New gluing methods and applications to nonlinear elliptic and parabolic equations

by

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Abstract

In this dissertation, we develop new gluing methods to construct concentration and blow-up solutions to some nonlinear elliptic and parabolic equations.

In Chapter 2, we construct line bubbling solutions along boundary geodesics for the supercritical Lane-Emden-Fowler problem in low dimensions 6 and 7 by devising a new infinite dimensional reduction method.

In Chapter 3, we construct type II finite time blow-up solutions to the energy critical heat equations in dimension 3, and the energy supercritical heat equation with cubic nonlinearity in dimensions 5, 6 and 7. The constructions rely on new inner-outer gluing method which aims at parabolic problems in low dimensions where slow decaying errors are present.

In Chapter 4, by developing a new fractional gluing method, we construct infinite and finite blow-up solutions to the fractional heat equation with the critical exponent.

In Chapter 5, we study the finite time singularity formation for the nematic liquid crystal flow in dimension two. We develop a new gluing method for this strongly coupled nonlinear system with non-variational structure and construct finite time blow-up solutions with precise profiles obtained.
Lay Summary

Concentration and blow-up phenomena are quite common in the study of nonlinear partial differential equations. The aim of this dissertation is to rigorously construct concentration and blow-up solutions to some nonlinear elliptic and parabolic equations. In this dissertation, we will develop new gluing methods to achieve the constructions.
Preface

This dissertation is based on original research articles by the author. We briefly explain as follows the contents of the papers that are published or submitted for publication in research journals from this dissertation and clarify the contributions of collaborators in each paper.

Chapter 2 is based on [27], which is submitted and under review. The research framework and methodology were discussed and developed by me, Guoyuan Chen, and Juncheng Wei, with each of us preparing and revising major sections of the manuscript.

Chapter 3 consists of two sections based on two different works [65, 67].

- Section 3.1 is based on [67], which is on arXiv (arXiv:2002.05765). The research methodology and preparation of the manuscript were carried out in equal parts by me, Manuel del Pino, Monica Musso, Juncheng Wei and Qidi Zhang. I contributed 20% of the new research framework, implementation of the new method, and detailed computations.

- Section 3.2 is based on [65], which has been put on arXiv (arXiv:2006.00716) and submitted for publication. The research scheme and manuscript composition were done in equal parts by me, Manuel del Pino, Chen-Chih Lai, Monica Musso and Juncheng Wei. I contributed 20% of the methodology, solutions, as well as the technical computations.

Chapter 4 consists of two parts based on [28, 155].

- A version of Section 4.1 has been published and appears in “M. Musso, Y. Sire, J. Wei, Y. Zheng, and Y. Zhou. Infinite time blow-up for the fractional
The research methodology was developed by me, Monica Musso, Yannick Sire, Juncheng Wei and Youquan Zheng, with each of us contributing 20% to the manuscript composition.

- A version of Section 4.2 has been published and appears in “G. Chen, J. Wei, and Y. Zhou. Finite time blow-up for the fractional critical heat equation in \( \mathbb{R}^n \). Nonlinear Anal., 193:111420, 2020”. The research and preparation of the manuscript were done by me, Guoyuan Chen and Juncheng Wei in equal parts.

Chapter 5 is based on [124], which has been put on arXiv (arXiv:1908.10955) and submitted for publication. The development of the new systematic method and manuscript composition were carried out together by me, Chen-Chih Lai, Fanghua Lin, Changyou Wang and Juncheng Wei. I contributed 20% to this original work, including the methodology, technical computations and manuscript composition.
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Chapter 1

Introduction

In the study of partial differential equations (PDEs), one of the fundamental issues is the qualitative property of the solutions. An important tool to study the qualitative properties of solutions is the so-called Lyapunov-Schmidt reduction method, gluing method in short. In this dissertation, we will develop new gluing method, which can be viewed as an infinite dimensional version of the Lyapunov-Schmidt reduction method, to construct concentration and blow-up solutions to some elliptic and parabolic problems.

The finite dimensional Lyapunov-Schmidt reduction method was first introduced by Floer and Weinstein ([81]) to construct single-bump solutions to the one dimensional nonlinear Schrödinger equation. This result was later generalized to higher dimensions by Oh ([157, 158]). In another aspect, the reduction method was developed by Bahri ([6]) and Bahri-Coron ([7]) to deal with problems involving Sobolev critical exponent. After these seminal works, there have been many developments on the finite dimensional reduction method, which has been applied to various nonlinear PDEs such as nonlinear Schrödinger equations, subcritical and critical elliptic problems.

While the finite dimensional reduction method is used to construct concentration with finitely many concentration points, the infinite dimensional reduction method is a generalization to handle concentration on higher dimensional sets such as curves and surfaces. The infinite dimensional reduction method was first de-
veloped by del Pino-Kowalczyk-Wei ([59]) to construct concentration on curves for two dimensional Schrödinger equation, and was successfully applied to construct counterexamples for the celebrated De Giorgi conjecture in higher dimensions $N \geq 9$ ([61]). Lots of new features have been found for the infinite dimensional reduction in the context of the Allen-Cahn equation, supercritical elliptic problems and many others.

Both finite and infinite dimensional reduction methods are well developed and widely applied to study the concentration phenomena in the elliptic settings. A natural question is whether a systematic method exists to study the finite or infinite time blow-ups for parabolic problems. The first systematic method, which is called inner-outer gluing method, has been recently developed by Dávila-del Pino-Wei ([55]) and Cortázar-del Pino-Musso ([43]) to construct finite time blow-up for the two dimensional harmonic map heat flow and infinite time blow-up for the energy critical heat equation, respectively. Roughly speaking, this parabolic gluing method consists of several major steps: we first find good approximate solutions and then reduce the original nonlinear problem to solving a basically decoupled inner-outer gluing system. In order to solve the system, delicate analysis is needed for the linear theories, which in turn require subtle choices of parameter functions ensuring the implementation of the construction. Finally, using the fixed point theorems, the full inner-outer gluing system is solved in well-chosen topology. On the other hand, new gluing methods have been successfully developed by Davila-del Pino-Musso-Wei ([53, 54]) to investigate the desingularization for the two dimensional Euler equation and also the helical filament for the three dimensional Euler equation. In the hyperbolic setting, new reduction method has recently been developed by del Pino-Jerrard-Musso ([64]) to study the interface dynamics of the wave Allen-Cahn equation. The parabolic gluing method can be viewed as an analog to the infinite dimensional reduction in the parabolic context, and its applications can be found in a lot of literatures such as the finite and infinite time blow-ups for the energy critical and supercritical heat equations [62, 63, 66, 69], infinite time bubble towers for the energy critical heat equation [68], infinite time blow-up for the Keller-Segel system [50], and others arising from geometry and fractional context [51, 173, 174].
Motivated by the references mentioned above, we will develop new gluing methods in this dissertation to construct solutions to the following problems:

1. **Supercritical Lane-Emden-Fowler problem**

   In Chapter 2, we study the supercritical elliptic problem in low dimensions

   \[
   \begin{align*}
   \Delta u + u^{\frac{n+4}{n-2}} \epsilon &= 0 \quad \text{in } \Omega, \\
   u &> 0 \quad \text{in } \Omega, \\
   u &= 0 \quad \text{on } \partial \Omega.
   \end{align*}
   \]  
   \hspace{1cm} (1.1)

   In [60], del Pino, Musso and Pacard constructed solutions to (1.1) with \( \epsilon > 0 \) sufficiently small and \( n \geq 8 \). To be more precise, they proved that if \( \partial \Omega \) contains a non-degenerate closed geodesic \( \Gamma \) with strictly negative inner normal curvature, then there exists a solution of (1.1) with concentration behavior as \( \epsilon \to 0^+ \) in the form of bubbling line which collapses to \( \Gamma \). Note that the argument in [60] relies crucially on the dimension restriction \( n \geq 8 \). By improving the linear theory and choosing new weighted topology, we construct line bubbling solutions in the low dimension case \( n = 6, 7 \).

2. **Fujita type equations**

   In Chapter 3, we investigate type II finite time blow-ups for the Fujita equation in \( \mathbb{R}^n \)

   \[ u_t = \Delta u + u^p. \]  
   \hspace{1cm} (1.2)

   • In Section 3.1, we consider the energy critical case \( p = 5 \) in dimension \( n = 3 \). We give a rigorous construction of type II finite time blow-up confirming all the blow-up rates formally predicted by Filippas-Herrero-Velázquez ([80]).

   • In Section 3.2, we deal with the energy supercritical case \( p = 3 \) in dimensions \( n = 5, 6, 7 \). We construct type II finite time singularities initiating on higher dimensional shrinking sphere with self-similar size \( \sqrt{T-t} \) which collapses to a point singularity as \( t \to T \). This is a completely new phenomenon which was first predicted in the context of the harmonic map heat flow [51].

3. **Fractional heat equations with critical exponent**

   3
In Chapter 4, we study fractional heat equations with critical exponent

$$u_t + (-\Delta)^s u = u^{\frac{n+2s}{n-2s}}, \quad (1.3)$$

where $(-\Delta)^s$ is the fractional Laplacian with $0 < s < 1$. Different from the local case, the tools that can be applied to deal with non-local (fractional) parabolic problems are limited. We shall design a new linear theory, which is the heart of the singularity formation, for fractional problem (1.3) to reveal the infinite and finite time blow-ups:

- In Section 4.1, we construct infinite time blow-up solutions to (1.3) with $n > 4s$.
- In Section 4.2, we construct finite time blow-up solutions to (1.3) with $4s < n < 6s$.

4. Nematic liquid crystal flow

In Chapter 5, we consider the initial-boundary value problem of the nematic liquid crystal flow in dimension two

\[
\begin{aligned}
\partial_t v + v \cdot \nabla v + \nabla P &= \Delta v - \varepsilon_0 \nabla \cdot (\nabla u \otimes \nabla u - \frac{1}{2} |\nabla u|^2 I) \quad \text{in} \quad \Omega \times (0,T), \\
\nabla \cdot v &= 0 \quad \text{in} \quad \Omega \times (0,T), \\
\partial_t u + v \cdot \nabla u &= \Delta u + |\nabla u|^2 u \quad \text{in} \quad \Omega \times (0,T),
\end{aligned}
\]

where $\Omega \subset \mathbb{R}^2$ is a smooth, bounded domain, $v : \Omega \times [0,T) \to \mathbb{R}^2$ is the fluid velocity field, $P : \Omega \times [0,T) \to \mathbb{R}$ is the fluid pressure function, $u : \Omega \times [0,T) \to S^2$ stands for the orientation field of nematic liquid crystal molecules. The parameter $\varepsilon_0 > 0$ represents the competition between kinetic energy and elastic energy. The model (1.4) was first proposed by Lin in [137] which is a simplified version of the Ericksen-Leslie system for the hydrodynamics flow of nematic liquid crystal material.

A long-standing problem is the existence of finite time singularities for (1.4) in dimension two (see [115] for the three dimensional case). For the critical dimension $n = 2$, the strong coupling between the incompressible Navier–Stokes equation with forcing $(1.4)_1-(1.4)_2$ and the transported harmonic map heat flow $(1.4)_3$
is very difficult to deal with. In a pioneering work [55], Dávila, del Pino and Wei constructed finite time singularities for the two dimensional harmonic map heat flow by using the inner–outer gluing method. However, the method developed in [55] cannot be directly applied to (1.4) since various substantial difficulties arise when dealing with the strongly coupled system:

• The natural scaling invariance suggests that usual perturbation argument seems not possible in the strongly coupled system (1.4). Instead, delicate analysis is required for both \( v \) and \( u \).

• Due to the singularity formation of \( u \), the forcing term in (1.4)_1 becomes singular. In order to carry out the inner–outer gluing scheme, the profile of the velocity near the singularity of \( u \) needs to be carefully analyzed.

• The nontrivial impact of the transported term \( v \cdot \nabla u \) on the harmonic map heat flow (1.4)_3 reflects in different Fourier modes of the inner problem where singularity takes place. This in turn requires careful adjustment of two new modulation parameters.

We will develop a new inner–outer gluing method for both \( v \) and \( u \), and constructed finite time singularities at prescribed points in \( \Omega \). Moreover, we obtain precise descriptions of the blow-up profiles.

In conclusion, the inner-outer gluing method turns out to be a systematic and powerful tool to handle various nonlinear PDEs arising from physics, fluid dynamics and differential geometry.
Chapter 2

New gluing methods for nonlinear supercritical elliptic problem

2.1 Introduction

In this Chapter, we consider the classical Lane-Emden-Fowler problem (see [83])

\[
\begin{cases}
\Delta u + u^p = 0 & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
v = 0 & \text{on } \partial \Omega,
\end{cases}
\]

(2.1)

where \( p > 1 \) and \( \Omega \) is a smooth bounded domain in \( \mathbb{R}^n \) with smooth boundary.

For \( 1 < p < \frac{n+2}{n-2} \), compactness of Sobolev embedding yields the existence of positive solution by minimizing the following functional

\[
S(p) = \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^2}{(\int_{\Omega} |u|^{p+1})^{\frac{1}{p+1}}}. 
\]

For subcritical case, an interesting phenomenon is the point bubbling if \( p = \frac{n+2}{n-2} - \varepsilon \) as \( \varepsilon \to 0^+ \). To be more precise, for \( \varepsilon > 0 \) small, there exists a solution to (2.1) in the form

\[
u_{\varepsilon}(x) = \mu_\varepsilon^{-\frac{n-2}{2}} w_n \left( \frac{x - x_\varepsilon}{\mu_\varepsilon} \right) + o(1), \quad \mu_\varepsilon \sim \varepsilon^{-\frac{1}{n-2}},\]

as \( \varepsilon \to 0^+ \). Here \( w_n \) is the Aubin-Talenti bubble

\[
\mathcal{w}_n(x) = \left( \frac{c_n}{1 + |x|^2} \right)^{\frac{n-2}{2}}
\]

which is the bounded radial solution of

\[
\Delta w + w^\frac{n+2}{n-2} = 0 \quad \text{in} \quad \mathbb{R}^n,
\]

where \( c_n = \sqrt{n(n-2)} \) (see [5, 183]). The blow-up point \( x_\varepsilon \) concentrates on a critical point \( x_0 \) of Robin’s function of \( \Omega \). For more related results see for example [8, 15, 57, 58, 82, 88, 103, 166] and the references therein.

For \( p \geq \frac{n+2}{n-2} \), Pohozaev identity [161] implies that there are no solutions of (2.1) if the domain \( \Omega \) is star-shaped. For \( \Omega \) with other geometric structures, solutions may exist. For example, in [119], Kazdan and Warner proved that if \( \Omega \) is a symmetric annulus, then the compactness of Sobolev embedding can be recovered for all \( p > 1 \) within the radial function space, which yields the existence of solutions to (2.1). In [7], Bahri and Coron obtained the existence of solutions to (2.1) for \( p = \frac{n+1}{n-2} \) if \( \Omega \) has nontrivial topology. On the other hand, in [14], Brezis and Nirenberg recovered the compactness by suitable linear perturbations for the critical exponent \( p = \frac{n+2}{n-2} \). For \( p > \frac{n+2}{n-2} \), variational method seems difficult to show the existence. A question raised by Rabinowitz, stated by Brezis in [13], is whether the nontrivial topology of the domain is sufficient for the solvability of (2.1) for \( p > \frac{n+2}{n-2} \). However, Passaseo [160] constructed a counterexample for this question by choosing the domain \( \Omega \) to be a thin tubular neighborhood of a copy of the unit sphere \( S^{n-2} \) in \( \mathbb{R}^n \) (\( n \geq 4 \)) with \( p \geq \frac{n+1}{n-3} \). Here \( \frac{n+1}{n-3} \) is called the second critical exponent, which is strictly larger than the critical exponent \( \frac{n+2}{n-2} \). We should mention that for an exterior domain of form \( \mathbb{R}^n \setminus \Omega \) with \( \Omega \) bounded, problem (2.1) always has infinitely many infinite energy (slow decay) solutions if \( p > \frac{n+2}{n-2} \) (see [45, 46]). Seeking finite energy (fast decay) solutions for supercritical exponents is much more difficult. In [46], Dávila, del Pino, Musso and Wei found fast decay solutions for \( p \) close to \( \frac{n+2}{n-2} \) from above.

In [60], del Pino, Musso and Pacard constructed solutions to (2.1) when \( p = \frac{n+1}{n-3} - \varepsilon \) with \( \varepsilon > 0 \) sufficiently small and \( n \geq 8 \). To be more precise, they proved
that if \( \partial \Omega \) contains a nondegenerate closed geodesic \( \Gamma \) with strictly negative inner normal curvature, then there exists a solution of (2.1) with a concentration behavior as \( \varepsilon \to 0^+ \) in the form of bubbling line which collapses to \( \Gamma \). A typical example of such domain is that \( \Omega \) has a convex hole. This phenomenon is called \textit{line bubbling}.

Note that the argument in [60] relies essentially on the dimension restriction \( n \geq 8 \). The line bubbling phenomenon has also been discovered in supercritical problems with \( p = \frac{n+1}{n-3} \pm \varepsilon \) on compact Riemannian manifold without boundary [49]. The concentration at higher dimensional boundary submanifolds was investigated in [72]. Also, the constructions in [49, 72] rely on a technical restriction that the codimension of the concentration submanifold is no less than \( 7 \).

We remark that point bubbling is determined by the Green’s function which relies on the global information of the domain \( \Omega \), whereas from the construction of [60], line bubbling only depends on the local structure of the domain near the concentration curve \( \Gamma \).

In this Chapter, we shall investigate the line bubbling for lower dimension case \( n = 6, 7 \). More precisely, we study the following problem

\[
\begin{aligned}
\Delta u + u^{\frac{n+1}{n-3} - \varepsilon} &= 0 \quad \text{in } \Omega, \\
u &> 0 \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

(2.4)

Let \( \Gamma \subset \partial \Omega \subset \mathbb{R}^n \) be a closed nondegenerate (see Definition 2.1.1 below) geodesic with global negative curvature. We assume that a nonresonance condition

\[
|k^2 \varepsilon^{\frac{2(n-3)}{n-1}} - \kappa^2| > \delta \varepsilon^{\frac{n-2}{n-3}}, \quad \forall k \in \mathbb{Z}_+.
\]

(2.5)

holds, where \( \delta > 0 \) is a constant, and \( \kappa > 0 \) is a constant depending only on \( \Gamma \) (see (2.113)). Note that the resonance phenomenon has been found in higher dimensional concentration for many elliptic problems, see for example [49] 59, 60, [44] and [143] and the references therein.

For simplicity, in the rest of this Chapter, we denote \( N = n - 1 \) and \( p = \frac{n+1}{n-3} = \frac{N+2}{N-2} \).

To describe our main result precisely, we introduce some geometric notations.
Let $\partial \Omega$ be endowed with the metric induced by the Euclidean metric and $\bar{\nabla}$ be the associated connection. Near the geodesic $\Gamma$, we introduce the following Fermi coordinates. Let $q \in \partial \Omega$ and we split $T_q \partial \Omega = T_q \Gamma \oplus N_q \Gamma$ into the tangent and normal bundles over $\Gamma$. Assume that $\Gamma$ is parameterized by arclength $x_0$ with $x_0 \to \gamma(x_0)$, and the length of $\Gamma$ is $2l$. Let $E_0$ be a unit tangent vector to $\Gamma$, $E_i, i = 1, \ldots, N - 1$, be an orthonormal basis of $N_q \Gamma$ which are parallel along $\Gamma$, namely $\bar{\nabla} E_0 E_i = 0, i = 1, \ldots, N - 1$. Since $\Gamma$ is geodesic, it holds that $\bar{\nabla} E_0 E_0 = 0$. In a neighborhood of $\Gamma$ in $\partial \Omega$, using exponential map $\exp_{\partial \Omega}$ on $\partial \Omega$, we introduce a local coordinates $F(x_0, \bar{x}) := \exp_{\gamma(x_0)}(x_i E_i), \bar{x} := (x_1, x_2, \ldots, x_{N-1})$, where we use Einstein summation over $i = 1, \ldots, N - 1$ for simplicity. In the neighborhood of $\Gamma$ in $\bar{\Omega}$, we give local coordinates $G(x_0, x) := F(x_0, \bar{x}) - x_N n(F(x_0, \bar{x})), x = (\bar{x}, x_N) \in \mathbb{R}^N$, where $x$ is in a small neighborhood of 0 and $n$ is the outward unit normal. Assume that the curvature of $\Gamma$ is given by $\partial_{x_0}^2 \gamma = \bar{h}_{00} n$, where $\bar{h}_{00}$ is a strictly positive function on $\Gamma$.

**Definition 2.1.1.** In local coordinates, we say that the geodesic $\Gamma$ is nondegenerate if

$$-\bar{d}_{x_0}^\nu + \sum_{j=1}^{N-1} (\bar{R}(E_0, E_j)E_0, E_k)\bar{d}_{j} = 0, x_0 \in [-l, l], k = 1, \ldots, N - 1, \quad (2.6)$$

has only the trivial $2l$-periodic solution $\bar{d} \equiv 0$, where $\bar{R}$ denotes the Ricci tensor on $\partial \Omega$.

The main theorem for this Chapter is as follows.

**Theorem 2.1.1.** Let $n = 6, 7$ and $\Omega \subset \mathbb{R}^n$ be a domain with smooth bounded boundary $\partial \Omega$. Assume $\Gamma \subset \partial \Omega$ is a closed nondegenerate geodesic with negative inner normal curvature. Then for all $\varepsilon > 0$ sufficiently small satisfying the nonresonance
condition (2.5) with $\delta > 0$ fixed, there exists a solution $u_\varepsilon$ of (2.4) such that

$$|\nabla u_\varepsilon|^2 \to S_{n-1}^{\frac{n-1}{2}} \delta_\Gamma \text{ as } \varepsilon \to 0^+,$$

in the measure sense, where $S_{n-1}$ is the best Sobolev constant in $\mathbb{R}^{n-1}$, and $\delta_\Gamma$ denotes the Dirac measure supported on $\Gamma$. Furthermore, $u_\varepsilon$ has the following form

$$u_\varepsilon(x_0,x) = \mu_\varepsilon \frac{N-2}{2\varepsilon} w \left( \frac{x - d_\varepsilon}{\mu_\varepsilon} \right) + o(1),$$

where $\mu_\varepsilon$ and $d_\varepsilon$ are defined in (2.16)-(2.18), $w : = w_N$ is the standard bubble given in (2.2) and $N = n - 1$.

The proof is based on the so-called inner-outer gluing procedure. One of the key ingredients in this method is to prove a priori estimates for associated linearized operators. Inspired by the linear theory in [60] and [55, Section 4], we develop a linear theory in the elliptic setting which shares similar flavor of parabolic blowing-up problems. Formally, in our problem, the tangential direction $y_0$ plays a similar role as the time variable. More precisely, we consider the projected equation

$$\begin{cases}
a_0 \partial_y^2 \phi + \Delta_y \phi + \omega \phi = h + \sum_{j=0}^{N+1} c_j(\rho y_0) Z_j(y) \text{ in } S_\rho \times D_R, \\
\phi(y_0, y) = 0, \forall (y_0, y) \in \partial (S_\rho \times D_R),
\end{cases}$$

(2.7)

where $a_0$ is a positive smooth $2l$-periodic function of $\rho y_0$, $\omega$ is a small coefficient operator given by Lemma 2.3.1 below, $\rho = \varepsilon^{\frac{N-1}{2}}$, $S_\rho$ is the circle parameterized by $y_0 \in [-\frac{l}{\rho}, \frac{l}{\rho}]$,

$$D_R = \{y = (\bar{y}, y_N) \in \mathbb{R}^N : |\bar{y}| < 2R, -\frac{d_{\varepsilon, N}}{\mu_\varepsilon}(\rho y_0) < y_N < 2R\}$$

with $R = R(\varepsilon) = \varepsilon^{-\theta_0}$ and $\theta_0$ as in (2.101), $Z_0$ is the first eigenfunction defined in (2.32), and $Z_i$ ($i = 1, \ldots, N, N+1$) are the bounded kernel functions of the linearized operator of equation (2.3) around the standard bubble $w = w_N$ defined in
\((2.2)\), namely,

\[ Z_j = \partial_j w \quad \text{for} \quad j = 1, \ldots, N, \quad \text{and} \quad Z_{N+1} = \frac{N-2}{2} w + x \cdot \nabla w, \]

(see for example [10]). Define the \(L^\infty\)-weighted norms

\[
\| \phi \|_\sigma := \sup_{S_\rho \times D_R} \langle y \rangle^\sigma |\phi(y_0, y)| + \sup_{S_\rho \times D_R} \langle y \rangle^{1+\sigma} |\nabla_y \phi(y_0, y)|,
\]

\[
\| h \|_{2+\sigma} := \sup_{S_\rho \times D_R} \langle y \rangle^{2+\sigma} |h(y_0, y)|,
\]

(2.8)

where \(0 < \sigma \leq N - 4\) with \(N = n - 1\) and \(\langle y \rangle := \sqrt{1 + |y|^2}\). It will be shown in Section 2.5 (see Proposition 2.5.2 below) that there exists a solution \(\phi\) to equation (2.7) satisfying the following estimate

\[
|\phi| \lesssim R^{5-\sigma} \langle y \rangle^{-\tau} \| h \|_{2+\sigma}. \tag{2.9}
\]

The main difference between Proposition 2.5.2 and the linear theory in [60] is as follows: to obtain similar a priori estimates as in [60], the authors assume that the solution \(\phi\) is orthogonal to \(Z_i\) \((i = 0, 1, \ldots, N, N + 1)\) in which a dimension restriction \(n \geq 8\) is needed to guarantee the integrability of orthogonality conditions in \(\mathbb{R}^N\), whereas Proposition 2.5.2 is established by first proving a fast decaying version (see Proposition 2.5.1) of the linear equation (2.7), then we apply it to the slow decaying version (see Proposition 2.5.2) and get the desired estimate (2.9). As a consequence, in the intermediate region \(|y| \sim R\), estimate (2.9) implies

\[
\| \phi \|_\sigma \lesssim \| h \|_{2+\sigma},
\]

while in the interior, the estimate we get for the solution \(\phi\) is deteriorated for low dimension case \(n = 6, 7\), namely that \(R^{5-\sigma}\) appears in front. However, by some further efforts, the linear theory is sufficient for us to carry out the inner-outer gluing scheme.

We remark that for dimensions \(n = 4, 5\), similar result of Proposition 2.5.2 does not hold and some new arguments, taking into account of global coupling, are needed. We will work on this problem in a future work.
This Chapter is organized as follows. In Section 2.2, we recall some basic geometric notations for local coordinates near the geodesic $\Gamma$. Section 2.3 is devoted to constructing the approximate solution and computing the size of the error. In Section 2.4, we set up the inner-outer gluing scheme and solve the outer problem. Before we solve the reduced projected inner problem in Section 2.6, we shall develop a linear theory of the associated linear problem in Section 2.5. Finally in Section 2.7, we adjust the parameter functions such that the reduced system $c_j(\rho y_0) = 0$ in (2.7) is satisfied for all $y_0$ and $j = 0, 1, \ldots, N + 1$ and prove Theorem 2.1.1.

Throughout this Chapter, the notation “$\lesssim$” always denotes “$\leq c$” where the constant $c > 0$ may differ from line to line but it is independent of $\varepsilon$.

### 2.2 Geometric Settings

In this section, we recall some geometric notations and results for the problem as in [60]. We refer the readers to [60] for detailed computations.

Consider the metric $\bar{g}$ on $\partial \Omega$ induced by the Euclidean metric in $\mathbb{R}^n$. Denote the associated connection by $\bar{\nabla}$. Then we introduce the Fermi coordinates in a neighborhood of $q \in \Gamma$ on $\partial \Omega$. Assume that $\Gamma$ is parameterized by the arclength $x_0$ with $x_0 \mapsto \gamma(x_0)$. Denote $E_0$ by the unit tangent vector to $\Gamma$. In a neighborhood of $q \in \Gamma$, we denote $E_1, \ldots, E_{N-1}$ by an orthonormal basis of $N_q \Gamma$. We can assume that for each $i = 1, \ldots, N-1$, $\bar{\nabla}_E E_i = 0$. Since $\Gamma$ is a geodesic, $\bar{\nabla}_{E_0} E_0 = 0$. Define

$$F(x_0, \bar{x}) = \exp_{\partial \Omega}^{(x_0)}(x_i E_i), \quad \bar{x} = (x_1, \ldots, x_{N-1}),$$

where $\exp_{\partial \Omega}$ is the exponential map on $\partial \Omega$ and the summation is from 1 to $N-1$.

The above Fermi coordinates are defined such that $\bar{g}_{ab} = \delta_{ab}$ along $\Gamma$, where $\bar{g}_{ab}$ is the coefficient of $\bar{g}$. Then the higher order terms in the Taylor expansion of the metric coefficients are estimated as follows, whose proof can be found in [60].

**Proposition 2.2.1.** At $q = F(x_0, \bar{x})$, the following estimates hold

$$\bar{g}_{00} = 1 + (\bar{R}(E_0, E_k)E_0, E_i)x_k x_l + \mathcal{O}(|\bar{x}|^3),$$
\[ g_{0i} = O(|\bar{x}|^2), \]
\[ \bar{g}_{ij} = \bar{\delta}_{ij} + \frac{1}{3} \langle \bar{R}(E_i, E_k)E_j, E_l \rangle x_k x_l + \bar{g}(|\bar{x}|^3), \]

where \( i, j, k, l = 1, \ldots, N - 1 \), \( \bar{R} \) is the curvature tensor and \( O(|\bar{x}|^s) \) is a smooth function not involving any term up to order \( s \) in \( x_i \), \( i = 1, \ldots, N - 1 \).

For simplicity, we denote
\[ R_{ijkl} = \langle \bar{R}(E_i, E_j)E_k, E_l \rangle. \] (2.10)

In order to parameterize a neighborhood near \( \Gamma \), we define the following coordinates \((x_0, x) \in \mathbb{R}^{N+1}\)
\[ G(x_0, x) = F(x_0, \bar{x}) - x_N n(F(x_0, \bar{x})), \quad x = (\bar{x}, x_N) \in \mathbb{R}^N, \]
with \( x \) close to 0 and \( n \) denotes the unit outward normal.

The coefficients of the Euclidean metric in these coordinates
\[ g_{ab} = \bar{g}_{ab} + 2 \bar{h}_{ab} x_N + \bar{k}_{ab} x_N^2 + O(x_N^3) \quad \text{for} \ a, b = 0, \ldots, N - 1, \] (2.11)

Moreover, it holds that
\[ g_{ab} = \bar{g}_{ab} + 2 \bar{h}_{ab} x_N + \bar{k}_{ab} x_N^2 + O(x_N^3) \quad \text{for} \ a, b = 0, \ldots, N - 1, \]

where \( \bar{g} \) is the metric on \( \partial \Omega \), \( \bar{h} \) is the second fundamental form of \( \partial \Omega \) with
\[ \bar{h}_{ab} = -E_b \cdot \nabla_{E_a} n = -E_a \cdot \nabla_{E_b} n, \] (2.12)
\[ \bar{k}_{ab} = (\bar{h} \otimes \bar{h})_{ab} = \sum_{c,d} \bar{h}_{ac} \bar{g}^{cd} \bar{h}_{db}. \] (2.13)

We remark that the normal curvature along the geodesic \( \Gamma \) in this setting is
\[ \partial_{x_0}^2 \gamma = \nabla_{E_0} E_0 = \bar{h}_{00} n. \]

Based on the above settings, we now compute the expansion of the Laplace-
Beltrami operator

\[ \Delta = \frac{1}{\sqrt{|g|}} \partial_{\alpha} (\sqrt{|g|} g^{\alpha \beta} \partial_{\beta}) = g^{\alpha \beta} \partial_{\alpha} \partial_{\beta} + \partial_{\alpha} g^{\alpha \beta} \partial_{\beta} + \frac{1}{2} \text{Tr}_{g} (\partial_{\alpha} g) g^{\alpha \beta} \partial_{\beta} \]

with the summation taken over \( \alpha, \beta = 0, \ldots, N \). By (2.11), we have

\[ \Delta = \partial^{2}_{x_{N}} + \frac{1}{2} \text{Tr}_{g} (\partial_{x_{N}} g) \partial_{x_{N}} + g^{ab} \partial_{a} \partial_{b} + \frac{1}{2} \text{Tr}_{g} (\partial_{x_{N}} g) g^{ab} \partial_{ab}, \]

where the summation is taken over \( a, b = 0, \ldots, N - 1 \).

Direct computations give the decomposition as follows

\[ \Delta = \partial^{2}_{x_{0}} + \sum_{j} \partial^{2}_{x_{j}} + \partial^{2}_{x_{N}} + A^{00} \partial_{x_{0}} \partial_{x_{0}} + \sum_{j} A^{0j} \partial_{x_{0}} \partial_{x_{j}} + \sum_{i,j} A^{ij} \partial_{x_{i}} \partial_{x_{j}} + B^{0} \partial_{x_{0}} + \sum_{j} A^{0j} \partial_{x_{j}} + B^{j} \partial_{x_{j}} + (\text{Tr}_{g} \tilde{h} - \text{Tr}_{g} \tilde{k} x_{N} + B^{N}) \partial_{x_{N}}, \]

where \( \tilde{R}, \tilde{g}, \tilde{h} \) and \( \tilde{k} \) only depend on \( x_{0} \). The metric \( \tilde{g} \), tensors \( \tilde{h} \) and \( \tilde{k} \) only depend on \( x_{0} \). All the rest functions \( A^{ij} \) and \( B^{j} \) depend on \( x_{0}, x_{1}, \ldots, x_{N} \) and have further decompositions as in [60, (4.13)]. For the reader’s convenience, we list below

\[ A^{00} = A^{00}_{N} x_{N} + \sum_{k,l} A^{00}_{kl} x_{k} x_{l}, \]

\[ A^{ij} = A^{ij}_{N} x_{N} + \left( \sum_{k} A^{ij}_{Nk} x_{k} \right) x_{N} + \sum_{k,l,m} A^{ij}_{kl} x_{k} x_{l} x_{m}, \]

\[ A^{0j} = A^{0j}_{N} x_{N} + \sum_{k,l} A^{0j}_{kl} x_{k} x_{l}, \]

\[ B^{0} = B^{0}_{N} x_{N} + \sum_{k} B^{0}_{k} x_{k}, \]

\[ B^{j} = B^{j}_{N} x_{N} + \sum_{k,l} B^{j}_{kl} x_{k} x_{l}, \]
\[ B^N = B_N^N x_N^2 + \left( \sum_k B_k^N x_k \right) x_N + \sum_j B_j^N x_j, \]  
(2.14)

where all the functions in (2.14) are smooth and depend on \( x_0, \ldots, x_N \).

### 2.3 Construction of the approximate solution

In this section, we shall construct an approximate solution to the following problem

\[
\begin{align*}
\Delta u + u^{\frac{N+2}{2} - \varepsilon} &= 0 & \text{in } \Omega, \\
u &= 0 & \text{on } \partial \Omega.
\end{align*}
\]

Our first approximate solution is based on the Aubin-Talenti bubble \( w \) satisfying

\[ \Delta w + w^{\frac{N+2}{2}} = 0 \text{ in } \mathbb{R}^N, \]

namely,

\[ w(x) = \left( \frac{c_N}{1 + |x|^2} \right)^{\frac{N-2}{2}}, \quad c_N = \sqrt{N(N - 2)}. \]

#### 2.3.1 Scaling coordinates near the geodesic

We shall rescale and translate the bubble along a curve close to the geodesic \( \Gamma \).

Let \((x_0, x) \in \mathbb{R}^{N+1}\) be the local coordinates near the geodesics \( \Gamma \). We perform the following change of variables

\[
(y_0, y) := \left( \frac{x_0}{\rho}, x - d_\varepsilon \right), \quad u(G(x_0, x)) = \mu_\varepsilon^{-\frac{N-2}{2}} v \left( \frac{x_0}{\rho}, \frac{x - d_\varepsilon}{\mu_\varepsilon} \right),
\]

(2.15)

where \( v = v(y_0, y), \rho = \varepsilon^{\frac{N-1}{2}}, \)

\[ \mu_\varepsilon(x_0) = \rho \tilde{\mu}_\varepsilon(x_0), \quad d_\varepsilon(x_0) = \varepsilon \tilde{d}_\varepsilon(x_0) \]

(2.16)

with

\[ \tilde{\mu}_\varepsilon(x_0) = \mu_\varepsilon^0(x_0) + \varepsilon \mu(x_0), \quad \tilde{d}_\varepsilon(x_0) = \varepsilon d_j(x_0) \text{ for } j = 1, \ldots, N - 1, \]

(2.17)
and \( d_{\varepsilon N}(x_0) = d_{\varepsilon N}^0(x_0) + \varepsilon d_N(x_0) \). In the above definitions, \( \mu_\varepsilon^0(x_0) \) and \( d_{\varepsilon N}(x_0) \) are given by

\[
\mu_\varepsilon^0(x_0) = \mu_0(x_0) + \varepsilon \frac{\mu_1}{\varepsilon N} \left( x_0 \right), \quad d_{\varepsilon N}^0(x_0) = d_{0 N}(x_0) + \varepsilon \frac{1}{\varepsilon N} d_{1 N}(x_0)
\]  

(2.18)

with \( \mu_0(x_0) = \frac{\alpha}{h_{00}(x_0)} \), \( d_{0 N}(x_0) = \frac{\beta}{h_{00}(x_0)} \), where positive constants \( \alpha \) and \( \beta \) only depend on \( N \) (see [60, (9.17)] for detailed computations), and \( \bar{h}_{00} \) is the normal curvature along \( \Gamma \). We assume \( \Gamma \) has globally negative curvature, namely, \( \partial_x^2 x_0 \gamma = \bar{h}_{00} n \), where \( \bar{h}_{00} \) is a smooth and strictly positive function along \( \Gamma \), and \( n \) is the outward unit normal.

The norms of the parameter functions \( \mu(x_0) \) and \( d(x_0) = (d_1, \ldots, d_N) \) in \((-l, l)\) are defined as follows

\[
\| \mu \|_a = \| \varepsilon \frac{\mu'}{\varepsilon N} \|_{\infty} + \| \varepsilon \frac{\mu''}{\varepsilon N} \|_{\infty} + \| \mu \|_{\infty},
\]

(2.19)

\[
\| d \|_d = \| d_N \|_b + \sum_{j=1}^{N-1} \| d_j \|_c,
\]

(2.20)

with

\[
\| d_N \|_b = \| \varepsilon d_N' \|_{\infty} + \| \varepsilon^\frac{1}{2} d_N'' \|_{\infty} + \| d_N \|_{\infty},
\]

(2.21)

\[
\| d_j \|_c = \| d_j' \|_{\infty} + \| d_j'' \|_{\infty} + \| d_j \|_{\infty}, \text{ for } j = 1, \ldots, N - 1.
\]

(2.22)

In the above definitions, the prime denotes \( \frac{d}{dx_0} \).

After the change of variables (2.15), the original cylinder close to \( \Gamma \) is transformed into the following region

\[
(y_0, y) \in \mathcal{D} := \left\{ (y_0, \bar{y}, y_N) : -\frac{d_{\varepsilon N}}{\mu_\varepsilon} < y_N < \frac{\delta}{\rho}, \ |\bar{y}| < \frac{\delta}{\rho} \right\},
\]

(2.23)

with some fixed \( \delta > 0 \).

After performing the change of variable (2.15), the Laplacian becomes

\[
\bar{\mu}_\varepsilon^{\frac{N+2}{2}} \Delta u = \mathcal{A}(v),
\]

where \( \mathcal{A} v = a_0 \partial_0^2 v + \Delta v + \mathcal{A} v \) with \( a_0 = \bar{\mu}_\varepsilon^2 \) and \( \bar{\mu}_\varepsilon \) is defined in (2.16). The
differential operator $\mathcal{A}$ can be expressed as

$$\mathcal{A} v = \sum_{\alpha, \beta} a_{\alpha, \beta} \partial_{\alpha, \beta} v + \sum_{\alpha} b_\alpha \partial_\alpha v + cv,$$

with $a_{\alpha, \beta} = O(\varepsilon + \rho^2 |y|^2)$, if $\alpha \neq 0$, $\beta \neq 0$, $a_{0, \beta} = O(\varepsilon)$, $a_{0, 0} = 0$ and $b_\alpha = \rho O(\varepsilon + \rho |y|)$, $c = \rho^2 O(1)$. The more specific expression of $\mathcal{A}$ is given by the following Lemma, whose proof can be found in [60, Lemma 5.1].

**Lemma 2.3.1.** After the change of variables (2.15), it holds that

$$\mu^2 \Delta u = \mathcal{A}(v) := a_0 \partial_0^2 v + \Delta_y v + \sum_{k=0}^5 \mathcal{A}_k v + \mathcal{B}(v),$$

where

$$\mathcal{A}_0(v) = (\mu'_e)^2 \left[D_{y,v}[y]^2 + 2(1 + \gamma)D_y v[y] + \gamma(1 + \gamma)v \right]$$

$$+ \mu'_e [D_{y,v}[y] + \gamma D_y v][d'_e] + D_{y,v}[d'_e]^2$$

$$- 2\mu'_e \left[ e^{-\frac{k}{\mu^2}} D_y (\partial_0 v)[\mu'_e + d'_e] + \gamma \mu'_e e^{-\frac{k}{\mu^2}} \partial_0 v \right]$$

$$- \mu_e D_y v[d''_e] - \mu_e \mu'_e (\gamma v + D_y v[y]),$$

$$\mathcal{A}_1 v = \sum_{i,j} \left[- \frac{1}{3} R_{ikj}(\mu_e y_k + d_{e,k})(\mu_e y_i + d_{e,i}) - 2h_{ij}(\mu_e y_N + d_{e,N}) \right]$$

$$+ \sum_k d_{ijk}(\mu_e y_k + d_{e,k})(\mu_e y_N + d_{e,N}) \partial_{ij} v,$$}

$$\mathcal{A}_2 v = \sum_j \left[ - \frac{4}{7} h_{ij}(\mu_e y_N + d_{e,N}) \right]$$

$$\times \left[ (D_y (\partial_0 v))d + \mu_e e^{-\frac{k}{\mu^2}} \partial_0 v - (\gamma v + D_y (\partial_0 v[y]) \mu'_e) \right],$$

$$\mathcal{A}_3 v = \left( \sum_k b^0_k (\mu_e y_k + d_{e,k}) + b^0_N (\mu_e y_N + d_{e,N}) \right)$$

$$\times \left\{ \mu_e \left[ - D_y v[d'_e] + \mu_e e^{-\frac{k}{\mu^2}} \partial_0 v - \mu'_e (\gamma v + D_y v[y]) \right] \right\},$$

$$\mathcal{A}_4 v = \sum_j \left[ \frac{2}{3} R_{ijk} + R_{0jk}(\mu_e y_k + d_{e,k}) + b^i_N (\mu_e y_N + d_{e,N}) \right] \mu_e \partial_{ij} v,$$
\[ \phi_S \nu = (\text{Tr}_g \tilde{h} - \text{Tr}_g \tilde{k}(\mu_{\varepsilon}y_N + d_{\varepsilon,N})) \mu_\varepsilon \partial_N \nu, \]  

(2.29)

\[ B(\nu) = O \left( (|\mu_{\varepsilon}y + \bar{d}_e|^2 + (\mu_{\varepsilon}y_N + d_{\varepsilon,N}) + (\mu_{\varepsilon}y_N + d_{\varepsilon,N})(\mu_{\varepsilon}y + \bar{d}_e)) \right) \phi_0(\nu) \]

\[ + O \left( (|\mu_{\varepsilon}y + \bar{d}_e|^2 + (\mu_{\varepsilon}y_N + d_{\varepsilon,N})|\mu_{\varepsilon}y + \bar{d}_e|^2 + (\mu_{\varepsilon}y_N + d_{\varepsilon,N})^2) \right) \partial_j \nu . \]

\[ + O \left( (|\mu_{\varepsilon}y + \bar{d}_e|^2 + (\mu_{\varepsilon}y_N + d_{\varepsilon,N})|\mu_{\varepsilon}y + \bar{d}_e| + (\mu_{\varepsilon}y_N + d_{\varepsilon,N})^2) \right) \times \left[ \mu_\varepsilon \varepsilon^{-\frac{N+1}{2}} \partial_0 \nu + \mu_\varepsilon \varepsilon^{-\frac{N+1}{2}} \partial_0 \nu - D_\gamma (\partial_j \nu) [d_e] \right. \]

\[ \left. - (\gamma \partial_j \nu + D_\gamma (\partial_j \nu) [\nu]) \mu_\varepsilon' - D_\gamma \nu \varepsilon' - \mu_\varepsilon'(\nu + D_\gamma \nu [\nu]) + \mu_\varepsilon \partial_j \nu \right] \]

(2.30)

\[ + O \left( (|\mu_{\varepsilon}y + \bar{d}_e|^2 + (\mu_{\varepsilon}y_N + d_{\varepsilon,N})|\mu_{\varepsilon}y + \bar{d}_e| + (\mu_{\varepsilon}y_N + d_{\varepsilon,N})^2) \right) \mu_\varepsilon \partial_N \nu . \]

In the above expressions (2.24)-(2.30), \( R_{ij} = \text{defined in (2.10), } \tilde{h}_{ij} \text{ is defined in (2.12), } \tilde{k} \text{ is defined in (2.13), and } d_{ij,N} \text{ is a smooth function of } \rho y_N \text{ with } A_{ij,N} = d_{ij,N}^2 x_N + O(x_N^2), \text{ where } A_{ij,N} \text{ is defined in (2.14). } b_{ij}^0 \text{ is a smooth function of } \rho y_N \text{ with } B_{ij}^0 = b_{ij}^0 x_N + O(x_N^2), \text{ where } B_{ij}^0 \text{ is defined in (2.14). } b_{ij}^0 \text{ is a smooth function of } \rho y_N \text{ with } B_{ij}^0 = b_{ij}^0 x_N + O(x_N^2), \text{ where } B_{ij}^0 \text{ is defined in (2.14). } \]

After performing the change of variable (2.15) under the local coordinates along \( \Gamma \), the original equation becomes

\[ \phi \nu + \mu_\varepsilon \varepsilon^{\frac{N+2}{2}} \frac{\nu^{N+2}}{\nu^{N+2}} = 0. \]

Define the error of \( w \) by

\[ S_{\varepsilon}(w) = \phi \nu + \mu_\varepsilon \varepsilon^{\frac{N+2}{2}} \frac{\nu^{N+2}}{\nu^{N+2}}. \]

### 2.3.2 The first approximate solution

We first define a smooth cut-off function \( \chi(s) = 1 \text{ for } s < \delta \) and \( \chi(s) = 0 \text{ for } s > 2\delta \), and \( \chi_{\varepsilon}(\nu) = \chi(\varepsilon^{\frac{1}{N+2}} |\nu|) \), where \( \delta \) in (2.23) is chosen such that

\[ \chi_{\varepsilon}(\nu, -\frac{d_{\varepsilon,N}}{\mu_\varepsilon}) = 0. \]

(2.31)
Denote $Z_0$ by the eigenfunction corresponding to the only negative eigenvalue $\lambda_0$ of the following eigenvalue problem

$$\Delta_y \phi + pw(y)^{p-1}\phi + \lambda \phi = 0, \phi \in L^\infty(\mathbb{R}^N),$$

namely,

$$\Delta_y Z_0 + pw(y)^{p-1}Z_0 + \lambda_0 Z_0 = 0 \quad (2.32)$$

with $\lambda_0 < 0$.

Our first approximate solution close to the geodesic $\Gamma$ is

$$w = \tilde{w} + e_\varepsilon(\rho y_0) \chi_{\varepsilon}(y) Z_0, \quad (2.33)$$

where $\tilde{w}$ is defined by $\tilde{w}(y) = (1 + \alpha_\varepsilon)[w(y) - \bar{w}(y)]$, with the Aubin-Talenti bubble $w$, $\alpha_\varepsilon := \mu_\varepsilon (N-2)^2\varepsilon^2 - 1$ and $\bar{w}(y) = w(\bar{y}, y_N + \frac{2d_N}{\mu_\varepsilon})$. Observe that $(1 + \alpha_\varepsilon)w$ satisfies

$$\Delta[(1 + \alpha_\varepsilon)w] + \mu_\varepsilon (N-2)^2||w||^{\frac{N-2}{2}} = 0 \quad \text{in } \mathbb{R}^N,$$

and $\tilde{w} = 0$ on $y_N = -\frac{d_N}{\mu_\varepsilon}$. In (2.33), $e_\varepsilon(\rho y_0)$ is defined as

$$e_\varepsilon = e\tilde{e}_\varepsilon \quad (2.34)$$

with

$$\tilde{e}_\varepsilon = e_0^\varepsilon + \varepsilon e_1 \quad \text{and} \quad e_0^\varepsilon = e_0 + \varepsilon^\frac{1}{N-2} e_1, \quad (2.35)$$

where $e_1$ is a smooth function uniformly bounded in $\varepsilon$, and

$$e_0 = \frac{2 \int_{\mathbb{R}^N} \partial_i w Z_0}{|\lambda_0|} (\text{Tr}_{\bar{g}} h - \bar{h}_{00}) d_0 N. \quad (2.36)$$

The purpose of the parameter function $e$ is to eliminate the resonance which can produce large noise in the tangential direction. We shall choose $e$ in the final section and the norm of $e$ is defined as

$$\|e\|_e = \|\varepsilon^{\frac{2}{N-2}} e''\|_\infty + \|\varepsilon^{\frac{1}{N-2}} e'\|_\infty + \|e\|_\infty. \quad (2.37)$$
Suppose that our parameter functions $\mu$, $d$ and $e$ satisfy

\[
\| (\mu, d, e) \| := \| \mu \|_a + \| d \|_d + \| e \|_e \leq c, \tag{2.38}
\]

where the definitions of the above norms are given in (2.19), (2.20) and (2.37), $c > 0$ is a constant independent of $\epsilon$.

By a similar computation as in [60, (5.33)], the expansion of the error $S_\epsilon(w)$ for small $\epsilon$ is given by

\[
S_\epsilon(w) = - p_w^{\rho - 1} \tilde{w} - \epsilon w^\rho \log w + \epsilon (-2\tilde{h}_{ij}d_{e,N}^\rho \partial_{ij}w + |\lambda_0| e_0^0 Z_0) + \epsilon^{\frac{N}{N-2}} \mu_0^0 (-2\tilde{h}_{ij}y_N \partial_{ij}w + \text{Tr}_g \tilde{h} \partial_N w) + \epsilon^2 \left( \rho^2 a_0 e'' + |\lambda_0| \epsilon Z_0 - 2\tilde{h}_{ij}d_N \partial_{ij}w \right)
\]

\[
+ \sum_{i,j} (d''_{ij} - \frac{1}{3} R_{i,jkl} d_k d_l + d_{ij,N} d_{e,N}^0 + 4\tilde{h}_{ij}d_w d_{e,N}^0) \partial_{ij}w + \gamma_\epsilon
\]

\[
+ \epsilon^{\frac{3N}{N-2}} \mu_0^0 \left[ - \sum_j \partial_j w \cdot d''_{ij} + \left( - \sum_{i,j} \frac{1}{3} R_{ijkl} y_k d_l \partial_{ij}w + 2d_{ij,N}^0 y_k d_{e,N}^0 \partial_{ij}w \right) \right]
\]

\[
+ \left( \frac{2}{3} R_{i,jk} + R_{0,jk} \right) d_k \partial_{ij}w + 4\tilde{h}_{ij}d_{e,N}^0 \partial_{ij}w \right] + \epsilon^{\frac{3N}{N-2}} \left[ - \mu'' \mu Z_{N+1}^0 + 2 \mu \mu_0^0 \left( - \frac{1}{3} R_{i,jkl} y_k y_l \partial_{ij}w \right)
\]

\[
+ \left( \frac{2}{3} R_{i,jk} + R_{0,jk} \right) y_k \partial_{ij}w + b_{N}^j y_N \partial_{ij}w - \text{Tr}_g \tilde{h} y_N \partial_N w \right] \right]
\]

\[
+ \epsilon^4 (\log \epsilon) r, \tag{2.39}
\]

where $\gamma_\epsilon = \gamma_0 + \epsilon^{\frac{1}{N-2}} \gamma_1^1$ with

\[
\gamma_0 = -2\tilde{h}_{ij} d_{0,N} e_0 \partial_{ij}Z_0 + p(p - 1)e_0^2 w^{\rho - 2} Z_0^2 + p e_0 w^{\rho - 1} \log w Z_0,
\]

\[
\gamma_1 = \gamma_2 + \epsilon^{\frac{1}{N-2}} \gamma_3^1 + \epsilon^2 \gamma_4^2 + \epsilon^3 \gamma_5^3 + \epsilon^4 \gamma_6^4.
\]
and $\Upsilon^1$ is a smooth function of the form $f_1(\rho y_0) f_2(\mu, d, e) f_3(\epsilon)$. In the above expression, $f_1$ is smooth and uniformly bounded in $\epsilon$, $f_2$ is smooth and uniformly bounded in $\epsilon$ and $f_3$ is smooth with $\sup_{\mathcal{D}} |f_3(\epsilon)| < +\infty$. Note that $f_2$ depends linearly on $\mu''$, $d''$ and $e''$. We refer the reader to [60, Appendix] for the detailed computations of (2.39).

From (2.39), we can write

$$S_\epsilon(w) = \epsilon S_0 + \epsilon^2 (\rho^2 a_0 e'' + |\lambda_0| e) \chi Z_0 + \epsilon^2 S_1,$$  \hspace{1cm} (2.40)

where

$$\epsilon S_0(\rho y_0) := -pw^{p-1} \bar{w} - \epsilon w^p \log w + \epsilon (-2\bar{h}_{ij} d_{e,N}^0 \partial_{ij} w + |\lambda_0| e_0^0 Z_0)$$

$$+ \epsilon \sum_{i=0}^{N+1} \mu_0^i (-2\bar{h}_{ij} y_n \partial_{ij} w + \text{Tr}_g \bar{h} \partial N w),$$

$S_0$ is smooth uniformly bounded in $\epsilon$ and $S_1$ depends on $\mu$, $d$ and $e$.

### 2.3.3 Correction of the approximate solution

Now we add an extra correction $\Pi$ to get rid of $\epsilon S_0$ in (2.40), namely, $\Pi$ solves the following linear problem

$$\begin{align*}
\begin{cases}
 a_0 \partial^2 \Pi + \Delta_s \Pi + \partial \bar{\mathcal{D}} \Pi + pw^{p-1} \Pi = -\epsilon S_0 + \sum_{i=0}^{N+1} \alpha_i Z_i & \text{in } \mathcal{D}, \\
 \Pi(y_0, y) = 0 & \text{on } \partial \mathcal{D}_{y_0} \text{ for all } y_0,
\end{cases}
\end{align*}$$

where $\mathcal{D}_{y_0} := \{ y \in \mathbb{R}^N : (y_0, y) \in \mathcal{D} \}$ with $\mathcal{D}$ defined in (2.23). By choosing suitable parameters $\mu$, $d$ and $e$ at main order, namely $\mu_0^0$, $d_{e,N}^0$ and $e_0^0$ (see (2.17), (2.35) and (2.36) for their definitions), the orthogonality conditions

$$\int_{\mathcal{D}_{y_0}} S_0 Z_i dy = 0 \text{ for all } y_0 \text{ and } i = 0, 1, \ldots, N + 1,$$

are achieved. The argument is the same as that of [60, Appendix]. We omit the details.

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We only need to consider the leading term in $\varepsilon S_0$

$$h_0 = -pw^p - \varepsilon w^p \log w + \varepsilon (-2\partial_i d_{0,N} \partial_j w + |\lambda_0| e_0^0 Z_0)$$

since other terms are of sufficiently fast decay. Note that

$$|w^p(y)| \lesssim \frac{1}{1+|y|^{\alpha}}, \quad | -2\partial_i d_{0,N} \partial_j w + |\lambda_0| e_0^0 Z_0| \lesssim \frac{1}{1+|y|^{\alpha}},$$

$$|w^{p-1}\bar{w}|(y, \varepsilon) \lesssim \frac{1}{1 + |y|^{4N}} \lesssim \frac{1}{1+|y|^{N-2} + (y + \varepsilon^{-\frac{1}{2}})^{N-2}}.$$ 

In order to apply the linear theory as in [60, Section 3], we can gain enough decay in $y$ by losing a little bit $\varepsilon$. To be more precise, for fixed $\vartheta > 0$ close to 0, it holds that

$$|w^{p-1}\bar{w}| = |(w^{p-1}\bar{w})\bar{w}| \lesssim \frac{\varepsilon^{1-\vartheta}}{1 + |y|^{4+(N-2)\vartheta}},$$

in $\vartheta$. We consider the problem

$$
\begin{cases}
  a_0 \partial_0^2 \Pi + \Delta \Pi + \varepsilon \partial_0 \Pi + pw^{p-1}\Pi = -\varepsilon S_0 + \sum_{i=0}^{N+1} \alpha_i Z_i \quad \text{in } \vartheta, \\
  \Pi(y_0, y) = 0 \quad \text{on } \partial \mathcal{D}, \\
  \int_{\partial \mathcal{D}} \Pi(y_0, y) Z_i dy = 0 \quad \text{for all } y_0 \text{ and } i = 0, 1, \ldots, N+1.
\end{cases}
$$

(2.41)

Since $\|\partial_0(\frac{d_{0,N}}{\mu})\|_{\infty} \lesssim \rho \varepsilon^{-\frac{1}{2}} (\|\mu'\|_{\infty} + \varepsilon \|d_{0,N}'\|_{\infty}) = o(1)$ and $\rho^{-1}\|\partial_0(\frac{d_{0,N}}{\mu})\|_{\infty} \lesssim \rho \varepsilon^{-\frac{1}{2}} (\|\mu'\|_{\infty} + \varepsilon \|d_{0,N}'\|_{\infty}) = o(1)$ as $\varepsilon \to 0$, problem (2.41) satisfies assumptions (2.73) and (2.74) in Proposition 2.5.1. Therefore, from Proposition 2.5.1, there exists a unique tuple $(\alpha_i, \Pi)$ solving the elliptic problem (2.41). Moreover, the solution $\Pi$ satisfies

$$|\Pi(y_0, y)| \lesssim \frac{\varepsilon^{1-\vartheta}}{1 + |y|^{2+(N-2)\vartheta}}, \quad \forall (y_0, y) \in \vartheta,$$

(2.42)

namely $\|\Pi\|_{\sigma} \lesssim \varepsilon^{1-\vartheta}$, where we have used the facts that $0 < \sigma \leq N - 4$ and $N = 5, 6$. Since $\Pi$ only depends on $\mu$ and $d$, by Lemma 2.3.1 we can estimate

$$
\|\Pi_{\mu_1,d_1} - \Pi_{\mu_2,d_2}\|_{\sigma} \leq c \varepsilon^{2-\vartheta} \|((\mu_1 - \mu_2) - (d_1 - d_2))\|.
$$

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Moreover, using similar computations as in [60, (5.47)], we have

$$\sup |\alpha_i| \leq o(1) \varepsilon^3.$$  

(2.43)

Let $\psi = \partial_0 \Pi$. Then by differentiating the equation (2.41) with respect to $y_0$, we have

$$\begin{cases}
a_0 \partial_0^2 \psi + \Delta_0 \psi + \omega_0^2 \psi + p \omega \partial_0 \psi = h \text{ in } \Omega, \\
f_{\partial_0} \psi(y_0, y) \partial_y = 0 \text{ for all } y_0 \text{ and } i = 0, 1, \ldots, N + 1, \\
\psi(y_0, \bar{y}, y_N) - \partial_0 \left( \frac{d \varepsilon}{\mu e} \right) \partial_N \Pi(y_0, \bar{y}, y_N) = 0 \text{ on } \partial \Omega \text{ for all } y_0,
\end{cases}$$

where $h = -\varepsilon \rho \omega_0 \partial_0 S_0 - (\partial_0 \omega_0^2) \Pi + \sum_{i=0}^{N+1} \partial_0 \alpha_i Z_i$. Then by a similar argument as above, we find that $\|\psi\|_\sigma \lesssim \rho \varepsilon^{1-\theta}$. Therefore, with the correction $\Pi$ which eliminates the term $\varepsilon S_0$ in (2.40), the new error for our new approximate solution $\hat{w} = w + \Pi$ is the following

$$\begin{align*}
S_\varepsilon(w) &= \varepsilon^2 S_1 + \varepsilon^2 (\rho^2 a_0 \omega - |\lambda_0| \varepsilon) \chi_\varepsilon Z_0 + N_1(\Pi) + \sum_{i=0}^{N+1} \alpha_i Z_i, \\
S_1 &\text{ given by (2.44) }
\end{align*}$$

where

$$N_1(\Pi) = \mu e \frac{N-2}{2} \left( (w + \Pi)^{p-\varepsilon} - w^{p-\varepsilon} \right) - p \omega \partial_0^{-1} \Pi.$$  

(2.45)

The dependence of $S_1$ on the parameter function $\mu, d$ and $e$ is given by

$$\|S_1(\mu_1, d_1, e_1) - S_1(\mu_2, d_2, e_2)\|_{2+\sigma} \leq c\|\mu_1 - \mu_2, d_1 - d_2, e_1 - e_2\|.$$  

Further, by (2.42), we know that $w \gtrsim |\Pi|$ gives $|y| \lesssim \varepsilon^{\frac{\theta-1}{\theta-2}}$ and vice versa. By the definition (2.8) and $R = R(\varepsilon) = e^{-\theta}$, with $\theta_\varepsilon$ given in (2.101), we see that $R \ll \varepsilon^{\frac{\theta-1}{\theta-2}}$, where we have used $\theta_\varepsilon \approx 0$. Thus by a direct Taylor expansion,
\[ \|N_1(\Pi)\|_{2+\sigma} \text{ can be estimated as follows} \]
\[
\|N_1(\Pi)\|_{2+\sigma} = \sup_{S_\rho \times D_R} (y)^{2+\sigma} |N_1(\Pi)| \lesssim \sup_{|y| \leq e^{\frac{\sigma}{N-4(N-2)\theta}}} (y)^{2+\sigma} w^{\rho-2} \Pi^2 \\
= \sup_{|y| \leq e^{\frac{\sigma}{N-4(N-2)\theta}}} (y)^{2+\sigma} (y)^{N-6} (y)^{-4-2(N-2)\theta} \varepsilon^{2(1-\theta)} \tag{2.46}
\]

This finishes the construction of the approximate solution \( \hat{\omega} \) and the estimate of the new error \( S_\varepsilon(\hat{\omega}) \).

### 2.4 The inner-outer gluing procedure

In this section, we shall apply the inner-outer gluing scheme to find a true solution based on the approximate solution \( \hat{\omega} \) we built in Section 2.3.

Letting \( u(z) = \rho^{-\frac{N-2}{2}} v(\frac{z}{\rho}) \), equation (2.4) becomes

\[
\begin{cases}
\Delta v + \rho^{\frac{(N-2)\varepsilon}{2}} v^{\rho-\varepsilon} = 0 & \text{in } \Omega_{\rho}, \\
v > 0 & \text{in } \Omega_{\rho}, \\
v = 0 & \text{on } \partial\Omega_{\rho},
\end{cases}
\]

where \( p = \frac{N+2}{N-2} \) and \( \Omega_{\rho} = \frac{1}{\rho} \Omega \). With a slight abuse of notation, we replace \( v \) by \( u \), namely,

\[
\begin{cases}
\Delta u + \rho^{\frac{(N-2)\varepsilon}{2}} u^{\rho-\varepsilon} = 0 & \text{in } \Omega_{\rho}, \\
u > 0 & \text{in } \Omega_{\rho}, \\
u = 0 & \text{on } \partial\Omega_{\rho}.
\end{cases}
\]

In this problem, we have local coordinates near the geodesic \( \Gamma \) and the corresponding Laplace-Beltrami operator encodes the geometric information of the geodesic \( \Gamma \), while in the region far away from the geodesic \( \Gamma \), we use the usual Euclidean coordinates. Therefore, after introducing some suitable cut-off functions, it is natural to decompose the full problem into the inner and outer parts in which the inner-outer gluing procedure can be carried out.
In a small neighborhood of the geodesic \( \Gamma \), we denote \( f(z) = \tilde{\mu}^{\frac{N-2}{2}}(\rho y_0) \tilde{f}(y_0, y) \), where \( z = \frac{1}{\rho} G(\rho y_0, \rho \tilde{\mu}(\rho y_0) y + \varepsilon \tilde{d}_e(\rho y_0)) \), or equivalently,

\[
\tilde{f}(y_0, y) = \mu^{\frac{N-2}{2}}(\rho y_0) f(z),
\]

where \( z \in \mathbb{R}^{N+1} \) is the original variable in \( \Omega_\rho \). Near the geodesic, the approximate solution we construct in previous section \( w \) now becomes \( \tilde{w} \) in this setting. Indeed, recalling that after the change of variables (2.15), we have

\[\mathcal{A}v + \mu^{\frac{(N-2)e}{2}} v^{-e} = 0.\]

Observe that \( \tilde{w} \) is only defined locally on a small neighborhood near the geodesic. In order to get a global approximate solution, we first introduce some cut-off functions. Let \( \delta > 0 \) be a fixed number with \( 4\delta < \hat{\delta} \), where \( \hat{\delta} \) is chosen in (2.31). Consider a standard cut-off function \( \zeta_\delta(s) \) with \( \zeta_\delta(s) = 1 \) for \( 0 < s < \delta \) and \( \zeta_\delta(s) = 0 \) for \( s > 2\delta \). We define

\[
\zeta_\delta^e(y_0, y) = \zeta_\delta(|G(\rho y_0, \tilde{\mu}(\rho y_0) \rho y + \varepsilon \tilde{d}_e(\rho y_0))|),
\]

and \( \eta_{\delta,2R}^e(z) = \zeta_\delta^e(y_0, y/R(\varepsilon)) \), where

\[
R(\varepsilon) = \varepsilon^{-\theta_e}, \text{ with } 0 < \theta_e < 1 \tag{2.47}
\]

which we will specify later. For simplicity, we write \( R(\varepsilon) \) as \( R \) in the sequel. By a zero extension of \( \tilde{w} \) far away from \( \frac{1}{\rho} \Gamma \), it is natural to choose our first global approximate solution as

\[
w(z) = \eta_{\delta,2R}^e(z) \tilde{w}(z).
\]

We are looking for a solution \( u = w + \Phi \) where \( \Phi = \eta_{\delta,2R}\phi + \psi \) and \( \phi \) is such that \( \phi \) is in principle defined only in \( \mathcal{D} \). Here \( \mathcal{D} \) is defined in (2.23). Then \( \Phi \) satisfies

\[
\begin{cases}
\Delta \Phi + p w^{p-1} \Phi + N(\Phi) + E = 0 & \text{in } \Omega_\rho, \\
\Phi = 0 & \text{on } \partial \Omega_\rho,
\end{cases}
\tag{2.48}
\]

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where
\[
N(\Phi) = \rho \left( N - 2 \right)^2 (w + \Phi)^p - w^p - p w^{p-1} \Phi \quad \text{and} \quad E = \Delta w + w^{p-1}.
\] (2.49)

Recall that near the stretched geodesic \( \Gamma_\rho = \frac{1}{\rho} \Gamma \), the error is
\[
S_\varepsilon(\tilde{w}) = A\tilde{w} + \mu (N^2 - 2) \varepsilon \tilde{w} + \mu (N^2 - 2) \varepsilon \tilde{w} + \mu \varepsilon - w \varepsilon - \mu \varepsilon - p \varepsilon - 1 \tilde{w}.
\]

Then near \( \Gamma_\rho \), \( S_\varepsilon(\tilde{u}) = 0 \) implies that the equation for \( \tilde{\Phi} \) is
\[
A\tilde{\Phi} + p\tilde{w}^{p-1} + N(\tilde{\Phi}) + S_\varepsilon(\tilde{w}) = 0,
\]
where \( N(\tilde{\Phi}) = \mu \varepsilon (N^2 - 2) \varepsilon \tilde{w} + \mu \varepsilon (N^2 - 2) \varepsilon \tilde{w} + \mu \varepsilon - w \varepsilon - \mu \varepsilon - p \varepsilon - 1 \tilde{w}.
\]

### 2.4.1 Inner and outer problems

It is direct to see that \( \Phi = \eta_{\delta, 2R} \overline{\phi} + \psi \) solves (2.48) if \( (\tilde{\phi}, \psi) \) solves the following two coupled equations

- \( \tilde{\phi} \) solves the inner problem
  \[
  \begin{cases}
  \mathcal{A}\tilde{\phi} + p\tilde{w}^{p-1}\tilde{\phi} = -N(\tilde{\phi} + \psi) - S_\varepsilon(\tilde{w}) - p\tilde{w}^{p-1} \psi & \text{in } S_{\rho} \times D_R, \\
  \tilde{\phi} = 0 & \text{on } \partial(S_{\rho} \times D_R).
  \end{cases}
  \] (2.50)

- \( \psi \) solves the outer problem
  \[
  \begin{cases}
  \Delta \psi = -(1 - \eta^\varepsilon_{\delta, 2R}) p \tilde{w}^{p-1} \psi - 2 \nabla \phi \cdot \nabla \eta^\varepsilon_{\delta, 2R} - \phi \Delta \eta^\varepsilon_{\delta, 2R} - (1 - \eta^\varepsilon_{\delta, 2R}) N(\eta^\varepsilon_{\delta, 2R} \phi + \psi) & \text{in } \Omega_{\rho}, \\
  \psi = 0 & \text{on } \partial \Omega_{\rho}.
  \end{cases}
  \] (2.51)

Define the following norm
\[
\|\phi\|_{\ast} := \sup_{S_\rho \times D_R} \left| R^{\sigma - \tau}(y)^\tau \phi(y_0, y) \right|
\] (2.52)
with \(0 < \sigma \leq N - 4\) and \(2 < \tau < N - 2\).

