Classifying Spaces for Topological Azumaya Algebras

by

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Abstract

In this thesis, we study a family of smooth varieties, whose members are denoted $B_n^r(\mathbb{C})$, that bears a similar relationship to topological Azumaya algebras as the Grassmannians $\mathrm{Gr}_{n,r}(\mathbb{C})$ do to complex vector bundles. Specifically, we will show that the varieties $B_n^r(\mathbb{C})$ form homotopical approximations to the classifying space $B\mathrm{PGL}_n(\mathbb{C})$. The varieties $B_n^r(\mathbb{C})$ are obtained by first considering the variety of r-tuples of $n \times n$ complex matrices that generate the matrix algebra $\mathrm{Mat}_n(\mathbb{C})$, and then taking the quotient by an evidently free $\mathrm{PGL}_n(\mathbb{C})$-action. The focus of this thesis is a computation of the singular cohomology groups of $B_n^r(\mathbb{C})$ when $n = 2$. We will show how these cohomological computations have applications in bounding the minimal number of generating sections of a topological Azumaya algebra over a paracompact space.

Lay Summary

The fields of algebra and topology are connected in various ways. Crudely, algebra is the study of mathematical structures with operations, or ways of combining elements, while topology is concerned with the shape or spatial arrangement of mathematical structures. This thesis focuses on one particular connection between algebra and topology; the story is as follows. One can often translate a given algebraic structure into a topological structure, a so-called “bundle,” that captures the same data as the original algebraic structure. It turns out that all bundles are merely shadows of a special bundle, called a “universal bundle.” Often in mathematics, objects with desirable properties are very hard to understand. The universal bundle is no exception. As a work-around, one can approximate the universal bundle by simpler bundles with similar properties. Remarkably, studying the topology of these approximations can shed light on the original algebraic objects.

Preface

Chapter 2 is expository. Chapter 3 is work known to B. Williams, Z. Reichstein, and U. First following a paper in production by these three authors. The material of Chapters 4, 5, and 6 is original, unpublished work by the author, S. Gant.

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Chapter 1

Introduction

Consider the following construction. For $1 \leq n \leq r$, one can form the *Stiefel variety* $V_{n,r}(\mathbb{C})$ of n-frames in r-space: the open subvariety of $\mathrm{Mat}_{n\times r}(\mathbb{C}) \cong \mathbb{A}^{rn}_{\mathbb{C}}$ consisting of full rank $n \times r$ matrices. Put another way, $V_{n,r}(\mathbb{C})$ is the space of r-tuples of vectors in n-space that generate $\mathbb{C}^n$ as a vector space. The Stiefel variety $V_{n,r}(\mathbb{C})$ admits a free $\mathrm{GL}_n(\mathbb{C})$-action, and the quotient $V_{n,r}(\mathbb{C})/\mathrm{GL}_n(\mathbb{C})$ is isomorphic to the Grassmannian $\mathrm{Gr}_{n,r}(\mathbb{C})$ of n-planes in r-space. Moreover, the map $V_{n,r}(\mathbb{C}) \to \mathrm{Gr}_{n,r}(\mathbb{C})$ is a principal $\mathrm{GL}_n(\mathbb{C})$-bundle. One can show that the homotopy groups $\pi_i(V_{n,r}(\mathbb{C}))$ are trivial for $i \leq 2(r-n)$. The effect is that, as r tends to infinity, the Stiefel varieties form better approximations to an $E\mathrm{GL}_n(\mathbb{C})$: a contractible CW complex that is the total space of a universal principal $\mathrm{GL}_n(\mathbb{C})$-bundle. As a result, the infinite Grassmannian $\mathrm{Gr}_{n,\infty}(\mathbb{C}) = \mathrm{colim}_r\,\mathrm{Gr}_{n,r}(\mathbb{C})$ is a model for the classifying space $B\mathrm{GL}_n(\mathbb{C})$.

In this thesis, we study two families of smooth $\mathbb{C}$-varieties, whose members are denoted $U_n^r(\mathbb{C})$ and $B_n^r(\mathbb{C})$, that bear a similar relationship to the group $\mathrm{PGL}_n(\mathbb{C})$ as the varieties $V_{n,r}(\mathbb{C})$ and $\mathrm{Gr}_{n,r}(\mathbb{C})$ do to $\mathrm{GL}_n(\mathbb{C})$. We define $U_n^r(\mathbb{C})$ to be the open subvariety of $\mathrm{Mat}_n^r(\mathbb{C})$—the variety of r-tuples of $n \times n$ complex matrices—consisting of those r-tuples that generate $\mathrm{Mat}_n(\mathbb{C})$ as a $\mathbb{C}$-algebra. The variety $U_n^r(\mathbb{C})$ admits a free $\mathrm{PGL}_n(\mathbb{C})$-action by simultaneous conjugation, and we denote the quotient by $B_n^r(\mathbb{C})$. As it happens, the quotient map $U_n^r(\mathbb{C}) \to B_n^r(\mathbb{C})$ is a principal $\mathrm{PGL}_n(\mathbb{C})$-bundle. Similar to the Stiefel varieties, we will show that the homotopy groups of $U_n^r(\mathbb{C})$ vanish below a certain degree that depends linearly on r for fixed n. As a consequence, the varieties $B_n^r(\mathbb{C})$ serve as approximations to the classifying space $B\mathrm{PGL}_n(\mathbb{C})$.

The Skolem–Noether theorem asserts that every automorphism of $\mathrm{Mat}_n(\mathbb{C})$ as a $\mathbb{C}$-algebra is given by conjugation. Consequently, the automorphism group of $\mathrm{Mat}_n(\mathbb{C})$ as an algebra is isomorphic to $\mathrm{PGL}_n(\mathbb{C})$. One therefore has a correspondence between principal $\mathrm{PGL}_n(\mathbb{C})$-bundles and bundles of matrix algebras (fibre bundles with fibre $\mathrm{Mat}_n(\mathbb{C})$ and structure group $\mathrm{PGL}_n(\mathbb{C})$). These latter objects are the so-called *topological Azumaya algebras*. We will show that $B_n^r(\mathbb{C})$ represents the functor defined on topological spaces that sends X to isomorphism classes of degree-n topological Azumaya algebras over X equipped with r globally generating sections.

The main goal of this thesis is to compute the singular cohomology groups $\mathrm{H}^*(B_2^r(\mathbb{C});\mathbb{Q})$ in the range $* \leq 4r-7$. We do so by way of the Leray–Serre spectral sequence associated to a particular fibration.

Outline. In Chapter 2, we will lay out the relevant tools that will be used in the succeeding chapters. Chapter 3 covers some of the homotopical properties of $U_n^r(\mathbb{C})$ and $B_n^r(\mathbb{C})$. We will show that the varieties $B_n^r(\mathbb{C})$ approximate the classifying space $B\mathrm{PGL}_n(\mathbb{C})$ in the sense that there is a $(2(r-1)(n-1)-1)$-equivalence $B_n^r(\mathbb{C}) \to B\mathrm{PGL}_n(\mathbb{C})$. We compute the singular cohomology groups $\mathrm{H}^*(U_2^r(\mathbb{C});\mathbb{Z})$ for $* \leq 4r-7$ in Chapter 4. In Chapter 5, the singular cohomology groups $\mathrm{H}^*(B_2^r(\mathbb{C});\mathbb{Q})$, for $* \leq 4r-7$, are computed. The primary motivation for these computations is Chapter 6. Here, we will discuss how these cohomological computations can be used to give obstructions to the generation by r global sections of a topological Azumaya algebra over a paracompact space.

Notational Conventions. Many of the topological spaces appearing in this thesis are subvarieties of $\mathbb{A}_{\mathbb{C}}^N$, and we will be jumping back and forth between the Euclidean and Zariski topologies. We therefore make the convention that all references to a topology, including computations of invariants, will be to the Euclidean topology unless otherwise specified: the word "Zariski" will be used. An *algebra* in this thesis will mean an associative and unital algebra. The term *manifold* will mean smooth, real manifold without boundary, and the term *manifold with boundary* will mean smooth, real manifold with possibly empty boundary. The complex conjugate of a complex number z will be denoted z^*, and we will reserve the "bar" notation for vectors, as in "$\bar{a} = (a_1, \ldots, a_r)$." Lastly, *dimension* will mean complex dimension unless otherwise specified: the word "real" or the symbol $\dim_{\mathbb{R}}$ will be used.

Chapter 2

Preliminaries

2.1 Some Fibre Bundle Theory

Given a G-space X, it will be important for us to know when the quotient map $X \to X/G$ has the structure of a principal G-bundle. Corollary 2.4 is to this end in the important case where G is a Lie group, X is a manifold, and the action is smooth. Recall that, for a G-space X, the action is *proper* if the map

$$\begin{aligned} \Theta\colon G \times X &\to X \times X \\ (g, x) &\mapsto (g \cdot x, x) \end{aligned}$$

is a proper map; i.e., the preimage of any compact set is compact.

Theorem 2.1 (Quotient Manifold Theorem [Lee13, Theorem 21.10])**.** *Let G be a Lie group that acts smoothly, freely, and properly on a manifold M (on the left). Then the orbit space M/G has a unique smooth structure making it a manifold of (real) dimension $\dim_{\mathbb{R}} M - \dim_{\mathbb{R}} G$ such that the quotient map $M \to M/G$ is a smooth submersion.*

Remark 2.2. If G is a compact Lie group acting on a manifold M, the action is proper.

Lemma 2.3 ([KMS93, Lemma 10.3])**.** *Let $f\colon M \to B$ be a surjective smooth submersion, and let G be a Lie group that acts smoothly and freely on M (on the left) such that the G-orbits are exactly the fibers of f. Then $f\colon M \to B$ is a principal G-bundle.*

The quotient manifold theorem and Lemma 2.3 together give the following corollary.

Corollary 2.4. *Let G be a Lie group acting smoothly, freely, and properly on a manifold M (on the left). Then the orbit space M/G has a unique smooth structure making it a manifold of (real) dimension $\dim_{\mathbb{R}} M - \dim_{\mathbb{R}} G$ such that the quotient map $M \to M/G$ is a smooth submersion and a principal G-bundle.*

We will also need the following facts concerning the cohomological Leray–Serre spectral sequence associated to a fibration.

Proposition 2.5 (Naturality of the Leray–Serre Spectral Sequence)**.** *Suppose R is a commutative ring, and*

$$\begin{tikzcd} F \arrow[r, "i"] \arrow[d, "f"] & E \arrow[r, "\pi"] \arrow[d, "e"] & B \arrow[d, "b"] \\ F' \arrow[r, "i'"] & E' \arrow[r, "\pi'"] & B' \end{tikzcd}$$

is a morphism of fibrations in which B, B' are path-connected and $\pi_1(B), \pi_1(B')$ act trivially on $\mathrm{H}^*(F; R), \mathrm{H}^*(F'; R)$ respectively. Let $E_k^{p,q}$ and $'E_k^{p,q}$ denote the terms in the Leray–Serre spectral sequences associated to the top and bottom fibrations respectively. Then

(a) There are induced maps $e_k^{p,q} \colon {}'E_k^{p,q} \to E_k^{p,q}$ that commute with the differentials. Moreover, $e_{k+1}^{p,q}$ is the induced map on homology by $e_k^{p,q}$.
(b) The maps $e_2^{p,q} \colon \mathrm{H}^p(B', \mathrm{H}^q(F'; R)) \to \mathrm{H}^p(B, \mathrm{H}^q(F; R))$ coincide with the maps induced by b and f.

This can be found in Section 23.1 of [FF16].

Remark 2.6. As a special case of (a), when the incoming and outgoing differentials for the terms $E_k^{p,q}, {}'E_k^{p,q}$ are all 0, then $e_k^{p,q} = e_{k+1}^{p,q}$.

Remark 2.7. As a special case of (b), the map $e_2^{0,q} \colon \mathrm{H}^q(F'; R) \to \mathrm{H}^q(F; R)$ coincides with f^*. If F and F' are connected, the map $e_2^{p,0} \colon \mathrm{H}^p(B'; R) \to \mathrm{H}^p(B; R)$ coincides with b^*.

2.2 Connectivity of the Complement of an Algebraic Set

We set out to prove Proposition 2.19, which says that, given a closed subvariety $Z \hookrightarrow \mathbb{A}_{\mathbb{C}}^N$ of codimension $d > 0$, the complement $\mathbb{A}_{\mathbb{C}}^N \setminus Z$ is $(2d-2)$-connected. That is, the homotopy groups $\pi_i(\mathbb{A}_{\mathbb{C}}^N \setminus Z)$ are trivial for $i \leq 2d-2$. Though this fact is well known, I have not been able to find it in the literature outside of the lecture notes [Ful07]. The proof given there is attributed to David Speyer. Here, we present a proof modeled on that one; it is due to B. Williams.

The following version of the Whitney Approximation Theorem is [Lee13, Theorem 6.26].

Theorem 2.8 (Whitney Approximation Theorem)**.** *Suppose $f \colon N \to M$ is a continuous map where N is a manifold with boundary and M is a manifold. Suppose $A \subseteq N$ is a (possibly empty) closed subset such that $f|_A$ is smooth. Then f is homotopic relative to A to a smooth map $N \to M$.*

Definition 2.9. Suppose $f, g\colon N \to M$ are two smooth maps of manifolds with boundary. A *smooth homotopy* from f to g is a map $H\colon N \times I \to M$ that restricts to f and g at 0 and 1 respectively, and such that H extends to a smooth map on some open neighborhood of $N \times I$ in $N \times \mathbb{R}$.

Lemma 6.28 in [Lee13] guarantees that smooth homotopy is an equivalence relation on smooth maps $N \to M$. The following is [Lee13, Lemma 6.29].

Lemma 2.10. *Suppose $f, g\colon N \to M$ are two smooth maps that are homotopic relative to some (possibly empty) closed set $A \subseteq N$. Then f and g are smoothly homotopic relative to A.*

Corollary 2.11 (Extension Lemma [Lee13, Corollary 6.27])**.** *Suppose N is a manifold with boundary and M is a manifold. Suppose $A \subseteq N$ is a closed subset and $f\colon N \to M$ is a smooth map. Then f has a smooth extension to N if and only if f has a continuous extension to N.*

Definition 2.12. Let M be a manifold with boundary and $m_0 \in M$ a basepoint. For an integer $n \geq 0$, let $\pi_k^{\mathrm{sm}}(M, m_0)$ denote the set of smooth homotopy classes of basepoint preserving maps $S^k \to M$.

Remark 2.13. There is a natural transformation $\pi_k^{\mathrm{sm}} \to \pi_k$. By Theorem 2.8, the map $\pi_k^{\mathrm{sm}}(M, m_0) \to \pi_k(M, m_0)$ is surjective, and by Lemma 2.10, this map is injective. We can therefore identify $\pi_k^{\mathrm{sm}}(M, m_0)$ with $\pi_k(M, m_0)$.

Definition 2.14. Suppose N is a manifold with boundary, and M is a manifold. Suppose $f\colon N \to M$ is a smooth map, $A \subseteq M$ is a submanifold, and $C \subseteq N$ is a subset. The map f is *transverse to A on C* if, for every $x \in f^{-1}A \cap C$,

$$T_{f(x)}A + df_x(T_xN) = T_{f(x)}M.$$

We say a f is *transverse to A* if f is transverse to A on N.

Lemma 2.15. *Suppose N is a manifold with boundary of (real) dimension n, that M is a manifold of (real) dimension m, and $A \subseteq M$ is a submanifold of (real) dimension a. Suppose $f\colon N \to M$ is a smooth map that is transverse to A. If $m > n + a$, then $f^{-1}A = \varnothing$.*

Proof. Suppose $x \in f^{-1}A$. The vector space $df_x(T_x(N))$ has (real) dimension no larger than n. So

$$m = \dim_{\mathbb{R}}(T_{f(x)}Z + df_x(T_x(N)) \leq a + n < m,$$

a contradiction. ∎

We will also need the “Extension Theorem” in [GP74, p. 72].

Theorem 2.16 (Extension Theorem)**.** *Let N be a manifold with boundary and M be a manifold, and suppose $A \subseteq M$ is a closed submanifold. Suppose $f\colon N \to M$ is a smooth map and $C \subseteq N$ is a closed subset such that f is transverse to A on C and $f|_{\partial N}$ is transverse to A on $C \cap \partial N$. Then there exists a smooth map $g\colon N \to M$, homotopic to f, such that g is transverse to A, $g|_{\partial N}$ is transverse to A, and g agrees with f on a neighborhood of C.*

Definition 2.17. A map $f\colon (X, x_0) \to (Y, y_0)$ of pointed spaces is an *n-equivalence* if the induced map

$$f_*\colon \pi_k(X, x_0) \to \pi_k(Y, y_0)$$

is an isomorphism for $k < n$ and is surjective for $k = n$. A pointed space (X, x_0) is *n-connected* if $\pi_k(X, x_0)$ is trivial for $k \leq n$.

Proposition 2.18. *Let M be a manifold of dimension m and $A \subseteq M$ a submanifold of real codimension $d > 1$. Let $m_0 \in M \setminus A$ be a basepoint. Then the inclusion $i\colon M \setminus A \to M$ is a $(d-1)$-equivalence.*

Proof. Following Remark 2.13, we may compute $\pi_k(M\setminus Z, m_0)$ and $\pi_k(M, m_0)$ using smooth basepoint preserving maps modulo smooth homotopy.

