A boundary local time for super-Brownian motion and new path properties for superprocess densities

by

Thomas J. Hughes

B.Sc., Queen’s University, 2014
M.Sc., University of British Columbia, 2016

A THESIS SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF

Doctor of Philosophy

in

THE FACULTY OF GRADUATE AND POSTDOCTORAL STUDIES
(Mathematics)

The University of British Columbia
(Vancouver)

August 2020

© Thomas J. Hughes, 2020
The following individuals certify that they have read, and recommend to the Faculty of Graduate and Postdoctoral Studies for acceptance, the thesis entitled:

A boundary local time for super-Brownian motion and new path properties for superprocess densities

submitted by Thomas J. Hughes in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics.

Examinining Committee:
Edwin Perkins, Mathematics
Supervisor

Martin Barlow, Mathematics
Supervisory Committee Member

Jun-cheng Wei, Mathematics
University Examiner

Nick Harvey, Computer Science
University Examiner

Additional Supervisory Committee Members:
Gordon Slade, Mathematics
Supervisory Committee Member
Abstract

We establish new fractal properties for superprocess densities.

Consider the density of super-Brownian motion in one dimension at a fixed time. We construct a random measure supported on the boundary of the zero set of the density, which we call the boundary local time, and prove that it is locally non-degenerate, in the sense that it almost surely has positive mass on every neighbourhood of the boundary of the support. We then use the boundary local time to give an almost sure lower bound on the Hausdorff dimension of the boundary of the zero set, confirming a conjecture of Mueller et al. [69], whose work on this problem motivated ours. The Hausdorff dimension is characterized in terms of the lead eigenvalue of a certain killed Ornstein-Uhlenbeck operator which also appears in the construction of the boundary local time. The proof of local non-degeneracy includes a novel analysis of the density near the right endpoint of its support, including a new bound on the expected mass in a neighbourhood of the endpoint.

We then consider the density of \((\alpha, \beta)\)-superprocess in dimension \(d\) with \(\alpha \in (0, 2), \beta \in (0, 1], \) and \(\beta d < \alpha\). A classical result called instantaneous propagation asserts that, at a fixed time, the density has positive mass on any open set almost surely when conditioned on survival. We give general conditions under which the integral the density against a measure \(\mu\) is positive almost surely when conditioned on survival. In particular, we show that this is determined by the relationship between the dimension of the measure, which we define in two ways, and a critical parameter determined by \(\alpha, \beta\) and \(d\). When the density does not charge \(\mu\) almost surely, we obtain upper and lower bounds on the probability that it does so. In both cases our results correspond to new existence and non-existence results for solutions to the partial differential equation which is dual to the process. In the regime where the density is continuous and positive at a fixed point almost surely, we prove that the density function is strictly positive.
We study two models for the density of a random spatial population. In the first model, individuals move locally in space and produce zero or two offspring in each generation. We construct a new tool for quantifying the interface between the region where the population density is zero and the region where it is positive and use it to compute the dimension (and hence size) of the interface.

In the second model, individuals move in space via potentially long-range jumps and can create large numbers of offspring simultaneously. In this case, we characterize the regions on which the density is locally positive in terms of the dimension of the region. In a special case, we show that the density is positive everywhere in space. These probabilistic results allow us to extend recent results in deterministic partial differential equations.
Preface

This thesis is based on three articles, two of which are published and one of which has been submitted for publication. Some minor changes have been made to the published articles.

A version of Chapter 2 has been published as A boundary local time for one-dimensional super-Brownian motion, of which I am the only author. It is published in The Electronic Journal of Probability, Volume 24, paper no. 54.

A version of Chapter 3 has been published as The boundary of the zero set of super-Brownian motion and its local time, by Ed Perkins (my thesis advisor) and myself. This work is published in Annales de l’Institut Henri Poincaré B: Probabilités et Statistiques, Volume 55-4. Our relative contributions to research and writing are as follows: two key auxiliary results are Proposition 3.3.5 and Proposition 3.3.6; the former is due to me and the latter is due to Ed Perkins. The remaining sections are the product of collaborative research. I prepared the first typed draft of Sections 3.4 and 3.5, and Ed Perkins prepared the first typed draft of Section 3.1. Everything not mentioned above was prepared jointly.

Chapter 4 is unpublished, original work of which I am the only author, and has been submitted for publication under the title The density of the \((\alpha, \beta)\)-superprocess and singular solutions to a fractional non-linear PDE.

Chapter 1 is an original introduction to the field, which I wrote, and includes some material from the other chapters.

The work in Chapters 2 and 3 is a continuation of a research program initiated by Mueller, Mytnik and Perkins (2017), and some results contained there were motivated by (and are resolutions to) conjectures of those authors. Methodologically, these chapters include both novel techniques and extensions of the existing methods, some developed independently by me and some in collaboration with my advisor. I independently developed the research program of Chapter 4.
# Table of Contents

Abstract ..................................................... iii

Lay Summary ................................................ iv

Preface ......................................................... v

Table of Contents ............................................ vi

Acknowledgments ............................................ ix

Dedication .................................................... x

1 Introduction ................................................ 1
   1.1 A critical branching particle system .................. 1
   1.2 Continuous state branching and superprocesses .... 3
      1.2.1 Warm up: a limit theorem for branching processes . 3
      1.2.2 Continuous state branching with general branching mechanisms . 5
      1.2.3 Spatial branching particle systems, superprocesses, and their duals . 11
   1.3 Two superprocesses and their properties ............... 15
      1.3.1 Super-Brownian motion .......................... 15
      1.3.2 The \((\alpha, \beta)\)-superprocess .................... 16
      1.3.3 Super-Brownian motion with \((1 + \beta)\)-stable branching .... 17
      1.3.4 Supports .......................................... 17
      1.3.5 Densities ......................................... 21
      1.3.6 The canonical measure and cluster decompositions .... 24
   1.4 Our Results ............................................. 26
      1.4.1 A boundary local time for super-Brownian motion .... 27
      1.4.2 New path properties for the \((\alpha, \beta)\)-superprocess .... 33
   1.5 Notes on the text ....................................... 38
# The boundary local time of super-Brownian motion I: construction and properties

- **2.1** Introduction & statement of main results ........................................... 41
- **2.2** Killed Ornstein-Uhlenbeck processes .................................................. 51
- **2.3** Some non-linear PDE ............................................................................. 60
- **2.4** Existence and properties of $L_t$ ............................................................. 65
- **2.5** Proof of Theorem 2.1.4 ............................................................................ 83
  - **2.5.1** PDE representations and preliminary bounds ....................................... 83
  - **2.5.2** Convergence ...................................................................................... 94
- **2.6** Proof of Lemma 2.5.5 ............................................................................. 103

# The boundary local time of super-Brownian motion II: 0-1 law

- **3.1** Introduction and statement of results .................................................... 118
- **3.2** Preliminaries .......................................................................................... 125
  - **3.2.1** Killed Ornstein-Uhlenbeck processes ................................................ 125
  - **3.2.2** Cluster and historical decompositions of super-Brownian motion .... 125
  - **3.2.3** Hitting probabilities of super-Brownian motion ................................ 127
- **3.3** Some semi-linear partial differential equations ........................................ 128
- **3.4** Proof of Theorem 3.1.2 .......................................................................... 141
- **3.5** Localization ........................................................................................... 152

# New properties for the density of the $(\alpha, \beta)$-superprocess

- **4.1** Introduction and statement of results .................................................... 156
- **4.2** Preliminaries .......................................................................................... 171
  - **4.2.1** Transition densities ........................................................................... 171
  - **4.2.2** The density of the $(\alpha, \beta)$-superprocess ........................................ 172
  - **4.2.3** The fractional PDE and $\mu(\lambda_t)$ .................................................. 176
  - **4.2.4** A Feynman-Kac formula .................................................................. 181
  - **4.2.5** Cluster decompositions .................................................................... 181
- **4.3** The density at a fixed point ................................................................... 182
- **4.4** Strict positivity of the density .................................................................. 188
- **4.5** Almost sure charging of (F1)-s measures when $\beta \leq \beta^*(\alpha, s)$ ........ 193
- **4.6** Decay of $N_x(\mu(\lambda_t) > 0)$ for (F2)-s measures when $\beta > \beta^*(\alpha, s)$ 198
- **4.7** The initial trace problem ........................................................................ 208

# Conclusion

- **5.1** Super-Brownian motion ......................................................................... 213
- **5.2** The $(\alpha, \beta)$-superprocess ................................................................. 214
Acknowledgments

First and foremost, I would like express my deep gratitude to my advisor, Ed Perkins, for his guidance, patience and encouragement over the past six years. Ed introduced me to the field, and his conscientious supervision kept me on track while allowing me to flourish and explore. His mentorship has been a great influence on me, mathematically and personally, and his generosity and wit are a joy and inspiration to those around him. Thank you, Ed.

I thank Martin Barlow and Gordon Slade for serving on my supervisory committee, Jun-cheng Wei and Nick Harvey for serving as university examiners, and Vitali Wachtel for serving as the external examiner. Thank you for lending your time and expertise.

I am grateful to the members of the UBC Mathematics Department for providing such an excellent working environment – the faculty, the staff, my fellow students, and my forever office mate Saraí. I am lucky to have been a member of UBC’s probability group, and I thank the past and present members of this group for the stimulating conversations, seminars and reading groups from which I learned so much. Thanks are also due to the organizers of the various schools and conferences which I have attended and from which I have benefited greatly. I am especially grateful to Alison Etheridge for generously inviting me to visit Oxford, and to Leonid Mytnik and Yaozhong Hu for inviting me to conferences at BIRS and the UofA, respectively.

Thanks to many friends old and new, near and far, mathematicians and otherwise, who have been there over the years. To my family: thank you for your support, which has been a constant comfort throughout my studies. And to Claire: thank you for everything – much of my share of happiness and success I owe to you.

Finally, I gratefully acknowledge the financial support I have received from NSERC, UBC, and the Li Tze Fong Memorial Fellowship.
To Claire and my family.
Chapter 1

Introduction

A Dawson-Watanabe superprocess, or simply a superprocess, is a measure-valued Markov process which models a spatial population evolving in time. The evolution has two essential sources of randomness: random spatial motion and random reproduction (branching). Critically, these degrees of randomness are independent, as are the evolutions of different individuals in the population.

In this Introduction, we first consider the motivating example of a continuous-time branching particle system in $d$-dimensional Euclidean space, $\mathbb{R}^d$. This is followed by a general discussion of non-spatial branching processes and their scaling limits, continuous state branching processes, after which we reintroduce spatial motion in order to define general superprocesses. We then specialize our discussion to the processes considered in this thesis and review their known properties before discussing our own contributions.

1.1 A critical branching particle system

We wish to describe a branching particle system whose heuristic description is as follows: a collection of particles undergo independent random motion in space; periodically, each individual has a reproduction event in which the individual (the parent) dies and is replaced with a random number of new individuals (the offspring) at the parent’s location. The distribution of the offspring is such that the mean number of offspring from each birth event equals 1. This condition is known as criticality and ensures that the expected number of individuals in the population is constant and equals the initial size of the population. For the present case, we assume that branching is binary: the parent will have either 0 or 2 offspring, each with probability $\frac{1}{2}$.

We fix a Markov process with state space $\mathbb{R}^d$ and denote its paths by $(\xi_t)_{t \geq 0}$. This Markov process is the spatial motion of the branching particle system. We fix $N \in \mathbb{N}$ and define the set of branch times $\{\frac{1}{N}, \frac{2}{N}, \ldots\}$. Now, consider a population which begins, at
time 0, with $N$ individuals located at the origin. Each individual evolves in space as an independent copy of $(\xi_t)_{t \geq 0}$ until the first branch time, $\frac{1}{N}$. At this time, a fair coin is flipped for each individual: if it comes up heads (with probability $\frac{1}{2}$) the individual dies. If it comes up tails, the individual branches into two offspring at the same location. The branching event occurs independently for each individual and produces a new generation of individuals which then evolve in space as independent copies of $(\xi_t)_{t \geq 0}$ until the next branching time, $\frac{2}{N}$. The reproduction procedure is then repeated, and the process continues to evolve in the same fashion.

By assigning a mass of $\frac{1}{N}$ to each individual, we can view the state at time $t > 0$ as a mass distribution. In particular, if there are $N_t$ surviving individuals at time $t$ with locations $x_i$ for $i = 1, \ldots, N_t$, we consider the empirical measure

$$X_t^{(N)} = \frac{1}{N} \sum_{i=1}^{N_t} \delta_{x_i},$$

where $\delta_x$ denotes the Dirac mass at $x \in \mathbb{R}^d$. (A proper analysis of this process requires a more sophisticated labelling system for the individuals, but we do not require this for our description here.) By fixing the particle mass at $\frac{1}{N}$, the expected total mass of the population remains constant for all $N$. The acceleration of the branching rate (i.e. branching events at every multiple of $\frac{1}{N}$) ensures that the variance of the total mass at time $t$ remains constant in $N$.

The larger $N$ is, the greater the size of the population and the smaller the contribution of a particular individual to the associated mass distribution. By shrinking the interval between branching times we are effectively accelerating time. Therefore for large values of $N$ we are observing a large population over long time-scales, normalized to have constant expected total mass.

A common theme in probability theory is to approximate the large scale behaviour of a discrete random system with a scaling limit, which is a continuum model that arises as the distributional limit of the rescaled discrete models. The most famous example is the central limit theorem. The class of processes we study in this thesis, superprocesses, are the scaling limits of branching particle systems such as the one described above. In addition to their connection with numerous discrete models (including more complex models than the one we have described), superprocesses are objects of independent interest. They also are deeply connected with certain families of non-linear elliptic and parabolic differential equations.
1.2 Continuous state branching and superprocesses

The goal of this section is to introduce and define the Dawson-Watanabe superprocess, our central object of study. We introduce superprocesses in the broader context of scaling limits for branching processes. This means that before considering spatial branching models and superprocesses we dedicate some discussion to non-spatial branching processes. The idea behind our presentation is that while most readers will be familiar with the theory of Markov processes in $\mathbb{R}^d$ (i.e. the processes which describe the spatial motion in our models), some may be unfamiliar with continuous-state branching.

1.2.1 Warm up: a limit theorem for branching processes

We begin with some classical limit theorems for branching processes. Let $\mu$ be a probability distribution on $\mathbb{N}$ with probability mass function (pmf) $\mu(\{n\}) = \mu_n$. For a function $f : \mathbb{N} \rightarrow \mathbb{R}$, we denote the expectation by $\mu(f) = \sum_{n=0}^{\infty} f(n)\mu_n$. We make the following assumptions on $\mu$:

- $\mu(Id) = \sum_{n=0}^{\infty} n\mu_n = 1$ (Criticality).
- $\sum_{n=0}^{\infty} n^2\mu_n < +\infty$ (Finite second moment).
- $\mu_1 < 1$ (Non-degeneracy).

Furthermore we suppose that $\mu$ has variance $\gamma > 0$.

A Galton-Watson branching process, or simply a branching process, with offspring distribution $\mu$ is a Markov chain with state space $\mathbb{N}$ with the following time-homogeneous transition probabilities:

$$P(Z_{n+1} = k \mid Z_n = m) = (\mu * \mu * \cdots * \mu)(k).$$

The state of the chain $Z_n$ denotes the number of individuals in the population at time $n$. Conditioned on $Z_n = m$, the distribution of $Z_{n+1}$ is the $m$-fold convolution of the offspring distribution $\mu$. Of course, this corresponds to the intuitive description that for each individual in the population at time $n$, we independently sample a number of offspring from $\mu$, and the size of the population at time $n+1$ is the total number of offspring produced by generation $n$.

The classical analysis of branching processes is often done using generating functions. Let $X$ be a random variable with distribution $\mu$. We denote the probability generating function of $X$ by $f(s) = E(s^X)$. It follows from an elementary conditioning argument that

$$f_n(s) = f(f_{n-1}(s))$$  \hspace{1cm} (1.2.1)
where \( f_n(s) = E(s^{Z_n}) \) is the (probability) generating function of \( Z_n \). In particular, if \( Z_0 = 1 \) deterministically, \( f_n(s) \) is equal to the \( n \)-fold composition of \( f \) with itself. This recursion is classically used to determine the extinction probability (that is, the probability that \( Z_n = 0 \) for some \( n \)) of the branching process, which can be shown to be equal to the smallest positive fixed point of \( f \), i.e. the smallest \( s_0 > 0 \) such that \( f(s_0) = s_0 \). We have assumed that our branching process is critical, in which case the extinction probability is one. More detailed information, such as the asymptotics of the survival probability, can also be obtained using this method. Suppose that \( Z_0 = 1 \). From (1.2.1), it follows that

\[
E(Z_n) = \mu(1)^n Z_0 = 1
\]

(1.2.2)

for all \( n \geq 1 \). We can also compute the second moment of \( Z_n \), which yields

\[
\text{Var}(Z_n) = n\gamma.
\]

(1.2.3)

By the second moment method, we deduce the following: when \( Z_0 = 1 \),

\[
P(Z_n > 0) \geq \frac{E(Z_n)^2}{E(Z_n^2)} \geq \frac{1^2}{n\gamma + 1^2} \approx \frac{1}{n\gamma}.
\]

(1.2.4)

This estimate is quite good and only differs from the actual asymptotics by a factor of 2. The exact asymptotics of the survival probability were originally established by Kolmogorov [44].

**Theorem 1.2.1.** If \( Z_0 = 1 \), then

\[
\lim_{n \to \infty} n P(Z_n > 0) = \frac{2}{\gamma}.
\]

This result can be proved using a more refined analysis of generating functions. See, for example, Theorem II.1.1 of [83].

We now present Feller’s Theorem, which describes the diffusion approximation, or scaling limit, of the branching process. The most famous diffusion approximation in probability theory is undoubtedly Donsker’s Theorem, which states that the scaling limit (with diffusive space-time rescaling) of a random walk with finite variance step distribution is Brownian motion. Feller’s Theorem is an analogous statement which characterizes the scaling limit of branching processes whose offspring distributions have finite second moments as the solution to a stochastic differential equation.

Let \( a > 0 \) and suppose \( \mu \) is an offspring distribution satisfying the assumptions stated at the beginning of the section. We define a sequence of stochastic processes \((Z_t^{(N)})_{t \geq 0}\) as follows. Let \( Z_0^{(N)} \) be a branching process with offspring distribution \( \mu \) and \( \tilde{Z}_0 = \lfloor aN \rfloor \). For
We will write $\mathbb{R}_+$ to denote the non-negative real axis $[0, \infty)$. For a Polish space $E$, denote by $\mathcal{D}(\mathbb{R}_+, E)$ the space of càdlàg (right-continuous with left limits) paths in $E$ with time index set $\mathbb{R}_+$, equipped with the Skorokhod topology. (For a general introduction to this space and topology, see Chapter 3 of Ethier and Kurtz [23].) The following is due to Feller [26].

**Theorem 1.2.2.** $(Z_t^{(N)})_{t \geq 0} \to (Z_t)_{t \geq 0}$ weakly on $\mathcal{D}(\mathbb{R}_+, \mathbb{R}_+)$, where $Z_t$ is the unique-in-law solution to

$$Z_t = a + \int_0^t \sqrt{\gamma Z_s} dB_s$$

(1.2.6)

The solution to the stochastic differential equation (SDE) (1.2.6) is called Feller’s branching diffusion. Solutions to the SDE are not only unique-in-law, but satisfy pathwise uniqueness by the Yamada-Watanabe theorem (see Theorem V.40 of [86]). The martingale problem formulation of (1.2.6) is a non-spatial version of the martingale problems which can be used to define some superprocesses; however, here we use an approach based on the Laplace functional.

### 1.2.2 Continuous state branching with general branching mechanisms

In this section we give a general limit theorem for branching processes and characterization of their limits, *continuous state branching processes* (CSBPs). The state space of a CSBP is $\mathbb{R}_+$ rather than $\mathbb{N}$, and the state is understood as the total mass of a population rather than the number of individuals. The continuous-state branching behaviour we discuss is closely related to the theory of infinitely divisible distributions and Lévy processes.

The first work considering continuous state branching in some generality was by Jiřina in 1958 [40]. Other early work includes articles of Lamperti [49, 50, 51] and Silverstein [87], some of which we discuss below. The construction we adopt is based on that used by Dawson [13] to construct superprocesses, which we employ again in the next section when we do the same. Here we restrict our attention to branching processes with a single type, but we note that continuous state branching models with multiple types have been studied extensively (early work includes [88] and [93]) and remain an area of active interest (some recent works include [47, 48]).

Consider a continuous-time branching process whose offspring distribution has probability mass function $(p_k)_{k \geq 0}$. The probability generating function of the offspring distribution is then

$$g(z) = \sum_{k=0}^{\infty} z^k p_k.$$  

(1.2.7)
In the continuous-time setting, individuals no longer reproduce simultaneously at each time step to produce the next generation; rather, we fix $\kappa > 0$ and attach an independent Exponential($\kappa$) clock to each individual. When the clock sounds, the individual is replaced with a random number of offspring sampled from the offspring distribution and to each offspring is attached an independent Exponential($\kappa$) clock.

The process equal to the number of individuals in the population is a strong Markov process with state space $\mathbb{N}$. We will denote it by $(Z_t)_{t \geq 0}$, and for the law and expectation of $(Z_t)_{t \geq 0}$ with $Z_0 = m \geq 1$, offspring generating function $g(z)$ and birth rate $\kappa$, we will write $P^m_\kappa$ and $E^m_\kappa$ respectively.

A critical tool for analysing many processes with branching behaviour is the Laplace functional, i.e. the quantity

$$L(m, \lambda, t) = E^m_\kappa \left( \exp(-\lambda Z_t) \right) \quad (1.2.8)$$

for $\lambda > 0$, $m \in \mathbb{N}$ and $t > 0$. We first remark that this quantity satisfies

$$L(m, \lambda, t) = L(1, \lambda, t)^m. \quad (1.2.9)$$

This is known as the branching property. It reflects, and is an immediate consequence of, the fact that the lineages extending from separate individuals in the branching process are independent. It implies that it is enough to analyse $L(1, \lambda, t)$, and so to denote this quantity we will simply write $L(\lambda, t)$. By conditioning on the time and size of the first branching event, it is elementary to show that

$$L(\lambda, t) = \exp(-\lambda \kappa t) + \int_0^t g(L(\lambda, t - s)) \kappa e^{-\kappa s} ds. \quad (1.2.10)$$

The fact that the Laplace functional of a branching process is related to the solution of an evolution equation is essential. Indeed, an analogous relationship holds for superprocesses and the methodology we adopt is centred around it. We will come to this in the next section.

The Laplace functional fully characterizes the distribution of $Z_t$ (and more generally of any $\mathbb{R}^+$-valued random variable) so it is a robust framework from which to study these processes and their distributional limits.

We consider a scaling regime in which $Z_0 = O(N)$ and the mass of each individual is $N^{-1}$. In addition to large population size, we wish to study the process over large time-scales which we achieve by using a branching rate $\kappa_N$ with $\lim_{N \to \infty} \kappa_N = \infty$.

A random variable is called infinitely divisible if for every $n \geq 1$, it can be written as a sum of $n$ iid random variables. If $Z_0 = N$, by the branching property we can write $Z_t$ as a sum of $N$ independent branching processes starting with one individual. We should
therefore expect that, once renormalized to ensure tightness, the scaling limit of $Z_t$ for any fixed $t > 0$ should be infinitely divisible. Indeed, this is the case. Infinite divisibility of the marginal distributions and the branching property are sufficient to characterize the scaling limits of branching processes, continuous state branching processes.

We now recall the Lévy-Khintchine formula for infinitely divisible random variables. (For a proof, see Section 9.5 of [4].)

**Theorem 1.2.3.** A random variable $X$ is infinitely divisible if and only if its characteristic function (Fourier transform) satisfies

$$E(\exp(iuX)) = \exp(-\Psi(u)), \quad (1.2.11)$$

where the characteristic exponent $\Psi$ is of the form

$$\Psi(u) = i bu + \frac{\gamma}{2} u^2 + \int_{-\infty}^{\infty} (1 - e^{iur} + iur1_{|r|\leq1})\nu(dr), \quad (1.2.12)$$

where $b \in \mathbb{R}$, $\gamma \geq 0$, $\nu$ is a measure on $\mathbb{R}$ with $\nu(\{0\}) = 0$ and $\int (1 \wedge |r|^2)\nu(dr) < \infty$.

The characteristic function is often used because it is well-defined regardless of the distributional properties of the variable. However, the variables and processes we consider are spectrally positive, which means that the Lévy measure $\nu$ is supported on $(0, \infty)$. This corresponds to a process with no negative jumps. In this case the Laplace transform is the extension of the Fourier transform from the imaginary axis to the negative real numbers. For a spectrally positive infinitely divisible random variable $X$, we have

$$E(\exp(-uX)) = \exp(-\psi(u)), \quad (1.2.13)$$

where the Laplace exponent $\psi(u)$ is of the form

$$\psi(u) = mu - \frac{\gamma}{2} u^2 + \int_{0}^{\infty} (1 - e^{-ur} - ur1_{|r|\leq1})\nu(dr), \quad (1.2.14)$$

with $m \in \mathbb{R}$, $\gamma \geq 0$ and $\nu$ a measure on $(0, \infty)$ satisfying $\int_0^\infty (1 \wedge r^2)\nu(dr) < \infty$. In the context of CSBPs, the Laplace exponent $\psi(u)$ is often called a branching mechanism. If, in addition, we have $\int_1^\infty r\nu(dr) < \infty$, we say that $\psi$ has finite mean. By redefining the constant $m$ we obtain that the general form of a branching mechanism with finite mean is

$$\psi(u) = mu - \frac{2}{2} u^2 + \int_{0}^{\infty} (1 - e^{-ur} - ur)\nu(dr), \quad (1.2.15)$$

with $\int_0^\infty r \wedge r^2 \nu(dr) < \infty$. As we discuss later, the value of $\psi'(0^+)$ gives the exponential rate
of growth of the mean of the CSBP. If $\psi$ has representation (1.2.15), then $\psi'(0^+) = m$ and the branching mechanism is called sub-critical, critical and super-critical corresponding to the cases $m < 0$, $m = 0$ and $m > 0$.

Our primary goal is to construct a broad class of CSBPs and thereby characterize the class of Markov processes which arise as scaling limits of branching processes. The limit theorem we state is general enough to construct every CSBP whose branching mechanism is critical (and therefore has finite mean). In order to construct these processes, we first choose the branching mechanism $\psi(u)$ and define a sequence of offspring distribution generating functions $g_N$ and branching rates $\kappa_N$, and show that the sequence of branching processes with parameters $(g_N, \kappa_N)$ converges in distribution to a limit process with branching mechanism $\psi$. In doing so, we restrict the offspring distributions under consideration. However, the method we adopt is sufficient to build all CSBPs in the class we consider. If a sequence of critical branching processes associated to a generic sequence $(g_N, \kappa_N)$ is tight, then its distributional (possibly subsequential) limit will be one of the CSBPs constructed in the sequel, and in this sense the sequence $(g_N, \kappa_N)$ is a perturbation of one of the sequences we consider.

Let $\psi(u)$ be a critical branching mechanism (i.e. $\psi$ is of form (1.2.15) with $m = 0$). In our choice of the scaling regime we follow Section 4 of Dawson [13]. For $N \in \mathbb{N}$, let

$$\kappa_N = -\psi'(N),$$

(1.2.16)

and

$$g_N(z) = \kappa_N^{-1}[\kappa_N z - N^{-1}\psi(N(1 - z))]$$

(1.2.17)

For $a > 0$ let $M_N$ be a Poisson$(aN)$ random variable. We wish to establish the convergence of the sequence of processes $(N^{-1}Z_t^{(N)})_{t \geq 0}$, where $Z_t^{(N)}$ has law $P_{M_N}^{g_N,\kappa_N}$. The following is the special case of Theorem 4.4.1 of [13] when there is no spatial motion.

**Theorem 1.2.4.** Let $\psi(u)$ be a critical branching mechanism, and for $N \geq 1$ and $a \geq 0$, let $\kappa_N$, $g_N$, $M_N$ be as above. Then the process $(N^{-1}Z_t^{(N)})_{t \geq 0}$, where $(Z_t^{(N)})_{t \geq 0}$ has law $P_{M_N}^{g_N,\kappa_N}$, converges in distribution to an $\mathbb{R}^+$-valued strong Markov process $(Y_t)_{t \geq 0}$ with $Y_0 = a$, where $Y_t$ is the strong Markov process with the following Laplace functional: for $x \geq 0$ and $\lambda > 0$,

$$E^Y_x(\exp(-\lambda Y_t)) = \exp(-xu^\lambda(t)),$$

(1.2.18)

where $u^\lambda(t)$ is the unique solution to

$$u' = \psi(u), \quad u(0) = \lambda.$$

(1.2.19)
The process \((Y_t)_{t \geq 0}\) in the limit above is called the \(\psi\)-continuous state branching process, or \(\psi\)-CSBP. Lamperti was the first to establish CSBPs as the distributional limits of branching processes [50, 51]. The limit theorem we have stated is restricted to critical branching mechanisms; however, for all \(\psi\) of the form (1.2.14), the \(\psi\)-CSBP exists and its Laplace functional is characterized by (1.2.18) and (1.2.19).

Let \(\psi\) be a finite mean branching mechanism with representation (1.2.15) and consider the \(\psi\)-CSBP \(Y_t\). The state \(Y_t\) represents the total mass of a population. As can be seen directly from (1.2.18), \(Y_t\) satisfies the branching property: for \(t, x, y, \lambda > 0\),

\[
E_{x+y}(\exp(-\lambda Y_t)) = E_x(\exp(-\lambda Y_t))E_y(\exp(-\lambda Y_t)).
\]

(1.2.20)

In particular, for every \(N \geq 1\) we can write

\[
E_x(\exp(-\lambda Y_t)) = (E_{x/N}(\exp(-\lambda Y_t)))^N,
\]

(1.2.21)

and hence \(Y_t\) is infinitely divisible.

The branching mechanism \(\psi\) encodes the properties of the \(\psi\)-CSBP. For example, a CSBP may demonstrate finite time extinction. Consider the extinction time \(\tau_0 = \inf\{t > 0 : Y_t = 0\}\). It was shown by Grey [31] that (for any \(a > 0\)), \(P_a(\tau_0 < \infty) = 1\) if and only if, for sufficiently large \(\theta > 0\), \(\psi(u) < 0\) for \(u \geq \theta\) and

\[
\int_{\theta}^{\infty} \frac{1}{|\psi(u)|} du < \infty.
\]

Blow-up behaviour is also governed by \(\psi\) [31]. However, this does not occur in the finite mean setting to which we have restricted.

We recall our earlier definition that \(\psi\) (and the \(\psi\)-CSBP) is sub-critical, critical, and super-critical respectively when \(m < 0\), \(m = 0\) and \(m > 0\). It can be shown via an elementary calculation using (1.2.18) and (1.2.19) that \(E_1^Y(Y_t) = e^{mt}\). Another result of Grey [31] states that \(P_a^Y(\tau_0 = \infty) > 0\) if and only if \(Y_t\) is super-critical. This is analogous to the situation for discrete branching processes, which survive forever with positive probability if and only if they are super-critical.

The Lévy-Khintchine formula characterizing infinitely divisible distributions is closely related to the theory of Lévy processes. A Lévy process is an \(\mathbb{R}^d\)-valued stochastic process with stationary, independent increments. An immediate consequence is that its distribution at time \(t > 0\), say \(W_t\), is infinitely divisible. Because the increments are stationary and independent, it follows that the distribution of a Lévy process is completely characterized by its characteristic exponent, which satisfies the Lévy-Khintchine formula. The following characterization of CSBPs as time-changed Lévy processes is due to Lamperti [49].
Theorem 1.2.5. Let $\psi$ be a branching mechanism, $W_t$ be the Lévy process with Laplace exponent $\psi$, and $\tilde{W}_t$ be the process $W_t$ stopped at 0. For $t > 0$ we define $U(t) = \inf\{\tau : \int_0^\tau \tilde{W}_s^{-1} ds = t\}$. Then $(\tilde{W}_{U(t)})_{t \geq 0}$ is a $\psi$-CSBP. Moreover, all continuous state branching processes can be obtained in this way from a Lévy process with no negative jumps.

We will primarily be interested in the two following critical finite mean branching mechanisms: $\psi(u) = -\gamma^2 u^2$ with $\gamma > 0$ and $\psi(u) = -u^{1+\beta}$, for $0 < \beta < 1$. We note that for both cases we have $\psi'(0^+) = 0$ and $\int_1^\infty |\psi|^{-1} < \infty$, so both are critical and go extinct in finite time almost surely by Grey’s result mentioned above.

Now consider the case $\psi(u) = -\gamma^2 u^2$. The $\psi$-CSBP is Feller’s branching diffusion with branching rate $\gamma$, which we have seen in the previous section. We saw that this process arises as the scaling limit of any branching process whose offspring distribution has finite variance, and that it is the solution to the stochastic differential equation (1.2.6). It is thus the Gaussian CSBP (while it is not a Gaussian process per se, it is driven by Brownian motion). In view of Theorem 1.2.5, Feller’s branching diffusion is a time-changed Brownian motion stopped at 0.

For $\psi(u) = -\gamma^2 u^2$, the branching is strictly binary. Although a CSBP describes an evolving population mass, and a priori does not have “individuals,” it is possible to study its genealogical structure and indeed recover its entire genealogical tree. In the case of $\psi(u) = -\gamma^2 u^2$, the genealogical tree is Aldous’ Brownian Continuum Random Tree, which is strictly binary almost surely. We discuss the genealogies of continuous state branching models at the end of Section 1.2.3.

Let us recall momentarily the general form of a finite mean branching mechanism:

$$\psi(u) = -\frac{\gamma}{2} u^2 + mu + \int_0^\infty (1 - e^{-ur} - ur) \nu(dr).$$

So $\psi(u)$ is a sum of three terms. The first is the one we have just considered in isolation and corresponds to binary branching. The second term $mu$ is a drift term; the sign of $m$ governs whether the process is critical, sub-critical or super-critical, and hence the asymptotics of the mean. The last term allows the most freedom and is determined by the Lévy measure, or jump measure, $\nu$. It follows that, much like Brownian motion (with drift) is the only Lévy process with continuous paths, Feller’s branching diffusion (possibly with drift) is the only CSBP with continuous paths. The rest have discontinuous birth events, i.e. positive jumps.

For $0 < \beta < 1$, one can obtain $\psi(u) = -u^{1+\beta}$ by taking $m = \gamma = 0$ and defining $\nu$ by $\nu(dr) = c\beta r^{-2-\beta} dr$ for $r > 0$. Let us denote the associated CSBP by $Y_t$. Since the only non-zero term in $\psi$ is $\nu$, $Y_t$ grows via upward jumps. In the $N$th branching process in the
approximation scheme we described above, the branching rate is

\[ \kappa_N = (1 + \beta)N^\beta. \]

This implies that the appropriate time-scale on which to observe such branching processes is different than in the finite variance case, which we saw in Theorem 1.2.2. We also have that for each \( N \geq 1 \),

\[ g_N(z) = z + \frac{1}{1 + \beta}(1 - z)^{1+\beta}. \]

This is the generating function of a heavy-tailed random variable in the domain of attraction of a positive stable random variable of index \( 1 + \beta \), whose \( p \)th moment is infinite for all \( p \geq 1 + \beta \). (For a complete description of the stable laws and their associated central limit theorems, see Chapter 9 of Breiman [4].)

The Lévy process associated to \( Y_t \) (c.f. Theorem 1.2.5) is a spectrally positive \((1 + \beta)\)-stable process. For this reason, we will call \( Y_t \) the \((1 + \beta)\)-stable branching CSBP. One consequence of the stable (heavy) tails of the jump measure of \( Y_t \) is that

\[ E_0(Y_t^{1+\theta}) = +\infty \]

for \( \theta \geq \beta \), which is expected in view of the offspring distribution of the associated branching processes. If \( \theta \in (0, \beta) \), the moment is finite. As is common throughout probability theory, the analysis of processes and variables with infinite second moments often requires different methods than those used to study processes with finite second moments.

### 1.2.3 Spatial branching particle systems, superprocesses, and their duals

We now describe our main object of study: superprocesses. Superprocesses are to spatial branching particle systems as CSBPs are to branching processes. We will first define the superprocess associated to a given spatial motion and branching mechanism, then briefly describe a class of branching particle systems which converge in distribution to the superprocess.

The state of a superprocess is a spatial mass distribution, so the natural state space is the space of measures. Let \( \mathcal{M}_F(\mathbb{R}^d) \) denote the space of finite measures on \( \mathbb{R}^d \), which we endow with the topology of weak convergence. Superprocess paths then reside in \( \mathcal{D}([0, \infty), \mathcal{M}_F(\mathbb{R}^d)) \).

We now describe the spatial motion underlying the superprocess. Let \( \xi_t \) be a time-homogeneous Borel strong Markov process on \( \mathbb{R}^d \) with transition semigroup \((S_t)_{t \geq 0}\). We will write \( \mathcal{B}_b \) to denote the space of bounded Borel functions on \( \mathbb{R}^d \), and \( \mathcal{B}_b^+ \) to denote those
that are non-negative. For $\phi \in \mathcal{B}^+_b$, we have

$$(S_t\phi)(x) = E^\xi_x(\phi(\xi_t)).$$

Generally, $S_t$ is a strongly continuous contraction semigroup on a Banach space (e.g. $L^p$ for $1 \leq p < \infty$). The infinitesimal generator of $\xi_t$, which we denote $A$, is a closed operator with dense domain $D(A)$ in that Banach space such that for all $\phi \in D(A)$,

$$\lim_{t \to 0^+} \frac{S_t\phi - \phi}{t} = A\phi.$$

Moreover,

$$\frac{d}{dt} S_t \phi = A S_t \phi = S_t A \phi.$$

We now define the superprocess associated to $A$ and $\psi$, where $A$ is as above and $\psi$ is as in (1.2.14). The definition we give specifies a measure-valued Markov process via its Markov transition kernel. For $\mu \in \mathcal{M}_F(R^d)$ and $\phi \in \mathcal{B}^+_b$, we write $\mu(\phi) = \int \phi(x) \mu(dx)$.

**Definition 1.2.6.** The $(A, \psi)$-superprocess is the $\mathcal{M}_F(R^d)$-valued Markov process $X_t$ with (time-homogeneous) Laplace functional characterized as follows: for $t > 0$, $X_0 \in \mathcal{M}_F(R^d)$ and $\phi \in \mathcal{B}^+_b$,

$$E^X_{X_0}(\exp(-X_t(\phi))) = \exp(-X_0(u_t)), \quad (1.2.22)$$

where $u_t(x)$ is the unique solution (with initial condition $u_0(x) = \phi(x)$) to

$$u_t(x) = (S_t\phi)(x) + \int_0^t S_{t-s}(\psi(u_s))(x) \, ds \quad (1.2.23)$$

for $(t, x) \in (0, \infty) \times R^d$.

For $\phi \in \mathcal{B}^+_b$, we will write $u^{\phi}_t(x)$ and $u^{\phi}(t, x)$ interchangeably to denote the unique solution to (1.2.23). The evolution equation (1.2.23) is a mild form of the partial differential equation (PDE)

$$\begin{cases}
\frac{\partial}{\partial t} u(t, x) = Au(t, x) + \psi(u(t, x)) \\
u(0, x) = \phi(x)
\end{cases} \quad (1.2.24)$$

for $(t, x) \in (0, \infty) \times R^d$. When a unique solution to (1.2.23) exists, generally it is also the unique solution to (1.2.24). Probabilists tend to work with the mild form to avoid technicalities such as regularity, but the differential form is useful for deriving heuristics.

If $u_0(x) = \phi(x) \equiv \lambda > 0$, we have $Au(t, x) \equiv 0$ for all $(t, x) \in (0, \infty) \times R^d$, and it follows that $u(t, x)$ is equal to the (constant in $x$) solution to (1.2.19). We therefore see that the total mass process $X_t(1)$ is a $\psi$-CSBP. All non-spatial properties such as criticality,
survival/extinction probabilities, and blow-up are completely determined by $\psi$.

The $(\mathcal{A}, \psi)$-superprocess has the branching property. Suppose that $\mu, \nu \in \mathcal{M}_F(\mathbb{R}^d)$, and consider $X_t$ with $X_0 = \mu + \nu$. It then follows from (1.2.22) that for all $\phi \in \mathcal{B}_b^+$ we have

$$E^{X}_{\mu+\nu}(\exp(-X_t(\phi))) = E^{X}_{\mu}(\exp(-X_t(\phi)))E^{X}_{\nu}(\exp(-X_t(\phi))).$$

An immediate consequence of the branching property is that $X_t$ is an infinitely divisible random measure. (Intuitively, a random measure $X$ is infinitely divisible if $X(\phi)$ is an infinitely divisible random variable for all suitable test functions $\phi$.)

On an heuristic level, the $(\mathcal{A}, \psi)$-superprocess is a $\psi$-CSBP which lives in $\mathbb{R}^d$, in which the “individuals” (or “particles”) move in space like independent copies of the Markov process generated by $\mathcal{A}$. The difficulty with this description is that a priori there are no particles in the continuous state space setting. One can envision the situation as infinitely many particles of infinitesimal mass which branch with infinite branching rate. In fact, this heuristic can be made rigorous using non-standard analysis. The use of non-standard models to study superprocesses was pioneered by Perkins in [78, 79, 80] and was also used in [14] and [90]. Other methods which recover the individual-based approach are discussed later in this section.

The first limit theorem establishing a superprocess as the limit of a branching particle system was from Watanabe [92] in 1968. (In fact, Watanabe proved the convergence of finite dimensional distributions; an early proof of tightness is due to Roelly-Coppoletta [85].) We will give a relatively general limit theorem which is due to Dawson [13], who constructed a broad array of superprocesses using duality and the Laplace functional. We note in passing that the class of superprocesses under consideration are those that at one point were called Dawson-Watanabe superprocesses.

Let $\psi$ be a critical branching mechanism. For $N \geq 1$ we define a continuous time branching process with branching rate and offspring generating function $\kappa_N$ and $g_N$ as in (1.2.16) and (1.2.17), only now the population lives in space. Fix $\mu \in \mathcal{M}_F(\mathbb{R}^d)$ and let $\mathcal{N}_N$ be a Poisson point process on $\mathbb{R}^d$ with intensity measure $N\mu$. The population begins with one individual at each point in $\mathcal{N}_N$. Each individual carries an $\text{Exponential}(\kappa_N)$ clock and moves like an independent copy of $\xi_t$ until the clock sounds. At this time, the individual is replaced with a random number of offspring sampled according to $g_N$ which appear at the parent’s location. They are all given clocks and evolve spatially like independent copies of $\xi_t$. We denote the empirical measure of the process at time $t \geq 0$, which we obtain by placing a point mass at the location of each living individual, by $X_t^{(N)}$. The following is stated in [13] as Theorem 4.4.1.

**Theorem 1.2.7.** Let $\mathcal{A}$ generate a Borel strong Markov process, $\psi$ be a critical branching
mechanism, and \((\kappa_N, g_N)\) be given by (1.2.16) and (1.2.17) for \(N \geq 1\). Then the process \((N^{-1}X_t^{(N)})_{t \geq 0}\) with initial configuration \(N_N\) converges in distribution (on \(\mathbb{D}([0, \infty), \mathcal{M}_F(\mathbb{R}^d))\)) to the \((\mathcal{A}, \psi)\)-superprocess with \(X_0 = \mu\).

The definition we have given is not the only way to characterize superprocesses, but it is one of the most general. Here we mention a few other approaches. One possibility is to define a superprocess as a Markov process directly by specifying its generator. The state is a measure, so the generator acts on functions of measures. See, for example, Section 4.2 of [22] or the introduction of [45]. The theory of martingale problems is also well-suited to defining and studying superprocesses. This is the approach used by Perkins in [83]. See also Chapter 9 of Ethier and Kurtz [23]. The martingale problem formulation is effective when proving that the density of a superprocess (when it exists) is the solution to a stochastic partial differential equation (SPDE); see [45] and [70].

It is often useful and interesting to consider not only the population size in a branching process but its genealogical structure. This is conceptually simple in the discrete case. One simply considers the entire family tree and keeps track of the ancestral data of each individual. There are no major mathematical difficulties in making this rigorous.

It is not as obvious how to do so, but the genealogical structure of continuous state branching models can be described. We briefly mention a few approaches. The first we discuss is the historical process of Dawson and Perkins [15]. The historical process \(H_t\) is an inhomogeneous Markov process with state space (at time \(t > 0\)) \(\mathcal{M}_F(\mathbb{D}([0, t], \mathbb{R}^d))\), the finite measures on càdlàg paths in \(\mathbb{R}^d\). The paths in \(H_t\) correspond to lineages of the superprocess which survive until time \(t\). The historical process therefore encodes all the genetic (historical) information of \(X_t\), and one can recover \(X_t\) by projecting \(H_t\) onto \(\mathbb{R}^d\) by identifying paths in \(\mathbb{D}([0, t], \mathbb{R}^d)\) with their endpoint.

The Brownian Continuum Random Tree (BCRT) was defined by Aldous in [1]. It is the distributional limit, in an appropriate space, of many families of discrete random trees, including the ancestral trees of branching processes which converge to Feller’s branching diffusion (the \(\psi\)-CSBP with \(\psi = -\frac{1}{2}u^2\)). In particular, the genealogical tree in binary branching CSBPs and superprocesses is the BCRT. In [2], Aldous proposed realizing the binary branching superprocess as a sort of tree-indexed Brownian motion relative to the BCRT as a means of describing the process in a way that incorporates genealogical information. This was ultimately achieved by Le Gall with the introduction of the Brownian snake in a series of works in the 1990s [52, 53]. Later work of Le Gall with Le Jan [58, 59] and Duquesne [18] constructed general Lévy trees, a family of continuum random trees generalizing the BCRT, and the associated Lévy snake, and used it to give a complete genealogical description of superprocesses with general branching mechanisms.

Both the historical process and the Brownian snake give a means of realizing the su-
perprocess in a way which includes genealogical data. The inclusion of ancestral paths recovers, in a sense, the notion that the mass in $X_t$ corresponds to the spatial distribution of massive individuals, although of course the number of individuals is infinite and their mass infinitesimal. Another method for “keeping track of the individuals” in continuous state branching models is the so-called “lookdown” model. In the lookdown approach, each individual has an associated “level,” or index, and individuals are ordered corresponding to the longevity of their ancestral lines. (Imposing this without looking into the future requires the use of exchangeability.) Donnelly and Kurtz introduced this method in [16] and applied it to superprocesses in [17]. The latter work included a construction of the historical process using a lookdown model. Many types of processes, including branching particle systems and genetic models, can be realized using a lookdown model. The variant of the lookdown approach developed by Kurtz and Rodriguez in [46] can also be applied to superprocesses.

This thesis is concerned with critical superprocesses when $X_0$ (and hence $X_t$) has finite mass. There is a literature concerned exclusively with the super-critical regime, often focused on asymptotic behaviour, but we do not discuss this here. In some scenarios super-critical superprocesses can be obtained from their critical counterparts via a Girsanov transformation for superprocesses; see Theorem 7.7.2 of [13]. Asymptotic and ergodic behaviour has also been considered in the regime with infinite mass and critical branching. One example is the persistence/extinction dichotomy proved by Dawson [11].

1.3 Two superprocesses and their properties

We now focus our attention on the two superprocesses which are studied in this thesis: super-Brownian motion and the $(\alpha, \beta)$-superprocess. We define these processes and review some of their fundamental properties.

1.3.1 Super-Brownian motion

Super-Brownian motion is the $(\mathcal{A}, \psi)$-superprocess with $\mathcal{A} = \frac{1}{2} \Delta$ and $\psi(u) = -\frac{\gamma}{2} u^2$, with $\gamma > 0$. Hereafter we fix $\gamma = 1$. We recognize this branching mechanism as corresponding to Feller’s branching diffusion, and so the branching is binary and the total mass process is continuous. The individuals move in space like Brownian motion, since $\frac{1}{2} \Delta$ generates Brownian motion in $\mathbb{R}^d$. Because binary branching and Brownian motion are of considerable interest, super-Brownian motion is of central importance in the study of superprocesses. It is also the best studied.

We will now specialize our notation so that $X_t$ denotes a super-Brownian motion (instead of a general superprocess), and for its law and expectation with initial value $X_0$ we will write
\( P_{X_0}^X \) and \( E_{X_0}^X \), respectively. In this case, (1.2.22) and (1.2.23) take the following form: for all \( \phi \in B_b^+ \),
\[
E_{X_0}^X(\exp(-X_t(\phi))) = \exp(-X_0(V_t^\phi)),
\]
where \( V_t^\phi(x) \) is the unique solution to
\[
V_t(x) = S_t\phi(x) - \frac{1}{2} \int_0^t S_{t-s}(V_s^2) \, ds,
\]
where \( S_t \) denotes the transition semigroup of Brownian motion (the heat semigroup). We will sometimes use the alternate notation \( V^{\phi}(t, x) \) to denote the unique solution to (1.3.2).

The evolution equation (1.3.2) is a mild form of the PDE
\[
\frac{\partial}{\partial t} V(t, x) = \frac{1}{2} \Delta V(t, x) - \frac{1}{2} V^2(t, x).
\]

### 1.3.2 The \((\alpha, \beta)\)-superprocess

For \( 0 < \alpha \leq 2 \) and \( 0 < \beta \leq 1 \), the \((\alpha, \beta)\)-superprocess is the superprocess with \( \psi(u) = -u^{1+\beta} \) and \( \mathcal{A} = -(\Delta)_{\frac{\alpha}{2}} \), the fractional Laplacian. We note that \( \alpha = 2 \) and \( \beta = 1 \) recovers super-Brownian motion, which we have discussed separately. For our purposes we will assume that \( \alpha < 2 \) but allow for \( \beta = 1 \), which corresponds to binary branching.

We recall from Section 1.2.2 that the branching mechanism \( \psi(u) = -u^{1+\beta} \) with \( \beta < 1 \) induces stable branching, which has two important properties: self-similarity and heavy tails. Recall as well that branching of this type has discontinuous birth events at which a positive mass is added to the total population. In the spatial setting, this means that birth events correspond to the addition of a point mass \( c\delta_x \) to the state, with mass \( c > 0 \) and at location \( x \in \mathbb{R}^d \), and that the set of jump times is dense in \([0, \tau_0)\), where \( \tau_0 \) is the extinction time (see Section 6.2.2 of [13]).

For \( 0 < \alpha < 2 \), the fractional Laplacian \( -(\Delta)_{\frac{\alpha}{2}} \) is the generator of the isotropic \( \alpha \)-stable process, which we will often simply call the \( \alpha \)-stable process. This process is a self-similar, pure jump Lévy process. It has discontinuous paths and its transition density has power-law decay. We will use the notation \( \Delta_\alpha = -(\Delta)_{\frac{\alpha}{2}} \).

We will denote the \((\alpha, \beta)\)-process by \( X_t \). With initial measure \( X_0 \in \mathcal{M}_F(\mathbb{R}^d) \), we will write \( P_{X_0}^X \) and \( E_{X_0}^X \) to denote its law and expectation. The Laplace functional is then expressed as follows: for \( \phi \in B_b^+ \),
\[
E_{X_0}^X(\exp(-X_t(\phi))) = \exp(-X_0(u_t^\phi)),
\]

16
where \( u^\phi_t(x) = u^\phi(t, x) \) is the unique solution to

\[
    u_t(x) = S_t^\phi(x) - \int_0^t S_{t-s}^\phi(u^{1+\beta}_s(x)) \, ds.
\]

(1.3.5)

Here \( S_t^\phi \) denotes the semigroup of the \( \alpha \)-stable process (the heat semigroup of \( \Delta_\alpha \)). The associated PDE is then

\[
    \frac{\partial}{\partial t} u_t(x) = \Delta_\alpha u_t(x) - u^{1+\beta}_t(x).
\]

(1.3.6)

1.3.3 Super-Brownian motion with \((1 + \beta)\)-stable branching

In the above we have excluded the natural case of \( A = \frac{1}{2} \Delta \) and \( \psi(u) = -u^{1+\beta} \) with \( \beta < 1 \). This superprocess is called super-Brownian motion with \((1 + \beta)\)-stable branching. We do not emphasize this process because none of the results in this thesis involve it.

1.3.4 Supports

Because superprocesses describe randomly evolving spatial mass distributions, their supports are of great interest. By support we mean the closed support and will write \( S(\mu) \) to denote the support of \( \mu \in \mathcal{M}_F(\mathbb{R}^d) \). In this section we review some classical results concerning the supports of super-Brownian motion and the \((\alpha, \beta)\)-superprocess. (We will also briefly discuss Borel supports.) In some cases, this gives us the opportunity to showcase the methodology we adopt to prove some of our own results.

For both \( X_t \) and \( X_t \), there is a parameter regime in which the measure is absolutely continuous with respect to Lebesgue measure and has a density. In particular, \( X_t \) is absolutely continuous if and only if \( d = 1 \), and \( X_t \) is absolutely continuous if and only if \( d < \frac{\alpha}{\beta} \) [27]. The new results of this thesis are about the densities and so exclusively hold in the absolutely continuous regime. We discuss the densities in detail in the next section. In the sequel, we discuss a variety of properties of the supports of \( X_t \) and \( X_t \), some of which are specific to the non-absolutely continuous regime and are therefore not germane to the results of this thesis. We include them nonetheless in the interest of situating our results within a broader context.

We begin with super-Brownian motion. For a Borel set \( A \subset \mathbb{R}^d \), we say that \( X \) ever charges \( A \) if \( X_t(A) > 0 \) for some \( t > 0 \). We will write \( B(x, R) \) to denote the open ball of radius \( R > 0 \) centered at \( x \), and \( \overline{B(x, R)} \) to denote its closure. The following is due to Iscoe [38].

**Theorem 1.3.1.** Let \( X_0 \in \mathcal{M}_F(\mathbb{R}^d) \) such that \( S(X_0) \subseteq \overline{B(0, R_0)} \), for \( 0 < R_0 < R \). Then

\[
P_{X_0}^X(\text{ever charges } B(0, R)^c) = 1 - \exp(-R^{-2} X_0(v(R^{-1})))
\]
where \( v : B(0, 1) \to \mathbb{R}^+ \) is the unique, positive, radial solution to

\[
\begin{aligned}
\Delta v(x) &= v^2(x), & |x| < 1 \\
v(x) &\to +\infty & \text{as } |x| \to 1.
\end{aligned}
\]  

(1.3.7)

**Corollary 1.3.2.** If \( X_0 \in \mathcal{M}_F(\mathbb{R}^d) \) has compact support, \( \cup_{t \geq 0} S(X_t) \) is compact \( P_{X_0}^X \)-almost surely.

Thus \( S(X_t) \) is compact\(^1\) and the probability that \( X \) ever charges a set is expressed in terms of the solution to a singular boundary problem for a non-linear elliptic equation.

To see how this arises, let us first consider the state at a fixed time \( t > 0 \). Suppose \( S(X_t) \subseteq B(0, R_0) \) and \( 0 \leq R_0 < R \). From (1.3.1), for \( \lambda > 0 \) we have

\[
E_{X_0}^X(\exp(-\lambda X_t(B(0, R)_c))) = \exp(-X_0(V_{\lambda,R}^X)),
\]

(1.3.8)

where \( V_{\lambda,R}^X \) is the solution to (1.3.3) with \( V_0 = \lambda 1_{B(0,R)_c} \). Taking \( \lambda \to \infty \) yields

\[
P_{X_0}^X(X_t(B(0, R)_c) = 0) = \exp(-X_0(V_{\infty,R}^\infty)),
\]

(1.3.9)

where \( V_{\infty,R}^\infty(x) = \lim_{\lambda \to \infty} V_{\lambda,R}^X(x) \). Informally, \( V_{\infty,R}^\infty \) is the solution to (1.3.3) with \( V_0 = \infty 1_{B(0,R)_c} \). The probability that \( X_t \) charges \( B(0, R)_c \) at fixed \( t > 0 \) is expressed in terms of the solution to a singular initial value problem to the dual equation.

The approach used to prove Theorem 1.3.1 is similar. It uses what is called the *occupation time* process of \( X_t \), namely

\[
\mathcal{X}_t(\phi) = \int_0^t X_s(\phi) \, ds.
\]

(1.3.10)

The event that \( X \) ever charges \( A \subset \mathbb{R}^d \) is equivalent to \( \mathcal{X}(A) > 0 \), where \( \mathcal{X} \) is the *integrated super-Brownian motion* \( \mathcal{X} = \lim_{t \to \infty} X_t \). The Laplace functional of \( X_t \) is connected to an inhomogeneous version of the regular dual equation (see [38]). The proof of Theorem 1.3.1

\(^1\)The compact support property is sometimes called finite speed propagation. From the perspective of Brownian motion, it is not surprising that \( S(X_t) \) is compact; Brownian paths are continuous, and every point in \( S(X_t) \) is the endpoint of a Brownian path. Of course, there are infinitely many such paths, so a priori this does guarantee that the collection of their endpoints is bounded. However, the strong dependence structure between them, namely the genealogical structure, causes \( S(X_t) \) to be compact.

Another perspective is as follows. If we consider super-Brownian motion with branching rate \( \gamma > 0 \), (i.e. with branching mechanism \( \psi(u) = -\frac{\gamma}{2} u^2 \)) the limit of the dynamics as \( \gamma \to 0 \) is simply heat flow, which famously has infinite-speed propagation. But for any \( \gamma > 0 \), the support is compact (although, on average, its diameter is larger for smaller values of \( \gamma \)). One can view this as the noise, that is the randomness associated to branching, slowing down the propagation of the mass distribution. An example of a similar phenomenon is a celebrated result showing that the speed of stochastic travelling waves solving a stochastic FKPP equation is slower than the classical travelling wave solutions of the FKPP equation [68].
considers a family of solutions to the inhomogeneous equation and ultimately establishes their convergence as \( t \to \infty \) to the solution of (1.3.7). There is also a direct connection between super-Brownian motion and the elliptic equation \( \Delta v = v^2 \) due to Le Gall [54] which uses the Brownian snake and exit measures to represent these solutions probabilistically. This development occurred after the original proof of the compact support property, but it gives a direct proof.

Besides \( X \) ever charging the complement of a large ball, this methodology can be adapted to other events. In the same paper, Iscoe considered the probability that \( X \) ever charged \( B(x, \epsilon) \) when \( X_0 = \delta_0 \) and determined its asymptotics for large \( |x| \) and small \( \epsilon \). In [14], Dawson, Iscoe and Perkins considered similar events but with \( Y_t \) rather than \( X \). They gave tight bounds on the probability that \( X_t \) charges a given set before some time \( t_0 > 0 \) which we use in a proof in this thesis.

The support process of super-Brownian motion is right continuous with a one-sided modulus of continuity. For \( A \subset \mathbb{R}^d \) and \( \delta > 0 \), let \( A^\delta = \{ x \in \mathbb{R}^d : d(x, A) \leq \delta \} \), where \( d(x, A) = \inf_{y \in A} |x - y| \). For \( r > 0 \), let \( h(r) = \sqrt{r \log(r^{-1})} \vee 1 \). The following is due to Dawson, Iscoe and Perkins [14].

**Theorem 1.3.3.** (a) For every \( c > 2 \), a.s. there exists \( \delta(c) > 0 \) such that if \( t, s \geq 0 \) with \( 0 < t - s \leq \delta(c) \), then \( S(X_t) \subset S(X_s)^{c h(t-s)} \).

(b) The map \( t \to S(X_t) \) is right-continuous with respect to the Hausdorff topology on compact sets.

Brownian motion is a two-dimensional object; if \( B_t \) is a Brownian motion in \( \mathbb{R}^d \) and \( R(B) = \{ B_t : t \in [0, 1] \} \), then \( \dim(R(B)) = 2 \wedge d \). The dimension of the space can play a limiting role in the dimension of \( R(B) \) (which occurs in dimension one) but otherwise \( R(B) \) is two-dimensional.\(^2\)

The state of super-Brownian motion at a fixed time is two-dimensional. The results indicating this take different forms for \( d = 1 \), \( d = 2 \), and \( d \geq 3 \). First, recall that for \( d = 1 \), \( X_t \) is absolutely continuous. Its size, in the sense of dimension, is as large as the space will allow.

Now consider \( d \geq 3 \). For an increasing, continuous function \( \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \), we will write \( \varphi - m(\cdot) \) to denote the \( \varphi \)-Hausdorff measure on \( \mathbb{R}^d \). Let \( \varphi_2(r) = r^2 \log^+ \log^+(r^{-1}) \). Suppose \( d \geq 3 \). Then there exist constants \( 0 < c(d) < C(d) \) such that \( P_{X_0}^{X} \)-a.s., for all Borel sets \( B \)

---

\(^2\)This fact has a beautiful extension called Kaufman’s Theorem, or the dimension doubling theorem. For \( A \subset [0, 1] \), let \( B(A) = \{ B_t : t \in A \} \), where \( B_t \) is a Brownian motion in dimension \( d \geq 2 \). Then for all \( A \subset [0, 1] \), \( \dim(B(A)) = 2 \dim(A) \) almost surely. What is most striking is that this holds simultaneously for all sets \( A \), rather than first fixing \( A \), which allows one to apply it to random sets which can themselves depend on the Brownian motion.
and all $t > 0$,

$$c(d)\varphi_2 - m(B \cap S(X_t)) \leq X_t(B) \leq C(d)\varphi_2 - m(B \cap S(X_t)).$$  \hspace{1cm} (1.3.11)

This very precise result is due to Perkins [79]. It should be emphasized that it holds for all $t$ simultaneously. For fixed $t$, (1.3.11) holds with $c(d) = C(d)$ almost surely (see Theorem 5.2 of [15]).

The situation for $d = 2$ is similar. The lower bound in (1.3.11) holds unchanged. In the upper bound, $\varphi_2 - m$ is replaced with $\varphi_2^* - m$, where $\varphi_2^*(r) = r^2(\log + r^{-1})^2$. For fixed $t > 0$, it was shown by Le Gall and Perkins [60] that $X_t(\cdot) = c\hat{\varphi}_2 - m(\cdot \cap S(X_t))$ a.s. for a deterministic constant $c > 0$, where $\hat{\varphi}_2(r) = r^2\log + r^{-1}\log + \log + (r^{-1})$.

For both $d = 2$ and $d \geq 3$, the above results imply that $S(X_t)$ is Lebesgue null, and so $X_t$ is singular with respect to $d$-dimensional Lebesgue measure. They also imply that $\dim(S(X_t)) = 2$. Furthermore, the fact that $X_t$ is equivalent (either up to multiplicative constants or exactly) to the $\varphi$-Hausdorff measure on its support for some $\varphi$ suggests that the study of $X_t$ can fundamentally be reduced to the study of $S(X_t)$. In [81], it is shown that, with probability one, for all $t > 0$ $X_t$ can be recovered as the limit of the (suitably renormalized) Lebesgue measure on a sequence of shrinking enlargements of $S(X_t)$.

Many interesting path and support properties concern the process as a whole rather than at a fixed time. This includes the study of polar sets and multiple points (see [14] and [80]). The support of the integrated process satisfies $\dim(S(X_t)) = 4 \wedge d$ and $X_t$ is absolutely continuous (under mild conditions on $X_0$) for $d \leq 3$ [89]. Section III of [83] gives a good overview of support properties of super-Brownian motion, including some mentioned above.

Let us now consider the $(\alpha, \beta)$-superprocess $X_t$. In some respects, the situation could not be more different. This is largely due to the discontinuity and long-range jumps of the $\alpha$-stable process. The following is originally due to Perkins [80] (see also Evans and Perkins [24]), with more recent work in a more general setting by Li and Zhou [61].

**Theorem 1.3.4.** For $X_0 \in \mathcal{M}_F(\mathbb{R}^d)$ and $t > 0$,

$$P_{X_0}^X(S(X_t) = \mathbb{R}^d \mid X_t \neq 0) = 1.$$ 

Conditioned on survival, the support of $X_t$ is unbounded and effectively non-random. This property is called *instantaneous propagation*, because when $X_0$ has compact support, the support of $X_t$ equals $\mathbb{R}^d$ a.s. for any $t > 0$ unless $X_t = 0$.

This theorem also implies that, in contrast with super-Brownian motion, $\dim(S(X_t)) = d$ a.s. on $\{X_t \neq 0\}$ regardless of the ambient dimension $d$. This should not be mistaken as implying that $X_t$ has simple or uninteresting support properties. What it really implies is that, unlike super-Brownian motion, the fractal properties of $X_t$ are not well represented by
its closed support. $S(\mathcal{X}_t) = \mathbb{R}^d$ means that every neighbourhood of every point has positive mass, but it does not tell us how this mass is distributed.

A Borel set $A \subset \mathbb{R}^d$ is a Borel support for $\mu$ if $\mu(A^c) = 0$. It was shown by Perkins [78] that when $d > \alpha$ and $\beta = 1$, $\mathcal{X}_t$ has a Borel support $\Lambda_t$ with $\dim(\Lambda_t) = \alpha$. Moreover, for $\varphi_\alpha(r) = r^\alpha \log^+ \log^+ r^{-1}$, we have the following: there exist universal constants $0 < c(\alpha, d) < C(\alpha, d)$ such that $P_{X_t^0}$-a.s., for all $t > 0$ there exists a Borel set $\Lambda_t$ such that for all Borel $B$,

$$c(\alpha, d) \varphi_\alpha - m(B \cap \Lambda_t) \leq \mathcal{X}_t(B) \leq C(\alpha, d) \varphi_\alpha - m(B \cap \Lambda_t). \quad (1.3.12)$$

So $\mathcal{X}_t$ is equivalent up to universal constants to its $\varphi_\alpha$-Hausdorff measures restricted to a particular Borel support $\Lambda_t$ (which Perkins defines explicitly) of dimension $\alpha$. Borel supports are highly non-unique, but this is the best case scenario, since Theorem 1.3.4 implies that no such result can hold with the closed support. When $d < \alpha$, the limiting effect of the ambient dimension takes over; in this case, $\mathcal{X}_t$ is absolutely continuous with respect to the Lebesgue measure.

For $\beta < 1$ the dimension of $\mathcal{X}_t$ is $\frac{\alpha}{\beta}$, but the known results are less precise. In [12], Dawson gives a proof that $\mathcal{X}_t$ has a Borel support of dimension $\frac{\alpha}{\beta}$, and for any random Borel set $A \subset \mathbb{R}^d$, $\mathcal{X}_t(A) = 0$ a.s. on $\{\dim(A) < \frac{\alpha}{\beta}\}$. See Theorem 7.3.1 of [12]. As we have noted, $\mathcal{X}_t$ is absolutely continuous if and only if $d < \frac{\alpha}{\beta}$, as we discuss presently.

### 1.3.5 Densities

We now discuss absolute continuity of $X_t$ and $\mathcal{X}_t$, which we have alluded to in the previous section. Let $0 < \alpha \leq 2$ and $0 < \beta \leq 1$ and consider the $(\alpha, \beta)$-superprocess. (This includes the case $(\alpha, \beta) = (2, 1)$, which is super-Brownian motion.) The following is originally due to Fleischmann [27].

**Theorem 1.3.5.** For $t > 0$, $\mathcal{X}_t$ is absolutely continuous almost surely if and only if $\beta < \frac{\alpha}{2}$.

Thus when $\beta < \frac{\alpha}{2}$, the superprocess has a density $\mathcal{X}_t(x)$ so that $\mathcal{X}_t(dx) = \mathcal{X}_t(x)dx$. For super-Brownian motion, this is the case if and only if $d = 1$. The new results in this thesis concern properties of the density and so hold in the $\beta < \frac{\alpha}{2}$ regime.

Given a density function, it is natural to ask about its regularity properties. We first discuss the regularity of the density of super-Brownian motion in dimension one, which we denote by $X_t(x)$. In this case, there is a version of the density which is jointly continuous in $(t, x)$. Moreover, it is $\eta$-Hölder continuous for all $\eta < \frac{1}{2}$ in space and all $\eta < \frac{1}{4}$ in time. That is, for each $\eta_x < \frac{1}{2}$ and $\eta_t < \frac{1}{4}$, for $t > 0$ there exists a random constant $c(\delta) > 0$ such
that if $|t - s| < \delta$ and $|x - y| < \delta$,

$$|X_t(x) - X_s(y)| < c(\delta)(|x - y|^{\eta_x} + |t - s|^{\eta_t}).$$  \hspace{1cm} (1.3.13)

The modulus of continuity for $X_t(x)$ is often proved as a corollary of another important result. It was shown by Konno and Shiga [45], and independently by Reimers [84], that $X_t(x)$ is a jointly continuous solution to the stochastic partial differential equation (SPDE)

$$\frac{\partial}{\partial t}X_t(x) = \frac{1}{2}\Delta X_t(x) + \frac{1}{2}\sqrt{X_t(x)}\dot{W}(t, x).$$  \hspace{1cm} (1.3.14)

In the above, $\dot{W}$ is a space-time white Gaussian noise. (Equivalently, $W$ is a cylindrical Wiener process on $\mathbb{R}^+ \times \mathbb{R}$ and $\dot{W}$ is its distributional derivative.) The proof that the density is jointly continuous uses a space-time Kolmogorov continuity criterion which implies the Hölder moduli in (1.3.13).

The SPDE (1.3.14) is of course formal, as its solution is non-differentiable. For a precise definition of a solution and a proof that the equation is solved by the density of super-Brownian motion, see [45] or Section III.4 of [83]. The notes of Walsh [91] are an excellent introduction to the classical theory of SPDE.

We now consider the density when $\beta < 1$. Because the stable branching mechanism occasions birth events when an atom is added to the measure, there is a random set of times at which $X_t$ is not absolutely continuous. Like the jumps of a stable process, the set of jump times is countable and dense, so the density cannot be jointly continuous. Most results concern the fixed-time regularity of $X_t(x)$.

The regularity of super-Brownian motion with $(1+\beta)$-stable branching was first studied by Mytnik and Perkins in [71]. As this is the $(\alpha, \beta)$-superprocess with $\alpha = 2$, the density exists if and only if $d < \frac{2}{\beta}$. They established a dichotomy: if $d = 1$, the density is continuous at a fixed time; if $d > 1$ and the density exists, it is locally unbounded. In this work the optimal modulus of continuity was not considered. It was also shown in [71] that for fixed $x \in \mathbb{R}^d$, $X_t(x)$ is locally unbounded in the temporal variable for all $1 \leq d < \frac{2}{\beta}$.

The above results have since been extended to cover all $\alpha \in (0, 2]$ and include moduli of continuity. See [76] for a comprehensive survey. The regularity properties are more nuanced for superprocesses with stable branching. We emphasize three articles in particular [28, 29, 75].

In [28], the continuity/local unboundedness dichotomy was extended for all $\alpha \in (0, 2]$, and the optimal modulus of continuity was determined.

**Theorem 1.3.6.** (a) If $d = 1$ and $\alpha > 1 + \beta$, then there is a version of the density $X_t(\cdot)$ which is locally $\eta$-Hölder continuous for all $\eta < \eta_c := \frac{\alpha}{1+\beta} - 1$. 

22
(b) If \( d > 1 \), or \( d = 1 \) and \( \alpha \leq 1 + \beta \), then \( \|1_U X_t(\cdot)\|_\infty = \infty \) for all open sets \( U \) almost surely on \( \{X_t \neq 0\} \).

Part (a) above is optimal; it was also shown that \( X_t(x) \) is not \( \eta \)-Hölder continuous for \( \eta \geq \eta_c \). Thus \( \eta_c \) is the optimal local modulus of continuity. The optimal modulus of continuity at a fixed point is better and was established in [29]. We say that \( f(x) \) is \( \eta \)-Hölder continuous at a point \( x_0 \) if there exists \( \delta > 0 \) such that

\[
\sup_{x: |x-x_0| \leq \delta} |x-x_0|^{-\eta} |f(x) - f(x_0)| < \infty.
\]

**Theorem 1.3.7.** Let \( d = 1 \), \( \alpha > 1 + \beta \), and fix \( t > 0 \) and \( x \in \mathbb{R} \). Then for any \( \eta < \eta^*_c \) := \( \min\{1+\alpha,1+\beta\} \), the continuous version of \( X_t(x) \) is almost surely \( \eta \)-Hölder continuous at \( x \).

The above modulus is again optimal. Finally, the so-called multifractal spectrum of moduli of continuity was determined in [75]. For all \( \eta \in [\eta_c,\eta^*_c) \), there is a set of points at which the density has optimal modulus of continuity \( \eta \). The Hausdorff dimension of this set of points was computed for each \( \eta \).

Let us briefly revisit the Laplace functional in the context of absolute continuity. For simplicity of notation we will consider \( X_t \) only, but the same principle applies for \( X_t \). The only requirement is that the density exists. We recall that \( E_{X_0}^X(\exp(-X_t(\phi))) = \exp(-X_0(u_0^\phi)) \)

for all \( \phi \in \mathcal{B}^+_1 \), where \( u_0^\phi(x) \) is the unique solution to the evolution equation (1.3.5) with \( u_0 = \phi \). (It is also the unique weak solution to (1.3.6) with \( u_0 = \phi \).) The dual pairing between finite measures and bounded functions make \( \mathcal{B}^+_0 \) the appropriate class of test functions to integrate against \( X_t \) when it is a measure. When \( X_t \) has a density, we have

\[
X_t(\phi) = \int X_t(x) \phi(x) \, dx.
\]

In the above, we can plausibly replace \( \phi(x) \, dx \) with a measure \( \mu(dx) \) in order to consider the pairing \( \mu(X_t) = \int X_t(x) \mu(dx) \) for certain measures \( \mu \). Indeed, this is the case for general finite measures, and there exist mild solutions to (1.3.6) with \( u_0 = \mu \in \mathcal{M}_F(\mathbb{R}^d) \) [7]. The dual relationship is preserved, and thus we can analyse functionals of the density like \( \mu(X_t) \) via solutions to the (1.3.6) with measure initial data.

Much of our work on the densities of \( X_t \) involves the analysis of solutions to the dual equation with measure-valued initial data. This is given the necessary rigorous treatment in the chapters of the thesis where we prove our results. We defer all discussions concerning the existence and uniqueness of solutions with measure-valued initial data and
will refer interchangeably to the solutions of the integral equation and partial differential equation.

### 1.3.6 The canonical measure and cluster decompositions

Every infinitely divisible measure has what is called a **canonical measure**. (For example, see Theorem 3.4.1 of [12].) It arises, roughly speaking, as the intensity measure in a Poissonization procedure. In this section we describe the canonical measure and cluster representations for superprocesses.

First we will consider super-Brownian motion \( \mathbb{X} \) with initial measure \( \mathbb{X}_0 \). By the branching property, for \( t > 0 \) and \( N \in \mathbb{N} \) we can realize \( \mathbb{X} \) as a sum of \( N \) independent super-Brownian motions with initial measure \( N^{-1}\mathbb{X}_0 \). In fact, this holds not just for fixed \( t \) but as a process; for all \( N \in \mathbb{N} \),

\[
\mathbb{X}_t \overset{\mathcal{D}}{=} \mathbb{X}_{N,1} + \cdots + \mathbb{X}_{N,N},
\]

where the \( \mathbb{X}_{N,i} \) are iid with law \( P_{N^{-1}\mathbb{X}_0} \) (see (II.7.1) of [83]). Fix \( t > 0 \). A simple argument using duality implies that

\[
P_{N^{-1}\mathbb{X}_0}(\mathbb{X}_t \neq 0) = p_N = 1 - \exp(N^{-1}\mathbb{X}_0(1)v(t))
\]

where \( v(t) \) is the solution to \( v' = -\frac{1}{2}v^2 \) with \( v \to \infty \) as \( t \to 0^+ \). In fact, \( v(t) = 2/t \). For large \( N \), we have

\[
p_N \sim N^{-1}\mathbb{X}_0(1)v(t),
\]

where \( a_N \sim b_N \) means that \( \lim_{N \to \infty} \frac{a_N}{b_N} = 1 \). The number of clusters in (1.3.15) which survive until time \( t \) is Binomial\((N,p_N)\). By the Poisson approximation to the binomial and (1.3.16), as \( N \to \infty \) the number of surviving clusters converges to a Poisson\((\mathbb{X}_0(1)v(t))\) random variable.

The convergence of the number of surviving clusters in the \( N \)th cluster decomposition (1.3.15) suggests that \( \mathbb{X} \) itself has a Poisson representation, and this is indeed the case. The **canonical measure** \( \mathbb{N}^\mathbb{X}_x \) of super-Brownian motion is a \( \sigma \)-finite measure on \( \mathbb{D}([0,\infty),\mathcal{M}_F(\mathbb{R}^d)) \) which, informally, describes the super-Brownian motion issuing from a single infinitesimal ancestor at location \( x \in \mathbb{R}^d \). It can be obtained via the limit

\[
\mathbb{N}^\mathbb{X}_x(\cdot) = \lim_{N \to \infty} N P_{N^{-1}\delta_x}^\mathbb{X}(\cdot),
\]

(For example, see Theorem II.7.3(a) of [83], which establishes that the renormalized laws of a sequence of branching particle systems converges to the canonical measure.) The canonical
measure is the intensity measure in a Poisson cluster representation of the superprocess. Let \( X_0 \in \mathcal{M}_F(\mathbb{R}^d) \) and let \( N_{X_0}^X \) be given by

\[
N_{X_0}^X(\cdot) = \int N^X_x(\cdot) X_0(dx).
\]

We then let \( \Theta(\cdot) \) be a Poisson point process on \( \mathbb{D}([0, \infty), \mathcal{M}_F(\mathbb{R}^d)) \) with intensity \( N_{X_0}^X \). Then the process defined by

\[
X_t(\cdot) = \begin{cases} 
\int \nu_t(\cdot) \Theta(d\nu), & t > 0 \\
X_0(\cdot), & t = 0
\end{cases}
\]

is a realization of super-Brownian motion with initial measure \( X_0 \), where for a path-valued “point” \( \nu \in \mathbb{D}([0, \infty), \mathcal{M}_F(\mathbb{R}^d)) \) in the point process \( \Theta \), \( \nu_t \in \mathcal{M}_F(\mathbb{R}^d) \) is the measure at time \( t \). For a proof, see Theorem II.7.3(a) of [83].

The canonical measure is also related to solutions to the dual equation. For \( \phi \in \mathcal{B}_b^+ \), if \( V_t^\phi \) is the solution to (1.3.2) then

\[
N_{X_0}^X(1 - \exp(-X_t(\phi))) = V_t^\phi(x).
\]

The above can be derived from a general formula for the Laplace functionals of Poisson point processes (for example, see Theorem 24.14 of [43]) and (1.3.1). It can be shown directly from (1.3.19) that the mass of \( \{X_t \neq 0\} \) under \( N_{X_0}^X \) is \( v(t) \), that is,

\[
N_{X_0}^X(X_t \neq 0) = \frac{2}{t}.
\]

This is particularly useful when we consider the Poisson cluster representation at a fixed time. Let \( X_t \) be a super-Brownian motion with initial measure \( X_0 \). From (1.3.18) it follows that

\[
X_t \overset{D}{=} \sum_{n=1}^N \hat{X}_t^i,
\]

where \( N \) is a Poisson(\( X_0(1)2t^{-1} \)) random variable and the clusters \( \hat{X}_t^i \) are iid with law \( N_{X_0}^X(X_t \in \cdot | X_t \neq 0) \). For \( 0 < s < t \), conditional on \( X_s \), by the above and the Markov property one can express \( X_t \) via a cluster decomposition with order \( X_s(1)(t-s)^{-1} \) clusters.

We now consider the \((\alpha, \beta)\)-superprocess. The heuristics are identical, so we simply state the relevant results. There is a \( \sigma \)-finite measure on \( \mathbb{D}([0, \infty), \mathcal{M}_F(\mathbb{R}^d)) \) called the canonical measure of the \((\alpha, \beta)\)-superprocess and denoted \( N_{X}^X \) such that

\[
N_{X}^X(1 - \exp(-X_t(\phi))) = u_t^\phi(x),
\]
where \( u_0^\phi(x) \) is the solution to (1.3.5) with \( u_0 = \phi \in B_0^+ \). Because the non-linearity of the dual equation is different, the asymptotics of the rate of surviving clusters differ from the super-Brownian case. In particular, choosing \( \phi \equiv \lambda \) in (1.3.22) and taking \( \lambda \to \infty \), it follows that
\[
N^X_\omega(\mathcal{X}_t \neq 0) = U_t,
\]
where
\[
U_t = \left( \frac{1}{\beta t} \right)^{\frac{1}{\beta}}.
\]
A Poisson point process realization analogous to (1.3.18) holds in this case as well (for example see Theorem 4.2.1 of [18]). If \( \mathcal{X}_t \) has law \( P^X_{\mathcal{X}_0} \), for fixed \( t > 0 \) we have
\[
\mathcal{X}_t \overset{D}{=} \sum_{n=1}^N \hat{X}^i_t,
\]
where \( N \) is Poisson((\( \beta t \)^{-\frac{1}{\beta}}) and the \( \hat{X}^i_t \) are iid with law \( N^X_{\mathcal{X}_0}(\mathcal{X}_t \in \cdot | \mathcal{X}_t \neq 0) \). The smaller \( \beta \) is, the more surviving clusters there are in a short time cluster decomposition of \( \mathcal{X}_t \). This fact is important in the proof of one of our main results about \( \mathcal{X}_t \).

### 1.4 Our Results

We now describe the new results proved in this thesis. Complete descriptions and proofs are given in the research chapters. The discussion is divided into two sections corresponding to our results on super-Brownian motion and the \((\alpha, \beta)\)-superprocess.

In both cases we are interested in the absolutely continuous regime, which we restrict to now. We study fine properties of the density and our analysis is based on the limiting behaviour of the Laplace functional. Recall the proof that super-Brownian motion has compact support which we sketched in Section 1.3.4. Restating the argument using canonical measure, for \( x \in \mathbb{R}^d \) and \( t, R, \lambda > 0 \), we have
\[
N^X_x(1 - (\exp(-\lambda X_t(B(0,R)^c))) = V_t^{\lambda,R}(x),
\]
where \( V_t^{\lambda,r} \) is the solution to (1.3.3) with \( V_0 = \lambda 1_{B(0,R)^c} \). This implies that
\[
N^X_x(X_t(B(0,R)^c > 0) = V_t^{\infty,R}(x) =: \lim_{\lambda \to \infty} V_t^{\lambda,R}(x),
\]
so the probability that \( X_t \) charges \( B(0,R)^c \) is governed by the (singular) limit of a family of solutions to the dual equation.

The same principle applies to general charging-type events, i.e. \( \{F(X_t) > 0\} \), where
\[ F(X_t) = X_t(\phi) \text{ for } \phi \in B^+ \text{ or } F(X_t) = \mu(X_t) \text{ for } \mu \in \mathcal{M}_F(\mathbb{R}^d), \]

where the quantity \[ \mu(X_t) = \int X_t(x) \mu(dx) \]

is well-defined when the density exists, as we discuss in Section 1.4.2. Because we are interested in the density, we focus on charging events for measures. We have that \[ N^X(\mu(X_t) > 0) \]

is equal to \[ \lim_{\lambda \to \infty} V^{\lambda \mu}(x), \]

where \[ V^\lambda \]

is the solution to the dual equation with initial data \[ \lambda \mu. \]

The same holds \emph{mutatis mutandis} for the \((\alpha, \beta)\)-superprocess.

While our results for the two processes are very different in nature, the proofs of both have this principle at their core.

In the case of the \((\alpha, \beta)\)-superprocess we are directly interested in the event \[ \{ \mu(X_t) > 0 \} \]

for various measures \[ \mu, \]

so the connection with the dual equation is obvious. It is a consequence of instantaneous propagation (recall Theorem 1.3.4) that the local properties of \[ \mu \]

determine the probability of \[ \{ \mu(X_t) > 0 \} \]

to a much greater extent than its large-scale spatial properties. Our characterization of \[ \{ \mu(X_t) > 0 \} \]

is based on fractal properties of \[ \mu, \]

in particular on some notions of its dimension, and we prove a dichotomy for the \((\alpha, \beta)\)-superprocess concerning when \[ X_t \]

almost surely charges a measure of a given dimension.

For super-Brownian motion in dimension one, if \[ \mu, X_0 \in \mathcal{M}_F(\mathbb{R}^d) \]

are both compactly supported and separated by some distance, the probability that \[ \mu(X_t) \]

is positive is largely governed by the relative locations of \[ S(X_0) \]

and \[ S(\mu) \]

because \[ X_t \]

has compact support and propagates with finite speed. Our interests in this case rather lie with the boundary of the zero set of the density, which is closely related to the asymptotics of the probability \[ P^X_{X_0}(0 < X_t(x) \leq a) \]

for small values of \[ a. \]

\[ P^X_{X_0}(X_t(x) > 0) \]

determined by \[ V^{\infty}_t(x), \]

where \[ V^{\infty} \]

is the limit as \[ \lambda \to \infty \]

of solutions to the dual equation with initial data \[ \lambda \delta_0. \]

Remarkably, the asymptotics of \[ P^X_{X_0}(0 < X_t(x) \leq a) \]

are also closely related to \[ V^{\infty}_t \], which we describe in the next section.

### 1.4.1 A boundary local time for super-Brownian motion

We now discuss our results concerning the density of super-Brownian motion. The work we discuss here is contained in Chapter 2 and Chapter 3.

Let \[ X_t \]

be a super-Brownian motion in dimension one. For \[ t > 0, \]

\[ X_t \]

has a density \[ X_t(x) \]

which is \[ \eta \]-Hölder continuous in space for all \[ \eta < \frac{1}{2}. \]

Furthermore, the density is compactly supported. Consider the zero set \[ Z_t = \{ x : X_t(x) = 0 \}. \]

Since \[ X_t \]

has compact support, \[ Z_t \]

contains intervals on which \[ X_t(x) \]

is obviously constant. However, these intervals are adjacent to the intervals where the density is positive, which is also where the density is rough (Hölder-\( \frac{1}{2} \)). The interface between these sets, which is the boundary of the zero set \[ \partial Z_t, \]

is a locus of very delicate behaviour.

One of our major results is the construction of a random measure \[ L_t \]

supported on \[ \partial Z_t \]

which we call the \textit{boundary local time} of \[ X_t. \]

The construction of \[ L_t \]

is based on an
approximation scheme via measures whose mass is concentrated where \( X_t(x) \) is positive but small. Consider the measure \( L^\lambda_t(dx) \) which is defined by its density

\[
L^\lambda_t(x) = w(\lambda)X_t(x)e^{-\lambda X_t(x)},
\]

(1.4.1)

where \( w(\lambda) \) is a scaling factor to be determined. The function \( f_\lambda(z) = ze^{-\lambda z} \) is most massive where \( z = O(\lambda^{-1}) \), and \( f_\lambda(0) = 0 \), and thus the mass of \( L^\lambda_t \) will concentrate where \( X_t(x) = O(\lambda^{-1}) \), roughly speaking on \( \{ x : 0 < X_t(x) \leq \lambda^{-1} \} \). Thus, if we take \( \lambda \to \infty \) and choose \( w(\lambda) \) in order to obtain a non-degenerate limit, the limiting measure will be supported on \( \partial Z_t \). The limiting measure is \( L_t \).

The proper scaling factor \( w(\lambda) \) was known from previous work of Mueller, Mytnik and Perkins [69]. This was the first work dedicated to close analysis of \( \partial Z_t \), and it is also the work in which the existence of the boundary local time was conjectured. The main contribution of [69] was to calculate the Hausdorff dimension of \( \partial Z_t \). An intermediate result determined the left tail of the density when conditioned to be positive. They showed that

\[
P_{X_0}^X(0 < X_t(x) \leq a) \approx a^{2\lambda_0 - 1}
\]

(1.4.2)

for \( a \in (0,1] \). Here \( \approx \) means equal up to multiplicative constants in a manner that will be made precise in Chapter 2. There is some dependence on \( x, t \) and \( X_0 \). The value \( \lambda_0 \in (0,1) \) is the lead eigenvalue associated to a particular Ornstein-Uhlenbeck operator related to \( X_t \). At the end of this section, after stating our main results, we describe \( \lambda_0 \) and how it enters into the analysis. The left tail (1.4.2) is closely related to the following: for \( \lambda \gg 0 \),

\[
E_{X_0}^X \left( \int X_t(x)e^{-\lambda X_t(x)} \, dx \right) \approx \lambda^{-2\lambda_0}.
\]

We therefore deduce that in order for \( L^\lambda_t(x) \) to converge to a finite, non-zero limit as \( \lambda \to \infty \), we should take \( w(\lambda) = \lambda^{2\lambda_0} \). The integral of a test function \( \phi \) against of \( L^\lambda_t \) is thus

\[
L^\lambda_t(\phi) = \lambda^{2\lambda_0} \int \phi(x)X_t(x)e^{-\lambda X_t(x)} \, dx.
\]

(1.4.3)

It was shown in [69] that \( E_{X_0}^X(L^\lambda_t(\phi)) \) converges as \( \lambda \to \infty \), and an explicit formula was given for the limit. This led the authors to conjecture that \( L^\lambda_t \) converges in \( \mathcal{M}_F(\mathbb{R}^d) \). We proved this conjecture in [35].