We will first solve the outer problem (2.51) for given \(\tilde{\phi}\) with sufficiently small \(\| \cdot \|_*\)-norm. After we get the outer solution \(\psi(\phi)\), the inner problem (2.50) is reduced to a nonlinear nonlocal problem. In order to solve the reduced inner problem, we shall develop a linear theory for the solvability of the associated model problem of the projected inner problem. Then by applying the linear theory and the Contraction Mapping Theorem, we solve the projected inner problem. Finally, we shall adjust our parameter functions \(\mu, d\) and \(e\) such that the reduced system \(c_j(\rho y_0) = 0\) is satisfied for all \(y_0\) and \(j = 0, 1, \ldots, N + 1\).

### 2.4.2 Solving the outer problem

Recall the definition of \(\| \cdot \|_*\) in (2.52). Given \(\phi\) such that \(\| \tilde{\phi} \|_* \lesssim \varepsilon^{2(1 - \rho)}\) in \(\mathcal{D}\), we solve the outer problem (2.51) first.

**Case 1.** Assume that \(\Omega\) is bounded. Then the following simple problem

\[
\begin{aligned}
-\Delta \psi &= h_1 \quad \text{in } \Omega_ho \\
\psi &= 0 \quad \text{on } \partial \Omega_ho
\end{aligned}
\]

has a unique solution \(\psi = (-\Delta)^{-1} h_1\) for \(h_1 \in L^\infty(\Omega_\rho)\). Further, we have \(\| \psi \|_\infty \lesssim \| h_1 \|_\infty\). Now we consider each term on the right hand side of the first equation of (2.51). Due to the effect of cut-off functions, we obtain from (2.52) that

\[
\| \tilde{\phi} \|_{L^\infty(\rho < |y| < 2R)} \sim R^{-\sigma} \| \tilde{\phi} \|_*
\]

\[
\| \Delta \eta_{\delta,2R} \|_\infty \lesssim \frac{1}{R^2} \| \tilde{\phi} \|_{L^\infty(\rho < |y| < 2R)} \lesssim \frac{1}{R^{2+\sigma}} \| \tilde{\phi} \|_*.
\]  

(2.53)

Similarly, we have

\[
\| \nabla \phi \cdot \nabla \eta_{\delta,2R} \|_\infty \lesssim \frac{1}{R^{2+\sigma}} \| \tilde{\phi} \|_*.
\]

(2.54)

According to decay of \(\phi\), we assume \(\| \psi \|_\infty \leq \Lambda R^{-\sigma} \| \tilde{\phi} \|_*\) with \(\Lambda\) fixed sufficiently large and set

\[
M(\psi) := (1 - \eta_{\delta,2R})N(\eta_{\delta,2R} \phi + \psi),
\]
where \( N(\eta_{\delta,2R}^\varepsilon \phi + \psi) \) is defined in (2.49). Then for \( \varepsilon \) small, we have

\[
\| M(\psi) \|_\infty \lesssim (1 + \Lambda \rho \varepsilon)^{\frac{1}{R^{\frac{s-2}{s-\sigma}}}} \| \phi \|_p^p, \tag{2.55}
\]

Moreover,

\[
\|(1 - \eta_{\delta,2R}^\varepsilon)p w^{\rho-1} \psi\|_\infty \lesssim \frac{1}{R^\sigma} \| \psi \|_\infty. \tag{2.56}
\]

By the Contraction Mapping Theorem, for \( R \) sufficiently large (by (2.47) this is possible as \( \varepsilon \) is small enough), the fixed point problem

\[
\psi = \mathcal{T}(\psi) := (-\Delta)^{-1} \left( M(\psi) + (1 - \eta_{\delta,2R}^\varepsilon)p w^{\rho-1} \psi + \phi \Delta \eta_{\delta,2R}^\varepsilon + 2\nabla \phi \cdot \nabla \eta_{\delta,2R}^\varepsilon \right) \tag{2.57}
\]

has a unique solution \( \psi = \psi(\phi) \) in the function space \( \mathcal{N} = \{ \psi : \| \psi \|_\infty \leq \Lambda R^{-\sigma} \| \tilde{\phi} \|_* \} \)

provided \( \| \tilde{\phi} \|_* \lesssim \varepsilon^{2(1-\theta)} \). Indeed, from (2.53)-(2.57) we have that

\[
\| \mathcal{T}(\psi) \|_\infty \lesssim R^{-\sigma} \| \phi \|_* \text{ for } N = 5, 6, \tag{2.58}
\]

where we have used the assumptions \( \| \tilde{\phi} \|_* \lesssim \varepsilon^{2(1-\theta)} \), \( R \) is sufficiently large and \( \varepsilon \) is small. Therefore, the mapping \( \mathcal{T}(\psi) \) maps \( \mathcal{N} \) to itself. On the other hand, from (2.55), we see that for \( \psi^{(1)}, \psi^{(2)} \in \mathcal{N} \)

\[
\| M(\psi^{(1)}) - M(\psi^{(2)}) \|_\infty \lesssim \left( \| \tilde{\phi} \|_{L^\infty(|\xi| < 2R)} + \frac{1}{R^{\frac{s-2}{s-\sigma}}} \| \phi \|_* \right)^{\rho-1} \| \psi^{(1)} - \psi^{(2)} \|_\infty
\]

\[
\lesssim (1 + \Lambda)^{\frac{1}{s-1}} \frac{1}{R^{\frac{2s-2}{s-\sigma}}} \| \tilde{\phi} \|_* \| \psi^{(1)} - \psi^{(2)} \|_\infty. \tag{2.59}
\]

Therefore, from (2.56) and (2.59) we have

\[
\| \mathcal{T}(\psi^{(1)}) - \mathcal{T}(\psi^{(2)}) \|_\infty \lesssim \left( (1 + \Lambda)^{\frac{1}{s-1}} \frac{1}{R^{\frac{2s-2}{s-\sigma}}} \| \tilde{\phi} \|_* + \frac{1}{R^{\rho-1}} \right) \| \psi^{(1)} - \psi^{(2)} \|_\infty,
\]

which implies \( \mathcal{T} \) is a contraction mapping in the space \( \mathcal{N} \) for \( \varepsilon \) sufficiently small and \( R \) sufficiently large. Thus, the Contraction Mapping Theorem implies the existence of such \( \psi \in \mathcal{N} \).

Moreover, from (2.57) and (2.58), the Lipschitz dependence of \( \psi \) on \( \phi \) is given
by

\[ \| \psi(\phi_1) - \psi(\phi_2) \|_\infty \lesssim R^{-\sigma} \| \phi_1 - \phi_2 \|_\ast \text{ for } N = 5, 6. \]

**Case 2.** Next we consider the unbounded case and let \( \Omega = \mathbb{R}^N \setminus \Upsilon \) with \( \Upsilon \) bounded.

Observe that the coupling term \(-2 \nabla \phi \cdot \nabla \eta^e_{\delta,2R} - \phi \Delta \eta^e_{\delta,2R} \) in the outer problem (2.51) is supported in \( \mathcal{S}_\rho \times (D_R \setminus D_{R/2}) \), where

\[ D_R \setminus D_{R/2} = \{ y \in \mathbb{R}^N : R < |y| < 2R \}. \]

So we decompose the outer problem into the following two equations

\[
\begin{cases}
\Delta \psi_1 = -2 \nabla \phi \cdot \nabla \eta^e_{\delta,2R} - \phi \Delta \eta^e_{\delta,2R} & \text{in } \mathcal{S}_\rho \times (D_R \setminus D_{R/2}), \\
\psi_1 = 0 & \text{on } \partial \mathcal{S}_\rho \times (D_R \setminus D_{R/2}),
\end{cases}
\]

(2.60)

\[
\begin{cases}
\Delta \psi_2 = -(1 - \eta^e_{\delta,2R})p w^{p-1} \psi_2 - (1 - \eta^e_{\delta,2R})N(\eta^e_{\delta,2R} \phi + \psi) & \text{in } \Omega_\rho, \\
\psi_2 = 0 & \text{on } \partial \Omega_\rho.
\end{cases}
\]

(2.61)

For equation (2.60), by a same argument as in the case 1, we obtain that there exists solution \( \psi_1 \in \mathcal{N} \), namely,

\[ \| \psi_1 \|_\infty \lesssim R^{-\sigma} \| \hat{\phi} \|_\ast. \]

(2.62)

We pull back the equation (2.61) for \( \psi_2 \) from \( \Omega_\rho \) to \( \Omega \). Define \( \hat{f}(z) = f(\rho^{\frac{p}{2}} z) \). Then the equation (2.61) becomes

\[
\begin{cases}
\Delta \hat{\psi}_2 = -\rho^{-2}(1 - \hat{\eta}^e_{\delta,2R})p \hat{w}^{p-1} \hat{\psi}_2 - \rho^{-2}(1 - \hat{\eta}^e_{\delta,2R})(\hat{\eta}^e_{\delta,2R} \hat{\phi} + \hat{\psi})^p & \text{in } \Omega, \\
\hat{\psi}_2 = 0 & \text{on } \partial \Omega,
\end{cases}
\]

namely,

\[
\begin{cases}
\Delta \hat{\psi}_2 = -\frac{O(\rho^{-2}(1 - \chi))\hat{\psi}_2}{\rho^{2} + |z|^2} - \rho^{-2}(1 - \chi)(O(R^{-\sigma}) \| \hat{\phi} \|_\ast \chi + \hat{\psi})^p & \text{in } \Omega, \\
\hat{\psi}_2 = 0 & \text{on } \partial \Omega,
\end{cases}
\]

(2.63)

where \( \chi \) is a smooth function with compact support. In the exterior domain, after a perform of Kelvin transform with respect to a given point \( q \) in the interior \( \tilde{\Upsilon} \subset \mathbb{R}^N \).
\( \mathbb{R}^{N+1} = \mathbb{R}^n, K \tilde{\psi}_2(x) = |x - q|^{2-(N+1)} \tilde{\psi}_2(z) \) with \( z = \frac{x - q}{|x - q|^2} \in \Omega \) and \( q \in \mathring{\Gamma} \), we see that the equation (2.63) becomes the following equation in the bounded domain \( K(\Omega) \)

\[
\begin{cases}
\Delta(K \tilde{\psi}_2(x)) = -\frac{O(p^2)}{1 + |x - q|^{4}} K \tilde{\psi}_2(x) \\
-\frac{1}{p^2|x - q|^{N+3}} [O(R^{-\sigma}) \| \tilde{\phi} \|_* \chi_3 + |x - q|^{N-1} K \tilde{\psi}(1 - \chi_4)]^p \text{ in } K(\Omega), \\
K \tilde{\psi}_2(x) = 0 \text{ on } \partial K(\Omega),
\end{cases}
\]

(2.64)

which has a solution \( K \tilde{\psi}_2 = (-\Delta)^{-1} h_2 \) with \( \| K \tilde{\psi}_2(x) \|_* \lesssim \| h_2(x) \|_* < +\infty, x \in K(\Omega) \), namely

\[
\| |z|^{N-1} \tilde{\psi}_2(z) \|_* \lesssim \| h_2(x) \|_* < +\infty, x \in K(\Omega),
\]

(2.65)

where \( h_2 \) is the nonhomogeneous term in (2.64). Here \( \chi_1 \) and \( \chi_4 \) are cut-off functions supported in neighborhood of the geodesic \( \Gamma \), \( \chi_2 \) and \( \chi_3 \) are two cut-off functions supported in bounded annulus. If \( \| \tilde{\psi}_2 \|_* \leq \Lambda R^{-\sigma} \| \tilde{\phi} \|_* \), we see that

\[
\| h_2(x) \|_* \sim \max \{ \rho^{-R^{-\sigma}} \| \tilde{\phi} \|_* , \rho^{-R^{-\sigma}} \| \tilde{\phi} \|_* \} \quad (2.66)
\]

for \( \epsilon \) small. Since \( K(\Omega) \) is bounded and \( \| \tilde{\phi} \|_* \lesssim \epsilon^{2(1-\theta)} \) with \( \theta > 0 \) close to 0, from (2.66) we get

\[
\| h_2 \|_* \lesssim R^{-\sigma} \| \tilde{\phi} \|_* \quad \text{for } N = 5, 6,
\]

where \( \theta_* \) cannot be chosen too small. For example, in the case \( N = 6 \), direct computation shows that \( \theta_* > 1/4 \). This is valid since we will choose \( \theta_* \) in (2.101) at last. By a similar fixed point argument, we can the problem (2.61) in the function space \( \mathcal{N} = \{ \psi : \| \psi \|_* \leq \Lambda R^{-\sigma} \| \tilde{\phi} \|_* \} \) whenever \( \| \tilde{\phi} \|_* \lesssim \epsilon^{2(1-\theta)} \) for \( \epsilon \) small, and the solution \( \psi_2 \) satisfies

\[
\| \psi_2 \|_* \lesssim R^{-\sigma} \| \tilde{\phi} \|_* \quad \text{for } N = 5, 6.
\]

(2.67)

Combining (2.58) and (2.67), we get that the solution \( \psi = \psi_1 + \psi_2 \) of the outer
problem (2.51) satisfies

$$\| \psi \|_{\infty} \lesssim R^{-\sigma} \| \tilde{\phi} \|_{\ast}$$ \quad for \( N = 5, 6 \). \quad (2.68)

This finishes the argument of the outer problem.

### 2.4.3 The reduced inner problem

As a conclusion, substituting \( \tilde{\psi} = \tilde{\psi}(\tilde{\phi}) \) in the inner problem (2.50), the full problem reduces to the following nonlinear nonlocal equation

$$\begin{cases}
\mathcal{A} \tilde{\phi} + p \tilde{\omega}^{p-1} \tilde{\phi} = -N(\zeta_{2d} \tilde{\phi} + \psi(\tilde{\phi})) - S_e(\tilde{\omega}) - p \tilde{\omega}^{p-1} \tilde{\psi}(\tilde{\phi}) & \text{in } S_\rho \times D_R, \\
\tilde{\phi} = 0 & \text{on } \partial(S_\rho \times D_R).
\end{cases} \quad (2.69)$$

Instead of directly solving (2.69), we shall solve the following projected problem

$$\begin{cases}
\mathcal{A} \tilde{\phi} + p \tilde{\omega}^{p-1} \tilde{\phi} = H(\tilde{\phi}, \tilde{\psi}, \mu, d, e) + \sum_{j=0}^{N+1} c_j(\rho y_0) Z_j(y) & \text{in } S_\rho \times D_R, \\
\tilde{\phi} = 0 & \text{on } \partial(S_\rho \times D_R),
\end{cases} \quad (2.70)$$

where

$$H(\tilde{\phi}, \tilde{\psi}, \mu, d, e) := -N(\zeta_{2d} \tilde{\phi} + \psi(\tilde{\phi})) - S_e(\tilde{\omega}) - p \tilde{\omega}^{p-1} \tilde{\psi}(\tilde{\phi}).$$

We shall develop a linear theory concerning the solvability for the associated model problem of (2.70) in Section 2.5. In Section 2.6, we will solve the projected inner problem (2.70) by the linear theory and the Contraction Mapping Theorem. In Section 2.7, we will derive and solve the reduced system of \( \mu, d \) and \( e \) such that \( c_j(\rho y_0) = 0 \) for \( j = 0, 1, \ldots, N + 1 \).
2.5 The linear theory for $N = 5$ and 6

In this section, we shall develop the linear theory concerning the existence and a priori estimates of the following linear problem

$$
\begin{aligned}
\mathcal{A} \phi + p\omega^{p-1} \phi &= h + \sum_{j=0}^{N+1} c_j(\rho y_0)Z_j(y) \quad \text{in } \mathcal{D}_1, \\
\phi &= 0 \quad \text{on } \partial \mathcal{D}_1,
\end{aligned}
$$

in the following domain

$$
\mathcal{D}_1 = \{(y_0,\bar{y},y_N) \in \mathbb{R}^{N+1} : -\frac{d_{e,N}}{\mu_{\varepsilon}}(\rho y_0) < y_N < M(\varepsilon), |\bar{y}| < M(\varepsilon)\},
$$

where $M(\varepsilon) > 0$ depends on $\varepsilon$. Here $h$ satisfies $\|h\|_{2+\sigma} < +\infty$ with $0 < \sigma \leq N - 4$, where $\| \cdot \|_{2+\sigma}$ norm is defined in (2.8) in the domain $\mathcal{D}_1$. Recall from Section 2.3 that for $(y_0,y) \in \mathcal{D}_1$,

$$
\mathcal{A} v = a_0 \partial_y^2 v + \Delta_y v + \mathcal{A'} v,
$$

where $a_0 = \bar{\mu}_{\varepsilon}^2 = (\mu_0 + \varepsilon \frac{\varepsilon^2}{\varepsilon} \mu_1 + \varepsilon \mu)^2$, $\mathcal{A'} v = \sum_{\alpha,\beta} a_{\alpha,\beta} \partial_{\alpha,\beta} v + \sum_{\alpha} b_{\alpha} \partial_{\alpha} v + cv$

with

$$
a_{\alpha,\beta} = O(\varepsilon + \rho^2 |y|^2) = O(\varepsilon) \quad \text{for } \alpha \neq 0, \beta \neq 0, \\
\quad a_{0,\beta} = O(\varepsilon) \quad \text{and} \quad a_{0,0} = 0, \\
b_{\alpha} = \rho O(\varepsilon + \rho |y|) = \rho O(\varepsilon) \quad \text{and} \quad c = \rho^2 O(1).
$$

The dimension restriction in the linear theory of [60, Section 3] is made such that the orthogonality conditions

$$
\int_{\mathbb{R}^n} \phi(y_0,y)Z_j(y)dy = 0, \quad j = 0,1,\ldots,N+1
$$

are well-defined. We have the following (see [60, Proposition 3.2])

**Proposition 2.5.1.** Assume that $N = 5,6$ and $\|\vec{h}\|_{2+\tau} < +\infty$ with $2 < \tau < N - 2$. 


For the linear projected problem

\[
\begin{cases}
\mathcal{A}\bar{\phi} + pw^{p-1}\bar{\phi} = \bar{h} + \sum_{j=0}^{N+1} \bar{c}_j (\rho y_0) Z_j(y) \text{ in } \mathcal{D}_1, \\
\bar{\phi} = 0 \text{ on } \partial \mathcal{D}_1, \\
\int_{\partial \mathcal{D}_1} \bar{\phi}(y_0, y) Z_j(y) dy = 0 \text{ for all } y_0 \in \mathbb{R}, j = 0, 1, \ldots, N + 1,
\end{cases}
\]  

(2.72)

if for all indices \(\alpha, \beta = 0, 1, \ldots, N + 1\),

\[
\| \partial_0 \left( \frac{d_{x,N}}{\mu} \right) \|_\infty + M(\epsilon) \| \partial_0 \left( \frac{d_{x,N}}{\mu} \right) \|_\infty + M(\epsilon) \| \partial_0 a_0 \|_\infty + \| a_{\alpha,\beta} \|_\infty + \| Da_{\alpha,\beta} \|_\infty + \| (y) b_{\alpha} \|_\infty + \| (y)^2 c \|_\infty < \delta,
\]

(2.73)

and

\[
\delta^{-1} < \frac{d_{x,N}(\rho y_0)}{\mu} < M(\epsilon) \delta \text{ for all } y_0 \in \mathbb{R}
\]

(2.74)

for some positive constant \(\delta\), then for any \(\| \bar{h} \|_{2+\tau} < +\infty\) there exists a unique solution \(\bar{\phi} = T(\bar{h})\) which defines a linear operator of \(\bar{h}\) with \(\| \bar{\phi} \|_\tau < +\infty\). Moreover, it holds that

\[
\| \bar{\phi} \|_\tau \lesssim \| \bar{h} \|_{2+\tau}.
\]

Denote \(h = R^{\tau-\sigma}\bar{h}\) and \(\phi = R^{\tau-\sigma}\bar{\phi}\), where \(2 < \tau < N - 2\). Since \(\tau > \sigma\) for \(N = 5, 6\), we see that

\[
\| \bar{h} \|_{2+\tau} \leq (y)^{2+\tau} R^{\sigma-\tau} (y)^{-2-\sigma} \| h \|_{2+\sigma} \leq \| h \|_{2+\sigma}.
\]

(2.75)

Since problem (2.72) is linear, multiplying equation (2.72) with \(R^{\tau-\sigma}\) yields that \(\bar{\phi}\) solves

\[
\begin{cases}
\mathcal{A}\phi + pw^{p-1}\phi = h + \sum_{j=0}^{N+1} c_j (\rho y_0) Z_j(y) \text{ in } \mathcal{D}_1, \\
\phi = 0 \text{ on } \partial \mathcal{D}_1, \\
\int_{\partial \mathcal{D}_1} \phi(y_0, y) Z_j(y) dy = 0 \text{ for all } y_0 \in \mathbb{R}, j = 0, 1, \ldots, N + 1,
\end{cases}
\]

(2.76)
with \( c_j(\rho y_0) = R^{r-\sigma} \tilde{c}_j(\rho y_0) \). From Proposition \ref{prop:251} and (2.75), we obtain
\[
|\phi(y_0, y)| \lesssim R^{r-\sigma}(y)^{-\tau} ||h||_{2+\sigma}.
\]
The above argument concludes the following proposition.

**Proposition 2.5.2.** Assume that \( 0 < \sigma \leq N - 4, 2 < \tau < N - 2, N = 5, 6 \) and \( ||h||_{2+\sigma} < +\infty \). If for all indices \( \alpha, \beta = 0, 1, \ldots, N+1 \),
\[
||\partial_0(\frac{d_{\varepsilon N}}{\mu_\varepsilon})(\varepsilon) + M(\varepsilon)||_{\infty} + ||\partial_{00}(\frac{d_{\varepsilon N}}{\mu_\varepsilon})(\varepsilon) + M(\varepsilon)||_{\infty} + ||a_{\alpha, \beta}||_{\infty} + ||Da_{\alpha, \beta}||_{\infty} + ||(y)b_{\alpha}||_{\infty} + ||(y)^2c||_{\infty} < \delta,
\]
and
\[
\delta^{-1} < \frac{d_{\varepsilon N}}{\mu_\varepsilon}(\rho y_0) < M(\varepsilon)\delta \text{ for all } y_0 \in \mathbb{R}
\]
for some positive constant \( \delta \), then there exists a unique solution \( \phi \) to equation (2.71) satisfying
\[
\int_{\mathbb{R}} \phi(y_0, y)Z_j(y)dy = 0, \forall y_0 \in \mathbb{R}, j = 0, 1, \ldots, N+1.
\]
Furthermore, one has
\[
|\phi(y_0, y)| \lesssim R^{r-\sigma}(y)^{-\tau} ||h||_{2+\sigma}.
\]
(2.76)

**Remark 2.5.1.**

1. *In the intermediate region \( |y| \sim R \), by Proposition 2.5.2 it follows that*
\[
||\phi||_{\sigma} \lesssim ||h||_{2+\sigma}.
\]
2. *The estimate (2.76) is deteriorated in the interior. However, when we apply the linear theory to solve the reduced inner problem (2.70), taking \( \tau \) close to 2 will be sufficient for our purpose.*
3. *An alternative proof can be carried out similarly as that of [43, Section 7].*
2.6 The inner problem

In this section we drop all the tildes in the projected inner problem (2.70) for simplicity. We solve problem (2.70) in a $2l/\rho$-period (in $y_0$) manner as follows

$$
\begin{align*}
\mathcal{L}^\prime(\phi) &= S_\varepsilon(w) + N(\phi) + \sum_{j=0}^{N+1} c_j(\rho y_0)Z_j(y) \quad \text{in } S_\rho \times D_R, \\
\phi(y_0, y) &= \phi(y_0 + \frac{2l}{\rho}, y) \quad \text{for all } (y_0, y) \in S_\rho \times D_R, \\
\dot{\phi} &= 0 \quad \text{on } \partial(S_\rho \times D_R),
\end{align*}
$$

where $\mathcal{L}^\prime(\phi) = \mathcal{A} \phi + p w^{p-1} \phi$ and

$$
N(\phi) = p(w^{p-1} - w^{p-1})\phi - N(\phi) + (\phi) + (\phi) - p w^{p-1} \psi(\phi)
$$

(2.77)

with $N(\phi) = \mu_e \frac{(N-2)e}{2}(w + \phi)^{p-\varepsilon} - \mu_e \frac{(N-2)e}{2} w^{p-\varepsilon} - p w^{p-1} \phi$.

Recall from (2.44) that

$$
S_\varepsilon(w) = S_\varepsilon(w) = \varepsilon^2 (\rho^2 a_0 e^{2s} + |\lambda_0| e(\rho y_0)) \chi_\varepsilon Z_0 + E,
$$

where $E := \varepsilon^2 S_1 + N_1(\Pi) + \sum_{j=0}^{N+1} \alpha_j Z_l$ with $N_1(\Pi)$ defined in (2.45).

We consider the inner problem

$$
\begin{align*}
\ &a_0 \Delta^2 \phi + \Delta \phi + p w^{p-1} \phi = H(\phi, \psi, \mu, d, e) + \sum_{j=0}^{N+1} c_j(\rho y_0)Z_j(y) \quad \text{in } S_\rho \times D_R, \\
\ &\phi(y_0, y) = \phi(y_0 + \frac{2l}{\rho}, y) \quad \text{for all } (y_0, y) \in S_\rho \times D_R, \\
\ &\dot{\phi}(y_0, y) = 0 \quad \text{on } \partial(S_\rho \times D_R), \\
\ &\int_{D_R} \phi(y_0, y)Z_i(y)dy = 0 \quad \text{for all } i = 0, 1, \ldots, N+1 \text{ and all } y_0 \in S_\rho,
\end{align*}
$$

(2.78)

where

$$
H(\phi, \psi, \mu, d, e) := S_\varepsilon(w) + N(\phi).
$$

Our aim is to find $\phi$ by using the linear theory we develop in Section 2.5. Here $\mu$, $d$ and $e$ satisfy (2.38). Note that the Lipschitz dependence of $E$ on $\mu, d$ and $e$ is

$$
\|E(\mu_1, d_1, e_1) - E(\mu_2, d_2, e_2)\|_\infty \lesssim \varepsilon^2 \|(\mu_1 - \mu_2, d_1 - d_2, e_1 - e_2)\|.
$$
Consider the fixed point problem

\[
\phi = T(S_\epsilon(\bar{w}) + N(\phi)) := \Lambda(\phi).
\]  

(2.79)

We solve the above problem in the function space

\[
M = \{ \phi : \| \phi \|_* \leq \Lambda_1 \epsilon^{2(1-\vartheta)} \},
\]

where the norm \( \| \cdot \|_* \) is defined as in (2.52) and \( \vartheta > 0 \) is close to 0. The first fact we show is that the mapping \( \Lambda \) maps \( M \) to itself. We estimate term by term as follows.

From (2.44) and (2.46), we observe that \( S_\epsilon(W) \) is independent of \( \phi \) and it satisfies

\[
\| S_\epsilon(W) \|_2 + \sigma \lesssim \epsilon^2 (1 - \vartheta).
\]  

(2.80)

The nonlinear term satisfies

\[
\| N(\phi) \|_{2+\sigma} \lesssim \| (w^{p-1}_p - w^{p-1})\phi \|_{2+\sigma} + \| \eta_{\delta,2R}^\epsilon (\eta_{\delta,2R}^\epsilon \phi + \psi(\phi)) \|_{2+\sigma} \\
+ \| (\eta_{\delta,2R}^\epsilon)^{p-1} w^{p-1}_p \psi(\phi) \|_{2+\sigma}.
\]

By (2.52), we estimate

\[
\| (w^{p-1}_p - w^{p-1})\phi \|_{2+\sigma} \lesssim \| w^{p-2}(\epsilon eZ_0 + \Pi)\phi \|_{2+\sigma} \\
\lesssim \sup_{S_p \times D_R} (y)^{2+\sigma} \langle y \rangle^N - 6 (\epsilon eZ_0 + \epsilon (\langle y \rangle)^{-\sigma}) \| \phi \| \\
\lesssim \epsilon R^{2-\sigma} \| \phi \|_*.
\]  

(2.81)

Since \( \| \psi \|_w \lesssim R^{-\sigma} \| \phi \|_* \), we get that for \( (y_0, y) \in S_p \times D_R \)

\[
| \eta_{\delta,2R}^\epsilon \phi + \psi| \leq (R^{-\sigma} + R^{2-\sigma} \langle y \rangle^{-\tau}) \| \phi \|_* \lesssim R^{2-\sigma} \langle y \rangle^{-\tau} \| \phi \|_*.
\]  

(2.82)

When \( N = 5, 6 \) we know that \( w \gtrsim | \eta_{\delta,2R}^\epsilon \phi + \psi | \) if \( |y| \lesssim R^{\frac{\sigma}{N-2-\tau}} | \phi |^\frac{1}{N-\tau-2} \), and vice versa. Recall that \( R = R(\epsilon) = \epsilon^{-\theta_*} \) with \( 0 < \theta_* < 1 \). We denote

\[
R_1 := R^{\frac{\sigma}{N-2-\tau}} \| \phi \|_*^{-\frac{1}{N-\tau-2}} \sim \epsilon^{\frac{\theta_*(1-\sigma)}{N-2-\tau} - \frac{2(1-\vartheta)}{N-1-2}}.
\]  

(2.83)
Therefore, we can perform the Taylor expansion as follows. Using (2.82), we obtain that

\[
\| \eta_{\delta,2R}^N(\eta_{\delta,2R}^N \phi + \psi(\phi)) \|_{2+\sigma} \lesssim \sup_{|y| \leq R_1} |(y)^{2+\sigma} w^{p-2}(\phi + \psi)^2| \nonumber
\]

\[
+ \sup_{R_1 \leq |y| \leq R} (y)^{2+\sigma} (R^{\tau-\sigma} (y)^{-\tau} \| \phi \|_s)^p \nonumber
\]

\[
\lesssim R^{2(\tau-\sigma)} \| \phi \|_s^2 + R^{2+\sigma-p\tau} R^{p(\tau-\sigma)} \| \phi \|_s^p . \tag{2.84}
\]

For the last term in the nonlinear term, since \( |\psi| \lesssim R-\sigma \| \phi \|_* \), we get

\[
\|(\eta_{\delta,2R}^N)^{p-1} w^{p-1} \psi(\phi) \|_{2+\sigma} \lesssim R^{-\sigma} \sup_{|y| \leq R} (y)^{p-2} \| \phi \|_s = R^{-\sigma} \| \phi \|_* . \tag{2.85}
\]

Collecting all the terms above from (2.81), (2.84) and (2.85), we obtain that for \( R(\varepsilon) = \varepsilon^{-\theta_*} \) with \( \theta_* \) small

\[
\| N(\phi) \|_{2+\sigma} \lesssim \varepsilon R^{\tau-\sigma} \| \phi \|_s + R^{2(\tau-\sigma)} \| \phi \|_s^2 + R^{2+\sigma-p\tau} R^{p(\tau-\sigma)} \| \phi \|_s^p + R^{-\sigma} \| \phi \|_* . \tag{2.86}
\]

Therefore, by (2.80), (2.86), (2.79) and Proposition 2.5.2, we have

\[
\| A(\phi) \|_s \lesssim \| H \|_{2+\sigma} \lesssim \epsilon^2 + \epsilon R^{\tau-\sigma} \| \phi \|_s + R^{2(\tau-\sigma)} \| \phi \|_s^2 + R^{2+\sigma-p\tau} R^{p(\tau-\sigma)} \| \phi \|_s^p
\]

\[
+ R^{-\sigma} \| \phi \|_* . \tag{2.87}
\]

We observe from (2.87) that, in order to make \( A \) map \( M \) to itself, the following relations should be satisfied

\[
\begin{aligned}
\epsilon R^{\tau-\sigma} & \leq C , \\
R^{2(\tau-\sigma)} \| \phi \|_s & \leq C , \\
R^{2+\sigma-p\tau} R^{p(\tau-\sigma)} \| \phi \|_s^{p-1} & \leq C ,
\end{aligned}
\]

for some constant \( C \) when \( \varepsilon \) is small. Recall that \( R = R(\varepsilon) = \varepsilon^{-\theta_*} \). By (2.83) and
∥φ∥∗ ∼ ε2(1−θ), we obtain that
\[
\begin{cases}
1 - \theta_\ast (\tau - \sigma) > 0, \\
2(1 - \vartheta) - 2\theta_\ast (\tau - \sigma) > 0, \\
\frac{\theta_\ast (\tau - \sigma) - 2(1 - \vartheta)}{N - 2 - \tau} (2 + \sigma - p\tau) - \theta_\ast p(\tau - \sigma) + 2(1 - \vartheta) (p - 1) > 0.
\end{cases}
\]

(2.88)

In our setting, τ is chosen slightly larger than 2 and θ > 0 is close to 0. When \( N = 5, \sigma \approx 1 \). When \( N = 6, \sigma \approx 2 \). Elementary computations show that (2.88) is satisfied if \( \theta_\ast < 1 \). Thus, by \( \theta_\ast < 1 \) and (2.87), it follows that for ε small
\[
\|A(\phi)\|_\ast \leq C\|\phi\|_\ast,
\]

where \( C \) is some positive constant. Therefore, we conclude that for \( \phi \in \mathcal{M} \) with \( \Lambda_1 \) fixed sufficiently large, \( A(\phi) \in \mathcal{M} \).

It remains to show that \( A \) is a contraction mapping in \( \mathcal{M} \). From (2.79), (2.86) and (2.87), we see that for \( \phi^{(1)}, \phi^{(2)} \in \mathcal{M} \)
\[
\|A(\phi^{(1)}) - A(\phi^{(2)})\|_\ast \lesssim \varepsilon R^{\tau - \sigma} \|\phi^{(1)} - \phi^{(2)}\|_\ast + \varepsilon^2 R^{2(\tau - \sigma)} \|\phi^{(1)} - \phi^{(2)}\|_\ast + \varepsilon^2 R^{2\tau - \sigma - p\tau} R^{p(\tau - \sigma)} \|\phi^{(1)} - \phi^{(2)}\|_\ast + R^{-\sigma} \|\phi^{(1)} - \phi^{(2)}\|_\ast.
\]

By our choices of \( \theta_\ast, \tau, \vartheta \) and \( \sigma \) in the system (2.88), we already see that
\[
\|A(\phi^{(1)}) - A(\phi^{(2)})\|_\ast \leq o(1) \|\phi^{(1)} - \phi^{(2)}\|_\ast,
\]

where \( o(1) \to 0 \) as \( \varepsilon \to 0^+ \). It then follows that \( A \) is a contraction mapping in the function space \( \mathcal{M} \) for \( \varepsilon \) sufficiently small. Therefore, the Contraction Mapping Theorem implies the existence of the solution \( \phi \).

Moreover, a similar argument as in [60], we find that the Lipschitz dependence of \( T \) on the parameters is given by
\[
\|T_{(\mu_1, d_1, e_1)} - T_{(\mu_2, d_2, e_2)}\| \lesssim R^{-\sigma} \|(\mu_1 - \mu_2, d_1 - d_2, e_1 - e_2)\|,
\]

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and for \( \| \phi \|_* \lesssim \varepsilon^{2(1-\sigma)} \), we have
\[
\| N(\mu_1, d_1, e_1)(\phi) - N(\mu_2, d_2, e_2)(\phi) \|_{2+\sigma} \lesssim R^{-\sigma} \varepsilon^{2(1-\sigma)} \| (\mu_1 - \mu_2, d_1 - d_2, e_1 - e_2) \|.
\]

By a fixed point argument, we see that
\[
\| \phi_{(\mu_1, d_1, e_1)} - \phi_{(\mu_2, d_2, e_2)} \|_* \lesssim R^{-\sigma} \varepsilon^{2(1-\sigma)} \| (\mu_1 - \mu_2, d_1 - d_2, e_1 - e_2) \|.
\]

We have thus proved the existence of the following

**Proposition 2.6.1.** For sufficiently small \( \varepsilon \) and suitable parameters \( \mu, d \) and \( e \) satisfying
\[
\| (\mu, d, e) \| = \| \mu \|_a + \| d \|_d + \| e \|_e \lesssim 1,
\]
the problem (2.78) has a unique solution \( \phi = \phi(\mu, d, e) \) satisfying
\[
\| \phi \|_* \lesssim \varepsilon^{2(1-\sigma)}.
\]
Furthermore, \( \phi \) depends Lipschitz continuously on \( \mu, d \) and \( e \) with
\[
\| \phi_{(\mu_1, d_1, e_1)} - \phi_{(\mu_2, d_2, e_2)} \|_* \lesssim R^{-\sigma} \varepsilon^{2(1-\sigma)} \| (\mu_1 - \mu_2, d_1 - d_2, e_1 - e_2) \|.
\]

### 2.7 Choice of the parameter functions \( \mu, d \) and \( e \)

In this section, we shall choose the parameter functions \( \mu, d \) and \( e \) such that
\[
c_j(\rho y_0) = 0, \ j = 0, 1, \ldots, N + 1.
\]

are satisfied. Multiplying equation (2.78) with \( Z_j \) and integrating in \( y \) over \( D_R \) imply that the reduced system (2.89) is equivalent to
\[
\int_{D_R} (a_0 \partial_0^2 \phi + \Delta_y \phi + \mathcal{A} \phi + p w^{p-1} \phi - H) Z_j dy = 0
\]
for all \( y_0 \in S_\rho \) and \( j = 0, 1, \ldots, N + 1 \). Recall from problem (2.78) that
\[
H = S_\varepsilon(\mathcal{W}) + N(\phi),
\]
where $S_{\varepsilon}(\bar{w})$ and $N(\phi)$ are defined in (2.44) and (2.77), respectively. Since $S_{\varepsilon}(\bar{w})$ and $N(\phi)$ involve $\mu$, $d$ and $e$, our full problem is reduced to a system involving these parameter functions. Recall that

$$N(\phi) = p(w^{p-1} - w^{p-1})\phi - \Pi(\eta_{\delta,2}\phi + \psi(\phi)) + (\eta_{\delta,2})^{p-1} p w^{p-1} \psi(\phi),$$

where

$$N(\phi) = \mu \frac{(N-2)e}{2} (w + \phi)^{p-\varepsilon} - \mu \frac{(N-2)e}{2} w^{p-\varepsilon} - \mu \frac{w}{p-\varepsilon} \phi,$$

$$S_{\varepsilon}(\bar{w}) = \varepsilon^2 S_1 + \varepsilon^2 (\rho^2 a_0 e''(\rho y_0) + \lambda_0 e(\rho y_0)) \chi \varepsilon Z_0 + \sum_{i=0}^{N+1} \alpha_i Z_i,$$

where $N_1(\Pi)$ is the nonlinear term defined as

$$N_1(\Pi) = \mu \frac{(N-2)e}{2} [(w + \Pi)^{p-\varepsilon} - \Pi w^{p-\varepsilon} - p w^{p-1} \Pi].$$

Next we expand (2.90) in terms of $\mu$, $d$, $e$.

### 2.7.1 Projections of $S_{\varepsilon}(\bar{w})$ on $Z_j$, $j = 0, 1, \ldots, N+1$

For $j = 0, 1, \ldots, N+1$, one has

$$\int_{D_r} S_{\varepsilon}(\bar{w}) Z_j = \varepsilon^2 \int_{D_r} S_1 Z_j + \varepsilon^2 \int_{D_r} S_2(\rho^2 a_0 e''(\rho y_0) + \lambda_0 e(\rho y_0)) \chi \varepsilon Z_0 Z_j$$

$$+ \int_{D_r} N_1(\Pi) Z_j + \alpha_j \int_{D_r} Z_j^2$$

$$= \int_{D_r} S_{\varepsilon}(\bar{w}) Z_j + \int_{D_r} N_1(\Pi) Z_j + \alpha_j \int_{D_r} Z_j^2,$$

where $S_{\varepsilon}(\bar{w})$ is defined in (2.40).

It turns out that the size of projections on $Z_0, Z_N$ and $Z_{N+1}$ are much larger than that of $Z_j$ directions with $j = 1, \ldots, N-1$. We proceed to compute the projections of $S_{\varepsilon}(\bar{w})$ and $N_1(\Pi)$ along different directions as follows. Here we mainly follow the results in [60, Section 5].

40
2.7.2 Projections of $S_\varepsilon(w)$ on $Z_j$, $j = 0, \ldots, N + 1$

We use the expansion of $S_\varepsilon(w)$ (2.39) to compute the projection of $S_\varepsilon(w)$ onto $Z_i$, $i = 0, 1, \ldots, N + 1$. Suppose the parameter functions $\mu$, $d$ and $e$ satisfy

$$\|p(\mu, d, e)\| = \|\mu\| + \|d\| + \|e\| \leq c,$$

where the norms are defined in (2.19), (2.20), (2.21), (2.22) and (2.37). With suitable choices of $\mu_0$ and $d_{e, N}$ as in (2.18) and $e_0$ as in (2.36), the main order terms in the projections of $S_\varepsilon(w)$ on $Z_j$ are eliminated, and thus we obtain

$$\int_{D_\varepsilon} S_\varepsilon(w) Z_k = \varepsilon^{2 + \frac{1}{4^2}} \left( \int_{\mathbb{R}^N} Z_k^2 \right) \left[ \mu_0 (-d_k'' + R_{0j0k} d_j) + \alpha_k (\rho y_0) \right. $$

$$\left. + \varepsilon \beta_0 (\rho y_0; \mu, d, e) \right) + \varepsilon^3 r, \quad (2.93)$$

$$\bar{\omega} \int_{D_\varepsilon} S_\varepsilon(w) Z_N = \varepsilon^{2 + \frac{1}{4^2}} \left[ B_{000} \mu + C_{000} d_N + \alpha_N (\rho y_0) + \varepsilon \beta_N (\rho y_0; \mu, d, e) \right]$$

$$\left. - \varepsilon^{3 + \frac{1}{4^2}} \left( \int_{\mathbb{R}^N} Z_N^2 \right) \mu_0 d_N' + \varepsilon^4 r, \quad (2.94) \right.$$
In the above expressions (2.93)-(2.96), \( R_{ijkl} \) is the component of the curvature tensor defined in (2.10), \( \alpha_k \) and \( \beta_k \) are smooth and uniformly bounded in \( \varepsilon \). Note that \( \alpha_k \) and \( \beta_k \) does not depend on \( \mu', d' \) and \( e' \). Function \( r \) is of the following form

\[
h_0(\rho y_0) \left[ h_1(\mu, d, e, \mu', d', e') + o(1)h_2(\mu, d, e, \mu', d', e', \mu'', d'', e'') \right],
\]

where \( h_0, h_1 \) and \( h_2 \) are smooth and uniformly bounded in \( \varepsilon \). \( A, B \) and \( C \) are constants depending only on the dimension with \( AC - B^2 > 0 \). \( \varpi \) is a constant depending on the dimension and the smooth functions \( \mu_0, d_0, e_0, \mu_1, d_1, e_1 : (-l, l) \to \mathbb{R} \) defined in (2.18), (2.34) and (2.36).

Since the projections (2.93)-(2.96) are exactly the same as that of [60], we omit the proof here. A proof can be found in [60, Appendix].

2.7.3 Projections of \( N_1(\Pi) \) and \( \sum \alpha_i Z_i \)

By the computations in Section 2.3.3 about the estimates of \( \Pi \), we have the following. For \( \varepsilon \) sufficiently small and \( N = 5, 6 \), it holds that

\[
\int_{D_R} N_1(\Pi) Z_j dy = \varepsilon^{2(1-\vartheta)} h_0(\rho y_0) \quad \text{for } j = 0, 1, \ldots, N + 1, \quad (2.97)
\]

where \( N_1(\Pi) \) is defined in (2.91), \( h_0(\rho y_0) \) is a smooth function of \( \rho y_0 \). Indeed, by (2.46) and the decay of \( Z_j \), \( j = 0, 1, \ldots, N + 1 \), we can easily get the desired estimate.

Note that in (2.97), \( h_0(\rho y_0) \) is independent of \( \mu, d \) and \( e \). From a quite similar argument as in [60, Appendix], we can eliminate the largest term \( \varepsilon^{2(1-\vartheta)} h_0(\rho y_0) \) by solving a system like [60, Appendix (9.17)]. Therefore, the new projection becomes

\[
\int_{D_R} N_1(\Pi) Z_j dy = o(1) \varepsilon^3 \quad \text{for } j = 0, 1, \ldots, N + 1. \quad (2.98)
\]

On the other hand, by (2.43) and the fact that \( \vartheta > 0 \) is close to 0, we see that

\[
\alpha_j \int_{D_R} Z_j^2 = o(1) \varepsilon^{3-\vartheta}, \quad (2.99)
\]

which is of smaller order compared with the projections of \( S_\varepsilon(w) \)’s for \( \varepsilon \) small.
2.7.4 Projections of $N(\phi)$

From (2.86), $\|\phi\|_* \sim \varepsilon^{2(1-\vartheta)}$ and the definition of $R = R(\varepsilon)$ as in (2.47), one has

$$\|N(\phi)\|_2 \lesssim \varepsilon^{3-2\vartheta-2\Theta_s(\tau-\sigma)} + \varepsilon^{4(1-\vartheta)-2\Theta_s(\tau-\sigma)}$$

$$+ \varepsilon^{\frac{\Theta_s(\tau-\sigma)-2(1-\vartheta)}{N-2}}(2+\sigma-\vartheta)^{p(\tau-\sigma) + 2(1-\vartheta)p} + \varepsilon^{2(1-\vartheta) + \sigma \Theta_s}.$$

Now we choose $\Theta_s$ such that the projections of $N(\phi)$ are of smaller order compared with the leading order of $\mathcal{S}_{\varepsilon}(\mathcal{w})$'s for $R$ sufficiently large (namely $\varepsilon$ sufficiently small). More precisely, by (2.93), the projection of $\mathcal{S}_{\varepsilon}(\mathcal{w})$ along $Z_j$ ($j = 1, \ldots, N - 1, N$) is of order $\varepsilon^{2+\frac{1}{N-2}}$. Thus, $\Theta_s$ satisfies the following inequalities

$$\begin{cases}
3 - 2\vartheta - \Theta_s(\tau - \sigma) > 2 + \frac{1}{N-2}, \\
4(1 - \vartheta) - 2\Theta_s(\tau - \sigma) > 2 + \frac{1}{N-2}, \\
\frac{\Theta_s(\tau-\sigma)-2(1-\vartheta)}{N-2} \left( 2 + \sigma - \vartheta \right) - \Theta_s p(\tau - \sigma) + 2(1-\vartheta)p > 2 + \frac{1}{N-2}, \\
2(1-\vartheta) + \sigma \Theta_s > 2 + \frac{1}{N-2}.
\end{cases} \quad (2.100)$$

We know that $\tau \approx 2$, $\sigma \approx N - 4$ and $\vartheta \approx 0$. To make the projection of $N(\phi)$ along $Z_j$ comparatively smaller than $\mathcal{S}_{\varepsilon}(\mathcal{w})$'s, a sound choice of $\Theta_s$ satisfying system (2.100) is

$$\Theta_s = \frac{1 + \nu}{(N-2)\sigma} \text{ with } R(\varepsilon) = \varepsilon^{-\Theta_s} \text{ and } \nu > 0 \text{ small.} \quad (2.101)$$

We can easily check the other directions $Z_0$ and $Z_{N+1}$ in a similar way.

In conclusion, with such $\Theta_s$ in (2.101), we obtain that for $i = 0, 1, \ldots, N, N+1$

$$\int_{D_R} N(\phi) Z_i dy = o(1) \int_{D_R} \mathcal{S}_{\varepsilon}(\mathcal{w}) Z_i dy. \quad (2.102)$$

2.7.5 Projections of $\mathcal{L}(\phi)$

Recall that $\mathcal{L}(\phi) = \mathcal{A}\phi + p w^{\theta-1} \phi$ and $\mathcal{A}\phi = a_0 \partial_y^2 \phi + \Delta_y \phi + \mathcal{A}\phi$. Since the differential operator $\mathcal{A}$ is a small perturbation of $\Delta_y$ of order $\varepsilon$, we have

$$|\mathcal{A}\phi| \lesssim \varepsilon |\phi| \lesssim \varepsilon R^{2-\sigma} \|\phi\|_*.$$
where we have used the definition of the norm $\| \cdot \|_*$ in (2.52). By $\| \phi \|_* \sim \varepsilon^{2(1-\sigma)}$ and the choice of $\theta_*$ in (2.101), we obtain

$$\left| \partial \tilde{\theta} \right| \lesssim \varepsilon^{3-2\sigma-\theta_*(\tau-\sigma)} = o(1) \varepsilon^{2+\frac{1}{1-\sigma}}$$ (2.103)

for $\varepsilon$ small. For the remaining terms, since the solution $\phi$ we get is orthogonal to $Z_j$ in $D_R$, we have

$$\int_{D_R} (a_0 \partial_0^2 \phi + \Delta_y \phi + pw^{p-1} \phi) Z_j dy = \int_{D_R} (\Delta_y \phi + pw^{p-1} \phi) Z_j dy$$

$$= \int_{D_R} (\Delta_y Z_j + pw^{p-1} Z_j) \phi dy + \int_{\partial D_R} Z_j \partial \nu \phi dS - \int_{\partial D_R} \phi \partial \nu Z_j dS$$

$$\lesssim R^{-\sigma} \| \phi \|_* = o(1) \varepsilon^{2+\frac{1}{1-\sigma}}$$ (2.104)

for $\varepsilon$ small, where we used the integration-by-parts formula and (2.101). In conclusion, combining (2.103) and (2.104), we have

$$\int_{D_R} \mathcal{L}(\phi) Z_j dy = o(1) \varepsilon^{2+\frac{1}{1-\sigma}}$$ (2.105)

for $j = 0, 1, \ldots, N+1$.

### 2.7.6 Reduced equations for $\mu, d, e$

By the computations above, the reduced system (2.90) are equivalent to a system of ODEs for $\mu, d, e$. We assume that

$$\| (\mu, d, e) \| := \| \mu \|_a + \| d \|_d + \| e \|_e \leq c.$$ (2.106)

By collecting (2.92), (2.93), (2.94), (2.95), (2.96), (2.98), (2.99), (2.102) and (2.105), we know that the reduced system (2.90) are achieved if $(e, d, \mu)$ satisfies
the following system of ODEs

\[
\begin{aligned}
\mathcal{L}_0(e) &:= \rho^2 a_0 e'' + |\lambda_0| e + \gamma_0 d_N \\
&= -\alpha_0(\rho y_0) - Q_0(d) + \varepsilon^2 M_0(\rho y_0; \mu, d, e), \\
\mathcal{L}_k(d_k) &:= -d_k'' + R_{010k} d_j \\
&= -\alpha_k(\rho y_0) + \varepsilon M_k(\rho y_0; \mu, d, e), \quad k = 1, \ldots, N - 1, \\
\mathcal{L}_N(d_N) &:= -\varepsilon C_N \sigma_\mu d_N'' + B_0^1 \mu + C_0^1 d_N \\
&= -\alpha_N(\rho y_0) + \varepsilon M_N(\rho y_0; \mu, d, e), \\
\mathcal{L}_{N+1}(\mu) &:= -\varepsilon \frac{\bar{\gamma}^2}{N} C_{N+1} \mu'' + A_0^1 \mu + B_0^1 d_N \\
&= -\alpha_{N+1}(\rho y_0) + \varepsilon M_{N+1}(\rho y_0; \mu, d, e),
\end{aligned}
\]

where

\[
C_N := \int_{\mathbb{R}^N} Z_N^2, \quad C_{N+1} := \int_{\mathbb{R}^N} Z_{N+1}^2, \\
\gamma_0 := -2(\text{Tr}_g \tilde{h} - \tilde{h}_0) \left( \int_{\mathbb{R}^N} \partial_i w Z_0 \right), \\
Q_0(d) = \sum_i [(d_i')^2 - \frac{1}{3} R_{3ii} d_i^2 d_i + a_{Nk} d_k d_{0, N} + 4 \tilde{h}_{0j} d_j d_{0, N}] \left( \int_{\mathbb{R}^N} \partial_i w Z_0 \right).
\]

For \( j = 0, 1, \ldots, N, N + 1 \), the operator \( M_j(\rho y_0; \mu, d, e) \) can be decomposed into the following form

\[
M_j(\rho y_0; \mu, d, e) = A_j(\rho y_0; \mu, d, e) + K_j(\rho y_0; \mu, d, e),
\]

where \( K_j \) is uniformly bounded in \( L^\infty(-l, l) \) for \((\mu, d, e)\) satisfying (2.106) and is compact, \( A_j \) depends on \((\mu, d, e, \mu', d', e', \mu'', d'', e'')\) and satisfies

\[
\|A_j(\mu_1, d_1, e_1) - A_j(\mu_2, d_2, e_2)\|_\infty \lesssim o(1)\|((\mu_1, d_1, e_1) - (\mu_2, d_2, e_2))\|,
\]

in which the dependence of \( A_j \) on \( \mu'', d'' \) and \( e'' \) is linear.

### 2.7.7 Linear theory for the ODE system (2.107)

For \( j = 0, 1, \ldots, N, N + 1 \), we first develop a linear theory concerning the invertibility of \( \mathcal{L}_j \) in a \( L^\infty \) manner.
We seek 2\(l\)-periodic solutions of the following problem

\[
\mathcal{L}_{N+1}(\mu) = h_1, \quad \mathcal{L}_N(d) = h_2,
\]

(2.109)

where \(\|h_1\|_\infty + \|h_2\|_\infty < +\infty\). We have the following existence and a priori estimate for the problem (2.109).

**Lemma 2.7.1.** Assume that \(A > 0\), \(C > 0\) and \(AC - B^2 > 0\). If \(\|h_1\|_\infty + \|h_2\|_\infty\) is bounded, then there exists a 2\(l\)-periodic solution \((\mu, d)\) of (2.109) such that

\[
\|\mu\|_\infty + \|d\|_\infty + \epsilon^{\frac{N}{N - 2}} \|\mu\|_\infty + \frac{1}{2} \|d\|_\infty \lesssim \|h_1\|_\infty + \|h_2\|_\infty.
\]

**Proof.** The associated energy functional for the operators \(\mathcal{L}_N\) and \(\mathcal{L}_{N+1}\) is given by

\[
F(\mu, d) = \int_{-l}^{l} \left[ \epsilon^{\frac{N}{N - 2}} \mu_0(\mu')^2 + \epsilon \mu_0(d')^2 + \epsilon^{\frac{N}{N - 2}} \mu_0' \mu_0' + \epsilon \mu_0'' d' \right. \\
+ \left. (A \mu^2 + 2Bd\mu + Cd^2)h_0 + h_1 \mu + h_2 d \right] dx_0.
\]

Since \(A > 0\), \(C > 0\) and \(AC - B^2 > 0\), for \(\epsilon > 0\) small, we have \(F(\mu, d) \geq c > 0\), and it is coercive and convex. Hence, the existence of solution to (2.109) follows from variational arguments.

The proof of the a priori estimate is the same as that in [60, Lemma 8.1]. \(\square\)

Now we consider the invertibility of

\[
\mathcal{L}_0(e) := \rho^2 a_0 e'' + |\lambda_0| e + \gamma_0 d = f.
\]

(2.110)

We perform the Liouville transform as follows.

\[
m = \int_{-l}^{l} \frac{1}{\sqrt{a_0(s)}} \, ds, \quad t = \frac{\pi \int_{-l}^{l} \left( \sqrt{a_0(\theta)} \right)^{-1} d\theta}{m},
\]

\[
\tilde{\lambda}_0 = \frac{m^2}{\pi^2} |\lambda_0|, \quad y(t) = a_0^{-\frac{1}{2}}(s) \tilde{e}(s), \quad q(t) = \frac{m^2}{\pi^2} \left( a_0^{\frac{1}{2}} \right)' a_0^{-\frac{3}{2}}.
\]
After the Liouville transform, equation (2.110) for $e$ gets reduced to

$$
\begin{cases}
\rho^2 (y'' + q(t)y) + \tilde{\lambda}_0 y = \tilde{f} \text{ in } (0, \pi), \\
y(0) = y(\pi), \ y'(0) = y'(\pi).
\end{cases}
$$

(2.111)

By directly applying the Sturm-Liouville theory to (2.111) together with the non-resonance condition

$$
|k^2 \epsilon^{\frac{N-1}{N-2}} - \kappa^2| > \delta \epsilon^{\frac{N-1}{N-2}}
$$

(2.112)

and

$$
\kappa = \frac{\sqrt{|\lambda_0|}}{2\pi} \int_{-l}^{l} \frac{1}{\sqrt{a_0(s)}} ds,
$$

(2.113)

we obtain the following existence and a priori estimates for $e$.

**Lemma 2.7.2 ([60]).** Assume that $\epsilon$ satisfies the non-resonance condition (2.112). If $f \in C(-l, l) \cap L^\infty(-l, l)$, then there exists a unique $2l$-periodic solution $e$ of (2.110) satisfying

$$
\rho^2 \|e''\|_\infty + \rho \|e'\|_\infty + \|e\|_\infty \lesssim \rho^{-1} \|f\|_\infty.
$$

Furthermore, if $f \in C^2(-l, l)$, then

$$
\rho^2 \|e''\|_\infty + \rho \|e'\|_\infty + \|e\|_\infty \lesssim \|f''\|_\infty + \|f'\|_\infty + \|f\|_\infty.
$$

**2.7.8 Final argument**

**Proof of Theorem 2.1.1.** From the nondegenerate condition of the geodesic $\Gamma$ (2.6), we have that for any $f \in L^\infty(-l, l)$, $k = 1, \ldots, N - 1$, there exists a $2l$-periodic function $d_k$ such that $\mathcal{L}_k(d_k) = f$ with

$$
\|d''_k\|_\infty + \|d'_k\|_\infty + \|d_k\|_\infty \lesssim \|f\|_\infty.
$$

(2.114)
Let \((\tilde{\mu}_0, \tilde{d}_{0,N}, \tilde{d}_{0,k})\) be a solution to

\[
\begin{cases}
\mathcal{L}_k(\tilde{d}_{0,k}) = \alpha_k, & k = 1, \ldots, N-1, \\
\mathcal{L}_N(\tilde{d}_{0,N}) = \alpha_N, \\
\mathcal{L}_{N+1}(\tilde{\mu}_0) = \alpha_{N+1}.
\end{cases}
\]

By Lemma 2.7.1 and (2.114), we obtain that

\[
\begin{align*}
\varepsilon \|\tilde{d}''_{0,N}\|_\infty + \varepsilon \|\tilde{d}'_{0,N}\|_\infty + \|\tilde{d}_{0,N}\|_\infty &\leq c, \\
\|\tilde{d}''_{0,k}\|_\infty + \|\tilde{d}'_{0,k}\|_\infty + \|\tilde{d}_{0,k}\|_\infty &\leq c, \\
\varepsilon^{N/2} \|\tilde{\mu}''_0\|_\infty + \varepsilon^{N-1/2} \|\tilde{\mu}'_0\|_\infty + \|\tilde{\mu}_0\|_\infty &\leq c.
\end{align*}
\] (2.115)

(2.116)

(2.117)

Now we consider

\[\mathcal{L}_0(\tilde{e}_0) = -\gamma_0 \tilde{d}_{0,N} - \alpha_0 - Q_0(\tilde{d}_0),\]

where \(\tilde{d}_0 = (\tilde{d}_{0,1}, \ldots, \tilde{d}_{0,N})\) and \(\gamma_0\) is defined in (2.108). Since \(\alpha_0\) and \(Q_0(\tilde{d}_0)\) are regular, by (2.115), (2.116) and Lemma 2.7.2, we have that

\[
\varepsilon^{2N/2} \|\tilde{e}''_0\|_\infty + \varepsilon^{N-1/2} \|\tilde{e}'_0\|_\infty + \|\tilde{e}_0\|_\infty \leq c.
\] (2.117)

Summarizing (2.115), (2.116) and (2.117), it holds that

\[
\| (\tilde{\mu}_0, \tilde{d}_0, \tilde{e}_0) \| \leq c.
\]

We assume that \(\mu = \tilde{\mu}_0 + \tilde{\mu}_1, d = \tilde{d}_0 + \tilde{d}_1, e = \tilde{e}_0 + \tilde{e}_1\). Then the original system (2.107) reduces to

\[
\begin{cases}
\mathcal{L}_0(\tilde{e}_1) = -\gamma_0 \tilde{d}_{1,N} + \varepsilon^2 M_0(\rho y_0; \mu, d, e), \\
\mathcal{L}_k(\tilde{d}_{1,k}) = \varepsilon M_k(\rho y_0; \mu, d, e), & k = 1, \ldots, N-1, \\
\mathcal{L}_N(\tilde{d}_{1,N}) = \varepsilon M_N(\rho y_0; \mu, d, e), \\
\mathcal{L}_{N+1}(\tilde{\mu}_1) = \varepsilon M_{N+1}(\rho y_0; \mu, d, e).
\end{cases}
\] (2.118)

A direct use of Shauder’s fixed point theorem clarifies the existence of \((\tilde{\mu}_1, \tilde{d}_1, \tilde{e}_1)\) solving system (2.118), whose proof can be found in [60]. We omit the details. \(\Box\)
Chapter 3

New gluing methods for nonlinear parabolic equations

In this Chapter, we will study the Fujita equation

$$u_t = \Delta u + u^p \text{ in } \Omega \times (0, T),$$ (3.1)

where $\Omega$ is the entire space $\mathbb{R}^n$ or a smooth domain in $\mathbb{R}^n$ and $0 < T \leq +\infty$. This semilinear heat equation with $p > 1$ has been widely studied since Fujita’s celebrated work [85]. Many literatures have been devoted to studying this problem about the singularity formation, especially the blow-up rates, profiles and sets. See, for instance, [39, 40, 90-95, 145-147, 154, 182] and references therein. Also, for a comprehensive survey in the literature, we refer the readers to the book of Quittner and Souplet [162].

For the finite time blow-up, it is said to be of

- **type I** if
  $$\limsup_{t \to T} (T - t)^{-1/p-1} \| u(\cdot,t) \|_\infty < +\infty,$$

- **type II** if
  $$\limsup_{t \to T} (T - t)^{-1/p-1} \| u(\cdot,t) \|_\infty = +\infty.$$

Type I blow-up is at most like that of the ODE $u_t = u^p$, while type II blow-up is
of the critical Sobolev exponent,

\[ p_S := \begin{cases} \frac{n+2}{n-2} & \text{if } n \geq 3, \\ +\infty & \text{if } n = 1,2. \end{cases} \]

Stability and genericity of type I blow-up have been considered for instance in [39, 147, 149]. In [92], Giga and Kohn first proved that for \( 1 < p < p_S \), only type I blow-up can occur in the case of convex domain. This was generalized to radial case [80] for the energy critical case \( p = p_S \). Solutions with multiple type I blow-up were first built in [148] in the real line. For the subcritical case \( p < p_S \), multiple-point, finite time type I blow-up solution was found and its stability was further studied in [149].