Suppose $k \leq d-1$. Let $[f] \in \pi_k(M, m_0)$ be a class represented by a smooth map $f\colon S^k \to M$. Using Theorem 2.16, with C the basepoint of S^k, we may assume f is transverse to A. By Lemma 2.15, we have $f(S^k) \cap A = \varnothing$ so that f factors as $S^k \xrightarrow{\bar{f}} M \setminus A \xrightarrow{i} M$. Then $[\bar{f}] \in \pi_k(M \setminus A, m_0)$ is a class mapping to $[f]$ under i_*. This shows that i_* is surjective.

Suppose $0 < k < d-1$. Let $[f] \in \pi_k(M \setminus Z, m_0)$ be a class represented by a smooth map $f\colon S^k \to M \setminus Z$ such that $i_*[f]$ is trivial. We show that $[f]$ is trivial. Since $[i \circ f]$ is trivial, there is a map $F\colon D^{k+1} \to M$ restricting to $i \circ f$ on $S^k = \partial D^{k+1}$. By way of the Whitney Approximation Theorem, we may replace F by a smooth map F' such that F' also restricts to $i \circ f$ on S^k. Then, using Theorem 2.16 with $N = D^{k+1}$ and $C = S^k$, we may replace F' by a smooth map $F''\colon D^{k+1} \to M$ such that $F''|_{S^k} = i \circ f$ and F'' is transverse to A. By Lemma 2.15 again, it follows that $F''(D^{k+1}) \cap A = \varnothing$. The existence of such an F'' implies that f is homotopic to the constant map at m_0 relative to m_0.

We have left to show $\pi_0(M \setminus Z, m_0) \to \pi_0(M, m_0)$ is injective. Suppose $x, y \in M \setminus Z$ are two points in the same component of M. Then there is a path $\gamma\colon I \to M$ from x to y. Using the Whitney Approximation Theorem, replace γ by a smooth path γ' from x to y. Then using Theorem 2.16 with $N = I$ and $C = \partial I$, we may replace γ' by a smooth path γ'' that is transverse to A. Since $d > 1$, the image of γ and Z do not intersect by Lemma 2.15. $\square$

Proposition 2.19. *Let $Z \hookrightarrow \mathbb{A}^N_{\mathbb{C}}$ be a closed subvariety of codimension $d > 0$. Then the inclusion $\mathbb{A}^N_{\mathbb{C}} \setminus Z \to \mathbb{A}^N_{\mathbb{C}}$ is a $(2d-1)$-equivalence. In particular, $\mathbb{A}^N_{\mathbb{C}} \setminus Z$ is $(2d-2)$-connected.*

Proof. There exists a stratification of Z with smooth strata of weakly increasing dimension [Whi65]. By induction on the stratification index, it suffices to treat the case $Z \hookrightarrow M$ is a smooth, closed subvariety of codimension d. That is, $Z \hookrightarrow M$ is a real codimension $2d$ submanifold of the manifold M. This case is handled by Proposition 2.18. $\square$

2.3 The Gysin Sequence

We refer to the long exact sequence of Theorem 2.26 as *the Gysin sequence*. The word "the" here is maybe misleading, as various other long exact sequences in cohomology go under the same name in the literature. In this thesis, the Gysin sequence refers to a long exact sequence in cohomology associated to a closed inclusion $N \to M$ of manifolds that relates the cohomology of N, M, and $M \setminus N$. We present a slightly different construction from that of [Dol80, Proposition 12.1], which constructs the sequence for topological manifolds. The Gysin sequence here is constructed in the smooth setting, which has the added bonus of making the naturality statement, Proposition 2.29, more apparent as one can make sense of transverse intersections. The analogous sequence for various oriented motivic cohomology theories is known as the *localization sequence*, where questions of naturality are perhaps better understood (or at least better referenced. See, for instance, [Pan09]).

In this section, all cohomology groups are computed with coefficients in an arbitrary commutative ring R; we suppress coefficients. For a pair $(X, X - Y)$, we denote the relative cohomology group $\mathrm{H}^n(X, X - Y)$ by $\mathrm{H}^n(X|Y)$.

Let M be a manifold. For a submanifold $A \subseteq M$, define the *normal bundle* $N_{A/M} \to A$ to be the unique vector bundle up to isomorphism fitting into the short exact sequence

$$0 \to TA \to TM|_A \to N_{A/M} \to 0$$

of vector bundles over A.

Definition 2.20. A *tubular neighborhood* of a submanifold $A \subseteq M$ is a pair (ϕ, V) where $\phi\colon N_{A/M} \to M$ is a smooth embedding such that $V = \phi(N_{A/M})$ is an open neighborhood of A and

$$\begin{tikzcd} N_{A/M} \arrow[rr, "\phi"] & & M \\ & A \arrow[ul, hook, "i_0"] \arrow[ur, hook] & \end{tikzcd}$$

commutes, where i_0 is the zero section.

Note that V inherits a vector bundle structure $V \to A$ via the diffeomorphism $\phi\colon N_{A/M} \to V$ with zero section the inclusion $A \hookrightarrow V$. The Tubular Neighborhood Theorem [Hir76, Theorem 5.2] guarantees the existence of a tubular neighborhood for $A \subseteq M$.

Notation 2.21. For a given tubular neighborhood (ϕ, V), we will sometimes refer to the open neighborhood V of A as a "tubular neighborhood."

Definition 2.22. Suppose $E \to A$ and $E' \to A'$ are vector bundles and $f\colon A' \to A$ is a smooth map of manifolds. A map $\tilde{f}\colon E' \to E$ is a *map of vector bundles over f* if

$$\begin{tikzcd} E' \arrow[r, "\tilde{f}"] \arrow[d] & E \arrow[d] \\ A' \arrow[r, "f"] & A \end{tikzcd}$$

commutes and the restriction $\tilde{f}_p\colon E'_p \to E_{f(p)}$ to the fibre over p is linear for each $p \in A'$. We say that $\tilde{f}$ is a *fibrewise isomorphism of vector bundles over f* if $\tilde{f}$ restricts to a linear isomorphism on the fibers.

Remark 2.23. A word of warning: in [Hir76], a fibrewise isomorphism over a map f is referred to as a "vector bundle map over f."

We will also need the following version of the Thom Isomorphism Theorem.

Theorem 2.24 (The Thom Isomorphism Theorem [MS16, Theorem 10.4])**.** *Suppose $\pi\colon E \to B$ is an oriented rank-d vector bundle. Identifying B with the image of the zero section, there is a unique class $\tau \in \mathrm{H}^d(E|B)$ such that:*

(a) τ restricts to the orientation generator in $\mathrm{H}^d(E_p|p)$ for each $p \in B$
(b) The map $\alpha \mapsto \alpha \smile \tau$ yields an isomorphism $\mathrm{H}^k(E) \to \mathrm{H}^{k+d}(E|B)$ for each k, where $\smile$ is the relative cup product $\mathrm{H}^k(E) \times \mathrm{H}^l(E|B) \to \mathrm{H}^{k+l}(E|B)$.

Notation 2.25. The retraction π induces an isomorphism $\pi^*\colon \mathrm{H}^n(B) \to \mathrm{H}^n(E)$ for each n. In what follows, we will refer to the isomorphism $(- \smile \tau)\pi^*\colon \mathrm{H}^n(B) \to \mathrm{H}^{n+d}(E|B)$ as the *Thom Isomorphism* and denote it Φ_E.

Theorem 2.26 (The Gysin Sequence)**.** *Suppose A is a closed, oriented submanifold of the oriented manifold M. There is a long exact sequence*

$$\cdots \longrightarrow \mathrm{H}^{k-d}(A) \longrightarrow \mathrm{H}^k(M) \longrightarrow \mathrm{H}^k(M - A) \longrightarrow \mathrm{H}^{k+1-d}(A) \longrightarrow \cdots$$

where d is the real codimension of A in M.

Proof. Choose a tubular neighborhood (ϕ, V) of $A \subseteq M$. Following the convention in [BT82, p.66], the orientations on A and M induce an orientation on the normal bundle $N_{A/M} \to A$. The diffeomorphism $\phi\colon N_{A/M} \to V$ then induces an orientation on the vector bundle $V \to A$, so we have a specified Thom isomorphism $\Phi_V\colon \mathrm{H}^k(A) \to \mathrm{H}^{k+d}(V|A)$. In the long exact sequence in cohomology for the pair $(M, M-A)$, replace the terms $\mathrm{H}^k(M|A)$ by $\mathrm{H}^k(V|A)$ via excision. We obtain the long exact sequence

$$\cdots \longrightarrow \mathrm{H}^k(V|A) \longrightarrow \mathrm{H}^k(M) \longrightarrow \mathrm{H}^k(M-A) \longrightarrow \mathrm{H}^{k+1}(V|A) \longrightarrow \cdots .$$

Replacing the terms $\mathrm{H}^k(V|A)$ with $\mathrm{H}^{k-d}(A)$ via the Thom isomorphism Φ_V, we have the desired long exact sequence. $\square$

We now turn to showing that, at least in a special case, the Gysin sequence is natural with respect to transverse intersections. To this end, we will need the technical Lemma 2.27.

For a given orthogonal structure on $N_{A/M}$, one can construct a closed disk subbundle of radius ϵ:

$$D_\epsilon(N_{A/M}) = \{(a,v) \in N_{A/M} : |v| \leq \epsilon\}.$$

If (ϕ, V) is a tubular neighborhood of A, we define a closed disk subbundle of $V \to A$ to be $\phi(D_\epsilon(N_{A/M}))$ for some $\epsilon > 0$. Note that an orthogonal structure on $N_{A/M}$ always exists. The following is Theorem 4.6.7 in [Hir76].

Lemma 2.27. *Let $f\colon N \to M$ be a smooth map of manifolds, and $A \hookrightarrow M$ a closed, embedded submanifold. Assume f is transverse to A. Given tubular neighborhoods U of $A' = f^{-1}A$ and V of A, and a closed disk subbundle $D \subseteq U$ such that $f(D) \subseteq V$, there exists a homotopy h_t from f to h such that:*

(a) $h|_D$ is the restriction of a fibrewise isomorphism of vector bundles $U \to V$ over $f|_{A'}$.
(b) $h_t = f$ on $A' \cup (N-U)$ for each $t \in I$.
(c) $h_t^{-1}(M-A) = N - A'$ for each $t \in I$.

Remark 2.28. In [Hir76], it is assumed that the submanifold A is compact, but that assumption is not used in the proof.

Proposition 2.29 (Naturality of the Gysin Sequence)**.** *Let A, N be orientable, closed, embedded submanifolds of the orientable manifold M intersecting transversely in a point p. Let $i\colon N \hookrightarrow M$ be the inclusion. Then there is an induced map of Gysin sequences: a commutative diagram*

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{H}^{k-d}(p) \arrow[r] & \mathrm{H}^k(N) \arrow[r] & \mathrm{H}^k(N-p) \arrow[r] & \mathrm{H}^{k+1-d}(p) \arrow[r] & \cdots \\
\cdots \arrow[r] & \mathrm{H}^{k-d}(A) \arrow[r] \arrow[u, "i^*"] & \mathrm{H}^k(M) \arrow[r] \arrow[u, "i^*"] & \mathrm{H}^k(M-A) \arrow[r] \arrow[u, "i^*"] & \mathrm{H}^{k+1-d}(A) \arrow[r] \arrow[u, "i^*"] & \cdots
\end{tikzcd}$$

where d is the real codimension of A in M (and p in N) and the vertical maps are all induced by i.

Proof. The crux of the proof is showing that there exist tubular neighborhoods V of A and U of p so that the composite

$$\mathrm{H}^{k-d}(A) \xrightarrow{\Phi_V} \mathrm{H}^k(V|A) \xrightarrow{i|_U^*} \mathrm{H}^k(U|p) \xrightarrow{\Phi_U^{-1}} \mathrm{H}^{k-d}(p)$$

coincides with $i|_p^*$ for each k. The only difficulty here is when $k = d$.

Choose a tubular neighborhood $(\phi, N_{A/M})$ of A, with $V = \phi(N_{A/M})$, sufficiently small so that $U = i^{-1}(V)$ is a tubular neighborhood of p in N, an open ball about p. This is possible since the intersection of N and A is transverse. Pick an orthogonal structure on the vector bundle $U \to p$ to construct a disk subbundle $D \subseteq U$. Lemma 2.27 asserts that there is some h, homotopic to i, satisfying (a)–(c) with respect to the disk bundle D. In particular, $h|_D$ is the restriction of a fibrewise isomorphism $\tilde{\imath}\colon U \to V$ over $i|_p$. The map $\tilde{\imath}$ induces a vector bundle isomorphism

$$\begin{tikzcd} U \arrow[rr, "\approx"] \arrow[dr] & & i^*V = V_p \arrow[dl] \\ & p & \end{tikzcd}$$

over p. Since A and M are orientable, so is the normal bundle $N_{A/M} \to A$. Pick an orientation on $N_{A/M} \to A$. The diffeomorphism $\phi\colon N_{A/M} \to V$ induces an orientation on the vector bundle $V \to A$, so we have a distinguished orientation generator $u_p \in \mathrm{H}^d(V_p|p)$. Give $U \to p$ an orientation so that $\tilde{\imath}^*\colon \mathrm{H}^d(V_p|p) \to \mathrm{H}^d(U|p)$ sends u_p to the orientation generator of $\mathrm{H}^d(U|p)$. Let τ_p^U and τ_A^V denote the Thom classes of the oriented vector bundles $U \to p$ and $V \to A$ respectively. Then $\tau_p^U \in \mathrm{H}^d(U|p)$ is the orientation generator, so that $\mathrm{H}^d(V_p|p) \xrightarrow{\tilde{\imath}^*} \mathrm{H}^d(U|p)$ sends u_p to τ_p^U.

Claim: $i|_U^*(\tau_A^V) = \tau_p^U$ By the property (c) in Lemma 2.27, $h_t|_D$ provides a homotopy from $i|_D$ to $h|_D$ as maps of pairs $(D, D-p) \to (V, V-A)$. From the commuting diagram of pairs

$$\begin{tikzcd} (U, U-p) \arrow[r, "\tilde{\imath}"] & (V_p, V_p - p) \arrow[d, hook] \\ (D, D-p) \arrow[u, hook, "\simeq"'] \arrow[r, "h|_D"] & (V, V-A), \end{tikzcd}$$

we have

$$\begin{tikzcd} \mathrm{H}^d(U|p) \arrow[d, "\cong"] & \mathrm{H}^d(V_p|p) \arrow[l, "\tilde{\imath}^*"'] \\ \mathrm{H}^d(D|p) & \mathrm{H}^d(V|A). \arrow[l, "h|_D^* = i|_D^*"'] \arrow[u] \end{tikzcd}$$

The map $\mathrm{H}^d(V|A) \to \mathrm{H}^d(V_p|p)$ sends τ_A^V to the orientation generator u_p by part (a) of the

Thom Isomorphism Theorem. Then $\tilde{i}^*(u_p) = \tau_p^U$ is sent to a generator $\alpha \in \mathrm{H}^d(D|p)$ via the leftmost map. So we have $i|_D^*(\tau_A^V) = \alpha$. From the commutative diagram

$$\begin{tikzcd}
\mathrm{H}^d(U|p) \arrow[d, "\cong"] & \mathrm{H}^d(V|A) \arrow[l, "i|_U^*"'] \arrow[dl, "i|_D^*"] \\
\mathrm{H}^d(D|p) &
\end{tikzcd}$$

it follows that $i|_U^*(\tau_A^V) = \tau_p^U$. This proves the claim.

The map of pairs $(N, N-p) \xrightarrow{i} (M, M-A)$ induces a long exact sequence in cohomology:

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{H}^k(N|p) \arrow[r] & \mathrm{H}^k(N) \arrow[r] & \mathrm{H}^k(N-p) \arrow[r] & \mathrm{H}^{k+1}(N|p) \arrow[r] & \cdots \\
\cdots \arrow[r] & \mathrm{H}^k(M|A) \arrow[r] \arrow[u, "i^*"] & \mathrm{H}^k(M) \arrow[r] \arrow[u, "i^*"] & \mathrm{H}^k(M-A) \arrow[r] \arrow[u, "i^*"] & \mathrm{H}^{k+1}(M|A) \arrow[r] \arrow[u, "i^*"] & \cdots
\end{tikzcd} \tag{2.1}$$

The commuting square

$$\begin{tikzcd}
(U, U-p) \arrow[r, "i|_U"] \arrow[d] & (V, V-A) \arrow[d] \\
(N, N-p) \arrow[r, "i"] & (M, M-A)
\end{tikzcd}$$

induces a square in cohomology

$$\begin{tikzcd}
\mathrm{H}^k(U|p) & \mathrm{H}^k(V|A) \arrow[l, "i|_U^*"'] \\
\mathrm{H}^k(N|p) \arrow[u, "\cong"] & \mathrm{H}^k(M|A) \arrow[l, "i^*"'] \arrow[u, "\cong"]
\end{tikzcd} \tag{2.2}$$

where the vertical maps are isomorphisms by excision. Via the commuting square (2.2), replace the maps $\mathrm{H}^k(M|A) \xrightarrow{i^*} \mathrm{H}^k(N|p)$ in (2.1) by $\mathrm{H}^k(V|A) \xrightarrow{i^*} \mathrm{H}^k(U|p)$.