**Theorem 1.4.1.** \( P_{X_0}^X \)-a.s. there exists a measure \( L_t \) supported on \( \partial Z_t \) such that \( L^\lambda_t \to L_t \) in probability in the metric space \( \mathcal{M}_F(\mathbb{R}^d) \), and \( L^\lambda_t(\phi) \to L_t(\phi) \) in \( \mathcal{L}^2 \) for all \( \phi \in \mathcal{B}_b(\mathbb{R}) \).

We also derive, as a by-product of our method of proof, explicit formulas for first and
second moments of $L_t$. These are crucial for deriving several other properties of $L_t$. First, let us revisit $\dim(\partial Z_t)$. In [69], the authors showed that

$$\dim(\partial Z_t) = 2 - 2\lambda_0 \in (0, 1) \quad (1.4.4)$$

with positive probability on $\{X_t \neq 0\}$. More specifically, they showed that $\dim(\partial Z_t) \leq 2 - 2\lambda_0$ a.s. and $\dim(\partial Z_t) \geq 2 - 2\lambda_0$ with positive probability. The lower bound was conjectured to hold a.s. on $\{X_t \neq 0\}$ but this was not shown. Completing the dimension result was a major motivation for the construction of the boundary local time.

**Theorem 1.4.2.** Under $P^X_{X_0}$, $\dim(\partial Z_t) = 2 - 2\lambda_0$ a.s. on $\{X_t \neq 0\}$.

In order to prove the above using $L_t$ we use Frostman’s Lemma. For $p > 0$ the $p$-energy of $\mu \in \mathcal{M}_F(\mathbb{R}^d)$ is

$$E_p(\mu) = \int |x - y|^{-p} \mu(dx)\mu(dy). \quad (1.4.5)$$

Frostman’s Lemma states that a Borel set $A$ supports a non-zero measure $\mu$ with finite $p$-energy if and only if $A$ has positive $p$-Hausdorff measure. Using the moment formulae we derive for $L_t$, we show that

$$E^X_{X_0}(E_p(L_t)) < \infty \quad (1.4.6)$$

for all $p < 2 - 2\lambda_0$, which implies that $\dim(\partial Z_t) \geq 2 - 2\lambda_0$ whenever $L_t \neq 0$.

To prove Theorem 1.4.2 then requires that $L_t(1) > 0$ a.s. on $\{X_t \neq 0\}$. While a natural and desirable property for $L_t$, the proof is non-trivial. It is based on a novel analysis of one-dimensional super-Brownian motion near the right endpoint of its support. The following was proved in joint work with Perkins [37].

**Theorem 1.4.3.** Under $P^X_{X_0}$, $L_t(1) > 0$ a.s. on $\{X_t \neq 0\}$.

We will sometimes refer to the above as the $0 - 1$ law. We also prove several other results concerning $\partial Z_t$ and $L_t$. These include the fact that $L_t$ is atomless almost surely and a representation for $L_t$ in terms of the boundary local times associated to each cluster in a cluster representation for super-Brownian motion. We also give a local version of the $0 - 1$ law, proving that $L_t$ has positive mass in every neighbourhood of $\partial S(X_t)$, the boundary of the support of the density. It follows from an elementary topological argument that $\partial S(X_t) \subseteq \partial Z_t$. In view of Theorem 1.4.1 and the above, we have

$$\partial S(X_t) \subseteq S(L_t) \subseteq \partial Z_t. \quad (1.4.7)$$

Whether these inclusions are strict or are in fact equalities of sets remains unresolved.
We now discuss the measures \( L^\lambda_t \) and the eigenvalue \( \lambda_0 \) in greater detail. \( L^\lambda_t \) is defined as above because it is amenable to analysis using solutions to the dual equation. We will proceed using canonical measure. First, we observe that for \( \lambda > 0 \),

\[
\mathbb{N}_0(1 - \exp(-\lambda X_t(x))) = V^\lambda_t(x),
\]

(1.4.8)

where \( V^\lambda_t \) is the unique solution to

\[
\frac{\partial}{\partial t} V_t(x) = \frac{1}{2} \Delta V_t(x) - \frac{1}{2} V_t(x)^2, \quad V_0 = \lambda \delta_0.
\]

(1.4.9)

To see that the above holds, one must extend the dual relationship from bounded functions to finite measures. As we have indicated, this can be done when the density exists. Differentiating the above with respect to \( \lambda \) yields

\[
\mathbb{N}_0(X_t(x) \exp(-\lambda X_t(x))) = \frac{\partial}{\partial \lambda} V^\lambda_t(x).
\]

(1.4.10)

Recall that \( X_t(x) e^{-\lambda X_t(x)} \) is the non-normalized density of \( L^\lambda_t \). Hence the above allows us to analyse moments of \( L^\lambda_t \) in terms of (the \( \lambda \)-derivative of) \( V^\lambda_t \). Formally differentiating (1.4.9) term by term with respect to \( \lambda \), one sees that \( W_t(x) = \frac{\partial}{\partial \lambda} V^\lambda_t(x) \) solves the initial value problem

\[
\frac{\partial}{\partial t} W_t(x) = \frac{1}{2} \Delta W_t(x) - W_t(x) V^\lambda_t(x), \quad W_0 = \delta_0.
\]

(1.4.11)

This differentiation can be made rigorous and in particular \( \frac{\partial}{\partial \lambda} V^\lambda_t(x) \) is the unique solution to (1.4.11). This is a linear equation and its solution has a (probabilistic) Feynman-Kac representation. (For a version of the Feynman-Kac formula, see Theorem 7.6 of Karatzas and Shreve [42].) From (1.4.10) and the Feynman-Kac formula, we have

\[
\mathbb{N}_0(X_t(x) \exp(-\lambda X_t(x))) = E^B_0 \left( \exp \left( - \int_0^t V^\lambda_{t-s}(B_s) ds \right) \right| B_t = x \),
\]

(1.4.12)

where \( B_t \) is a standard Brownian motion. Via some changes of variables and scaling arguments, we can obtain a new expression for the above:

\[
\mathbb{N}_0(X_t(x) \exp(-\lambda X_t(x))) = E^Y_0 \left( \exp \left( - \int_0^{\log(\lambda^2 t)} V^\lambda_1 e^{s/2} (Y_{\log(\lambda^2 t) - s}) ds \right) \right| Y_{\log(\lambda^2 t)} = x \).
\]

(1.4.13)

Here \( Y \) is a standard Ornstein-Uhlenbeck process on \( \mathbb{R} \). We will return to the Ornstein-Uhlenbeck process shortly.

From (1.4.8), it follows that \( V^\lambda_t(x) \leq \mathbb{N}_0(X_t \neq 0) = 2/t \) for all \( \lambda > 0 \). We can therefore
take $\lambda \to \infty$ in (1.4.8) and define $V_t^\infty(x)$ by

$$V_t^\infty(x) = \lim_{\lambda \to \infty} V_\lambda^t(x) = \mathbb{N}_0^X(X_t(x) > 0). \tag{1.4.14}$$

To analyse (1.4.13) we would like to replace $V_1^{e^{t/2}}$ with $V_1^\infty$. Clearly we obtain a lower bound for (1.4.13) if we make this replacement directly. On the other hand, in [69] it was shown using preliminary estimates on the rate of convergence of $V_\lambda^1$ to $V_1^\infty$ that

$$\sup_{T>0} \exp \left( \int_0^T \sup_{y \in \mathbb{R}} |V_1^\infty(y) - V_1^{e^{s/2}}(y)| ds \right) < \infty.$$  

This implies that, up to multiplicative constants, we have

$$\mathbb{N}_0(X_t(x) \exp(-\lambda X_t(x))) \approx E_0^Y \left( \exp \left( -\int_0^{\log(\lambda^2 t)} F(Y_s) ds \right) \left| Y_{\log(\lambda^2 t)} = x \right. \right), \tag{1.4.15}$$

where $F(y) = V_1^\infty(y)$.

To determine the asymptotics of (1.4.15) as $\lambda \to \infty$ we appeal to the spectral theory of killed diffusions. Let $A$ be the generator of a standard Ornstein-Uhlenbeck process, $Y_t$, on $\mathbb{R}$. That is, for a twice differentiable test function $f$,

$$Af(x) = -\frac{x^2}{2} f'(x) + \frac{1}{2} f''(x).$$

For a bounded, continuous, non-negative function $\phi$, the operator

$$A^\phi f(x) = Af(x) - \phi(x) f(x)$$

is the generator of a killed Ornstein-Uhlenbeck process, $Y_t^\phi$, with killing rate given by $\phi$. That is, if we continue to denote the un killed Ornstein-Uhlenbeck process by $Y_t$, the instantaneous rate of killing at time $t$ is $\phi(Y_t)$. If $\rho$ denotes the death time of the process, we can express the survival probability up to time $T > 0$ conditional on $Y$ as

$$P(\rho \geq T | Y) = \exp \left( -\int_0^T \phi(Y_s) ds \right). \tag{1.4.16}$$

The operator $A$ is spectrally negative and has discrete spectrum. In particular, there is a sequence of eigenvalues $0 \geq -\lambda_0^\phi \geq -\lambda_1^\phi \geq -\lambda_2^\phi \geq \cdots \to -\infty$. It is possible to express the transition density (from time 0 to time $T > 0$) of the killed diffusion in terms of an
eigenfunction expansion which shows that

\[ P(\rho \geq T) \approx e^{-\lambda_0^T T}. \]  

(1.4.17)

We observe that (1.4.15) can be interpreted as a survival probability as in (1.4.16) with killing function \( F \). One must handle the conditioning on the endpoint, but it essentially follows from the (1.4.15), (1.4.16) and (1.4.17) that

\[ N_0(\rho \geq T) \approx e^{-\lambda_0 T}. \]

(1.4.18)

where \( \lambda_0 = \lambda_0^F \), whence we derive the scaling factor \( w(\lambda) = \lambda^2 \lambda_0 \) for \( L^\lambda_t \). The asymptotics of \( L^\lambda_t(x) \) as \( \lambda \to \infty \) are closely related to the left tail of \( X_t(x) \). The derivation of (1.4.2) in [69] uses (1.4.18) and a Tauberian theorem.

The sketch above indicates how \( \lambda_0 \) appears in the analysis of the first moments of \( L^\lambda_t \) and the left tail of \( X_t(x) \) (and therefore of \( \partial \rho_t \)). Our work, in particular the proof of Theorem 1.4.1, is based on the convergence of second moments of \( L^\lambda_t \), i.e. moments of the form \( N_0(L^\lambda_t(\phi)L^\lambda_t'(\varphi)) \) for \( \phi, \varphi \in B^+_0 \) and \( \lambda, \lambda' > 0 \). Note that

\[ L^\lambda_t(\phi)L^\lambda_t'(\varphi) = (\lambda' \lambda)^{2\lambda_0} \int X_t(x) X_t(y) \phi(x) \varphi(y) \exp(-\lambda X_t(x) - \lambda' X_t(y)) dxdy. \]

To compute the expectation of the above, the essential quantity is

\[ N_0(X_t(x) X_t(y) \exp(-\lambda X_t(x) - \lambda' X_t(y))). \]

We can study this in a similar fashion to (1.4.10) using duality and differentiating with respect to \( \lambda \) and \( \lambda' \). We have

\[ N_0(X_t(x) X_t(y) \exp(-\lambda X_t(x) - \lambda' X_t(y))) = \frac{\partial^2}{\partial \lambda \partial \lambda'} V_t^{\lambda \delta_x + \lambda' \delta_y}(0). \]

(1.4.19)

Here, \( V_t^{\lambda \delta_x + \lambda' \delta_y} \) is the solution to the dual equation with \( V_0 = \lambda \delta_x + \lambda' \delta_y \). These two-pointed solutions (and their limits as \( \lambda, \lambda' \to \infty \)) are critical to understanding the limiting behaviour of second moments of \( L^\lambda_t \). The second derivative in (1.4.19) admits a probabilistic representation analogous to but more complicated than (1.4.12), which involves a branching Brownian path. This can then be rescaled to obtain Ornstein-Uhlenbeck processes as was done for first moments. However, establishing the asymptotic size (i.e. \( (\lambda' \lambda)^{-2\lambda_0} \)) of (1.4.19) does not suffice for our purposes; to show that \( L^\lambda_t \) converges in the space of measures in probability, we need to explicitly compute the limit of (1.4.19) as \( \lambda, \lambda' \to \infty \), a technical computation of considerable length. In doing so, we obtain a second moment formula for
the boundary local time which we use to show many of our other results, including the finite energy result (1.4.6).

In parallel with our work [35, 37] (the latter with Perkins) and that of Mueller, Mytnik and Perkins [69] on the boundary of the zero set of \( X_t \), the boundary of the range of super-Brownian motion has been considered. We recall from Section 1.3.4 that the integrated super-Brownian motion \( \mathbb{X}(\cdot) = \int_0^\infty X_t(\cdot) dt \) has a density \( \mathbb{X}(x) \) in dimensions \( d = 1, 2, 3 \).

Mytnik and Perkins computed the Hausdorff dimension of the boundary of the zero set of \( \mathbb{X}(x) \) in [73]. With Hong, they improved this result by showing that \( \partial(S(\mathbb{X})) \) has the same dimension [33]. (This remains open at fixed time; that is, it has not been shown that \( \partial(S(X_t)) \) has the same dimension as \( \partial(Z_t) \).) Very recently and in the spirit of our work, Hong [34] has constructed a boundary local time supported on \( \partial(S(\mathbb{X})) \); the tools he uses in this higher dimensional setting are quite distinct from those described above.

### 1.4.2 New path properties for the \((\alpha, \beta)\)-superprocess

In this section our object of study is the \((\alpha, \beta)\)-superprocess, \( X_t \), on \( \mathbb{R}^d \) with \( \alpha < 2 \) in the absolutely continuous regime \( \beta < \frac{\alpha}{d} \). We recall from our earlier discussion that for \( t > 0 \) there is a continuous version of the density \( X_t(x) \) if and only if \( d = 1 \) and \( \alpha > 1 + \beta \), and the density is locally unbounded in the discontinuous case. Remarkably, with one exception our results are completely agnostic to this dichotomy.

We recall that for all \( \beta \in (0, 1] \), the \((\alpha, \beta)\)-superprocess with \( \alpha < 2 \) has instantaneous propagation: for \( t > 0 \), \( \text{supp}(X_t) = \mathbb{R}^d \) almost surely when conditioned on survival. Thus the event that \( X_t \) has positive mass on a set of positive \( d \)-dimensional Lebesgue measure is trivial.

Our results are in the spirit of instantaneous propagation, in that we classify certain trivial events for the density. For \( \mu \in \mathcal{M}_F(\mathbb{R}^d) \), we define

\[
\mu(X_t) = \int X_t(x) \mu(dx). \tag{1.4.20}
\]

A priori this object is not well-defined, because the density is only unique up to Lebesgue null sets, and, in view of instantaneous propagation, we are interested in \( \mu(X_t) \) for measures \( \mu \) which are singular with respect to the Lebesgue measure. We show that \( \mu(X_t) \) is well-defined for any finite measure \( \mu \) by first arguing that the density exists almost surely at any fixed point.

A natural starting point is to consider \( \mu = \delta_x \), so that \( \mu(X_t) = X_t(x) \). We define

\[
\beta^*(\alpha) = \frac{\alpha}{d + \alpha}.
\]

**Theorem 1.4.4.** The following hold under \( N_0^X \) and \( P_{X_0}^X \) for \( X_0 \in \mathcal{M}_F(\mathbb{R}^d) \).

(a) Let \( \beta \leq \beta^*(\alpha) \). Then for fixed \( x \in \mathbb{R}^d \) and \( t > 0 \), \( X_t(x) > 0 \) almost surely on \( \{X_t \neq 0\} \).
(b) Let $\beta^*(\alpha) < \beta < \frac{\alpha}{d}$. For fixed $x \in \mathbb{R}^d$ and $t > 0$, $X_t(x) = 0$ with positive probability on \{$X_t \neq 0$\}. Moreover,

$$C_1 \frac{t^{-\frac{1}{\pi}}}{1 + |t^{-\frac{1}{\pi}x}|^{d+\alpha}} \leq N_0^X(X_t(x) > 0) \leq C_2 \frac{t^{-\frac{1}{\pi}} \log(e + |t^{-\frac{1}{\pi}x}|)}{1 + |t^{-\frac{1}{\pi}x}|^{d+\alpha}}$$

(1.4.21)

for constants $0 < C_1 < C_2$.

The above result is due to Chen, Véron, and Wang [7] for $\beta < \beta^*(\alpha)$ and the first two authors for $\beta = \beta^*(\alpha)$ [8]. In fact, these authors’ work was on solutions to the dual equation. Their main theorem, which establishes a dichotomy concerning the existence of very singular solutions to the dual equation, is equivalent to the above statement about the density of the $(\alpha, \beta)$-superprocess.

The above theorem and instantaneous propagation characterize the event \{µ(\X_t) > 0\} when $\mu = \delta_x$ and $\mu = m_U$, the Lebesgue measure restricted to an open set $U$. When conditioned on survival, $m_U(\X_t) > 0$ a.s. for all parameters and $\delta_x(\X_t) = \X_t(x) > 0$ a.s. if and only if $\beta \leq \beta^*(\alpha)$. The point mass $\delta_x$ and Lebesgue measure on an open set are, locally speaking, the most concentrated and most spread out measures on $\mathbb{R}^d$. Our results interpolate between these two and hold for general measures.

For $\delta \in [0, d]$, let $\beta^*(\alpha, \delta) = \frac{\alpha}{d - \delta + \alpha}$. As we will see, this is the critical parameter for a measure which is “of dimension $\delta$.” We write $\mathcal{H}^\delta(A)$ to denote the $x^\delta$-Hausdorff measure of a set $A$. (See [25] or [62] for an introduction to Hausdorff measure.) The Hausdorff measure and Hausdorff dimension of a measure’s support are closely related to mass conditions on small balls for the measure. For example, if $\mu \in \mathcal{M}_F(\mathbb{R}^d)$ satisfies

$$\sup_{x \in \mathbb{R}^d, r > 0} r^{-\delta} \mu(B(x, r)) < \infty,$$

(1.4.22)

then $\mu$ is at least $\delta$-dimensional in the sense that $\mathcal{H}^\delta(S(\mu)) > 0$, and hence $\dim(S(\mu)) \geq \delta$. (This is called Frostman’s Lemma. It is discussed in the introduction of Chapter 4; see also Theorem 8.8 of [62].) A similar result holds for lower bounds; if

$$\inf_{x \in S(\mu), 0 < r < 1} r^{-\delta} \mu(B(x, r)) > 0,$$

(1.4.23)

then $\dim(S(\mu)) \leq \delta$. The following is a partial statement of our results on this problem. For $x \in \mathbb{R}^d$ and $A$ closed, we let $d(x, A) = \inf_{y \in A} |y - x|$.

**Theorem 1.4.5.** Let $\delta \in [0, d]$ and $X_0 \in \mathbb{R}^d$. For $t > 0$ we consider $X_t$ when conditioned on survival under $P_{X_0}^X$ and $N_0^X$.

(a) Suppose that $\mu \in \mathcal{M}_F(\mathbb{R}^d)$ satisfies $\mathcal{H}^\delta(S(\mu)) > 0$. Then if $\beta \leq \beta^*(\alpha, \delta)$, $\mu(X_t) > 0$
almost surely.

(b) Suppose that $\mu$ has compact support and satisfies (1.4.23). Then if $\beta > \beta^*(\alpha, \delta)$, $\mu(\mathcal{X}_t) = 0$ with positive (conditional) probability. Moreover, for all $t > 0$ and $x \in \mathbb{R}^d$ we have

$$\frac{C_1 t^{-\frac{1}{\beta}}}{1 + |t^{-\frac{1}{\alpha}} d(x, S(\mu))|^{d+\alpha}} \leq N^X_x(\mu(\mathcal{X}_t) > 0) \leq C_2 \left[ t^{-\frac{1}{\beta}} \vee t^{-\frac{1}{\delta}} \right] \frac{\log(e + t^{-\frac{1}{\delta}} d(x, S(\mu)))}{1 + |t^{-\frac{1}{\alpha}} d(x, S(\mu))|^{d+\alpha}}$$

for constants $0 < C_1 < C_2$ which depend only on $\alpha$, $d$ and $\mu$.

The $(\alpha, \beta)$-superprocess therefore exhibits remarkably reliable behaviour on a microscopic level. The parameters determine a scale beyond which all charging events involving the density at a fixed time are trivial. In the non-trivial regime (corresponding to part (b) in the above theorem), (1.4.24) gives asymptotic bounds on $N^X_x(\mu(\mathcal{X}_t) > 0)$ as $d(x, S(\mu)) \to \infty$ (for fixed $t > 0$) and $t \downarrow 0$ (for fixed $x$). Under additional assumptions on $\mu$, we improve the lower bound in (1.4.24) so that the power of $t$ matches the power of $t$ in the upper bound for $t \leq 1$ (see Theorem 4.1.20 in Chapter 4). In this case our estimates are nearly sharp. The methods we use to prove parts (a) and (b) in Theorem 1.4.5 are completely different. The proof of (a) is probabilistic in nature and has an underlying cluster heuristic. As such, it also gives a new simple probabilistic proof of Theorem 1.4.4(a) by taking $\mu = \delta_x$. To see how the point case is recovered from our general result, we observe that the dimension $\delta$ of $\mu = \delta_x$ is 0 (in particular, $H^0(S(\delta_x)) = 1 > 0$ for part (a), and $\mu = \delta_x$ satisfies (1.4.23) with $\delta = 0$ for part (b)) and that $\beta^*(\alpha, 0) = \beta^*(\alpha)$.

Before discussing the proof of Theorem 1.4.5(a), we expand on the connection between charging type events for $\mathcal{X}_t$ and the limiting behaviour of the Laplace functional, which we have alluded to previously. Let $\lambda > 0$ and $\mu \in \mathcal{M}_F(\mathbb{R}^d)$. Then

$$N^X_0(1 - \exp(-\mu(\mathcal{X}_t))) = u^\lambda \mu_t(x),$$

where $u^\lambda \mu_t(x)$ is the unique solution to (1.3.6) with measure-valued initial data $\lambda \mu$. (A priori, $u^\lambda \mu_t(x)$ is a solution to the integral equation when $\phi$ is replaced with $\lambda \mu$; it does, in fact, coincide with a weak solution to the (1.3.6). We handle this technicality in the main body of the thesis.) We have $u^\lambda \mu_t(x) \leq U_t$ for all $\lambda > 0$, where we recall from (1.3.23) and (1.3.24) that

$$N^X_0(\mathcal{X}_t \neq 0) = U_t = \left( \frac{1}{\beta t} \right)^{\frac{1}{\beta}}.$$  

(1.4.26)

It follows from (1.4.25) that $u^\infty \mu_t(x) = \lim_{\lambda \to \infty} u^{\lambda \mu}_t(x)$ exists and satisfies

$$u^\infty \mu_t(x) = N^X_0(\mu(\mathcal{X}_t) > 0).$$  

(1.4.27)
This is in the same spirit as (1.4.14) in the previous section. From (1.4.26) and (1.4.27), we conclude the following:

\[ u_t^\infty(x) \equiv U_t \iff \mu(\mathcal{X}_t) > 0 \text{ a.s. on } \{\mathcal{X}_t \neq 0\}. \tag{1.4.28} \]

If \( u_t^\infty(x) \equiv U_t \), we say that \( u_t^\infty \) is flat. Otherwise we say it is non-flat. The event that \( \mathcal{X}_t \) charges \( \mu \) is trivial (i.e. \( \mu(\mathcal{X}_t) > 0 \) a.s. when \( \mathcal{X}_t \) is conditioned on survival) if and only if \( u_t^\infty \) is flat.

To illustrate the above connection, let us return to Theorem 1.4.4, which concerns the density at a fixed point. In [7] the PDE version of Theorem 1.4.4 was motivated by the study of very singular solutions (VSS) to (1.3.6). Informally, the VSS is the solution with initial data equal to \( \infty \cdot \delta_0 \).

A standard approach, and the one they use, is to try to construct the VSS as the limit of \( u_\lambda^\infty(x) \), where \( u_\lambda^\infty(x) \) is the solution of

\[
\begin{cases}
\frac{\partial}{\partial t} u_t(x) = \Delta_\alpha u_t(x) - u_t^{1+\beta}(x) \\
u_0 = \lambda \delta_0.
\end{cases}
\tag{1.4.29}
\]

Let \( u_\infty^\infty(x) = \lim_{\lambda \to \infty} u_\lambda^\infty(x) \). In [7] it was found that \( u_\infty^\infty(x) \) was a very singular solution if and only if \( \beta \in (\beta^*(\alpha), \frac{\alpha}{d}) \), where we recall that \( \beta^*(\alpha) = \frac{d}{d+\alpha} \). For \( \beta \leq \beta^*(\alpha) \), \( u_\infty^\infty(x) \equiv U_t \). The claims of Theorem 1.4.4 then follow under canonical measure from (1.4.27) with \( \mu = \delta_0 \), since for \( x \in \mathbb{R}^d \) we have

\[ \mathbb{N}_0^V(\mathcal{X}_t(x) > 0) = u_t^\infty(x). \tag{1.4.30} \]

The results can be extended to \( \mathcal{X}_t \) under \( P_{\mathcal{X}_0}^V \) via a cluster argument. The dichotomy concerning the existence of a VSS for the equation (1.3.6) and its probabilistic consequences were what originally led to our work on the \((\alpha,\beta)\)-superprocess. Our work generalizes from \( \mu = \delta_x \) to general measures \( \mu \) and gives a dichotomy in terms of the “dimension” of measure as we have noted.

We now sketch the proof of Theorem 1.4.5(a) using a cluster argument. Let \( \mu \in \mathcal{M}_F(\mathbb{R}^d) \) and suppose that \( B \) is a closed ball of radius 1 such that \( \mu(B) > 0 \). By instantaneous propagation, for \( t > 0 \), when \( \mathcal{X}_t \) is conditioned on survival we have \( \mathcal{X}_t(B) > 0 \) a.s. By conditioning, one can restrict to the event \( \{\mathcal{X}_t(B) \geq \epsilon\} \) for \( \epsilon > 0 \).

We consider \( P(\mu(\mathcal{X}_t) > 0 \mid \mathcal{F}_s) \), where \( \mathcal{F}_s = \sigma(\{\mathcal{X}_u: u \leq s\}) \). Conditional on \( \mathcal{F}_s \), by the Markov property we can realize \( \mathcal{X}_t \) via a cluster decomposition as in (1.3.25). On the event

\[ \text{The solution } V_t^\infty(x) \text{ from the previous section is the VSS of } \partial_t V = \frac{1}{2} \Delta V - \frac{1}{2} V^2, \text{ which was originally shown to exist by Brezis, Peletier and Terman [6]. Their results built on seminal work of Brezis and Friedman establishing the existence of solutions with measure-valued initial data [5]. The first work on measure-valued initial data and the VSS for the fractional equation was done thirty years after the same for the non-fractional equation.} \]
\{X_t(B) \geq \epsilon\}, one can appeal to fixed time continuity results to show that \(\{X_s(B) \geq \epsilon/2\}\) for \(s\) sufficiently close to \(t\). Therefore, the intensity of clusters originating in \(B\) in the time \(s\) cluster decomposition of \(X_t\) is at least of order \(\epsilon(t-s)^{-1/\beta}\) (recall from (1.3.23) and (1.3.24) that \(N_x^X(X_{t-s} \neq 0) \approx (t-s)^{-1/\beta}\) as \(s \to t^-\)).

Each cluster \(X^i\) originating in \(B\) at time \(s\) has some probability that \(\mu(X^i) > 0\). We now suppose that for some \(\delta \in [0,d]\),

\[
\mu(B(x,r)) \leq Cr^\delta \tag{1.4.31}
\]

for some \(C > 0\) for all \(x \in \mathbb{R}^d\) and \(r > 0\). The condition quantifies how spread out the mass of \(\mu\) is; the larger the value of \(\delta\), the larger the set over which \(\mu\) spreads its mass (note that (1.4.31) corresponds to (1.4.22), a notion that \(\mu\) is at least \(\delta\)-dimensional) and hence the more likely it is that a cluster \(X^i\) charges \(\mu\).

From (1.4.27), we observe that a lower bound on the probability that a canonical cluster charges \(\mu\) is equivalent to a lower bound on \(u^\infty_\mu(x)\). (Since we ultimately show that \(u^\infty_\mu = U_t\), this is a preliminary lower bound.) As in the previous section, we analyse \(u^\lambda_\mu(x)\) by differentiating \(u^\lambda_\mu(x)\) with respect to \(\lambda\) and using the Feynman-Kac formula. In this case, for \(\lambda > 0\) we obtain

\[
\frac{\partial}{\partial \lambda} u^\lambda_\mu(x) = \int E^W_x (\exp \left( -(1+\beta) \int_0^t u^\lambda_{\mu-u}(W_u)^2 du \right) \bigg| W_t = y) \mu(dy),
\]

where \(W_t\) is a symmetric \(\alpha\)-stable process. From the above, one can see that an upper bound for \(u^\lambda_\mu(x)\) can be used in the Feynman-Kac equation to obtain a lower bound for \(\frac{\partial}{\partial \lambda} u^\lambda_\mu(x)\), and hence for \(u^\infty_\mu(x)\). In order to bound \(u^\lambda_\mu(x)\) above, we use a comparison with the solution to the linear fractional heat equation, which itself bounded by analysing certain potential integrals involving the fractional heat kernel (i.e. the transition density of the stable process with index \(\alpha\)) and \(\mu\). This is of course where the condition (1.4.31) is essential. Ultimately, using this method we show that for all \(x \in B\),

\[
N_x^X(\mu(X_{t-s}) > 0 \mid X_{t-s} \neq 0) \geq c(t-s)^{-\frac{\alpha}{d-\beta+\alpha}} \tag{1.4.32}
\]

for a constant \(c > 0\). Hence in the time \(s\) cluster decomposition of \(X_t\) we have order \((t-s)^{-1/\beta}\) independent trials (clusters), each with success probability bounded below as in (1.4.32).

It follows that the probability of success converges to 1 as \(s \to t^-\) when \(\beta < \beta^*(\alpha,\delta)\). The case of equality, that is \(\beta = \beta^*(\alpha,\delta)\), is more delicate, but we show that the probability converges to 1 in this case as well.

The above argument proves Theorem 1.4.5(a) in the case that \(\mu\) satisfies (1.4.31). We are ultimately able to extend this to all measures \(\mu\) which satisfy \(H^\delta(S(\mu)) > 0\) by the
following argument. Frostman’s Lemma, which we discussed above, asserts that a Borel set \( A \subset \mathbb{R}^d \) satisfies \( H^\delta(A) > 0 \) if and only if \( A \) supports a measure satisfying (1.4.31).

Thus, given a measure \( \mu \) such that \( H^\delta(S(\mu)) > 0 \), there is a measure \( \mu' \) which satisfies \( S(\mu') \subseteq S(\mu) \) and (1.4.31), and the proof above applies to \( \mu' \). A PDE argument then allows us to transfer the conclusion from \( \mu' \) to \( \mu \). This is somewhat surprising, as it implies that it is strictly the closed support of \( \mu \), and not how \( \mu \) locally distributes mass, which determines if it is charged a.s. by \( \mathcal{K} \).

The proof of Theorem 1.4.5(b) uses a very different argument and is analytical rather than probabilistic. It is based on, and is a generalization of, the proof used in [7] to prove that \( u^\infty_t \) is non-flat when \( \beta > \beta^*(\alpha,d) \) (that is to prove the existence of a VSS for the dual equation). We do not describe it in detail here. Both sides of the dichotomy we prove (i.e. parts (a) and (b) of Theorem 1.4.5) correspond to new existence and non-existence results for solutions to (1.3.6) with very singular initial conditions. Our analytic results belong to the so-called initial trace theory of (1.3.6), which was introduced in [8] and is still in its early stages. These results, as well as several probabilistic corollaries of Theorem 1.4.5, are stated in the introduction to Chapter 4.

We conclude the section with the statement of another new theorem. By Theorem 1.4.4, if \( \beta \leq \beta^*(\alpha) \) then \( \mathcal{K}_i(x) > 0 \) for a.e. \( x \in \mathbb{R}^d \) when \( \mathcal{K}_i \neq 0 \). It is natural to ask if this can be improved from a.e. \( x \) to all \( x \in \mathbb{R}^d \). We show that when the density is continuous, this is the case.

**Theorem 1.4.6.** Suppose that \( d = 1 \), \( \alpha > 1 + \beta \) and \( \beta < \beta^*(\alpha) \). Then with probability one on \( \{ \mathcal{K}_i \neq 0 \} \),

\[ \mathcal{K}_i(x) > 0 \text{ for all } x \in \mathbb{R}. \]

The proof combines the tools used in our (probabilistic) proofs of of Theorem 1.4.4(a) and Theorem 1.4.5(a) with the regularity of the density established by Fleischmann et. al [28].

### 1.5 Notes on the text

The material on super-Brownian motion is divided into two chapters, Chapter 2 and Chapter 3, which correspond to the articles [35] and [37], respectively. Roughly speaking, we construct the boundary local time of super-Brownian motion (Theorem 1.4.1) in Chapter 2 and prove that it is non-zero almost surely when conditioned on survival (Theorem 1.4.3) in Chapter 3. Because both of these are required to prove Theorem 1.4.2, this result can be seen as being proved over both chapters. It is stated in Chapter 3.

The material on the \((\alpha,\beta)\)-superprocess is contained in Chapter 4.
We have done our best to ensure that, where possible, the notation in these chapters matches the notation used in the introduction. For any discrepancies, the notation defined within the chapter should be used.
Chapter 2

The boundary local time of super-Brownian motion I: construction and properties

Summary. For a one-dimensional super-Brownian motion with density $X(t, x)$, we construct a random measure $L_t$ called the boundary local time which is supported on $BZ_t := \partial\{x : X(t, x) = 0\}$, thus confirming a conjecture of Mueller, Mytnik and Perkins [69]. $L_t$ is analogous to the local time at 0 of solutions to an SDE. We establish first and second moment formulas for $L_t$, some basic properties, and a representation in terms of a cluster decomposition. Via the moment measures and the energy method we give a more direct proof that $\dim(BZ_t) = 2 - 2\lambda_0 > 0$ with positive probability, a recent result of Mueller, Mytnik and Perkins [69], where $-\lambda_0$ is the lead eigenvalue of a killed Ornstein-Uhlenbeck operator that characterizes the left tail of $X(t, x)$. In Chapter 3, we use the boundary local time and some of its properties proved here to show that $\dim(BZ_t) = 2 - 2\lambda_0$ a.s. on $\{X_t(\mathbb{R}) > 0\}$. 
2.1 Introduction & statement of main results

Super-Brownian motion is a Markov process taking values in the space of finite measures on $\mathbb{R}^d$, $\mathcal{M}_F(\mathbb{R}^d)$, equipped with the topology of weak convergence. We denote this process by $X = (X_t : t \geq 0)$ and denote by $P^X_{X_0}$ and $E^X_{X_0}$, respectively, a probability and its expectation under which $X$ is a super-Brownian motion with initial data $X_0 \in \mathcal{M}_F(\mathbb{R}^d)$. In one dimension, $X_t$ is almost surely an absolutely continuous random measure and thus has a density we denote by $X(t,x)$. The density is jointly continuous (and will exist) for $t > 0$, and is continuous with Hölder index $\frac{1}{2} - \epsilon$ in the spatial variable for all $\epsilon > 0$ (see [83], for example, where this is implicit in the proof of Theorem III.4.2). It was shown by Konno and Shiga in [45] and independently by Reimers in [84] that $X(t,x)$ satisfies the following stochastic partial differential equation (SPDE):

$$\frac{\partial X(t,x)}{\partial t} = \frac{\Delta X(t,x)}{2} + \sqrt{X(t,x)} \dot{W}(t,x), \quad (2.1.1)$$

where $\dot{W}(t,x)$ is a space-time white noise. For a complete discussion of such equations, including the precise definition of a solution, see [91] and [45].

Before discussing our results, we give a brief introduction to the canonical measure $N_0$ of super-Brownian motion. $N_0$ is a $\sigma$-finite measure on $C([0,\infty),\mathcal{M}_F(\mathbb{R}))\setminus\{0\}$. It describes the behaviour of a single cluster, that is, the descendants of a single ancestor, of super-Brownian motion started at the origin. (Likewise $N_x$ is a cluster started from $x$ and is just a translation of $N_0$.) In fact, one way of obtaining $N_0$ is as a weak limit of branching particle systems starting with a single particle, as in Theorem II.7.3 of [83]. Although $N_0$ itself is an infinite measure, when restricted to $\{X_t > 0\}$ for $t > 0$ it is finite; in particular we have $N_0(\{X_t > 0\}) = 2/t$ (see Theorem II.7.2 of [83]). A fact of central importance about the canonical measure is that super-Brownian motion under $P^X_{X_0}$ can be understood as a superposition of canonical clusters. This is discussed in greater detail later on (see (2.1.15)). We will use the notation $X_t$ and $X(t,x)$ to denote the superprocess and its density, respectively, under both $P^X_{X_0}$ and $N_0$. The law of the process will always be clear from context. For a complete overview of the canonical measure, including proofs of the properties just stated, see Section II.7 of [83].

In a recent work by Mueller, Mytnik and Perkins [69], the authors studied the small-scale asymptotic behaviour of $X(t,x)$, as well as the boundary of its zero set. We define the random set $Z_t = \{x \in \mathbb{R} : X(t,x) = 0\}$. The boundary of the zero set $BZ_t$ is then defined as

$$BZ_t := \partial Z_t = \{x \in Z_t : (x-\epsilon, x+\epsilon) \cap Z_t^c \neq \emptyset \ \forall \epsilon > 0\},$$

where the second equality holds by continuity of the density. The results in [69] involve an
eigenvalue \( \lambda_0 \in (\frac{1}{2}, 1) \) which we describe in greater detail shortly. The authors of [69] show that the left tail of the distribution of \( X(t, x) \) behaves like

\[
P^X_{x_0}(0 < X(t, x) < a) \propto t^{-\frac{1}{2} - \lambda_0} a^{2\lambda_0 - 1}
\]

as \( a \downarrow 0 \), where \( f(a) \propto g(a) \) means that \( f(a) \) is bounded above and below by \( cg(a) \) for different constants \( c \). Clearly for the above to be true we must take \( t \geq t_0 \) for some \( t_0 > 0 \).

The upper bound is uniform in \( x \) and the lower bound required a localizing assumption. For details, see Section 4 and in particular Theorem 4.8 of [69]. Let \( \dim(B) \) denote the Hausdorff dimension of a set \( B \subseteq \mathbb{R} \).

**Theorem 2.A.** [Mueller, Mytnik, Perkins (2017)] Under \( P^X_{x_0} \), \( \dim(BZ_t) \leq 2 - 2\lambda_0 \) almost surely on \( \{X_t > 0\} \) and \( \dim(BZ_t) \geq 2 - 2\lambda_0 \) with positive probability.

Because \( \lambda_0 \in (1/2, 1) \), the dimension satisfies \( 2 - 2\lambda_0 \in (0, 1) \). The lower bound was conjectured to hold with full probability on \( \{X_t > 0\} \), implying that \( \dim(BZ_t) = 2 - 2\lambda_0 \) almost surely on \( \{X_t > 0\} \). The difficulty in proving that the lower bound for the dimension holds with probability one on \( \{X_t > 0\} \) is owing to the delicate nature of the \( BZ_t \). It is not monotone in the initial conditions nor in the measure \( X_t \) itself.

We will construct a random measure \( L_t \), which we call the boundary local time of \( X_t \), supported on \( BZ_t \). (See Theorems 2.1.1 and 2.1.2.) The existence of \( L_t \) was conjectured in Section 5.1 of [69]. Once we have constructed \( L_t \), we use it to give a simpler alternative proof of the lower bound in Theorem 2.A. Our method is to show that \( L_t \) has finite \( p \)-energy for all \( p < 2 - 2\lambda_0 \); in particular, see Theorem 2.1.3 below. In Chapter 3, \( L_t \) and several of its properties derived here, including Theorem 2.1.2(a), Proposition 2.1.6 and Theorem 2.1.9, will be used to resolve the problem left open in Theorem 2.A and Theorem 2.1.3, showing that \( \dim(BZ_t) = 2 - 2\lambda_0 \) almost surely on \( \{X_t > 0\} \).

We now give a description of \( \lambda_0 \). Define a function \( F(x) \) by

\[
F(x) := -\log P^X_{x_0}(\{X(1, x) = 0\}) = N_0(\{X(1, x) > 0\}) > 0.
\]

The second equality is standard and is a consequence of (2.1.14) below. Section 2.3, from (2.3.5) to (2.3.14), provides a thorough overview of \( F \) as the limit as \( \lambda \to \infty \) of the family of functions \( \{V^\lambda_t\}_{\lambda>0} \) which characterize the Laplace transform of the density \( X(t, x) \). Let \( A^\phi f = \frac{1}{2}f''(x) - \frac{2}{x}f'(x) \) denote the infinitesimal generator of a standard, one-dimensional Ornstein-Uhlenbeck process \( Y \). For a bounded, continuous and non-negative function \( \phi \) with limits at infinity (\( F \) is such a function), \( A^\phi f = Af - \phi f \) is the generator of an Ornstein-Uhlenbeck process with Markovian killing corresponding to \( \phi \); that is, for a sample path \( \{Y_s : s \in [0, \infty)\} \in C([0, \infty); \mathbb{R}) \), we define the lifetime of the process as \( \rho^\phi \), after which it
is “killed,” or put into an inert cemetery state. The distribution of $\rho^\phi$ is given by

$$P(\rho > t \mid Y) = \exp \left( - \int_0^t \phi(Y_s) \, ds \right) \quad \text{for } t > 0. \quad (2.1.4)$$

Section 2.2 develops the relevant theory for these processes and their generators. In particular, Theorem 2.2.1 states that $A^\phi$, taken as an operator on the appropriate Hilbert space, has a countable orthonormal family of eigenfunctions $\{\psi_n^\phi\}_{n=0}^\infty$ with corresponding discrete spectrum $0 \geq -\lambda_0^\phi \geq -\lambda_1^\phi \geq \cdots \to -\infty$. We define $\lambda_0 = \lambda_F^> > 0$. As we have noted, it was shown in [69] that $\lambda_0 \in (1/2, 1)$. Numerical estimates by Zhu [96], for which the stated digits are expected to be accurate, suggest that $\lambda_0 \approx 0.8882$. This implies that the value of $\dim(BZ_t)$ from Theorem 2.1.2 is approximately 0.224. A more detailed discussion of the numerics can be found in the introduction of Chapter 3.

The method the authors of [69] used to show (2.1.2) involved computing the asymptotic behaviour of the Laplace transform of the density. In particular (see Proposition 4.5 of that work),

$$\lim_{\lambda \to \infty} t^{\lambda_0} \lambda^{2\lambda_0} E_{X_0}^X \left( \int \phi(x) X(t, x) e^{-\lambda X(t, x)} dx \right) \quad (2.1.5)$$

for every bounded Borel function $\phi$, where $m(dz)$ denotes the unit variance Gaussian measure in one dimension, $c_0$ is a positive constant and $\psi_0^F$ is the lead eigenfunction of $A^F$. For a super-Brownian motion with density $X(t, x)$, for $\lambda > 0$ we define the measure $L_t^\lambda \in \mathcal{MF}(\mathbb{R})$ by $dL_t^\lambda(x) = \lambda^{2\lambda_0} e^{-\lambda X(t, x)} X(t, x) dx$. That is, for a bounded measurable function $\phi : \mathbb{R} \to \mathbb{R}$, we define

$$L_t^\lambda(\phi) = \lambda^{2\lambda_0} \int \phi(x) X(t, x) e^{-\lambda X(t, x)} dx. \quad (2.1.6)$$

$L_t^\lambda$ is defined the same way under $P_{X_0}^X$ and $\mathbb{N}_0$. The scaling factor of $\lambda^{2\lambda_0}$ can be deduced from (2.1.5). The convergence of $E_{X_0}^X(L_t^\lambda(\phi))$ as $\lambda \to \infty$, noted in (2.1.5), led the authors of [69] to conjecture (Section 5.1 of that reference) that there is a random measure $L_t$ on $\mathbb{R}$ such that $L_t^\lambda \to L_t$ in $\mathcal{MF}(\mathbb{R})$ in probability. Our main result is the verification of this conjecture. In all that follows, $X_0 \in \mathcal{MF}(\mathbb{R})$.

**Theorem 2.1.1** (Boundary local time: existence and convergence). Let $t > 0$. Under both $P_{X_0}^X$ and $\mathbb{N}_0$ there is a random measure $L_t(dx) \in \mathcal{MF}(\mathbb{R})$, supported on $BZ_t$, such that $L_t^\lambda \to L_t$ in measure as $\lambda \to \infty$, and there is a sequence $\lambda_n \to \infty$ such that $L_t^{\lambda_n} \to L_t$
a.s. as \( n \to \infty \). Moreover, under \( P_{X_0}^X \) or \( \mathbb{N}_0 \), for all bounded and continuous functions \( \phi \), \( L_\lambda^\lambda(\phi) \to L_t(\phi) \) in \( L^2 \) as \( \lambda \to \infty \).

By almost sure convergence, we mean that for \( P_{X_0}^X \) or \( \mathbb{N}_0 \)-a.e. \( \omega \), \( L_\lambda^\lambda(\omega) \to L_t(\omega) \) weakly in \( \mathcal{M}_F(\mathbb{R}) \). Convergence in measure means with respect to any metric on \( \mathcal{M}_F(\mathbb{R}) \) which induces the weak topology, e.g. the Wasserstein metric (see for example p. 48 of [82]).

**Theorem 2.1.2** (Properties of \( L_t \)). (a) For all \( t > 0 \), we have \( P_{X_0}^X(L_t > 0 \mid X_t > 0) > 0 \) and \( \mathbb{N}_0(L_t > 0 \mid X_t > 0) \geq 1 - \lambda_0^2 \).

(b) \( L_t \) is atomless almost surely under \( P_{X_0}^X \) and \( \mathbb{N}_0 \).

**Definition.** \( L_t \) is the boundary local time of \( X_t \).

We note that \( Z_t \) will contain intervals, unlike the zero set of a Brownian motion (which is equal to its boundary). It is easy to see that \( L_t \) is supported on \( BZ_t \) from the fact that as \( \lambda \) gets large, \( L_\lambda^\lambda \) concentrates on \( \{ x : 0 < X(t,x) = O(\lambda^{-1}) \} \), and properties of the weak topology on \( \mathcal{M}_F(\mathbb{R}) \) (see the proof of Theorem 2.1.1 in Section 2.4). For fixed \( t > 0 \), \( x \to X(t,x) \) is a continuous path taking values in \( \mathbb{R}^+ = [0, \infty) \). \( BZ_t \) is the set of points where this path begins and ends its excursions from 0. As \( L_t \) is supported on \( BZ_t \), in this sense \( L_t \) is a local time of \( x \to X(t,x) \) on these excursion endpoints, and hence the boundary local time of \( X(t, \cdot) \).

The existence of a measure supported on \( BZ_t \) allows us to use the energy method to study its dimension. We will provide a second moment formula for \( L_t \), with which we compute the expectation of energy integrals of the form

\[
\iint |x - y|^{-p} \, dL_t(x) \, dL_t(y).
\]

If \( L_t > 0 \) and the above energy is finite, then \( \dim(\text{supp}(L_t)) \geq p \) by Frostman’s connection between energy integrals and Hausdorff dimension (see Theorem 4.27 of Mörters and Peres [65]). We introduce some notation. For \( h : \mathbb{R}^2 \to \mathbb{R} \), define \( (L_t \times L_t)(h) \) by

\[
(L_t \times L_t)(h) = \iint h(x,y) \, dL_t(x) \, dL_t(y).
\]

For \( p > 0 \), we define \( h_p(x,y) = |x - y|^{-p} \). The second moment formula for \( L_t \) allows us to establish the following.

**Theorem 2.1.3** (Finite energy and Hausdorff dimension). Both \( E_{X_0}^X((L_t \times L_t)(h_p)) \) and \( \mathbb{N}_0((L_t \times L_t)(h_p)) \) are finite for all \( p < 2 - 2\lambda_0 \). Moreover, \( \dim(BZ_t) = 2 - 2\lambda_0 \) almost surely on \( \{L_t > 0\} \) under both measures.
The fact that \( \dim(BZ_t) \leq 2 - 2\lambda_0 \) \( P_{X_0}^X \)-a.s. is already known from Theorem 2.A, and from this it follows easily under \( \mathbb{N}_0 \), as we point out in the proof of Theorem 2.1.3. By the above, the lower bound, i.e. \( \dim(BZ_t) \geq 2 - 2\lambda_0 \), holds with at least the probability that \( L_t > 0 \), as in Theorem 2.1.2(a). This plays an important role in Chapter 3; in Theorem 3.1.2 of that chapter we show that with respect to both \( P_{X_0}^X \) and \( \mathbb{N}_0 \), \( L_t > 0 \) almost surely on \( \{X_t > 0\} \), thus improving part (a) of Theorem 2.1.2 above and establishing almost sure non-degeneracy of \( L_t \). Combined with Theorem 2.1.3, this completes the dimension result.

There are a number of other potential uses for such a local time. We now discuss some possibilities. By sampling a point from \( L_t \), we are able to “view \( X_t \) from the perspective of a typical point in \( BZ_t \).” More precisely, one can define \( Q_{X_0}(\{(Z,X_t) \in A\}) = E_{X_0}^X(\int 1_A(z,X_t) dL_t(z)) \) and study properties of the Palm measure \( Q_{X_0}(X_t \in \cdot | Z = z) \). The behaviour of \( X_t \) near \( BZ_t \) is complex and there is still much that is not understood about it. For example, the density has an improved modulus of continuity and is nearly Lipzschitz (i.e. Hölder \( 1 - \eta \) for all \( \eta > 0 \)) at points in \( BZ_t \) (see Theorem 2.3 of [72]). This suggests that \( BZ_t \) would be small, but despite this \( BZ_t \) has positive dimension. Constructing and studying the Palm measure described above would give a more structured approach for investigating this phenomenon.

As a local time, \( L_t \) has the potential to study pathwise uniqueness in the SPDE (2.1.1), a problem which remains open, assuming a similar role as that of the semi-martingale local time in the Yamada-Watanabe Theorem for one-dimensional SDEs (see Theorem V.40 of Rogers and Williams [86]). It may also provide insight in the behaviour of some discrete processes; super-Brownian motion in high dimensions is the scaling limit of a number of lattice models and interacting particle systems. In dimension one, it is still the scaling limit of branching random walk (for example see [92] or Theorem II.5.1(iii) of [83]). One could obtain information about the boundaries of such approximating processes by proving a limit theorem establishing weak convergence of the laws of their discrete local times to that of \( L_t \). Of course, \( L_t \) allows for us to study \( BZ_t \) more directly, as we have done in Theorem 2.1.3. In fact, with \( L_t \) it may be possible to determine the exact Hausdorff measure function of \( BZ_t \).

We now discuss the method of our proof. Upper bounds on second moments of \( L_t^\lambda \) were obtained in Section 5.1 of [69], but in order to establish the existence of \( L_t \) we require exact asymptotics, which are more delicate. The main ingredient is the following convergence result. In order to state it we need to introduce some notation. Recall that \( m(dx) \) denotes the centred unit variance Gaussian measure. Let \( \psi_0 = \psi_0^F \) (the eigenfunction of \( A^F \) corresponding to eigenvalue \( -\lambda_0 \)). The constant \( C_{2.1.4} \) is given explicitly in (2.5.46), and the function \( \rho \) is defined in (2.5.47). The function \( V_t^{\infty,\infty} \) is defined in Section 2.3 as \( V_t^{\infty,\infty}(x_1,x_2) = \mathbb{N}_0(\{X(t,x_1) > 0\} \cup \{X(t,x_2) > 0\}) \) (see (2.3.25)). Finally, we will denote
by $P^B_x$ and $E^B_x$ the law and associated expectation of a standard Brownian motion $B_t$ with initial value $B_0 = x$.

**Theorem 2.1.4** (Convergence of second moments of $L^\lambda_t$). There exists a constant $C_{2.1.4} > 0$ and continuous function $\rho : \mathbb{R} \times \mathbb{R} \to (0, 1]$ such that for bounded Borel $h : \mathbb{R}^2 \to \mathbb{R}$,

$$
\lim_{\lambda, \lambda' \to \infty} \mathbb{N}_0((L^\lambda_t \times L^{\lambda'}_t)(h)) = C_{2.1.4}^2 \int_0^t (t-s)^{-2\lambda_0} \left[ \iint E^B_0 \left( h(\sqrt{t-s} z_1 + B_s, \sqrt{t-s} z_2 + B_s) \right) \times \exp \left( -\int_0^s V^\infty_{t-u} \langle \sqrt{t-s} z_1 + B_s - B_u, \sqrt{t-s} z_2 + B_s - B_u \rangle \, du \right) \times \rho(z_1, z_2) \psi_0(z_1) \psi_0(z_2) \, dm(z_1) \, dm(z_2) \right] \, ds.
$$

Moreover, the limit is finite for all bounded $h$.

That the formula above is finite is not obvious, as $\lambda_0 > 1/2$; we discuss this in more detail shortly. From the above we can deduce that $\{L^\lambda_t(\phi)\}_{\lambda > 0}$ is Cauchy in $L^2(\mathbb{N}_0)$ and therefore has a limit by completeness; in particular see Corollary 2.4.1 and its proof. We then argue that the limit is in fact the integral with respect to a unique measure, which is $L_t$. The proof of Theorem 2.1.4 is long and technical; Section 2.5 is entirely devoted to it. We use the Laplace functional to obtain a Feynman-Kac type representation for $\mathbb{N}_0(L^\lambda_t(\phi) L^{\lambda'}_t(\phi))$ and then establish its convergence. The reason we do so under $\mathbb{N}_0$ is because the Feynman-Kac formulas are simpler in this setting. We now present first and second moment formulas for $L_t$ under $\mathbb{N}_0$; as one would expect, the second moment formula in part (b) agrees with the limit of $\mathbb{N}_0((L^\lambda_t \times L^{\lambda'}_t)(h))$ given in Theorem 2.1.4. The terms $C_{2.1.4}$ and $\rho$ are the same that appeared in that result.

**Theorem 2.1.5** (Moments of $L_t$ under $\mathbb{N}_0$). (a) For a bounded or non-negative Borel function $\phi : \mathbb{R} \to \mathbb{R}$,

$$
\mathbb{N}_0(L_t(\phi)) = C_{2.1.4} t^{-\lambda_0} \int \phi(\sqrt{t} z) \psi_0(z) \, dm(z). \quad (2.1.8)
$$
(b) For measurable $h : \mathbb{R}^2 \to \mathbb{R}$, either bounded or non-negative,

\[
\mathbb{N}_0((L_t \times L_t)(h)) = C^2_{2.1.4} \int_0^t (t-s)^{-2\lambda_0} \left[ \int \int E^B_0 \left( h(\sqrt{t-s} z_1 + B_s, \sqrt{t-s} z_2 + B_s) \right) \times \exp \left( - \int_0^s V^\infty_{t-u} (\sqrt{t-s} z_1 + B_s - B_u, \sqrt{t-s} z_2 + B_s - B_u) \, du \right) \times \rho(z_1, z_2) \psi_0(z_1) \psi_0(z_2) \, dm(z_1) \, dm(z_2) \right] \, ds.
\]

Moreover, (2.1.9) is finite for all bounded $h$.

As we noted earlier, finiteness of (2.1.9) is not obvious since $\lambda_0 > 1/2$ (although it is implicit in the proof of Theorem 2.1.4), which can make (2.1.9) hard to use; for applications, the following upper bound for second moments is easier to apply than the exact formula. The value $\theta$ is defined as $\theta = \int \psi_0 \, dm$. $Y$ is an Ornstein-Uhlenbeck process started at $z_1$ with corresponding expectation $E^Y_{z_1}$. The exponential term in the first bound of the following proposition can be interpreted as a survival probability of $Y$, producing a $w^{\lambda_0}$ term which makes the integral finite. (The proofs of Theorem 2.1.3 and Theorem 2.1.2(b) in Section 2.4 both use this technique.)

**Proposition 2.1.6** (Second moment bounds under $\mathbb{N}_0$). For a non-negative Borel function $h : \mathbb{R}^2 \to \mathbb{R}$,

\[
\mathbb{N}_0((L_t \times L_t)(h)) \leq C^2_{2.1.4} \int_0^t w^{-2\lambda_0} \left[ \int \int E^Y_{z_1} \left( \exp \left( - \int_0^{\log(t/w)} F(Y_u) \, du \right) \times h(\sqrt{t \log(t/w)}, \sqrt{t \log(t/w)} + \sqrt{w(z_2 - z_1)}) \right) \psi_0(z_1) \psi_0(z_2) \, dm(z_1) \, dm(z_2) \right] \, dw.
\]

Moreover,

\[
\mathbb{N}_0(L_t(1)^2) \leq \frac{C^2_{2.1.4} \theta^2}{1 - \lambda_0} t^{1-2\lambda_0}.
\]

As we have alluded to, applying (2.1.10) with $h(x, y) = |x - y|^{-p}$ gives an upper bound for the expectation of energy integrals of the form (2.1.7), which is how we prove Theorem 2.1.3.

Thus far, we have not commented on the proofs of existence and properties of $L_t$ under $P_{X_0}^X$. The proofs rely on the conditional representation in terms of canonical clusters, which we will discuss shortly. First, in order to keep the moment results together, we state our results regarding the moments of $L_t$ under $P_{X_0}^X$. 

47
Theorem 2.1.7 (Moments of $L_t$ under $P^X_{X_0}$). For a bounded or non-negative Borel function $\phi : \mathbb{R} \to \mathbb{R}$,

$$E^X_{X_0}(L_t(\phi)) = C_{2.1.4} t^{-\lambda_0} \int \phi(x_0 + \sqrt{t}z) \exp \left( -\frac{1}{t} \int F(z + t^{-1/2}(x_0 - y_0)) dX_0(y_0) \right) \times \psi_0(z) dm(z) dX_0(x_0).$$ \hfill (2.1.12)

(b) There is a constant $C_{2.1.7}$ such that

$$E^X_{X_0}(L_t(1)^2) \leq C_{2.1.7} \left( X_0(1)^{2(1-2\lambda_0)} + X_0(1)^2 t^{2\lambda_0} \right).$$ \hfill (2.1.13)

We note that the right hand side of (2.1.12) is equal to that of (2.1.5), and so was originally computed in Proposition 1.5 of [69] as $\lim_{\lambda \to \infty} E^X_{X_0}(L_{\lambda t}(\phi))$. The fact that the same formula gives the mean measure of $L_t$ then follows from the $L^2$ convergence of $L_{\lambda t}(\phi)$, as in Theorem 2.1.1.

We first establish the existence of $L_t$, as well as its properties, under the measure $N_0$, owing to the fact that the second moments of $L_{\lambda t}$ admit simpler formulas in this case. In order to prove the same for super-Brownian motion, we need to use the relationship between super-Brownian motion under $P^X_{X_0}$ and the canonical measure, which we now describe. We recall that $N_x$ is a $\sigma$-finite measure such that $N_x(\{X_t > 0\}) = 2/t$ which describes the “law” of a single cluster of super-Brownian motion started at $x$; that is, the descendants of a single ancestor at $x$. More precisely, super-Brownian motion is a superposition of canonical clusters; for a bounded, non-negative Borel function $\phi : \mathbb{R} \to \mathbb{R}$,

$$E^X_{X_0}(\exp(-X_t(\phi))) = \exp \left( -\int \int 1 - e^{-\mu_t(\phi)} dN_{x_0}(\mu) dX_0(x_0) \right).$$ \hfill (2.1.14)

This expression for the Laplace functional is in fact a consequence of a distributional equality between super-Brownian motion under $P^X_{X_0}$ and a Poisson point process of canonical clusters. For $X_0 \in \mathcal{M}_F(\mathbb{R})$, let $N_{x_0}(\cdot) = \int N_x(\cdot) dX_0(x)$ and let $\Theta_{X_0}$ be a Poisson point process on $C([0, \infty), \mathcal{M}_F(\mathbb{R}))$ with intensity $N_{X_0}$. We define a $\mathcal{M}_F(\mathbb{R})$-valued process $(X_t : t \geq 0)$ by

$$X_t(\cdot) = \begin{cases} \int \mu_t(\cdot) d\Theta_{X_0}(\mu) & \text{if } t > 0, \\ X_0(\cdot) & \text{if } t = 0. \end{cases}$$ \hfill (2.1.15)

By Theorem 4 of Section IV.3 of [57], $(X_t : t \geq 0)$ is a super-Brownian motion with initial measure $X_0$. The “points” of the point process $\Theta_{X_0}$ are the clusters of $X$. For fixed $t > 0$, (2.1.15) leads to

$$X_t = \sum_{j \in I_t} \mu_t^j,$$
where \{\mu^j_t : j \in I_t\} are the points of a Poisson point process with finite intensity \(N_{X_0}(\mu_t \in \cdot | \mu_t > 0)\). Let \(X_0(\cdot) = X_0(\cdot)/X_0(1)\). Assuming our probability space is rich enough to allow us to choose random relabellings of these points, by the above we can write

\[
X_t = \sum_{i=1}^{N} X^i_t,
\]

where \(N\) is Poisson\((2X_0(1)/t)\) and, given \(N\), \(\{X^i_t : i = 1, \ldots, N\}\) are iid with distribution \(N_{\bar{X}_0}(X_t \in \cdot | X_t > 0)\). We can and do condition on the values of the initial points of the clusters, denoted by \(x_1, \ldots, x_N\), which are iid points with distribution \(\bar{X}_0\), in which case \(X^i_t\) has conditional distribution \(N_{x_i}(X_t \in \cdot | X_t > 0)\). In order to prove the existence and properties of \(L_t\) with respect to a super-Brownian motion \(X_t\), we realize the super-Brownian motion as a point process and express \(X_t\) as above. Conditioning on \(N\) and applying (2.1.16), we can write \(L^\lambda_t(\phi)\) as

\[
L^\lambda_t(\phi) = \lambda^{2\lambda_0} \int \left[ \sum_{i=1}^{N} X^i(t, x) \right] e^{-\lambda \sum_{i=1}^{N} X^i(t, x)} \phi(x) dx.
\]

The almost sure existence of boundary local times corresponding to the canonical clusters allows us to take this limit quite easily and so establish that \(L_t\) exists under \(P^X_{X_0}\) (i.e. Theorem 2.1.1). Furthermore, we obtain a conditional representation for \(L_t\) in terms of its clusters; this allows us to transfer the properties of \(L_t\) under \(N_0\) to \(L_t\) under \(P^X_{X_0}\). Let \(L^i_t\) denote the boundary local time of \(X^i_t\). In the statement that follows, we assume that we have realized \(X_t\) using (2.1.16).

**Theorem 2.1.8 (Cluster decomposition).** Let \(X_t\) be super-Brownian motion under \(P^X_{X_0}\) and \(L_t\) be its boundary local time. Conditional on \(N\), we have

\[
dL_t(x) = \sum_{i=1}^{N} \left[ \sum_{j \neq i} 1(\sum_{j \neq i} X^j(t, x) = 0) dL^i_t(x) \right]
\]

\[
= 1(X(t, x) = 0) \sum_{j=1}^{N} dL^j_t(x).
\]

**Remark.** Given the nature of \(BZ_t\), we expect this behaviour. In the cluster decomposition, each cluster has a boundary local time of its own. Since each is supported on the boundary of its respective zero set, the local time \(L_t\) of \(X_t\) will be equal to the sum of cluster local times, except the boundary of the zero set of one cluster may be “swallowed” by the support of another, hence the indicator functions.
The idea of representing the boundary local time of $X_t$ in terms of the boundary local time of its clusters is not restricted to a super-Brownian motion and its comprising canonical clusters. The following formulation of the same principle will be useful in Chapter 3. Recall that a sum of independent super-Brownian motions is a super-Brownian motion.

**Theorem 2.1.9 (General cluster decomposition).** Suppose $X^1, \ldots, X^n$ are independent super-Brownian motions with corresponding boundary local times $L^i_t$ at time $t > 0$, for $i = 1, \ldots, n$. Let $X = \sum_{i=1}^n X^i$ and let $L_t$ be the boundary local time of $X_t$. Then

$$dL_t(x) = \sum_{i=1}^n 1(\sum_{j \neq i} X^j(t, x) = 0) dL^i_t(x)$$

$$= 1(X(t, x) = 0) \sum_{i=1}^n dL^i_t(x).$$

One example of superprocesses satisfying the above conditions follows from (III.1.3) of [83]. Let $X_0 \in \mathcal{M}_F(\mathbb{R})$ and suppose that $\{A_1, \ldots, A_n\}$ is a Borel partition of $\mathbb{R}$. Define $X^i$ as the contribution to $X$ from ancestors at time 0 which are in $A_i$. (This makes $X^i$ a super-Brownian motion with initial measure $X_0(\cdot \cap A_i)$; a precise definition of $X^i$ may be given in terms of the historical process as in the above reference.) Then $X = \sum_{i=1}^N X^i$ satisfies the conditions of the above theorem.

**Notations.** We will make use of the common convention that $C$ denotes any positive constant whose value is not important. The value of $C$ may change line to line in a derivation; to bring attention to the fact that the constant has changed, we will sometimes label the new constant $C'$. We write $f \sim g$ if $\lim_{x} f(x)/g(x) = 1$, where the limit will be clear from context. As the reader has probably inferred, we will write $\mu > 0$ when a measure has positive mass (that is, to indicate that $\mu(1) > 0$). For an interval $I \subseteq \mathbb{R}$, let $C(I, \mathbb{R})$ denote the space of continuous maps from $I$ to $\mathbb{R}$.

Let $S_t$ denote the semi-group of Brownian motion and $p_t$ the associated heat kernel (the Gaussian density of variance $t$). Let $\mathcal{N}(x_0, \sigma^2)$ denote the law of a one-dimensional Gaussian with mean $x_0$ and variance $\sigma^2$.

**Organization of the chapter.** This chapter is organized as follows. Section 2.2 gives a brief overview of the theory of one-dimensional Ornstein-Uhlenbeck processes with Markovian killing. Our method relies on a change of variables which allow us to express certain quantities in terms of eigenvalue problems involving these processes’ generators.

Section 2.3 describes fundamental background connecting the Laplace functional of super-Brownian motion to a family of semi-linear PDEs. We also introduce the families $V^\lambda$ and $V^\lambda X$, which play a key role in our analysis.

In Section 2.4, we assume Theorem 2.1.4 (the $L^2$-convergence result) and proceed to
prove our main results, including existence and properties of $L_t$ and the cluster representations. First we prove the existence of $L_t$ under $\mathbb{N}_0$ (Theorem 2.1.1 for $\mathbb{N}_0$) and the formulae for its first and second moments (Theorem 2.1.5). Next, we use the cluster decomposition to prove the existence of $L_t$ under $P_{X_0}^X$ (Theorem 2.1.1 for $P_{X_0}^X$) and its representation in terms of clusters (Theorems 2.1.8 and 2.1.9). We then establish the upper bound on second moments of $L_t$ under $\mathbb{N}_0$ (Proposition 2.1.6), which allows us to prove the remaining results, including Theorem 2.1.3, the dimension result.

Section 2.5 contains the proof of Theorem 2.1.4, with the proof of one technical lemma given in Section 2.6.

2.2 Killed Ornstein-Uhlenbeck processes

As above, we define the operator $A$ by $Af(x) = \frac{f''(x)}{2} - \frac{xf'(x)}{2}$. The Markov process generated by $A$ is a one-dimensional Ornstein-Uhlenbeck process with mean zero. We denote this process by $Y$, denote its law when started at $x$ by $P_x^Y$ with corresponding expectation $E_x^Y$.

For general initial conditions $Y_0 \sim \mu \in M_1(\mathbb{R})$ (the space of probability measures on $\mathbb{R}$), we write its law as $P_\mu^Y$. $Y$ has a stationary measure, the unit variance Gaussian measure, $\mu$. When $Y_0 \sim \mu$, the process is reversible and can be defined for time values in $\mathbb{R}$. We will denote the law of this stationary process on $\mathbb{R}$ by $P_\mu^Y$.

We now introduce the notions of killing and lifetime for the process $(Y_t : t \geq 0)$. Let $\phi \in C^+([-\infty, \infty], \mathbb{R})$, the space of non-negative continuous functions with limits at $\pm \infty$. Such functions are also bounded. We will call this family of functions killing functions. Let $A^\phi f(x) = Af(x) - f(x)\phi(x)$. $A^\phi$ is the generator of an Ornstein-Uhlenbeck process subjected to Markovian killing at rate $\phi(Y_t)$. The lifetime of the killed process is $\rho^\phi = \inf\{t > 0 : \int_0^t \phi(Y_s) \, ds > e\}$, where $e$ is an independent Exp(1) random variable. We recall that the distribution of $\rho^\phi$ is given by (2.1.4).

The generators $A$ and $A^\phi$ correspond to strongly continuous contraction semigroups on $L^2(\mu)$. The following theorem is proved in [69], where it is stated as Theorem 2.3. We note that the statement of the result in that paper had a misprint when describing the convergence of the transition densities, which appeared in part (c). We have corrected the statement, which is in part (b) of the following.

**Theorem 2.2.1.** For $\phi \in C^+([-\infty, \infty], \mathbb{R})$, the following statements hold.

(a) $A^\phi$ has a complete orthonormal family of $C^2$ eigenfunctions $\{\psi_n : n \geq 1\}$ in $L^2(\mu)$ satisfying $A^\phi \psi_n = -\lambda_n \psi_n$, where $0 \leq \lambda_0 \leq \lambda_1 \leq \cdots \to \infty$. Furthermore, $-\lambda_0$ is a simple eigenvalue and $\psi_0 > 0$.

(b) For $t > 0$, the diffusion $Y$ generated by $A^\phi$ has a jointly continuous transition density
$q_t(x, y)$ with respect to $m$, given by

$$q_t(x, y) = \sum_{n=0}^{\infty} e^{-\lambda_n t} \psi_n(x) \psi_n(y),$$

(2.2.1)

where the series converges in $L^2(m \times m)$ and uniformly absolutely on sets of the form $[\epsilon, \infty) \times [-\epsilon^{-1}, \epsilon^{-1}]^2$ for all $\epsilon > 0$.

(c) For $0 < \delta < \frac{1}{2}$, there exists a constant $c_\delta > 0$ such that

$$q_t(x, y) \leq c_\delta e^{-\lambda_0 t} e^{\delta(x^2+y^2)} \text{ for all } t \geq s^*(\delta),$$

(2.2.2)

where $s^*(\delta) > 0$ is the solution of

$$2\delta = \frac{e^{-s^*/2} - e^{-s^*}}{1 - e^{-s^*}}.$$  

(2.2.3)

(d) Denote $\theta = \int \psi_0 \, dm$. For all $t \geq 0$ and $x \in \mathbb{R}$,

$$e^{\lambda_0 t} P_x(\rho \phi > t) = \theta \psi_0(x) + r(t, x),$$

(2.2.4)

where, for any $\delta > 0$, there is a constant $c_\delta > 0$ such that

$$\psi_0(x) \leq c_\delta e^{\delta x^2},$$

(2.2.5)

$$|r(t, x)| \leq c_\delta e^{\delta x^2} e^{-(\lambda_1 - \lambda_0) t}.$$  

(2.2.6)

(e) As $T \to \infty$, $P_x(Y \in \cdot | \rho \phi > T) \to P_x^{Y, \infty}$ weakly on $C([0, \infty), \mathbb{R})$, where $P_x^{Y, \infty}$ is the law of the diffusion with the transition density

$$\tilde{q}_t(x, y) \equiv q_t(x, y) \frac{\psi_0(y)}{\psi_0(x)} e^{\lambda_0 t}$$

(2.2.7)

with respect to $m$.

The bounds in part (d) of the above easily imply the following estimates, which we will often use. For $0 < \delta < 1/2$, there is a constant $C_\delta > 0$ such that

$$P_x^{Y}(\rho \phi > t) \leq C_\delta e^{\delta x^2} e^{-\lambda_0 t} \quad \forall \, x \in \mathbb{R}, \, t > 0.$$  

(2.2.8)

This implies that there is a constant $C > 0$ such that

$$P_m^{Y}(\rho \phi > t) \leq C e^{-\lambda_0 t} \quad \forall \, t > 0.$$  

(2.2.9)
The following limit result is a simple consequence of the eigenfunction expansion for \( q_t(x, y) \).

**Lemma 2.2.2.** For all \( x, y \in \mathbb{R} \),

\[
\lim_{t \to \infty} e^{\lambda_0 t} q_t(x, y) = \psi_0(x) \psi_0(y).
\]

The convergence is uniform on compact sets.

**Proof.** For all \( t > 0 \) and \( x, y \in \mathbb{R} \), from (2.2.1), we have

\[
e^{\lambda_0 t} q_t(x, y) = \psi_0(x) \psi_0(y) + \sum_{n=1}^{\infty} e^{-(\lambda_n - \lambda_0) t} \psi_n(x) \psi_n(y). \tag{2.2.10}
\]

As we are taking \( t \to \infty \) we can restrict to \( t \geq 1 \), in which case the absolute value of the sum above is bounded above by

\[
e^{-(\lambda_1 - \lambda_0)(t-1)} \sum_{n=1}^{\infty} e^{-(\lambda_n - \lambda_0)} |\psi_n(x) \psi_n(y)|.
\]

By Theorem 2.2.1(b) with \( t = 1 \), the series in the above is convergent, and the convergence is uniform on compact sets. Part (a) of the same theorem states that \(-\lambda_0\) is a simple eigenvalue. Hence \( \lambda_1 - \lambda_0 > 0 \) and the above vanishes as \( t \to \infty \); in fact, because the series converges uniformly on compacts to a continuous limit, the above vanishes uniformly on compacts as \( t \to \infty \), so (2.2.10) gives the result. \[\square\]

It will be useful for us to study the distribution of the process \( Y \) when conditioned on survival and its endpoint. Hereafter we assume that \( Y \) has killing function \( \phi \in C^+(\mathbb{R}) \) and we denote its lifetime by \( \rho \). For fixed \( T > 0 \) and \( z \in \mathbb{R} \), consider the \([0, T]\)-indexed inhomogeneous Markov process taking values in \( \mathbb{R} \) with transition density (with respect to \( dm(y_2) \))

\[
\hat{q}_{s,t}(y_1, y_2) = \frac{q_{t-s}(y_1, y_2) q_{T-t}(y_2, z)}{q_{T-s}(y_1, z)} \tag{2.2.11}
\]

for \( 0 \leq s < t < T \). (The kernels are degenerate when \( t = T \), since \( Y_T = z \).) Below we verify that the finite dimensional distributions defined by this transition kernel have an extension to a (necessarily) unique law on \( C([0, T], \mathbb{R}) \), which we denote by \( P_{x}^{Y}(\cdot \mid \rho > T, Y_T = z) \) when the initial point is \( x \in \mathbb{R} \), and show that it gives an explicit version of the suggested regular conditional distribution for all \( z \in \mathbb{R} \). We then establish that for fixed \( S > 0 \), \( P_{x}^{Y}(Y_{[0,S]} \in \cdot \mid \rho > T, Y_T = z) \) converges weakly to \( P_{x}^{Y,\infty}(Y_{[0,S]} \in \cdot) \) as \( T \to \infty \) for all \( z \in \mathbb{R} \).

**Lemma 2.2.3.** (a) Let \( x \in \mathbb{R} \) and \( T > 0 \). For all \( z \in \mathbb{R} \), the finite dimensional distributions described in (2.2.11), with initial value \( x \), have a unique extension to \( C([0, T], \mathbb{R}) \). The
resulting laws \( P^Y_x (\cdot | \rho > T, Y_T = z) \) are continuous in \( z \) and define a regular conditional probability for \( Y_{[0,T]} | \) under \( P^Y_x \) conditioned on \( Y_T \).

(b) Let \( x, z \in \mathbb{R}, S > 0 \) be fixed. Then \( P_x (Y_{[0,S]} | \in \cdot | \rho > T, Y_T = z) \) converges weakly on \( C([0, S], \mathbb{R}) \) to \( P^Y_x (Y_{[0,S]} | \in \cdot) \) as \( T \to \infty \).

(c) For all \( S, K > 0 \), \( \{ P_x (Y_{[0,S]} | \in \cdot | \rho > T, Y_T = z) : |x|, |z| \leq K, T \geq S \} \) is tight on \( C([0, S], \mathbb{R}) \).

Before proving the lemma, we make an observation concerning time reversals of \( Y \) under \( P^Y_x (\cdot | \rho > T, Y_T = z) \). For \( T > 0 \) and \( t \in [0, T] \), define \( \hat{Y}_t = Y_{T-t} \). Let \( x, z \in \mathbb{R} \). For \( 0 < t_1 < t_2 < T \) and \( \phi_1, \phi_2 \) bounded Borel functions, we have

\[
E^Y_x (\phi_1 (\hat{Y}_{t_1}) \phi_2 (\hat{Y}_{t_2}) | \rho > T, Y_T = z) = \frac{1}{q_T (x, z)} \int \phi_1 (y_1) \phi_2 (y_2) q_{T-t_2} (x, y_2) q_{t_2-t_1} (y_2, y_1) q_{t_1} (y_1, z) dm (y_1) dm (y_2)
\]

where the last equality uses \( q_t (x, y) = q_t (y, x) \). The above equality of distributions can be extended to general finite dimensional distributions. Because the extension of the finite dimensional distributions to a law on \( C([0, T], \mathbb{R}) \) (i.e. from Lemma 2.2.3(a)) is unique, we therefore have that for all \( x, z \in \mathbb{R} \),

\[
P^Y_x (Y_{[0,T]} | \in \cdot | \rho > T, Y_T = z) = P^Y_x (Y_{[0,T]} | \in \cdot | \rho > T, Y_T = x).
\]  

As a last note, we will sometimes denote the law \( P^Y_x (\cdot | \rho > T, Y_T = z) \) simply by \( P^Y_x (\cdot | Y_T = z) \) when it is clear from context that we are working with the killed process.

Proof of Lemma 2.2.3. Let \( x, z \in \mathbb{R} \) and \( T > 0 \). We define a distribution, which we denote by \( P^Y_x (\cdot | \rho > T, Y_T = z) \), on finite (time-indexed) collections of random variables, which describes the finite dimensional distributions (FDDs) of the inhomogeneous Markov process with transition density (2.2.11). For \( 0 = t_0 < t_1 < \cdots < t_n < T \) and bounded, continuous functions \( \phi_1, \ldots, \phi_n \), we define the \( n \)-dimensional FDD of \( (Y_{t_1}, \ldots, Y_{t_n}) \) under \( P^Y_x (\cdot | \rho > T, Y_T = z) \) as

\[
E^Y_x \left( \prod_{i=1}^n \phi_i (Y_{t_i}) \right | (\rho > T, Y_T = z) = \frac{1}{q_T (x, z)} \int \left[ \prod_{i=1}^n \phi_i (y_i) q_{t_n-t_{n-1}} (y_{n-1}, y_n) \right] q_{t_n} (y_n, z) \prod_{i=1}^n dm (y_i).
\]  

where we use the convention \( y_0 = x \). We note that (2.2.13) also defines the FDDs of a
regular conditional distribution of \((Y_t : t \in [0, T])\) under \(P_x^Y\) conditioned on \(Y_T = z\) (which is why we have used this notation). Thus when we have established that these laws extend to a probability on \(C([0, T], \mathbb{R})\), we will have explicitly constructed a version of the regular conditional distribution.

To prove that \(P_x^Y(\cdot \mid \rho > T, Y_T = z)\) extends to a probability on \(C([0, T], \mathbb{R})\), we will establish a tightness criterion. We consider the fourth moments of increments of \(Y\). Let \(0 < s < t < T\). Expanding using (2.2.13), we have

\[
E_x^Y((Y_t - Y_s)^4 \mid \rho > T, Y_T = z) = \frac{1}{q_T(x, z)} \int \int (y_2 - y_1)^4 q_s(x, y_1) q_{t-s}(y_1, y_2) q_{T-t}(y_2, z) \, dm(y_1) \, dm(y_2). \tag{2.2.14}
\]

We now collect some elementary bounds and inequalities which will allow us to obtain a useful upper bound for the above. First, we note that while \(q_t(x, y)\) is a transition density with respect to \(m\), it will sometimes be useful to express it as a density with respect to the Lebesgue measure. Since \(p_1(\cdot)\) is the density of \(m\), we have

\[
q_t(x, y) \, dm(y) = q_t(x, y) \, p_1(y) \, dy. \tag{2.2.15}
\]

We will use a comparison with an un-killed Ornstein-Uhlenbeck process. The transition kernel of a standard Ornstein-Uhlenbeck process is described by, for \(0 \leq s < t\),

\[
(Y_t - Y_s \mid Y_s = y) \sim \mathcal{N}(e^{-(t-s)/2}y, 1 - e^{-(t-s)}).
\]

Let \(k_t(x, y)\) denote the transition density of an un-killed Ornstein-Uhlenbeck process with respect to Lebesgue measure. Then for \(x, y \in \mathbb{R}\) and \(t > 0\),

\[
k_t(x, y) = \frac{(2\pi)^{-1/2}}{\sqrt{1 - e^{-t}}} \exp \left( -\frac{(y - e^{-t/2}x)^2}{2(1 - e^{-t})} \right). \tag{2.2.16}
\]

The transition densities of the killed Ornstein-Uhlenbeck process are bounded above by those of the un-killed process. This implies that

\[
(i) \quad q_t(x, y) \, dm(y) \leq k_t(x, y) \, dy, \quad (ii) \quad q_t(x, y) \, p_1(y) \leq k_t(x, y). \tag{2.2.17}
\]

It is easy to establish from (2.2.16) that there is a constant \(c > 0\) such that

\[
k_t(x, y) \leq cp_t(y - xe^{-t/2}) \quad \text{for all} \ t \leq 2 \ \text{and} \ x, y \in \mathbb{R}. \tag{2.2.18}
\]

where we recall that \(p_t(\cdot)\) is the Gaussian density of variance \(t\). Let \(K > 0\). From (2.2.17)(ii)
and \((2.2.18)\) it follows that there is a constant \(C_1(K)\) such that
\[
q_{T-t}(y_2, z) \leq k_{T-t}(y_2, z) p_1(z)^{-1} \leq \frac{C_1(K)}{\sqrt{t-T}} \quad \forall\; y_2 \in \mathbb{R}, \; z \in [-K, K], \; \text{and} \; t \leq T' < T.
\]
(2.2.19)

Next, we note that it holds by elementary formulas for moments of Gaussians that there is a constant \(c > 0\) such that
\[
\int (y_2 - y_1)^4 p_t(y_2) \, dy_2 \leq c \left( t^2 + |y_1|^4 \right) \quad \forall\; y_1 \in \mathbb{R}, \; t > 0.
\]
(2.2.20)

Finally, observe that \(q_t(\cdot, \cdot)\) is bounded below by the transition density of \(Y\) with constant killing function \(\|\phi\|_{\infty}\). Thus for all \(K > 0\) and \(M \geq 1\), from \((2.2.15)\) we have
\[
q_t(x, z) \geq e^{-\|\phi\|_{\infty}T} k_T(x, z) p_1(z)^{-1} \geq \delta(K, M) \quad \forall\; x, z \in [-K, K], \; T \in [M^{-1}, M]
\]
(2.2.21)

for a sufficiently small constant \(\delta(K, M) > 0\).

Let \(0 < T' < T\) and suppose that \(0 < s < t \leq T'\) such that \(t-s \leq 1\). Let \(K > 0\) and suppose that \(x, z \in [-K, K]\). Using \((2.2.17)(i)\) to bound \(q_s(x, y_1) dm(y_1)\) and \(q_{t-s}(y_1, y_2) dm(y_2)\), and \((2.2.19)\) to bound \(q_{T-t}(y_2, z)\), from \((2.2.14)\) we obtain that
\[
\begin{align*}
E_x^Y((Y_t - Y_s)^4 \mid \rho > T, Y_T = z) & \leq \frac{C_1(K) (T - T')^{-1/2}}{q_T(x, z)} \int k_s(x, y_1) \left[ \int \int (y_2 - y_1)^4 k_{t-s}(y_1, y_2) \, dy_2 \right] dy_1 \\
& \leq \frac{C_1(K) (T - T')^{-1/2}}{q_T(x, z)} \int k_s(x, y_1) \left[ \int c(y_2 - y_1)^4 p_{t-s}(y_2 - e^{-(t-s)/2} y_1) \, dy_2 \right] dy_1,
\end{align*}
\]
(2.2.22)

where the second inequality uses \((2.2.18)\). Changing variables and applying \((2.2.20)\), we obtain that
\[
\int c(y_2 - y_1)^4 p_{t-s}(y_2 - e^{-(t-s)/2} y_1) \, dy_2 \leq C(|y_1|^4 (1 - e^{-(t-s)/2})^4 + (t-s)^2) \\
\leq C(1 + |y_1|^4)(t-s)^2,
\]
(2.2.23)

where in the second inequality we have \(1 - e^{-x} \leq x\) for \(x \geq 0\) and \((t-s)^4 \leq (t-s)^2\) (since \(t-s \leq 1\)). Substituting this into \((2.2.22)\), we obtain that
\[
E_x^Y((Y_t - Y_s)^4 \mid Y_T = z) \leq \frac{C_1(K) (T - T')^{-1/2}}{q_T(x, z)} (t-s)^2 \int C k_s(x, y_1) (|y_1|^4 + 1) \, dy_1.
\]
(2.2.24)

Recall that we have assumed \(x, z \in [-K, K]\). By \((2.2.16)\) it is clear that for \(K > 0\,
the integral is bounded above by some constant $C_2(K) > 0$ for all $x \in [-K, K]$ and $s > 0$. Using this along with (2.2.21), with a choice of $M \geq 1$ for which $T \in [M^{-1}, M]$, from the above we deduce the following:

For all $x, z \in [-K, K]$, $0 < s < t \leq T'$ such that $t - s \leq 1$,

$$E_x^Y((Y_t - Y_s)^4 \mid Y_T = z) \leq C_2(K) \delta(K, M)^{-1} C_1(K) (T - T')^{-1/2} \times (t - s)^2. \quad (2.2.25)$$

Let $T' = 2T/3$. Hereafter we consider increments of size at most $1 \wedge T/3$. We have that (2.2.25) holds for all $0 < s < t \leq 2T/3$ such that $t - s \leq 1 \wedge T/3$ with constant $C_2(K)\delta(K, M)^{-1} C_1(K)(T/3)^{-1/2}$. It remains to show that it also holds on $[2T/3, T]$. To do so, we make use of reversibility. Suppose $T/3 \leq s < t < T$. Then

$$E_x^Y((Y_t - Y_s)^4 \mid Y_T = z) = \frac{1}{q_T(x, z)} \int \int (y_2 - y_1)^4 q_s(x, y_1) q_{t-s}(y_1, y_2) q_{T-t}(y_2, z) \, dm(y_1) \, dm(y_2) = E_x^Y((Y_{T-s} - Y_{T-t})^4 \mid Y_T = x), \quad (2.2.26)$$

where the last equality uses $q_t(x, y) = q_t(y, x)$ (a consequence of (2.2.1)) and (2.2.13). Since $0 < T - t < T - s \leq 2T/3$, by (2.2.25) and (2.2.26) we have that for all $x, z \in [-K, K]$ and $T/3 \leq s < t < T$ such that $t - s \leq 1 \wedge T/3$,

$$E_x^Y((Y_t - Y_s)^4 \mid \rho > T, Y_T = z) = C_2(K) \delta(K, T)^{-1} C_1(K)(T/3)^{-1/2} \times (t - s)^2.$$

Combined with the previous statement that this holds for all $0 < s < t \leq 2T/3$, we have that

For all $x, z \in [-K, K]$, $0 < s < t < T$ such that $t - s \leq 1 \wedge T/3$,

$$E_x^Y((Y_t - Y_s)^4 \mid Y_T = z) \leq C_2(K) \delta(K, M)^{-1} C_1(K)(T/3)^{-1/2} \times (t - s)^2. \quad (2.2.27)$$

The above proof can be easily modified to obtain the same bound (with a potential change to the constant) for increments in which $s = 0$ or $t = T$, and we omit it. Thus by (2.2.27) and the Kolmogorov Continuity Theorem, $P_x^Y(\cdot \mid \rho > T, Y_T = z)$ has a unique extension to a probability on $C([0, T], \mathbb{R})$, also denoted by $P_x^Y(\cdot \mid \rho > T, Y_T = z)$. As we noted earlier, this gives an explicit construction of the regular conditional distribution $(Y_t : t \in [0, T])$ under $P_x^Y$ given $\rho > T$ and $Y_T = z$, Additionally, suppose that $z_n \to z$ and that $z_n \in [-K, K]$ for all $n \geq 1$. From (2.2.27), $\{P_x^Y(\cdot \mid \rho > T, Y_T = z_n) : n \geq 1\}$ is tight. It is clear from (2.2.13) and continuity of $q_t(\cdot, \cdot)$ that the FDDs of $P_x^Y(\cdot \mid \rho > T, Y_T = z_n)$ converge to those of $P_x^Y(\cdot \mid \rho > T, Y_T = z)$. Thus the aforementioned tightness proves that $P_x^Y(\cdot \mid \rho > T, Y_T = z_n)$
converges to $P^Y_x(\cdot | \rho > T, Y_T = z)$ as a law on $C([0, T], \mathbb{R})$. Thus we have proved part (a).