Solutions with type II blow-up are in fact much harder to detect. The first example was discovered by Herrero-Velázquez [106, 107], for \( p > p_{JL} \) where \( p_{JL} \) is the Joseph-Lundgren exponent [118],

\[ p_{JL} = \begin{cases} 1 + \frac{4}{n-4-2\sqrt{n-1}} & \text{if } n \geq 11, \\ +\infty & \text{if } n \leq 10. \end{cases} \]

In fact, no type II blow-up is present for radial solutions if \( p_S < p < p_{JL} \) in the case of a ball or in entire space under additional assumptions, see [145, 146, 153]. For radial positive solutions this is not possible if \( p = p_S \) [80]. Examples of nonradial positive blow-up solutions for \( p > p_{JL} \) have been found in [37, 40]. Various different scenarios have been discovered or discarded in the supercritical case. See for instance [37, 40, 63, 106, 107, 145, 147, 154] and the book [162].

For the critical case \( p = p_S \), solutions were classified near the ground state of the energy critical heat equation in \( \mathbb{R}^n \) with \( n \geq 7 \) in [38]. Using matched asymptotic expansions, Filippas, Herrero and Velazquez [80] formally obtained...
sign-changing solutions with type II blow-up for $p = p_S$ in lower dimensions $n = 3, 4, 5, 6$. The blow-up of radial blow-up solutions found in [80] are given by

$$
\|u\|_{L^\infty(\mathbb{R}^n)} \sim \begin{cases} 
(T - t)^{-k}, & n = 3, \\
(T - t)^{-k}|\log(T - t)|^{\frac{2k}{n-1}}, & n = 4, \\
(T - t)^{-k}, & n = 5, \\
(T - t)^{-\frac{1}{2}}|\log(T - t)|^{-\frac{15}{2}}, & n = 6,
\end{cases}
$$

which are corrected to

$$
\|u\|_{L^\infty(\mathbb{R}^n)} \sim \begin{cases} 
(T - t)^{-k}, & n = 3, \\
(T - t)^{-k}|\log(T - t)|^{\frac{2k}{n-1}}, & n = 4, \\
(T - t)^{-3k}, & n = 5, \\
(T - t)^{-\frac{1}{2}}|\log(T - t)|^{-\frac{15}{2}}, & n = 6,
\end{cases}
$$

by Harada [104]. The first rigorous radial example was constructed in [167] for $n = 4$ and $k = 1$. See also [70] for nonradial constructions of blow-up at multiple points for the case $n = 4$ and $k = 1$. There is a deep connection between four dimensional energy critical heat equation and two-dimensional harmonic map heat flows ([55, 164]). Faster blow-up rates in the case $n = 4$, $k \geq 2$ are established in [165]. In dimension $n = 5$, del Pino, Musso and Wei [63] gave a rigorous construction of type II blow-up in both radial and nonradial cases confirming the case of $k = 1$. Higher speed blow-up solutions for $n = 5$, $k \geq 2$ were constructed by Harada [104]. Very recently Harada succeeded in establishing the type II blow-up in dimension $n = 6$ in [105].

The singularity formation for supercritical case $p > p_S$ is much more intricate. Herrero and Velázquez [106, 107] found type II blow-up solution in the radial class for $n \geq 11$ and $p > p_{JL}$. The solution locally resembles a asymptotically singular scaling of a radial and positive solution to the stationary problem

$$
\Delta u + u^p = 0 \text{ in } \mathbb{R}^n.
$$

See also [152] for the case that $\Omega$ is a ball, and [37] for the case of general domain.
with the restriction that $p$ is an odd integer. For the borderline case $p = p_{JL}$ and $n \geq 11$, the existence of type II blow-up was shown in [168]. In the Matano-Merle range $p_S < p < p_{JL}$ which complements the Herrero-Velázquez range, no type II blow-up can occur in radially symmetric class in the case of a ball or in entire space under additional assumptions [145, 146, 153]. In [62], the authors successfully constructed non-radial type II blow-up solution to (3.1) in the Matano-Merle range $p = \frac{n+1}{n-3} \in (p_S, p_{JL})$. More precisely, the solution constructed in [62] blows up along a certain curve with axial symmetry in the sense that the energy density approaches to the Dirac measure along the curve as $t \searrow T$. See also [40] for another kind of anisotropic blow-up for the case $n \geq 12$ and $p > p_{JL}$.

Singularity formation triggered by criticality and super-criticality in many other literatures has also been widely studied, such as the Schrödinger map, wave equations, Yang-Mills problems, geometric flows such as harmonic map heat flows, mean curvature flows and Yamabe flows. We refer the readers for instance to [3, 11, 26, 38, 44, 75, 86, 89, 98, 116, 120–123, 150, 156, 163], and the references therein.

The aim of this Chapter is to construct type II finite time blow-up solutions to (3.1) in the following cases

- **Energy critical case**: $p = 5$ and $n = 3$. We deal with this last remaining case as predicted in [80] in Section 3.1.
- **Energy supercritical case**: $p = 3$ and $n = 5, 6, 7$. We will construct in Section 3.2 type II finite time blow-up initiating along the shrinking sphere with the self-similar size $\sqrt{T-t}$ which collapses to a point as $t \to T$.

The constructions in this Chapter are based on the inner–outer gluing method, which has been a very powerful tool in constructing solutions in many elliptic problems, see for instance [49, 59–61] and the references therein. Also, this method has been successfully applied to various parabolic problems and Euler equation recently, such as the infinite time and finite time blow-ups for energy critical heat equations [43, 62, 63, 66, 68, 69], singularity formation for the harmonic map heat flows [51, 55, 173], the vortex dynamics in Euler flows [53], and infinite time blow-up for the half-harmonic map flow [174]. We refer the interested readers to a survey by del Pino [56] for more results in parabolic settings.
Throughout this Chapter, we shall use the symbol “$\lesssim$” to denote “$\leq C$” for a positive constant $C$ independent of $t$ and $T$. Here $C$ might be different from line to line.

### 3.1 Type II finite time blow-up for the energy critical heat equation in dimension three

#### 3.1.1 Introduction

We consider the Fujita nonlinear heat equation

$$
\begin{cases}
  u_t = \Delta u + |u|^{p-1}u & \text{in } \mathbb{R}^n \times (0, T), \\
  u(\cdot, 0) = u_0 & \text{in } \mathbb{R}^n.
\end{cases}
$$

(3.3)

In this section, we shall give rigorous constructions of type II finite time blow-up solutions as predicted by [80] in the critical case $n = 3$ and $p = p_5 = 5$.

**Theorem 3.1.1.** Let $n = 3$ and $k \in \mathbb{Z}_+$. For each $T > 0$ sufficiently small there exists an initial condition $u_0$ such that the solution of Problem (3.3) blows up at time $T$ which looks like at main order

$$
u(x, t) = \eta \left( \frac{x}{r \sqrt{T-t}} \right) \left[ \mu^{-\frac{1}{2}}(t) w \left( \frac{x}{\mu(t)} \right) + 2 \mu'(t) \mu^{\frac{3}{2}}(t) J \left( \frac{x}{\mu(t)} \right) \right]$$

$$+ \left( 1 - \eta \left( \frac{x}{r \sqrt{T-t}} \right) \right) \eta \left( \frac{x}{r_2 \sqrt{T-t}} \right) (T-t)^{k} \frac{1}{|x|} C_k H_{2k} \left( \frac{|x|}{2 \sqrt{T-t}} \right)$$

$$+ \theta(x, t),$$

where $\eta$ is the smooth cut-off function defined in (3.15), $w(y) = 3^\frac{1}{2} \left( 1 + |y|^2 \right)^{-\frac{1}{2}},$ $C_k = \frac{(-1)^{k_1} \sqrt{3}}{(2k)!},$ $H_{2k}$ is the Hermite polynomial defined in (3.6), $J$ is defined in (3.10), $r > 0$ is a small constant, $r_2 > 3r$ and $\|\theta\|_{L^\infty} \leq T^a$ for some $a > 0$. Moreover, the blow-up rate $\mu(t)$ satisfies

$$\mu(t) \sim \mu_0(t) = 3^\frac{1}{2} A (T-t)^{2k}, \quad k \in \mathbb{Z}_+$$

where $A$ is any positive constant.
The method of this section is close in spirit to the analysis in the works \[43, 55, 62, 63, 66\], where the inner-outer gluing method is employed. The main obstacle in proving finite time blow-up in dimension three is the very slow decaying behavior of the kernel \(Z_0(y) \sim \frac{1}{|y|}\) which is not even in \(L^2\). To overcome this difficulty we follow the ideas in \[66\] in which the authors constructed infinite time blow-up for (3.3) with fast decaying initial condition. First we use the matched asymptotic expansion of \[80\] to construct a good inner and outer expansions. But this is not good enough as the error still carries slow decaying. As in \[55\] and \[66\], we use a global term to correct the slow decaying error. This global term carries all the information needed to solve the scaling parameter. We then use the inner-outer gluing procedure to find a true solution. Interestingly the remainder of the scaling parameter solves a nonlocal ODE of the following form

\[
\int_t^T \frac{\alpha(s)}{\sqrt{T-s}} ds = h(t), \quad t < T, \quad \alpha(T) = 0
\]

which is Caputo derivative of \(\frac{1}{2}\). This is the nonlocal feature for three dimensional problem. As far as we know this seems to be the first construction of finite time blow-up for energy critical heat equation in \(\mathbb{R}^3\).

### 3.1.2 First approximation and matching

In this section, following the matched asymptotic expansions first developed in \[80\], we derive the initial blow-up rate of finite blow-up solutions to

\[
u_t = \Delta u + |u|^4 u, \quad u(x, 0) = u_0(x), \quad x \in \mathbb{R}^3, \quad t > 0,
\]

where the initial value \(u_0\) will be determined later.

**Approximate solutions**

In the self-similar variable

\[z = \frac{x}{\sqrt{T-t}}, \quad \tau = -\log(T-t), \quad \Phi(z, \tau) = (T-t)^{\frac{3}{2}} u(x,t),\]

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problem (3.4) reads as

\[ \Phi_\tau = \Delta \Phi - \frac{1}{2} (z \cdot \nabla \Phi) - \frac{\Phi}{4} + |\Phi|^4 \Phi. \]

We next find good approximate solution for the above equation in both inner and outer regimes.

For the outer part, the first profile will be chosen as the solution to the linear problem

\[ \partial_\tau \Phi_{\text{out}} = \Delta \Phi_{\text{out}} - \frac{1}{2} (z \cdot \nabla \Phi_{\text{out}}) - \frac{\Phi_{\text{out}}}{4}. \] (3.5)

Writing \( \Phi_{\text{out}} = e^{\gamma \tau} m(z) \), we look for radially symmetric solutions to the following ODE

\[ m'' + \left( \frac{2}{z} - \frac{1}{2} z \right) m' - (\gamma + \frac{1}{4}) m = 0, \]

which turns out to be an eigenvalue problem ([80]). In order to get even solutions with polynomial growth rate in the outer regime, we take \( \gamma = \frac{1}{4} - k, \ k \in \mathbb{Z}_+ \), and then

\[ m(z) = A \frac{1}{z} C_k H_{2k}(\frac{z}{2}), \]

where \( H_{2k} \) is Hermite polynomial

\[ H_{2k}(\frac{z}{2}) = \frac{(-1)^k (2k)!}{k!} \left( 1 - k \frac{z^2}{2} + \cdots + a_k z^{2k} \right), \quad C_k = \frac{(-1)^k k! \sqrt{3}}{(2k)!}, \] (3.6)

\( a_k \) is constant depending on \( k \), and \( A \) is any positive constant. Therefore, we get special solutions of (3.5)

\[ \Phi_{\text{out}}(z, \tau) = A \frac{1}{z} e^{\left( \frac{1}{4} - k \right) \tau} C_k \frac{1}{z} H_{2k}(\frac{z}{2}), \]

which are sufficient for the matching process to be carried out below, and we take the outer approximate solution of (3.4) to be

\[ u_{\text{out}}(x, t) = (T - t)^{-\frac{1}{4}} \Phi_{\text{out}}(z, \tau) = A \frac{1}{x} (T - t)^{\frac{1}{4}} C_k H_{2k}(\frac{1}{2} \frac{x}{\sqrt{T - t}}). \] (3.7)
It can be directly checked that in the original variable $u_{out}$ satisfies

$$\partial_t u_{out} = \Delta u_{out}.$$  

For the inner part, we choose the approximate solution to be

$$\Phi_{in}(z, \tau) = \epsilon^{-1/2} \left( w(y) + \sigma(\tau) J(y) \right),$$

where $w(y) = 3^\frac{3}{4} (1 + |y|^2)^{-\frac{1}{2}}$, $y = \frac{x}{\mu}$, $\sigma(\tau) = 2 \epsilon(\tau) \partial_t \epsilon(\tau) - \epsilon^2(\tau)$, and $\epsilon(\tau)$ is a positive function to be determined later. Here $J$ is the radial solution of

$$\Delta J + 5w^4 J + \frac{1}{2} y \cdot \nabla w + \frac{w}{4} = 0, \quad J(0) = 0, \quad J'(0) = 0.$$  \hspace{1cm} (3.8)

Set $\mu(t) := \sqrt{T - t} \epsilon$ and $y = \frac{x}{\mu}$. Then one has

$$\bar{u}_{in}(x,t) = (T - t)^{-\frac{1}{4}} \Phi_{in}(z, \tau) = \mu^{-\frac{1}{2}} \left[ w\left(\frac{x}{\mu}\right) + 2 \mu \mu' J\left(\frac{x}{\mu}\right) \right]$$

since $\sigma = 2 \mu \mu'$. Denote

$$Z_0(y) := -\left[ y \cdot \nabla w + \frac{w}{2} \right] = \frac{3^\frac{3}{4}}{2} \frac{|y|^2 - 1}{(1 + |y|^2)^{\frac{3}{2}}},$$  \hspace{1cm} (3.9)

which is a radial kernel of the homogeneous part of (3.8). Then it is easy to see that

$$J(y) = Z_0(y) \int_0^y \frac{1}{Z_0^2(s)} s^{-2} \int_s^y Z_0(u) u^2 \frac{Z_0(u)}{2} du \, ds,$$  \hspace{1cm} (3.10)

and thus $J(y) \sim \frac{3^\frac{3}{4}}{8} y$ as $y \to \infty$.

**Matching inner and outer solutions**

In the region $\mu(t) \ll |x| \ll \sqrt{T - t}$, the inner and outer solutions have the following asymptotic behaviors respectively

$$\bar{u}_{in} \sim 3^\frac{3}{4} \mu^\frac{1}{2} |x|^{-1} + \frac{3^\frac{3}{4}}{4} \mu^{-\frac{1}{2}} \mu' |x|,$$  \hspace{1cm} (3.11)
Matching the inner and outer solutions, we get
\[ \mu \sim 3^{1/2}A(T-t)^{2k}, \quad k \in \mathbb{Z}_+ \]
so it is natural to choose
\[ \mu_0 := 3^{1/2}A(T-t)^{2k} \]
as the leading order of the scaling parameter \( \mu(t) \).

We next choose
\[ u_{in}(x,t) = \mu^{-1/2}w\left(\frac{x}{\mu}\right) + 2\mu_0 \mu^{1/2}J\left(\frac{x}{\mu}\right) \]  \hspace{1cm} (3.13)
and take the first approximate solution as follows
\[ U_1(x,t) := \eta\left(\frac{|x|}{r\sqrt{T-t}}\right)u_{in} + \left(1 - \eta\left(\frac{|x|}{r_1(T-t)^{\zeta_1}}\right)\right)\eta\left(\frac{|x|}{r_2(T-t)^{\zeta_2}}\right)u_{out}, \]  \hspace{1cm} (3.14)
where \( \eta \) is the smooth cut-off function satisfying
\[ \eta(t) = 1 \text{ for } t \in [0,1] \text{ and } \eta(t) = 0 \text{ for } t \in [2,\infty), \]  \hspace{1cm} (3.15)
and later in Section 3.1.6 we shall choose \( \zeta_1 = \zeta_2 = 1/2, r = r_1 > 0, r_2 > 3r \). For simplicity, we write
\[ \eta_1(x,t) := \eta\left(\frac{|x|}{r\sqrt{T-t}}\right), \quad \eta_{o1}(x,t) = \eta\left(\frac{|x|}{r_1(T-t)^{\zeta_1}}\right), \quad \eta_{o2}(x,t) = \eta\left(\frac{|x|}{r_2(T-t)^{\zeta_2}}\right). \]

The purposes of the cut-off \( 1 - \eta_{o1} \eta_{o2} \) in front of \( u_{out} \) are to prevent the approximation from being singular near the origin and to restrict the approximation in the self-similar scale.
Error of the first approximation

We define the error function

\[ S(u) := -\partial_t u + \Delta_x u + u^5, \]

and compute

\[ S(U_1) = \eta_1(-\partial_t u_{in} + \Delta_x u_{in} + u_{in}^5) + (1 - \eta_{o1}) \eta_{o2}(-\partial_t u_{out} + \Delta_x u_{out} + u_{out}^5) \]

\[ - \partial_t \eta_1 u_{in} + \Delta_x \eta_1 u_{in} + 2 \nabla_x \eta_1 \nabla_x u_{in} \]

\[ - \partial_t [(1 - \eta_{o1}) \eta_{o2}] u_{out} + \Delta_x [(1 - \eta_{o1}) \eta_{o2}] u_{out} + 2 \nabla_x [(1 - \eta_{o1}) \eta_{o2}] \nabla_x u_{out} \]

\[ + [\eta_1 u_{in} + (1 - \eta_{o1}) \eta_{o2} u_{out}]^5 - \eta_1 u_{in}^5 - (1 - \eta_{o1}) \eta_{o2} u_{out}^5. \]

Let \( S_{in} := -\partial_t u_{in} + \Delta_x u_{in} + u_{in}^5 \) and \( S_{out} := -\partial_t u_{out} + \Delta_x u_{out} + u_{out}^5 \), where we compute

\[ \partial_t u_{in} = -\mu^{-\frac{1}{2}} \mu' \left( \frac{w}{2} + \nabla_y w \cdot \frac{x}{\mu} \right) + 2 \mu_0'' \mu \frac{1}{2} J \left( \frac{x}{\mu} \right) \]

\[ + \mu_0' \mu^{-\frac{3}{2}} \mu' J \left( \frac{x}{\mu} \right) - 2 \mu_0' \mu^{-\frac{1}{2}} \mu' \nabla_y J \left( \frac{x}{\mu} \right) \cdot \frac{x}{\mu}, \]

\[ \partial_t \left( \mu \frac{1}{2} J \left( \frac{x}{\mu} \right) \right) = \mu^{-\frac{1}{2}} \left[ \frac{1}{2} J \left( \frac{x}{\mu} \right) - \nabla_y J \left( \frac{x}{\mu} \right) \cdot \frac{x}{\mu} \right]. \]

Therefore, we have

\[ S_{in} = \mu^{-\frac{1}{2}} (\mu - \mu_0)' \left( \frac{w}{2} + \nabla_y w \cdot \frac{x}{\mu} \right) - 2 \mu_0'' \mu \frac{1}{2} J \left( \frac{x}{\mu} \right) \]

\[ - \mu_0' \mu^{-\frac{3}{2}} \mu' J \left( \frac{x}{\mu} \right) + 2 \mu_0' \mu^{-\frac{1}{2}} \mu' \nabla_y J \left( \frac{x}{\mu} \right) \cdot \frac{x}{\mu} \]

\[ + \left[ \mu^{-\frac{1}{2}} w \left( \frac{x}{\mu} \right) + 2 \mu_0' \mu \frac{1}{2} J \left( \frac{x}{\mu} \right) \right] - \left( \mu^{-\frac{1}{2}} w \left( \frac{x}{\mu} \right) \right)^5 - 5 \left( \mu^{-\frac{1}{2}} w \left( \frac{x}{\mu} \right) \right)^4 2 \mu_0' \mu \frac{1}{2} J \left( \frac{x}{\mu} \right), \]

where the first term can be written as

\[ \mu^{-\frac{1}{2}} (\mu - \mu_0)' \left( \frac{w}{2} + \nabla_y w \cdot \frac{x}{\mu} \right) = \mu^{-\frac{1}{2}} (\mu - \mu_0)' \frac{3}{2} \frac{1 - |y|^2}{(1 + |y|^2)^{\frac{3}{2}}}. \]
We notice that

\[
(\mu' - \mu_0')\mu^{-\frac{1}{2}} = 2(\mu^{\frac{1}{2}} - \mu_0^{\frac{1}{2}})' + (\mu^{\frac{1}{2}} - \mu_0^{\frac{1}{2}})\mu_0^{-1}\mu_0' - (\mu^{\frac{1}{2}} - \mu_0^{\frac{1}{2}})^2\mu^{-\frac{1}{2}}\mu_0^{-1}\mu_0',
\]

\[
\frac{3\frac{1}{2} - 1 - |y|^2}{2(1 + |y|^2)} = \frac{3\frac{1}{2}}{2} \frac{\mu^2 - |x|^2}{(\mu^2 + |x|^2)^{\frac{3}{2}}} = \frac{3\frac{1}{2}}{2} \frac{\mu^3}{(\mu^2 + |x|^2)^{\frac{3}{2}}} - \frac{3\frac{1}{2}}{2} \frac{\mu}{(\mu^2 + |x|^2)^{\frac{3}{2}}}.
\]

So the leading term is

\[
\eta_1 [2(\mu^{\frac{1}{2}} - \mu_0^{\frac{1}{2}})' + (\mu^{\frac{1}{2}} - \mu_0^{\frac{1}{2}})\mu_0^{-1}\mu_0'] \mu^{-1}(-\frac{3\frac{1}{2}}{2} \frac{\mu}{(\mu^2 + |x|^2)^{\frac{3}{2}}}) = \eta_1 \frac{\alpha(t)}{(\mu^2 + |x|^2)^{\frac{3}{2}}}.
\]

Here we define

\[
\mu(t) := \mu_0(t)(1 + \Lambda(t))^2,
\]

\[
\alpha(t) := (-3\frac{1}{2}\mu_0^{-1}t)(\mu_0(t)\Lambda(t))'.
\]

For the last term in \(S_{in}\), we compare the size of \(\mu^{-\frac{1}{2}}w(\frac{x}{\mu})\) and \(2\mu_0^{\frac{1}{2}}J(\frac{x}{\mu})\) in the regime \(\frac{|x|}{r\sqrt{T-t}} \leq 2\) thanks to the cut-off \(\eta_1\): if \(\frac{|x|}{\mu} \ll 1\), we have

\[
\mu^{-\frac{1}{2}} \gg \mu_0^{\frac{1}{2}} \Leftrightarrow \mu^{-1} \gg \mu_0' \Leftrightarrow 3^{-\frac{1}{2}}A^{-1}(T-t)^{-2k} \gg 3^{-\frac{1}{2}}A2k(T-t)^{2k-1}.
\]

Thus we get \(\mu^{-\frac{1}{2}}w(\frac{x}{\mu}) \gg 2\mu_0^{\frac{1}{2}}J(\frac{x}{\mu})\). If \(\frac{|x|}{\mu} \gg 1\), we only need to check

\[
\frac{\mu^{-\frac{1}{2}}}{|x|} \gg \mu_0^{\frac{1}{2}} \frac{|x|}{\mu} \Leftrightarrow \mu(\mu_0^{-1})^{-1} \gg |x|^2 \Leftrightarrow (T-t) \gg |x|^2
\]

since \(\frac{|x|}{r\sqrt{T-t}} \leq 2\) and \(r \ll 1\). So \(\mu^{-\frac{1}{2}}w(\frac{x}{\mu}) \gg 2\mu_0^{\frac{1}{2}}J(\frac{x}{\mu})\) is also satisfied. Therefore, the last term in \(S_{in}\) can be expanded as

\[
\left[\mu^{-\frac{1}{2}}w(\frac{x}{\mu}) + 2\mu_0^{\frac{1}{2}}J(\frac{x}{\mu})\right]^5 - (\mu^{-\frac{1}{2}}w(\frac{x}{\mu}))^5 - 5(\mu^{-\frac{1}{2}}w(\frac{x}{\mu}))^42\mu_0^{\frac{1}{2}}J(\frac{x}{\mu}) \\
\lesssim \left[\mu^{-\frac{1}{2}}w(\frac{x}{\mu})\right]^3 \left[2\mu_0^{\frac{1}{2}}J(\frac{x}{\mu})\right]^2,
\]

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where \( \theta_1, \theta_2 \in [0, 1] \) and

\[
|J(y)| \leq h(y) := \begin{cases} y^2 & \text{if } y \to 0, \\ y & \text{if } y \to \infty. \end{cases}
\]  

(3.16)

We define that \( \chi(x) = 1 \) if \( |x| \leq 1 \) and \( \chi(x) = 0 \) otherwise. Therefore, we have the following estimate

\[
\eta_1 S - \chi(\frac{x}{c_0(T-t)^{\frac{3}{2}}} \mu) \alpha(t) \leq |\alpha(t)| \frac{\mu^2}{(\mu^2 + |x|^2)^{\frac{3}{2}}} \eta_1 + |\alpha(t)| \frac{\mu^2}{(\mu^2 + |x|^2)^{\frac{3}{2}}} \chi(r \leq \frac{|x|}{\sqrt{T-t}} \leq c_0) \\
+ \Lambda^2 \mu^{-\frac{3}{2}} \mu_0 \left[ 3 \frac{\mu^2}{(\mu^2 + |x|^2)^{\frac{3}{2}}} - \frac{3}{2} \frac{1}{(\mu^2 + |x|^2)^{\frac{3}{2}}} \right] \eta_1 \\
+ 2 \mu_0 \mu^{-\frac{3}{2}} \mu_0' \frac{x}{\mu} \eta_1 + 2 \mu_0 \mu^{-\frac{3}{2}} \mu_0' h(\frac{x}{\mu}) \eta_1.
\]  

(3.17)

We decompose \( u = U_1 + \Phi_1 + \Phi_2 \) and compute

\[
S(U_1 + \Phi_1 + \Phi_2) = S(U_1) - \chi(\frac{x}{c_0(T-t)^{\frac{3}{2}}} \mu) \frac{\alpha(t)}{(\mu^2 + |x|^2)^{\frac{3}{2}}} \chi(\frac{x}{c_0(T-t)^{\frac{3}{2}}} \mu) \frac{\alpha(t)}{(\mu^2 + |x|^2)^{\frac{3}{2}}} - \partial_t \Phi_1 \\
- \partial_t \Phi_2 + \Delta_\tau \Phi_1 + \Delta_\tau \Phi_2 + (U_1 + \Phi_1 + \Phi_2)^5 - U_1^5,
\]

where \( \chi(x) = 1 \) if \( |x| \leq 1 \) and \( \chi(x) = 0 \) otherwise, \( c_0 \) is some constant, \( \Phi_1 \) and \( \Phi_2 \) are perturbations to be determined later.

**Nonlocal correction: second approximation**

Observe that in the error there is a slow decaying term \( \frac{\alpha(t)}{(\mu^2 + |x|^2)^{\frac{3}{2}}} \). Following the idea in [66], we now introduce a nonlocal correction \( \Phi_1 \) solving

\[
\partial_t \Phi_1 = \Delta_\tau \Phi_1 + \chi(\frac{x}{c_0(T-t)^{\frac{3}{2}}} \mu) \frac{\alpha(t)}{(\mu^2 + |x|^2)^{\frac{3}{2}}}.
\]  

(3.18)
Choosing the initial data, we get from Duhamel’s formula

\[
\Phi_1(x,t) = \sum_{j=1}^{k} c_j \mathcal{R}^{(j)}(x,t) + \int_0^t \int_{\mathbb{R}^3} (\frac{1}{2\sqrt{\pi}})^3 (t-s)^{-\frac{3}{2}} e^{-\frac{|x-\xi|^2}{4(t-s)}}
\times \chi\left(\frac{\xi}{c_0(T-s)^{\frac{1}{2}}}\right) \frac{\alpha(s)}{(\mu^2(s) + |\xi|^2)^{1/2}} d\xi ds,
\]

(3.19)

where \(c_j\) are constants and \(\mathcal{R}^{(j)}\) satisfy heat equation

\[
\partial_t \mathcal{R}^{(j)}(x,t) = \Delta_x \mathcal{R}^{(j)}(x,t)
\]

which will be determined later when solving \(\alpha(t)\) from the reduced equation in Section 3.1.5.

Then the new error becomes

\[
S(U_1 + \Phi_1) = S(U_1) - \chi\left(\frac{x}{c_0(T-t)^{\frac{1}{2}}}\right) \frac{\alpha(t)}{(\mu^2 + |x|^2)^{\frac{1}{2}}} + (U_1 + \Phi_1)^5 - U_1^5.
\]

Let \(\Phi = \Phi_1 + \Phi_2\). We have

\[
S(U_1 + \Phi_1 + \Phi_2)
= S(U_1) - \chi\left(\frac{x}{c_0(T-t)^{\frac{1}{2}}}\right) \frac{\alpha(t)}{(\mu^2 + |x|^2)^{\frac{1}{2}}} + (U_1 + \Phi_1)^5 - U_1^5
- \partial_t \Phi_2 + \Delta_x \Phi_2 + 5(U_1 + \Phi_1)^4 \Phi_2 + (U_1 + \Phi_1 + \Phi_2)^5 - (U_1 + \Phi_1)^5
- 5(U_1 + \Phi_1)^4 \Phi_2.
\]

### 3.1.3 Inner–outer gluing system

In this section, we set up the inner–outer gluing scheme for the nonlinear problem. We look for perturbation of the following form

\[
\Phi_2(x,t) = \psi(x,t) + \eta R \mu_0^{-\frac{1}{2}} \phi\left(\frac{x}{\mu_0},t\right),
\]

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where \( \dot{\phi}(x, t) := \mu_0^{-\frac{1}{2}} \phi(\frac{x}{\mu_0}, t) \), \( \eta_R := \eta(\frac{x}{R^\mu_0}) \), and \( R(t) = \mu_0^{-\beta}(t) \) with \( \beta \in (0, 1/2) \).

We next compute

\[
\partial_t \Phi_2(x, t) = \partial_t \psi(x, t) + \partial_t \eta_R \mu_0^{-\frac{1}{2}} \phi(\frac{x}{\mu_0}, t) + \eta_R \partial_t (\mu_0^{-\frac{1}{2}} \phi(\frac{x}{\mu_0}, t)),
\]

where

\[
\partial_t (\mu_0^{-\frac{1}{2}} \phi(\frac{x}{\mu_0}, t)) = -\mu_0^{-\frac{1}{2}} \mu'_0 \left[ \frac{1}{2} \phi(\frac{x}{\mu_0}, t) + \nabla_x \phi(\frac{x}{\mu_0}, t) \cdot \frac{x}{\mu_0} \right] + \mu_0^{-\frac{1}{2}} \partial_t \phi(\frac{x}{\mu_0}, t),
\]

\[
\Delta_x \Phi_2(x, t) = \Delta_x \psi(x, t) + \Delta_x \eta_R \mu_0^{-\frac{1}{2}} \phi(\frac{x}{\mu_0}, t) + 2 \nabla_x \eta_R \mu_0^{-\frac{3}{2}} \nabla_x \phi(\frac{x}{\mu_0}, t) + \eta_R \mu_0^{-\frac{3}{2}} \Delta_x \phi(\frac{x}{\mu_0}, t).
\]

Therefore, we obtain

\[
S(U_1 + \Phi_1 + \Phi_2)
= S(U_1) - \chi(\frac{x}{c_0(T - t)^{\frac{1}{2}}}) \frac{\alpha(t)}{(\mu^2 + |x|^2)^{\frac{1}{2}}} + (U_1 + \Phi_1)^5 - U_1^5
- \partial_t \psi(x, t) - \partial_t \eta_R \mu_0^{-\frac{1}{2}} \phi(\frac{x}{\mu_0}, t)
+ \eta_R \mu_0^{-\frac{3}{2}} \mu'_0 \left[ \frac{1}{2} \phi(\frac{x}{\mu_0}, t) + \nabla_x \phi(\frac{x}{\mu_0}, t) \cdot \frac{x}{\mu_0} \right] - \eta_R \mu_0^{-\frac{1}{2}} \partial_t \phi(\frac{x}{\mu_0}, t)
+ \Delta_x \psi(x, t) + \Delta_x \eta_R \mu_0^{-\frac{1}{2}} \phi(\frac{x}{\mu_0}, t) + 2 \nabla_x \eta_R \cdot \mu_0^{-\frac{3}{2}} \nabla_x \phi(\frac{x}{\mu_0}, t)
+ \eta_R \mu_0^{-\frac{3}{2}} \Delta_x \phi(\frac{x}{\mu_0}, t) + 5(U_1 + \Phi_1)^4 \psi(x, t) + \eta_R \mu_0^{-\frac{1}{2}} \phi(\frac{x}{\mu_0}, t)
+ (U_1 + \Phi_1 + \Phi_2)^5 - (U_1 + \Phi_1)^5 - 5(U_1 + \Phi_1)^4 \Phi_2.
\]

It can be directly checked that \( S(U_1 + \Phi_1 + \Phi_2) = 0 \) if \( \psi \) solves

\[
- \partial_t \psi(x, t) + \Delta_x \psi(x, t) - \partial_t \eta_R \mu_0^{-\frac{1}{2}} \phi(\frac{x}{\mu_0}, t) + \Delta_x \eta_R \mu_0^{-\frac{1}{2}} \phi(\frac{x}{\mu_0}, t)
+ 2 \nabla_x \eta_R \cdot \mu_0^{-\frac{3}{2}} \nabla_x \phi(\frac{x}{\mu_0}, t) + 5(U_1 + \Phi_1)^4 (1 - \eta R) \psi(x, t)
+ 5[(U_1 + \Phi_1)^4 - (\mu^{-\frac{1}{2}} w(\frac{x}{\mu}))^4] \psi(x, t) \eta_R
+ (U_1 + \Phi_1 + \Phi_2)^5 - (U_1 + \Phi_1)^5 - 5(U_1 + \Phi_1)^4 \Phi_2
\]

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and $\phi$ solves

$$
\eta_R \mu_0^{-\frac{3}{2}} \left[ \frac{1}{2} \phi \left( \frac{x}{\mu_0}, t \right) + \nabla_x \phi \left( \frac{x}{\mu_0}, t \right) \cdot \frac{x}{\mu_0} \right] - \eta_R \mu_0^{-\frac{3}{2}} \partial_t \phi \left( \frac{x}{\mu_0}, t \right)
+ \eta_R \mu_0^{-\frac{1}{2}} \Delta_x \phi \left( \frac{x}{\mu_0}, t \right) + 5 \left( \mu^{-\frac{1}{2}} w \left( \frac{x}{\mu} \right) \right)^4 \psi(x,t) \eta_R + 5 (U_1 + \Phi_1)^4 \mu_0^{-\frac{3}{2}} \phi \left( \frac{x}{\mu_0}, t \right) \eta_R
+ \eta_R \left[ S(U_1) - \chi \left( \frac{x}{c_0(T - t)^\frac{1}{2}} \right) \frac{\alpha(t)}{\mu^2 + |x|^2} \right] + [(U_1 + \Phi_1)^5 - U_1^5] \eta_R = 0.
$$

(3.21)

Recall that in the inner regime one has

$$
U_1 = u_{in} = \mu^{-\frac{1}{2}} w \left( \frac{x}{\mu} \right) + 2 \mu_0' \mu \frac{1}{2} J \left( \frac{x}{\mu} \right).
$$

Writing $y = \frac{x}{\mu_0}$, we rearrange problems (3.20) and (3.21) and get

- The outer problem:

$$
\partial_t \psi(x,t) = \Delta_x \psi(x,t) + \mathcal{G}(\phi, \psi, \alpha),
$$

(3.22)

where

$$
\mathcal{G}(\phi, \psi, \alpha)
= - \partial_t \eta_R \mu_0^{-\frac{3}{2}} \phi \left( \frac{x}{\mu_0}, t \right) + \Delta_x \eta_R \mu_0^{-\frac{3}{2}} \phi \left( \frac{x}{\mu_0}, t \right) + 2 \nabla_x \psi \cdot \mu_0^{-\frac{3}{2}} \nabla_x \phi \left( \frac{x}{\mu_0}, t \right)
+ \left( 5 (U_1 + \Phi_1)^4 (1 - \eta_R) + 5 \left[ (U_1 + \Phi_1)^4 \left( \mu^{-\frac{1}{2}} w \left( \frac{x}{\mu} \right) \right)^4 \eta_R \right] \psi
+ (U_1 + \Phi_1 + \Phi_2)^5 - (U_1 + \Phi_1)^5 - 5 (U_1 + \Phi_1)^4 \Phi_2
+ \left[ S(U_1) - \chi \left( \frac{x}{c_0(T - t)^\frac{1}{2}} \right) \frac{\alpha(t)}{\mu^2 + |x|^2} \right] \left( 1 - \eta_R \right)
+ [(U_1 + \Phi_1)^5 - U_1^5] \right) \eta_R.
$$

(3.23)
The inner problem:

$$
\mu_0^2 \partial_t \phi(y,t) = \Delta y \phi(y,t) + 5w^4(y)\phi(y,t) + \mathcal{H}(\phi, \psi, \alpha) \quad \text{in} \quad B_{2R} \times (0,T),
$$

(3.24)

where

$$
\mathcal{H}(\phi, \psi, \alpha) = 5 \left[ (u_{in}(\mu_0 y, t) + \Phi_1(\mu_0 y, t))^4 - \mu_0^{-2} w^4(y) \right] \mu_0^2 \phi(y,t)
$$

$$
+ 5 \mu_0^4 (1 + \Lambda)^{-4} w^4 \left( \frac{y}{(1 + \Lambda)^2} \right) \psi(\mu_0 y, t)
$$

$$
+ \mu_0 \mu_0^4 \left[ \frac{1}{2} \phi(y,t) + \nabla_y \phi(y,t) \cdot y \right]
$$

$$
+ \mu_0^2 \left[ S(U_1) - \chi \left( \frac{\mu_0 y}{c_0(T - t)^{1/2}} \right) \frac{\alpha(t)}{(\mu^2 + |\mu_0 y|^2)} \right]
$$

$$
+ \mu_0^3 \left[ (u_{in}(\mu_0 y, t) + \Phi_1(\mu_0 y, t))^5 - u_{in}^5(\mu_0 y, t) \right].
$$

(3.25)

We choose \( \zeta_1 = \zeta_2 = \frac{1}{2} \) so that \( (1 - \eta \left( \frac{|x|}{r_1(T-t)^{\frac{1}{2}}} \right)) \eta \left( \frac{|x|}{r_2(T-t)^{\frac{3}{2}}} \right) u_{out} = 0 \) in the inner region \( \{x : |x| \leq 2R\mu_0\} \)

$$
r_1(T-t)^{\zeta_1} \gg 2R\mu_0
$$

for \( T \ll 1 \) since \( R(t) = \mu_0^{-\beta} \) with \( \beta \in (0, 1/2) \).

In the sequel, we shall solve the inner–outer gluing system (3.22) and (3.24) by developing suitable linear theories and the fixed point argument.

3.1.4 Linear theories

Linear theory for the inner problem

By using the Fourier decomposition and delicate analysis, the authors in [66] developed the linear theory for the inner problem (3.24) in dimension 3. Denote \( Z^- \) by the positive radial bounded eigenfunction associated to the only negative eigenvalue \( \lambda_- \) to

$$
\Delta \phi + 5w^4 \phi + \lambda_- \phi = 0, \quad \phi \in L^\infty(\mathbb{R}^3).
$$
It is further known that $\lambda_-$ is simple and $Z_-$ satisfies

$$Z_-(y) \sim |y|^{-1} e^{-\sqrt{|\lambda_0|}|y|} \text{ as } |y| \to \infty.$$ 

Consider an orthonormal basis $\{\Theta_i\}_{i=0}^{\infty}$ made up of spherical harmonics in $L^2(S^2)$, i.e.

$$\Delta_{S^2} \Theta_i + \lambda_i \Theta_i = 0 \text{ in } S^2$$

with $0 = \lambda_0 < \lambda_1 = \lambda_2 = \lambda_3 = 2 < \lambda_4 \leq \cdots$. More precisely, $\Theta_0(y) = a_0$, $\Theta_i(y) = a_1 y_i$, $i = 1, \ldots, 3$ for two constants $a_0, a_1$. For $h \in L^2(B_{2R})$, we decompose

$$h(y,t) = \sum_{j=0}^{\infty} h_j(r,t) \Theta_j(y/r), \quad r = |y|, \quad h_j(r,t) = \int_{S^2} h(r\theta,t) \Theta_j(\theta) d\theta$$

and write $h = h^0 + h^1 + h^\perp$ with

$$h^0 = h_0(r,t), \quad h^1 = \sum_{j=1}^{3} h_j(r,t) \Theta_j, \quad h^\perp = \sum_{j=4}^{\infty} h_j(r,t) \Theta_j.$$ 

Also, we decompose $\phi = \phi^0 + \phi^1 + \phi^\perp$ in a similar form. Define

$$\|h\|_{v,2+\sigma} := \sup_{(y,t) \in B_{2R} \times (0,T)} \mu_0^{-v}(t)(1 + |y|^{2+\sigma}) |h(y,t)|.$$ \quad (3.26)

**Proposition 3.1.1 ([66]).** Let $v$, $\sigma$ be given positive numbers with $0 < \sigma < 2$. Then, for all sufficiently large $R > 0$ and any $h = h(y,t)$ with $\|h\|_{v,2+\sigma} < +\infty$ that satisfies for all $j = 0, 1, \ldots, 3$

$$\int_{B_{2R}} h(y,t) Z_j(y) dy = 0 \text{ for all } t \in (0,T)$$

there exist $\phi = \phi[h]$ and $e_0 = e_0[h]$ which solve

$$\mu_0^2 \phi_t = \Delta \phi + 5w^4 \phi + h(y,t) \text{ in } B_{2R} \times (0,T), \quad \phi(y,0) = e_0 Z_-(y) \text{ in } B_{2R}.$$
Moreover, they define linear operators of \( h \) that satisfy the estimates

\[
|\phi(y,t)| \lessapprox \mu_0^\nu(t) \left[ \frac{R^{4-\sigma}}{1+|y|^4} \| h^0 \|_{v.2+\sigma} + \frac{R^{4-\sigma}}{1+|y|^5} \| h^1 \|_{v.2+\sigma} + \| h \|_{v.2+\sigma} \right],
\]

\[
|\nabla_y \phi(y,t)| \lessapprox \mu_0^\nu(t) \left[ \frac{R^{4-\sigma}}{1+|y|^4} \| h^0 \|_{v.2+\sigma} + \frac{R^{4-\sigma}}{1+|y|^5} \| h^1 \|_{v.2+\sigma} + \| h \|_{v.2+\sigma} \right],
\]

\[
|e_0[h]| \lessapprox \| h \|_{v.2+\sigma}.
\]

**Remark 3.1.1.**

1. Since the blow-up profile constructed in this section is radially symmetric, we only consider the case of mode 0 in the Fourier decomposition. The construction of nonradial solutions can be carried out in a similar manner as in [66, Section 10] with nonradial perturbation.

2. Using the method of supersolution as in [55, Lemma 7.3], one can improve the linear theory at mode 0 in the interior region:

\[
|\phi^0(y,t)| + (1 + |y|)|\nabla \phi^0(y,t)| \\
\lessapprox \mu_0^\nu(t) R^{4-\sigma} \| h^0 \|_{v.2+\sigma} \min \left\{ 1, (1 + |y|)^{-2} R^{2+\sigma/3} \right\}.
\]

If we define

\[
\| \phi \|_{0,v.\sigma} := \sup_{(y,t) \in B_2 R \times (0,T)} \frac{1 + |y|}{\mu_0^\nu(t) R^{4-\sigma}(t)} [ |\phi(y,t)| + (1 + |y|)|\nabla \phi(y,t)| ],
\]

then under the assumptions of Proposition 3.1.1 we have

\[
\| \phi^0 \|_{0,v.\sigma} \lessapprox \| h^0 \|_{v.2+\sigma}.
\]

3. Under the assumptions in Proposition 3.1.1, if the right hand side \( h(y,t) \) further satisfies \( h(y,t) \in C^{2k-2+2\varepsilon,k-1+\varepsilon}_{y,t}(B_4 R \times (0,T)) \) for some \( 0 < \varepsilon < 1 \), then

\[
\| \phi \|_{C^{2k(1-\varepsilon)}_{y,t}(B_2 R \times (0,T))} \lessapprox T^\varepsilon.
\]

This is a consequence of scaling argument and the parabolic Schauder estimate.
Linear theory for the outer problem

In this section, we develop the linear theory for the outer problem. The model problem is

\[
\begin{aligned}
\psi_t &= \Delta \psi + f \quad \text{in} \quad \mathbb{R}^3 \times (0, T), \\
\psi(\cdot, 0) &= \psi_0 \quad \text{in} \quad \mathbb{R}^3,
\end{aligned}
\]

where the non-homogeneous term \( f \) in (3.28) is assumed to be bounded with respect to the weights appearing in the outer problem (3.22). Define the weights

\[
\begin{aligned}
\rho_1 &= \mu_0^{\nu - \frac{2}{3}}(t) R^{-2 - a}(t) \chi_{\{|x| \leq 2\mu_0 R\}}, \\
\rho_2 &= \frac{\mu_0^{\nu_2}}{R^2} \chi_{\{|x| \geq \mu_0 R\}}, \\
\rho_3 &= 1,
\end{aligned}
\]

where \( \nu > 0, \ 0 < a < 1, \ 1 \leq a_2 \leq 2, \ \nu_2 > 0, \) and we choose \( R(t) = \mu_0^{-\beta}(t) \) for \( \beta \in (0, 1/2) \) throughout this section. We define the norms

\[
\| f \|_{**} := \sup_{(x,t) \in \mathbb{R}^3 \times (0,T)} \left( \sum_{i=1}^{3} \rho_i(x,t) \right)^{\frac{-1}{2}} |f(x,t)|,
\]

\[
\| \psi \|_* := \mu_0^{\frac{1}{2} - \nu}(0) R^{\alpha}(0) \| \psi \|_{L^\infty(\mathbb{R}^3 \times (0,T))} + \mu_0^{\frac{3}{2} - \nu}(0) R^{1+\alpha}(0) \| \nabla \psi \|_{L^\infty(\mathbb{R}^3 \times (0,T))}
\]

\[
+ \sup_{(x,t) \in \mathbb{R}^3 \times (0,T)} \left[ \mu_0^{\frac{1}{2} - \nu}(t) R^{\alpha}(t) |\psi(x,t) - \psi(x,T)| \right]
\]

\[
+ \sup_{(x,t) \in \mathbb{R}^3 \times (0,T)} \left[ \mu_0^{\frac{3}{2} - \nu}(t) R^{1+\alpha}(t) |\nabla \psi(x,t) - \nabla \psi(x,T)| \right]
\]

\[
+ \sup_{\mathbb{R}^3 \times IT} \left[ \mu_0^{2\gamma + \frac{1}{2} - \nu}(t_2) R^{2\gamma + \alpha}(t_2) \right] \frac{1}{(t_2 - t_1)^{\gamma}} |\psi(x,t_2) - \psi(x,t_1)|,
\]

where \( \nu > 0, \ 0 < a, \ \gamma < 1, \) and the last supremum is taken over

\[
\mathbb{R}^3 \times IT = \left\{(x,t_1,t_2) : x \in \mathbb{R}^3, \ 0 \leq t_1 \leq t_2 \leq T, \ t_2 - t_1 \leq \frac{1}{10}(T-t_2) \right\}.
\]

For problem (3.28), we have the following estimates.
Proposition 3.1.2. Let $\psi$ be the solution to problem (3.28) with $\|f\|_{**} < +\infty$. Then it holds that

$$
\|\psi\|_* \lesssim \|f\|_{**}.
$$

Proposition 3.1.2 is established by using parabolic regularity theory and the Duhamel’s formula similarly as in [55].

Remark 3.1.2. Under the assumptions of Proposition 3.1.2, if we further assume that $f(x,t) \in C^{k-2+\epsilon}_{x,t}(\mathbb{R}^3 \times (0,T))$, then $\|\psi\|_{C^{k-2, k-1+\epsilon}_{x,t}(B_{\mu_0}(0) \times (0,T))} \lesssim T^\epsilon$ for some $0 < \epsilon < 1$.

3.1.5 The reduced equation for $\alpha(t)$

From the linear theory for the inner problem (3.24) in Section 3.1.4, orthogonality condition is required to guarantee the existence of solutions with sufficient space-time decay. In this section, we will adjust the scaling parameter $\mu(t)$ by such orthogonality condition.

Recall that the slow decaying kernel for the linearized operator

$$
Z_0(y) = -[y \cdot \nabla w + \frac{w}{2}] = \frac{3^\frac{1}{2}}{2} \frac{|y|^2 - 1}{(1 + |y|^2)^{\frac{3}{2}}},
$$

and

$$
\mu(t) = \mu_0(t)(1 + \Lambda(t))^2, \quad \alpha(t) = (-3^{\frac{1}{2}})^{-\frac{1}{2}}(\mu_0(t)\Lambda(t))'^.'
$$

Since $\mu_0(t) \sim (T-t)^{2k}$, our aim is to look for

$$
\alpha(t) \sim (T-t)^{k-1}\Lambda(t), \quad k \in \mathbb{Z}_+,
$$

where $\Lambda(t) \to 0$ as $t \to T$.

From the linear theory in Section 3.1.4, the inner solution can be found in suitable topology if the following orthogonality condition is satisfied

$$
\int_{B_{2R}} \mathfrak{H}'(\phi, \psi, \alpha)Z_0(y)dy = 0 \text{ for all } t \in (0,T), \quad (3.32)
$$
where \( \mathcal{H}(\phi, \psi, \alpha) \) is defined in (3.25). Define

\[
\|h\|_\delta := \sup_{t \in (0, T)} |(T - t)^{-\delta} h(t)|. \tag{3.33}
\]

It turns out that the reduced problem (3.32) is a problem involving the following nonlocal operator

\[
\int_0^t \frac{\alpha(s)}{(t - s)^{1/2}} ds,
\]

which is the nonlocal feature inherited from the slow decaying kernel \( Z_0(y) \).

**A linear theory for the reduced equation**

Before we consider the reduced problem (3.32), we first develop a key linear theory for the following problem

\[
\int_0^t \frac{\alpha(s)}{(t - s)^{1/2}} ds = k \sum_{j=1}^k c_j \mathcal{B}^{(j)}(0, t) + h(t), \tag{3.34}
\]

where \( h(t) \in C^k_t(0, T) \), and for \( j = 1, \ldots, k \), \( c_j \) are constants and \( \mathcal{B}^{(j)} \) are smooth functions to be determined. We have the following lemma concerning the solvability of problem (3.34), which enables us to solve \( \alpha(t) \) with sufficiently fast decay in problem (3.32).

**Lemma 3.1.1.** For problem (3.34), if \( h(t) \in C^k_t(0, T) \), then there exist constants \( c_j \) and smooth functions \( \mathcal{B}^{(j)} \) such that problem (3.34) has a solution satisfying

\[
\alpha(t) \sim (T - t)^{k-1} \Lambda(t), \quad k \in \mathbb{Z}_+,
\]

where \( \Lambda(t) \to 0 \) as \( t \to T \).

**Proof.** In order to find \( \alpha(t) \) with the above vanishing order, it suffices to show that

\[
\alpha(T) = \alpha'(T) = \alpha''(T) = \cdots = \alpha^{(k-1)}(T) = 0.
\]

We shall choose \( c_j \) and \( \mathcal{B}^{(j)} \) using solutions for heat equations as building blocks.
We first find solutions \( B_j(x,t) \) to the following heat equation

\[
\begin{aligned}
\dot{B}_j &= \Delta B_j \quad \text{in} \quad \mathbb{R}^3 \times (0,T), \\
B_j(x,0) &= B_{0,j} \quad \text{in} \quad \mathbb{R}^3,
\end{aligned}
\] (3.35)

where the decaying initial data \( B_{0,j} \) will be chosen. Using Duhamel’s formula in problem (3.35), we write

\[
B_j(0,t) = \int_{\mathbb{R}^3} e^{-\frac{\xi^2}{4t}} \frac{B_{0,j}(\xi)}{(4\pi t)^{3/2}} \, d\xi.
\]

Let us choose the initial condition \( B_{0,j}(|x|) = e^{-\kappa_j |x|^2} \). Then

\[
B_j(0,t) = \frac{1}{(4\pi)^{3/2}} \int_{\mathbb{R}^3} t^{-3/2} e^{-\frac{\xi^2}{4t}} e^{-\kappa_j |\xi|^2} \, d\xi = \frac{1}{8} \left( \kappa_j t + \frac{1}{4} \right)^{-3/2}
\]

and

\[
\begin{aligned}
\int_0^t B_j(0,s) \frac{ds}{(t-s)^{1/2}} &= 2 \int_0^{t^{1/2}} B_j(0,t-u^2) \, du \\
&= \frac{1}{4} \int_0^{t^{1/2}} \left( \kappa_j (t-u^2) + \frac{1}{4} \right)^{-3/2} \, du \\
&= \frac{2t^{1/2}}{4\kappa_j t + 1}.
\end{aligned}
\]

We consider the linear combination of the initial data

\[
\mathcal{B}^{(i)}_0(x) = \sum_{j=1}^k \ell_j^{(i)} B_{0,j}(|x|)
\]

so that the corresponding solution at \( x = 0 \) is

\[
\mathcal{B}^{(i)}(0,t) = \sum_{j=1}^k \ell_j^{(i)} \frac{1}{8} \left( \kappa_j t + \frac{1}{4} \right)^{-3/2}
\]
and

\[
\int_0^t \mathcal{B}^{(i)}(0,s) \frac{2t^{1/2}}{4kt + 1} \, ds = \sum_{j=1}^k \ell_j^{(i)} \frac{t^{1/2}}{4kj + 1}, \tag{3.36}
\]

Rearranging the constants \( \tilde{\ell}_j^{(i)} = \frac{\ell_j^{(i)}}{4kj} \) and \( \tilde{\kappa}_j = \frac{1}{4kj} \), we denote

\[
\Upsilon_j(t) = \frac{t^{1/2}}{t + \tilde{\kappa}_j}, \quad \Upsilon^{(i)}(t) = \sum_{j=1}^k \tilde{\ell}_j^{(i)} \frac{t^{1/2}}{t + \tilde{\kappa}_j}. \tag{3.37}
\]

By adjusting free parameters \( \tilde{\ell}_j^{(i)} \) and \( \tilde{\kappa}_j \), we can find solutions \( \Upsilon^{(i)}(t) \) with vanishing order \((T - t)^i\) near \( T \). Indeed, writing \( \tilde{\Upsilon}_j^{(i)}(t) = \sum_{j=1}^k \tilde{\ell}_j^{(i)} \frac{t^{1/2}}{t + \tilde{\kappa}_j} \), \( \Upsilon^{(i)}(t) \sim (T - t)^i \) is equivalent to showing the invertibility of the following system

\[
Mv = e_i, \tag{3.38}
\]

where

\[
M = \begin{pmatrix}
\frac{1}{T + \kappa_1} & \frac{1}{T + \kappa_2} & \cdots & \frac{1}{T + \kappa_k} \\
\frac{1}{(T + \kappa_1)^2} & \frac{1}{(T + \kappa_2)^2} & \cdots & \frac{1}{(T + \kappa_k)^2} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{(-1)^k}{(T + \kappa_1)^k} & \frac{(-1)^k}{(T + \kappa_2)^k} & \cdots & \frac{(-1)^k}{(T + \kappa_k)^k}
\end{pmatrix},
\]

\[
v = (\tilde{\ell}_1^{(i)}, \tilde{\ell}_2^{(i)}, \ldots, \tilde{\ell}_k^{(i)})^T, \quad e_i = (0, \cdots, 0, \frac{1}{i - \text{th entry}}, 0, \cdots, 0)^T.
\]

Letting \( \kappa_1, \ldots, \kappa_k \) be different from each other, it is obvious that system (3.38) is invertible. So we construct the solutions \( \Upsilon^{(i)}(t) \) with vanishing order \((T - t)^i\) near \( T \). Choosing a linear combination of \( k \) such functions and plugging them into the reduced equation, we obtain

\[
\int_0^t \alpha(s) \frac{2s^{1/2}}{(t-s)^{1/2}} \, ds = \sum_{j=1}^k c_j \mathcal{B}^{(j)}(0,t) + h(t),
\]

where \( c_j \) are free parameters in the initial data to be adjusted below. Then from
Lemma 3.42 we have

$$\int_0^t \alpha(s) ds = \frac{1}{(\Gamma(\frac{1}{2}))^2} \left( \sum_{j=1}^k c_j \int_0^t \mathcal{R}^{(j)}(0, s) ds + \int_0^t h(s) \right) \frac{1}{(t-s)^{1/2}} ds + \int_0^t h(s) \frac{1}{(t-s)^{1/2}} ds \right).$$  (3.39)

We write

$$\int_0^t h(s) \frac{1}{(t-s)^{1/2}} ds = d_0(T) + \sum_{j=1}^k d_j(T-t)^j + \text{h.o.t.}$$

We can choose $c_j$ such that problem (3.39) has solution $\alpha(t) = o((T-t)^{k-1})$. In other words, our aim is to show

$$\alpha(T) = \alpha'(T) = \alpha''(T) = \cdots = \alpha^{(k-1)}(T) = 0.$$  (3.40)

From (3.39), (3.36) and (3.37), we obtain

$$\int_0^t \alpha(s) ds = \frac{1}{(\Gamma(\frac{1}{2}))^2} \left( \sum_{j=1}^k c_j \mathcal{Y}^{(j)}(t) + \sum_{j=1}^k d_j(T-t)^j \right) + \text{h.o.t.}$$

By the vanishing order of $\mathcal{Y}^{(j)}(t)$ and choosing $c_j = d_j$, we conclude the validity of (3.40). The proof is complete. \hfill \square

**Reduced equation for $\alpha(t)$**

By the orthogonality condition (3.32), we directly compute

$$\int_{B_{2R}} 5 \left( u_{\mu_0}(\mu_0 y, t(\tau)) + \Phi_1(\mu_0 y, t(\tau)) \right)^4 - \mu_0^{-2} w^4(y) \mu_0^2 \Phi(y, \tau) Z_0(y) dy$$

$$+ \int_{B_{2R}} 5 \mu_0^4 (1 + \Lambda)^{-4} w^4 \left( \frac{y}{1 + \Lambda} \right) \psi(\mu_0 y, t(\tau)) Z_0(y) dy$$

$$+ \int_{B_{2R}} \mu_0 \mu_0' \left[ \frac{1}{2} \phi(y, \tau) + \nabla \phi(y, \tau) \cdot y \right] Z_0(y) dy$$

$$+ \int_{B_{2R}} \mu_0^2 \left[ S(U_1) - \chi \left( \frac{\mu_0 y}{c_0(T-t)^2} \right) \frac{\alpha(t)}{(\mu^2 + |\mu_0 y|^2)^{1/2}} \right] Z_0(y) dy$$

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\[ + \int_{B_{2R}} \mu_0^2 \left[ (u_{in}(\mu_0y, t(\tau)) + \Phi_1(\mu_0y, t(\tau)))^5 - u_{in}^5(\mu_0y, t(\tau)) \right] Z_0(y) \, dy = 0. \]

Since

\[ u_{in}(x, t) = \mu^{-\frac{1}{2}}w(x) + 2\mu_0^2 \mu^2 J(x) = \mu^{-\frac{1}{2}}w(y) + 2\mu_0^2 \mu^2 J(y), \]

we obtain

\[
\begin{align*}
\int_{B_{2R}} 5 \left[ (\mu^{-\frac{1}{2}}w(y) + 2\mu_0^2 \mu^2 J(y) + \Phi_1(\mu_0y, t(\tau)))^4 \right.
&\left. - \mu_0^{-2}w^4(y) \right] \mu_0^2 \phi(y, \tau) Z_0(y) \, dy \\
&+ \int_{B_{2R}} 5 \mu_0^2 (1 + \Lambda)^{-4} \Phi_1(0, t(\tau)) Z_0(y) \, dy \\
&+ \int_{B_{2R}} \mu_0 \mu_0^2 \left[ \frac{1}{2} \phi(y, \tau) + \nabla_y \phi(y, \tau) \cdot y \right] Z_0(y) \, dy \\
&+ \int_{B_{2R}} \mu_0^2 \left[ S(U_1) - \chi \left( \frac{\mu_0y}{c_0(T - t)^{\frac{1}{2}}} \right) \frac{\alpha(t)}{(\mu^2 + |\mu_0y|^2)^{\frac{1}{2}}} \right] Z_0(y) \, dy \\
&+ \int_{B_{2R}} \mu_0^2 \left[ (u_{in} + \Phi_1(\mu_0y, t(\tau)))^5 - u_{in}^5 \right] Z_0(y) \, dy = 0,
\end{align*}
\]

where we expand

\[
\begin{align*}
(u_{in} + \Phi_1(\mu_0y, t(\tau)))^5 - u_{in}^5
&= 5(\mu_0^{-\frac{1}{2}}w(y))^4 \Phi_1(0, t(\tau)) + 5(\mu_0^{-\frac{1}{2}}w(y))^4 (\Phi_1(\mu_0y, t(\tau)) - \Phi_1(0, t(\tau))) \\
&\quad + 5 \left[ u_{in}^4 - (\mu_0^{-\frac{1}{2}}w(y))^4 \right] \Phi_1(\mu_0y, t(\tau)) \\
&\quad + \Phi_1^2(\mu_0y, t(\tau)) \int_0^1 \Phi_1(\mu_0y, t(\tau)) \, ds
\end{align*}
\]
and we shall prove the leading term is $5(\mu_0^{-\frac{5}{2}}w(y))^4\Phi_1(0, t(\tau))$, and all other terms have sufficiently fast decay. Indeed, we simplify the above equation and evaluate

\[
\int_{B_{2R}} \left[ (\mu^{-\frac{1}{2}}w(y) + 2\mu_0^2 \mu^2 J(y) + \Phi_1(\mu_0 y, t(\tau))^4 \right.
\]

\[-\mu_0^{-2}w(y) \frac{3}{2} \Phi(y, \tau)Z_0(y) dy
\]

\[+ \int_{B_{2R}} 5w^4(y)\psi(\mu_0 y, t(\tau))Z_0(y) dy + \int_{B_{2R}} 5 \left[ (1 + \Lambda)^{-4}w^4(\frac{y}{(1 + \Lambda)^2}) - w^4(y) \right]
\]

\[\times \psi(\mu_0 y, t(\tau))Z_0(y) dy
\]

\[+ \int_{B_{2R}} \frac{1}{2} \left[ \frac{1}{2} \phi(y, \tau) + \nabla \phi(y, \tau) \cdot \mathbf{y} \right] Z_0(y) dy
\]

\[+ \int_{B_{2R}} \mu_0^2 \left[ S(U_1) - \chi(\frac{\mu_0 y}{c_0(T - t)^\frac{1}{2}}) \left( \frac{\alpha(t)}{(\mu^2 + |\mu_0 y|^2)^\frac{1}{2}} \right) \right] Z_0(y) dy
\]

\[+ \int_{B_{2R}} 5w^4(y)\Phi_1(0, t(\tau))Z_0(y) dy
\]

\[+ \int_{B_{2R}} \mu_0^2 \left[ (\mu_0 y + \Phi_1(\mu_0 y, t(\tau)))^5 - \mu_0^3 - 5(\mu_0^{-\frac{1}{2}}w(y))^4\Phi_1(0, t(\tau)) \right] Z_0(y) dy = 0.
\]

Recall that $Z_0(y) = \frac{\mu_0}{R} \left( \frac{1}{2} - \frac{|\mathbf{y}|^2 - 1}{(1 + |\mathbf{y}|^2)^\frac{3}{2}} \right)$, $w(y) = 3^{\frac{1}{2}}(1 + |\mathbf{y}|^2)^{-\frac{1}{2}}$. So

\[\int_{B_{2R}} 5w^4(y)Z_0(y) dy = 10\pi 3^{\frac{3}{2}}(\frac{1}{15} + O(R^{-2})).
\]

Next we consider the nonlocal term

\[\Phi_1(0, t)
\]

\[= \sum_{j=1}^{k} c_{j} \mathcal{Z}^{(j)}(0, t) + \int_{0}^{t} \int_{\mathbb{R}^3} \left( \frac{1}{2\sqrt{\pi}} \right)^3 (t - s)^{-\frac{3}{2}} e^{-\frac{|\xi|^2}{4(t - s)^2}} \chi(\frac{\xi}{c_0(T - s)^\frac{1}{2}}) \frac{\alpha(s)}{|\xi|} d\xi ds
\]

\[+ \int_{0}^{t} \int_{\mathbb{R}^3} \left( \frac{1}{2\sqrt{\pi}} \right)^3 (t - s)^{-\frac{1}{2}} e^{-\frac{|\xi|^2}{4(t - s)^2}} \chi(\frac{\xi}{c_0(T - s)^\frac{1}{2}}) \alpha(s) \left( \frac{1}{\mu(s) + |\xi|} - \frac{1}{|\xi|} \right) d\xi ds.
\]
Notice
\[
\int_0^t \int_{\mathbb{R}^3} \left( \frac{1}{2\sqrt{\pi}} \right)^3 (t - s)^{-\frac{3}{2}} e^{-\frac{|\xi|^2}{4(T-s)^2}} \chi \left( \frac{\xi}{c_0(T-s)^\frac{1}{2}} \right) \frac{\alpha(s)}{|\xi|} d\xi ds
= \pi^{-\frac{1}{2}} \int_0^t (t - s)^{-\frac{1}{2}} \alpha(s) \left[ 1 - e^{-\frac{c_0^2(t-s)}{4(T-s)^2}} \right] ds.
\]

Then
\[
\Phi_1(0,t) = \sum_{j=1}^k c_j \Phi_j(0,t) + \pi^{-\frac{1}{2}} \int_0^t (t - s)^{-\frac{1}{2}} \alpha(s) \left[ 1 - e^{-\frac{c_0^2(t-s)}{4(T-s)^2}} \right] ds
+ \int_0^t \int_{\mathbb{R}^3} \left( \frac{1}{2\sqrt{\pi}} \right)^3 (t - s)^{-\frac{3}{2}} e^{-\frac{|\xi|^2}{4(T-s)^2}} \chi \left( \frac{\xi}{c_0(T-s)^\frac{1}{2}} \right) \alpha(s) \left( \frac{1}{\mu(s) + |\xi|} - \frac{1}{|\xi|} \right) d\xi ds
\]
and thus
\[
\int_{B_{2R}} 5 \left[ \left( \mu^{-\frac{1}{2}} w \left( \frac{y}{(1+\Lambda)^2} \right) + 2\mu_0 \mu^2 J \left( \frac{y}{(1+\Lambda)^2} \right) + \Phi_1(\mu_0 y, t(\tau)) \right)^4 - \mu_0^{-2} w^4(y) \right] d\mu_0^{\frac{3}{2}} \Phi(y, \tau) Z_0(y) dy
+ \int_{B_{2R}} 5 w^4(y) \psi(\mu_0 y, t(\tau)) Z_0(y) dy + \int_{B_{2R}} 5 \left[ (1+\Lambda)^{-4} w^4 \left( \frac{y}{(1+\Lambda)^2} \right) - w^4(y) \right]
\times \psi(\mu_0 y, t(\tau)) Z_0(y) dy
+ \int_{B_{2R}} \mu_0^{\frac{1}{2}} \mu_0^4 \left[ \frac{1}{2} \phi(y, \tau) + \nabla_y \phi(y, \tau) \cdot y \right] Z_0(y) dy
+ \int_{B_{2R}} \mu_0^3 \left[ S(U_1) - \chi \left( \frac{\mu_0 y}{c_0(T-t)^{\frac{1}{2}}} \right) \frac{\alpha(t)}{(\mu^2 + |\mu_0 y|^2)^\frac{1}{2}} \right] Z_0(y) dy
+ 10\pi^{-\frac{3}{2}} \left( \frac{1}{15} + O(R^{-2}) \right) \left( \sum_{j=1}^k c_j \Phi_j(0,t) \right)
+ 10\pi^{-\frac{3}{2}} \left( \frac{1}{15} + O(R^{-2}) \right) \pi^{-\frac{1}{2}} \int_0^t (t - s)^{-\frac{1}{2}} \alpha(s) \left[ 1 - e^{-\frac{c_0^2(t-s)}{4(T-s)^2}} \right] ds
\]
\[ + 10\pi^3 \left( \frac{1}{15} + O(R^{-2}) \right) \int_0^t \int_{\mathbb{R}^3} \left( \frac{1}{2\sqrt{\pi}} \right)^3 (t-s)^{-\frac{1}{2}} e^{-\frac{|\xi|^2}{4(t-s)}} \chi \left( \frac{\xi}{c_0(T-s)^{\frac{1}{2}}} \right) \]
\[
\times \left( \frac{1}{\mu(s) + |\xi|} - \frac{1}{|\xi|} \right) d\xi \, ds
\]
\[ + \int_{B_{2R}} \mu_0^2 \left[ (u_{in} + \Phi_1(\mu_0y, t(\tau)))^5 - u_{in}^5 - 5(\mu_0 \frac{1}{2} w(y))^4 \Phi_1(0, t(\tau)) \right] Z_0(y) \, dy = 0. \]

The ansatz for the parameter function is
\[ \alpha(t) := (-3^\frac{1}{2} \mu_0 \frac{1}{2} (\mu_0 \Lambda(t))') \to 0 \quad \text{as} \quad t \to T \]

which is possible provided
\[ \alpha(t) \sim (T-t)^{-k} (T-t)^{-2k} \Lambda(t)(T-t)^{-1} \sim (T-t)^{k-1} \Lambda(t). \]

Next we compute
\[
\int_0^t (t-s)^{-\frac{1}{2}} \alpha(s) \, ds
\]
\[ = \pi^\frac{1}{2} \sum_{j=1}^k c_j \mathcal{B}^{(j)}(0, t) + O(R^{-2}) \pi^\frac{1}{2} \sum_{j=1}^k c_j \mathcal{B}^{(j)}(0, t) + \int_0^t (t-s)^{-\frac{1}{2}} \alpha(s) e^{-\frac{\xi^2}{4(t-s)}} \, ds
\]
\[ - \int_{B_{2R}} \frac{5}{2\pi^\frac{1}{2} 3^\frac{1}{2}} \left[ (\mu_0 \frac{1}{2} w \left( \frac{y}{1+\Lambda} \right)^2 + 2\mu_0 \frac{1}{2} \frac{1}{2} J \left( \frac{y}{1+\Lambda} \right) + \Phi_1(\mu_0y, t(\tau)) \right]^4
\]
\[ - \mu_0 \frac{1}{2} w^4(y) \right) x \mu_0 \frac{1}{2} \Phi(\mu_0y, t(\tau)) Z_0(y) \, dy - \int_{B_{2R}} \frac{5}{2\pi^\frac{1}{2} 3^\frac{1}{2}} w^4(y) \psi(\mu_0y, t(\tau)) Z_0(y) \, dy
\]
\[ - \int_{B_{2R}} \frac{1}{2\pi^\frac{1}{2} 3^\frac{1}{2}} \mu_0 \frac{1}{2} \mu_0 \left[ \frac{1}{2} \Phi(\mu_0y, t(\tau)) + \nabla \Phi(\mu_0y, t(\tau)) \right] Z_0(y) \, dy
\]
\[ + \int_{B_{2R}} \frac{1}{2\pi^\frac{1}{2} 3^\frac{1}{2}} \mu_0 \left[ S(U_1) - \chi \left( \frac{\mu_0y}{c_0(T-t)^{\frac{1}{2}}} \right) \frac{\alpha(t)}{(\mu^2 + |\mu_0y|^2)^{\frac{1}{2}}} \right] Z_0(y) \, dy
\]
\[ - O(R^{-2}) \int_0^t (t-s)^{-\frac{1}{2}} \alpha(s) \left[ 1 - e^{-\frac{\xi^2}{4(t-s)}} \right] ds \]
\[-(1 + O(R^{-2}))\pi^{\frac{1}{2}} \int_0^t \int_{\mathbb{R}^3} \left( \frac{1}{2\sqrt{\pi}} \right)^3 (t - s)^{-\frac{3}{2}} e^{-\frac{\|\xi\|^2}{2(t - s)}} \chi\left(\frac{\xi}{c_0(T - s)^{\frac{1}{2}}}\right) \times \alpha(s) \left( \frac{1}{\mu(s) + |\xi|} - \frac{1}{|\xi|} \right) d\xi ds \]

\[-\int_{B_{2R}} \frac{\mu_0^\alpha}{2\pi^{\frac{3}{2}}} \left[ (u_m + \Phi_1(\mu_0^2, t(\tau)))^s - u_m^s - 5(\mu_0^2)w(y) \Phi_1(0, t(\tau)) \right] Z_0(y) dy.