The above claim shows that

$$\begin{tikzcd}
\mathrm{H}^0(p) \arrow[d, "\Phi_U"] & \mathrm{H}^0(A) \arrow[l, "i^*"'] \arrow[d, "\Phi_V"] \\
\mathrm{H}^d(U|p) & \mathrm{H}^d(V|A) \arrow[l, "i|_U^*"']
\end{tikzcd}$$

commutes where the vertical maps are the respective Thom isomorphisms. For each k, the square

$$\begin{tikzcd}
\mathrm{H}^k(p) \arrow[d, "\Phi_U"] & \mathrm{H}^k(A) \arrow[l, "i^*"'] \arrow[d, "\Phi_V"] \\
\mathrm{H}^{k+d}(U|p) & \mathrm{H}^{k+d}(V|A) \arrow[l, "i|_U^*"']
\end{tikzcd}$$

commutes. Replacing $\mathrm{H}^{k+d}(V|A) \xrightarrow{i|_U^*} \mathrm{H}^{k+d}(U|p)$ in the long exact sequence by $\mathrm{H}^k(A) \xrightarrow{i^*} \mathrm{H}^k(p)$ via this commuting square, we obtain the desired diagram. $\square$

Chapter 3

Basic Properties of $U_n^r(\mathbb{C})$ and $B_n^r(\mathbb{C})$

Let k be a field. We say that a set $S \subseteq \mathrm{Mat}_n(k)$ *generates a subalgebra* $\mathscr{A} \subseteq \mathrm{Mat}_n(k)$, or simply S *generates* $\mathscr{A}$, if $\mathscr{A}$ is the smallest k-subalgebra of $\mathrm{Mat}_n(k)$ containing S.

Notation 3.1. Let $U_n^r(\mathbb{C})$ denote the set

$$\{(A_1, \ldots, A_r) \in \mathrm{Mat}_n^r(\mathbb{C}) : \{A_1, \ldots, A_r\} \text{ generates } \mathrm{Mat}_n(\mathbb{C})\}$$

and let $Z_n^r(\mathbb{C}) = \mathrm{Mat}_n^r(\mathbb{C}) \setminus U_n^r(\mathbb{C})$.

If $n > 1$ and $r > 1$, then $U_n^r(\mathbb{C})$ is nonempty since $\mathrm{Mat}_n(\mathbb{C})$ can be generated by two elements; take the matrices

$$A = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & & & 0 \\ 0 & 1 & \ddots & & \vdots \\ \vdots & & & & \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Any matrix with a 1 in some entry and 0s elsewhere is of the form $B^k A B^l$, and such matrices taken together generate $\mathrm{Mat}_n(\mathbb{C})$. However, if $n > 1$ and $r = 1$, then $U_n^r(\mathbb{C})$ is empty since the algebra that a single element generates is commutative—a proper subalgebra of $\mathrm{Mat}_n(\mathbb{C})$. And if $n = 1$, any element of $\mathrm{Mat}_n^r(\mathbb{C})$ generates $\mathrm{Mat}_n(\mathbb{C})$. To avoid these pathologies, we will assume $r > 1$ and $n > 1$ in what follows.

For an arbitrary field k, note that after choosing the standard basis of k^n, the algebra $\mathrm{Mat}_n(k)$ can be identified with the algebra of linear endomorphisms of k^n.

Notation 3.2. Let $S \subseteq \mathrm{Mat}_n(k)$ be a subset. We will say that *the matrices* $A \in S$ *have a common invariant subspace* if there is a proper, nontrivial linear subspace invariant under each $A \in S$. That is, there is a linear subspace $L \subseteq k^n$ of dimension m, with $1 \leq m \leq n-1$, such that

$$AL \subseteq L$$

for each $A \in S$. If the matrices in S have a common invariant subspace, we will also occasionally say the set S *has an invariant subspace*.

We will make repeated use of the following classical theorem due to Burnside. An elementary proof can be found in [LR04].

Theorem 3.3 (Burnside's Theorem)**.** *Let k be an algebraically closed field and $n > 1$. A set $S \subseteq \mathrm{Mat}_n(k)$ generates the matrix algebra $\mathrm{Mat}_n(k)$ if and only if the matrices in S do not have a common invariant subspace.*

Remark 3.4. The assumption that k is algebraically closed is necessary. For instance, let $S = \{A_1, A_2\}$ where A_1, A_2 are two rotation matrices in $\mathbb{R}^2$. These two matrices have no common 1-dimensional eigenspaces, and the algebra they generate is commutative.

Proposition 3.5. *The subspace $Z_n^r(\mathbb{C}) \hookrightarrow \mathrm{Mat}_n^r(\mathbb{C})$ is Zariski closed.*

Proof. This proof is due to Z. Reichstein. There are countably many monomials $\{p_i\}_{i=1}^{\infty}$ in r non-commuting variables. An r-tuple $\bar{A} = (A_1, \ldots, A_r) \in \mathrm{Mat}_n^r(\mathbb{C})$ does not generate the matrix algebra $\mathrm{Mat}_n(\mathbb{C})$ if and only if the matrices $p_i(A_1, \ldots, A_r)$ do not span $\mathrm{Mat}_n(\mathbb{C})$ as a vector space. That is,

$$\dim(\mathrm{span}(p_1(\bar{A}), p_2(\bar{A}), \ldots, p_m(\bar{A}))) \leq n^2 - 1 \tag{3.1}$$

for every $m \geq 1$. This is a Zariski closed condition in $\mathrm{Mat}_n^r(\mathbb{C})$ for each m. Indeed, (3.1) is equivalent to the $n^2 \times m$ matrix

$$\begin{pmatrix} p_1(\bar{A}) & \cdots & p_m(\bar{A}) \end{pmatrix}$$

having rank $\leq n^2 - 1$, which is a Zariski closed condition in $\mathrm{Mat}_{n^2 \times m}(\mathbb{C})$, and the map

$$\begin{aligned} \mathrm{Mat}_n^r(\mathbb{C}) &\to \mathrm{Mat}_{n^2 \times m}(\mathbb{C}) \\ \bar{A} &\mapsto \begin{pmatrix} p_1(\bar{A}) & \cdots & p_m(\bar{A}) \end{pmatrix} \end{aligned}$$

is clearly regular. If Z_m denotes the set of $\bar{A} \in \mathrm{Mat}_n^r(\mathbb{C})$ satisfying (3.1), then

$$Z_n^r(\mathbb{C}) = \bigcap_{m=1}^{\infty} Z_m \,.$$

$\square$

As a consequence of Proposition 3.5, the subspace $U_n^r(\mathbb{C})$ is an open subvariety of $\mathrm{Mat}_n^r(\mathbb{C})$ and is, in particular, smooth. Our next aim is to show that the variety $U_n^r(\mathbb{C})$ is highly-connected (in a sense made precise by Corollary 3.9). The proof is just an application of Proposition 2.19, so we need to compute the dimension of $Z_n^r(\mathbb{C})$.

Lemma 3.6. *The subspace*

$$\Sigma_m = \{(A_1, \ldots, A_r, L) \in \mathrm{Mat}_n^r(\mathbb{C}) \times \mathrm{Gr}_{m,n}(\mathbb{C}) : A_i L \subseteq L \textit{ for each } i\}$$

is Zariski closed in $\mathrm{Mat}_n^r(\mathbb{C}) \times \mathrm{Gr}_{m,n}(\mathbb{C})$ *for* $m = 1, \ldots, n-1$.

Proof. Let $S \to \mathrm{Gr}_{m,n}(\mathbb{C})$ be the tautological bundle over the Grassmannian of m-planes in $\mathbb{C}^n$ (with the Zariski topology) and $T' \to \mathrm{Gr}_{m,n}(\mathbb{C})$ be the trivial bundle of complex rank n. Consider the short exact sequence $0 \to S \to T' \to Q \to 0$ of vector bundles over $\mathrm{Gr}_{m,n}(\mathbb{C})$.

If T is the trivial complex rank-n bundle over $\mathrm{Mat}_n^r(\mathbb{C}) \times \mathrm{Gr}_{m,n}(\mathbb{C})$, then for each $i = 1, \ldots, r$ we have a vector bundle map

$$\begin{aligned} \psi_i \colon T &\to T \\ ((A_1, \ldots, A_r), L, v) &\mapsto ((A_1, \ldots, A_r), L, A_i v). \end{aligned}$$

Let $\pi_2 \colon \mathrm{Mat}_n^r(\mathbb{C}) \times \mathrm{Gr}_{m,n}(\mathbb{C}) \to \mathrm{Gr}_{m,n}(\mathbb{C})$ be the projection onto the second factor, and consider the sequence of vector bundle maps

$$\Psi \colon (\pi_2^* S)^{\oplus r} \longrightarrow T^{\oplus r} \xrightarrow{(\psi_1, \ldots, \psi_r)} T^{\oplus r} \longrightarrow (\pi_2^* Q)^{\oplus r}$$

over $\mathrm{Mat}_n^r(\mathbb{C}) \times \mathrm{Gr}_{m,n}(\mathbb{C})$. Note that $T \cong \pi_2^* T'$ as vector bundles; the first and last maps of the composite Ψ are induced by pullback. Then $(\bar{A}, L) \in \Sigma_m$ if and only if $\Psi_{(\bar{A}, L)}$, the restriction of Ψ to the fibre above $(\bar{A}, L)$, is a rank-0 linear map. Since the map $\mathrm{rank}_\Psi \colon \mathrm{Mat}_n^r(\mathbb{C}) \times \mathrm{Gr}_{m,n}(\mathbb{C}) \to \mathbb{Z}$ given by $(\bar{A}, L) \mapsto \mathrm{rank}\, \Psi_{(\bar{A}, L)}$ is lower semicontinuous and $\mathrm{Mat}_n^r(\mathbb{C}) \times \mathrm{Gr}_{m,n}(\mathbb{C})$ is endowed with the Zariski topology, the subspace Σ_m is Zariski closed in $\mathrm{Mat}_n^r(\mathbb{C}) \times \mathrm{Gr}_{m,n}(\mathbb{C})$. $\square$

Lemma 3.7. *The variety* Σ_m *is irreducible*

Proof. Let $\overline{\pi}_1$ and $\overline{\pi}_2$ denote the restrictions of $\pi_1 \colon \mathrm{Mat}_n^r(\mathbb{C}) \times \mathrm{Gr}_{m,n}(\mathbb{C}) \to \mathrm{Mat}_n^r(\mathbb{C})$ and $\pi_2 \colon \mathrm{Mat}_n^r(\mathbb{C}) \times \mathrm{Gr}_{m,n}(\mathbb{C}) \to \mathrm{Gr}_{m,n}(\mathbb{C})$ to Σ_m respectively. Let $X_m = \pi_1(\Sigma_m)$: the subvariety of $Z_n^r(\mathbb{C})$ consisting of those r-tuples that have an invariant m-dimensional linear subspace.

We have a commuting diagram

$$\begin{tikzcd}
\Sigma_m \arrow[dr, hook] \arrow[drr, bend left, "\overline{\pi}_2"] \arrow[dd, "\overline{\pi}_1"'] & & \\
& \mathrm{Mat}_n^r(\mathbb{C}) \times \mathrm{Gr}_{m,n}(\mathbb{C}) \arrow[r, "\pi_2"] \arrow[dd, "\pi_1"] & \mathrm{Gr}_{m,n}(\mathbb{C}) \\
X_m \arrow[dr, hook] & & \\
& \mathrm{Mat}_n^r(\mathbb{C}) &
\end{tikzcd}$$

The group $\mathrm{PGL}_n(\mathbb{C})$ acts on $\mathrm{Mat}_n(\mathbb{C})$ (on the left) by conjugation, say $g \cdot A = gAg^{-1}$ for definiteness, and on $\mathrm{Gr}_{m,n}(\mathbb{C})$ in the usual way. Endow $\mathrm{Mat}_n^r(\mathbb{C}) \times \mathrm{Gr}_{m,n}(\mathbb{C})$ with the diagonal action. The variety Σ_m is invariant under this action: if a linear subspace L is invariant under a matrix A, then gL is invariant under gAg^{-1} for any $g \in \mathrm{PGL}_n(\mathbb{C})$. Observe that the map $\overline{\pi}_2$ is $\mathrm{PGL}_n(\mathbb{C})$-equivariant with respect to these actions.

Now, consider the space of $n \times n$ matrices that have $L_0 = \langle e_1, \ldots, e_m \rangle$ as an invariant linear subspace. Thus is readily seen to be isomorphic to $\mathbb{A}_{\mathbb{C}}^{n^2 - m(n-m)}$. Hence the fibre $F = \overline{\pi}_2^{-1}(L_0)$ is isomorphic to $\mathbb{A}_{\mathbb{C}}^{r(n^2 - m(n-m))}$. Since the action is transitive on $\mathrm{Gr}_{m,n}(\mathbb{C})$, the map $\overline{\pi}_2$ is surjective and the fibres are all isomorphic. Moreover, the restriction of the action map $F \times \mathrm{PGL}_n(\mathbb{C}) \to \Sigma_m$ is surjective. Since the source of this map is an irreducible variety, Σ_m is irreducible. $\square$

Proposition 3.8. *The dimension of $Z_n^r(\mathbb{C})$ is $rn^2 - (r-1)(n-1)$.*

Proof. First we compute the dimension of Σ_m. There is a Zariski dense open $U \subseteq \mathrm{Gr}_{m,n}(\mathbb{C})$ over which $\overline{\pi}_2$ is flat. Hence the dimension of $\overline{\pi}_2^{-1} U$ is

$$\begin{aligned}
\dim F + \dim \mathrm{Gr}_{m,n}(\mathbb{C}) &= r(n^2 - m(n-m)) + m(n-m) \\
&= rn^2 - m(r-1)(n-m).
\end{aligned}$$

Since Σ_m is irreducible, the Zariski closure of $\overline{\pi}_2^{-1} U$ is Σ_m so that

$$\dim \Sigma_m = rn^2 - m(r-1)(n-m).$$

Next we compute the dimension of the irreducible component $X_m = \pi_1(\Sigma_m)$ of $Z_n^r(\mathbb{C})$. To see that X_m is in fact closed in $Z_n^r(\mathbb{C})$, note that since $\mathrm{Gr}_{m,n}(\mathbb{C})$ is compact, the projection

π_1 is a closed map. We also have

$$Z_n^r(\mathbb{C}) = \bigcup_{m=1}^{n-1} X_m$$

as a consequence of Burnside's theorem (this is another proof that $Z_n^r(\mathbb{C}) \hookrightarrow \mathrm{Mat}_n^r(\mathbb{C})$ is Zariski closed).

Let $V_1 \subseteq \mathrm{Mat}_n^r(\mathbb{C})$ be the Zariski open subset consisting of r-tuples $(A_1, \ldots, A_r)$ such that the matrix A_1 has distinct eigenvalues. Note that V_1 is in fact Zariski open since an r-tuple is in V_1 if and only if the discriminant of the characteristic polynomial of A_1 is nonzero. The variety V_1 meets X_m: any r-tuple of diagonal matrices such that the first matrix has distinct eigenvalues has an invariant m-dimensional linear subspace. So $X_m \cap V_1$ is Zariski dense in X_m. Since a matrix with distinct eigenvalues has finitely many invariant linear subspaces, the restriction of $\overline{\pi}_1$ to $\overline{\pi}_1^{-1}(X_m \cap V_1)$ has finite fibres. It follows from Grothendieck's version of Zariski's Main Theorem [Mum88, p. 289] that the varieties $\overline{\pi}_1^{-1}(X_m \cap V_1)$ and $X_m \cap V_1$ have the same dimension. These two varieties are Zariski dense and open in Σ_m and X_m respectively, so Σ_m and X_m have the same dimension. We see that the largest-dimensional irreducible components of $Z_n^r(\mathbb{C})$ are X_1 and X_{n-1}, each of dimension $rn^2 - (r-1)(n-1)$. □

Corollary 3.9. *The variety $U_n^r(\mathbb{C})$ is $(2(r-1)(n-1)-2)$-connected.*

Proof. Apply Proposition 2.19 to the codimension-$(r-1)(n-1)$ inclusion

$$Z_n^r(\mathbb{C}) \hookrightarrow \mathrm{Mat}_n^r(\mathbb{C}) \cong \mathbb{A}_{\mathbb{C}}^{rn^2}.$$

□

We see in particular that the connectedness of $U_n^r(\mathbb{C})$ increases with r. As previously mentioned, the group $\mathrm{PGL}_n(\mathbb{C})$ acts on $\mathrm{Mat}_n^r(\mathbb{C})$ by simultaneous conjugation:

$$g \cdot (A_1, \ldots, A_r) = (gA_1g^{-1}, \ldots, gA_rg^{-1}).$$

The open subvariety $U_n^r(\mathbb{C}) \hookrightarrow \mathrm{Mat}_n^r(\mathbb{C})$ is invariant under this action. This follows from the observation that if an r-tuple of linear endomorphisms has no invariant subspace in the standard basis, then that r-tuple has no invariant subspace in any basis representation. Moreover, the $\mathrm{PGL}_n(\mathbb{C})$-action on $U_n^r(\mathbb{C})$ is free. Indeed, if $g \in \mathrm{PGL}_n(\mathbb{C})$ fixes some $(A_1, \ldots, A_r) \in U_n^r(\mathbb{C})$, then conjugation by g fixes any polynomial in the A_is. Since the A_is are generating, g must be the identity element.