Before proving (b), we note the following consequence of (2.2.27) and its proof. Let $S, K > 0$ and fix $M > 1$ such that $S \in [M^{-1}, M]$. By considering increments of $(Y_s : s \in [0, S])$ but allowing the time $T$ at which we condition $Y_T = z$ to take values in $[S, M]$, we have that

$$\{P^Y_x(Y|_{[0, S]} \in \cdot | \rho > T, Y_T = z) : |x|, |z| \leq K, T \in [S, M]\}$$

is tight. (2.2.28)

Next we turn to part (b). Fix $S > 0$ and $x, z \in \mathbb{R}$. We now check that the FDDs of $(Y_s : s \in [0, S])$ under $P^Y_x(\cdot | \rho > T, Y_T = z)$ converge to those of $(Y_s : s \in [0, S])$ under $P^{Y, \infty}$ as $T \to \infty$. Let $0 < t_1 < t_2 \leq S$ and let $\phi_1$ and $\phi_2$ be bounded and continuous functions. Then from (2.2.13), we have

$$E^Y_x(\phi_1(Y_{t_1}) \phi_2(Y_{t_2}) | \rho > T, Y_T = z)$$

$$= \frac{1}{q_T(x, z)} \int \phi_1(y_1) \phi_2(y_2) q_{t_1}(x, y_1) q_{t_2-t_1}(y_1, y_2) q_{T-t_2}(y_2, z) \, dm(y_1) \, dm(y_2)$$

$$= \frac{e^{\lambda_0 t_2}}{e^{\lambda_0 T} q_T(x, z)} \int \phi_1(y_1) \phi_2(y_2) q_{t_1}(x, y_1) q_{t_2-t_1}(y_1, y_2)$$

$$\times e^{\lambda_0(T-t_2)} q_{T-t_2}(y_2, z) \, dm(y_1) \, dm(y_2).$$

(2.2.29)

By Lemma 2.2.2, we have

$$\lim_{T \to \infty} e^{\lambda_0(T-t)} q_{T-t}(y_2, z) = \psi_0(y_2) \psi_0(z), \quad \lim_{T \to \infty} e^{\lambda_0 T} q_T(x, z) = \psi_0(x) \psi_0(z).$$

(2.2.30)

Moreover, applying (2.2.2) with $\delta = 1/8$, we have that

$$e^{\lambda_0(T-t)} q_{T-t}(y_2, z) \leq c e^{y_2^2/8 + z^2/8} \forall y_2, z \in \mathbb{R}, t \in (0, S] \text{ and } T \geq S + s^*(1/8),$$

(2.2.31)

where $s^*(1/8)$ is as in (2.2.3). Using (2.2.31) (replacing $t$ with $t_2$) and (2.2.17)(i) we obtain the following bound for the integrand in (2.2.29):

$$|\phi_1(Y_{t_1}) \phi_2(Y_{t_2}) q_{t_1}(x, y_1) q_{t_2-t_1}(y_1, y_2) e^{\lambda_0(T-t_2)} q_{T-t_2}(y_2, z)| \, dm(y_1) \, dm(y_2)$$

$$\leq c e^{z^2/8} \|\phi_1\|_\infty \|\phi_2\|_\infty e^{y_2^2/8} k_{t_1}(x, y_1) k_{t_2-t_1}(y_1, y_2) \, dy_1 \, dy_2$$

for all $T \geq S + s^*(1/8)$. By (2.2.16), $k_{t_1}(x, y_1)$ and $k_{t_2-t_1}(y_1, y_2)$ are Gaussians with variance at most 1, and so a short argument shows that the above quantity is integrable. This allows
us to use Dominated Convergence in (2.2.29), so by (2.2.30) we have

$$\lim_{T \to \infty} E_x(\phi_1(Y_{t_1}) \phi_2(Y_{t_2}) \mid Y_T = z)$$

$$= \frac{e^{\lambda_0 t_2}}{\psi_0(x)\psi_0(z)} \int \phi_1(y_1) \phi_2(y_2) q_{t_1}(x, y_1) q_{t_2 - t_1}(y_1, y_2) \psi_0(y_2) \psi_0(z) dm(y_1) dm(y_2)$$

$$= \int \phi_1(y_1) \phi_2(y_2) \left[ e^{\lambda_0 t_1} q_{t_1}(x, y_1) \psi_0(y_1) \right] \left[ e^{\lambda_0 (t_2 - t_1)} q_{t_2 - t_1}(y_1, y_2) \psi_0(y_2) \right] dm(y_1) dm(y_2)$$

$$= \int \phi_1(y_1) \phi_2(y_2) \hat{q}_{t_1}(x, y_1) \hat{q}_{t_2 - t_1}(y_1, y_2) dm(y_1) dm(y_2)$$

$$= E^Y_{x, \infty}(\phi_1(Y_{t_1}) \phi_2(Y_{t_2})).$$

The above argument can be easily generalized to $n$-fold FDDs for all $n \geq 2$, (the $\delta = 1/8$ in (2.2.31) can be reduced to handle larger $n$) and thus we have the desired convergence of the FDDs as $T \to \infty$. In order to obtain weak convergence of the laws on $C([0, S], \mathbb{R})$, we need tightness of the distributions as $T \to \infty$. To prove that the distributions are tight we will analyse the fourth moments of increments, as in (2.2.14), but first we obtain one more bound. We note that by Lemma 2.2.2 and joint continuity of $(T, (x, y)) \to q_T(x, y)$, it holds that for all $K > 0$,

$$e^{\lambda_0 T} q_T(x, z) \geq \delta(K) > 0 \ \forall x, z \in [-K, K], T \geq 1$$

(2.2.32)

for sufficiently small $\delta(K) > 0$. Let $K > 0$ and $x, z \in [-K, K]$. In (2.2.14), we bound $q_{t - \lambda}(y, z)$ above using (2.2.31) and bound the other transition densities using (2.2.17), which gives

$$E^Y_x((Y_t - Y_0)^4 \mid \rho > T, Y_T = z)$$

$$\leq \frac{e^{\lambda_0 t + s^2/8}}{e^{\lambda_0 T} q_T(x, z)} \int k_s(x, y_1) \left[ \int (y_2 - y_1)^4 k_{t-s}(y_1, y_2) e^{y_2^2/8} dy_2 \right] dy_1$$

$$\leq e^{\lambda_0 S + K^2/8} \delta(K)^{-1} \int k_s(x, y_1) e^{y_1^2/4} \left[ \int c(y - y_1(1 - e^{-(t-s)/2}))^4 p_{t-s}(y) e^{y^2/4} dy \right] dy_1$$

$$\leq e^{\lambda_0 S + K^2/8} \delta(K)^{-1} \int c' k_s(x, y_1) e^{y_1^2/4} \left[ \int c'(y - y_1(1 - e^{-(t-s)/2}))^4 p_{2(t-s)}(y) dy \right] dy_1$$

for all $T \geq S + s^* (1/8)$. In the second inequality we have used (2.2.32) as well as (2.2.18) and a change of variables. The third follows from a short calculation and the fact that $t - s \leq 1$. Applying (2.2.20) to the above and arguing as in (2.2.23), we obtain that, for all
\[ T \geq S + s^*(1/8), \]
\[
E^Y_x((Y_t - Y_s)^4 \mid \rho > T, Y_T = z) \\
\leq e^{\lambda_0 S + K^{2/8}} \delta(K)^{-1} (t-s)^2 C \int k_s(x, y_1) e^{y_1^2/4} (1 + |y_1|^4) dy_1, \\
\leq C_3(S, K) (t-s)^2 \quad \forall x, z \in [-K, K], 0 \leq s < t \leq S \text{ such that } t-s \leq 1, \tag{2.2.33}
\]
for a constant \( C_3(S, K) > 0 \), where to see that the integral is bounded uniformly for \( |x| \leq K \), we use the fact, from (2.2.16), that \( k_s(x, y_1) \) is Gaussian with mean of absolute value bounded above by \( |x| \) and variance less than 1. The fact that (2.2.33) holds for all \( T \geq S + s^*(1/8) \) implies that the laws \( P^Y_x(Y_{[0,S]} \in \cdot \mid \rho > T, Y_T = z) \) are tight as \( T \to \infty \). Combined with the convergence of the FDDs to those of \( P^Y_x \), this proves (b).

Observe that (2.2.33) proves part (c) if we restrict to \( T \geq S + s^*(1/8) \). If we choose \( M \geq 1 \) such that \( M^{-1} < S < S + s^*(1/8) < M \), then (2.2.28) gives tightness of the laws for \( T \in [S, S + s^*(1/8)] \). Combining these two cases gives the desired tightness and proves (c).

\[ \square \]

### 2.3 Some non-linear PDE

Let \( B_{b+}(\mathbb{R}) \) denote the space of bounded, non-negative Borel functions. Recall that \( S_t \) denotes the semigroup of Brownian motion. By Theorem III.5 of [57], for \( \phi \in B_{b+}(\mathbb{R}) \), there exists a unique non-negative solution, denoted \( V^\phi_t(x) \), to the evolution equation

\[
V_t = S_t(\phi) - \frac{1}{2} \left( \int_0^t S_{t-s}(V^2_s) ds \right), \tag{2.3.1}
\]
such that

\[
E^X_{X_0}(e^{-X_t(\phi)}) = e^{-X_0(V^\phi_t)} \tag{2.3.2}
\]
for all \( X_0 \in \mathcal{M}_F(\mathbb{R}) \). Applying the above with \( X_0 = \delta_x \), (2.1.14) gives

\[
N_x(1 - e^{-X_t(\phi)}) = V^\phi_t(x). \tag{2.3.3}
\]

We are interested in the case when the initial data is a measure, and also in the differential form of the equation. The integral equation (2.3.1) has a corresponding PDE, which is the following:

\[
\frac{\partial V}{\partial t} = \frac{1}{2} \frac{\partial^2 V}{\partial x^2} - \frac{V^2}{2} \quad \text{for } (t, x) \in (0, \infty) \times \mathbb{R}, \quad V_t \to \phi \text{ as } t \downarrow 0. \tag{2.3.4}
\]
In [5], this equation was shown to have a unique $C^{1,2}$ solution when $\phi \in \mathcal{M}_F(\mathbb{R})$, where $V_t \to \phi$ is understood as weak convergence of measures. That is, we identify the function $V_t$ with the measure $V_t(x)\,dx$, which converges weakly to $\phi$. By Lemma 2.1 of [70], the solution of (2.3.4) is also the unique solution to (2.3.1). We denote the unique solution to (2.3.1) and (2.3.4) by $V_{\phi t}$. Part (d) of the same lemma establishes that if $\phi_n \to \phi$ weakly as $n \to \infty$, then $V_t^{\phi_n}(x) \to V_t^{\phi}(x)$ for all $t > 0, x \in \mathbb{R}$. We note from (2.3.1) that $V_t^{\phi_n} \leq S_t \phi_n \leq ct^{-1/2} \phi_n(\mathbb{R})$. Using this and the fact that $X_t$ has a bounded, continuous density, if we approximate measures by functions in $B_{b+}(\mathbb{R})$, we can take bounded limits in (2.3.2) and (2.3.3) to establish that (2.3.2) and (2.3.3) hold for $V_t^{\phi}$ when $\phi \in \mathcal{M}_F(\mathbb{R})$.

**Notation.** As $X_t$ is absolutely continuous, when $\phi \in \mathcal{M}_F(\mathbb{R})$ we interpret $X_t(\phi)$ as $\int X(t,x)\,d\phi(x)$.

We now state some useful properties of solutions to (2.3.4). For a proof, see Lemma 2.6 in [70].

**Proposition 2.3.1.** Let $\phi, \psi \in \mathcal{M}_F(\mathbb{R})$.

(a) (Monotonicity) If $\phi \leq \psi$, then $0 \leq V_t^{\phi} \leq V_t^{\psi}$ for all $t > 0$.

(b) (Sub-additivity) $V_t^{\phi+\psi} \leq V_t^{\phi} + V_t^{\psi}$ for all $t > 0$.

Next we fix $\phi = \lambda \delta_x \in \mathcal{M}_F(\mathbb{R})$ for $\lambda > 0$, so that $X_t(\phi) = \lambda X(t,x)$. Denote by $V_t^\lambda$ the unique, non-negative $C^{2,1}$ solution to the initial value problem

$$
\frac{\partial V}{\partial t} = \frac{1}{2} \frac{\partial^2 V}{\partial x^2} - \frac{V^2}{2} \quad \text{for } (t,x) \in (0,\infty) \times \mathbb{R}, \quad V_t \to \lambda \delta_0 \text{ weakly as } t \downarrow 0. \tag{2.3.5}
$$

This family was originally studied in [41]. It is an exercise to use (2.3.5) or the scaling properties of super-Brownian motion to show that $V_t^\lambda(x)$ satisfies the following space-time scaling relationship. For $\lambda, r > 0$, we have

$$
V_{t^r}^{\lambda x}(x) = \lambda^2 V_{t^r}^\lambda(\lambda x). \tag{2.3.6}
$$

By translation invariance in the initial conditions of (2.3.5), and by (2.3.2) and (2.3.3) we have

$$
E_{\delta_0}^X(e^{-\lambda X(t,x)}) = e^{-V_t^\lambda(x)}, \tag{2.3.7}
$$

$$
N_0(1 - e^{-\lambda X(t,x)}) = V_t^\lambda(x) \tag{2.3.8}
$$

for all $x \in \mathbb{R}$ and $t > 0$. It is clear from (2.3.7) that $V_t^\lambda$ increases to a limit as $\lambda \to \infty$. In the PDE literature this was established in [41], where it was shown that $V^\lambda$ converges locally uniformly as $\lambda \to \infty$ to a function $V_t^\infty$ on $(0,\infty) \times \mathbb{R}$. Heuristically, $V_t^\infty$ is the solution of
(2.3.5) when \( \lambda = +\infty \). Rigorously, it is the unique solution to the following problem:

\[
\frac{\partial V}{\partial t} = \frac{1}{2} \frac{\partial^2 V}{\partial x^2} - \frac{V^2}{2} \quad \text{for} \ (t, x) \in (0, \infty) \times \mathbb{R},
\]

\[
\lim_{t \downarrow 0} V_t(x) = 0 \ \forall x \neq 0, \quad \lim_{t \downarrow 0} \int_{B_\epsilon} V_t(x) \, dx = +\infty \ \forall \epsilon > 0,
\]

(2.3.9)

where \( B_\epsilon = B(0, \epsilon) \), the ball with radius \( \epsilon \) centered at the origin. \( V_t^{\infty} \) was introduced and shown to solve (2.3.9) in [6]; uniqueness of the solution is a consequence of Theorem 3.5 of [64]. Taking \( \lambda \to \infty \) in (2.3.8), we see that \( V_t^{\infty} \) satisfies

\[
V_t^{\infty}(x) = N_0(\{X(t, x) > 0\}).
\]

(2.3.10)

We recall that (see Theorem II.7.2 of [83])

\[
N_0(\{X_t > 0\}) = 2/t.
\]

(2.3.11)

Thus (2.3.10) implies that

\[
V_t^{\infty}(x) \leq 2/t \ \forall x.
\]

(2.3.12)

Taking \( \lambda^2 = 1/t \) and letting \( r \to \infty \) in (2.3.6), one obtains that \( V_t^{\infty}(x) = t^{-1}V_1^{\infty}(t^{-1/2}x) \).

**Definition.** Define \( F : \mathbb{R} \to \mathbb{R}^+ \) by

\[
F(x) := V_1^{\infty}(x).
\]

(2.3.13)

It follows that \( V_t^{\infty}(x) = t^{-1}F(t^{-1/2}x) \). It was shown in [6] that \( F \) is the solution to an ODE problem. (In fact, their PDEs and ODEs have different (constant) coefficients, but Section 3 of [69] shows that \( F \) is a rescaled version of the function they study.) \( F \) is the unique solution of

\[
(i) \ F''(x) + xF'(x) + F(x)(2 - F(x)) = 0
\]

\[
(ii) \ F > 0, F \in C^2(\mathbb{R})
\]

\[
(iii) \ F'(0) = 0, F(x) \sim c_1|x|e^{-x^2/2} \ \text{as} \ |x| \to \infty
\]

(2.3.14)

for some \( c_1 > 0 \). We recall that \( f(x) \sim h(x) \) means \( f(x)/h(x) \to 1 \) as \( x \to \infty \). This \( F \) is the function we discussed in the introduction, for which \( -\lambda_0 \) is the lead eigenvalue of the operator \( A^F \). In particular, by evaluating (2.3.10) at \( t = 1 \) we can recover (2.1.3), our preliminary definition.

As part of the proof of Theorem 2.A, the authors of [69] computed the rate of convergence
of $V_t^\lambda$ to $V_t^\infty$. In particular, Proposition 4.6 of that reference states that

$$\sup_{x \in \mathbb{R}} \left[ V_t^\infty(x) - V_t^\lambda(x) \right] \leq Ct^{-1/2-\lambda_0} \lambda^{1-2\lambda_0}$$

(2.3.15)

for some constant $C$. (This is closely connected to (2.1.5).) A similar lower bound with the same power of $\lambda$ is established in the same proposition, although in this case one must be careful when $t$ is close to zero. We will make frequent use of (2.3.15) in this chapter to bound error terms arising when we make approximations to obtain an eigenvalue problem. Let $Y$ be an Ornstein-Uhlenbeck process. We define $Z_T(Y)$ as

$$Z_T(Y) = \exp \left( \int_0^T F(Y_s) - V_t^{\epsilon s/2}(Y_s) \, ds \right).$$

(2.3.16)

Since $V_t^{\epsilon s/2} \uparrow V_t^\infty = F$ as $s \to \infty$, the integrand converges to zero as $s \to \infty$. As $Z_T(Y)$ is increasing in $T$, we can define $Z_\infty(Y) := \lim_{T \to \infty} Z_T(Y)$. By (2.3.15), we can easily deduce that the (monotone) limit

$$Z_\infty(Y) := \lim_{T \to \infty} Z_T(Y)$$

(2.3.17)

exists and is finite, and that moreover there is a constant $C_Z > 0$ such that, uniformly for all $Y$,

$$Z_T(Y) \leq Z_\infty(Y) \leq C_Z < \infty \quad \forall \, T > 0.$$

(2.3.18)

Finally, we introduce another family of solutions to (2.3.4), which arise when we compute second moments of $L_t^\lambda$; we will evaluate expressions that involve the density at two points $x_1, x_2 \in \mathbb{R}$. Let $V_t^{(\lambda,\lambda'),(x_1,x_2)}$ denote $V_t^\phi$ when $\phi = \lambda \delta_{x_1} + \lambda' \delta_{x_2} \in \mathcal{M}_F(\mathbb{R})$, so that, by (2.3.3),

$$V_t^{(\lambda,\lambda'),(x_1,x_2)}(y) = N_y \left( 1 - e^{-\lambda X(t,x_1) - \lambda' X(t,x_2)} \right).$$

(2.3.19)

When evaluating this function at 0, we will denote it by $V_t^{\lambda,\lambda'}(x_1,x_2) = V_t^{(\lambda,\lambda'),(x_1,x_2)}(0)$. In other words,

$$V^{\lambda,\lambda'}(x_1,x_2) = N_0 \left( 1 - e^{-\lambda X(t,x_1) - \lambda' X(t,x_2)} \right).$$

(2.3.20)

By (2.3.19), (2.3.20) and translation invariance of the canonical measure, these families satisfy

$$V_t^{(\lambda,\lambda'),(x_1,x_2)}(y) = V_t^{(\lambda,\lambda'),(x_1-y,x_2-y)}(0) = V_t^{\lambda,\lambda'}(x_1 - y, x_2 - y).$$

(2.3.21)

Lastly, as can be readily seen from (2.3.20) and the symmetry of the canonical measure,

$$V_t^{\lambda,\lambda'}(x_1,x_2) = V_t^{\lambda,\lambda'}(-x_1,-x_2)$$

(2.3.22)
for all \(x_1, x_2 \in \mathbb{R}\). This family also satisfies a following scaling relationship which can be derived from studying the associated PDE directly. In particular,

\[
V_t^{r, c, \lambda'}(x_1, x_2) = \lambda^2 V_{\lambda'^2 t}^{r, c, \lambda}(\lambda x_1, \lambda x_2) = (\lambda')^2 V_{(\lambda')^2 t}^{r, c, \lambda'}(\lambda' x_1, \lambda' x_2),
\]

(2.3.23)

for all \(\lambda, \lambda', r, c > 0\) and \(x_1, x_2 \in \mathbb{R}\). Taking limits and applying bounded convergence in (2.3.2), we see that \(V_t^{\lambda, \lambda'}(x_1, x_2)\) has a monotone limit as \(\lambda, \lambda' \to \infty\) (by Proposition 2.3.1(a)). We denote this limit \(V_t^{\infty, \infty}(x_1, x_2)\). In agreement with our previous notation we define the following.

**Definition.** We define \(F_2 : \mathbb{R}^2 \to \mathbb{R}^+\) by

\[
F_2(x_1, x_2) := V_1^{\infty, \infty}(x_1, x_2).
\]

(2.3.24)

By taking the limit as \(\lambda, \lambda' \to \infty\) in (2.3.20) (and in (2.3.2) with \(\phi = \lambda \delta_{x_1} + \lambda' \delta_{x_2}\)) we obtain that

\[
V_t^{\infty, \infty}(x_1, x_2) = N_0(\{X(t, x_1) > 0\} \cup \{X(t, x_2) > 0\}) = -\log P_0(X(t, x_1) = X(t, x_2) = 0).
\]

(2.3.25)

We conclude by stating a version of (2.3.15) for the functions \(V_t^{\lambda, \lambda'}\).

**Lemma 2.3.2.** There is a positive constant \(C\) such that for all \(t, \lambda, \lambda' > 0\),

\[
\sup_{x_1, x_2 \in \mathbb{R}} \left[ V_t^{\infty, \infty}(x_1, x_2) - V_t^{\lambda, \lambda'}(x_1, x_2) \right] \leq Ct^{-1/2-\lambda_0} \left[ \lambda'^{1-2\lambda_0} + \lambda^{1-2\lambda_0} \right].
\]

**Proof.** Let \(x_1, x_2 \in \mathbb{R}\) and \(t, \lambda, \lambda' > 0\). We write

\[
V_t^{\infty, \infty}(x_1, x_2) - V_t^{\lambda, \lambda'}(x_1, x_2) = \left[ V_t^{\infty, \infty}(x_1, x_2) - V_t^{\infty, \infty}(x_1, x_2) \right] + \left[ V_t^{\lambda, \infty}(x_1, x_2) - V_t^{\lambda, \lambda'}(x_1, x_2) \right].
\]

(2.3.26)

Using (2.3.25) and (2.3.20) and taking \(\lambda' \to \infty\) in the latter, it follows that the first term in the above is equal to

\[
N_0 \left( 1 - 1(X(t, x_1) = X(t, x_2) = 0) \right) - N_0 \left( 1 - e^{-\lambda X(t, x_1)}1(X(t, x_2) = 0) \right)
\]

\[
= N_0 \left( 1(X(t, x_2) = 0) \left( e^{-\lambda X(t, x_1)} - 1(X(t, x_1) = 0) \right) \right)
\]

\[
\leq N_0 \left( e^{-\lambda X(t, x_1)} - 1(X(t, x_1) = 0) \right)
\]

\[
= V_t^{\infty}(x_1) - V_t^{\lambda}(x_1)
\]

\[
\leq Ct^{-1/2-\lambda_0} \lambda^{1-2\lambda_0},
\]
where the second last line follows from (2.3.10) and (2.3.8), and the final inequality is by (2.3.15). We use similar reasoning to bound the second term of (2.3.26) by the same expression with \( \lambda' \) replacing \( \lambda \), which gives the desired result.

\[ \square \]

### 2.4 Existence and properties of \( L_t \)

As stated in the introduction, our method first establishes the existence and properties of \( L_t \) under \( N_0 \) and then uses the cluster decomposition to establish them under \( P_X X_0 \). The main ingredient in the proof of Theorem 2.1.1 is the convergence of second moments of \( L_{\lambda t} (\phi) \) as \( \lambda \to \infty \). For a bounded Borel function \( \phi \), we show that \( N_0(L_{\lambda t} (\phi)^2) \) converges as \( \lambda \to \infty \). In fact, we prove convergence of second moments of general functions of two variables. For \( h: \mathbb{R}^2 \to \mathbb{R} \) we recall the notation

\[
(L_{\lambda t}^\lambda \times L_{\lambda t}^\lambda)(h) = \int h(x, y) dL_{\lambda t}^\lambda(x) dL_{\lambda t}^\lambda(y).
\]

\( L_{\lambda t}^\lambda (\phi)^2 \) is easily recovered by taking \( h(x, y) = \phi(x)\phi(y) \). The following result is the workhorse of this chapter.

**Theorem 2.1.4.** There exists a constant \( C_{2.1.4} > 0 \) and continuous function \( \rho: \mathbb{R} \times \mathbb{R} \to (0, 1] \) such that for bounded Borel \( h: \mathbb{R}^2 \to \mathbb{R} \),

\[
\lim_{\lambda, \lambda' \to \infty} N_0((L_{\lambda t}^\lambda \times L_{\lambda t}^\lambda')(h)) = C_{2.1.4}^2 \int_0^t (t - s)^{-2\lambda_0} \left[ \int \int \mathbb{E}^B_0 \left( h(\sqrt{t - s} z_1 + B_s, \sqrt{t - s} z_2 + B_s) \right) \times \exp \left( -\int_0^s V_{t-u}^{\infty, \infty} (\sqrt{t - s} z_1 + B_s - B_u, \sqrt{t - s} z_2 + B_s - B_u) du \right) \times \rho(z_1, z_2) \psi_0(z_1) \psi_0(z_2) dm(z_1) dm(z_2) \right] ds.
\]

**Corollary 2.4.1.** For a bounded Borel function \( \phi \), \( L_{\lambda t}^\lambda (\phi) \) converges in \( L^2(N_0) \) as \( \lambda \to \infty \).

**Proof.** Since \( L^2(N_0) \) is complete, it is enough to show that \( \{L_{\lambda t}^\lambda (\phi)\}_{\lambda > 0} \) is Cauchy in \( L^2(N_0) \). For \( \lambda, \lambda' > 0 \), we have

\[
N_0((L_{\lambda t}^\lambda (\phi) - L_{\lambda t}^{\lambda'}(\phi))^2) = N_0((L_{\lambda t}^\lambda (\phi))^2) + N_0((L_{\lambda t}^{\lambda'}(\phi))^2) - 2N_0(L_{\lambda t}^\lambda (\phi)L_{\lambda t}^{\lambda'}(\phi)).
\]

By Theorem 2.1.4, this converges to 0 as \( \lambda, \lambda' \to \infty \). \( \square \)

The proof of Theorem 2.1.4 is long and technical. We defer it to Section 2.5, which is devoted to its proof. For now, we assume the result and use it to establish our other main
results, the first being the existence of $L_t$ under $\mathbb{N}_0$.

**Proof of Theorem 2.1.1 for $\mathbb{N}_0$.** Fix $t > 0$. Because $X_t = 0$ implies that $L_t^\lambda = 0$ for all $\lambda > 0$, without loss of generality we can work under the finite measure $\mathbb{N}_0 \cap \{X_t > 0\}$. By Corollary 2.4.1, for a bounded continuous function $\phi$, there exists a random variable $l(t, \phi)$ such that $L_t^\lambda(\phi) \to l(t, \phi)$ in $\mathcal{L}^2(\mathbb{N}_0)$ as $\lambda \to \infty$. It follows that $L_t^\lambda(\phi) \to l(t, \phi)$ in measure. We will now establish that there exists a unique random measure $L_t$ such that the random variable $l(t, \phi)$ is the integral of $\phi$ with respect to a random measure $L_t$, i.e. $l(t, \phi) = L_t(\phi)$ for all continuous and bounded functions $\phi$.

We need to establish that the measures $\{L_t^\lambda : \lambda > 0\}$ are tight $\mathbb{N}_0$-almost surely. To see that this is true, we recall that $X(t, \cdot)$ is compactly supported $\mathbb{N}_0$-a.s., (see Corollary III.1.4 of [83] for the result under $P_{\delta_0}^X$, condition the cluster representation on $N = 1$ to get it for $\mathbb{N}_0$) and hence the mass of $X_t$ is contained in a ball $B(0, R)$ for some $R = R(\omega) > 0$. Since $L_t^\lambda(A) = \lambda^{2\lambda_0} \int A X(t, x)e^{-\lambda X(t, x)} dx$, this implies that the mass of $L_t^\lambda$ is contained in $B(0, R)$ for all $\lambda > 0$, which implies that $\{L_t^\lambda(\omega) : \lambda > 0\}$ is tight.

Let $\{\phi_n\}_{n=1}^\infty$ be a countable determining class for $\mathcal{M}_F(\mathbb{R})$ consisting of bounded, continuous functions. We choose $\phi_1 = 1$. $\mathcal{L}^1$-boundedness of the total mass and tightness are sufficient conditions for a family in $\mathcal{M}_F(\mathbb{R})$ (with the weak topology) to be relatively compact. By Corollary 2.4.1, $\{L_t^\lambda(1) : \lambda > 0\}$ is $\mathcal{L}^2(\mathbb{N}_0)$-bounded, and hence $\mathcal{L}^1(\mathbb{N}_0)$-bounded, and so from the above we see that

$$\{L_t^\lambda : \lambda > 0\} \text{ is relatively compact } \mathbb{N}_0\text{-a.s.}$$

As we have noted, $L_t^\lambda(\phi_n) \to l(t, \phi_n)$ in measure as $\lambda \to \infty$. Using the fact that convergence in measure implies almost sure convergence along a subsequence, we can iteratively define subsequences and take a diagonal subsequence $\{\lambda_m\}_{m=1}^\infty$ which satisfies

$$L_t^{\lambda_m}(\phi_n) \to l(t, \phi_n) \text{ as } m \to \infty \text{ for all } n \geq 1 \text{ } \mathbb{N}_0\text{-a.s.} \quad (2.4.1)$$

As shown above, $\{L_t^{\lambda_m}\}_{m=1}^\infty$ is relatively compact $\mathbb{N}_0$-almost surely. Combined with (2.4.1), this means that for $\mathbb{N}_0$-a.a. $\omega$ we have the above convergence for all $n \geq 1$ and relative compactness of the measures $\{L_t^{\lambda_m}\}_{m=1}^\infty$. Choose such an $\omega$. By relative compactness of $\{L_t^\lambda\}_{\lambda>0}$, any subsequence of $\{\lambda_m\}_{m=1}^\infty$ admits a further sequence along which the measures converge in the weak topology. It remains to show that all subsequential limits coincide. Suppose $L_t(\omega)$ and $L_t'(\omega)$ are two such limit measures. Since $\omega$ has been chosen so that (2.4.1) holds, we have that $L_t(\omega)(\phi_n) = L_t'(\omega)(\phi_n)$ for all $n$. Since the family $\{\phi_n\}_{n \geq 1}$ are a determining class, this implies that $L_t(\omega) = L_t'(\omega)$. Hence all subsequences admit a further
subsequence with the same limit \( L_t(\omega) \) in the weak topology. Since the weak topology on \( \mathcal{M}_F(\mathbb{R}) \) is metrizable, the “every subsequence admits a further converging subsequence” criterion for convergence applies, and we have \( L_t^{\lambda_m}(\omega) \) converges to \( L_t(\omega) \in \mathcal{M}_F(\mathbb{R}) \) as \( m \to \infty \). This gives the almost sure convergence along \( \{\lambda_m\}_{m=1}^{\infty} \).

We now check that \( L^t_1 \to L_t \) in measure as \( \lambda \to \infty \). First note that we can restrict to the finite measure \( \mathbb{N}_0(\cdot \cap \{X_t > 0\}) \), since \( L^t_1 = L_t = 0 \) for all \( \lambda > 0 \) on \( \{X_t = 0\} \). Let \( d(\mu, \nu) \) be a metric which metrizes the weak topology on \( \mathcal{M}_F(\mathbb{R}) \), e.g. the Wasserstein metric (see p. 48 of [82]). If \( L^t_1 \) did not converge to \( L_t \) in measure, then there would be a sequence \( \{\lambda_k\}_{k=1}^{\infty} \) and \( \epsilon, \delta > 0 \) such that \( \mathbb{N}_0(\{d(L^t_1^{\lambda_k}, L_t^t) > \epsilon \} \cap \{X_t > 0\}) > \delta \) for all \( k \geq 1 \). However, using the previous argument we can obtain a subsequence on which the measures converge to \( L_t \) \( \mathbb{N}_0(\cdot \cap \{X_t > 0\}) \)-a.s., which, because \( \mathbb{N}_0(\cdot \cap \{X_t > 0\}) \) is a finite measure, contradicts the previous statement. Hence we must have that \( L^t_1 \to L_t \) in measure.

Next, we observe that for continuous and bounded \( \phi \), \( L_t(\phi) = l(t, \phi) \). To see this, recall that \( L^t_1^{\lambda_0}(\phi) \) converges to \( l(t, \phi) \) in \( L^2(\mathbb{N}_0) \). As we have just shown that \( \lim_{m \to \infty} L^t_1^{\lambda_m}(\phi) = L_t(\phi) \) \( \mathbb{N}_0 \)-a.s., it must hold that \( L_t(\phi) = l(t, \phi) \). This implies that \( L^t_1^{\lambda}(\phi) \to L_t(\phi) \) in \( L^2(\mathbb{N}_0) \) by Corollary 2.4.1.

Finally, we verify that \( L_t \) is supported on \( BZ_t \). We fix \( \omega \) outside of a null set such that \( L^t_1^{\lambda_m} \to L_t \) in \( \mathcal{M}_F(\mathbb{R}) \) as \( m \to \infty \). For an open set \( U \), \( L_t(U) \leq \liminf_{m \to \infty} L^t_1^{\lambda_m}(U) \) (a consequence of the Portmanteau theorem). From (2.1.6), we have \( L^t_1^{\lambda_m}(Z_t) = 0 \) for all \( m \geq 1 \), which implies that \( L_t(\text{int}(Z_t)) = 0 \). Moreover, \( X(t, x) > 0 \) implies that \( \lambda^{2\lambda_0} X(t, x) e^{-\lambda_m X(t, x)} \to 0 \) as \( m \to \infty \), so for \( \epsilon > 0 \), \( L_t(\{x : X(t, x) > \epsilon\}) = 0 \), and hence \( L_t(Z_t^c) = 0 \). Since \( L_t(\text{int}(Z_t) \cup Z_t^c) = 0 \), we must have \( \text{supp}(L_t) \subseteq BZ_t \). \( \square \)

**Proof of Theorem 2.1.5.** To prove (b), by Theorem 2.1.4 it is enough to show that \( \mathbb{N}_0((L_t \times L_t)^{\lambda_{n}}(h)) = \lim_{n \to \infty} \mathbb{N}_0((L_t^{\lambda_{n}} \times L_t^{\lambda_{n}})(h)) \) for a sequence \( \lambda_n \to \infty \), which we choose to be the sequence from Theorem 2.1.1 on which \( L^{\lambda_n}_t \to L_t \) almost surely. Because \( L_t = 0 \) when \( X_t = 0 \), we can work on the probability measure \( \mathbb{N}_0(\cdot | X_t > 0) \). For bounded and continuous \( h : \mathbb{R}^2 \to \mathbb{R} \), \( |(L_t^{\lambda_m} \times L_t^{\lambda_n})(h)| \leq \|h\|_{\infty} L_t^{\lambda_n}(1)^2 \). By Theorem 2.1.1, \( L_t^{\lambda_n}(1) \) converges in probability and in \( L^2(\mathbb{N}_0(\cdot | X_t > 0)) \) to \( L_t(1) \), which implies that \( L_t^{\lambda_n}(1)^2 \) and hence \( (L_t^{\lambda_n} \times L_t^{\lambda_n})(h) \) are uniformly integrable (see, e.g. Theorem 4.6.3 of [19]). We can therefore exchange limit with expectation, giving

\[
\mathbb{N}_0(\lim_{n \to \infty} (L_t^{\lambda_n} \times L_t^{\lambda_n})(h)) = \lim_{n \to \infty} \mathbb{N}_0((L_t^{\lambda_n} \times L_t^{\lambda_n})(h)).
\]

Since \( L_t^{\lambda_n} \to L_t \) in \( \mathcal{M}_F(\mathbb{R}) \) and \( h \) is bounded and continuous, the integrand on the left hand side is equal to \( (L_t \times L_t)(h) \), which gives the result. By a Monotone Class Theorem (e.g. Corollary 4.4 in the Appendix of Ethier and Kurtz [23]), the same holds for all bounded and measurable \( h \).
We now turn to part (a). Let \( \phi : \mathbb{R} \to \mathbb{R} \) be bounded and Borel. We recall from the Introduction (see (2.1.5)) that Proposition 4.5 of [69] states that

\[
\lim_{\lambda \to \infty} t^{\lambda_0} E_{\lambda}^{X_{X_0}} (L_t^\lambda (\phi)) = C_{2.1.4} \int \int \phi(x_0 + \sqrt{t}z) \exp \left( -\frac{1}{t} \int F(z + t^{-1/2}(x_0 - y_0)) dX_0(y_0) \right) \psi_0(z) dm(z) dX_0(x_0).
\]  

(2.4.2)

(The fact that the constant appearing in Proposition 4.5 of [69] equals \( C_{2.1.4} \) is implicit in the proof.) The proof uses the Palm measure formula for \( X_t \) under \( P_{X_0} \); see Theorem 4.1.3 of Dawson-Perkins [15]. The corresponding Palm measure formula for the superprocess under \( N_0 \) is in fact simpler, and the same proof shows that

\[
\lim_{\lambda \to \infty} N_0(L_t^\lambda (\phi)) = C_{2.1.4} t^{-\lambda_0} \int \int \phi(\sqrt{t}z) \psi_0(z) dm(z).
\]  

(2.4.3)

Consider now a bounded and continuous function \( \phi \); we can also clearly assume that \( \phi \geq 0 \). By Theorem 2.1.1 (under \( N_0 \)), \( L_t^\lambda (\phi) \) converges in \( L^2 \) with respect to the probability measure \( N_0(X_t \in \cdot | X_t > 0) \), which implies that it also converges in \( L^1 \), allowing us to exchange limit and expectation in (2.4.3), which gives part (a) for bounded and continuous \( \phi \). This extends to all bounded and measurable \( \phi \) by a monotone class argument (as above for part (b)). Finally, it is clear that both (a) and (b) hold for general non-negative functions by the Monotone Convergence Theorem.

We now describe how to ascertain the existence of \( L_t \) when \( X_t \) is a super-Brownian motion under \( P_{X_0} \) via the cluster representation. In particular, we recall (2.1.15) and (2.1.16). Let \( X_0 \in \mathcal{M}_F(\mathbb{R}) \) and \( t > 0 \).

**Proof of Theorem 2.1.1** for \( P_{X_0} \). Let \( N, x_1, \ldots, x_N, X_1^t, \ldots, X_N^t \) be as in the cluster decomposition (2.1.16). For \( \lambda > 0 \), define the measure \( L_t^\lambda \) via (2.1.6) using \( X_t \). For \( i = 1, \ldots, N \), let \( L_t^{i,\lambda} \) denote the measure defined in (2.1.6) corresponding to \( X_i^t \). By Theorem 2.1.1 for \( N_0 \) and translation invariance, \( N_{x_i}(X_i^t \in \cdot | X_i^t > 0) \)-a.s. there exists \( L_t^i \) such that \( L_t^{i,\lambda} \to L_t^i \) in \( \mathcal{M}_F(\mathbb{R}) \) in measure. Define \( L_t \in \mathcal{M}_F(\mathbb{R}) \) by (2.1.17). That is,

\[
dL_t(x) = \sum_{i=1}^N 1 \left( \sum_{j \neq i} X_j^t(t, x) = 0 \right) dL_t^i(x).
\]

Let \( \phi : \mathbb{R} \to \mathbb{R} \) be bounded and continuous. We will show that

\[
L_t^\lambda (\phi) \to L_t(\phi) \text{ in probability as } \lambda \to \infty.
\]  

(2.4.4)
Once we establish (2.4.4), the proof of Theorem 2.1.1 for $\mathbb{N}_0$ applies and shows that $L_t^\lambda \to L_t$ in probability in $\mathcal{M}_F(\mathbb{R})$ as $\lambda \to \infty$. With the exception of $L^2$ convergence, which we show afterward, this proves Theorem 2.1.1 for $P_{X_0}^X$.

Turning to (2.4.4), we will argue conditionally on $(N, x_1, \ldots, x_N)$. That is, we argue under the regular conditional distribution for $(X_1^1, \ldots, X_i^N)$ given $(N, x_1, \ldots, x_N)$. As such, we treat $N \geq 1$ and $x_1, \ldots, x_N \in \mathbb{R}$ as fixed, and $X_1^1, \ldots, X_i^N$ are independent random measures with respective laws $\mathbb{N}_x(X_t \in \cdot | X_t > 0)$ for $i = 1, \ldots, N$. Let $E$ denote the expectation of a probability realizing this conditional representation for $X_t$. Expanding $L_t^1(\phi)$ in terms of the clusters, we have

$$L_t^1(\phi) = \int \lambda e^{-\lambda X(t, x)} \phi(x) dx = \int \lambda \left[ \sum_{i=1}^N X_i(t, x) \right] (x) dx = \sum_{i=1}^N \int \lambda X_i(t, x) e^{-\lambda X_i(t, x)} \left[ e^{-\lambda \sum_{j \neq i} X_j(t, x)} \phi(x) \right] dx = \sum_{i=1}^N L_t^1(\phi e^{-\lambda Z_i(t, \cdot)}), \quad (2.4.5)$$

where we define $Z_i^N(t, x) = \sum_{j \neq i} X_j^i(t, x)$, in which the indices are understood to sum from 1 to $N$. Using this notation, $L_t(\phi) = \sum_{i=1}^N L_t^1(\phi 1(Z_i^N(t, \cdot) = 0))$. Thus by (2.4.5), to prove (2.4.4) it is clearly enough to show that for any $1 \leq i \leq N$,

$$L_t^1(\phi e^{-\lambda Z_i^N(t, \cdot)}) \to L_t^1(\phi 1(Z_i^N(t, \cdot) = 0)) \text{ in probability as } \lambda \to \infty. \quad (2.4.6)$$

Without loss of generality, assume that $\lambda > 1$. Let $1 \leq \lambda' \leq \lambda$. Then

$$|L_t^i(\phi e^{-\lambda Z_i^N(t, \cdot)}) - L_t^i(\phi 1(Z_i^N(t, \cdot) = 0))| \
\leq |L_t^i(\phi e^{-\lambda Z_i^N(t, \cdot)} - e^{-\lambda' Z_i^N(t, \cdot)})| + |L_t^i(\phi e^{-\lambda' Z_i^N(t, \cdot)}) - L_t^i(\phi e^{-\lambda' Z_i^N(t, \cdot)})| \
+ |L_t^i(\phi (e^{-\lambda'} Z_i^N(t, \cdot) - 1(Z_i^N(t, \cdot) = 0))| \
\leq \|\phi\|_\infty |L_t^i(\phi e^{-\lambda' Z_i^N(t, \cdot)} 1(Z_i^N(t, \cdot) > 0))| + |L_t^i(\phi e^{-\lambda' Z_i^N(t, \cdot)}) - L_t^i(\phi e^{-\lambda' Z_i^N(t, \cdot)})| \
+ \|\phi\|_\infty L_t^i(1(Z_i^N(t, \cdot) > 0)) \
=: \|\phi\|_\infty R_1(\lambda', \lambda) + R_2(\phi, \lambda') + \|\phi\|_\infty R_3(\lambda', \lambda). \quad (2.4.7)$$

We first consider $R_1$. Since $X_i^i$ and $Z_i^N(t, \cdot)$ are independent and $L_t^i$ is a measurable function
of $X_i^t$, conditional on $X_i^t$ we have, for all $\lambda > 1$ and $1 \leq \lambda' < \lambda$,

$$E(R_1(\lambda', \lambda) | X_i^t) = \int E(e^{-\lambda' Z^i_N(t,x)}1(Z^i_N(t,x) > 0)) | X_i^t) dL_t^{i,\lambda}(x)$$

$$= \int E(e^{-\lambda' \sum_{j \neq i} X_j^i(t,x)} \sum_{j \neq i} X_j^i(t,x) > 0)) dL_t^{i,\lambda}(x)$$

$$\leq \sum_{j \neq i} \int \mathbb{N}_{x_j}(e^{-\lambda' X_j^i(t,x)}1(X_j^i(t,x) > 0) | X_i^t > 0) dL_t^{i,\lambda}(x)$$

$$= \sum_{j \neq i} \mathbb{N}_{x_j}(X_j^i > 0)^{-1} \int \mathbb{N}_{x_j}(e^{-\lambda' X_j^i(t,x)} - 1(X_j^i(t,x) = 0)) dL_t^{i,\lambda}(x)$$

$$= \frac{t}{2} \sum_{j \neq i} \int \mathbb{N}_{x_j}(1 - 1(X_j^i(t,x) = 0)) - \mathbb{N}_{x_j}(1 - e^{-\lambda' X_j^i(t,x)}) dL_t^{i,\lambda}(x)$$

$$= \frac{t}{2} \sum_{j \neq i} \int V_t^{\infty}(x - x_j) - V_t^{\lambda'}(x - x_j) dL_t^{i,\lambda}(x), \quad (2.4.8)$$

where in the second last line we have used (2.3.11), and the last follows from (2.3.8), (2.3.10), and translation invariance. We apply (2.3.15) to the integrand and take the expectation of the above to obtain that

$$E(R_1(X, \lambda)) \leq C \frac{N - 1}{2} t^{1/2 - \lambda_0} \lambda^{-(2\lambda_0 - 1)} \mathbb{N}_{x_i}(L_t^{i,\lambda}(1) | X_i^t > 0)$$

$$= C \frac{N - 1}{4} t^{3/2 - \lambda_0} \lambda^{-(2\lambda_0 - 1)} \mathbb{N}_{x_i}(L_t^{i,\lambda}(1)) \quad \text{(by (2.3.11))}$$

$$\leq C(t, N) \lambda^{-(2\lambda_0 - 1)}, \quad (2.4.9)$$

for all $\lambda > 1$ and $1 \leq \lambda' < \lambda$, where the last inequality is by Theorem 2.1.5(a) and the fact that $L_t^{i,\lambda}(1) \to L_t^i(1)$ in $L^2(\mathbb{N}_{x_i})$ (from Theorem 2.1.1). Next we consider $R_3$. Note that we can expand and bound this term in exactly the same way as we did $R_1$ in (2.4.7) but with $L_t^i$ replacing $L_t^{i,\lambda}$. Taking the expectation and proceeding as above then gives

$$E(R_3) \leq C \frac{N - 1}{4} t^{3/2 - \lambda_0} \mathbb{N}_{x_i}(L_t^i(1)) \lambda^{-(2\lambda_0 - 1)}. \quad (2.4.10)$$

Fix $\delta > 0$. By (2.4.9) and (2.4.10) and Markov’s inequality there exists $\lambda(\delta)$ such that for $\lambda' \geq \lambda(\delta)$,

$$P(R_1(X', \lambda) > \delta) + P(R_3(\lambda', \lambda) > \delta) < C'(t, N) \lambda^{-(2\lambda_0 - 1)}/\delta. \quad (2.4.11)$$

Now consider $R_2(\phi)$. Since $\phi \cdot e^{-\lambda' Z^i_N(t,\cdot)}$ is a bounded, continuous function for all $\lambda' \geq 1$, by Theorem 2.1.1 for $\mathbb{N}_{x_i}$, $R_2(\phi, \lambda') \to 0$ in probability as $\lambda \to \infty$ for all $\lambda' \geq 1$. From this and (2.4.11) we conclude, by choosing $\lambda' \leq \lambda$ sufficiently large, that (2.4.7) converges to 0 in probability as $\lambda \to \infty$. As we noted in (2.4.6), this is sufficient to prove the result.
It remains to show that $L_i^\lambda(\phi) \to L_i(\phi)$ in $L^2(P_{X_0}^X)$ for all continuous and bounded functions $\phi$. Let $\phi$ be such a function, and suppose that $X_t$ is realized as in (2.1.16) under a probability $P_{X_0}^X$. Under $P_{X_0}^X(\cdot \mid N)$, from (2.4.5) and (2.1.7) we have

$$
(L_i^\lambda(\phi) - L_i(\phi))^2 = \left( \sum_{i=1}^N L_i^\lambda(e^{-\lambda Z_N(t, \cdot)} \cdot \phi) - L_i(1(Z_N(t, \cdot) = 0) \cdot \phi) \right)^2 
\leq N \sum_{i=1}^N [L_i^\lambda(e^{-\lambda Z_N(t, \cdot)} \cdot \phi) - L_i(1(Z_N(t, \cdot) = 0) \cdot \phi)]^2.
$$

(2.4.12)

We recall that $X_1^t, \ldots, X_N^t$ are iid with distribution $\mathbb{N}_{X_0}(X_t \in \cdot \mid X_t > 0)$, where $X_0 = X(\cdot)/X_0(1)$ and $\mathbb{N}_{X_0}(\cdot) = \int \mathbb{N}_x(\cdot) dX_0(x)$. This implies that the $N$ summands in (2.4.12) are identically distributed; in particular, conditional on $N$ we define identically distributed random variables $e_i^{N,\lambda} \geq 0$, for $i = 1, \ldots, N$, by

$$
e_i^{N,\lambda} = [L_i^\lambda(e^{-\lambda Z_N(t, \cdot)} \cdot \phi) - L_i(1(Z_N(t, \cdot) = 0) \cdot \phi)]^2.
$$

(2.4.13)

By (2.4.6), $e_i^{N,\lambda}$ converges to 0 in probability as $\lambda \to \infty$ when conditioned on $(x_1, \ldots, x_N)$. However, one can integrate the conditional probabilities over $(x_1, \ldots, x_N) \in \mathbb{R}_N$ to determine that

$$
e_i^{N,\lambda} \to 0 \text{ in probability under } P_{X_0}^X(\cdot \mid N) \text{ as } \lambda \to \infty.
$$

(2.4.14)

It is clear from (2.4.13) that for all $\lambda > 0$,

$$
e_i^{N,\lambda} \leq 2\|\phi\|_\infty^2(\sum_{i=1}^N L_i^\lambda(1)^2 + L_i(1)^2) \quad \forall i = 1, \ldots, N, \forall N \geq 1.
$$

(2.4.15)

By Theorem 2.1.1 for $\mathbb{N}_0$, $L_i^\lambda(1)^2 \to L_i(1)^2$ in probability under $\mathbb{N}_{X_0}(\cdot \mid X_t > 0)$ and hence under $P_{X_0}^X(\cdot \mid N)$. Furthermore, since $L_i^\lambda(1) \to L_i(1)$ in $L^2(\mathbb{N}_{X_0}(\cdot \mid X_t > 0))$ (by Theorem 2.1.1 for $\mathbb{N}_0$), it follows from Cauchy-Schwarz that $L_i^\lambda(1)^2 \to L_i(1)^2$ in $L^1(\mathbb{N}_{X_0}(\cdot \mid X_t > 0))$; since $X_t$ has distribution $\mathbb{N}_{X_0}(X_t \in \cdot \mid X_t > 0)$ under $P_{X_0}^X(\cdot \mid N)$, this implies $L_i^\lambda(1)^2 \to L_i(1)^2$ in $L^1(P_{X_0}^X(\cdot \mid N))$. Hence $\{2\|\phi\|_\infty^2(\sum_{i=1}^N L_i^\lambda(1)^2 + L_i(1)^2) : \lambda \geq 1\}$ is uniformly integrable. Thus by (2.4.15), $\{e_i^{N,\lambda} : \lambda \geq 1\}$ is uniformly integrable, and by (2.4.14) we have $L^1$ convergence. That is,

$$
E_{X_0}^X(e_i^{N,\lambda} \mid N) \to 0 \text{ as } \lambda \to \infty.
$$

(2.4.16)

Conditioning on $N = n$ and summing over $n \in \mathbb{N}$, by (2.4.12) and Fubini’s Theorem we
have
\[ E_{X_0}^X((L_\lambda^X(\phi) - L_t(\phi))^2) \leq \sum_{n=1}^{\infty} P_{X_0}^X(N = n) n \sum_{i=1}^{n} E_{X_0}^X(e_{i,n,\lambda}^X | N = n). \]

Since \( E_{X_0}^X(e_{i,n,\lambda}^X | N) \leq 2\|\phi\|_{\infty} E_{X_0}^X((L_{\lambda}^X(1))^2 + L_t(1)^2) \leq C(t,\phi) \) for all \( \lambda \geq 1 \), for some constant \( C(t,\phi) > 0 \) (by uniform integrability), the \( n \)-th term in the above is bounded above by \( C(t,\phi) P_{X_0}^X(N = n)n^2 \). Dominated Convergence therefore allows us to exchange limit and summation in the above, which by (2.4.16) gives the result.

**Proof of Theorems 2.1.8 and 2.1.9.** The proof of Theorem 2.1.8 is in fact implicit in the above proof of Theorem 2.1.1 for \( P_{X_0}^X \). Conditionally on the number of clusters \( N \), \( L_t \) was defined under \( P_{X_0}^X \) by (2.1.17), so by construction it has the claimed conditional representation.

The proof of Theorem 2.1.9 is virtually identical to that of Theorem 2.1.8, except in this case we already know that \( L_t \) exists and \( L_{\lambda}^X \to L_t \) in \( M_F(\mathbb{R}) \). One can then decompose \( X_t \) in terms of the different contributions and show that \( L_t \) has the desired representation using the same argument as appears above, making the obvious changes between the law of super-Brownian motion and canonical measure where necessary.

As we have commented on, the expression in Theorem 2.1.4, which is the same as (2.1.9) in Theorem 2.1.5(b), is finite for all bounded \( h \), despite the appearance of non-integrability (since \( \lambda_0 > 1/2 \)). Proposition 2.1.6, which we restate here for convenience, provides a useful upper bound on second moments which is our main tool for studying \( L_t \). The bound is not difficult to obtain. Its derivation relies only on trivial upper bounds and several changes of variables. Recall that \( E_z^Y \) denotes the expectation of a standard Ornstein-Uhlenbeck process \( Y \) with \( Y_0 = z \).

**Proposition 2.1.6.** For a non-negative Borel function \( h : \mathbb{R}^2 \to \mathbb{R} \),
\[
\mathbb{N}_0((L_t \times L_t)(h)) \leq C_{2.1.4}^2 \int_0^t w^{-2\lambda_0} \left[ \int \int E_{z_1}^Y \left( \exp \left( -\int_0^{\log(t/w)} F(Y_u) \, du \right) \psi_0(z_1) \psi_0(z_2) \, dm(z_1) \, dm(z_2) \right) dw \right] \times h(\sqrt{7} Y_{\log(t/w)}^0, \sqrt{7} Y_{\log(t/w)}^0 + \sqrt{w}(z_2 - z_1)) \psi_0(z_1) \psi_0(z_2) \, dm(z_1) \, dm(z_2). \tag{2.1.10}
\]
Moreover,
\[
\mathbb{N}_0(L_t(1)^2) \leq \frac{C_{2.1.4}^2 \theta^2}{1 - \lambda_0} t^{1-2\lambda_0}. \tag{2.1.11}
\]

**Proof.** Let \( h : \mathbb{R}^2 \to \mathbb{R} \) be Borel measurable and non-negative. We use the formula for
\[ \mathbb{N}_0((L_t \times L_t)(h)) \] given by (2.1.9). We recall that \( \rho(z_1, z_2) \leq 1 \) and use this bound, and we bound above by using \( V_\infty \) in the exponential. This gives

\[ \mathbb{N}_0((L_t \times L_t)(h)) \leq C^2_{2.1.4} \int_0^t (t-s)^{-2\lambda_0} \left[ \int \int E_0^B \left( \exp \left( - \int_0^s V_t^\infty \left( \sqrt{t-s} z_1 + B_s - B_u \right) du \right) \right) \times h(\sqrt{t-s} z_1 + B_s, \sqrt{t-s} z_2 + B_s) \psi_0(z_1) \psi_0(z_2) \ dm(z_1) \ dm(z_2) \right] \ ds. \]

Since \( z_1 \sim m, \sqrt{t-s} z_1 \) has a normal distribution with variance \( t-s \), and we interpret it as the Brownian increment \( B_t - B_s \). Hence the above is equal to

\[ C^2_{2.1.4} \int_0^t (t-s)^{-2\lambda_0} \left[ \int \int E_0^W \left( \exp \left( - \int_0^s V_t^\infty (W_u) du \right) \right) h(W_t, \sqrt{w} z_2 + W_t - W_0) \times \psi_0(W_w/W_0) \psi_0(z_2) \ dm(z_2) \right] \ dw, \]

where in the second line we have used \( w = t-s \) and defined \( W_u = B_t - B_{t-u} \). Hence \( W_u \) is a standard Brownian motion under \( \mathbb{P}_0^W \). Recall that \( V_t^\infty(x) = u^{-1} F(u^{-1/2}x) \). Applying this and letting \( u = e^r \) in the integral, we obtain that the above is equal to

\[ C^2_{2.1.4} \int_0^t w^{-2\lambda_0} \left[ \int E_0^W \left( \exp \left( - \int_{\log W}^{\log t} F(e^{-r/2} W_\epsilon) \ d\epsilon \right) \right) h(W_t, \sqrt{w} z_2 + W_t - W_0) \times \psi_0(W_w/W_0) \psi_0(z_2) \ dm(z_2) \right] \ dw. \]

We now define a stationary Ornstein-Uhlenbeck process \( Y \) (with stationary measure \( m \)) by \( Y_r = e^{-r/2} W_\epsilon \) for \( r \in \mathbb{R} \). Recall that we denote its law by \( E^Y \). The above is therefore equal to

\[ C^2_{2.1.4} \int_0^t w^{-2\lambda_0} \left[ \int E^Y \left( \exp \left( - \int_{\log W}^{\log t} F(Y_u) \ du \right) \right) h(\sqrt{t} Y_{\log t}, \sqrt{w} z_2 + \sqrt{t} Y_{\log t} - \sqrt{w} Y_{\log w}) \times \psi_0(Y_{\log w}) \psi_0(z_2) \ dm(z_2) \right] \ dw. \]

73
By stationarity of $Y$, we can shift time by $\log w$ in the above to obtain

\[
C_{2.1.4}^2 \int_0^t w^{-2\lambda_0} \left[ \int E^Y \left( \exp \left( - \int_0^{\log(t/w)} F(Y_u) \, du \right) \right) h(\sqrt{t} Y_{\log(t/w)}; \sqrt{w} z_2 + \sqrt{t} Y_{\log(t/w)} - \sqrt{w} Y_0) \times \psi_0(Y_0) \psi_0(z_2) \, dm(z_2) \right] \, dw.
\]

$Y_0$ has distribution $m$, so we condition on the value of $Y_0$ and call it $z_1$. This gives the desired expression and proves that (2.1.10) holds. The proof of (2.1.11) is a consequence of the following lemma.

**Lemma 2.4.2.** For $t > 0$,

\[
\int P^Y_z(\rho^F > t) \psi_0(z) \, dm(z) = \theta e^{-\lambda_0 t}.
\]

Returning to (2.1.11), we apply (2.1.10) with $h = 1$. Separating the integrals, we obtain that

\[
N_0(L_t(1)^2) \leq C_{2.1.4}^2 \theta \int_0^t w^{-2\lambda_0} \left( \int P^Y_z(\rho^F > \log(t/w)) \psi_0(z) \, dm(z) \right) \, dw,
\]

where we have used $\int \psi_0 \, dm = \theta$. The inequality (2.1.11) now readily follows from Lemma 2.4.2, which completes the proof of Proposition 2.1.6.

**Proof of Lemma 2.4.2.** Expanding in terms of the transition densities, we have

\[
\int P^Y_z(\rho^F > t) \psi_0(z) \, dm(z) = \int \left( \int q_t(z, y) \, dm(y) \right) \psi_0(z) \, dm(z)
= \langle q_t, 1 \otimes \psi_0 \rangle_{L^2(m \times m)},
\]

where $\langle \cdot, \cdot \rangle_{L^2(m \times m)}$ denotes the inner product on $L^2(m \times m)$ and $\otimes$ is the tensor product of functions. Recall from that Theorem 2.2.1(a) that the eigenfunction expansion (2.2.1) converges in $L^2(m \times m)$ to $q_t(\cdot, \cdot)$, and that $\|\psi_0\|_{L^2(m)} = 1$. Thus by the above and Fubini’s theorem, (2.4.17) is equal to

\[
\sum_{n=0}^{\infty} e^{-\lambda_n t} \langle \psi_n \otimes \psi_n, 1 \otimes \psi_0 \rangle_{L^2(m \times m)} = e^{-\lambda_0 t} \langle \psi_0 \otimes \psi_0, 1 \otimes \psi_0 \rangle_{L^2(m \times m)}
= e^{-\lambda_0 t} \int \psi_0^2 \, dm \int \psi_0 \, dm = \theta e^{-\lambda_0 t},
\]

where the first equality follows from orthogonality of the eigenfunctions, which implies that $\int \psi_n \psi_0 \, dm = 0$ for all $n \geq 1$. The last line uses $\int \psi_0 \, dm = \theta$ and $\int \psi_0^2 \, dm = 1$. \qed
We now use the bounds in Proposition 2.1.6 to derive the remaining properties of $L_t$ and their consequences. In order of presentation, we now prove Theorem 2.1.3, Theorem 2.1.7, and Theorem 1.2.

**Proof of Theorem 2.1.3.** Recall that for $p > 0$, $h_p(x, y) = |x - y|^{-p}$. We first establish that

$$N_0((L_t \times L_t)(h_p)) < \infty \quad (2.4.18)$$

for all $p < 2 - 2\lambda_0$. Applying (2.1.10) with $h_p$, we have

$$N_0((L_t \times L_t)(h_p)) \leq C_{2.1.4}^2 \int_0^t w^{-2\lambda_0} \left[ \int \int E^Y_{z_1} \left( \exp \left( - \int_0^{\log(t/w)} F(Y_u) \, du \right) \right) \right. \\
\left. \times \left| \sqrt{w(z_2 - z_1)} \right|^{-p} \psi_0(z_1) \psi_0(z_2) \, dm(z_1) \, dm(z_2) \right] \, dw \leq C_{2.1.4}^2 \int_0^t w^{-2\lambda_0 - p/2} \left[ \int \int E^Y_{z_1} \left( \exp \left( - \int_0^{\log(t/w)} F(Y_u) \, du \right) \right) \right. \\
\left. \times \left| z_1 - z_2 \right|^{-p} \psi_0(z_1) \psi_0(z_2) \, dm(z_1) \, dm(z_2) \right] \, dw.
$$

Recalling (2.1.4), the expectation is equal to the survival probability $P^Y_{z_1}(\rho^F > \log(t/w))$, so the above equals

$$C_{2.1.4}^2 \int_0^t w^{-\lambda_0 - p/2} \left[ \int \int P^Y_{z_1}(\rho^F > \log(t/w)) \left| z_1 - z_2 \right|^{-p} \psi_0(z_1) \psi_0(z_2) \, dm(z_1) \, dm(z_2) \right] \, dw.
$$

Applying (2.2.5) and (2.2.8), both with $\delta = 1/8$, this is bounded above by

$$C \int_0^t w^{-\lambda_0 - p/2} \left[ \int \int \left| z_1 - z_2 \right|^{-p} t^{-\lambda_0} w^{\lambda_0} e^{z_1^2/4} e^{z_2^2/8} \, dm(z_1) \, dm(z_2) \right] \, dw \\
= C(p) t^{-\lambda_0} \int_0^t w^{-\lambda_0 - p/2} \, dw.
$$

The second line follows because the integrand has Gaussian tails in $z_1$ and $z_2$ and $p < 2 - 2\lambda_0 < 1$. Finally, the integral in the final line is finite because $-\lambda_0 - p/2 > -\lambda_0 - \lambda_0 + 1 > -1$, which proves (2.4.18). In fact, we have shown that

$$N_0((L_t \times L_t)(h_p)) \leq C(p) t^{1 - 2\lambda_0 - p/2}. \quad (2.4.19)$$

Next, we establish the same under $P^X_{X_0}$. That is, we will show that

$$E^X_{X_0}((L_t \times L_t)(h_p)) < \infty \quad (2.4.20)$$

75
for $p < 2 - 2\lambda_0$. We use the cluster decomposition and argue conditionally as in the proof of Theorem 2.1.1 (for $P_{X_{X_0}}^X$) above. Suppose that $P_{X_{X_0}}^X$ is a probability under which $X_t$ is realized as in (2.1.16). Conditioning on $N, x_1, \ldots, x_N$, by (2.1.17) we have

$$dL_t(x) \leq \sum_{i=1}^N dL_t^i(x).$$

Thus we obtain that

$$\int \int |x - y|^{-p} dL_t(x) dL_t(y) \leq \int \int |x - y|^{-p} \left( \sum_{i=1}^N dL_t^i(x) \right) \left( \sum_{i=1}^N dL_t^i(y) \right) = \sum_{i=1}^N \int \int |x - y|^{-p} dL_t^i(x) dL_t^i(y) + \sum_{i=1}^N \sum_{j \neq i} \int \int |x - y|^{-p} dL_t^i(x) dL_t^j(y).$$

(2.4.21)

Recall that the $X_t^i$ are independent with distributions $\mathbb{N}_{x_1}(X_t^i \in \cdot \mid X_t > 0)$. By (2.3.11) and (2.4.19), we therefore have

$$\mathbb{N}_{x_1}^1 \left( \int \int |x - y|^{-p} dL_t^1(x) dL_t^1(y) \bigg| X_t^1 > 0 \right) = C(p) t^{1 - 2\lambda_0 - p/2} (2/t)^{-1} =: C_1(p) t^{2 - 2\lambda_0 - p/2},$$

(2.4.22)

which provides a bound for the summands in the first term of (2.4.21). We now consider the mixed integrals in (2.4.21), that is, the summands in the second term. Without loss of generality, let $i = 1$ and $j = 2$, and denote their (independent) distributions by $\mathbb{N}_{x_1}^1(X_t^1 \in \cdot \mid X_t^1 > 0), \mathbb{N}_{x_2}^2(X_t^2 \in \cdot \mid X_t^2 > 0)$. Because the integrands are non-negative, we can change the order of integration and obtain

$$\mathbb{N}_{x_1}^1 \otimes \mathbb{N}_{x_2}^2 \left( \int \int |x - y|^{-p} dL_t^1(x) dL_t^2(y) \bigg| X_t^1 > 0, X_t^2 > 0 \right)$$

$$= \mathbb{N}_{x_1}^1 \left( \int \mathbb{N}_{x_2}^2 \left( \int |x - y|^{-p} dL_t^2(y) \bigg| X_t^2 > 0 \right) dL_t^1(x) \bigg| X_t^1 > 0 \right).$$

(2.4.23)
To compute the inner expectation we apply translation invariance and (2.3.11), which gives
\[
\begin{align*}
N_{z_2}^2 \left( \int \left| y - x \right|^{-p} dL_t^2(y) \right| X_t^2 > 0 \\
= (t/2) N_0 \left( \int \left| y - x \right|^{-p} dL_t(y) \right) \\
= (t/2) N_0 \left( \int \left| y - x + x_2 \right|^{-p} dL_t(y) \right) \\
= C_{2.1.4}(t/2) t^{-\lambda_0} \int \left| \sqrt{t} - (x - x_2) \right|^{-p} \psi_0(z) \, dm(z),
\end{align*}
\]
where the last line follows from the mean measure formula (2.1.8). By (2.2.5) with \( \delta = 1/4 \), we have that \( \psi_0(z_2) \, dm(z_2) \leq c e^{-z_2^2/4} dz_2 \). Thus the above is bounded above by
\[
C t^{1-\lambda_0} \int \left( \left| \sqrt{t} - (x - x_2) \right|^{-p} \vee 1 \right)e^{-z^2/4} dz \\
= C t^{1-\lambda_0} \int \left( \left| w - (x - x_2) \right|^{-p} \vee 1 \right) t^{-1/2} e^{-w^2/4t} \, dw \\
\leq C t^{1-\lambda_0} t^{-1/2} \int \left| w - (x - x_2) \right|^{-p} \, dw + C t^{1-\lambda_0} \int t^{-1/2} e^{-w^2/4t} \, dw \\
= C'(p) t^{1/2-\lambda_0} + C t^{1-\lambda_0} < \infty.
\]

By the above bound and another application of (2.1.8), (2.4.23) is bounded above by
\[
\left[ C'(p) t^{1/2-\lambda_0} + C t^{1-\lambda_0} \right] N_{z_1}^1 \left( L_t^1(1) \left| X_t^1 > 0 \right. \right) =: C_2(p) \left[ t^{3/2-2\lambda_0} + t^{2-2\lambda_0} \right]. \tag{2.4.24}
\]
We note that both (2.4.22) and (2.4.24) are independent of the points \( x_1, \ldots, x_N \). Therefore by these bounds and (2.4.21) we have shown that
\[
E_{X_0}( (L_t \times L_t)(h_p) \left| N \right) \leq C_1(p) N t^{2-2\lambda_0-p/2} + C_2(p) (N^2 - N) \left[ t^{3/2-2\lambda_0} + t^{2-2\lambda_0} \right].
\]
Taking the expectation above with respect to \( N \), which we recall is Poisson with mean \( 2X_0(1)/t \), gives
\[
E_{X_0}( (L_t \times L_t)(h_p) ) \leq C_1(p) X_0(1) t^{1-2\lambda_0-p/2} + C_2(p) X_0(1)^2 \left[ t^{-1/2-2\lambda_0} + t^{-2\lambda_0} \right] < \infty, \tag{2.4.25}
\]
which proves (2.4.20).

Under both \( P_{X_0}^X \) and \( N_0 \), we have shown that the \( p \)-energy of \( L_t \) has finite expectation, and hence \( L_t \) has finite \( p \)-energy almost surely, for all \( p < 2-2\lambda_0 \). By the energy method (see, for example, Theorem 4.27 of Mörters and Peres [65]), this implies that \( \dim(BZ_t) \geq 2-2\lambda_0 \) a.s. on \( \{ L_t > 0 \} \) under \( P_{X_0}^X \) and \( N_0 \). Combined with Theorem 2.A, this completes the proof
of Theorem 2.1.3 for $P_{X_0}^X$. To see that the upper bound on the dimension holds for $N_0$ follows from the cluster decomposition. Consider $X_t$ under $P_{X_0}^X$. In the cluster decomposition of $X_t$, the probability that $N = 1$ is positive. Conditioning on this event, $X_t$ is equal to $X_1^t$, which has law $N_0(X_1^t \in \cdot | X_t > 0)$. Because $\dim(BZ_t) \leq 2 - 2\lambda_0$ a.s. on this event, we therefore have $N_0(\{\dim(BZ_t) \leq 2 - 2\lambda_0 \} | X_t > 0) = 1$. In particular, this implies that $N_0(\{\dim(BZ_t) \leq 2 - 2\lambda_0 \} | L_t > 0) = 1$, since $\{L_t > 0\} \subseteq \{X_t > 0\}$. We note that conditioning on the event $\{L_t > 0\}$ is valid under both $P_{X_0}^X$ and $N_0$ by Theorem 2.1.2(a). This completes the proof.

**Proof of Theorem 2.1.7.** To see part (a), we note that (2.4.2) gives an expression for $\lim_{\lambda \rightarrow \infty} E_{X_0}^X(L_t^\lambda(\phi))$. On the subsequence $\{\lambda_n\}_{n=1}^\infty$ from Theorem 2.1.1, $L_t^{\lambda_n}(\phi) \rightarrow L_t(\phi)$ a.s. for bounded and continuous $\phi$, so it is enough to show that $\lim_{n \rightarrow \infty} E_{X_0}^X(L_t^{\lambda_n}(\phi)) = E_{X_0}^X(\lim_{n \rightarrow \infty} L_t^{\lambda_n}(\phi))$. By Theorem 2.1.1, $L_t^{\lambda}(\phi)$ converges in $L^2(P_{X_0}^X)$ and hence is bounded in $L^2(P_{X_0}^X)$. It is therefore uniformly integrable, which justifies the above exchange of limit and integration. This proves the result for bounded and continuous $\phi$. We extend the moment formula to bounded measurable functions by a Monotone Class Lemma and to non-negative measurable functions by Monotone convergence.

We now prove part (b). Suppose we realize $X_t$ under a probability $P_{X_0}^X$ such that (2.1.16) holds. Conditionally on $N$, by (2.1.17) we have

$$L_t(1)^2 \leq \left( \sum_{i=1}^N L_i^t(1) \right)^2 = \sum_{i=1}^N L_i^t(1)^2 + \sum_{i=1}^N \sum_{j \neq i} L_i^t(1)L_j^t(1).$$

The clusters are independent with laws $\mathbb{N}_{X_0}(X_i^t \in \cdot | X_i^t > 0) = (t/2)\mathbb{N}_{X_0}(\{X_i^t > 0, X_i^t \in \cdot\})$, the equality by (2.3.11). Thus, applying Theorem 2.1.5(a) and Proposition 2.1.6(b) to the above and using independence, we obtain

$$E_{X_0}^X(L_t(1)^2 | N) \leq CN(t/2)t^{1-2\lambda_0} + C(N^2 - N)(t/2)^2t^{-2\lambda_0}. \quad (2.4.26)$$

As in the proof of Theorem 2.1.3, we take the expectation with respect to $N$, which has a Poisson$(2X_0(1)/t)$ distribution. This proves part (b).

It remains to prove Theorem 2.1.2. We will derive part (a) below using Proposition 2.1.6; part (b) requires a few lemmas which we now discuss.

We say that $L_t$ has an atom of mass $c > 0$ at $x$ if $L_t(\{x\}) = c$. We decompose $L_t$ as

$$L_t = \tilde{L}_t + \nu_t, \quad (2.4.27)$$

where $\tilde{L}_t$ is atomless and $\nu_t$ is strictly atomic. We begin with an elementary observation.
which provides an upper bound for the mass of the atoms of a measure. Let $M \in \mathbb{N}$. Let $I^n_k = [-M, -M + 2^{-n}]$, and for $k = 2, 3, \ldots, 2M2^n$, define the dyadic interval $I^n_k = (-M + (k-1)2^{-n}, -M + k2^{-n})$. Then $\{I^n_k : k \leq 2M2^n\}$ is a partition of $[-M, M]$ into disjoint intervals of length $2^{-n}$. The following lemma is elementary.

**Lemma 2.4.3.** Fix $M \in \mathbb{N}$ and suppose that $\mu$ is a finite measure supported on $[-M, M]$ with decomposition $\mu = \rho + \nu$, where $\rho$ is atomless and $\nu = \sum_{i \in I} c_i \delta_{x_i}$ is strictly atomic. Then for every $n \geq 1$,

$$\sum_{k=1}^{2M2^n} \mu(I^n_k)^2 \geq \sum_{i \in I} c_i^2.$$

The next lemma gives an upper bound for the second moment of $L_t$ on a ball. We denote by $B(x, r)$ the ball of radius $r > 0$ centred at $x \in \mathbb{R}$. We recall $s^*(\delta)$ from Theorem 2.2.1(c); in what follows we use $\delta = 1/8$, and $s^*$ denotes $s^*(1/8)$.

**Lemma 2.4.4** (Second moments on balls). There is a constant $C_{2.4.4} > 0$ and $t$-dependent constant $C_{2.4.4}(t) > 0$ such that for all $x \in \mathbb{R}$ and $r < e^{-s^*}t$,

$$\mathbb{N}_0(L_t(B(x, r)^2)) \leq C_{2.4.4} \left[ t^{-\lambda_0 r^{2-2\lambda_0}} P^W_0(W_{t/3} \in B(x, r)) + t^{-3\lambda_0 + 1/2} r P^W_0(W_t \in B(x, r)) \right] \leq C_{2.4.4}(t) \left[ r^{3-2\lambda_0} + r^2 \right],$$

where $W$ is a standard Brownian motion under $P^W_0$.

We delay the proof of this lemma to the end of the section and first prove Theorem 2.1.2.

**Proof of Theorem 2.1.2.** First consider part (a). For canonical measure, via the second moment method we have

$$\mathbb{N}_0(L_t(1) > 0) \geq \frac{(\mathbb{N}_0(L_t(1))^2}{\mathbb{N}_0(L_t(1)^2)} \geq \frac{C^2_{2.1.4}}{C^2_{2.1.4}} \frac{\theta^2 t^{-2\lambda_0}}{\theta^2 t^{1-2\theta_0} (1 - \lambda_0)^{-1}} = \frac{1 - \lambda_0}{t},$$

where we recall that $\int \psi_0 \, dm = \theta$ and we have used Theorem 2.1.5(a) and (2.1.11). We recall that $\mathbb{N}_0(X_t > 0) = 2/t$, which implies that $\mathbb{N}_0(L_t > 0 \mid X_t > 0) = \frac{1 - \lambda_0}{2t}$. This proves the result for $\mathbb{N}_0$.

To see that $P^X_{X_0}(L_t > 0) > 0$, we realize $X_t$ under $P^X_{X_0}$ via a cluster decomposition. The event that the number of clusters $N$ is exactly one has some positive probability $p > 0$; restricted to this event, $X_t$ is equal to a single canonical cluster conditioned on survival (as in the proof of Theorem 2.1.3), which we just showed has probability at least $\frac{1 - \lambda_0}{2t}$ that $L_t > 0$. Hence $P^X_{X_0}(L_t > 0) \geq p \frac{1 - \lambda_0}{2t} > 0$.
We now prove part (b). First consider $L_t$ under $\mathbb{N}_0$ and recall the decomposition (2.4.27), i.e. $L_t = \tilde{L}_t + \nu_t$, the latter strictly atomic. Fix $M \in \mathbb{N}$ and consider the restriction of $L_t$ to $[-M, M]$, i.e. $dL_t^{(M)}(x) := 1_{[-M, M]}(x) \, dL_t(x)$, with decomposition $L_t^{(M)} = \tilde{L}_t^{(M)} + \nu_t^{(M)}$. Note that the radius of the dyadic intervals is $r(I^n_k) = 2^{-n+1}$. By Lemma 2.4.4, we have

$$N_0 \left( \sum_{k=1}^{2M2^n} L_t^{(M)}(I^n_k)^2 \right) = \sum_{k=1}^{2M2^n} N_0 \left( L_t^{(M)}(I^n_k)^2 \right) \leq C(t) 2M2^n \left[ (2^{-(n+1)})^3 - 2\lambda_0 + (2^{-(n+1)})^2 \right] \leq C(t) 2M \left[ (2^{-n})^2 - 2\lambda_0 + 2^{-n} \right] \to 0 \text{ as } n \to \infty$$

because $2 - 2\lambda_0 > 0$. Moreover, by Lemma 2.4.3, the first expression is greater than or equal to the expectation (under $N_0$) of the sum of the squares of the atoms of $L_t^{(M)}$. The above implies that this expectation must in fact be zero, so $\nu_t^{(M)} = 0$ $N_0$-a.s. As this holds for all $M$, $\nu_t = 0$ and $L_t$ is atomless under $N_0$. To obtain the result under $P_{X_0}$, we note from the cluster decomposition and (2.1.17) that (conditionally) $L_t$ is a sum of $N$ measures which are atomless by the above, and hence is atomless.

**Proof of Lemma 2.4.4.** We apply (2.1.10) with $h(z_1, z_2) = 1_{B(x, r)}(z_1) 1_{B(x, r)}(z_2)$. This gives

$$N_0(L_t(B(x, r))^2) = C \int_0^t w^{-2\lambda_0} \left[ \int \int E_{z_1}^Y \left( \exp \left( - \int_0^{\log(t/w)} F(Y_u) \, du \right) \right. \right. \times 1_{B(x, r)}(\sqrt{Y_{\log(t/w)}}) 1_{B(x, r)}(\sqrt{Y_{\log(t/w)} + \sqrt{w}(z_2 - z_1)}) \left. \psi_0(z_1) \psi_0(z_2) \, dm(z_1) \, dm(z_2) \right] \, dw. \quad (2.4.28)$$

We now divide the above into two cases depending on the size of $w$. We first consider the singular case, where $w$ is small.

**Case 1:** $w < e^{-s^* t}$. We interpret the exponential in (2.4.28) as the probability that $Y$ survives until time $\log(t/w)$ when it is subject to Markovian killing with rate $F(Y_u)$. Because this probability is equal to the integral of the transition density over all of $\mathbb{R}$, the portion of the integral
If \( w \in [0, e^{-s^*} t] \) equals
\[
C \int_0^{e^{-s^*} t} w^{-2\lambda_0} \left[ \int \int q_{\log(t/w)}(z_1, y) 1_B(x, r)(\sqrt{ty}) 1_B(x, r)(\sqrt{ty} + \sqrt{w}(z_2 - z_1)) \times \psi_0(z_1) \psi_0(z_2) \, dm(z_1) \, dm(z_2) \right] \, dw.
\]
\[
\leq C \int_0^{e^{-s^*} t} w^{-2\lambda_0} \left[ \int \int e^{-\lambda_0 \log(t/w)} e^{z_1^2/8} e^{y^2/8} 1_B(x, r)(\sqrt{ty}) 1_B(x, r)(\sqrt{ty} + \sqrt{w}(z_2 - z_1)) \times \psi_0(z_1) \psi_0(z_2) \, dm(z_1) \, dm(z_2) \right] \, dw.
\]
\[
\leq Ct^{-\lambda_0} \int_0^{e^{-s^*} t} w^{-\lambda_0} \left[ \int \int e^{z_1^2/4} e^{z_2^2/8} 1_B(x, r)(\sqrt{ty} + \sqrt{w}(z_2 - z_1)) \, dm(z_1) \, dm(z_2) \right] \, dm(y) \, dw.
\]

The first inequality uses (2.2.2) with \( \delta = 1/8 \), which applies because \( \log(t/w) > s^* \) for all \( w \) in the above integral, and the second uses (2.2.5), both with \( \delta = 1/8 \). In the integral in the last line we collect all the Gaussian terms. The square-bracketed term is equal to
\[
C \int \int 1_B(x, r)(\sqrt{ty} + \sqrt{w}(z_2 - z_1)) e^{-z_1^2/4} e^{-3z_2^2/8} \, dz_1 \, dz_2
\]
\[
= Ct^{-\lambda_0} \int 1_B(x, r)(\sqrt{ty} + \sqrt{w}z) e^{-3z^2/20} \, dz.
\]

We have used the convolution property for independent Gaussians. We define Gaussian random variables \( g_1 \sim \mathcal{N}(0, 4t/3) \) and \( g_2 \sim \mathcal{N}(0, 10/3) \). Substituting the last expression into (2.4.29), we obtain
\[
Ct^{-\lambda_0} \int_0^{e^{-s^*} t} w^{-\lambda_0} \left[ \int \int 1_B(x, r)(\sqrt{ty}) 1_B(x, r)(\sqrt{ty} + \sqrt{w}z) e^{-3z^2/20} e^{-3y^2/8} \, dz \, dy \right] \, dw
\]
\[
= C't^{-\lambda_0} \int_0^{e^{-s^*} t} w^{-\lambda_0} \left[ P(g_1 \in B(x, r), g_1 + \sqrt{w}g_2 \in B(x, r)) \right] \, dw
\]
\[
\leq Ct^{-\lambda_0} \int_0^{e^{-s^*} t} w^{-\lambda_0} \left[ P(g_1 \in B(x, r)) P(\sqrt{w}g_2 \in B(0, 2r)) \right] \, dw
\]
\[
= Ct^{-\lambda_0} P(g_1 \in B(x, r)) \int_0^{e^{-s^*} t} w^{-\lambda_0} P(g_2 \in B(0, 2rw^{-1/2})) \, dw.
\]

Suppose that \( 4r^2 < e^{-s^*} t \). If \( 2rw^{-1/2} > 1 \), we bound the probability in the integral above by 1. If \( 2rw^{-1/2} \leq 1 \), the probability is simply bounded by the diameter of the ball, \( 4rw^{-1/2} \).
Finally, note that if $4r^2 \geq e^{-s^*}t$, then (2.4.30) is bounded above by
\[
Ct^{-\lambda_0}P(g_1 \in B(x,r)) \left[ \int_0^{4r^2} w^{-\lambda_0} dw + 4r \int_{4r^2}^{e^{-s^*}t} w^{-\lambda_0-1/2} dw \right] 
\leq Ct^{-\lambda_0}P(g_1 \in B(x,r))t^{2-2\lambda_0}.
\] (2.4.31)

Finally, note that if $4r^2 \geq e^{-s^*}t$, then (2.4.30) is bounded above by
\[
Ct^{-\lambda_0}P(g_1 \in B(x,r)) \int_0^{e^{-s^*}t} w^{-\lambda_0} dw \leq Ct^{-\lambda_0}P(g_1 \in B(x,r))(e^{-s^*}t)^{1-\lambda_0} 
\leq Ct^{-\lambda_0}P(g_1 \in B(x,r))t^{2-2\lambda_0},
\]
so the upper bound for (2.4.29) obtained in (2.4.31) holds in this case as well.

**Case 2:** $w \in (e^{-s^*}t,t]$.

In this case we simply bound the exponential term in (2.4.28) above by 1, effectively ignoring the killing, in which case $Y_{\log(t/w)} \sim m$. We also use (2.2.5) with $\delta = \frac{1}{2}$. Hence the contribution to (2.4.28) from the $w \in (e^{-s^*}t,t]$ case is bounded above by
\[
C \int_{e^{-s^*}t}^t w^{-2\lambda_0} \left[ \int \int \int 1_{B(x,r)}(\sqrt{ty}) 1_{B(x,r)}(\sqrt{ty} + \sqrt{w}(z_2 - z_1)) 	imes e^{-\frac{z_1^2}{4}} e^{-\frac{z_2^2}{4}} dm(z_1) dm(z_2) dm(y) \right] dw 
\leq C \int_{e^{-s^*}t}^t w^{-2\lambda_0} \left[ \int \int 1_{B(x,r)}(\sqrt{ty}) 1_{B(x,r)}(\sqrt{ty} + \sqrt{w}(z_2 - z_1)) 	imes e^{-\frac{z_1^2}{4}} e^{-\frac{z_2^2}{4}} dz_1 dz_2 dm(y) \right] dw 
\leq C \int_{e^{-s^*}t}^t w^{-2\lambda_0} \left[ \int \int 1_{B(x,r)}(\sqrt{ty}) 1_{B(x,r)}(\sqrt{ty} + \sqrt{w^2}) e^{-\frac{z^2}{2}} dz dm(y) \right] dw 
\leq Ct^{-\lambda_0}P(g_3 \in B(x,r)) \int_{e^{-s^*}t}^t w^{-2\lambda_0} P(g_4 \in B(0,2rw^{-1/2})) dw.
\]

In the above, $g_3 \sim \mathcal{N}(0,t)$ and $g_4 \sim \mathcal{N}(0,1)$. The third line follows by the convolution property of Gaussians. We again bound the probability in the integral by the size diameter
of the ball, which gives the following upper bound for the above:

\[
C t^{-\lambda_0} P(g_3 \in B(x, r)) 4r \int_{e^{-s^*}}^{t} w^{-2\lambda_0 - 1/2} dw.
\]

\[
\leq C t^{-\lambda_0} P(g_3 \in B(x, r)) 4r (e^{-s^*} t)^{-2\lambda_0 + 1/2}
\]

\[
= C t^{-3\lambda_0 + 1/2} P(g_3 \in B(x, r)) r.
\]

(2.4.32)

By combining (2.4.31) and (2.4.32) and interpreting the Gaussian probabilities in terms of Brownian motion, we obtain the first inequality of the result. The second bound is obtained by bounding the Brownian density above by its maximum value.

\[\square\]

2.5 Proof of Theorem 2.1.4

The proof of Theorem 2.1.4 is split up into two main parts. In the first, we obtain representations for \(N_0((L_t^\lambda \times L_t^{\lambda'}) (h))\) in terms of solutions to (2.3.4), in particular the family \(V^{\lambda,\lambda'}\) introduced in Section 2.3. In the second part, using these representations, we establish convergence of \(N_0((L_t^\lambda \times L_t^{\lambda'}) (h))\) as \(\lambda, \lambda' \to \infty\). The proof of a technical lemma (Lemma 2.5.5) is given in Section 2.6.

2.5.1 PDE representations and preliminary bounds

We begin by deriving an expression for second moments of \(L_t^\lambda\) under the canonical measure. In particular, we study \(N_0((L_t^\lambda \times, L_t^{\lambda'}) (h))\) for \(\lambda, \lambda' > 0\). The formula we obtain is naturally suggested by a branching particle heuristic. Its proof uses PDE methods. Let \(E^B_x\) denote the expectation of a Brownian motion started at \(x\). \(E^B_{(x,y)}\) denotes the law of two independent Brownian motions \(B^1\) and \(B^2\) started from points \(x\) and \(y\) respectively. We recall the definition of \(V_{t}^{\lambda,\lambda'}\) from (2.3.20).