From the theory on Riemann-Liouville fractional differential operator (see [74] for instance), we have

\[
\frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_0^t (t - y)^{-\frac{1}{2}} \frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_0^y (y - s)^{-\frac{1}{2}} \alpha(s) ds dy
\]

\[
= \frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_0^t (t - y)^{-\frac{1}{2}} \frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_0^y \chi[s \leq y] (y - s)^{-\frac{1}{2}} \alpha(s) ds dy
\]

\[
= \frac{1}{(\Gamma\left(\frac{1}{2}\right))^2} \int_0^t \int_0^t (t - s)^{-1} \left( \frac{t - y}{t - s} \right)^{-\frac{1}{2}} \left( \frac{y - s}{t - s} \right)^{-\frac{1}{2}} \alpha(s) dy ds
\]

\[
= \int_0^t \alpha(s) ds.
\]

Therefore, changing the variable \((t - s)^{1/2} = u\), we obtain

\[
\int_0^t \alpha(s) ds = \frac{1}{\Gamma\left(\frac{1}{2}\right)} \sum_{j=1}^{k} c_j \int_0^t \mathcal{B}^{(j)}(0, a) \left( \frac{t - a}{t} \right)^{1/2} da + \sum_{\ell=1}^{\mathcal{J}_2} \mathcal{J}_\ell,
\]

where

\[
\mathcal{J}_1 = \frac{2}{\Gamma\left(\frac{1}{2}\right)} \sum_{j=1}^{k} c_j \int_0^t \mathcal{O}(R^{-2}(t - u^2)) \mathcal{B}^{(j)}(0, t - u^2) du,
\]

\[
\mathcal{J}_2 = \frac{2}{\Gamma^2\left(\frac{1}{2}\right)} \int_0^t (t - u^2)^{-1/2} \alpha(s) e^{-\frac{\|\xi\|^2}{4(t - u^2)}} ds du,
\]

\[
\mathcal{J}_3 = -\frac{5}{\pi^{\frac{3}{2}}} \int_0^t \int_{B_{2R}(t - u^2)} \left[ (\mu_0^2 (t - u^2)w(y) \left( \frac{y}{1 + \Lambda(t - u^2)^2} \right)^2 \right]
\]

\[
+ 2\mu_0(t - u^2)\mu_0^2 (t - u^2) \mathcal{J}(\frac{y}{1 + \Lambda(t - u^2)^2}) + \Phi_1(\mu_0(t - u^2) y, t - u^2)\right)^4.
\]
Now we check the differentiability of right hand sides $\mathcal{J}_\ell$ ($\ell = 1, \ldots, 10$) in (3.43).
term by term. First, we consider $J_1$. This term is smooth about $t$ near $T$ and we have the following estimate

$$\frac{\partial}{\partial t} \left( \int_0^{\frac{1}{2}} O(R^{-2}(t-u^2)) \mathcal{B}^{(j)}(0,t-u^2) \, du \right) \bigg|_{t=T} \sim O(T^{\frac{1}{2}-i}).$$

Next, for $J_2$ we have

$$\frac{\partial}{\partial t} \left( \int_0^{\frac{1}{2}} \int_0^{t-u^2} (t-u^2-s)^{-\frac{1}{2}} \alpha(s) e^{-\frac{\gamma(t-s)}{4(t-u^2-s)}} \, ds \, du \right)$$

$$= \int_0^{\frac{1}{2}} \int_0^{t-u^2} \frac{\partial}{\partial t} \left( (t-u^2-s)^{-\frac{1}{2}} \alpha(s) e^{-\frac{\gamma(t-s)}{4(t-u^2-s)}} \right) \, ds \, du.$$

Similarly, we can check that this term is smooth about $t$ near $T$ and have the following estimate

$$\frac{\partial}{\partial t} \left( \int_0^{\frac{1}{2}} \int_0^{t-u^2} (t-u^2-s)^{-\frac{1}{2}} \alpha(s) e^{-\frac{\gamma(t-s)}{4(t-u^2-s)}} \, ds \, du \right) \bigg|_{t=T} \sim o(T^{\frac{1}{2}-i}).$$

Then, the term $J_3$ is $C^k_t$ for $t$ near $T$ if we have $\phi \in C^{2k}_{x,t}$, and we have the following estimate

$$\frac{\partial}{\partial t} \left( \int_0^{\frac{1}{2}} \int_{B_{2R(t-u^2)}} \left[ (\mu^{-\frac{1}{2}}(t-u^2)w(y) \frac{y}{(1+\Lambda(t-u^2))^2}) \\
+ 2\mu_0(t-u^2)\mu^{-\frac{1}{2}}(t-u^2)J(y) \frac{y}{(1+\Lambda(t-u^2))^2}) \\
+ \Phi_1(\mu_0(t-u^2),y,t-u^2))^4 - \mu_0^{-2}(t-u^2)^4 w(y) \right] \\
\times \mu_0^2(t-u^2) \phi(y,t-u^2)Z_0(y) \, dy \, du \right|_{t=T} = o(T^{\frac{1}{2}-i}).$$

The term $J_4$ is $C^k_t$ for $t$ near $T$ if we have $\psi \in C^{2k}_{x,t}$. Further, we have the estimate

$$\frac{\partial}{\partial t} \left( \int_0^{\frac{1}{2}} \int_{B_{2R(t-u^2)}} w^4(y) \psi(\mu_0(t-u^2),y,t-u^2)Z_0(y) \, dy \, du \right|_{t=T} = o(T^{\frac{1}{2}-i}).$$

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Next, we consider $\mathcal{S}_5$, which is $C^k$ if we assume $\psi \in C^{2k,k}_{x,t}$, and

$$
\partial_t^{(i)} \int_0^1 \int_{B_{2r(t-u^2)}} \left[ (1 + \Lambda(t-u^2))^{-4} w^4 \left( \frac{y}{(1 + \Lambda(t-u^2))^2} \right) - w^4(y) \right] dy \, dx \times \psi(y) \mu_{0}(y) \bigg|_{t=T} \sim O(T^{\frac{3}{2}-i}).
$$

(3.44)

Next we consider $\mathcal{S}_6$. It is $C^k$ about $t$ near $T$ if we have $\phi \in C^{2k,k}_{x,t}$ and we have the following estimate

$$
\partial_t^{(i)} \int_0^1 \int_{B_{2r(t-u^2)}} \left[ \phi(y,t-u^2) \mu_{0}(y) \right] \left( \frac{1}{2} \phi(y,t-u^2) \right)
$$

$$
+ \nabla \phi(y,t-u^2) \cdot \frac{1}{2} \phi(y,t-u^2) \bigg|_{t=T} \sim o(T^{\frac{3}{2}-i}).
$$

Next, for the term $\mathcal{S}_7$, it is $C^k$ if we assume $\Lambda \in C^{k+1}_t$, and we have

$$
\partial_t^{(i)} \int_0^1 \int_{B_{2r(t-u^2)}} \left[ S(U_1)(\mu_{0}(y) \mu_{0}(t-u^2)) - \chi \left( \frac{\mu_{0}(t-u^2)y}{\mu(t-u^2)} \right) \right]
$$

$$
\times \left[ \frac{\alpha(t-u^2)}{(\mu^2(t-u^2) + |\mu_{0}(t-u^2)y|^2)^2} \right] \bigg|_{t=T} \sim o(T^{\frac{3}{2}-i}).
$$

We next consider $\mathcal{S}_8$. It is smooth for $t$ near $T$ and we have the following estimate

$$
\partial_t^{(i)} \int_0^1 \left[ O(R^{-2}(t-u^2)) \right] \int_0^T \left[ \alpha(s) \right] \left[ 1 - e^{-\frac{\alpha(t-u^2)}{4(t-u^2)}} \right] ds \, dx \bigg|_{t=T} \sim o(T^{\frac{3}{2}-i}).
$$

Next, for $\mathcal{S}_9$, it is $C^k$ about $t$ near $T$ if we assume $\Lambda \in C^{k+1}_t$ and the following estimate holds

$$
\partial_t^{(i)} \int_0^1 \left[ O(R^{-2}(t-u^2)) \right] \int_0^T \left( \frac{1}{2\sqrt{\pi}} \right)^3 (t-u^2-s)^{-\frac{3}{2}} e^{-\frac{\alpha(t-u^2)}{4(t-u^2-s)}}
$$

$$
\times \chi \left( \frac{\xi}{\mu(t-u^2)} \right) \left[ \frac{1}{\mu(s)} \right] \left[ \frac{1}{|\xi|} \right] \left[ \frac{1}{|\xi|} \right] \, d\xi \, ds \, dx \bigg|_{t=T} \sim o(T^{\frac{3}{2}-i}).
$$
Finally, for \( \mathcal{S}_{10} \), it is \( C^k \) about \( t \) near \( T \) if \( \Lambda \in C^k_t \), and

\[
\partial_t^{(i)} \int_0^{\frac{T}{2}} \int_{B_{2R_0}(0^2)} \mu_0^2(t-u^2) \times \left[ (u_{in}(y,t-u^2) + \Phi_1(\mu(t-u^2)y,t-u^2)) \right. \\
- u_{in}^5(y,t-u^2) - 5(\mu_0^{\frac{1}{2}}(t-u^2)w(y))^4 \Phi_1(0,t-u^2) \left. \right] Z_0(y) \, dy \, du \bigg|_{t=T} \sim o(T^{\frac{1}{2}-i}).
\]

Then, we have the following equation

\[
\alpha(t) = \partial_t \left[ \frac{1}{\Gamma(\frac{1}{2})} \sum_{j=1}^{k} c_j \int_0^t \mathcal{B}^{(j)}(0,a) \frac{1}{(t-a)^{\frac{1}{2}}} \, da + \frac{2}{\Gamma^2(\frac{1}{2})} \int_0^{\frac{T}{2}} h[c, \Lambda, \psi, \phi](t-u^2) \, du \right]
\]

(3.46)

and thus

\[
\Lambda(t) = \mu_0^{-1}(t) \int_t^T 3^{\frac{1}{2}} \mu_0^{\frac{1}{2}}(b) \partial_b \left[ \frac{1}{\Gamma(\frac{1}{2})} \sum_{j=1}^{k} c_j \int_0^b (b-a)^{-\frac{1}{2}} \mathcal{B}^{(j)}(0,a) \, da \\
+ \frac{2}{\Gamma^2(\frac{1}{2})} \int_0^{b^2} h[c, \Lambda, \psi, \phi](b-u^2) \, du \right] \, db.
\]

(3.47)

Define the space

\[
\mathcal{X}_{\Lambda} := \left\{ \Lambda(t) : \| \Lambda(t)(T-t)^{-1+\epsilon} \|_{L^\infty} \leq C_{0\Lambda}, \| \Lambda^{(1)}(t)(T-t)^{\epsilon} \|_{L^\infty} \leq C_{1\Lambda}, \ldots, \right. \\
\left. \| \Lambda^{(k)}(t)(T-t)^{k-1+\epsilon} \|_{L^\infty} \leq C_{k\Lambda}, \Lambda(t) \in C^{k+1,p}(0,T-\delta), \text{ for all } \delta \in (0,T) \right\},
\]

(3.48)

where \( C_{0\Lambda}, C_{1\Lambda}, \ldots, C_{k\Lambda} \) are some fixed small constants and \( \epsilon \) is a small positive constant. Define

\[
\mathcal{X}_c := \left\{ \tilde{c} = (c_1, c_2, \ldots, c_k) : |c_j| \leq C_c T^{\frac{1}{2}-j-\epsilon}, \ j = 1, 2, \ldots, k \right\},
\]

(3.49)

where \( C_c \) is a fixed constant.

We aim to solve (3.47) for \( \Lambda \in \mathcal{X}_{\Lambda}, \tilde{c} \in \mathcal{X}_c \) by Schauder fixed point theorem. For all \( \Lambda_1 \in \mathcal{X}_{\Lambda}, \tilde{c}_1 \in \mathcal{X}_c \), we want to find the unique \( \tilde{c}_2 \) to get \( \Lambda_2 \in \mathcal{X}_{\Lambda} \) satisfying
the following equation

\[
\Lambda_2(t) = \mu_0^{-1}(t) \int_{t}^{T} 3^{-\frac{1}{2}} \mu_0^{\frac{1}{2}}(b) \partial_b \left[ \frac{1}{\Gamma\left(\frac{1}{2}\right)} \sum_{j=1}^{k} c_{2j} \int_{0}^{b} (b-a)^{-\frac{1}{2}} \mathcal{B}^{(j)}(0,a) da \right. \\
+ \frac{2}{\Gamma^2\left(\frac{1}{2}\right)} \int_{0}^{b^{\frac{1}{2}}} h[\bar{c}_1, \Lambda_1, \psi[\Lambda_1], \phi[\Lambda_1]](b-u^2) du \bigg] \, db. \quad (3.50)
\]

Since we expect \( \Lambda_2 \in \mathcal{X}_\Lambda \), we need to choose suitable \( c_{2j} \) to cancel the lower power of \( T - t \) on the right hand side

\[
c_{2j} = -\frac{2}{\Gamma\left(\frac{1}{2}\right)} j! b^{\frac{1}{2}} \left( \int_{0}^{b} h[\bar{c}_1, \Lambda_1, \psi[\Lambda_1], \phi[\Lambda_1]](b-u^2) du \right) \bigg|_{b=T} = O(T^{\frac{1}{2}-j}).
\]

The higher derivatives of \( h \) are well defined here since \( \Lambda \in \mathcal{X}_\Lambda \) and \( \alpha(t) \in C^k_x(0, T - \delta) \). We can use Schauder estimate to improve the regularity of \( \phi, \psi \) to \( C^{2k+2+2\rho,k+1+\rho}_{x,t} \) for \( t \in (0, T - \delta) \) for any small \( \delta \). So \( h \) also has higher derivatives up to order \( k \). Taking higher derivatives for (3.51) and choosing \( T \) small enough, we have \( \Lambda_2 \in \mathcal{X}_\Lambda \).

Next, we show the Lipschitz continuity of \( \Lambda_2 \). For any \( 0 \leq t_1 < t_2 \leq T - \delta \),

\[
|\Lambda_2(t_1) - \Lambda_2(t_2)|
= \left| \mu_0^{-1}(t_1) \int_{t_1}^{t_2} 3^{-\frac{1}{2}} \mu_0^{\frac{1}{2}}(b) \partial_b \left[ \frac{1}{\Gamma\left(\frac{1}{2}\right)} \sum_{j=1}^{k} c_{2j} \int_{0}^{b} (b-a)^{-\frac{1}{2}} \mathcal{B}^{(j)}(0,a) da \right. \\
+ \frac{2}{\Gamma^2\left(\frac{1}{2}\right)} \int_{0}^{b^{\frac{1}{2}}} h[\bar{c}_1, \Lambda_1, \psi[\Lambda_1], \phi[\Lambda_1]](b-u^2) du \bigg] \, db \\
+ \left. (\mu_0^{-1}(t_1) - \mu_0^{-1}(t_2)) \times \int_{t_2}^{T} 3^{-\frac{1}{2}} \mu_0^{\frac{1}{2}}(b) \partial_b \left[ \frac{1}{\Gamma\left(\frac{1}{2}\right)} \sum_{j=1}^{k} c_{2j} \int_{0}^{b} (b-a)^{-\frac{1}{2}} \mathcal{B}^{(j)}(0,a) da \right. \\
+ \frac{2}{\Gamma^2\left(\frac{1}{2}\right)} \int_{0}^{b^{\frac{1}{2}}} h[\bar{c}_1, \Lambda_1, \psi[\Lambda_1], \phi[\Lambda_1]](b-u^2) du \bigg] \, db \right| \\
\lesssim (T-t_1)^{-2k} \int_{t_1}^{t_2} (T-b)^{2k} \, db + (T - \theta t_1 - (1 - \theta)t_2)^{-2k-1} \\
\times (t_2 - t_1) \int_{t_2}^{T} (T-b)^{2k} \, db
\]

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\[ t_2 - t_1. \]  
\[ (3.51) \]

Similarly, we can take \( k \)-th derivative of (3.51) to prove \( \Lambda_2^{(k)}(t) \in \text{Lip}(0, T - \delta) \), \( \forall \delta \in (0, T) \).

We shall solve the full inner–outer gluing system together with the reduced problem (3.32) in Section 3.1.6.

3.1.6 Solving the inner–outer gluing system

In this section, we will solve the inner outer gluing system (3.22) and (3.24) by using the linear theories developed in Section 3.1.4 together with the fixed point argument. We first estimate the right hand sides of problems (3.22) and (3.24) under the topology chosen in Section 3.1.4.

The outer problem: estimate of \( \mathcal{G} \)

Recall (3.23). We write

\[ g_1 := -\partial_t \eta R \mu_0^{-\frac{1}{2}} \phi \left( \frac{x}{\mu_0}, t \right) + \Delta \eta R \mu_0^{-\frac{1}{2}} \phi \left( \frac{x}{\mu_0}, t \right) + 2 \nabla \eta R \cdot \mu_0^{-\frac{3}{2}} \nabla y \phi \left( \frac{x}{\mu_0}, t \right), \]

\[ g_2 := \left( 5(U_1 + \Phi_1)^4(1 - \eta_R) + 5 \left[ (U_1 + \Phi_1)^4 - \left( \mu_0^{-\frac{2}{5}} \phi \left( \frac{x}{\mu_0}, t \right) \right) \right] \eta_R \right) \psi, \]

\[ g_3 := (U_1 + \Phi_1 + \Phi_2)^5 - (U_1 + \Phi_1)^5 - 5(U_1 + \Phi_1)^4 \Phi_2, \]

\[ g_4 := \left[ S(U_1) - \chi \left( \frac{x}{c_0(T - t)^\frac{1}{2}} \left( \mu_0^2 + |x|^2 \right)^\frac{1}{2} \right) \frac{\alpha(t)}{(1 - \eta_R)} ight] (1 - \eta_R) + [(U_1 + \Phi_1)^5 - U_1^5] (1 - \eta_R). \]

We next estimate \( g_1, g_2, g_3, g_4 \). Since \( \| \phi \|_{0, \nu, \sigma} < +\infty \), we have

\[ |g_1| \lesssim \mu_0^{-\frac{3}{2}}(t)R^{-1 - \sigma}(t)\chi_{\{|x| \sim \mu_0R\}} \]

and thus for some \( \varepsilon > 0 \)

\[ \|g_1\|_{**} \lesssim T^\varepsilon (1 + \| \phi \|_{0, \nu, \sigma}) \]  
\[ (3.52) \]
provided

\[ 1 + a - \sigma < 0. \]  

(3.53)

Since \( \| \psi \|_* < +\infty \) and we choose the initial data such that \( \psi(0,T) = 0 \), we have

\[ |\psi(x,t)| = |\psi(x,t) - \psi(0,T)| \]
\[ \leq |\psi(x,t) - \psi(x,T)| + |\psi(x,T) - \psi(0,T)| \]
\[ \lesssim \mu_0^{-\frac{1}{2}}(t)R^{-a}(t)\|\psi\|_*. \]

Then thanks to the cut-off \( \eta_R \), we have

\[ |g_2| \lesssim (1 - \eta_R) \frac{1}{1 + |y|^2} |\psi| \lesssim \frac{\mu_0^{v-a}(t)R^{a-4}(t)}{|x|^{a_2}} \|\psi\|_* \chi_{\{x| \geq \mu_0 R\}}. \]

So we have that for some \( \varepsilon > 0 \)

\[ \|g_2\|_* \lesssim T^\varepsilon \|\psi\|_* \]  

(3.54)

provided

\[ \nu - \nu_2 - \frac{1}{2} + a_2 - \beta(a_2 - a - 4) > 0. \]  

(3.55)

We evaluate

\[ |g_3| \lesssim |(U_1 + \Phi_1)^3 \Phi_2^2| \]
\[ \lesssim \mu_0^{2\nu - \frac{3}{2}}(t) R^{-2a}(t) \|\psi\|_*^2 \chi_{\{x| \leq \mu_0 R\}} + \mu_0^{2\nu - \frac{3}{2}}(t) R^{-2a}(t) \|\phi\|_{0,\nu,\sigma}^2 \chi_{\{\mu_0 R \leq x \leq \sqrt{T-t}\}} \]
\[ + \mu_0^{2\nu - \frac{3}{2} + a_2}(t) R^{a_2 - a - 3}(t) \frac{1}{|x|^{a_2}} \|\psi\|_*^2 \chi_{\{\mu_0 R \leq x \leq \sqrt{T-t}\}}. \]

Then for some \( \varepsilon > 0 \)

\[ \|g_3\|_* \lesssim T^\varepsilon (1 + \|\psi\|_*^2 + \|\phi\|_{0,\nu,\sigma}^2) \]  

(3.56)

provided

\[ \nu - \beta(2 - a) > 0, \quad \nu - \beta(\frac{14 - 2\sigma}{3} + a) > 0, \]
\[ 2\nu - \frac{5}{2} + a_2 - \beta(a_2 - 2a - 3) - \nu_2 > 0. \]  

(3.57)
To estimate $g_4$, we first estimate

$$|S(U_1) - \mathcal{K}(\frac{x}{c_0(T-t)^{1/2}} \frac{\alpha(t)}{(\mu^2 + |x|^2)^{1/2}})(1-\eta_R)|$$

$$\lesssim \left| \eta_1S_{in} - \mathcal{K}(\frac{x}{c_0(T-t)^{1/2}} \frac{\alpha(t)}{(\mu^2 + |x|^2)^{1/2}})(1-\eta_R) \right|$$

$$+ |(1-\eta_0)\eta_0(-\partial_t u_{out} + \Delta u_{out} + u_{out}^5)|$$

$$+ |\partial_t \eta_1 u_{in} + \Delta \eta_1 u_{in} + 2\nabla_x \eta_1 \nabla_x u_{in} - \partial_t [(1-\eta_0)\eta_0] u_{out}$$

$$+ \Delta_x [(1-\eta_0)\eta_0] u_{out} + 2\nabla_x [(1-\eta_0)\eta_0] \nabla_x u_{out}$$

$$+ |\eta_1 u_{in} + (1-\eta_0) \eta_0 u_{out}|^5 - \eta_1 u_{in}^5 - (1-\eta_0) \eta_0 u_{out}^5|.$$  \hspace{1cm} (3.58)

From (3.17), we have

$$\left| \frac{S_{in}}{S_{out}} - \frac{\alpha(t)}{\mu^2 + |x|^2} \right| (1-\eta_R) \lesssim \frac{\mu_0^{-\frac{1}{2} + a_2}}{|x|^{a_2}} R^{\frac{3}{2}} \mathcal{K}_{\{x \geq \mu_0 R\}} \left\| \Lambda \right\|_{\infty} + \frac{\mu_0^{-\frac{1}{2}}}{|x|} \mathcal{K}_{\{x \geq \mu_0 R\}} \left\| \Lambda \right\|_{\infty}$$

$$+ \frac{\mu_0^{-\frac{1}{2} + a_2 - 3}}{|x|^{a_2}} R^{\frac{3}{2}} \mathcal{K}_{\{x \geq \mu_0 R\}} + \frac{\mu_0^{-\frac{1}{2} + a_2 - 5}}{|x|^{a_2}} R^{\frac{3}{2}} \mathcal{K}_{\{x \geq \mu_0 R\}},$$  \hspace{1cm} (3.59)

where the function $h$ is defined in (3.16). Similarly, we have the following estimates for the rest terms. We evaluate the term

$$\left\| S_{out} \right\| = \left\| -\partial_t u_{out} + \Delta u_{out} + u_{out}^5 \right\| \lesssim (T-t)^{\frac{5}{2}(k-\xi_1)}.$$  \hspace{1cm} (3.60)

If we choose $\xi_1 = \xi_2 = \frac{1}{2}$, $r = r_1$ and $r_2 > 3r$, then we have

$$\left| -\partial_t \eta_1 u_{in} + \Delta \eta_1 u_{in} + 2\nabla_x \eta_1 \nabla_x u_{in} - \partial_t [(1-\eta_0)\eta_0] u_{out}$$

$$+ \Delta_x [(1-\eta_0)\eta_0] u_{out} + 2\nabla_x [(1-\eta_0)\eta_0] \nabla_x u_{out} \right|$$

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\( \lesssim (T-t)^{k-\frac{1}{2}}, \)  

(3.61)

where we have used the cancellation in the matching (3.11)-(3.12). Since \( \zeta_1 = \zeta_2 = \frac{1}{2}, \) we have

\[
\left| \eta_1 u_{in} + (1 - \eta_{o1}) \eta_{o2} u_{out} \right| - \eta_1 u_{in}^5 - (1 - \eta_{o1}) \eta_{o2} u_{out}^5 \lesssim \left| u_{out} \right|^5 \chi_{\{|x| \sim \sqrt{T-t}\}} \lesssim (T-t)^{5(k-\frac{1}{2})}.
\]

(3.62)

We choose initial condition of \( \Phi_1 \) such that \( \Phi_1(0,T) = 0. \) Then by Duhamel’s formula, we can show that

\[
\left| \Phi_1(x,t) \right| \lesssim \alpha(t) (T-t)^{3/2}.
\]

Therefore, we obtain

\[
\left| \left[ (U_1 + \Phi_1)^5 - U_1^5 \right] (1 - \eta_R) \right| \lesssim (1 - \eta_R) |U_1^5 \Phi_1| \lesssim (1 - \eta_R) \frac{\mu_0^{-2}(t)}{1+|y|^4} \alpha(t)(T-t)^{3/2}.
\]

(3.63)

Collecting estimates (3.58)-(3.63), we conclude that

\[
\| g_4 \| \lesssim T^\varepsilon (1 + \| \Lambda \|_\infty)
\]

(3.64)

provided

\[
a_2 - \frac{1}{2} - \frac{1}{2k} + \beta(3-a_2) - \nu_2 > 0, \quad \frac{1}{2} - \frac{1}{2k} - \nu_2 > 0,
\]

\[
\frac{1}{2} + a_2 - \frac{3}{4k} - \nu_2 > 0, \quad 5\left( \frac{1}{2} - \frac{1}{4k} \right) - \nu + \frac{5}{2} - \beta(2+a) > 0,
\]

\[
2 - \nu - \frac{1}{4k} - \beta(2+a) > 0, \quad a_2 + \frac{1}{4k} - \frac{3}{2} + \beta(4-a_2) - \nu_2 > 0.
\]

(3.65)

In conclusion, from (3.52), (3.53), (3.54), (3.55), (3.56), (3.57), (3.64) and (3.65), we obtain that for some \( \varepsilon > 0 \)

\[
\| \mathcal{G} \| \lesssim T^\varepsilon (1 + \| \Psi \|_\varepsilon + \| \Phi \|_{0,\nu,\sigma} + \| \Lambda \|_\infty)
\]

(3.66)

provided

\[
1 + a - \sigma < 0, \quad \nu - \nu_2 - \frac{1}{2} + a_2 - \beta(a_2-a-4) > 0,
\]
\[
\begin{align*}
\nu - \beta (2 - a) > 0, \\
2\nu - \frac{5}{2} + a_2 - \beta (a_2 - 2a - 3) - \nu_2 > 0, \\
\frac{1}{2} - \frac{1}{2k} - \nu_2 > 0, \\
5\left(\frac{1}{2} - \frac{1}{4k}\right) - \nu + \frac{5}{2} - \beta (2 + a) > 0, \\
a_2 + \frac{1}{4k} - \frac{3}{2} + \beta (4 - a_2) - \nu_2 > 0.
\end{align*}
\]

(3.67)

**The inner problem: estimate of \( H \)**

Recall (3.25). We evaluate

\[
\begin{align*}
5 \left[ (u_{in}(\mu_0y, t) + \Phi_1(\mu_0y, t))^4 - \mu_0^{-2} w^4(y) \right] \mu_0^2 \phi(y, t) \\
\lesssim \frac{\mu_0^2 \mu_0^3}{1 + |y|^2} \| \phi \|_{0, \nu, \sigma}, \\
\lesssim \frac{\mu_0^0 \mu_0^3}{1 + |y|^4} \| \psi \|_s + \frac{\mu_0 \mu_0^0 \mu_0^3 R^{\nu + \sigma}}{1 + |y|} \| \phi \|_{0, \nu, \sigma}.
\end{align*}
\]

(3.68)

(3.69)

By (3.17), we have

\[
\begin{align*}
\mu_0^2 \left[ S(U_1) - \chi \left( \frac{\mu_0 y}{C_0(T - t)} \right) \frac{\alpha(t)}{(\mu^2 + |\mu_0 y|^2)^2} \right] \\
\lesssim \frac{\mu_0 \mu_0^0 \| \Lambda \|_\infty}{1 + |y|^2} + \mu_0 \mu_0^0 \frac{\mu_0^3}{1 + |y|^4} + \frac{(\mu_0^0)^2 \mu_0^3}{1 + |y|^2}.
\end{align*}
\]

(3.70)

\[
\begin{align*}
\mu_0^2 \left[ (u_{in}(\mu_0 y, t) + \Phi_1(\mu_0 y, t))^5 - \mu_0^5 \alpha(t) \right] \\
\lesssim \frac{\mu_0^0 \mu_0^0 \| \Lambda \|_\infty}{1 + |y|^4}.
\end{align*}
\]

(3.71)

From estimates (3.68)–(3.71), we obtain that for some \( \varepsilon > 0 \)

\[
\| H \|_{v, 2 + \sigma} \lesssim T^\varepsilon \left( 1 + \| \phi \|_{0, \nu, \sigma} + \| \psi \|_s + \| \Lambda \|_\infty \right)
\]

(3.72)

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provided

\[
1 + \frac{1}{4k} - \frac{\beta(4 - \sigma)}{3} > 0, \quad 0 < \sigma < 2,
\]
\[
a > 0, \quad 2 - \frac{1}{2k} - \frac{\beta(2\sigma + 7)}{3} > 0,
\]
\[
2 - \frac{1}{2k} - \beta(\sigma - 1) > 0, \quad 4 - \frac{1}{k} - \beta(3 + \sigma) > 0,
\]
\[
1 + \frac{1}{4k} - \nu > 0. \quad (3.73)
\]

**The fixed point formulation**

The inner–outer gluing system (3.22) and (3.24) can be formulated as a fixed point problem for operators we shall describe below.

We first define the following function spaces

\[
X_{\phi} := \left\{ \phi \in L^\infty(B_{2R} \times (0, T)) \cap C_{y,t}^{2k+2\rho,2\rho}(B_{2R} \times (0, T - \delta)) : \| \phi \|_{0,\nu,\sigma} < +\infty \right\},
\]
\[
X_{\psi} := \left\{ \psi \in L^\infty(\mathbb{R}^3 \times (0, T)) \cap C_{x,t}^{2k+2\rho,2\rho}(\mathbb{R}^3 \times (0, T - \delta)) : \| \psi \|_* < +\infty \right\},
\]
\[
X_{\Lambda} := \left\{ \Lambda(t) : \| \Lambda(t)(T - t)^{-1+\epsilon}\|_{L^\infty} \leq C_{0\Lambda}, \| \Lambda^{(1)}(t)(T - t)^\epsilon\|_{L^\infty} \leq C_{1\Lambda}, \ldots, \right. \\
\left. \| \Lambda^{(k)}(t)(T - t)^{k-1+\epsilon}\|_{L^\infty} \leq C_{k\Lambda}, \Lambda(t) \in C^{k+1,\rho}(0, T - \delta) \text{ for all } \delta \in (0, T) \right\},
\]
\[
X_{\vec{c}} := \left\{ \vec{c} = (c_1, c_2, \ldots, c_k) : |c_j| \leq C_t T^{1-j-\epsilon}, j = 1, 2, \ldots, k \right\}. \quad (3.74)
\]

Define

\[
\mathcal{X} = X_{\phi} \times X_{\psi} \times X_{\Lambda} \times X_{\vec{c}}. \quad (3.75)
\]

We shall solve the inner–outer gluing system in a closed ball \( B \) in \( \phi, \psi, \Lambda, \vec{c} \) \( \in \mathcal{X} \).

The inner–outer gluing system (3.22) and (3.24) can be formulated as a fixed point problem, where we define an operator \( \mathcal{F} \) which returns the solution from \( B \) to \( \mathcal{X} \)

\[
\mathcal{F} : B \subset \mathcal{X} \rightarrow \mathcal{X}
\]

\[
v \mapsto \mathcal{F}(v) = (\mathcal{F}_\phi(v), \mathcal{F}_\psi(v), \mathcal{F}_\Lambda(v), \mathcal{F}_{\vec{c}}(v))
\]

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with
\[ F(\phi, \psi, \Lambda, \vec{c}) = T(\phi, \psi, \Lambda), \]
\[ F(\psi, \psi, \Lambda, \vec{c}) = T(\psi, \psi, \Lambda), \]
\[ F(\Lambda, \psi, \Lambda, \vec{c}) = T(\Lambda, \psi, \Lambda), \]
\[ F(\vec{c}, \phi, \psi, \Lambda, \vec{c}) = T(\vec{c}, \phi, \psi, \Lambda). \]

Here \( T \) is the operators given from Proposition 3.1.1 which solves the inner problem (3.24). The operator \( T \) defined by Proposition 3.1.2 deals with the outer problem (3.22). Operators \( T \) and \( T \) handle the reduced equation (3.32).

**Choice of constants**

In this section, we list all the constraints of the parameters which are sufficient for the inner–outer gluing scheme to work.

We first indicate all the parameters used in different norms.

- \( R(t) = \mu_0^{-\beta}(t) \) with \( \beta \in (0, 1/2) \).
- The norm for \( \phi \) solving the inner problem (3.24) is \( \| \cdot \|_{0, \nu, \sigma} \) which is defined in (3.27), where we require that \( \nu > 0, 0 < \sigma < 2 \).
- The norm for \( \psi \) solving the outer problem (3.22) is \( \| \cdot \|_* \) which is defined in (3.31), while the \( \| \cdot \|_{*,*} \)-norm for the right hand side of the outer problem (3.22) is defined in (3.30). Here we require that \( \nu, \nu_2 > 0 \) and \( a, \gamma \in (0, 1) \). Also, as mentioned in Remark 3.1.2, we require \( \nu + 2a_2/4k > \nu - 1/2 + a \beta \) such that the \( \| \cdot \|_* \)-norm is well defined.

In order to get the desired estimates for the outer problem (3.22), by (3.67), we need the following restrictions

\[ 1 + a - \sigma < 0, \quad v - \nu_2 - \frac{1}{2} + a_2 - \beta(a_2 - a - 4) > 0, \]
\[ v - \beta(2 - a) > 0, \quad v - \beta\left(\frac{14 - 2\sigma}{3} + a\right) > 0, \]
\[ 2v - \frac{5}{2} + a_2 - \beta(a_2 - 2a - 3) - \nu_2 > 0, \quad a_2 - \frac{1}{2} - \frac{1}{2k} + \beta(3 - a_2) - \nu_2 > 0, \]
\[ \frac{1}{2} - \frac{1}{2k} - \nu_2 > 0, \quad \frac{1}{2} + \frac{a_2 - 3}{4k} - \nu_2 > 0, \]
\[ 5\left(\frac{1}{2} - \frac{1}{4k}\right) - \nu + \frac{5}{2} - \beta(2 + a) > 0, \quad 2 - \nu - \frac{1}{4k} - \beta(2 + a) > 0, \]
\[ a_2 + \frac{1}{4k} - \frac{3}{2} + \beta(4 - a_2) - \nu_2 > 0. \]

In order to get the desired estimates for the inner problem (3.24), by (3.73), we need
\[ 1 + \frac{1}{4k} - \frac{\beta(4 - \sigma)}{3} > 0, \quad 0 < \sigma < 2, \]
\[ a > 0, \quad 2 - \frac{1}{2k} - \frac{\beta(2\sigma + 7)}{3} > 0, \]
\[ 2 - \frac{1}{2k} - \beta(\sigma - 1) > 0, \quad 4 - \frac{1}{k} - \beta(3 + \sigma) > 0, \]
\[ 1 + \frac{1}{4k} - \nu > 0. \]

Elementary computations show that suitable choices of the parameters satisfying all the restrictions in this section can be found, which ensures the implementation of the gluing procedure.

**Proof of Theorem 3.1.1**

Consider the operator
\[
\mathcal{F} = (\mathcal{F}_\phi, \mathcal{F}_\psi, \mathcal{F}_\Lambda, \mathcal{F}_\bar{c})
\]
given in (3.76). To prove Theorem 3.1.1, our strategy is to show the existence of a fixed point for the operator \( \mathcal{F} \) in \( B \) by the Schauder fixed point theorem. By collecting the estimates (3.66), (3.72), and using Proposition 3.1.2, Proposition 3.1.1 and discussions in Section 3.1.5 we conclude that for \( (\phi, \psi, \Lambda, \bar{c}) \in B \)
\[
\begin{align*}
||\mathcal{F}_\phi(\phi, \psi, \Lambda, \bar{c})||_{0, \nu, \sigma} & \leq CT^\varepsilon, \\
||\mathcal{F}_\psi(\phi, \psi, \Lambda, \bar{c})||_* & \leq CT^\varepsilon, \\
||\mathcal{F}_\Lambda(\phi, \psi, \Lambda, \bar{c})||_\Lambda & \leq CT^\varepsilon, \\
||\mathcal{F}_\bar{c}(\phi, \psi, \Lambda, \bar{c})||_{\bar{c}} & \leq CT^\varepsilon,
\end{align*}
\]
where \( C > 0 \) is a constant independent of \( T \), and \( \varepsilon > 0 \) is a small fixed number. On the other hand, compactness of the operator \( \mathcal{F} \) defined in (3.77) can be proved

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by proper variants of (3.78). Indeed, if we vary the parameters slightly such that all the restrictions in Section 3.1.6 are still satisfied, then we get (3.78) with the norms in the left hand side defined by the new parameters, while the closed ball $B$ remains the same. To be more specific, for fixed $\nu', a'$ which are close to $\nu, a$, one can show that if $(\phi, \psi, \Lambda) \in B$, then

$$\|F_\phi(\phi, \psi, \Lambda, \vec{c})\|_{0, \nu', \sigma'} \leq CT^\nu'.$$

Furthermore, one can show that for $\nu' > \nu$ and $\nu' - \frac{\beta(4-\sigma')}{3} > \nu - \frac{\beta(4-\sigma)}{3}$, one has a compact embedding in the sense that if a sequence $\{\phi_0^n\}$ is bounded in the $\|\cdot\|_{0, \nu, \sigma}$-norm, then there exists a subsequence which converges in the $\|\cdot\|_{0, \nu, \sigma}$-norm. Thus, the compactness follows directly from a standard diagonal argument by Arzelà–Ascoli’s theorem. Arguing in a similar manner, the compactness for the rest operators can be proved. Therefore, the existence of the desired blow-up solution is concluded from the Schauder fixed point theorem.

### 3.1.7 Nonradial case: blow-up at multiple points

As a by-product, the inner–outer gluing method carried out above can be applied to construct non-radial type II blow-up at multiple $N$ points for the first blow-up rate $k = 1$. To be more precise, we take the first approximation to be

$$U_N = \sum_{j=1}^{N} \mu_j^{-\frac{1}{2}}(t) w \left( \frac{x - \xi_j(t)}{\mu_j(t)} \right),$$

where we expect that the scaling and translation parameters satisfy

$$\mu_j(t) \to 0, \quad \xi_j(t) \to q_j \quad \text{as} \quad t \to T$$

for $j = 1, \ldots, N$, where $q_j$ are given points in $\mathbb{R}^3$ with $\max_{k,l=1, \ldots, N} |q_k - q_l| > \delta$ for uniform $\delta > 0$. Formally, the error of $U_N$ behaves like

$$S(U_N) \sim \sum_{j=1}^{N} \left( \mu_j^{-\frac{3}{2}}(t) \bar{\mu}_j(t) Z_0(y_j) + \mu_j^{-\frac{1}{2}}(t) \bar{\xi}_j \cdot \nabla w(y_j) \right) := \sum_{j=1}^{N} (\varepsilon_{0,j} + \varepsilon_{1,j}),$$

where
where $y_j = \frac{x - \xi_j(t)}{\mu_j(t)}$. To cancel out the slow decay error at mode 0 near each point $q_j$, we introduce the correction $\Phi_j$ solving
\[ \partial_t \Phi_j = \Delta \Phi_j + \varepsilon_{0,j} \quad \text{for } j = 1, \ldots, N \]
so that the corrected approximation is
\[ u^* = \sum_{j=1}^{N} \mu_j^{-\frac{1}{2}}(t)w\left(\frac{x - \xi_j(t)}{\mu_j(t)}\right) + \Phi_j. \]
We then look for the solution
\[ u = u^* + \sum_{j=1}^{N} \mu_j^{-\frac{1}{2}}(t)\eta_{R,j} \phi(y_j,t) + \psi(x,t). \]
Let us emphasize that in the non-radial case, the blow-up rate for $k = 1$ will be obtained by orthogonality condition instead of the matching process in the general case $k \geq 2$. The orthogonality condition at scaling mode 0 is basically
\[ \int_{B_2R} 5w^4 \Phi_j Z_0(y)dy + \int_{B_2R} 5w^4 \psi Z_0(y)dy \approx 0 \]
which turns out to be a nonlocal equation like before, and using the method in Section 3.1.5 we have
\[ \mu_j(t) \sim (T - t)^2 \quad \text{for } j = 1, \ldots, N. \]
Indeed, the orthogonality condition at mode 0 gives a nonlocal reduced equation of the following form:
\[ \int_0^t \mu_j^{-1/2}(s) \dot{\mu}_j(s) \frac{1}{(t-s)^{1/2}} ds = c_{*,j}, \]
where $c_{*,j} < 0$ is some constant coming from the initial data. We rewrite the above integro-differential equation as
\[ \int_0^t \frac{\dot{\psi}_j(s)}{(t-s)^{1/2}} ds = c_{*,j}, \]
where \( v_j(t) = 2\mu_j^{1/2}(t) \). Imposing \( v_j(T) = 0 \) and using (3.42), we obtain that for some \( c > 0 \)

\[
v_j(t) - v_j(T) = v_j(t) = c(T^{1/2} - t^{1/2}) \sim T - t
\]

and thus

\[
\mu_j(t) \sim (T - t)^2
\]

which is precisely the first rate \((k = 1)\) predicted in [80].

On the other hand, the orthogonality condition at translation mode 1

\[
\int_{B_{2R}} \xi_{1,j} Z_\ell(y) dy \approx 0
\]

simply implies \( \xi_j(t) \sim q_j \) for \( j = 1, \ldots, N \), where \( Z_\ell(y) = \partial_{y_\ell} w \), \( \ell = 1, 2, 3 \). We will not elaborate on the details.

### 3.2 Type II finite time blow-up for energy supercritical heat equation

#### 3.2.1 Introduction

In this section, we are concerned with the energy supercritical heat equation with the \((n - 3)\)-th Sobolev exponent \( p = 3 \)

\[
\begin{aligned}
  u_t &= \Delta u + u^3 & \text{in } \Omega \times (0, T), \\
  u(x, t) &= u|_{\partial\Omega} & \text{on } \partial\Omega \times (0, T), \\
  u(x, 0) &= u_0(x) & \text{in } \Omega,
\end{aligned}
\]

where \( 5 \leq n \leq 7 \), \( \Omega \) is either \( \mathbb{R}^n \) or a smooth, bounded domain in \( \mathbb{R}^n \) enjoying the symmetry that \( \Omega \) is invariant under the orthogonal transformations

- \( Q(x_1, \ldots, x_n) = (R(x_1, \ldots, x_{n-3}), x_{n-2}, x_{n-1}, x_n) \) with
  \[
  x = (x_1, \ldots, x_n) \in \Omega, \quad R \in SO(n - 3),
  \]
  where \( SO(n - 3) \) is the classical rotation group,
\begin{itemize}
  \item \(\pi_i(x_1, \ldots, x_i, \ldots, x_n) = (x_1, \ldots, -x_i, \ldots, x_n)\) with \(i = n-2, n-1, n\),
\end{itemize}

namely

\[
Q(\Omega) = \Omega, \quad \pi_i(\Omega) = \Omega \quad \text{for} \quad i = n-2, n-1, n.
\] (3.80)

In other words, \(\Omega\) is a radial domain in the first \(n-3\) coordinates and even in the remaining \(3\) coordinates.

We denote Aubin-Talenti bubbles by

\[
U_{\lambda, \xi}(x) = \lambda^{-\frac{n-2}{2}} U \left( \frac{x - \xi}{\lambda} \right) = \sigma_n \left( \frac{\lambda}{\lambda^2 + |x - \xi|^2} \right)^{\frac{n-2}{2}}, \quad \sigma_n = (n(n-2))^{\frac{n-2}{2}}
\] (3.81)

which solve the Yamabe problem

\[
\Delta U + U^\frac{n+2}{n-2} = 0 \quad \text{in} \quad \mathbb{R}^n.
\]

The aim of this section is to construct type II finite time blow-up solution which blows up along an \((n-4)\)-dimensional sphere with shrinking size \(\sqrt{T-t}\). To state the main result, we write \(x = (x^*, x^{**}) \in \Omega\) with \(x^* \in \mathbb{R}^{n-3}\) and \(x^{**} \in \mathbb{R}^3\), and denote \(|x^*| = r, x^{**} = z\). We look for solution \(u(x, t)\) with the same symmetry as \(\Omega\)'s

\[
u(x, t) = \tilde{u}(r, z, t)
\] (3.82)

for a function \(\tilde{u}\) defined in \(\mathcal{D} \times (0, T)\), where

\[
\mathcal{D} := \left\{ (r, z) : r = \sqrt{x_1^2 + \cdots + x_{n-3}^2}, z = (x_{n-2}, x_{n-1}, x_n), \ x = (x_1, \ldots, x_n) \in \Omega \right\}.
\] (3.83)

More precisely, we have

**Theorem 3.2.1.** Let \(\Omega\) be \(\mathbb{R}^n\) or a smooth, bounded domain in \(\mathbb{R}^n\) in the symmetry class (3.80) with \(5 \leq n \leq 7\). Then for \(T > 0\) sufficiently small, there exist initial and boundary conditions such that the solution \(u(x, t)\) to problem (3.79) blows up along a shrinking sphere, with the profile of the form

\[
u(x, t) = \lambda^{-1}(t) U \left( \frac{(r, z) - (\xi_r(t), \xi_z(t))}{\lambda(t)} \right) (1 + o(1)) \quad \text{as} \quad t \nearrow T,
\]
where $U$ is the standard bubble (3.81) in $\mathbb{R}^4$, and for some $\kappa > 0$, $z_0 \in \mathbb{R}^3$,

$$\lambda(t) = \kappa \left( \frac{T-t}{\log(T-t)} \right)^2 (1+o(1)) \quad \text{as} \quad t \nearrow T;$$

$$\xi(t) = (\xi_r(t), \xi_z(t)) = \left( \sqrt{2(n-4)(T-t)(1+o(1))}, z_0(1+o(1)) \right) \quad \text{as} \quad t \nearrow T.$$

Theorem 3.2.1 exhibits a completely new blow-up phenomenon where the type II blow-up takes place along a higher dimensional manifold with shrinking size. More precisely, from the characterization of $\xi(t) = (\xi_r(t), \xi_z(t))$, the blow-up position of the solution $u(x,t)$ we construct is a copy of $S^{n-4}$, and the $(n-4)$-dimensional sphere shrinks with self-similar size $\sqrt{T-t}$ and asymptotically collapses to a point $(0, z_0)$ in $\Omega$ as $t \nearrow T$. Roughly speaking, the shape of the solution we construct looks like a “thin tube” in the $(r,z)$-coordinate for $n = 5$. This type of concentration set with shrinking size was first conjectured to exist in [51] in the context of harmonic map heat flow, where the authors considered the case that the singularity set is a fixed circle. This intriguing question in the setting of harmonic map heat flow has recently been studied in [52]. Also, in the setting of energy supercritical heat equation (3.1) with $p = \frac{n+1}{n-3}$ (the second Sobolev exponent), type II finite time blow-up solution concentrating on a fixed circle was constructed in [62]. In another aspect, the higher dimensional blow-ups for parabolic problems can be regarded as the parallel with bubbling phenomena in the elliptic setting, see [59–61] for example. It is worth mentioning that the boundary bubbling driven by the geometry of the boundary in [60] was also conjectured to be true for parabolic problems in [62].

In fact, there are similar phenomena as shown in Theorem 3.2.1 in other literatures. In [184], a very similar “neck pinch” blow-up with self-similar size $\sqrt{T-t}$ was already found for the finite time singularity formation of the generalized Euler equations in dimension three (see [184, Theorem 1.11]). A similar relation $\xi_r \sim \frac{1}{\xi_z}$, which is the key to obtain novel dynamics of the shrinking concentration set in Theorem 3.2.1, seems to be also crucially used to produce such “neck pinch” blow-up in [184, Section 7]. In the context of the finite time blow-up for nonlinear Schrödinger equations (NLS), the shrinking (or collapsing) concentration sets have been found in [109, 151] (see also [79] for the numerical simulations).
Note that the solutions constructed in [109, 151] are essentially radially symmetric and the shrinking concentration sets there are not of self-similar size $\sqrt{T-t}$, while the solution constructed here in Theorem 3.2.1 is cylindrically symmetric and $\xi_r(t) \sim \sqrt{T-t}$. At the level of NLS with cylindrical symmetry, the standing ring (with fixed radius) blow-ups have been investigated in [108, 190] for example.

In the symmetry class (3.82), problem (3.79) is reduced to solving

$$
\begin{align*}
\ddot{u} &= \Delta_{(r,z)} \dot{u} + \frac{n-4}{r} \dot{u}_r + \dot{u}^3 \\
\dot{u}_r &= 0 \\
\ddot{u} &= \ddot{u} |_{\partial \Omega} \\
\ddot{u}(\cdot,0) &= \ddot{u}_0
\end{align*}
$$

(3.84)

where $\mathcal{D}$ is defined in (3.83) and $\Delta_{(r,z)} := \partial^2_r + \Delta_z$ is the Laplacian in $\mathbb{R}^4$. Note that $p = 3$ is the critical exponent in $\mathbb{R}^4$. So problem (3.84) can be viewed as the energy critical problem in $\mathbb{R}^4$ with a perturbation $\frac{n-4}{r} \dot{u}_r$. It turns out that the term $\frac{n-4}{r} \dot{u}_r$ plays a crucial role in producing the shrinking concentration set. The first step of the construction consists of choosing a suitable approximate solution with sufficiently small error and then decomposing the original problem into inner and outer problems, where the inner problem is essentially the linearization around the bubble supported in a well-chosen ball. The inner and outer problems will be solved by developing linear theories for the associated linear problems and the Schauder fixed point theorem. Since the desired blow-up solution is located exactly at the self-similar regime $r = O(\sqrt{T-t})$, the estimates for the linear outer problem are much more delicate than the case that the concentration set is fixed ($r = O(1)$). On the other hand, it is a difficult issue to control the term of type $\frac{n-4}{r} \dot{u}_r$ due to the shrinking effect $r \sim \sqrt{T-t} \to 0$ as $t \nearrow T$. As a consequence, in order to carry out the fixed point argument under suitable topology, the estimates for the inner and outer problems should be very refined. For the outer problem, we achieve this by using the Duhamel’s formula in $\mathbb{R}^n$, while for the inner problem, motivated by [43, 55], we improve the linear theory by decomposing the linearized problem into three different modes: scaling mode, translation mode and higher modes. The most difficult modes are scaling mode and translation mode. To find inner solution with
proper space-time decay, we shall carry out a new inner–outer gluing scheme for the scaling mode, while the estimates for the translation mode are obtained by the blow-up argument. As mentioned, the inner problem is supported in a ball with radius \( R = R(t) \) in terms of the translated and scaled variable. To ensure the inner–outer gluing to be carried out, the radius \( R(t) \) and the parameters in the norms will be very carefully chosen, which results in the dimension restriction \( 5 \leq n \leq 7 \). This seems to be reasonable since the singularity takes place exactly at the critical level \( r \sim \sqrt{T-t} \), and from our computations, the estimates for the key coupling terms in the inner–outer gluing system get worse as \( n \) increases. In other words, upper bound of the dimension \( n \) should be present in this setting to ensure the implementation of the inner–outer gluing procedure. We believe that the blow-up on higher dimensional shrinking sphere with \( n \geq 8 \) should exist, presumably with more complicated blow-up rate.

Note that the supercritical problem (3.79) is in the Matano–Merle range \( n+2 \over n-2 < p < p_{JL} \). It was proved that no type II blow-up is present for radial solutions in the case of a ball or in entire space under additional assumptions [145, 146, 153], while the solution constructed in Theorem 3.2.1 is certainly not radially symmetric as the translation mode also plays a crucial role in the construction resulting in the novel dynamics of the concentration set. In this aspect, the results in Theorem 3.2.1 share a similar flavor as that of [62].

We close the introduction by mentioning that for problem (3.79), type II finite time blow-up on a sphere with fixed instead of shrinking size also exists, and the upper bound for the dimension \( n \) for this phenomenon to exist should be greater than 8. We will not elaborate on this problem since it is a rather direct consequence of our construction here.

3.2.2 Approximate solutions and error estimates

In the symmetry class (3.82), problem (3.79) becomes

\[
\begin{align*}
    u_t &= u_{rr} + \frac{n-4}{r} u_r + \Delta_x u + u^3,
\end{align*}
\]
where \((r,z) \in D\) with \(D\) defined in (3.83). Define the error operator
\[
\mathcal{S}(u) := -u_t + \Delta_{(r,z)} u + \frac{n-4}{r} u_r + u^3,
\]
where \(\Delta_{(r,z)} := \partial_r^2 + \Delta_r\) is the Laplacian in \(\mathbb{R}^4\). Our first approximate solution is based on the Aubin–Talenti bubble (see \([5, 183]\))
\[
U(y) = \frac{\alpha_0}{1 + |y|^2}
\] (3.85)
which solves the Yamabe problem
\[
\Delta y U + U^3 = 0 \text{ in } \mathbb{R}^4.
\]
Here \(\alpha_0 = 2\sqrt{2}\). It is well-known that the linearized operator around the bubble
\[
L_0(\phi) := \Delta \phi + 3U^2 \phi
\] (3.86)
is non-degenerate in the sense that all bounded solutions to \(L_0(\phi) = 0\) are the linear combination of
\[
Z_i(y) := \partial_{y_i} U(y), \quad i = 1, 2, 3, 4, \quad Z_5(y) := U(y) + \nabla U(y) \cdot y.
\] (3.87)

**First approximate solution**

We define
\[
U_{\lambda(t), \xi(t)}(r,z) = \lambda^{-1}(t) U \left( \frac{(r,z) - \xi(t)}{\lambda(t)} \right),
\]
where
\[
\xi(t) = (\xi_r(t), \xi_z(t))
\]
with
\[
\xi_r(t) = \xi_{r,*}(t) + \xi_{r,1}(t), \quad \xi_{r,1}(t) = o(\xi_{r,*}(t)),
\]
\[
\xi_z(t) = \xi_{z,*}(t) + \xi_{z,1}(t), \quad |\xi_{z,1}(t)| = o(|\xi_{z,*}(t)|).
\]
In the sequel, we denote
\[ y = \frac{(r, z) - \xi(t)}{\lambda(t)}. \]
Now we choose the first approximate solution as
\[ U^* = \eta_* U_{\lambda(t)}, \xi(t), \]
where the cut-off function \( \eta_* \) is defined by
\[ \eta_*(s) = \begin{cases} 
1, & s < 1, \\
0, & s > 2. 
\end{cases} \]
with the positive constant \( \delta \) fixed sufficiently small. Here the smooth cut-off function \( \eta \) is defined by
\[ \eta(s) = \begin{cases} 
1, & s < 1, \\
0, & s > 2. 
\end{cases} \]
Then the first error of \( U^* \) is
\[
\mathcal{S}(U^*) = -\eta_* \partial_t U_{\lambda, \xi} + U_{\lambda, \xi}(\Delta_{(r,z)} \eta_* - \partial_t \eta_* + 2\nabla \eta_* \cdot \nabla U_{\lambda, \xi} + (\eta_*^3 - \eta_*) U_{\lambda, \xi}^3 \\
+ \frac{n-4}{r} \partial_t (\eta_* U_{\lambda, \xi}) \\
= \eta_* \left[ \lambda^{-2} \left( t \lambda(t) Z_3(y) + \lambda^{-2} \Delta U(y) \cdot \xi(t) \right) + U_{\lambda, \xi}(\Delta_{(r,z)} \eta_* - \partial_t \eta_* + 2\nabla \eta_* \cdot \nabla U_{\lambda, \xi} + (\eta_*^3 - \eta_*) U_{\lambda, \xi}^3 \\
+ \frac{n-4}{r} \partial_t (\eta_* U_{\lambda, \xi}) \right] \\
= \eta_* \left\{ \lambda^{-2} \left( t \lambda(t) \left( -\frac{a_0}{1 + |y|^2} + \frac{2a_0}{(1 + |y|^2)^2} \right) + \lambda^{-2}(t) \Delta U(y) \cdot \xi(t) \right) + U_{\lambda, \xi}(\Delta_{(r,z)} \eta_* - \partial_t \eta_* + 2\nabla \eta_* \cdot \nabla U_{\lambda, \xi} + (\eta_*^3 - \eta_*) U_{\lambda, \xi}^3 \\
+ \frac{n-4}{r} \partial_t (\eta_* U_{\lambda, \xi}) \right\}. \]

where \( \nabla := (\partial_r, \nabla_z) \).
Corrected approximate solution

Observe that the slow decaying error in (3.90) is

\[
\tilde{\varepsilon}_0 = -\frac{\alpha_0 \dot{\lambda}(t)}{\lambda^2(t) + \rho^2} \approx -\frac{\alpha_0 \dot{\lambda}(t)}{\rho^2},
\]

(3.91)

where \(\rho = |(r,z) - \xi(t)|\). In order to reduce the size of the first error, we shall choose \(\Psi^0\) to be an approximate solution of

\[
\partial_t u = \Delta_{(r,z)} u + \varepsilon_0 \text{ in } \mathbb{R}^4 \times (0,T).
\]

To achieve this, we consider

\[
\partial_t \psi^0 = \psi^0 \rho \partial_\rho + \frac{3}{\rho} \psi^0 - \frac{\alpha_0 \dot{\lambda}(t)}{\rho^2}.
\]

We perform the change of variable \(\phi_0 = \rho \psi^0\) and get

\[
\partial_t \phi_0 = (\phi_0)_{\rho \rho} + \frac{1}{\rho} (\phi_0)_\rho - \frac{1}{\rho^2} \phi_0 - \frac{\alpha_0 \dot{\lambda}(t)}{\rho^2}.
\]

(3.92)

From the same computations as in [55, Section 4], an explicit solution to problem (3.92) is given by the Duhamel’s formula

\[
\phi_0 = -\alpha_0 \rho \int_{-T}^{t} \dot{\lambda}(s) k(\rho, t-s) ds
\]

with

\[
k(\rho, t) := \frac{1 - e^{-\frac{t^2}{\rho^2}}}{\rho^2}.
\]

(3.93)

Therefore, by \(\psi^0 = \rho^{-1} \phi_0\), we get

\[
\psi^0 = -\alpha_0 \int_{-T}^{t} \dot{\lambda}(s) k(\rho, t-s) ds.
\]
We regularize the above $\psi^0$ and choose a good approximation $\Psi^0$ to be

$$
\Psi_0(r,z,t) = \Psi_0(|(r,z) - \xi(t)|,t) = \Psi_0(\rho,t) = -\alpha_0 \int_{-T}^t \lambda(s)k(\xi,\rho,t - s)ds.
$$

(3.94)

where

$$
\zeta(\rho,t) = \sqrt{\rho^2 + \lambda^2(t)}.
$$

(3.95)

Now we compute the new error produced by $\Psi_0$ and get

$$
\partial_t \Psi_0 - \Delta_{(r,z)} \Psi_0 - \phi_0
= \alpha_0 \left[ \frac{(r - \xi_0) \xi_r + (z - \xi_0) \xi_z - \lambda(t) \dot{\lambda}(t)}{\zeta} \right] \int_{-T}^t \lambda(s) k_\zeta(\zeta, t - s)ds
+ \alpha_0 \int_{-T}^t \lambda(s) \left[ -k_\zeta(\zeta, t - s) + \frac{\rho^2}{\xi^2} k_{\zeta\zeta}(\zeta, t - s) + \frac{3}{\xi} k_\zeta(\zeta, t - s)
+ \frac{\lambda^2(t)}{\xi^3} k_\zeta(\xi, t - s) \right] ds.
$$

Observe from (3.93) that $k(\zeta, t)$ satisfies $-k_\zeta + k_{\zeta\zeta} + \frac{3}{\xi} k_\zeta = 0$. Therefore, we obtain

$$
\partial_t \Psi_0 - \Delta_{(r,z)} \Psi_0 - \phi_0
= \alpha_0 \left[ \frac{(r - \xi_0) \xi_r + (z - \xi_0) \xi_z - \lambda(t) \dot{\lambda}(t)}{\lambda(t)(1 + |y|^2)^{1/2}} \right] \int_{-T}^t \lambda(s) k_\zeta(\zeta, t - s)ds
+ \frac{\alpha_0}{\lambda(t)(1 + |y|^2)^{3/2}} \int_{-T}^t \lambda(s) \left[ -\zeta k_{\zeta\zeta}(\zeta, t - s) + k_\zeta(\zeta, t - s) \right] ds
:= \mathcal{R}[\lambda].
$$

(3.96)

Now we choose the corrected approximation as

$$
u^* = U^* + \eta_* \Psi_0$$

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and its error is given by

\[ \mathcal{S}(u^*) = \mathcal{H}[\lambda, \xi] + U_{\lambda, \xi}(\Delta_{(r)} \eta_s - \partial_r \eta_s) + 2\nabla \eta_s \cdot \nabla U_{\lambda, \xi} + \left(\eta_s^3 - \eta_s\right)U_{\lambda, \xi} + U_{\lambda, \xi} \frac{n-4}{r} \partial_r \eta_s = -\Psi_0 \partial_r \eta_s + \Psi_0 \Delta_{(r)} \eta_s \]

+ \left(\frac{2\alpha_0}{r} \lambda^{-2} \dot{\lambda}(t) + \lambda^{-2} \nabla U(y) \cdot \dot{\xi}(t) \right)

+ \left(\frac{n-4}{\lambda(t)y_1 + \xi_r(t)} \lambda^{-2} \partial_y U(y) - \mathcal{H}[\lambda] \right) \tag{3.97}

where \( \mathcal{H}[\lambda, \xi] \) is defined as

\[ \mathcal{H}[\lambda, \xi] := \eta_s \left[ \frac{2\alpha_0}{r} \lambda^{-2} \dot{\lambda}(t) \right] + \lambda^{-2} \nabla U(y) \cdot \dot{\xi}(t) \]

+ \left(\frac{n-4}{\lambda(t)y_1 + \xi_r(t)} \lambda^{-2} \partial_y U(y) \right) \tag{3.98}

We define

\[ \mathcal{S}_{\text{out}}[\lambda, \xi] := U_{\lambda, \xi}(\Delta_{(r)} \eta_s - \partial_r \eta_s) + 2\nabla \eta_s \cdot \nabla U_{\lambda, \xi} + \left(\eta_s^3 - \eta_s\right)U_{\lambda, \xi} \]

+ \left(\frac{2\alpha_0}{r} \lambda^{-2} \dot{\lambda}(t) + \lambda^{-2} \nabla U(y) \cdot \dot{\xi}(t) \right)

+ \left(\frac{n-4}{\lambda(t)y_1 + \xi_r(t)} \lambda^{-2} \partial_y U(y) \right) \tag{3.99}

where \( \eta_R \) is a cut-off in (3.101). To further reduce the size of the error, we let \( \Psi_1 \) be the solution solving

\[
\begin{align*}
\begin{cases}
\partial_t \Psi_1 = \Delta_{(r)} \Psi_1 + \frac{n-4}{r} \partial_r \Psi_1 + \mathcal{S}_{\text{out}}[\lambda_s, \xi_s] & \text{in } \mathcal{D} \times (0, T), \\
\Psi_1 = 0 & \text{on } (\partial \mathcal{D} \setminus \{r = 0\}) \times (0, T), \\
\partial_r \Psi_1 = 0 & \text{on } (\mathcal{D} \cap \{r = 0\}) \times (0, T), \\
\Psi_1(r, z, 0) = 0 & \text{in } \mathcal{D},
\end{cases}
\end{align*}
\]

(3.100)

where \( \mathcal{S}_{\text{out}}[\lambda_s, \xi_s] \) is defined by replacing \( \lambda, \xi \) in \( \mathcal{S}_{\text{out}}[\lambda, \xi] \) by \( \lambda_s \) and \( \xi_s \), respectively. Here \( \lambda_s \) and \( \xi_s \) are leading orders of \( \lambda \) and \( \xi \), which will be derived in Section 3.2.4. Note that the new error produced by \( \Psi_1 \) turns out to be smaller order.
and will not change the leading orders $\lambda_*, \xi_*$. We shall show this in Section 3.2.7.