Let $B_n^r(\mathbb{C})$ denote the orbit space $U_n^r(\mathbb{C})/\mathrm{PGL}_n(\mathbb{C})$ endowed with the quotient topology. We wish to show that the quotient map $U_n^r(\mathbb{C}) \to B_n^r(\mathbb{C})$ is a principal $\mathrm{PGL}_n(\mathbb{C})$-bundle. In light of Corollary 2.4, all we have left to show is that the action is proper.

Proposition 3.10. *The action of* $\mathrm{PGL}_n(\mathbb{C})$ *on* $U_n^r(\mathbb{C})$ *is proper.*

Proof. It suffices to show that, for any compact $K \subseteq U_n^r(\mathbb{C})$, the set

$$\mathrm{PGL}_n(\mathbb{C})_K = \{g \in \mathrm{PGL}_n(\mathbb{C}) : K \cap g \cdot K \neq \varnothing\}$$

is compact [Lee13, Proposition 21.5].

Claim: $\mathrm{PGL}_n(\mathbb{C})_K \subseteq \mathrm{PGL}_n(\mathbb{C})$ *is closed.* Let (g_m) be a sequence in $\mathrm{PGL}_n(\mathbb{C})_K$ converging to $g' \in \mathrm{PGL}_n(\mathbb{C})$. For each m, there is some $x_m \in K$ such that $g_m \cdot x_m \in K$. Since K is compact, some subsequence (x_{m_h}) converges to $x' \in K$. Consider the (continuous) map

$$\begin{aligned} \Theta\colon \mathrm{PGL}_n(\mathbb{C}) \times U_n^r(\mathbb{C}) &\to U_n^r(\mathbb{C}) \times U_n^r(\mathbb{C}) \\ (g, x) &\mapsto (g \cdot x, x) \end{aligned}$$

The sequence (g_{m_h}, x_{m_h}) converges to (g', x') while the sequence $\Theta(g_{m_h}, x_{m_h}) = (g_{m_h} \cdot x_{m_h}, x_{m_h})$ is in the compact set $K \times K$ and converges to $\Theta(g', x') = (g' \cdot x', x') \in K \times K$. In particular, $g' \cdot x' \in K \cap g' \cdot K$. This proves the claim.

For each $\bar{A} = (A_1, \ldots, A_r) \in K$, we construct a function $f_{\bar{A}}$ as follows. Let E_{ij} be the $n \times n$ matrix with 1 in the (i,j)th entry and 0 elsewhere. Since $\bar{A}$ is generating, there are polynomials $p_{ij}^{\bar{A}}$ such that $E_{ij} = p_{ij}^{\bar{A}}(A_1, \ldots, A_r)$. Put

$$f_{\bar{A}} = (p_{11}^{\bar{A}}, p_{12}^{\bar{A}}, \ldots, p_{nn}^{\bar{A}})\colon U_n^r(\mathbb{C}) \to \mathrm{Mat}_n(\mathbb{C})^{n^2} \approx \mathrm{Mat}_{n^2}(\mathbb{C}).$$

The identification $\mathrm{Mat}_n(\mathbb{C})^{n^2} \approx \mathrm{Mat}_{n^2}(\mathbb{C})$ here is such that the ith coordinate of $\mathrm{Mat}_n(\mathbb{C})^{n^2}$ corresponds to the ith column of the matrices in $\mathrm{Mat}_{n^2}(\mathbb{C})$. If $V_{n^2,n^2}(\mathbb{C})$ denotes the Stiefel manifold of n^2-frames in $\mathbb{C}^{n^2}$ (this is simply $\mathrm{GL}_{n^2}(\mathbb{C})$), then $f_{\bar{A}}(\bar{A}) \in V_{n^2,n^2}(\mathbb{C})$, as the matrices E_{ij} form a basis for the vector space $\mathrm{Mat}_n(\mathbb{C})$. Since $f_{\bar{A}}$ is continuous and $V_{n^2,n^2}(\mathbb{C}) \subseteq \mathrm{Mat}_n(\mathbb{C})^{n^2}$ is open, there is a neighborhood $W_{\bar{A}}$ of $\bar{A}$ such that $f_{\bar{A}}(W_{\bar{A}}) \subseteq V_{n^2,n^2}(\mathbb{C})$. Finitely many open sets $W_{\bar{A}_1}, \ldots, W_{\bar{A}_l}$ cover K. Write $W_i = W_{\bar{A}_i}$ and $f_i = f_{\bar{A}_i}$.

Consider the function

$$f = (f_1, \ldots, f_l)\colon U_n^r(\mathbb{C}) \to \mathrm{Mat}_{n^2}(\mathbb{C})^l \approx \mathrm{Mat}_{n^2 \times ln^2}(\mathbb{C}).$$

For every $\bar{A} \in K$, there is some $i \in \{1, \ldots, l\}$ such that $f_i(\bar{A}) \in V_{n^2,n^2}(\mathbb{C})$. Hence $f(\bar{A})$ is full-rank. This is to say that the image of K under f lies in the Stiefel manifold $V_{n^2,ln^2}(\mathbb{C})$. Note that the group $\mathrm{GL}(\mathrm{Mat}_n(\mathbb{C})) \cong \mathrm{GL}_{n^2}(\mathbb{C})$ acts on $V_{n^2,ln^2}(\mathbb{C})$ properly, the quotient being the Grassmannian $\mathrm{Gr}_{n^2,ln^2}(\mathbb{C})$.

It is not difficult to see that the set of invertible linear maps $T\colon \mathrm{Mat}_n(\mathbb{C}) \to \mathrm{Mat}_n(\mathbb{C})$ satisfying $T(AB) = T(A)T(B)$ for every $A, B \in \mathrm{Mat}_n(\mathbb{C})$ is closed in $\mathrm{GL}(\mathrm{Mat}_n(\mathbb{C}))$. Phrased differently, this says that the action of $\mathrm{PGL}_n(\mathbb{C})$ on $\mathrm{Mat}_n(\mathbb{C})$ by conjugation gives rise to a closed embedding $\rho\colon \mathrm{PGL}_n(\mathbb{C}) \hookrightarrow \mathrm{GL}(\mathrm{Mat}_n(\mathbb{C}))$.

Next, consider the sets

$$\begin{aligned}\mathrm{PGL}_n(\mathbb{C})_{f(K)} &= \{g \in \mathrm{PGL}_n(\mathbb{C}) : f(K) \cap \rho(g)\cdot f(K) \neq \varnothing\},\\ \mathrm{GL}(\mathrm{Mat}_n(\mathbb{C}))_{f(K)} &= \{g \in \mathrm{GL}(\mathrm{Mat}_n(\mathbb{C})) : f(K) \cap g\cdot f(K) \neq \varnothing\}.\end{aligned}$$

One can check that

$$f(g \cdot \bar{A}) = \rho(g) \cdot f(\bar{A})$$

for any $\bar{A} \in K$ and $g \in \mathrm{PGL}_n(\mathbb{C})$. This follows from the fact that the polynomials $p_{ij}^{\bar{A}_i}$ that make up the components of f satisfy a similar equation. The inclusion

$$\mathrm{PGL}_n(\mathbb{C})_K \subseteq \mathrm{PGL}_n(\mathbb{C})_{f(K)}$$

follows. We also have

$$\mathrm{PGL}_n(\mathbb{C})_{f(K)} = \mathrm{GL}(\mathrm{Mat}_n(\mathbb{C}))_{f(K)} \cap \mathrm{PGL}_n(\mathbb{C}).$$

From the claim and the fact that $\mathrm{PGL}_n(\mathbb{C}) \subseteq \mathrm{GL}(\mathrm{Mat}_n(\mathbb{C}))$ is closed, the subset

$$\mathrm{PGL}_n(\mathbb{C})_K \subseteq \mathrm{GL}(\mathrm{Mat}_n(\mathbb{C}))_{f(K)}$$

is closed. Since the action of $\mathrm{GL}(\mathrm{Mat}_n(\mathbb{C}))$ on $V_{n^2,ln^2}(\mathbb{C})$ is proper, the set $\mathrm{GL}(\mathrm{Mat}_n(\mathbb{C}))_{f(K)}$ is compact. □

Corollary 3.11. *The map $U_n^r(\mathbb{C}) \to B_n^r(\mathbb{C})$ is a principal $\mathrm{PGL}_n(\mathbb{C})$-bundle, and $B_n^r(\mathbb{C})$ is a manifold of real dimension $2n^2(r-1)+2$.*

Proof. Apply Corollary 2.4. □

Proposition 3.12. *There is a $(2(r-1)(n-1)-1)$-equivalence $B_n^r(\mathbb{C}) \to B\mathrm{PGL}_n(\mathbb{C})$.*

Proof. Consider the long exact sequence in homotopy groups associated to the delooping $U_n^r(\mathbb{C}) \to B_n^r(\mathbb{C}) \to B\mathrm{PGL}_n(\mathbb{C})$. Note that each of the spaces appearing in this sequence are path-connected ($r > 1$ and $n > 1$). Since $\pi_i(U_n^r(\mathbb{C})) = 0$ for $i \leq 2(r-1)(n-1) - 2$, the result follows. $\square$

Chapter 4

The Cohomology of $U_2^r(\mathbb{C})$

The goal of this chapter is to compute the singular cohomology groups $\mathrm{H}^*(U_2^r(\mathbb{C});\mathbb{Z})$ in the range $* \leq 4r-7$. The computation uses many of the tools discussed in Chapter 2.

Notation 4.1. Let $U(r) = U_2^r(\mathbb{C})$ and $Z(r) = Z_2^r(\mathbb{C}) = \mathrm{Mat}_2^r(\mathbb{C}) \setminus U_2^r(\mathbb{C})$.

We begin with a few definitions:

- $T(r)$—the set of r-tuples $(A_1, \ldots, A_r) \in \mathrm{Mat}_2^r(\mathbb{C})$ such that the A_is pairwise commute: $[A_i, A_j] = 0$ for each $i, j \in \{1, \ldots, r\}$.
- $M^o(r) = \mathrm{Mat}_2^r(\mathbb{C}) \setminus T(r)$.
- $W(r) = Z(r) \setminus T(r)$, those r-tuples $(A_1, \ldots, A_r) \in Z(r)$ that do not pairwise commute: there are some $i, j \in \{1, \ldots, r\}$ such that $[A_i, A_j] \neq 0$.
- $K(r)$—the set of r-tuples $(A_1, \ldots, A_r) \in \mathrm{Mat}_2^r(\mathbb{C})$ such that $\bigcap_{i=1}^r \ker A_i$ is 1-dimensional.

Recall that $Z(r)$ is a closed subvariety of $\mathrm{Mat}_2^r(\mathbb{C})$. We have $T(r) \subseteq Z(r)$ since the algebra generated by a pairwise-commuting r-tuple is commutative, a proper subalgebra of $\mathrm{Mat}_2(\mathbb{C})$. Also, $T(r)$ is a closed subvariety of $\mathrm{Mat}_2^r(\mathbb{C})$; the condition that the matrices in an r-tuple commute is Zariski closed. Then $W(r)$ is also a variety since it is (Zariski) locally closed in $\mathrm{Mat}_2^r(\mathbb{C})$.

There is another characterization of $W(r)$. An r-tuple $(A_1, \ldots, A_r)$ lies in $W(r)$ if and only if the A_is share a unique common 1-dimensional eigenspace and do not pairwise commute. To see this, note that any r-tuple in $W(r)$ must have a common eigenvector as a consequence of Burnside's Theorem. If an r-tuple has two common 1-dimensional eigenspaces, then the matrices are simultaneously diagonalizable, and so the r-tuple lies in $T(r)$.

Proposition 4.2. *The space $K(r)$ is a quasi-affine $\mathbb{C}$-variety.*

Proof. Let $U = \mathrm{Mat}_2^r(\mathbb{C}) \setminus \{\bar{0}\}$. Consider the incidence variety

$$\Sigma = \{((A_1, \dots, A_r), L) \in U \times \mathbb{P}^1_{\mathbb{C}} : A_i L = 0 \text{ for each } i\}.$$

As one can check, the condition $A_i L = 0$ is Zariski closed, so Σ is a closed subvariety of $U \times \mathbb{P}^1_{\mathbb{C}}$. Since $\mathbb{P}^1_{\mathbb{C}}$ is compact, the projection $U \times \mathbb{P}^1_{\mathbb{C}} \to U$ is a closed map of varieties. The image of Σ under this projection is $K(r)$. Hence $K(r)$ is Zariski closed in U. $\square$

Proposition 4.3. *There is a regular map $p\colon W(r) \to \mathbb{P}^1_{\mathbb{C}}$ which sends an r-tuple $(A_1, \dots, A_r)$ to its unique invariant line.*

Proof. We exhibit p as the composition of two regular maps $W(r) \xrightarrow{f} K(\binom{r}{2}) \xrightarrow{g} \mathbb{P}^1_{\mathbb{C}}$. First, consider the map $f\colon W(r) \to \mathrm{Mat}_n^{\binom{r}{2}}(\mathbb{C})$ given by

$$(A_1, \dots, A_r) \mapsto ([A_1, A_2], [A_1, A_3], \dots, [A_{r-1}, A_r]).$$

We claim that f factors as $W(r) \to K(\binom{r}{2}) \hookrightarrow \mathrm{Mat}_n^{\binom{r}{2}}(\mathbb{C})$. If $(A_1, \dots, A_r) \in W(r)$ and L is the common 1-dimensional eigenspace to the A_is, then $L \subseteq \ker[A_i, A_j]$ for each $i < j$. And for some $i < j$, the commutator $[A_i, A_j]$ is nonzero, so $\ker[A_i, A_j]$ is 1-dimensional. Hence,

$$L = \bigcap_{i<j} \ker[A_i, A_j].$$

The map f is clearly regular.

Next, consider the map $g\colon K(r) \to \mathbb{P}^1_{\mathbb{C}}$ which sends an r-tuple $(B_1, \dots, B_r)$ to the line $\bigcap_{i=1}^r \ker B_i$. We need to show that g is regular. Let $U_i = \{[z_0 : z_1] \in \mathbb{P}^1_{\mathbb{C}} : z_i = 1\}$ be the standard open cover of $\mathbb{P}^1_{\mathbb{C}}$, and suppose coordinates for $K(r)$ are given by

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \dots, \begin{pmatrix} a_r & b_r \\ c_r & d_r \end{pmatrix}.$$

An r-tuple $\bar{B} = (B_1, \dots, B_r)$ lies in $g^{-1}U_0$ if and only if $e_2 \notin \bigcap_{i=1}^r \ker B_i$, which is to say that $b_i \neq 0$ for some i or $d_i \neq 0$ for some i. Note that $g^{-1}U_0$ is Zariski open. On the set $D(b_i) = \{\bar{B} \in K(r) : b_i \neq 0\}$, define p by $\bar{B} \mapsto [1 : -a_i/b_i]$, and on $D(d_i)$, define p by $\bar{B} \mapsto [1 : -c_i/d_i]$. On $D(b_i)$, the matrix B_i is nonzero, and $[1 : -a_i/b_i]$ is the unique line in $\ker B_i$. Similarly for $D(d_i)$. So the maps are well defined and agree on intersections since they each pick out the unique line $\bigcap_{i=1}^r \ker B_i$. This defines g as a regular map on $g^{-1}U_0$. Similarly, define g on $D(a_i)$ by $\bar{B} \mapsto [-b_i/a_i : 1]$, and on $D(c_i)$ by $\bar{B} \mapsto [-d_i/c_i : 1]$. This defines g as a regular map on $g^{-1}U_1$, and these definitions agree on $g^{-1}U_0 \cap g^{-1}U_1$. $\square$

Proposition 4.4. *There is a fibre bundle $F \to W(r) \xrightarrow{p} \mathbb{P}^1_{\mathbb{C}}$ in the category of $\mathbb{C}$-varieties,* *where F is isomorphic to the variety of upper triangular matrices that do not pairwise commute.*

Proof. The map p is $\mathrm{PGL}_2(\mathbb{C})$-equivariant, where $\mathrm{PGL}_2(\mathbb{C})$ acts on $\mathbb{P}^1_{\mathbb{C}}$ in the usual way. Since the action is transitive on $\mathbb{P}^1_{\mathbb{C}}$, the fibres are all isomorphic to the fibre $F = p^{-1}([1:0])$: those r-tuples $(A_1, \dots, A_r) \in W(r)$ where each A_i is upper triangular. Note that F is Zariski open in the variety of r-tuples of upper triangular matrices.