**Proposition 2.5.1.** Let \(h : \mathbb{R}^2 \to \mathbb{R}\) be a bounded Borel function and \(\lambda, \lambda', t > 0\). Then

\[
N_0((L_t^\lambda \times L_t^{\lambda'}) (h)) = (\lambda\lambda')^{2\lambda_0} \int_{0}^{t} E^B_{0} \left( E^B_{(0,0)} \left[ h(B_s + B_{t-s}^1, B_s + B_{t-s}^2) \right. \right.
\]

\[
\times \exp \left( - \int_{0}^{s} V_{t-u}^{\lambda,\lambda'}(B_s^1 + B_{t-s}^1 - B_u + B_{t-s}^2 - B_u) du \right)
\]

\[
\times \exp \left( - \int_{0}^{t-s} V_{r}^{\lambda,\lambda'}(B_r^1 + B_{t-s}^1 - B_r^2 - B_{t-s}^2) dr \right)
\]

\[
\times \exp \left( - \int_{0}^{t-s} V_{r}^{\lambda,\lambda'}(B_r^2 + B_{t-s}^2 - B_r^1, B_{t-s}^2) dr \right) \bigg] ds.
\]

The proof of Proposition 2.5.1 requires the following lemma.
Lemma 2.5.2. Let \( \varphi \in \mathcal{M}_F(\mathbb{R}) \) and \( \varphi_1, \varphi_2 \in \mathcal{L}^1(\mathbb{R}) \) be non-negative and continuous. Then

\[
\mathbb{N}_0 \left( X_t(\varphi_1) X_t(\varphi_2) e^{-X_t(\varphi)} \right) = \int_0^t E_0^B \left( \exp \left( - \int_0^s V_{t-u}^\varphi(B_u) \, ds \right) \right) \times \prod_{i=1,2} E_0^{B_i} \left[ \exp \left( - \int_0^{t-s} V_{t-s-r}^\varphi(B_s + B_r^i) \, dr \right) \varphi_i(B_s + B_{t-s}^i) \right] \, ds.
\]

Proof. Let \( \epsilon_1, \epsilon_2 > 0 \) and \( \varphi, \varphi_1 \) and \( \varphi_2 \) be as in the statement. Viewing \( \varphi_1 \) and \( \varphi_2 \) as the density functions of the finite measures they induce (i.e. \( \varphi_i(A) = \int_A \varphi_1(x) \, dx \)), let \( V_t^{\varphi, \epsilon_1, \epsilon_2} \) denote the solution to (2.3.4) when \( \phi = \varphi + \epsilon_1 \varphi_1 + \epsilon_2 \varphi_2 \in \mathcal{M}_F(\mathbb{R}) \). By (2.3.3) and the discussion below (2.3.4),

\[
\mathbb{N}_0 \left( 1 - e^{-X_t(\varphi + \epsilon_1 \varphi_1 + \epsilon_2 \varphi_2)} \right) = V_t^{\varphi, \epsilon_1, \epsilon_2}(0).
\]

We differentiate this expression once with respect to \( \epsilon_1 \) and once with respect to \( \epsilon_2 \). The derivatives of the inner expression of the left hand side are bounded above by integrable quantities (i.e. \( X_t(\varphi_1) \) and \( X_t(\varphi_1) X_t(\varphi_2) \)) so we can take the differentiation inside the expectation in the probabilistic representation, and the derivatives of the right hand side exist. The resulting equation is the following:

\[
\mathbb{N}_0 \left( X_t(\varphi_1) X_t(\varphi_2) e^{-X_t(\varphi + \epsilon_1 \varphi_1 + \epsilon_2 \varphi_2)} \right) = -\frac{\partial^2}{\partial \epsilon_1 \partial \epsilon_2} V_t^{\varphi, \epsilon_1, \epsilon_2}(0). \tag{2.5.1}
\]

We note that the limit of the left hand side as \( \epsilon_1, \epsilon_2 \downarrow 0 \) is the desired expression. We now obtain an expression for the first derivatives of \( V_t^{\varphi, \epsilon_1, \epsilon_2}(0) \) with respect to \( \epsilon_1 \) and \( \epsilon_2 \). Consider the following partial differential equation:

\[
\frac{\partial u_t}{\partial t} = \frac{\Delta}{2} u_t - V_t^{\varphi, \epsilon_1, \epsilon_2} u_t \quad \text{for } (t, x) \in (0, \infty) \times \mathbb{R}, \quad u_t \to \varphi_1 \text{ as } t \downarrow 0, \tag{2.5.2}
\]

where the \( u_t \to \varphi_1 \) in the sense of weak convergence of measures. The above can be obtained heuristically by formally differentiating (2.3.4) with respect to \( \epsilon_1 \) when the initial conditions are \( \varphi + \epsilon_1 \varphi_1 + \epsilon_2 \varphi_2 \). By Lemmas 2.3 and 2.5 of [70], (2.5.2) has a unique solution, which we denote by \( U_t^{1, \epsilon_1, \epsilon_2} \), which satisfies

\[
V_t^{\varphi, \epsilon_1, \epsilon_2}(x) = V_t^{\varphi, 0, \epsilon_2}(x) + \int_0^{\epsilon_1} U_t^{1, \epsilon, \epsilon_2}(x) \, d\epsilon.
\]

Thus \( U_t^{1, \epsilon_1, \epsilon_2} = \frac{\partial}{\partial \epsilon_1} V_t^{\varphi, \epsilon_1, \epsilon_2} \). We can apply the same argument to obtain a similar representation for \( \frac{\partial}{\partial \epsilon_2} V_t^{\varphi, \epsilon_1, \epsilon_2} \), which we denote by \( U_t^{2, \epsilon_1, \epsilon_2} \). Both \( U_t^{1, \epsilon_1, \epsilon_2} \) and \( U_t^{2, \epsilon_1, \epsilon_2} \) have Feynman-Kac representations; for example, see Theorem 7.6 of Karatzas and Shreve [42].
We take the expression for $i = 1$ and differentiate it with respect to $\epsilon_2$. We obtain

$$
- \frac{\partial^2}{\partial \epsilon_2 \partial \epsilon_1} V_t^{\varphi,\epsilon_1,\epsilon_2}(x)
\begin{aligned}
&= E_x^B \left( \varphi_1(B_t) \exp \left( - \int_0^t V_{t-s}^{\varphi,\epsilon_1,\epsilon_2}(B_s) \, ds \right) \right) \\
&= E_x^B \left( \varphi_1(B_t) \exp \left( - \int_0^t V_{t-s}^{\varphi,\epsilon_1,\epsilon_2}(B_s) \, ds \right) \right) \\
&\quad \times \int_0^t E_0^{B^2} \left( \varphi_2(B_s + B_{t-s}^2) \exp \left( - \int_0^{t-s} V_{t-s-r}^{\varphi,\epsilon_1,\epsilon_2}(B_s + B_{t-r}^2) \, dr \right) \right) \, ds.
\end{aligned}
$$

where the final line follows from another application of (2.5.3), this time with $i = 2$. First we note that all the terms are non-negative, so we can take the internal integral over time outside the expectation. For $s < t$, the integrand then describes one Brownian motion started at 0 and run to time $t$, and a second which branches from the first at time $s$ and evolves independently. By applying the Markov property at time $s$ we equivalently view it as a Brownian path that branches at time $s$ into two independent Brownian motions $B_1$ and $B_2$ which themselves run for a duration of $t - s$. This formulation combined with the independence of the Brownian motions gives us

$$
\frac{\partial^2}{\partial \epsilon_2 \partial \epsilon_1} V_t^{\varphi,\epsilon_1,\epsilon_2}(x) = - \int_0^t E_x^B \left( \exp \left( - \int_0^s V_{t-u}^{\varphi,\epsilon_1,\epsilon_2}(B_u) \, ds \right) \right) \\
\times \prod_{i=1,2} E_0^{B_i} \left[ \varphi_i(B_s + B_{t-s}^i) \exp \left( - \int_0^{t-s} V_{t-s-r}^{\varphi,\epsilon_1,\epsilon_2}(B_s + B_{t-r}^i) \, dr \right) \right] \, ds.
$$

The derivatives in $\epsilon_1$ and $\epsilon_2$ are one-sided at 0 so we cannot exactly evaluate at $\epsilon_1 = \epsilon_2 = 0$. However, $V_t^{\varphi,\epsilon_1,\epsilon_2}(x)$ is continuous in $\epsilon_1$ and $\epsilon_2$ and the integrand is bounded above by $||\varphi_1||_\infty ||\varphi_2||_\infty$ so we can apply bounded convergence. As $\epsilon_1, \epsilon_2 \downarrow 0$, $V_t^{\varphi,\epsilon_1,\epsilon_2} \rightarrow V_t^\varphi$ by Lemma 2.1(d) of [70]. We also take $\epsilon_1, \epsilon_2 \downarrow 0$ in the left hand side of (2.5.1) and apply Dominated Convergence. Evaluating at $x = 0$ gives the result.

**Proof of Proposition 2.5.1.** We will prove the result for functions of product form, i.e. $h(x, y) = \varphi_1(x) \varphi_2(y)$, and then use a monotone class theorem. Let $x_1, x_2 \in \mathbb{R}$ and $\lambda, \lambda' > 0$. Consider the expression from Lemma 2.5.2 with $\varphi = \lambda \delta_{x_1} + \lambda' \delta_{x_2}$. For now we simply let $\varphi_1$ and $\varphi_2$ be functions satisfying the assumptions of Lemma 2.5.2, but we will shortly choose.
As noted, the absolute value of the expression inside \( N \) is finite. Thus we can apply Fubini and rewrite (2.5.5) as

\[
\mathbb{N}_0 \left( X_t(\varphi_1) X_t(\varphi_2) e^{-\lambda X(t,x_1) - \lambda' X(t,x_2)} \right) = \int_0^t E_0^{B'} \left( \exp \left( - \int_0^s V_{t-u}^{\lambda,\lambda'} (B_u - x_1, B_u - x_2) \, du \right) \right) \prod_{i=1,2} E_0^{B'} \left[ \exp \left( - \int_0^{t-s} V_{t-s-r}^{\lambda,\lambda'} (B_s + B_r^i - x_1, B_s + B_r^i - x_2) \, dr \right) \varphi_i(B_s + B_r^i) \right] \, ds,
\]

(2.5.4)

where we have also used (2.3.21), i.e., translation invariance of \( V^{\lambda,\lambda'}(x_1, x_2) \). Now let \( \varphi_i = p_\delta(\cdot - x_i) \), where we recall that \( p_\delta(\cdot) \) denotes the Gaussian density of variance \( \delta \). Let \( \phi_1, \phi_2 \) be bounded, continuous functions and integrate \( \phi_1(x_1) \phi_1(x_2) \) multiplied by the above over \( x_1 \) and \( x_2 \). The left hand side is then

\[
\iint \phi_1(x_1) \phi_2(x_2) \mathbb{N}_0 \left( X_t(p_\delta(\cdot - x_1)) X_t(p_\delta(\cdot - x_2)) e^{-\lambda X(t,x_1) - \lambda' X(t,x_2)} \right) \, dx_1 \, dx_2.
\]

(2.5.5)

The absolute value of (2.5.5) is bounded above by

\[
\| \phi_1 \|_\infty \| \phi_2 \|_\infty \mathbb{N}_0 \left( \int X_t(p_\delta(\cdot - x_1)) \, dx_1 \int X_t(p_\delta(\cdot - x_2)) \, dx_2 \right),
\]

(2.5.6)

where the change of order of integration follows because all the terms are non-negative once we bound \( |\phi_i(x_i)| \) by \( \| \phi_i \|_\infty \). Now we note that

\[
\int X_t(p_\delta(\cdot - x_i)) \, dx_i = \iint X(t, y) p_\delta(x_i - y) \, dy \, dx_i = \int X(t, y) \left( \int p_\delta(x_i - y) \, dx_i \right) \, dy = X_t(1).
\]

Combined with (2.5.6), this implies that the expression in (2.5.5) is integrable and its absolute value is bounded above by \( \| \phi_1 \|_\infty \| \phi_2 \|_\infty \mathbb{N}_0(X_t(1)^2) \). We note that \( \mathbb{N}_0(X_t(1)^2) < \infty \). To see this, first observe that \( E_{\delta_0}^{\lambda}(X_t(1)^2) \) is finite and in fact equals \( 1 + t \), which follows from the martingale problem for super-Brownian motion (see Section II.5 of [83]). By the cluster decomposition (2.1.16), \( E_{\delta_0}^{X}(X_t(1)^2) \) is equal to the mean of a Poisson random variable multiplied by \( \mathbb{N}_0(X_t(1)^2 | X_t > 0) \), and so the latter quantity, and hence \( \mathbb{N}_0(X_t(1)^2) \), are also finite. Thus we can apply Fubini and rewrite (2.5.5) as

\[
\mathbb{N}_0 \left( \iint \phi_1(x_1) \phi_2(x_2) X_t(p_\delta(\cdot - x_1)) X_t(p_\delta(\cdot - x_2)) e^{-\lambda X(t,x_1) - \lambda' X(t,x_2)} \, dx_1 \, dx_2 \right).
\]

(2.5.7)

As noted, the absolute value of the expression inside \( \mathbb{N}_0 \) is bounded above by \( \| \phi_1 \|_\infty \| \phi_2 \|_\infty X_t(1)^2 \), which is integrable under \( \mathbb{N}_0 \), for all \( \delta \). We take \( \delta \downarrow 0 \) and apply
Dominated Convergence to obtain that the limit of (2.5.7) as $\delta \downarrow 0$ is equal to
\[
\mathbb{N}_0 \left( \lim_{\delta \to 0^+} \int \int \phi_1(x_1) \phi_2(x_2) X_t(p_\delta(\cdot - x_1)) X_t(p_\delta(\cdot - x_2)) e^{\lambda X(t,x_1) - \lambda' X(t,x_2)} \, dx_1 \, dx_2 \right)
= \mathbb{N}_0 \left( \lim_{\delta \to 0^+} \int \phi_1(x_1) X_t(p_\delta(\cdot - x_1)) e^{-\lambda X(t,x_1)} \, dx_1 \right) \times \left( \int \phi_2(x_2) X_t(p_\delta(\cdot - x_2)) e^{-\lambda' X(t,x_2)} \, dx_2 \right)
\]

(2.5.8)

We know that
\[
X_t(p_\delta(\cdot - x_i)) = \int X(t,y)p_\delta(y-x_i) \, dy = X_t*p_\delta(x_i).
\]

Moreover, $X(t,\cdot) \in C_c(\mathbb{R})$ (i.e. $X(t,\cdot)$ is continuous with compact support) $\mathbb{N}_0$-a.s. and $\{p_\delta\}_{\delta > 0}$ are an approximate identity family, which together with the above imply that $X_t(p_\delta(\cdot - x_i)) \to X(t,x_i)$ as $\delta \downarrow 0$. We note that for all $\delta > 0$, $|X_t(p_\delta(\cdot - x_i))| = |X_t*p_\delta(x_i)| \leq \|X(t,\cdot)\|_\infty$. Choose $K \in \mathbb{N}$ such that $\text{supp}(X_t) \subseteq [-K,K]$. On the set $[-2K,2K]$ we bound $|X_t(p_\delta(\cdot - x_i))|$ above by $\|X(t,\cdot)\|_\infty$. For $|x_i| > 2K$, a short calculation shows that $|X_t(p_\delta(\cdot - x_i))| \leq C(K)X_t(1) p_1(|x_i| - K)$ for some constant $C(K) > 0$ for all $0 < \delta \leq 1/2$. Hence $|X_t(p_\delta(\cdot - x_i))|$ has an upper bound which is a bounded function with Gaussian tails, uniformly for $0 < \delta \leq 1/2$. Using this bound and boundedness of $\phi_i$, we can apply Dominated Convergence in (2.5.8), which gives that the limit of (2.5.5) as $\delta \downarrow 0$ equals
\[
\mathbb{N}_0 \left( \left( \int \phi_1(x_1) X(t,x) e^{-\lambda X(t,x_1)} \, dx_1 \right) \left( \int \phi_2(x_2) X(t,x_2) e^{-\lambda' X(t,x_2)} \, dx_2 \right) \right).
\]

When rescaled by $(\lambda\lambda')^{2\lambda_0}$ this is equal to $\mathbb{N}_0(L^\lambda_t(\phi_1)L^\lambda_t(\phi_2))$. We now turn our attention to the right hand side of (2.5.4). With $\varphi_i = p_\delta(\cdot - x_i)$, integrating against $\phi(x_1)\phi(x_2)dx_1dx_2$, we have
\[
\int \int \phi_1(x_1) \phi_2(x_2) \left( \int_0^t E_0^B \left( E_{(0,0)}^{B^1,B^2} \left[ \exp \left( - \int_0^u V_{t-u}^{\lambda',\lambda'}(B_s + B^1_{t-s} - x_1, B_s + B^1_{t-s} - x_2) \, ds \right) \right] \right) \right) \, dx_1 \, dx_2.
\]

Since the above is equal to (2.5.5), which we have shown is integrable, we can take the spatial integrals inside the expectations. At this point we note that we are integrating a bounded function of $x_1$ and $x_2$ with respect to the densities $p_\delta(B_s + B^1_{t-s} - x_i)$, which, because $p_\delta$ is the kernel of the Brownian semigroup, is the same as viewing $x_i$ as $B_s + B^i_{t-s+\delta}$. Hence
the above is equal to

\[
\int_0^t E_0^{B_i} \left( E_{(0,0)}^{B_1^i, B_2^i} \left[ \phi_1(B_s + B_{t-s}^1) \phi_2(B_s + B_{t-s}^2) \right. \right. \\
\left. \left. \times \exp \left( - \int_0^s V_{t-u}^{\lambda, \lambda'}(B_u - B_s - B_{t-s}^1, B_u - B_s - B_{t-s}^2) \, du \right) \right) \times \exp \left( - \int_0^{t-s} V_{t_s-r}^{\lambda, \lambda'}(B_r^1 - B_{t-s}^1, B_r^1 - B_{t-s}^2) \, dr \right) \right) \\
\left. \left. \times \exp \left( - \int_0^{t-s} V_{t_s-r}^{\lambda, \lambda'}(B_r^2 - B_{t-s}^1, B_r^2 - B_{t-s}^2) \, dr \right) \right) \right) ds. \tag{2.5.9}
\]

Taking \( \delta \downarrow 0 \) and applying Dominated Convergence, we note that because \( B_{t-s}^i \to B_{t-s}^i \) and \( \phi_1, \phi_2 \) and \( V_{t-s}^{\lambda, \lambda'} \) are continuous, the limit is equal to the above with \( \delta = 0 \). To obtain the desired form we make a time reversal of the Brownian motions. Define \( \hat{B}_u^i = B_{t-s}^i - B_{t-s}^i \). We note that the \( \hat{B}_i \) are standard Brownian motions and that \( \hat{B}_0^i = 0 \) and \( B_{t-s}^i = -\hat{B}_{t-s}^i \). Making this substitution shows that (2.5.9) with \( \delta = 0 \) is equal to

\[
\int_0^t E_0^{\hat{B}_i} \left( E_{(0,0)}^{\hat{B}_1, \hat{B}_2} \left[ \phi_1(B_s + \hat{B}_{t-s}^1) \phi_2(B_s + \hat{B}_{t-s}^2) \right. \right. \\
\left. \left. \times \exp \left( - \int_0^s V_{t-u}^{\lambda, \lambda'}(B_u - B_s - \hat{B}_{t-s}^1, B_u - B_s - \hat{B}_{t-s}^2) \, du \right) \right) \times \exp \left( - \int_0^{t-s} V_{t_s-r}^{\lambda, \lambda'}(\hat{B}_r^1 - \hat{B}_{t-s}^1, \hat{B}_r^1 - \hat{B}_{t-s}^2) \, dr \right) \right) \\
\left. \left. \times \exp \left( - \int_0^{t-s} V_{t_s-r}^{\lambda, \lambda'}(\hat{B}_r^2 - \hat{B}_{t-s}^1, \hat{B}_r^2 - \hat{B}_{t-s}^2) \, dr \right) \right) \right) ds.
\]

The time index of the Brownian motions now matches the time index of the function \( V_{t-s}^{\lambda, \lambda'} \) in the last two lines, allowing us to reverse the time of the integrals for a simpler expression. To obtain the desired expression we now use (2.3.22), i.e. \( V_{t}^{\lambda, \lambda'}(a, b) = V_{t}^{\lambda, \lambda'}(-a, -b) \), and relabel \( \hat{B}_i \) to be simply \( B_i \). This proves the result for \( h(x_1, x_2) = \phi_1(x_1) \phi_2(x_2) \) when \( \phi_1, \phi_2 \) are bounded and continuous. The result for general bounded measurable \( h : \mathbb{R}^2 \to \mathbb{R} \) now follows from a standard monotone class argument such as Corollary 4.4 in the Appendix of Ethier and Kurtz [23].

**Definition.** Let \( \Gamma^{\lambda, \lambda'}(s) \) denote the integrand in Proposition 2.5.1, so that the proposition states that

\[
N_0((L_t^\lambda \times L_t^{\lambda'})(h)) = (\lambda')^{2\lambda} \int_0^t \Gamma^{\lambda, \lambda'}(s) ds. \tag{2.5.10}
\]

\( \Gamma^{\lambda, \lambda'}(s) \) also depends on \( h \), but we omit this. The next lemma changes variables to obtain
an expression involving Ornstein-Uhlenbeck processes. We first introduce some notation. For bounded and measurable $h : \mathbb{R}^2 \to \mathbb{R}$ and a (continuous) path $(B_u : u \in [0, s])$, define $\Psi_{B,s}^{\lambda,N}(\cdot, \cdot)$ by

$$
\Psi_{B,s}^{\lambda,N}(x, y) = h(x + B_s, y + B_s) \exp \left(- \int_0^s V_{t-u}^{\lambda,N} (x + B_s - B_u, y + B_s - B_u) \, du \right).
$$

We define $H^\epsilon_u$ as a scaling of $V_{t}^{\lambda,N}$:

$$
H^\epsilon_u(x, y) = uV^1,\epsilon(\sqrt{u}x, \sqrt{u}y) = V^\epsilon(x, y).
$$

The scaling in the following lemma cannot be done uniformly for all $s \in [0, t]$ because it requires $\lambda^2 > (t-s)^{-1}$ and $\lambda'^2 > (t-s)^{-1}$. We derive an expression for $\Gamma_{\lambda,N}(s)$ in terms of two independent Ornstein-Uhlenbeck processes which we denote $Y^1$ and $Y^2$, for which we denote the joint (independent) expectation $E^{Y^1,Y^2}$.

**Lemma 2.5.3.** Let $0 < s < t$, $T_1 = T_1(s) = \log(\lambda^2(t-s))$, $T_2 = T_2(s) = \log(\lambda'^2(t-s))$. Then for all $\lambda > (t-s)^{-1/2}$ and $\lambda' > (t-s)^{-1/2}$, we have

$$
\Gamma_{\lambda,N}(s) = E_0^B \left[ E_{(0,0)}^{B^1, B^2} \left[ \Psi_{B,s}^{\lambda,N}(\sqrt{t-s}Y^1_1, \sqrt{t-s}Y^2_2) \times \exp \left(- \int_0^1 V^1,\lambda/\lambda(B_u, B_u + e^{T_1/2}(Y^2_2 - Y^1_1)) + V^1,\lambda'/\lambda'(B_u, B_u + e^{T_2/2}(Y^2_2 - Y^1_1)) \, du \right) \times \exp \left(- \int_0^{T_1} H_{e^u}^{\lambda,N}(Y^1_u, Y^1_u + e^{T_1-u/2}(Y^2_2 - Y^1_1)) \, du \right) \times \exp \left(- \int_0^{T_2} H_{e^u}^{\lambda,N}(Y^2_u, Y^2_u + e^{T_2-u/2}(Y^1_1 - Y^2_2)) \, du \right) \right] \right].
$$

**Proof.** We begin with the expression from Proposition 2.5.1. We observe that $\Psi_{B,s}^{\lambda,N}$ appears and we may write the quantities in the first two lines as $\Psi_{B,s}^{\lambda,N}(B^1_{t-s}, B^2_{t-s})$. In the third and fourth lines we apply (2.3.23) to obtain

$$
\Gamma_{\lambda,N}(s) = E_0^B \left[ E_{(0,0)}^{B^1, B^2} \left[ \Psi_{B,s}^{\lambda,N}(B^1_{t-s}, B^2_{t-s}) \times \exp \left(- \int_0^{t-s} \lambda^2 V^1,\lambda/\lambda(B_r, \lambda(B^1_B + B^2_{t-s} - B^1_{t-s})) \, dr \right) \times \exp \left(- \int_0^{t-s} \lambda'^2 V^{1,\lambda'/\lambda'}(\lambda'(B^1_B + B^2_{t-s} - B^1_{t-s}), \lambda'B^2_{t-s}) \, dr \right) \right] \right].
$$

We define $\hat{B}^1_u = \lambda B^1_{\lambda-2u}$ and $\hat{B}^2_u = \lambda' B^2_{\lambda'-2u}$, which are both standard Brownian motions.
Making a time change in the integrals (i.e. letting $u = \lambda^2 r$ or $\lambda'^2 r$) gives

$$
\Gamma^{\lambda,\lambda'}(s) = E^B_0 \left( E^B_{(0,0)} \left[ \Psi_{B,s}^{\lambda,\lambda'} \left( \lambda^{-1} \dot{B}^1_{\lambda^2(t-s)}, \lambda'^{-1} \dot{B}^2_{\lambda'^2(t-s)} \right) \right]
\times \exp \left( - \int_0^{\lambda^2(t-s)} V_u^{1,\lambda^2} (\dot{B}^1_u, \dot{B}^1_u + \frac{\lambda}{\lambda^2} \dot{B}^2_{\lambda'^2(t-s)} - \dot{B}^1_{\lambda^2(t-s)}) \, du \right)
\times \exp \left( - \int_0^{\lambda'^2(t-s)} V_u^{\lambda',1} (\dot{B}^2_u + \frac{\lambda'}{\lambda'} \dot{B}^1_{\lambda'^2(t-s)} - \dot{B}^2_{\lambda^2(t-s)}), \dot{B}^2_u \, du \right) \right) \right).
$$

Because we have assumed $\lambda, \lambda' > (t-s)^{-1/2}$, the upper bounds of integration in the integrals are greater than 1. We now apply the Markov property for $\dot{B}^i$ at time $u = 1$. We collect the portions of the integrals from the second and third lines on the interval $[0,1]$, leaving the integrals from $1$ to $\lambda^2(t-s)$ and $\lambda'^2(t-s)$. Conditional on $\dot{B}^i_1$, the Brownian motions in the integrands' arguments are Brownian motions with initial position $\dot{B}^i_1$. If we denote these by $\tilde{B}^i_u$ (in which case, essentially, $\tilde{B}^i_u = \tilde{B}^i_u$), we obtain

$$
\Gamma^{\lambda,\lambda'}(s) = E^B_0 \left( E^B_{(0,0)} \left[ \Psi_{B,s}^{\lambda,\lambda'} \left( \lambda^{-1} \dot{B}^1_{\lambda^2(t-s)}, \lambda'^{-1} \dot{B}^2_{\lambda'^2(t-s)} \right) \right]
\times \exp \left( - \int_0^{1} V_u^{1,\lambda^2} (\tilde{B}^1_u, \tilde{B}^1_u + \frac{\lambda}{\lambda^2} \tilde{B}^2_{\lambda'^2(t-s)} - \tilde{B}^1_{\lambda^2(t-s)}) \, du \right)
\times \exp \left( - \int_0^{\lambda^2(t-s)-1} V_u^{1,\lambda^2} (\tilde{B}^1_u, \tilde{B}^1_u + \frac{\lambda}{\lambda^2} \tilde{B}^2_{\lambda'^2(t-s)} - \tilde{B}^1_{\lambda^2(t-s)}), \tilde{B}^2_u \, du \right) \right)
\times \exp \left( - \int_0^{\lambda'^2(t-s)-1} V_u^{\lambda',1} (\tilde{B}^2_u + \frac{\lambda'}{\lambda'} \tilde{B}^1_{\lambda'^2(t-s)} - \tilde{B}^2_{\lambda^2(t-s)}), \tilde{B}^2_u \, du \right) \right).
$$

Recall that if a process $Y$ is defined by

$$
Y_r = e^{-r/2} B_{e^r-1}
$$

where $B$ is a standard Brownian motion, then $Y$ is a standard one-dimensional Ornstein-Uhlenbeck process with $Y_0 = B_0$. For $i = 1, 2$ we let $Y^i_r = e^{-r/2} B^i_{e^r-1}$. Recall that $T_1 = \log(\lambda^2(t-s))$ and $T_2 = \log(\lambda'^2(t-s))$. We therefore have that

$$
\tilde{B}^1_{\lambda^2(t-s)-1} = e^{T_1/2} Y^1_{T_1}, \quad \tilde{B}^2_{\lambda'^2(t-s)-1} = e^{T_2/2} Y^2_{T_2}.
$$
Expressing \( \lambda \) and \( \lambda' \) in terms of \( T_1, T_2 \) shows that

\[
\frac{\lambda}{\lambda'} \hat{B}_{\lambda'(t-s)-1}^2 = e^{T_1/2} Y_{T_2}^2, \quad \frac{\lambda'}{\lambda} \hat{B}_{\lambda(t-s)-1}^2 = e^{T_2/2} Y_{T_1}^1.
\]

Likewise, we express the argument of \( \Psi_{B,s}^{\lambda,\lambda'} \) in terms of \( Y_i \) and \( T_i \). We substitute \( u = e^r - 1 \) and apply the above in (2.5.13) to obtain

\[
\Gamma^{\lambda,\lambda'}(s) = E^B_0 \left( E^{B^1,\hat{B}^2}_{(0,0)} \left[ E^{Y_1,Y_2}_{(B,B)}^{\lambda,\lambda'}(\sqrt{t-s} Y_{T_1}^1, \sqrt{t-s} Y_{T_2}^2) \times \exp \left( - \int_0^1 V_{t_i}^{1,Y_1}(\hat{B}_{u_1}^{\lambda'}, \hat{B}_{u_1}^{\lambda} + e^{T_1/2}(Y_{T_2}^2 - Y_{T_1}^1)) + V_{t_i}^{\lambda,Y_1,1}(\hat{B}_{u_1}^{\lambda',1}, \hat{B}_{u_1}^{\lambda,1}) du \right) \times \exp \left( - \int_0^{T_1} e^{r_i} V_{e^{r_i}}^{1,Y_1}(e^{r_i/2} Y_{T_2}^2 - Y_{T_1}^1)) dr \right) \times \exp \left( - \int_0^{T_2} e^{r_i} V_{e^{r_i}}^{\lambda,Y_1,1}(e^{r_i/2} Y_{T_2}^2 - Y_{T_1}^1)) dr \right) \right] \right).
\]

We now apply (2.3.23) and (2.5.12) in the third and fourth lines. In the third line this gives

\[
e^{r_i} V_{e^{r_i}}^{1,Y_1}(e^{r_i/2} Y_{T_2}^2 - Y_{T_1}^1)) = V_{r_i}^{r_i/2,e^{r_i/2}\lambda}(Y_{Y_{T_2}^2}^1, Y_{T_1}^1 + e^{r_i/2}(Y_{T_2}^2 - Y_{T_1}^1))
\]

and similar in the fourth. Noting that \( V_t^{c,d}(a,b) = V_t^{d,c}(b,a) \), we have obtained the desired expression.

We now obtain an upper bound for \( \Gamma^{\lambda,\lambda'}(s) \) and show that the contribution to \( N_0((\lambda' \leq 1, \text{ and let } h : [0,\infty) \rightarrow \mathbb{R} \text{ be bounded and measurable. There is a constant } C_{2.5.4} > 0 \text{ such that the following hold.}

(a) For all \( \epsilon > (t-s)^{-1/2} \),

\[
(\lambda\lambda')^{2\epsilon_0} |\Gamma^{\lambda,\lambda'}(s)| \leq C_{2.5.4} \|h\|_s t^{-\lambda_0} (t-s)^{-\lambda_0}.
\]

(b) For \( 0 < \epsilon < t \),

\[
(\lambda\lambda')^{2\epsilon_0} \int_{t-\epsilon}^t |\Gamma^{\lambda,\lambda'}(s)| ds \leq C_{2.5.4} \|h\|_s t^{-\lambda_0} (\epsilon^{1-\lambda_0} + \lambda^{2(1-\lambda_0)}).
\]

**Proof.** To begin we use \( |h| \leq \|h\|_{\infty} \) and apply monotonicity (Proposition 2.3.1(a)), i.e.
\[ V^\lambda(x), V^{\lambda'}(y) \leq V^{\lambda, \lambda'}(x, y), \] to obtain
\[
(\lambda\lambda')^{2\lambda_0} |\Gamma^{\lambda, \lambda'}(s)| \\
\leq \|h\|_{\infty} (\lambda\lambda') \mathbb{E}_0^{B^1} \left( \exp \left( - \int_0^s V^\lambda_{t-u} (B^1_{t-u} + B_u) \, du \right) \times \exp \left( - \int_0^{t-s} V^\lambda_r (B^1_r) \, dr \right) \exp \left( - \int_0^{t-s} V^{\lambda'}_r (B^2_r) \, dr \right) \right)
\]
\[
= \|h\|_{\infty} (\lambda\lambda') \mathbb{E}_0^{B^1} \left( \exp \left( - \int_0^t V^\lambda_u (B^1_u) \, du \right) \mathbb{E}_0^{B_2} \left( \exp \left( - \int_0^{t-s} V^{\lambda'}_u (B^2_u) \, du \right) \right) \right),
\]
where the final line follows from a time reversal of \( B \) and concatenating the time-reversed \( B \) with \( B^1 \). Applying (2.3.6) twice and changing the time variable, the above is equal to
\[
\|h\|_{\infty} (\lambda\lambda') \mathbb{E}_0^{B^1} \left( \exp \left( - \int_0^{t^2} V^\lambda_u (\lambda B^1_{\lambda^{-2}u}) \, du \right) \times \mathbb{E}_0^{B_2} \left( \exp \left( - \int_0^{t^{\lambda^2(t-s)}} V^{\lambda'}_u (\lambda B^2_{\lambda^{-2}u}) \, du \right) \right) \right).
\]
The rescaled Brownian motions in the above are themselves standard Brownian motions which we will denote by \( \hat{B}^1, \hat{B}^2 \). We next let \( e' = u \) in both integrals and apply (2.3.6) to see that the above equals
\[
\|h\|_{\infty} (\lambda\lambda') \mathbb{E}_0^{B^1} \left( \exp \left( - \int_0^{\log(\lambda^2 t)} V^\lambda_{e'/2} (e^{-r/2} \hat{B}^1_{e'}) \, dr \right) \times \mathbb{E}_0^{B_2} \left( \exp \left( - \int_0^{\log(\lambda^2(t-s))} V^{\lambda'}_{e'/2} (e^{-r/2} \hat{B}^2_{e'}) \, dr \right) \right) \right) \, ds
\]
\[
\leq \|h\|_{\infty} (\lambda\lambda') \mathbb{E}_m^{Y^1} \left( \exp \left( - \int_0^{\log(\lambda^2(t-s))} V^\lambda_{e'/2} (Y^1_{e'}) \, dr \right) \right) \times \mathbb{E}_m^{Y^2} \left( \exp \left( - \int_0^{\log(\lambda^2(t-s))} V^{\lambda'}_{e'/2} (Y^2_{e'}) \, dr \right) \right), \quad (2.5.14)
\]
where \( Y^i = e^{-r/2} \hat{B}^i_{e'} \), which makes \( Y^i \) a stationary Ornstein-Uhlenbeck process for \( u \in \mathbb{R} \), and we recall our assumption that \( \lambda^2 t \geq 1 \). We condition on the value of \( Y^1_0 \), which has distribution \( m \).

We first use the above to prove (a). Assuming that \( \lambda' > (t-s)^{-1/2} \), the upper endpoint
We can approximate the first expectation with the survival probability of above by \(C\lambda,\lambda'(s)\), we know that when \(\lambda,\lambda'(t-s)\) using the above in (2.5.16), which is an upper bound for \((killed\ at\ rate \ F)\ recognizing\ the\ expectations\ as\ survival\ probabilities\ of\ killed\ Ornstein-Uhlenbeck\ processes\ we\ add\ and\ subtract F(Y^2)\ in\ the\ integrals.\ Recalling\ the\ definition\ of\ Z_T(Y)\ from\ (2.3.16),\ we\ define\ Z^1_T(Y^1),\ Z^2_T(Y^2)\ in\ the\ same\ way.\ Thus\ (2.5.15)\ is\ equal\ to

\[
\|h\|_\infty (\lambda\lambda')^{2\lambda_0} E^Y_m \left( Z^1_{\log(\lambda^2 t)}(Y^1) \exp \left( - \int_0^{\log(\lambda^2 t)} F(Y^1) du \right) \right) \\
\times E^Y_m \left( Z^2_{\log(\lambda^2(t-s))}(Y^2) \exp \left( - \int_0^{\log(\lambda^2(t-s))} F(Y^2) dr \right) \right) \\
\leq \|h\|_\infty (\lambda\lambda')^{2\lambda_0} C Z^1 E^Y_m \left( \exp \left( - \int_0^{\log(\lambda^2 t)} F(Y^1) du \right) \right) \\
\times C Z^2 E^Y_m \left( \exp \left( - \int_0^{\log(\lambda^2(t-s))} F(Y^2) dr \right) \right) ds \\
= C\|h\|_\infty (\lambda\lambda')^{2\lambda_0} P^1_m (\rho^F > \log(\lambda^2 t)) P^2_m (\rho^F > \log(\lambda^2(t-s))). \tag{2.5.16}
\]

In the first inequality we have used (2.3.18) twice, and the second equality follows by recognizing the expectations as survival probabilities of killed Ornstein-Uhlenbeck processes killed at rate \(F(Y^i)\). By (2.2.9), we have

\[
P^1_m (\rho^F > \log(\lambda^2 t)) \leq Ct^{-\lambda_0} \lambda^{-2\lambda_0}, \quad P^2_m (\rho^F > \log(\lambda^2(t-s))) \leq C(t-s)^{-\lambda_0} \lambda^{-2\lambda_0}.
\]

Using the above in (2.5.16), which is an upper bound for \((\lambda\lambda')^{2\lambda_0} |\Gamma_{\lambda,\lambda'}(s)|\), proves (a).

We now show (b). Let \(0 < \epsilon < t\). Using (2.5.14) we obtain that

\[
(\lambda\lambda')^{2\lambda_0} \int_{t-\epsilon}^t |\Gamma_{\lambda,\lambda'}(s)| ds \leq \|h\|_\infty (\lambda\lambda')^{2\lambda_0} E^Y_m \left( \exp \left( - \int_0^{\log(\lambda^2 t)} V^1 e^{\epsilon/2} (Y^1) dr \right) \right) \\
\times \int_{t-\epsilon}^t E^Y_m \left( \exp \left( - \int_0^{\log(\lambda^2(t-s))} V^1 e^{\epsilon/2} (Y^2) dr \right) \right) ds. \tag{2.5.17}
\]

We can approximate the first expectation with the survival probability of \(Y^1\), just as we did in the proof of (a), and bound it above by \(C\lambda^{-2\lambda_0} t^{-\lambda_0}\). Furthermore, by the proof of part (a), we know that when \(\lambda' > (t-s)^{-1/2}\) the expectation in the integral above is bounded above by \(C(\lambda')^{-2\lambda_0} (t-s)^{-\lambda_0}\). When this is not the case we bound it above by 1. Thus the
right hand side of (2.5.17) is bounded above by
\[ C\|h\|_{\infty} t^{-\lambda_0} \left[ 1(\lambda' \geq \epsilon^{-1/2}) \int_{t-\epsilon}^{t-\lambda' \epsilon^{-2}} (t-s)^{-\lambda_0} ds 
+ (\lambda')^{2\lambda_0} \int_{t-\lambda' \epsilon^{-2}}^{t} \Gamma^{\lambda \lambda'}(s) ds \right] \leq C\|h\|_{\infty} t^{-\lambda_0} \left[ \epsilon^{1-\lambda_0} + \lambda' \epsilon^{-2(1-\lambda_0)} \right]. \]
The result now follows. \qed

2.5.2 Convergence

We now show that the expressions for \(N_0((L^\lambda_t \times L^\lambda'_t)(h))\) obtained in the previous section converge as \(\lambda, \lambda' \to \infty\). We do so by computing the limit explicitly. Let \(h : \mathbb{R}^2 \to \mathbb{R}\) be bounded and measurable. Clearly we may assume without loss of generality that \(h \geq 0\).

We recall from (2.5.10) and Proposition 2.5.1 that
\[ N_0((L^\lambda_t \times L^\lambda'_t)(h)) = \int_0^t (\lambda \lambda')^{2\lambda_0} \Gamma^{\lambda \lambda'}(s) ds, \]
where \(h \geq 0\) implies \(\Gamma^{\lambda \lambda'}(s) \geq 0\). Our strategy is to compute the limit of \((\lambda \lambda')^{2\lambda_0} \Gamma^{\lambda \lambda'}(s)\) as \(\lambda, \lambda' \to \infty\) and pass the limit through the integral. However, the scaling we use cannot be done uniformly in \(s\). In order to handle this and the singularity at \(s = t\), we fix \(\epsilon > 0\) and analyse the integral on \([t - \epsilon, t]\) separately. We have
\[ N_0((L^\lambda_t \times L^\lambda'_t)(h)) = \int_0^{t-\epsilon} (\lambda \lambda')^{2\lambda_0} \Gamma^{\lambda \lambda'}(s) ds + (\lambda \lambda')^{2\lambda_0} \int_{t-\epsilon}^t \Gamma^{\lambda \lambda'}(s) ds. \quad (2.5.18) \]

By Lemma 2.5.4(b), the limit superior of the absolute value of the second term as \(\lambda' \to \infty\) is bounded above by \(C\|h\|_{\infty} t^{-\lambda_0} \epsilon^{1-\lambda_0}\). Hence, if
\[ \lim_{\lambda, \lambda' \to \infty} \int_0^{t-\epsilon} (\lambda \lambda')^{2\lambda_0} \Gamma^{\lambda \lambda'}(s) ds \]
exists for all \(\epsilon > 0\), then by the Cauchy condition \(\lim_{\lambda, \lambda' \to \infty} N_0((L^\lambda_t \times L^\lambda'_t)(h))\) exists and is the limit of the above as \(\epsilon \downarrow 0\). Thus it suffices to fix \(\epsilon > 0\) and establish the convergence of, and find the limit of, the first term of (2.5.18), first as \(\lambda, \lambda' \to \infty\) and then as \(\epsilon \downarrow 0\). By Lemma 2.5.4(a), we have
\[ (\lambda \lambda')^{2\lambda_0} |\Gamma^{\lambda \lambda'}(s)| \leq g(s) \quad \text{for all } s \in [0, t-\epsilon] \]
for all $\lambda, \lambda' > \epsilon^{-1/2}$ for a function $g(s) \geq 0$ satisfying $\int_0^{t-\epsilon} g(s) ds < \infty$. It follows that, if $(\lambda\lambda')^{2\lambda_0} \Gamma^{\lambda,\lambda'}(s)$ converges as $\lambda, \lambda' \to \infty$, then Dominated Convergence implies that

$$
\lim_{\lambda, \lambda' \to \infty} \mathbb{N}_0((L^\lambda \times L^\lambda')(h)) = \lim_{\epsilon \to 0^+} \lim_{\lambda, \lambda' \to \infty} \int_0^{t-\epsilon} (\lambda\lambda')^{2\lambda_0} \Gamma^{\lambda,\lambda'}(s) ds 
$$

$$
= \lim_{\epsilon \to 0^+} \int_0^{t-\epsilon} \lim_{\lambda, \lambda' \to \infty} (\lambda\lambda')^{2\lambda_0} \Gamma^{\lambda,\lambda'}(s) ds, \quad (2.5.19)
$$

and so it suffices to find the limit of $(\lambda\lambda')^{2\lambda_0} \Gamma^{\lambda,\lambda'}(s)$ as $\lambda, \lambda' \to \infty$.

Let $s \in (0, t)$ and assume $\lambda, \lambda' > (t-s)^{-1/2}$. By Lemma 2.5.3,

$$(\lambda\lambda')^{2\lambda_0} \Gamma^{\lambda,\lambda'}(s) = (\lambda\lambda')^{2\lambda_0} E_0^B \left( E^{B1,B2}_{\lambda,\lambda'} \left[ \left( \frac{Y}{\sqrt{t-s}} Y_{T_1}^{1/2} - Y_{T_1}^{1/2} \right) \right] \right)
$$

$$
\times \exp \left( - \int_0^{T_1} V_{u}^{1,\lambda'/\lambda}(B_{u}^{1}, B_{u}^{1} + e^{T_1/2}(Y_{T_2}^{2} - Y_{T_1}^{1})) + V_{u}^{1,\lambda'/\lambda}(B_{u}^{2}, B_{u}^{2} + e^{T_2/2}(Y_{T_1}^{1} - Y_{T_2}^{2})) du \right)
$$

$$
\times \exp \left( - \int_0^{T_1} H_{e^u}^{\lambda'/\lambda}(Y_{u}^{1}, Y_{u}^{1} + e^{T_1/2}(Y_{T_2}^{2} - Y_{T_1}^{1})) du \right)
$$

$$
\times \exp \left( - \int_0^{T_1} F(Y_{u}^{1}) du \right)
$$

$$
\times \left[ \frac{Y_{T_1}^{1/2}}{\sqrt{t-s} Y_{T_1}^{1/2}} - \frac{Y_{T_2}^{2}}{\sqrt{t-s} Y_{T_1}^{1/2}} \right] \right], \quad (2.5.20)
$$

where $T_1 = T_1(s) = \log(\lambda^2(t-s))$, $T_2 = T_2(s) = \log(\lambda^2(t-s))$. Inside the integral in the third term we add and subtract $F(Y_{u}^{1})$ and decompose as follows

$$
\exp \left( - \int_0^{T_1} H_{e^u}^{\lambda'/\lambda}(Y_{u}^{1}, Y_{u}^{1} + e^{T_1/2}(Y_{T_2}^{2} - Y_{T_1}^{1})) du \right)
$$

$$
= \exp \left( - \int_0^{T_1} F(Y_{u}^{1}) du \right) \exp \left( \int_0^{T_1} F(Y_{u}^{1}) - H_{e^u}^{\lambda'/\lambda}(Y_{u}^{1}, Y_{u}^{1} + e^{T_1/2}(Y_{T_2}^{2} - Y_{T_1}^{1})) du \right).
$$

We do the same to the fourth term with the obvious changes of indices. The first term in the above is the probability that the Ornstein-Uhlenbeck process $Y^1$ with killing function $F$ survives until time $T_1$. We extract a similar term from the symmetric term corresponding to $Y^2$ and $T_2$. Weighting the expectation of a functional with this survival probability is equivalent to restricting the expectation to the event that the process survives; in our case, we restrict to the event that $Y^1$ and $Y^2$ survive until $T_1$ and $T_2$, respectively. Thus (2.5.20)
is equal to

\[
(\lambda')^{2\lambda_0} E_0^B \left( \int E_{(0,0)}^{B^1,B^2} \left( \int \Psi_{B,s}^{\lambda',\lambda} (\sqrt{t-s} Y^1_{(T_1)}, \sqrt{t-s} Y^2_{(T_2)}) \times \exp \left( - \int_0^1 V^1_{u}^{1,\lambda'/\lambda} (B^1_u, B^1_u + e^{T_{1/2}} (Y^1_{(T_1)} - Y^1_u)) + V^2_{u}^{1,\lambda'/\lambda} (B^2_u, B^2_u + e^{T_{2/2}} (Y^2_{(T_2)} - Y^2_u)) du \right) \times \exp \left( \int_0^{T_{1}} F(Y^1_u) - H^{\lambda'/\lambda}_{e^{\lambda/2}} (Y^1_u, Y^1_u + e^{T_{1/2}} (z_1 - z_2)) du \right) \left| \rho_1 > T_1, Y^1_{(T_1)} = z_1 \right. \right) \times \exp \left( \int_0^{T_{2}} F(Y^2_u) - H^{\lambda'/\lambda}_{e^{\lambda/2}} (Y^2_u, Y^2_u + e^{T_{2/2}} (z_1 - z_2)) du \right) \left| \rho_2 > T_2, Y^2_{(T_2)} = z_2 \right. \right) \times q_{T_1} (B^1_1, z_1) q_{T_2} (B^2_1, z_2) dm(z_1) dm(z_2) \right) \right),
\]

(2.5.21)

where \( \rho_i = \rho^F_i \) is the lifetime of the killed process \( Y^i \). Recall the transition density \( q_t(\cdot, \cdot) \) (with respect to \( m \)) of the killed diffusion. We condition on the endpoints \( Y^i_{(T_i)} = z_i \) (recall from Lemma 2.2.3(a) that the regular conditional distributions exist for all \( z_i \in \mathbb{R} \) and integrate against \( q_{T_i}(\cdot, z_i) dm(z_i) \) to obtain that (2.5.21) is equal to

\[
(\lambda')^{2\lambda_0} E_0^B \left( \int E_{(0,0)}^{B^1,B^2} \left( \int \Psi_{B,s}^{\lambda',\lambda} (\sqrt{t-s} z_1, \sqrt{t-s} z_2) \times \exp \left( - \int_0^1 V^1_{u}^{1,\lambda'/\lambda} (B^1_u, B^1_u + e^{T_{1/2}} (z_2 - z_1)) + V^2_{u}^{1,\lambda'/\lambda} (B^2_u, B^2_u + e^{T_{2/2}} (z_1 - z_2)) du \right) \times \exp \left( \int_0^{T_{1}} F(Y^1_u) - H^{\lambda'/\lambda}_{e^{\lambda/2}} (Y^1_u, Y^1_u + e^{T_{1/2}} (z_2 - z_1)) du \right) \left| \rho_1 > T_1, Y^1_{(T_1)} = z_1 \right. \right) \times \exp \left( \int_0^{T_{2}} F(Y^2_u) - H^{\lambda'/\lambda}_{e^{\lambda/2}} (Y^2_u, Y^2_u + e^{T_{2/2}} (z_1 - z_2)) du \right) \left| \rho_2 > T_2, Y^2_{(T_2)} = z_2 \right. \right) \times q_{T_1} (B^1_1, z_1) q_{T_2} (B^2_1, z_2) dm(z_1) dm(z_2) \right) \right) =: \int \int (\lambda')^{2\lambda_0} \left[ E_0^B \left( E_{(0,0)}^{B^1,B^2} \left( G(\lambda, \lambda', s, B, B^1, B^2, z_1, z_2) \times q_{T_1} (B^1_1, z_1) q_{T_2} (B^2_1, z_2) \right) \right) \right] dm(z_1) dm(z_2).
\]

(2.5.22)

The function \( G \) is defined implicitly. The conditional probabilities that appear are the same that are defined in Section 2.2, in particular Lemma 2.2.3. We have used that the terms in the third and fourth lines are independent conditional on the endpoints. Hereafter, \( Y^1 \) and \( Y^2 \), and their respective laws, refer to killed Ornstein-Uhlenbeck processes with killing function \( F \). Furthermore, after this point we will suppress the conditioning on \( \rho_i > T_i \), as it is implicit in the conditioning \( Y^i_{(T_i)} = z_i \) that \( \rho_i > T_i \).
We introduce notation for the terms appearing in \( G(\lambda, \lambda', s, B, B^1, B^2, z_1, z_2) \). We define

\[
Q(\lambda, \lambda', B^1, B^2, z_1, z_2)
:= \exp \left( - \int_0^1 V_u^{1,\lambda'/\lambda}(B_u^1, B_u^1 + e^{T_1/2}(z_2 - z_1)) + V_u^{1,\lambda'/\lambda}(B_u^2, B_u^2 + e^{T_2/2}(z_1 - z_2)) \, du \right),
\]

and

\[
\tilde{Z}_{T_1}^1 = \tilde{Z}_{T_1}^1(Y^1, z_1, z_2, \lambda'/\lambda)
:= \exp \left( \int_0^{T_1} F(Y_u^1) - H_{e_u}^{\lambda'/\lambda}(Y_u^1, Y_u^1 + e^{T_1-u/2}(z_2 - z_1)) \, du \right),
\]

\[
\tilde{Z}_{T_2}^2 = \tilde{Z}_{T_2}^2(Y^2, z_2, z_1, \lambda'/\lambda)
:= \exp \left( \int_0^{T_2} F(Y_u^2) - H_{e_u}^{\lambda'/\lambda}(Y_u^2, Y_u^2 + e^{T_2-u/2}(z_1 - z_2)) \, du \right).
\]

Recall that \( \Psi_{B,s}^{\lambda,\lambda'}(\sqrt{t-s}z_1, \sqrt{t-s}z_2) \) was defined in (2.5.11). From (2.5.22) we have

\[
G(\lambda, \lambda', s, B, B^1, B^2, z_1, z_2)
= \Psi_{B,s}^{\lambda,\lambda'}(\sqrt{t-s}z_1, \sqrt{t-s}z_2) Q(\lambda, \lambda', B^1, B^2, z_1, z_2) E_{B_1}^Y(\tilde{Z}_{T_1}^1|Y_{T_1}^1 = z_1) E_{B_2}^Y(\tilde{Z}_{T_2}^2|Y_{T_2}^2 = z_2).
\]

We note that \( \tilde{Z}_{T_1}^1 \) and \( \tilde{Z}_{T_2}^2 \) are perturbations of the corresponding \( Z_{T_i}^i \) terms. In particular, we defined \( Z_{T_i}^i \) by

\[
Z_{T_i}^i(Y^i) = \exp \left( \int_0^{T_i} F(Y_u^i) - V_1^{e_u/2}(Y_u^i) \, du \right).
\]

By Proposition 2.3.1(a) and (2.5.12), we have that \( H_{e_u}^e(x, y) \geq V_1^{e_u/2}(x) \), and hence

\[
\tilde{Z}_{T_i}^i \leq Z_{T_i}^i(Y^i) \leq C_Z,
\]

where the second inequality is by (2.3.18). Using \( Q(\lambda, \lambda', B^1, B^2, z_1, z_2) \leq 1 \) and \( |\Psi_{B,s}^{\lambda,\lambda'}| \leq \|h\|_\infty \), both of which are obvious from these terms’ definitions, we therefore obtain that for a constant \( C_1 > 0 \),

\[
|G(\lambda, \lambda', s, B, B^1, B^2, z_1, z_2)| \leq C_1
\]
uniformly in its arguments. We now define \( \Theta(\lambda, \lambda', s, B, B^1, B^2, z_1, z_2) \) as the function in the square-bracketed term in (2.5.22) multiplied by the scaling factor \((\lambda\lambda')^{2\lambda_0}\). That is,

\[
\Theta(\lambda, \lambda', s, B, B^1, B^2, z_1, z_2)
:= G(\lambda, \lambda', s, B, B_1^1, B_1^2, z_1, z_2) (\lambda\lambda')^{2\lambda_0} q_{T_1}(B_1^1, z_1) q_{T_2}(B_1^2, z_2).
\]
Note from (2.5.22) that
\[ \Gamma^{\lambda,\lambda}(s) = \int_0^s \int E_0^B(E_0^{B^1,B^2}(\Theta(\lambda,\lambda',s,B,B^1,B^2,z_1,z_2))) \, dm(z_1) \, dm(z_2). \] (2.5.31)

Recall that \( T_1 = \log(\lambda^2(t-s)) \) and \( T_2 = \log(\lambda^2(t-s)) \). Taking \( s^*(1/8) \) as in Theorem 2.2.1(c), we note that if \( \lambda, \lambda' > e^{s^*/2}(t-s)^{-1/2} \), then \( T_1, T_2 \geq s^*(1/8) \). We define \( \lambda(s) \) as
\[ \lambda(s) = \left[ e^{s^*/2}(t-s)^{-1/2} \right] \vee 1 \] (2.5.32)
and \( \tau(s) \) by
\[ \tau(s) = \log(\lambda(s)^2(t-s)). \] (2.5.33)

Applying (2.2.2) with \( \delta = 1/8 \), we obtain
\[ q_{t_1}(b_1,z_1) q_{t_2}(b_2,z_2) \leq C(t-s)^{-2\lambda_0}(\lambda\lambda')^{-2\lambda_0}e^{1/8(b_1^2+b_2^2+z_1^2+z_2^2)} \]
for all \( T_1, T_2 > \tau(s) \) (equivalently, \( \lambda, \lambda' > \bar{\lambda}(s) \)). Using the above and (2.5.29), we obtain
\[ |\Theta(\lambda,\lambda',s,B,B^1,B^2,z_1,z_2)| \leq C(t-s)^{-2\lambda_0} \exp\left( \left[ (B_1^1)^2 + (B_1^1)^2 + z_1^2 + z_2^2 \right] /8 \right) \quad \text{for all } \lambda, \lambda' > \bar{\lambda}(s). \] (2.5.34)

Since \( B_1^1 \sim m \), (2.5.34) implies that \( \Theta \) has a (uniform in \( \lambda, \lambda' > \bar{\lambda}(s) \)) upper bound which is integrable with respect to \( dP_0^B dP_0^{B^1} dP_0^{B^2} dm(z_1) \, dm(z_2) \). From (2.5.31), this implies that \( (\lambda\lambda')^{2\lambda_0} \Gamma^{\lambda,\lambda}(s) \) is bounded for \( \lambda, \lambda' > \bar{\lambda}(s) \) (for fixed \( s < t \)). Moreover, if \( \lim_{\lambda,\lambda' \to \infty} \Theta(\lambda,\lambda',s,B,B^1,B^2,z_1,z_2) \) exists for \( P_0^B \otimes P_0^{B^1,B^2} \)-a.a. \( \omega \) and Lebesgue-a.a. \( z_1, z_2 \in \mathbb{R} \), then by (2.5.31) and Dominated Convergence (using (2.5.34)), we have
\[ \lim_{\lambda,\lambda' \to \infty} (\lambda\lambda')^{2\lambda_0} \Gamma^{\lambda,\lambda}(s) = \int_0^t \int E_0^B(E_0^{B^1,B^2}[\lim_{\lambda,\lambda' \to \infty} \Theta(\lambda,\lambda',s,B,B^1,B^2,z_1,z_2)]) \, dm(z_1) \, dm(z_2). \] (2.5.35)

In view of (2.5.19), the above implies the following:

If \( \lim_{\lambda,\lambda' \to \infty} \Theta(\lambda,\lambda',s,B,B^1,B^2,z_2,z_2) \) exists \( P_0^B \otimes P_0^{B^1,B^2} \)-a.s. for a.e. \( z_1, z_2 \in \mathbb{R} \),
then
\[ \lim_{\lambda,\lambda' \to \infty} N_0((L_t^\lambda \times L_t^{\lambda'})(h)) = \int_0^t \left[ \int_0^s \int E_0^B(E_0^{B^1,B^2}[\lim_{\lambda,\lambda' \to \infty} \Theta(\lambda,\lambda',s,B,B^1,B^2,z_1,z_2)]) \, dm(z_1) \, dm(z_2) \right] ds. \] (2.5.36)
As \( h \geq 0 \), and hence \( \Gamma^{\lambda',\lambda}(s) \geq 0 \), the right hand side of the above is equal to the last expression of (2.5.19) (provided \( \Theta \) converges) by Monotone Convergence. Thus it suffices to compute the limit of \( \Theta(\lambda, \lambda', s, B, B^1, B^2, z_2, z_2) \) as \( \lambda, \lambda' \to \infty \). As we only need to find the limit a.e. in \((z_1, z_2)\), we will hereafter assume that \( z_1 \neq z_2 \). We also take this opportunity to reiterate our assumptions about \( \lambda \) and \( \lambda' \). Originally we assumed \( \lambda, \lambda' > (t - s)^{-1/2} \); in view of the above, we augment the assumption to \( \lambda, \lambda' > \bar{\lambda}(s) \), or equivalently, \( T_1, T_2 > \tau(s) \) (see (2.5.32) and (2.5.33)). This implies that \( \lambda, \lambda' > 1 \) and \( T_1, T_2 > s'(1/8) \).

\( \Theta \) is the product of the function \( G \) and the rescaled transition densities, that is, \( \lambda^{2\lambda_0} q_{T_1}(B^1_1, z_1) \) and \( \lambda^{2\lambda_0} q_{T_2}(B^2_1, z_2) \). We will show that both of these approach finite limits as \( \lambda, \lambda' \to \infty \). First, let us handle the transition densities. By Lemma 2.2.2,

\[
\lim_{T_i \to \infty} e^{\lambda_0 T_i} q_{T_i}(B^i_1, z_i) = \psi_0(B^i_1) \psi_0(z_i)
\]

for \( i = 1, 2 \). Using the definitions of \( T_1 \) and \( T_2 \) (e.g. \( T_1 = \log(\lambda^2(t - s)) \)), we readily obtain from the above that

\[
\lambda^{2\lambda_0} q_{T_1}(B^1_1, z_1) \to (t - s)^{-\lambda_0} \psi_0(B^1_1) \psi_0(z_1) \quad \text{as} \quad \lambda \to \infty, \quad \text{and} \n
\lambda^{2\lambda_0} q_{T_2}(B^2_1, z_2) \to (t - s)^{-\lambda_0} \psi_0(B^2_1) \psi_0(z_2) \quad \text{as} \quad \lambda' \to \infty
\]

(2.5.37)

for all \( B^1_1, B^2_1, z_1, z_2 \in \mathbb{R} \).

We now compute the limit of \( G \), whose definition we recall from (2.5.26). We proceed by computing the limit of each constituent term. The analysis is most technical for the conditional expectations of \( \tilde{Z}^i_{T_i} \), for \( i = 1, 2 \), which were defined in (2.5.24) and (2.5.25). These two quantities are essentially the same with different parameters. To avoid excessive and cumbersome notation we define a variable \( \tilde{Z}_T \). Let \( \lambda, \lambda' > 0 \) and \( T = \log(\lambda^2(t - s)) \), and \( x, z_2, z_2 \in \mathbb{R} \). For a killed Ornstein-Uhlenbeck process \( Y \), we define

\[
\tilde{Z}_T = \tilde{Z}_T(Y, z_1, z_2, \lambda'/\lambda) := \exp \left( \int_0^T F(Y_u) - H^{\lambda'/\lambda}_u(Y_u, Y_u + e^{T_u} (z_2 - z_1)) \, du \right). \quad (2.5.38)
\]

In order to characterize the limit of the conditional expectation of \( \tilde{Z}_T \) (as in (2.5.26)), we introduce a quantity \( W_S(Y, z) \). For \( S > 0 \) and \( z \in \mathbb{R} \), we define

\[
W_S(Y, z) = \exp \left( \int_0^S F(Y_u) - F_2(Y_u, Y_u - e^{u/2}(z - Y_0)) \, du \right), \quad (2.5.39)
\]

where we recall from (2.3.24) that \( F_2(a, b) = V^\infty_1(a, b) \). By Proposition 2.3.1(a), \( F(a) \leq F_2(a, b) \) for all \( a, b \in \mathbb{R} \), so the integrand is non-positive and hence \( W_S(Y, z) \leq 1 \) and is non-increasing in \( S \). Since it is bounded below by 0, we can define its monotone limit as
\[ W_\infty(Y, z) = \lim_{S \to \infty} W_S(Y, z) \leq 1. \]

**Lemma 2.5.5.** Let \( x, z_1, z_2 \in \mathbb{R} \) such that \( z_1 \neq z_2 \). Then

\[
\lim_{\lambda, \lambda' \to \infty} E^Y_x(\tilde{Z}_T(Y, z_1, z_2, \lambda'/\lambda) \mid Y_T = z_1) = E^Y_{x}(Z_\infty(Y)) E^Y_{z_1}(W_\infty(Y, z_2)).
\]

Recall that \( E^Y_{x} \) is the expectation under the law of the killed process \( Y \) with \( Y_0 = x \) conditioned to survive for all time, as defined in Theorem 2.2.1(e). \( Z_T(Y) \) is as defined in (2.3.16) and we recall from (2.3.17) and (2.3.18) that \( Z_\infty(Y) = \lim_{T \to \infty} Z_T(Y) \) exists and is bounded by \( C_Z \).

Heuristically, the \( Z_\infty \) term in the limiting expression in Lemma 2.5.5 comes from the early (small \( u \)) part of the integral in \( \tilde{Z}_T \), and the \( W_\infty \) term comes from the tail part (i.e. \( u \) near \( T \)), and these two contributions are asymptotically independent. Since the time at which we condition is \( T \) and \( T \) goes to infinity, in the limit the expectations are computed under the measure of the process conditioned to survive forever.

Section 2.6 is devoted to the proof of Lemma 2.5.5. For now, we carry on with the proof of Theorem 2.1.4. Returning to \( \tilde{Z}^1_{T_1} \) and \( \tilde{Z}^2_{T_2} \), it follows from Lemma 2.5.5 that

For all \( B^1, B^2 \in \mathbb{R} \) and all \( z_1, z_2 \in \mathbb{R} \) such that \( z_1 \neq z_2 \),

\[
\lim_{\lambda, \lambda' \to \infty} E^Y_{B^1}(\tilde{Z}^1_{T_1}(Y^1, z_1, z_2, \lambda'/\lambda) \mid Y^1_{T_1} = z_1) = E^Y_{B^1}(Z_\infty(Y)) E^Y_{z_1}(W_\infty(Y, z_2)),
\]

and

\[
\lim_{\lambda, \lambda' \to \infty} E^Y_{B^1}(\tilde{Z}^2_{T_2}(Y^2, z_2, z_1, \lambda'/\lambda) \mid Y^2_{T_2} = z_2) = E^Y_{B^1}(Z_\infty(Y)) E^Y_{z_2}(W_\infty(Y, z_1)).
\]

(2.40)

To find the limit of \( G \) it remains to identify the limits of \( \Psi_{B, s}^{\lambda, \lambda'}(\sqrt{t - s}z_1, \sqrt{t - s}z_2) \) and \( Q(\lambda_1, \lambda_2, B^1, B^2, z_1, z_2) \). From (2.5.11) we recall that the former prelimit is defined as

\[
\Psi_{B, s}^{\lambda, \lambda'}(\sqrt{t - s}z_1, \sqrt{t - s}z_2) = h(\sqrt{t - s}z_1 + B_s, \sqrt{t - s}z_2 + B_s)
\]

\[
\times \exp \left( -\int_0^s V_{t-u}^{\lambda, \lambda'}(\sqrt{t - s}z_1 + B_s - B_u, \sqrt{t - s}z_2 + B_s - B_u) \, du \right).
\]

For all \( z_1, z_2 \in \mathbb{R} \) and all Brownian paths \( (B_u, u \in [0, s]) \), the obvious limit of the above as \( \lambda, \lambda' \to \infty \) is obtained by replacing \( V^{\lambda, \lambda'}_{t-u} \) with \( V^{\infty, \infty}_{t-u} \). By monotonicity (in \( \lambda, \lambda' \)) of the integral and continuity of the exponential we can take the limit inside. Denoting the limit by \( \Psi_{B, s}^{\infty, \infty}(\sqrt{t - s}z_1, \sqrt{t - s}z_2) \), we have

\[
\lim_{\lambda, \lambda' \to \infty} \Psi_{B, s}^{\lambda, \lambda'}(\sqrt{t - s}z_1, \sqrt{t - s}z_2) = \Psi_{B, s}^{\infty, \infty}(\sqrt{t - s}z_1, \sqrt{t - s}z_2).
\]

(2.41)
This leaves \( Q(\lambda', B^1, B^2, z_1, z_2) \), which we recall from (2.5.23) is defined by

\[
Q(\lambda', B^1, B^2, z_1, z_2) = \exp \left( - \int_0^1 V_u^{1, \lambda'/\lambda}(B_u^1, B_u^1 + e^{T_1/2}(z_2 - z_1)) + V_u^{1, \lambda'/\lambda}(B_u^2, B_u^2 + e^{T_2/2}(z_1 - z_2)) \, du \right).
\]

The integrand is the sum of two terms that are very similar; for now we restrict our attention to the first. In particular, we will show that

\[
\lim_{T_1 \to \infty} \sup_{\lambda' > (t-s)^{-1/2}} \left| \exp \left( - \int_0^1 V_u^{1, \lambda'/\lambda}(B_u^1, B_u^1 + e^{T_1/2}(z_2 - z_1)) \, du \right) - \exp \left( - \int_0^1 V_u^{1}(B_u^1) \, du \right) \right| = 0. \tag{2.5.42}
\]

We claim that since the second argument of the integrand goes to infinity, asymptotically the function resembles \( V_u^1(B_u^1) \). To see this use both parts of Proposition 2.3.1 to conclude that

\[
0 \leq \left[ V_u^{1,c}(B_u^1, B_u^1 + e^{T_1/2}(z_2 - z_1)) - V_u^1(B_u^1) \right] \leq V_u^\infty(B_u^1 + e^{T_1/2}(z_2 - z_1))
\]

for all \( c > 0 \). \( P_0^B \)-a.s., there is a constant \( R(\omega) > 0 \) such that \( |B_u^1(\omega)| \leq R(\omega) \) for all \( u \in [0, s] \). Provided \( z_1 \neq z_2 \), for sufficiently large \( \lambda, e^{T_1/2}|z_2 - z_1| \geq 2R \). Then for \( \lambda \) sufficiently large and \( \lambda, \lambda' > \lambda(s) \),

\[
\left| \exp \left( - \int_0^1 V_u^{1, \lambda'/\lambda}(B_u^1, B_u^1 + e^{T_1/2}(z_2 - z_1)) \, du \right) - \exp \left( - \int_0^1 V_u^{1}(B_u^1) \, du \right) \right| \leq \int_0^1 V_u^\infty(e^{T_1/2}(z_2 - z_1) - R) \, du.
\]

The integrand is bounded above by \( V_u^\infty(R) \). Since \( V_u^\infty(R) = u^{-1}F(u^{-1/2}R) \), (from (2.3.13)) by (2.3.14)(iii) we have \( V_u^\infty(R) \leq cu^{-3/2}Re^{-u^{1-R/2}} \), which is bounded on \( [0, 1] \). We take \( \lambda \to \infty \) and apply Dominated Convergence; since \( V_u^\infty(y) = u^{-1}F(u^{-1/2}y) \), applying (2.3.14)(iii) again gives that \( \lim_{|y| \to \infty} V_u^\infty(y) = 0 \), and hence limit of the above as \( \lambda \to \infty \) (i.e. as \( T_1 \to \infty \)) is zero. Thus (2.5.42) holds. We handle the second term in the integral in \( Q(\lambda', B^1, B^2, z_1, z_2) \) in an identical fashion, now with the roles of \( \lambda \) and \( \lambda' \) reversed, thereby establishing that

\[
\lim_{\lambda, \lambda' \to \infty} Q(\lambda, \lambda', B^1, B^2, z_1, z_2) = \exp \left( - \int_0^1 V_u^{1}(B_u^1) \, du \right) \exp \left( - \int_0^1 V_u^{1}(B_u^2) \, du \right). \tag{2.5.43}
\]
We have therefore found the limit of $G$ and hence of $\Theta$. In particular, recall the definitions (2.5.26) and (2.5.30). From (2.5.37), (2.5.40), (2.5.41) and (2.5.43), we have shown that $dP_0^B dP_0^{B_1} dP_0^{B_2}$ a.s., for all $z_1, z_2 \in \mathbb{R}$ such that $z_1 \neq z_2$,

$$
\lim_{\lambda, \lambda' \to \infty} \Theta(\lambda, \lambda', s, B, B^1, B^2, z_1, z_2) = (t-s)^{-2\lambda_0} \Psi_{B,s}^{\infty}(\sqrt{t-s} z_1, \sqrt{t-s} z_2) 
\times E_{z_1}^{\infty}(W_\infty(Y, z_2)) E_{z_2}^{\infty}(W_\infty(Y, z_1)) E_{B_1}^{Y,\infty}(Z_\infty(Y)) E_{B_2}^{Y,\infty}(Z_\infty(Y)) 
\times \exp \left( \int_0^1 V_u^1(B_u^1) - V_u^1(B_u^2) \, du \right) \psi_0(B_1) \psi_0(B_2) \psi_0(z_1) \psi_0(z_2). \quad (2.5.44)
$$

Thus by (2.5.36), $\lim_{\lambda, \lambda' \to \infty} N_0((L^\lambda_t \times L^{\lambda'}_t)(h))$ exists and satisfies

$$
\lim_{\lambda, \lambda' \to \infty} N_0((L^\lambda_t \times L^{\lambda'}_t)(h)) = \int_0^t \int_0^t E_0^B \left( E_{(0,0)}^{B_1, B_2} \left[ \lim_{\lambda, \lambda' \to \infty} \Theta(\lambda, \lambda', s, B, B^1, B^2, z_1, z_2) \right] \right) dm(z_1) dm(z_2) \, ds. \quad (2.5.45)
$$

To obtain the desired expression, we note that the terms in (2.5.44) that depend on $B^1$ and $B^2$ can be collected in a constant. In particular, we define a constant $C_{2.1.4} > 0$ by

$$
C_{2.1.4}^2 = E_{(0,0)}^{B_1, B_2} \left( \exp \left( - \int_0^1 V_u^1(B_u^1) + V_u^1(B_u^2) \, du \right) \right.
\times E_{B_1}^{Y,\infty}(Z_\infty(Y)) E_{B_2}^{Y,\infty}(Z_\infty(Y)) \psi_0(B_1) \psi_0(B_2) \bigg) 
= \left[ E_0^B \left( \exp \left( - \int_0^1 V_u^1(B_u) \, du \right) E_{B_1}^{Y,\infty}(Z_\infty(Y)) \psi_0(B_1) \right) \right]^2. \quad (2.5.46)
$$

We also define a function $\rho(\cdot, \cdot)$ by

$$
\rho(z_1, z_2) = E_{z_1}^{Y,\infty}(W_\infty(Y, z_2)) E_{z_2}^{Y,\infty}(W_\infty(Y, z_1)). \quad (2.5.47)
$$

It is clear that $\rho(\cdot, \cdot)$ is jointly continuous and bounded by 1 from the definition of $W_\infty(Y, z)$. Thus by (2.5.44),

$$
E_{(0,0)}^{B_1, B_2} \left[ \lim_{\lambda, \lambda' \to \infty} \Theta(\lambda, \lambda', s, B, B^1, B^2, z_1, z_2) \right] = C_{2.1.4}^2 (t-s)^{-2\lambda_0} \Psi_{B,s}^{\infty}(\sqrt{t-s} z_1, \sqrt{t-s} z_2) \rho(z_1, z_2) \psi_0(z_1) \psi_0(z_2).
$$

Substituting the above into (2.5.45) completes the proof of Theorem 2.1.4. \(\square\)
2.6 Proof of Lemma 2.5.5

Recalling the statement of the lemma, we will show that

$$\lim_{\lambda, \lambda' \to \infty} E_x^Y \left( \tilde{Z}_T(Y, z_1, z_2, \lambda'/\lambda) \mid Y_T = z_1 \right) = E_x^Y(\tilde{Z}_\infty(Y)) E_{z_1}^Y(W_{\infty}(Y, z_2))$$ (2.6.1)

for all $x, z_1, z_2 \in \mathbb{R}$ such that $z_1 \neq z_2$, where the law of $Y$ on the left hand side is that of a killed Ornstein-Uhlenbeck process with killing function $F$, with $Y_0 = x$. Recall that $E_x^Y$ is the expectation under the law of the killed process $Y$ with $Y_0 = x$ conditioned to survive for all time. For convenience, we now recall the definitions of the quantities above: from (2.3.16), (2.5.38) and (2.5.39),

$$\tilde{Z}_T(Y, z_1, z_2, \lambda'/\lambda) = \exp \left( \int_0^T F(Y_u) - H^{' \lambda'/\lambda}_u(Y_u, Y_u + e^{T-u}(z_2 - z_1)) \, du \right),$$

$$W_S(Y, z) = \exp \left( \int_0^S F(Y_u) - F_2(Y_u, Y_u - e^{S/2}(z - Y_0)) \, du \right),$$

$$Z_T(Y) = \exp \left( \int_0^T F(Y_u) - V_{e^{u/2}}(Y_u) \, du \right).$$

As we have previously discussed, $Z_\infty(Y)$ and $W_\infty(Y, z)$ are the respective limits of $Z_T(Y)$ and $W_S(Y, z)$ as $T, S \to \infty$, both of which exist. Recall from (2.3.18) that $Z_\infty(Y) \leq C_Z$. Because $z_1$ and $z_2$ are fixed, we will hereafter suppress the dependence of $\tilde{Z}_T(Y, z_1, z_2, \lambda'/\lambda)$ on $z_1$ and $z_2$ and simply write $\tilde{Z}_T(Y, \lambda'/\lambda)$. Finally, we recall our working assumption that $\lambda, \lambda' > \tilde{\lambda}(s)$, (see (2.5.32)) which implies that $\lambda, \lambda' > 1$ and $T > s^*(1/8)$.

Let $0 < K < T/2$. We apply the Markov property to $E_x^Y(\tilde{Z}_T(Y, \lambda'/\lambda) \mid Y_T = z_1)$ at times $K$ and $T - K$ and expand in terms of the joint density of $(Y_K, Y_{T-K})$. As in (2.2.11), the joint density of $(Y_K, Y_{T-K})$ at $(w, y)$ with respect to $m \times m$ under $P_x^Y(Y \in \cdot \mid Y_T = z_1)$ is

$$q_K(x, w)q_{T-2K}(w, y)q_K(y, z_1) q_T(x, z_1).$$

Thus we obtain the following:

$$E_x^Y \left( \tilde{Z}_T(Y, \lambda'/\lambda) \mid Y_T = z_1 \right) = E_x^Y \left( \exp \left( \int_0^T F(Y_u) - H^{' \lambda'/\lambda}_u(Y_u, Y_u + e^{T-u}(z_2 - z_1)) \, du \right) \mid Y_T = z_1 \right)$$
\[
\begin{align*}
&= \int \int E^Y_x \left( \exp \left( \int_0^K F(Y_u) - H^{Y'/\lambda}_{e^{w/\lambda}}(Y_u, Y_u + e^{w/\lambda}(z_2 - z_1)) \right| Y_K = w \right) \\
&\quad \times E^Y_w \left( \exp \left( \int_0^{T-2K} F(Y_u) - H^{Y'/\lambda}_{e^{w/\lambda}}(Y_u, Y_u + e^{w/\lambda}(z_2 - z_1)) \right| Y_{T-2K} = y \right) \\
&\quad \times E^Y_y \left( \exp \left( \int_0^K F(Y_u) - H^{Y'/\lambda}_{e^{w/\lambda}}(Y_u, Y_u + e^{w/\lambda}(z_2 - z_1)) \right| Y_K = z_1 \right) \\
&\quad \times \frac{q_K(x, w) q_{T-2K}(w, y) q_K(y, z_1)}{q_T(x, z_1)} \, dm(w) \, dm(y).
\end{align*}
\] (2.6.2)

Denote the three conditional expectations by \(A_1(x, w, \lambda, \lambda', K)\), \(A_2(w, y, \lambda, \lambda', K)\) and \(A_3(y, z_1, \lambda, \lambda', K)\). That is,
\[
\begin{align*}
A_1(x, w, \lambda, \lambda', K) &= E^Y_x \left( \exp \left( \int_0^K F(Y_u) - H^{Y'/\lambda}_{e^{w/\lambda}}(Y_u, Y_u + e^{w/\lambda}(z_2 - z_1)) \right| Y_K = w \right), \\
A_2(w, y, \lambda, \lambda', K) &= E^Y_w \left( \exp \left( \int_0^{T-2K} F(Y_u) - H^{Y'/\lambda}_{e^{w/\lambda}}(Y_u, Y_u + e^{w/\lambda}(z_2 - z_1)) \right| Y_{T-2K} = y \right), \\
A_3(y, z_1, \lambda, \lambda', K) &= E^Y_y \left( \exp \left( \int_0^K F(Y_u) - H^{Y'/\lambda}_{e^{w/\lambda}}(Y_u, Y_u + e^{w/\lambda}(z_2 - z_1)) \right| Y_K = z_1 \right).
\end{align*}
\] (2.6.3) (2.6.4) (2.6.5)

We observe that \(A_1, A_2\) and \(A_3\) all depend on \(z_1\) and \(z_2\) in addition to their listed arguments, as these values appear in their integrands. Again, as \(z_1\) and \(z_2\) are fixed, we omit this additional dependence. Noting that the integrand is bounded above by \(F(Y_u) - V_1^{e^{w/2}}(Y_u)\) in each case, from (2.3.18) we have \(A_i \leq C_Z\) for \(i = 1, 2, 3\). In terms of the \(A_i\), \(2.6.2\) can be rewritten as
\[
\begin{align*}
E^Y_x \left( \tilde{Z}_T(Y, \lambda'/\lambda) | Y_T = z_1 \right) &= \int \int A_1(x, w, \lambda, \lambda', K) A_2(w, y, \lambda, \lambda', K) A_3(y, z_1, \lambda, \lambda', K) \\
&\quad \times \frac{q_K(x, w) q_{T-2K}(w, y) q_K(y, z_1)}{q_T(x, z_1)} \, dm(w) \, dm(y).
\end{align*}
\] (2.6.6)

There are two main contributions in the \(A_i\). The first comes from \(F\) and the first argument of the \(H\) function, and is approximately equal to \(F(Y_u) - V_1^{e^{w/2}}(Y_u)\); the second comes from the second argument of the \(H\) function. We will see that, asymptotically, \(A_1\) is only affected by the first contribution and gives the \(Z_\infty(Y)\) term in (2.6.1); \(A_3\) is only affected by the second contribution and gives the \(W_\infty(Y, z_2)\) term in (2.6.1). The contribution of \(A_2\) will be seen to be negligible. We first show that \(A_2\) is arbitrarily close to 1 as \(K\) is made large, uniformly in \(T\) sufficiently large depending on \(K\). Define \(Z_T^0(Y, \lambda'/\lambda, K)\) as \(\tilde{Z}_T(Y, \lambda'/\lambda, K)\)
with $A_2$ replaced by 1; that is,

$$Z_T^Y(Y, X'/\lambda, K) = \exp \left( \int_0^T F(Y_u) - H_{e^{u/K}}^{X'/\lambda}(Y_u, Y_u + e^{\frac{T-u}{2}}(z_2 - z_1)) \, du \right)$$

$$\times \exp \left( \int_{T-K}^T F(Y_u) - H_{e^{u/K}}^{X'/\lambda}(Y_u, Y_u + e^{\frac{T-u}{2}}(z_2 - z_1)) \, du \right).$$  \hfill (2.6.7)

As in (2.6.2) and (2.6.6), we therefore have

$$E_x^Y \left( \frac{Z_T^Y(Y, X'/\lambda, K)}{Z_T^Y(Y, X'/\lambda, K)} \right | Y_T = z_1)$$

$$= \int \int A_1(x, w, \lambda, X, K)A_3(y, z_1, \lambda, X, K) \frac{qK(x, w)q_{T-2K}(w, y)qK(y, z_1)}{q_T(x, z_1)} \, dm(w) \, dm(y).$$  \hfill (2.6.8)

By monotonicity (Proposition 2.3.1(a)) and (2.3.15) we have

$$F(Y_u) - H_{e^{u/K+u}}^{X'/\lambda}(Y_u, Y_u + e^{\frac{T-u}{2}}(z_2 - z_1)) \leq F(Y_u) - V_1^{(K+u)/2}(Y_u) \leq C e^{-(K+u)(2\lambda_0-1)/2}$$

uniformly in $T > 2K$. Integrating this over $u$ shows that the exponent in $A_2$ is bounded above $C' e^{-(2\lambda_0-1)K/2}$ for a constant $C'$, uniformly in $T > 2K$. We choose $K$ large enough so that exponent in $A_2$ is smaller than 2. Then by (2.6.6) and (2.6.7), applying the mean value theorem, we have

$$|E_x^Y \left( \frac{Z_T^Y(Y, X'/\lambda) - Z_T^Y(Y, X'/\lambda, K)}{Z_T^Y(Y, X'/\lambda, K)} \right | Y_T = z_1)$$

$$\leq \frac{1}{q_T(x, z_1)} \int \int A_1(x, w, \lambda, X', K) \left | A_2(w, y, \lambda, X', K) - 1 \right | A_3(y, z_1, \lambda, X', K)$$

$$\times qK(x, w)q_{T-2K}(w, y)qK(y, z_1) \, dm(w) \, dm(y)$$

$$\leq \frac{e^2 C_Z^2}{q_T(x, z_1)} \int \int E_u^Y \left( \int_0^{T-2K} |F(Y_u) - H_{e^{u/K+u}}^{X'/\lambda}(Y_u, Y_u + e^{\frac{T-u}{2}}(z_2 - z_1))| \, du \right | Y_T = z_1)$$

$$\times qK(x, w)q_{T-2K}(w, y)qK(y, z_1) \, dm(w) \, dm(y).$$ \hfill (2.6.10)

uniformly for all $T > 2K$, where we have also used $A_1A_3 \leq C_Z^2$. The term in the absolute value inside the integral can be positive or negative; (2.6.9) provides an upper bound for $F - H_{e^{u/K+u}}^{X'/\lambda}$. To obtain a lower bound, we note that $H_{e^{u/K+u}}^{X'/\lambda}(a, b) \leq F_2(a, b) \leq F(a) + F(b)$ by Proposition 2.3.1 (using part (a) and then part (b)). This bound implies that

$$F(Y_u) - H_{e^{u/K+u}}^{X'/\lambda}(Y_u, Y_u + e^{\frac{T-u}{2}}(z_2 - z_1)) \geq -F(Y_u + e^{\frac{T-u}{2}}(z_2 - z_1)).$$  \hfill (2.6.11)

Together, (2.6.9) and (2.6.11) imply that the absolute value appearing in the integral in
(2.6.10) is bounded above by
\[ C e^{-(K+u)(2\lambda_0-1)/2} + F(Y_u + e^{\frac{T-K-u}{2}}(z_2 - z_1)). \]
We have already noted that when integrated over \( u \), the first term is bounded by \( C' e^{-K(2\lambda_0-1)/2} \) (uniformly in \( T \)). The first term has no dependence on the spatial parameters \( w \) and \( y \), so in (2.6.10) the transition densities and can be integrated and cancelled with the denominator.

We get that for all \( T > 2K \), (2.6.10) is bounded above by
\[ C' e^{-K(2\lambda_0-1)/2} + \frac{C}{q_T(x, z_1)} \int \int E^Y_w \left( \int_0^{T-2K} F(Y_u + e^{\frac{T-K-u}{2}}(z_2 - z_1)) \, du \right) \bigg| Y_{T-2K} = y \]
\[ \times q_K(x, w)q_{T-2K}(w, y)q_K(y, z_1) \, dm(w) \, dm(y). \]

We consider the time reversed process in the above and apply (2.2.12), which implies that the above is equal to, and hence for all \( T > 2K \), (2.6.10) is bounded above by
\[ C' e^{-K(2\lambda_0-1)/2} + \frac{C}{q_T(x, z_1)} \int \int E^Y_w \left( \int_0^{T-2K} F(Y_u + e^{\frac{K+u}{2}}(z_2 - z_1)) \, du \right) \bigg| Y_{T-2K} = w \]
\[ \times q_K(x, w)q_{T-2K}(w, y)q_K(y, z_1) \, dm(w) \, dm(y). \] (2.6.12)

We recall the asymptotic behaviour of \( F \) from (2.3.14)(iii), i.e. that \( F(x) \sim c_1 |x| e^{-x^2/2} \) as \( |x| \to \infty \). This implies there is a constant \( c_2 > 0 \) such that
\[ F(x) \leq c_2 (1 + |x|) e^{-x^2/2} \text{ for all } x \in \mathbb{R}. \] (2.6.13)

In order for this to give a useful upper bound in (2.6.12), we’ll need to show that the argument of \( F \) is large in absolute value. It is enough to show that \( |Y_u| \lesssim e^{\frac{K+u}{2}} |z_2 - z_1| \) with high probability when conditioned on its endpoint. Recall that we have assumed \( z_1 \neq z_2 \). We bound the integrand over the two cases mentioned above and exchange the integral and expectation, which is justifiable since \( F \) is positive. We have
\[ E^Y_y \left( \int_0^{T-2K} F(Y_u + e^{\frac{K+u}{2}}(z_2 - z_1)) \, du \right) \bigg| Y_{T-2K} = w \]
\[ \leq E^Y_y \left( \int_0^{T-2K} F(e^{\frac{K+u}{2}}|z_2 - z_1|) + F(0)1(|Y_u| \geq e^{\frac{K+u}{2}}|z_2 - z_1|) \, du \right) \bigg| Y_{T-2K} = w \]
\[ \leq c_2 \int_0^\infty \left( 1 + e^{\frac{K+u}{4}}|z_2 - z_1| \right) e^{-e^{\frac{K+u}{2}}|z_2 - z_1|^2/2} \, du \]
\[ + F(0) \int_0^{T-2K} P^Y_y(|Y_u| \geq e^{\frac{K+u}{4}}|z_2 - z_1|) \bigg| Y_{T-2K} = w \) \, du, \] (2.6.14)
where we have used (2.6.13) and the fact that $F$ is radially decreasing. A simple substitution shows that

$$c_2 \int_0^\infty (1 + e^{K/4}|z_2 - z_1|)e^{-e^{K/4}|z_2 - z_1|^2/2} du$$

$$\leq 4c_1 \int_{e^{K/4}|z_2 - z_1|}^\infty (1 + a^{-1})e^{-a^2/2} da$$

$$\leq C \int_{e^{K/4}|z_2 - z_1|}^\infty e^{-a^2/2} da + C1(e^{K/4}|z_2 - z_1| < 1) \int_{e^{K/4}|z_2 - z_1|}^1 a^{-1}e^{-a^2/2} da$$

$$\leq C \int_{e^{K/4}|z_2 - z_1|}^\infty e^{-a^2/2} da - C \left[ \log(e^{K/4}|z_2 - z_1|) \wedge 0 \right].$$

(2.6.15)

To bound the second term in (2.6.14) we expand the probability of the large excursion in terms of the transition densities. There are two cases, which we handle in the following lemma. In what follows, $s^* = s^*(1/8)$ from Theorem 2.2.1(c).