In conclusion, the corrected approximation we finally choose is

$$u_* = U* + \eta_* \Psi_0 + \Psi_1.$$  

In the sequel, we shall find a perturbation $P$ such that $u = u_* + P$ is the desired solution, namely, $\mathcal{S}(u_* + P) = 0$.

### 3.2.3 The inner–outer gluing scheme

We look for solution of the form $u = U* + w$, where $w$ is a small perturbation consisting of inner and outer parts

$$w = \varphi_{in} + \varphi_{out}$$

with

$$\varphi_{in} = \lambda^{-1}(t) \eta_R \Phi(y,t), \quad \varphi_{out} = \eta_* \Psi_0(r,z,t) + \Psi_1(r,z,t) + \psi(r,z,t) + Z^*(x,t).$$

Here

$$\eta_R = \eta_{R(t)}(r,z,t) = \eta\left(\frac{|(r,z) - \xi_*(t)|}{\lambda(t) R(t)}\right), \quad (3.101)$$

the smooth cut-off function $\eta$ is defined by (3.89), and $Z^*$ satisfies

$$\begin{cases} 
Z^*_t = \Delta x Z^* & \text{in } \Omega \times (0,T), \\
Z^*(\cdot,t) = 0 & \text{on } \partial \Omega \times (0,T), \\
Z^*(\cdot,0) = Z^*_0 & \text{in } \Omega,
\end{cases} \quad (3.102)$$

in the original variables $x \in \mathbb{R}^n$. Throughout this section, we choose $R(t)$ such that $\lambda(t) R(t) \ll \sqrt{T-t}$ for $T \ll 1$. Denote

$$\mathcal{D}_{2R} = \{(r,z) \in \mathbb{R}^4 : |(r,z) - (\xi_r, \xi_z)| \leq 2\lambda R\}$$

and $\Psi^* = \psi + Z^*$. Then $u$ is a solution to the original problem (3.79) if
• \( \phi \) solves the inner problem

\[
\lambda^2 \phi_t = \Delta_y \phi + 3U^2(y)\phi + \mathcal{H}(\phi, \psi, \lambda, \xi) \quad \text{in} \quad D_R \times (0,T), \tag{3.103}
\]

where

\[
\mathcal{H}(\phi, \psi, \lambda, \xi)(y, t) := 3\lambda U^2(y)[\eta_r \Psi_0 + \psi + Z^*]\left(\lambda y + \xi, t\right) + \frac{(n-4)\lambda}{\lambda y_1 + \xi} \phi_{y_1} \\
+ \lambda \left[\lambda (\nabla_y \phi \cdot y + \phi) + \nabla_y \phi \cdot \xi\right] + \frac{(n-4)\lambda^3}{r} \eta_r \partial_r \psi_0 \\
+ \lambda^3 \mathcal{N}(w) + \lambda^3 \mathcal{K}[\lambda, \xi] + 3\lambda U^2(y)\Psi_1 \tag{3.104}
\]

with \( \mathcal{H}[\lambda, \xi] \) defined in (3.98).

• \( \psi \) solves the outer problem

\[
\psi_t = \Delta_{(\cdot, \cdot)} \psi + \frac{n-4}{r} \partial_r \psi + \mathcal{G}(\phi, \psi, \lambda, \xi) \quad \text{in} \quad D \times (0,T) \tag{3.105}
\]

with

\[
\mathcal{G}(\phi, \psi, \lambda, \xi) := 3\lambda^{-2}(1 - \eta_R)U^2(y)(\psi + Z^* + \eta_r \Psi_0 + \Psi_1) \\
+ \lambda^{-3} \left[\Delta_y \eta_R \phi + 2\nabla_y \eta_R \cdot \nabla_y \phi - \lambda^2 \phi \partial_R \eta_R\right] \\
+ \frac{(n-4)\lambda^{-1}}{r} \phi \partial_r \eta_R + (1 - \eta_R) \mathcal{K}_1[\lambda, \xi] \\
+ \mathcal{S}_{\text{out}}[\lambda, \xi] - \mathcal{S}_{\text{out}}[\lambda_s, \xi_s] + (1 - \eta_R) \mathcal{N}(w), 
\]

where

\[
\mathcal{K}_1[\lambda, \xi] := \eta_s \left[\frac{2\alpha_0 \lambda^{-2}(t)\lambda(t)}{(1 + |y|^2)^2} - \mathcal{G}[\lambda]\right], \tag{3.107}
\]

\( \mathcal{S}_{\text{out}} \) is defined in (3.99) and

\[
\mathcal{N}(w) := (U^* + w)^3 - (U^*)^3 - 3U^2_{\lambda, \xi} w. \tag{3.108}
\]

We now describe our strategy to solve the inner and outer problems. We shall first
develop linear theories for the associated linear problems of (3.105) and (3.103). Since the solution we want to construct concentrates on an \((n - 4)\)-dimensional sphere with shrinking size \(\sqrt{T - t}\), suitable estimates for the outer solution \(\psi\) are very delicate to find. To achieve this, we find solutions in the symmetry class (3.82) by using the Duhamel’s formula in \(\mathbb{R}^n\). For the linear inner problem, we want to find inner solution with proper space-time decay. Since the inner–outer gluing relies on delicate analysis of the space-time decay of solutions, we shall further decompose the inner problem (3.103) into three different spherical harmonic modes and construct solution in each mode. To get more refined estimates for the gluing to work, we carry out a new inner–outer gluing scheme for the linear inner problem, where certain orthogonality conditions are of course needed due to the existence of the nontrivial kernels (see (3.87)) of the linearized operator \(L_0\) in (3.86). This will give us the reduced equations for the parameter functions \(\lambda(t)\) and \(\xi(t)\). The reduced equation for \(\xi(t)\) will be easy to solve. However, the reduced equation for \(\lambda(t)\) turns out to be an integro-differential equation due to the non-local correction \(\Psi_0\) in (3.94), and it is more involved. Thanks to [55], we solve \(\lambda(t)\) by following a similar procedure since the integro-differential equation for \(\lambda(t)\) is close in spirit to that of [55]. Finally, by using the Schauder fixed point theorem, we solve the inner–outer gluing system and prove the existence of the desired blow-up solution.

The rest of this section is organized as follows. In Section 3.2.4, we derive the leading orders for the parameter functions \(\lambda(t)\) and \(\xi(t)\). In Section 3.2.5, we establish the estimates for the linear outer problem with different right hand sides which appear in \(\mathcal{G}\) defined in (3.106). The proof is postponed to the Section 3.2.8. In Section 3.2.6, we develop the linear theory for the inner problem by spherical harmonic decomposition. In Section 3.2.7, the inner–outer gluing system is formulated, and we shall solve \((\phi, \psi, \lambda, \xi)\) from the full system by the linear theories developed in Section 3.2.5, Section 3.2.6 and the Schauder fixed point theorem.

3.2.4 The choices of \(\lambda_*\) and \(\xi_*\)

In this section, we shall choose the leading orders \(\lambda_*(t), \xi_*(t) = (\xi_{r,*}(t), \xi_{z,*}(t))\) of the parameter functions \(\lambda(t)\) and \(\xi(t)\). In Section 3.2.6, a linear theory for inner
problem concerning the solvability and estimates of the associated linear problem will be developed, where approximately the following orthogonality conditions

\[ \int_{\mathbb{R}^4} \mathcal{K}(\phi, \psi, \lambda, \xi) Z_j(y) dy = 0 \quad \text{for all } j = 1, \ldots, 5, \, t \in (0, T) \quad (3.109) \]

are needed to guarantee the existence of inner solution \( \phi \) with desired space-time decay. Here \( Z_j \) are the kernel functions (c.f. (3.87)) of the linearized operator \( L_0 \). Basically, we will derive the scaling and translation parameters \( \lambda(t) \) and \( \xi(t) \) at main order from the orthogonality conditions (3.109).

Recall that \( \mathcal{K}(\phi, \psi, \lambda, \xi)(y, t) \) is defined in (3.104). In this section, we shall single out the leading term \( \mathcal{K}_s \) in \( \mathcal{K} \) to derive \( \lambda_s \) and \( \xi_s \). We define

\[ \mathcal{K}_s[\lambda, \xi, \Psi^*] := 3\lambda U^2(y)[\eta_0 \Psi_0 + \Psi^*](\lambda y + \xi, t) + \lambda^3 \mathcal{K}[\lambda, \xi] \\
= 3\lambda U^2(y)[\eta_0 \Psi_0 + \Psi^*](\lambda y + \xi, t) + \frac{2\alpha_0 \eta_0 \lambda(t) \dot{\lambda}(t)}{(1 + |y|^2)^2} \]

\[ + \lambda(t) \eta_0 \nabla U(y) : \dot{\xi}(t) + \frac{(n-4)\lambda(t) \eta_0}{\lambda(t) y_1 + \xi_r(t)} \partial_y U(y) \]

\[ - \frac{\alpha_0 \lambda^2(t) \eta_0}{(1 + |y|^2)^{3/2}} \int_{-T}^t \dot{\lambda}(s) \left[ -\zeta k_\zeta \zeta^t_{\zeta, t-s} + k_\zeta(\zeta, t-s) \right] ds \]

\[ - \alpha_0 \eta_0 \lambda^2(t) \left[ \frac{(r - \xi_r(\xi_r(t))^2 + (z - \xi_z(\xi_z(t)^2 - \lambda(t) \dot{\lambda}(t) \right]}{(1 + |y|^2)^{1/2}} \right] \]

\[ \times \int_{-T}^t \dot{\lambda}(s) k_\zeta(\zeta, t-s) ds, \]

where \( \Psi^* = \psi + Z^* \). The contribution of the rest terms \( \mathcal{K} - \mathcal{K}_s \) in the orthogonality conditions turns out to be negligible compared to the leading term \( \mathcal{K}_s \). We shall deal with this in Section 3.2.7 when we finally solve the inner–outer gluing system.

Since \( R(t) \) is chosen such that \( \lambda(t) R(t) \ll \sqrt{T-t} \), we see that \( \text{supp}(\eta_R) \subset \text{supp}(\eta_s) \) for \( T \ll 1 \). So in the inner region \( \mathcal{D}_{2R}, \eta_s \equiv 1 \). Then

\[ \int_{\mathbb{R}^4} \mathcal{K}_s[\lambda, \xi, \Psi^*] Z_1(y) dy = 0 \]

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implies that
\[ \dot{\xi}_r(t) + \frac{n-4}{\xi_r(t)} = o(1), \]
where \( o(1) \to 0 \) as \( t \nearrow T \). So the choice of \( \xi_r(t) \) at main order is
\[ \xi_{r,*}(t) = \sqrt{2(n-4)(T-t)}. \]
Similarly, the orthogonality conditions
\[ \int_{\mathbb{R}^4} \mathcal{H}_s[\lambda, \xi, \Psi^*]Z_j(y)dy = 0, \quad j = 2, 3, 4 \]
imply that
\[ \dot{\xi}_z(t) = o(1), \quad j = 2, 3, 4, \]
where \( o(1) \to 0 \) as \( t \nearrow T \). So at main order, the choice of \( \xi_z(t) \) is
\[ \xi_{z,*}(t) = z_0, \]
where \( z_0 \) is a given point in \( \mathbb{R}^3 \). For convenience, we take \( z_0 \equiv (0, 0, 0) \).

In order to get the reduced equation for \( \lambda(t) \) from
\[ \int_{\mathbb{R}^4} \mathcal{H}_s[\lambda, \xi, \Psi^*]Z_5(y)dy = 0, \]
we first evaluate
\[
\int_{\mathbb{R}^4} \mathcal{H}[\lambda]Z_5(y)dy
= \frac{\alpha_0}{\lambda(t)} \int_{\mathbb{R}^4} \frac{Z_5(y)}{1+|y|^2}  \left( \int_{-T}^{t} \lambda(s)[k_{\xi}(\xi, t-s) - \xi k_{\xi}(\xi, t-s)]ds \right) dy
- \alpha_0 \lambda(t) \int_{\mathbb{R}^4} \frac{Z_5(y)}{1+|y|^2} \left( \int_{-T}^{t} \lambda(s)k_{\xi}(\xi, t-s)ds \right) dy.
\]
Let
\[ \Upsilon = \frac{\xi^2}{t-s} = \frac{\lambda^2(t)(1+|y|^2)}{t-s}, \quad \tau = \frac{\lambda^2(t)}{t-s} \quad (3.110) \]
and $K(\Upsilon) = \frac{1-e^{-\frac{\Upsilon}{t}}}{\Upsilon}$. Then, recalling from (3.93), we get

$$k_\zeta(\zeta, t-s) - \zeta k_\zeta(\zeta, t-s) = -\frac{8(1-e^{-\frac{z^2}{4(t-s)}})}{\zeta^3} - \frac{2e^{-\frac{z^2}{4(t-s)}}}{\zeta(t-s)} + \frac{\zeta e^{-\frac{z^2}{4(t-s)}}}{4(t-s)^2}$$

$$= -4 \left(\frac{\Upsilon}{t-s}\right)^{3/2} K_{YY}(\Upsilon)$$

and also

$$k_\zeta(\zeta, t-s) = \frac{2\sqrt{\Upsilon}}{(t-s)^{3/2}} K_{Y}(\Upsilon).$$

Therefore, we obtain

$$\int_{\mathbb{R}^4} H[\lambda] Z_5(y) dy = -\frac{4\alpha_0}{\lambda^2(t)} \int_{\mathbb{R}^4} \frac{Z_5(y)}{(1+|y|^2)^2} \left( \int_{-T}^t \frac{\dot{\lambda}(s)}{t-s} Y^2 K_{YY}(\Upsilon) ds \right) dy$$

$$- \frac{2\alpha_0 \dot{\lambda}(t)}{\lambda(t)} \int_{\mathbb{R}^4} \frac{Z_5(y)}{(1+|y|^2)^2} \left( \int_{-T}^t \frac{\dot{\lambda}(s)}{t-s} Y K_{Y}(\Upsilon) ds \right) dy.$$  

(3.111)

We expand $Z^*(\lambda y + \xi, t)$ and $\psi(\lambda y + \xi, t)$ at $q = (0, z_0)$

$$Z^*(\lambda y + \xi, t) = Z^*_0(q) + o(1), \quad \psi(\lambda y + \xi, t) = \psi(q, 0) + o(1).$$

On the other hand, from (3.94) and (3.110), we have

$$\int_{\mathbb{R}^4} 3\lambda(t) U^2(y) Z_5(y) \Psi_0(\rho, t) dy$$

$$= -3\alpha_0 \lambda(t) \int_{\mathbb{R}^4} U^2(y) Z_5(y) \left( \int_{-T}^t \frac{\dot{\lambda}(s)}{t-s} K(\Upsilon) ds \right) dy.$$  

(3.112)

Then, the orthogonality condition

$$\int_{\mathbb{R}^4} \mathcal{H}_e[\lambda, \xi, \Psi^*] Z_5(y) dy = 0$$

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gives

\[ \int_{\mathbb{R}^4} \left( 3U^2(y)[\psi_0(\rho, t) + \psi(q, 0) + Z^*_0(q)] + \lambda^2(t) \mathcal{X}^{[\lambda, \xi]} \right) Z_5(y) dy + o(1) = 0. \]  

(3.113)

By (3.113), (3.98), (3.111), (3.112) and some computations, we obtain

\[ 4\alpha_0 \int_{\mathbb{R}^4} \frac{Z_5(y)}{1 + |y|^2} \left( \int_{T-t}^t \frac{\dot{\lambda}(s)}{t-s} K_{\gamma Y} ds \right) dy \]

\[ -3\alpha_0 \int_{\mathbb{R}^4} U^2(y)Z_5(y) \left( \int_{T-t}^t \frac{\dot{\lambda}(s)}{t-s} K(\gamma) ds \right) dy \]

\[ + 2\alpha_0 \dot{\lambda}(t) \int_{\mathbb{R}^4} \frac{Z_5(y)}{1 + |y|^2} \left( \int_{T-t}^t \frac{\dot{\lambda}(s)}{t-s} YK_{\gamma Y} ds \right) dy \]

\[ + 2\alpha_0 \dot{\lambda}(t) \int_{\mathbb{R}^4} \frac{Z_5(y)}{1 + |y|^2} dy + 3[Z^*_0(q) + \psi(q, 0)] \int_{\mathbb{R}^4} U^2(y)Z_5(y) dy + o(1) = 0. \]  

(3.114)

The scaling parameter \( \lambda(t) \) should be decreasing to 0 as \( t \nearrow T \) so that a blow-up solution can be constructed. So we may impose \( \dot{\lambda}(t) = o(1) \) as \( t \nearrow T \). Then (3.114) becomes

\[ 4\alpha_0 \int_{\mathbb{R}^4} \frac{Z_5(y)}{1 + |y|^2} \left( \int_{T-t}^t \frac{\dot{\lambda}(s)}{t-s} K_{\gamma Y} ds \right) dy \]

\[ -3\alpha_0 \int_{\mathbb{R}^4} U^2(y)Z_5(y) \left( \int_{T-t}^t \frac{\dot{\lambda}(s)}{t-s} K(\gamma) ds \right) dy \]

\[ = -3[Z^*_0(q) + \psi(q, 0)] \int_{\mathbb{R}^4} U^2(y)Z_5(y) dy + o(1). \]  

(3.115)

We define

\[ 4\alpha_0 \int_{\mathbb{R}^4} \frac{Z_5(y)}{1 + |y|^2} \left( \int_{T-t}^t \frac{\dot{\lambda}(s)}{t-s} K_{\gamma Y} ds \right) dy \]

\[ -3\alpha_0 \int_{\mathbb{R}^4} U^2(y)Z_5(y) \left( \int_{T-t}^t \frac{\dot{\lambda}(s)}{t-s} K(\gamma) ds \right) dy := \int_{T-t}^t \dot{\lambda}(s) \left( \frac{\lambda^2(t)}{t-s} \right) ds \]
with
\[ \Gamma(\tau) := \alpha_0 |S^3| \int_0^\infty \left( \frac{4Z_5(y)|y|^3}{(1 + |y|^2)^2} Y^2 K_{\beta\gamma}(Y) - 3U^2(y)Z_5(y)|y|^3 K(Y) \right) \bigg|_{Y = \tau(1 + |y|^2)} d|y|, \]
(3.116)

where $|S^3|$ is the area of the unit sphere $S^3$. Now we need to analyze the behavior of $\Gamma(\tau)$ as $\tau \ll 1$ and $\tau \gg 1$. By the definition of $U(y)$ and $Z_5(y)$ as in (3.85) and (3.87) respectively, we write $\Gamma(\tau)$ explicitly as
\[ \Gamma(\tau) = \alpha_0^2 |S^3| \int_0^\infty \left( \frac{(1 - |y|^2)|y|^3}{(1 + |y|^2)^4} \left[ 4Y^2 K_{\beta\gamma}(Y) - 3\alpha_0^2 K(Y) \right] \right) \bigg|_{Y = \tau(1 + |y|^2)} d|y|. \]

Since
\[ 4Y^2 K_{\beta\gamma}(Y) - 3\alpha_0^2 K(Y) = -\frac{Y e^{-\frac{Y}{4}}}{4} - 2e^{-\frac{Y}{4}} + (8 - 3\alpha_0^2) \frac{1 - e^{-\frac{Y}{4}}}{Y}, \]
with $\alpha_0 = 2\sqrt{2}$, we obtain
\[ \Gamma(\tau) = \begin{cases} c_* + O(\tau), & \text{for } \tau < 1, \\ O\left(\frac{1}{\tau}\right), & \text{for } \tau > 1, \end{cases} \]
where $c_* > 0$ is a constant. Therefore, (3.115) is reduced to
\[ c_* \int_{T}^{T - \lambda^2(t)} \frac{\dot{\lambda}(s)}{t - s} ds = -3c_0 [Z_0^*(q) + \psi(q, 0)] + o(1), \]
(3.117)
where $c_0 := \int_{\mathbb{R}^3} U^2(y)Z_5(y) dy < 0$. Since $\lambda(t)$ decreases to 0 as $t \nearrow T$, we impose
\[ a_* := Z_0^*(q) + \psi(q, 0) < 0. \]

Now we claim that a good choice of $\dot{\lambda}(t)$ at main order is
\[ \dot{\lambda}(t) = -\frac{c}{\log(T - t)^2}, \]
(3.118)
where $c > 0$ is a constant to be determined later. Indeed, by substituting, we get

\[
\int_{-T}^{t - \lambda^2(t)} \frac{\dot{\lambda}(s)}{t - s} ds
= \int_{-T}^{t - (T - t)} \frac{\dot{\lambda}(s)}{t - s} ds + \int_{t - (T - t)}^{t - \lambda^2(t)} \frac{\dot{\lambda}(t) - \dot{\lambda}(s)}{t - s} ds
= \int_{-T}^{t - (T - t)} \frac{\dot{\lambda}(s)}{t - s} ds + \dot{\lambda}(t)(\log(T - t) - 2\log(\lambda(t))) - \int_{t - (T - t)}^{t - \lambda^2(t)} \frac{\dot{\lambda}(t) - \dot{\lambda}(s)}{t - s} ds
\approx \int_{-T}^{t} \frac{\dot{\lambda}(s)}{T - s} ds - \dot{\lambda}(t)\log(T - t) := \beta(t)
\]
as $t \searrow T$. By (3.118), we then see that

\[
\log(T - t) \frac{d\beta}{dt}(t) = \frac{d}{dt} \left( -\log^2(T - t)\dot{\lambda}(t) \right) = 0,
\]
which means $\beta(t)$ is a constant. Thus, equation (3.117) can be approximately solved for

\[
\dot{\lambda}(t) = -\frac{c}{|\log(T - t)|^2}
\]
with the constant $c$ chosen as

\[
-c \int_{-T}^{T} \frac{ds}{(T - s)|\log(T - s)|^2} = \kappa_*,
\]
where $\kappa_* := -\frac{3\log a_t}{c_*}$. At main order, we obtain $\dot{\lambda}_*(t) = \kappa_* \dot{\lambda}_*(t)$ with $\dot{\lambda}_*(t) = -\frac{|\log T|}{|\log(T - t)|^2}$. By imposing $\dot{\lambda}_*(T) = 0$, we finally get

\[
\dot{\lambda}_*(t) = \frac{|\log T|(T - t)}{|\log(T - t)|^2} (1 + o(1)) \text{ as } t \searrow T.
\]
3.2.5 Linear theory for the outer problem

In order to solve the outer problem (3.105), we need a linear theory for the associated linear problem. We consider

\[
\begin{aligned}
\psi_t &= \Delta_{(r,z)} \psi + \frac{n-4}{r} \partial_r \psi + f_{\text{out}} \quad \text{in } D \times (0, T), \\
\psi &= 0 \quad \text{on } (\partial D \setminus \{r = 0\}) \times (0, T), \\
\psi_r &= 0 \quad \text{on } (D \cap \{r = 0\}) \times (0, T), \\
\psi(r, z, 0) &= 0 \quad \text{in } D,
\end{aligned}
\] (3.119)

where the non-homogeneous term \( f_{\text{out}} \) in (3.119) is assumed to be bounded with respect to the weights appearing in the outer problem (3.105). Define the weights

\[
\begin{aligned}
\rho_1 &:= \lambda_n^{v-3}(t) R^{-2-\alpha(t)} \chi_{\{\|r, z\| \leq 2 \lambda_n R\}}, \\
\rho_2 &:= \frac{\lambda_n^{v_2}}{|(r, z) - \xi(t)|^{\frac{3}{2}}} \chi_{\{\lambda_n R \leq \|r, z\| \leq 2 \delta \sqrt{T-t}\}}, \\
\rho_3 &:= 1,
\end{aligned}
\] (3.120)

where we choose \( R(t) = \lambda_n^{-\beta}(t) \) for \( \beta \in (0, 1/2) \). Define the norms

\[
\|f\|_{**} := \sup_{(r, z, t) \in D \times (0, T)} \left( \sum_{i=1}^{3} \rho_i(r, z, t) \right)^{-1} |f(r, z, t)|,
\] (3.121)

\[
\|\psi\|_* := \frac{\lambda_n^{2-v-\frac{2}{3}}(0) R^{2+\frac{8}{3}}(0)}{|\log T|} \|\psi\|_{L^\infty(\Omega \times (0, T))} \\
+ \frac{\lambda_n^{\frac{1}{2}-v-\frac{2}{3}}(0) R^{2+\alpha-\frac{8}{3}}(0)}{|\log T|} \|\nabla \psi\|_{L^\infty(\Omega \times (0, T))} \\
+ \sup_{(r, z, t) \in D \times (0, T)} \left[ \frac{\lambda_n^{2-v-\frac{2}{3}}(t) R^{2+\alpha-\frac{8}{3}}(t)}{|\log(T-t)|} |\psi(r, z, t) - \psi(r, z, T)| \right].
\]
\[
+ \sup_{(r,z,t)\in \mathcal{D} \times (0,T)} \left[ \lambda_s^{\frac{2}{2-v}} \frac{R^2+\alpha-\frac{4}{2}}{\log(T-t)} \left| \nabla \psi(r,z,t) - \nabla \psi(r,z,T) \right| \right] \\
+ \sup_{\mathcal{D} \times I_T} \frac{\lambda_s^{2-v+\gamma} (t_2) R^2+\alpha(t_2)}{(t_2-t_1)\gamma} |\psi(r,z,t_2) - \psi(r,z,t_1)|,
\]

where \( y = \frac{(r,z)-\xi(t)}{\lambda_s(t)} \), \( \nu, \alpha \in (0,1) \), \( \gamma \in (0,1) \), and the last supremum is taken over

\[ \mathcal{D} \times I_T = \left\{ (r,z,t_1,t_2) : (r,z) \in \mathcal{D}, 0 \leq t_1 \leq t_2 \leq T, t_2-t_1 \leq \frac{1}{10}(T-t_2) \right\}. \]

For problem (3.119), we have the following proposition.

**Proposition 3.2.1.** Let \( \psi \) be the solution to problem (3.119) with \( \|f_{\text{out}}\|_* < +\infty \). Then it holds that

\[ \|\psi\|_* \lesssim \|f_{\text{out}}\|_* \cdot \] \hspace{1cm} (3.123)

In order to establish Proposition 3.2.1, we consider

\[
\begin{cases}
\psi_t = \Delta_{\mathbb{R}^n} \psi + f & \text{in } \Omega \times (0,T), \\
\psi = 0 & \text{on } \partial \Omega \times (0,T), \\
\psi(x,0) = 0 & \text{in } \Omega,
\end{cases}
\]

which is defined in \( \mathbb{R}^n \) in the symmetry class (3.82). For problem (3.124), we prove the following three lemmas concerning the a priori estimates with different right hand sides.

**Lemma 3.2.1.** Let \( \psi \) solve problem (3.124) with right hand side

\[ |f(x,t)| \lesssim \lambda_s^{v-3}(t) R^{2-\alpha} (t) \chi_{\{(r,z)-\xi(t)\leq 2\lambda_s R\}}. \]

If \( \nu - 3 + \beta(2+\alpha) < 0 \), \( \nu - \beta(2-\alpha) > 0 \), then

\[
|\psi(x,t)| \lesssim \lambda_s^{\frac{2}{2-v} + \frac{4}{\beta}} (0) R^{-2-\alpha+\frac{4}{2}} (0) \log T, \hspace{1cm} (3.125)
\]

\[
|\psi(x,t) - \psi(x,T)| \lesssim \lambda_s^{\frac{2}{2-v} + \frac{4}{\beta}} (t) R^{-2-\alpha+\frac{4}{2}} (t) \log (T-t), \hspace{1cm} (3.126)
\]

\[
|\nabla \psi(x,t)| \lesssim \lambda_s^{\frac{2}{2-v} + \frac{4}{\beta}} (0) R^{-2-\alpha+\frac{4}{2}} (0) \log T, \hspace{1cm} (3.127)
\]

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\[ |\nabla \psi(x,t) - \nabla \psi(x,T)| \lesssim \lambda_\alpha^{-\frac{3}{2}}(t)R^{1/2}(t)\log(T-t)|, \] (3.128)
\[ |\psi(x,t_2) - \psi(x,t_1)| \lesssim \lambda_\alpha^{-\frac{3}{2}}(t_2)R^{1/2}(t_2)(t_2-t_1)^{1-\mu/2}, \] (3.129)
where \(0 \leq t_1 \leq t_2 \leq T\) with \(t_2 - t_1 \leq \frac{1}{10}(T-t_2)\) and \(\mu \in (0,1)\).

**Lemma 3.2.2.** Let \(\psi\) solve problem (3.124) with right hand side

\[ |f(x,t)| \lesssim \frac{\lambda_\alpha^{\nu_2}}{|(r,z) - \xi(t)|^2} x_{\lambda_\alpha R \leq |(r,z) - \xi(t)| \leq 2\delta \sqrt{T-t}}, \]
where \(\nu_2 \in (0,1), \delta > 0\). Then

\[ |\psi(x,t)| \lesssim \lambda_\alpha^{\nu_2-1}(0)R^{-2}(0)| \log T|, \] (3.130)
\[ |\psi(x,t) - \psi(x,T)| \lesssim \lambda_\alpha^{\nu_2-1}(t)R^{-2}(t)| \log(T-t)|, \] (3.131)
\[ |\nabla \psi(x,t)| \lesssim \lambda_\alpha^{\nu_2-2}(t)R^{-2}(t)\sqrt{T}, \] (3.132)
\[ |\nabla \psi(x,t) - \nabla \psi(x,T)| \lesssim \lambda_\alpha^{\nu_2-2}(t_2)R^{-2}(t_2)| \log(T-t)|, \] (3.133)
\[ |\psi(x,t_2) - \psi(x,t_1)| \lesssim \lambda_\alpha^{\nu_2-1-\gamma}(t_2)R^{-2}(t_2)(t_2-t_1)^{\gamma}, \] (3.134)
where \(0 \leq t_1 \leq t_2 \leq T\) with \(t_2 - t_1 \leq \frac{1}{10}(T-t_2)\) and \(\gamma \in (0,1)\).

**Lemma 3.2.3.** Let \(\psi\) solve problem (3.124) with right hand side

\[ |f(x,t)| \lesssim 1. \]

Then

\[ |\psi(x,t)| \lesssim t \] (3.135)
\[ |\psi(x,t)| \lesssim (T-t)| \log(T-t)|, \] (3.136)
\[ |\nabla \psi(x,t)| \lesssim T^{1/2}, \] (3.137)
\[ |\nabla \psi(x,t) - \nabla \psi(x,T)| \lesssim (T-t)^{1/2} \] (3.138)
\[ |\psi(x,t_2) - \psi(x,t_1)| \lesssim (t_2-t_1)| \log(t_2-t_1)|, \] (3.139)
where \(0 \leq t_1 \leq t_2 \leq T\) with \(t_2 - t_1 \leq \frac{1}{10}(T-t_2)\).
Proof of Proposition 3.2.1. We denote $\psi[f_{\text{out}}]$ by the solution to problem (3.124) with the right hand side $f_{\text{out}}$ satisfying $\|f_{\text{out}}\|_{\ast\ast} < +\infty$. Decompose $f_{\text{out}} = \sum_{i=1}^{3} f_i$ with $|f_i| \lesssim \rho_i |f_i|_{\ast\ast}$.

Let $1 - \frac{\mu}{2} = \gamma$ in Lemma 3.2.1 Then by the linearity, (3.123) follows from Lemma 3.2.1, Lemma 3.2.2 and Lemma 3.2.3.

The proofs of Lemma 3.2.1, Lemma 3.2.2 and Lemma 3.2.3 are postponed to Section 3.2.8.

Remark 3.2.1. Let us point out the reason why we use the $\|\cdot\|_{\ast}\ast$-norm of $\psi$ ((3.122)) only involving $v$ but not $v_2$ appearing in Lemma 3.2.2. Lemma 3.2.2 is needed to deal with the right hand side of outer problem with cut-off $1 - \eta R$ in front. For convenience, when we carry out the inner–outer gluing procedure to bound right hand sides in the chosen topology, we will adjust $v_2$ such that the control of $\psi$ in Lemma 3.2.2 is better than that of Lemma 3.2.1. This will result in a constraint for the parameters $v_2 + 1 > v + \frac{2}{n} + \beta(\alpha - \frac{4}{n})$. In fact, the above constraint will be satisfied by the choices of parameters in Section 3.2.7.

3.2.6 Linear theory for the inner problem

In this section, we develop a linear theory concerning the estimates for the associated linear problem of the inner problem under certain topology.

In order to solve the inner problem (3.103), we consider the associated linear problem

$$\lambda^2 \phi_t = \Delta \phi + 3U^2(y)\phi + h(y,t) \quad \text{in} \quad \mathcal{D}_{2R} \times (0,T).$$

(3.140)

Recall that the linearized operator $L_0 = \Delta + 3U^2$ has only one positive eigenvalue $\mu_0$ such that

$$L_0(Z_0) = \mu_0 Z_0, \quad Z_0 \in L^\infty(\mathbb{R}^4),$$

where the corresponding eigenfunction $Z_0$ is radially symmetric with the asymptotic behavior

$$Z_0(y) \sim |y|^{-3/2} e^{-\sqrt{\mu_0}|y|} \quad \text{as} \quad |y| \to +\infty.$$

(3.141)
Multiplying equation (3.140) by $Z_0$ and integrating over $\mathbb{R}^4$, we obtain that

$$\lambda^2(t)p'(t) - \mu_0 p(t) = q(t),$$

where $p(t) = \int_{\mathbb{R}^4} \phi(y, t) Z_0(y) dy$ and $q(t) = \int_{\mathbb{R}^4} h(y, t) Z_0(y) dy$. Then we have

$$p(t) = e^{\int_0^t \mu_0 \lambda^{-2}(r) dr} \left( p(0) + \int_0^t q(\eta) \lambda^{-2}(\eta) e^{-\int_0^\eta \mu_0 \lambda^{-2}(r) dr} d\eta \right).$$

In order to get a decaying solution, the initial condition

$$p(0) = -\int_0^T q(\eta) \lambda^{-2}(\eta) e^{-\int_0^\eta \mu_0 \lambda^{-2}(r) dr} d\eta$$

is required. The above formal argument suggests that a linear constraint should be imposed on the initial value $\phi(y, 0)$. Therefore, we consider the associated linear Cauchy problem of the inner problem (3.103)

$$\begin{cases}
\lambda^2 \phi_t = \Delta_y \phi + 3U^2(y) \phi + h(y, t) & \text{in } D_{2R} \times (0, T), \\
\phi(y, 0) = e_0 Z_0(y) & \text{in } D_{2R(0)},
\end{cases}$$

(3.142)

where $R = R(t) = \lambda_+^{-\beta}(t)$ for $\beta \in (0, 1/2)$. On the other hand, the parabolic operator $-\lambda^2 \partial_t + L_0$ is certainly not invertible since all the time independent elements in the 5 dimensional kernel of $L_0$ (see (3.87)) also belong to the kernel of $-\lambda^2 \partial_t + L_0$.

In order to construct solution to (3.142) with suitable space-time decay, some orthogonality conditions are expected to hold. So we consider the projected problem

$$\begin{cases}
\lambda^2 \phi_t = \Delta_y \phi + 3U^2(y) \phi + h(y, t) + c_0(t) Z_5(y) + \sum_{\ell=1}^4 c_\ell(t) Z_\ell(y) & \text{in } D_{2R} \times (0, T), \\
\phi(y, 0) = e_0 Z_0(y) & \text{in } D_{2R(0)}.
\end{cases}$$

(3.143)

Our aim is to find suitable solution to problem (3.143) with space-time decay of the following type

$$\|\phi\|_{\infty, v, a} := \sup_{y \in D_{2R}} \lambda_+^{-v}(t)(1 + |y|^a) \left[\|\phi(y, t)\| + (1 + |y|) |\nabla \phi(y, t)|\right],$$

(3.144)
and the right hand side of problem (3.143)
\[ \| h \|_{v,a} := \sup_{y \in \mathcal{D}_R} \lambda_{\gamma}^{v}(t) (1 + |y|^a) |h(y,t)|. \tag{3.145} \]

The construction of such solution is carried out by decomposing the equation into different spherical harmonic modes. Let an orthonormal basis \( \{ \Theta_i \}_{i=0}^\infty \) made up of spherical harmonics in \( L^2(S^3) \), i.e.
\[ \Delta_{S^3} \Theta_i + \tilde{\mu}_i \Theta_i = 0 \quad \text{in} \quad S^3 \]
with \( 0 = \tilde{\mu}_0 < \tilde{\mu}_1 = \cdots = \tilde{\mu}_4 = 3 < \tilde{\mu}_5 \leq \cdots \). More precisely, \( \Theta_0(y) = c_0, \Theta_i(y) = c_1 y_j, j = 1, \ldots, 4 \) for two constants \( c_0, c_1 \) and \( \tilde{\mu}_i \) takes the general form \( i(2 + i) \) with multiplicity \( \frac{(3+i)!}{6i!} \) for \( i \geq 0 \).

For \( h \in L^2(\mathcal{D}_R) \), we decompose it into
\[ h(y,t) = \sum_{j=0}^{\infty} h_j(r,t) \Theta_j(y/r), \quad r = |y|, \quad h_j(r,t) = \int_{S^3} h(r \theta, t) \Theta_j(\theta) d\theta \]
and write \( h = h^0 + h^1 + h^\perp \) with
\[ h^0 = h_0(r,t), \quad h^1 = \sum_{j=1}^{4} h_j(r,t) \Theta_j, \quad h^\perp = \sum_{j=5}^{\infty} h_j(r,t) \Theta_j. \]

Also, we decompose \( \phi = \phi^0 + \phi^1 + \phi^\perp \) in a similar form. Then finding a solution to problem (3.143) is equivalent to finding the pairs \( (\phi^0, h^0), (\phi^1, h^1), (\phi^\perp, h^\perp) \) in each mode.

**Proposition 3.2.2.** Let constants \( a, v, v_1, \sigma \in (0, 1) \) and \( a_1 \in (1, 2) \). For \( T > 0 \) sufficiently small and any \( h(y,t) \) satisfying \( \| h^0 \|_{v,2+a} < +\infty, \| h^1 \|_{v_1,2+a_1} < +\infty, \| h^\perp \|_{v,2+a} < +\infty \), there exists a solution \( (\phi, c^0, c^1, e_0) \) solving (3.143) and
\[ (\phi, c^0, c^1, e_0) = (\phi[h], c^0[h^0], c^1[h^1], e_0[h]) \]
defines a linear operator of \( h \) that satisfies the estimates
For $|y| \leq 2R^\sigma$, 
\[
\begin{aligned}
|\phi(y,t)| + (1 + |y|)|\nabla \phi(y,t)| 
\lesssim 
& \left[ \frac{\lambda^y_\nu(t) R^{(4-a)} \log R}{1 + |y|^4} \|h_0\|_{v,2+a} + \frac{\lambda^y_\nu(t)}{1 + |y|^a} \|h^1\|_{v,1,2+a} \\
+ & \frac{\lambda^y(t)}{1 + |y|^a} \|h^1\|_{v,2+a} \right], 
\end{aligned}
\] (3.146)

For $2R^\sigma \leq |y| \leq 2R$, 
\[
\begin{aligned}
|\phi(y,t)| + (1 + |y|)|\nabla \phi(y,t)| 
\lesssim 
& \left[ \frac{\lambda^y_\nu(t) \log R}{1 + |y|^a} \|h_0\|_{v,2+a} + \frac{\lambda^y_\nu(t)}{1 + |y|^a} \|h^1\|_{v,1,2+a} \\
+ & \frac{\lambda^y(t)}{1 + |y|^a} \|h^1\|_{v,2+a} \right], 
\end{aligned}
\] (3.147)

c^0(t) = - \int_{\mathcal{D}_{2R}} h_0 Z_5 dy - \mathcal{O}[h^0], \quad c^\ell(t) = - \int_{\mathcal{D}_{2R}} h^\ell Z_5 dy \quad \text{for } \ell = 1, \ldots, 4, \quad (3.148)

where $R_\nu = R^\sigma$ and $\mathcal{O}[h^0]$ is a linear operator of $h^0$ satisfying 
\[
|\mathcal{O}[h^0]| \lesssim \lambda^\nu_\nu R_\nu^{a'} \log R \|h^0\|_{v,2+a}
\]
for $a' \in (0, a)$. Moreover, 
\[
|e_0[h]| \lesssim \lambda^\nu_\nu \left( \|h_0\|_{v,2+a} + \|h^1\|_{v,1,2+a} + \|h^1\|_{v,2+a} \right).
\]

We devote the rest of this section to proving Proposition 3.2.2. Our strategy is to construct $\phi = \phi^0 + \phi^1 + \phi^\perp$ mode by mode.

1. **Construction at mode 0.**

We construct $\phi^0$ solving the linearized problem at mode 0 
\[
\begin{aligned}
\lambda^2 \phi^0 &= \Delta \phi^0 + 3U^2(y)\phi^0 + h_0(y,t) + \tilde{c}^0(t)Z_5(y) \quad \text{in } \mathcal{D}_{2R} \times (0, T), \\
\phi^0(y, 0) &= e_0 Z_0(y) \quad \text{in } \mathcal{D}_{2R(0)}. 
\end{aligned}
\] (3.149)
The linear theory for mode 0 is the following

**Proposition 3.2.3.** Let \( \nu, a, \sigma \in (0, 1) \). Suppose \( \|h^0\|_{v, 2+a} < +\infty \). Then there exists a solution \((\phi^0, c^0, e_0)\) to problem (3.149), which depends on \( h^0 \) linearly such that

\[
|\phi^0(y, t)| + (1 + |y|)|\nabla \phi^0(y, t)| \lesssim \lambda_*^\nu \log R \|h^0\|_{v, 2+a} \begin{cases} \frac{R^{\sigma(4-a)}}{1+|y|^\sigma} & \text{for } |y| \leq 2R^\sigma, \\ \frac{1}{1+|y|^\sigma} & \text{for } 2R^\sigma \leq |y| \leq 2R, \end{cases}
\]

\[
\tilde{c}^0[h^0](t) = -\int_{\mathcal{D}_R} h^0 Z_5 dy - \mathcal{O}[h^0],
\]

where \( \mathcal{O}[h^0] \) is a linear operator of \( h^0 \) satisfying

\[
|\mathcal{O}[h^0]| \lesssim \lambda_*^\nu R^{a'-a} \log R \|h^0\|_{v, 2+a}
\]

for \( a' \in (0, a) \). Furthermore, it holds that

\[
|e_0[h^0]| \lesssim \lambda_*^\nu \|h^0\|_{v, 2+a}.
\]

**Remark 3.2.2.** If we define

\[
\|\phi^0\|_{0, \sigma, \nu, a} := \sup_{(y, t) \in \mathcal{D}_R \times (0, T)} \frac{1 + |y|^4}{\lambda_*^\nu(t) R^{\sigma(4-a)(t)}(t) \log R} \left( |\phi^0(y, t)| + (1 + |y|)|\nabla \phi^0(y, t)| \right),
\]

then Proposition 3.2.3 implies that

\[
\|\phi^0\|_{0, \sigma, \nu, a} \lesssim \|h^0\|_{v, 2+a}.
\]

The strategy to prove Proposition 3.2.3 is a new inner–outer gluing scheme. We shall decompose \( \phi^0 \) into inner and outer profiles to get more refined estimates. Before we prove Proposition 3.2.3, we first state a result for the following problem

\[
\begin{aligned}
\lambda^2 \phi = \Delta y \phi + 3U^2(y) \phi + h(y, t) + c^0(t)Z_5 - c(t)Z_0 & \quad & \text{in } \mathcal{D}_R \times (0, T), \\
\phi(y, 0) = 0 & \quad & \text{in } \mathcal{D}_R(0).
\end{aligned}
\]

(3.151)
Proposition 3.2.4. Let $\nu, a \in (0, 1)$. Then for sufficiently large $R$ and any $h$ satisfying $\|h\|_{v, 2+a} < +\infty$, there exists a solution $(\phi, \varepsilon^0, c)$ to (3.151) which is linear in $h$ such that

$$|\phi(y,t)| + (1 + |y|)|\nabla \phi(y,t)| \lesssim \lambda_\nu^R \frac{R^{1-a}\log R}{1 + |y|^4} \|h\|_{v, 2+a}.$$  \hfill (3.152)

$$\varepsilon^0(t) = -\frac{\int_{\mathcal{D}_2R} hZ_5 dy}{\int_{\mathcal{D}_2R} |Z_5|^2 dy},$$

$$|\varepsilon(t) - \int_{\mathcal{D}_2R} hZ_0| \lesssim \lambda_\nu^v(t) \left( \left\|h - Z_0 \int_{\mathcal{D}_2R} hZ_0 \right\|_{v, 2+a} + e^{-\theta R} \|h\|_{v, 2+a} \right). \hfill (3.153)$$

The proof of Proposition 3.2.4 can be carried out similar to that of [43, Section 7]. Proposition 3.2.4 will be needed to describe the inner profile of $\phi^0$ when the inner–outer gluing scheme is carried out.

Proof of Proposition 3.2.3. Suppose

$$\phi^0(y,t) = \phi_1^0 + e(t)Z_0(y)$$

with $\phi_1^0$ solving problem

$$\begin{cases}
\lambda^2 \phi_1 = \Delta_y \phi + 3U^2(y)\phi + h^0(y,t) + \varepsilon^0(t)Z_5 - c(t)Z_0 & \text{in } \mathcal{D}_{2R} \times (0,T), \\
\phi(y,0) = 0 & \text{in } \mathcal{D}_{2R(0)}. 
\end{cases} \hfill (3.154)$$

For $e \in C^1((0,T))$, we get

$$\lambda^2 \phi_1^0 = \Delta_y \phi_1^0 + 3U^2 \phi_1^0 + h^0(y,t) + \varepsilon^0(t)Z_5 + [\lambda^2 \varepsilon(t) - \mu_0 e(t) - c(t)]Z_0(y),$$

from which we see that a natural choice of bounded solution $e(t)$ to

$$\lambda^2 \varepsilon(t) - \mu_0 e(t) - c(t) = 0, \ t \in (0,T)$$

is

$$e(t) = -\int_t^T \exp \left( -\int_t^\eta \frac{\mu_0}{\lambda^2(s)} ds \right) \frac{c(\eta)}{\lambda^2(\eta)} d\eta. \hfill (3.155)$$

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Therefore, $\phi^0$ solves problem (3.149) with the initial condition $\phi^0(y, 0) = e(0)Z_0(y)$. It is clear from (3.155) and (3.153) that

$$|e_0| \lesssim \lambda_3^\nu \|h^0\|_{\nu, 2+a}.$$ 

So, to solve (3.149), we only need to consider (3.154).

Now we carry out an inner–outer gluing scheme for the mode 0. Consider

$$
\begin{cases}
  \lambda^2 \phi_t = \Delta_y \phi + 3U^2(y)\phi + h^0(y, t) + c^0(t)Z_5 - c(t)Z_0 & \text{in } D_{2R} \times (0, T), \\
  \phi(y, 0) = 0 & \text{in } D_{2R(0)}, \\
  \phi = 0 & \text{on } \partial D_{2R} \times (0, T).
\end{cases}
$$

(3.156)

We shall construct $\phi^0$ solving (3.156) of the form

$$\phi^0 = \phi^0_{\text{out}} + \eta_R \phi^0_{\text{in}},$$

where $\eta_R := \eta \left( \frac{|y|}{R} \right)$ with $\eta$ defined in (3.89) and $R_\sigma = R^\sigma$ for $\sigma \in (0, 1)$. A solution $\phi^0$ to (3.156) is found if $\phi^0_{\text{out}}$ and $\phi^0_{\text{in}}$ solve the system

$$
\begin{cases}
  \lambda^2 \partial_t \phi^0_{\text{out}} = \Delta_y \phi^0_{\text{out}} + 3(1 - \eta_R)U^2(y)\phi^0_{\text{out}} + C[\phi^0_{\text{in}}] + (1 - \eta_R)h^0 & \text{in } D_{2R} \times (0, T), \\
  \phi^0_{\text{out}}(y, 0) = 0 & \text{in } D_{2R(0)}, \\
  \phi^0_{\text{out}} = 0 & \text{on } \partial D_{2R} \times (0, T),
\end{cases}
$$

(3.157)

$$
\begin{cases}
  \lambda^2 \partial_t \phi^0_{\text{in}} = \Delta_y \phi^0_{\text{in}} + 3U^2(y)\phi^0_{\text{in}} + 3U^2(y)\phi^0_{\text{out}} + h^0 + c^0Z_5 - cZ_0 & \text{in } D_{2R_\sigma} \times (0, T), \\
  \phi^0_{\text{in}}(y, 0) = 0 & \text{in } D_{2R_\sigma(0)},
\end{cases}
$$

(3.158)

where $C[\phi^0_{\text{in}}] := \phi^0_{\text{in}}(\Delta \eta_R - \lambda^2 \partial_t \eta_R) + 2\nabla \eta_R \cdot \nabla \phi^0_{\text{in}}$. We first consider the outer part (3.157). For the model problem

$$
\begin{cases}
  \lambda^2 \partial_t \psi = \Delta \psi + h^0 & \text{in } D_{2R} \times (0, T), \\
  \psi(y, 0) = 0 & \text{in } D_{2R(0)}, \\
  \psi = 0 & \text{on } \partial D_{2R} \times (0, T),
\end{cases}
$$

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we have
\[ \| \psi \|_{v,a} \lesssim \| h^0 \|_{v,2+a} \] (3.159)

by \( \beta \in (0, 1/2) \) and the parabolic comparison. Then we apply the above estimate to the following problem
\[
\begin{aligned}
\lambda^2 \partial_t \psi &= \Delta \psi + 3(1 - \eta_{R_\ast})U^2 \psi + h^0 \quad \text{in } D_{2R} \times (0, T), \\
\psi(y, 0) &= 0 \quad \text{in } D_{2R(0)}, \\
\psi &= 0 \quad \text{on } \partial D_{2R} \times (0, T),
\end{aligned}
\] (3.160)

and we claim that the solution \( \psi \) to (3.160) satisfies
\[ \| \psi \|_{v,a} \lesssim \| h^0 \|_{v,2+a}. \]

Indeed, by (3.159), we only need to estimate
\[ 3(1 - \eta_{R_\ast})U^2 \psi \lesssim (1 - \eta_{R_\ast}) \frac{\lambda^v_y}{1 + |y|^{a+2}} \| \psi \|_{v,a} \lesssim R_*^{-2} \frac{\lambda^v_y}{1 + |y|^{2+a}} \| \psi \|_{v,a} \]

and we conclude that
\[ \| 3(1 - \eta_{R_\ast})U^2 \psi \|_{v,2+a} \lesssim R_*^{-2} \| \psi \|_{v,a}. \] (3.161)

So from (3.159) and (3.161), we obtain
\[ \| \psi \|_{v,a} \lesssim \| 3(1 - \eta_{R_\ast})U^2 \psi + h^0 \|_{v,2+a} \lesssim R_*^{-2} \| \psi \|_{v,a} + \| h^0 \|_{v,2+a} \]

and for \( R_* \) sufficiently large, it follows that
\[ \| \psi \|_{v,a} \lesssim \| h^0 \|_{v,2+a} \] (3.162)

as desired.

We look for a solution \( \phi_{out}^0 \) to problem (3.157). By (3.162), we get
\[ \| \phi_{out}^0 \|_{v,a'} \lesssim C[\phi_{in}^0]\| v,2+a' \| + (1 - \eta_{R_\ast})h^0 \|_{v,2+a'}, \] (3.163)
where $a' \in (0, a)$. Here $\phi_{out}^0$ defines a linear operator of $\phi_{in}^0$ and $h^0$. We write it as $\phi_{out}^0[\phi_{in}^0, h^0]$. Now we need to find $\phi_{in}^0$ solving the inner part

$$
\begin{align*}
\begin{cases}
\lambda^2 \partial_t \phi_{in}^0 = \Delta \phi_{in}^0 + 3U^2(y)\phi_{in}^0 + 3U^2(y)\phi_{out}^0[\phi_{in}^0, h^0] \\
\phi_{in}^0(y, 0) = 0 \quad \text{in } \Omega_{2R_\ast, (0)}.
\end{cases}
\end{align*}
$$

(3.164)

To solve the inner part (3.164), we consider the fixed point problem

$$
\phi_{in}^0 = \mathcal{F} [3U^2(y)\phi_{out}^0[\phi_{in}^0, h^0] + h^0]
$$

in the function space equipped with the norm

$$
||\phi_{in}^0||_{0, \ast} := \sup_{(y, t) \in \Omega_{2R_\ast} \times (0, T)} \lambda_{\ast}^{-1}(t)R_{\ast}^{d-4}(\log R)^{-1}(1 + |y|^4) \left[ ||\phi_{in}^0|| + (1 + |y|)|\nabla \phi_{in}^0| \right].
$$

We apply Proposition 3.2.4 in the inner regime $\Omega_{2R_\ast} \times (0, T)$, then (3.152) gives

$$
||\mathcal{F}[g]||_{0, \ast} \lesssim ||g||_{v_{2+a}}.
$$

(3.165)

We claim that

$$
||C[\phi_{in}^0]||_{v_{2+a'}} \lesssim R_{\ast}^d - a \log R||\phi_{in}^0||_{0, \ast}.
$$

(3.166)

Indeed, we evaluate

$$
\begin{align*}
|C[\phi_{in}^0]| &= |\phi_{in}^0(\Delta \eta_{R_\ast} - \lambda^2 \partial_t \eta_{R_\ast}) + 2\nabla \eta_{R_\ast} \cdot \nabla \phi_{in}^0| \\
&\lesssim R_{\ast}^{-2} \left| \eta''(y') \right| \left| \lambda_{\ast}^{-1}(t)R_{\ast}^{d-4}(\log R)^{-1}(1 + |y|^4) \right| ||\phi_{in}^0||_{0, \ast} \\
&\lesssim \frac{\lambda_{\ast}^{-1}(t)R_{\ast}^{d-4}(\log R)^{-1}(1 + |y|^4)}{1 + |y|^2 + a'} \lesssim \lambda_{\ast}^{-1}(t)R_{\ast}^{d-4}(\log R)^{-1}(1 + |y|^4) ||\phi_{in}^0||_{0, \ast}.
\end{align*}
$$

which proves (3.166). From (3.163) and (3.166), we then get that

$$
\begin{align*}
||\phi_{out}^0||_{v_{2+a'}} \lesssim R_{\ast}^d - a \log R||\phi_{in}^0||_{0, \ast} + ||(1 - \eta_{R_\ast})h^0||_{v_{2+a'}} \\
&\lesssim R_{\ast}^d - a \log R||\phi_{in}^0||_{0, \ast} + R_{\ast}^d - a' ||h^0||_{v_{2+a'}}.
\end{align*}
$$

(3.167)
Next, we compute
\[ |3U^2(y)\phi_{out}^0| \lesssim \frac{\lambda_v^v}{1 + |y|^{4+a'}} \|\phi_{out}^0\|_{v,a'} \lesssim \frac{\lambda_v^v}{1 + |y|^{2+a}} \|\phi_{out}^0\|_{v,a'} . \]
So we get
\[ \|3U^2(y)\phi_{out}^0\|_{v,2+a} \lesssim \|\phi_{out}^0\|_{v,a'} . \] (3.168)

By (3.167)–(3.168), we obtain
\[ \|3U^2(y)\phi_{out}^0\|_{v,2+a} \lesssim R^{d''-a} \log R \|\phi_{in}^0\|_{0,*} + R^{d''-a} \|h^0\|_{v,2+a} . \] (3.169)

Therefore, we conclude from (3.165) that
\[ \|T[3U^2(y)\phi_{out}^0[\phi_{in}^0,h^0] + h^0]\|_{0,*} \lesssim R^{d''-a} \log R \|\phi_{in}^0\|_{0,*} + \|h^0\|_{v,2+a} , \]
which shows that the operator
\[ \phi_{in}^0 \mapsto T[3U^2(y)\phi_{out}^0[\phi_{in}^0,h^0] + h^0] \]
is a contraction if \( R \) is sufficiently large. A unique fixed point \( \phi_{in}^0 \) thus exists and
\[ \|\phi_{in}^0\|_{0,*} \lesssim \|h^0\|_{v,2+a} . \] (3.170)

Replacing \( a' \) by \( a \) in the computations of (3.163) and (3.166), we obtain
\[ \|\phi_{out}^0\|_{v,a} \lesssim \log R \|h^0\|_{v,2+a} . \] (3.171)

Recalling \( \phi^0 = \phi_{out}^0 + \eta_R, \phi_{in}^0 \) and combining (3.170) and (3.171), we conclude
\[ \|\phi^0(y,t) + (1 + |y|)\nabla \phi^0(y,t)\| \lesssim \lambda_v^v \log R \|h^0\|_{v,2+a} \left\{ \begin{array}{ll}
\frac{R^{d''-a}}{1 + |y|^r} & \text{for } |y| \leq 2R^\sigma, \\
\frac{1}{1 + |y|^r} & \text{for } 2R^\sigma \leq |y| \leq 2R.
\end{array} \right. \]

Finally, we prove the estimate of \( c^0 \). By Proposition 3.2.4, we get
\[ c^0(t) = -\frac{\int_{\partial \mathcal{D}_R} (h^0 + 3U^2(y)\phi_{out}^0[\phi_{in}^0,h^0])Z_5 dy}{\int_{\partial \mathcal{D}_R} |Z_5|^2 dy} . \]

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Notice that $3U^2(y)\phi_{\text{out}}^0[\phi_m^0,h^0]$ is linear in $h^0$. By (3.169)–(3.170), we conclude that
\[
\left| \int_{\mathcal{D}_2 R} 3U^2(y)\phi_{\text{out}}^0[\phi_m^0,h^0]Z_5 dy \right| \lesssim \lambda^\nu \left( R^{d-a}_s \log R \|\phi_m^0\|_{0,a} + R^{d-a}_s \|h^0\|_{v,2+a} \right)
\lesssim \lambda^\nu R^{d-a}_s \log R \|h^0\|_{v,2+a}.
\]

The proof is complete. \qed

2. Construction at modes 1 to 4.

As we can see in mode 0, the estimates are somewhat deteriorated inside the inner regime, and this will result in difficulties when solving the inner problem. One can observe that, for modes 1 to 4, the kernel function for the corresponding linearized operator has faster decay than mode 0 which suggests that the estimates at modes 1 to 4 should be better than mode 0’s. Inspired by the argument in [55, Section 7], we shall carry out the construction for modes 1 to 4 by means of the blow-up argument.

We perform the change of variable
\[
\tau = \tau_\lambda(t) = \tau_0 + \int_0^t \frac{ds}{\lambda^2(s)}
\]
so that $\tau \sim \tau_0 + \frac{|\log(T-t)|^2}{\lambda^2 |\log T|}$. We choose the constant $v'_1 > 0$ so that $\tau^{-v'_1} \sim \lambda^v_1$ for $v_1 \in (0,1)$. We have

**Proposition 3.2.5.** Assume $a_1 \in (1,2), v_1 \in (0,1), \|h^1\|_{v_1,2+a_1} < +\infty$, and
\[
\int_{\mathcal{D}_2 R} h^1(y,\tau)Z_i(y)dy = 0 \text{ for all } \tau \in (\tau_0, +\infty), i = 1, \ldots, 4.
\]

For sufficiently large $R$, there exists a pair $(\phi^1, e_0)$ solving
\[
\begin{align*}
\partial_\tau \phi^1 &= \Delta \phi^1 + 3U^2 \phi^1 + h^1(y,\tau) \quad y \in \mathcal{D}_2 R \times (\tau_0,\infty), \\
\phi^1(y,\tau_0) &= e_0 Z_0(y) \quad y \in \mathcal{D}_2 R,
\end{align*}
\]
and $(\phi^1, e_0) = (\phi^1[h^1], e_0[h^1])$ defines a linear operator of $h^1$ that satisfies
\[
\|\phi^1\|_{v_1,1+a_1} \lesssim \|h^1\|_{v_1,2+a_1},
\]

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In order to prove Proposition 3.2.5, we consider the following Cauchy problem

\[
\begin{cases}
\partial_\tau \phi^1 = \Delta \phi^1 + 3U^2 \phi^1 + h^1(y, \tau) - c(\tau)Z_0(y) & \text{for} \quad y \in \mathbb{R}^4, \quad \tau \geq \tau_0, \\
\phi^1(y, \tau_0) = 0 & \text{for} \quad y \in \mathbb{R}^4,
\end{cases}
\]

with \( h^1 \) supported in \( \mathcal{D}_{2R} \times (\tau_0, +\infty) \) and \( \| h^1 \|_{v_1, 2 + a_1} < +\infty \) in the \((y, \tau)\) variable, where \( v_1 \in (0, 1) \) and \( a_1 \in (1, 2) \). For notational convenience, we denote \( \phi^1 \) by \( \phi \) and \( h^1 \) by \( h \) in the following lemma.

**Lemma 3.2.4.** Assume \( a_1 \in (1, 2), v_1 \in (0, 1), \| h \|_{v_1, 2 + a_1} < +\infty, \) and

\[
\int_{\mathbb{R}^4} h(y, \tau)Z_0(y)dy = 0 \quad \text{for all} \quad \tau \in (\tau_0, +\infty), \quad i = 1, \ldots, 4.
\]

For \( \tau_1 \) sufficiently large, the solution \( \phi(y, \tau) \) to

\[
\begin{cases}
\partial_\tau \phi = \Delta \phi + 3U^2 \phi + h(y, \tau) - c(\tau)Z_0(y) & \text{for} \quad y \in \mathbb{R}^4, \quad \tau \geq \tau_0, \\
\int_{\mathbb{R}^4} \phi(y, \tau)Z_0(y)dy = 0 & \text{for all} \quad \tau \in (\tau_0, +\infty), \\
\phi(y, \tau_0) = 0 & \text{for} \quad y \in \mathbb{R}^4,
\end{cases}
\]

satisfies

\[
\| \phi(y, \tau) \|_{a_1, \tau_1} \lesssim \| h \|_{2 + a_1, \tau_1}.
\]  

Further,

\[
|c(\tau)| \lesssim \tau^{-v_1} R^{a_1} \| h \|_{2 + a_1, \tau_1} \quad \text{for} \quad \tau \in [\tau_0, \tau_1),
\]

where \( \| h \|_{h, \tau} := \sup_{\tau \in [\tau_0, \tau_1)} \tau^{v_1} \sup_{y \in \mathbb{R}^4} (1 + |y|^6)|h(y, \tau)|. \)

**Proof.** Note that problem (3.173) is equivalent to problem (3.172) for

\[
|c(\tau)| \lesssim \tau^{-v_1} R^{a_1} \| h \|_{2 + a_1, \tau_1}.
\]  

By the time decay of \( h \) and spatial decay of \( Z_0 \) (see (3.141)), we have

\[
|c(\tau)| \lesssim \tau^{-v_1} R^{a_1} \| h \|_{2 + a_1, \tau_1}.
\]
Now we prove (3.174) by blow-up argument.