We construct an isomorphism of varieties $p^{-1}U_0 \to F \times U_0$ over U_0. There is a map of varieties $h\colon U_0 \to \mathrm{PGL}_2(\mathbb{C})$ given by

$$[1:x] \mapsto \begin{bmatrix} 1 & 0 \\ -x & 1 \end{bmatrix}.$$

Let ϕ be the composition

$$p^{-1}U_0 \xrightarrow{id \times p} p^{-1}U_0 \times U_0 \xrightarrow{id \times h} p^{-1}U_0 \times \mathrm{PGL}_2(\mathbb{C}) \xrightarrow{\quad a \quad} F$$

where a is the action map $((A_1, \dots, A_r), g) \mapsto (gA_1g^{-1}, \dots, gA_rg^{-1})$. One can check that an r-tuple in the image of ϕ has e_1 as an eigenvector. Then

$$\phi \times p\colon p^{-1}U_0 \to F \times U_0$$

gives the desired isomorphism. If $h' = ih$, where $i\colon \mathrm{PGL}_2(\mathbb{C}) \to \mathrm{PGL}_2(\mathbb{C})$ is the inversion map, then

$$F \times U_0 \xrightarrow{id \times h'} F \times \mathrm{PGL}_2(\mathbb{C}) \xrightarrow{\quad a \quad} p^{-1}U_0$$

provides an inverse for $\phi \times p$. The proof that $p^{-1}U_1 \xrightarrow{\approx} F \times U_1$ over U_1 is similar. $\square$

Proposition 4.5. *The variety $W(r)$ is smooth.*

Proof. The fibre F is smooth since it is an open subvariety of the variety of r-tuples of upper triangular matrices, which is isomorphic to $\mathbb{A}^{3r}_{\mathbb{C}}$. The fibre bundle constructed above shows that $W(r)$ admits an open covering by two smooth varieties: $p^{-1}U_0$ and $p^{-1}U_1$, each isomorphic to $F \times \mathbb{A}^1_{\mathbb{C}}$. $\square$

An outline of the computation of $\mathrm{H}^*(U(r); \mathbb{Z})$ in the range $* \leq 4r - 7$ is as follows. After discarding the high-codimension locus $T(r)$ from $\mathrm{Mat}^r_2(\mathbb{C})$, we are left with the highly-connected variety $M^o(r)$. By means of the Leray–Serre spectral sequence associated to the fibre bundle $F \to W(r) \to \mathbb{C}P^1$, we compute $\mathrm{H}^*(W(r); \mathbb{Z})$ in a range. The Gysin sequence associated to the inclusion $W(r) \hookrightarrow M^o(r)$ then relates the cohomology of $W(r)$, $M^o(r)$, and $M^o(r) \setminus W(r) = U(r)$.

Lemma 4.6. *The dimension of $T(r)$ is $2r+2$.*

Proof. Let Y_i be the open subvariety of $T(r)$ consisting of r-tuples $(A_1, \dots, A_r) \in T(r)$ such that the minimal and characteristic polynomials of A_i coincide; i.e., A_i is nonscalar. We claim that if $(A_1, \dots, A_r) \in Y_i$, then A_j commutes with A_i if and only if A_j can be written as a polynomial in A_i of degree at most 1. To see this, consider the ring $R = \mathbb{C}[t]/m_{A_i}(t)$ where $m_{A_i}(t)$ is the (degree 2) minimal polynomial of A_i. The vector space $\mathbb{C}^2$ has R-module structure, where $p(t) \in R$ acts by $v \mapsto p(A_i)v$. Since A_j commutes with A_i, the matrix A_j defines an R-module endomorphism of $\mathbb{C}^2$. And since the minimal polynomial and the characteristic polynomial of A_i coincide, $\mathbb{C}^2$ is a cyclic R-module. Hence any R-module endomorphism of $\mathbb{C}^2$ is given by multiplication by some $r \in R$, which is to say that A_j is a polynomial in A_i of degree at most 1.

Let $S \subseteq \mathrm{Mat}_2(\mathbb{C})$ be the closed subvariety of scalar matrices. There is a map of varieties $(\mathrm{Mat}_2(\mathbb{C}) \setminus S) \times (\mathbb{A}^2_{\mathbb{C}})^{r-1} \to Y_i$ given by

$$(A_i, (a_1, b_1), \dots, \widehat{(a_i, b_i)}, \dots, (a_r, b_r)) \mapsto (a_1 A_i + b_1 I_2, \dots, A_i, \dots, a_r A_i + b_r I_2)$$

where the "hat" indicates an omitted entry and I_2 is the identity matrix. The discussion in the previous paragraph makes it clear that this map is a bijection. The source of this map, being an open subvariety of affine space, is irreducible, so Y_i is also irreducible. As a consequence of generic flatness, we have

$$\dim Y_i = \dim(\mathrm{Mat}_2(\mathbb{C}) \setminus S) \times (\mathbb{A}^2_{\mathbb{C}})^{r-1} = 4 + 2(r-1) = 2r+2.$$

Hence the Zariski closure $\overline{Y}_i \subseteq T(r)$ has dimension $2r+2$. The complement of $\bigcup_{i=1}^r Y_i$ in $T(r)$ is the irreducible component of $T(r)$ consisting of r-tuples of scalar matrices. This component has dimension r. We have exhausted the irreducible components of $T(r)$; the result follows. □

Lemma 4.7. *The variety $M^o(r)$ is $(4r-6)$-connected.*

Proof. Apply Proposition 2.19 to the inclusion $T(r) \hookrightarrow \mathrm{Mat}_2^r(\mathbb{C})$. □

Proposition 4.8. *For $r \geq 3$, the cohomology groups $\mathrm{H}^i(W(r); \mathbb{Z})$, in the range $i \leq 2r-4$, are given by*

$$\mathrm{H}^i(W(r); \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & i = 0, 2 \\ 0 & \textit{otherwise (with } i \leq 2r-4\textit{)} \end{cases}.$$

Proof. Identify the variety of upper triangular matrices with $\mathbb{A}^{3r}_{\mathbb{C}}$. By a similar argument as in the proof of Lemma 4.6, the dimension of $\mathbb{A}^{3r}_{\mathbb{C}} \setminus F$ is $3 + 2(r-1) = 2r+1$. By Proposition 2.19, the variety F is $(2r-4)$-connected so that $\widetilde{H}^i(F; \mathbb{Z}) = 0$ for $i \leq 2r-4$.

A portion of the E_2 page of the Leray–Serre spectral sequence associated to the fibration $F \to W(r) \to \mathbb{C}P^1$ is

$$
\begin{array}{c|ccccc}
2r-4 & 0 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
1 & 0 & 0 & 0 & 0 & \cdots \\
0 & \mathbb{Z} & 0 & \mathbb{Z} & 0 & \cdots \\
\hline
 & 0 & 1 & 2 & 3 & \cdots
\end{array}
$$

(here we are using the $r \geq 3$ assumption). The differentials d_k on succeeding pages with source or target in the range $p + q \leq 2r - 4$ are all zero, so $E_2^{p,q} \cong E_\infty^{p,q}$ in this range. This yields the result. $\square$

Proposition 4.9. *For $r \geq 3$, the cohomology groups $\mathrm{H}^i(U(r); \mathbb{Z})$, in the range $i \leq 4r - 7$, are given by*

$$
\mathrm{H}^i(U(r); \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & i = 0, 2r-3, 2r-1 \\ 0 & \textit{otherwise (with } i \leq 4r-7\textit{)} \end{cases}.
$$

Proof. As a consequence of Lemma 4.7, $\widetilde{H}^i(M^o(r); \mathbb{Z}) = 0$ for $i \leq 4r - 6$. Note also that $M^o(r)$, being an open subvariety of $\mathrm{Mat}_2^r(\mathbb{C})$, is smooth and has dimension $4r$. The variety F is an open subvariety of $\mathbb{A}_{\mathbb{C}}^{3r}$ and so has dimension $3r$. From the fibre bundle $F \to W(r) \to \mathbb{P}_{\mathbb{C}}^1$, we see that the dimension of $W(r)$ is $3r + 1$. Hence $W(r)$ has real codimension $2r - 2$ in $M^o(r)$. Since $W(r) \hookrightarrow M^o(r)$ is a closed inclusion of smooth $\mathbb{C}$-varieties, there is a Gysin sequence

$$
\begin{tikzcd}
\cdots \arrow[r] & \mathrm{H}^{k-(2r-2)}(W(r)) \arrow[r] & \mathrm{H}^k(M^o(r)) \arrow[r] & \mathrm{H}^k(M^o(r) \setminus W(r)) \arrow[dll, rounded corners] \\
 & \mathrm{H}^{k+1-(2r-2)}(W(r)) \arrow[r] & \mathrm{H}^{k+1}(M^o(r)) \arrow[r] & \cdots
\end{tikzcd}
$$

Noting that $M^o(r) \setminus W(r) = U(r)$, the result follows. $\square$

Chapter 5

The Cohomology of $B_2^r(\mathbb{C})$

For $r \geq 3$, we compute the singular cohomology groups $\mathrm{H}^i(B_2^r(\mathbb{C}); \mathbb{Q})$ in the range $i \leq 4r-7$ via the Leray–Serre spectral sequence associated to a fibration. The fibration in question is $U_2^r(\mathbb{C}) \to B_2^r(\mathbb{C}) \to B\mathrm{PGL}_2(\mathbb{C})$, which is obtained by delooping the principal $\mathrm{PGL}_2(\mathbb{C})$-bundle $\mathrm{PGL}_2(\mathbb{C}) \to U_2^r(\mathbb{C}) \to B_2^r(\mathbb{C})$.

Notation 5.1. We let $B(r)$ denote $B_2^r(\mathbb{C})$. Throughout this chapter, all cohomology groups are taken with rational coefficients, and we suppress coefficients.

The computation breaks into two parts. Section 5.1 is concerned with the case when r is odd and Section 5.2 the case when r is even.

To begin, the inclusion $\mathrm{SO}(3) \approx \mathrm{PU}_2 \hookrightarrow \mathrm{PGL}_2(\mathbb{C})$ is a homotopy equivalence, so the induced map $B\mathrm{SO}(3) = B\mathrm{PU}_2 \to B\mathrm{PGL}_2(\mathbb{C})$ is a homotopy equivalence. Also,

$$\mathrm{H}^*(B\mathrm{SO}(3)) \cong \mathbb{Q}[p_1]$$

where $|p_1| = 4$. This follows from a result in [Bro82], where the cohomology groups of $B\mathrm{SO}(3)$ are computed with integer coefficients. Note also that $B\mathrm{PGL}_2(\mathbb{C})$ is simply-connected since $\mathrm{PGL}_2(\mathbb{C})$ is path-connected, so the system of local coefficients on $B\mathrm{PGL}_2(\mathbb{C})$ is simple. For the purpose of these computations, we will restrict our attention to terms $E_k^{p,q}$ with $p + q \leq 4r - 7$.

5.1 Case 1: r odd

Theorem 5.2. *When r is odd,*

$$\mathrm{H}^i(B(r)) \cong \begin{cases} \mathbb{Q} & i \leq 2r-6 \textit{ and } i \equiv 0 \pmod 4 \\ \mathbb{Q} & 2r-1 \leq i \leq 4r-7 \textit{ and } i \equiv 1 \pmod 4 \\ 0 & \textit{otherwise with } i \leq 4r-7 \end{cases}.$$

The terms on the second page of the Leray–Serre spectral sequence associated to $U(r) \to B(r) \to B\mathrm{PGL}_2(\mathbb{C})$ are given by

$$E_2^{p,q} = \mathrm{H}^p(B\mathrm{PGL}_2(\mathbb{C})) \otimes_{\mathbb{Q}} \mathrm{H}^q(U(r)).$$

A portion of the E_2-page is thus

$$\begin{tikzcd}
4r-7 & & & & & & \\
\vdots & & & & & & \\
2r-1 & \mathbb{Q} & \mathbb{Q} & \cdots & \mathbb{Q} & \mathbb{Q} & \\
2r-2 & & & & & & \\
2r-3 & \mathbb{Q} \arrow[ddrrr, dashed, "d_{2r-2}"] & \mathbb{Q} & \cdots & \mathbb{Q} & \mathbb{Q} & \cdots \\
\vdots & & & & & & \\
0 & \mathbb{Q} & \mathbb{Q} & \cdots & \mathbb{Q} & \mathbb{Q} & \\
 & 0 & 4 & \cdots & 2r-2 & 2r+2 &
\end{tikzcd}$$

where the empty entries are 0. The first differentials with nontrivial source and target in the range $p+q \leq 4r-7$ occur on the E_{2r-2}-page, so $E_2^{p,q} = E_{2r-2}^{p,q}$ in this range. We compute the differential $d_{2r-2} \colon E_{2r-2}^{0,2r-3} \to E_{2r-2}^{2r-2,0}$.

The First Comparison. Embed $S^1 \hookrightarrow \mathbb{C}$ as complex numbers of modulus 1. There is a closed inclusion of Lie groups $\rho \colon S^1 \hookrightarrow \mathrm{PGL}_2(\mathbb{C})$ given by $z \mapsto \begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix}$. This yields a free action of S^1 on $U(r)$, where $z \in S^1$ acts by

$$\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \ldots, \begin{pmatrix} a_r & b_r \\ c_r & d_r \end{pmatrix} \right) \mapsto \left(\begin{pmatrix} a_1 & z^* b_1 \\ z c_1 & d_1 \end{pmatrix}, \ldots, \begin{pmatrix} a_r & z^* b_r \\ z c_r & d_r \end{pmatrix} \right). \tag{5.1}$$

By Corollary 2.4, $S^1 \to U(r) \to U(r)/S^1$ is a principal S^1-bundle. The morphism of fibrations

$$\begin{tikzcd}
\mathrm{PGL}_2(\mathbb{C}) \arrow[r] & U(r) \arrow[r] & B(r) \\
S^1 \arrow[u, "\rho"] \arrow[r] & U(r) \arrow[u, equal] \arrow[r] & U(r)/S^1 \arrow[u]
\end{tikzcd}$$

induces a morphism of deloopings

$$\begin{tikzcd}
U(r) \arrow[r] & B(r) \arrow[r] & B\mathrm{PGL}_2(\mathbb{C}) \\
U(r) \arrow[u, equal] \arrow[r] & U(r)/S^1 \arrow[u] \arrow[r] & BS^1. \arrow[u, "B\rho"']
\end{tikzcd}$$

The Second Comparison. Let M be the subset of $\mathrm{Mat}_2^r(\mathbb{C})$ consisting of r-tuples where the last matrix is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Suppose

$$\bar{A} = \left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \dots, \begin{pmatrix} a_{r-1} & b_{r-1} \\ c_{r-1} & d_{r-1} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right)$$

is an r-tuple in M. The only possible 1-dimensional eigenspaces common to each of the matrices are $\langle e_1 \rangle$ and $\langle e_2 \rangle$, as these are the eigenspaces of the last matrix. Hence, $\bar{A} \in U(r) \cap M$ if and only if $\bar{c} = (c_1, \dots, c_{r-1}) \in \mathbb{C}^{r-1} \setminus \{\bar{0}\}$ and $\bar{b} = (b_1, \dots, b_{r-1}) \in \mathbb{C}^{r-1} \setminus \{\bar{0}\}$. From (5.1), it is clear that $U(r) \cap M$ is invariant under the S^1-action. Moreover, as an open subset of $M \approx \mathbb{C}^{4r-4}$, the space $U(r) \cap M$ is a manifold. From Corollary 2.4 it follows that $S^1 \to U(r) \cap M \to (U(r) \cap M)/S^1$ is a principal S^1-bundle. We obtain a fibration $U(r) \cap M \to (U(r) \cap M)/S^1 \to BS^1$ such that, if $i \colon U(r) \cap M \to U(r)$ is the inclusion, the diagram

$$\begin{tikzcd}
U(r) \arrow[r] & U(r)/S^1 \arrow[r] & BS^1 \\
U(r) \cap M \arrow[u, "i"] \arrow[r] & (U(r) \cap M)/S^1 \arrow[u] \arrow[r] & BS^1 \arrow[u, equal]
\end{tikzcd}$$

commutes.