**Lemma 2.6.1.** Let $M > 0$ and $w,y \in \mathbb{R}$.

(a) There is a constant $C > 0$ such that for $u,S > 0$ satisfying $u,S - u \geq s^*$,

$$P^Y_y (|Y_u| \geq M \mid Y_S = w) \leq \frac{C}{q_S(y,w)} e^{-\lambda_0 S} e^{(y^2 + w^2)/8} \left[ e^{-M^2/4} \frac{e^{-M^2/4}}{M} \wedge 1 \right].$$

(b) For fixed $u_0 > 0$ the families

$$\{P^Y_y (Y_u \in \cdot \mid Y_S = w) : S \geq u_0, 0 \leq u \leq u_0 \} \text{ and }$$

$$\{P^Y_y (Y_{S-u} \in \cdot \mid Y_S = w) : S \geq u_0, 0 \leq u \leq u_0 \}$$

are tight.

**Proof.** To see (a), we use (2.2.11) and (2.2.2) with $\delta = 1/8$ to obtain that for $u,S - u \geq s^*$,

$$P^Y_y (Y_u \geq M \mid Y_S = w) = \int_M^\infty \frac{q_u(y,a)q_{S-u}(a,w)}{q_S(y,w)} dm(a)$$

$$\leq \frac{e^2}{q_S(y,w)} e^{-\lambda_0 S} \int_M^\infty e^{(y^2 + 2a^2 + w^2)/8} dm(a)$$

$$\leq \frac{C}{q_S(y,w)} e^{-\lambda_0 S} e^{(y^2 + w^2)/8} \left[ e^{-M^2/4} \frac{e^{-M^2/4}}{M} \wedge 1 \right],$$

where the last line uses a standard upper bound on Gaussian tails and bounds the integral above by a constant when $M$ is small. The bound for $Y_u < -M$ is the same. The first family in part (b) is tight as a consequence of Lemma 2.2.3(c). To see that the second family is tight we consider the time reversal of $Y$ and use (2.2.12), from which tightness
now also follows from Lemma 2.2.3(c).

Applying Lemma 2.6.1(a), using (2.6.15) and separating the integrals depending if \( u, S - u \geq s^* \) or not, we have that (2.6.14) is bounded above by

\[
C \int_{e^{K/4}}^{\infty} e^{-a^2/2} da - C \left[ \log(e^{K/4}|z_2 - z_1|) \wedge 0 \right]
\]

\[
+ \frac{C}{qT-2K(y,w)} e^{-\lambda_0(T-2K)} e^{y^2/8} e^{w^2/8} \int_{s^*}^{T-2K-s^*} \left[ \frac{e^{-\frac{K+u}{2}|z_2 - z_1|^2/4}}{e^{-\frac{K+u}{4}|z_2 - z_1|}} \wedge 1 \right] du
\]

\[
+ C \left( \int_0^{s^*} + \int_{T-2K-s^*}^{T-2K} \right) P_y^Y (|Y_u| \geq e^{K+u/4}|z_2 - z_1| \big| Y_{T-2K} = w) du.
\]  

(2.6.16)

As the above is an upper bound for the expectation appearing in the second term of (2.6.12), and (2.6.12) is an upper bound for (2.6.10), we have

\[
|E_x^Y (\tilde{Z}_T(Y, \lambda'/\lambda) - Z_x^a(Y, \lambda'/\lambda, K) \big| Y_T = z_1)|
\]

\[
\leq C e^{-K(2\lambda_0 - 1)/2} + \frac{C}{qT(x,z_1)} \int \left[ \int_{e^{K/4}|z_2 - z_1|}^{\infty} e^{-a^2/2} da - C \left[ \log(e^{K/4}|z_2 - z_1|) \wedge 0 \right] \right]
\]

\[
+ \frac{1}{qT-2K(y,w)} e^{-\lambda_0(T-2K)} e^{y^2/8} e^{w^2/8} \int_{s^*}^{T-2K-s^*} \left[ \frac{e^{-\frac{K+u}{2}|z_2 - z_1|^2/4}}{e^{-\frac{K+u}{4}|z_2 - z_1|}} \wedge 1 \right] du
\]

\[
+ \left( \int_0^{s^*} + \int_{T-2K-s^*}^{T-2K} \right) P_y^Y (|Y_u| \geq e^{K+u/4}|z_2 - z_1| \big| Y_{T-2K} = w) du \right]
\]

\[
\times q_K(x,w)qT-2K(w,y)qK(y,z_1) \ dm(w) \ dm(y).
\]  

(2.6.17)

Note that the first two terms in the integral with respect to \( y \) and \( w \) are independent of these variables. We can therefore integrate them out; using the fact that

\[
\int \int q_K(x,w)qT-2K(w,y)qK(y,z_1) \ dm(w) \ dm(y) = q_T(x,z_1)
\]
(and an obvious cancellation) we obtain that

\[
|E^Y_x (\hat{Z}_T(Y, \lambda'/\lambda) - Z^q_T(Y, \lambda'/\lambda, K) | Y_T = z_1)|
\]

\[
\leq C e^{-K(2\lambda_0 - 1)/2} + C \int_0^\infty e^{-a^2/2da} - C \left[ \log(e^{K/4}|z_2 - z_1|) \wedge 0 \right]
\]

\[
+ \frac{C}{q_T(x, z_1)} e^{-\lambda_0(T-2K)} \int e^{y^2/8} e^{w^2/8} q_K(x, w) q_K(y, z_1) dm(w) dm(y)
\]

\[
\times \left( \int_{s^*}^{T-2K-s^*} \left[ \frac{e^{-K/4 |z_2 - z_1|^2/4}}{e^{K/4 |z_2 - z_1|^2/4}} \wedge 1 \right] du \right)
\]

\[
+ \frac{C}{q_T(x, z_1)} \int \int \left[ \left( \int_0^{s^*} + \int_{T-2K-s^*}^{T-2K} \right) P^Y_y (|Y_u| \geq e^{K/4}|z_2 - z_1| | Y_{T-2K} = w) du \right]
\]

\[
\times q_K(x, w) q_T(x, z_1) dm(w) dm(y)
\]

\[
=: \delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5, \tag{2.6.18}
\]

where \( \delta_i = \delta_i(T, K, z_1, z_2) \) and is defined by the obvious correspondence. We first note that

\[
\delta_i(T, K, z_1, z_2) \to 0 \text{ as } K \to \infty \text{ (uniformly in } T \geq 2K \text{) for } i = 1, 2, 3.
\]

Turning to \( \delta_4 \) and \( \delta_5 \), we observe that by Lemma 2.2.2, \( e^{\lambda_0 T} q_T(x, z_1) \to \psi_0(x) \psi_0(z_1) \) as \( \lambda \to \infty \). Since \( T \to q_T(x, z_1) \) is continuous, \( q_T(x, z_1) > 0 \) for all \( T \geq \tau(s) \) and \( \psi_0(x) \psi_0(z_1) > 0 \), this implies that there exists \( \beta(x, z_1) = \beta > 0 \) such that

\[
q_T(x, z_1) \geq \beta e^{-\lambda_0 T} \psi_0(x) \psi_0(z_1) \quad \forall \ T \geq \tau(s). \tag{2.6.19}
\]

Applying (2.2.2) twice with \( \delta = 1/8 \) and using (2.6.19), we have

\[
|\delta_4(T, K, z_1, z_2)|
\]

\[
\leq \frac{C \beta^{-1}}{\psi_0(x) \psi_0(z_1)} e^{2\lambda_0 K} e^{-2\lambda_0 K} e^{(x^2 + z_1^2)/8} \int e^{y^2/4} e^{w^2/4} dm(w) dm(y)
\]

\[
\times \left( \int_{s^*}^{T-2K-s^*} \frac{e^{-K/4 |z_2 - z_1|^2/4}}{e^{K/4 |z_2 - z_1|^2/4}} du \right)
\]

\[
\leq \frac{C e^{(x^2 + z_1^2)/8}}{\psi_0(x) \psi_0(z_1)} \left( \int_{s^*}^{T-2K-s^*} \frac{e^{-K/4 |z_2 - z_1|^2/4}}{e^{K/4 |z_2 - z_1|^2/4}} du \right)
\]

\[
= \frac{4C e^{(x^2 + z_1^2)/8}}{\psi_0(x) \psi_0(z_1)} \int_{s^*/4 e^{K/4 |z_2 - z_1|}}^{\infty} \frac{e^{-a^2/4}}{a^2} da, \tag{2.6.21}
\]

where the last line follows from a simple substitution. Thus we have \( \delta_4(T, K, z_1, z_2) \to 0 \) as
$K \to \infty$, and again we note that convergence is uniform in $T > 2K$. It remains to handle $\delta_5$. By three applications of (2.2.2) with $\delta = 1/8$ and (2.6.19), we have

$$|\delta_5(T, K, z_1, z_2)| \leq \frac{C}{\psi_0(x)\psi_0(z_1)} \iint \left[ \left( \int_0^{s^*} + \int_{T-2K-s^*}^T \right) P_y(|Y_u| \geq e^{\frac{K+u}{2}}|z_2 - z_1| | Y_{T-2K} = w) \, du \right]$$

$$\times e^{x^2/8}e^{w^2/4}e^{y^2/4}e^{s^2/8}dm(w) \, dm(y).$$

The square bracketed term vanishes as $K \to \infty$ uniformly in $T \geq 2K + s^*(1/8)$ by Lemma 2.6.1(b). The probabilities are bounded so the integrand obviously has a uniformly integrable upper bound. By Dominated Convergence, we have that $\delta_5(T, K, z_1, z_2) \to 0$ as $K \to \infty$, uniformly in $T \geq 2K + s^*(1/8)$. We have therefore shown that $\sum_{i=1}^5 \delta_i(T, K, z_1, z_2)$ is arbitrarily small as $K \to \infty$, uniformly in $T \geq 2K + s^*(1/8)$ and in $\lambda' > \lambda(s)$, where we recall that we have assumed $\lambda, \lambda' > \bar{\lambda}(s)$. From (2.5.33), $\lambda > \bar{\lambda}(s)$ is equivalent to $T > \tau(s)$. As $\tau(s) \geq s^*(1/8)$, $T \geq 2K + \tau(s)$ implies that $T \geq 2K + s^*(1/8)$. Thus by (2.6.18) we have proved the following. Recall that $\tilde{Z}_T(Y, \lambda'/\lambda) = \tilde{Z}_T(Y, z_1, z_2, \lambda'/\lambda)$.

**Lemma 2.6.2.** For all $x, z_1, z_2 \in \mathbb{R}$ such that $z_1 \neq z_2$,

$$\delta^{\infty}_a(K) = \sup_{T \geq 2K + \tau(s), \lambda' > \bar{\lambda}(s)} |E_x^Y (\tilde{Z}_T(Y, \lambda'/\lambda) - Z^a_T(Y, \lambda'/\lambda, K) | Y_T = z_1)|$$

satisfies $\lim_{K \to \infty} \delta^{\infty}_a(K) = 0$.

Given this lemma, it suffices to find the limit of (the conditional expectation of) $Z^a_T(Y, \lambda'/\lambda, K)$, and so $A_2$ has been replaced by 1.

Next we consider $A_3(y, z_1, \lambda, \lambda', K)$, which we recall from (2.6.5) is defined as

$$E_y^Y \left( \exp \left( \int_0^K F(Y_u) - H_{e^{K+u}}(Y_u, Y_u + e^{\frac{K+u}{2}}(z_2 - z_1)) \, du \right) | Y_K = z_1 \right).$$

We will show that in the limit as $\lambda, \lambda' \to \infty$, the integrand will be $F - F_2$. Define $A_3^*(y, z_1, K)$ by

$$A_3^*(y, z_1, K) = E_y^Y \left( \exp \left( \int_0^K F(Y_u) - F_2(Y_u, Y_u + e^{\frac{K+u}{2}}(z_2 - z_1)) \, du \right) | Y_K = z_1 \right).$$

(2.6.22)

The difference between the integrands of $A_3$ and $A_3^*$ is equal to $(F_2 - H_{e^{K+u}})(Y_u, Y_u + e^{\frac{K+u}{2}}(z_2 - z_1))$, which is non-negative by monotonicity. To obtain an upper bound we apply
Lemma 2.3.2 to obtain

\[
(F_2 - H_{e^{T-K+u}}^{X/Y}(Y_u, Y_u + e^{K-u/2}(z_2 - z_1))) \
\leq C \left[ e^{-(T-K+u)(2\lambda_0-1)/2} + \left( \frac{\lambda'}{\lambda} \right)^{-(2\lambda_0-1)} e^{-(T-K+u)(2\lambda_0-1)/2} \right] \
\leq C e^{(K-u)(2\lambda_0-1)/2(t-s)} \frac{1}{(2\lambda_0-1)/2} \left[ \lambda^{-(2\lambda_0-1)} + \lambda'^{-(2\lambda_0-1)} \right].
\]

(2.6.23)

The first line uses the definition of \( H^c \), which we recall from (2.5.12), and in the second line we have used that \( T = \log(\lambda^2(t-s)) \). Since \( \lambda, \lambda' > \tilde{\lambda}(s) \geq 1 \), the last expression in (2.6.23) is bounded by \( C e^{K/2(t-s)}(2\lambda_0-1)/2 \) for all \( u \in [0, K] \). Thus, using \( |e^x - e^y| \leq (e^x \vee e^y)|x-y| \) and (2.6.23), we have

\[
|A_3^b(y,z_1,K) - A_3(y,z_1,\lambda,\lambda',K)| \
\leq \exp \left( C K e^{K/2(t-s)-(2\lambda_0-1)/2} \right) \
\times E^Y_x \left( \int_0^K (F_2 - H_{e^{T-K+u}}^{X/Y}(Y_u, Y_u + e^{K/2}(z_2 - z_1))) \, du \bigg| Y_K = z_1 \right) \
\leq \exp \left( C K e^{K/2(t-s)-(2\lambda_0-1)/2} (t-s)^{(2\lambda_0-1)/2} \right) \
\times \left[ \lambda^{-(2\lambda_0-1)} + \lambda'^{-(2\lambda_0-1)} \right] \int_0^K C e^{(K-u)(2\lambda_0-1)/2} \, du \
\leq C(K,t-s) \left[ \lambda^{-(2\lambda_0-1)} + \lambda'^{-(2\lambda_0-1)} \right].
\]

(2.6.24)

for some constant \( C(K,t-s) > 0 \). Define \( Z^b_T(Y,\lambda',\lambda,K) \) as we defined \( Z^b_T(Y,\lambda'/\lambda,K) \) in (2.6.7) but with \( F - F_2 \) replacing the integrand in the second term. That is,

\[
Z^b_T(Y,\lambda'/\lambda,K) = \exp \left( \int_0^K F(Y_u) - H_{e^{T-K+u}}^{X/Y}(Y_u, Y_u + e^{T-K+u/2}(z_2 - z_1))) \, du \right) \
\times \exp \left( \int_{T-K}^T F(Y_u) - F_2(Y_u, Y_u + e^{T-K+u/2}(z_2 - z_1))) \, du \right).
\]

(2.6.25)

In particular, we have

\[
E^Y_x \left( Z^b_T(Y,\lambda'/\lambda,K) \bigg| Y_T = z_1 \right) \
= \iint A_1(x, w, \lambda, \lambda', K) A_3^b(y,z_1,K) \frac{q_K(x,w) q_T-2K(w,y) q_K(y,z_1)}{q_T(x,z_1)} \, dm(w) \, dm(y).
\]

(2.6.26)

Because (2.6.24) is uniform in \( y \) and \( z_1 \) and \( |A_1| \leq C_Z \), we can integrate out the transition densities to obtain the following.
Lemma 2.6.3. For $K > 0$ and $s \in [0, t)$, there is a constant $C(K, t - s)$ such that

$$
\delta^b_3(K, \lambda, \lambda') = |E^Y_x(Z^b_T(Y, \lambda'/\lambda, K) - Z^b_T(Y, \lambda'/\lambda, K) \mid Y_T = z_1)|
$$

$$
\leq C(K, t - s) \left[ \lambda^{-(2\lambda_0-1)} + \lambda'^{-(2\lambda_0-1)} \right]
$$

for all $\lambda, \lambda' > \bar{\lambda}(s)$.

We now analyse $A^*_3$ in greater detail. In particular, we perform a time reversal on the process $Y$. By (2.2.12), we have

$$
A^*_3(y, z_1, K) = E^Y_{z_1}(\exp \left( \int_0^K F(Y_u) - F_2(Y_u, Y_u + e^{\frac{z_2 - z_1}{2}}) du \right) \mid Y_K = y).
$$

This is the term that in (2.6.1) we claimed converges to $E^Y_{z_1}(W_\infty(Y, z_2))$, defined in (2.5.39), in the limit. However, the above expectation is still conditional on the endpoint. We now show that the contribution from the tail of the integral is vanishing, making the quantity asymptotically independent of the endpoint $y$. Let $0 < M < K$. Define $A^*_3(y, z_1, M, K)$ by truncating the integral in (2.6.22) at time $M$. That is,

$$
A^*_3(y, z_1, M, K) = E^Y_{z_1}(\exp \left( \int_0^M F(Y_u) - F_2(Y_u, Y_u + e^{\frac{z_2 - z_1}{2}}) du \right) \mid Y_K = y). \ (2.6.27)
$$

We now define $Z^c(Y, \lambda'/\lambda, M, K)$ by truncating the corresponding integral in $Z^b(Y, \lambda'/\lambda, K)$ (the integral over $[T - K, T]$ in (2.6.25) becomes the integral over $[T - M, T]$) so that $A^*_3(y, z_1, M, K)$ replaces $A^*_3(y, z_1, K)$ in the conditional expectation.

Lemma 2.6.4. For all $x, z_1, z_2 \in \mathbb{R}$ such that $z_1 \neq z_2$,

$$
\delta^b_3(M) = \sup_{K \geq M + s^*} \sup_{T \geq 2K + \tau(s)} \sup_{\lambda' > \bar{\lambda}(s)} |E^Y_x(Z^b_T(Y, \lambda'/\lambda, K) - Z^b_T(Y, \lambda'/\lambda, M, K) \mid Y_T = z_1)|
$$

satisfies $\lim_{M \to \infty} \delta^b_3(M) = 0$.

Proof. Using the inequality $|e^{-x} - e^{-y}| \leq |x - y|$ for $x, y \geq 0$, we have

$$
|A^*_3(y, z_1, K) - A^*_3(y, z_1, M, K)|
$$

$$
\leq E^Y_{z_1}(\left( \int_M^K |F(Y_u) - F_2(Y_u, Y_u + e^{u/2}(z_2 - z_1))| du \right) \mid Y_K = y). \ (2.6.28)
$$

By Proposition 2.3.1(b), the absolute value of the above integrand is at most $F(Y_u + e^{u/2}(z_2 - z_1))$ (for a similar argument see (2.6.11)). Exchanging expectation and integration, we proceed as in (2.6.14), (2.6.15), and (2.6.16), and apply Lemma 2.6.1 to bound (2.6.28).
above by
\[
c_1 \int_0^{K-M} \left( 1 + e^{M+u/4} |z_2 - z_1| \right) e^{-e^{M+u/4} |z_2 - z_1|^2/2} \, du \\
+ F(0) \int_M^K P_{z_1}^Y (|Y_u| \geq e^{u/4} |z_2 - z_1| \, |Y_K = y) \, du \\
\leq 4c_1 \int_0^{\infty} (1 + a^{-1}) e^{-a^2/2} \, da \\
+ Ce^{-\lambda_0 K} q_K(z_1, y) e^{(z_1^2 + y^2)/8} \int_0^{K-M-s^*} \left[ e^{-e^{M+u/4} |z_2 - z_1|^2} / e^{M+u/4} |z_2 - z_1|^2 \right] \, du \\
+ C \int_{K-s^*}^{K} P_{z_1}^Y (|Y_u| \geq e^{M+u/4} |z_2 - z_1| \, |Y_K = y) \, du. \tag{2.6.29}
\]

Expanding in terms of transition densities and using \(|A_1| \leq C_Z\), we have
\[
\left| E_{z_1}^Y \left( Z_T^b(Y, \lambda'/\lambda, K) - Z_T^c(Y, \lambda'/\lambda, M, K) \mid Y_T = z_1 \right) \right| \\
\leq \frac{C}{q_T(x, z_1)} \int |A_3^s(y, z_1, K) - A_3^s(y, z_1, M, K)q_T(x, y) q_K(y, z_1) \, dm(y).
\]

Using (2.6.29) as an upper bound for the integrand, we obtain an expression which closely resembles (2.6.18); in particular, four terms appear, directly corresponding to \(\delta_2, \delta_3, \delta_4\) and \(\delta_5\) of that expression. Moreover, they can be handled using the exact same arguments, as in the proof of Lemma 2.6.2, but with \((M, K)\) playing the roles of \((K, T)\). Because the arguments are the same, we omit them.

We now establish the limit of \(A_3^s(y, z_1, M, K)\) as \(K \to \infty\). Recalling (2.6.27) and the definition of \(W_M\) in (2.5.39), we have
\[
A_3^s(y, z_1, M, K) = E_{z_1}^Y \left( \exp \left( \int_0^M F(Y_u) - F_2(Y_u, Y_u + e^{u/2}(z_2 - z_1)) \, du \right) \bigg| Y_K = y \right) \\
= E_{z_1}^Y (W_M(Y, z_2) \mid Y_K = y).
\]
The functional \(W_M(Y, z_2)\) is a bounded continuous function of \(Y_{[0, M]}\). By Lemma 2.2.3(b), we have
\[
\forall M > 0 \lim_{K \to \infty} A_3^s(y, z_1, M, K) = E_{z_1}^{Y, \infty} (W_M(Y, z_2)) \tag{2.6.30}
\]
We define $Z^d_T(Y, \lambda'/\lambda, M, K)$ by

$$Z^d_T(Y, \lambda'/\lambda, M, K) := \exp \left( \int_0^K F(Y_u) - H^\lambda_{e^u} (Y_u, Y_u + e^u (z_2 - z_1)) \, du \right) E_{z_1}^{Y,\infty} (W_M(Y, z_2)). \tag{2.6.31}$$

Note that the second term is now deterministic; it no longer depends on the original Ornstein-Uhlenbeck process $Y$ or the spatial variable $y$. We then have

$$E_x^Y (Z^d_T(Y, \lambda'/\lambda, M, K) \mid Z_T = z_1) = E_{z_1}^{Y,\infty} (W_M(Y, z_2)) \int A_1(x, w, \lambda, \lambda', M, K) \frac{q_K(x, w) q_{\lambda - K}(w, z_2)}{q_T(x, z_1)} \, dm(w).$$

Bounding $A_1 \leq C_Z$ and integrating out the transition densities, by (2.6.30) we obtain the following.

**Lemma 2.6.5.** For all $x, z_1, z_2 \in \mathbb{R},$

$$\delta^d(M, K) = \sup_{T \geq 2K + \tau(s), \lambda > \lambda(s)} \left| E_x^Y (Z^d_T(Y, \lambda'/\lambda, M, K) - Z^d_T(Y, \lambda'/\lambda, M, K) \mid Y_T = z_1) \right|$$

satisfies $\lim_{K \to \infty} \delta^d(M, K) = 0$ for each fixed $M > 0$.

From our starting expression for $\tilde{Z}_T(Y, \lambda'/\lambda)$ in (2.6.6), all that remains to be handled in $Z^d_T(Y, \lambda'/\lambda, M, K)$ is the $A_1$ term, whose definition we recall from (2.6.3) is

$$A_1(x, w, \lambda, \lambda', K) = E_x^Y \left( \exp \left( \int_0^K F(Y_u) - H^\lambda_{e^u} (Y_u, Y_u + e^u (z_2 - z_1)) \, du \right) \bigg| Y_K = w \right).$$

The dominant contribution to the integral in $A_1$ resembles $F(Y_u) - V_1 e^{\alpha/2}(Y_u)$. By Proposition 2.3.1 we have the following upper and lower bounds for the difference of this quantity and the integrand:

$$0 \leq \left[ F(Y_u) - V_1 e^{\alpha/2}(Y_u) \right] - \left[ F(Y_u) - H^\lambda_{e^u} (Y_u, Y_u + e^u (z_2 - z_1)) \right] \leq F(Y_u + e^u (z_2 - z_1)). \tag{2.6.32}$$

Recall from (2.3.16) that $Z_K(Y)$ is defined as

$$Z_K(Y) = \exp \left( \int_0^K F(Y_u) - V_1 e^{\alpha/2}(Y_u) \, du \right).$$

Because the exponentials in both $A_1$ and $Z_K(Y)$ are bounded above by $C_Z$, by (2.6.32) we
have

\[ |A_1(x, w, \lambda, \lambda', K) - E_x^Y(Z_K(Y) \mid Y_K = w)| \]
\[
\leq C_Z E_x^Y \left( \int_0^K F(Y_u + e^{T-x_u}(z_2 - z_1)) \, du \right) \bigg| Y_K = w \bigg). \quad (2.6.33)
\]

We define \( Z^c(Y, M, K) \) by

\[ Z^c(Y, M, K) = Z_K(Y) \times E_{z_1}^{Y,\infty}(W_M(Y, z_2)). \quad (2.6.34) \]

Using (2.6.33) and proceeding as in the proofs of Lemmas 2.6.2 and 2.6.4, we obtain the following.

**Lemma 2.6.6.** For all \( x, z_1, z_2 \in \mathbb{R} \) such that \( z_1 \neq z_2 \),

\[ \delta^d_c(T, M, K) = \sup_{\lambda' > \bar{\lambda}(s)} |E_x^Y(Z_T^d(Y, \lambda'/\lambda, M, K) - Z^c(Y, M, K) \mid Y_T = z_1)| \]

satisfies \( \lim_{T \to \infty} \delta^d_c(T, M, K) = 0 \) for all fixed \( M \) and \( K \) such that \( 0 < M < K \).

From (2.6.34), we have

\[ E_x^Y(Z^c(Y, M, K) \mid Y_T = z_1) = E_x^Y(Z_K(Y) \mid Y_T = z_1) E_{z_1}^{Y,\infty}(W_M(Y, z_2)). \]

Thus by Lemma 2.2.3(b) and the fact that \( Z_K(Y) \leq C_Z \) (and is a continuous function of \( Y \)) we have the following.

**Lemma 2.6.7.** For all \( x, z_1, z_2 \in \mathbb{R} \),

\[ \delta^c_f(T, M, K) = \left| E_x^Y(Z^c(Y, M, K) \mid Y_T = z_1) - E_x^{Y,\infty}(Z_K(Y)) E_{z_1}^{Y,\infty}(W_M(Y, z_2)) \right| \]

satisfies \( \lim_{T \to \infty} \delta^c_f(T, M, K) = 0 \) for all fixed \( M \) and \( K \) such that \( 0 < M < K \).

We are now ready to establish the limiting form of \( E_x^Y(\tilde{Z}_T(Y, \lambda'/\lambda) \mid Y_T = z_1) \) (provided \( z_1 \neq z_2 \)). Let \( M > 0, K > M, T \geq 2K + \tau(s) \) and \( \lambda' > \bar{\lambda}(s) \). Bounding above by the sum of the \( \delta \) terms in Lemmas 2.6.2-2.6.7, we have that

\[ \left| E_x^Y(\tilde{Z}_T(Y, \lambda'/\lambda) \mid Y_T = z_1) - E_x^{Y,\infty}(Z_K(Y)) E_{z_1}^{Y,\infty}(W_M(Y, z_2)) \right| \]
\[
\leq \delta^c_a(K) + \delta^c_b(K, \lambda, \lambda') + \delta^c_c(M) + \delta^c_f(T, M, K) + \delta^c_e(T, M, K). \quad (2.6.35)
\]

Let \( \epsilon > 0 \). By Lemma 2.6.4, we can choose \( M_0 > 0 \) to be sufficiently large such that \( \delta^c_c(M) < \epsilon/4 \) for all \( M \geq M_0 \), and choose some \( M \geq M_0 \). By Lemma 2.6.2 and Lemma 2.6.5,
we can then choose $K_0$ to be large enough such that $\delta_{\epsilon}^g(K) + \delta_{\epsilon}^g(M, K) < \epsilon/4$ for all $K \geq K_0$. Fix $K > K_0$. Next, by Lemmas 2.6.6 and 2.6.7 we can choose $T_0 > 2K + \tau(s)$ such that for all $T \geq T_0$, $\delta^d(T, M, K) + \delta^d_j(T, M, K) < \epsilon/4$. Finally, Lemma 2.6.3 allows us to choose $\lambda(\epsilon) > \bar{\lambda}(s)$ such that $T = \log(\lambda^2(t-s)) \geq T_0$ and $\delta^d(\epsilon) < \epsilon/4$ for all $\lambda, \lambda' \geq \lambda(\epsilon)$.

We therefore obtain from (2.6.35) that

$$\limsup_{\lambda, \lambda' \to \infty} \left| E_x^Y(\tilde{Z}_T(Y, \lambda'/\lambda) \mid Y_T = z_1) - E_x^Y(\tilde{Z}_T(Y) \rangle) E_{x_{z_1}}^Y(W_M(Y, z_2)) \right| < \epsilon$$

for the $M$ and $K$ chosen above. This holds for all $\epsilon > 0$ for sufficiently large $M$ and $K$ (with $M < K$). It therefore holds that if $\lim_{M, K \to \infty, K > M} E_x^{Y, \infty}(Z_K(Y)) E_{x_{z_1}}^{Y, \infty}(W_M(Y, z_2))$ exists, then $\lim_{\lambda, \lambda' \to \infty} E_x^Y(\tilde{Z}_T(Y, \lambda'/\lambda) \mid Y_T = z_1)$ exists and is equal to it. Thus it suffices to find the limit of $E_x^{Y, \infty}(Z_K(Y)) E_{x_{z_1}}^{Y, \infty}(W_M(Y, z_2))$ as $M, K \to \infty$ with $M < K$. As the first term depends only on $K$ and the second depends only on $M$, we can consider the limits independently. First consider $E_x^{Y, \infty}(Z_K(Y))$. By (2.3.18), $Z_K \uparrow Z_\infty \leq C_\infty$, so the limit of the first term as $K \to \infty$ is $E_x^{Y, \infty}(Z_\infty(Y))$ by Monotone Convergence. We recall the definition of $W_M$ from (2.5.39). The integral in $W_M$ is monotone in $M$ and hence converges to the integral on $[0, \infty]$ as $M \to \infty$. Using the fact that $|W_M(Y, z_2)| \leq 1$ for all $M$ and continuity of the exponential, we can bring the limit inside, and $E_{x_{z_1}}^{Y, \infty}(W_M(Y, z_2)) \to E_{x_{z_1}}^{Y, \infty}(W_\infty(Y, z_2))$ as $M \to \infty$. Thus we have shown that

$$\lim_{\lambda, \lambda' \to \infty} E_x^Y(\tilde{Z}_T(Y, \lambda'/\lambda) \mid Y_T = z_1) = E_x^{Y, \infty}(Z_\infty(Y)) E_{x_{z_1}}^{Y, \infty}(W_\infty(Y, z_2)).$$

This is (2.6.1), which is what we wanted to show, so the proof of Lemma 2.5.5 is complete.
Chapter 3

The boundary local time of super-Brownian motion II: 0-1 law

**Summary.** If $X(t, x)$ is the density of one-dimensional super-Brownian motion, we prove that

$$\dim(\partial \{x : X(t, x) > 0\}) = 2 - 2\lambda_0 \in (0, 1) \text{ a.s. on } \{X_t \neq 0\},$$

where $-\lambda_0 \in (-1, -1/2)$ is the lead eigenvalue of a killed Ornstein-Uhlenbeck process. This confirms a conjecture of Mueller, Mytnik and Perkins [69] who proved the above with positive probability. To establish this result we derive some new basic properties of the boundary local time introduced in Chapter 2 analyse the behaviour of $X(t, \cdot)$ near the upper edge of its support. Numerical estimates of $\lambda_0$ suggest that the above Hausdorff dimension is approximately .224.
3.1 Introduction and statement of results

Let $(X_t, t \geq 0)$ denote a super-Brownian motion on the line starting at $X_0 \neq 0$ under $P_{X_0}^X$. Here $X_0 \in \mathcal{M}_F(\mathbb{R})$, the space of finite measures on $\mathbb{R}$ with the topology of weak convergence, and $P_{X_0}^X$ will denote any probability under which $X$ has the above law. Our branching rate is chosen to be one so that the jointly continuous density, $X(t, x)$, of $X_t$ for $t > 0$, is the unique in law solution of the stochastic partial differential equation (SPDE)

$$\frac{\partial X}{\partial t}(t, x) = \frac{1}{2} \frac{\partial^2 X}{\partial x^2}(t, x) + \sqrt{X(t, x)} \dot{W}(t, x), \quad X \geq 0, \quad X(0) = X_0 \quad (3.1.1)$$

(see Section III.4 of [83]). Here $\dot{W}$ is a space-time white noise on $[0, \infty) \times \mathbb{R}$, and the initial condition means that $X_t(dx) = X(t, x)dx \to X_0(dx)$ in $\mathcal{M}_F(\mathbb{R})$ as $t \downarrow 0$.

The boundary of the zero set of $X_t$,

$$\partial Z_t = \partial\{x : X(t, x) = 0\} = \partial\{x : X(t, x) > 0\}, \quad (3.1.2)$$

was studied in [69]. The increased regularity of $X$ on and near this set has played an important role in the study of SPDE such as (3.1.1) (see [74] and [72]). Mytnik and Perkins (unpublished) had obtained side conditions on $X$ which would give pathwise uniqueness in (3.1.1) but which would imply that $\dim(\partial Z_t)$, the Hausdorff dimension of $\partial Z_t$, is zero. The intuition here is that solutions to (3.1.1) should only separate in their respective zero sets since these are the only points at which the noise coefficient is non-Lipschitz. So the smaller this set is, the harder it will be for solutions to separate. In [69] it was shown that if $-\lambda_0 \in (-1, -1/2)$ is the lead eigenvalue of the killed Ornstein-Uhlenbeck operator described below, then (see Theorem 1.3 of [69]) in fact

$$P_{X_0}^X(\dim(\partial Z_t) = 2 - 2\lambda_0) > 0. \quad (3.1.3)$$

Here it was also conjectured (see the comment following Theorem 1.3 in [69]) that

$$\dim(\partial Z_t) = 2 - 2\lambda_0 \text{ a.s. on } \{X_t \neq 0\}. \quad (3.1.4)$$

In any case, the rigorous bounds on $\lambda_0$ mentioned above imply the dimension of $\partial Z_t$ is in $(0, 1)$, at least with positive probability, and the aforementioned pathwise uniqueness problem remains unresolved in spite of a recent negative result in Chen [9]. Here pathwise non-uniqueness to (3.1.1) was shown if an innocent looking immigration term of the form $\psi(x)$ ($\psi$ smooth, non-negative and compactly supported) is added to the right-hand side of (3.1.1). The immigration term, however, gives $\partial Z_t$ positive Lebesgue measure and this is what allows Chen to establish separation of solutions.
The boundary set itself is rather delicate as small perturbations of $X$ will of course completely change the nature of $\partial Z_t$. In particular, it is a non-monotone function of the initial condition. This is one reason some of the standard zero-one arguments (see, e.g., the proof of Theorem 1.3 of [80] for the dimension of the range of $X$) were not able to resolve the conjecture (3.1.4). Our main result (Theorem (3.1.1) below) will use the boundary local time $L_t(dx)$ of $\partial Z_t$, which was constructed in Chapter 2, to confirm (3.1.4). The local time is a random measure supported by $\partial Z_t$ which we are just beginning to understand, and some of its basic properties derived here will play a central role in our arguments. As a random measure supported on the set of points where solutions to (3.1.1) can separate, $L_t(dx)$ has the potential of playing the same role in the study of SPDE arising from population models that ordinary local time does for stochastic differential equations. Of course, one would need to construct $L_t$ for a much larger class of random processes.

Numerical estimates of $\lambda_0$ due to Peiyuan Zhu suggest that (3.1.4) implies

$$\dim(\partial Z_t) \approx .224 \text{ a.s. on } \{X_t(1) > 0\},$$

perhaps larger than one may think given that $X(t, \cdot)$ is Hölder $1 - \eta$ in space near its zero set for any $\eta > 0$ (see Theorem 2.3 in [72]). We briefly discuss this approximation below and give some evidence for the accuracy of the estimate to the digits given.

It will often be more convenient to work with the canonical measure of super-Brownian motion, $N_x$, which is a more fundamental object in many ways. Recall that $X$ arises as the scaling limit of the empirical measures of critical branching random walk. $N_x$ is a $\sigma$-finite measure on $C([0, \infty), \mathcal{M}_F(\mathbb{R}))$ (the space of continuous measure-valued paths) describing the behaviour of the descendants of a single ancestor at $x$ at time $0$ (see Theorem II.7.3 of [83]). A super-Brownian motion under $P^X_{X_0}$ may be constructed as the integral of a Poisson point process with intensity $N_{X_0}(\cdot) = \int N_x(\cdot) dX_0(x)$ (see (3.2.5) below). In particular, if we write $X_t(\phi) = \int \phi(x) X_t(dx)$, then for $\phi \geq 0$,

$$E^{X}_{X_0}(e^{-X_t(\phi)}) = \exp\left(-\int 1 - e^{-\nu(\phi)} dN_{X_0}(\nu)\right).$$

(3.1.6)

Our next job is to describe $\lambda_0$ more carefully. We let

$$F(x) = -\log(P^X_{\delta_0}(X(1, x) = 0)) = N_0(X(1, x) > 0),$$

(3.1.7)

where the last equality is a simple consequence of (3.1.6) with $\phi = \infty \delta_x$ and $X_0 = \delta_0$ (see Proposition 3.3 of [69]). Then $F$ is the unique positive symmetric $C^2$ solution to

$$\frac{F''}{2}(y) + \frac{y}{2} F'(y) + F(y) - \frac{F(y)^2}{2} = 0,$$

(3.1.8)
and
\[ F'(0) = 0, \quad \lim_{y \to \infty} y^2 F(y) = 0. \]  
(3.1.9)

(See (1.10) and (1.12) of [69] and the discussion in Section 3 of the same reference.) Let
\[ Af(y) = f''(y) - \frac{y}{2} f'(y) \]
be the generator of the Ornstein-Uhlenbeck process, \( Y \), on the line.

For \( \phi \in C([-\infty, \infty]) \), the space of continuous functions on \( \mathbb{R} \) with finite limits at \( \pm \infty \), we let
\[ A^\phi(f) = Af - \phi f \]
be the generator of the Ornstein-Uhlenbeck process \( Y^\phi \), now killed when
\[ \int_0^t \phi(Y_s) \, ds \]
exceeds an independent exponential mean one r.v. If \( m \) denotes the standard normal law on \( \mathbb{R} \), the resolvent of \( A^\phi \) is a Hilbert-Schmidt integral operator on the Hilbert space of square integrable functions with respect to \( m \), \( L^2(m) \). Therefore \( A^\phi \) has a complete orthonormal system of eigenfunctions \( \{ \psi_n^\phi : n \geq 0 \} \) with non-positive eigenvalues \( \{-\lambda_n^\phi\} \) ordered so that \(-\lambda_n^\phi \) decreases to \(-\infty \). The lead eigenvalue \(-\lambda_0^\phi \leq 0 \) is simple and so has a unique normalized eigenfunction \( \psi_0^\phi \). See Theorem 3.2.1 below for this and related information. If we set \( \phi = F \), then our eigenvalue \(-\lambda_0^F \) is \(-\lambda_0^F \), which is in \((-1, -1/2)\) by an elementary calculation in Proposition 3.4(b) of [69], using the fact (Proposition 3.4(b) of [69]) that
\[ \lambda_0^F/2 = 1/2. \]  
(3.1.10)

Here then is our main result.

**Theorem 3.1.1.** For any \( X_0 \in \mathcal{M}_F(\mathbb{R}) \setminus \{0\} \) and \( t > 0 \),
\[ \dim(\partial Z_t) = 2 - 2\lambda_0 \in (0, 1) \quad P_{X_0}^X - \text{a.s. and } \mathbb{N}_0 - \text{a.e.} \text{ on } \{X_t > 0\}. \]  
(3.1.11)

In fact Theorem 1.3(a) in [69] already gives
\[ \dim(\partial Z_t) \leq 2 - 2\lambda_0 \quad P_{X_0}^X - \text{a.s. and } \mathbb{N}_0 - \text{a.e.} \]  
(3.1.12)

Although the above reference only considers \( P_{X_0}^X \), the result for \( \mathbb{N}_0 \) then follows easily by the Poisson point process decomposition mentioned above (see (2.5) below), just as in the last six lines of the proof of Theorem 3.1.2 at the end of Section 3.4. Therefore it is the lower bound on \( \dim(\partial Z_t) \) that we must consider. The lower bound on the dimension was attained with positive probability in Theorem 5.5 of [69] by first deriving a sufficient capacity condition for \( \partial Z_t \) to intersect a given set, \( A \), with positive probability (Theorem 5.2 of [69]) and then taking \( A \) to be the range of an appropriate Lévy process. As was already noted, the authors were unable to use this approach to establish the lower bound a.s. The standard approach to lower bounds on Hausdorff dimension is through the energy method. That is,
first construct a finite random measure or local time, $L_t$, supported by $\partial Z_t$ such that

$$E\left(\int \int |x-y|^{-\alpha} dL_t(x)dL_t(y)\right) < \infty \quad \forall \ 0 < \alpha < 2 - 2\lambda_0. \tag{3.1.13}$$

The energy method (see Theorem 4.27 of [65]) would then imply

$$\dim(\partial Z_t) \geq 2 - 2\lambda_0 \text{ a.s. on } \{L_t \neq 0\}. \tag{3.1.14}$$

The existence of such a boundary local time was established in [35], confirming a construction conjectured in Section 5 of [69], which we briefly describe now. Define a measure $L_t^\lambda \in M_F(\mathbb{R})$ by

$$L_t^\lambda(\phi) = \int \phi(x) \lambda^{2\lambda_0} X(t,x) e^{-\lambda X(t,x)} dx \tag{3.1.15}$$

for bounded Borel functions $\phi$. Note that as $\lambda$ gets large $L_t^\lambda$ becomes concentrated on the set of points $x$ where $0 < X(t,x) = O(1/\lambda)$. The normalization of $\lambda^{2\lambda_0}$ comes from the left tail behaviour of $X(t,x)$ in Theorem 1.2 of [69]. The following result is taken from Chapter 2, more specifically it is included in Theorems 2.1.1, 2.1.2, 2.1.3 and 2.1.5, and Proposition 2.1.6.

**Theorem 3.A.**  
(a) There is a finite atomless random measure, $L_t$, on the line such that under $N_0$ or $P_{X_0}$,

$$L_t^\lambda \to L_t \text{ in measure in the metric space } M_F(\mathbb{R}) \text{ as } \lambda \to \infty.$$  

Moreover $L_t$ is supported on $\partial Z_t$ a.s.

(b) There is a positive constant $C_A$ such that for any Borel $\phi : \mathbb{R} \to [0, \infty)$,

$$\int L_t(\phi) dN_0 = C_A t^{-\lambda_0} \int \phi(\sqrt{t}z) \psi_{\lambda_0}^F(z) dm(z). \tag{3.1.16}$$

(c) (3.1.13) holds under both $N_0$ and $P_{X_0}$.

(d) There is a constant $C_B$ such that

$$\int L_t(1)^2 dN_0 \leq C_B t^{1-2\lambda_0}. \tag{3.1.17}$$

Let $S(X_t) = \{x : X(t,x) > 0\}$ be the closed support of $X_t$ and define $U_t = \sup(S(X_t))$ to be the upper most point of the support. It now follows from Theorem 3.A that (3.1.14) holds under both $P_{X_0}$ and $N_0$ (for the latter one can work under the probability $N_0(\cdot | X_t \neq 0)$). And so Theorem 3.1.1 is immediate from (3.1.12) and the following:
Theorem 3.1.2. Under the measures $\mathbb{N}_0$ and $P^X_{X_0}$, $L_t > 0$ almost surely on $\{X_t > 0\}$. In fact, almost surely on $\{X_t > 0\}$, $L_t((U_t - \delta, U_t)) > 0$ for all $\delta > 0$.

This theorem shows that as long as $X_t$ has not gone extinct, the part of $\partial Z_t$ at its upper edge will have positive $L_t$ measure, and, in particular, $L_t$ itself is not equal to the zero measure. It is natural to consider a local version of the above and show that $L_t$ will charge any open interval which contains points in $\partial Z_t$. This clearly fails (note from Theorem 3.A that $L_t$ is atomless) if $X_t(\cdot)$ has isolated zeros, which clearly would be in $\partial Z_t$. An elementary argument shows that $\partial S(X_t) \subseteq \partial Z_t$ and the former set clearly will not contain isolated zeros of $X_t(\cdot)$. Given that the existence of isolated zeros of $X_t(\cdot)$ remains unresolved (we conjecture that they do not exist), here then is our local version of Theorem 3.1.2:

Theorem 3.1.3. For $t > 0$, $P^X_{X_0}$-a.s. and $\mathbb{N}_0$-a.e., for any $a < b$, $(a, b) \cap \partial S(X_t) \neq \emptyset$ implies $L_t((a, b)) > 0$.

Evidently we do not know whether or not $\partial Z_t \setminus \partial S(X_t)$ is non-empty; isolated zeros are not the only possible points in this set—see Lemma 3.5.1 below. Nonetheless we make the following conjecture:

Conjecture. $L_t$ is supported on $\partial S(X_t)$ and so $\dim(\partial S(X_t)) = 2 - 2\lambda_0$ on $\{X_t \neq 0\}$ $P^X_{X_0}$-a.s. and $\mathbb{N}_0$-a.e.,

the last conclusion being immediate from the first by (3.1.12), Theorem 3.1.2, Theorem 3.A(a,b) and the energy method described above.

Corollary 3.1.4. For $t > 0$, $P^X_{X_0}$-a.s. and $\mathbb{N}_0$-a.e., for any $a < b$, $(a, b) \cap \partial S(X_t) \neq \emptyset$ implies $\dim(\partial Z_t \cap (a, b)) = 2 - 2\lambda_0$.

Proof. By considering rational values we may fix $a$ and $b$ and work under either $P^X_{X_0}$ or $\mathbb{N}_0(\cdot|X_t \neq 0)$. Assume $(a, b) \cap \partial S(X_t) \neq \emptyset$. In view of (3.1.13) we may apply the energy method to $L_t|_{(a,b)}$, which is a.s. non-zero by Theorem 3.1.3, and so conclude that $\dim(\partial Z_t \cap (a, b)) \geq 2 - 2\lambda_0$ a.s. on $\{(a, b) \cap \partial(S(X_t)) \neq 0\}$. The corresponding upper bound is immediate from (3.1.12).

We comment briefly on the numerical approximation of $\lambda_0$ carried out by Peiyuan Zhu in [96]. One first needs to numerically approximate $F$ using an ODE solver and the “shooting method” to find the minimal value of $c$ so that $F_c(0) = c$, $F'_c(0) = 0$ and $F_c$ satisfying (3.1.8) remains non-negative. It is known that $F_c = F$ (see, e.g., [6]). One then approximates this numerically generated $F$ by a linear combination of Gaussians $\hat{F}$ (with varying means and variances). We estimate $-\lambda_0^F$ by $-\lambda_0^\hat{F}$, the lead eigenvalue of the Ornstein-Uhlenbeck
operator with $\hat{F}$-killing on a large interval $[0,K]$ with Neumann boundary conditions. $K$ must be taken sufficiently large to approximate the corresponding operator on $[0,\infty)$. The final step is then to use CHEBFUN software to estimate $\lambda_0^F$. One could also obtain $\hat{F}$ by interpolating between the numerically generated grid points using Chebychev polynomials—the results agree to the given accuracy. We have some faith in the resulting approximation of $\lambda_0^F \approx .8882$ because if we replace $F$ with $F/2$, the same method leads to $\lambda_0^{F/2} \approx .5000$. This compares well with the exact (known) value in (3.1.10).

The proof of Theorem 3.1.2 includes some input from the semilinear pde’s associated with super-Brownian motion (such as (3.1.19) below) which are carried out in Section 3.3. This is then used in Section 3.4, to study $X_t(dx)$ near the upper end of its support, $U_t$. For $\epsilon > 0$, define

$$\tau^\epsilon = \tau^\epsilon(t) = \inf\{x \in \mathbb{R} : X_t([x,\infty)) < \epsilon\}. \quad (3.1.18)$$

In particular, if $X_t(1) < \epsilon$, then $\tau^\epsilon = -\infty$. The following result gives some insight into the behaviour of $X_t$ near the upper edge of its support and so the following first moment bound, which is proved in Section 3.4, may be of independent interest.

**Proposition 3.1.5.** There is a non-increasing function, $c_{3.1.5}(t)$, such that for all $t, \epsilon > 0$ and $u > 0$:

(a) For any $X_0 \in \mathcal{M}_F(\mathbb{R})$, $E^X_0 \left( \int_{\tau^\epsilon(t) - u}^{\tau^\epsilon(t) + u} X(t,x)dx \right) \leq c_{3.1.5}(t)X_0(1)(u^2 \vee \epsilon)$.

(b) $\mathbb{P}_0 \left( \int_{\tau^\epsilon(t) - u}^{\tau^\epsilon(t) + u} X_t(dx) \right) \leq c_{3.1.5}(t)(u^2 \vee \epsilon)$.

One can understand the important $u^2$ behaviour in the above for small $u, \epsilon$ from the improved modulus of continuity of $X(t, \cdot)$ near its zero set (mentioned above). Theorem 2.3 of [72] shows that for $\eta > 0$ there is $\delta(\omega) > 0$ so that $|X(t,x) - X(t,x + h)| \leq |h|^{1-\eta}$ for $X(t,x) \leq |h| \leq \delta(\omega)$. This readily leads to (for $\epsilon, u$ small) $X(t, \tau^\epsilon(t)) \leq \epsilon^{5-\eta}$ and after a short argument (consider $u \geq \epsilon^{5-\eta}$ and $u < \epsilon^{5-\eta}$ separately) that

$$\int_{\tau^\epsilon(t) - u}^{\tau^\epsilon(t) + u} X(t,x)dx = \int_{\tau^\epsilon(t) - u}^{\tau^\epsilon(t) + u} X(t,x)dx + \epsilon \leq c(\epsilon^{1-2\eta} + u^{2-\eta}),$$

which comes close to the above mean behaviour. The actual proof uses the unique non-negative solution, $v^\infty_t(x) = v^\infty(t,x)$, in $C^{1,2}((0,\infty) \times \mathbb{R})$ of

$$\frac{\partial v^\infty}{\partial t}(t,x) = \frac{1}{2} \frac{\partial^2 v^\infty}{\partial x^2} - \frac{(v^\infty)^2}{2}, \quad v^\infty_0 = \infty 1_{(-\infty,0)}.$$  \hspace{1cm} (3.1.19)

Such semilinear parabolic equations arise of course as exponential dual functions for super-Brownian motion—see Section 3.3 for more on this in general, and Theorem 3.3.3 for more information on the particular equation above, including its precise meaning. More specifically, the proof uses $G(x) = v^\infty_1(x)$ which also is the unique $C^\infty$ solution of (3.1.8) but now
with the boundary conditions (see Lemma 3.3.4(c))

\[
\lim_{x \to \infty} x^2 G(x) = 0, \quad \lim_{x \to -\infty} G(x) = 2. \tag{3.1.20}
\]

Using a Palm measure formula for \(X_t\) (Theorem 4.1.3 from [15]), the Feynman-Kac Formula and some pde bounds (notably Proposition 3.3.7), we show (see (3.4.10)) that for \(u^2 \geq \epsilon\) (from which the general case follows easily),

\[
\mathbb{N}_0 \left( \int_{\tau(t)-u}^\infty X(t, x) \, dx \right) \leq c(t) \mathbb{E}_m^Y \left( \exp \left( - \int_0^{\log(1/u^2)} G(Y_s) \, ds \right) \right), \tag{3.1.21}
\]

where \(Y\) is an (unkilled) Ornstein-Uhlenbeck process with initial law \(m\) under \(P^Y_m\). So, as in [69], one can use the spectral decomposition of \(A^G\) to see that the right-hand side of (3.1.21) is at most \(c(t) e^{-\lambda_0^G \log(1/u^2)} = c(t) u^{2\lambda_0^G}\). Unlike \(\lambda_F^0\), we can identify the eigenfunction for \(\lambda_0^G\) and verify that \(\lambda_0^G = 1\) (Proposition 3.3.5), and hence obtain the required bound in Proposition 3.1.5(b).

Turning to Theorem 3.1.2 itself, Theorem 3.A(b),(d) and the second moment method easily give (Lemma 3.4.1)

\[
\mathbb{N}_0(L_\delta([3\sqrt{\delta}, \infty)) > 0 | X_\delta \neq 0) \geq p > 0 \quad \forall \delta > 0.
\]

One can then use this to conclude that the right-most ancestor, say at \(x\), at time \(t - \delta\) of the population at time \(t\) will have descendants at time \(t\) with a positive boundary local time on \([x + 3\sqrt{\delta}, \infty)\) with conditional (on \(\mathcal{F}_{t-\delta}\)) probability at least \(p\). Now one must show that the descendants of the other ancestors at time \(t - \delta\) do not flood into the boundary region of the right-most ancestor and hence remove it from the overall boundary. This issue captures the delicate and non-monotone character of the boundary. To resolve it we use a classical hitting estimate for \(X\) from [14] (see Theorem 3.2.3 below) and Proposition 3.1.5. This will lead to a uniform lower bound on \(P^X_{X_0}(L_t > 0 | \mathcal{F}_{t-\delta})\) with high probability at least on \(\{X_t \neq 0\}\) and the martingale convergence theorem then shows \(L_t > 0\) with high probability on \(\{X_t \neq 0\}\).

Theorem 3.1.3 is proved in Section 3.5. Section 3.2 reviews a number of standard tools we will need in the proofs including the spectral decomposition of the killed Ornstein-Uhlenbeck processes, some cluster decompositions of super-Brownian motion based on historical information, and the aforementioned hitting estimate for super-Brownian motion.
3.2 Preliminaries

3.2.1 Killed Ornstein-Uhlenbeck processes

Recall that $Y$ is an Ornstein-Uhlenbeck process with generator $A$, starting at $x$ under $P_x^Y$ and starting with the standard normal law $m$ under $P_m^Y$. As above for $\phi \in C([-\infty, \infty])$, $\phi \geq 0$, $A^\phi$ is the generator of the Ornstein-Uhlenbeck process, $Y^\phi$, killed at time $\rho_\phi = \inf\{t : \int_0^t \phi(Y_s)ds > e\}$, where $e$ denotes an independent exponential r.v. with mean one. The result below is standard, and included in Theorem 2.3 of Mueller, Mytnik and Perkins [69].

**Theorem 3.2.1.** (a) $A^\phi$ has a complete orthonormal family $\{\psi_n : n \geq 1\}$ of $C^2$ eigenfunctions of $L^2(m)$ satisfying $A^\phi \psi_n = -\lambda_n \psi_n$, where $\{-\lambda_n\}_{n=1}^\infty$ is a non-increasing sequence of non-positive eigenvalues such that $\lambda_n \to \infty$. Furthermore, $-\lambda_0$ is a simple eigenvalue and $\psi_0 > 0$.

(b) Let $\theta = \int \psi_0 dm$. For all $0 < \delta$, there exists $c_\delta$ such that for all $x \in \mathbb{R}$,

\[
|e^{\lambda_0 t} P_x^Y(\rho_\phi > t) - \theta \psi_0(x)| \leq c_\delta e^{\delta x^2} e^{-(\lambda_1 - \lambda_0)t}, \tag{3.2.1}
\]

and

\[
\psi_0(x) \leq c_\delta e^{\delta x^2}. \tag{3.2.2}
\]

In particular,

\[
P_x^Y(\rho_\phi > t) \leq C e^{\delta x^2} e^{-\lambda_0 t}. \tag{3.2.3}
\]

and

\[
P_m^Y(\rho_\phi > t) \leq C e^{-\lambda_0 t}. \tag{3.2.4}
\]

3.2.2 Cluster and historical decompositions of super-Brownian motion

We recall the cluster decomposition of super-Brownian motion from Theorem 4 in Section IV.3 of [57]. If $X_0 \in \mathcal{M}_F(\mathbb{R})$, let $\Xi_{X_0}$ be a Poisson point process on the space $C([0, \infty), \mathcal{M}_F(\mathbb{R}))$ of continuous measure-valued paths with intensity $N_{X_0}(\cdot) = \int N_x(\cdot) dX_0(x)$. Then

\[
X_t(\cdot) = \begin{cases} 
\int \nu_t(\cdot) \Xi_{X_0}(d\nu) & \text{if } t > 0 \\
X_0(\cdot) & \text{if } t = 0
\end{cases} \tag{3.2.5}
\]

defines a super-Brownian motion with initial state $X_0$. In particular this shows that for $t > 0$, ($\overset{\text{D}}{=} \text{denotes equality in law}$)

\[
X_t \overset{\text{D}}{=} \sum_{i=1}^N X^i_t, \tag{3.2.6}
\]

125
where $N$ has a Poisson law with mean $2X_0(1)/t = \mathbb{N}_{X_0}(X_t > 0)$, and given $N$, $\{X_i^t : i \leq N\}$ are iid random measures with law $\mathbb{N}_{X_0}(X_t \in \cdot | X_t > 0)$. The summands in (3.2.6) correspond to the contributions to $X_t$ from each of the finite number of ancestors at time 0 of the population at time $t$.

We will also make use of the historical process associated with a super-Brownian motion. The historical process encodes the genealogical information of the super-Brownian motion $X$. Good introductions may be found in [15], or Sections II.8 and III.1 of [83]. Let $C([0, \infty), \mathbb{R})$ denote the space of continuous $\mathbb{R}$-valued paths on $[0, \infty)$, endowed with the compact-open topology. The historical process $(H_t : t \geq 0)$ is a measure-valued time-inhomogeneous Markov process taking values in $M_F(C([0, \infty), \mathbb{R}))$ such that $y = y_{t\uparrow}$ for $H_t$-a.a. $y$ for all $t \geq 0$ a.s. If we identify constant paths with $\mathbb{R}$, then, viewing $H_0$ as an element of $M_F(\mathbb{R})$, we can recover the super-Brownian motion $X$ starting at $X_0 = H_0$ from its associated historical process $H_t$ by projecting $H_t$ onto time $t$, that is, $X_t(\cdot) = H_t(\{y \in C([0, \infty), \mathbb{R}) : y_t \in \cdot\})$. Intuitively, $(y_s, s \leq t)$ gives the historical path of the particle $y_t$ in the support of $X_t$. We will use a modulus of continuity for the paths $y$ governed by $H_t$. Let $S(H_t)$ denote the closed support of $H_t$ and set $h(r) = (r \log(1/r))^{1/2}$. For $c > 0$ and $\delta > 0$, define $K(c, \delta)$ by

$$K(c, \delta) = \{y \in C([0, \infty), \mathbb{R}) : |y_r - y_s| \leq ch(r - s) \forall r, s \geq 0 \text{ s.t. } |r - s| \leq \delta\}. \quad (3.2.7)$$

By Theorem III.1.3(a) of [83], if $c > 2$ and $T > 0$, then $P_{X_0}^X$-a.a. $\omega$, there exists $\delta = \delta(T, c, \omega) > 0$ a.s. such that

$$S(H_t) \subset K(c, \delta) \text{ for all } t \in [0, T]. \quad (3.2.8)$$

Moreover, the proof of the above shows that for any $c > 2, T > 0$ there are $\rho(c) > 0$ and $C(T)$ such that

$$P_{X_0}^X(\delta(T, c) \leq r) \leq C_{3.2.9}(T)r^{\rho(c)} \text{ for all } r \in (0, 1], \quad (3.2.9)$$

where

$$\lim_{c \to \infty} \rho(c) = \infty. \quad (3.2.10)$$

A second decomposition of a superprocess based on historical information will also play an important role in our arguments. Let $(\mathcal{F}_t)$ be the usual right-continuous completed filtration generated by $H$ and assume $0 \leq \delta \leq t$ are fixed. Assume $\tau \in [-\infty, \infty]$ is a $\sigma(X_{t-\delta})$-measurable random variable. We decompose $X_{t-\delta}$ into the sum of two random

126
measures:

\[ X_{t-\delta}(dx) = X_{t-\delta}^{R,\tau,\delta}(dx) = 1_{\{x \geq \tau\}}X_{t-\delta}(dx) \quad \text{and} \quad X_{t-\delta}(dx) = X_{t-\delta}^{L,\tau,\delta}(dx) = 1_{\{x < \tau\}}X_{t-\delta}(dx). \]  

(3.2.11)

We then track the descendants of each of these populations at future times and so define measure-valued processes by

\[
\hat{X}_s^R(\phi) = \hat{X}_s^{R,\tau,\delta}(\phi) = \int \phi(y_{t-\delta+s})1(y_{t-\delta} \geq \tau)H_{t-\delta+s}(dy)
\]

\[
\hat{X}_s^L(\phi) = \hat{X}_s^{L,\tau,\delta}(\phi) = \int \phi(y_{t-\delta+s})1(y_{t-\delta} < \tau)H_{t-\delta+s}(dy).
\]

Clearly we have

\[ \hat{X}_s^R + \hat{X}_s^L = X_{t-\delta+s} \quad \text{for all} \quad s \geq 0, \quad \text{and (if} \quad s = 0) \quad X_{t-\delta}^R + X_{t-\delta}^L = X_{t-\delta}. \]

(3.2.12)

By (III.1.3) on p. 193 of [83] and the Markov property of \( H \), we get:

Conditional on \( F_{t-\delta} \), \((\hat{X}_s^R)\) and \((\hat{X}_s^L)\) are independent \((F_{t-\delta+s})\)-super-Brownian \( (3.2.13) \) motions with initial laws \( X_{t-\delta}^R \) and \( X_{t-\delta}^L \), respectively.

Given the above decompositions of super-Brownian motion into a sum of independent super-Brownian motions, it is not surprising that we will also need to know how the corresponding boundary local time, \( L_t \), decomposes. Recall that a sum of \( n \) independent super-Brownian motions with initial conditions \( X_0^1, \ldots, X_0^n \in \mathcal{M}_F(\mathbb{R}) \) and with boundary local times \( L_0^1, \ldots, L_0^n \). The next result is Theorem 1.9 of [35].

**Theorem 3.2.2.** Suppose \( X_0^1, \ldots, X_0^n \) are independent one-dimensional super-Brownian motions, starting at \( X_0^1, \ldots, X_0^n \in \mathcal{M}_F(\mathbb{R}) \) and with boundary local times \( L_0^1, \ldots, L_0^n \). Let \( X = \sum_{i=1}^n X_i \) and \( L_t \) be the boundary local time of \( X \). Then

\[
dL_t(x) = \sum_{i=1}^n 1\left( \sum_{j \neq i} X_j(t, x) = 0 \right) dL_t^i(x) = \sum_{i=1}^n 1(X(t, x) = 0) dL_t^i(x). \]  

(3.2.14)

### 3.2.3 Hitting probabilities of super-Brownian motion

The proofs of our main theorems will make use of bounds on hitting probabilities for super-Brownian motion.

**Theorem 3.2.3.** There exists a universal constant \( c_{3.2.3} < \infty \) such that:
(i) For $R > 2\sqrt{t}$,
\[
N_0(X_s([R, \infty))) > 0 \text{ for some } s \leq t \leq c_{3.2.3} R^{-2} \left( \frac{R}{\sqrt{t}} \right)^3 e^{-R^2/2t}.
\]

(ii) For all $X_0 \in \mathcal{M}_F(\mathbb{R})$ such that $X_0$ is supported on $(-\infty, 0]$ and for all $R > 2\sqrt{t}$, we have
\[
P_{X_0}^X(X_s([R, \infty))) = 0 \text{ for all } s \leq t 
\geq \exp \left( -c_{3.2.3} \int_{-\infty}^{0} (R-x)^{-2} \left( \frac{R-x}{\sqrt{t}} \right)^3 e^{-(R-x)^2/2t} dX_0(x) \right).
\]

Proof. (i) is a simple consequence of Theorem 3.3(b) of [14] with $d = 1$ (and its proof) and (3.2.5).

We derive (ii) as a consequence of (i) by using (3.2.5). Indeed, this result and well-known formulas for the Laplace transform of a Poisson point process (see, for example, Theorem 24.14 of [43]) imply that for $R > 2\sqrt{t}$ and $\theta > 0$, we have
\[
E_{X_0}^X \left( \exp \left( -\theta \int_0^t X_s([R, \infty)) \, ds \right) \right)
= \exp \left( -\int N_x \left( 1 - \exp \left( -\theta \int_0^t X_s([R, \infty)) \, ds \right) \right) \right) dX_0(x).
\]
A simple application of Dominated Convergence allows us to let $\theta \to \infty$ and conclude that
\[
P_{X_0}^X(X_s([R, \infty))) = 0 \text{ for all } s < t
= \exp \left( -\int N_x (X_s([R, \infty})) > 0 \text{ for some } s \leq t \right) dX_0(x).
\]
Part (ii) follows by applying (i) and translation invariance. \qed

3.3 Some semi-linear partial differential equations

We recall the relationship of the Laplace functional of super-Brownian motion with solutions of a semi-linear partial differential equation (PDE). We first present the integral form of the equation. Let $B_b^+(\mathbb{R})$ denote the space of non-negative bounded Borel functions on the line. Let $E_x^B$ denote the expectation of standard Brownian motion with $B_0 = x$, and denote the Brownian semigroup by $S_t$, i.e. $S_t \phi(x) = E_x^B(\phi(B_t))$. By Theorem II.5.11 of [83], for
\( \phi \in \mathcal{B}_{b_+}(\mathbb{R}) \) there exists a unique non-negative solution to the integral equation

\[
v_t = S_t \phi - \int_0^t S_{t-s}(v_s^2/2) \, ds \quad \text{for} \ (t, x) \in [0, \infty) \times \mathbb{R},
\]

which we denote by \( V_t^\phi(x) \), such that for all \( X_0 \in \mathcal{M}_F(\mathbb{R}) \),

\[
E^{X}_{X_0}(e^{-X_t(\phi)}) = e^{-X_0(V_t^\phi)}.
\]

It follows from (3.2.5) and the above with \( X_0 = \delta_x \) that

\[
\mathbb{N}_x(1 - e^{-X_t(\phi)}) = V_t^\phi(x).
\]

It is clear from (3.3.1) that \( V_t^\phi(x) \leq S_t \phi(x) \leq \|\phi\|_\infty \), and so \( V_t^\phi(x) - S_t \phi(x) \to 0 \) as \( t \downarrow 0 \) pointwise in \( x \). This readily implies that

\[
V_t^\phi \xrightarrow{v} \phi = V_0^\phi \quad \text{as} \ t \downarrow 0,
\]

where \( \xrightarrow{v} \) denotes vague convergence of the Radon measure \( V_t^\phi(x)dx \) to \( \phi(x)dx \). (3.3.1) is known as the mild form of the PDE

\[
\frac{\partial v_t}{\partial t} = \frac{1}{2} \frac{\partial^2 v_t}{\partial x^2} - \frac{v_t^2}{2} \quad \text{for} \ (t, x) \in (0, \infty) \times \mathbb{R}, \quad v_t \xrightarrow{v} \phi = v_0 \quad \text{as} \ t \downarrow 0,
\]

where it will be understood that solutions of (3.3.4) will be in the space \( C^{1,2}((0, \infty) \times \mathbb{R}) \) of functions with continuous partial derivatives up to order 1 in time and 2 in space on the given open set. This formulation allows one to consider initial conditions which are measures. In this context Marcus and Véron [64] (Theorem 3.5) proved existence and uniqueness of a (non-negative) solution, \( \bar{V}^\phi \), to (3.3.4) as a rather special case of more general initial conditions which they classify with their initial trace theory. The use of their general theory may seem like overkill, but it will soon be convenient to use a stability result in [64]. It is easy to show that their solutions also satisfy the mild form (3.3.1) as we now sketch. First, monotonicity of \( \bar{V}^\phi \) in \( \phi \) (e.g. Theorem 3.4 of [64]) and comparison with the elementary solution with initial (constant) value \( \|\phi\|_\infty \) show that

\[
\bar{V}^\phi(t, x) \leq \|\phi\|_\infty.
\]

For \( \varepsilon > 0 \), \( \bar{V}^\phi_{t+\varepsilon} := \bar{V}^\phi_{t+\varepsilon} \) defines the unique solution to (3.3.4) with \( C^2 \) initial data \( \bar{V}^\phi_{t+\varepsilon} \) and evidently the solution is now in \( C^{1,2}([0, \infty) \times \mathbb{R}) \). Such strong solutions are known to be solutions of the mild equation (3.3.1) (see, e.g., the outline following Proposition II.5.10 in

129
and use the above boundedness). We therefore have

\[ \bar{V}^\phi_{t+\epsilon} = S_t \bar{V}^\phi_t - \int_0^t S_{t-s}(\bar{V}^\phi_{s+\epsilon}^2/2) \, ds \quad \text{for } (t, x) \in [0, \infty) \times \mathbb{R}. \]

It is easy to justify taking the limit pointwise as \( \epsilon \downarrow 0 \) (use (3.3.5)), which shows that \( \bar{V}^\phi_t \) solves the integral equation (3.3.1). By uniqueness of solutions to (3.3.1) we conclude that \( \bar{V}^\phi_t = V^\phi_t \). We therefore have that for \( \phi \in B_{b+}(\mathbb{R}) \), there exists a unique non-negative solution \( V^\phi_t \) to (3.3.4) (also satisfying (3.3.1)) such that (3.3.2) and (3.3.3) hold.

For \( \lambda > 0 \), we denote by \( v^\lambda_t \) the unique non-negative solution of

\[ \frac{\partial v_t}{\partial t} = \frac{1}{2} \frac{\partial^2 v_t}{\partial x^2} - \frac{v_t^2}{2}, \quad \text{for } (t, x) \in (0, \infty) \times \mathbb{R}, \quad v_t \to \lambda 1_{(-\infty,0]} = v_0^\lambda \text{ as } t \downarrow 0. \]  

(3.3.6)

Given the above discussion, \( v^\lambda_t \) is also a solution of (3.3.1) with \( \phi = \lambda 1_{(-\infty,0]} \). We will sometimes write \( v^\lambda_t(x) = v^\lambda(t,x) \). By (3.3.3), translation invariance and symmetry, \( v^\lambda_t \) satisfies for all \( t > 0 \),

\[ v^\lambda_t(x) = N_0(1 - e^{-\lambda X_t([x,\infty))}) = N_0(1 - e^{-\lambda X_t([x,\infty))}) \]  

(3.3.7)

Similarly, by (3.3.2) we also have for \( t > 0 \),

\[ E_{\delta_0} \left( e^{-\lambda X_t([x,\infty))} \right) = e^{-v^\lambda_t(x)}. \]  

(3.3.8)

It is an exercise to use uniqueness in (3.3.6) or scaling properties of super-Brownian motion to show that \( v^\lambda \) satisfies the following scaling relationship:

\[ v^\lambda(t,x) = rv^{\lambda/r}(rt, \sqrt{rt}x) \quad \forall r, \lambda > 0. \]  

(3.3.9)

Take \( r = \lambda \) to see that

\[ v^\lambda(t,x) = \lambda v^1(\lambda t, \sqrt{\lambda x}) \]  

(3.3.10)

and \( r = 1/t \) to obtain

\[ v^\lambda(t,x) = t^{-1} v^\lambda(1,t^{-1/2}x). \]  

(3.3.11)

The following monotonicity properties are clear from (3.3.7).

\[ \text{Lemma 3.3.1. The map } x \to v^\lambda_t(x) \text{ is decreasing in } x, \text{ and } \lambda \to v^\lambda_t(x) \text{ is increasing in } \lambda. \]

We may let \( \lambda \to \infty \) in (3.3.7) and (3.3.8) and use Dominated Convergence (for this in (3.3.7), use \( 1 - e^{-\lambda X_t([x,\infty))} \leq 1(X_t([x,\infty)) > 0) \), which is integrable with respect to \( N_0 \) to

130
see that $v^\lambda(t, x) \uparrow v^\infty(t, x)$ as $\lambda \uparrow \infty$, where

$$v^\infty_t(x) = N_0(X_t([x, \infty)) > 0) = -\log \left( P_{\delta_0}^X(X_t([x, \infty)) = 0) \right). \tag{3.3.12}$$

Note that

$$v^\infty_t(x) = N_0(X_t([x, \infty)) > 0) \leq N_0(X_t(1) > 0) = 2/t, \tag{3.3.13}$$

(see Theorem II.7.2(iii) of [83]) and in particular $v^\infty_t$ is finite for $t > 0$.

**Proposition 3.3.2.** $v^\lambda_t(x) \to v^\infty_t(x)$ uniformly on compact sets in $(0, \infty) \times \mathbb{R}$. In fact, there is uniform convergence for $(t, x) \in [a, \infty) \times [-R, R]$, for any $a > 0$ and $R > 0$.

**Proof.** Taking $\lambda \to \infty$ in (3.3.11), we obtain that

$$v^\infty_t(x) = t^{-1} v_1^\infty(t^{-1/2} x). \tag{3.3.14}$$

This fact and (3.3.11) imply that

$$v^\infty_t(x) - v^\lambda_t(x) = t^{-1} \left[ v_1^\infty(t^{-1/2} x) - v_1^\lambda(t^{-1/2} x) \right].$$

Let $0 < a, R$. Then by the above, for $t \geq a$ we have

$$v^\infty_t(x) - v^\lambda_t(x) \leq a^{-1} \left[ v_1^\infty(t^{-1/2} x) - v_1^\lambda(a(t^{-1/2} x)) \right],$$

where we have used monotonicity in $\lambda$. Thus

$$\sup_{t \geq a} \sup_{|x| \leq R} v^\infty_t(x) - v^\lambda_t(x) \leq a^{-1} \sup_{t \geq a} \sup_{|x| \leq R} v_1^\infty(t^{-1/2} x) - v_1^\lambda(t^{-1/2} x) \leq a^{-1} \sup_{|x| \leq a^{-1/2} R} v^\infty_t(x) - v^\lambda_t(x).$$

The continuity of $v^\infty_1$ (e.g., from (3.3.12)) and Dini’s Theorem imply that $v^\lambda_t \uparrow v^\infty_t$ uniformly on compact sets, and the result follows. \hfill \Box

**Theorem 3.3.3.** $v^\infty_t(x) \in C^{1,2}((0, \infty) \times \mathbb{R})$ and is the unique non-negative solution to the PDE

$$(i) \quad \frac{\partial v_t}{\partial t} = \frac{1}{2} \frac{\partial^2 v_t}{\partial x^2} - \frac{v_t^2}{2} \quad \text{on } (t, x) \in (0, \infty) \times \mathbb{R},$$

$$(ii) \quad \lim_{t \downarrow 0} \int_U v^\infty_t(x) \, dx = +\infty \quad \forall U \subseteq \mathbb{R} \text{ open such that } U \cap (-\infty, 0] \neq \emptyset, \tag{3.3.15}$$

$$\lim_{t \downarrow 0} \int_K v^\infty_t(x) \, dx = 0 \quad \forall K \subseteq \mathbb{R} \text{ compact such that } K \subset (0, \infty).$$
Proof. From Proposition 3.3.2 we have the local uniform convergence of $v^\lambda_t$ to $v^\infty_t$. The family $\{v^\lambda_t\}$ therefore satisfies the conditions of Theorem 3.10 of [64], which shows that $v^\infty(t, x)$ solves (3.3.15). Uniqueness follows by Theorem 3.5 of the same paper.

Recall (see (3.1.20)) that $G(x) = v^\infty_1(x)$.

Lemma 3.3.4. (a) For all $t > 0$, $v^\infty_t(x) = t^{-1}G(t^{-1/2}x)$ for all $x \in \mathbb{R}$.

(b) $G > 0$ and is decreasing.

(c) $G \in C^\infty(\mathbb{R})$ and is the unique positive $C^2$ solution to the ordinary differential equation

$$\frac{1}{2}G''(x) + \frac{x}{2}G'(x) + G(x) - \frac{1}{2}G(x)^2 = 0$$

(3.3.16)

with boundary conditions $\lim_{x \to \infty} x^2G(x) = 0$, $\lim_{x \to -\infty} G(x) = 2$.

(d) There is a constant $c_{3.3.4}$ such that:

(i) $G(x) \leq c_{3.3.4} |x|e^{-x^2/2} \forall x > 2$,

$0 \leq 2 - G(x) \leq c_{3.3.4} |x|e^{-x^2/2} \forall x < -2$.

(ii) $G' \leq 0$, and

$$|G'(x)| \leq c_{3.3.4}xe^{-x^2/2} \forall x > 2$$

$$|G'(x)| \leq c_{3.3.4}e^{-x^2/2} \forall x \leq 0.$$

Proof. (a) is a restatement of (3.3.14) and (b) is obvious from (3.3.12).

We now prove (c). By Theorem 3.3.3, $G = v^\infty_1$ is $C^2$. Noting that $v^\infty_t(x) = t^{-1}G(t^{-1/2}x)$ solves (3.3.15)(i), we can use the chain rule to see

$$-\frac{1}{t^2}G(t^{-1/2}x) - \frac{1}{2t^2}(t^{-1/2}x)G'(t^{-1/2}x) = \frac{1}{2t^2}G''(t^{-1/2}x) - \frac{1}{2t^2}G(t^{-1/2}x)^2$$

$$-G(y) - \frac{y}{2}G'(y) = \frac{1}{2}G''(y) - \frac{1}{2}G(y)^2,$$

This proves that $G$ solves (3.3.16). To see that it is $C^3$, we note that when we solve (3.3.16) for $G''$, the expression is differentiable because $G$ and $G'$ are differentiable. Proceeding by induction we see that $G$ is $C^\infty$. The boundary conditions will clearly follow from (d) below. It remains to prove uniqueness. Let $H$ be any positive $C^2$ solution of (3.3.16) satisfying the given boundary conditions and set $u(t, x) = t^{-1}H(t^{-1/2}x)$. Then, reversing the above steps one easily sees that $u$ is a $C^2$ solution of (3.3.15)(i). Let $0 < a < b$ and choose $t > 0$ small
enough so that \( y^2 H(y) < \epsilon a \) for \( y \geq at^{-1/2} \). Then

\[
\int_a^b v(t, x) dx = t^{-1/2} \int_a^b H(t^{-1/2} x) t^{-1/2} dx \leq t^{-1/2} \int_{at^{-1/2}}^\infty H(y) dy \\
\leq \epsilon at^{-1/2} \int_{at^{-1/2}}^\infty y^{-2} dy \\
= \epsilon.
\]

This proves the second boundary condition in (3.3.15)(ii). The first boundary condition is even easier to establish. So by the uniqueness in Theorem 3.3.3, \( H(x) = u(1, x) = v^\infty(1, x) = G(x) \).

Next, we prove part (d)(i). To deduce the bound for positive \( x \), we note that

\[
G(x) = N_0(X_1([x, \infty)) > 0) \\
\leq N_0(X_s([x, \infty)) > 0 \text{ for some } s \leq 1) \\
\leq c_{3.2.3}|x|e^{-x^2/2},
\]

for all \( x > 2 \), by Theorem 3.2.3(i). The lower bound on \( 2 - G(x) \) is immediate from (3.3.13) (for all \( x \)). For \( x < -2 \), we have

\[
2 - G(x) = N_0(X_1(1) > 0) - N_0(X_1([x, \infty)) > 0) \\
\leq N_0(X_1((-\infty, x]) > 0) \\
\leq c_{3.2.3}|x|e^{-x^2/2},
\]

again using Theorem 3.2.3(i) and symmetry.

Finally consider part (d)(ii). By (b) \( G' \leq 0 \). Now note that (3.3.16) can be rewritten as

\[
\left( e^{x^2/2}G'(x) \right)' = e^{x^2/2}G(x)(G(x) - 2).
\]

Integrating the above, for \( x_0, x \in \mathbb{R} \) we get

\[
G'(x) = e^{-x^2/2} \left[ e^{x^2/2}G'(x_0) + \int_{x_0}^x e^{y^2/2}G(y)(G(y) - 2) dy \right].
\]

For \( x > x_0 \geq 2 \), both terms in the above are non-positive, and, if \( c \) is the provisional
constant arising in (i), we can use part (i) to deduce that

\[ |G'(x)| \leq e^{-x^2/2} \left[ e^{x_0^2/2} |G'(x_0)| + 2e \int_{x_0}^{x} |y| \, dy \right] \leq (c_1(x_0) + c_2 x^2) e^{-x^2/2} \leq c'e^x e^{x^2/2} \]

for \( x > 2 \). For \( x \leq 0 = x_0 \), we note that the integral in (3.3.18) has its sign reversed, so is positive. Because \( G'(x) \leq 0 \), \( |G'(x)| \) is bounded above by the absolute value of the first term in (3.3.18), which gives the required bound.

Recall from Section 3.2.1 that if \( G \) is as above, then \( A^G \) is the generator of a killed Ornstein-Uhlenbeck process with killing function \( G \), \( -\lambda_0^G \) is its lead eigenvalue, and \( \psi_0^G \) denotes its corresponding unit eigenfunction in \( L^2(m) \).

**Proposition 3.3.5.** For some constant \( c_{3.3.5} > 0 \), \( \psi_0^G(x) = -c_{3.3.5} e^{x^2/2} G'(x) \) with corresponding eigenvalue \( -\lambda_0^G = -1 \).

**Proof.** Recall that \( G \) is the \( C^\infty \) solution of (3.3.16). Rearranging the equation, we can write \( G''(x) = -x G'(x) - 2G(x) + G(x)^2 \). \( G \) is \( C^\infty \), so we can differentiate again to obtain a new ODE:

\[
\frac{1}{2} G''' + \frac{1}{2} G' + \frac{1}{2} x G'' + G' - \frac{1}{2} 2G G' = 0 \\
\iff \frac{1}{2} G''' + \frac{1}{2} x G'' + \frac{3}{2} G' - GG' = 0 \quad (3.3.19)
\]

Let \( \psi(x) = e^{x^2/2} G'(x) \). Let us first observe that \( \psi \in L^2(m) \) because

\[
\int \psi(x)^2 \, dm(x) = (2\pi)^{-1/2} \int G'(x)^2 e^{x^2/2} \, dx < \infty,
\]

where the integral converges by Lemma 3.3.4(d)(ii). We compute the first and second derivatives of \( \psi \):

\[
\psi'(x) = xe^{x^2/2} G'(x) + e^{x^2/2} G''(x)
\]

and

\[
\psi''(x) = x^2 e^{x^2/2} G'(x) + e^{x^2/2} G'(x) + xe^{x^2/2} G''(x) + xe^{x^2/2} G''(x) + e^{x^2/2} G'''(x) = x^2 e^{x^2/2} G'(x) + e^{x^2/2} G'(x) + 2xe^{x^2/2} G''(x) + e^{x^2/2} G'''(x).
\]
Using the above, we evaluate $A^G\psi$.

$$A^G\psi = \frac{1}{2}\psi''(x) - \frac{1}{2}x\psi'(x) - \psi(x)G(x)$$

$$= e^{x^2/2} \left[ \frac{1}{2}x^2 G'(x) + \frac{1}{2}G''(x) + xG'''(x) + \frac{1}{2}G'''(x) \right]$$

$$- e^{x^2/2} \left[ \frac{1}{2}x^2 G'(x) + \frac{1}{2}xG''(x) + \frac{1}{2}G'(x) - G(x)G'(x) \right]$$

$$= e^{x^2/2} \left[ \frac{1}{2}G'''(x) + \frac{1}{2}xG''(x) + \frac{3}{2}G'(x) - G(x)G'(x) \right]$$

$$- e^{x^2/2}G'(x)$$

$$= -\psi(x),$$

where the last equality is due to (3.3.19). Moreover, $G'(x) \leq 0$ for all $x$, so $-e^{x^2/2}G'(x) \geq 0$, and we have already seen that it is in $L^2(m)$. Therefore $\psi$ is a non-positive eigenfunction of $A^G$ with eigenvalue $-1$. Clearly $\psi$ cannot be orthogonal to the lead eigenfunction $\psi_0^G > 0$ (recall Theorem 3.2.1(a)). It follows that $\psi_0^G = -c_{3.3.5}\psi$ for some normalizing constant $c_{3.3.5} > 0$ and hence the corresponding lead eigenvalue is $-1$. □

The next result gives a bound on the left tail of the distribution of $X_t([x, \infty))$ which will play an important role in the proof of Proposition 3.1.5. We do not know what the “correct” power law behaviour is, but see the remark at the end of this section for a possible answer.

**Proposition 3.3.6.** For $0 < p < 1/6$ and $t > 0$ there is a constant $C_{3.3.6} = C_{3.3.6}(p,t)$ such that

$$P_{\delta_0}^X \left( 0 < X_t([x, \infty)) \leq \frac{1}{\lambda} \right) \leq C_{3.3.6} \lambda^{-p} \text{ for all } x \in \mathbb{R} \text{ and } \lambda > 0.$$  

**Proof.** It clearly suffices to consider $\lambda \geq 1$, which is assumed until otherwise indicated. Let $0 < p < 1/6$ and $\varepsilon = \varepsilon(p) \in (0,1/6 - p)$. Assume we are working under a probability, $P$, for which $H$ is a historical process defining the super-Brownian motion $X$ starting at $\delta_0$ and let $\mathcal{F}_t$ be the right-continuous completed filtration generated by $H$. $E$ will denote expectation with respect to $P$. Recall $h(r), \rho(c)$ and $\delta(t,c)$ are as in (3.2.8) and (3.2.9). By (3.2.10) we may choose $c = c(p)$ large enough so that $\rho(c)\varepsilon \geq p$ and so by (3.2.9),

$$P(\delta(t+1,c) \leq \lambda^{-\varepsilon}) \leq C_{3.2.9}(t+1)\lambda^{-\varepsilon}\rho(c) \leq C_{3.2.9}(t+1)\lambda^{-p} \text{ for all } \lambda \geq 1. \quad (3.3.20)$$
By (II.5.11) and (II.5.12) of [83],

$$E_{X_0}(e^{-\lambda X_t(1)}) = \exp\left(\frac{-2\lambda X_0(1)}{2 + \lambda t}\right); \quad P_{X_0}(X_t(1) = 0) = \exp(-2X_0(1)/t),$$

(3.3.21)

and so for $\lambda \geq 1$,

$$P(0 < X_t(1) \leq \lambda^{-1/6}) \leq eE(1(X_t(1) > 0) \exp(-\lambda^{1/6}X_t(1)))$$

$$= e\left[\exp\left(\frac{-2\lambda^{1/6}}{2 + \lambda^{1/6}t}\right) - \exp(-2/t)\right]$$

$$\leq 2e\left[\frac{1}{t} - \frac{\lambda^{1/6}}{2 + \lambda^{1/6}t}\right]$$

$$\leq 4et^{-2}\lambda^{-1/6}.$$  

(3.3.22)

Let

$$E = E_{x,\lambda} = \{0 < X_t([x, \infty)) \leq 1/\lambda, \delta(t + 1, c) > \lambda^{-\varepsilon}, X_t(1) > \lambda^{-1/6}\}.$$

Then by (3.3.20) and (3.3.22), it suffices to show

$$P(E_{x,\lambda}) \leq C(t,p)\lambda^{-p} \quad \text{for all } x \in \mathbb{R} \text{ and } \lambda \geq 1.$$  

(3.3.23)

Assume for now that $x \in \mathbb{R}$ and $\lambda \geq 1$. Note that if $\tau(\lambda) = \tau^{\lambda^{-2/3}}(t)$ (recall $\tau^\varepsilon$ is as in (3.1.18)), then

on $E_{x,\lambda}$ we have, $-\infty < \tau(\lambda) < x$ and $X_t([\tau(\lambda), \infty)) = \lambda^{-2/3}$.