By standard parabolic theory, for any $R' > 0$, there exists a constant $K$ depending on $R'$ and $\tau_1$ such that

$$|\phi(y, \tau)| \leq K, \text{ in } B_{R'} \times [\tau_0, \tau_1].$$

It is easy to check that $\bar{\phi} = C_1 + |y| a_1$ is a super-solution to the original equation (3.173). Thus, $\|\phi\|_{a_1, \tau_1} < +\infty$. We claim that

$$\int \phi Z_i = 0 \text{ for all } \tau \in [\tau_0, \tau_1], \ i = 1, \ldots, 4.$$  

Indeed, we multiply (3.173) by $Z_i \eta_{R'}$, where $\eta_{R'} = \eta(|y| R')$ and $\eta$ is the standard cut-off function defined in (3.89). Then we have

$$\int \phi(\cdot, \tau) \cdot Z_i \eta_{R'} = \int ds \int R^4 (\phi(\cdot, s) \cdot L_0[\eta_{R'} Z_i] + hZ_i \eta_{R'} - c(s) Z_0 Z_i \eta_{R'}).$$

Further computation gives

$$\int R^4 [\phi(\cdot, s) \cdot L_0[\eta_{R'} Z_i] + hZ_i \eta_{R'} - c(s) Z_0 Z_i \eta_{R'}]$$

$$= \int R^4 [\eta_{R'} L_0[Z_i] + Z_i \Delta \eta_{R'} + 2\nabla \eta_{R'} \cdot \nabla Z_i] + hZ_i \eta_{R'} - c(s) Z_0 Z_i \eta_{R'}$$

$$= O((R')^{-\varepsilon})$$

for some $\varepsilon > 0$. By taking $R' \to +\infty$, we get the desired result.

Now we want to prove

$$\|\phi\|_{a_1, \tau_1} \lesssim \|h\|_{2+a_1, \tau_1}.$$  

We prove by contradiction. Suppose that there exist sequences $\tau_i \to +\infty$ and $\phi_k, h_k, c_k$ satisfying

$$\left\{ \begin{array}{l}
 \partial_\tau \phi_k = \Delta \phi_k + 3U^2(y) \phi_k + h_k - c_k(\tau) Z_0(y), \ y \in \mathbb{R}^4, \ \tau \geq \tau_0, \\
 \int R^4 \phi_k(y, \tau) \cdot Z_i(y) dy = 0 \text{ for all } \tau \in [\tau_0, \tau_i], \ i = 0, 1, \ldots, 4, \\
 \phi_k(y, \tau_0) = 0, \ y \in \mathbb{R}^4,
 \end{array} \right.$$
and
\[ \| \Phi_k \|_{a_1, \tau_i^1} = 1, \quad \| h_k \|_{2 + a_1, \tau_i^1} \to 0. \tag{3.176} \]

By (3.175), we know \( \sup_{\tau \in (\tau_0, \tau_i^1)} \tau^\epsilon \| c_k(\tau) \| \to 0. \) We claim that
\[ \sup_{\tau_0 < \tau < \tau_i^1} \tau^\epsilon \| \phi_k(y, \tau) \| \to 0 \tag{3.177} \]
uniformly on compact subsets of \( \mathbb{R}^4. \) We prove (3.177) by contradiction.

**Case 1.** For some \( |y_k| \leq M \) and \( \tau_0 < \tau_2^k < \tau_1^k, \) if
\[ (\tau_2^k)^\epsilon \| \phi_k(y_k, \tau_2^k) \| \geq \frac{1}{2}, \]
then we know that \( \tau_2^k \to +\infty. \) Define
\[ \tilde{\phi}_k(y, \tau) = (\tau_2^k)^\epsilon \phi_k(y, \tau_2^k + \tau). \]

Then
\[ \partial_\tau \tilde{\phi}_k = D_0[\tilde{\phi}_k] + \tilde{h}_k - \tilde{c}_k(\tau)Z_0(y) \text{ in } \mathbb{R}^4 \times (\tau_0 - \tau_2^k, 0]. \]

Due to the spatial decay of \( h \) and \( c, \) we know \( \tilde{h}_k \to 0, \tilde{c}_k \to 0. \) By comparison, we get
\[ |\tilde{\phi}_k(y, \tau)| \leq \frac{1}{1 + |y|^{a_1}} \text{ in } \mathbb{R}^4 \times (\tau_0 - \tau_2^k, 0]. \]

Hence, up to a subsequence, \( \tilde{\phi}_k \to \tilde{\phi} \) uniformly on compact subsets with \( \tilde{\phi} \neq 0 \) and
\[
\begin{cases}
\partial_\tau \tilde{\phi} = \Delta \tilde{\phi} + 3U^2(y)\tilde{\Phi} \text{ in } \mathbb{R}^4 \times (-\infty, 0], \\
\int_{\mathbb{R}^4} \tilde{\phi}(y, \tau) \cdot Z_j(y) dy = 0 \text{ for all } \tau \in (-\infty, 0], j = 0, 1, \ldots, 4, \\
|\tilde{\phi}(y, \tau)| \leq \frac{1}{1 + |y|^{a_1}} \text{ in } \mathbb{R}^4 \times (-\infty, 0], \\
\tilde{\phi}(y, \tau_0) = 0, y \in \mathbb{R}^4. \tag{3.178}
\end{cases}
\]

Note that the orthogonality conditions above are well-defined if \( a_1 > 1. \) We now claim that \( \tilde{\phi} = 0. \) Indeed, by parabolic regularity theory, \( \tilde{\phi}(y, \tau) \) is smooth. By
scaling argument, we get
\[
\frac{1}{1 + |y|} \| \nabla \tilde{\phi} \| + |\tilde{\phi}_\tau| + |\Delta \tilde{\phi}| \lesssim \frac{1}{1 + |y|^{2+a_1}}.
\]

Differentiating (3.178) with respect to \( \tau \), we get
\[
\frac{1}{1 + |y|} \| \nabla \tilde{\phi}_\tau \| + |\tilde{\phi}_{\tau \tau}| + |\Delta \tilde{\phi}| \lesssim \frac{1}{1 + |y|^{4+a_1}}.
\]

Differentiating (3.178) with respect to \( \tau \) and integrating, we get
\[
\frac{1}{2} \partial_\tau \int_{\mathbb{R}^4} |\tilde{\phi}_\tau|^2 + B(\tilde{\phi}_\tau, \tilde{\phi}_\tau) = 0,
\]
where
\[
B(\tilde{\phi}, \tilde{\phi}) = \int_{\mathbb{R}^4} |\nabla \tilde{\phi}|^2 - 3U^2(y)|\tilde{\phi}|^2 dy.
\]

Since \( \int_{\mathbb{R}^4} \tilde{\phi}(y, \tau) : Z_j(y) dy = 0 \) for all \( \tau \in (-\infty, 0] \), \( j = 0, 1, \ldots, 4 \), \( B(\tilde{\phi}, \tilde{\phi}) \geq 0 \). Also, we have
\[
\int_{\mathbb{R}^4} |\tilde{\phi}_\tau|^2 = -\frac{1}{2} \partial_\tau B(\tilde{\phi}, \tilde{\phi}).
\]

From above, we get
\[
\partial_\tau \int_{\mathbb{R}^4} |\tilde{\phi}_\tau|^2 \leq 0, \quad \int_{-\infty}^0 d\tau \int_{\mathbb{R}^4} |\tilde{\phi}_\tau|^2 < +\infty.
\]

Hence \( \tilde{\phi}_\tau = 0 \). So \( \tilde{\phi} \) is independent of \( \tau \) and \( L_0[\tilde{\phi}] = 0 \). Since \( \tilde{\phi} \) is bounded, by the non-degeneracy of \( L_0 \), \( \tilde{\phi} \) is a linear combination of \( Z_j \), \( j = 1, \ldots, 4 \). From orthogonality conditions \( \int_{\mathbb{R}^4} \tilde{\phi} \cdot Z_j = 0 \), \( j = 1, \ldots, 4 \), we obtain \( \tilde{\phi} = 0 \), a contradiction. Thus,
\[
\sup_{\tau_0 < \tau < \tau_k^0} \tau_k^j|\phi_k(y, \tau)| \rightarrow 0.
\]

**Case 2.** Suppose there exists \( y_k \) with \( |y_k| \rightarrow +\infty \) such that
\[
\left( \tau_k^j \right)^{\nu_j} (1 + |y_k|^{a_1}) |\phi_k(y_k, \tau_k^j)| \geq \frac{1}{2}.
\]
Let
\[ \tilde{\phi}_k(z, \tau) := (\tau^2_k)^{\nu'_i}|y_k|^{a_1}\phi_k(y_k + |y_k|z, |y_k|^2\tau + \tau^2_k). \]
Then
\[ \partial_\tau\tilde{\phi}_k = \Delta \tilde{\phi}_k + a_k \tilde{\phi}_k + \tilde{h}_k(z, \tau), \]
where \( a_k = 3U^2(y_k + |y_k|z), \)
\[ \tilde{h}_k(z, \tau) = (\tau^2_k)^{\nu'_i}|y_k|^{2+a_1}h_k(y_k + |y_k|z, |y_k|^2\tau + \tau^2_k) \]
\[ - (\tau^2_k)^{\nu'_i}|y_k|^{2+a_1}c(|y_k|^2\tau + \tau^2_k)Z_0(y_k + |y_k|z). \]

By the definition of \( h_k, \)
\[ |\tilde{h}_k(z, \tau)| \lesssim o(1) \frac{((\tau^2_k)^{-1}|y_k|^2\tau + 1)^{-\nu'_i}}{|\hat{y}_k + z|^{2+a_1}} \]
with \( \hat{y}_k = \frac{y_k}{|y_k|} \to -\hat{e} \) and \( |\hat{e}| = 1. \) Thus \( \tilde{h}_k(z, \tau) \to 0 \) uniformly on compact subsets of \( \mathbb{R}^4 \setminus \{\hat{e}\} \times (-\infty, 0] \) and \( a_k \) has the same property. Moreover, \( |\tilde{\phi}_k(0, \tau_0)| \geq \frac{1}{2} \) and
\[ |\tilde{\phi}_k(z, \tau)| \lesssim \frac{((\tau^2_k)^{-1}|y_k|^2\tau + 1)^{-\nu'_i}}{|\hat{y}_k + z|^{a_1}}. \]
Hence we may assume \( \tilde{\phi}_k \to \tilde{\phi} \neq 0 \) uniformly on compact subsets of \( \mathbb{R}^4 \setminus \{\hat{e}\} \times (-\infty, 0] \) with \( \tilde{\phi} \) satisfying
\[ \tilde{\phi}_\tau = \Delta \tilde{\phi} \text{ in } \mathbb{R}^4 \setminus \{\hat{e}\} \times (-\infty, 0] \quad (3.179) \]
and
\[ |\tilde{\phi}(z, \tau)| \leq |z - e|^{-a_1} \text{ in } \mathbb{R}^4 \setminus \{\hat{e}\} \times (-\infty, 0]. \quad (3.180) \]

**Claim**: functions \( \tilde{\phi} \) satisfying (3.179) and (3.180) are 0.

Without loss of generality, we assume \( \hat{e} = 0. \) Then
\[ \begin{cases} \tilde{\phi}_\tau = \Delta \tilde{\phi} \text{ in } \mathbb{R}^4 \setminus \{0\} \times (-\infty, 0], \\ |\tilde{\phi}(z, \tau)| \leq |z|^{-a_1} \text{ in } \mathbb{R}^4 \setminus \{0\} \times (-\infty, 0]. \end{cases} \quad (3.181) \]
We consider the function \( \tilde{u}(\rho, \tau) = (\rho^2 + c\tau)^{-\frac{a_1}{2} + \epsilon\rho^{-2}} \) for some constant \( c > 0. \)
Direct computations give us
\[
\bar{u}_\tau - \Delta \bar{u} = a_1 (\rho^2 + c\tau)^{-\frac{a_1}{2} - 2} \left[ (2-a_1 - \frac{c}{2})\rho^2 + (4c - \frac{c^2}{2})\tau \right].
\]

Then we know that if \( a_1 < 2 \), we can always find \( c > 0 \) such that \( \bar{u}(\rho, \tau + M) \) is a super-solution, where \( M \) is a large constant. Thus, \( |\tilde{\phi}| \leq 2\bar{u}(\rho, \tau + M) \). By letting \( M \to \infty \) and the arbitrariness of \( \varepsilon \), we get \( \tilde{\phi} = 0 \), a contradiction. The proof is complete. \( \square \)

**Proof of Proposition 3.2.5** From Lemma 3.2.4, for any \( \tau_1 > \tau_0 \) with \( \tau_0 \) fixed sufficiently large, we have
\[
|\phi^1(y, \tau)| \lesssim \tau^{-v_1} (1 + |y|)^{-a_1}\|h^1\|_{2-a_1, \tau} \text{ for all } \tau \in (\tau_0, \tau_1), y \in \mathbb{R}^4,
\]
\[
|c(\tau)| \lesssim \tau^{-v_1} R^{a_1}\|h^1\|_{2-a_1, \tau} \text{ for all } \tau \in (\tau_0, \tau_1).
\]

By assumption, \( \|h^1\|_{1,2+a_1} < +\infty \) and \( \|h^1\|_{2+a_1, \tau} \leq \|h^1\|_{1,2+a_1} \) for an arbitrary \( \tau_1 \). It then follows that
\[
|\phi^1(y, \tau)| \lesssim \tau^{-v_1} (1 + |y|)^{-a_1}\|h^1\|_{1,2+a_1} \text{ for all } \tau \in (\tau_0, \tau_1), y \in \mathbb{R}^4,
\]
\[
|c(\tau)| \lesssim \tau^{-v_1} R^{a_1}\|h^1\|_{1,2+a_1} \text{ for all } \tau \in (\tau_0, \tau_1).
\]

By the arbitrariness of \( \tau_1 \), we have
\[
|\phi^1(y, \tau)| \lesssim \tau^{-v_1} (1 + |y|)^{-a_1}\|h^1\|_{1,2+a_1} \text{ for all } \tau \in (\tau_0, +\infty), y \in \mathbb{R}^4,
\]
\[
|c(\tau)| \lesssim \tau^{-v_1} R^{a_1}\|h^1\|_{1,2+a_1} \text{ for all } \tau \in (\tau_0, +\infty).
\]

The gradient estimates follows from the scaling argument and the standard parabolic theory. The proof is complete. \( \square \)

**3. Construction at higher modes \( j \geq 5 \).**

For higher modes \( j \geq 5 \), we recall that
\[
h^1 = \sum_{j=5}^{\infty} h_j(r,\tau)\Theta_j, \quad \phi^1(h^1) = \sum_{j=5}^{\infty} \phi_j(r,\tau)\Theta_j
\]

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and let $\phi^\perp[h^\perp]$ solve the following problem

$$
\begin{cases}
\lambda^2 \phi_t = \Delta y \phi + 3U^2(y)\phi + h^\perp & \text{in } D_2 \times (0, T), \\
\phi = 0 & \text{on } \partial D_2 \times (0, T), \\
\phi(\cdot, 0) = 0 & \text{in } D_2.
\end{cases}
$$

Similarly, it follows from [43, Section 7] that

$$
|\phi^\perp(y, t)| + (1 + |y|)|\nabla \phi^\perp(y, t)| \lesssim \lambda^\ast \frac{1}{1 + |y|^a} \|h^\perp\|_{\nu, 2 + a}.
$$

**Proof of Proposition 3.2.2** Recall that $\phi[h] = \phi^0[h^0] + \phi^1[h^1] + \phi^\perp[h^\perp]$. The validity of Proposition 3.2.2 is concluded from Proposition 3.2.3, Proposition 3.2.5 and (3.182). The proof is complete.

### 3.2.7 Solving the inner–outer gluing system

In this section, we shall solve the inner–outer gluing system by the linear theories developed in Section 3.2.5 and Section 3.2.6, and the Schauder fixed point theorem. Our goal is to find a solution $(\phi^0, \phi^1, \phi^\perp, \psi, \lambda, \xi)$ to the inner–outer gluing system in Section 3.2.3 so that the desired blow-up solution is constructed. We shall solve the inner–outer gluing system in the function space $X$ defined in (3.258). First, we make some assumptions about the parameter functions.

We write $$\lambda_\ast(t) = \frac{|\log T| |(T - t)|}{|\log(T - t)|^2}$$

and assume that for some numbers $c_1, c_2 > 0$,

$$c_1 |\lambda_\ast(t)| \leq |\dot{\lambda}(t)| \leq c_2 |\lambda_\ast(t)| \text{ for all } t \in (0, T).$$

Recall that we take $R(t) = \lambda_\ast^{-\beta}(t)$ for $\beta \in (0, 1/2)$.

For $\|\phi^0\|_{a, \sigma, v, a}, \|\phi^1\|_{in, v_1, a_1}, \|\phi^\perp\|_{in, v, a}, \|\psi\|_\ast, \|Z^\ast\|_{\infty}, \|\lambda\|_F, \|\xi\|_G$ bounded, we shall first estimate right hand sides $G(\phi, \psi, \lambda, \xi), H(\phi, \psi, \lambda, \xi)$ in the inner and outer problems. Here the above norms are defined in (3.150), (3.144), (3.122), (3.256) and (3.257).
The outer problem: estimates of $\mathcal{G}$

Recall from (3.105) that the outer problem

$$\psi_t = \Delta_{(r,z)} \psi + \frac{n - 4}{r} \partial_r \psi + \mathcal{G}(\phi, \psi, \lambda, \xi) \quad \text{in} \quad \mathcal{D} \times (0, T),$$

where $\mathcal{G}(\phi, \psi, \lambda, \xi)$ is defined in (3.106). In order to apply the linear theory Proposition 3.2.1, we estimate all the terms in $\mathcal{G}(\phi, \psi, \lambda, \xi)$. Define

$$\mathcal{G}(\phi, \psi, \lambda, \xi) = g_1 + g_2 + g_3$$

with

$$g_1 := 3\lambda^{-2}(1 - \eta_R)U^2(y)(\psi + Z^* + \eta_0 \Psi_0 + \Psi_1),$$

$$g_2 := \lambda^{-3} \left[ (\Delta y \eta_R) \phi + 2 \nabla_y \eta_R \cdot \nabla \phi - \lambda^2 \phi \partial_t \eta_R \right] + \frac{(n - 4)\lambda^{-1}}{r} \phi \partial_r \eta_R,$$

$$g_3 := (1 - \eta_R) \mathcal{N}[\lambda, \xi] + \mathcal{S}_{\text{out}}[\lambda, \xi] - \mathcal{S}_{\text{out}}[\lambda_0, \xi_0] + (1 - \eta_R) \mathcal{N}(w).$$

To estimate $g_1$, we need to estimate the corrections $\Psi_0$ and $\partial_r \Psi_0$ defined in (3.94) and (3.100), respectively.

**Estimates of $\Psi_0$ and $\partial_r \Psi_0$**

We first estimate the size of $\Psi_0$. Decompose

$$\Psi_0 = -\alpha_0 \int_{-T}^{t} \hat{\lambda}(s) \frac{1 - e^{-\frac{\zeta^2}{4(t-s)}}}{\zeta^2} ds = -\alpha_0 \left( \int_{-T}^{t} + \int_{t}^{t + \frac{T}{4}} \right) \hat{\lambda}(s) \frac{1 - e^{-\frac{\zeta^2}{4(t-s)}}}{\zeta^2} ds.$$

(3.183)

For the first integral above, we have two cases

- For $T - t > \frac{\zeta^2}{4}$, we further decompose

$$\int_{-T}^{t - \frac{\zeta^2}{4}} \hat{\lambda}(s) \frac{1 - e^{-\frac{\zeta^2}{4(t-s)}}}{\zeta^2} ds = \left( \int_{-T}^{t - (T-t)} + \int_{t - (T-t)}^{t - \frac{\zeta^2}{4}} \right) \hat{\lambda}(s) \frac{1 - e^{-\frac{\zeta^2}{4(t-s)}}}{\zeta^2} ds.$$

Since $T - s < 2(t - s)$ and $\frac{\zeta^2}{4(t-s)} < 1$, the first integral above can be evaluated
\[
\int_{-T}^{t-(T-t)} \hat{\lambda}(s) \frac{1 - e^{-\frac{s^2}{2(T-s)}}}{\xi^2} ds \lesssim \int_{-T}^{t-(T-t)} \frac{\hat{\lambda}(s)}{T-s} ds
\]  
(3.184)

\[
\lesssim |\log T| \int_{-T}^{t-(T-t)} \frac{1}{(T-s)|\log(T-s)|^2} ds \lesssim 1.
\]

Similarly, for the second integral

\[
\int_{t-(T-t)}^{t-\frac{\xi^2}{4}} \hat{\lambda}(s) \frac{1 - e^{-\frac{s^2}{2(T-s)}}}{\xi^2} ds \lesssim \int_{t-(T-t)}^{t-\frac{\xi^2}{4}} \frac{|\log T|}{(T-s)|\log(T-s)|^2} ds
\]  
(3.185)

\[
\lesssim \frac{|\log T|}{|\log(T-t)|^2} \left| \log \left( \frac{s^2}{4} \right) - \log(T-t) \right| 
\lesssim |\hat{\lambda}| \left[ |\log(p^2 + \lambda^2)| + 1 \right].
\]

- For \(T-t < \frac{\xi^2}{4}\), since \(s < t - \frac{\xi^2}{4} < t - (T-t)\), we compute

\[
\int_{-T}^{t-\frac{\xi^2}{4}} \frac{\hat{\lambda}(s)}{t-s} ds \lesssim \int_{-T}^{t-(T-t)} \frac{\hat{\lambda}(s)}{T-s} ds \lesssim 1.
\]  
(3.186)

Then we evaluate

\[
\int_{t-\frac{\xi^2}{4}}^{t} \hat{\lambda}(s) \frac{1 - e^{-\frac{s^2}{2(T-s)}}}{\xi^2} ds \lesssim \frac{1}{\xi^2} \int_{t-\frac{\xi^2}{4}}^{t} |\hat{\lambda}(s)| ds \lesssim 1.
\]  
(3.187)

Combining (3.183)–(3.187), we conclude that

\[
|\Psi_0| \lesssim |\hat{\lambda}| \left[ |\log(p^2 + \lambda^2)| + 1 \right].
\]  
(3.188)
Then we want to compute the size of $\partial_r \Psi_0$. Similarly, we have

$$
\partial_r \Psi_0 = -\alpha_0 \int_{-T}^{t} \hat{\lambda}(s) k_{\xi}(\xi(p,t), t-s) \xi, ds \\
\lesssim (r - \sqrt{2(n-4)(T-t)}) \left( \int_{T}^{t} \frac{\xi^2}{\xi} ds + \int_{t}^{T} \frac{\xi^2}{\xi} ds \right) \hat{\lambda}(s) \\
\times \frac{\left( \xi^2 e^{-\frac{\xi^2}{2(t-s)}} - 2 + 2e^{-\frac{\xi^2}{4(t-s)}} \right)}{\xi^4} \right) ds.
\tag{3.189}
$$

For the first integral, we decompose

$$
\int_{-T}^{t} \frac{\xi^2}{\xi} \hat{\lambda}(s) \left( \xi^2 e^{-\frac{\xi^2}{2(t-s)}} - 2 + 2e^{-\frac{\xi^2}{4(t-s)}} \right) ds \\
= \left( \int_{-T}^{t-(T-t)} + \int_{t-(T-t)}^{t} \right) \frac{\xi^2}{\xi} \hat{\lambda}(s) \left( \xi^2 e^{-\frac{\xi^2}{2(t-s)}} - 2 + 2e^{-\frac{\xi^2}{4(t-s)}} \right) ds \\
\lesssim \int_{-T}^{t-(T-t)} \frac{\hat{\lambda}(s)}{(T-s)^2} ds + \int_{t-(T-t)}^{t} \frac{\hat{\lambda}(s)}{(t-s)^2} ds \\
\lesssim \frac{|\hat{\lambda}|}{T-t} + \frac{|\hat{\lambda}|}{\rho^2 + \lambda^2}. \tag{3.190}
$$

On the other hand, one has

$$
\int_{t}^{\xi} \frac{\xi^2}{\xi} \hat{\lambda}(s) \left( \xi^2 e^{-\frac{\xi^2}{2(t-s)}} - 2 + 2e^{-\frac{\xi^2}{4(t-s)}} \right) ds \lesssim \frac{|\hat{\lambda}|}{\rho^2 + \lambda^2}. \tag{3.191}
$$

By (3.189)–(3.191), we obtain

$$
\partial_r \Psi_0 \lesssim (r - \sqrt{2(n-4)(T-t)}) \max \left\{ \frac{|\hat{\lambda}|}{T-t}, \frac{|\hat{\lambda}|}{\rho^2 + \lambda^2} \right\}. \tag{3.192}
$$

**Estimate of $\Psi_1$**

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Similar to the proof of Lemma 3.2.1, Lemma 3.2.2, and Lemma 3.2.3 in Section 3.2.8, the correction term \( \Psi_1 \) given by (3.100) can be estimated as

\[
|\Psi_1| \lesssim \int_0^t \int_{\mathbb{R}^n} e^{-\frac{|w-w|}{|y_0(y)|}} \mathcal{J}_{\text{out}}[\lambda_*, \xi_*](w, s) dwdsl,
\]

where

\[
\mathcal{J}_{\text{out}}[\lambda_*, \xi_*] := U_{\lambda_*, \xi_*}(\Delta_{(r,z)} \eta_* - \partial_t \eta_*) + 2\nabla \eta_* \cdot \nabla U_{\lambda_*, \xi_*} + (\eta_*^3 - \eta_*) U_{\lambda_*, \xi_*}^3, \quad U_{\lambda_*, \xi_*} \frac{n-4}{r} \partial_r \eta_*
\]

\[
+ \frac{n-4}{r} \Psi_0[\lambda_*] \partial_r \eta_* + \Psi_0[\lambda_*] \Delta_{(r,z)} \eta_* + 2\nabla \eta_* \cdot \nabla \Psi_0[\lambda_*] - \Psi_0[\lambda_*] \partial_t \eta_*
\]

\[
+ \frac{n-4}{r} \eta_* (1 - \eta_R) \partial_r \Psi_0[\lambda_*] + \eta_* (1 - \eta_R) \frac{n-4}{\lambda_* y_1 + \xi_* y_1} \lambda_*^{-2} \partial_r U(y)
\]

\[
+ \eta_* (1 - \eta_R) \lambda_*^{-2} \nabla U \cdot \xi_*.
\]

Here we take the term

\[
U_{\lambda_*, \xi_*}(\Delta_{(r,z)} \eta_* - \partial_t \eta_*) + \frac{n-4}{r} \eta_* (1 - \eta_R) \partial_r \Psi_0[\lambda_*]
\]

\[
+ \eta_* (1 - \eta_R) \left( \frac{n-4}{\lambda_* y_1 + \xi_* y_1} \lambda_*^{-2} \partial_r U(y) + \lambda_*^{-2} \nabla U \cdot \xi_* \right)
\]

in \( \mathcal{J}_{\text{out}}[\lambda_*, \xi_*] \) as an example, and all the estimates for other terms can be carried out in a similar manner. By the definition of \( \eta_* \) in (3.88) and (3.192), we have

\[
\left| U_{\lambda_*, \xi_*}(\Delta_{(r,z)} \eta_* - \partial_t \eta_*) + \frac{n-4}{r} \eta_* (1 - \eta_R) \partial_r \Psi_0[\lambda_*] \right| \lesssim \frac{\lambda_*}{(T-t)^{\frac{3}{2}} \mathcal{X}(\rho \sim \sqrt{T-t}) + \frac{1}{\lambda_* R \sqrt{T-t}} \mathcal{X}(\lambda_* R^{\rho} \leq \sqrt{T-t})}.
\]

On the other hand, thanks to our choice \( \xi_{r_*}(t) = \sqrt{2(n-4)(T-t)} \), we have

\[
\dot{\xi}_{r_*} + \frac{n-4}{\lambda_* y_1 + \xi_* y_1} = -\frac{(n-4)\lambda_* y_1}{\xi_{r_*}(\lambda_* y_1 + \xi_* y_1)},
\]

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and thus
\[
\frac{n-4\lambda\eta R}{\lambda_y+|\xi|^2} - \lambda^{-2} \partial_y^2 U(y) + \lambda^{-2} \nabla U \cdot \dot{\xi}
\]
\[\leq \frac{\lambda^{-1} y^2}{\xi R(\lambda_y+|\xi|^2)^2} \chi_{E} \leq \frac{\lambda^{-1}}{(T-t)R^2}.
\]
Then from (3.193), we obtain
\[
|\Psi_1| \leq \int_0^T \frac{\lambda_y(t)}{(T-s)^2} + \frac{1}{\lambda_y(s)R(s)\sqrt{T-s}} + \frac{\lambda^{-1}(s)}{(T-s)R^2(s)} ds
\]
\[\leq |\log(T-t)|\lambda^{-2-1}(t)
\]
where we have used \( R(t) = \lambda^{-\beta}(t) \).

**Estimate of \( g_1 \).**

Since we shall solve \( \psi \) in the function space \( X_\psi \) defined in (3.253), we get
\[
g_1 = 3\lambda^{-2}(1-\eta R)U(y)(\psi + Z + \eta \Psi_0 + \Psi_1)
\]
\[\leq \frac{R^{-2}(t)\lambda^{v+2+\frac{\nu}{2}}(0)R^{-2-\alpha+\frac{\nu}{2}}(0)|\log T|\chi_{E}(|r,z|-|\xi|\geq \lambda, R)}{|(r,z)-\xi|^2} \|\psi\|_*
\]
\[+ \frac{R^{-2}(t)|\log(T-t)|\chi_{E}(|r,z|-|\xi|\geq \lambda, R)}{|(r,z)-\xi|^2} \|Z\|_*
\]
\[+ \frac{R^{-2}(t)\lambda^{2\beta-1}(t)|\log(T-t)|\chi_{E}(|r,z|-|\xi|\geq \lambda, R)}{|(r,z)-\xi|^2} \|\lambda\|_*
\]
\[+ \frac{R^{-2}(t)\lambda^{2\beta-1}(t)|\log(T-t)|\chi_{E}(|r,z|-|\xi|\geq \lambda, R)}{|(r,z)-\xi|^2} \|\lambda\|_*
\]
by using (3.194). So by the choice of the weight \( \rho_2 \) as in (3.120), we have
\[
\|g_1\|_* \leq T^{\rho_2}(|\psi|_* + |\lambda\|_* + 1) \]
\[\leq T^{\rho_2}(|\psi|_* + |\lambda\|_* + 1) \]
provided
\[v - 2 + \frac{4}{n} + 2\beta(4 + \alpha - \frac{8}{n}) - v_2 > 0, \quad 2\beta - v_2 > 0,
\]
\[4\beta - 1 - v_2 > 0.
\]
Here $\varepsilon_0$ is a small positive number.

**Estimate of $g_2$.**

Thanks to the cut-off, $g_2$ is supported in

$$\{(r, z, t) \in \mathcal{D} \times (0, T) : \exists s \in (0, T), t \in (0, T) \}$$

and we have

$$g_2 = \lambda^{-3} \left[ (\Delta_y \eta_R) \phi + 2 \nabla_y \eta_R \cdot \nabla_y \phi - \lambda^2 (\partial_t \eta_R) \phi \right] + \frac{(n-4) \lambda^{-1}}{r} \phi \partial_r \eta_R \lesssim (R^{a-\alpha} \log R + \lambda a^{1/2} \log R + \lambda^{\nu+1/2} R^{1+a-a_1}) \times \rho_1 \left( \| \phi^0 \|_{0, v, a} + \| \phi^1 \|_{\text{in}, v_1, a_1} + \| \phi^\perp \|_{\text{in}, v, a} \right).$$

So it follows that

$$\|g_2\|_{s, \sigma} \lesssim T^{\varepsilon_0} \left( \| \phi^0 \|_{0, v, a} + \| \phi^1 \|_{\text{in}, v_1, a_1} + \| \phi^\perp \|_{\text{in}, v, a} \right)$$

provided

$$0 < \alpha < a < 1, \quad \beta < \frac{1}{2},$$

$$\beta(1 + \alpha - a_1) + \nu - v_1 - \frac{1}{2} < 0.$$ 

(3.198)

Here $\varepsilon_0$ is a small positive number.

**Estimate of $g_3$.**

We want to estimate

$$g_3 = (1 - \eta_R) \mathcal{J}_1[\lambda, \xi] + \mathcal{J}_\text{out}[\lambda, \xi] - \mathcal{J}_\text{out}[\lambda_*, \xi_\ast] + (1 - \eta_R).$$

Recall (3.107). We have

$$\int_{-T}^t \lambda(s) k_\xi (\zeta, t-s) ds = \int_{-T}^t \frac{e^{-\zeta^2 / 2 (t-s)}}{2 \zeta (t-s)} \left[ \frac{2 \left( 1 - e^{-\zeta^2 / 2 (t-s)} \right)}{\zeta^3} \right] ds$$

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\[
\begin{aligned}
&\lesssim \left( \int_{t-T}^{t-(T-t)} + \int_{t-(T-t)}^{t-(T-t/2)} \right) \dot{\lambda}(s) \frac{\zeta}{(t-s)^2} ds \\
&\quad + \int_{t-(T-t/2)}^{t} \frac{\dot{\lambda}(s)}{\zeta^3} ds \\
&\lesssim \frac{|\dot{\lambda}|\zeta}{T-t} = \frac{\dot{\lambda}}{\zeta}
\end{aligned}
\] (3.199)

and
\[
\begin{aligned}
&\lesssim \left( \int_{t-T}^{t-(T-t)} + \int_{t-(T-t)}^{t-(T-t/2)} \right) \dot{\lambda}(s) \left( \frac{\zeta}{(t-s)^2} + \frac{\zeta^3}{(t-s)^3} \right) ds + \int_{t-(T-t/2)}^{t} \frac{\dot{\lambda}(s)}{\zeta^3} ds \\
&\lesssim \frac{|\dot{\lambda}|\zeta}{T-t} + \frac{|\dot{\lambda}|\zeta^3}{T-t} + \frac{\dot{\lambda}}{\zeta}.
\end{aligned}
\] (3.200)

From (3.199)–(3.200), we obtain
\[
\mathcal{R}[\lambda] \lesssim |\dot{\lambda}| \left( \frac{1}{(T-t)(1 + |y|^2)} + \frac{\lambda^2}{(T-t)^2} + \frac{1}{\lambda^2(1 + |y|^2)^2} \right)
\]
\[
+ \frac{\lambda(y \cdot \dot{\xi} + \dot{\lambda})}{T-t} + \frac{y \cdot \dot{\xi} + \dot{\lambda}}{\lambda(1 + |y|^2)}.
\] (3.201)

Now we evaluate the first term \((1 - \eta_R)\mathcal{X}_1[\lambda, \xi] \) in \(g_3\) by using (3.201). Thanks to the cut-off \((1 - \eta_R)\eta_\ast\), the term \((1 - \eta_R)\mathcal{X}_1[\lambda, \xi] \) is supported in
\[
\{(r, z, t) \in \mathcal{D} \times (0, T) : \lambda_\ast R \leq |(r, z) - \xi(t)| \leq 2\delta \sqrt{T-t}\}. 
\]
So we get
\[
(1 - \eta_R)\mathcal{X}_1[\lambda, \xi] = (1 - \eta_R)\eta_\ast \left[ 2\alpha_0 \lambda^{-2}(t) \dot{\lambda}(t) \right. \\
\left. - \alpha_0 \lambda^{-2}(t) \dot{\lambda}(t) \right] \\
\lesssim R^{-2} \lambda^{-2} \|\lambda\|_{\mathcal{P}_2} + \|\lambda\|_{\mathcal{P}_2} \lambda^{-1} |\xi| \|\lambda\|_{\mathcal{P}_2} (1 + |\xi|) + T^0 \|\lambda\|_{\mathcal{P}_2},
\]

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where the $\| \cdot \|_G$-norm is defined in (3.257). We see that if

$$v_2 - 2\beta < 0, \quad v_2 - 1 < 0,$$

then one has

$$\| (1 - \eta_R) \mathcal{K} [\lambda, \xi] \|_{* *} \lesssim T^\varepsilon_0 (\| \lambda \|_{\infty} + \| \xi \|_G + 1)$$

for some $\varepsilon_0 > 0$.

Next, we consider $\mathcal{S}_{out} [\lambda, \xi] - \mathcal{S}_{out} [\lambda_\ast, \xi_\ast]$. Similarly, direct computations yield that

$$| \mathcal{S}_{out} [\lambda, \xi] - \mathcal{S}_{out} [\lambda_\ast, \xi_\ast] | \lesssim \frac{\lambda_\ast \sigma'(t)}{T - t} (\mathcal{K} (\| \xi(t) \|_{\infty} - \xi)}{\| \mathcal{F}_{\xi(t)} \|_{\lambda, \xi}}$$

$$\lesssim T^\varepsilon_0 \rho_2,$$

where $\sigma', \varepsilon > 0$. So we get

$$\| \mathcal{S}_{out} [\lambda, \xi] - \mathcal{S}_{out} [\lambda_\ast, \xi_\ast] \|_{* *} \lesssim T^\varepsilon_0.$$ (3.204)

Finally, we compute the nonlinear terms

$$(1 - \eta_R) \mathcal{N}(w) = (1 - \eta_R) \left( (U^* + w)^3 - (U^*)^3 - 3U^2 \lambda \xi w \right) \lesssim (1 - \eta_R) U^* w^2$$

$$\lesssim \lambda \nu R^{2\sigma(4-a)+\alpha} (\log R)^2 \| \phi \|_{0, v, \alpha}^2 \rho_1$$

$$+ \lambda^{v} R^\alpha \| \phi \|_{v}^2 \| \lambda \|_{v, \alpha} \rho_1$$

$$+ \lambda^v R^\alpha \| \phi \|_{v}^2 \| \lambda \|_{v, \alpha} \rho_1 + | \log(T - t)/2 | \| \lambda \|_{2}^2 \rho_2$$

$$+ \lambda^v \rho_2 \| Z^* \|_{\infty}^2 + \lambda^{\alpha - v_2} | \log(T - t)/2 | \rho_2.$$
Therefore, we obtain that for $\varepsilon_0 > 0$

$$
\|(1 - \eta_k)\mathcal{N}(w)\|_{**} \lesssim T^{\varepsilon_0} \left( \|\phi^0\|_{0,\sigma,v,a}^2 + \|\phi^1\|_{in,v_1,a_1}^2 + \|\phi^1\|_{in,v,a}^2 \\
+ \|\psi\|_s^2 + \|\mathcal{Z}\|_\alpha^2 + \|\lambda\|_\infty + 1 \right)
$$  \hspace{1cm} (3.205)

provided

$$
v - \beta (2\sigma (4 - a) + \alpha - 8) > 0, \quad 2v_1 - \beta (\alpha - 2a_1) > 0, \\
v - \beta (\alpha - 2a) > 0, \quad v_2 < 1, \\
2v - 3 + \frac{8}{n} - v_2 + \beta (4 + 2\alpha - \frac{16}{n}) > 0, \quad 4\beta - v_2 - 1 > 0. \hspace{1cm} (3.206)
$$

Collecting (3.195)–(3.198), and (3.202)–(3.206), we conclude that for a fixed number $\varepsilon_0 > 0$

$$
\|\mathcal{G}\|_{**} \lesssim T^{\varepsilon_0} \left( \|\psi\|_s + \|\mathcal{Z}\|_\infty + \|\phi^0\|_{0,\sigma,v,a} + \|\phi^1\|_{in,v_1,a_1} + \|\phi^1\|_{in,v,a} + \|\lambda\|_\infty + \|\xi\|_G + 1 \right)
$$  \hspace{1cm} (3.207)

with the parameters $\beta, a, a_1, \alpha, v, v_1, v_2, \sigma$ chosen in the following range

$$
v - 2 + \frac{4}{n} + \beta (4 + \alpha - \frac{8}{n}) - v_2 > 0, \quad 2\beta - v_2 > 0, \\
4\beta - v_2 - 1 > 0, \quad 0 < \alpha < a < 1, \\
\beta < \frac{1}{2}, \quad \beta (1 + \alpha - a_1) + v - v_1 - \frac{1}{2} < 0, \\
v - \beta (2\sigma (4 - a) + \alpha - 8) > 0, \quad 2v_1 - \beta (\alpha - 2a_1) > 0, \\
v - \beta (\alpha - 2a) > 0, \quad v_2 < 1, \\
2v - 3 + \frac{8}{n} - v_2 + \beta (4 + 2\alpha - \frac{16}{n}) > 0. \hspace{1cm} (3.208)
$$

The inner problems: estimates of $\mathcal{H}^0$, $\mathcal{H}^1$ and $\mathcal{H}^\perp$

Recall from (3.103) that the inner problem is the following

$$
\lambda^2 \phi_t = \Delta_y \phi + 3U^2(y)\phi + \mathcal{H}(\phi, \psi, \lambda, \xi) \quad \text{in} \; \mathcal{D}_2R \times (0, T),
$$

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where \( \mathcal{H}(\phi, \psi, \lambda, \xi) \) is defined in (3.104). Since the inner–outer gluing relies on delicate analysis of the space-time decay of solutions, we further decompose the inner problem (3.103) into three different spherical harmonic modes

\[
\begin{align*}
\lambda^2 \phi_0^i &= \Delta_0 \phi_0 + 3U^2(y)\phi_0 + \mathcal{H}^0(\phi, \psi, \lambda, \xi) \quad \text{in } \mathscr{D}_2 \times (0, T), \\
\phi_0^i(\cdot, 0) &= 0
\end{align*}
\]

\[
\begin{align*}
\lambda^2 \phi_1^i &= \Delta_0 \phi_1 + 3U^2(y)\phi_1 + \mathcal{H}^1(\phi, \psi, \lambda, \xi) \quad \text{in } \mathscr{D}_2 \times (0, T), \\
\phi_1^i(\cdot, 0) &= 0
\end{align*}
\]

\[
\begin{align*}
\lambda^2 \phi^\perp_i &= \Delta_0 \phi^\perp + 3U^2(y)\phi^\perp + \mathcal{H}^\perp(\phi, \psi, \lambda, \xi) \quad \text{in } \mathscr{D}_2 \times (0, T), \\
\phi^\perp(\cdot, 0) &= 0
\end{align*}
\]

with

\[
\begin{align*}
\mathcal{H}^0(\phi, \psi, \lambda, \xi) &= \int_{S^3} \mathcal{H}(\phi, \psi, \lambda, \xi) \Theta_0(\theta) d\theta, \\
\mathcal{H}^1(\phi, \psi, \lambda, \xi) &= \sum_{j=1}^{4} \left( \int_{S^3} \mathcal{H}(\phi, \psi, \lambda, \xi) \Theta_j(\theta) d\theta \right) \Theta_j, \\
\mathcal{H}^\perp(\phi, \psi, \lambda, \xi) &= \sum_{j \geq 5} \left( \int_{S^3} \mathcal{H}(\phi, \psi, \lambda, \xi) \Theta_j(\theta) d\theta \right) \Theta_j,
\end{align*}
\]

where \( \Theta_j \) \((j = 0, 1, \ldots)\) are spherical harmonics. From the linear theory in Section 3.2.6 we know that for \( \mathcal{H} = \mathcal{H}^0 + \mathcal{H}^1 + \mathcal{H}^\perp \) satisfying

\[
\| \mathcal{H}^0 \|_{v, 2+a}, \| \mathcal{H}^1 \|_{v_1, 2+a_1}, \| \mathcal{H}^\perp \|_{v, 2+a} < +\infty,
\]

there exists a solution \((\phi^0, \phi^1, \phi^\perp, c^0, c^\ell) \ (\ell = 1, \ldots, 4)\) solving the projected inner problems

\[
\begin{align*}
\lambda^2 \phi_0^i &= \Delta_0 \phi_0 + 3U^2(y)\phi_0 + \mathcal{H}^0(\phi, \psi, \lambda, \xi) + c^0 Z_5 \quad \text{in } \mathscr{D}_2 \times (0, T), \\
\phi_0^i(\cdot, 0) &= 0
\end{align*}
\]
\[
\begin{align*}
\begin{cases}
\lambda^2 \phi_t = \Delta_y \phi + 3U^2(y)\phi + \mathcal{H}^1(\phi, \psi, \lambda, \xi) + \sum_{\ell=1}^{4} c^\ell Z_{\ell} & \text{in } D_2 \times (0, T), \\
\phi(\cdot, 0) = 0 & \text{in } D_2,
\end{cases}
\end{align*}
\] (3.213)

\[
\begin{align*}
\begin{cases}
\lambda^2 \phi_t = \Delta_y \phi + 3U^2(y)\phi + \mathcal{H}^\perp(\phi, \psi, \lambda, \xi) & \text{in } D_2 \times (0, T), \\
\phi(\cdot, 0) = 0 & \text{in } D_2,
\end{cases}
\end{align*}
\] (3.214)

and the inner solution \( \phi[\mathcal{H}] = \phi^0[\mathcal{H}^0] + \phi^1[\mathcal{H}^1] + \phi^\perp[\mathcal{H}^\perp] \) with proper space-time decay can be found ensuring the inner–outer gluing to be carried out. First, we choose all the parameters such that

\[
\|\mathcal{H}^0\|_{0, 2+a}, \|\mathcal{H}^1\|_{0, 2+a}, \|\mathcal{H}^\perp\|_{0, 2+a} < +\infty.
\]

To this end, we first give some estimates for \( \mathcal{H} \).

- By (3.188), we have

\[
\begin{align*}
& |3\lambda U^2(y)|_\eta_0 \mathcal{Psi}_0(\lambda y + \xi, t) + \psi(\lambda y + \xi, t) + Z^*(\lambda y + \xi, t)| \\
& \lesssim \frac{\lambda_s(t)}{1 + |y|^a} \left[ |\lambda_s|/(\log \lambda_s + \log(1 + |y|)) \right. \\
& \left. + \lambda_s^{-2+\frac{4}{a}}(0)R^{-2-a+\frac{4}{a}(0)}|\log T||\psi||_* + ||Z^*||_\infty \right].
\end{align*}
\] (3.215)

- By (3.192), we obtain

\[
\begin{align*}
& \left| \lambda \left[ \lambda \nabla_y \phi \cdot y + \phi + \nabla_y \phi \cdot \xi \right] + \frac{(n-4)\lambda}{\lambda y_1 + \xi_r} \phi_{y_1} + \frac{(n-4)\lambda^3}{r} \eta_s \mathcal{Psi}_0 \right| \\
& \lesssim \lambda_s |\lambda_s| \left( \frac{\lambda_s^4 R^{(4-a)}(\log R)}{1 + |y|^4} \|\phi^0\|_{0, \sigma, v, a} + \frac{\lambda_s^4}{1 + |y|^{a_1}} \|\phi^1\|_{0, v, 1, a_1} \right. \\
& \left. + \frac{\lambda_s^4}{1 + |y|^a} \|\phi^\perp\|_{0, v, a, a} \right) + \frac{\lambda_s}{\sqrt{T-t}} \left( r - \sqrt{2(n-4)(T-t)} \right) \frac{|\lambda_s|}{1 + |y|^2}.
\end{align*}
\] (3.216)

- Using (3.188), (3.194) and (3.201), we evaluate

\[
\left| \lambda^3 \mathcal{N}(\psi) + \lambda^3 \mathcal{H}[\lambda, \xi] + 3\lambda U^2(y)\Psi_1 \right|
\]
\[
\begin{align*}
&\lesssim \frac{\lambda_t^{2\nu} R^{2(4-a)} (\log R)^2}{1 + |y|^2} \|\phi^0\|^2_{0,\sigma,v,a} + \frac{\lambda_t^{2\nu_1}}{1 + |y|^{2+2a_1}} \|\phi^1\|^2_{0,\sigma,v_1,a_1} \\
&\quad + \frac{\lambda_t^{2\nu}}{1 + |y|^{2+2a}} \|\phi^\perp\|^2_{0,\sigma,v,a} + \frac{\lambda_t^{2}}{1 + |y|^2} |\dot{\lambda_t}| \log(T-t)^2 \\
&\quad + \frac{\lambda_t^{2}(t) \lambda_{\nu}^{2\nu-4+\frac{3}{n}}(0) R^{-4-2\alpha + \frac{4n}{\nu}}(0) \log T^2}{1 + |y|^2} \|\psi\|^2_{0,\sigma,v,a} + \frac{\lambda_{\nu}}{\sqrt{T-t}} \frac{y_1}{1 + |y|^2} \\
&\quad + \frac{\lambda_{\nu}}{1 + |y|^2} \|Z^+\|^2_{\infty} + \frac{\lambda_{\nu}^{4\beta} \log(T-t)^2}{1 + |y|^2} + \frac{\lambda_{\nu} \hat{\lambda}}{1 + |y|^2} + \frac{\lambda_{\nu} |\dot{\lambda}|}{1 + |y|^2} + \frac{\lambda_{\nu}^4 \dot{\lambda}}{(T-t)^2} + \frac{\lambda_{\nu}^3 (y \cdot \dot{\xi} + \dot{\lambda})}{1 + |y|^2} \\
&\quad + \frac{\lambda_{\nu}^5}{(T-t)^2} + \frac{\lambda_{\nu}^4 (y \cdot \dot{\xi} + \dot{\lambda})}{1 + |y|^2} + \frac{\lambda_{\nu}^3 (y \cdot \dot{\xi} + \dot{\lambda})}{1 + |y|^2}.
\end{align*}
\]

(3.217)

- In the spherical coordinates, the projection of

\[
\lambda^{-2}(t) \nabla U(y) \cdot \dot{\xi}(t) + \frac{n-4}{\lambda(t)y_1 + \xi_r(t)} \lambda^{-2}(t) \partial_{y_1} U(y)
\]

on mode 0 is given by

\[
\int_{S^3} \frac{n-4}{\lambda(t)y_1 + \xi_r(t)} \lambda^{-2}(t) \partial_{y_1} U(y) \Theta_0 \, d\theta = C \frac{\lambda^{-2}}{(1 + |y|^2)^2} \int_0^\pi |y| \cos \theta \sin^2 \theta \, d\theta
\]

\[
= -C \frac{\lambda^{-2}}{(1 + |y|^2)^2} \frac{\lambda |y|^2 \pi}{2 (\xi_r + \sqrt{\xi_r^2 - (\lambda |y|)^2})^2},
\]

(3.218)

where \(C\) is a constant. Note that since our choice of \(\xi_r\) is

\[
\xi_r \sim \sqrt{2(n-4)(T-t)},
\]

the above projection on mode 0 behaves exactly like the first error \(E_0\) defined in (3.91), and direct computations show that the sum of these two terms does not vanish. So we can deal with (3.218) by slightly modifying the first correction \(\Psi_0\). Here we omit the details.
Similarly, the projection of
\[ \lambda^{-2}(t) \nabla U(y) \cdot \hat{\xi}(t) + \frac{n-4}{\lambda(t)y_1 + \xi_r(t)} \lambda^{-2}(t) \partial_{y_1} U(y) \]
on mode 1 can be computed as
\[
\int_{S^3} \lambda^{-2}(t) \partial_{y_1} U(y) \left( \frac{n-4}{\lambda(t)y_1 + \xi_r(t)} + \hat{\xi}_r \right) \Theta_1(\theta) d\theta \\
= C' \lambda^{-1}(t)|y|^2 \int_0^\pi \frac{\cos^3 \theta \sin^2 \theta}{\lambda|y| \cos \theta + \xi_r} d\theta \\
= C' \lambda^{-1}(t)|y|^2 \left( \frac{(\lambda|y|)^4 + 4(\lambda|y|)^2 \xi_r^2 - 8\xi_r^4 + 8\xi_r^3 \sqrt{\xi_r^2 - (\lambda|y|)^2}}{8(\lambda|y|)^5} \right),
\]
where \( C' \) is a constant and we have used that \( \hat{\xi}_r \sim -\frac{n-4}{\xi_r} \). Note that in \( \mathcal{D}_{2R} \), namely \( |y| \leq 2R \), we have \( \lambda|y| \ll \xi_r \) for \( T \) sufficiently small. Therefore, by directly expanding the above expression, we obtain
\[
\int_{S^3} \lambda^{-2}(t) \partial_{y_1} U(y) \left( \frac{n-4}{\lambda(t)y_1 + \xi_r(t)} + \hat{\xi}_r \right) \Theta_1(\theta) d\theta \lesssim \frac{1}{\xi_r^3 (1 + |y|)^3}.
\] (3.219)

Then we estimate \( \mathcal{H} \) in three different modes.

**Estimate of \( \mathcal{H}^0 \).**

By (3.215)–(3.218), we obtain
\[
\| \mathcal{H}^0 \|_{v,2+a} \lesssim \lambda_{s}^{-1-v} |\lambda_{s}| \| \log (T - t) \| + \lambda_{s}^{-1-v}(t) \lambda_{s}^{-2 + \frac{2}{5}}(0) R^{-2 - a + \frac{5}{2}}(0) \| \log T \| \| \psi \|_{s} \\
+ \lambda_{s}^{-1-v} \| Z^* \|_{\infty} + \lambda_{s}^{\frac{1}{2}} R^{\sigma(4-a)} \log R \| \phi^0 \|_{0, \sigma, v, a} \\
+ \lambda_{s}^{v} R^{2\sigma(4-a)} \| \log R \| \| \phi^0 \|_{0, \sigma, v, a} + \lambda_{s} |\lambda_{s}| R^{\sigma(4-a)} \log R \| \phi^0 \|_{0, \sigma, v, a} \\
+ \lambda_{s}^{-2-v}(t) R^a(t) \lambda_{s}^{2\nu - 4 + \frac{2}{5}}(0) R^{4 - 2\alpha + \frac{16}{5}}(0) \| \log T \| \| \psi \|_{s}^2 \\
+ \lambda_{s}^{-2-v} |\lambda_{s}|^2 R^{2a} \| \log (T - t) \|^2 + \lambda_{s}^{-2-v} R^a \| Z^* \|_{s}^2 + \lambda_{s}^{4\beta - V R^a} \| \log (T - t) \|^2 \\
+ \lambda_{s}^{-1-v} |\lambda_{s}| + |\lambda_{s}| \left( \frac{\lambda_{s}^{2-v}}{(T - t)^{2+a}} + \lambda_{s}^{1-v} + \frac{\lambda_{s}^{1-v} R^{2+a} \| \log (T - t) \|^2}{T - t} \right)
\]

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\[ + \lambda_s^{2-v} R^d(R|\dot{\xi}| + \dot{\lambda}) \],

from which we conclude that

\[ \| \mathcal{H}^0 \|_{\nu,2+a} \lesssim T^{s_0} \left( \| \phi^0 \|_{0,\sigma,\nu,a} + \| \psi \|_1 + \| Z^* \|_\infty + \| \lambda \|_\infty + \| \xi \|_G + 1 \right) \] (3.220)

provided

\[ \nu < 1, \quad \frac{4}{n} - 1 + \beta \left( 2 + \alpha - \frac{8}{n} \right) > 0, \]

\[ \frac{1}{2} - \beta \sigma (4-a) > 0, \quad 1 - \beta \sigma (4-a) > 0, \]

\[ \nu - 2\beta \sigma (4-a) > 0, \quad 2 - \nu - a\beta > 0, \]

\[ \nu - 2 + \frac{8}{n} > 2 + 8 + \beta (4 + 2\alpha - \frac{16}{n}) > 0, \quad 2 - \nu - a\beta > 0, \]

\[ (4-a)\beta - \nu > 0, \quad \frac{3}{2} - \nu - \beta (1+a) > 0. \] (3.221)

**Estimate of \( \mathcal{H}^1 \).**

From (3.215)–(3.217) and (3.219), we have

\[ \| \mathcal{H}^1 \|_{\nu_1,2+a_1} \]

\[ \lesssim \lambda_s^{1-v_1} |\log(T-t)|||\dot{\lambda}_s| + \lambda_s^{1-v_1}(t) \lambda_s^{v-2+\frac{8}{n}}(0) R^{-2-\alpha+\frac{8}{n}}(0) |\log T| \| \psi \|, \]

\[ + \lambda_s^{1-v_1} \| Z^* \|_\infty + \lambda_s \| R^2 \|_{1,v_1,a_1} + \lambda_s \| \dot{\lambda}_s \| R^2 \| \phi \|_{1,v_1,a_1} \]

\[ + \lambda_s^{\frac{2-v}{1}} \| \dot{\lambda}_s \| R^{1+a_1} + \lambda_s^{v_1} \| \phi \|_{2,v_1,a_1} + \lambda_s^{2-v_1} \| \dot{\lambda}_s \| R^{a_1} \| \log(T-t) \|^2 \]

\[ + \lambda_s^{2-v_1} (t) R^{a_1}(t) \lambda_s^{2v-4+\frac{8}{n}}(0) R^{-4-2\alpha+\frac{16}{n}}(0) |\log T|^2 \| \psi \|_s^2 \]

\[ + \lambda_s^{2-v_1} R^{a_1} \| Z^* \|_\infty + \lambda_s^{4+\frac{8}{n}} R^{a_1} \| \log(T-t) \|^2 + \lambda_s^{\frac{3}{2}-v_1} R^{1+a_1} \]

\[ + |\dot{\lambda}_s| \left( \frac{\lambda_s^{5-v_1}}{(T-t)^2} R^{2+a_1} + \lambda_s^{1-v_1} + \frac{\lambda_s^{4-v_1} R^{2+a_1} (R|\dot{\xi}| + \dot{\lambda})}{T-t} \right) + \lambda_s^{2-v_1} R^{a_1} (R|\dot{\xi}| + \dot{\lambda}) \].
Therefore, we obtain that for \( \varepsilon_0 > 0 \)

\[
\|\mathcal{H}^{-1}\|_{v,2+a_1} \lesssim T^{\varepsilon_0} \left( \|\phi^1\|_{\text{in},v,a_1} + \|\psi\|_\ast + \|Z^\ast\|_\infty + \|\lambda\|_\infty + \|\xi\|_G + 1 \right) (3.222)
\]

provided

\[
\begin{aligned}
\varepsilon_1 &< 1, \\
1 - 2\beta &> 0, \\
\varepsilon_1 &> 0, \\
2\varepsilon_1 - 2 + \frac{8}{n} + \beta(4 + 2\alpha - a_1 - \frac{16}{n}) &> 0, \\
(4 - a_1)\beta - \varepsilon_1 &> 0, & \frac{3}{2} - \varepsilon_1 - \beta(1 + a_1) &> 0.
\end{aligned}
\]

(3.223)

**Estimate of \( \mathcal{H}^\perp \).**

Using (3.215)–(3.217), we get

\[
\|\mathcal{H}^\perp\|_{v,2+a} \\
\lesssim \lambda_s^{1-v} \dot{\lambda}_a \|\log(T-t)\| + \lambda_s^{1-v}(t) \lambda_s^{v-2+\frac{4}{n}}(0) R^{-2-\alpha} + \lambda_s^{2-v} \log(T-t) \|
\]

\[
+ \lambda_s^{1-v} \|\phi^1\|_{\text{in},v,a} + \lambda_s^{2-v} \|\lambda^\ast\|^2 R^\alpha \log(T-t)^2
\]

\[
+ \lambda_s^{2-v} R^\alpha \|Z^\ast\|_\infty + \lambda_s^{4\beta - v} R^\alpha \log(T-t)^2 + |\dot{\lambda}_a| \left( \frac{\lambda_s^{5-v}}{(T-t)^2} + \lambda_s^{2-v} R^\alpha \right)
\]

Thus, one has

\[
\|\mathcal{H}^\perp\|_{v,2+a} \lesssim T^{\varepsilon_0} \left( \|\phi^1\|_{\text{in},v,a} + \|\psi\|_\ast + \|Z^\ast\|_\infty + \|\lambda\|_\infty + \|\xi\|_G + 1 \right) (3.224)
\]
provided
\[
0 < \nu < 1, \quad \frac{4}{n} - 1 + \beta (2 + \alpha - \frac{8}{n}) > 0,
\]
\[
1 - 2\beta > 0, \quad 2 - \nu > 0, \quad 2 - \nu > 0,
\]
\[
\nu - 2 + \frac{8}{n} + \beta (4 + 2\alpha - \frac{16}{n}) > 0, \quad 2 - \nu > 0,
\]
\[
(4 - a)\beta - \nu > 0, \quad \frac{3}{2} - \nu - \beta (1 + a) > 0. \quad (3.225)
\]
Collecting (3.220)–(3.225), we conclude that for some \( \varepsilon_0 > 0 \)
\[
\|\mathcal{K}_0\|_{v,2+a} + \|\mathcal{K}_1\|_{v_1,2+a} + \|\mathcal{K}_2\|_{v,2+a} \lesssim T_0 \left(\|\phi\|_{v,0} + \|\phi\|_{v_1,0} + \|\phi\|_{v,0} + \|\phi\|_{v_1,0} + \|\psi\|_{v_1} + \|\lambda\|_{\infty} + \|\xi\|_G + \|Z\|_{\infty} + 1\right) \quad (3.226)
\]
provided the parameters \( \beta, a, a_1, \alpha, \nu, v_1, \sigma \) satisfy
\[
0 < \nu < 1, \quad \frac{4}{n} - 1 + \beta (2 + \alpha - \frac{8}{n}) > 0,
\]
\[
\frac{1}{2} - \beta \sigma (4 - a) > 0, \quad \nu - 2\beta \sigma (4 - a) > 0,
\]
\[
2 - \nu - a\beta > 0, \quad 2 - \nu - a\beta > 0,
\]
\[
\nu - 2 + \frac{8}{n} + \beta (4 + 2\alpha - \frac{16}{n}) > 0, \quad 2 - \nu > 0,
\]
\[
(4 - a)\beta - \nu > 0, \quad \frac{3}{2} - \nu - \beta (1 + a) > 0,
\]
\[
0 < v_1 < 1, \quad \nu - v_1 - 1 + \frac{4}{n} + \beta (2 + \alpha - \frac{8}{n}) > 0,
\]
\[
1 - 2\beta > 0, \quad \frac{3}{2} - v_1 - \beta (1 + a_1) > 0,
\]
\[
2 - v_1 - a_1\beta > 0, \quad 2 - v_1 - a_1\beta > 0,
\]
\[
2 - v_1 - a_1\beta > 0, \quad 2 - v_1 - a_1\beta > 0. \quad (3.227)
\]
The parameter problems

From (3.212)–(3.214), we need to adjust the parameter functions \( \lambda(t) \), \( \xi(t) \) such that

\[
c^0[\lambda, \xi, \Psi^r] = 0, \quad c^\ell[\lambda, \xi, \Psi^r] = 0, \quad \ell = 1, \ldots, 4,
\]

where

\[
c^0[\lambda, \xi, \Psi^r] = -\frac{\int_{\Omega^0} \mathcal{H}^0 Z_5 dy}{\int_{\Omega^0} |Z_5|^2 dy} - O[\mathcal{H}^0], \quad (3.228)
\]

\[
c^\ell[\lambda, \xi, \Psi^r] = -\frac{\int_{\Omega^\ell} \mathcal{H}^1 Z_\ell dy}{\int_{\Omega^\ell} |Z_\ell|^2 dy} \quad \text{for } \ell = 1, \ldots, 4. \quad (3.229)
\]

It turns out that we can easily achieve at the translation mode (3.229), but the scaling mode (3.228) is more complicated.

The reduced problem of \( \xi(t) \)

We first consider the reduced equation for \( \xi(t) = (\xi_r(t), \xi_c(t)) \). Notice that (3.229) is equivalent to

\[
\int_{\Omega^i} \mathcal{H}^1(\phi, \psi, \lambda, \xi)(y,t) Z_i(y) dy = 0 \quad \text{for all } t \in (0, T), \ i = 1, \ldots, 4.
\]

Recall that \( \xi_r(t) = \sqrt{2(n-4)(T-t)} + \xi_{r,1}(t), \ \xi_c(t) = z_0 + \xi_{c,1}(t) \) and write \( \Psi^r = \psi + Z^r \). Then for \( i = 1, \ldots, 4, \)

\[
\int_{\Omega^i} \mathcal{H}^1(\phi, \psi, \lambda, \xi)(y,t) Z_i(y) dy = 0
\]

yield that

\[
\begin{aligned}
\dot{\xi}_r + \frac{n-4}{\xi_r} &= b_r[\lambda, \xi, \phi, \Psi^r], \\
\dot{\xi}_c &= b_c[\lambda, \xi, \phi, \Psi^r],
\end{aligned} \quad (3.230)
\]

where

\[
b_r[\lambda, \xi, \phi, \Psi^r] = \int_{\Omega^r} \mathcal{H}_r[\lambda, \xi, \phi, \Psi^r](y,t) Z_r(y) dy,
\]

\[
b_c[\lambda, \xi, \phi, \Psi^r] = \int_{\Omega^c} \mathcal{H}_c[\lambda, \xi, \phi, \Psi^r](y,t) Z_c(y) dy \quad \text{for } j = 2, 3, 4,
\]

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with

\[ \mathcal{H}_r[\lambda, \xi, \phi, \Psi^*](y, t) = \left[ \int_{\mathbb{R}^3} \left( \mathcal{H} - \lambda \mathbf{U}_{y_1} \left( \frac{\xi_r}{\lambda y_1 + \xi_r} \right) \right) \Theta_1(\theta) d\theta \right] \Theta_1, \]

\[ \mathcal{H}_z[\lambda, \xi, \phi, \Psi^*] = \sum_{j=2}^{4} \left[ \int_{\mathbb{R}^3} \left( \mathcal{H} - \lambda \mathbf{U}_{y_j} \eta_j \right) \Theta_j(\theta) d\theta \right] \Theta_j. \]

Here \( \Theta_j \) \((j = 1, \ldots, 4)\) are the eigenfunctions corresponding to the second eigenvalue of \( -\Delta_{\mathbb{R}^3} \). Now we want to evaluate the sizes of \( b_r[\lambda, \xi, \phi, \Psi^*] \), \( b_z[\lambda, \xi, \phi, \Psi^*] \).

