R/R_{crit})^{1/2}')
title('Energy and Linear Stability Bounds')
legend('Energy Stability Bound (from roots 3 and 4)',...
                                                 'Linear Stability Bound')
```

And the code for the figure corresponding to oscillatory convection:

```
syms a C2 A Rrho tau

k = pi;
tau = 0.5;
Le = 1/tau;
Rrho1 = 0:0.02:tau^3;
Rrho2 = 0.12:0.02:1;
Rrho = 0:0.02:1;
Rcrit = (k^2+pi^2)^2.*(1+tau)./(k^2.*(1-Rrho));
D = (k^2+pi^2)^2*(1+tau)/k^2;
X1 = (k^2+pi^2)^2./k^2./(1-Rrho1.*Le);
X2 = (1-tau)./((1-(Rrho2.*tau).^(1/2)).^2)*D;

plot(Rrho,Rcrit,'ok')
hold on
plot(Rrho1,X1,'.r')
hold on
plot(Rrho2,X2,'.r')
hold off
ylim([30 1000])
xlim([0 1])
```

```
grid on
line([tau^3 tau^3],[30 1000])
line([tau tau],[30 1000])

xlabel('R_\rho')
ylabel('R_{crit}')
title('Critical Rayleigh numbers for Linear and Energy Analyses')
legend({'R_{crit Linear}','R_{crit Energy}'},'Location','southeast')
legend('boxoff')
```

E.2 Code for Figures in Section 3.3

```
syms a C2 A Rrho tau root4

k = pi;
tau = 0.2;
Le = 1/tau;
Rrho = tau:0.01:1;
Rcrit = (k^2+pi^2)^2./(k^2.*(Rrho.*Le-1));

root4 = 2*((-Rrho-tau.^2)-sqrt((tau.^3-Rrho).*(tau-Rrho)))/((1+tau).^2);

C2root4 = -root4.*(1-tau).*(2*tau+root4.*(1+tau))./((2*tau).*...
                (2.*Rrho+root4.*(1+tau)));
aroot4 = (-1-root4+C2root4);
min4=(2.*(C2root4).^(1/2))./(aroot4);

plot(Rrho,min4.^2,'ok')
hold on
plot(Rrho,((Rcrit)./(4*pi^2)),'.k')
hold off

xlabel('R_\rho')
ylabel('R/R_{crit}')
title('Energy and Linear Stability Bounds')
legend('Energy Stability Bound (from root 4)','Linear Stability Bound')
```

E.3 Code for Landau Coefficients for Oscillatory Convection

Below is the Matlab code used to solve for the coefficients of the Landau equation in section 2.4.2. First is the file containing governing equations. Followed by the code executing the calculations:

```
%equito.m

psi = (epi*A+epi^3*A3)*sin(pi*z)*exp(1i*k*x+1i*o*t);
psi = psi + (epi*conj(A)+epi^3*conj(A3))*sin(pi*z)*exp(-1i*k*x-1i*o*t);
th = (epi*B+epi^3*B3)*sin(pi*z)*exp(1i*k*x+1i*o*t)+epi^2*D1*sin(2*pi*z);
th = th + (epi*conj(B)+epi^3*conj(B3))*sin(pi*z)*exp(-1i*k*x-1i*o*t);
si = (epi*C+epi^3*C3)*sin(pi*z)*exp(1i*k*x+1i*o*t)+epi^2*D2*sin(2*pi*z);
si = si + (epi*conj(C)+epi^3*conj(C3))*sin(pi*z)*exp(-1i*k*x-1i*o*t);

w = -diff(psi,x); u = diff(psi,z);

thT = (epi*BT+epi^3*B3)*sin(pi*z)*exp(1i*k*x+1i*o*t);
thT = thT+(epi*conj(BT)+epi^3*conj(B3))*sin(pi*z)*exp(-1i*k*x-1i*o*t);
siT = (epi*CT+epi^3*C3)*sin(pi*z)*exp(1i*k*x+1i*o*t);
siT= siT+(epi*conj(CT)+epi^3*conj(C3))*sin(pi*z)*exp(-1i*k*x-1i*o*t);

eq1 = diff(psi,x,2)+diff(psi,z,2)+R*diff(th,x)-R*Rrh*diff(si,x);
eq2 = epi^2*thT+diff(th,t)-w-diff(th,x,2)-diff(th,z,2)+u*(diff(th,x))+...
                w*(diff(th,z));
eq3 = epi^2*siT+diff(si,t)-w-tau*diff(si,x,2)-tau*diff(si,z,2)+...
                u*(diff(si,x))+w*(diff(si,z));