(3.3.24)

Introduce

$$E_{x,\lambda}^1 = E \cap \{x - \tau(\lambda) \geq \lambda^{-1/6}\} \quad \text{and} \quad E_{x,\lambda}^2 = E \cap \{x - \tau(\lambda) < \lambda^{-1/6}\}.$$

We consider $E^1$ first. Set

$$\beta = \frac{1}{3} + \varepsilon.$$

Then there is a $\lambda = \lambda(c, \varepsilon, t) = \lambda(p, t) \geq 1$ such that

$$2ch(\lambda^{-\beta}) < \lambda^{-1/6} \text{ and } \lambda^{-\beta} < t/2 \text{ for } \lambda \geq \lambda.$$

(3.3.25)

Until otherwise indicated we will assume now that $\lambda \geq \lambda$. Define

$$\zeta_{t-\lambda^{-\beta}} = \inf\{s \geq 0 : H_{s+t-\lambda^{-\beta}}(\{y : y_{t-\lambda^{-\beta}} \geq x - ch(\lambda^{-\beta})\}) = 0\}.$$
It follows from (3.2.13) that for $u = 0$ or $\lambda^{-\beta}$,

conditional on $F_{t-u}$, $Z_s = H_{t-u+s}(\{y : y_{t-\lambda^{-\beta}} \geq x - ch(\lambda^{-\beta})\})$ $(s \geq 0)$ is equal in law to the Feller diffusion $(X_s(1), s \geq 0)$ starting at $H_{t-u}(\{y : y_{t-\lambda^{-\beta}} \geq x - ch(\lambda^{-\beta})\})$. (3.3.26)

Throughout this proof we will assume the Feller diffusion $X_s(1)$ starts at $x_0 \geq 0$ under $P_{x_0}$. On $E^1$ we have $\lambda^{-\beta} \leq \lambda^{-\varepsilon} < \delta(t + 1, c)$ and so by the modulus of continuity (3.2.8),

$$H_t(\{y : y_{t-\lambda^{-\beta}} \geq x - ch(\lambda^{-\beta})\}) \geq H_t(\{y : y_t \geq x\}) = X_t([x, \infty)) > 0.$$  

This implies that (use (3.3.26) with $u = \lambda^{-\beta}$ to see that $Z_s$ sticks at zero when it hits zero)

$$\zeta_{t-\lambda^{-\beta}} > \lambda^{-\beta} \text{ on } E^1_{x, \lambda}.$$ (3.3.27)

Now again use the modulus of continuity and then (3.3.25) to that on $E^1$,

$$H_t(\{y : y_{t-\lambda^{-\beta}} \geq x - ch(\lambda^{-\beta})\}) \leq H_t(\{y : y_t \geq x - 2ch(\lambda^{-\beta})\})$$

$$\leq X_t([x - \lambda^{-1/6}, \infty)) \quad \text{(by (3.3.25))}$$

$$\leq X_t([\tau(\lambda), \infty)) \quad \text{(since } x - \tau(\lambda) \geq \lambda^{-1/6} \text{ on } E^1)$$

$$= \lambda^{-2/3},$$

the last by (3.3.24). Use the above fact that $H_t(\{y : y_{t-\lambda^{-\beta}} \geq x - ch(\lambda^{-\beta})\}) \leq \lambda^{-2/3}$ on $E^1$ and condition on $F_t$ (recall (3.3.26) with $u = 0$) to conclude that

$$P(E^1 \cap \{\zeta_{t-\lambda^{-\beta}} > \lambda^{-\beta} + \lambda^{-1/2}\})$$

$$\leq E(1H_t(\{y : y_{t-\lambda^{-\beta}} \geq x - ch(\lambda^{-\beta})\}) \leq \lambda^{-2/3})P(Z_{t+\lambda^{-1/2}} > 0|F_t))$$

$$\leq P_{\lambda^{-2/3}}(X_{\lambda^{-1/2}}(1) > 0)$$

$$= 1 - \exp\left(-\frac{2\lambda^{-2/3}}{\lambda^{-1/2}}\right) \quad \text{(by (3.3.21))}$$

$$\leq 2\lambda^{-1/6}.$$ (3.3.28)

So (3.3.27) and (3.3.28) show that

$$P(E^1 \cap \{\zeta_{t-\lambda^{-\beta}} \notin [\lambda^{-\beta}, \lambda^{-\beta} + \lambda^{-1/2}]\}) \leq 2\lambda^{-1/6} \text{ for } \lambda \geq \Lambda(p, t).$$ (3.3.29)

Let

$$M(\omega) = X_{t-\lambda^{-\beta}}([x - ch(\lambda^{-\beta}), \infty)).$$

If $\zeta = \inf\{s \geq 0 : X_s(1) = 0\}$ is the lifetime of the Feller diffusion $X_s(1)$, then we may apply
(3.3.26) with \( u = \lambda^{-\beta} \) to see that

\[
P(\zeta_t - \lambda^{-\beta} \in [\lambda^{-\beta}, \lambda^{-\beta} + \lambda^{-1/2}])
\]

\[
= E(E_M(\zeta \in [\lambda^{-\beta}, \lambda^{-\beta} + \lambda^{-1/2}]))
\]

\[
= E(P(\lambda^{-\beta} + \lambda^{-1/2} = 0) - P(\lambda^{-\beta} = 0))
\]

\[
= E\left(\exp\left(\frac{-2M}{\lambda^{-\beta} + \lambda^{-1/2}}\right) - \exp\left(\frac{-2M}{\lambda^{-\beta}}\right)\right) \quad \text{(by (3.3.21))}
\]

\[
\leq E\left(\exp(-2M/(\lambda^{-\beta} + \lambda^{-1/2}))2M\right)\left[1/\lambda^{-\beta} - 1/(\lambda^{-\beta} + \lambda^{-1/2})\right]
\]

\[
= E\left(\exp(-2M/(\lambda^{-\beta} + \lambda^{-1/2}))2M/(\lambda^{-\beta} + \lambda^{-1/2})\right)[\lambda^{-1/2}/\lambda^{-\beta}]
\]

\[
\leq \lambda^{-(1/2)-\beta} = \lambda^{-1/6+\varepsilon} \leq \lambda^{-p}, \quad (3.3.30)
\]

where we have used \( \sup_{x \geq 0} xe^{-x} = e^{-1} \leq 1 \) in the last line. Combining (3.3.29) and (3.3.30) we arrive at

\[
P(E_{1,x,\lambda}) \leq 3\lambda^{-p} \quad \text{for all } \lambda \geq \lambda(p, t), \quad x \in \mathbb{R}.
\]

This then implies that for some \( c_{3.3.31} = c_{3.3.31}(p, t) \),

\[
P(E_{1,x,\lambda}) \leq c_{3.3.31}(p, t)\lambda^{-p} \quad \text{for all } \lambda \geq 1, \quad x \in \mathbb{R}. \quad (3.3.31)
\]

Consider next \( E^2 = E_{2,x,\lambda} \) where for now \( \lambda \geq 1 \) and of course \( x \in \mathbb{R} \). Recall that \( U_s = \sup(S(X_s)) \). On \( E^2 \), we have \( \lambda^{-5/6} \leq \lambda^{-\varepsilon} \leq \delta(t+1,c) \) and so by the modulus of continuity (3.2.8),

\[
P(E^2 \cap \{U_{t+\lambda^{-5/6}} \geq x + ch(\lambda^{-5/6})\}) \leq P(E^2 \cap \{H_{t+\lambda^{-5/6}}(\{y : y_t \geq x\}) > 0\})
\]

\[
\leq P(\lambda^{-1}(X_{\lambda^{-5/6}}(1) > 0),
\]

where we have used (3.2.13) with \( \delta = 0 \), and \( H_t(\{y : y_t \geq x\}) \leq 1/\lambda \) on \( E \) in the last line. Now use (3.3.21) to see that the above equals \( 1 - \exp(-2\lambda^{-1}\lambda^{5/6}) \leq 2\lambda^{-1/6} \), and so conclude that

\[
P(E^2 \cap \{U_{t+\lambda^{-5/6}} \geq x + ch(\lambda^{-5/6})\}) \leq 2\lambda^{-1/6} \quad \text{for all } \lambda \geq 1, \quad x \in \mathbb{R}. \quad (3.3.32)
\]
The modulus of continuity also implies

\[ P(E^2 \cap \{ U_{t+\lambda^{-5/6}} \leq x - \lambda^{-1/6} - ch(\lambda^{-5/6}) \} ) \]
\[ \leq P(E^2 \cap \{ H_{t+\lambda^{-5/6}}(\{ y : yt \geq x - \lambda^{-1/6} \} = 0) \} ) \]
\[ = P(E^2 \cap \{ H_{t+\lambda^{-5/6}}(\{ y : yt \geq \tau(\lambda) \} = 0) \} ) \quad \text{(recall } x - \tau(\lambda) < \lambda^{-1/6} \text{ on } E^2) \]
\[ = P_{\lambda^{-2/3}}(X_{\lambda^{-5/6}}(1) = 0) \quad \text{(by (3.2.13) with } \delta = 0, \text{ and (3.3.24))} \]
\[ = \exp \left( \frac{-2\lambda^{-2/3}}{\lambda^{-5/6}} \right) = \exp(-2\lambda^{1/6}) \leq \lambda^{-1/6} , \]
the last since } \lambda \geq 1. \text{ The above inequality and (3.3.32) imply that }
\[ P(E^2 \cap \{ U_{t+\lambda^{-5/6}} \notin (x - \lambda^{-1/6} - ch(\lambda^{-5/6}), x + ch(\lambda^{-5/6})) \} \leq 3\lambda^{-1/6} \quad \forall \lambda \geq 1, x \in \mathbb{R}. \] (3.3.33)

Differentiate both sides of the scaling relationship in Lemma 3.3.4(a) and so get

\[ \left| \frac{\partial}{\partial x} v^\infty(t, x) \right| \leq t^{-3/2}\|G'\|_{\infty}. \] (3.3.34)

If } t' = t + \lambda^{-5/6}, x_1 = x - \lambda^{-1/6} - ch(\lambda^{-5/6}), x_2 = x + ch(\lambda^{-5/6}), \text{ and } \lambda \geq \Lambda(p, t), \text{ then }
\[ P(U_{t'} \in (x_1, x_2]) = P(X_{t'}([x_2, \infty)) = 0) - P(X_{t'}([x_1, \infty)) = 0) \]
\[ = e^{-v^\infty(t_2)} - e^{-v^\infty(t_1)} \quad \text{(by (3.3.12))} \]
\[ \leq t^{-3/2}\|G'\|_{\infty}(x_2 - x_1) \quad \text{(by (3.3.34))} \]
\[ = t^{-3/2}\|G'\|_{\infty}(\lambda^{-1/6} + 2ch(\lambda^{-5/6})) \]
\[ \leq 2t^{-3/2}\|G'\|_{\infty}\lambda^{-1/6}, \]

where in the last line we used (3.3.25). The above, together with (3.3.33), implies that

\[ P(E^2_{x, \lambda}) \leq (2t^{-3/2}\|G'\|_{\infty} + 3)\lambda^{-1/6} \quad \text{for all } x \in \mathbb{R}, \lambda \geq \Lambda(p, t). \]

This in turn shows that for some } c_{3.3.35}(p, t),
\[ P(E^2_{x, \lambda}) \leq c_{3.3.35}(p, t)\lambda^{-1/6} \quad \text{for all } x \in \mathbb{R}, \lambda \geq 1. \] (3.3.35)

Combining (3.3.31) and (3.3.35), we derive (3.3.23), as required. \qed

An easy consequence of the above is a rate of convergence of } v^\lambda \text{ to } v^\infty \text{ as } \lambda \to \infty. \text{ This will play an important role in the proof of Theorem 3.1.2 given in the next section.
Proposition 3.3.7. For any $0 < p < 1/6$ there is a $C_{3,3,7}(p)$ such that

$$\sup_x |v_t^\infty(x) - v_1^\lambda(x)| \leq C_{3,3,7}(p)t^{-p-1}\lambda^{-p} \quad \text{for all } \lambda, t > 0.$$  

Proof. Let $0 < p < 1/6$. By (3.3.8) and (3.3.12),

$$e^{-v_1^\lambda(x)} - e^{-v_t^\infty(x)} = E_{\delta_0}^X(e^{-\lambda X_t([x,\infty))}1(X_t([x,\infty)) > 0))$$

$$= E_{\delta_0}^X\left(\int_0^\infty 1(0 < X_t([x,\infty)) \leq u)e^{-\lambda u}\lambda du\right)$$

$$\leq C_{3,3,6}(p,t)\int_0^\infty u^p e^{-\lambda u}\lambda du \quad \text{(Proposition 3.3.6)}$$

$$= \Gamma(p + 1)C_{3,3,6}(p,1)\lambda^{-p}. \quad \text{(3.3.36)}$$

Recalling from (3.3.13) that $v_t^\infty(x) \leq 2/t$, we also have

$$e^{-v_1^\lambda(x)} - e^{-v_t^\infty(x)} \geq e^{-v_t^\infty(x)}(v_t^\infty(x) - v_1^\lambda(x)) \geq e^{-2/t}(v_t^\infty(x) - v_1^\lambda(x)). \quad \text{(3.3.37)}$$

Combine (3.3.36) and (3.3.37) and set $t = 1$ to see that

$$\sup_x |v_1^\infty(x) - v_1^\lambda(x)| \leq e^2\Gamma(p + 1)C_{3,3,6}(p,1)\lambda^{-p}.$$  

The required relation is now immediate from the scaling relations (3.3.11) and Lemma 3.3.4(a).

Remark. We do not believe $p = 1/6$ is sharp in any way. Theorem 1.5 of [69] studies solutions of

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - \frac{u^2}{2}, \quad u_0 = \lambda\delta_0.$$  

In particular this paper shows (via a Feynman-Kac argument) that for some $0 < \underline{C}(K) \leq \overline{C} < \infty$,

$$\underline{C}(K)t^{-(1/2) - \lambda_0}\lambda^{-(2\lambda_0 - 1)} \leq u_t^\infty(x) - u_t^\lambda(x) \leq \overline{C}t^{-(1/2) - \lambda_0}\lambda^{-(2\lambda_0 - 1)},$$

where $\lambda_0 = \lambda_0^K$ (as in Theorem 3.1.1) and the lower bound is valid for $\lambda \geq t^{-1/2}$ and $|x| \leq K\sqrt{t}$. So naively changing $u_1^\lambda$ to $v_1^\lambda$ leads to replacing $F = u_t^\infty$ with $G = v_t^\infty$, and one might think that (the $t$ dependence is by scaling (3.3.11))

$$v_t^\infty(x) - v_t^\lambda(x) \approx Ct^{-2\lambda_0^K}\lambda^{-(2\lambda_0^K - 1)} = Ct^{-2}\lambda^{-1} \quad \text{as } \lambda \to \infty, \quad \text{(3.3.38)}$$

where $\approx$ means bounded below and above for perhaps differing positive constants $C$. This
rate does hold if \( x = -\infty \), where (by (3.3.21) and (3.3.12))

\[
v_t^\infty(-\infty) - v_t^\lambda(-\infty) \approx \frac{2\lambda}{2 + \lambda t} \sim 4t^{-2}\lambda^{-1} \quad \text{as} \quad \lambda \to \infty.
\]

However the proof in [69] relies on the scaling of \( u^\lambda \), which differs from that of \( v^\lambda \). Moreover there is some evidence that the convergence when \( x \gg 0 \) is slower. In fact a heuristic argument suggests that the correct rate at \(+\infty\) is given by \( p = G(0) - 1 \in (0, 1) \). The last upper bound is obvious because \( G(0) < G(-\infty) = 2 \). For the lower bound on \( G(0) \), note that by (3.3.12) we have

\[
G(0) = \frac{N_0}{L_t([k\sqrt{t}, \infty))} = \frac{1}{C_A t^{-\lambda_0}} \int k\sqrt{t} z \geq k\sqrt{t} \psi_0(z) \, dm(z) = C_A t^{-\lambda_0} \int_k^\infty \psi_0 \, dm = c(k) t^{-\lambda_0},
\]

where \( c(k) > 0 \). Thus by the second moment method, we have

\[
N_0(L_t([k\sqrt{t}, \infty])) \geq \frac{N_0(L_t([k\sqrt{t}, \infty]))}{N_0(L_t([k\sqrt{t}, \infty])^2) \geq \frac{(c(k) t^{-\lambda_0})^2}{C_{3.4.1} t^{1-2\lambda_0}} =: 2c_{3.4.1}(k) t^{-1}.
\]

### 3.4 Proof of Theorem 3.1.2

We first establish a lower bound on the probability that \( L_t \) has positive mass at distances of order \( \sqrt{t} \) away from zero under canonical measure. This follows readily from moment calculations in [35].

**Lemma 3.4.1.** There is a finite constant \( C_{3.4.1} \) and for all \( k \geq 0 \), positive constants \( c_{3.4.1}(k) \), such that for all \( t > 0 \) and \( k \geq 0 \),

\[
N_0(L_t([k\sqrt{t}, \infty]))^2 \leq C_{3.4.1} t^{1-2\lambda_0}, \tag{3.4.1}
\]

and

\[
N_0(L_t([k\sqrt{t}, \infty])) \geq c_{3.4.1}(k). \tag{3.4.2}
\]

**Proof.** The first claim is immediate from Theorem 3.A(d). The second claim is an easy application of the second moment method as we now show. By Theorem 3.A(b) the first moment of \( L_t([k\sqrt{t}, \infty])) \) is

\[
N_0(L_t([k\sqrt{t}, \infty])) = C_A t^{-\lambda_0} \int \{k\sqrt{t} z \geq k\sqrt{t} \} \psi_0(z) \, dm(z) = C_A t^{-\lambda_0} \int_k^\infty \psi_0 \, dm = c(k) t^{-\lambda_0},
\]

where \( c(k) > 0 \). Thus by the second moment method, we have

\[
N_0(L_t([k\sqrt{t}, \infty])) \geq \frac{N_0(L_t([k\sqrt{t}, \infty]))^2}{N_0(L_t([k\sqrt{t}, \infty])^2) \geq \frac{(c(k) t^{-\lambda_0})^2}{C_{3.4.1} t^{1-2\lambda_0}} =: 2c_{3.4.1}(k) t^{-1}.
\]
Because $L_t = 0$ when $X_t = 0$ and $\mathbb{N}_0(\{X_t > 0\}) = 2/t$, this implies that

$$\mathbb{N}_0(\{L_t(k\sqrt{t}) > 0 \mid X_t > 0\}) \geq c_{3.4.1}(k).$$

We begin the study of $X_t$ near the upper edge of its support. Recall the notation $\tau^\epsilon(t)$ from (3.1.18) in the Introduction. We first obtain a preliminary upper bound for the mass of $X_t$ near $\tau^\epsilon(t)$.

**Lemma 3.4.2.** Let $t, \epsilon > 0$ and $u > 0$. Then

$$\mathbb{N}_0 \left( \int_{\tau^\epsilon(t) - u}^{\infty} X(t, x) \, dx \right) \leq eE_0^B \left( \exp \left\{ - \int_0^t v_s^{-1}(B_s + u) \, ds \right\} \right),$$

where $B$ is a standard Brownian motion under $P_0^B$ and $v_s^{-1}$ is as in (3.3.6). If $X_0 \in \mathcal{M}_F(\mathbb{R}) \setminus \{0\}$, then $E_{X_0}^X \left( \int_{\tau^\epsilon(t) - u}^{\infty} X(t, x) \, dx \right) / X_0(1)$ is bounded by the same expression.

**Proof.** As $t$ is fixed we will write $\tau^\epsilon$ for $\tau^\epsilon(t)$. We begin by examining $\mathbb{N}_0 \left( \int_{\tau^\epsilon(t) - u}^{\infty} X(t, x) \, dx \right)$. It is equal to

$$\mathbb{N}_0 \left( \int 1(x + u > \tau^\epsilon) X(t, x) \, dx \right) = \mathbb{N}_0 \left( \int 1(X_t([x + u, \infty)) < \epsilon) X(t, x) \, dx \right) \leq e\mathbb{N}_0 \left( \int e^{-\epsilon^{-1}X_t([x+u,\infty))]X(t, x) \, dx \right).$$

(We note that the above is true when $\tau^\epsilon \in \mathbb{R}$ and when $\tau^\epsilon = -\infty$, in which case $X_t(x + u, \infty) \leq \epsilon$ for all $x \in \mathbb{R}$.) By Theorem 4.1.3 of Dawson and Perkins [15] and translation invariance, the above is equal to

$$eE_0^B \left( \exp \left( - \int_0^t u_{t-s}^{-1}(W_s - W_t - u) \, ds \right) \right), \quad (3.4.3)$$

where $W$ is a standard Brownian motion under $P_0^B$ and $u_t^\lambda(x)$ solves (3.3.6) but with $u_0^\lambda = \lambda 1_{[0, \infty)}$. Clearly $u_t^\lambda(x) = v_t^\lambda(-x)$ ($v_t^\lambda$ as in (3.3.6)). So if $B_s = -W_{t-s} + W_t$, a new standard Brownian motion under $P_0^B$, the above equals

$$eE_0^B \left( \exp \left\{ - \int_0^t u_{t-s}^{-1}(-B_{t-s} - u) \, ds \right\} \right) = eE_0^B \left( \exp \left\{ - \int_0^t v_s^{-1}(B_s + u) \, ds \right\} \right). \quad (3.4.4)$$

Consider next $P_{X_0}^X$, for a non-zero initial condition $X_0$. As above, $E_{X_0}^X \left( \int_{\tau^\epsilon+u}^{\infty} X(t, x) \, dx \right)$
equals
\[ E^{X}_{X_0} \left( \int 1(X_t([x + u, \infty))) < \epsilon \right) X(t, x)dx \],
which by Theorem 4.1.1 of [15] is bounded by
\[ \int\int 1(X_t([x + u, \infty)) < \epsilon) dN_{x_0} dX_0(x_0) \leq X_0(1) e E^B_0 \left( \exp \left( - \int_0^t v_s^{-1} (B_s + u) ds \right) \right). \]

To obtain the left-hand side of the above, we have ignored the contribution to \( X_t \) from particles unrelated to the individual selected at \( x \) by \( X_t \) (the quoted theorem in [15] giving the rigorous justification), and the inequality follows from the bound (3.4.4) and the fact that the above calculation applies, where now \( W_0 = x_0 \), because \( B \) remains a Brownian motion starting at 0.

We can now give the proof of Proposition 3.1.5 (restated below for convenience). The quantity of interest is bounded in terms of the survival probability of an Ornstein-Uhlenbeck process \( Y \) killed at rate \( G(Y_s) \), for which we know the lead eigenvalue is \(-1\) by Proposition 3.3.5. This leads to the \( u^2 \) term in upper bound. Proposition 3.3.7 allows us to make the approximations which lead to the eigenvalue problem.

**Proposition 3.1.5.** There is a non-increasing function, \( c_{3.1.5}(t) \), such that for all \( t, \epsilon > 0 \) and \( u > 0 \):
(a) For any \( X_0 \in M_F(\mathbb{R}) \), \( E^{X}_{X_0} \left( \int_{s^*(t) - u}^{\infty} X(t, x) dx \right) \leq c_{3.1.5}(t) X_0(1) (u^2 \vee \epsilon) \).
(b) \( \mathbb{N}_0 \left( \int_{s^*(t) - u}^{\infty} X_t(dx) \right) \leq c_{3.1.5}(t) (u^2 \vee \epsilon) \).

**Proof.** The results are trivial if \( u > 1 \) so we may assume \( u \leq 1 \). Suppose first that \( 1 \geq u^2 \geq \epsilon \). By Lemma 3.4.2 it suffices to show
\[ E^B_0 \left( \exp \left( - \int_0^t v_s^{-1} (B_s + u) ds \right) \right) \leq c_{3.1.5}(t) (u^2 \vee \epsilon). \] (3.4.5)

By the scaling relation (3.3.11) the left-hand side of the above equals
\[ E^B_0 \left( \exp \left( - \int_0^t v_1^{-1} \left( \frac{B_s + u}{\sqrt{s}} \right) ds \right) \right). \] (3.4.6)

We define \( \hat{Y}_s = e^{-s/2} B e^s \), which defines a stationary Ornstein-Uhlenbeck process on \( \mathbb{R} \). As this process is reversible with respect to its stationary measure \( m \), \( Y_s = \hat{Y}_{-s} = e^{s/2} B e^{-s} \) is also a stationary Ornstein-Uhlenbeck process. We denote its expectation by \( E^Y \). An
exponential time change \((s = e^{-\hat{s}})\) shows that (3.4.6) is equal to

\[
E^Y \left( \exp \left( - \int_{-\log t}^{\infty} v_1^{e^{-\hat{s}}} \left( Y_{\hat{s}} + ue^{s/2} \right) d\hat{s} \right) \right) \tag{3.4.7}
\]

\[
= E^Y_m \left( \exp \left( - \int_{0}^{\infty} v_1^{e^{-s'}} \left( Y_{s'} + ut^{-1/2} e^{s'/2} \right) ds' \right) \right).
\]

The equality follows from changing variables to \(s' = \hat{s} + \log t\) and the stationarity of \(Y\).

We next truncate the integral and then add and subtract a \(v_1^\infty\) term. This shows that if \(p \in (0, 1/6)\), then (3.4.6) is at most

\[
E^Y_m \left( \exp \left( \int_{0}^{\log(1/u^2)} v_1^{\infty} - v_1^{e^{-s}} \left( Y_s + ut^{-1/2} e^{s/2} \right) ds \right) \right.
\]

\[
\times \exp \left( - \int_{0}^{\log(1/u^2)} v_1^{\infty} \left( Y_s + ut^{-1/2} e^{s/2} \right) ds \right) \right)
\]

\[
\leq E^Y_m \left( \exp \left( C_{3.3.7}(p) \int_{0}^{\log(1/u^2)} (\epsilon^{-1} e^{-s})^{-p} ds \right) \right.
\]

\[
\times \exp \left( - \int_{0}^{\log(1/u^2)} v_1^{\infty} \left( Y_s + ut^{-1/2} e^{s/2} \right) ds \right) \right). \tag{3.4.8}
\]

The inequality follows by Proposition 3.3.7. Moreover, since \(u^2 \geq \epsilon\),

\[
\int_{0}^{\log(1/u^2)} (\epsilon^{-1} te^{-s})^{-p} ds = t^{-p} \epsilon^{-p} e^{ps/p} \int_{0}^{\log(1/u^2)} \left( \frac{\epsilon}{u^2} \right)^p \leq \frac{t^{-p}}{p} \left( \frac{\epsilon}{u^2} \right)^p \leq \frac{t^{-p}}{p}.
\]

This bounds (3.4.6) above by

\[
e^{-C_{3.3.7}t^{-p}/p} E^Y_m \left( \exp \left( - \int_{0}^{\log(1/u^2)} G \left( Y_s + ut^{-1/2} e^{s/2} \right) ds \right) \right), \tag{3.4.9}
\]

where \(C_{3.3.7} = C_{3.3.7}(p)\) and we recall \(G = v_1^\infty\). Define \(\Delta(s)\) by

\[
\Delta(s) = |G(Y_s) - G \left( Y_s + ut^{-1/2} e^{s/2} \right)|.
\]

\(G'\) is continuous and has limit \(0\) at \(\pm \infty\), thus \(\|G'\|_\infty < \infty\). By the Mean Value Theorem,

\[
\Delta(s) \leq \|G'\|_\infty t^{-1/2} ue^{s/2}.
\]

144
Thus (3.4.9) is bounded above by

\[ e^{C_{3.3.7} t^{-p/p}} E_m^Y \left( \exp \left( \int_0^{\log(1/u^2)} \Delta(s) \, ds \right) \exp \left( - \int_0^{\log(1/u^2)} G(Y_s) \, ds \right) \right) \]

\[ \leq e^{C_{3.3.7} t^{-p/p}} E_m^Y \left( \exp \left( \|G\|_{\infty} t^{-1/2} u \int_0^{\log(1/u^2)} e^{s/2} \, ds \right) \right) \]

\[ \leq e^{C_{3.3.7} t^{-p/p} + 2\|G\|_{\infty} t^{-1/2}} E_m^Y \left( \exp \left( - \int_0^{\log(1/u^2)} G(Y_s) \, ds \right) \right). \]  \hspace{1cm} (3.4.10)

Let \( c(t) = e^{C_{3.3.7} t^{-p/p} + 2\|G\|_{\infty} t^{-1/2}} \). The remaining term is the probability that an Ornstein-Uhlenbeck process killed at rate \( G(Y_s) \) survives until time \( \log(1/u^2) \). If \( \rho^G \) is the lifetime of this process, we have bounded (3.4.6) by

\[ c(t) E_m^Y \left( \exp \left( - \int_0^{\log(1/u^2)} G(Y_s) \, ds \right) \right) = c(t) P_m^Y (\rho^G > \log(1/u^2)) \]

\[ \leq c(t) e^{-\lambda t^5} (\log(1/u^2)) = c_{3.1.5}(t) u^2. \]

The inequality follows from (3.2.4) in Theorem 3.2.1(b) and the final equality is by Proposition 3.3.5 and setting \( c_{3.1.5}(t) = Cc(t) \). This completes the proof when \( u^2 \geq \epsilon \). If \( u^2 < \epsilon \), we have for (b), say,

\[ N_0 \left( \int_{\tau - u}^{\infty} X_t(dx) \right) \leq N_0 \left( \int_{\tau - \sqrt{\epsilon}}^{\infty} X_t(dx) \right) \leq c_{3.1.5}(t) \epsilon, \]

where the final inequality follows by applying the \((u')^2 \geq \epsilon\) case with \( u' = \sqrt{\epsilon} \). The argument for (a) is the same. \( \square \)

We are now ready to give the proof of Theorem 3.1.2. As suggested in the Introduction, the method of proof is to decompose the measure \( X_{t-\delta} \) into two measures, to the right and left of \( \tau^\delta(t - \delta) \). We then show that there is a uniformly positive probability that, the measure to the right of \( \tau^\delta(t - \delta) \) produces positive mass (at time \( t \)) in \( L_t \) on a set far enough to the right that the mass from the measure to the left of \( \tau^\delta(t - \delta) \) does not interfere with it.

**Proof of Theorem 3.1.2.** First, consider \( P_{X_0}^X \). Let \( (\mathcal{F}_t) \) denote the usual completed right-continuous filtration generated by the associated historical process, \( H \). Fix \( t > 0 \). Let \( \delta_n = 2^{-n} \) and only consider \( n \) so that \( \delta_n < t/2 \). We will show that the martingale \( P_{X_0}^X (L_t > 0 | \mathcal{F}_{t-\delta_n}) \) is bounded below by a positive number a.s. on \( \{X_t > 0\} \), and so, as it converges to \( 1_{\{L_t > 0\}} \) a.s., the latter must be 1 a.s. on \( \{X_t > 0\} \).
Set \( \tau_n = \tau_n^\delta_n(t - \delta_n) \), that is,

\[
\tau_n = \inf \{ x \in \mathbb{R} : X_{t-\delta_n}([x, \infty)) < \delta_n \} \geq -\infty.
\]

Now invoke the decomposition in (3.2.12) and (3.2.13) with \( \tau = \tau_n \) and \( \delta = \delta_n \). That is, we define random measures by

\[
X_{t-\delta_n}^R(dx) = 1_{\{x \geq \tau_n\}} X_{t-\delta_n}(dx), \quad X_{t-\delta_n}^L(dx) = 1_{\{x < \tau_n\}} X_{t-\delta_n}(dx), \quad (3.4.11)
\]

and define measure-valued processes by

\[
\dot{X}_s^R(\phi) = \dot{X}_s^R \tau_n, \delta_n(\phi) = \int \phi(y_{t-\delta_n} + s)1(y_{t-\delta_n} \geq \tau_n) H_{t-\delta_n+s}(dy),
\]

\[
\dot{X}_s^L(\phi) = \dot{X}_s^L \tau_n, \delta_n(\phi) = \int \phi(y_{t-\delta_n} + s)1(y_{t-\delta_n} < \tau_n) H_{t-\delta_n+s}(dy),
\]

\[
\dot{X}_s(\phi) = \dot{X}_s^R(\phi) + \dot{X}_s^L(\phi) (= X_{t-\delta_n+s}(\phi)).
\]

Therefore by (3.2.12) and (3.2.13),

\[
X_t = \dot{X}_{\delta_n} = \dot{X}_{\delta_n}^R + \dot{X}_{\delta_n}^L, \quad \text{where conditional on } \mathcal{F}_{t-\delta_n}, \ \dot{X}^R \text{ and } \dot{X}^L \text{ are independent super-Brownian motions with initial states } X_{t-\delta_n}^R \text{ and } X_{t-\delta_n}^L, \text{ respectively.} \quad (3.4.12)
\]

Below we will argue conditionally on \( \mathcal{F}_{t-\delta_n} \) and hence work with this pair of independent super-Brownian motions, \( \dot{X}^R \) and \( \dot{X}^L \), with initial laws \( X_{t-\delta_n}^R \) and \( X_{t-\delta_n}^L \). We apply the cluster decomposition (3.2.6) to each of these super-Brownian motions to conclude

\[
\dot{X}_{\delta_n}^R \overset{\mathcal{D}}{=} \sum_{i=1}^{N_R} \dot{X}_{\delta_n}^{R,i}, \quad \dot{X}_{\delta_n}^L \overset{\mathcal{D}}{=} \sum_{i=1}^{N_L} \dot{X}_{\delta_n}^{L,i}, \quad (3.4.13)
\]

where \( N_R \) is a Poisson r.v. with rate \( 2X_{t-\delta_n}^R(1)/\delta_n \), \( N_L \) is an independent Poisson r.v. with rate \( 2X_{t-\delta_n}^L(1)/\delta_n \), and, conditional on \( (N_R, N_L) \), the clusters \( \{\dot{X}_{\delta_n}^{R,i} : i \leq N_R\} \) are iid with law \( \int N_x(X_{\delta_n} \in \cdot | X_{\delta_n} > 0) X_{t-\delta_n}^R(dx)/X_{t-\delta_n}^R(1) \) and similarly \( \{\dot{X}_{\delta_n}^{L,i} : i \leq N_L\} \) are iid with law \( \int N_x(X_{\delta_n} \in \cdot | X_{\delta_n} > 0) X_{t-\delta_n}^L(dx)/X_{t-\delta_n}^L(1) \). These last two collections are also conditionally independent. (Note also that if \( X_{t-\delta_n}^L = 0 \), say, then \( N_L = 0 \) and so there are no clusters to describe.) Let \( \dot{L}_{\delta_n}^R, \dot{L}_{\delta_n}^L \) and \( \dot{L}_{\delta_n} \) denote the boundary local times of \( \dot{X}_{\delta_n}^R, \dot{X}_{\delta_n}^L \) and \( \dot{X}_{\delta_n} \), respectively. By (3.4.12) and applying Theorem 3.2.2 conditionally on \( \mathcal{F}_{t-\delta_n} \) we
have
\[ \hat{L}_{\delta_n}(dx) = 1_{\{X_{\delta_n}^L(x) = 0\}} \hat{L}_n^R(dx) + 1_{\{X_{\delta_n}^R(x) = 0\}} \hat{L}_n^L(dx). \] (3.4.14)

We also let \( \hat{L}_{\delta_n}^{R,i} \) denote the boundary local time of \( \hat{X}_{\delta_n}^{R,i} \). If \( \mu \otimes \nu \) denotes product measure, it follows that

\[
P_{X_0}^X(L_t([\tau_n + 3\sqrt{\delta_n}, \infty)) > 0| \mathcal{F}_{t-\delta_n})
= P_{X_0}^X(\hat{L}_{\delta_n}([\tau_n + 3\sqrt{\delta_n}, \infty)) > 0| \mathcal{F}_{t-\delta_n})
\geq P_{X_{t-\delta_n}}^X \otimes P_{X_{t-\delta_n}}^X \left( \int 1(x \geq \tau_n + 3\sqrt{\delta_n})1(X_{\delta_n}(x) = 0) \hat{L}_n^R(dx) > 0 \right)
\geq P_{X_{t-\delta_n}}^X (\hat{L}_n^R([\tau_n + 3\sqrt{\delta_n}, \infty)) > 0)P_{X_{t-\delta_n}}^X (X_{\delta_n}(\tau_n + 3\sqrt{\delta_n}, \infty)) = 0), \] (3.4.15)

where in the third line we use (3.4.12) and (3.4.14). Now work on \( \{\tau_n > -\infty\} \in \mathcal{F}_{t-\delta_n} \) and consider the first term in (3.4.15). In this case \( X_{t-\delta_n}^R(1) = \delta_n \), and so \( N_R \) is Poisson with mean 2. Therefore by restricting to \( \{N_R = 1\} \) and noting that in this case \( \hat{L}_{\delta_n}^R = \hat{L}_{\delta_n}^{R,1} \), we have

\[
P_{X_{t-\delta_n}}^X (\hat{L}_n^R([\tau_n + 3\sqrt{\delta_n}, \infty)) > 0)
\geq 2e^{-2}P_{X_{t-\delta_n}}^X (\hat{L}_n^R([\tau_n + 3\sqrt{\delta_n}, \infty)) > 0| N_R = 1)
= 2e^{-2} \int_{\tau_n}^{\infty} N_0(L_{\delta_n}([\tau_n + 3\sqrt{\delta_n}, \infty)) > 0| X_{\delta_n} > 0)X_{t-\delta_n}^R(dx)/\delta_n
\geq 2e^{-2}N_0(L_{\delta_n}([3\sqrt{\delta_n}, \infty)) > 0| X_{\delta_n} > 0),
\]

where the last line again uses \( X_{t-\delta_n}^R(1) = \delta_n \) on \( \{\tau_n > -\infty\} \). Therefore Lemma 3.4.1 and (3.4.15) now imply that on \( \{\tau_n > -\infty\} \),

\[
P_{X_0}^X(L_t([\tau_n + 3\sqrt{\delta_n}, \infty)) > 0| \mathcal{F}_{t-\delta_n}) \geq 2e^{-2}c_{3.4.1} \int P_{X_{t-\delta_n}}^X (X_{\delta_n}^L([\tau_n + 3\sqrt{\delta_n}, \infty)) = 0).
\] (3.4.16)

It remains to handle the final probability. We will consider events on which it has a uniform lower bound and which will occur infinitely often in \( n \). For \( K \in \mathbb{N} \), define an event \( A_{K,n} \in \mathcal{F}_{t-\delta_n} \) by

\[
A_{K,n} = \left\{ \int_{3}^{\infty} we^{-w^2/2} X_{t-\delta_n}(\tau_n - (w - 3)\sqrt{\delta_n}) dw \leq K\sqrt{\delta_n} \right\}.
\] (3.4.17)

(Note that the \( A_{K,n} \) depends only on mass to the left of \( \tau_n \), and so the measure in the
integral is equal to $X_{t-\delta_n}^L$. Noting that $X_{t-\delta_n}(-\infty) = 0$, we see that $\{\tau_n = -\infty\} \subseteq A_{K,n}$. On $A_{K,n}$, we have the following lower bound on the probability on the right-hand side of (3.4.16):

$$P_{X_{t-\delta_n}}^X(\hat{X}_n([\tau_n + 3\sqrt{\delta_n}, \infty)) = 0) \geq e^{-c \cdot 3.2.3 K} =: q_K.$$ (3.4.18)

To prove (3.4.18), first note it is trivial when $\tau_n = -\infty$, because in this case $X_{t-\delta_n}^L = 0$. To see it when $\tau_n > -\infty$ we apply Theorem 3.2.3(ii) with $R = 3\sqrt{\delta_n}$ and initial state $X_{t-\delta_n}^L$, along with translation invariance and the change of variables $w = (\tau_n + 3\sqrt{\delta_n} - x)/\sqrt{\delta_n}$, to obtain (on $A_{K,n}$).

$$P_{X_{t-\delta_n}}^X(\hat{X}_n([\tau_n + 3\sqrt{\delta_n}, \infty)) = 0) \geq \exp\left(-c \cdot 3.2.3 \int_{-\infty}^{\tau_n} (\tau_n + 3\sqrt{\delta_n} - x)^2 \left(\frac{\tau_n + 3\sqrt{\delta_n} - x}{\sqrt{\delta_n}}\right)^3 \times \exp(-3\delta_n) X_{t-\delta_n}(x) dx\right)$$

$$= \exp\left(-c \cdot 3.2.3 \frac{1}{\sqrt{\delta_n}} \int_{-\infty}^{\infty} e^{-w^2/2} X_{t-\delta_n}(\tau_n - (w - 3)\sqrt{\delta_n}) dw\right)$$

$$\geq e^{-c \cdot 3.2.3 K},$$ (3.4.19)

which proves (3.4.18), with the final inequality using the fact that $\omega \in A_{K,n}$.

Let $\Lambda_K = \{A_{K,n} \cap \{\tau_n < -\infty\} \text{ infinitely often in } n\}$. That is,

$$\Lambda_K = \bigcap_{M=1}^{\infty} \bigcup_{n \geq M} (A_{K,n} \cap \{\tau_n < -\infty\}).$$ (3.4.20)

By (3.4.16) and (3.4.18), for all $\omega \in \Lambda_K$, we have

$$P_{X_0}^X(L_t([\tau_n + 3\sqrt{\delta_n}, \infty]) > 0 \mid \mathcal{F}_{t-\delta_n}) \geq 2e^{-2c \cdot 3.4.1(3)} q_K =: p_K \text{ for infinitely many } n.$$ 

It follows that

$$\limsup_{n \to \infty} P_{X_0}^X(L_t > 0 \mid \mathcal{F}_{t-\delta_n}) \geq \limsup_{n \to \infty} P_{X_0}^X(L_t([\tau_n + 3\sqrt{\delta_n}, \infty]) > 0 \mid \mathcal{F}_{t-\delta_n}) \geq p_K \text{ a.s. on } \Lambda_K.$$ (3.4.21)

Moreover, $P_{X_0}^X(L_t > 0 \mid \mathcal{F}_{t-\delta_n})$ is a bounded martingale and converges a.s. to $P_{X_0}^X(L_t > 0 \mid \mathcal{F}_{t-})$. Because $s \to X_s$ is a continuous map, we have $X_t = X_{t-}$ and so $X_t$ is measurable with respect to $\mathcal{F}_{t-}$. Moreover, $L_t$ is defined as a measurable functional of $X_t$ (recall
Theorem 3.A(a)). Thus we have

\[ P^X_{X_0}(L_t > 0 \mid \mathcal{F}_{t-\delta_n}) \to P^X_{X_0}(L_t > 0 \mid \mathcal{F}_{t-}) = 1_{\{L_t > 0\}} \quad \text{a.s.} \]  \hspace{1cm} (3.4.22)

By (3.4.21), this implies that \( 1_{\{L_t > 0\}}(\omega) \geq p_K > 0 \) a.s. on \( \Lambda_K \), and hence

\[ L_t > 0 \text{ almost surely on } \Lambda_K. \] \hspace{1cm} (3.4.23)

The final ingredient of the proof is to show that \( \Lambda_K \uparrow \{X_t > 0\} \) as \( K \to \infty \) a.s. We proceed by bounding the probability of \( A^c_{K,n} \). Using (3.4.17) and Markov’s inequality, we have

\[ P^X_{X_0}(A^c_{K,n}) = P^X_{X_0}(A^c_{K,n} \cap \{\tau_n > -\infty\}) \leq \frac{1}{K\sqrt{\delta_n}} \mathbb{E}^X_{X_0} \left( 1_{\{\tau_n > -\infty\}} \int_{3}^{\infty} w e^{-w^2/2} X_{t-\delta_n}(\tau_n - (w - 3)\sqrt{\delta_n}) dw \right). \] \hspace{1cm} (3.4.24)

The first equality follows because \( A^c_{K,n} \subseteq \{\tau_n > -\infty\} \). We proceed by integration by parts. For \( w > 3 \), define \( g(w) \) by

\[ g(w) = \frac{1}{\sqrt{\delta_n}} X_{t-\delta_n}([\tau_n - (w - 3)\sqrt{\delta_n}, \tau_n]) = \left( \int_{3}^{w} X_{t-\delta_n}(\tau_n - (u - 3)\sqrt{\delta_n}) du \right). \]

The last expression, which follows from a change of variable, makes it clear that

\[ g'(w) = X_{t-\delta_n}(\tau_n - (w - 3)\sqrt{\delta_n}). \]

Clearly \( g(3) = 0 \). Taking \( f(w) = we^{-w^2/2} \) and integrating by parts, on \( \{\tau_n > -\infty\} \) we have

\[ \int_{3}^{\infty} w e^{-w^2/2} X_{t-\delta_n}(\tau_n - (w - 3)\sqrt{\delta_n}) dw = \left[ f(w)g(w) \right]_{3}^{\infty} - \int_{3}^{\infty} f'(w)g(w) dw = 0 + \frac{1}{\sqrt{\delta_n}} \int_{3}^{\infty} (w^2 - 1)e^{-w^2/2} X_{t-\delta_n}([\tau_n - (w - 3)\sqrt{\delta_n}, \tau_n]) dw. \]

We substitute this into (3.4.24) and exchange the order of integration (the integrand is positive) to obtain

\[ P^X_{X_0}(A^c_{K,n}) \leq \frac{1}{\delta_n K} \int_{3}^{\infty} (w^2 - 1)e^{-w^2/2} P^X_{X_0}(1_{\{\tau_n > -\infty\}} X_{t-\delta_n}([\tau_n - (w - 3)\sqrt{\delta_n}, \tau_n])) dw. \] \hspace{1cm} (3.4.25)

We now note that the mass term appearance in the integral can be controlled by Proposition
3.1.5. We have for \( w \geq 3 \)

\[
P_{X_0}^X(1_{\{\tau_n > -\infty\}} X_{t-\delta_n} (\tau_n - (w - 3) \sqrt{\delta_n}, \tau_n)) \leq P_{X_0}^X \left( \int_{\tau_n - \sqrt{\delta_n}(w-3)}^{\infty} X_{t-\delta_n} (x) \, dx \right)
\]

\[
\leq c_{3.1.5} (t/2) X_0(1) (\delta_n + ((w - 3) \sqrt{\delta_n})^2)
= c_{3.1.5} (t/2) X_0(1) \delta_n (1 + (w - 3)^2).
\]

(3.4.26)

The second inequality is by Proposition 3.1.5 and our initial assumption that \( \delta_n < t/2 \).

Using (3.4.26) in (3.4.25), we obtain for \( n \geq n_0 \),

\[
P_{X_0}^X(A_{K,n}^c) \leq c_{3.1.5} (t/2) X_0(1)
\]

(3.4.27)

This allows us to bound \( P_{X_0}^X(\{X_t > 0\} \cap \Lambda_K^c) \) as follows:

\[
P_{X_0}^X(\{X_t > 0\} \cap \Lambda_K^c) = \lim_{M \to \infty} P_{X_0}^X(\{X_t > 0\} \cap \left[ \bigcap_{n=M}^{\infty} A_{K,n}^c \cup \{\tau_n = -\infty\} \right])
\]

\[
\leq \lim_{M \to \infty} P_{X_0}^X(\{X_t > 0\} \cap \left[ \bigcap_{n=M}^{\infty} A_{K,n}^c \right] \cup \left[ \bigcup_{n=M}^{\infty} \{\tau_n = -\infty\} \right])
\]

\[
\leq \lim_{M \to \infty} P_{X_0}^X\left( \bigcap_{n=M}^{\infty} A_{K,n}^c \right) \cup P_{X_0}^X(\{X_t(1) > 0\} \cap \{X_{t-\delta_n} \leq \delta_n \text{ i.o.}\})
\]

\[
\leq \frac{c_0}{K}.
\]

(3.4.28)

The second term vanishes because \( s \to X_s(1) \) is continuous almost surely, and the bound on the first is by (3.4.27). We therefore have that

\[
P_{X_0}^X(\{X_t > 0\} \setminus \Lambda_K) \leq \frac{c_0}{K},
\]

(3.4.29)

and hence for \( P_{X_0}^X \)-almost all \( \omega \in \{X_t > 0\} \), \( \omega \in \Lambda_K \) for \( K \) sufficiently large. Here we also use the fact that \( \Lambda_K \) is increasing in \( K \). This and (3.4.23) completes the proof that \( L_t > 0 \) a.s. on \( \{X_t > 0\} \).

The claim that \( L_t((U_t - \delta, U_t)) > 0 \) almost surely on \( \{X_t > 0\} \) now follows from two elementary lemmas, the second of which is left as a standard exercise (a variant is known as Hunt’s Lemma).  

150
Lemma 3.4.3. For all $\delta > 0$, almost surely we have
\[ \limsup_{n \to \infty} 1_{\{L_t([\tau_n + 3\sqrt{\delta_n}, \infty)) > 0\}} \leq 1_{\{L_t((U_t - \delta, U_t)) > 0\}}. \tag{3.4.30} \]

Lemma 3.4.4. Let $(\mathcal{F}_n)_{n \in \mathbb{N}}$ be an arbitrary filtration, $\mathcal{F}_\infty$ the minimal $\sigma$-algebra containing $\mathcal{F}_n$ for all $n$, and let $\{Y_n\}_{n \in \mathbb{N}}$ be a sequence of random variables such that $|Y_n| \leq W$ for all $n \in \mathbb{N}$ for some integrable $W$. Then
\[ \limsup_{n \to \infty} E(Y_n \mid \mathcal{F}_n) \leq E(\limsup_{n \to \infty} Y_n \mid \mathcal{F}_\infty). \]

Lemma 3.4.3 is proved at the end of the section. First we see how they complete the proof of Theorem 3.1.2 for $P_{X_0}^Y$. Applying (3.4.21) and the Lemmas, with $Y_n = 1_{\{L_t([\tau_n + 3\sqrt{\delta_n}, \infty)) > 0\}}$ in Lemma 3.4.4, we have
\[ 1_{\{L_t((U_t - \delta, U_t)) > 0\}} \geq \limsup_{n \to \infty} E_{X_0}^Y(1_{\{L_t([\tau_n + 3\sqrt{\delta_n}, \infty)) > 0\}} \mid \mathcal{F}_{t - \delta_n}) \geq P_K \text{ a.s. on } \Lambda_K. \]

So by (3.4.29), $L_t((U_t - \delta, U_t)) > 0$ $P_{X_0}^Y$-a.s. on $\{X_t > 0\}$. \hfill \qed

Proof of Lemma 3.4.3. We may work on $\{X_t > 0\}$ as both sides are zero if $X_t = 0$. By Dominated Convergence we have
\[ \lim_{n \to \infty} \int_{U_t - \delta}^{U_t} X_{t - \delta_n}(x) \, dx = \int_{U_t - \delta}^{U_t} X(t, x) \, dx > 0 \text{ on } \{X_t > 0\}. \]
This implies that for $n$ large enough, $\int_{U_t - \delta}^{U_t} X_{t - \delta_n}(x) \, dx > 2^{-n}$, and so
\[ \tau_n + 3\sqrt{\delta_n} > \tau_n > U_t - \delta \text{ for } n \text{ large}. \]
Therefore for $n$ sufficiently large,
\[ L_t([\tau_n + 3\sqrt{\delta_n}, \infty)) \leq L_t((U_t - \delta, \infty)) = L_t((U_t - \delta, U_t)), \]
and the result follows. \hfill \qed

This completes the proof of Theorem 3.1.2 under $P_{X_0}^Y$. To see that the same holds under $N_0$, we apply the above result with $X_0 = \delta_0$. We may assume that $X_t$ is defined by the right-hand side of the cluster decomposition (3.2.6). So $X_t$ is a sum of $N \sim \text{Poisson}(2/t)$ independent canonical clusters with law $N_0(X_t \in \cdot \mid X_t > 0)$, and $N = 1$ with probability
$2^{-1}e^{-2/t} > 0$. In particular we can condition on $N = 1$, which gives

$$
\mathbb{N}_0(X_t((U_t - \delta, U_t)) > 0 \text{ for all } \delta > 0 \mid X_t > 0)
= P_{\delta_0}^X(X_t((U_t - \delta, U_t)) > 0 \text{ for all } \delta > 0 \mid N = 1) = 1
$$

by the result under $P_{\delta_0}^X$ and the inclusion $\{N = 1\} \subset \{X_t > 0\}$. Thus the result also holds under $\mathbb{N}_0$.

### 3.5 Localization

In this section we prove Theorem 3.1.3, which states that $L_t$ has positive mass on any neighborhood of any point in $\partial S(X_t)$ almost surely. The proof uses both decompositions from Section 3.2.2, Theorem 3.1.2, and the elementary topological fact that if $x \in \partial S(X_t)$, there is a sequence of open “holes” in the support near $x$ (Lemma 3.5.1 below).

Let $d_H$ denote the Hausdorff metric on non-empty compact subsets of $\mathbb{R}$. That is, $d_H(K_1, K_2) = d_0(K_1, K_2) + d_0(K_2, K_1)$, where $d_0(K_1, K_2) = \inf\{\delta > 0 : K_1 \subset K_2^\delta\}$ and $K_2^\delta$ is the set of points which are less than distance $\delta$ from $K_2$.

**Lemma 3.5.1.** For $x_0 \in \mathbb{R}$, $x_0 \in \partial S(X_t)$ if and only if there exists two sequences of non-empty open intervals $I_m$ and $J_m$ such that $d_H(I_m, \{x_0\}), d_H(J_m, \{x_0\}) \to 0$ as $m \to \infty$, which satisfy $X_t(\cdot) | I_m = 0$ and $X_t(\cdot) | J_m > 0$ for all $m$.

**Proof.** Let $x_0 \in \partial S(X_t)$. A sequence $(J_m)_{m=1}^\infty$ with the described conditions must exist because $X_t(\cdot)$ is continuous. We know $B(x_0, 2^{-m}) \not\subset \{X_t > 0\}$ because $x_0$ is not an interior point of $\{X_t > 0\}$. So we may choose an open interval $I_m$ inside the non-empty open set $B(x_0, 2^{-m}) \cap \{X_t > 0\}^c$ which is contained in $B(x_0, 2^{-m}) \cap \{X_t = 0\}$, as required. We leave the converse as an easy exercise.

**Proof of Theorem 3.1.3.** We first work under $P_{X_0}^X$ and may assume $X_t = H_t(\{y_t \in \cdot\})$ where $H$ is an associated historical process. Let $t > 0$, $q \in \mathbb{Q}$, and $\delta_n = 2^{-n}$ where we may consider only $\delta_n < t$. We again use the decomposition (3.2.12), now with $\tau = q$ and $\delta = \delta_n$, that is

$$
\hat{X}_s^{L,q,\delta_n}(\phi) = \int \phi(y_{t-\delta_n+s}) 1(y_{t-\delta_n} < q) H_{t-\delta_n+s}(dy)
$$

$$
\hat{X}_s^{R,q,\delta_n}(\phi) = \int \phi(y_{t-\delta_n+s}) 1(y_{t-\delta_n} \geq q) H_{t-\delta_n+s}(dy).
$$

(3.5.1)

As $\hat{X}_s^{L,q,\delta_n} + \hat{X}_s^{R,q,\delta_n} = X_{t-\delta_n+s}$, both $\hat{X}_s^{L,q,\delta_n}$ and $\hat{X}_s^{R,q,\delta_n}$ have densities, which we denote by $\hat{X}_s^{L,q,\delta_n}(x)$ and $\hat{X}_s^{R,q,\delta_n}(x)$. Recall from (3.2.11) that $X_{t-\delta_n}(A) = X_{t-\delta_n}(A \cap (-\infty, q))$ and $X_{t-\delta_n}(A) = X_{t-\delta_n}(A \cap [q, \infty))$ for measurable $A \subset \mathbb{R}$. Let $\mathcal{F}_t$ be the usual right
Recall the definitions of $S$, therefore by (3.5.3) and (3.5.4), there are sequences $(I_n)$ and $(J_n)$ such that $\{X_{\delta_n}^{I_n}\}$ and $\{X_{\delta_n}^{J_n}\}$ are independent super-Brownian motions with initial laws $X^{L,q}\delta_n$ and $X^{R,q}\delta_n$, respectively.

Therefore by Theorem 3.1, for each $s > 0$, $X_{\delta_n}^{R,q}\delta_n$ and $X_{\delta_n}^{L,q}\delta_n$ each have a boundary local time, which we denote by $\hat{L}_{X_{\delta_n}^{L,q}\delta_n}$ and $\hat{L}_{X_{\delta_n}^{R,q}\delta_n}$, respectively. By (3.5.2) and applying Theorem 3.2.2 conditionally on $F_{t-\delta_n}$, we have the following decomposition of $L_t$:

$$L_t(\phi) = \int \phi(x)1(\hat{X}_{\delta_n}^{R,q}\delta_n(x) = 0) \, d\hat{L}_{X_{\delta_n}^{L,q}\delta_n}(x) + \int \phi(x)1(\hat{X}_{\delta_n}^{L,q}\delta_n(x) = 0) \, d\hat{L}_{X_{\delta_n}^{R,q}\delta_n}(x).$$

(3.5.3)

Let $U^q\delta_n = \sup S(\hat{X}_{\delta_n}^{L,q}\delta_n)$. By (3.5.2) and applying Theorem 3.1.2 conditionally on $F_{t-\delta_n}$, we see that $\hat{L}_{X_{\delta_n}^{L,q}\delta_n}(U^q\delta_n - \delta, U^q\delta_n) > 0$ for all $\delta > 0$ almost surely on $\{\hat{X}_{\delta_n}^{L,q}\delta_n > 0\}$.

Taking a union over countable events, this implies that

$$\left(\forall n \in \mathbb{N}, \forall q \in \mathbb{Q}, \hat{X}_{\delta_n}^{L,q}\delta_n(1) > 0 \Rightarrow \hat{L}_{X_{\delta_n}^{L,q}\delta_n}(U^q\delta_n - \delta, U^q\delta_n) > 0 \forall \delta > 0\right) \text{ } P_{X_0}^X \text{-a.s.}$$

(3.5.4)

The fact that $S(\hat{L}_{X_{\delta_n}^{R,q}\delta_n}) \subset \partial\{x : \hat{X}_{\delta_n}^{R,q}\delta_n(x) > 0\}$ implies that

$$\hat{X}_{\delta_n}^{R,q}\delta_n((\infty, U^q\delta_n)) = 0 \Rightarrow \hat{L}_{X_{\delta_n}^{R,q}\delta_n}(U^q\delta_n - \delta, U^q\delta_n) = 0 \text{ a.s.}.$$  

Therefore by (3.5.3) and (3.5.4),

$$\left(\forall n \in \mathbb{N}, \forall q \in \mathbb{Q}, \hat{X}_{\delta_n}^{L,q}\delta_n(1) > 0 \text{ and } \hat{X}_{\delta_n}^{R,q}\delta_n((\infty, U^q\delta_n)) = 0 \text{ imply } \hat{L}_{X_{\delta_n}^{L,q}\delta_n}(U^q\delta_n - \delta, U^q\delta_n) > 0 \forall \delta > 0\right) \text{ } P_{X_0}^X \text{-a.s.}$$

(3.5.5)

Recall the definitions of $h$ and $K(c, \delta)$ from Section 3.2.2 (see (2.7)). By the modulus of continuity for $S(H_t)$, (3.2.8), if $c > 2$, then $P_{X_0}^X$-a.a. $\omega$, there exists $\delta = \delta(c, \omega) > 0$ such that $S(H_t) \subset K(c, \delta)$. Thus there exists $\Omega_0 \subset \Omega$ such that $P_{X_0}^X(\Omega_0^c) = 0$ and for all $\omega \in \Omega_0$, the event in (3.5.5) and $S(H_t) \subset K(3, \delta)$ for some $\delta(3, \omega) > 0$ both hold. Let $\omega \in \Omega_0$. Let $a < b$ and suppose that $(a, b) \cap \partial S(X_t) \neq \emptyset$. Then there exists $x_0 \in \partial S(X_t) \cap (a, b)$. By Lemma 3.5.1 there are sequences $(I_m)_{m=1}^\infty$, $(J_m)_{m=1}^\infty$ of non-empty open intervals converging to $x_0$ with respect to $d_H$ such that $X_{I_m}(\cdot)|_{I_m} = 0$ and $X_{J_m}(\cdot)|_{J_m} > 0$. Suppose $I_m = (a_m, b_m)$ and $J_m = (d_m, e_m)$. Since $I_m, J_m \to \{x_0\}$, we can consider $m$ large enough so that $\overline{I_m, J_m} \subset (a, b)$. Without loss of generality we assume that $J_m$ lies to the left of $I_m$, i.e. that $e_m < a_m$. (If $J_m$ lies to the right of $I_m$, a symmetrical argument dealing with the left-hand endpoints $L^r\delta_n$ of the supports of $X^r\delta_n$ applies).
Let $I'_m$ be the open middle third of $I_m$, i.e., $b_m - a_m = l_m$ and $I'_m = (a_m + l_m/3, a_m + 2l_m/3)$. Choose $q \in \mathbb{Q} \cap I'_m$ and $n \in \mathbb{N}$ large enough so that $3h(\delta_n) < l_m/3$ and $\delta_n < \delta(3, \omega)$. By (3.5.1) and the modulus of continuity,

$$S(\hat{X}_{\delta_n}^{R,q,\delta_n}) \subseteq (q - 3h(\delta_n), \infty) \subseteq (a_m, \infty),$$

and hence

$$\hat{X}_{\delta_n}^{R,q,\delta_n}((\infty, a_m]) = 0. \tag{3.5.6}$$

Moreover, because $(\hat{X}_{\delta_n}^{L,q,\delta_n} + \hat{X}_{\delta_n}^{R,q,\delta_n})(\cdot) = X_t(\cdot) > 0$ on $J_m = (d_m, e_m)$, and $e_m < a_m$, we have that

$$\hat{X}_{\delta_n}^{L,q,\delta_n}((d_m, a_m)) > 0. \tag{3.5.7}$$

Furthermore, the modulus of continuity also implies that

$$S(\hat{X}_{\delta_n}^{L,q,\delta_n}) \subseteq (\infty, q + 3h(\delta_n)) \subseteq (\infty, b_m),$$

where the last inclusion holds since $3h(\delta_n) < l_m/3$ and $q \in I'_m$. The above, together with $X_t((a_m, b_m)) = 0$, implies that $S(\hat{X}_{\delta_n}^{L,q,\delta_n}) \subseteq (-\infty, a_m]$. This and (3.5.7) imply $U_{q,\delta_n} \in (d_m, a_m] \subset (a, b)$. By (3.5.6) this implies that $\hat{X}_{\delta_n}^{R,q,\delta_n}((-\infty, U_{q,\delta_n})) = 0$ and so by (3.5.7) we may apply (3.5.5) and conclude that $L_t((U_{q,\delta_n} - \delta, U_{q,\delta_n})) > 0$ for all $\delta > 0$. Since $U_{q,\delta_n} \in (d_m, a_m]$, choosing $\delta = (d_m - a)/2 > 0$ (the last by $J_m \subset (a, b)$) gives $(U_{q,\delta_n} - \delta, U_{q,\delta_n}) \subset (a, b)$, and hence $L_t((a, b)) > 0$. This proves the result for $P_{X_{\delta_n}}^X$.

To see that the same holds under $N_0$, we proceed as in the proof of Theorem 3.1.2 and condition that the Poisson number of clusters, $N$ in (3.2.6), is one to get

$$N_0((a, b) \cap \partial S(X_t) \neq \emptyset \Rightarrow L_t((a, b)) > 0 \mid X_t > 0) = P_{\delta_0}^X((a, b) \cap \partial S(X_t) \neq \emptyset \Rightarrow L_t((a, b)) > 0 \mid N = 1) = 1.$$

Thus, under $N_0$ the result holds almost surely on $\{X_t > 0\}$ for all rational $a, b$, and hence holds almost surely. \[\square\]
Chapter 4

New properties for the density of the \((\alpha, \beta)\)-superprocess

Summary. We consider the density \(X_t(x)\) of the critical \((\alpha, \beta)\)-superprocess in \(\mathbb{R}^d\) with \(\alpha \in (0, 2)\) and \(\beta < \frac{2}{d}\). A recent result from partial differential equations [7] implies the following dichotomy: for fixed \(x \in \mathbb{R}^d\), \(X_t(x) > 0\) a.s. on \(\{X_t \neq 0\}\) if and only if \(\beta \leq \beta^*(\alpha) := \frac{\alpha}{d + \alpha}\).

We strengthen this by proving that if \(\beta < \beta^*(\alpha)\) and the density is continuous, which holds if and only if \(d = 1\) and \(\alpha > 1 + \beta\), then \(X_t(x) > 0\) for all \(x \in \mathbb{R}\) a.s. on \(\{X_t \neq 0\}\).

In the \(d\)-dimensional case when \(\beta < \frac{2}{d}\) we then give close to sharp conditions on a measure \(\mu\) such that \(\mu(X_t) := \int X_t(x) \mu(dx) > 0\) a.s. on \(\{X_t \neq 0\}\). Our characterization is based on the size of the support of \(\mu\), in the sense of Hausdorff measure and dimension.

For \(s \in [0, d]\), if \(\beta \leq \beta^*(\alpha, s) = \frac{\alpha}{d - s + \alpha}\) and \(\text{supp}(\mu)\) has positive \(x^s\)-Hausdorff measure, then \(\mu(X_t) > 0\) a.s. on \(\{X_t \neq 0\}\); and when \(\beta > \beta^*(\alpha, s)\), if \(\mu\) satisfies a uniform lower density condition which implies \(\text{dim}(\text{supp}(\mu)) < s\), then \(P(\mu(X_t) = 0 | X_t \neq 0) > 0\).

Our methods also give new results for the fractional non-linear equation which is dual to the superprocess, i.e.

\[
\partial_t u(t, x) = -(-\Delta)^{\frac{s}{2}} u(t, x) - u(t, x)^{1+\beta}
\]

with domain \((t, x) \in (0, \infty) \times \mathbb{R}^d\). The initial trace of a solution \(u_t(x)\) (see [8]) is a pair \((S, \nu)\), where the singular set \(S\) is a closed set around which local integrals of \(u_t(x)\) diverge as \(t \to 0\), and \(\nu\) is a Radon measure which gives the limiting behaviour of \(u_t(x)\) on \(S^c\) as \(t \to 0\). For \(\beta < \frac{s}{d}\) we characterize the problem of existence of solutions with initial trace \((S, 0)\) in terms of a critical dimension \(d_c = d + \alpha(1 - \beta^{-1})\). For \(S \neq \mathbb{R}^d\) with \(\text{dim}(S) > d_c\) (and in some cases \(\text{dim}(S) = d_c\)) we prove that no such solution exists. When \(\text{dim}(S) < d_c\) and \(S\) is the compact support of a measure satisfying a uniform lower density condition, we prove that a solution exists.
4.1 Introduction and statement of results

In this chapter we study some path properties of the \((\alpha, \beta, d)\)-superprocess. For parameters \(\alpha \in (0, 2)\), \(\beta \in (0, 1]\), \(d \in \mathbb{N}\), the \((\alpha, \beta, d)\)-superprocess, or simply the \((\alpha, \beta)\)-superprocess when the dimension is fixed, is a strong Markov process taking values in \(\mathcal{M}_F(\mathbb{R}^d)\), the space of finite measures on \(\mathbb{R}^d\) equipped with the topology of weak convergence. We will denote it by \(X = (X_t : t \geq 0)\), so that \(X_t \in \mathcal{M}_F(\mathbb{R}^d)\). The spatial Markov process associated to \(X\) is a rotationally symmetric \(\alpha\)-stable process in \(\mathbb{R}^d\) and, for \(\beta \in (0, 1)\), the branching mechanism is that of a continuous state branching process with \((1 + \beta)\)-stable branching. When \(\beta = 1\) the superprocess is binary branching and the associated continuous state branching process is Feller’s branching diffusion.

The paths of \(X\) live in \(D([0, \infty), \mathcal{M}_F(\mathbb{R}^d))\), the space of càdlàg paths in \(\mathcal{M}_F(\mathbb{R}^d)\). The Markov transition kernel of \(X\) is defined by its Laplace functional, which is characterized via a dual relationship with a fractional non-linear evolution equation. For a general probability space realizing \(X\) with initial value \(X_0 \in \mathcal{M}_F(\mathbb{R}^d)\), we will write \((\Omega, \mathcal{F}, P_{X_0})\), and denote the associated expectation by \(E_{X_0}\). Let \(\mathcal{B}_b^+(\mathbb{R}^d) = B_b^+\) denote the space of bounded, measurable functions on \(\mathbb{R}^d\). Then for every \(\phi \in \mathcal{B}_b^+\),

\[
E_{X_0}^X(\exp(-X_t(\phi))) = \exp(-X_0(u_t^\phi)), \tag{4.1.1}
\]

where \(X_t(\phi) = \int \phi(x)X_t(dx)\), and \(u_t^\phi(x)\) is the unique solution of the evolution equation

\[
u_t(x) = S_t \phi(x) - \int_0^t S_{t-s}(u_s^{1+\beta}) ds
\]

for \((t, x) \in Q := (0, \infty) \times \mathbb{R}^d\). \((S_t)_{t \geq 0}\) denotes the transition semigroup of the isotropic (rotationally symmetric) \(\alpha\)-stable process in \(\mathbb{R}^d\). (See Theorem 4.4.1 of [13] for the existence and uniqueness of solutions to (4.1.2) and the derivation of (4.1.1).) The generator of the \(\alpha\)-stable process is the fractional Laplacian \(\Delta_\alpha = -(-\Delta)^{\frac{\alpha}{2}}\). We will use the probabilistic convention, so that \(\Delta_\alpha\) corresponds to the \(\alpha\)-stable process; the actual “fraction” associated to this operator, and the parameter commonly used in the partial differential equation (PDE) literature, is then \(\frac{\alpha}{2} \in (0, 1)\). In the singular integral formulation, \(\Delta_\alpha\) is defined as

\[
\Delta_\alpha f(x) := \lim_{\epsilon \to 0^+} -a_{\alpha,d} \int_{\mathbb{R}^d} \frac{f(x) - f(y)}{|x - y|^{d+\alpha}} \chi_\epsilon(|x - y|) \, dy
\]

for a constant \(a_{\alpha,d} > 0\), where

\[
\chi_\epsilon(r) = \begin{cases} 
1 & \text{if } r > \epsilon, \\
0 & \text{if } r \in [0, \epsilon].
\end{cases}
\]
The integral equation (4.1.2) corresponds to the fractional PDE
\[ \partial_t u = \Delta_\alpha u - u^{1+\beta}. \] (4.1.3)

When they exist, solutions of (4.1.3) and (4.1.2) generally coincide. In Section 4.2.3 we define weak solutions to (4.1.3) (see Definition 4.2.4) and make this correspondence rigorous in certain cases. In addition to studying solutions of (4.1.3) (or (4.1.2)) as a means of proving properties of \( X \), we also prove novel results concerning the existence and non-existence of solutions to (4.1.3) with very singular initial conditions.

We are interested in the density of \( X_t \). It is a classical result of Fleischmann [27] that \( X_t \) is absolutely continuous if and only if \( \beta < \frac{\alpha}{d} \). This work is concerned only with absolutely continuous case, and we restrict to it now.

**Assumption.** For the remainder of this chapter, we assume \( \beta < \frac{\alpha}{d} \).

Under this assumption, \( X_t \) has a density \( X_t(x) \), so that \( X_t(dx) = X_t(x)dx \). A priori, a generic density is only defined up to Lebesgue-null sets. Much of this work concerns the behaviour of the density on such sets, so this is not sufficient. In particular, we need the object
\[ \mu(X_t) := \int_{\mathbb{R}^d} X_t(x) \mu(dx) \] (4.1.4)
to be well-defined for \( \mu \in \mathcal{M}_F(\mathbb{R}^d) \). Let \( B(x, r) \) denote the closed ball of radius \( r > 0 \) around \( x \in \mathbb{R}^d \). We define
\[ X^\epsilon_t(x) = \frac{X_t(B(x, \epsilon))}{|B(x, \epsilon)|}, \] (4.1.5)
where \( |A| \) denotes the Lebesgue measure of \( A \subset \mathbb{R}^d \). By the Lebesgue Differentiation Theorem, \( X^\epsilon_t(x) \) converges as \( \epsilon \downarrow 0 \) for Lebesgue-a.e. \( x \in \mathbb{R}^d \), and consequently
\[ X_t(x) := \liminf_{\epsilon \downarrow 0} X^\epsilon_t(x) \] (4.1.6)
is a density for \( X_t \). In fact, more holds.

**Lemma 4.1.1.** For fixed \( x \in \mathbb{R}^d \), \( X^\epsilon_t(x) \to X_t(x) \) \( P_{X_0}^X \)-a.s as \( \epsilon \downarrow 0 \). Moreover, for \( \mu \in \mathcal{M}_F(\mathbb{R}^d) \), \( X^\epsilon_t(x) \to X_t(x) \) for \( \mu \)-a.e. \( x \) almost surely and \( \mu(X^\epsilon_t) \to \mu(X_t) \) in \( L^1(P_{X_0}^X) \) as \( \epsilon \downarrow 0 \).

The above is in fact an abridged version of Lemma 4.2.1, which is proved in Section 4.2.2. Alongside this, as we discuss in Section 4.2.3, the evolution equation has a unique solution with initial condition given by a finite measure. Consequently, both sides of (4.1.1) are well-defined when \( \phi \) is replaced with a finite measure. The measure case can then be summarized.
as follows: (see Lemma 4.2.9 for a precise statement) for \( \mu \in \mathcal{M}_F(\mathbb{R}^d) \), there is a unique solution \( u^\mu_t(x) \) to (4.1.3) on \( Q \) such that \( u^\mu_t \to \mu \) weakly in the sense of measures as \( t \downarrow 0 \). The solution \( u^\mu_t \) satisfies

\[
E^{\mathcal{X}}_{\mathcal{X}_0}(\exp(-\mu(\mathcal{X}_t))) = \exp(-\mathcal{X}_0(u^\mu_t)).
\]  
(4.1.7)

In the above, \( \mu(\mathcal{X}_t) \) is defined by (4.1.4) with the density \( \mathcal{X}_t(x) \) from (4.1.6).

Another approach to specifying a canonical version of the density is via a Green’s function representation for \( \mathcal{X}_t \), which is the method used by Fleischmann, Mytnik and Wachtel in [28] and other works. This version of the density is given by

\[
\mathcal{X}_t(x) = \mathcal{X}_0 * p_t(x) + \int_{(0,t] \times \mathbb{R}^d} p_{t-s}(y-x) M(ds,dy).
\]  
(4.1.8)

In the above, \( p_t \) is the transition density of the symmetric \( \alpha \)-stable process (see Section 4.2.1) and the measure \( M \) is a compensated stable martingale measure associated to \( \mathcal{X} \). Although we do not show it here, the density above will agree with the version we use.

The canonical measure associated with the \((\alpha, \beta)\)-superprocess, which we denote \( \mathcal{N}_0 \), is a \( \sigma \)-finite measure supported on \( \mathbb{D}([0,\infty), \mathcal{M}_F(\mathbb{R}^d)) \), the space of càdlàg \( \mathcal{M}_F(\mathbb{R}^d) \)-valued paths. In the construction of superprocesses as scaling limits of discrete spatial branching models, one takes the number of individuals in the population to infinity while their masses are simultaneously scaled to 0. The canonical measure \( \mathcal{N}_0 \) is then the “law” of the superprocess when started with a single (infinitesimal) ancestor at the origin. Likewise, \( \mathcal{N}_x \) describes the superprocess descending from an ancestor located at \( x \in \mathbb{R}^d \). In Section 4.2.5 we describe the relationship between canonical measure and the superprocess with law \( P^{\mathcal{X}}_{\mathcal{X}_0} \).

The following formula will be used quite frequently: for any \( x \in \mathbb{R}^d \), for \( t > 0 \),

\[
\mathcal{N}_x(\mathcal{X}_t \neq 0) = U_t := \left( \frac{1}{\beta t} \right)^\frac{1}{\beta}.
\]  
(4.1.9)

(This is shown, for example, in (5.4.2) of [12].) We remark that \( U_t \) is the maximal solution on \((0, \infty)\) to the ODE \( u' = -u^{1+\beta} \). By (4.1.9), when considering \( \mathcal{X}_t \) under \( \mathcal{N}_x \) for some fixed \( t > 0 \), one is essentially working with a finite measure. This formula is a consequence of the fact that the canonical measure also shares a close relationship with the dual evolution equation. If \( u_t^\phi(x) \) is as in (4.1.2) for \( \phi \in \mathcal{B}_+^d \), then

\[
\mathcal{N}_x(1 - \exp(-\mathcal{X}_t(\phi))) = u_t^\phi(x).
\]  
(4.1.10)

As with \( P^{\mathcal{X}}_{\mathcal{X}_0} \), the above relationship can be generalized to include measures when \( \mathcal{X}_t \) has a density. If \( u^\mu_t \) is the unique solution to (4.1.3) with initial condition \( \mu \in \mathcal{M}_F(\mathbb{R}^d) \), (see
Lemma 4.2.9 for details) then
\[ \mathbb{N}_x(1 - \exp(-\mu(X_t))) = u_t^d(x). \] (4.1.11)

We now motivate our results with the statement of two theorems. The first is a fundamental result about superprocesses associated to $\alpha$-stable spatial motions. The result is originally due to Perkins, whose proof of the $\beta = 1$ case appeared in [80]. The proof for $\beta < 1$ appears in a more recent work of Li and Zhou [61]. We define $\text{supp}(\mu)$ to be the closed support of $\mu \in \mathcal{M}_F(\mathbb{R}^d)$ and will often refer to it as the support of $\mu$.

**Theorem 4.A.** [Perkins (1990); Li, Zhou (2008)] For $X_0 \in \mathcal{M}_F(\mathbb{R}^d)$ and $t > 0$,
\[ P_{X_0}(\text{supp}(X_t) = \mathbb{R}^d \text{ or } \emptyset) = 1. \]
Similarly, $\mathbb{N}_0(\text{supp}(X_t) = \mathbb{R}^d \mid X_t \neq 0) = 1$.

The superprocesses we study are critical and therefore go extinct almost surely. That is, $P_{X_0}(X_t \neq 0 \text{ for all } t > 0) = 0$. (This is analogous to the almost sure extinction of critical branching processes, whose spatial analogues have scaling limits given by superprocesses.) Thus, the above statement can be understood to say that, conditioned on non-extinction at time $t > 0$, (i.e. conditioned on $\{X_t \neq 0\}$) $\text{supp}(X_t) = \mathbb{R}^d$ a.s.

Theorem 4.A is sometimes called *instantaneous propagation*. This is because, regardless of the choice of initial measure, $X_t$ has mass “everywhere” in $\mathbb{R}^d$. So for $X_0 = \delta_x$, varying $x$ over $\mathbb{R}^d$ has no influence on the support of $X_t$, which is $\mathbb{R}^d$ almost surely on $\{X_t \neq 0\}$ for any choice of $x$. The condition $\text{supp}(X_t) = \mathbb{R}^d$ is equivalent to: $X_t(U) > 0$ for every open set $U \subset \mathbb{R}^d$, where $X_t(U) = (X_t, 1_U)$ and $1_U$ is the indicator function of $U$. So for open $U$, $X_t \neq 0$ implies $X_t(U) > 0$ almost surely. For $\lambda > 0$, consider now $u^{\lambda 1_U}$, which is the solution to
\[ \partial_t u = \Delta_\alpha u - u^{1+\beta}, \quad u_0 = \lambda 1_U. \]

By (4.1.10),
\[ \mathbb{N}_x(1 - \exp(-\lambda X_t(U))) = -u_t^{\lambda 1_U}(x). \]
Taking $\lambda \to \infty$ with standard limiting arguments, we observe that
\[ \lim_{\lambda \to \infty} \mathbb{N}_x(1 - \exp(-\lambda X_t(U))) = \mathbb{N}_x(X_t(U) > 0). \]

By instantaneous propagation, the right hand side is equal to $\mathbb{N}_x(X_t \neq 0)$, which, by trans-
lation invariance, is equal for any choice of $x$. The implication is that
\[ \lim_{\lambda \to \infty} u_t^{\lambda U^1}(x) = N_0(\mathcal{X}_t \neq 0) \]
for all $x \in \mathbb{R}^d$. Hence $u_t^{\infty U^1} = \lim_{\lambda \to \infty} u_t^{\lambda U^1}$ is constant in space for each $t > 0$. In particular, by (4.1.9), for all $x \in \mathbb{R}^d$ we have $u_t^{\infty U^1}(x) = U_t$.

**Remark 4.1.2.** Using the same argument, it follows that $N_0(\mathcal{X}_t(\phi) > 0) = U_t$ for any measurable function $\phi \geq 0$ which is positive on a set of positive Lebesgue measure. In particular, $\lim_{\lambda \to \infty} u_t^{\lambda \phi} = U_t$ for any such function. These probabilistic results, which we believe are not widely known in the PDE community, imply the non-existence of positive solutions to (4.1.3) whose singular sets (see (4.1.24)) have positive Lebesgue measure.

The other result motivating ours is a recent result in the PDE literature which, to our knowledge, is not yet widely known in the probability community. It is due to Chen, Véron, and Wang [7]. We first give the result originally stated for PDE, then interpret probabilistically. Let $u_t^\lambda$ denote the solution to (4.1.3) with initial data $\lambda \delta_0$ and let $u_t^{\infty} = \lim_{\lambda \to \infty} u_t^\lambda$.

Let $\beta^*(\alpha) = \frac{\alpha}{d + \alpha}$. We will generally view $d$ as fixed and therefore omit the dependence of $\beta^*(\alpha)$ on $d$.

**Theorem 4.B.** [Chen, Véron, Wang (2017); Chen, Véron (2019)] Let $t > 0$ and $x \in \mathbb{R}^d$.

(a) Let $\beta \leq \beta^*(\alpha)$. Then $u_t^{\infty} = U_t$.

(b) Let $\beta^*(\alpha) < \beta < \frac{\alpha}{d}$. Then $u_t^{\infty}$ satisfies
\[
C_1 \frac{t^{-\frac{1}{\beta}}}{1 + |t^{-\frac{1}{\beta}} x|^{d + \alpha}} \leq u_t^{\infty}(x) \leq C_2 \frac{t^{-\frac{1}{\beta}} \log(e + |t^{-\frac{1}{\beta}} x|)}{1 + |t^{-\frac{1}{\beta}} x|^{d + \alpha}}
\]
for constants $0 < C_1 < C_2$.

We introduce some terminology: if a solution to (4.1.3), or a limit of solutions is equal to $U_t$, then we will call the solution (or limit) flat, because these solutions are constant for fixed $t > 0$. Otherwise it is non-flat. Thus $u_t^{\infty}$ is flat when $\beta \leq \beta^*(\alpha)$ and non-flat when $\beta^*(\alpha) < \beta < \frac{\alpha}{d}$.

With the exception of the case that $\beta = \beta^*(\alpha)$, this result was proved in [7], while the $\beta = \beta^*(\alpha)$ case was proved by the first two authors of that paper in [8]. Rather surprisingly, depending on the parameters $(\alpha, \beta, d)$, $u_t^{\infty}(x)$ is either flat or has almost the same asymptotic decay as the heat kernel associated to $\Delta_\alpha$. We interpret the above probabilistically. From
(4.1.11) with \( \mu = \lambda \delta_x \), we have

\[
N_0(1 - \exp(-\lambda \mathcal{X}_t(x))) = u^L_t(x),
\]

where \( \mathcal{X}_t(x) \) is the density of \( \mathcal{X}_t \) at \( x \). Taking \( \lambda \to \infty \) yields

\[
N_0(\mathcal{X}_t(x) > 0) = u^\infty_t(x),
\]

from which the following is an immediate consequence of Theorem 4.B.

**Theorem 4.1.3.** Fix \( t > 0 \). The following hold under \( N_0 \) and \( P_{X_0}^{Y} \) for \( X_0 \in \mathcal{M}_F(\mathbb{R}^d) \).

(a) Let \( \beta \leq \beta^*(\alpha) \). Then for fixed \( x \in \mathbb{R}^d \), \( \mathcal{X}_t(x) > 0 \) almost surely on \( \{ \mathcal{X}_t \neq 0 \} \). In particular, \( \mathcal{X}_t(x) > 0 \) for Lebesgue-a.e. \( x \in \mathbb{R}^d \) almost surely on \( \{ \mathcal{X}_t \neq 0 \} \).

(b) Let \( \beta^*(\alpha) < \beta < \alpha/d \). Then \( \{ x : \mathcal{X}_t(x) > 0 \} \) has finite Lebesgue measure almost surely. Moreover, we have

\[
C_1 \frac{t^{-\frac{1}{\alpha}}}{1 + |t^{-\frac{1}{\alpha}}x|^{d+\alpha}} \leq N_0(\mathcal{X}_t(x) > 0) \leq C_2 \frac{t^{-\frac{1}{\alpha}} \log(e + |t^{-\frac{1}{\alpha}}x|)}{1 + |t^{-\frac{1}{\alpha}}x|^{d+\alpha}}
\]

for constants \( 0 < C_1 < C_2 \).

Theorem 4.B(a) implies part (a) above as follows: by (4.1.9) and (4.1.13), when \( \beta \leq \beta^*(\alpha) \) we have

\[
N_0(\mathcal{X}_t(x) > 0) = N_0(\mathcal{X}_t \neq 0).
\]

Since \( \{ \mathcal{X}_t(x) > 0 \} \subseteq \{ \mathcal{X}_t \neq 0 \} \), it follows that \( N_x(\mathcal{X}_t(x) = 0 \mid \mathcal{X}_t \neq 0) = 0 \) and Fubini’s theorem implies that \( \mathcal{X}_t(x) > 0 \) for Lebesgue-a.e. \( x \) almost surely under \( N_0(\cdot \mid \mathcal{X}_t \neq 0) \). In part (b), the fact that \( \{ x : \mathcal{X}_t(x) > 0 \} \) has finite Lebesgue measure under \( N_0 \) follows from integrability of the upper bound in (4.1.14) and Fubini’s theorem. For \( P_{X_0}^{Y} \), the result then follows from the cluster representation for the superprocess (see Section 4.2.5).

In keeping with the terminology of instantaneous propagation for the behaviour from Theorem 4.A, we propose to call strong instantaneous propagation the property described for \( \beta \leq \beta^*(\alpha) \). Both results say, in some sense, that \( \mathcal{X}_t \) has mass “everywhere.” Instantaneous propagation describes this on the level of mass on open sets, whereas strong instantaneous propagation concerns the density at a fixed point.

A priori, taken as an immediate consequence of Theorem 4.B, we lack a probabilistic intuition for this result, as Theorem 4.B is proved using analytical methods. In Section 4.3 we give a probabilistic proof of part (a). The arguments there are prototypical of other arguments used later on to prove some of our other results.
**Open Problem.** Give a probabilistic proof of Theorem 4.1.3(b).

We now strengthen one of the conclusions of Theorem 4.1.3 when a continuous version of the density \( X_t(\cdot) \) exists. The starting point for this is another dichotomy for the \((\alpha, \beta)\)-superprocess, which was proved by Fleischmann, Mytnik and Wachtel [28].

**Theorem 4.C.** [Fleischmann, Mytnik, Wachtel (2010)] Fix \( t > 0 \) and consider \( X_t \) under \( P_X \) with \( X_0 \in \mathcal{M}_F(\mathbb{R}^d) \).

(a) If \( d = 1 \) and \( \alpha > 1 + \beta \), then there is a version of the density \( X_t(\cdot) \) which is locally \( \eta \)-H"older continuous for all \( \eta < \frac{\alpha}{1+\beta} - 1 \).

(b) If \( d > 1 \), or \( d = 1 \) and \( \alpha \leq 1 + \beta \), then \( \|1_U X_t(\cdot)\|_\infty = \infty \) for all open sets \( U \) almost surely on \( \{X_t \neq 0\} \).

We will refer to the parameter regime \( d = 1, \alpha > 1 + \beta \) as the *continuous case*. In this case it is understood that we will always work with the continuous version of the density \( x \to X_t(x) \), and it is easy to see that this version agrees with the version defined in (4.1.6). The case from Theorem 4.C(b) is then the *discontinuous case*. For \( d = 1 \), in the parameter regime with \( \alpha > 1 + \beta \) and \( \beta < \beta^*(\alpha) \), the density enjoys both continuity and strong instantaneous propagation, and we are able to show the following strict positivity result.

**Theorem 4.1.4.** Let \( d = 1, \alpha > 1 + \beta \) and \( \beta < \beta^*(\alpha) \). Then for \( t > 0 \),

\[ X_t(x) > 0 \quad \text{for all } x \in \mathbb{R} \text{ almost surely on } \{X_t \neq 0\} \]

under both \( P_X \) and \( N_0 \).

**Remark 4.1.5.** Strong instantaneous propagation still holds when \( \beta = \beta^*(\alpha) \), but the proof of the above does not, and this case is left open. Of course, \( \{x : X_t(x) > 0\} \) has finite Lebesgue measure when \( \beta > \beta^*(\alpha) \) by Theorem 4.1.3(b).

This is an interesting property and is perhaps best understood in the context of similar results for non-negative solutions to stochastic PDE (SPDE). Consider first the stochastic heat equation

\[ \partial_t Y_t(x) = \Delta Y_t(x) + Y_t(x)^{\gamma} \dot{W}(t,x) \quad (4.1.15) \]

on \((0, \infty) \times \mathbb{R}, \) where \( \dot{W}(t,x) \) is space-time white Gaussian noise and \( \gamma > 0 \). It was shown by Mueller in [66] that if \( \gamma \geq 1 \), then a non-negative solution \( Y_t(x) \) satisfies \( Y_t(x) > 0 \) for all \((t,x) \in (0, \infty) \times \mathbb{R}. \) On the other hand, Perkins and Mueller [67] have shown that when \( \gamma < 1 \), if \( Y_0 \) has compact support then \( Y_t \) has compact support for all \( t > 0 \) a.s. The case
\( \gamma = \frac{1}{2} \) corresponds to the density of super-Brownian motion, in which case the result was originally due to Iscoe [39].

In the SPDE associated to the \((\alpha, \beta)\)-superprocess with \( \beta < 1 \) in dimension one, the diffusion term is replaced with fractional diffusion and the noise is stable rather than Gaussian. In particular, the density \( \mathcal{X}_t(x) \) of the \((\alpha, \beta)\)-superprocess solves the SPDE

\[
\partial_t \mathcal{X}_t(x) = \Delta_\alpha \mathcal{X}_t(x) + \mathcal{X}_t(x)^{1+\beta} \dot{L}(t,x), \tag{4.1.16}
\]

where \( \dot{L}(t,x) \) is a spectrally positive space-time stable noise of index \( 1 + \beta \). The Green’s function representation (4.1.8) established in [28] can be viewed as a mild form of solution but only applies at fixed times. Alternatively, weak solutions to the SPDE obtained by replacing \( \Delta_\alpha \) with \( \Delta \) in (4.1.16) were studied in [70]. The key results of that work (Propositions 4.1 and 5.1) could be generalized to construct weak solutions to (4.1.16).