By direct computations, we get

\[ |b_r[\lambda, \xi, \phi, \Psi^*]| \leq \left( \lambda_s \| \dot{\lambda}_s \| \log(T - t) + \lambda_s \| Z^* \|_\infty \left( 1 + O(R^{-3}) \right) \right. \]

\[ \left. + \lambda_s(t) \lambda_s^{\nu_2 - \frac{2}{n}}(0) R^{2 - 2\alpha + \frac{2}{n}}(0) \| \log T \| \| \psi \|_\infty \left( 1 + O(R^{-3}) \right) \right) \]

\[ + \lambda_s^{2\nu_1} \| \xi \|_G \| \phi \|_{\text{in,}v_1} \left( 1 + O(R^{-a_1}) \right) \]

\[ + \lambda_s^{2\nu_1} \| \dot{\phi} \|_{\text{in,}v_1} \left( 1 + O(R^{-a_1}) \right) \]

\[ + \lambda_s^{2\nu_2} \| R \| \left( 1 + O(R^{-1}) \right) + \lambda_s^{2\nu_2} \| \phi \|_{\text{in,}v_2} \left( 1 + O(R^{-2a_1 - 1}) \right) \]

\[ + \lambda_s^{2\nu_2} \| R \| \left( 1 + O(R^{-1}) \right) + \lambda_s^{2\nu_1} \| \phi \|_{\text{in,}v_2} \left( 1 + O(R^{-2a_1 - 1}) \right) \]

\[ + \lambda_s^{2\nu_1} \| \dot{\phi} \|_{\text{in,}v_2} \left( 1 + O(R^{-a_1}) \right) \]

\[ + \lambda_s^{2\nu_2} \| R \| \left( 1 + O(R^{-1}) \right) + \lambda_s^{2\nu_1} \| \phi \|_{\text{in,}v_2} \left( 1 + O(R^{-2a_1 - 1}) \right) \]

\[ + \lambda_s^{2\nu_1} \| \dot{\phi} \|_{\text{in,}v_2} \left( 1 + O(R^{-a_1}) \right) \]

\[ + \lambda_s^{2\nu_2} \| R \| \xi \|_G \left( 1 + O(R^{-1}) \right), \quad (3.231) \]

and

\[ |b_z[\lambda, \xi, \phi, \Psi^*]| \leq \left( \lambda_s \| \dot{\lambda}_s \| \log(T - t) + \lambda_s \| Z^* \|_\infty \left( 1 + O(R^{-3}) \right) \right. \]

\[ \left. + \lambda_s(t) \lambda_s^{\nu_2 - \frac{2}{n}}(0) R^{2 - 2\alpha + \frac{2}{n}}(0) \| \log T \| \| \psi \|_\infty \left( 1 + O(R^{-3}) \right) \right) \]

\[ + \lambda_s^{2\nu_1} \| \phi \|_{\text{in,}v_1} \left( 1 + O(R^{-a_1}) \right) \]

\[ + \lambda_s^{2\nu_1} \| \dot{\phi} \|_{\text{in,}v_1} \left( 1 + O(R^{-a_1}) \right) \]

\[ + \lambda_s^{2\nu_1} \| \xi \|_G \| \phi \|_{\text{in,}v_1} \left( 1 + O(R^{-a_1}) \right) \]

\[ + \lambda_s^{2\nu_1} \| \dot{\phi} \|_{\text{in,}v_1} \left( 1 + O(R^{-a_1}) \right) \]

\[ + \lambda_s^{2\nu_1} \| \xi \|_G \| \phi \|_{\text{in,}v_1} \left( 1 + O(R^{-a_1}) \right) \]

\[ + \lambda_s^{2\nu_1} \| \dot{\phi} \|_{\text{in,}v_1} \left( 1 + O(R^{-a_1}) \right) \]

\[ + \lambda_s^{2\nu_1} \| \xi \|_G \| \phi \|_{\text{in,}v_1} \left( 1 + O(R^{-a_1}) \right) \]

\[ + \lambda_s^{2\nu_1} \| \dot{\phi} \|_{\text{in,}v_1} \left( 1 + O(R^{-a_1}) \right) \]

\[ + \lambda_s^{2\nu_1} \| \xi \|_G \| \phi \|_{\text{in,}v_1} \left( 1 + O(R^{-a_1}) \right) \]

\[ + \lambda_s^{2\nu_1} \| \dot{\phi} \|_{\text{in,}v_1} \left( 1 + O(R^{-a_1}) \right) \]
Then we analyze the reduced problem (3.233), which defines operators \( \Xi \). We shall solve

\[
\begin{align*}
&+ \lambda_0^2 |\lambda_n| (1 + O(R^{-1})) + \lambda_0^2 \| \phi \|_{\infty, v_1, a_1}^2 (1 + O(R^{-2a_1-1})) \\
&+ \lambda_0^2 (t) \lambda_0^{2^*} \phi (0) R^{-4 - 2a + \frac{1}{2}} (0) \log T^2 \| \psi \|_2^2 (1 + O(R^{-1})) \\
&+ \lambda_0^2 |Z^*|_2 (1 + O(R^{-1})) + \lambda_0^2 |\dot{\lambda}_n|^2 |\log(T - t)|^2 (1 + O(R^{-1})) \\
&+ \lambda_0^{4\beta} |\log(T - t)|^2 (1 + O(R^{-1})) + \lambda_n |\dot{\lambda}_n| (1 + O(R^{-3})) \\
&+ \lambda_0^2 |\lambda_n| (1 + O(R^{-1})) + \lambda_0^2 |\dot{\lambda}_n| R + \lambda_0^{3+u} |\lambda_n| R^2 \| \xi \|_G \\
&+ \lambda_0^{2+u} |\dot{\lambda}_n| R \| \xi \|_G (1 + O(R^{-1})).
\end{align*}
\]

Since \( \xi_r(t) = \sqrt{2(n-4)(T-t)} + \xi_r(t) \), problem (3.230) becomes

\[
\begin{align*}
\dot{\xi}_{r,1} - \frac{(n-4) \xi_{r,1}}{\sqrt{2(n-4)(T-t)} (\sqrt{2(n-4)(T-t)} + \xi_{r,1})} &= b_r [\lambda, \xi, \phi, \Psi^*], \\
\dot{\xi}_{c,j} &= b_{c,j} [\lambda, \xi, \phi, \Psi^*].
\end{align*}
\]

Then we analyze the reduced problem (3.233), which defines operators \( \Xi_r \) and \( \Xi_{c,j} \) \((j = 2, 3, 4)\) that return the solutions \( \xi_{r,1} \) and \( \xi_{c,j} \) respectively. Here we write

\[
\Xi = (\Xi_r, \Xi_{c2}, \Xi_{c3}, \Xi_{c4}).
\]

We shall solve \((\xi_{r,1}, \xi_{c,1})\) under the norm

\[
\| \xi \|_G = \sup_{t \in (0, T)} \left[ (T-t)^{-\frac{1}{2} + v} |\xi_{r,1}(t)| + M_1 (T-t)^{\frac{1}{2} - v} |\xi_{c,1}(t)| \\
+ |\xi_{c,j}(t)| + (T-t)^{-u} |\dot{\xi}_{c,j}(t)| \right]
\]

for \( v > 0 \) and \( 0 < M_1 < 1 \). From (3.233), we have

\[
|\xi_{r,1}(t)| \leq \left( \frac{(T-t)^{\frac{1}{2} + u} \| \xi \|_G}{2(T-t)} + \| b_r [\lambda, \xi, \phi, \Psi^*] \|_{L^2(0,T)} \right) (T-t),
\]

\[
|\xi_{c,j}(t)| \leq |z_0| + \| b_{c,j} [\lambda, \xi, \phi, \Psi^*] \|_{L^2(0,T)} (T-t).
\]
Therefore, we obtain
\[
\|\Xi_r\|_G \leq \frac{1+M_1}{2} \|\xi\|_G + (1+M_1)(T-t)^{\frac{1}{2}-\nu} \|b_r[\lambda, \xi, \phi, \Psi^*]\|_{L^\infty(0,T)}, \tag{3.235}
\]
\[
\|\Xi_{z_j}\|_G \leq |z_0| + (T-t)^{-\nu} \|b_{z_j}[\lambda, \xi, \phi, \Psi^*]\|_{L^\infty(0,T)}. \tag{3.236}
\]
By (3.231), (3.232), (3.235) and (3.236), we conclude that for some constant \(C > 0\)
\[
\|\Xi_r\|_G \leq \left[ \frac{1+M_1}{2} + C(1+M_1)(T-t)^{\frac{1}{2}-\nu} \left( \lambda^1_{\text{v}} + \lambda^2_{\text{v}} |\lambda|_R \right) \right] \|\xi\|_G
+ C(1+M_1)(T-t)^{\frac{1}{2}-\nu} \left[ \lambda(t) \lambda^2_{\text{v}}(0) R^{-2-\alpha+\frac{\gamma}{8}} \|\Psi\|_s + \lambda_s \|Z^*\|_{L^\infty(0,T)} \right.
+ \left( \lambda^1_{\text{v}} + \lambda^2_{\text{v}} \right) \|\phi^1\|_{L^\infty(0,T)} + \lambda_s \|\lambda\|_s + \lambda^s_{\text{v}} \right], \tag{3.237}
\]
\[
\|\Xi_{z_j}\|_G \leq |z_0| + C(T-t)^{-\nu} \left[ \lambda(t) \lambda^2_{\text{v}}(0) R^{-2-\alpha+\frac{\gamma}{8}} \|\Psi\|_s + \lambda_s \|Z^*\|_{L^\infty(0,T)} \right.
+ \left( \lambda^1_{\text{v}} + \lambda^2_{\text{v}} \right) \|\phi^1\|_{L^\infty(0,T)} + \lambda_s \|\lambda\|_s + \lambda^s_{\text{v}} \right]
+ \left( \lambda^1_{\text{v}} + \lambda^2_{\text{v}} \lambda_s \|\phi^1\|_{L^\infty(0,T)} \right) \|\xi\|_G \right]. \tag{3.238}
\]

**The reduced problem of \(\lambda(t)\)**

Since the reduced problem of \(\lambda\) is essentially the same as that of [55], we shall follow the strategy and logic in [55, Section 8]. From direct computations, we see that (3.228) gives a non-local integro-differential equation
\[
\int_{-T}^{t} \frac{\dot{\lambda}(s)}{t-s} \left( \frac{\lambda^2(t)}{t-s} \right) ds + c_0 \dot{\lambda} = a[\lambda, \xi, \Psi^*](t) + a_r[\lambda, \xi, \phi, \Psi^*](t), \tag{3.239}
\]
where
\[
c_0 = 2\alpha_0 \int_{\mathbb{R}^2} \frac{Z_5(y)}{(1+|y|^2)^2} dy, \]
\[
a[\lambda, \xi, \Psi^*] = - \int_{\mathcal{D}_{2R}} 3U^2(y) (\Psi_0 + \Psi^*) Z_5(y) dy, \tag{3.240}
\]
\[
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\]
and the remainder term \( a_r[\lambda, \xi, \phi, \Psi^*](t) \) turns out to be smaller order and has the following bound

\[
|a_r[\lambda, \xi, \phi, \Psi^*](t)| \leq \left[ \lambda^y R^{\sigma(4-a)}(1 + |\lambda|) + \lambda^{y-\frac{1}{2}} R^{\sigma(4-a)} \|\xi\|_G \right] \log R \|\phi^0\|_{0,0,0,0} (1 + O(R^{-2}))
\]

\[
+ \lambda^\frac{1}{2} R^{\lambda} \|\log(T-t)\| + \lambda^{2v-1} R^{2\sigma(4-a)}(\log R)^2 \|\phi^0\|_{0,0,0,0} (1 + O(R^{-8}))
\]

\[
+ \lambda(t) \lambda^{2v-4+\frac{1}{2}} (0) R^{-4-2\alpha+\frac{16}{3}} (\log(T-t))^2 \|\Psi\|_s^2
\]

\[
+ \lambda |\lambda|^2 |\log(T-t)|^3 + \lambda_a |\log(T-t)| \|Z^*\|_{\infty}^2 + \lambda^{2\beta-1} |\log(T-t)|^3.
\]

We first introduce the following norms

\[
\|f\|_{0,l} := \sup_{t \in [0, T]} \frac{|\log(T-t)|^l}{(T-t)^\Theta} |f(t)|,
\]

where \( f \in C([-T, T]; \mathbb{R}) \) with \( f(T) = 0 \), and \( \Theta \in (0, 1) \), \( l \in \mathbb{R} \).

\[
[g]_{r,m,l} := \sup_{t \in I_T} \frac{|\log(T-t)|^l}{(T-t)^m(t-s)^l} |g(t) - g(s)|,
\]

where \( I_T = \{0 \leq s \leq t \leq T : t-s \leq \frac{1}{10}(T-t)\} \), \( g \in C([-T, T]; \mathbb{R}) \) with \( g(T) = 0 \) and \( 0 < \gamma < 1 \), \( m > 0 \), \( l \in \mathbb{R} \). Also, we define

\[
\mathcal{B}_0[\lambda](t) := \int_{-T}^t \lambda(s) \Gamma \left( \frac{\lambda^2(t)}{t-s} \right) ds + c_0 \lambda
\]

and write

\[
e^0[\mathcal{A}] = \mathcal{B}_0[\lambda] - (a[\lambda, \xi, \Psi^*] + a_r[\lambda, \xi, \phi, \Psi^*]).
\]

A key proposition concerning the solvability of \( \lambda \) is stated as follows.

**Proposition 3.2.6 ([55]).** Let \( \omega, \Theta \in (0, \frac{1}{2}) \), \( \gamma \in (0, 1) \), \( m \leq \Theta - \gamma \) and \( l \in \mathbb{R} \). If \( a(t) \) satisfies \( a(T) < 0 \) with \( 1/C \leq a(T) \leq C \) for some constant \( C > 1 \), and

\[
T^{\Theta} |\log T|^{1+\gamma-\Theta} |a(\cdot) - a(T)|_{0,l-1} + |a|_{r,m,l-1} \leq C_l
\]

(3.243)
for some \( c > 0 \), then there exist two operators \( \mathcal{P} \) and \( \mathcal{B}_0 \) such that \( \lambda = \mathcal{P}[a] : [-T, T] \to \mathbb{R} \) satisfies

\[
\mathcal{B}_0[\lambda](t) = a(t) + \mathcal{B}_0[a](t)
\]  

(3.244)

with

\[
|\mathcal{B}_0[a](t)| \lesssim \left( T^{1+c} + T^\Theta \log \frac{|\log T|}{|\log T|} \right) \left( a(\cdot) - a(T) \| \Theta_{l-1} + [a]_{\gamma,m,l-1} \right) \frac{(T-t)^{m+(1+\omega)\gamma}}{|\log(T-t)|^l}.
\]

The proof of Proposition 3.2.6 is in [55]. The idea of the proof is to observe that

\[
\mathcal{B}_0[\lambda] \approx \int_{-T}^{t-\lambda^2(t)} \frac{\dot{\lambda}(s)}{t-s} ds,
\]

and we decompose

\[
\mathcal{B}_0[\lambda] = \mathcal{B}_0^* [\lambda] + \mathcal{S}[\dot{\lambda}] + \mathcal{R}_\omega[\dot{\lambda}],
\]

where

\[
\mathcal{B}_0^* [\lambda] := \mathcal{B}_0[\lambda] - \int_{-T}^{t-\lambda^2(t)} \frac{\dot{\lambda}(s)}{t-s} ds,
\]

\[
\mathcal{S}[\dot{\lambda}] := \dot{\lambda} \left[ (1 + \omega) \log(T-t) - 2 \log \lambda_s(t) \right] + \int_{-T}^{t-(T-t)^{1+\omega}} \frac{\dot{\lambda}(s)}{t-s} ds,
\]

(3.245)

\[
\mathcal{R}_\omega[\dot{\lambda}] := - \int_{t-(T-t)^{1+\omega}}^{t-\lambda^2(t)} \frac{\dot{\lambda}(t) - \dot{\lambda}(s)}{t-s} ds.
\]

Here \( \omega > 0 \) is a fixed number. We solve a modified equation where we drop \( \mathcal{R}_\omega[\dot{\lambda}] \) in (3.244), and thus the remainder \( \mathcal{B}_0 \) is essentially \( \mathcal{B}_0^* [\lambda] \) and \( a_r[\lambda, \xi, \phi, \Psi^*] \).

In another aspect, we modify problem (3.239) replacing \( a[\lambda, \xi, \Psi^*] \) by its main term. To this end, we define

\[
a[\lambda, \xi, \Psi^*] = a^0[\lambda, \xi, \Psi^*] + a^1[\lambda, \xi, \Psi^*] + a^1_r[\lambda, \xi, \Psi^*]
\]

with

\[
a^0[\lambda, \xi, \Psi^*] = - \int_{\mathcal{S}_{2R} \setminus \mathcal{D}} \hat{L}^0[\Psi] Z_5(y) dy,
\]

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\[ a^1[\lambda, \xi, \Psi^*] = - \int_{\mathcal{D}_2} \tilde{L}^1[\Psi]Z_5(y)dy, \]
\[ a^1[\lambda, \xi, \Psi] = - \int_{\mathcal{D}_2} \tilde{L}^1[\Psi]Z_5(y)dy, \]

where \( \tilde{L}[\Psi] := 3U^2(\Psi_0 + \Psi^*), \) \( \tilde{L}_0[\Psi] \) is the projection of \( \tilde{L}[\Psi] \) on mode 0, \( \tilde{L}^1[\Psi] \) is the projection of \( \tilde{L}[\Psi] \) on modes 1 to 4, and \( \tilde{L}^j[\Psi] \) is the projection of \( \tilde{L}[\Psi] \) on higher modes \( j \geq 5. \)

We define
\[
c_0^*[\lambda, \xi, \Psi^*](t) := \frac{\mathcal{R}_0 \left[ a^0[\lambda, \xi, \Psi^*] \right](t) + a^1[\lambda, \xi, \Psi^*](t) + a^j[\lambda, \xi, \Psi^*](t)}{\int_{\mathcal{D}_2} |Z_5(y)|^2dy} \]
\[ \quad - (c_0^*[\mathcal{H}[\lambda, \xi, \Psi^*]] - c_0^*[\mathcal{H}_0[\lambda, \xi, \Psi^*]]), \tag{3.246} \]

where \( \mathcal{R}_0 \) is the operator given in Proposition 3.2.6, \( c_0^0 \) is defined in (3.242), and \( c_0^0 \) is the operator given in Proposition 3.2.3. The reason for choosing such \( c_0^0 \) is the following. By Proposition 3.2.6 the equation we solve is
\[
\mathcal{R}_0[\lambda](t) = a^0[\lambda, \xi, \Psi^*](t) + \mathcal{R}_0[a^0[\lambda, \xi, \Psi^*]],
\]
which is equivalent to
\[
c_0^*[\mathcal{H}_0] = \frac{\mathcal{R}_0 \left[ a^0[\lambda, \xi, \Psi^*] \right](t) + a^1[\lambda, \xi, \Psi^*](t) + a^j[\lambda, \xi, \Psi^*](t)}{\int_{\mathcal{D}_2} |Z_5(y)|^2dy}.
\]

We shall consider the following reduced equation
\[
c_0^*[\mathcal{H}_0] = c_0^0[\lambda, \xi, \Psi^*],
\]
from which we get (3.246).

By (3.240), Proposition 3.2.6 and Proposition 3.2.1 it is natural to choose
\[
\Theta = v - 2 + \frac{4}{n} + \beta(2 + \alpha - \frac{8}{n}), \quad m = v - 2 - \gamma + \beta(2 + \alpha).
\]

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In order for \( \|a(\cdot) - a(T)\|_{\Theta,l-1} \) and \([a]_{\gamma,m,l-1}\) to be finite, we require
\[
l < \max\{1 + 2\Theta, 1 + 2m\}
\]
and it then follows that
\[
\|a(\cdot) - a(T)\|_{\Theta,l-1} \lesssim |\log T|^{l-\Theta-1}, \quad [a]_{\gamma,m,l-1} \lesssim |\log T|^{l-m-1}.
\]

Another assumption \( m < \Theta - \gamma \) in Proposition 3.2.6 is valid since \( \beta < 1/2 \). Finally, in order to make the remainder \( \mathcal{R}_0[a] \) small, we impose
\[
m + (1 + \omega)\gamma > \Theta,
\]
which implies that
\[
\beta > \frac{1}{2} - \frac{\omega \gamma m}{8}.
\]

**Inner–outer gluing system**

By the discussions in Section 3.2.7, we transform the inner–outer problems (3.103), (3.105) into the problems of finding solutions \((\psi, \phi^0, \phi^1, \phi, \lambda, \xi)\) solving the following inner–outer gluing system

\[
\begin{align*}
\psi_t &= \Delta_{(r,z)} \psi + \frac{n-4}{r} \partial_r \psi + \mathcal{G}(\phi^0 + \phi^1 + \phi^+, \psi + Z^*, \lambda, \xi) \quad \text{in } \mathcal{D} \times (0,T), \\
\psi &= -\Psi^0 \quad \text{on } (\partial \mathcal{D}\setminus\{r = 0\}) \times (0,T), \\
\psi_r &= 0 \quad \text{on } (\mathcal{D} \cap \{r = 0\}) \times (0,T), \\
\psi(r,z,0) &= -(1 - \eta_x)\Psi^0 \quad \text{in } \mathcal{D},
\end{align*}
\]
\[
\begin{align*}
\lambda^2 \phi^0 &= \Delta_y \phi^0 + 3U^2(y)\phi^0 + \mathcal{H}^0(\phi, \psi, \lambda, \xi) + c^0[\mathcal{H}^0]Z_5 \quad \text{in } \mathcal{D}_{2R} \times (0,T), \\
\phi^0(\cdot,0) &= 0 \quad \text{in } \mathcal{D}_{2R},
\end{align*}
\]
\[
\begin{align*}
\lambda^2 \phi^1 &= \Delta_y \phi^1 + 3U^2(y)\phi^1 + \mathcal{H}^1(\phi, \psi, \lambda, \xi) + \sum_{\ell=1}^4 c^\ell[\mathcal{H}^1]Z_\ell \quad \text{in } \mathcal{D}_{2R} \times (0,T), \\
\phi^1(\cdot,0) &= 0 \quad \text{in } \mathcal{D}_{2R},
\end{align*}
\]
\[
\begin{aligned}
\lambda^2 \phi^\perp_t &= \Delta_y \phi^\perp + 3U^2(y)\phi^\perp + \mathcal{H}^\perp(\phi, \psi, \lambda, \xi) + c^0[\lambda, \xi, \Psi^*]Z_5 \quad \text{in } \mathcal{D}_2 \times (0, T), \\
\phi^\perp(\cdot, 0) &= 0 \quad \text{in } \mathcal{D}_2,
\end{aligned}
\]  
(3.250)

\[
\begin{aligned}
e^0[\mathcal{H}](t) - e^0[\lambda, \xi, \Psi^*](t) &= 0 \quad \text{for all } t \in (0, T), \\
e^1[\mathcal{H}](t) &= 0 \quad \text{for all } t \in (0, T),
\end{aligned}
\]  
(3.251)

where \( \eta_* \) is defined in (3.88), \( \mathcal{I} \) is defined in (3.106), \( \mathcal{H}^0, \mathcal{H}^1, \mathcal{H}^\perp \) are the projections on different modes defined in (3.209)–(3.211).

It is direct to see that if \((\psi, \phi^0, \phi^1, \phi^\perp, \lambda, \xi)\) satisfies the system (3.247)–(3.252), then

\[
\Psi^* = \psi + Z^*, \quad \phi = \phi^0 + \phi^1 + \phi^\perp
\]
solve the inner–outer problems (3.103), (3.105), and thus the desired blow-up solution is found.

The fixed point formulation

The inner–outer gluing system (3.247)–(3.252) can be formulated as a fixed point problem for operators we shall describe below.

We first define the following function spaces

\[
X_{\phi^0} := \{ \phi^0 \in L^\infty(\mathcal{D}_2 \times (0, T)) : \| \phi^0 \|_{0, \sigma, \nu, a} < +\infty \},
X_{\phi^1} := \{ \phi^1 \in L^\infty(\mathcal{D}_2 \times (0, T)) : \| \phi^1 \|_{0, \sigma, \nu, a} < +\infty \},
X_{\phi^\perp} := \{ \phi^\perp \in L^\infty(\mathcal{D}_2 \times (0, T)) : \| \phi^\perp \|_{0, \sigma, \nu, a} < +\infty \},
X_{\psi} := \{ \psi \in L^\infty(\mathcal{D} \times (0, T)) : \| \psi \|_* < +\infty, \quad \psi \text{ is Lipschitz continuous with respect to } (r, z) \text{ in } \mathcal{D} \times (0, T) \}. \]  
(3.253)

In order to introduce the space for the parameter function \( \lambda(t) \), we recall from (3.241) that the integral operator \( \mathcal{B}_0 \) takes the following approximate form

\[
\mathcal{B}_0[\lambda] = \int_{-T}^t \frac{\lambda(s)}{t-s} ds + O(\| \lambda \|_\infty).
\]
Proposition 3.2.6 provides an approximate inverse operator $P$ of the integral operator $B_0$ such that for $a(t)$ satisfying (3.243), $\lambda := P[a]$ satisfies

$$B_0[\lambda] = a + R_0[a] \text{ in } [-T, T],$$

where $R_0[a]$ is a small remainder. Also, the proof in [55] gives the following decomposition

$$P[a] = \lambda_0,\kappa + P_1[a] \quad (3.254)$$

with

$$\lambda_0,\kappa := \kappa \log T \int_T^t \frac{1}{\log(T-s)^2} ds, \quad t \leq T,$$

$\kappa = \kappa[a] \in \mathbb{R}$, and the function $\lambda_1 = P_1[a]$ satisfies

$$\|\lambda_1\|_{*,3-t} \lesssim \|\log T\|^{1-t} \log^2(\|\log T\|) \quad (3.255)$$

for $0 < t < 1$, where the $\|\cdot\|_{*,3-t}$-norm is defined as follows

$$\|f\|_{*,k} := \sup_{t \in [-T,T]} |\log(T-t)|^k |\dot{f}(t)|.$$

So we define

$$X_\lambda := \{ \lambda_1 \in C^1([-T,T]) : \lambda_1(T) = 0, \|\lambda_1\|_{*,3-t} < \infty \}.$$

Here by $(\kappa, \lambda_1)$, we represent $\lambda$ in the form

$$\lambda = \lambda_0,\kappa + \lambda_1,$$

and from [55], one can write the norm

$$\|\lambda\|_F = |\kappa| + \|\lambda_1\|_{*,3-t}. \quad (3.256)$$

Recall that $\xi(t) = (\xi_r(t), \xi_z(t))$ with $\xi_r(t) = \sqrt{2(n-4)(T-t)} + \xi_{r,1}(t)$, $\xi_z(t) = z_0 + \xi_{z,1}(t)$ and write $\xi(t) = \xi_r(t) + \xi_z(t)$. We define the following space for
with

\[\|\xi\|_G := \sup_{t \in (0,T)} \left[ (T-t)^{-\frac{1}{2}-\nu}|\xi_{r,1}(t)| + M_1(T-t)^{\frac{1}{2}-\nu}|\dot{\xi}_{r,1}(t)| \right.\]

\[\left. + |\xi_{z,j}(t)| + (T-t)^{-\nu}|\dot{\xi}_{z,j}(t)| \right] \] (3.257)

for some $0 < \nu < 1$.

Define

\[\mathcal{X} = X_{\phi^0} \times X_{\phi^1} \times X_{\phi^\perp} \times X_{\psi} \times \mathbb{R} \times X_{\lambda} \times X_{\xi}.\] (3.258)

We shall solve the inner–outer gluing system in a closed ball $B$ in which

\[(\phi^0, \phi^1, \phi^\perp, \psi, \kappa, \lambda_1, \xi_1) \in \mathcal{X}^\ast\]

satisfies

\[
\begin{cases}
\|\phi^0\|_{0,\sigma,v,a} + \|\phi^1\|_{\text{in},v_1,a_1} + \|\phi^\perp\|_{\text{in},v,a} \leq 1, \\
\|\psi\|_{\ast} \leq 1, \\
|\kappa - \kappa_0| \leq |\log T|^{-1/2}, \\
|\lambda_1|_{+3, a_1} \leq C|\log T|^{1-a} \log^2(|\log T|), \\
\|\xi\|_G \leq 1,
\end{cases}\] (3.259)

for some large and fixed constant $C$, where $\kappa_0 = Z_0^\ast(0)$.

The inner–outer gluing system (3.247)–(3.252) can be formulated as a fixed point problem, where we define an operator $\mathcal{F}$ which returns the solution from $B$ to $\mathcal{X}$

\[\mathcal{F} : B \subset \mathcal{X} \rightarrow \mathcal{X}\]

\[v \mapsto \mathcal{F}(v) = (\mathcal{F}_{\phi^0}(v), \mathcal{F}_{\phi^1}(v), \mathcal{F}_{\phi^\perp}(v), \mathcal{F}_{\psi}(v), \mathcal{F}_{\kappa}(v), \mathcal{F}_{\lambda_1}(v), \mathcal{F}_{\xi}(v))\]

with

\[\mathcal{F}_{\phi^0}(\phi^0, \phi^1, \phi^\perp, \psi, \kappa, \lambda_1, \xi_1) = \mathcal{T}_0(\mathcal{X}_0[\lambda, \xi, \Psi^\ast]),\]
\[ F(\phi^0, \phi^1, \psi, \kappa, \lambda_1, \xi_1) = \mathcal{T}_1(\mathscr{H}^1[\lambda, \xi, \Psi^*]), \]
\[ F(\phi^\perp, \phi^0, \psi, \kappa, \lambda_1, \xi_1) = \mathcal{T}_\perp \left( \mathscr{H}^\perp[\lambda, \xi, \Psi^*] + c^0[\lambda, \xi, \Psi^*] Z_5 \right), \]
\[ F(\psi^0, \psi^1, \kappa, \lambda_1, \xi_1) = \mathcal{T}_\psi \left( \mathcal{G}(\phi^0 + \phi^1 + \phi^\perp, \lambda, \xi) \right), \]
\[ F(\kappa^0, \phi^0, \psi^1, \kappa, \lambda_1, \xi_1) = \kappa[a^0[\lambda, \xi, \Psi^*]], \]
\[ F(\lambda_1, \phi^0, \phi^\perp, \psi, \kappa, \lambda_1, \xi_1) = \mathcal{P}_1[a^0[\lambda, \xi, \Psi^*]], \]
\[ F(\xi^0, \phi^1, \psi, \kappa, \lambda_1, \xi_1) = \Xi(\phi^0, \phi^1, \phi^\perp, \psi, \lambda, \xi). \]  

(3.260)

Here \( \mathcal{T}_0, \mathcal{T}_1 \) and \( \mathcal{T}_\perp \) are the operators given from Proposition 3.2.2 which solve different modes of the inner problems (3.248)–(3.250). The operator \( \mathcal{T}_\psi \) defined by Proposition 3.2.1 deals with the outer problem (3.247). Operators \( \kappa[a], \mathcal{P}_1 \) and \( \Xi \) handle the equations for \( \lambda \) and \( \xi \) which are defined in Proposition 3.2.6, (3.254) and (3.234).

**Choice of constants**

In this section, we list all the constraints of the parameters \( \beta, \alpha, a, a_1, \nu, v_1, v_2, \sigma \) which are sufficient for the inner–outer gluing scheme to work.

We first indicate all the parameters used in different norms.

- \( R(t) = \lambda_e^{-\beta(t)} \) with \( \beta \in (0, 1/2) \).
- The norm for \( \phi^0 \) solving mode 0 of the inner problem (3.248) is \( \| \cdot \|_{0, \sigma, v, a} \) which is defined in (3.150), where we require that \( v, a \in (0, 1) \), and \( \sigma > 0 \) is fixed and sufficiently small.
- The norm for \( \phi^1 \) solving modes 1 to 4 of the inner problem (3.249) is \( \| \cdot \|_{in, v_1, a_1} \) which is defined by (3.144), where we require that \( v_1 \in (0, 1) \) and \( a_1 \in (1, 2) \).
- The norm for \( \phi^\perp \) solving higher modes \( \{j \geq 5\} \) of the inner problem (3.250) is \( \| \cdot \|_{in, v, a} \) which is defined in (3.144), where \( v, a \in (0, 1) \).
- The norm for \( \psi \) solving the outer problem (3.247) is \( \| \cdot \|_s \) which is defined in (3.122), while the \( \| \cdot \|_{s^*} \)-norm for the right hand side of the outer problem (3.247) is defined in (3.121). Here we require that \( v, \alpha, v_2 \in (0, 1) \) and \( \gamma \in \ldots \).
(0, 1). Also, in Remark 3.2.1, we require $v_2 + 1 > v + \frac{3}{n} + \beta(\alpha - 2) n$ such that the \(\|\cdot\|_\ast\)-norm of $\psi$ is well-defined.

- In Proposition 3.2.6, we have the parameters $\omega, \Theta, m, l, \gamma$. Here $\omega$ is the parameter used to describe the remainder $R_\omega$ in (3.245) and $\omega \in (0, 1/2)$. To apply Proposition 3.2.6 in our setting, we let

$$
\Theta = v - 2 + \frac{4}{n} + \beta(2 + \alpha - \frac{8}{n}),
$$

$$
m = v - 2 - \gamma + \beta(2 + \alpha), \quad l < 1 + 2m,
$$

and require that

$$
\beta > \frac{1}{2} - \frac{\omega \gamma n}{8}
$$

such that $m + (1 + \omega)\gamma > \Theta$ is guaranteed. Also, we need

$$
v - 2 + \frac{4}{n} + \beta(2 + \alpha - \frac{8}{n}) > 0
$$

to ensure that $\Theta > 0$.

In order to get the desired estimates for the outer problem (3.247), by the computations in Section 3.2.7, we need the following restrictions

$$
v - 2 + \frac{4}{n} + \beta(4 + \alpha - \frac{8}{n}) - v_2 > 0, \quad 2\beta - v_2 > 0,
$$

$$
4\beta - v_2 - 1 > 0, \quad 0 < \alpha < a < 1,
$$

$$
\beta < \frac{1}{2}, \quad \beta(1 + \alpha - a_1) + v - v_1 - \frac{1}{2} < 0,
$$

$$
v - \beta(2\sigma(4 - a) + \alpha - 8) > 0, \quad 2v_1 - v - \beta(\alpha - 2a_1) > 0,
$$

$$
v - \beta(\alpha - 2a) > 0, \quad v_2 < 1,
$$

$$
2v - 3 + \frac{8}{n} - v_2 + \beta(4 + 2\alpha - \frac{16}{n}) > 0.
$$

In order to get the desired estimates for the inner problems at different modes (3.248)–(3.250), by the computations in Section 3.2.7, we need

$$
0 < v < 1, \quad \frac{4}{n} - 1 + \beta(2 + \alpha - \frac{8}{n}) > 0,
$$

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It turns out that suitable choices of the parameters satisfying all the restrictions in this section can be found for the space dimensions \( n = 5, 6, 7 \). Here we give specific example for each case. Sound choices are listed as follows.

- **\( n = 5 \)**: \( \beta = \frac{1}{2} - \varepsilon, \alpha = 4\varepsilon, a = \frac{9}{2}\varepsilon, a_1 = 2 - 2\varepsilon, v_1 = 3\varepsilon, v_2 < 1, \sigma > 0 \text{ small} \),

- **\( n = 6 \)**: \( \beta = \frac{1}{2} - \varepsilon, \alpha = 4\varepsilon, a = \frac{13}{4}\varepsilon, a_1 = 2 - 2\varepsilon, v_1 = 3\varepsilon, v_2 < 1, \sigma > 0 \text{ small} \),

- **\( n = 7 \)**: \( \beta = \frac{1}{2} - \varepsilon, \alpha = 4\varepsilon, a = \frac{57}{14}\varepsilon, a_1 = 2 - 2\varepsilon, v_1 = 3\varepsilon, v_2 < 1, \sigma > 0 \text{ small} \),

where \( \varepsilon > 0 \) is fixed and sufficiently small.

**Proof of Theorem 3.2.1**

Consider the operator

\[
\mathcal{F} = (\mathcal{F}_\phi, \mathcal{F}_\psi, \mathcal{F}_\kappa, \mathcal{F}_{\xi})
\]

given in (3.260).

To prove Theorem 3.2.1, our strategy is to show that the operator \( \mathcal{F} \) has a fixed point in \( \mathcal{B} \) by the Schauder fixed point theorem. Here the closed ball \( \mathcal{B} \) is defined in (3.259). By collecting the estimates (3.207), (3.220), (3.222), (3.224), (3.237),
we conclude that for \((\phi^0, \phi^1, \phi^\perp, \psi, \kappa, \lambda_1, \xi_1) \in \mathcal{B}\)

\[
\begin{aligned}
&\|\mathcal{F}_{\phi^0}(\phi^0, \phi^1, \phi^\perp, \psi, \kappa, \lambda_1, \xi_1)\|_{0, \sigma, \nu, a} \leq C\varepsilon, \\
&\|\mathcal{F}_{\phi^1}(\phi^0, \phi^1, \phi^\perp, \psi, \kappa, \lambda_1, \xi_1)\|_{\nu_1, \alpha} \leq C\varepsilon, \\
&\|\mathcal{F}_{\phi^\perp}(\phi^0, \phi^1, \phi^\perp, \psi, \kappa, \lambda_1, \xi_1)\|_{\nu, a} \leq C\varepsilon, \\
&\|\mathcal{F}_{\psi}(\phi^0, \phi^1, \phi^\perp, \psi, \kappa, \lambda_1, \xi_1)\|_* \leq C\varepsilon, \\
&\|\mathcal{F}_{\kappa}(\phi^0, \phi^1, \phi^\perp, \psi, \kappa, \lambda_1, \xi_1) - \kappa_0\| \leq C|\log T|^{-1}, \\
&\|\mathcal{F}_{\lambda_1}(\phi^0, \phi^1, \phi^\perp, \psi, \kappa, \lambda_1, \xi_1)\|_{3-1} \leq C|\log T|^{-1} \log^2(|\log T|), \\
&\|\mathcal{F}_{\xi}(\phi^0, \phi^1, \phi^\perp, \psi, \kappa, \lambda_1, \xi_1)\|_G \leq C\varepsilon,
\end{aligned}
\]

(3.262)

where \(C > 0\) is a constant independent of \(T\), and \(\varepsilon > 0\) is a small fixed number. On
the other hand, compactness of the operator \(\mathcal{F}\) defined in (3.261) can be proved by
a similar argument as in the proof of Theorem 3.1.1 (see Section 3.1.6). Then the
existence of the desired solution follows from the Schauder fixed point theorem.

The proof is complete. \(\square\)

### 3.2.8 Proofs of technical Lemmas

**Proof of Lemma 3.2.1** The proof is achieved by considering the following Cauchy
problem in \(\mathbb{R}^n\)

\[
\begin{aligned}
&\partial_t \psi_0 = \Delta \psi_0 + f \quad \text{in} \quad \mathbb{R}^n \times (0, T), \\
&\psi_0(x, 0) = 0 \quad \text{in} \quad \mathbb{R}^n.
\end{aligned}
\]

(3.263)

If we decompose the solution to (3.124) in the form \(\psi = \psi_0 + \psi_1\), then \(\psi_1\) solves
the homogeneous heat equation in \(\Omega \times (0, T)\) with boundary condition \(-\psi_1\). By
standard parabolic estimates, it suffices to establish the estimates (3.125)–(3.129)
for \(\psi_0\). In the sequel, we denote \(\psi\) by the solution to (3.263) given by Duhamel’s
formula

\[
\psi(x, t) = \int_0^t \int_{\mathbb{R}^n} e^{-\frac{|x-w|^2}{4\pi(t-s)}} f(w, s) dw ds
\]

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where \(w = (w_1, \ldots, w_n)\), \(r = \sqrt{w_1^2 + \cdots + w_{n-3}^2}\) and \(z = (w_{n-2}, w_{n-1}, w_n)\). We decompose

\[
I_1 \leq \int_0^t \lambda_s^{-3}(s) R^{2-\alpha}(s) \int_{|t-r-\xi(s)| \leq 2 \lambda_s(s) R(s)} e^{\varepsilon(t-s) \frac{|w|^2}{4(t-s)}} \frac{e^{-\frac{|w|^2}{4(t-s)}}}{(4\pi(t-s))^{n/2}} dw ds,
\]

for some \(\delta_1 \geq 1\) to be found. Here we recall that \(\xi(t) \sim (\sqrt{(n-4)(T-t)}, z_0)\) and take \(z_0 = (0, 0, 0)\) for convenience. Directly integrating, we obtain

\[
I_{11} = \int_0^t (T-t) \lambda_s^{-3}(s) R^{2-\alpha}(s) \int_{|t-r-\xi(s)| \leq 2 \lambda_s(s) R(s)} e^{\varepsilon(t-s) \frac{|w|^2}{4(t-s)}} dw ds
\]

\[
\lesssim \left( \int_0^t (T-t) \lambda_s^{-3}(s) R^{2-\alpha}(s) \int_{|\tilde{r}-\tilde{\xi}(s)| \leq 2 \lambda_s(s) R(s)} e^{\varepsilon(t-s) \frac{|w|^2}{4(t-s)}} dw dv ds \right)
\]

\[
\lesssim \left( \int_0^t (T-t) \lambda_s^{-3}(s) R^{2-\alpha}(s) \int_{|\tilde{r}-\tilde{\xi}(s)| \leq 2 \lambda_s(s) R(s)} (\tilde{r})^{n-4} d\tilde{r} d\tilde{z} ds \right)
\]

\[
\lesssim \int_0^t (T-t) \lambda_s^{-3}(s) R^{2-\alpha}(s) \frac{(\lambda_s(s) R(s))^{n/2}}{(T-s)^{n/2}} \left( (\lambda_s(s) R(s))^{2} - T \right)^{\frac{n-4}{2}} ds
\]

\[
\lesssim \lambda_s^{-1}(0) R^{2-\alpha}(0),
\]

where \(\tilde{x} = x(t-s)^{-1/2}\), \(\tilde{w} = w(t-s)^{-1/2}\), \(\tilde{r} = r(t-s)^{-1/2}\), \(\tilde{z} = z(t-s)^{-1/2}\), and for the third inequality above, we have used the fact that \(\sqrt{T-s} \gg \lambda_s(s) R(s)\) for \(T \ll 1\) since \(\lambda_s(s) R(s) = \lambda_s^{1-\beta}(s)\) with \(0 < \beta < 1/2\). Then similarly we compute
\[
I_{13} \lesssim \lambda_s^{v-3+\delta_1} (0) R^{\alpha+\frac{8}{n}} (0) |\log T|.
\]
(3.265)

Similarly, to prove (3.126), we decompose
\[
|\psi(x, t) - \psi(x, T)| \leq I_{21} + I_{22} + I_{23}
\]
with
\[
I_{21} = \int_0^{t-(T-t)} \int_{\mathbb{R}^n} |G(x-w, t-s) - G(x-w, T-s)| |f(w, s)| dw ds,
\]
\[
I_{22} = \int_{t-(T-t)}^t \int_{\mathbb{R}^n} |G(x-w, t-s) - G(x-w, T-s)| |f(w, s)| dw ds,
\]
\[
I_{23} = \int_t^T \int_{\mathbb{R}^n} |G(x-w, T-s)| |f(w, s)| dw ds,
\]
where \(G(x,t)\) is the heat kernel
\[
G(x,t) = \frac{e^{-\frac{|x|^2}{4t}}}{(4\pi t)^{n/2}}.
\]
(3.266)

For the first integral \(I_{21}\), we have
\[
I_{21} \leq (T-t) \int_0^1 \int_0^{t-(T-t)} \int_{\{|w-w_v| - \xi(s) \leq 2\lambda_s R(s)\}} |\partial_t G(x-w, t_v-s)| \\
\times \lambda_s^{v-3}(s) R^{2-\alpha}(s) dw ds dv
\]
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where \((\overline{w}_r, \overline{w}_z) = (\sqrt{\overline{w}_1^2 + \cdots + \overline{w}_{n-3}^2}, \overline{w}_{n-2}, \overline{w}_{n-1}, \overline{w}_n)\) and \(\overline{t}_v = vT + (1 - v)t\). Changing variables \(\overline{w}_v = (\overline{w}_{r,v}, \overline{w}_{z,v}) = (\overline{w}_r(t_v - s)^{-1/2}, \overline{w}_z(t_v - s)^{-1/2})\), we evaluate

\[
\int_{|w_v - w_v| < 2\lambda_v(s)R(s)} |\partial_t G(x - w, t_v - s)| dw_v \leq \int_{|w_v - w_v| < 2\lambda_v(s)R(s)} e^{-\frac{|w_v|^2}{4}} \left(1 + |w_v|^2\right) \frac{1}{t_v - s} \, dw_v
\]

and thus

\[
\int_0^{t-(T-t)} \int_{|w_v - w_v| < 2\lambda_v(s)R(s)} |\partial_t G(x - w, t_v - s)| \lambda_v^{-3}(s) R^{-2-\alpha}(s) \, dw_v \, ds
\]

\[
\leq \int_0^{t-(T-t)} \lambda_v^{-3}(s) R^{-2-\alpha}(s) \left(\frac{T-s}{t_v-s}\right)^{\frac{\alpha}{2}} \left(\frac{\lambda_v(s)R(s)}{t_v-s}\right)^4 \, ds
\]

\[
\leq \lambda_v^{-1}(t) R^{2-\alpha}(t),
\]

from which we conclude that

\[
I_{21} \lesssim \lambda_v^{-1}(t) R^{2-\alpha}(t) |\log(T-t)|. \tag{3.267}
\]

For \(I_{22}\), we have

\[
I_{22} \leq \int_{t-(T-t)}^{t} \int_{|w_v - w_v| < 2\lambda_v(s)R(s)} |G(x - w, t - s)| \lambda_v^{-3}(s) R^{-2-\alpha}(s) \, dw_v \, ds
\]

\[
+ \int_{t-(T-t)}^{t} \int_{|w_v - w_v| < 2\lambda_v(s)R(s)} |G(x - w, T - s)| \lambda_v^{-3}(s) R^{-2-\alpha}(s) \, dw_v \, ds.
\]

The first integral above can be estimated as

\[
\int_{t-(T-t)}^{t} \int_{|w_v - w_v| < 2\lambda_v(s)R(s)} |G(x - w, t - s)| \lambda_v^{-3}(s) R^{-2-\alpha}(s) \, dw_v \, ds
\]

\[
= \left(\int_{t-(T-t)}^{t-\lambda_v^E(t)} + \int_{t-\lambda_v^E(t)}^{t}\right) |G(x - w, t - s)| \lambda_v^{-3}(s) R^{-2-\alpha}(s) \, dw_v \, ds.
\]

Notice that we already estimate the above integral in (3.264) and (3.265). So with
the choice $\delta_1 = \frac{n+4-8\beta}{n}$, one has
\[
\int_{t=(T-t)}^t \int_{[(w_r, w_z)-\xi(s)] \leq 2\lambda_r(s)R(s)} |G(x-w, t-s)\lambda_r^{v-3}(s)R^{-2-\alpha}(s)dwds
\lesssim \lambda_r^{v-2+\frac{4}{n}}(t)R^{-2-\alpha+\frac{8}{n}}(t)|\log(T-t)|.
\]

Similarly, it holds that
\[
\int_{t=(T-t)}^t \int_{[(w_r, w_z)-\xi(s)] \leq 2\lambda_r(s)R(s)} |G(x-w, T-s)\lambda_r^{v-3}(s)R^{-2-\alpha}(s)dwds
\lesssim \lambda_r^{v-2+\frac{4}{n}}(t)R^{-2-\alpha+\frac{8}{n}}(t)|\log(T-t)|.
\]

Therefore, we obtain
\[
I_{22} \lesssim \lambda_r^{v-2+\frac{4}{n}}(t)R^{-2-\alpha+\frac{8}{n}}(t)|\log(T-t)|, \quad (3.268)
\]

For $I_{23}$, changing variables $\tilde{x} = x(T-s)^{-1/2}$, $\tilde{w} = w(T-s)^{-1/2}$ and $(\tilde{w}_r, \tilde{w}_z) = (w_r(T-s)^{-1/2}, w_z(T-s)^{-1/2})$, one has
\[
I_{23} \lesssim \int_t^T \int_{[(w_r, w_z)-\xi(s)] \leq 2\lambda_r(s)R(s)} e^{-\frac{|x-w|}{nR(s)}}\lambda_r^{v-3}(s)R^{-2-\alpha}(s)dwds
\lesssim \int_t^T \int_{[(\tilde{w}_r, \tilde{w}_z)-\frac{\xi(s)}{\sqrt{s}}] \leq 2\lambda_r(s)R(s)} e^{-\frac{|\tilde{x}-\tilde{w}|\sqrt{s}}{n\sqrt{R(s)}}}\lambda_r^{v-3}(s)R^{-2-\alpha}(s)\tilde{d}\tilde{w}
\lesssim \int_t^T \frac{|\log(T)|^{v+1-\beta(2-\alpha)}(T-s)^{v-1-\beta(2-\alpha)}}{|\log(T-s)|^{2(v+1-\beta(2-\alpha))}}ds
\lesssim \lambda_r^v(t)R^{-\alpha}(t) \quad (3.269)
\]

provided $v - \beta(2-\alpha) > 0$. Collecting (3.267), (3.268) and (3.269), we conclude the validity of (3.126).

Then we prove the gradient estimate (3.127). By the heat kernel, we get
\[
|\nabla\psi(x,t)| \lesssim \int_0^t \frac{\lambda_r^{v-3}(s)R^{-2-\alpha}(s)}{(t-s)^{n/2}} \int_{[(w_r, w_z)-\xi(s)] \leq 2\lambda_r(s)R(s)} e^{-\frac{|x-w|}{n(\sqrt{s})^{1/2}}} |x-w|dwds
\lesssim \int_0^t \frac{\lambda_r^{v-3}(s)R^{-2-\alpha}(s)}{(t-s)^{1/2}} \int_{[(\tilde{w}_r, \tilde{w}_z)-\frac{\xi(s)}{\sqrt{s}}] \leq 2\lambda_r(s)R(s)} e^{-\frac{|\tilde{x}-\tilde{w}|}{\sqrt{s}}} (1+|\tilde{w}|)\tilde{d}\tilde{w},
\]
where \( \tilde{x} = x(t - s)^{-1/2}, (w_r, w_z) = \left(\sqrt{w_1^2 + \cdots + w_{n-3}^2, w_{n-2}, w_{n-1}, w_n}\right) \), and \((\tilde{w}_r, \tilde{w}_z) = \left(\sqrt{\tilde{w}_1^2 + \cdots + \tilde{w}_{n-3}^2, \tilde{w}_{n-2}, \tilde{w}_{n-1}, \tilde{w}_n}\right)\). First, we compute

\[
\int_0^{t-(T-t)} \frac{\lambda_s^{v-3}(s)R^{-2-\alpha}(s)}{(t-s)^{1/2}} \int_{(\tilde{w}_r, \tilde{w}_z) - \tilde{\xi}(s)} \left| \frac{\xi(s)}{\sqrt{\lambda_s}} \right| \leq 2\xi(s) R(s) e^{-\frac{|\xi|^2}{\tilde{\lambda_s} R(s)}} (1 + |\tilde{w}|) \tilde{w} ds
\]

\[
\lesssim \int_0^{t-(T-t)} \frac{\lambda_s^{v+1}(s)R^{2-\alpha}(s)(T-s)^{n/2}}{(t-s)^{n+1}} ds
\]

\[
\lesssim \int_0^{t-(T-t)} \lambda_s^{v+1}(s)R^{2-\alpha}(s)(T-s)^{-\frac{n}{2}} ds
\]

\[
\lesssim \lambda_s^{v+1}(0)R^{2-\alpha}(0) |\log T|.
\]  

(3.270)

Then we compute

\[
\int_{t-(T-t)}^{t-\tilde{\lambda}_s^* (t)} \frac{\lambda_s^{v-3}(s)R^{-2-\alpha}(s)}{(t-s)^{1/2}} \int_{(\tilde{w}_r, \tilde{w}_z) - \tilde{\xi}(s)} \left| \frac{\xi(s)}{\sqrt{\lambda_s}} \right| \leq 2\xi(s) R(s) e^{-\frac{|\xi|^2}{\tilde{\lambda_s} R(s)}} (1 + |\tilde{w}|) \tilde{w} ds
\]

\[
\lesssim \int_{t-(T-t)}^{t-\tilde{\lambda}_s^* (t)} \frac{\lambda_s^{v+1}(s)R^{2-\alpha}(s)(T-s)^{n/2}}{(t-s)^{n+1}} ds
\]

\[
\lesssim \lambda_s^{v+\frac{n}{2} + (1-\alpha)\delta_2} (t)R^{2-\alpha}(t) |\log(T-t)|.
\]  

(3.271)

where \( \delta_2 \geq 1 \) is a constant to be determined. On the other hand, we have

\[
\int_{t-(T-t)}^{t} \frac{\lambda_s^{v-3}(s)R^{-2-\alpha}(s)}{(t-s)^{1/2}} \int_{(w_r, w_z) - \xi(s)} \left| \frac{\xi(s)}{R(s)} \right| \leq 2\lambda_s R(s) e^{-\frac{|\xi|^2}{\lambda_s R(s)}} |x - w| dw ds
\]

\[
\lesssim \int_{t-(T-t)}^{t} \frac{\lambda_s^{v-3}(s)R^{2-\alpha}(s)}{(t-s)^{1/2}} ds \lesssim \lambda_s^{v+\frac{n}{2} \delta_2} (t)R^{2-\alpha}(t). \]

(3.272)

By choosing \( \delta_2 = \frac{n+4-8\beta}{n} \) and combining (3.270)–(3.272), we prove the validity of the gradient estimate (3.127). The proof of (3.128) is similar to that of (3.126). We omit the details.

To prove the Hölder estimate (3.129), we decompose

\[
|\psi(x,t_2) - \psi(x,t_1)| \leq J_{11} + J_{12} + J_{13}
\]

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with

\[ J_{11} = \int_0^{t_1-(t_2-t_1)} \int_{\mathbb{R}^n} |G(x-w,t_2-s) - G(x-w,t_1-s)|f(w,s)dwds, \]

\[ J_{12} = \int_{t_1-(t_2-t_1)}^{t_1} \int_{\mathbb{R}^n} |G(x-w,t_2-s) - G(x-w,t_1-s)|f(w,s)dwds, \]

\[ J_{13} = \int_{t_1}^{t_2} \int_{\mathbb{R}^e} G(x-w,t_2-s)f(w,s)dwds, \]

where \(G(x,t)\) is the heat kernel (3.266). Here we assume that \(0 < t_1 < t_2 < T\) with \(t_2 < 2t_1\). For \(J_{11}\), by letting \(t_v = vt_2 + (1-v)t_1\), we have

\[ J_{11} \leq (t_2 - t_1) \int_0^{t_1-(t_2-t_1)} \int_{\mathbb{R}^n} |\partial_xG(x-w,t_v-s)|f(w,s)dwdsdv \]

\[ \lesssim (t_2 - t_1) \int_0^{t_1-(t_2-t_1)} \int_{\mathbb{R}^n} e^{-\frac{|w|^2}{4(t_v-s)}} \frac{|x-w|^2}{(t_v-s)^{\frac{n-1}{2}}} \lambda_s^{-3}(s)R^{-2-\alpha}(s)dwdsdv, \]

where \((w_r, w_z) = \left(\sqrt{w_1^2 + \cdots + w_n^2}, w_1, w_2, \ldots, w_n\right)\). Taking \(x_v = x(t_v-s)^{-1/2}\), \(w_v = w(t_v-s)^{-1/2}\), \((w_{r,v}, w_{z,v}) = (w_v(t_v-s)^{-1/2}, w_z(t_v-s)^{-1/2})\), we get

\[ \int_{\{(w_r, w_z) - \xi(s)\leq 2\lambda_v(s)R(s)\}} e^{-\frac{|w|^2}{4(t_v-s)}} \frac{|x-w|^2}{(t_v-s)^{\frac{n+1}{2}}} \lambda_s^{-3}(s)R^{-2-\alpha}(s)dw \]

\[ = \int_{\{(w_r, w_z) - \xi(s)\leq \frac{2\lambda_v(s)R(s)}{\sqrt{t_v-s}}\}} e^{-\frac{|w-w_v|^2}{4(t_v-s)}} (1 + \frac{|x_v-w_v|^2}{t_v-s}) \lambda_s^{-3}(s)R^{-2-\alpha}(s)dw_v, \]

Observing that for any \(\mu \in (0,1)\), we have

\[ \int_{\{(w_r, w_z) - \xi(s)\leq \frac{2\lambda_v(s)R(s)}{\sqrt{t_v-s}}\}} e^{-\frac{|w-w_v|^2}{4(t_v-s)}} (1 + \frac{|x_v-w_v|^2}{t_v-s}) dw_v \lesssim \left(\frac{\sqrt{T-s}}{\sqrt{t_v-s}}\right)^\mu, \]

where we have used the facts that \(\xi(s) \sim (\sqrt{2(n-4)(T-s)},0,0,0)\) and \(\sqrt{T-s} \gg 169\).
Recalling that $\lambda(s)R(s)$ for $T \ll 1$. Thus, one has

$$J_{11} \lesssim (t_2 - t_1) \int_{t_1 - (t_2 - t_1)}^{t_1 - (t_2 - t_1)} \frac{\lambda^{v-3}(s)R^{-2-\alpha}(s)(T-s)^\frac{\mu}{2}}{(t_2 - s)^{1 + \frac{\mu}{2}}} ds.$$

Recalling that $R(t) = \lambda e^{-\beta t}$ for $\beta \in (0, 1/2)$, we have the following two cases

- If $v - 3 + \beta(2 + \alpha) + \frac{\mu}{2} < 0$, then we have

  $$\int_{t_1 - (t_2 - t_1)}^{t_1 - (t_2 - t_1)} \frac{\lambda^{v-3}(s)R^{-2-\alpha}(s)(T-s)^\frac{\mu}{2}}{(t_2 - s)^{1 + \frac{\mu}{2}}} ds \lesssim \lambda^{v-3+\frac{\mu}{2}}(t_1)R^{-2-\alpha}(t_1)|\log(T-t_1)| \int_{t_1 - (t_2 - t_1)}^{t_1 - (t_2 - t_1)} \frac{1}{(t_2 - s)^{1 + \frac{\mu}{2}}} ds \lesssim \lambda^{v-3+\frac{\mu}{2}}(t_1)R^{-2-\alpha}(t_1)|\log(T-t_1)|(t_2 - t_1)^{-\frac{\mu}{2}}.$$

- If $v - 3 + \beta(2 + \alpha) + \frac{\mu}{2} \geq 0$, then we decompose

  $$\int_{t_1 - (t_2 - t_1)}^{t_1 - (t_2 - t_1)} \frac{\lambda^{v-3}(s)R^{-2-\alpha}(s)(T-s)^\frac{\mu}{2}}{(t_2 - s)^{1 + \frac{\mu}{2}}} ds = \left( \int_{0}^{t_1 - (T-t_1)} + \int_{t_1 - (T-t_1)}^{t_1 - (t_2 - t_1)} \right) \frac{\lambda^{v-3}(s)R^{-2-\alpha}(s)(T-s)^\frac{\mu}{2}}{(t_2 - s)^{1 + \frac{\mu}{2}}} ds.$$

Assuming $v - 3 + \beta(2 + \alpha) < 0$, we obtain that

$$\int_{0}^{t_1 - (T-t_1)} \frac{\lambda^{v-3}(s)R^{-2-\alpha}(s)(T-s)^\frac{\mu}{2}}{(t_2 - s)^{1 + \frac{\mu}{2}}} ds \lesssim \lambda^{v+\frac{\mu}{2}-3}(t_2)R^{-2-\alpha}(t_2)(t_2 - t_1)^{-\frac{\mu}{2}}$$

and similarly

$$\int_{t_1 - (T-t_1)}^{t_1 - (t_2 - t_1)} \frac{\lambda^{v-3}(s)R^{-2-\alpha}(s)(T-s)^\frac{\mu}{2}}{(t_2 - s)^{1 + \frac{\mu}{2}}} ds \lesssim \lambda^{v+\frac{\mu}{2}-3}(t_2)R^{-2-\alpha}(t_2)(t_2 - t_1)^{-\frac{\mu}{2}}.$$

In both cases, we have

$$J_{11} \lesssim \lambda^{v+\frac{\mu}{2}-3}(t_2)R^{-2-\alpha}(t_2)(t_2 - t_1)^{1-\frac{\mu}{2}}.$$
For $J_{12}$, we evaluate
\[
\int_{t_1-(t_2-t_1)}^{t_1} \int_{\mathbb{R}^n} |G(x-w, t_1-s)| f(w, s) dwds \\
\lesssim \int_{t_1-(t_2-t_1)}^{t_1} \lambda_*^{v-3}(s) R^{-2-\alpha}(s) \left( \frac{\sqrt{T-s}}{\sqrt{t_1-s}} \right)^{\mu} ds \\
\lesssim \int_{t_1-(t_2-t_1)}^{t_1} \lambda_*^{v+\frac{\mu}{2}-3}(t_2) R^{-2-\alpha}(t_2)(t_2-t_1)^{1-\mu/2},
\]
where we have changed variables $\tilde{x} = x(t_1-s)^{-1/2}$, $\tilde{w} = w(t_1-s)^{-1/2}$, and $\mu \in (0, 1)$. Similarly, we have
\[
\int_{t_1-(t_2-t_1)}^{t_1} \int_{\mathbb{R}^n} |G(x-w, t_2-s)| f(w, s) dwds \lesssim \lambda_*^{v+\frac{\mu}{2}-3}(t_2) R^{-2-\alpha}(t_2)(t_2-t_1)^{1-\mu/2}.
\]
Thus we conclude that
\[
J_{12} \lesssim \lambda_*^{v+\frac{\mu}{2}-3}(t_2) R^{-2-\alpha}(t_2)(t_2-t_1)^{1-\mu/2}.
\]
Finally, for $J_{13}$,
\[
J_{13} = \int_{t_2}^{t_1} \int_{\mathbb{R}^n} G(x-w, t_2-s) f(w, s) dwds \\
\lesssim \int_{t_1}^{t_2} \lambda_*^{v-3}(s) R^{-2-\alpha}(s) \left( \frac{\sqrt{T-s}}{\sqrt{t_1-s}} \right)^{\mu} ds \\
\lesssim \lambda_*^{v+\frac{4}{2}-3}(t_2) R^{-2-\alpha}(t_2)(t_2-t_1)^{1-\mu/2}
\]
follows from the same argument as before, where $\tilde{x} = x(t_2-s)^{-1/2}$ and $\tilde{w} = w(t_2-s)^{-1/2}$. This completes the proof of (3.129). \qed
Chapter 4

New gluing methods for the fractional problems

In this Chapter, we consider the fractional heat equation with critical exponent

\[ u_t + (-\Delta)^s u = u^{\frac{n+2s}{n-2s}} \quad \text{in } \Omega \times (0, \infty), \]

where \( \Omega \) is either a smooth, bounded domain in \( \mathbb{R}^n \) \( (n > 2s) \) or \( \Omega = \mathbb{R}^n \). Compared with the local case \( s = 1 \), the infinite and finite time bubbling for fractional parabolic problems is much more intricate. For the semilinear equation, Sugitani [181] proved non-existence of global solutions below the Fujita exponent \( p_* = 1 + \frac{2s}{n} \). The case of global existence above the exponent remains open since all the known techniques fail in this case. As far as blow-up is concerned, a theory in the spirit of the one developed for instance by Giga and Kohn [92] is missing. A crucial step in these approaches is to exhibit a monotone quantity. For the operator \( \partial_t + (-\Delta)^s \), such a quantity is missing (See [71] for a recent progress on the monotonicity formula in the subcritical case with \( s = 1/2 \)). On the other hand, the ODE techniques are no longer applicable in the fractional setting. Instead, we shall develop a new inner–outer gluing method for fractional parabolic problems to construct infinite and finite time bubbling solutions described in the following two sections.
4.1 Infinite time blow-up for the critical heat equation with fractional Laplacian

4.1.1 Introduction

Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^n$, $n \geq 1$. We consider the fractional heat equation with critical exponent

\[
\begin{cases}
    u_t = -(-\Delta)^s u + u^{\frac{n+2s}{n-2s}} & \text{in } \Omega \times (0, \infty), \\
    u = 0 & \text{on } (\mathbb{R}^n \setminus \Omega) \times (0, \infty), \\
    u(\cdot, 0) = u_0 & \text{in } \mathbb{R}^n,
\end{cases}
\]

(4.1)

for a function $u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ and a smooth, positive initial datum $u_0$ satisfying $u_0|_{\mathbb{R}^n \setminus \Omega} = 0$, $s \in (0, 1)$. Here, for any point $x \in \mathbb{R}^n$, the fractional Laplace operator $(-\Delta)^s u(x)$ is defined as

\[
(-\Delta)^s u(x) := C(n,s) \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy
\]

with a suitable positive normalizing constant $C(n,s)$. We refer to [73] for an introduction to the fractional Laplace operator and to [48] Appendix for a heuristic physical motivation in nonlocal quantum mechanics of the fractional operator considered here.

Parabolic problems like (4.1) and related ones have attracted much attention in recent years, for example, [9, 12, 19, 20, 22–24, 35, 77, 78, 171, 172] and the references therein. As in the case of $s = 1$, problem (4.1) is the formal negative $L^2$-gradient flow of the functional

\[
E(u) = \frac{1}{2} \int_{\mathbb{R}^n} |(-\Delta)^s u|^2 dx - \frac{n - 2s}{2} \int_{\Omega} |u|^{\frac{2n}{n-2s}} dx
\]

in

\[
H_0^s(\Omega) := \left\{ u \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} |(-\Delta)^s u|^2 dx < +\infty \text{ and } u = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega \right\},
\]

i.e., $\frac{d}{dt} E(u(\cdot,t)) = -\int_{\mathbb{R}^n} |u_t|^2 dx$. If the function $u(x,t)$ is independent of $t$, (4.1)
is a semilinear elliptic problem with fractional Laplacian, which has been studied extensively, for instance, in [36, 170].