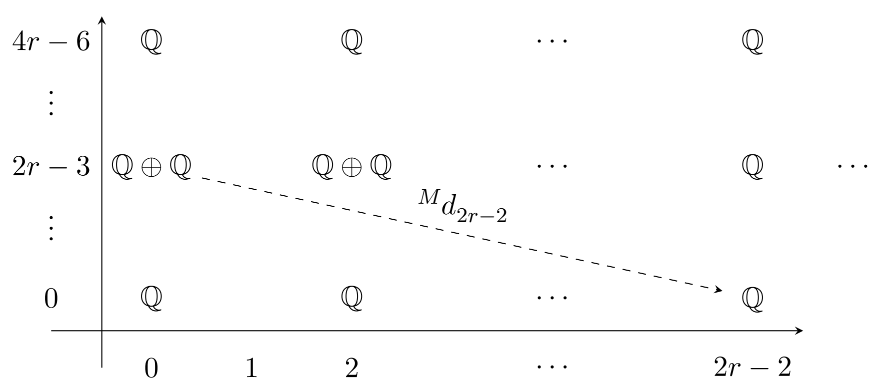
The Final Comparison. There is a deformation retraction from $U(r) \cap M$ onto a subspace homeomorphic to $S^{2r-3} \times S^{2r-3}$ given by

$$\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \dots, \begin{pmatrix} a_{r-1} & b_{r-1} \\ c_{r-1} & d_{r-1} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \mapsto$$
$$\left(\begin{pmatrix} (1-t)a_1 & \frac{b_1}{1+t(|\bar{b}|-1)} \\ \frac{c_1}{1+t(|\bar{c}|-1)} & (1-t)d_1 \end{pmatrix}, \dots, \begin{pmatrix} (1-t)a_{r-1} & \frac{b_{r-1}}{1+t(|\bar{b}|-1)} \\ \frac{c_{r-1}}{1+t(|\bar{c}|-1)} & (1-t)d_{r-1} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right).$$

Let $f\colon U(r)\cap M \to S^{2r-3}\times S^{2r-3}$ be the retract at $t=1$. The product $S^{2r-3}\times S^{2r-3}$ carries an S^1-action given by $z\cdot(\bar{b},\bar{c}) = (z\bar{b}, z^*\bar{c})$. Evidently, f is equivariant with respect to this action. Let π_1,π_2 be the two (equivariant) projection maps $S^{2r-3}\times S^{2r-3}\to S^{2r-3}$ onto the first and second factors respectively. The quotient of the action of S^1 on S^{2r-3} in either case is $\mathbb{C}P^{r-2}$, and the S^1-equivariant map $\pi_1 f$ induces a morphism of fibrations

$$\begin{tikzcd} U(r)\cap M \arrow[r] \arrow[d, "\pi_1 f"] & (U(r)\cap M)/S^1 \arrow[r] \arrow[d] & BS^1 \arrow[d, equal] \\ S^{2r-3} \arrow[r] & \mathbb{C}P^{r-2} \arrow[r] & BS^1. \end{tikzcd} \tag{5.2}$$

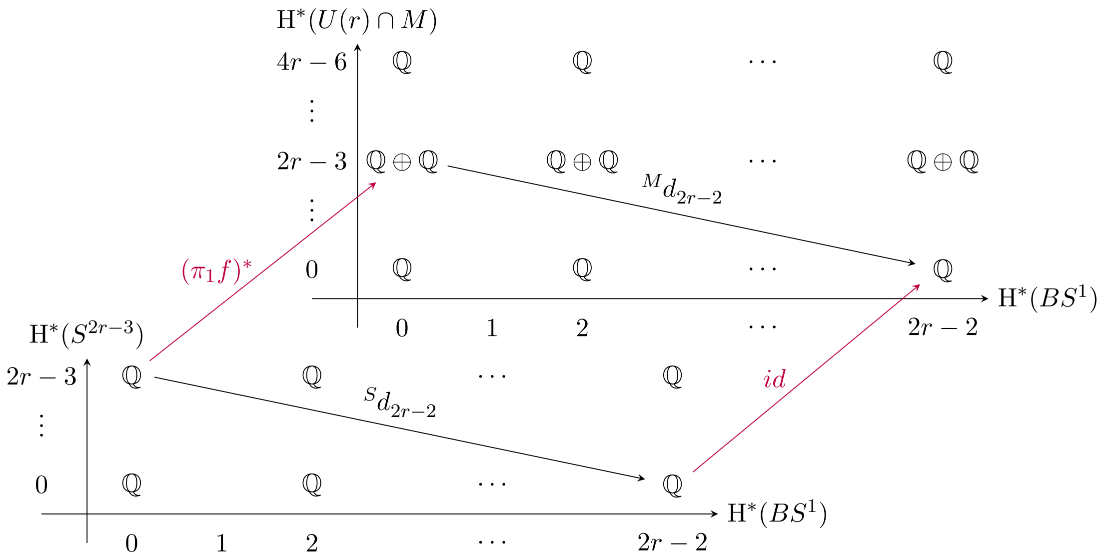
With all of our comparisons in hand, we now work backwards to compute d_{2r-2}. Let ${}^{M}E_k^{p,q}$, ${}^{M}d_k$ denote the terms and differentials in the spectral sequence associated to the top fibration, and let ${}^{S}E_k^{p,q}$, ${}^{S}d_k$ denote the terms and differentials in the spectral sequence associated to the bottom fibration in (5.2). The ${}^{M}E_2$-page is given by



We see that the first possibly nontrivial differential is on the ${}^{M}E_{2r-2}$-page. Identify ${}^{M}E_{2r-2}^{0,2r-3} = \mathrm{H}^{2r-3}(U(r)\cap M)$ with $\mathrm{H}^{2r-3}(S^{2r-3})\oplus \mathrm{H}^{2r-3}(S^{2r-3})$ via the Künneth formula.

Lemma 5.3. *The differential ${}^{M}d_{2r-2}\colon {}^{M}E_{2r-2}^{0,2r-3}\to {}^{M}E_{2r-2}^{0,2r-2}$ sends (β,γ) to a generator provided $\beta\neq-\gamma$.*

Proof. First we show that ${}^{M}d_{2r-2}$ is nonzero. Let ${}^{S,M}e_k^{p,q}\colon {}^{S}E_k^{p,q}\to {}^{M}E_k^{p,q}$ be the map of spectral sequences induced by (5.2). We have that ${}^{S,M}e_2^{0,2r-3} = (\pi_1 f)^*$, and ${}^{S,M}e_2^{2r-2,0} = id_{\mathrm{H}^{2r-2}(BS^1)}$ by Proposition 2.5. Portions of the E_2 (and E_{2r-2}) pages of the two spectral sequences, with the maps ${}^{S,M}e_2^{0,2r-3}$ and ${}^{S,M}e_2^{2r-2,0}$ indicated, are given by



Note that ${}^{S,M}e_{2r-2}^{0,2r-3} = {}^{S,M}e_2^{0,2r-3} = (\pi_1 f)^*$ and ${}^{S,M}e_{2r-2}^{2r-2,0} = {}^{S,M}e_2^{2r-2,0} = id_{H^{2r-2}(BS^1)}$ by Remark 2.6. Up to homotopy, the map $\pi_1 f$ is the projection $S^{2r-3}\times S^{2r-3}\to S^{2r-3}$ onto the first factor. If α is a generator of $H^{2r-3}(S^{2r-3})$, the induced map on cohomology sends α to $(\alpha,0)$. The spectral sequence for the lower fibration converges to $H^*(\mathbb{C}P^{r-2})$, which is trivial in degrees $*\geq 2r-3$, so ${}^{S}d_{2r-2}\colon {}^{S}E_{2r-2}^{0,2r-3}\to {}^{S}E_{2r-2}^{2r-2,0}$ is an isomorphism. We see that ${}^{M}d_{2r-2}$ sends $(\alpha,0)$ to a generator of ${}^{M}E_{2r-2}^{2r-2,0}$.

Since the kernel of ${}^{M}d_{2r-2}$ is 1-dimensional, to prove the lemma it is enough to show that ${}^{M}d_{2r-2}(\beta,\gamma) = {}^{M}d_{2r-2}(\gamma,\beta)$ for any $\beta,\gamma\in H^{2r-3}(S^{2r-3})$, as this would imply that $\ker {}^{M}d_{2r-2} = \langle(\alpha,-\alpha)\rangle$. Let $\sigma\colon U(r)\cap M\to U(r)\cap M$ be the map

$$\left(\begin{pmatrix} a_1 & b_1\\ c_1 & d_1\end{pmatrix},\dots,\begin{pmatrix} a_{r-1} & b_{r-1}\\ c_{r-1} & d_{r-1}\end{pmatrix},\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right)\mapsto$$
$$\left(\begin{pmatrix} a_1 & c_1^*\\ b_1^* & d_1\end{pmatrix},\dots,\begin{pmatrix} a_{r-1} & c_{r-1}^*\\ b_{r-1}^* & d_{r-1}\end{pmatrix},\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right).$$

It is a simple check that σ is S^1-equivariant, so we have a commuting diagram

$$\begin{tikzcd}
U(r)\cap M \arrow[r] \arrow[d, "\sigma"] & (U(r)\cap M)/S^1 \arrow[r] \arrow[d] & BS^1 \arrow[d, equal]\\
U(r)\cap M \arrow[r] & (U(r)\cap M)/S^1 \arrow[r] & BS^1.
\end{tikzcd}$$

There is an induced self-map of the spectral sequence ${}^{M}E$; in particular, the square

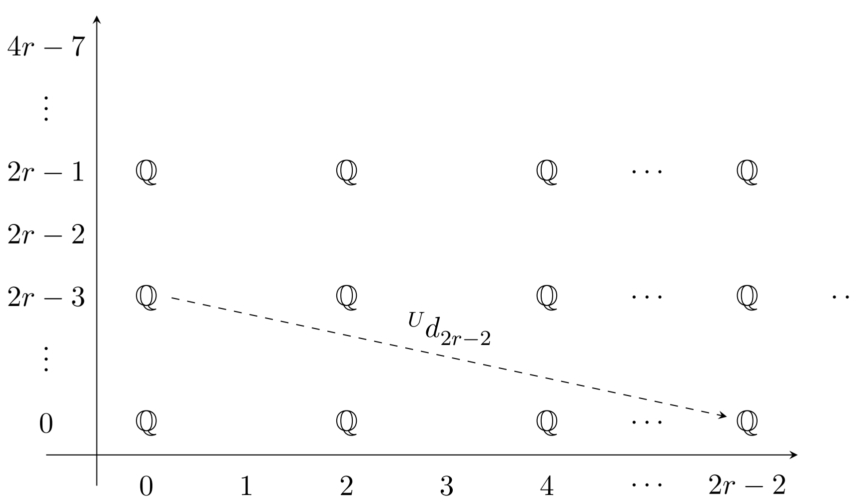
$$\begin{tikzcd}
{}^{M}E_{2r-2}^{0,2r-3} \arrow[r, "{}^{M}d_{2r-2}"] \arrow[d, "\sigma^*"] & {}^{M}E_{2r-2}^{2r-2,0} \arrow[d, "id"] \\
{}^{M}E_{2r-2}^{0,2r-3} \arrow[r, "{}^{M}d_{2r-2}"] & {}^{M}E_{2r-2}^{2r-2,0}
\end{tikzcd} \tag{5.3}$$

commutes. The map $S^{2r-3} \to S^{2r-3}$ given by $(a_1, \dots, a_{r-1}) \mapsto (a_1^*, \dots, a_{r-1}^*)$ induces the trivial map on cohomology since it is the composition of an even number of reflections. So σ induces the same map on cohomology as the map $S^{2r-3} \times S^{2r-3} \to S^{2r-3} \times S^{2r-3}$ that interchanges factors. That is, $\sigma^*\colon \mathrm{H}^{2r-3}(U(r) \cap M) \to \mathrm{H}^{2r-3}(U(r) \cap M)$ also interchanges factors. From the commutativity of (5.3), we have ${}^{M}d_{2r-2}(\beta, \gamma) = {}^{M}d_{2r-2}(\gamma, \beta)$. $\square$

Denote the terms and differentials for the spectral sequence associated to $U(r) \to U(r)/S^1 \to BS^1$ by ${}^{U}E_k^{p,q}$ and ${}^{U}d_k$ respectively.

Lemma 5.4. *The differential* ${}^{U}d_{2r-2}\colon {}^{U}E_{2r-2}^{0,2r-3} \cong \mathbb{Q} \to \mathbb{Q} \cong {}^{U}E_{2r-2}^{2r-2,0}$ *is an isomorphism.*

Proof. We have ${}^{U}E_2^{p,q} = \mathrm{H}^p(BS^1) \otimes_{\mathbb{Q}} \mathrm{H}^q(U(r))$, so a portion of the ${}^{U}E_2$-page is



We see that the first possibly nontrivial differential in the range $p + q \leq 4r - 7$ is on the ${}^{U}E_{2r-2}$-page.

The second comparison

$$\begin{tikzcd} U(r) \arrow[r] & U(r)/S^1 \arrow[r] & BS^1 \\ U(r)\cap M \arrow[r] \arrow[u, "i"] & U(r)\cap M/S^1 \arrow[r] \arrow[u] & BS^1 \arrow[u, equal] \end{tikzcd} \tag{5.4}$$

gives rise to a morphism of spectral sequences ${}^{U,M}e_k^{p,q}\colon {}^{U}E_k^{p,q} \to {}^{M}E_k^{p,q}$. By naturality, ${}^{U,M}e_{2r-2}^{0,2r-3} = i^*\colon \mathrm{H}^{2r-3}(U(r)) \to \mathrm{H}^{2r-3}(U(r)\cap M)$, and ${}^{U,M}e_{2r-2}^{2r-2,0} = id_{\mathrm{H}^{2r-2}(BS^1)}$. We have a commuting diagram

$$\begin{tikzcd} {}^{U}E_{2r-2}^{0,2r-3} \cong \mathbb{Q} \arrow[r, "{}^{U}d_{2r-2}"] \arrow[d, "i^*"'] & \mathbb{Q} \cong {}^{U}E_{2r-2}^{2r-2,0} \arrow[d, "\cong"] \\ {}^{M}E_{2r-2}^{0,2r-3} \cong \mathbb{Q}\oplus\mathbb{Q} \arrow[r, "{}^{M}d_{2r-2}"] & \mathbb{Q} \cong {}^{M}E_{2r-2}^{2r-2,0}. \end{tikzcd}$$

Thus ${}^{U}d_{2r-2}$ is an isomorphism provided the image of a generator δ under i^* is not of the form $(\beta, -\beta)$ for some $\beta \in \mathrm{H}^{2r-3}(S^{2r-3})$. This is accomplished by the following lemma. $\square$

Lemma 5.5. *The map* $i^*\colon \mathrm{H}^{2r-3}(U(r)) \to \mathrm{H}^{2r-3}(U(r)\cap M)$ *sends a generator* δ *to* (β, γ), *where* $\gamma \neq -\beta$.

Proof. First we show that i^* is nonzero by exploiting naturality of the Gysin sequence. Following the notation of Chapter 4, there is an inclusion $\tilde{\jmath}\colon \mathbb{C}^{r-1} \to M^o(r) \to \mathrm{Mat}_2^r(\mathbb{C})$ given by

$$(b_1, \dots, b_{r-1}) \mapsto \left(\begin{pmatrix} 0 & b_1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & b_2 \\ 0 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & b_{r-1} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right).$$

Indeed for an r-tuple in $\operatorname{im}\tilde{\jmath}$, the first and last matrices do not commute, so $\operatorname{im}\tilde{\jmath} \subseteq M^o(r)$. Recall that $W(r) \hookrightarrow M^o(r)$ is a closed inclusion of smooth $\mathbb{C}$-varieties. We claim there is a pullback square

$$\begin{tikzcd} \{\bar{0}\} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & W(r) \arrow[d] \\ \mathbb{C}^{r-1} \arrow[r, "\tilde{\jmath}"] & M^o(r). \end{tikzcd}$$

For any $\bar{b} \in \mathbb{C}^{r-1} \setminus \{\bar{0}\}$, we have $\tilde{\jmath}(\bar{b}) \in U(r) = M^o(r) \setminus W(r)$; as long as one of the b_is is nonzero, the matrices in $\tilde{\jmath}(\bar{b})$ do not share an eigenvector. Also, the matrices in the r-tuple $\tilde{\jmath}(\bar{0})$ share the eigenvector e_2 (and do not pairwise commute), so $\tilde{\jmath}(\bar{0}) \in W(r)$. Hence, $\tilde{\jmath}^{-1}W(r) = \{\bar{0}\}$.

Next we need to verify that the map $\tilde{\jmath}$ is transverse to $W(r)$. Let $g\colon (-\epsilon,\epsilon) \to W(r)$ be a smooth path such that $g(0) = \tilde{\jmath}(\bar{0})$. Write g as

$$t \mapsto \left(\begin{pmatrix} a_1(t) & b_1(t) \\ 1+c_1(t) & d_1(t) \end{pmatrix} \begin{pmatrix} a_2(t) & b_2(t) \\ c_2(t) & d_2(t) \end{pmatrix}, \dots, \begin{pmatrix} a_{r-1}(t) & b_{r-1}(t) \\ c_{r-1}(t) & d_{r-1}(t) \end{pmatrix}, \begin{pmatrix} 1+a_r(t) & b_r(t) \\ c_r(t) & d_r(t)-1 \end{pmatrix} \right)$$

for some smooth functions $a_i, b_i, c_i, d_i\colon (-\epsilon,\epsilon) \to \mathbb{R}^2 \approx \mathbb{C}$ that all evaluate to 0 at $t = 0$. The map $\tilde{\jmath}$ is transverse to $W(r)$ if, for all such paths, $b_i'(0) = 0$ for $i \in \{1,\dots,r-1\}$. Recall from Chapter 4 that there is a smooth map $p\colon W(r) \to \mathbb{C}P^1$ that sends an r-tuple to its unique invariant 1-dimensional eigenspace. We may assume that the image of g lies in $p^{-1}U_1$, where $U_1 = \{[z_0 : 1] \in \mathbb{C}P^1\} \approx \mathbb{C}$. The composition $p \circ g$ is a smooth path, given explicitly by $t \mapsto [\mu(t) : 1]$ for some smooth path μ in $\mathbb{C}$ such that $\mu(0) = 0$. Requiring that the first matrix in $g(t)$ has $[\mu(t) : 1]$ as an eigenvector is equivalent to

$$a_1\mu + b_1 = (1 + c_1)\mu^2 + d_1\mu,$$

where we have dropped the variable t. After taking a derivative and evaluating at $t = 0$, we find that $b_1'(0) = 0$. Similarly, requiring that the ith matrix in $g(t)$, for $i \in \{2,\dots,r-1\}$, has $[\mu(t) : 1]$ as an eigenvector is equivalent to

$$a_i\mu + b_i = c_i\mu^2 + d_i\mu.$$

Again we find that $b_i'(0) = 0$ after taking a derivative and evaluating at $t = 0$.