In this context, Theorem 4.1.4 is a (fixed time) strict positivity result for a fractional SPDE with stable noise, and it is the first such result that we are aware of. Furthermore, because \( \{ x : \mathcal{X}_t(x) > 0 \} \) has finite Lebesgue measure when \( \beta > \beta^*(\alpha) \), it is apparent that the interplay of fractional diffusion and stable noise with non-Lipschitz coefficients lead to non-trivial behaviour which is not seen in Gaussian SPDE like (4.1.15) (c.f. the results from [66] and [67] discussed above).

In order to best understand the results that follow, it will be useful to view Theorem 4.A and Theorem 4.1.3 from a unified perspective. Let \( m_U \) denote the Lebesgue measure restricted to \( U \subset \mathbb{R}^d \). Theorem 4.A states that if \( U \) is open (so \( m_U \) is non-zero), \( \int \mathcal{X}_t(x)m_U(dx) > 0 \) a.s. on \( \{ \mathcal{X}_t \neq 0 \} \). Thus \( \mathcal{X}_t(\cdot) \) has mass everywhere in \( \mathbb{R}^d \) at a macroscopic level—the level of open sets. In terms of the PDE (4.1.3), the equivalent statement is that \( \lim_{\lambda \to \infty} u^\lambda_{m_U}(x) \) is equal to the flat solution \( U_t \). The Lebesgue measure on an open set \( U \) is, locally speaking, the most spread out measure on \( \mathbb{R}^d \). The least spread out measure, that is the most concentrated measure, is \( \delta_x \) for some \( x \in \mathbb{R}^d \). Theorem 4.1.3 states that \( \mathcal{X}_t(\cdot) \) integrated against the measure \( \delta_x \), i.e. \( \mathcal{X}_t(x) \), is positive a.s. on \( \{ \mathcal{X}_t \neq 0 \} \) if and only if \( \beta \leq \frac{\alpha}{d+\alpha} \), again with an equivalent interpretation that \( \lim_{\lambda \to \infty} u^\lambda_{\delta_x} = U_t \). In this case, \( \mathcal{X}_t(\cdot) \) has mass everywhere at a microscopic level—at a single point. Taken together, these two results describe the almost sure behaviour of \( \mathcal{X}_t(\cdot) \) on both the most concentrated and least concentrated measures on \( \mathbb{R}^d \): \( \delta_x \) and \( m_U \), respectively. Our results that follow interpolate between these two extremes to describe the almost sure behaviour of \( \mu(\mathcal{X}_t) \) for general measures \( \mu \). We are therefore also able to answer some questions about flatness and non-flatness of solutions to (4.1.3) that are the limits as \( \lambda \to \infty \) of solutions with initial condition \( \lambda \mu \). Indeed, a general principle underlying this work can be summarized as follows:
\[ \mu(X_t) > 0 \text{ almost surely on } \{X_t \neq 0\} \text{ if and only if } \lim_{\lambda \to \infty} u_t^{\lambda \mu} = U_t. \]

A condition which quantifies the size of a measure, in the sense of the size of its support, is the following mass distribution property. A measure \( \mu \in \mathcal{M}(\mathbb{R}^d) \) satisfies condition (F1) with parameter \( s \in [0, d] \), or simply (F1)-s, if:

(F1)-s \quad For some constant \( \mathcal{C} > 0 \), for all \( x \in \mathbb{R}^d \) and \( r > 0 \),

\[ \mu(B(x, r)) \leq \mathcal{C} r^s. \]

This condition means that \( \mu \) is spread out in the sense that its support is large, i.e. at least \( s \)-dimensional. In particular, if \( \mu \) satisfies (F1)-s, then \( \dim(\text{supp}(\mu)) \geq s \), where \( \dim(A) \) denotes the Hausdorff dimension of \( A \subset \mathbb{R}^d \). (See Frostman’s Lemma below.)

We view \( \{X_t(x) > 0\} \) as the event that \( X_t(\cdot) \) charges the measure \( \delta_x \). Our results give a partial answer the question: which measures does \( X_t(\cdot) \) charge almost surely?

**Definition 4.1.6.** For \( s \in [0, d] \), let \( \beta^*(\alpha, s) = \frac{\alpha}{(d-s)+\alpha} \).

We remark that \( \beta^*(\alpha, 0) = \beta^*(\alpha) \), the critical parameter from Theorem 4.1.3. Furthermore, \( \beta^*(\alpha, d) = 1 \), which is the critical parameter implicit in Theorem A, since the \((\alpha, \beta)\)-superprocess has instantaneous propagation for all \( \beta \leq 1 \). Recall from (4.1.11) that

\[ N_x(1 - \exp(-\lambda \mu(X_t))) = u_t^{\lambda \mu}(x). \]

As \( \lambda \to \infty \) the left hand side increases to \( N_x(\mu(X_t) > 0) < \infty \).

**Definition 4.1.7.** Let \( u_t^{\infty \mu}(x) = \lim_{\lambda \to \infty} u_t^{\lambda \mu}(x) \).

We therefore have

\[ N_x(\mu(X_t) > 0) = u_t^{\infty \mu}(x). \] (4.1.17)

Since \( \{\mu(X_t) > 0\} \subseteq \{X_t \neq 0\} \), (4.1.9) and (4.1.17) imply that

\[ \sup_x u_t^{\infty \mu}(x) \leq U_t < \infty. \] (4.1.18)

The results that follow are fixed time results. For Theorems 4.1.8, 4.1.9, 4.1.10(a) and 4.1.12, and the discussion of these results, we fix \( t > 0 \).

**Theorem 4.1.8.** Suppose that \( \mu \in \mathcal{M}(\mathbb{R}^d) \) satisfies (F1)-s for some \( s \in [0, d] \) and let \( \beta \leq \beta^*(\alpha, s) \).

(a) \( \mu(X_t) > 0 \) a.s. on \( \{X_t \neq 0\} \) under \( N_0 \) and \( P_{X_0}^X \). Equivalently, we have \( u_t^{\infty \mu} = U_t \).

(b) For any \( \nu \in \mathcal{M}(\mathbb{R}^d) \) with \( \text{supp}(\mu) \subseteq \text{supp}(\nu) \), \( \nu(X_t) > 0 \) a.s. on \( \{X_t \neq 0\} \) under \( N_0 \) and \( P_{X_0}^X \), and \( u_t^{\infty \nu} = U_t \).
Thus if \( \mu \) is spread out in the sense of (F1)-s and \( \beta \leq \beta^*(\alpha, s) \), the density charges \( \mu \) almost surely when \( X_t \) is conditioned on survival. What is more surprising is part (b), which states that it is the closed support of a measure \( \nu \) rather than its particular properties which ensure that \( \nu(X_t) > 0 \) almost surely on \( \{X_t \neq 0\} \). We next show how the above leads to a more general result using Frostman’s Lemma. For \( S \subset \mathbb{R}^d \), let \( H^s(S) \) denote the \( x^s \)-Hausdorff measure of \( S \) and recall that \( \dim(S) \) denotes the Hausdorff dimension of \( S \). Let \( M_F(S) \) denote the space of finite measures \( \mu \) with \( \text{supp}(\mu) \subseteq S \).

**Frostman’s Lemma.** Suppose that \( S \subset \mathbb{R}^d \) is Borel. Then \( H^s(S) > 0 \) if and only if \( \exists \mu \in M_F(S) \) satisfying (F1)-s.

See Theorem 8.8 in [62] for a proof. The following is an immediate consequence of Theorem 4.1.8 and Frostman’s Lemma.

**Theorem 4.1.9.** Let \( s \in [0, d] \) and suppose \( \beta \leq \beta^*(\alpha, s) \). If \( \mu \in M_F(\mathbb{R}^d) \) satisfies \( H^s(\text{supp}(\mu)) > 0 \), then \( \mu(X_t) > 0 \) a.s. on \( \{X_t \neq 0\} \) under \( P_{X_0}^X \) and \( N_0 \), and \( u_t^\infty_\mu = U_t \).

Recall that if \( \text{dim}(\text{supp}(\mu)) = s \), then \( H^{s'}(\text{supp}(\mu)) = +\infty \) for all \( s' < s \). In particular, when \( \beta < \beta^*(\alpha, s) \) the conclusions above hold when \( \text{dim}(\text{supp}(\mu)) \geq s \). The assertions in Theorems 4.1.8 and 4.1.9 that \( u_t^\infty_\mu = U_t \) complement Theorem G of [8], which proves a similar result when \( s \) is an integer and the set (in this setting \( \text{supp}(\mu) \)) is a line or hyperplane. Their result is stated in the language of initial traces, which we discuss later. Our critical parameter \( \beta^*(\alpha, s) \) agrees with the critical parameter of their result.

Observe that \( \beta^*(\alpha, \alpha) = \frac{\alpha}{d} \). Since \( \beta^*(\alpha, s) \) is increasing in \( s \), it follows that, if \( \beta < \frac{\alpha}{d} \), then \( \beta < \beta^*(\alpha, s) \) for all \( s \geq \alpha \). The first condition is required for the existence of the density, and hence if \( s \geq \alpha \), \( \beta < \beta^*(\alpha, s) \) holds over the entire parameter set in which we are interested. In other words, whenever the density exists, almost surely it charges any \( \mu \in M_F(\mathbb{R}^d) \) such that \( \text{dim}(\text{supp}(\mu)) \geq \alpha \). (This also complements an observation made in [8].)

The results of Theorems 4.1.8 and 4.1.9 are sharp. When \( \beta > \beta^*(\alpha, s) \), under complimentary assumptions on the measure \( \mu \), \( u_t^\infty_\mu \) is non-flat and \( \mu(X_t) = 0 \) with positive probability on \( \{X_t \neq 0\} \). In order to state this result precisely we introduce the condition (F2). We say that \( \mu \in M_F(\mathbb{R}^d) \) satisfies property (F2) with parameter \( s \in [0, d] \), or (F2)-s, if:

\[(F2)-s \quad \text{For some constant } \underline{C} > 0, \text{ for all } x \in \text{supp}(\mu) \text{ and } r \in (0, 1], \quad \mu(B(x, r)) \geq \underline{C} r^s.\]
In contrast with (F1), which implies that the mass of \( \mu \) is spread out in a certain sense, (F2) tells us that the mass of \( \mu \) is not too spread out in the same sense. In particular, (F2)-s implies that \( \dim(\text{supp}(\mu)) \leq s \) (for example, see Section 8.7 of [32]).

Let \( d(x,S) = \inf_{y \in S} |x - y| \) denote the distance between \( x \in \mathbb{R}^d \) and a set \( S \subset \mathbb{R}^d \). The restriction \( s < \alpha \) in the following is because this is required for \( (\beta^*(\alpha, s), \frac{\alpha}{4}) \) to be non-empty.

**Theorem 4.1.10.** Suppose that \( \mu \in M_F(\mathbb{R}^d) \) satisfies \((F2)\)-s for some \( s \in [0, \alpha) \) and has compact support \( \text{supp}(\mu) = S \). Let \( \beta^*(\alpha, s) < \beta < \frac{\alpha}{4} \).

(a) For \( x \in \mathbb{R}^d \) and \( X_0 \in M_F(\mathbb{R}^d) \), \( N_x(\mu(X_i) = 0 | X_i \neq 0) > 0 \) and \( P^X_{X_0}(\mu(X_t) = 0 | X_t \neq 0) > 0 \).

(b) \( N_x(\mu(X_i) > 0) = u^\infty_t(\mu(x)) \) satisfies the following: there are constants \( c_{4.1.19} > c_{4.1.19} > 0 \) such that for all \((t, x) \in Q\),

\[
\frac{c_{4.1.19} t^{-\frac{1}{\beta}}}{1 + |t^{-\frac{1}{\beta}} d(x, S)|^{d + \alpha}} \leq N_x(\mu(X_i) > 0) \leq C_{4.1.19} \left[ t^{-\frac{1}{\beta}} \vee t^{-\frac{1}{\beta}} \right] \frac{\log(e + t^{-\frac{1}{\beta}} d(x, S))}{1 + |t^{-\frac{1}{\beta}} d(x, S)|^{d + \alpha}}.
\]

(4.1.19)

In particular, \( N_x(\mu(X_i) > 0) \) vanishes uniformly on \( \{x : d(x, S) \geq \rho\} \) as \( t \downarrow 0 \), for all \( \rho > 0 \).

(c) If, in addition, \( \mu \) satisfies \((F1)\)-s, then there is a constant \( c_{4.1.20} > 0 \) such that for all \( x \in \mathbb{R}^d \) and \( t \in (0, 1] \),

\[
N_x(\mu(X_i) > 0) \geq c_{4.1.20} \left[ \frac{t^{-\frac{1}{\beta}}}{1 + |t^{-\frac{1}{\beta}} [d(x, S) + \text{diam}(S)]|^{d + \alpha}} \right].
\]

(4.1.20)

(d) For any \( \nu \in M_F(S) \), the conclusions of part (a) hold when \( \mu \) is replaced with \( \nu \). Furthermore, for constants \( C_{4.1.21} > c_{4.1.21} > 0 \) we have

\[
\frac{c_{4.1.21} t^{-\frac{1}{\beta}}}{1 + |t^{-\frac{1}{\beta}} d(x, \text{supp}(\nu))|^{d + \alpha}} \leq N_x(\nu(X_i) > 0) \leq C_{4.1.21} \left[ t^{-\frac{1}{\beta}} \vee t^{-\frac{1}{\beta}} \right] \frac{\log(e + t^{-\frac{1}{\beta}} d(x, S))}{1 + |t^{-\frac{1}{\beta}} d(x, S)|^{d + \alpha}}.
\]

(4.1.21)

Theorems 4.1.9 and 4.1.10 give a sharp picture of the behaviour of \( X_t(\cdot) \) when integrated against an \( s \)-dimensional measure. Restrict to the event \( \{X_t \neq 0\} \). Then we have that when \( \mu \) is at least \( s \)-dimensional (i.e. \( \mathcal{H}^s(\text{supp}(\mu)) > 0 \)) and \( \beta \leq \beta^*(\alpha, s) \), \( X_t(\cdot) \) charges \( \mu \) a.s., and when \( \mu \) is at most \( s \)-dimensional (by this we mean \( \mu \) satisfies \((F2)\)-s) and \( \beta > \beta^*(\alpha, s) \), the probability that \( X_t(\cdot) \) charges \( \mu \) has spatial decay similar to an \( \alpha \)-stable heat kernel. Theorem 4.A and Theorem 4.1.3 are the special cases of \( s = d \) and \( s = 0 \), respectively; our results cover \( s \in [0, d] \). Furthermore, for \( s \in [0, \alpha) \) there is a non-trivial transition as we vary \( \beta \) over the critical value \( \beta^*(\alpha, s) \). For \( s \in [\alpha, d] \) the density charges any \( s \)-dimensional set or measure almost surely for all \( \beta < \frac{\alpha}{4} \).
In the flat case, by Theorem 4.1.8(b) and Frostman’s Lemma we were able to generalize the results from (F1)-s measures to general measures whose supports have positive $\mathcal{H}^s$-measure (Theorem 4.1.9). In the non-flat case, our most general result is Theorem 4.1.10(d), which holds for all $\nu$ whose support is contained in that of an (F2)-s measure and is therefore at most s-dimensional. It is an attractive open problem to show that this can be extended to measures supported on general compact sets of dimension at most $s$. With the exception of the critical case $\beta = \beta^*(\alpha, s)$, this would provide a complete characterization of $\mathbb{N}_\pm(\mu(X_t) > 0)$ for compactly supported $\mu$.

**Remark 4.1.11.** For the special case of $\beta = 1$, which requires $d = 1$ and $\alpha \in (1, 2)$ in order for the density to exist, we observe that $\beta^*(\alpha, s) < 1$ for all $s \in [0, 1)$. By Theorem 4.1.10, this implies that for any $\mu \in \mathcal{M}_F(\mathbb{R}^d)$ with $\text{dim}(\text{supp}(\mu)) < 1$, the $(\alpha, 1)$-superprocess in $\mathbb{R}^1$ fails to charge $\mu$ with positive probability on $\{X_t \neq 0\}$.

To obtain matching upper and lower bounds on $\mathbb{N}_\pm(\mu(X_t) > 0)$ as $t \downarrow 0$ in the non-flat case requires $\mu$ to satisfy both (F1)-s and (F2)-s (see Theorem 4.1.10(c)). This is a strong condition which is sometimes called Ahlfors-David regularity. Still, the properties (F1)-s and (F2)-s are satisfied by many measures which are, in an appropriate sense, uniform over some $s$-dimensional set $S$. First consider the case where $s$ is an integer. If $S$ is a rectifiable curve in $\mathbb{R}^d$ and $\mu$ is its length measure, then $\mu$ satisfies (F1)-1 and (F2)-1. If $S$ is a surface, or two-dimensional manifold, then its surface measure $\mu$ satisfies (F1)-2 and (F2)-2. (In view of the discussion following Theorem 4.1.8, this implies that for every $\beta < \frac{d}{r}$, if $\mu$ is the surface or volume measure for a manifold of dimension at least two, then $\mu(X_t) > 0$ a.s. when conditioned on survival). For non-integer $s$, it is known that if $S$ is a self-similar Cantor set of dimension $s$, then its uniform measure $\mu$ satisfies (F1)-s and (F2)-s. (For example, see the discussion in Section 3 in [63] or Section 1.2 of [10].) Our results can be applied directly to all of the examples above.

Many random sets support measures satisfying (F1) and/or (F2). To illustrate our dichotomy for the $(\alpha, \beta)$-superprocess we consider the range of an independent fractional Brownian motion with Hurst parameter $H \in (0, 1)$ in $\mathbb{R}^d$. We denote this process by $(B_t)_{t \geq 0}$ and let $R(B) = \{B_t : t \in [0, 1]\}$. In order to attain both sides of the dichotomy we assume that $H > d^{-1} \lor \alpha^{-1}$, which requires $d \geq 2$ and $\alpha \in (1, 2)$. (As we discuss below, $\text{dim}(R(B)) = H^{-1} \wedge d$. Our assumption guarantees that $\text{dim}(R(B)) = H^{-1} < \alpha$, and we have $\beta^*(\alpha, H^{-1}) < \frac{\alpha}{d}$.) The paths of $B_t$ are $\eta$-Hölder continuous for $\eta \in (0, H)$, (for example see Proposition 1.6 of [77]) from which it can be easily shown that the measure $\mu_B$ defined by $\mu_B(A) = \int_0^1 1_A(B_t) \, dt$ satisfies (F2)-s almost surely (with a random constant $\underline{C}(\omega)$) for all $s < H^{-1}$. Hence if $\beta > \beta^*(\alpha, H^{-1})$, by Theorem 4.1.10, $\mu_B(X_t) = 0$ with positive probability on $\{X_t \neq 0\}$. On the other hand, $\text{dim}(R(B)) = H^{-1} \wedge d$. This is a special case of Theorem 2.1 of [94]. Given our assumption on $H$, $\text{dim}(R(B)) = H^{-1} < \alpha$. Hence by
Theorem 4.1.9, (and because $\text{supp}(\mu_B) = R(B)$) if $\beta < \beta^*(\alpha, H^{-1})$ then $\mu_B(X_t) > 0$ a.s. on $\{X_t \neq 0\}$.

Finally, we specialize the flatness and non-flatness results to the continuous case, which we recall from Theorem 4.C is when $d = 1$ and $\alpha > 1 + \beta$. It is an elementary consequence of continuity of $X_t(\cdot)$ that $\mu(X_t) > 0$ if and only if $X_t(x) > 0$ for some $x \in \text{supp}(\mu)$. Theorem 4.1.9 and 4.1.10 then imply the following. (Part (b) is mainly included for contrast.)

**Theorem 4.1.12.** Let $d = 1$, $\alpha > 1 + \beta$ and $s \in [0, d]$.

(a) Suppose $\beta \in (\beta^*(\alpha, s), \frac{3}{2})$ (which implies $s < \alpha$) and let $S = \text{supp}(\mu)$ for a compactly supported measure $\mu \in \mathcal{M}_F(\mathbb{R}^d)$ satisfying (F2)-s. Then $X_t(x) = 0$ for all $x \in S$ with positive probability on $\{X_t \neq 0\}$ under $P_{X_0}^Y$ and $N_0$. Moreover, (4.1.19) holds when $N_x(\mu(X_t) > 0)$ is replaced with $N_x(\{X_t(x) > 0 \text{ for some } x \in S\})$, and if $\mu$ satisfies (F1)-s, then so does (4.1.20).

(b) Suppose $\beta \leq \beta^*(\alpha, s)$ and let $S \subset \mathbb{R}^d$ be Borel with $\mathcal{H}^s(S) > 0$. Then almost surely on $\{X_t \neq 0\}$ there is a point $x \in S$ such that $X_t(x) > 0$.

Thus far we have generally viewed $\alpha$ as fixed and $\beta$ as a variable parameter, with the critical $\beta$ (i.e. $\beta^*(\alpha, s)$) depending on $s$, the dimension of $S$ or $\text{supp}(\mu)$. We would also like to draw attention to another perspective. We define the saturation dimension associated to the parameters $(\alpha, \beta, d)$.

**Definition 4.1.13.** For fixed parameters $(\alpha, \beta, d)$, the saturation dimension $d_{\text{sat}}$ is

$$d_{\text{sat}} = d_{\text{sat}}(\alpha, \beta, d) = d + \alpha - \frac{\alpha}{\beta}.$$  (4.1.22)

Thus $d_{\text{sat}}$ is simply the value of $s$ for which $\beta = \beta^*(\alpha, s)$, and therefore it is the critical dimension of the set or measure as pertains to this problem. It is the maximum dimension of a set which the density can fail to charge with positive probability: if $\mathcal{H}^{d_{\text{sat}}}(S) > 0$, implying $\text{dim}(S) \geq d_{\text{sat}}$, then the behaviour of $X_t$ is trivial on $S$ the sense that $\mu(X_t) > 0$ a.s. on $\{X_t \neq 0\}$ for every $\mu \in \mathcal{M}_F(\mathbb{R}^d)$ with $\text{supp}(\mu) = S$. The fact that $d_{\text{sat}} \leq d$ corresponds exactly to instantaneous propagation, i.e. Theorem 4.A. On the other hand, strong instantaneous propagation (i.e. the conclusions of Theorem 4.1.3(a)) corresponds to parameters for which $d_{\text{sat}} = 0$ (where if $d_{\text{sat}} < 0$ we simply define it to be $0$). A summary of some of the results above in these terms is as follows:

- If $\mu \in \mathcal{M}_F(\mathbb{R}^d)$ with $\mathcal{H}^{d_{\text{sat}}}(\text{supp}(\mu)) > 0$, then $X_t(\cdot)$ charges $\mu$ almost surely on $\{X_t \neq 0\}$.

- If $S = \text{supp}(\mu)$ for some $\mu \in \mathcal{M}_F(\mathbb{R}^d)$ which satisfies (F2)-s with $s < d_{\text{sat}}$ (implying $\text{dim}(S) < d_{\text{sat}}$), then for every $\nu \in \mathcal{M}_F(S)$, $\nu(X_t) = 0$ with positive probability on $\{X_t \neq 0\}$.  

168
We now turn our attention to the PDE (4.1.3) and state some new results concerning the initial trace theory for positive solutions to this equation. We emphasize that all of our results apply only when \( \beta < \frac{\alpha}{d} \), which in the PDE literature is sometimes called the subcritical absorption regime. Following [8], we define the initial trace. For \( A \subseteq \mathbb{R}^d \), let \( C_c(A) \) denote the space of continuous, compactly supported functions on \( A \). A positive solution \( u \) to (4.1.3) has initial trace \((\mathcal{S}, \nu)\), where \( \mathcal{S} \subseteq \mathbb{R}^d \) is closed and \( \nu \) is a Radon measure satisfying \( \nu(\mathcal{S}) = 0 \), if

\[
\begin{align*}
&\text{• For all } \xi \in C_c(\mathcal{S}^c), \\
&\lim_{t \to 0} \int \xi(x) u_t(x) \, dx = \int \xi \, d\nu. \tag{4.1.23}
\end{align*}
\]

\[
\begin{align*}
&\text{• For every } z \in \mathcal{S} \text{ and } \rho > 0, \\
&\lim_{t \to 0} \int_{B(z, \rho)} u_t(x) \, dx = +\infty. \tag{4.1.24}
\end{align*}
\]

The set \( \mathcal{S} \) is called the singular set of \( u \), whereas \( \mathcal{S}^c \) is called the regular set. Theorems A and B of [8] give general conditions under which a positive solution to (4.1.3) can be associated to an initial trace \((\mathcal{S}, \nu)\). Our results concern the converse problem of determining if there exists a solution with a given initial trace.

The non-fractional analogue of (4.1.3) is

\[
\partial_t u = \Delta u - u^p. \tag{4.1.25}
\]

The initial trace theory for (4.1.25) is well understood. For an analytic approach which covers a wide range of parameters \( p \) (including \( p = 1 + \beta \) for \( \beta \in (0, 1] \)), see Marcus and Véron [64]. Because of the dual relationship with super-Brownian motion, the problem is also amenable to probabilistic analysis. For the case \( p = 2 \), Le Gall [55] characterized the positive solutions to (4.1.25) using the Brownian snake.\(^1\)

The theory for the fractional equation (4.1.3) is more recent and is far from complete. Theorem B hints at this, as it shows that existence of a solution with initial trace \((\{0\}, 0)\) (sometimes called a very singular solution) depends on the values of the parameters. A one-point singular set is the smallest non-trivial singular set, and so a consequence of the theorem is that when \( \beta \leq \beta^*(\alpha) \), the singular initial trace problem is trivial in the sense that if \( \mathcal{S} \) is non-empty, the solution equals or exceeds \( U_t \) (see also Theorem F of [8]). This issue does not arise for solutions to (4.1.25) with \( p \in (1, 2] \) because the associated superprocess

\(^1\)Probabilistic approaches using superprocesses have been very effective in studying the elliptic equation related to (4.1.25), \( \Delta u = u^p \) with \( p \in (1, 2] \). The \( p = 2 \) case has been studied by Le Gall in [54, 56], again using the Brownian snake. The work of Dynkin and Kuznetsov (e.g. [20, 21]) gives results covering \( p \in (1, 2] \), which is the entire range for which the superprocess approach applies.
has compact support which is localized near its initial conditions, and hence the event that
the range of the superprocess does not intersect a given closed set (or that the superprocess
does not charge a measure supported on that set in an appropriate sense) always has positive
probability.

Our results about flatness and non-flatness of $u_t^\infty\mu$ allow us to advance the theory of
existence for solutions to (4.1.3) with a given initial trace. At this stage we are largely
cconcerned with existence and we do not consider the regularity of solutions. For this reason
we study weak solutions. A precise definition of a weak solution to (4.1.3) with initial trace
$(S, \nu)$ is given in Section 4.7. In that section we also restrict our attention to solutions
$u(t, x)$ which are bounded above by $U_t$, i.e. $u(t, x) \leq U_t$. We define $U = U(Q)$ by
\[
U = \{ u : Q \to [0, \infty) : u_t \leq U_t \ \forall \ t > 0 \}. \tag{4.1.26}
\]

By duality, in particular (4.1.9), (4.1.10) and (4.1.11), $U$ includes all solutions which admit a
probabilistic representation in terms of the $(\alpha, \beta)$-superprocess, which includes all solutions
obtained as the limit of solutions with initial data in $\mathcal{M}_F(\mathbb{R}^d)$ or $B^+_d$. See also Theorem
D of [8], which proves (for classical solutions) that solutions satisfying a mild integrability
property are bounded by $U_t$ and hence belong to $U$. It remains unresolved if there exist
positive solutions to (4.1.3) which do not belong to $U$; because they cannot be obtained as
the limits of solutions with truncated function-valued initial conditions, it is unclear how
one might construct one.

**Theorem 4.1.14.** (a) (Non-existence) Let $S \subset \mathbb{R}^d$ be closed with $\mathcal{H}^{d_{\text{sat}}}(S) > 0$. If the
singular set of a solution $u$ to (4.1.3) in $U$ contains $S$, then $u_t = U_t$. In particular, if
$S \neq \mathbb{R}^d$ there is no solution to (4.1.3) in $U$ with initial trace $(S, \nu)$ for any Radon measure
$\nu$.

(b) (Existence) Suppose that $S \subset \mathbb{R}^d$ is compact and there exists $\mu \in \mathcal{M}_F(\mathbb{R}^d)$ satisfying
$(F2)$-s with $s < d_{\text{sat}}$ such that $S = \text{supp}(\mu)$. Then there exists a weak solution to (4.1.3) in
$U$ with initial trace $(S, 0)$. The solution is given by $\lim_{\lambda \to \infty} u^{\lambda\mu}_t(x) = \mathbb{N}_x(\mu(\mathcal{X}_t) > 0)$ and
satisfies (4.1.19).

The problem of existence for solutions with initial trace $(S, 0)$ is therefore characterized
by the saturation dimension. This generalizes Theorem B(a) (and Theorem F of [8]), which
can be viewed as the special case of the above when $d_{\text{sat}} = 0$. When $d_{\text{sat}} = 0$ there is no
transition: all non-empty singular sets fall into the non-existence regime.

**Organization of the chapter.** The rest of the chapter is organized as follows. Section 4.2 provides background information and preliminary results; the key subsections are
Section 4.2.2, which covers the density of $\mathcal{X}_t$ and $\mu(\mathcal{X}_t)$, and Section 4.2.3, which discusses
solutions to (4.1.3) with measure-valued initial data and extends the dual relationship to include finite measures. In Section 4.3 we give a probabilistic proof of Theorem 4.1.3(a), and in Section 4.4 we prove Theorem 4.1.4. Sections 4.5 and 4.6 cover, respectively, flatness and non-flatness of $u^\infty_\mu$ for general measures $\mu$, in particular Theorems 4.1.8 and 4.1.10. Finally, we discuss solutions to (4.1.3) with non-empty singular sets and prove Theorem 4.1.14 in Section 4.7.

4.2 Preliminaries

4.2.1 Transition densities

We denote by $p_t(x)$ the fundamental solution (or heat kernel) to the fractional heat equation on $\mathbb{R}^d$. That is, $p_t(x)$ is the solution to

$$\partial_t u = \Delta_\alpha u, \quad u_0 = \delta_0$$

on $Q = (0, \infty) \times \mathbb{R}^d$. The semigroup $(S_t)_{t\geq 0}$ which we have already introduced is a convolution semigroup with kernel $p_t$, i.e.

$$S_t \phi(x) = \phi * p_t(x) = \int \phi(y)p_t(x-y)dy.$$

We note that for $\mu \in \mathcal{M}_F(\mathbb{R}^d)$, $S_t\mu(x)$ can be defined in the same way with no difficulty. The kernel $p_t$ is radial and radially decreasing. In a slight abuse of notation, for $\rho > 0$ we will sometimes write $p_t(\rho)$ to mean $p_t(x)$, where $|x| = \rho$.

$(S_t)_{t\geq 0}$ is the transition semigroup of the (rotationally) symmetric $\alpha$-stable process, so we have the following: if $W$ is a symmetric $\alpha$-stable process with law and expectation $P^W_x$ and $E^W_x$, respectively, when started at $x$, then for an appropriate class of functions (e.g. bounded and measurable),

$$E^W_x (\phi(W_t)) = S_t \phi(x).$$

In particular, $p_t$ is also the transition density of $W$, that is

$$P^W_x(W_t \in dy) = p_t(y-x)dy.$$

The symmetric $\alpha$-stable process is self-similar, which is reflected by the scaling property for $p_t$:

$$p_t(x) = t^{-\frac{d}{\alpha}} p_1(t^{-\frac{\alpha}{d}} x).$$

Finally we recall the asymptotic decay of the transition density. There are universal con-
It is occasionally useful to write the above as
\[
\left( t^{-\frac{d}{\alpha}} \wedge \frac{t}{|x|^{d+\alpha}} \right) \leq p_t(x) \leq \left( t^{-\frac{d}{\alpha}} \wedge \frac{t}{|x|^{d+\alpha}} \right),
\]
where in order to do so one may have to adjust the constants. Without loss of generality we will fix \( c_{4.2.1} \) and \( C_{4.2.1} \) so that both bounds hold.

### 4.2.2 The density of the \((\alpha, \beta)\)-superprocess

In this section we give an overview of the density of \( X_t \). The main purpose is to show that we can take a version of the density which is defined almost surely at any fixed point \( x \in \mathbb{R}^d \) and that with this version the quantity \( \mu(X_t) \) from (4.1.4) is well-defined almost surely. We also discuss the regularity properties of the density in the continuous regime.

Recall from (4.1.5) and (4.1.6) that for \( x \in \mathbb{R}^d \) and \( \epsilon > 0 \),
\[
X_{\epsilon t}(x) = \frac{X_t(B(x, \epsilon))}{|B(x, \epsilon)|}
\]
and
\[
X_t(x) := \liminf_{\epsilon \downarrow 0} X_{\epsilon t}(x).
\]
Since \( X_t \) is absolutely continuous, \( x \rightarrow X_t(x) \) is a density for it by the Lebesgue differentiation theorem (for example see Theorem 3.21 of [30]). Because we are interested in the behaviour of the density at fixed points and more generally on Lebesgue null sets, we require more than the standard a.e.-convergence of \( X_{\epsilon t}(\cdot) \) to \( X_t(\cdot) \). The desired conditions are shown to hold in the next lemma.

For \( \psi : \mathbb{R}^d \rightarrow \mathbb{R} \) and \( \epsilon > 0 \), let \( \psi_{\epsilon}(x) := \epsilon^{-d} \psi(\epsilon^{-1} x) \).

**Lemma 4.2.1.** Let \( X_0 \in \mathcal{M}_F(\mathbb{R}^d) \) and \( t > 0 \) and consider \( X_t \) under \( P_{X_0}^X \).

(a) For every \( x \in \mathbb{R}^d \), \( X_{\epsilon t}(x) \rightarrow X_t(x) \) a.s. and in \( L^1 \) as \( \epsilon \downarrow 0 \).

(b) For every \( \mu \in \mathcal{M}_F(\mathbb{R}^d) \), \( X_{\epsilon t}(x) \rightarrow X_t(x) \) for \( \mu \)-a.e. \( x \) almost surely and \( \mu(X_{\epsilon t}) \rightarrow \mu(X_t) \) in \( L^1 \), where \( \mu(X_t) = \int X_t(x) \mu(dx) \). Moreover, we have
\[
E_{X_0}^X(\mu(X_t)) = \mu(S_t X_0).
\]

(c) If \( \psi : \mathbb{R}^d \rightarrow [0, \infty) \) satisfies \( \int \psi = 1 \) and \( \sup_x |\psi(x)|(1 + |x|)^{d+\delta} < \infty \) for some \( \delta > 0 \),
then the conclusions of part (a) and (b) hold when $X_t^e$ is replaced with $X_t \ast \psi_e$.

We prove this lemma at the end of the section. Part (c) is included because it is sometimes useful to approximate $X_t(x)$ using convolutions with general kernels. A particularly useful example is when $\psi = p_1$, in which case $\psi_{\frac{1}{\alpha}} = p_e$.

We now turn our attention to the Hölder regularity of the density in the continuous regime, i.e. $d = 1$ and $\alpha > 1 + \beta$ (c.f. Theorem 4.C). The following was proved in [28].

**Theorem 4.D.** [Fleischmann, Mytnik and Wachtel (2010)] Let $d = 1$ and $\alpha > 1 + \beta$. Let $X_0 \in \mathcal{M}_F(\mathbb{R}^d)$ and $t > 0$. Under $P_{X_0}^X$, there is a continuous version $X_t(\cdot)$ of the density such that for every $\eta < \eta_c = \frac{\alpha}{1+\beta} - 1, X_t(\cdot)$ is locally Hölder of index $\eta$, i.e.

$$\sup_{x_1, x_2 \in K, x_1 \neq x_2} \frac{|X_t(x_1) - X_t(x_2)|}{|x_1 - x_2|^\eta} < \infty \quad \text{for all compact } K \subset \mathbb{R}.$$  

Furthermore, the value $\eta_c$ is optimal in that, for any $\eta > \eta_c$, with probability one, for any open $U \subset \mathbb{R},$

$$\sup_{x_1, x_2 \in U, x_1 \neq x_2} \frac{|X_t(x_1) - X_t(x_2)|}{|x_1 - x_2|^\eta} = \infty \quad \text{whenever } X_t \neq 0.$$  

As we have noted in the introduction, when $d = 1$ and $\alpha > 1 + \beta$, the density we define in (4.1.6) is the same as the continuous version. The statement in the above implies the following. Let $\eta < \frac{\alpha}{1+\beta} - 1$ and $X_0 \in \mathcal{M}_F(\mathbb{R})$. Then

$$P_{X_0}^X \text{-a.s.}, \text{ for all compact } K \subset \mathbb{R}, \text{ there exists } C(K, \eta, \omega) > 0 \text{ such that for all } x, y \in K,$$

$$|X_t(x) - X_t(y)| \leq C(K, \eta, \omega)|x - y|^\eta. \quad (4.2.4)$$

We conclude the section with the proof of Lemma 4.2.1. The proof uses a classical result concerning absolute continuity of the laws of superprocesses. As with Theorem 4.A, the result originates in [24], where it was proved for binary branching superprocesses. The proof for $(1 + \beta)$-stable branching superprocesses appears in [61]. We do not state the result in full generality, but refer the reader to Theorems 1.1 and 2.2, respectively, of [24] and [61]. For any $X_0, \hat{X}_0 \in \mathcal{M}_F(\mathbb{R}^d)$ and $t > 0$, we have

The laws of $X_t$ under $P_{X_0}^X$ and $P_{\hat{X}_0}^X$ are mutually absolutely continuous. \quad (4.2.5)

We use the above when $\hat{X}_0$ is a translation of $X_0$. In particular, for $x \in \mathbb{R}^d$ suppose that $\hat{X}_0 = X_0 + x$, where for a measure $\mu$ we define $\mu + x$ by $(\mu + x)(A) = \int 1_A(y - x)\mu(dy)$. Applying (4.2.5), the laws of $X_t$ under $P_{X_0}^X$ and $P_{X_0 + x}^X$ are mutually absolutely continuous.
On the other hand, by translation invariance of the superprocess, the law of $\mathcal{X}_t$ under $P_{\mathcal{X}_0}^{x+x}$ is equal to the law of $\mathcal{X}_t - x$ under $P_{\mathcal{X}_0}^x$. Consequently, we have that for $x \in \mathbb{R}^d$ and $t > 0$,

$$\text{The laws of } \mathcal{X}_t \text{ and } \mathcal{X}_t + x \text{ are mutually absolutely continuous under } P_{\mathcal{X}_0}^x. \quad (4.2.6)$$

We will also use the following moment bound.

**Lemma 4.2.2.** Let $0 < \theta < \beta$ and $\phi \geq 0$. Then

$$E_{\mathcal{X}_0}^X(\mathcal{X}_t(\phi)^{1+\theta}) \leq 1 + C \int_0^t \mathcal{X}_0(S_{t-s}((S_s\phi)^{1+\beta}))ds + \mathcal{X}_0((S_t\phi)^{1+\beta})$$

for a constant $C = C(\alpha, d, \theta) > 0$.

The $\alpha = 2$ case of this result appears as Lemma 2.1 of [71]. The same argument used in [71] also works in the $\alpha \in (0, 2)$ when one replaces the heat semigroup with the fractional heat semigroup, so we omit the proof and continue with the proof of Lemma 4.2.1. The arguments used to bound moments below are also borrowed from [71].

**Proof of Lemma 4.2.1.** Fix $t > 0$ and $\mathcal{X}_0 \in \mathcal{M}_F(\mathbb{R}^d)$. Let $\psi : \mathbb{R}^d \to [0, \infty)$ satisfy $\int \psi = 1$ and $\sup_x |\psi(x)|(1 + |x|)^{d+\delta} < \infty$ for some $\delta > 0$ and recall that $\psi_\epsilon(x) = \epsilon^{-d}\psi(\epsilon^{-1}x)$. These conditions include the case where $\psi$ is the normalized indicator function of the unit ball. A priori this is the case we are interested in, but we carry out the analysis for the general case as this is required for part (c).

Let $\mathcal{X}_t^{\epsilon, \psi}(x) = \mathcal{X}_t * \psi_\epsilon(x)$. Since $\mathcal{X}_t$ is absolutely continuous almost surely, we have

$$P_{\mathcal{X}_0}^{\epsilon} \text{-almost surely, } \mathcal{X}_t^{\epsilon, \psi}(x) \text{ converges for Lebesgue-a.e. } x \in \mathbb{R}^d \text{ as } \epsilon \downarrow 0. \quad (4.2.7)$$

If $\psi$ is the normalized indicator function of the unit ball, the above is simply Lebesgue’s differentiation theorem. Otherwise, the result follows from Theorem 8.15 of [30].

We now show that the convergence holds pointwise almost surely. Suppose there exists $x_0 \in \mathbb{R}^d$ such that $P_{\mathcal{X}_0}^x(\{\mathcal{X}_t^{\epsilon, \psi}(x_0) \text{ does not converge as } \epsilon \downarrow 0\}) > 0$. By (4.2.6) this implies that $P_{\mathcal{X}_0}^x(\{\mathcal{X}_t^{\epsilon, \psi}(x) \text{ does not converge as } \epsilon \downarrow 0\}) > 0$ for all $x \in \mathbb{R}^d$. However, by Fubini’s theorem this implies that the set of points for which $\mathcal{X}_t^{\epsilon, \psi}(x)$ does not converge has positive Lebesgue measure a.s., which contradicts (4.2.7). Hence we must have that

$$\mathcal{X}_t^{\epsilon, \psi}(x) \text{ converges } P_{\mathcal{X}_0}^X \text{-a.s. as } \epsilon \downarrow 0 \text{ for all } x \in \mathbb{R}^d. \quad (4.2.8)$$

Thus we have shown the almost sure convergence stated in part (a). Before proving the $L^1$ convergence and part (b), we show how absolute continuity also implies the a.s. convergence in part (c). When $\psi$ is the normalized indicator function of a unit ball, we denote $\mathcal{X}_t^{\epsilon, \psi}(x)$
by $\mathcal{X}_t^\epsilon(x)$. Let $\psi$ be any other function satisfying the conditions as above. If there exists $x_0 \in \mathbb{R}^d$ such that $P_{\mathcal{X}_0}^X(\liminf_{t \downarrow 0} \mathcal{X}_t^\epsilon(x_0) \neq \liminf_{t \downarrow 0} \mathcal{X}_t^\epsilon,\psi(x_0)) > 0$, then (4.2.6) implies that the probability is positive when we replace $x_0$ with $x$ for all $x \in \mathbb{R}^d$. By Fubini’s theorem, $\liminf_{t \downarrow 0} \mathcal{X}_t^\epsilon(x) \neq \liminf_{t \downarrow 0} \mathcal{X}_t^\epsilon,\psi(x)$ on a set of positive Lebesgue measure almost surely. Since $\liminf_{t \downarrow 0} \mathcal{X}_t^\epsilon(\cdot)$ and $\liminf_{t \downarrow 0} \mathcal{X}_t^\epsilon,\psi(\cdot)$ are both densities for $\mathcal{X}_t$, this is a contradiction. Hence $\liminf_{t \downarrow 0} \mathcal{X}_t^\epsilon(x) = \liminf_{t \downarrow 0} \mathcal{X}_t^\epsilon,\psi(x)$ a.s. for every $x \in \mathbb{R}^d$. The left hand side is simply equal to $\mathcal{X}_t(x)$, and so for all $\psi$ satisfying the stated conditions, (4.2.8) can be improved to

$$
\mathcal{X}_t^\epsilon,\psi(x) \rightarrow \mathcal{X}_t(x) \text{ $P_{\mathcal{X}_0}^X$-a.s. as } \epsilon \downarrow 0 \text{ for all } x \in \mathbb{R}^d,
$$

which proves that the almost sure convergence in (a) holds for all $\psi$ from (c). Furthermore, the above implies that the limit of $\mathcal{X}_t^\epsilon,\psi(x)$ does not depend on the choice of $\psi$. Given this, we can work with general $\psi$ and prove the remaining claims of (a) and (b) simultaneously with (c).

In order to establish that $\mathcal{X}_t^\epsilon,\psi(x) \rightarrow \mathcal{X}_t(x)$ in $L^1$, we will show $L^p$-boundedness of the quantity $\mathcal{X}_t^\epsilon,\psi(x)$ for $p = 1 + \theta$ with $\theta < \beta$. To do so we apply Lemma 4.2.2. Let $\psi^x(\cdot) = \psi_x(\cdot - x)$ and note that $\mathcal{X}_t^\epsilon,\psi(x) = \mathcal{X}_t(\psi^x)$. For $\theta < \beta$, we have

$$
E_{\mathcal{X}_0}^X((\mathcal{X}_t^\epsilon,\psi(x))^{1+\theta}) \leq 1 + C \left[ \int_0^1 \mathcal{X}_0(S_{t-s}((S_s\psi^x)^{1+\beta})) ds + \mathcal{X}_0((S_t\psi^x)^{1+\beta}) \right]. \quad (4.2.9)
$$

Since $\int \psi^x = 1$, we can view $\psi^x$ as the density of a random variable, which we will denote $Y$. Thus for $s > 0$,

$$
(S_s\psi^x)(z)^{1+\beta} = \left[ \int p_s(y-z)x(\psi^x(y))dy \right]^{1+\beta} = E_Y(p_s(Y-z))^{1+\beta} \leq E_Y(p_s(Y-z)^{1+\beta}) = \int p_s(y-z)^{1+\beta}(\psi^x(y))dy, \quad (4.2.10)
$$

We also have

$$
p_s(y-z)^{1+\beta} = p_s(y-z)p_s(y-z) \leq C p_s(y-z) s^{-\frac{\beta}{\alpha}}
$$

by (4.2.2). Hence by (4.2.10) and the above,

$$
\mathcal{X}_0(S_{t-s}((S_s\psi^x)^{1+\beta})) \leq C s^{-\frac{\beta}{\alpha}} \int \int p_{t-s}(z-w)p_s(y-z)(\psi^x(y))dy dz \mathcal{X}_0(dw)
\leq C \mathcal{X}_0(1)s^{-\frac{\beta}{\alpha}} t^{-\frac{\beta}{\alpha}}, \quad (4.2.11)
$$

175
and measurable. The mean measure formula for superprocesses gives

\[ S \]

We have

\[ \mu \]

Because

\[ (4.2.12) \]

the above implies

\[ \mu \]

where

\[ \mu \]

This implies that

\[ a.s. \text{ for every } x \]

boundedness of

\[ (4.2.11) \]

does not depend on

\[ L \]

convergence is also in \( L^1(P_{X_0}^X) \).

We now fix \( \mu \in \mathcal{M}_F(\mathbb{R}^d) \) and show \( L^1 \) convergence of \( \mu(X_t^\epsilon) \) to \( \mu(X_t) \). Note that the bound in \( (4.2.11) \) does not depend on \( x \). Thus, rather than consider \( L^{1+\theta} \)-boundedness of \( X_t(\psi_\epsilon^\epsilon) \) for fixed \( x \) with respect to \( P_{X_0}^X \), we can fix \( \mu \in \mathcal{M}_F(\mathbb{R}^d) \) and consider \( L^{1+\theta} \)-boundedness of \( (\omega, x) \to X_t(\psi_\epsilon^\epsilon)(\omega) \) with respect to \( P_{X_0}^X \otimes \mu \). Because the bounds are uniform in \( x \) and \( \mu \) is finite, the argument requires no modification. Since \( X_t(\psi_\epsilon^\epsilon) \to X_t(x) \) a.s. for every \( x \), it follows that \( X_t(\psi_\epsilon^\epsilon) \to X_t(x) \) in \( L^1(P_{X_0}^X \otimes \mu) \) (as a function of \( (\omega, x) \)). This implies that \( \mu(X_t^\epsilon, \psi) \to \mu(X_t) \) in \( L^1 \).

It remains to show the moment formula \( (4.2.3) \). First, let \( \phi : \mathbb{R}^d \to [0, \infty) \) be bounded and measurable. The mean measure formula for superprocesses gives

\[ E_{X_0}^X(X_t(\phi)) = X_0(S_t \phi). \tag{4.2.12} \]

Now fix \( \mu \in \mathcal{M}_F(\mathbb{R}^d) \). We have

\[ \mu(X_t^\epsilon, \psi) = \int \int \psi_\epsilon(x - y)X_t(dy)\mu(dx) = \int \int \mu_\epsilon(x)X_t(dx) = X_t(\mu_\epsilon), \]

where \( \mu_\epsilon = \mu \ast \hat{\psi}_\epsilon \) with \( \hat{\psi}_\epsilon(\cdot) = \psi_\epsilon(\cdot - \cdot) \). Since \( \mu_\epsilon \) is a bounded and measurable function, by \( (4.2.12) \) the above implies

\[ E_{X_0}^X(\mu(X_t^\epsilon, \psi)) = X_0(S_t \mu_\epsilon). \]

Because \( \mu(X_t^\epsilon, \psi) \to \mu(X_t) \) in \( L^1(P_{X_0}^X) \), the left hand side converges to \( E_{X_0}^X(\mu(X_t)) \) as \( \epsilon \downarrow 0 \). We have \( S_t \mu_\epsilon \leq \mu(1) t^{-\frac{d}{\alpha}} \) for all \( \epsilon > 0 \), so the right hand side converges to \( X_0(S_t \mu) \) by Dominated Convergence. Since \( X_0(S_t \mu) = X_0(S_t X_0) \), this proves \( (4.2.3) \) and the proof is complete.

\[ \square \]

### 4.2.3 The fractional PDE and \( \mu(X_t) \)

In this section we extend the duality between \( X_t \) and solutions to the evolution equation \((4.1.2)\). Recall from \((4.1.1)\) that in its basic form, the dual relationship states that

\[ E_{X_0}^X(\exp(-X_t(\phi))) = \exp(-X_0(u_t^\phi)), \]

where for bounded and measurable \( \phi \geq 0 \), \( u_t^\phi \) is the unique solution to \((4.1.2)\) with initial data \( \phi \). The purpose of this section is to extend this relationship to allow \( \phi \) to be replaced with a finite measure when \( \beta < \frac{d}{\alpha} \). We also introduce weak solutions to \((4.1.3)\) and some of their properties.
The integral equation (4.1.2) is a mild form of the PDE (4.1.3). We will work with weak solutions. Recall that $Q = (0, \infty) \times \mathbb{R}^d$. Let $C^{1,2}_c(Q)$ denote the space of compactly supported functions on $Q$ which are once and twice continuously differentiable in time and space, respectively. For $T > 0$, let $Q_T = (0, T) \times \mathbb{R}^d$ and $\bar{Q}_T = [0, T] \times \mathbb{R}^d$, and let $C^{1,2}_b(\bar{Q}_T)$ denote the space of bounded functions on $\bar{Q}_T$ with bounded, continuous derivatives up to order one in time and order two in space. For $p \geq 1$, we let $L^p_{\text{loc}}(Q)$ denote the space of functions $\phi$ such that $\int_K |\phi|^p < \infty$ for every compact $K \subset Q$.

**Definition 4.2.3.** A function $u : Q \to \mathbb{R}$ is a weak solution to (4.1.3) if $(t, x) \to u(t, x)$ is continuous, $u \in L^{1+\beta}_{\text{loc}}(Q)$, and

$$
\int_Q (u(t, x)[-\partial_t \xi(t, x) - \Delta_\alpha \xi(t, x)] + u(t, x)^{1+\beta} \xi(t, x)) \, dx \, dt = 0
$$

(4.2.13)

for all $\xi \in C^{1,2}_c(Q)$.

For measure-valued initial data, the PDE problem of interest is

$$
\begin{cases}
\partial_t u = \Delta_\alpha u - u^{1+\beta} & \text{for } (t, x) \in Q, \\
u_0 = \mu
\end{cases}
$$

(4.2.14)

for $\mu \in \mathcal{M}_F(\mathbb{R}^d)$. In the following, recall that by convergence in $\mathcal{M}_F(\mathbb{R}^d)$ we mean weak convergence of measures.

**Definition 4.2.4.** For $\mu \in \mathcal{M}_F(\mathbb{R}^d)$, we say that $u : Q \to \mathbb{R}^+$ is a weak solution to (4.2.14) if it is a weak solution to (4.1.3), $u \in L^1(Q_T) \cap L^{1+\beta}(Q_T)$ for all $T > 0$, and $u_t \to \mu$ in $\mathcal{M}_F(\mathbb{R}^d)$ as $t \downarrow 0$.

**Proposition 4.2.5.** For $\mu \in \mathcal{M}_F(\mathbb{R}^d)$, there exists a unique weak solution $u : Q \to [0, \infty)$ to (4.2.14). Moreover, for $T > 0$ we have

$$
\int_{Q_T} (u(t, x)[-\partial_t \xi(t, x) - \Delta_\alpha \xi(t, x)] + u(t, x)^{1+\beta} \xi(t, x)) \, dx \, dt
$$

$$
= \int_{\mathbb{R}^d} \xi(0, x) \mu(dx) - \int_{\mathbb{R}^d} u(T, x) \xi(T, x) \, dx
$$

(4.2.15)

for all $\xi \in C^{1,2}_b(\bar{Q}_T)$.

We will denote the unique solution to (4.2.14) by $u^{\mu}_t(x)$ or $u^\mu(t, x)$. The above is proved in Theorem 1.1 of [7]. The authors of that work use a slightly different definition which incorporates (4.2.15). However, a short argument which we omit shows that given a solution in the sense of Definition 4.2.4, (4.2.15) holds for all $\xi \in C^{1,2}_b(\bar{Q}_T)$. The definition
of a solution used in [7] also does not require continuity, but it can be verified that $u_t^\mu(x)$ is jointly continuous, for example using its correspondence with the solution to the integral equation given in Lemma 4.2.7(b) below. (The proof of Lemma 4.2.7(b) requires only a small modification if one does not assume a priori that $u_t^\mu(x)$ is continuous.) The following stability result holds as a consequence of Theorem 1.1 of [7].

**Lemma 4.2.6.** (a) If $\mu \in \mathcal{M}_F(\mathbb{R}^d)$ and $(\mu_n)_{n \geq 1}$ is a sequence of measures such that $\mu_n \to \mu$ in $\mathcal{M}_F(\mathbb{R}^d)$, then $u_t^{\mu_n}(x) \to u_t^\mu(x)$ locally uniformly in $Q$.

(b) The map $\mu \to u_t^\mu$ is increasing.

Solutions of (4.2.14) are bounded above by solutions of the homogeneous fractional heat equation with the same initial data. Again by Theorem 1.1 of [7], we have

$$u_t^\mu(x) \leq S_t\mu(x) \text{ for } (t, x) \in Q,$$

(4.2.16)

where $(t, x) \to S_t\mu(x)$ is the unique solution to $\partial_t v = \Delta_\alpha v$ on $Q$ with $v_0 = \mu$ (see Theorems 3.1 and 5.1 of [3]).

**Lemma 4.2.7.** Let $\mu \in \mathcal{M}_F(\mathbb{R}^d)$. (a) The integral equation

$$u_t(x) = S_t\mu(x) - \int_0^t S_{t-s}(u_s^{1+\beta})(x) ds, \quad (t, x) \in Q$$

(4.2.17)

has a unique, non-negative, jointly continuous solution in $\mathcal{L}^1(Q_T) \cap \mathcal{L}^{1+\beta}(Q_T)$ for all $T > 0$.

(b) The weak solution to (4.2.14) with initial data $\mu$ and the solution to (4.2.17) are equal.

We will therefore use the notation $u_t^\mu(x)$ and $u^\mu(t, x)$ to refer to the unique solution to (4.2.14) and (4.2.17). With the exception of continuity, part (a) of the above is proved in Lemma A.2 of [27], and continuity can be shown by a direct calculation which we omit.

**Proof of Lemma 4.2.7(b).** Let $\mu \in \mathcal{M}_F(\mathbb{R}^d)$ and let $u(t, x) = u^\mu(t, x)$, the unique weak solution to (4.2.14). For $T > 0$, $x_0 \in \mathbb{R}^d$ and $n \in \mathbb{N}$, we take $\xi(t, x) = p_{T+n-1-t}(x - x_0) \in C_b^{1,2}(Q)$ in (4.2.15). Since $p_t$ solves $\partial_t p_t(x) = \Delta_\alpha p_t(x)$, we have

$$\int_{Q_T} u(t, x)^{1+\beta} p_{T+n-1-t}(x - x_0) \, dx \, dt$$

$$= \int_{\mathbb{R}^d} p_{T+n-1}(x - x_0) \mu(dx) - \int_{\mathbb{R}^d} u(T, x) p_{n-1}(x - x_0) \, dx.$$  

(4.2.18)

For all $n \geq 1$, $\|p_{T+n-1}\|_\infty \leq CT^{-\frac{d}{\alpha}}$ by (4.2.2), and so by bounded convergence the first term in the second line converges to $S_T\mu(x_0)$. The second term converges to $u(T, x_0)$ by continuity of $u$. Now consider the first line. Let $\epsilon > 0$. For $n \geq 1$ and $t \leq T - \epsilon$,
\[ \|p_{T+n^{-1}-t}\|_\infty \leq C e^{-\frac{d}{\alpha}}. \] Since \( u \in \mathcal{L}^{1+\beta}(Q_T) \), we can apply Dominated Convergence to obtain that
\[
\lim_{n \to \infty} \int_0^{T-\epsilon} \int_{\mathbb{R}^d} u(t, x)^{1+\beta} p_{T+n^{-1}-t}(x-x_0) \, dx \, dt = \int_0^{T-\epsilon} \int_{\mathbb{R}^d} u(t, x)^{1+\beta} p_{T-\epsilon}(x-x_0) \, dx \, dt.
\]

On the other hand, by (4.2.16) there is a constant \( K > 0 \) such that \( u(t, x)^{1+\beta} \leq K \) for all \( x \in \mathbb{R}^d \) and \( t \in [T-\epsilon, T] \). Consequently, we have
\[
\lim_{\epsilon \downarrow 0} \sup_{n \geq 1} \left| \int_0^T \int_{\mathbb{R}^d} u(t, x)^{1+\beta} p_{T+n^{-1}-t}(x-x_0) \, dx \, dt \right| = 0.
\]

Combining everything above, we take \( n \to \infty \) in (4.2.18) to conclude that for all \((T, x_0) \in Q\),
\[
u(T, x_0) = S_T \mu(x_0) - \int_{Q_T} u(t, x)^{1+\beta} p_{T-\epsilon}(x-x_0) \, dx \, dt,
\]
and hence \( u \) is equal to the solution of (4.2.17). \( \square \)

**Remark 4.2.8.** Along the same lines as the above, it can be shown that if \( u(t, x) \) is a bounded weak solution to (4.1.3) and \( u(t, x) \to \phi(x) \) a.e. as \( t \downarrow 0 \) for \( \phi \in \mathcal{B}^+_b \), then \( u(t, x) = u^\phi(t, x) \), the solution to (4.1.2) with \( u_0 = \phi \). This also implies that such a solution is unique and has the probabilistic representation \( u(t, x) = \mathbb{N}_x(1-\exp(-\mathcal{X}_t(\phi))) \) by (4.1.10).

We now extend the dual relationship with the \((\alpha, \beta)\)-superprocess to measures.

**Lemma 4.2.9.** Let \( \mu \in \mathcal{M}_F(\mathbb{R}^d) \). Then for \( x \in \mathbb{R}^d \) and \( t > 0 \),
\[
\mathbb{N}_x(1-\exp(-\mu(\mathcal{X}_t))) = u^\mu_t(x),
\]
and for \( \mathcal{X}_0 \in \mathcal{M}_F(\mathbb{R}^d) \),
\[
E^{\mathcal{X}}_{\mathcal{X}_0}(\exp(-\mu(\mathcal{X}_t))) = \exp(-\mathcal{X}_0(u^\mu_t)).
\]

**Proof.** We give the proof under \( P^{\mathcal{X}}_{\mathcal{X}_0} \) and note that it follows by essentially the same argument for the canonical measure. (One can restrict to the event \( \{\mathcal{X}_t \neq 0\} \) because \( 1-\exp(-\mu(\mathcal{X}_t)) = 0 \) on \( \{\mathcal{X}_t = 0\} \), which allows us to treat \( \mathbb{N}_x \) as a finite measure.)

Fix \( \mu \in \mathcal{M}_F(\mathbb{R}^d) \) and define \( \mu_n = S_{n^{-1}} \mu. \) Then \( \mu_n \) is smooth, bounded and positive, so by (4.1.1) we have
\[
E^{\mathcal{X}}_{\mathcal{X}_0}(\exp(-\mathcal{X}_t(\mu_n))) = \exp(-\mathcal{X}_0(u^\mu_n)).
\]

Since \( \mu_n \to \mu \) in \( \mathcal{M}_F(\mathbb{R}^d) \) as \( n \to \infty \), by Lemma 4.2.6(a) it follows that \( u^{\mu_n} \to u^\mu \) locally.
uniformly. In particular, $u_{tn}^\mu \to u^\mu_t$ pointwise. By (4.2.16) and (4.2.2), we have
\[ |u_{tn}^\mu(x)| \leq C\mu(1)t^{-\frac{d}{\alpha}} \]
for all $n \geq 1$ and $x \in \mathbb{R}^d$. Hence $X_0(u_{tn}^\mu) \to X_0(u^\mu_t)$ by Dominated Convergence, and consequently
\[ \lim_{n \to \infty} \exp(-X_0(u_{tn}^\mu)) = \exp(-X_0(u^\mu_t)). \tag{4.2.22} \]
Consider now the left hand side of (4.2.21). Expanding $(X_t, \mu_n)$, we have
\[ X_t(\mu_n) = \int S_{n-1}s_{\alpha}^{\beta \gamma}(x)X_t(x)dx. \]
We have used the symmetry of $S_{n-1}$. Note that $S_{n-1}X_t(x) = X_t * p_{n-1}(x)$ is an approximation of the density which satisfies the conditions of Lemma 4.2.1(c), and hence $X_t(\mu_n) \to \mu(X_t)$ in $L^1(P_{X_0}^X)$. In particular, this implies the convergence of the left hand side of (4.2.21) to $E_{X_0}^X(\exp(-\mu(X_t)))$. Combined with (4.2.22), this implies (4.2.20) and completes the proof.

\[ \square \]

**Remark 4.2.10.** Note that (4.2.19) immediately implies that $\mu \to u^\mu_t$ is increasing, which we have already stated as Lemma 4.2.6(b). We will refer to this monotonicity in initial conditions of solutions to (4.1.3) as the comparison principle. Given Remark 4.2.8, the comparison principle also holds for bounded weak solutions with initial data in $B_0^+$. Solutions to (4.1.3) satisfy a useful scaling property. For $\lambda > 0$ and $\mu \in \mathcal{M}_F(\mathbb{R}^d)$, one can verify directly using elementary methods and uniqueness of solutions to problem (4.2.14) the following formula:
\[ u^\lambda_{\mu}(t, x) = \lambda^\gamma u^\mu(\cdot/\lambda^\gamma)(\lambda^{\alpha \gamma}t, \lambda^{\beta \gamma}x), \tag{4.2.23} \]
where $\gamma = \frac{1}{\alpha - \beta}$, and the measure $\mu(\cdot/\lambda^{\beta \gamma})$ is the dilation of $\mu$ defined by
\[ \mu(\cdot/\lambda^{\beta \gamma})(A) = \mu(A/\lambda^{\beta \gamma}) = \int 1(\lambda^{\beta \gamma}x \in A)\mu(dx) \]
for measurable $A \subseteq \mathbb{R}^d$. This leads to a very useful expression when we scale out the time variable $t$ to obtain an expression involving a solution at time 1; in particular, we have
\[ u^\mu_t(x) = t^{-\frac{\gamma}{\alpha} - \frac{\beta}{\beta \gamma}} u^\mu(\cdot/\lambda^{\beta \gamma})(1, t^{-\frac{1}{\beta \gamma}}x). \tag{4.2.24} \]
It follows that $u_t^{\infty \mu}(x) = \lim_{\lambda \to \infty} u_t^{\lambda \mu}(x)$ satisfies

$$u_t^{\infty \mu}(x) = t^{-\frac{1}{\alpha}} u_t^{\infty \mu(\cdot/t^{-\frac{1}{\alpha}})}(1, t^{-\frac{1}{\alpha}}x).$$  \hspace{1cm} (4.2.25)

### 4.2.4 A Feynman-Kac formula

We now state a Feynman-Kac formula for some functions related to solutions of (4.1.2). First, for $\phi, \psi \in B_b^+$, we formally define

$$z^{\phi, \psi}(t, x) = \frac{\partial}{\partial \epsilon} u^{\phi + \epsilon \psi}(t, x) \bigg|_{\epsilon = 0} = \lim_{\epsilon \downarrow 0} \frac{u^{\phi + \epsilon \psi}(t, x) - u^\phi(t, x)}{\epsilon},$$  \hspace{1cm} (4.2.26)

where $u^\phi(t, x)$ is taken to be the solution to (4.1.2) with $u_0 = \phi$. By (4.1.10) and the Dominated Convergence Theorem, it follows that the derivative with respect to $\epsilon$ exists and

$$z^{\phi, \psi}(t, x) = N_x(\mathcal{X}_t(\psi)) \exp(-\mathcal{X}_t(\phi)).$$

Recall that $E^W_x$ denotes the expectation associated to an $\alpha$-stable process $W_t$ with $W_0 = x$.

**Lemma 4.2.11.** (a) For $\phi, \psi \in B_b^+$,

$$z^{\phi, \psi}(t, x) = E^W_x \left( \psi(W_t) \exp \left( -(1 + \beta) \int_0^t u^\phi(W_s) \beta \, ds \right) \right).$$  \hspace{1cm} (4.2.27)

(b) For $\phi \in B_b^+$ and $\lambda > 0$, $\frac{\partial}{\partial \lambda} u^{\lambda \phi}_t(x)$ exists for all $(t, x) \in Q$ and

$$\frac{\partial}{\partial \lambda} u^{\lambda \phi}_t(x) = E^W_x \left( \phi(W_t) \exp \left( -(1 + \beta) \int_0^t u^{\lambda \phi}_t(W_s) \beta \, ds \right) \right)$$  \hspace{1cm} (4.2.28)

In particular, for $(t, x) \in Q$ and $\Lambda > 0$ we have

$$u^{\Lambda \phi}_t(x) = \int_0^\Lambda \frac{\partial}{\partial \lambda} u^{\lambda \phi}_t(x) \, d\lambda.$$  \hspace{1cm} (4.2.29)

A proof of part (a) is implicit in the proof of Theorem 6.3.1 in [12], where the representation (4.2.27) is established by showing that $z^{\phi, \psi}(t, x)$ solves a certain linear evolution equation. To see that (b) follows from (a), it suffices to consider $z^{\lambda \phi, \phi}(t, x)$ for $\phi \in B_b^+$ and $\lambda > 0$.

### 4.2.5 Cluster decompositions

The connection between the $(\alpha, \beta)$-superprocess and its canonical measure is via a cluster representation in which the superprocess is given by a Poisson superposition of clusters
whose intensity is canonical measure. To make this precise, for $\mathcal{X}_0 \in \mathcal{M}_F(\mathbb{R}^d)$ we define

$$N_{\mathcal{X}_0}(\cdot) = \int N_x(\cdot) \mathcal{X}_0(dx).$$

Let $\Xi(\cdot)$ be a Poisson point process on $\mathbb{D}([0, \infty), \mathcal{M}_F(\mathbb{R}^d))$ with intensity $N_{\mathcal{X}_0}$. Then the process

$$\mathcal{X}_t(\cdot) = \begin{cases} \int \nu_t(\cdot) \Xi(d\nu) & \text{if } t > 0, \\ \mathcal{X}_0(\cdot) & \text{if } t = 0 \end{cases}$$

is an $(\alpha, \beta)$-superprocess with law $P_{\mathcal{X}_0}^X$. This is a consequence of Theorem 4.2.1 of [18]. For fixed $t > 0$ the above implies that

$$\mathcal{X}_t \overset{\mathcal{D}}{=} \sum_{i=1}^N \mathcal{X}^i_t,$$  \hspace{1cm} (4.2.30)

where $\overset{\mathcal{D}}{=}$ indicates equality of distribution. In the above, $N$ is a Poisson random variable with mean $\mathcal{X}_0(1)(\beta t)^{-\frac{1}{\beta}}$ and the $\mathcal{X}^i_t$ are iid random measures with distribution $N_{\mathcal{X}_0}(\mathcal{X}_t \in \cdot | \mathcal{X}_t \neq 0)$. This representation gives us a convenient way to compare path properties under $P_{\mathcal{X}_0}^X$ and $N_x$. In particular, suppose we realize $\mathcal{X}_t$ under $P_{\mathcal{X}_0}^X$ via (4.2.30). Since the probability that $N = 1$ is positive, we can condition on this event, and it follows that

$$N_x(\mathcal{X}_t \in \cdot | \mathcal{X}_t \neq 0) = P_{\mathcal{X}_0}^X(\mathcal{X}_t \in \cdot | N = 1).$$ \hspace{1cm} (4.2.31)

Consequently, for measurable $A \subset \mathcal{M}_F(\mathbb{R}^d)$ we have

$$\text{If } P_{\mathcal{X}_0}^X(\mathcal{X}_t \in A | \mathcal{X}_t \neq 0) = 1, \text{ then } N_x(\mathcal{X}_t \in A | \mathcal{X}_t \neq 0) = 1. \hspace{1cm} (4.2.32)$$

### 4.3 The density at a fixed point

In the introduction, we noted that Theorem 4.1.3 is equivalent to Theorem 4.B, which is proved analytically. In this section we give a probabilistic proof of Theorem 4.1.3(a), and therefore of Theorem 4.B(a). We define $u^\lambda_t(x)$ as the solution to (4.2.14) with initial measure $\lambda \delta_0$. Then by translation invariance of the equation (4.1.3) and (4.2.20), for $x \in \mathbb{R}^d$ we have

$$E_{\mathcal{X}_0}^X(\exp(-\lambda \mathcal{X}_t(x))) = \exp \left( - \int u^\lambda_t(y - x) \mathcal{X}_0(dy) \right) \hspace{1cm} (4.3.1)$$

We define $u^\infty_t = \lim_{\lambda \to \infty} u^\lambda_t$ and observe that, by taking $\lambda \to \infty$ in (4.3.1),

$$P_{\mathcal{X}_0}^X(\mathcal{X}_t(x) = 0) = \exp \left( - \int u^\infty_t(y - x) \mathcal{X}_0(dy) \right). \hspace{1cm} (4.3.2)$$
The main purpose of this section is to show the following.

**Proposition 4.3.1.** If $\beta \leq \beta^*(\alpha) = \frac{\alpha}{d+\alpha}$, then for fixed $x \in \mathbb{R}^d$, $\mathcal{X}_t(x) > 0$ a.s. on $\{\mathcal{X}_t \neq 0\}$ under $P_{\mathcal{X}_0}^x$ and $\mathbb{N}_0$. Hence $u_t^\infty = U_t$.

In fact, this result is a consequence of the more general Theorem 4.1.8, which we prove in Section 4.5. However, we state and prove it separately for a few reasons. First, while Theorem 4.1.8 (and several other results) concern the behaviour of $\mu(\mathcal{X}_t)$ for certain families of measures, the measure $\delta_x$ is of particular interest because it corresponds to the density at a fixed point. The other reason is that, while the method used to prove Proposition 4.3.1 and Theorem 4.1.8 is largely the same, the proof of the latter involves technicalities that do not arise in the former. We therefore opt to include the simpler proof in the case which is particularly interesting.

Recall that we will sometimes write $u^\lambda(t, x)$ to denote $u^\lambda_t(x)$. We note the particular form that the scaling relationship (4.2.24) takes for the family $u^\lambda(t, x)$. Since $\delta_0(\cdot/r) = \delta_0$, it follows that

$$u^\lambda(t, x) = t^{\frac{1}{\alpha}} u^t \frac{\alpha-\beta}{\alpha} \lambda(1, t^{-\frac{1}{\alpha}} x)$$

(4.3.3)

Consequently, $u^\infty(t, x)$ satisfies

$$u^\infty(t, x) = t^{-\frac{1}{\alpha}} u^\infty(1, t^{-\frac{1}{\alpha}} x).$$

(4.3.4)

The following lemma gives a lower bound for $u^\lambda_1$ (for $\lambda \geq 1$) in terms of the heat kernel $p_1$ of the symmetric $\alpha$-stable process and holds for all $0 < \beta < \frac{\alpha}{d}$. The statement of this result for $u^\infty_1$ appeared in [7], where it was called Lemma 5.3. Here we give a probabilistic proof.

**Lemma 4.3.2.** There is a constant $c_{4.3.2} = c_{4.3.2}(\alpha, \beta, d) > 0$ such that for all $\lambda \geq 1$,

$$u^\lambda_1(x) \geq c_{4.3.2} p_1(x).$$

In particular, $u^\infty_1(x) = \mathbb{N}_0(\mathcal{X}_1(x) > 0) \geq c_{4.3.2} p_1(x)$.

*Proof.* For $\lambda > 0$ and $\epsilon > 0$, consider $u^{\lambda \epsilon}(t, x)$, the unique solution to (4.1.2) with $u_0 = \lambda p_\epsilon$.

By Lemma 4.2.11(b), we have

$$\frac{\partial}{\partial \lambda} u^{\lambda \epsilon}(1, x) = E_x^W \left( p_\epsilon(W_1) \exp \left( -(1 + \beta) \int_0^1 u^{\lambda \epsilon}(1 - s, W_s)^\beta ds \right) \right).$$

183
By (4.2.16), $u_{s}^{\lambda_{p_{\epsilon}}} \leq \lambda S_{p_{\epsilon}} = \lambda p_{s+\epsilon}$, which implies that

$$\frac{\partial}{\partial \lambda} u_{s}^{\lambda_{p_{\epsilon}}} (1, x) \geq W_{x} (p_{\epsilon}(W_{1})) \exp \left( -\lambda^{\beta} (1 + \beta) \int_{0}^{1} C_{4.2.1}^{\beta} (s + \epsilon)^{-\frac{d\alpha}{\alpha}} ds \right).$$

In the second line we have used the fact that $p_{1-s+\epsilon}$ is radially decreasing, hence $p_{1-s+\epsilon}(W_{s}) \geq p_{1-s+\epsilon}(0)$, and removed the exponential from the expectation because it no longer depends on $W$. We note that $E_{x}^{W} (p_{\epsilon}(W_{1})) = S_{1} p_{\epsilon}(x) = p_{1+\epsilon}(x)$ from the semigroup property. Using (4.2.2) and changing variables in the integral, we then have

$$\frac{\partial}{\partial \lambda} u_{s}^{\lambda_{p_{\epsilon}}} (1, x) \geq p_{1+\epsilon}(x) \exp \left( -\lambda^{\beta} (1 + \beta) \int_{0}^{1} C_{4.2.1}^{\beta} (s + \epsilon)^{-\frac{d\alpha}{\alpha}} ds \right).$$

Because $\beta < \frac{d}{\alpha}$, the integral remains bounded as $\epsilon \downarrow 0$. It follows that for a constant $C > 0$, for all $\epsilon \in (0, 1]$,

$$\frac{\partial}{\partial \lambda} u_{s}^{\lambda_{p_{\epsilon}}} (1, x) \geq p_{1+\epsilon}(x) \exp \left( -C \lambda^{\beta} \right).$$

In particular, (4.2.29) and the above imply that

$$u^{p_{\epsilon}}(1, x) \geq c_{0} p_{1+\epsilon}(x)$$

for a constant $c_{0} > 0$. By Lemma 4.2.6(a), the left hand side converges to $u^{1}(1, x)$ as $\epsilon \downarrow 0$, and the right hand side converges to $p_{1}(x)$. Thus we have $u^{1}(1, x) \geq c_{0} p_{1}(x)$. Since $\lambda \rightarrow u^{\lambda}(1, x)$ is increasing, this implies the result. \qed

By (4.3.4) and Lemma 4.3.2, one obtains that

$$u_{t}^{\infty}(x) \geq c_{4.3.2} t^{-\frac{1}{\beta}} p_{1}(t^{-\frac{1}{\alpha}} x). \quad (4.3.5)$$

For fixed $x \neq 0$, by (4.2.2) we then have

$$\liminf_{t \downarrow 0} \frac{u_{t}^{\infty}(x)}{t^{-\frac{1}{\beta} + \frac{d\gamma}{\alpha}}} > 0.$$ 

It is therefore immediate that $\lim_{t \downarrow 0} u_{t}^{\infty}(x) = \infty$ for all $x \in \mathbb{R}^{d}$ when $\beta < \beta^{*}(\alpha)$. It does not give the same conclusion when $\beta = \beta^{*}(\alpha)$, and in neither case is it immediate that $u_{t}^{\infty}$ is flat.

Let $(\mathcal{F}_{t})_{t \geq 0}$ denote the standard right-continuous filtration associated to $\mathcal{X} = \{\mathcal{X}_{t} : t \geq 0\}$. For a $P_{\mathcal{X}_{0}}^{\mathcal{X}}$-integrable function $f(\mathcal{X})$, to denote its conditional expectation we will omit the sub- and superscripts and simply write $E(f(\mathcal{X}) | \mathcal{F})$. The (one-dimensional) Markov
property for $\mathcal{X}$ is then expressed as

$$E(f(\mathcal{X}_{t+s}) \mid \mathcal{F}_s)(\omega) = E^X_{\mathcal{X}_t(\omega)}(f(\mathcal{X}_t)).$$

Because $\mathcal{X}$ is càdlàg, $P^X_{\mathcal{X}_0}(\mathcal{X}_t = \mathcal{X}_t+ \forall t > 0) = 1$. The following lemma gives almost surely left continuity at a fixed time.

**Lemma 4.3.3.** Fix $t > 0$ and $\mathcal{X}_0 \in \mathcal{M}_F(\mathbb{R}^d)$. Almost surely under $P^X_{\mathcal{X}_0}$ there is no discontinuity of $s \to \mathcal{X}_s$ at time $t$, and hence $\mathcal{X}_s \to \mathcal{X}_t$ in $\mathcal{M}_F(\mathbb{R}^d)$ as $s \uparrow t$. Moreover, for any open or closed ball $B$, $\lim_{s \uparrow t} \mathcal{X}_s(B) = \mathcal{X}_t(B)$ almost surely.

**Proof.** Fix $t > 0$. The claim is a consequence of Lemma 1.6 of [28]. Part (a) of that lemma states that the discontinuities of $t \to \mathcal{X}_t$ are described by a jump measure $N(d(s, x, r))$, and the form of the compensator of $N$ given in part (b) implies that a.s. there is no jump at time $t$. Hence $\mathcal{X}_s \to \mathcal{X}_t$ weakly as $s \uparrow t$. For an open or closed ball $B$, the fact that $\mathcal{X}_s(B) \to \mathcal{X}_t(B)$ as $s \uparrow t$ follows from weak convergence and the fact that $\mathcal{X}_t(\partial B) = 0$ (by absolute continuity). 

**Proof of Proposition 4.3.1.** Let $\mathcal{X}_0 \in \mathcal{M}_F(\mathbb{R}^d)$, $x \in \mathbb{R}^d$ and $t > 0$. Our method is to show that $P^X_{\mathcal{X}_0}(\mathcal{X}_t(x) = 0, \mathcal{X}_t \neq 0) = 0$. This implies that $P^X_{\mathcal{X}_0}(\mathcal{X}_t(x) > 0, \mathcal{X}_t \neq 0) = P^X_{\mathcal{X}_0}(\mathcal{X}_t \neq 0)$.

At the end of the proof we discuss the case for canonical measure and show that $u^\infty_t = U_t$.

Recall that $B(x, r) = \{y \in \mathbb{R}^d : |y - x| \leq r\}$. From Theorem 4.A, we have $\text{supp}(\mathcal{X}_t) = \mathbb{R}^d$ a.s. on $\{\mathcal{X}_t \neq 0\}$. In particular, $P^X_{\mathcal{X}_0}$-a.s. on $\{\mathcal{X}_t \neq 0\}$ we have $\mathcal{X}_t(B(x, 1)) > 0$. Thus it suffices to show that

$$P^X_{\mathcal{X}_0}(\{\mathcal{X}_t(x) = 0\} \cap A^\kappa) = 0$$

for every $\kappa > 0$, where we define the event

$$A^\kappa = \{\mathcal{X}_t(B(x, 1)) \geq \kappa\}.$$

Let $\kappa > 0$ and consider the event $A^{2\kappa}$. By Lemma 4.3.3, $\lim_{s \uparrow t} \mathcal{X}_s(B(x, 1)) = \mathcal{X}_t(B(x, 1)) \geq 2\kappa$. We let $\delta_n = 2^{-n}$ and note that the previous statement implies that $\mathcal{X}_{t-\delta_n}(B(x, 1)) \geq \kappa$ for $n$ sufficiently large (depending on $\omega$). That is, if we define $E_n^\kappa$ by

$$E_n^\kappa = \{\mathcal{X}_{t-\delta_n}(B(x, 1)) \geq \kappa\},$$

then

$$\text{for a.e. } \omega \in A^{2\kappa}, \exists N = N(\omega) \text{ such that } \omega \in \cap_{n=1}^\infty E_n^\kappa.$$

185
In particular, this implies that
\[ \lim_{n \to \infty} P_{X_0}(A^2 \cap (E_n)^c) = 0. \] (4.3.8)

Consider the conditional probability \( P(\mathcal{X}_t(x) = 0 \mid \mathcal{F}_{t-\delta_n}) \). Applying the Markov property, we obtain
\[
P(\mathcal{X}_t(x) = 0 \mid \mathcal{F}_{t-\delta_n}) = P_{X_{t-\delta_n}}(\mathcal{X}_\delta_n(x) = 0)
= \exp \left( - \int u_{\delta_n}^\infty (y - x) \mathcal{X}_{t-\delta_n}(dy) \right),
\]
where the second equality uses (4.3.2). We now bound above by ignoring all the mass of \( \mathcal{X}_{t-\delta_n} \) outside of \( B(x,1) \). This gives
\[
P(\mathcal{X}_t(x) = 0 \mid \mathcal{F}_{t-\delta_n}) \leq \exp \left( -c_{4.3.2} \int_{B(x,1)} \delta_n^\frac{\beta}{\alpha} p_1(\delta_n^{\frac{-1}{\alpha}} (y - x)) \mathcal{X}_{t-\delta_n}(dy) \right),
\]
where the second inequality uses (4.3.5). Since \( p_1 \) is radially decreasing, the minimum value it can attain in the integral above is \( p_1(2\delta_n^{\frac{1}{\alpha}}) \), and \( p_1(2\delta_n^{\frac{1}{\alpha}}) \geq c_1 \delta_n^{\frac{d+\alpha}{\alpha}} \) for some \( c_1 > 0 \) by (4.2.2). We thus obtain that for \( c_2 = c_1 \cdot c_{4.3.2} > 0 \),
\[
P(\mathcal{X}_t(x) = 0 \mid \mathcal{F}_{t-\delta_n}) \leq \exp \left( -c_2 \delta_n^{\frac{1}{\beta} + \frac{d+\alpha}{\alpha}} \mathcal{X}_{t-\delta_n}(B(x,1)) \right).
\]

Now suppose that \( \omega \in E_\kappa \). Then \( \mathcal{X}_{t-\delta_n}(B(x,1)) \geq \kappa \), and hence
\[
\text{For } P_{X_0}^\kappa \text{-a.e. } \omega \in E_\kappa, \text{ we have}
P^\kappa_{X_0}(\mathcal{X}_t(x) = 0 \mid \mathcal{F}_{t-\delta_n})(\omega) \leq \exp \left( -c_2 \kappa \delta_n^{\frac{1}{\beta} + \frac{d+\alpha}{\alpha}} \right). \] (4.3.9)

First suppose that \( \beta < \beta^*(\alpha) = \frac{\alpha}{d+\alpha} \). In this case, the exponent of \( \delta_n \) in (4.3.9) is negative.
and so the right hand side of (4.3.9) converges to 0 as \( n \to \infty \). We thus obtain that

\[
P_{\mathcal{X}_0}(\{X_t(x) = 0\} \cap A_{2\kappa}) \\
\leq E_{\mathcal{X}_0}(P(\{X_t(x) = 0\} \cap E_n^\kappa \mid \mathcal{F}_{t-\delta_n})) \\
+ P_{\mathcal{X}_0}(A_{2\kappa} \cap (E_n^\kappa)') \\
\leq P_{\mathcal{X}_0}(E_n^\kappa) \exp \left( -c_2 \kappa \delta_n \frac{1}{n} + \frac{d + \alpha}{\alpha} \right) \\
+ P_{\mathcal{X}_0}(A_{2\kappa} \cap (E_n^\kappa)') \\
\to 0 \quad \text{as} \quad n \to \infty,
\]

where we have used (4.3.8) and (4.3.9). We have therefore shown that

\[
P_{\mathcal{X}_0}(\{X_t(x) = 0\} \cap A_{2\kappa}) = 0.
\]

This suffices to prove the result, so the proof is complete for \( \beta < \frac{\alpha}{d + \alpha} \).

Now suppose that \( \beta = \frac{\alpha}{d + \alpha} \). Here we use a martingale argument. The process \( P(\mathcal{X}_t(x) > 0 \mid \mathcal{F}_{t-\delta_n}) \) is a bounded martingale with respect to the increasing sequence of \( \sigma \)-algebras \( \{\mathcal{F}_{t-\delta_n}\}_{n=1}^\infty \). By the martingale convergence theorem it follows that

\[
\lim_{n \to \infty} P(\mathcal{X}_t(x) > 0 \mid \mathcal{F}_{t-\delta_n})(\omega) = P(\mathcal{X}_t(x) > 0 \mid \mathcal{F}_{t-})(\omega) \tag{4.3.10}
\]

for \( P_{\mathcal{X}_0} \)-a.e. \( \omega \), where \( \mathcal{F}_{t-\delta_n} \uparrow \mathcal{F}_{t-} \coloneqq \sigma(\mathcal{X}_s : 0 \leq s < t) \). We now show that the right hand side of the above is equal to \( 1(\mathcal{X}_t(x) > 0)(\omega) \) almost surely. By (4.1.6) we have

\[
\mathcal{X}_t(x) = \liminf_{\epsilon \downarrow 0} \frac{\mathcal{X}_t(B(x, \epsilon))}{|B(x, \epsilon)|} = \liminf_{\epsilon \downarrow 0} \frac{\mathcal{X}_{t-}(B(x, \epsilon))}{|B(x, \epsilon)|} \quad \text{a.s.,}
\]

where \( \mathcal{X}_{t-}(B(x, \epsilon)) = \lim_{s \uparrow t} \mathcal{X}_s(B(x, \epsilon)) \) exists and equals \( \mathcal{X}_t(B(x, \epsilon)) \) a.s. for all \( \epsilon > 0 \) by Lemma 4.3.3. In a slight abuse of notation, let us denote by \( \mathcal{X}_{t-}(x) \) the quantity on the right hand side of the above. Then \( \mathcal{X}_{t-}(x) = \mathcal{X}_t(x) \) almost surely and \( \mathcal{X}_{t-}(x) \) is \( \mathcal{F}_{t-} \)-measurable. We therefore have, for \( P_{\mathcal{X}_0} \)-a.e. \( \omega \),

\[
P(\mathcal{X}_t(x) > 0 \mid \mathcal{F}_{t-})(\omega) = P(\mathcal{X}_{t-}(x) > 0 \mid \mathcal{F}_{t-})(\omega) \\
= 1(\mathcal{X}_{t-}(x) > 0)(\omega) \\
= 1(\mathcal{X}_t(x) > 0)(\omega).
\]

Hence (4.3.10) implies that

\[
\lim_{n \to \infty} P(\mathcal{X}_t(x) > 0 \mid \mathcal{F}_{t-\delta_n})(\omega) = 1(\mathcal{X}_t(x) > 0)(\omega) \quad P_{\mathcal{X}_0} \text{-a.s.} \tag{4.3.11}
\]
Since $\beta = \frac{\alpha}{d+\alpha}$, the exponent of $\delta_n$ in (4.3.9) is zero, and hence the right hand side of the inequality in (4.3.9) is bounded above by $\exp(-c_2\kappa) =: c(\kappa) < 1$. From (4.3.7), (4.3.9) and (4.3.11), we can fix $\omega$ outside a null set such that (i) $E^\kappa_n$ occurs for sufficiently large $n$ on $A^{2n}$, (ii) $P(X_t(x) > 0 | F_{t-\delta_n})(\omega) \to 1(X_t(x) > 0)(\omega)$ as $n \to \infty$, and (iii) for all $n \geq 1$, if $\omega \in E^\kappa_n$ we have

$$P(X_t(x) = 0 | F_{t-\delta_n})(\omega) \leq c(\kappa) < 1.$$ 

This third condition is equivalent to

$$P(X_t(x) > 0 | F_{t-\delta_n})(\omega) \geq 1 - c(\kappa) > 0.$$ 

From (i) and (iii), $P(X_t(x) > 0 | F_{t-\delta_n})(\omega) \geq 1 - c(\kappa)$ for $n$ sufficiently large. From (ii) we conclude that $1(X_t(x) > 0)(\omega) \geq 1 - c(\kappa)$, and hence $X_t(x) > 0$. We have therefore shown that $P^X_{\lambda_0}(\{X_t(x) = 0\} \cap A_{2n}) = 0$, that is (4.3.6) holds, for all $\kappa > 0$, which proves the result. Hence the proof is complete for the case $\beta = \frac{\alpha}{d+\alpha}$, and we are done.