%landauo.m

% Variables
syms R0 R2 Rrh tau x z t epi k pi D1 D2 A3 o T real
syms A AT  complex

p2 = k^2+pi^2;
R0 = p2^2*(tau+1)/k^2/(1-Rrh);

R = R0+epi^2*R2;
```

```
% linear mode
B = -A*1i*k/(1i*o+p2);
C = -A*1i*k/(p2*tau+1i*o);

BT = -1i*k*AT/(1i*o+p2);
CT = -1i*k*AT/(1i*o+p2*tau);

% homogeneous solution at O(epi^3); must be eliminated at the end
% with the solvability condition, del^2(eq1)+R*(eq2)-R*Rrh/tau*(eq3)
B3 = 0; C3 =  0;

equito

% linear equations for R=R0
eq11=limit(diff(eq1,epi),epi,0);
eq22=limit(diff(eq2,epi),epi,0);
eq33=limit(diff(eq3,epi),epi,0);

% Check linear solution at O(epi)
eq11a = simplify(eq11/sin(pi*z)/exp(1i*o*t+1i*k*x)/A);
eq22a = simplify(eq22/sin(pi*z)/exp(1i*o*t+1i*k*x));
eq33a = simplify(eq33/sin(pi*z)/exp(1i*o*t+1i*k*x));

% Find mean corrections at O(epi^2)
eq222=simplify(limit(diff(eq2,epi,2)/2,epi,0)/sin(2*pi*z));
eq333=simplify(limit(diff(eq3,epi,2)/2,epi,0)/sin(2*pi*z));

% Solve for amplitudes of mean corrections to theta and sigma
D1a=solve(eq222,D1);
D2a=solve(eq333,D2);

D1 = -k^2*p2*abs(A)^2/2/pi/(o^2+p2^2);
D2 = -k^2*p2*abs(A)^2/pi/2/(o^2+p2^2*tau^2);

rr=simplify(D1a-D1); rr %check that the code is correct;
rr=simplify(D2a-D2); rr
```

```
equito

% Check correct D1 D2
eq111=simplify(limit(diff(eq1,epi,2)/2/sin(2*pi*z),epi,0));
eq222=simplify(limit(diff(eq2,epi,2)/2/sin(2*pi*z),epi,0));
eq333=simplify(limit(diff(eq3,epi,2)/2/sin(2*pi*z),epi,0));

% Solvability condition
% Extract O(epi^3) terms
eq2 = limit(diff(eq2,epi,3)/6,epi,0); eq2=simplify(eq2);

eq2 = simplify(int(eq2*sin(pi*z),z,0,1));
B3 = (1i*k*(abs(A)^2*A*p2*k^2/2/(o^2+p2^2)-A3)-BT)/(1i*o+p2);

equito
eq2 = limit(diff(eq2,epi,3)/6,epi,0); eq2=simplify(eq2);
eq2 = simplify(int(eq2*sin(pi*z),z,0,1));
eq2;

eq3 = limit(diff(eq3,epi,3)/6,epi,0); eq3=simplify(eq3);
% extract sin(pi*z) parts
eq3 = simplify(int(eq3*sin(pi*z),z,0,1));
C3 = (1i*k*(abs(A)^2*A*p2*k^2/2/(o^2+p2^2*tau^2)-A3)-CT)/(1i*o+p2*tau);

equito

eq3 = limit(diff(eq3,epi,3)/6,epi,0); eq3=simplify(eq3);
eq3 = simplify(int(eq3*sin(pi*z),z,0,1));
eq3;

B3 = (1i*k*(abs(A)^2*A*p2*k^2/4/(o^2+p2^2))-BT)/(1i*o+p2);
C3 = (1i*k*(abs(A)^2*A*p2*k^2/4/(o^2+p2^2*tau^2))-CT)/(1i*o+p2*tau);
solva = (R0*(-B3+Rrh*C3)+R2*(-B+Rrh*C));

ATfac = diff(solva,AT); ATfaca=simplify(ATfac); ATfaca;%A_T coeffs
R2fac = diff(solva,R2); R2faca=simplify(R2fac); R2faca;%A*R2 coeffs
A3fac = limit(limit(solva,AT,0),R2,0);
```