The inclusions $\{\bar{0}\} \hookrightarrow \mathbb{C}^{r-1}$ and $W(r) \hookrightarrow M^o(r)$ are both closed, codimension-$(r-1)$ inclusions of smooth $\mathbb{C}$-varieties. Let j denote the restriction of $\tilde{\jmath}$ to $\mathbb{C}^{r-1} \setminus \{\bar{0}\}$. A portion of the induced map on Gysin sequences (Proposition 2.29) is

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}^{2r-3}(\mathbb{C}^{r-1} \setminus \{\bar{0}\}) \arrow[r, "\cong"] & \mathrm{H}^0(\mathrm{pt}) \arrow[r] & 0 \\
\mathrm{H}^{2r-3}(M^o(r)) \arrow[r] \arrow[u, "\tilde{\jmath}^*"] & \mathrm{H}^{2r-3}(U(r)) \arrow[r] \arrow[u, "j^*_{2r-3}"] & \mathrm{H}^0(W(r)) \arrow[r] \arrow[u, "\tilde{\jmath}|^*_{\{\bar{0}\}}", "\cong"'] & \mathrm{H}^{2r-2}(M^o(r)). \arrow[u, "\tilde{\jmath}^*"]
\end{tikzcd}$$

Recall that $M^o(r)$ is $(4r-6)$-connected and $W(r)$ is connected (Lemma 4.7 and Proposition 4.8 respectively). So the map $\mathrm{H}^{2r-3}(U(r)) \cong \mathbb{Q} \to \mathbb{Q} \cong \mathrm{H}^0(W)$ is an isomorphism. It follows that j^*_{2r-3} is an isomorphism. Since j factors as $\mathbb{C}^{r-1} \setminus \{\bar{0}\} \to U(r) \cap M \xrightarrow{i} U(r)$, the map i^*_{2r-3} is nonzero.

Let δ be a generator of $\mathrm{H}^{2r-3}(U(r))$, and put $i^*(\delta) = (\beta, \gamma)$. We show that $\beta = \gamma$. Consider the map $\eta\colon U(r) \to U(r)$ given by conjugation by the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Let $-\eta$ be the composition of η with the antipodal map $U(r) \to U(r)$. Explicitly, $-\eta$ is the map

$$\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \dots \begin{pmatrix} a_r & b_r \\ c_r & d_r \end{pmatrix}\right) \mapsto \left(\begin{pmatrix} -d_1 & c_1 \\ b_1 & -a_1 \end{pmatrix}, \dots, \begin{pmatrix} -d_r & c_r \\ b_r & -a_r \end{pmatrix}\right).$$

Observe that $U(r) \cap M$ is invariant under $-\eta$. The two maps $i \circ -\eta|_{U(r)\cap M}$ and $-\eta \circ i$ obviously coincide. Up to homotopy, $-\eta|_{U(r)\cap M}$ is the map $S^{2r-3} \times S^{2r-3} \to S^{2r-3} \times S^{2r-3}$ that interchanges factors. Thus, in degree $2r-3$, the induced map $-\eta|^*_{U(r)\cap M}$ also interchanges factors. We have

$$(i \circ -\eta|_{U(r)\cap M})^*(\delta) = -\eta|^*_{U(r)\cap M}(\beta, \gamma) = (\gamma, \beta).$$

On the other hand, since $\mathrm{PGL}_2(\mathbb{C})$ is path-connected, the map η is homotopic to $id_{U(r)}$ (the identity map can be thought of as conjugation by the identity matrix). In a similar vain, the path-connected group $\mathbb{C}^\times$ acts on $U(r)$ by scaling, so the antipodal map is homotopic to $id_{U(r)}$. It follows that $-\eta \simeq id_{U(r)}$, so that

$$(-\eta i)^*(\delta) = i^*(\delta) = (\beta, \gamma).$$

$\square$

Recall that the terms and differentials for our original fibration $U(r) \to B(r) \to B\mathrm{PGL}_n(\mathbb{C})$ are denoted by $E_k^{p,q}$ and d_k respectively.

Lemma 5.6. *The differential $d_{2r-2}\colon E_{2r-2}^{0,2r-3} \to E_{2r-2}^{2r-2,0}$ is an isomorphism.*

Proof. From the first comparison

$$\begin{tikzcd} U(r) \arrow[r] & B(r) \arrow[r] & B\mathrm{PGL}_2(\mathbb{C}) \\ U(r) \arrow[r] \arrow[u, equal] & U(r)/S^1 \arrow[r] \arrow[u] & BS^1 \arrow[u, "B\rho"'] \end{tikzcd}$$

we have a commuting square

$$\begin{tikzcd} E_{2r-2}^{0,2r-3} \arrow[r, "d_{2r-2}"] \arrow[d, "\cong"] & E_{2r-2}^{2r-2,0} \arrow[d, "B\rho^*"] \\ {}^U E_{2r-2}^{0,2r-3} \arrow[r, "{}^U d_{2r-2}"] & {}^U E_{2r-2}^{2r-2,0} \end{tikzcd}$$

where all objects are isomorphic to $\mathbb{Q}$. The differential ${}^U d_{2r-2}$ is an isomorphism by Lemma 5.4, so d_{2r-2} is an isomorphism. $\square$

Suppose α is a generator of $E_{2r-2}^{0,2r-3}$. The class p_1^k is a generator of $E_{2r-2}^{4k,0}$, so αp_1^k is a generator of $E_{2r-2}^{4k,2r-3}$ as a consequence of the multiplicative structure on the spectral sequence. Then

$$d_{2r-2}(\alpha p_1^k) = d_{2r-2}(\alpha)p_1^k + (-1)^{2r-3}\alpha d_{2r-2}(p_1^k) = d_{2r-2}(\alpha)p_1^k$$

is a generator of $E_{2r-2}^{4k+2r-2,0}$ since $d_{2r-2}(\alpha)$ is a generator of $E_{2r-2}^{2r-2,0}$. In other words, the differential $d_{2r-2}\colon E_{2r-2}^{4k,2r-3} \to E_{2r-2}^{4k+2r-2,0}$ is an isomorphism for every $k \geq 0$. The relevant portion of the E_{2r-1}-page is thus

$$\begin{array}{c|cccccccc}
4r-7 & & & & & & & \\
\vdots & & p+q \leq 4r-7 & & & & & \\
2r-1 & \mathbb{Q} & \mathbb{Q} & \cdots & \mathbb{Q} & \mathbb{Q} & \mathbb{Q} & \cdots \\
2r-2 & & & & & & & \\
2r-3 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\
\vdots & & & & & & & \\
0 & \mathbb{Q} & \mathbb{Q} & \cdots & \mathbb{Q} & 0 & 0 & \cdots \\
\hline
 & 0 & 4 & \cdots & 2r-6 & 2r-2 & 2r+2 &
\end{array} \tag{5.5}$$

where some zeros are added for clarity. For any of the nonzero terms under the dashed line, all outgoing differentials on succeeding pages have trivial target, and all incoming differentials on succeeding pages have trivial source. So $E_{2r-1}^{p,q} = E_\infty^{p,q}$ in the range $p+q \leq 4r-7$. We can now read off the terms $\mathrm{H}^i(B(r)) = \bigoplus_{p+q=i} E_\infty^{p,q}$, for $i \leq 4r-7$, from (5.5).

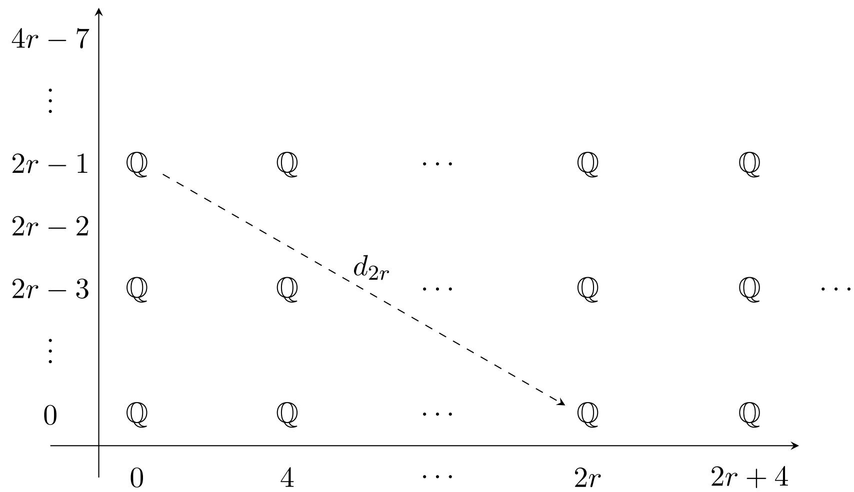
5.2 Case 2: r even

Theorem 5.7. *When r is even,*

$$\mathrm{H}^i(B(r)) \cong \begin{cases} \mathbb{Q} & i \leq 2r-4 \text{ and } i \equiv 0 \pmod 4 \\ \mathbb{Q} & 2r-3 \leq i \leq 4r-7 \text{ and } i \equiv 1 \pmod 4 \\ 0 & \text{otherwise with } i \leq 4r-7 \end{cases}.$$

The E_2-page of the Leray–Serre spectral sequence associated to the fibration $U(r) \to$

$B(r) \to B\mathrm{PGL}_2(\mathbb{C})$ is

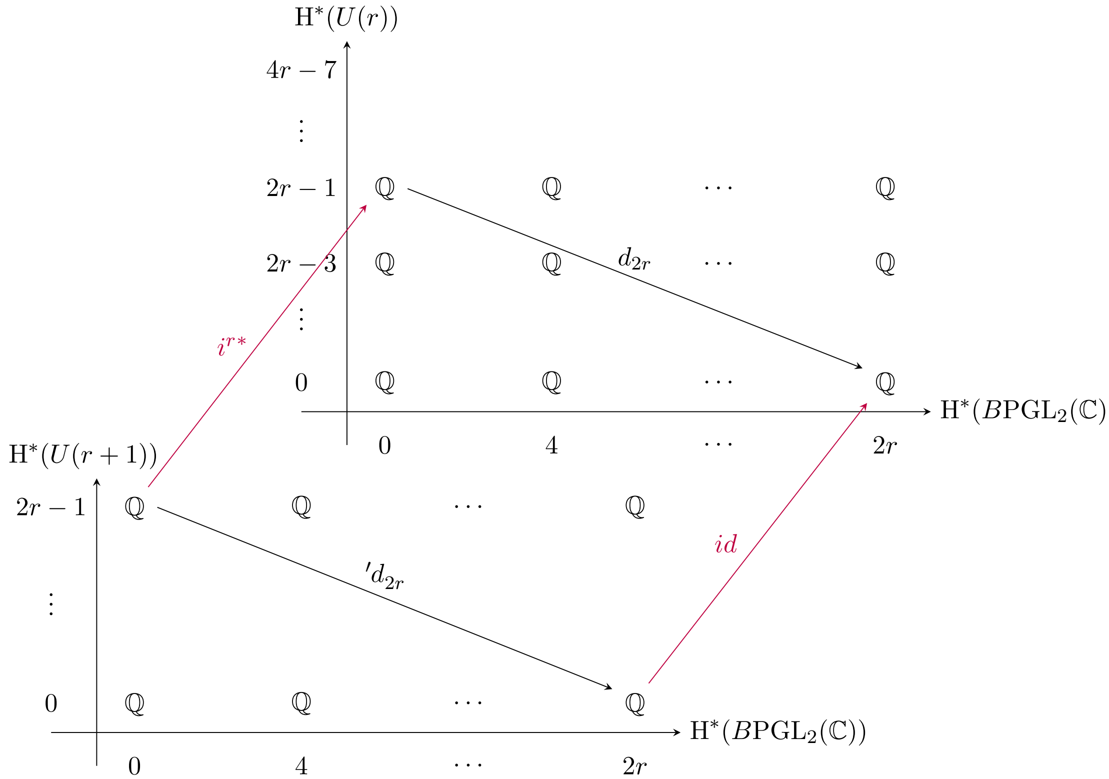


Note that the differentials $d_{2r-2}\colon E_{2r-2}^{4i,2r-3} \to E_{2r-2}^{4i+2r-2,0}$ have target 0 since $2r-2 \equiv 2 \pmod 4$. The first differentials with nontrivial source and target in the relevant range occur on the E_{2r}-page. We compute $d_{2r}\colon E_{2r}^{0,2r-1} \to E_{2r}^{2r,0}$.

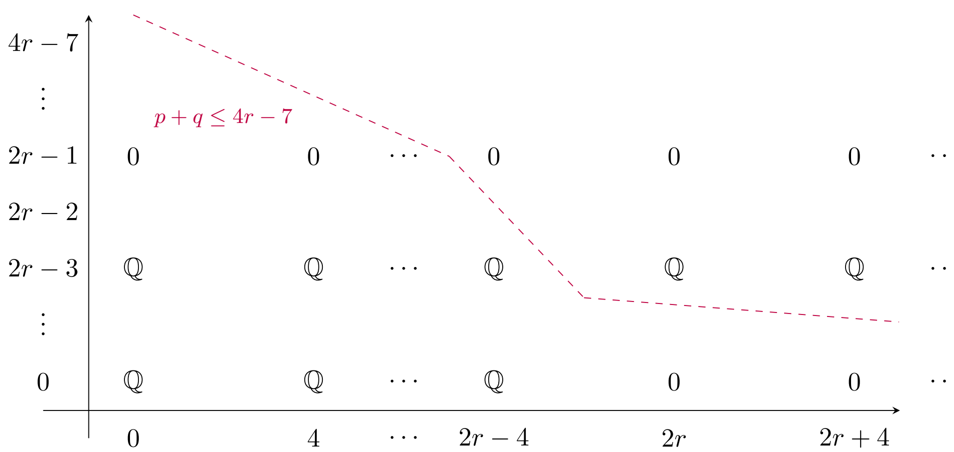
There is an inclusion $i^r\colon U(r) \to U(r+1)$ given by setting the last matrix equal to the 0 matrix. This map i^r is clearly $\mathrm{PGL}_2(\mathbb{C})$-equivariant, so there is an induced morphism of fibrations

$$\begin{tikzcd}
U(r) \arrow[r] \arrow[d, "i^r"] & B(r) \arrow[r] \arrow[d] & B\mathrm{PGL}_2(\mathbb{C}) \arrow[d, equal] \\
U(r+1) \arrow[r] & B(r+1) \arrow[r] & B\mathrm{PGL}_2(\mathbb{C}).
\end{tikzcd}$$

Denote the terms and differentials for the spectral sequence associated to the lower fibration by ${}'E_k^{p,q}$ and ${}'d_k$ respectively. We have an induced map of spectral sequences ${}'E \to E$. In particular, the square in



commutes. The differential $'d_{2r}$ is an isomorphism by Lemma 5.6, so both i^{r*} and d_{2r} are isomorphisms. As before, this implies that the differentials $d_{2r}\colon E_{2r}^{4i,2r-1} \to E_{2r}^{4i+2r,0}$ are isomorphisms for every $i \geq 0$. The E_{2r+1}-page is thus



(5.6)

Again, for terms below the dashed line, all the incoming and outgoing differentials on successive pages are trivial. So $E_{2r+1}^{p,q} = E_{\infty}^{p,q}$ for $p + q \leq 4r - 7$. The result follows.

Chapter 6

The Number of Generators of a Topological Azumaya Algebra

As mentioned in the introduction, the isomorphism

$$\mathrm{Aut}_{\mathbb{C}\text{-alg}}(\mathrm{Mat}_n(\mathbb{C})) \cong \mathrm{PGL}_n(\mathbb{C})$$

is a direct consequence of the Skolem–Noether theorem. With this isomorphism in mind, we make the following definition.

Definition 6.1. Let X be a topological space. A *topological Azumaya algebra of degree-n over* X is a fibre bundle $\mathcal{A} \to X$ with structure group $\mathrm{PGL}_n(\mathbb{C})$ and fibre $\mathrm{Mat}_n(\mathbb{C})$.

In other words, a topological Azumaya algebra is a (locally trivial) bundle of matrix algebras over X.

Definition 6.2. A *morphism of degree-n topological Azumaya algebras* $(f, \tilde{f})$ from $p\colon \mathcal{A} \to X$ to $p'\colon \mathcal{A}' \to Y$ is morphism of fibre bundles with fibre $\mathrm{Mat}_n(\mathbb{C})$ and structure group $\mathrm{PGL}_n(\mathbb{C})$. That is,

(*i*) The diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\tilde{f}} & \mathcal{A}' \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

commutes.