Having shown that $P^X_{\lambda_0}(X_t(x) > 0 | X_t \neq 0) = 1$, the result under $N_0$ then follows by (4.2.32). In particular, we obtain that $N_0(X_t(x) > 0) = N_0(X_t \neq 0)$, and hence that $u_t^\infty(x) = U_t$. \hfill \Box

### 4.4 Strict positivity of the density

In this section we prove Theorem 4.1.4, which states that the density is strictly positive under certain conditions in the continuous case. In particular, in dimension one ($d = 1$) with $\alpha > 1 + \beta$ (continuity) and $\beta < ^*(\alpha)$ (strong instantaneous propagation), we have

$X_t(x) > 0$ for all $x \in \mathbb{R}$ almost surely on $\{X_t \neq 0\}$

at a fixed time $t > 0$. The proof of the result hinges in part on the following result, which gives an exponential rate of decay for the left tail of the density conditional on non-negligible nearby mass. Its proof shares many ideas with the proof of Proposition 4.3.1.

The following holds for general dimensions $d \in \mathbb{N}$. We denote $B_R = \{x \in \mathbb{R}^d : |x| \leq R\}$.

**Lemma 4.4.1.** Let $\beta < \frac{\alpha}{d+\alpha}$. Let $R \geq 1$ and $t \in (0,1]$. There is a constant $c_{4.4.1} > 0$ which depends only on $(\alpha, \beta, d)$ such that for any $X_0 \in \mathcal{M}_F(\mathbb{R}^d)$,

$$P^X_{X_0}(X_t(x) \leq t^{\frac{\alpha-\beta}{\alpha}}) \leq e \cdot \exp\left(-c_{4.4.1} \frac{X_0(B_R)}{R^{\alpha+\alpha}} t^{-q}\right)$$

for all $x \in B_R$, where $q = \frac{1}{\beta} - \frac{d+\alpha}{\alpha} > 0$.

**Proof.** Fix $t \in (0,1]$, $R \geq 1$ and $X_0 \in \mathcal{M}_F(\mathbb{R}^d)$. Let $\lambda = t^{\frac{\alpha-\beta}{\alpha+\beta}}$. One can verify directly
that
\[1(\mathcal{X}_t(x) \leq \lambda^{-1}) \leq \exp(1 - \lambda \mathcal{X}_t(x)).\]

Using the above and applying (4.3.1), we have
\[
P_{\mathcal{X}_0}^X(\mathcal{X}_t(x) \leq \lambda^{-1}) \leq e \cdot E_{\mathcal{X}_0}^X(\exp(\lambda \mathcal{X}_t(x)))
= e \cdot \exp\left(- \int u^\lambda(t, y - x)\mathcal{X}_0(dy)\right)
\leq e \cdot \exp\left(- \int_{B_R} u^\lambda(t, y - x)\mathcal{X}_0(dy)\right).
\]
(4.4.1)

In the last line we simply disregard the mass of \(\mu\) outside of \(B_R\). Next we obtain a lower bound on the integrand in the above. From (4.3.3), we have
\[
u(t, y - x) = t^{-\frac{1}{\beta}}u^\lambda(t, y - x) = t^{-\frac{1}{\beta}}u^1(t, y - x) \geq c_{4.3.2}t^{-\frac{1}{\beta}}p_1(t^{-\frac{1}{\beta}}(y - x)).
\]

The second line uses the fact that \(\lambda t^{\alpha-\beta} = 1\) and the third follows from Lemma 4.3.2. Finally, for all \(x, y \in B_R\), we use (4.2.2) and the above to obtain that
\[
u(t, y - x) \geq c t^{-\frac{1}{\beta} + \frac{d + \alpha}{2d + \alpha}} \forall x, y \in B_R
\]for a constant \(c > 0\) depending only on \((\alpha, \beta, d)\). Using the above in (4.4.1), we obtain that
\[
P_{\mathcal{X}_0}^X(\mathcal{X}_t(x) \leq \lambda^{-1}) \leq e \cdot \exp\left(-c\mathcal{X}_0(B_R)t^{-\frac{1}{\beta} + \frac{d + \alpha}{2d + \alpha}}\right).
\]

Since \(\lambda = t^{-\frac{\alpha-\beta}{\alpha\beta}}\), this completes the proof.

Besides the above, the other main ingredient in the proof of Theorem 4.1.4 is Hölder continuity of \(\mathcal{X}_t(x)\), which we discussed in Section 4.2.2. In particular we will use (4.2.4). As can be seen from the proof, the actual index of Hölder continuity is irrelevant. Any positive index works.

Proof of Theorem 4.1.4. Let \(d = 1\) and \(\alpha > 1 + \beta\). Let \(\mathcal{X}_0 \in \mathcal{M}_F(\mathbb{R}^d)\). By scaling it is sufficient to consider the time \(t = 1\). We will show that \(\mathcal{X}_1(x) > 0\) for all \(x \in [-R, R]\) \(P_{\mathcal{X}_0}^X\)-a.s. for every \(R \geq 1\), and hence that \(\mathcal{X}_1(x) > 0\) for all \(x \in \mathbb{R}\).

Fix \(R \in \mathbb{N}\). As in the proof of Proposition 4.3.1, we will use instantaneous propagation.
In particular, by Theorem 4.A we have

\[ P_{X_0}^X(X_1([-R, R]) > 0 \mid X_1 \neq 0) = 1. \]

Let \( E^{R,\kappa} = \{X_1([-R, R]) \geq \kappa\} \). By the above, it suffices to show that

\[ X_1(x) > 0 \text{ for all } x \in [-R, R] \text{ a.s. on } E^{R,\kappa} \]

(4.4.2)

for all \( \kappa > 0 \). We fix \( \kappa > 0 \) and consider the event \( E^{R,2\kappa} \). For a sequence \( \{\delta_n\}_{n \in \mathbb{N}} = \{2^{-\gamma n}\}_{n \in \mathbb{N}} \), with \( \gamma > 0 \) to be specified later, we define events \( B_n^{R,\kappa} \) by

\[ B_n^{R,\kappa} = \{X_1([-R, R]) \geq \kappa\}. \]

By Lemma 4.3.3, \( \lim_{s \uparrow 1} X_s([-R, R]) = X_1([-R, R]) \geq 2\kappa \) a.s. on \( E^{R,2\kappa} \), and hence for a.e. \( \omega \in E^{R,2\kappa} \) there is \( s_0(\omega) < 1 \) such that \( X_s([-R, R]) \geq \kappa \) for all \( s \in [s_0(\omega), 1] \). Hence \( B_n^{R,\kappa} \) occurs for sufficiently large \( n \), that is,

\[ E^{R,2\kappa} \subseteq \{B_n^{R,\kappa} \text{ eventually}\}, \]

(4.4.3)

where

\[ \{B_n^{R,\kappa} \text{ eventually}\} = \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} B_n^{R,\kappa}. \]

For \( n \in \mathbb{N} \), let \( \Lambda_n \) denote the set of dyadic lattice points at scale \( 2^{-n} \), i.e. \( \Lambda_n = 2^{-n}\mathbb{Z} \). Recall that \( R \in \mathbb{N} \). We then let \( \Lambda_n^R = [-R, R] \cap \Lambda_n \), that is

\[ \Lambda_n = \{-R + k2^{-n} : k = 0, 1, \ldots, 2R2^n\}. \]

Next, we define

\[ F_n = F_n(R) = \{X_1(x) > \delta_n^{\alpha\beta} \forall x \in \Lambda_n^R\}. \]

(4.4.4)

The first step of our proof is to show that

\[ P_{X_0}^X(E^{R,2\kappa} \cap \{F_n^c \text{ i.o.}\}) = 0, \]

(4.4.5)

where \( \text{i.o.} \) is short for \( \text{infinitely often} \), meaning

\[ \{F_n^c \text{ i.o.}\} = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} F_n^c. \]
We now show that (4.4.5) holds. By (4.4.3), we have

\[ P_{X_0}(E^{R,2\kappa} \cap \{ F_n^c \text{ i.o.} \}) \leq P_{X_0}(\{ B_n^{R,\kappa} \text{ eventually} \} \cap \{ F_n^c \text{ i.o.} \}) \]

Suppose that \( \omega \) is in the event on the right hand side of the above; then (i) there is \( N(\omega) \) such that \( \omega \in B_n^{R,\kappa} \) for all \( n \geq N(\omega) \), and (ii) for any \( m \in \mathbb{N} \), there is \( n > m \) so that \( \omega \in F_n^c \). Together, (i) and (ii) imply that for any \( m \geq N(\omega) \), there is \( n > m \) such that \( \omega \in B_n^{R,\kappa} \cap F_n^c \). That is, the above event is a sub-event of \( \{ B_n^{R,\kappa} \cap F_n^c \} \). Hence

\[ P_{X_0}(E^{R,2\kappa} \cap \{ F_n^c \text{ i.o.} \}) \leq P_{X_0}(\{ B_n^{R,\kappa} \cap F_n^c \} \cap \{ B_n^{R,\kappa} \cap F_n^c \text{ eventually} \}) \]

(4.4.6)

We bound the probabilities arising in the final term using Lemma 4.4.1. We condition on \( F_{1-\delta_n} \) and note that \( B_n^{R,\kappa} \in F_{1-\delta_n} \). By the Markov property, we have

\[ P(B_n^{R,\kappa} \cap F_n^c \mid F_{1-\delta_n}) = 1(B_n^{R,\kappa}) P_{X_{1-\delta_n}}(F_n^c). \]

\[ \leq \sum_{x \in \Lambda_1^R} 1(B_n^{R,\kappa}) P_{X_{1-\delta_n}}(X(\delta_n, x) \leq \delta_n^{\alpha/\beta}). \]

By the definition of \( B_n^{R,\kappa} \), in the above we need only compute the probability for such \( X_{1-\delta_n} \) as satisfy \( X_{1-\delta_n}([-R, R]) \geq \kappa \), in which case we can apply Lemma 4.4.1 for each \( x \in \Lambda_1^R \). From this we obtain (recall that \( |\Lambda_1^R| = 2R2^n \))

\[ P(B_n^{R,\kappa} \cap F_n^c \mid F_{1-\delta_n}) \leq C_1(R) 2^n \exp \left( -c_{4.4.1} \frac{\kappa}{R^{1+\alpha}} \delta_n^{-\gamma} \right), \]

where \( C_1(R) = 2R \). Recall that we have chosen \( \delta_n = 2^{-\gamma n} \), and so, substituting the above into (4.4.6), we obtain

\[ P_{X_0}(E^{R,2\kappa} \cap \{ F_n^c \text{ i.o.} \}) \leq \lim_{N \to \infty} \sum_{n=N}^{\infty} C_1(R) 2^n \exp \left( -c_{4.4.1} \frac{\kappa}{R^{1+\alpha}} 2^{-\gamma n} \right) \]

\[ = 0. \]

Thus we have shown that (4.4.5) holds, implying that \( \{ F_n \text{ eventually} \} \) occurs a.s. on \( E^{R,2\kappa} \).
Recalling the definition of $F_n$ from (4.4.4) and that $\delta_n = 2^{-\gamma n}$, it therefore holds that

$$X_1(x) > 2^{-\gamma \frac{\alpha}{\alpha - \beta} n} \text{ for each } x \in \Lambda^R_n. \quad (4.4.7)$$

Next we use the Hölder continuity of $X_t(\cdot)$. Let $0 < \eta < \frac{\alpha}{1+\beta} - 1$. By (4.2.4) with $K = [-R, R]$, for a random constant $C_2(R) = C_2(R, \eta, \omega) > 0$,

$$|X_1(x_1) - X_1(x_2)| \leq C_2(R)|x_1 - x_2|^{\eta} \text{ for all } x_1, x_2 \in [-R, R]. \quad (4.4.8)$$

Having chosen $\eta$, we can now choose a corresponding value of $\gamma$. Let $\gamma = \frac{\alpha \beta \eta}{\alpha - \beta}$. Then by (4.4.7), for $n$ sufficiently large,

$$X_1(x) > 2^{-\frac{\eta}{2} n} \text{ for each } x \in \Lambda^R_n. \quad (4.4.9)$$

Let $y \in [-R, R]$. We define $[y]_n = \min \{x \in \Lambda^R_n : x \geq y\}$. That is, if $y \notin \Lambda^R_n$, then $[y]_n$ the point in $\Lambda^R_n$ nearest to $y$ on the right; if $y \in \Lambda^R_n$, then $[y]_n = y$. Note that $|y - [y]_n| < 2^{-n}$ for all $y \in [-R, R]$ by the definition of $\Lambda^R_n$. Hence by (4.4.8),

$$\sup_{y \in [-R, R]} |X_1(y) - X_1([y]_n)| \leq C_2(R) 2^{-\eta n}. \quad (4.4.10)$$

By the triangle inequality, for $y \in [-R, R]$,

$$X_1(y) \geq X_1([y]_n) - |X_1(y) - X_1([y]_n)|.$$

Note that $\{[y]_n : y \in [-R, R]\} = \Lambda^R_n$. Taking the infimum of the above over $[-R, R]$ and applying (4.4.10), we obtain that

$$\inf_{y \in [-R, R]} X_1(y) \geq \left( \inf_{y \in [-R, R]} X_1([y]_n) \right) - \sup_{y \in [-R, R]} |X_1(y) - X_1([y]_n)|$$

$$\geq \inf_{x \in \Lambda^R_n} X_1(x) - C_2(R) 2^{-\eta n}.$$

By (4.4.9), for all sufficiently large $n$ we therefore have

$$\inf_{y \in [-R, R]} X_1(y) \geq 2^{-\frac{\eta}{2} n} - C_2(R) 2^{-\eta n}$$

$$= 2^{-\frac{\eta}{2} n} \left( 1 - C_2(R) 2^{-\frac{\eta}{2} n} \right).$$

By taking $n$ to be large enough in comparison to $C_2(R)$, the right hand side is positive.
This proves that the density is strictly positive on $[-R, R]$ a.s. on $E^{R, 2k}$. Hence (4.4.2) holds and the proof is complete.

4.5 Almost sure charging of (F1)-s measures when $eta \leq \beta^*(\alpha, s)$

In this section we prove that, under some conditions on $\alpha$ and $\beta$, $\mu(\mathcal{X}_t) > 0$ almost surely on ${\mathcal{X}_t \neq 0}$ for certain measures $\mu$, which is equivalent to $N_\nu(\mu(\mathcal{X}_t) > 0) = u_t^\infty \mu = U_t$. More precisely, this section contains the proof of Theorem 4.1.8(a). We recall the Frostman condition (F1) for a measure: for $s \in [0, d)$, $\mu \in M_F(\mathbb{R}^d)$ satisfies (F1)-s if

\[(F1)-s \quad \text{For some constant } C, \text{ for all } x \in \mathbb{R}^d \text{ and } r > 0, \]

\[\mu(B(x, r)) \leq Cr^s.\]

Theorem 4.1.8(a) states that if $\mu \in M_F(\mathbb{R}^d)$ satisfies (F1)-s and $\beta \leq \beta^*(\alpha, s) = \frac{\alpha}{(d-s)+\alpha}$, then $\mu(\mathcal{X}_t) > 0$ almost surely on ${\mathcal{X}_t \neq 0}$ and, equivalently, $u_t^\infty \mu = U_t$.

Without loss of generality, we can assume that $\text{supp}(\mu)$ is bounded. Indeed, if (F1)-s holds for $\mu$, then it also holds for the restriction of $\mu$ to a bounded set. Furthermore, if $\mu'$ denotes this restriction, then $u_t^{\lambda \mu'} \leq u_t^{\lambda \mu}$ for $\lambda > 0$ by the comparison principle, and hence it suffices to show that $u_t^\infty \mu' = U_t$. We will further assume that $\text{supp}(\mu) \subseteq B_1 = \{x \in \mathbb{R}^d : |x| \leq 1\}$. This is allowable because, by translation invariance of (4.1.3), $u_t^\infty \mu$ is flat if and only if $u_t^\infty \mu_z$ is flat, where $\mu_z$ is the translate of $\mu$ by $z \in \mathbb{R}^d$. We can therefore translate $\mu$ so that it has positive mass in $B_1$, then discard the mass outside $B_1$ by the previous argument.

We set the following standing assumption: for the remainder of this section, let $\mu \in M_F(\mathbb{R}^d)$ satisfy (F1)-s for some $s \in [0, d)$ and $\mu(B_1^c) = 0$. Without loss of generality we will suppose that $\mu(B_1) = \mu(\mathbb{R}^d) = 1$.

The proof of Theorem 4.1.8(a) uses a similar argument to the proof of Proposition 4.3.1. Recall that the bound $w(1, x) \geq cp_1(x)$ from Lemma 4.3.2 played a critical role in that result. We use a similar bound here, which however is adapted to $u_t^\infty \mu$ for $\mu$ satisfying (F1)-s. By monotonicity of $\lambda \mapsto u_t^{\lambda \mu}$, we have the trivial bound that $u_t^{\infty \mu} \geq u_t^{\lambda \mu}$ for all $\lambda > 0$. As we will also be using scaling properties of these solutions, which involve rescaling the initial measure (see (4.2.23)), the critical scale turns out to be $u^r \mu(\cdot/r)(1, x)$, where we recall that for $r > 0$, $\mu(\cdot/r)$ is the measure given by $\mu(A/r) = \int 1_A(rx) d\mu(x)$. If $\mu$ has support $S$, then the support of $\mu(\cdot/r)$ is $rS = \{rx : x \in S\}$. The next result, which is analogous to Lemma 4.3.2, gives a lower bound for $u^r \mu(\cdot/r)(1, x)$.

\[\]
Lemma 4.5.1. Let $\mu \in \mathcal{M}_F(\mathbb{R}^d)$ satisfy $(F1)$-s for some $s \in [0, d]$. Then there is a constant $c_{4.5.1} = c_{4.5.1}(\mu, \alpha, \beta, d) > 0$ such that for all $r \geq 1$,

$$u^{s}\mu(\cdot/r)(1, x) \geq c_{4.5.1} r^s S_1(\mu(\cdot/r))(x).$$

The proof of Lemma 4.5.1 requires the following boundedness result.

Lemma 4.5.2. Let $\mu \in \mathcal{M}_F(\mathbb{R}^d)$ satisfy $(F1)$-s for $s \in [0, d]$ and fix $\alpha \in (0, 2)$. Then there is a constant $C_{4.5.2} = C_{4.5.2}(\mu, \alpha, d) > 0$ such that for all $t > 0$,

$$\sup_{r \geq 1} \sup_{y \in \mathbb{R}^d} S_t(r^s \mu(\cdot/r))(y) \leq C_{4.5.2} t^{-\frac{(d-s)}{\alpha}}.$$

Proof. Fix $t, r > 0$. For $y \in \mathbb{R}^d$, we have

$$S_t(r^s \mu(\cdot/r))(y) = r^s \int p_t(\tilde{z} - y) d\mu(\tilde{z}/r)$$

$$= r^s \int p_t(rz - y) d\mu(z)$$

$$= r^s \int_{0}^{\infty} \mu(\{z : p_t(rz - y) \geq k\}) dk,$$

where the last line uses Fubini’s theorem. By (4.2.2), it follows that

$$\{z : p_t(rz - y) \geq k\} \subseteq \{z : C_{4.5.2} t^{-\frac{d}{\alpha}} \geq k\} \cap \{C_{4.5.2} t^{-\frac{d}{\alpha}} \geq k\}.$$

Note that the first set on the right hand side is equal to $B(y/r, c(k^{-1} t^{\frac{1}{d+\alpha}} r^{-1})$ with $c = C_{4.2.1}^{\frac{1}{d+\alpha}}$. Using this and the fact that $\mu$ satisfies $(F1)$-s, we have

$$S_t(r^s \mu(\cdot/r))(y) \leq r^s \int_{0}^{C_{4.2.1} t^{-\frac{d}{\alpha}}} \mu \left( B(y/r, c(k^{-1} t^{\frac{1}{d+\alpha}} r^{-1}) \right) dk$$

$$\leq C r^s r^{-s} t^{\frac{d}{d+\alpha}} \int_{0}^{C_{4.2.1} t^{-\frac{d}{\alpha}}} k^{-\frac{s}{d+\alpha}} dk$$

$$\leq C t^{-\frac{(d-s)}{\alpha}},$$

where we recall that $s \leq d$ and so $C$ depends only on $C$, $d$ and $\alpha$. 

Proof of Lemma 4.5.1. As in the proof of Lemma 4.3.2, we will use the Feynman-Kac formula from Section 4.2.4. We cannot apply Lemma 4.2.11 directly to $\frac{\partial}{\partial \lambda} u^{s, \mu(\cdot/r)}$ because
$r^s \mu(\cdot/r)$ is not a function, so for $\epsilon > 0$ we define

$$\psi_{\epsilon,r} = S_\epsilon (r^s \mu(\cdot/r)) = r^s (\mu(\cdot/r) * p_\epsilon).$$  \hfill (4.5.1)

Then $\psi_{\epsilon,r}$ is a smooth, bounded, non-negative function, so by Lemma 4.2.11(b), $w^{\lambda \psi_{\epsilon,r}}(t, x) = \frac{\partial}{\partial \lambda} u^{\lambda \psi_{\epsilon,r}}(t, x)$ exists and

$$w^{\lambda \psi_{\epsilon,r}}(t, x) = E_x^W \left( \psi_{\epsilon,r}(W_t) \exp \left( -(1 + \beta) \int_0^t u^{\lambda \psi_{\epsilon,r}}(t - \tau, W_\tau)^\beta d\tau \right) \right). \hfill (4.5.2)$$

By (4.2.16), we have

$$u^{\lambda \psi_{\epsilon,r}}(\tau) \leq S_{\tau}(\lambda \psi_{\epsilon,r}) = \lambda S_{\tau}(\psi_{\epsilon,r})$$

for $\tau > 0$. Hence by (4.5.2) with $t = 1$ we have

$$w^{\lambda \psi_{\epsilon,r}}(1, x) \geq E_x^W \left( \psi_{\epsilon,r}(W_1) \exp \left( -(1 + \beta) \lambda \beta \int_0^1 [S_{1-\tau}(\psi_{\epsilon,r})(W_\tau)]^\beta d\tau \right) \right)$$

$$= E_x^W \left( \psi_{\epsilon,r}(W_1) \exp \left( -(1 + \beta) \lambda \beta \int_0^1 [S_{1-\tau+\epsilon}(r^s \mu(\cdot/r))(W_\tau)]^\beta d\tau \right) \right),$$

where to obtain the final expression we have used (4.5.1) and the semigroup property. By Lemma 4.5.2,

$$[S_{1-\tau+\epsilon}(r^s \mu(\cdot/r))(W_\tau)]^\beta \leq C_{4.5.2}^\beta (1 - \tau + \epsilon)^{-\frac{(d-s)\beta}{\alpha}},$$

and in particular there is a constant $C > 0$ such that for $0 < \epsilon \leq 1,$

$$[S_{1-\tau+\epsilon}(r^s \mu(\cdot/r))(W_\tau)]^\beta \leq C(1 - \tau + \epsilon)^{-\frac{d\beta}{\alpha}}.$$

Using this bound and changing variables, we obtain that

$$w^{\lambda \psi_{\epsilon,r}}(1, x) \geq E_x^W (\psi_{\epsilon,r}(W_1)) \exp \left( -C(1 + \beta) \lambda \beta \int_0^1 (u + \epsilon)^{-\frac{d\beta}{\alpha}} du \right).$$

Since $\beta < \frac{d}{\alpha}$ the integral remains bounded as $\epsilon \downarrow 0$. Hence for a new constant $C > 0$, we obtain that for all $0 < \epsilon \leq 1,$

$$w^{\lambda \psi_{\epsilon,r}}(1, x) \geq E_x^W (\psi_{\epsilon,r}(W_1)) \exp \left( -C \lambda \beta \right). \hfill (4.5.3)$$

We now integrate over $\lambda$ to obtain a lower bound for $u^{\psi_{\epsilon,r}}(1, x).$ Since $w^{\lambda \psi_{\epsilon,r}}(1, x) =$
\[ \frac{\partial}{\partial \lambda} u^{\lambda \psi_{\epsilon,r}}(1,x) \]

by (4.2.29) and (4.5.3) we have

\[ u^{\psi_{\epsilon,r}}(1,x) = \int_0^1 w^{\lambda \psi_{\epsilon,r}}(1,x) \, d\lambda \]

\[ \geq E_x^W(\psi_{\epsilon,r}(W_1)) \int_0^1 \exp\left(-C\lambda^2\right) \, d\lambda \]

\[ \geq c_0 E_x^W(\psi_{\epsilon,r}(W_1)) \]

(4.5.4)

for a constant \( c_0 > 0 \). It remains to show that the left and right hand sides of the above converge to the desired quantities when \( \epsilon \downarrow 0 \). It follows from Lemma 4.2.6 that

\[ \lim_{\epsilon \to 0} u^{\psi_{\epsilon,r}}(1,x) = u^{r^s \mu(\cdot/r)}(1,x). \]

(4.5.5)

Turning to the right hand side of (4.5.4), we first observe that

\[ E_x^W(\psi_{\epsilon,r}(W_1)) = S_1 \psi_{\epsilon,r}(x). \]

Thus we may use the Dominated Convergence Theorem to see that

\[ \lim_{\epsilon \to 0} E_x^W(\psi_{\epsilon,r}) = S_1(r^s \mu(\cdot/r))(x). \]

(4.5.6)

Letting \( \epsilon \downarrow 0 \) in (4.5.4), from (4.5.5) and (4.5.6) we obtain that

\[ u^{r^s \mu(\cdot/r)}(1,x) \geq c_0 r^s S_1(\mu(\cdot/r))(x) \]

for all \( x \in \mathbb{R}^d \), which completes the proof. \( \square \)

We now have all the tools we need to prove Theorem 4.1.8(a). Part (b) is proved in Section 4.7.

**Proof of Theorem 4.1.8(a).** Fix \( X_0 \in \mathcal{M}_F(\mathbb{R}^d) \). Let \( \mu \in \mathcal{M}_F(\mathbb{R}^d) \) satisfy (F1)-s as well as our assumptions that \( \mu(1) = 1 \) and \( \mu(B^c_1) = 0 \). By Lemma 4.2.9, in particular using (4.2.20) with \( u_t^{\lambda \mu} \) and taking \( \lambda \to \infty \), we obtain

\[ P^X_{X_0}(\mu(X_t) = 0) = \exp(-\mathcal{X}_0(u_t^{\infty \mu})). \]

(4.5.7)

Let \( \delta_n = 2^{-n} \). Assume that \( n \) is large enough so that \( \delta_n < t \), and consider the conditional probability \( P(\mu(X_t) = 0 \mid \mathcal{F}_{t-\delta_n}) \). Applying the Markov property and using (4.5.7), we
obtain that

\[
P(\mu(\mathcal{X}_t) = 0 \mid \mathcal{F}_{t-\delta_n}) = P_{\mathcal{X}_{t-\delta_n}}^\mu(\mu(\delta_n) = 0) \\
= \exp\left(-\int u^{\infty \mu}(\delta_n, x) d\mathcal{X}_{t-\delta_n}(x)\right) \\
\leq \exp\left(-\int_{B_2} u^{\infty \mu}(\delta_n, x) d\mathcal{X}_{t-\delta_n}(x)\right),
\]

(4.5.8)

where \(B_2 = \{x \in \mathbb{R}^d : |x| \leq 2\}\). Using monotonicity of \(\lambda \rightarrow u^{\lambda \mu}(\delta_n, x)\) and the scaling relationship (4.2.24), we have

\[
u^{\infty \mu}(\delta_n, x) \geq \delta_n^{-\frac{\alpha}{\alpha - d}} u^{\delta_n^{-\frac{1}{\alpha}} \mu(\cdot / \delta_n^{-\frac{1}{\alpha}})}(1, \delta_n^{-\frac{1}{\alpha}} x) \\
\geq c_{4.5.1} \delta_n^{-\frac{1}{\alpha} - \frac{\beta}{\alpha}} S_1(\mu(\cdot / \delta_n^{-\frac{1}{\alpha}})) (\delta_n^{-\frac{1}{\alpha}} x).
\]

(4.5.9)

The final inequality follows from Lemma 4.5.1. We expand the semigroup term in the above as a convolution with \(p_1\). After a change of variables, we have

\[
S_1(\mu(\cdot / \delta_n^{-\frac{1}{\alpha}})) (\delta_n^{-\frac{1}{\alpha}} x) = \int p_1(\delta_n^{-\frac{1}{\alpha}}(x - y)) d\mu(y) \\
\geq \mu(1) p_1(\delta_n^{-\frac{1}{\alpha}} d(x, \mathcal{S})),
\]

(4.5.10)

where \(\mathcal{S} = \text{supp}(\mu)\) and \(d(x, \mathcal{S}) = \inf_{y \in \mathcal{S}} |x - y|\), and we recall that for \(\rho > 0\), \(p_1(\rho)\) denotes \(p_1(|z|)\) with \(|z| = \rho\). Because \(\mu\) is supported on \(B_1\), for any \(x \in B_2\) we have \(d(x, \mathcal{S}) \leq 3\). In particular, using this in (4.5.10) and substituting it into (4.5.9), we obtain

\[
u^{\infty \mu}(\delta_n, x) \geq c_{4.5.1} \delta_n^{-\frac{1}{\alpha} - \frac{\beta}{\alpha}} p_1(3\delta_n^{-\frac{1}{\alpha}}) \quad \text{for all } x \in B_2,
\]

where \(\mu(1)\) does not appear because it equals one. Using (4.2.2) to bound \(p_1\) below, we conclude that for a constant \(c_1 > 0\),

\[
u^{\infty \mu}(\delta_n, x) \geq c_1 \delta_n^{-\frac{1}{\alpha} - \frac{s}{\alpha} - \frac{\beta}{\alpha} - (d + \alpha)} \\
= c_1 \delta_n^{-q} \quad \text{for all } x \in B_2,
\]

(4.5.11)

where \(q := \frac{1}{\beta} - \frac{d + s + \alpha}{\alpha}\). Using (4.5.11) in (4.5.8), we obtain the following:

\[
P(\mu(\mathcal{X}_t) = 0 \mid \mathcal{F}_{t-\delta_n}) \leq \exp \left(-c_1 \mathcal{X}_{t-\delta_n}(B_2) \delta_n^{-q}\right).
\]

(4.5.12)
From this point, the proof is identical to that of Proposition 4.3.1. By instantaneous propagation, \( X_t(B_2) > 0 \) almost surely on \( \{ X_\tau \neq 0 \} \). One considers the event \( A_{2\kappa} = \{ X_t(B_2) \geq 2\kappa \} \) for \( \kappa > 0 \) and notes that \( X_{t-\delta_n}(B_2) \geq \kappa \) eventually a.s. on \( A_{2\kappa} \). This leads to a statement analogous to (4.3.9). One then finishes the proof in the same way: by direct computation when \( \beta < \beta^*(\alpha, s) \) and using martingale convergence when \( \beta = \beta^*(\alpha, s) \). This completes the proof that \( P_{X_0}^X(\mu(X_t) = 0 \mid X_t > 0) \). The result under \( N_x \) follows from (4.2.32), which implies that
\[
N_x(\mu(X_t) = 0 \mid X_t \neq 0) = 0.
\]
In particular, we have \( N_x(\mu(X_t) > 0) = N_x(\mu(X_t) \neq 0) = U_t \). Since \( u_t^\infty(\mu) = N_x(\mu(X_t) > 0) \), this proves the last claim.

4.6 Decay of \( N_x(\mu(X_t) > 0) \) for (F2)-s measures when \( \beta > \beta^*(\alpha, s) \)

This section is concerned with establishing conditions under which \( N_x(\mu(X_t) > 0) = u_t^\infty(\mu) \) is non-flat (and hence \( \mu(X_t) = 0 \) with positive probability on \( \{ X_\tau \neq 0 \} \)) and quantifying its asymptotic behaviour under these conditions. The main result we prove is Theorem 4.1.10. The proofs are analytic and we pose it as an open problem to prove the same results using probabilistic arguments.

We will show that \( u_t^\infty = N_x(\mu(X_t) > 0) \) is non-flat when \( \beta > \beta^*(\alpha, s) = \frac{\alpha}{d-s+\alpha} \), where \( s \in [0, d] \) and \( \mu \) has compact support and satisfies (F2)-s, which we recall is the condition that

(F2)-s  For some constant \( \underline{C} > 0 \), for all \( x \in \text{supp}(\mu) \) and \( r \in (0, 1) \),
\[
\mu(B(x, r)) \geq \underline{C} r^s.
\]

The method we use is to show the existence certain barrier functions for the equation (4.1.3). A function \( h : Q \to \mathbb{R}^+ \) is a barrier function for \( \mu \) if it is a super-solution to (4.1.3) on \( Q \) that explodes on \( \text{supp}(\mu) \) with order \( t^{-\frac{\alpha}{d}} \) and vanishes on \( \text{supp}(\mu)^c \) as \( t \downarrow 0 \). Our method is based on, and adapted from, a similar argument in [7].

First, we define \( W : \mathbb{R}^+ \to \mathbb{R}^+ \) by
\[
W(r) = \frac{\log(e + r^2)}{1 + r^{d+\alpha}}.
\]
We also introduce \( V : \mathbb{R}^d \to \mathbb{R}^+ \), given by
\[
V(x) = W(|x|) = \frac{\log(e + |x|^2)}{1 + |x|^{d+\alpha}}.
\] (4.6.2)

For \((t, x) \in Q\), we then define \( w_t(x) \) by
\[
w_t(x) = t^{-\frac{d}{2}}(1 + t^{-\frac{\alpha}{2}})V(t^{-\frac{1}{2}}x).
\] (4.6.3)

Finally, for \( k > 0 \) and \( \mu \in \mathcal{M}_F(\mathbb{R}^d) \), let \( h_k(t, x) \) be given by
\[
h_k(t, x) = k (w_t * \mu)(x).
\] (4.6.4)

Note that \( w_t \in C^{1,2}(Q) \), the space of functions which are once continuously differentiable in time and twice continuously differentiable in space. Consequently, we also have that \( h_k \in C^{1,2}(Q) \). Recall that \( \beta^*(\alpha, s) = \frac{\alpha}{(d-s)+\alpha} \). In what follows, we restrict to \( s \in [0, \alpha) \), since this is required to have \( \beta^*(\alpha, s) < \frac{\alpha}{d} \).

For closed \( S \subset \mathbb{R}^d \), recall that \( \mathcal{M}_F(S) \) is the space of measures \( \mu \in \mathcal{M}_F(\mathbb{R}^d) \) with \( \text{supp}(\mu) \subseteq S \), and that \( d(x, S) = \inf_{y \in S} |x-y| \).

**Proposition 4.6.1.** Suppose that \( \mu \in \mathcal{M}_F(\mathbb{R}^d) \) satisfies (F2)-s for some \( s \in [0, \alpha) \) and has compact support \( S \subset \mathbb{R}^d \). Let \( \beta^*(\alpha, s) < \beta < \frac{\alpha}{d} \).

(a) There exists \( \Lambda_0 > 0 \) such that if \( k \geq \Lambda_0 \), \( h_k \) is a (strong) supersolution to (4.1.3) on \( Q \), in the sense that for all \((t, x) \in Q\),
\[
(\partial_t - \Delta_\alpha)h_k(t, x) + h_k(t, x)^{1+\beta} \geq 0.
\] (4.6.5)

(b) For \( x \in S \) and \( t > 0 \),
\[
h_k(t, x) \geq c_{4.6.6}k t^{-\frac{1}{2}},
\] (4.6.6)

where \( c_{4.6.6} = C \cdot c_0 \) and \( c_0 > 0 \) depends only on \((\alpha, d)\). For all \((t, x) \in Q\),
\[
h_k(t, x) \leq k\mu(1)[t^{-\frac{1}{2}} - \frac{\alpha}{d} \lor t^{-\frac{1}{2}}]W(t^{-\frac{1}{2}}d(x, S)).
\] (4.6.7)

In particular, \( \lim_{t \to 0} h_k(t, x) = \infty \) for \( x \in S \), and \( h_k(t, \cdot) \) vanishes uniformly on \( \{ x : d(x, S) \geq \rho \} \) as \( t \downarrow 0 \) for all \( \rho > 0 \).

(c) For \((t, x) \in Q\) we have
\[
h_k(t, x) \geq k c_{4.6.8} t^{-\frac{1}{2}}W(t^{-\frac{1}{2}}d(x, S)),
\] (4.6.8)

where \( c_{4.6.8} = C \cdot c_1 \) and \( c_1 > 0 \) depends only on \((\alpha, d)\).
(d) For any $\nu \in M_F(S)$, if $k \geq \Lambda_0$ then $h_k(t, x) \geq u^\nu(t, x)$ on $Q$.

This proposition is the main result underlying Theorem 4.1.10 (as well as part (b) of Theorem 4.1.14). Before proving it, we comment on the technique. By and large, our method is adapted from the argument used by Chen, Veron and Wang in [7] to prove the result we called Theorem 4.B(b) in the introduction. Our barrier function is modelled after theirs and we make use of some of their intermediate results. Define $\tilde{w}_t$ by

$$\tilde{w}_t(x) = t^{-\frac{1}{d}} W(t^{-\frac{1}{d}} |x|).$$

In [7], it is shown that $k\tilde{w}_t(x)$ is a supersolution to (4.1.3) for sufficiently large $k$. This barrier function is what the authors use to prove that $\lim_{\lambda \to \infty} u^{\lambda \delta_0}(t, x)$ is non-flat when $\beta > \frac{\alpha}{d + \alpha}$. Part of their proof was a detailed analysis of $-\Delta_\alpha V$. In equation (5.11) of [7], the following bound is established: there is a constant $c_1 > 0$ such that for $x \in \mathbb{R}^d$ with $|x| \geq 2$,

$$-\Delta_\alpha V(x) \geq -\frac{c_1}{1 + |x|^{d + \alpha}}. \quad (4.6.9)$$

This bound is critical to their argument and it is equally critical in ours which follows. As can be seen from (4.6.4), the function $h_k(t, x)$ is essentially $\tilde{w}_t$ spread out over $S = \text{supp}(\mu)$ via a convolution with $\mu$, with an additional power of $t$ to locally normalize mass of $\mu$ when $t \leq 1$. By spreading out $\tilde{w}_t$ over $S$, we construct a supersolution which is singular on $S$ as $t \downarrow 0$.

We make a few observations about the functions we have introduced. The function $W(z)$ is not globally decreasing for positive $z$, and correspondingly $V$ and $w_t$ are not globally radially decreasing. However, for any $d \geq 1$ and $\alpha \in (0, 2)$, $W$ attains its maximum value at some $r_0 \in [0, 1)$ and $W(r)$ is decreasing for $r \geq r_0$. Furthermore, one can verify that $\min_{r \in [0, 1]} W(r) = W(1)$, and so for all $d \geq 1$ and $\alpha \in (0, 2)$, $W$ is weakly decreasing in the sense that

$$\min_{r' \in [0, r]} W(r') = W(r) \text{ for all } r \geq 1. \quad (4.6.10)$$

$V$ inherits this as a form of weak radial decreasing, i.e.

$$\min_{x \in B(0, R)} V(x) = W(R) \text{ for all } R \geq 1. \quad (4.6.11)$$

Finally, one can show that there is a constant $c_{4.6.12} > 0$ such that

$$\text{For all } R \geq 0, \sup_{|x| \geq R} V(x) \leq c_{4.6.12} W(R). \quad (4.6.12)$$

Proof of Proposition 4.6.1(a). First let us consider the time derivative of $w_t(x)$. Expanding
directly using (4.6.2) and (4.6.3), for \( z = t^{-\frac{1}{\alpha}}|x| \) we have

\[
\partial_t(w_t(x)) &= -\frac{1}{\beta} t^{-\frac{1}{\beta} - 1} W(t^{-\frac{1}{\alpha}}|x|) + \left( -\frac{1}{\beta} - \frac{s}{\alpha} \right) t^{-\frac{1}{\beta} - \frac{s}{\alpha} - 1} W(t^{-\frac{1}{\alpha}}|x|) \\
&\quad - \frac{1}{\alpha} t^{-\frac{1}{\beta} - 1}(1 + t^{-\frac{1}{\alpha}})(t^{-\frac{1}{\alpha}}|x|) W'(t^{-\frac{1}{\alpha}}|x|) \\
&= t^{-\frac{1}{\beta} - \frac{s}{\alpha} - 1} \left[ \left( -\frac{1}{\beta} - \frac{s}{\alpha} \right) W(z) - \frac{1}{\alpha} z W'(z) \right] \\
&\quad + t^{-\frac{1}{\beta} - 1} \left[ -\frac{1}{\beta} W(z) - \frac{1}{\alpha} z W'(z) \right].
\] (4.6.13)

Computing \( W' \) directly, we obtain that

\[
W'(z) = \frac{2z}{e + z^2} \frac{1}{1 + z^{d+\alpha}} - (d + \alpha) \frac{\log(e + z^2)z^{d+\alpha-1}}{(1 + z^{d+\alpha})^2}
\]

From (4.6.13), it follows that

\[
\partial_t(w_t(x)) = t^{-\frac{1}{\beta} - \frac{s}{\alpha} - 1} W(z) \left[ \left( -\frac{1}{\beta} - \frac{s}{\alpha} \right) - \frac{1}{\alpha} \frac{2z^2(e + z^2)^{-1}}{\log(e + z^2)} + \frac{d + \alpha}{\alpha} \frac{z^{d+\alpha}}{1 + z^{d+\alpha}} \right] \\
&\quad + t^{-\frac{1}{\beta} - 1} W(z) \left[ -\frac{1}{\beta} - \frac{2z^2(e + z^2)^{-1}}{\alpha \log(e + z^2)} + \frac{d + \alpha}{\alpha} \frac{z^{d+\alpha}}{1 + z^{d+\alpha}} \right] \\
&= t^{-\frac{1}{\beta} - \frac{s}{\alpha} - 1} W(z)f_1(z) + t^{-\frac{1}{\beta} - 1} W(z)f_2(z). \tag{4.6.14}
\]

Consider \( f_1(z) \), the square bracketed quantity in the first line. The second term in \( f_1(z) \) vanishes as \( z \to \infty \), and the third converges to \( \frac{\alpha z^{d+\alpha}}{d + \alpha} \). The assumption \( \beta > \frac{\alpha}{d + \alpha} \) is equivalent to \( \frac{1}{\beta} + \frac{s}{\alpha} < \frac{d + \alpha}{\alpha} \); it follows that \( f_1(z) \) is positive for sufficiently large \( z \). Moreover, there are constants \( R_0 > 0 \) and \( c_2 > 0 \) such that the for \( z \geq R_0 \), \( f_1(z) \geq c_2 \). Since \( f_2(z) > f_1(z) \) for all \( z \geq 0 \), we also have \( f_2(z) \geq c_2 \) for \( z \geq R_0 \), and from (4.6.14) we have the following: for all \( (t, x) \) satisfying \( t^{-\frac{1}{\alpha}}|x| \geq R_0 \),

\[
\partial_t(w_t(x)) \geq c_2 t^{-\frac{1}{\beta} - 1}(1 + t^{-\frac{s}{\alpha}})W(t^{-\frac{1}{\alpha}}|x|). \tag{4.6.15}
\]

Next we consider \( \Delta_\alpha w_t \). By the scaling of the \( \alpha \)-stable process, if \( g^\lambda(x) = g(\lambda x) \), then \( \Delta_\alpha g^\lambda(x) = \lambda^\alpha (\Delta_\alpha g)(\lambda x) \). Using this and (4.6.3), it follows that

\[
\Delta_\alpha w_t(x) = t^{-\frac{1}{\beta} - 1}(1 + t^{-\frac{s}{\alpha}})(\Delta_\alpha V)(t^{-\frac{1}{\alpha}}|x|). \tag{4.6.16}
\]

Using the above and (4.6.9), for all \( (t, x) \) such that \( t^{-\frac{1}{\alpha}}|x| \geq 2 \),

\[
- \Delta_\alpha w_t(x) \geq -c_1 t^{-\frac{1}{\beta} - 1}(1 + t^{-\frac{s}{\alpha}}) \frac{1}{1 + |t^{-\frac{1}{\alpha}}|^{d+\alpha}}. \tag{4.6.17}
\]
This bound allows a direct comparison with $\partial_t (w_t(x))$. In particular, by (4.6.15) and (4.6.17), (and recalling the definition of $W$ from (4.6.1)) we have, for $|t^{-\frac{1}{\alpha}}x| \geq 2 \vee R_0$,

$$(\partial_t - \Delta_\alpha) w_t(x) \geq \frac{t^{-\frac{1}{\alpha}}(1 + t^{-\frac{1}{\alpha}})}{1 + |t^{-\frac{1}{\alpha}}x|^{d+\alpha}} \left[ c_2 \log(e + |t^{-\frac{1}{\alpha}}x|^2) - c_1 \right].$$

It follows that for some $R_1 \geq R_0 \vee 2$,

$$(\partial_t - \Delta_\alpha) w_t(x) \geq 0 \text{ for all } (t,x) \text{ satisfying } |t^{-\frac{1}{\alpha}}x| \geq R_1. \quad (4.6.18)$$

Now consider $h_k(t,x)$. We can take differentiation under the integral in (4.6.4) to obtain

$$(\partial_t - \Delta_\alpha) h_k(t,x) = k \int (\partial_t - \Delta_\alpha) w_t(x - y) d\mu(y). \quad (4.6.19)$$

Recall that $d(x,S)$ denotes the distance from $x \in \mathbb{R}^d$ to the set $S$. If $t^{-\frac{1}{\alpha}} d(x,S) \geq R_1$, from (4.6.18) the integrand in (4.6.19) is positive for all $y \in S$, i.e. for all $y \in \text{supp}(\mu)$, and hence

$$(\partial_t - \Delta_\alpha) h_k(t,x) \geq 0 \text{ for all } (t,x) \text{ satisfying } t^{-\frac{1}{\alpha}} d(x,S) \geq R_1. \quad (4.6.20)$$

The condition on $(t,x)$ that $t^{-\frac{1}{\alpha}} d(x,S) \geq R_1$ will be important, so we introduce

$$Q^{\geq R_1} = \{(t,x) \in Q : t^{-\frac{1}{\alpha}} d(x,S) \geq R_1\}.$$  

The statement (4.6.20) then reads that

$$(\partial_t - \Delta_\alpha) h_k(t,x) \geq 0 \text{ for all } (t,x) \in Q^{\geq R_1}. \quad (4.6.21)$$

We also introduce

$$Q^{< R_1} = \{(t,x) \in Q : t^{-\frac{1}{\alpha}} d(x,S) < R_1\},$$

and now consider the behaviour of $(\partial_t - \Delta_\alpha) h_k$ on $Q^{< R_1}$. We can apply (4.6.18) to the integrand in (4.6.19) to obtain that

$$(\partial_t - \Delta_\alpha) h_k(t,x) \geq k \int_{B(0,t^{\frac{1}{\alpha}} R_1)} (\partial_t - \Delta_\alpha) w_t(y) d\mu(y - x). \quad (4.6.22)$$
By (4.6.13) and (4.6.16), for \( u = t^{-\frac{1}{\beta}}x \) we have

\[
(\partial_t - \Delta_\alpha)w_t(y) = t^{-\frac{1}{\beta} - \frac{s}{\alpha} - 1} \left[ \left( -\frac{1}{\beta} - \frac{s}{\alpha} \right) W(|u|) - \frac{1}{\alpha} |u| W'(|u|) - \Delta_\alpha V(u) \right],
\]

\[
+ t^{-\frac{1}{\beta} - 1} \left[ -\frac{1}{\beta} W(|u|) - \frac{1}{\alpha} |u| W'(|u|) - \Delta_\alpha V(u) \right].
\]

Since all the terms in above are continuous functions of \( u \) and \(|u| \leq R_1\) on \( \{ y : |y| \leq t^{\frac{1}{\beta} R_1} \} \), the square-bracketed terms above are bounded on this set. Thus

\[
\sup_{y : |y| \leq t^{\frac{1}{\beta} R_1}} |(\partial_t - \Delta_\alpha w_t(y)| \leq c_3 t^{-\frac{1}{\beta} - 1}[t^{-\frac{s}{\alpha}} \vee 1]
\]

for some \( 0 < c_3 < \infty \). Using this in (4.6.22) we obtain that

\[
(\partial_t - \Delta_\alpha)h_k(t, x) \geq -kc_3 t^{-\frac{1}{\beta} - 1} \left[ \mu(B(x, t^{\frac{1}{\beta} R_1})) \right]
\]

for all \((t, x) \in Q\). We now must show that the non-linear term in (4.1.3), given by \( h_k(t, x)^{1+\beta} \), is sufficiently large on \( Q^{<R_1} \) so that \( h_k(t, x) \) is a super-solution to (4.1.3) even in the case of the worst-case bound given in (4.6.23). From (4.6.3) and (4.6.4), we have

\[
\frac{1}{k^{1+\beta}}h_k(t, x)^{1+\beta} = \left[ \int w_t(x-y) d\mu(y) \right]^{1+\beta}
\]

\[
= \left[ t^{-\frac{1}{\beta}}(1 + t^{-\frac{s}{\alpha}}) \int V(t^{-\frac{1}{\beta}}(x-y)) d\mu(y) \right]^{1+\beta}
\]

\[
\geq \left[ (t^{-\frac{1}{\beta} - \frac{s}{\alpha}} \vee t^{-\frac{1}{\beta}}) \int \mu(B(x, t^{\frac{1}{\beta} R_1+1})) V(t^{-\frac{1}{\beta}}(x-y)) d\mu(y) \right]^{1+\beta}
\]

where the final inequality holds because \( V \geq 0 \). Restricted to \( B(x, t^{\frac{1}{\beta} (R_1+1)}) \), \( V(t^{-\frac{1}{\beta}}(x-y)) \) is bounded below by \( W(R_1 + 1) \) by (4.6.11). This implies that

\[
\frac{1}{k^{1+\beta}}h_k(t, x)^{1+\beta} \geq c_4 t^{-\frac{1}{\beta} - 1} \left[ \mu(B(x, t^{\frac{1}{\beta} (R_1 + 1)})) \right]^{1+\beta}
\]

on \( Q^{<R_1} \), where \( c_4 = W(R + 1)^{1+\beta} \). Combining (4.6.23) and (4.6.24), we obtain that for
for $x \in Q_{T}^{<R_1}$,

$$\begin{align*}
(\partial_t - \Delta_\alpha)h_k(t,x) + h_k(t,x)^{1+\beta} &\geq kt^{-\frac{1}{\alpha} - 1}s_\alpha \left[ \frac{\mu(B(x, t^{\frac{1}{\alpha}} R_1 + 1)))}{t^{\frac{1}{\alpha}} \wedge 1} \right]^{1+\beta} - c_3 \left[ \frac{\mu(B(x, t^{\frac{1}{\alpha}} R_1))}{t^{\frac{1}{\alpha}} \wedge 1} \right] \\
&\geq kt^{-\frac{1}{\alpha} - 1}s_\alpha \left[ \frac{\mu(B(x, t^{\frac{1}{\alpha}} R_1))}{t^{\frac{1}{\alpha}} \wedge 1} \right] \left( c_4 k^{\beta} - c_3 \right). \quad (4.6.25)
\end{align*}$$

Since $(t,x) \in Q^{<R_1}$, there must be a point $y_0 \in S$ such that $B(y_0, t^{\frac{1}{\alpha}}) \subset B(x, t^{\frac{1}{\alpha}} R_1 + 1))$. In particular, since $\mu$ satisfies $(F2)$-s we have

$$\mu(B(x, t^{\frac{1}{\alpha}} R_1 + 1))) \geq \mu(B(y_0, t^{\frac{1}{\alpha}})) \geq C[t^{\frac{1}{\alpha}} \wedge 1].$$

The minimum above appears since for $y_0 \in S$ and $t \geq 1$, $\mu(B(y_0, t^{\frac{1}{\alpha}})) \geq \mu(B(y_0, 1)) \geq C$. Using this in (4.6.25), we obtain

$$\begin{align*}
(\partial_t - \Delta_\alpha)h_k(t,x) - h_k(t,x)^{1+\beta} &\geq kt^{-\frac{1}{\alpha} - 1}s_\alpha \left[ \frac{\mu(B(x, t^{\frac{1}{\alpha}} R_1))}{t^{\frac{1}{\alpha}} \wedge 1} \right] \left( c_4' k^{\beta} - c_3 \right)
\end{align*}$$

for all $(t,x) \in Q^{<R_1}$, where $c_4' = C^\beta c_4$. For sufficiently large $k$ the above is non-negative, and hence $h_k(t,x)$ is a super-solution to (4.1.3) on $Q^{<R_1}$. On the other hand, because $h_k(t,x)^{1+\beta} \geq 0$, (4.6.21) implies that $h_k(t,x)$ is a super-solution on $Q^{\geq R_1}$. Thus we have shown that for sufficiently large $k$, $h_k(t,x)$ is a super-solution to (4.1.3) on $Q^{<R_1} \cup Q^{\geq R_1} = Q$.

Proof of Proposition 4.6.1(b)-(d). We first show part (c). For fixed $x$, since $S$ is closed, there is a point $y_0 \in S$ such that $|x - y_0| = d(x, S)$. Hence from (4.6.4),

$$\begin{align*}
h_k(t,x) &\geq k \int_{B(y_0, t^{\frac{1}{\alpha}})} w_t(x-y) d\mu(y) \\
&= kt^{-\frac{1}{\alpha} + (1 - t^{-\frac{1}{\alpha}})} \int_{B(y_0, t^{\frac{1}{\alpha}})} W(t^{-\frac{1}{\alpha}} |x-y|) d\mu(y) \\
&\geq k[t^{-\frac{1}{\alpha}} \wedge t^{\frac{1}{\alpha}} (t^{\frac{1}{\alpha}} - \frac{1}{\alpha})] W(t^{-\frac{1}{\alpha}} d(x, S) + 1) \mu(B(y_0, t^{\frac{1}{\alpha}})),
\end{align*}$$

where the last line has used the triangle inequality and (4.6.10). Since $\mu$ satisfies $(F2)$-s, $\mu(B(y_0, t^{\frac{1}{\alpha}})) \geq C t^{\frac{1}{\alpha}}$ for $t \leq 1$ and $\mu(B(y_0, t^{\frac{1}{\alpha}})) \geq C$ for $t > 1$. This implies that above is bounded below by $kC t^{-\frac{1}{\alpha}} W(t^{-\frac{1}{\alpha}} d(x, S) + 1)$. The result then follows from (4.6.12).

Next we prove part (b). The claim for $x \in S$ follows from part (c). Now let $x \in \mathbb{R}^d$. 204
Applying (4.6.12), we obtain that for every \( y \in S \),

\[
 w_t(x - y) \leq c_{4.6.12}[t^{-\frac{1}{\beta} - \frac{s}{\alpha}} \vee t^{-\frac{1}{\beta}}]W(t^{-\frac{1}{\alpha}}d(x, S)).
\]

One then uses this bound directly in the convolution defining \( h_k(t, x) \), i.e. (4.6.4), to obtain (4.6.7). Using formula (4.6.1) for \( W \) and the fact that \( \frac{1}{\beta} + \frac{s}{\alpha} < \frac{d+\alpha}{\alpha} \), the uniform convergence as \( t \downarrow 0 \) follows.

We now prove (d). Suppose that \( \nu \in \mathcal{M} F(S) \). (This includes the case \( \nu = \mu \).) In order to show that \( u^\nu(t, x) \leq h_k(t, x) \), we will need to consider a sequence of solutions corresponding to a sequence of approximations of \( \nu \). Let \( Z > 0 \) be the normalizing constant such that

\[
 Z^{-1} \int V(x)dx = 1.
\]

Let \( \tilde{V}(\cdot) = Z^{-1}V(\cdot) \) and define a sequence of \( C^2 \) mollifiers by \( \phi_n(x) = n^d\tilde{V}(nx) \) for \( n \geq 1 \). Then it is immediate that \( \nu * \phi_n \to \nu \) in the weak sense of measures as \( n \to \infty \). Let \( u_n(t, x) = u^{\nu*\phi_n}(t, x) \). By Lemma 4.2.6(a),

\[
 \lim_{n \to \infty} u_n(t, x) = u^\nu(t, x) \tag{4.6.26}
\]

for all \((t, x) \in Q\). For \( x \in \mathbb{R}^d \),

\[
 \nu * \phi_n(x) = \frac{1}{Z} n^d \int W(n|y - x|)d\nu(y) \leq \frac{1}{Z} \nu(1)n^d \sup_{y \in S} W(n|x - y|)
\]

Hence from (4.6.12) we obtain that

\[
 \nu * \phi_n(x) \leq \frac{c_{4.6.12}\nu(1)}{Z}[n^dW(nd(x, S))]. \tag{4.6.27}
\]

Let \( t_n = n^{-\alpha} \). By (4.6.8), we have

\[
 h_k(t_n, x) \geq c_{4.6.8}kn^\frac{d}{\beta}W(nd(x, S)) = c_{4.6.8}kn^\frac{d}{\beta} - d[n^dW(nd(x, S))] \tag{4.6.28}
\]

By part (a), if \( k \geq A_0 \), then \((t, x) \to h_k(t_n + t, x)\) is a super-solution to (4.1.3) on \( Q \) with initial value \( h_k(t_n, \cdot) \). Since \( \frac{d}{\beta} > d \), it follows from (4.6.27) and (4.6.28) that, for sufficiently large \( n \),

\[
 \nu * \phi_n(x) \leq h_k(t_n, x).
\]
Since $u_n$ has initial data $\nu * \phi_n$ and $(t, x) \to h_k(t_n + t, x)$ is a super-solution, the above and the comparison principle imply that

$$u_n(t, x) \leq h_k(t_n + t, x)$$

for all $(t, x) \in Q$. Taking $n \to \infty$ on both sides and using (4.6.26), we obtain that

$$u^\nu(t, x) \leq h_k(t, x)$$

for all $(t, x) \in Q$.

The final ingredient in the proof of Theorem 4.1.10 is the following lower bound on $u^\infty_{\mu_t}(x)$. We recall that $u^\infty_{\mu_t}(x) = \lim_{\lambda \to \infty} u^{\lambda\mu_t}_{\lambda}(x)$.

**Lemma 4.6.2.** Let $\mu \in M_F(\mathbb{R}^d)$. Then for every $z \in \text{supp}(\mu)$,

$$u^\infty_{\mu_t}(x) \geq u^\infty_{\mu_t}(x - z) \geq c_{4.6.2} t^{-\frac{1}{\beta}} p_1(t^{-\frac{1}{\alpha}}(x - z))$$

for all $(t, x) \in Q$, where $c_{4.6.2} > 0$ depends only on $(\alpha, \beta, d)$.

We defer the proof to the end of the section.

**Proof of Theorem 4.1.10.** Fix $s \in [0, \alpha)$ and let $\mu$ satisfy (F2)-$s$ and have compact support $S$. Let $\beta^*(\alpha, s) < \beta < \frac{\alpha}{d}$. We begin with part (b). We need to show that the upper bound in (4.1.19) holds, i.e. for some constant $C > 0$,

$$u^\infty_{\mu_t}(x) \leq C[t^{-\frac{1}{\beta} - \frac{s}{\alpha}} \vee t^{-\frac{1}{\beta}}] W(t^{-\frac{1}{\alpha}}d(x, S))$$

on $Q$. Let $\Lambda_0 > 0$ be as in Proposition 4.6.1(a). Then by Proposition 4.6.1(d), $u^{\lambda\mu}_{t}(x) \leq h_{\Lambda_0}(t, x)$ for every $\lambda > 0$, and hence $u^\infty_{\mu_t}(x) = \lim_{\lambda \to \infty} u^{\lambda\mu}_{t}(x) \leq h_{\Lambda_0}(t, x)$. The above bound then follows from Proposition 4.6.1(b) with $C = \mu(1)\Lambda_0$, which proves the upper bound in (4.1.19). The lower bound in (4.1.19) follows from Lemma 4.6.2.

Next we prove part (a). From the upper bound in (4.1.19), it is clear that for $t > 0$, $u^\infty_{\mu_t}(x) \to 0$ as $d(x, S) \to \infty$. We can thus take $x_0 \in \mathbb{R}^d$ so that $u^\infty_{\mu_t}(x_0) < U_t$, which is equivalent to $\mathbb{N}_{x_0}(\mu(\mathcal{X}_t) = 0 | \mathcal{X}_t \neq 0) > 0$. By (4.2.31), we have $P^X_{x_0}(\mu(\mathcal{X}_t) = 0 | \mathcal{X}_t \neq 0) > 0$. Appealing to (4.2.5), it follows that

$$P^X_{x_0}(\mu(\mathcal{X}_t) = 0 | \mathcal{X}_t \neq 0) > 0$$

for every $\mathcal{X}_0 \in M_F(\mathbb{R}^d)$. In particular, we can take $\mathcal{X}_0 = \delta_x$ and use (4.2.31) again to see that $\mathbb{N}_{x}(\mu(\mathcal{X}_t) = 0 | \mathcal{X}_t \neq 0) > 0$ for every $x \in \mathbb{R}^d$. This completes the proof of (a).
To see that part (d) holds, recall that Proposition 4.6.1(d) applies to any \( \nu \in \mathcal{M}_F(S) \). The upper bound in (4.1.21) then follows by the same argument used to prove the upper bound in (4.1.19) above. The argument used to prove (a) can then be used to prove that the claims from (a) hold when \( \nu \) is replaced with \( \mu \). Finally, as in the proof of part (b), the lower bound in (4.1.21) follows from Lemma 4.6.2.

It remains to show part (c). Recall from (4.2.25) that

\[
 u_\infty^\mu(x) = t^{-\frac{1}{\beta}} u_1^{\infty \mu(\cdot/t^{-\frac{1}{\beta}})}(t^{-\frac{1}{\beta}}x).
\]

If \( \mu \) satisfies (F1)-s, for \( t \leq 1 \) we apply Lemma 4.5.1 to the right hand side of the above to obtain

\[
 u_\infty^\mu(x) \geq c_{4.5.1} t^{-\frac{1}{\beta} - \frac{\alpha}{\alpha}} \int p_1(y - t^{-\frac{1}{\alpha}}x)d\mu(y/t^{-\frac{1}{\alpha}}) \\
 = c_{4.5.1} t^{-\frac{1}{\beta} - \frac{\alpha}{\alpha}} \int p_1(t^{-\frac{1}{\alpha}}(y - x))d\mu(y).
\]

For every \( y \in S \), \( |y - x| \leq d(x, S) + \text{diam}(S) \), so from the above, for \( x \in \mathbb{R}^d \) and \( t \leq 1 \) we have

\[
 u_\infty^\mu(x) \geq c_{4.5.1} \mu(1)t^{-\frac{1}{\beta} - \frac{\alpha}{\alpha}} p_1(t^{-\frac{1}{\alpha}}(d(x, S) + \text{diam}(S))).
\]

The lower bound on \( p_1 \) from (4.2.1) then implies (4.1.20).

**Proof of Lemma 4.6.2.** Let \( \mu \in \mathcal{M}_F(\mathbb{R}^d) \) and \( z \in \text{supp}(\mu) \). Then \( \mu(B(z, \rho)) > 0 \) for every \( \rho > 0 \). For \( k > 0 \) and \( \rho > 0 \), we have

\[
 \lim_{t \to 0} \int_{B(z, \rho)} u_t^{k \mu}(x) dx \geq k\mu(B(z, \rho/2)).
\]

Since \( u_t^{\infty \mu} \geq u_t^{k \mu} \) for all \( k > 0 \), it follows that

\[
 \lim_{t \to 0} \int_{B(z, \rho)} u_t^{\infty \mu}(x) dx = +\infty \quad \text{for all} \quad z \in \text{supp}(\mu) \quad \text{and} \quad \rho > 0. \tag{4.6.29}
\]

We now fix \( \lambda > 0 \) and \( z \in \text{supp}(\mu) \). By (4.6.29), there exists \((t_n, \rho_n)\) such that \( t_n, \rho_n \to 0 \) and

\[
 \int_{B(z, \rho_n)} u_t^{\infty \mu}(x) dx = \lambda
\]

for all \( n \geq 1 \). Let \( \phi_n(x) = u_{t_n}^{\infty \mu}(x)1_{B(z, \rho_n)}(x) \). Note that \( \phi_n \to \lambda \delta_0(\cdot - z) \) in the weak sense of measures as \( n \to \infty \), so by Lemma 4.2.6(a), \( \lim_{n \to \infty} u_t^{\phi_n}(x) = u_t^\lambda(x - z) \). By (4.1.10), we have

\[
 u_t^{\phi_n}(x) = N_x(1 - \exp(\mathcal{N}_t(\phi_n))). \tag{4.6.30}
\]
On the other hand, we remark that for every $k > 0$, by (4.2.19) and the Markov property,

\[
 u^k_{t+t_n}(x) = N_x(1 - \exp(k\mu(X_{t+t_n})) \\
 = N_x(1 - \exp(X_{t+t_n}^{k\mu})). 
\]

Taking $k \to \infty$, we obtain

\[
 u^\infty_{t+t_n}(x) = N_x(1 - \exp(X_{t}^{\infty\mu})). \tag{4.6.31} 
\]

By the definition of $\phi_n$, we have $\phi_n \leq u^\infty_{t+t_n}$ for all $n \in \mathbb{N}$. Hence, from (4.6.30) and (4.6.31), we have

\[
 u^{\phi_n}_t(x) \leq u^\infty_{t+t_n}(x) 
\]

for all $(t,x) \in Q$ and all $n \in \mathbb{N}$. As noted above, the left hand side converges to $u^\lambda_t(x-z)$ as $n \to \infty$. Taking $n \to \infty$, we obtain that

\[
 u^\lambda_t(x-z) \leq u^\infty_t(x). 
\]

Taking $\lambda \to \infty$, we obtain $u^\infty_t(x-z) \leq u^\infty_t(x)$. This proves the first inequality in the lemma, and the second then follows from (4.3.5). \hfill \Box

### 4.7 The initial trace problem

We now apply the results of Sections 4.5 and 4.6 to the initial trace theory for (4.1.3). Recall that the initial trace of a solution $u_t(x)$ to (4.1.3) was defined in (4.1.23) and (4.1.24). We restate the definition here for convenience. A pair $(\mathcal{S},\nu)$ with $\mathcal{S} \subset \mathbb{R}^d$ a closed set and $\nu$ a Radon measure with $\nu(\mathcal{S}) = 0$ is the initial trace of $u_t(x)$ if:

- For all $\xi \in C_c(\mathcal{S}^c)$.
  
  \[
  \lim_{t \to 0} \int \xi(x) u_t(x) dx = \int \xi d\nu. \tag{4.7.1} 
  \]

- For every $z \in \mathcal{S}$ and $\rho > 0$,

  \[
  \lim_{t \to 0} \int_{B(z,\rho)} u_t(x) dx = +\infty. \tag{4.7.2} 
  \]

Our contribution is to the problem of determining when a solution with a given initial trace exists. We consider weak solutions. First, recall that we have defined a weak solution to (4.1.3) in Definition 4.2.3. Although we only consider the problem for initial traces in which the regular component (i.e. the Radon measure) is null, our definition applies for general
initial traces.

**Definition 4.7.1.** For closed \( S \subset \mathbb{R}^d \) and a Radon measure \( \nu \) satisfying \( \nu(S) = 0 \), we say that \( u : Q \to [0, \infty) \) is a weak solution to the initial trace problem with initial trace \((S, \nu)\) if:

- \( u \) is a weak solution to (4.1.3) in the sense of Definition 4.2.3.
- The initial trace \((S, \nu)\) is attained in the sense that (4.7.1) and (4.7.2) hold.

Our main result about the initial trace problem is Theorem 4.1.14, which has two parts: non-existence and existence. Most of the work has already been carried out in Sections 4.5 and 4.6. Both proofs require the following lemma.

**Lemma 4.7.2.** For \( \mu \in \mathcal{M}_F(\mathbb{R}^d) \), \( u_t^\infty(\mu) = \lim_{\lambda \to \infty} u_t^\lambda(\mu) \) is a weak solution to (4.1.3). Furthermore, for all \( \rho > 0 \) and \( z \in \text{supp}(\mu) \),

\[
\lim_{t \to 0} \int_{B(z, \rho)} u_t^\infty(\mu) \, dx = +\infty,
\]

(4.7.3)

and hence the singular set of \( u_t \) contains \( \text{supp}(\mu) \).

**Proof.** Let \( \mu \in \mathcal{M}_F(\mathbb{R}^d) \). We must show that \( u_t^\infty(t, x) \) satisfies (4.2.13) for every \( \xi \in C_c^{1,2}(Q) \). For \( \lambda > 0 \), \( u_t^\lambda(t, x) \) is a solution to the problem (4.2.14) with \( u_0 = \lambda \mu \), so by definition we have

\[
\int_Q (u_t^\lambda(t, x)[-\partial_t \xi(t, x) - \Delta \xi(t, x)]) + u_t^\lambda(t, x)^{1+\beta} \xi(t, x) \, dx \, dt = 0.
\]

Since \( \xi \) has compact support, by (4.1.18) the bound \( u_t^\lambda(t, x) \leq u_t^\infty(t, x) \leq U_t \) allows us to apply Dominated Convergence and conclude that \( u_t^\infty(t, x) \) satisfies (4.2.13) for all \( \xi \in C_c^{1,2}(Q) \). Similarly, \( u_t^\infty(t, x) \leq U_t \) implies that \( u_t^\infty(t, x) \) is bounded on \( (\epsilon, \infty) \times \mathbb{R}^d \) for all \( \epsilon > 0 \) and hence \( u_t^\infty \in L_{1+\beta}^{\text{loc}}(Q) \). To see the \( u_t^\infty(x) \) is continuous, we note that for any \( t_0 > 0 \), \((t, x) \to u_t^\infty(t_0 + t, x)\) is a weak solution to (4.1.3) which is globally bounded by \( U_{t_0} \). In particular, (recall Remark 4.2.8) it is a solution to the integral equation (4.1.2) with initial data \( u_{t_0}^\infty \in \mathcal{B}_b^+ \) and hence is continuous. It follows that \( u_t^\infty(x) \) is continuous on \([t_0, \infty) \times \mathbb{R}^d \) for all \( t_0 > 0 \) and hence on \( Q \). Thus \( u_t^\infty(x) \) is a weak solution to (4.1.3).

The fact that (4.7.3) holds has already been shown in the proof of Lemma 4.6.2; see (4.6.29).

Our main result concerning flatness and non-existence is the following. We recall from (4.1.26) the space \( \mathcal{U} \) of positive functions on \( Q \) bounded above by \( U_t \), and that we have restricted our attention to solutions in \( \mathcal{U} \).
Theorem 4.7.3. Suppose that \( S \subset \mathbb{R}^d \) is closed and supports a measure \( \mu \in \mathcal{M}_F(\mathbb{R}^d) \) for which \( \lim_{\lambda \to \infty} u^{\lambda \mu}_t = U_t \). If \( S \) is contained in the singular set of a solution \( u \) to (4.1.3) in \( U \), then \( u_t = U_t \). In particular, if \( S \neq \mathbb{R}^d \) there is no solution to (4.1.3) in \( U \) with singular set \( S \).

We obtain Theorem 4.1.8(b) and Theorem 4.1.14(a) as corollaries as follows.

Proof of Theorem 4.1.8(b). Let \( s \in [0, d] \), \( \beta \leq \beta^*(\alpha, s) \) and suppose that \( \mu \in \mathcal{M}_F(\mathbb{R}^d) \) satisfies (F1)-s. Then \( u^{\infty \mu}_t(x) = U_t \) by Theorem 4.1.8(a). Hence \( S = \text{supp}(\mu) \) supports a measure \( \mu \) for which \( \lim_{\lambda \to \infty} u^{\lambda \mu}_t(x) = U_t \). Now let \( \nu \in \mathcal{M}_F(\mathbb{R}^d) \) be any measure such that \( \text{supp}(\mu) \subseteq \text{supp}(\nu) \). By Lemma 4.7.2, \( u^{\infty \nu}_t \) is a weak solution to (4.1.3) whose singular set contains \( \text{supp}(\nu) \), which itself contains \( S \). The result follows from Theorem 4.7.3.

Proof of Theorem 4.1.14(a). If \( S \subset \mathbb{R}^d \) is closed with \( \mathcal{H}^{d_{\text{sat}}}(S) > 0 \), then by Frostman’s Lemma there exists \( \mu \in \mathcal{M}_F(S) \) which satisfies (F1)-d_{sat}. By Theorem 4.1.8(a), \( u^{\infty \mu}_t = U_t \), and the result follows from Theorem 4.7.3.

The proof of Theorem 4.7.3 requires the following pointwise estimate for solutions with a given singular set. A more general version of this result proved for classical solutions in [8], where it was called Theorem C. We include a short proof for the sake of completeness. The argument is essentially the same as the argument in the proof of Lemma 4.6.2. Recall that \( u^{\infty}_t(x) = \lim_{\lambda \to \infty} u^{\lambda \delta_0}_t(x) \).

Proposition 4.7.4. Suppose that \( u(t, x) \) is a weak solution to (4.1.3) in \( U \) whose singular set contains \( S \subset \mathbb{R}^d \). Then for every \( z \in S \),

\[
u(t, x) \geq u^{\infty}_t(t, x - z) \geq c_{4.7.4} t^{-\frac{1}{\beta} p_1(t^{-\frac{1}{\alpha}}(x - z))}
\]

for all \( (t, x) \in Q \), where \( c_{4.7.4} > 0 \) depends only on \( (\alpha, \beta, d) \).

Proof. Suppose that \( z \in S \), the singular set of \( u(t, x) \). Let \( \lambda > 0 \). Since (4.7.2) holds, there must be sequences \( (t_n)_{n \geq 1} \) and \( (\rho_n)_{n \geq 1} \) such that \( t_n, \rho_n \to 0 \) and

\[
\int_{B(z, \rho_n)} u(t_n, x) \, dx = \lambda.
\]

Let \( \phi_n(x) = 1_{B(z, \rho_n)}(x) u(t_n, x) \). We then have

\[
u(t_n, x) \geq \phi_n(x)
\]
for all $x \in \mathbb{R}^d$. Since $u(t_n, \cdot)$ and $\phi_n$ are both bounded (because $u \in \mathcal{U}$), it follows from the comparison principle (recall Remark 4.2.10) that

$$u(t_n + t, x) \geq u^{\phi_n}(t, x) \quad (4.7.4)$$

for all $(t, x) \in Q$. Note that $\phi_n \to \lambda \delta_z$ in $M_F(\mathbb{R}^d)$, so by Lemma 4.2.6(a) and translation invariance, $u^{\phi_n}(t, x) \to u^\lambda(t, x - z)$. (Recall that $u^\lambda(t, x) = u^{\lambda \delta_0}(t, x)$.) Of course, $\lim_{n \to \infty} u(t_n + t, x) = u(t, x)$, and so taking $n \to \infty$ in (4.7.4) we obtain

$$u(t, x) \geq u^\lambda(t, x - z).$$

Since this holds for all $\lambda > 0$, it follows that $u(t, x) \geq u^\infty(t, x - z)$. The second inequality in the result then follows from (4.3.5). \qed

**Proof of Theorem 4.7.3.** Let $\mathcal{S}$ and $\mu$ be as in the statement of the theorem. Suppose that $u_t(x)$ is a solution to (4.1.3) in $\mathcal{U}$ whose singular set contains $\mathcal{S}$. By Proposition 4.7.4, for some constant $c > 0$ we have

$$u_t(x) \geq \sup_{z \in \mathcal{S}} ct \frac{1}{p_1(t - \frac{1}{\alpha}d(x, \mathcal{S})))}$$

where the second equality holds because $\mathcal{S}$ is closed and $p_1$ is continuous and radially decreasing. By assumption, $\mathcal{S}$ supports a finite measure $\mu$ such that $\lim_{\lambda \to \infty} u^{\lambda \mu}_t = U_t$. Fix $\lambda > 0$. By (4.2.16),

$$u^{\lambda \mu}_t(x) \leq \lambda S_t \mu(x)$$

$$\leq \lambda \mu(1)p_1(d(x, \mathcal{S})))$$

$$= \lambda \mu(1)t^{-\frac{d}{\alpha}}p_1(t - \frac{1}{\alpha}d(x, \mathcal{S}))) \quad (4.7.6)$$

Since $\beta < \frac{d}{\alpha}$, (4.7.5) and (4.7.6) imply that there exists $t_0(\lambda) > 0$ such that

$$u^{\lambda \mu}_t(x) \leq u_t(x) \quad \text{for all } x \in \mathbb{R}^d \text{ and } t \in (0, t_0(\lambda)]. \quad (4.7.7)$$

Applying the comparison principle (as in Remark 4.2.10) at time $t_0(\lambda)$, it follows that

$$u^{\lambda \mu}_t(x) \leq u_t(x) \quad \text{for all } (t, x) \in Q. \quad (4.7.8)$$

The above holds for all $\lambda > 0$. Since $\lim_{\lambda \to \infty} u^{\lambda \mu}_t = U_t$ and $u \in \mathcal{U}$, it follows that $u_t(x) = U_t$. \qed
It remains to prove our existence result, Theorem 4.1.14(b), which states that if $\mathcal{S} \subset \mathbb{R}^d$ is compact and $\mathcal{S} = \text{supp}(\mu)$ for $\mu \in \mathcal{M}_F(\mathbb{R}^d)$ satisfying (F2)-s for some $s < d_{\text{sat}}$, then there exists a weak solution of (4.1.3) with initial trace $(\mathcal{S}, 0)$ if $\beta^*(\alpha, s) < \beta < \frac{\alpha}{d}$. This solution is $u_t^{\infty \mu}$.

**Proof of Theorem 4.1.14(b).** Let $\mu \in \mathcal{M}_F(\mathbb{R}^d)$ satisfy (F2)-s and have compact support $\mathcal{S}$. By Lemma 4.7.2, $u_t^{\infty \mu}(x)$ is a weak solution to (4.1.3) and $\mathcal{S}$ is contained in the singular set of $u_t^{\infty \mu}$. By (4.1.19), $u_t^{\infty \mu}(x)$ vanishes uniformly on $\{x : d(x, \mathcal{S}) \geq \epsilon\}$ as $t \downarrow 0$ for any $\epsilon > 0$. It follows that for any $\xi \in C_c(\mathcal{S}^c)$,

$$\lim_{t \to 0} \int \xi(x) u_t^{\infty \mu}(x) dx = 0.$$ 

Hence the singular set of $u_t^{\infty \mu}$ is no larger than $\mathcal{S}$ and (4.1.23) holds with measure $\nu = 0$, which implies that $u_t^{\infty \mu}$ has initial trace $(\mathcal{S}, 0)$. \qed
Chapter 5

Conclusion

We conclude with a discussion of our results in a broader context. We also discuss possible extensions and future work.

5.1 Super-Brownian motion

In the chapters on super-Brownian motion, we mentioned that the interest in \( \partial Z_t \) was due to an important open problem concerning SPDE: establishing that pathwise uniqueness holds (or does not hold) for the stochastic equation

\[
\frac{\partial}{\partial t} X_t(x) = \frac{\Delta}{2} X_t(x) + \sqrt{X_t(x)} \dot{W}(t,x),
\]  

(5.1.1)

where \( \dot{W}(t,x) \) is a space-time white Gaussian noise. We recall that the density of super-Brownian motion in one dimension is the unique in law solution to (5.1.1). It remains unclear how the local time itself, as constructed in this thesis, could resolve pathwise uniqueness. However, the resolution of the dimension problem is useful here due to the importance of \( \partial Z_t \) to this problem and could potentially be applied to proving pathwise uniqueness for SPDE related to (5.1.1).

An open question of some interest is whether or not the results about \( \partial Z_t \) hold for \( \partial S(X_t) \), the boundary of the support. From a geometric standpoint \( \partial S(X_t) \) may be the more natural set. Some natural problems are to determine if \( \partial S(X_t) \) has dimension \( 2 - 2\lambda_0 \) a.s. on \( \{X_t \neq 0\} \), and if \( \partial S(X_t) \) is a support for \( L_t \). That is, if \( L_t(S(X_t)^c) = 0 \) almost surely. Clearly, showing that \( \partial Z_t = \partial S(X_t) \) solves both of these problems, since, as we recall from (1.4.7), \( \partial S(X_t) \subseteq S(L_t) \subseteq \partial Z_t \). This is related to other open problems about \( \partial Z_t \). For example, it is unknown if \( \partial Z_t \) can have isolated points. An isolated point in \( \partial Z_t \) is not in \( \partial S(X_t) \), so the existence of isolated points would imply that the sets are not the same. Furthermore, since \( L_t \) has no atoms, the existence of isolated points would imply...
that $S(L_t)$ is strictly smaller than $\partial Z_t$.

As we noted in Chapter 1, there has been a parallel investigation into the boundary of the range of super-Brownian motion, and in this case the improvement from the boundary of the zero set to the boundary of the support has been successful [73, 33]. In these works, the zero set and support are defined with respect to the density $\mathcal{X}(x)$ of the integrated superprocess. (See (1.3.10) and the discussion in Section 1.3.4.) However, the methods in this case are very different and rely on the special Markov property for exit measures of super-Brownian motion, a tool which does not appear to apply in our setting.

It would be interesting to revisit the problems considered in this thesis for super-Brownian motion with $(1 + \beta)$-stable branching. We denote this process by $X_t$ and recall that $X_t$ has a density if and only if $\beta < \frac{2}{d}$. It is possible to describe the left tail of the density in terms of an eigenvalue defined analogously to $\lambda_0$. However, the dimension of $\partial Z_t$ has not been computed, and there has been no attempt to construct the boundary local time. Despite this, more progress has been made on the problem of pathwise uniqueness in this setting. Yang and Zhou considered the regularity to solutions of SPDE driven by stable noise, including solutions to the SPDE

$$\frac{\partial}{\partial t} X_t(x) = \frac{\Delta}{2} X_t(x) + X_t(x)^{1+\beta} \dot{F}(t, x),$$

(5.1.2)

where $\dot{F}(t, x)$ is a one-sided stable noise of index $1 + \beta$. It is known from work of Mytnik [70] that the unique in law solution to (5.1.2) is the density of $(1 + \beta)$-stable branching super-Brownian motion. Yang and Zhou [95] proved that pathwise uniqueness is satisfied for (5.1.2) when $0 < \beta < \sqrt{5} - 2$. Remarkably, their work did not make use of the so-called improved modulus of regularity at the zero set, which has been important in other proofs of pathwise uniqueness (e.g. [74]). It is worth investigating if such improved moduli can be established in case of stable noise, and if so, if they could be applied to prove that pathwise uniqueness holds in (5.1.2) for values of $\beta$ greater than $\sqrt{5} - 2$.

### 5.2 The $(\alpha, \beta)$-superprocess

There are a number of questions related to our work on the $(\alpha, \beta)$-superprocess which we believe merit attention. A very natural problem is to complete the classification of measures $\mu$ for which $\mu(\mathcal{X}_t) > 0$ almost surely. In view of Theorems 4.1.9 and 4.1.10, the main problem here is to determine what happens when $S(\mu)$ has dimension $d_{\text{sat}} = d + \alpha - \frac{\alpha}{\beta}$ but is $\mathcal{H}^{d_{\text{sat}}}$-null. (Recall that $\mathcal{H}^d$ denotes the $x^d$-Hausdorff measure, and we showed that $\mu(\mathcal{X}_t) > 0$ a.s. on $\{\mathcal{X}_t \neq 0\}$ when $\mathcal{H}^{d_{\text{sat}}}(S(\mu)) > 0$.) Furthermore, we would like very much to extend the non-flat result (Theorem 4.1.10) to all measures with compact support of dimension less than $d_{\text{sat}}$, not simply those whose support is contained in that of an (F2) measure.
One could also consider the case where \( \text{supp}(\mu) \) is non-compact. Our results, as well as instantaneous propagation, suggest that the local properties, rather than the global ones, determine if \( \mu(\mathcal{X}_t) > 0 \) a.s. or not. However, it would be very interesting to know if there exist measures for which global properties “override” local properties, that is if there exists a (potentially infinite) measure \( \mu \) with unbounded support such that \( \dim(S(\mu)) < d_{\text{sat}} \), but \( \mu(\mathcal{X}_t) > 0 \) a.s. on \( \{\mathcal{X}_t \neq 0\} \). On a related note, Theorem L of [8] implies that for functions \( \phi \) with sufficiently fast growth at infinity, \( \mathcal{X}_t(\phi) = +\infty \) a.s. on \( \{\mathcal{X}_t \neq 0\} \). In a sense, this posits an almost sure global property analogous to the almost sure local properties we have considered.

The initial trace theory of the dual equation, that is the fractional PDE

\[
\frac{\partial}{\partial t} u_t(x) = \Delta_{\alpha} u_t(x) - u^{1+\beta}(x),
\]

is still in its initial stages, having been introduced in 2019 [8]. Our results advanced this theory by characterizing the admissible singular sets in terms of their Hausdorff dimension. Our theorem is not complete, the extensions we mentioned above for our results about \( \mathcal{X}_t \) translate to the following in the PDE setting. They are (i) to extend the existence result (Theorem 4.1.14) by removing the condition that \( S \) is the support of an (F2) measure to allow \( S \) to be a compact set with \( \dim(S) < d_{\text{sat}} \); and (ii) to understand the “critical” case, that is to determine if solutions exist when \( \dim(S) = d_{\text{sat}} \) with \( \mathcal{H}^{d_{\text{sat}}}(S) = 0 \).

In the super-Brownian case, Le Gall [55] gave a probabilistic characterization of the admissible initial traces for the dual equation and furthermore gave probabilistic representations for the solutions. This was done using the Brownian snake. Our existence theorem does give probabilistic representations; however, while we do show that the admissible singular sets are roughly characterized by their Hausdorff dimension, the admissible initial traces have not been fully classified, probabilistically or otherwise. An obvious and attractive extension of our existence result (Theorem 4.1.14(b)) would be to allow non-zero regular components. Because the main difficulty in this problem is due to the singular set, we believe it should be possible to generalize the non-singular behaviour without much difficulty.

It is natural to ask if the snake approach would work in our setting, and if it could be used to improve our results. The development of the Lévy snake (in particular the \((1+\beta)\)-stable snake) and the associated superprocess representation [18] offers a potentially powerful tool for investigating the charging properties of \( \mathcal{X}_t \) and the initial trace theory of the dual equation. A potential obstacle is that Lévy snakes in general are harder to analyse than the Brownian snake because the height process is non-Markovian. Of course, the issue of “flatness” is novel in this setting and does not occur in the super-Brownian case. It is
not clear if the snake approach is well-suited to handling this behaviour.

Next we discuss a few problems related to our result on the strict positivity of \( \mathcal{X}_t(x) \). Recall that we showed that, in dimension one, \( \mathcal{X}_t(x) > 0 \) for all \( x \in \mathbb{R} \) a.s. on \( \{ \mathcal{X}_t \neq 0 \} \) if \( \beta < \beta^*(\alpha) = \frac{\alpha}{d + \alpha} \) and \( \alpha > 1 + \beta \), the latter condition implying that \( \mathcal{X}_t \) is continuous. While this is known to be false for \( \beta > \beta^*(\alpha) \), the case of \( \beta = \beta^*(\alpha) \) remains open. To prove strict positivity in this critical case would require the development of new methods. On the other hand, if it turns out that \( \mathcal{X}_t \) is not strictly positive, the fractal properties of its zero set may be of interest.

One might also ask what sort of positivity results can be proved when the density is not continuous (i.e. when \( d > 1 \) and/or \( \alpha \leq 1 + \beta \)). Strict positivity in the sense above cannot be proved here because the density is defined up to Lebesgue null sets. However, the conditional estimate Lemma 4.4.1 which is central to the proof of strict positivity holds irrespective of the regularity of the density, which is why there is hope that an analogous result will hold in the discontinuous regime for \( \beta < \beta^*(\alpha) \). “Strict” positivity for a discontinuous density could take the following form: for a compact set \( K \subset \mathbb{R}^d \), almost surely on \( \{ \mathcal{X}_t \neq 0 \} \) there exists \( \epsilon = \epsilon(K, \omega) > 0 \) such that \( \mathcal{X}_t(x) \geq \epsilon \) for Lebesgue-a.e. \( x \in K \).

Finally, another direction to consider is the non-absolutely continuous regime. For \( \beta \geq \frac{\alpha}{d} \), the admissible initial measures for (5.2.1) have been exactly classified via Bessel capacities [8]. This is interesting from a probabilistic perspective because it apparently characterizes the measures \( \mu \) for which it is possible to make sense of the integral \( \int d\mu d\mathcal{X}_t \) when \( \mathcal{X}_t \) itself is singular with respect to the Lebesgue measure. (Any work on this would require defining what is meant by such an integral.) This is related to the underlying dimension of \( \mathcal{X}_t \) and could be useful for determining exact measure results analogous to those discussed in Section 1.3.4 in the binary branching setting. Furthermore, in analogy with [55] (see Proposition 2 of that work), there may be a classification of admissible measures involving the potential theory of the \( \alpha \)-stable process.
Bibliography