```
A3faca=simplify(A3fac/A/abs(A)^2); A3faca;%A^3 coeffs

ATfac = -1i*k*R0*(1/(1i*o+p2)^2-Rrh/(1i*o+tau*p2)^2);%output from above
R2fac = 1i*k*A*(1/(1i*o+p2)-Rrh/(1i*o+tau*p2));
A3fac = -1i/4*p2*k^3*R0*(1/(p2^2+o^2)/(1i*o+p2)-...
Rrh/(p2^2*tau^2+o^2)/(1i*o+tau*p2));

check = simplify(ATfac-ATfaca); check
check = simplify(R2fac-R2faca); check
check = simplify(A3fac-A3faca); check

o=p2*sqrt((Rrh-tau^2)/(1-Rrh));
R0 = p2^2*(1+tau)/k^2/(1-Rrh);

ATfac = -1i*k*R0*(1/(1i*o+p2)^2-Rrh/(1i*o+tau*p2)^2);
R2fac = 1i*k*A*(1/(1i*o+p2)-Rrh/(1i*o+tau*p2));
A3fac = -1i/4*p2*k^3*R0*(1/(p2^2+o^2)/(1i*o+p2)-...
Rrh/(p2^2*tau^2+o^2)/(1i*o+tau*p2));

% Equation: AT + c1 * R2 * A + c2 * |A|^2 A = 0
c1 = (R2fac/A/ATfac); c1;
c2 = (A3fac/ATfac);   c2;
```

E.4 Code for Steady Solution Plot

```
function SS

tau = 1/80; %for sugar-salt, this is 1/3
mu = 0.1;
k0 = 1;
H = pi/k0/mu;
Rt = k0^2*(mu^2+1)^2;
R = Rt + 0.05;
A = sqrt((R-Rt)*8*tau^2/(1-tau^2)/(mu^2+1)^3);
C2 = -A^2*(mu^2+1)/2;
zspan = [0:100]/100*H;
R = Rt + 0.05;
```

```
A2 = (R-Rt)*8*tau^2/(1-tau^2)/(mu^2+1)^3;
C2 = -(1+mu^2)/2*A2

solinit = bvpinit(zspan,@guess,C2);
sol = bvp4c(@odefcn,@bcs,solinit);
sol.parameters
plot(sol.y(1,:),sol.x,'c')
pbaspect([1 2 3])
ylim([0 H+7])
ylabel('z_0')
xlabel('T_0(z_0)')
title({'Steady Solutions of T_0(z_0) at'; 'Varying Rayleigh Numbers'})
hold on
grid on

R = Rt + 0.1;
sol = bvp4c(@odefcn,@bcs,sol);
sol.parameters
plot(sol.y(1,:),sol.x,'b')
hold on

R = Rt + 0.2;
sol = bvp4c(@odefcn,@bcs,sol);
sol.parameters
plot(sol.y(1,:),sol.x,'k')
legend('R = R_0 + 0.05','R = R_0 + 0.1','R = R_0 + 0.2')
legend('boxoff')
hold off

function dTdz = odefcn(z,T,lam)

T7p=3*k0^2*T(5) - 3*k0^4*T(3) + k0^6*T(1) +...
R*k0^2*(1/2)*(1/tau/tau - 1)*(T(3) - k0^2*T(1))*...
             (T(3)*T(1) - k0^2*T(1)^2 - lam)+R*k0^2*(T(3)-k0^2*T(1));

dTdz = [T(2) %first deriv
T(3) %2nd
```

```
T(4) %3rd
T(5) %4th
T(6) %5th
T7p  %6th
(T(3)*T(1)-T(1)^2*k0^2)/H];
end

function res = bcs(Ta,Tb,lam)
res = [Ta(1) %T top = 0
Tb(1) %T bottom = 0
Ta(3) - k0^2*Ta(1) %w top = 0
Tb(3) - k0^2*Tb(1) %w bottom = 0
    Ta(5)-2*k0^2*Ta(3) %condition on T, from 5.0.12
Tb(5)-2*k0^2*Tb(3)
Ta(7)
Tb(7)-lam];
end

function g = guess(z)
g = A*[sin(pi*z/H)
pi/H*cos(pi*z/H)
-pi^2/H^2*sin(pi*z/H)
-pi^3/H^3*cos(pi*z/H)
pi^4/H^4*sin(pi*z/H)
0
0];
end
end
```