(*ii*) For any trivializations (V, ϕ) and (V', ϕ') of $\mathcal{A} \to X$ and $\mathcal{A}' \to Y$ respectively such that $x \in V$ and $f(x) \in V'$, the composite

$$\{x\} \times \mathrm{Mat}_n(\mathbb{C}) \xrightarrow{\phi^{-1}} p^{-1}(x) \xrightarrow{\tilde{f}} p'^{-1}(f(x)) \xrightarrow{\phi'} \{f(x)\} \times \mathrm{Mat}_n(\mathbb{C})$$

is given by the action of some $\theta_{\phi\phi'}(x) \in \mathrm{PGL}_n(\mathbb{C})$. Moreover, the assignment $x \mapsto \theta_{\phi\phi'}(x)$ defines a continuous map $V \cap f^{-1}V' \to \mathrm{PGL}_n(\mathbb{C})$.

Remark 6.3. As defined, a morphism of topological Azumaya algebras is a $\mathbb{C}$-algebra isomorphism on the fibres. This definition is nonstandard. One might instead define a morphism of topological Azumaya algebras to be a $\mathbb{C}$-algebra endomorphism on the fibres. But since $\mathrm{Mat}_n(\mathbb{C})$ is a simple ring, any map of $\mathbb{C}$-algebras $\mathrm{Mat}_n(\mathbb{C}) \to \mathrm{Mat}_n(\mathbb{C})$ is an isomorphism.

Remark 6.4. A morphism of topological Azumaya algebras over the identity map is an isomorphism of topological Azumaya algebras. This follows from the observations that such a morphism of topological Azumaya algebras is, in particular, a map of rank-n^2 complex vector bundles which is a linear isomorphism on the fibres. And such a map of vector bundles is a vector bundle isomorphism. It is not hard to see that the inverse map is not only a vector bundle map, but a morphism of topological Azumaya algebras.

Let $\mathrm{Az}_n(X)$ denote the set of isomorphism classes of degree-n topological Azumaya algebras over X. There is a natural correspondence between $\mathrm{Az}_n(X)$ and isomorphism classes of principal $\mathrm{PGL}_n(\mathbb{C})$-bundles over X. If we suppose further that X is paracompact, we are led to the natural correspondence

$$\mathrm{Az}_n(X) \cong [X, B\mathrm{PGL}_n(\mathbb{C})].$$

For each r there is a map $g_r : B_n^r(\mathbb{C}) \to B\mathrm{PGL}_n(\mathbb{C})$, well-defined up to homotopy, that classifies the principal $\mathrm{PGL}_n(\mathbb{C})$-bundle $U_n^r(\mathbb{C}) \to B_n^r(\mathbb{C})$. The motivating question in what follows is: given a degree-n topological Azumaya algebra $\mathcal{A} \to X$ classified by a map $X \to B\mathrm{PGL}_n(\mathbb{C})$, what does it mean for that map to factor, up to homotopy, as $X \to B_n^r(\mathbb{C}) \xrightarrow{g_r} B\mathrm{PGL}_n(\mathbb{C})$ for some r? Before addressing this question, we need a couple of definitions.

Let $p\colon \mathcal{A} \to X$ be a degree-n topological Azumaya algebra and suppose $s_1, \dots, s_r$ are (ordered) sections of p. We say that $s_1, \dots, s_r$ are *generating* if, for any $x \in X$, the r-tuple $(s_1(x), \dots, s_r(x))$ generates the fibre as a $\mathbb{C}$-algebra. Precisely, the sections are generating if, given any $x \in X$ and any trivialization (V, ϕ) with $x \in V$:

$$\begin{tikzcd} p^{-1}(V) \arrow[rr, "\phi", "\approx"'] \arrow[dr, "p"'] & & V \times \mathrm{Mat}_n(\mathbb{C}) \arrow[dl, "\pi_1"] \\ & V & \end{tikzcd},$$

one has $(\phi s_1(x), \dots, \phi s_r(x)) \in \{x\} \times U_n^r(\mathbb{C}) \approx U_n^r(\mathbb{C})$. In this case, we will call the data of $(\mathcal{A} \to X, \{s_i\}_{i=1}^r)$ a *topological Azumaya algebra over X with r generating sections.* Two topological Azumaya algebras with r generating sections $(\mathcal{A} \to X, \{s_i\}_{i=1}^r)$ and $(\mathcal{A}' \to X, \{s_i'\}_{i=1}^r)$ are *isomorphic* if there is an isomorphism of topological Azumaya algebras

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "h"] \arrow[dr] & & \mathcal{A}' \arrow[dl] \\ & X & \end{tikzcd}$$

such that

$$hs_i = s_i'$$

for each i.

Notation 6.5. We denote the set of isomorphism classes of degree-n topological Azumaya algebras over X with r generating sections by $\mathrm{Az}_n^r(X)$. For topological spaces Y and Z, let $\mathscr{C}(Y, Z)$ denote the set of continuous maps from Y to Z.

Proposition 6.6. *Let X be a topological space. There is a natural bijective correspondence*

$$\mathscr{C}(X, B_n^r(\mathbb{C})) \cong \mathrm{Az}_n^r(X).$$

Proof. First, we construct a function $\Phi\colon \mathrm{Az}_n^r(X) \to \mathscr{C}(X, B_n^r(\mathbb{C}))$. Let $\mathcal{A} \to X$ be a topological Azumaya algebra equipped with r generating sections $s_1, \ldots, s_r$. For each trivializing neighborhood (V_j, ϕ_j), define a map $f_j\colon V_j \to U_n^r(\mathbb{C})$ by $x \mapsto (\phi_j s_1(x), \ldots, \phi_j s_r(x))$. The maps f_j may not agree on intersections of trivializing neighborhoods, but they only differ by an element of the structure group $\mathrm{PGL}_n(\mathbb{C})$. After composing with the projection $q\colon U_n^r(\mathbb{C}) \to B_n^r(\mathbb{C})$, the maps agree on intersections of trivializing neighborhoods. Let $\Phi(\mathcal{A} \to X, \{s_i\}_{i=1}^r)$ be the unique map $X \to B_n^r(\mathbb{C})$ obtained from gluing the maps qf_j.

Define $\Psi\colon \mathscr{C}(X, B_n^r(\mathbb{C})) \to \mathrm{Az}_n^r(X)$ as follows. Let $\mathcal{E} = U_n^r(\mathbb{C}) \times_{\mathrm{PGL}_n(\mathbb{C})} \mathrm{Mat}_n(\mathbb{C})$ be the quotient of $U_n^r(\mathbb{C}) \times \mathrm{Mat}_n(\mathbb{C})$ by the diagonal action, and let $p\colon \mathcal{E} \to B_n^r(\mathbb{C})$ be the topological Azumaya algebra associated to the principal $\mathrm{PGL}_n(\mathbb{C})$-bundle $q\colon U_n^r(\mathbb{C}) \to B_n^r(\mathbb{C})$. The map $\tilde{s_i}\colon U_n^r(\mathbb{C}) \to U_n^r(\mathbb{C}) \times \mathrm{Mat}_n(\mathbb{C})$ given by $\bar{A} = (A_1, \ldots, A_r) \mapsto (\bar{A}, A_i)$ is $\mathrm{PGL}_n(\mathbb{C})$-equivariant, so there is an induced map s_i on quotients. It is clear that the sections $s_1, \ldots, s_r$ of $p\colon \mathcal{E} \to B_n^r(\mathbb{C})$ are generating. Given $f\colon X \to B_n^r(\mathbb{C})$, let $\Psi(f)$ be the pullback bundle $f^*\mathcal{E} \to X$ equipped with the sections $f^*s_1, \ldots, f^*s_r$. If we identify $f^*\mathcal{E}$ with a subspace of $X \times \mathcal{E}$ in the usual way, then f^*s_i is given by $x \mapsto (x, s_i f(x))$. It is clear that that these sections are also generating.

$\Phi \circ \Psi = id$. Given $f\colon X \to B_n^r(\mathbb{C})$, we want to show that $\Phi(f^*\mathcal{E} \to X, \{f^*s_i\}_{i=1}^r)$ coincides with the map f. Let $x \in X$, and set $f(x) = [\bar{A}]$ where $\bar{A} = (A_1, \ldots, A_r)$. Then $f^*s_i(x) = (x, s_i f(x)) = (x, [\bar{A}, A_i])$. Suppose (V, ϕ) is a trivializing neighborhood of x for the pullback bundle $f^*\mathcal{E} \to X$. Then, tracing through some definitions,

$$\phi \circ f^* s_i(x) = \phi(x, [\bar{A}, A_i]) = (x, g \cdot A_i)$$

for some $g \in \mathrm{PGL}_n(\mathbb{C})$ that depends only on x. Hence, $(\phi f^* s_1(x), \ldots, \phi f^* s_r(x)) = (x, g \cdot \bar{A})$, so $\Phi \circ \Psi(f)$ sends x to the class $[g \cdot \bar{A}] = f(x)$.

$\Psi \circ \Phi = id$. Let $p' \colon \mathcal{A} \to X$ be a topological Azumaya algebra with r generating sections $t_1, \ldots, t_r$. Using the function Φ, construct the map $f \colon X \to B_n^r(\mathbb{C})$. We want to show that $(\mathcal{A} \to X, \{t_i\}_{i=1}^r)$ is isomorphic to $(f^*\mathcal{E} \to X, \{f^* s_i\}_{i=1}^r)$ as topological Azumaya algebras with r generating sections. Define a map $\tilde{f} \colon \mathcal{A} \to \mathcal{E}$ on generators of the fibres by $t_i(x) \mapsto s_i f(x)$. We need to verify that this defines a map of topological Azumaya algebras over f. Choose trivializations (ϕ_j, V_j) and (ϕ, V) of $\mathcal{A} \to X$ and $\mathcal{E} \to X$ respectively. The trivialization (ϕ, V) is associated to a trivialization (ϕ_{a}, V) for the principal $\mathrm{PGL}_n(\mathbb{C})$-bundle $q \colon U_n^r(\mathbb{C}) \to B_n^r(\mathbb{C})$:

$$\begin{tikzcd} q^{-1}V \arrow[rr, "\phi_{\mathrm{a}}"] \arrow[dr, "q"'] & & V \times \mathrm{PGL}_n(\mathbb{C}) \arrow[dl, "\pi_1"] \\ & V & \end{tikzcd} .$$

For $\bar{A} \in q^{-1}V$, write $\phi_{\mathrm{a}}(\bar{A}) = ([\bar{A}], g_{\bar{A}})$. Then, by definition of the associated bundle, $\phi \colon p^{-1}V \to V \times \mathrm{Mat}_n(\mathbb{C})$ is the map $[\bar{A}, B] \mapsto ([\bar{A}], {g_{\bar{A}}}^{-1}.B)$.

First we verify (ii) of Definition 6.2. Suppose $x \in V_j$ and $f(x) \in V$. Note that $f(x) = [f_j(x)] = [(\phi_j t_1(x), \ldots, \phi_j t_r(x))]$. The composite

$$\{x\} \times \mathrm{Mat}_n(\mathbb{C}) \xrightarrow{\phi_j^{-1}} p'^{-1}(x) \xrightarrow{\tilde{f}} p^{-1}(f(x)) \xrightarrow{\phi} \{f(x)\} \times \mathrm{Mat}_n(\mathbb{C})$$

is given by $(x, B) \mapsto (f(x), {g_{f_j(x)}}^{-1}.B)$. The map $\theta_{\phi_j \phi} : V_j \cap f^{-1}V \to \mathrm{PGL}_n(\mathbb{C})$ which sends x to ${g_{f_j(x)}}^{-1}$ is continuous since it is the composite

$$V_j \cap f^{-1}V \xrightarrow{f_j} q^{-1}V \xrightarrow{\phi_{\mathrm{a}}} V \times \mathrm{PGL}_n(\mathbb{C}) \xrightarrow{\pi_2} \mathrm{PGL}_n(\mathbb{C}) \xrightarrow{i} \mathrm{PGL}_n(\mathbb{C})$$

where i is the inversion map.

To see (i) of Definition 6.2, observe that $\tilde{f}$ is given locally as the product of

$$f\pi_1 \colon V_j \cap f^{-1}V \times \mathrm{Mat}_n(\mathbb{C}) \to V$$

with the function

$$V_j \cap f^{-1}V \times \mathrm{Mat}_n(\mathbb{C}) \xrightarrow{\theta_{\phi_j\phi} \times id} \mathrm{PGL}_n(\mathbb{C}) \times \mathrm{Mat}_n(\mathbb{C}) \xrightarrow{a} \mathrm{Mat}_n(\mathbb{C}),$$

so $\tilde{f}$ is continuous.

The morphism of topological Azumaya algebras

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\tilde{f}} & \mathcal{E} \\ \big\downarrow & & \big\downarrow \\ X & \xrightarrow{f} & B_n^r(\mathbb{C}) \end{array}$$

induces a map $h\colon \mathcal{A} \to f^*\mathcal{E}$ which is an isomorphism of topological Azumaya algebras over X by Remark 6.4. Moreover, $f^*s_i(x) = (x, s_i f(x)) = (x, \tilde{f}t_i(x)) = ht_i(x)$. Naturality of Ψ follows from naturality of the pullback construction (with sections). $\square$

Proposition 6.6 can be used to give obstructions to the generation of a topological Azumaya algebra by r sections in the following way. Suppose X is paracompact and $\mathcal{A} \to X$ is a topological Azumaya algebra classified by a map $f\colon X \to B\mathrm{PGL}_n(\mathbb{C})$. Suppose also that, for some i, the induced map $f^*\colon\ \mathrm{H}^i(B\mathrm{PGL}_n(\mathbb{C}); R) \to \mathrm{H}^i(X; R)$ is injective while the map $g_r^*\colon\ \mathrm{H}^i(B\mathrm{PGL}_n(\mathbb{C}); R) \to \mathrm{H}^i(B_n^r(\mathbb{C}); R)$ is not injective (recall $g_r\colon B_n^r(\mathbb{C}) \to B\mathrm{PGL}_n(\mathbb{C})$ classifies the principal $\mathrm{PGL}_n(\mathbb{C})$-bundle $U_n^r(\mathbb{C}) \to B_n^r(\mathbb{C})$). In light of Proposition 6.6, the topological Azumaya algebra $\mathcal{A} \to X$ can be generated by r global sections if and only if the homotopy class of f factors through $B_n^r(\mathbb{C})$. We would then have a factorization of f^* through a non-injective map, which is impossible.

Chapter 7

Conclusion

After introducing the varieties $U_n^r(\mathbb{C})$ and $B_n^r(\mathbb{C})$, we were able to show in Proposition 3.12 that the spaces $B_n^r(\mathbb{C})$ form homotopical approximations to the classifying space $B\mathrm{PGL}_n(\mathbb{C})$. Moreover, the quotient map $U_n^r(\mathbb{C}) \to B_n^r(\mathbb{C})$ is a principal $\mathrm{PGL}_n(\mathbb{C})$-bundle. Using the techniques discussed in Chapter 2, we arrived at the computation

$$\mathrm{H}^i(U_2^r(\mathbb{C});\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & i = 0, 2r-3, 2r-1 \\ 0 & \text{otherwise (with } i \leq 4r-7) \end{cases}.$$

Then, assembling this data into the Leray–Serre spectral sequence associated to the fibration $U_2^r(\mathbb{C}) \to B_2^r(\mathbb{C}) \to B\mathrm{PGL}_2(\mathbb{C})$, we found that

$$\mathrm{H}^i(B_2^r(\mathbb{C});\mathbb{Q}) \cong \begin{cases} \mathbb{Q} & i \leq 2r-6 \text{ and } i \equiv 0 \pmod 4 \\ \mathbb{Q} & 2r-1 \leq i \leq 4r-7 \text{ and } i \equiv 1 \pmod 4 \\ 0 & \text{otherwise with } i \leq 4r-7 \end{cases}$$

when r is odd. And when r is even,

$$\mathrm{H}^i(B_2^r(\mathbb{C});\mathbb{Q}) \cong \begin{cases} \mathbb{Q} & i \leq 2r-4 \text{ and } i \equiv 0 \pmod 4 \\ \mathbb{Q} & 2r-3 \leq i \leq 4r-7 \text{ and } i \equiv 1 \pmod 4 \\ 0 & \text{otherwise with } i \leq 4r-7 \end{cases}.$$

These latter two computations being the main technical results of this paper. The purpose of these computations is twofold. On the one hand, the computations give an indication of how well the spaces $B_2^r(\mathbb{C})$ approximate the classifying space $B\mathrm{PGL}_2(\mathbb{C})$. Secondly, and perhaps more importantly, by measuring non-injectivity of the map $H^*(B\mathrm{PGL}_n(\mathbb{C})) \to H^*(B_n^r(\mathbb{C}))$, one can give obstructions to the generation by r global sections of a topological Azumaya algebra over a paracompact space, as discussed in Chapter 6.

There is considerable literature devoted to bounding the minimal number of generators of an algebra. This thesis fits in this context; the techniques of Chapter 6 can be carried out in the algebraic setting to give bounds on the minimal number of generators of an Azumaya algebra over a commutative ring in the sense of [AG60]. This is discussed in an upcoming paper of B. Williams, U. First, and Z. Reichstein, where cohomological computations of the varieties B_n^r are considered for $n \geq 3$. For more on this topic, see for instance [SW20] for the case of étale algebras and [FR17] for techniques that apply to more general kinds of algebras.

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