When \( s = 1 \), problem (4.1) is the classical parabolic equation with critical exponent

\[
\begin{cases}
    u_t = \Delta u + u^{\frac{n+2}{n-2}} & \text{in } \Omega \times (0, \infty), \\
    u = 0 & \text{on } \partial \Omega \times (0, \infty), \\
    u(\cdot, 0) = u_0 & \text{in } \Omega.
\end{cases}
\]

(4.2)

Many authors are interested in the blow-up phenomenon of (4.2), for example, [43, 62, 66, 85, 147, 149, 162, 167]. In [43], Cortázar, del Pino and Musso obtained the following result. Suppose \( n > 4 \), let \( \hat{G}(x, y) \) be the Green’s function of \( -\Delta \) in \( \Omega \) with Dirichlet boundary value and \( \hat{H}(x, y) \) be its regular part. Given \( k \) distinct points \( q_1, \ldots, q_k \) in \( \Omega \) such that the matrix

\[
\hat{G}(q) = \begin{bmatrix}
\hat{H}(q_1, q_1) & -\hat{G}(q_1, q_2) & \cdots & -\hat{G}(q_1, q_k) \\
-\hat{G}(q_2, q_1) & \hat{H}(q_2, q_2) & \cdots & -\hat{G}(q_2, q_k) \\
\vdots & \vdots & \ddots & \vdots \\
-\hat{G}(q_k, q_1) & -\hat{G}(q_k, q_2) & \cdots & \hat{H}(q_k, q_k)
\end{bmatrix}
\]

is positive definite, they proved the existence of an initial datum \( u_0 \) and smooth parameter functions \( \xi_j(t) \to q_j \), \( 0 < \mu_j(t) \to 0 \), as \( t \to +\infty \), \( j = 1, \ldots, k \), such that there exists an infinite time blow-up solution \( u_q \) of (4.2) which has the approximate form

\[
u_q \approx \sum_{j=1}^{k} \alpha_n \left( \frac{\mu_j(t)}{\mu_j^2(t) + |x - \xi_j(t)|^2} \right)^{\frac{n-2}{2}}
\]

with \( \mu_j(t) = \beta_j t^{-\frac{1}{n-1}} (1 + o(1)) \) for certain positive constants \( \beta_j \). The aim of this section is to show that this phenomenon also occurs in problem (4.1). Our starting point is the positive entire solutions of the equation

\[-(-\Delta)^s U + U^{\frac{n+2}{n-2}} = 0 \quad \text{in } \mathbb{R}^n,\]
which are given by the bubbles

\[ U_{\mu, \xi}(x) = \mu^{-\frac{n-2s}{2}} U_0 \left( \frac{x - \xi}{\mu} \right) = \alpha_{n,s} \left( \frac{\mu}{\mu^2 + |x - \xi|^2} \right)^{\frac{n-2s}{2}}, \tag{4.3} \]

where

\[ U_0(y) = \alpha_{n,s} \left( \frac{1}{1 + |y|^2} \right)^{\frac{n-2s}{2}} \]

and \( \alpha_{n,s} \) is a constant depending only on \( n \) and \( s \), see, [1, 29, 130]. Let \( G(x,y) \) be the Green’s function for the following nonlocal problem

\[
\begin{aligned}
(-\Delta)^s G(x,y) &= c(n,s) \delta(x - y) \quad \text{in } \Omega, \\
G(\cdot, y) &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega,
\end{aligned}
\]

where \( \delta(x) \) denotes the Dirac measure at the origin and \( c(n,s) \) satisfies

\[ (-\Delta)^s \Gamma(x) = c(n,s) \delta(x), \quad \Gamma(x) = \frac{\alpha_{n,s}}{|x|^{n-2s}}. \]

The regular part of \( G(x,y) \) is denoted by \( H(x,y) \), namely \( H(x,y) \) solves the following problem

\[
\begin{aligned}
(-\Delta)^s H(x,y) &= 0 \quad \text{in } \Omega, \\
H(\cdot, y) &= \Gamma(\cdot - y) \quad \text{in } \mathbb{R}^n \setminus \Omega.
\end{aligned}
\]

Let \( q = (q_1, \ldots, q_k) \) be the collection of \( k \) distinct points in \( \Omega \) and define

\[
\mathcal{G}(q) := \begin{bmatrix}
H(q_1, q_1) & -G(q_1, q_2) & \cdots & -G(q_1, q_k) \\
-G(q_2, q_1) & H(q_2, q_2) & \cdots & -G(q_2, q_k) \\
\vdots & \ddots & \ddots & \vdots \\
-G(q_k, q_1) & -G(q_k, q_2) & \cdots & H(q_k, q_k)
\end{bmatrix}. \tag{4.5}
\]

**Theorem 4.1.1.** Assume \( n > 4s, \ s \in (0,1) \) and \( q_1, \ldots, q_k \) are distinct points in \( \Omega \) such that the matrix (4.5) is positive definite. Then there exist \( u_0 \) and smooth parameter functions \( \xi_j(t) \to q_j, \ 0 < \mu_j(t) \to 0, \) as \( t \to +\infty, \ j = 1, \ldots, k, \) such that
there exists solution \( u_q \) to problem (4.1) with the form

\[
 u_q = \sum_{j=1}^{k} \alpha_n, s \left( \frac{\mu_j(t)}{\mu_j^2(t) + |x - \xi_j(t)|^2} \right)^{\frac{n-2s}{2}} - \mu_j^{\frac{n-2s}{2}}(t)H(x,q_j) + \mu_j^{\frac{n-2s}{2}}(t)\varphi(x,t),
\]

where \( \varphi(x,t) \) is bounded satisfying \( \varphi(x,t) \to 0 \) as \( t \to +\infty \), uniformly away from \( q_j \). Furthermore, there exists a submanifold \( \mathcal{M} \) with codimension \( k \) in \( X := \{ u \in C^1(\mathbb{R}^n) : u|_{\mathbb{R}^n \setminus \Omega} = 0 \} \) containing \( u_q(x,0) \) such that, if \( u_0 \) is a small perturbation of \( u_q(x,0) \) in \( \mathcal{M} \), then the solution \( u(x,t) \) of (4.1) still has the form

\[
 u(x,t) = \sum_{j=1}^{k} \alpha_n, s \left( \frac{\bar{\mu}_j(t)}{\bar{\mu}_j^2(t) + |x - \bar{\xi}_j(t)|^2} \right)^{\frac{n-2s}{2}} - \bar{\mu}_j^{\frac{n-2s}{2}}(t)H(x,\bar{q}_j) + \bar{\mu}_j^{\frac{n-2s}{2}}(t)\bar{\varphi}(x,t),
\]

where the point \( \bar{q}_j \) is close to \( q_j \) for \( j = 1, \ldots, k \).

**Remark 4.1.1.** By the definition of \( H(x,y) \) in (4.4), points \( q_1, \ldots, q_k \) can be chosen sufficiently close to the smooth boundary such that the matrix (4.5) is diagonally dominant and thus positive definite. For a special case \( k = 1 \), the maximum principle implies \( H(q,q) > 0 \) where the concentration point \( q \in \Omega \).

In order to prove this theorem, we shall develop a new **inner-outer gluing scheme** for fractional parabolic problems. When dealing with parabolic problems, a crucial step in the scheme is to find a solution of the linearized parabolic equation around the bubble with sufficiently fast decay. However, it seems that the local argument in [43] for the classical critical heat equation does not work in the fractional case. Inspired by Lemma 4.5 of [55] and the linear theory of [174], we will use a blow-up argument based on the nondegeneracy of bubbles and a removable singularity property for the corresponding limit equations. (See Section 4.1.5 below.)
4.1.2 Construction of the approximation

Setting up the problem.

Let $t_0 > 0$. We consider the following evolution problem

\[
\begin{aligned}
    u_t &= -(-\Delta)^s u + u^{\frac{n+2s}{n-2s}} & \quad & \text{in } \Omega \times (t_0, \infty), \\
    u &= 0 & \quad & \text{in } (\mathbb{R}^n \setminus \Omega) \times (t_0, \infty),
\end{aligned}
\]

which provides a solution $u(x,t) = u(x,t-t_0)$ to (4.1). Given $k$ points $q_1, \ldots, q_k \in \mathbb{R}^n$, our aim is to find a solution $u(x,t)$ of (4.6) in the following approximate form

\[
u(x,t) \approx \sum_{j=1}^{k} U_{\mu_j(t), \xi_j(t)}(x)
\]

with $\xi_j(t) \to q_j$, $\mu_j(t) \to 0$ as $t \to \infty$ for all $j = 1, \ldots, k$ and $U_{\mu_j(t), \xi_j(t)}(x)$ is defined in (4.3). Denote the error operator as

\[
S(u) := -u_t - (-\Delta)^s u + u^p,
\]

where $p = \frac{n+2s}{n-2s}$. Then the error of $U_{\mu_j(t), \xi_j(t)}(x)$ is

\[
S(U_{\mu_j(t), \xi_j(t)}) = -\frac{\partial}{\partial t} U_{\mu_j, \xi_j}(x) = \mu_j^{\frac{n-2s}{2}} \left( \frac{\dot{\mu}_j}{\mu_j} Z_{n+1}(y_j) + \frac{\dot{\xi}_j}{\mu_j} \nabla U(y_j) \right)
\]

\[
= \mu_j^{\frac{n-2s}{2} - 1} \dot{\mu}_j Z_{n+1}(y_j) + \mu_j^{\frac{n-2s}{2} - 1} \dot{\xi}_j \cdot \nabla U(y_j).
\]

Here $y_j = \frac{x - \xi_j(t)}{\mu_j(t)}$. It turns out that the terms $\mu_j^{\frac{n-2s}{2} - 1} \dot{\mu}_j Z_{n+1}(y_j)$ and $\mu_j^{\frac{n-2s}{2} - 1} \dot{\xi}_j \cdot \nabla U(y_j)$ do not have enough decay to perform the gluing method, so we add nonlocal terms to cancel them out at main order. Note that the main order of

\[
Z_{n+1}(y) = \frac{n-2s}{2} \alpha_{n,s} \frac{1 - |y|^2}{(1 + |y|^2)^{\frac{n-2s}{2}+1}}
\]

is

\[
- \frac{n-2s}{2} \alpha_{n,s} \frac{1}{(1 + |y|^2)^{\frac{n-2s}{2}}}.
\]
Therefore, we consider the equation

$$-\varphi_t - (-\Delta)^s \varphi - \frac{n - 2s}{2} \alpha_{n,s} \frac{\mu_j}{\mu_j} \left( \frac{\mu_j^{-(n-2s)}}{1 + \left| \frac{x - \xi_j}{\mu_j} \right|^2} \right) = 0 \text{ in } \mathbb{R}^n \times (t_0, +\infty). \tag{4.7}$$

Then

$$\Phi^0_j(x, t) = -\int_{t_0}^t \int_{\mathbb{R}^n} p(t - \tilde{s}, x - y) \frac{\mu_j(\tilde{s})}{\mu_j(y)} \frac{\mu_j^{-(n-2s)}(\tilde{s})}{1 + \left| \frac{y - \xi_j(\tilde{s})}{\mu_j(\tilde{s})} \right|^2} \frac{|y_j|^2}{(1 + |y_j|^2)^{2 - 2s/2}} dyd\tilde{s},$$

is a bounded solution for (4.7). Here the function $p(t, x)$ is the heat kernel for the fractional heat operator $-\partial_t^{\alpha} - (-\Delta)^s$, see [16] for its definition and properties.

Using the super-sub solution argument (see Lemma 4.1.3), it is easy to see that $\Phi^0_j(x, t)$ satisfies the estimate $\Phi^0_j(x, t) \sim \frac{\mu_j^{n+4s}}{\mu_j(1 + |y_j|^n)^{n-4s}}$.

Similarly, for $y_j = \frac{x_i}{\mu_i}$, we consider the equation

$$-\varphi_t - (-\Delta)^s \varphi + \alpha_{n,s} (n-2s) \mu_j^{-(n-2s)-1} \frac{|y_j|^2}{(1 + |y_j|^2)^{2 - 2s/2}} \tilde{\xi}_j \cdot y_j = 0 \text{ in } \mathbb{R}^n \times (t_0, +\infty). \tag{4.8}$$

Its solution defined by

$$\Phi^1_j(x, t) = -\int_{t_0}^t \int_{\mathbb{R}^n} p(t - \tilde{s}, x - y) \mu_j^{-(n-2s)}(\tilde{s}) \frac{\tilde{\xi}_j(\tilde{s}) \cdot \frac{y_j - \xi_j(\tilde{s})}{\mu_j(\tilde{s})}}{\mu_j(\tilde{s})} \times \left( \frac{|y_j - \xi_j(\tilde{s})|}{\mu_j(\tilde{s})} \right)^2 \frac{2^{-2s/2}}{(1 + \left| \frac{y_j - \xi_j(\tilde{s})}{\mu_j(\tilde{s})} \right|^2)^2} dyd\tilde{s},$$

satisfies the estimate $\Phi^1_j(x, t) \sim \frac{|\tilde{\xi}_j|}{\mu_j} \frac{\mu_j^{n+4s}}{1 + |y_j|^n}$. Define $\Phi^*_j(x, t) = \Phi^0_j(x, t) + \Phi^1_j(x, t)$. 

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Since \( u = 0 \) in \( \mathbb{R}^n \setminus \Omega \), a better approximation than \( \sum_{j=1}^{k} U_{\mu_j(t),\xi_j(t)}(x) \) is

\[
u(x,t) = \sum_{j=1}^{k} u_j(x,t) \text{ with } u_j(x,t) := U_{\mu_j(t),\xi_j(t)}(x) + \mu_j^{\frac{n-2s}{2}} \Phi_j(x,t) - \mu_j^{\frac{n-2s}{2}} H(x,q_j). \tag{4.9}
\]

The error of \( \nu \) can be computed as

\[
S(\nu) = - \sum_{i=1}^{k} \partial_i u_i + \left( \sum_{i=1}^{k} u_i \right)^p - \sum_{i=1}^{k} U_{\mu_i, \xi_i} - \sum_{i=1}^{k} \mu_i^{\frac{n-2s}{2}} (-\Delta)^s \Phi_i(x,t). \tag{4.10}
\]

The error \( S(\nu) \).

Near a given point \( q_j \), we have the following estimate.

Lemma 4.1.1. Consider the region \( |x - q_j| \leq \frac{1}{2} \min_{i \neq j} |q_i - q_j| \) for a fixed index \( j \), denote \( x = \xi_j + \mu_j y_j \), then we have

\[
S(\nu) = \mu_j^{\frac{n-2s}{2}} (\mu_j E_0 + \mu_j E_1 + \mathcal{R})
\]

with

\[
E_0 = p U(y_j)^{p-1} \left[ -\mu_j^{n-2s} H(q_j, q_j) + \sum_{i \neq j} \mu_i^{\frac{n-2s}{2}} \mu_j^{\frac{n-2s}{2}} G(q_j, q_i) \right] + p U(y_j)^{p-1} \mu_j^{n-2s} \Phi_j^0(q_j, t) + \mu_j^{2s-2} \mu_j \left( Z_{n+1}(y_j) + \frac{n - 2s}{2} \alpha_{n,s} \frac{1}{(1 + |y_j|^2)^{\frac{n-2s}{2}}} \right),
\]

\[
E_1 = p U(y_j)^{p-1} \left[ -\mu_j^{n-2s} \nabla H(q_j, q_j) + \sum_{i \neq j} \mu_j^{\frac{n-2s}{2}} \mu_i^{\frac{n-2s}{2}} \nabla G(q_j, q_i) \right] \cdot y_j + p U(y_j)^{p-1} \mu_j^{n-2s} \Phi_j^1(q_j, t) + \alpha_{n,s}(n - 2s) \mu_j^{2s-2} \frac{\dot{\xi}_j \cdot y_j}{(1 + |y_j|^2)^{\frac{n-2s}{2} + 1}},
\]

\[
\mathcal{R} = \frac{\mu_0^{n-2s+2} g}{1 + |y_j|^{4s-2}} + \frac{\mu_0^{n-2s} \overline{g} \cdot (\xi_j - q_j) + \mu_0^{n+2s} f + \mu_0^{n-1} \sum_{i=1}^{k} \mu_i f_i + \mu_0^{k} \sum_{i=1}^{k} \ddot{\xi}_i \cdot f_i}{1 + |y_j|^{4s}},
\]

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where \( f, f_i, \tilde{f}_i \) are smooth, bounded functions depending on \((y, \mu_{0^{-1}} \mu, \xi, \mu_j y_j)\), and \( g, \tilde{g} \) depend on \((y, \mu_{0^{-1}} \mu, \xi)\).

**Proof.** We write

\[
 u_{\mu, \xi}(x, t) = \sum_{i=1}^{k} \mu_{i}^{-\frac{n-2s}{2}} U(y_i) + \mu_{i}^{-\frac{n-2s}{2}} \Phi_i^s(x, t) - \mu_{i}^{-\frac{n-2s}{2}} H(x, q_i), \quad y_i = \frac{x - \xi_i}{\mu_i}
\]

and

\[
 S(u_{\mu, \xi}) = S_1 + S_2,
\]

where

\[
 S_1 := \sum_{i=1}^{k} \left( \mu_{i}^{-\frac{n-2s}{2}} \mu_{i}^{-\frac{n-2s}{2}} \xi_i \cdot \nabla U(y_i) + \mu_{i}^{-\frac{n-2s}{2}} \mu_{i}^{-\frac{n-2s}{2}} Z_{u+1}(y_i) \\
+ \frac{n-2s}{2} \mu_{i}^{-\frac{n-2s}{2}} \mu_{i}^{-\frac{n-2s}{2}} H(x, q_i) \right) - \sum_{i=1}^{k} \partial_i \left( \mu_{i}^{-\frac{n-2s}{2}} \Phi_i^s(x, t) \right),
\]

\[
 S_2 := \left( \sum_{i=1}^{k} \mu_{i}^{-\frac{n-2s}{2}} U(y_i) + \mu_{i}^{-\frac{n-2s}{2}} \Phi_i^s(x, t) - \mu_{i}^{-\frac{n-2s}{2}} H(x, q_i) \right)^p \\
- \sum_{i=1}^{k} \mu_{i}^{-\frac{n-2s}{2}} U(y_i)^p - \sum_{i=1}^{k} \mu_{i}^{-\frac{n-2s}{2}} (-\Delta)^s \Phi_i^s(x, t).
\]

Let

\[
 S_2 = S_{21} + S_{22}
\]

with

\[
 S_{21} = \mu_{j}^{-\frac{n-2s}{2}} \left[ (U(y_j) + \Theta_j)^p - U(y_j)^p \right],
\]

\[
 S_{22} = - \sum_{i \neq j} \mu_{i}^{-\frac{n-2s}{2}} U(y_i)^p - \sum_{i=1}^{k} \mu_{i}^{-\frac{n-2s}{2}} (-\Delta)^s \Phi_i^s(x, t),
\]

\[
 \Theta_j = - \mu_{j}^{-\frac{n-2s}{2}} (H(x, q_j) - \Phi_j^s(x, t)) \\
+ \sum_{i \neq j} \left[ (\mu_{j} \mu_{i}^{-1})^{-\frac{n-2s}{2}} U(y_i) - (\mu_{j} \mu_{i}^{-1})^{-\frac{n-2s}{2}} (H(x, q_i) - \Phi_i^s(x, t)) \right]. \tag{4.11}
\]

Observe that \(|\Theta_j| \lesssim \mu_0^{n-2s}\) uniformly in small \(\delta\), we assume \(U(y_j)^{-1}|\Theta_j| < \frac{1}{2}\) in
the considered region. By Taylor expansion, we have

\[
S_{21} = \mu_j^{-\frac{n+2s}{2}} \left[ pU(y_j)^{p-1} \Theta_j + p(p-1) \int_0^1 (1-s) (U(y_j) + s \Theta_j)^{p-2} ds \Theta_j \right].
\]

For \( i \neq j \),

\[
U(y_i) = U \left( \frac{\mu_j y_j + \xi_j - \xi_i}{\mu_i} \right) = \frac{\alpha_{n,s} \mu_j^{-2s}}{(\mu_i^2 + |\mu_j y_j + \xi_j - \xi_i|^2)^{\frac{n-2s}{2}}} f(\xi_j, \mu, \mu_j y_j)
\]

where \( f \) is smooth in its arguments and \( f(q,0,0) = 0 \). Then

\[
\Theta_j = -\mu_j^{-2s} \left( H(\mu_j y_j + \xi_j, q_j) - \Phi_j^* (\mu_j y_j + \xi_j, \xi_j) \right)
+ \sum_{i \neq j} \left( (\mu_j \mu_i)^{\frac{n-2s}{2}} G(\mu_j y_j + \xi_j, q_i) + \mu_i^{n-2s+2} f(\xi_j, \mu, \mu_j y_j)
+ (\mu_j \mu_i)^{\frac{n-2s}{2}} \Phi_j^* (\mu_j y_j + \xi_j, t) \right).
\]

By further expansion, we get

\[
\Theta_j = -\mu_j^{-2s} \left( H(q_j, q_j) - \Phi_j^* (q_j, t) \right) + \sum_{i \neq j} (\mu_j \mu_i)^{\frac{n-2s}{2}} G(q_j, q_i)
+ \mu_j^{n-2s+2} f(\xi_j, \mu, \mu_j y_j) + (\mu_j \mu_i)^{\frac{n-2s}{2}} \Phi_j^* (\mu_j y_j + \xi_j, t) + (\mu_j y_j + \xi_j - q_j) \cdot
\]

\[
\left[ -\mu_j^{-2s} \nabla (H(q_j, q_j) - \Phi_j^* (q_j, t)) + \sum_{i \neq j} (\mu_j \mu_i)^{\frac{n-2s}{2}} \nabla G(q_j, q_i) \right]
+ \int_0^1 \left\{ -\mu_j^{-2s} D^2_x (H - \Phi_j^* ) (q_j + s(\mu_j y_j + \xi_j - q_j), q_j)
+ \sum_{i \neq j} (\mu_j \mu_i)^{\frac{n-2s}{2}} D^2_x G(q_j + s(\mu_j y_j + \xi_j - q_j), q_i) \right\} [\mu_j y_j + \xi_j - q_j]^2 (1-s) ds.
\]
We conclude that
\[
\Theta_j = -\mu_j^{n-2s} (H(q_j, q_j) - \Phi_j^* (q_j, t)) + \sum_{i \neq j} (\mu_j \mu_i)^{\frac{n-2s}{2}} G(q_j, q_i) \\
+ \left[-\mu_j^{n-2s+1} \nabla H(q_j, q_j) + \sum_{i \neq j} \mu_j^{\frac{n-2s}{2}+1} \mu_i^{\frac{n-2s}{2}} \nabla G(q_j, q_i) \right] \cdot y_j \\
+ \mu_0^{n-2s} (\xi - q_j) \cdot f(\xi, \mu_j y_j, \mu_0^{-1} \mu) + \mu_0^{n-2s+2} F(\xi, \mu_j y_j, \mu_0^{-1} \mu)[y_j]^2 \\
+ \mu_i^{n-2s+2} f(\xi, \mu, \mu_j y_j),
\]
where \( f \) and \( F \) are smooth and bounded. On the other hand, we have
\[
S_{22} = -\sum_{i \neq j} \mu_i^{\frac{n-2s}{2}} U(y_i)^p - \sum_{i = 1}^k \mu_i^{\frac{n-2s}{2}} (-\Delta)^s \Phi_i^*(x, t) \\
= -\sum_{i \neq j} \frac{\alpha_{i,j} \mu_i^{\frac{n-2s}{2}}}{|q_j - q_i|^n} + \mu_i^{\frac{n-2s}{2}} f(\xi, \mu, \mu_j y_i) - \sum_{i = 1}^k \mu_i^{\frac{n-2s}{2}} (-\Delta)^s \Phi_i^*(x, t),
\]
so
\[
S_{22} = \mu_0^{n+2s} f(\xi, \mu_0^{-1} \mu, \mu_j y_j) - \sum_{i = 1}^k \mu_i^{\frac{n-2s}{2}} (-\Delta)^s \Phi_i^*(x, t),
\]
where \( f \) is smooth in its arguments and \( f(q, 0, 0) = 0 \).

Decompose \( S_1 = S_{11} + S_{12} \), where
\[
S_{11} := \mu_j^{\frac{n-2s}{2}+1} \xi_j \cdot \nabla U(y_j) + \mu_j^{\frac{n-2s}{2}+1} \mu_j Z_{n+1}(y_j) - \mu_j^{\frac{n-2s}{2}} \partial_t \Phi_j^*(x, t),
\]
\[
S_{12} := \sum_{i \neq j} \mu_i^{\frac{n-2s}{2}+1} \xi_i \cdot \nabla U(y_i) - \left( \partial_t \mu_j^{\frac{n-2s}{2}} \right) \Phi_j^*(x, t) \\
+ \mu_i^{\frac{n-2s}{2}+1} \mu_i Z_{n+1}(y_i) + \sum_{i = 1}^k \frac{n-2s}{2} \mu_i^{\frac{n-2s}{2}-1} \mu_i H(x, y_i) \\
- \sum_{i \neq j} \partial_t \left( \mu_i^{\frac{n-2s}{2}} \Phi_i^*(x, t) \right).
\]

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We write

\[
S_{12} = \sum_{i \neq j} -\alpha_{n,s} (n - 2s) \mu_j^{\frac{n-2s}{2}} \hat{\xi}_i \cdot \left[ \frac{q_i - q_j}{|q_i - q_j|^{n-2s} + 2} + \tilde{f}_i(\xi, \mu, \mu) \right] \\
+ \sum_{i \neq j} \mu_i^{\frac{n-2s}{2}} \mu_j^{\frac{n-2s}{2}} \left[ \frac{c_{n,s}}{|q_i - q_j|^{n-2s}} + f_i(\xi, \mu, \mu) \right] \\
+ \sum_{i=1}^{k} \frac{n-2s}{2} \mu_i^{\frac{n-2s}{2}} \mu_j^{\frac{n-2s}{2}} \left[ (H(q_j, q_i) - \Phi^*_i(q_j,t)) + \tilde{f}_i(\mu, \xi) \right],
\]

where \( \tilde{f}_i \) are smooth in their arguments vanishing in the limit. In total, we can write

\[
S_{12} = \mu_0^{\frac{n-2s}{2}} \sum_{i=1}^{k} \mu_i f_{i0}(\mu_0^{-1} \mu, \xi, \mu) + \mu_0^{\frac{n-2s}{2}} \sum_{i=1}^{k} \hat{\xi}_i \cdot \tilde{f}_{i1}(\mu_0^{-1} \mu, \xi, \mu) 
\]

for functions \( f_{i0}, \tilde{f}_{i1} \) smooth. This concludes the proof of the lemma.

Next, we try to find a solution with the following form

\[
u(x, t) = u_{\mu, \xi}(x, t) + \tilde{\phi}(x, t),
\]

where \( \tilde{\phi} \) is a small term. By \( S(u_{\mu, \xi} + \tilde{\phi}) = 0 \), our main equation can be expressed with respect to \( \tilde{\phi} \) as

\[
- \partial_t \tilde{\phi} - (-\Delta)^s \tilde{\phi} + pu^{p-1}_{\mu, \xi} \tilde{\phi} + S(u_{\mu, \xi}) + \tilde{N}_{\mu, \xi}(\tilde{\phi}),
\]

where

\[
\tilde{N}_{\mu, \xi}(\tilde{\phi}) = (u_{\mu, \xi} + \tilde{\phi})^p - u_{\mu, \xi} - pu^{p-1}_{\mu, \xi} \tilde{\phi}.
\]

Write \( \tilde{\phi}(x, t) \) in self-similar form around \( q_j \)

\[
\tilde{\phi}(x, t) = \mu_j^{-\frac{n-2s}{2}} \phi \left( \frac{x - \xi_j}{\mu_j} \right).
\]

By a direct computation, we obtain from (4.12)-(4.14) that

\[
0 = \mu_j^{\frac{n-2s}{2}} S(u_{\mu, \xi} + \tilde{\phi}) = -(-\Delta)^s \phi + pU(y)^{p-1} \phi + \mu_j^{\frac{n-2s}{2}} S(u_{\mu, \xi}) + A[\phi]
\]
with
\[
A[\phi] = -\mu^2 \partial_t \phi + \mu^{2s-1} \mu_j \left[ \frac{n-2s}{2} \phi + y \cdot \nabla y \phi \right] + \nabla y \phi \cdot \mu^{2s-1} \xi_j
+ p \left[ (U(y) + \Theta_j)^{p-1} - U(y)^{p-1} \right] \phi + (U(y) + \Theta_j + \phi)^p
- (U(y) + \Theta_j)^p - p (U(y) + \Theta_j)^{p-1} \phi,
\]
where \( \Theta_j \) is defined in (4.11). We assume that \( \phi \) decays in the \( y \) variable and \( A[\phi] \) is small when \( t \) is large.

### Improvement of the approximation.

To improve the approximation, \( \phi(y,t) \) should be equal to the solution \( \phi_0(y,t) \) of the following elliptic type equation of order \( 2s \)

\[
-\left( -\Delta \right)^s \phi_0 + p U(y)^{p-1} \phi_0 = -\mu \frac{n+2s}{j} \nabla S(u,\xi_j) \text{ in } \mathbb{R}^n, \quad \phi_0(y,t) \to 0 \text{ as } |y| \to \infty
\]

at main order. Equation (4.16) is a special case of

\[
L_0[\psi] := -\left( -\Delta \right)^s \psi + p U(y)^{p-1} \psi = h(y) \text{ in } \mathbb{R}^n, \quad \psi(y) \to 0 \text{ as } |y| \to \infty. \quad (4.17)
\]

It is well known (see [47]) that every bounded solution of \( L_0[\psi] = 0 \) in \( \mathbb{R}^n \) is the linear combination of the functions

\[ Z_1, \ldots, Z_{n+1}, \]

where

\[ Z_i(y) := \frac{\partial U}{\partial y_i}(y), \quad i = 1, \ldots, n, \quad Z_{n+1}(y) := \frac{n-2s}{2} U(y) + y \cdot \nabla U(y). \]

Furthermore, problem (4.17) is solvable for \( h(y) = O(|y|^{-m}) \), \( m > 2s \), if it holds that

\[
\int_{\mathbb{R}^n} h(y) Z_i(y) dy = 0 \quad \text{for all } \quad i = 1, \ldots, n+1.
\]
First, we consider the solvability condition for (4.16),

\[
\int_{\mathbb{R}^n} \mu_j^{n+2s} S(u_{\mu, \xi})(y, t)Z_{n+1}(y) dy = 0. \tag{4.18}
\]

We claim that by choosing \( \mu_{0j} = b_j \mu_0(t) \) for suitable positive constant \( b_j \), \( j = 1, \ldots, k \), \( \mu_0(t) = c_{n,s} t^{-\frac{1}{n+2s}} \) with \( c_{n,s} \) be a positive constant depending only on \( n \) and \( s \), this identity can be achieved at main order. Note that, we have \( \dot{\mu}_0(t) = -\frac{1}{(n-4s)Z_n} \mu_0^{n-4s+1}(t) \). The main contribution to the integral comes from the term

\[
E_{0j} = pU(y_j)^{p-1} \left[ -\mu_j^{n-2s-1} H(q_j, q_j) + \sum_{i \neq j} \mu_j^{n-2s-1} \mu_i^{n-2s} G(q_j, q_i) \right] \\
+ pU(y_j)^{p-1} \mu_j^{n-2s-1} \Phi_0^j(q_j, t) \\
+ \mu_j^{2s-2} \dot{\mu}_j \left( Z_{n+1}(y_j) + \frac{n-2s}{2} \alpha_{n,s} \frac{1}{(1 + |q_j|^{2})^{\frac{n-2s}{2}}} \right).
\]

Now, let us compute the nonlocal term \( \Phi_0^j(q_j, t) \). Since the heat kernel function \( p(t, x) \) satisfies

\[
p(t - \tilde{s}, q_j - y) \asymp \frac{t - \tilde{s}}{[(t - \tilde{s})^{\frac{1}{2}} + |q_j - y|^{2}]^{\frac{n+2s}{2}}},
\]

we have

\[
p(t - \tilde{s}, q_j - y) = (t - \tilde{s})^{-\frac{n}{2}} p \left( 1, \frac{|q_j - y|}{(t - \tilde{s})^{\frac{1}{2}}} \right)
\]
a small constant \( \delta \) where

\[
\Phi_0^0(q_j, t) = - \int_0^t \int_\mathbb{R}^n p(t - \delta, q_j - y) \frac{\mu_j(\delta)}{\mu_j(\delta)} \frac{\mu_j^{-(n-2s)}(\delta)}{1 + \frac{|y - \delta|^2}{\mu_j(\delta)^2}} dyd\delta.
\]

We claim that

\[
\Phi_0^0(q_j, t) = -(1 + o(1)) \int_0^t \int_\mathbb{R}^n p(t - \delta, q_j - y) \frac{\mu_j(\delta)}{\mu_j(\delta)} \frac{\mu_j^{-(n-2s)}(\delta)}{1 + \frac{|y - \delta|^2}{\mu_j(\delta)^2}} dyd\delta,
\]

where

\[
F(a) = \int_\mathbb{R}^n p(1, x) \frac{1}{(1 + a^2 |x|^2)^{\frac{n-2s}{2}}} dx.
\]

We claim that

\[
\Phi_0^0(q_j, t) = -(1 + o(1)) \int_0^t \int_\mathbb{R}^n p(t - \delta, q_j - y) \frac{\mu_j(\delta)}{\mu_j(\delta)} \frac{\mu_j^{-(n-2s)}(\delta)}{1 + \frac{|y - \delta|^2}{\mu_j(\delta)^2}} dyd\delta = c(1 + o(1))
\]

for a suitable positive constant \( c \) depending on \( n, s \) and \( b_j, j = 1, \ldots, k \). Indeed, for a small constant \( \delta > 0 \), we decompose the integral

\[
\int_0^t \int_\mathbb{R}^n p(t - \delta, q_j - y) \frac{\mu_j(\delta)}{\mu_j(\delta)} \frac{\mu_j^{-(n-2s)}(\delta)}{1 + \frac{|y - \delta|^2}{\mu_j(\delta)^2}} dyd\delta
\]
\[
\int_{t_0}^{t} \frac{\mu_j(\tilde{s})}{\mu_j(\tilde{s})} \mu_j^{-2s}(\tilde{s}) F \left( \frac{(t-\tilde{s})^{\frac{1}{n}}}{\mu_j(\tilde{s})} \right) d\tilde{s} = \int_{t_0}^{t-\delta} \frac{\mu_j(\tilde{s})}{\mu_j(\tilde{s})} \mu_j^{-2s}(\tilde{s}) F \left( \frac{(t-\tilde{s})^{\frac{1}{n}}}{\mu_j(\tilde{s})} \right) d\tilde{s}
\]

\[
+ \int_{t-\delta}^{t} \frac{\mu_j(\tilde{s})}{\mu_j(\tilde{s})} \mu_j^{-2s}(\tilde{s}) F \left( \frac{(t-\tilde{s})^{\frac{1}{n}}}{\mu_j(\tilde{s})} \right) d\tilde{s}
\]

\[:= I_1 + I_2.\]

For the first integral we have \( t - \tilde{s} > \delta \), therefore

\[
0 \leq -I_1 \leq \int_{t_0}^{t-\delta} \frac{\mu_j(\tilde{s})}{\mu_j(\tilde{s})} \mu_j^{-2s}(\tilde{s}) F \left( \frac{(t-\tilde{s})^{\frac{1}{n}}}{\mu_j(\tilde{s})} \right) d\tilde{s} \leq C \int_{t_0}^{t-\delta} \frac{\mu_j^{-2s}(\tilde{s})}{\mu_j(\tilde{s})} \left( t-\tilde{s} \right)^{-\left(n-2s\right)} d\tilde{s}
\]

\[= Cb_j^{n-4s}c_{n,s}^{n-4s} \int_{t_0}^{t-\delta} \frac{1}{\tilde{s}} \frac{1}{(t-\tilde{s})^{\frac{n-2s}{2s}}} d\tilde{s} \leq \frac{Cb_j^{n-4s}c_{n,s}^{n-4s}}{t_0} \frac{1}{n-4s} \delta^{\frac{n-2s}{2s}}.
\]

Here we have used the ansatz \( \mu_{0j} = b_jc_{n,s}t^{-\frac{n}{2s}} \) and the fact that \(|a|^{n-2s}F(a) \leq C\).

For the second integral, we have

\[
I_2 = \int_{t-\delta}^{t} \frac{\mu_j(\tilde{s})}{\mu_j(\tilde{s})} \mu_j^{-2s}(\tilde{s}) F \left( \frac{(t-\tilde{s})^{\frac{1}{n}}}{\mu_j(\tilde{s})} \right) d\tilde{s}
\]

\[= \int_{0}^{\frac{\delta}{t-\tilde{s}}} \frac{\mu_j(\tilde{s})}{\mu_j(\tilde{s})} \mu_j^{-2s}(\tilde{s}) F \left( \frac{\mu_j(\tilde{s})}{\mu_j(\tilde{s})} \right) \frac{\mu_j(\tilde{s})}{\frac{1}{2n} (t-\tilde{s})^{-\frac{n-2s}{2s}} + 1} d\tilde{s}.
\]

Note that \( \frac{1}{2n} (t-\tilde{s})^{-\frac{n-2s}{2s}} + 1 = \frac{1}{2n} (t-\tilde{s})^{-\frac{n-2s}{2s}} \left( 1 - 2s (\frac{t-\tilde{s}}{\delta}) \right) > \frac{1}{2n} (t-\tilde{s})^{-\frac{n-2s}{2s}} \left( 1 - \frac{2s}{(n-4s)} \delta \right), \)

\[d\tilde{s} = \frac{\mu_j(\tilde{s})}{\frac{1}{2n} (t-\tilde{s})^{-\frac{n-2s}{2s}} + 1} (1 + O(\delta)) d\tilde{s} \text{ for } \delta \text{ small and}
\]

\[
I_2 = -\frac{2sb_j^{4s-n}}{(n-4s)c_{n,s}^{n-4s}} \left( \int_{0}^{\frac{\delta}{t-\tilde{s}}} \frac{1}{s^{2s-1}} F(\tilde{s}) d\tilde{s} + o(1) \right) = -\frac{2sb_j^{4s-n}}{(n-4s)c_{n,s}^{n-4s}} A + o(1)
\]

as long as \( \frac{\delta}{\mu_j(t-\tilde{s})} \) is large. Here \( A = \int_{0}^{\frac{\delta}{t}} \tilde{s}^{2s-1} F(\tilde{s}) d\tilde{s} < +\infty \) since \( n > 4s \). Thus we
have

\[
\Phi_j^0(q_j, t) = -(1 + o(1)) \int_{t_0}^t \mu_j(\hat{s}) \mu_j^{-(n-2s)}(\hat{s}) F \left( \frac{(t-\hat{s})^{\frac{s}{2}}}{\mu_j(\hat{s})} \right) d\hat{s}
\]

\[
= \frac{2s b_j^{4s-n}}{(n-4s)c_{n,s}^{-4s}} A + o(1) := B b_j^{4s-n} + o(1)
\] (4.20)

when \( t_0 \) is chosen sufficiently large. Here \( B = B_{n,s} := \frac{2s}{(n-4s)c_{n,s}^{-4s}} A \). This proves (4.19).

By direct computations, we have

\[
\mu_j^{-(n-2s-1)}(t) \int_{\mathbb{R}^n} E_{0j}(y, t) Z_{n+1}(y) dy
\]

\[
\approx c_1 \left[ b_j^{n-2s-1} H(q_j, q_j) - \sum_{i \neq j} b_i^{n-2s} G(q_j, q_i) \right] - \frac{2sc_1 A + c_2}{(n-4s)c_{n,s}^{-4s}} b_j^{2s-1}
\] (4.21)

with

\[
c_1 = -p \int_{\mathbb{R}^n} \frac{U(y)^{n-1} Z_{n+1}(y) dy}{2 \alpha_{n,s} (1 + |y|^{2s})^{\frac{n-2s}{2}}}
\]

\[
c_2 = \int_{\mathbb{R}^n} \left( Z_{n+1}(y) + \frac{n-2s}{2} \frac{1}{\alpha_{n,s} (1 + |y|^{2s})^{\frac{n-2s}{2}}} \right) Z_{n+1}(y) dy.
\]

Denote

\[
\mu_j(t) = b_j \mu_0(t) = b_j c_{n,s}^{- \frac{1}{n-2s}}.
\]

Then the solvability conditions (4.18) can be achieved at main order if

\[
b_j^{n-2s} H(q_j, q_j) - \sum_{i \neq j} (b_i b_j)^{\frac{n-2s}{2}} G(q_j, q_i) - \frac{2sc_1 A + c_2}{(n-4s)c_{n,s}^{-4s}} b_j^{2s} = 0 \text{ for all } j = 1, \ldots, k.
\] (4.22)

By imposing \(- \frac{2sc_1 A + c_2}{(n-4s)c_{n,s}^{-4s}} c_1 = - \frac{2s}{n-2s}, \) namely

\[
c_{n,s} = \left[ \frac{(2sc_1 A + c_2)(n-2s)}{2s(n-4s)c_1} \right]^{-\frac{1}{n}}
\]

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we have
\[ \dot{\mu}_0(t) = -\frac{2sc_1}{(2sc_1A + c_2)(n - 2s)} \mu_0^{n-4s+1}(t). \] (4.23)

From (4.22) and (4.23), the constants \( b_j \) satisfy the system
\[ b_j^{n-2s-1}H(q_j, q_j) - \sum_{i \neq j} b_j^{n-2s-1} b_i^{n-2s} G(q_j, q_i) = \frac{2s b_j^{2s-1}}{n - 2s} \text{ for all } j = 1, \ldots, k. \] (4.24)

System (4.24) is equivalent to the variational problem \( \nabla_\mu I(b) = 0 \) with
\[ I(b) := \frac{1}{n - 2s} \left[ \sum_{j=1}^k b_j^{n-2s} H(q_j, q_j) - \sum_{i \neq j} b_j^{n-2s} b_i^{n-2s} G(q_j, q_i) - \sum_{j=1}^k b_j^{2s} \right]. \] (4.25)

Let \( \Lambda_j = b_j^{\frac{n-2s}{2}} \). Then
\[ (n - 2s)I(b) = \tilde{I}(\Lambda) = \left[ \sum_{j=1}^k H(q_j, q_j)\Lambda_j^2 - \sum_{i \neq j} G(q_j, q_i)\Lambda_i\Lambda_j - \sum_{j=1}^k \Lambda_j^{4s-2s} \right]. \]

Standard argument shows that system (4.24) has a unique solution if and only if the following matrix
\[ \mathcal{G}(q) = \begin{bmatrix} H(q_1, q_1) & -G(q_1, q_2) & \cdots & -G(q_1, q_k) \\ -G(q_2, q_1) & H(q_2, q_2) & \cdots & -G(q_2, q_k) \\ \vdots & \vdots & \ddots & \vdots \\ -G(q_k, q_1) & -G(q_k, q_2) & \cdots & H(q_k, q_k) \end{bmatrix} \]

is positive definite.

Next, we consider the solvability conditions for (4.16),
\[ \int_{\mathbb{R}^n} \mu_j^{n-2s} S(u_{\mu, \xi}) (y, t) Z_i(y) dy = 0, \quad i = 1, \ldots, n. \]

It is clear that these conditions are fulfilled at main order by simply setting \( \xi_0 j = q_j \).

Now we fix the function \( \mu_0(t) \) and the positive constants \( b_j \) satisfying (4.24) and denote
\[ \bar{\mu}_0 = (\mu_0, \ldots, \mu_k) = (b_1 \mu_0, \ldots, b_k \mu_0). \]
Let $\Phi_j$ be the unique solution to (4.16) for $\mu = \bar{\mu}_0$. Then we have

$$-(\Delta_j^0 + \mu_0 E_{0j}) \Phi_j(y) = -\mu_0 E_{0j} [\bar{\mu}_0, \xi_j] \Phi_j(y, t) \to 0 \text{ as } |y| \to \infty.$$  

From the definition of $\mu_0$ and $b_j$, one has

$$\mu_0 E_{0j} = -\gamma_j \mu_0^{n-2s} q_0(y),$$

where the constant $\gamma_j$ is positive and $q_0(y) = \left( \frac{pU(y)}{p^{2s}} - 1 \right) c_1^{n-4s} c_1 + \frac{b_j^{2s}}{(n-4s)c_1^n c_1} \left( \frac{Z_{n+1}(y)}{n+1} + \frac{n-2s}{2} \alpha_{n,s} \frac{1}{(1+|y|^2)^{\frac{n+2}{2}}} \right), \quad (4.26)$$

with $\int_{\mathbb{R}^n} q_0(y) Z_{n+1} dy = 0$.

Let $p_0 = p_0(|y|)$ be the radial solution of $L_0(p_0) = q_0$. Then $p_0(|y|) = O(|y|^{-2s})$ as $|y| \to \infty$ since we have (4.26). Therefore, $\Phi_j(y, t) = \gamma_j \mu_0^{n-2s} p_0(y). \quad (4.27)$

Thus we can define the corrected approximation as

$$u_{\mu, \xi}^*(x, t) = u_{\mu, \xi}(x, t) + \tilde{\Phi}(x, t) \quad (4.28)$$

with

$$\tilde{\Phi}(x, t) = \sum_{j=1}^{k} \mu_j^{-\frac{n-2s}{2}} \Phi_j \left( \frac{x - \xi_j}{\mu_j}, t \right).$$

**Estimating the error $S(u_{\mu, \xi}^*)$.**

In the region $|x - q_i| > \delta$ for all $i = 1, \ldots, k$, $S(u_{\mu, \xi}^*)$ can be described as

$$S(u_{\mu, \xi}^*)(x, t) = \mu_0^{\frac{n-2s}{2} - 1} \sum_{j=1}^{k} \mu_j f_j + \mu_0^{\frac{n-2s}{2}} \sum_{j=1}^{k} \xi_j \cdot \bar{f}_j + \mu_0^{\frac{n-2s}{2}} f, \quad (4.29)$$

where the functions $f_j, \bar{f}_j, f$ are smooth bounded and depend on $(x, \mu_0^{-1} \mu, \xi)$. Now we consider the region near each of the points $q_j$. By direct computations,
\( S(u_{\mu, \xi}) \) is given by

\[
S(u_{\mu, \xi}) = S(u_{\mu, \xi}) - \sum_{j} c_{j} \xi_{j} + \sum_{j} \xi_{j} \frac{\mu_{j}^{n+2s}}{\mu_{0}^{n+2s}} \mu_{0} \frac{\mu_{j}^{n+2s}}{\mu_{0}^{n+2s}} \xi_{j} \}
\]

where \( y_{j} = x - q_{j} \).

From (4.15), for a given fixed \( j \) and \( |x - q_{j}| \leq \delta \), we have

\[
\mu_{j}^{n+2s} S(u_{\mu, \xi}) = \mu_{j}^{n+2s} S(u_{\mu, \xi}) - \mu_{0} E_{0j}[\mu_{0}, \mu_{0j}] + A_{j}(y).
\]

After direct computations,

\[
A_{j} = \mu_{0}^{n+4s} f(\mu_{0}^{-1}, \xi, \mu_{j}, y) + \frac{\mu_{0}^{2n-4s}}{1 + |y|^{2s}} g(\mu_{0}^{-1}, \xi, \mu_{j}, y), \quad y_{j} = \frac{x - q_{j}}{\mu_{j}},
\]

where \( f \) and \( g \) are smooth and bounded. Finally we set

\[
\mu(t) = \mu_{0} + \lambda(t) \quad \text{with} \quad \lambda(t) = (\lambda_{1}(t), \ldots, \lambda_{s}(t)).
\]

From (4.31), we have

\[
S(u_{\mu, \xi}) = \mu_{j}^{n+2s} \left\{ \mu_{0j} \left( E_{0j}[\mu, \mu_{j}] - E_{0j}[\mu_{0}, \mu_{0j}] \right) + \lambda_{j} E_{0j}[\mu, \mu_{j}] \right. \]

\[
\left. + \mu_{j} E_{1j}[\mu, \xi_{j}] + R_{j} + A_{j} \right\}.
\]

Recall Lemma 4.1.1 that

\[
E_{0j}[\mu, \mu_{j}] = p U(\gamma)^{p-1} \left[ -\mu_{j}^{n-2s-1} H(q_{j}, q_{i}) + \sum_{j \neq j} \mu_{j}^{n-2s-1} \mu_{i}^{n-2s} G(q_{j}, q_{i}) \right]
\]

\[
+ p U(\gamma)^{p-1} \mu_{j}^{n-2s-1} \Phi_{0}^{0}(q_{j}, t) + \mu_{j}^{2s-2} \mu_{j} \left( Z_{n+1}(y) + \frac{n-2s}{2} \alpha_{n,s} \frac{1}{1 + |y|^{2s}} \right).
\]

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Note that $\Phi_0^j$ depends on $\mu$, $\dot{\mu}$ and

\[
\mu_0^{n-2s-1} \Phi_0^j[\mu_0 + \lambda, b_j \mu_0 + \dot{\lambda}_j](q_j, t) - \mu_0^{n-2s-1} \Phi_0^j[\mu_0, b_j \mu_0](q_j, t) = - (b_j \mu_0)^{2s-2} 2sA \dot{\lambda}_j - \mu_0^{n-2s-2} (n-4s+1) b_j^{2s-2} B \dot{\lambda}_j
\]

which can be deduced by similar arguments as (4.20), we have

\[
E_{0j}[\mu_0 + \lambda, b_j \mu_0 + \dot{\lambda}_j] - E_{0j}[\mu_0, b_j \mu_0]
\]

\[
= (b_j \mu_0)^{2s-2} \dot{\lambda}_j \left( Z_{n+1}(y) + \frac{n-2s}{2} \alpha_{n,s} \frac{1}{(1 + |y|^2)^{\frac{n-2s}{2}}} \right)
\]

\[
+ (2s-2) (b_j \mu_0)^{2s-3} \lambda_j (b_j \mu_0 + \dot{\lambda}_j) \left( Z_{n+1}(y) + \frac{n-2s}{2} \alpha_{n,s} \frac{1}{(1 + |y|^2)^{\frac{n-2s}{2}}} \right)
\]

\[
- \mu_0^{n-2s-2} pU(y)^{p-1} \left[ \sum_{i=1}^k M_{ij} \lambda_i + \sum_{i,j=1}^k f_{ijl}(\mu_0^{-1} \lambda) \lambda_i \lambda_l \right]
\]

\[
+ \mu_0^{n-2s-2} pU(y)^{p-1} (n-2s-1) b_j^{2s-2} B \dot{\lambda}_j
\]

\[
- pU(y)^{p-1} (b_j \mu_0)^{2s-2} 2sA \dot{\lambda}_j - \mu_0^{n-2s-2} pU(y)^{p-1} (n-4s+1) b_j^{2s-2} B \dot{\lambda}_j,
\]

where $f_{ijl}$ are smooth functions and for $i = j$,

\[
M_{ij} = (n-2s-1) b_j^{n-2s-2} H(q_j, q_j) - \left( \frac{n-2s}{2} - 1 \right) \sum_{i \neq j} b_j^{\frac{n-2s}{2}} \frac{\alpha_{n,s}^{-1}}{\alpha_{n,s}^{-1}} G(q_j, q_i),
\]

for $i \neq j$,

\[
M_{ij} = - \frac{n-2s}{2} \sum_{i \neq j} b_j^{\frac{n-2s}{2}} b_i^{\frac{n-2s}{2}} G(q_j, q_i).
\]

$M = D^2 I_0(b)$ with

\[
I_0(b) := \frac{1}{n-2s} \left[ \sum_{j=1}^k b_j^{n-2s} H(q_j, q_j) - \sum_{i \neq j} b_j^{\frac{n-2s}{2}} b_i^{\frac{n-2s}{2}} G(q_j, q_i) \right].
\]

Since $D^2 I(b)$ is positive definite, we denote its positive eigenvalues corresponding to the eigenvectors $\tilde{w}_j$ by $\tilde{\sigma}_j$ for $j = 1, \ldots, k$. Thus

\[
D^2 I(b) = P^T \text{diag}(\tilde{\sigma}_1, \ldots, \tilde{\sigma}_k) P \quad (4.33)
\]

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and $P$ is the $k \times k$ matrix given by $P := (\tilde{w}_1, \ldots, \tilde{w}_k)$. From the definition of $b_j$ in (4.24), one has

$$M = P^T \text{diag}(\bar{\sigma}_1 + \frac{2s(2s-1)}{n-2s}b_1^{2s-2}, \ldots, \bar{\sigma}_k + \frac{2s(2s-1)}{n-2s}b_k^{2s-2})P.$$ 

Now we estimate $\lambda_j E_{0j} |\mu, \mu_j|$. Indeed, we have

$$\lambda_j E_{0j} |\mu, \mu_j| = (b_j \mu_0)^{2s-2} \lambda_j \lambda_j \left( \sum_{i=1}^k f_{ij}(\mu_0^{-1} \lambda_j) \lambda_j \xi_j \right)$$

where functions $f_{ij}$ are smooth in its arguments.

Collecting all the above estimates, we have the full expansion for $S(u^*_{\mu, \xi}).$

**Lemma 4.1.2.** We consider the region $|x - q_j| \leq \frac{1}{2} \min_{i \neq j} |q_i - q_j|$ for a fixed index $j$. Let $\mu = \bar{\mu}_0 + \bar{\sigma}$ and $|\lambda(t)| \leq \mu_0(t)^{1+\sigma}$ for some $0 < \sigma < \bar{\sigma}$ with $\bar{\sigma}$ be a constant satisfying $0 < \bar{\sigma} \leq \frac{n-2s}{2s} \bar{\sigma}_j b_j^{2s-2}$, $j = 1, \ldots, k$. Then for large $t$, $S(u^*_{\mu, \xi})$ can be expressed as

$$S(u^*_{\mu, \xi}) = \sum_{j=1}^k \mu_j^{\frac{n-2s}{2s}} \left\{ \mu_0(b_j \mu_0)^{2s-2} \lambda_j \left( Z_{n+1}(y_j) + \frac{n-2s}{2} \alpha_{n,s} \frac{1}{(1 + |y_j|^2)^{n/2}} - 2sA p U(y_j)^{p-1} \right) \right\}$$

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\[ -\mu_0 \mu_0^{n-2s-2} p U(y_j)^{p-1} \sum_{i=1}^{k} M_{ij} \lambda_i + \mu_j^{2s-1} \alpha_{n,s}(n-2s) \frac{\xi_j \cdot y_j}{(1 + |y_j|^2)^{\frac{n-2s}{2}+1}} \]

\[ + \mu_j p U(y_j)^{p-1} \left[ -\mu_j^{n-2s} \nabla H(q_j, q_j) + \sum_{i \neq j} \mu_j^{n-2s} \mu_i^{n-2s} \nabla G(q_j, q_i) \right] \cdot y_j \]

\[ + \sum_{j=1}^{k} \frac{\mu_j}{\mu_0} \lambda_j b_j^{2s-1} \left[ (2s-1) \mu_0^{2s-2} \mu_0 \left( Z_{n+1}(y_j) + \frac{n-2s}{2} \alpha_{n,s} \frac{1}{(1 + |y_j|^2)^{\frac{n-2s}{2}}} \right) \right. \]

\[ + \left. p U(y_j)^{p-1} \mu_0^{n-2s-1} \left( -b_j^{n-4s} H(q_j, q_j) + \sum_{i \neq j} b_j^{n-4s} b_i^{n-4s} G(q_j, q_i) \right) \right] \]

\[ + p U(y_j)^{p-1} \mu_0^{n-2s-1}(2s-1)B \]

\[ + \mu_0^{-\frac{n+2s}{2}} \left[ \sum_{j=1}^{k} \frac{\mu_0^{n-2s+2} g_j}{1 + |y_j|^{4s-2}} + \sum_{j=1}^{k} \frac{\mu_0^{n-4s} g_j}{1 + |y_j|^{2s}} + \sum_{j=1}^{k} \frac{\mu_0^{n-2s} g_j}{1 + |y_j|^{4s}} \lambda_j \right] \]

\[ + \mu_0^{-\frac{n+2s}{2}} \left[ \sum_{j=1}^{k} \frac{\mu_0^{n-2s} \tilde{g}_j}{1 + |y_j|^{4s}} \cdot (\xi_j - q_j) \right] \]

\[ + \mu_0^{-\frac{n+2s}{2}} \left[ \mu_0^{n-2s-2} \sum_{i,j,l=1}^{k} p U(y_j)^{p-1} f_{ijl} \lambda_i \lambda_l + \sum_{i,j,l=1}^{k} \frac{f_{ijl}}{1 + |y_j|^{n-2s}} \lambda_i \lambda_l \right] \]

\[ + \mu_0^{-\frac{n+2s}{2}} \left[ \mu_0^{n-2s} f + \mu_0^{n-1} \sum_{i=1}^{k} \tilde{f}_i + \mu_0^{n} \sum_{i=1}^{k} \tilde{\xi}_i \right], \]

where \( x = \xi_j + \mu_j y_j, \tilde{f}_i, f, f_{ijl} \) are smooth and bounded functions depending on \( (\mu_0^{-1} \mu, \xi, x) \) and \( g_j, \tilde{g}_j \) depend on \( (\mu_0^{-1} \mu, \xi, y_j) \).

### 4.1.3 The inner-outer gluing procedure

We are looking for a solution to (4.6) with the form

\[ u = u^*_\mu, \xi + \tilde{\phi} \]

when \( t_0 \) is sufficiently large. The function \( \tilde{\phi}(x, t) \) is a smaller term and we will find it by means of the inner-outer gluing procedure.
Let us write
\[ \tilde{\phi}(x,t) = \psi(x,t) + \phi^i(x,t) \]
where \( \phi^i(x,t) := \sum_{j=1}^{k} \eta_j R(x,t) \tilde{\phi}_j(x,t) \)
with
\[ \tilde{\phi}_j(x,t) := \mu_{0j}^{\frac{n-2s}{2}} \frac{x-\xi_j}{\mu_{0j}}, \quad \mu_{0j}(t) = b_j \mu_0(t), \]
\[ \eta_j(x,t) = \eta \left( \frac{|x-\xi_j|}{R \mu_{0j}} \right). \]

Here \( \eta(\tau) \) is a smooth cut-off function defined on \([0, \infty)\) with \( \eta(\tau) = 1 \) for \( 0 \leq \tau < 1 \) and \( \eta(\tau) = 0 \) for \( \tau > 2 \). The number \( R \) is defined as
\[ R = \rho_0^p \quad \text{with} \quad 0 < \rho \ll 1. \]  

Problem (4.6) can be written with respect to \( \tilde{\phi} \) as
\[
\begin{cases}
\partial_t \tilde{\phi} = -(-\Delta)^{s} \tilde{\phi} + p(u_{\mu,\xi}^*)^{p-1} \tilde{\phi} + \tilde{N}(\tilde{\phi}) + S(u_{\mu,\xi}^*) \quad \text{in} \Omega \times (t_0, \infty), \\
\tilde{\phi} = -u_{\mu,\xi}^* \quad \text{in} (\mathbb{R}^n \setminus \Omega) \times (t_0, \infty),
\end{cases}
\]
where
\[ \tilde{N}(\tilde{\phi}) = (u_{\mu,\xi}^* + \tilde{\phi})^p - p(u_{\mu,\xi}^*)^{p-1} \tilde{\phi} - (u_{\mu,\xi}^*)^p, \]
\[ S(u_{\mu,\xi}^*) = -\partial_t u_{\mu,\xi}^* - (-\Delta)^{s} u_{\mu,\xi}^* + (u_{\mu,\xi}^*)^p. \]

According to Lemma 4.1.2 we let \( y_j = \frac{x-\xi_j}{\mu_j} \) and denote
\[ S(u_{\mu,\xi}^*) = \sum_{j=1}^{k} S_{\mu,\xi,j} + S^{(2)}_{\mu,\xi}, \]
where
\[ S_{\mu,\xi,j} = \mu_j^{\frac{n-2s}{2}} \mu_{0j}(b_j \mu_0)^{2s-2} \lambda_j \]
\[ \times \left( Z_{n+1}(y_j) + \frac{n-2s}{2} \alpha_{0,s} \frac{1}{(1 + |y_j|^2)^{\frac{n-2s}{2}}} - 2sA_{\nu U}(y_j)^{p-1} \right) \]
\[ -\mu_0 j_0^{n-2s-2} pU(y_j) p^{-1} \sum_{i=1}^k M_{ij} \lambda_i \]

\[ + \lambda_j b_j^{2s-1} \left[ (2s-1)\mu_0^{2s-2} \mu_0 \left( Z_{n+1}(y_j) + \frac{n-2s}{2} \alpha_{n,s} \frac{1}{(1 + |y_j|^2)^{\frac{n+2s}{2}}} \right) \right] \]

\[ + pU(y_j) p^{-1} \mu_0^{n-2s-1} \left( -b_j^{n-4s} H(q_j, q_j) + \sum_{i \neq j} b_j^{n-2s} b_i^{n-2s} G(q_j, q_i) \right) \]

\[ + pU(y_j) p^{-1} \mu_0^{n-2s-1} (2s-1)B \]

\[ + \mu_j \left[ \mu_j^{2s-2} \alpha_{n,s} (n-2s) \frac{\dot{\xi}_j \cdot y_j}{(1 + |y_j|^2)^{\frac{n+2s}{2} + 1}} \right] \]

\[ + pU(y_j) p^{-1} \left( -\mu_j^{n-2s} \nabla H(q_j, q_j) + \sum_{i \neq j} \mu_j^{n-2s} \mu_i^{n-2s} \nabla G(q_j, q_i) \cdot y_j \right) \}.

Set

\[ V_{\mu, \xi} = p \sum_{j=1}^k \left( (u^{\mu, \xi}_{\mu, \xi})^p - \left[ \mu_j^{\frac{n-2s}{2}} U \left( \frac{x - \xi_j}{\mu_j} \right) \right]^{p^{-1}} \right) \eta_{j,R}, \] (4.36)

\[ + p(1 - \sum_{j=1}^k \eta_{j,R})(u^{\mu, \xi}_{\mu, \xi})^p. \]

Then \( \tilde{\phi} \) solves problem (4.35) if

(1) \( \psi \) solves the outer problem

\[ \partial_t \psi = -(-\Delta)^s \psi + V_{\mu, \xi} \psi \]

\[ + \sum_{j=1}^k \left\{ \left[ (-\Delta)^s \eta_{j,R}, (-\Delta)^s \tilde{\phi}_j \right] + \tilde{\phi}_j (-(-\Delta)^s - \partial_t) \eta_{j,R} \right\} \] (4.37)

\[ + \tilde{N}_{\mu, \xi}(\tilde{\phi}) + S_{out} \quad \text{in } \Omega \times (t_0, \infty), \]

\[ \psi = -u^{\mu, \xi}_{\mu, \xi} \quad \text{in } \left( \mathbb{R}^n \setminus \Omega \right) \times (t_0, \infty), \]

where

\[ S_{out} = S^{(2)}_{\mu, \xi} + \sum_{j=1}^k (1 - \eta_{j,R}) S_{\mu, \xi, j} \] (4.38)
and

$$[-(-\Delta)^{\frac{s}{2}} \eta_{j,R}, -(-\Delta)^{\frac{s}{2}} \tilde{\phi}_j] := C_n s \int_{\mathbb{R}^n} \frac{[\eta_{j,R}(y) - \eta_{j,R}(x)][\tilde{\phi}_j(x) - \tilde{\phi}_j(y)]}{|y|^{n+2s}} dy.$$  (4.39)

(2) \(\tilde{\phi}_j\) solves

$$\eta_{j,R} \partial_t \tilde{\phi}_j = \eta_{j,R} \left[ -((-\Delta)^s \tilde{\phi}_j + pU_j^{p-1} \phi_j + pU_j^{p-1} \psi + S_{\mu, \xi, j} \right] \text{ in } \mathbb{R}^n \times (t_0, \infty),$$  (4.40)

for all \(j = 1, \ldots, k\), \(U_j := \mu_j^{-\frac{n-2s}{n}} U \left( \frac{x - \xi_j}{\mu_j} \right)\). In terms of \(\phi_j(y,t)\), (4.40) becomes the inner problem

$$\left\{ \begin{array}{l}
\mu_0^{2s} \partial_t \phi_j = -(-\Delta)^{s} \phi_j + pU_j^{p-1}(y) \phi_j \\
+ \left\{ \frac{n+2s}{2} \mu_0 \nabla \cdot (\nabla \phi_j + \mu_0 \nabla \phi_j) + p \mu_0 \frac{n+2s}{2} \mu_j U_j^{p-1}(y) \psi(\tilde{\xi}_j + \mu_0 y, t) \\
+ B_j[\phi_j] + B_0^j[\phi_j] \right\} \chi_{B_2R(0)}(y) \text{ in } \mathbb{R}^n \times (t_0, \infty),
\end{array} \right.$$  (4.41)

for \(j = 1, \ldots, k\), where

$$B_j[\phi_j] := \mu_0^{2s-1} \mu_0 \left( \frac{n-2s}{2} \phi_j + y \cdot \nabla \phi_j \right) + \mu_0^{2s-1} \nabla \phi_j \cdot \tilde{\xi}_j$$  (4.42)

and

$$B_0^j[\phi_j] := p \left[ U_j^{p-1} \left( \frac{\mu_0}{\mu_j} y \right) - U_j^{p-1}(y) \right] \phi_j + p \left[ \mu_0^{2s}(u_{\mu, \xi}^{p-1} - U_j^{p-1} \left( \frac{\mu_0}{\mu_j} y \right) \right] \phi_j.$$  (4.43)

Here \(\chi_{B_2R(0)}(y)\) is the characteristic function of \(B_2R(0)\).

The rest of this section is organized as follows. In Section 4.1.4, we solve the outer problem (4.37). In Section 4.1.5, a linear theory for the inner problem (4.41) is developed. We study the solvability conditions for (4.41) in Section 4.1.5 and the full problem is finally solved in Section 4.1.6.
4.1.4 The outer problem

In this section, we shall solve the outer problem for a given smooth function $\phi$ which is sufficiently small. We consider the initial boundary value problem

\[
\begin{aligned}
\partial_t \psi &= -(-\Delta)^s \psi + V_{\mu, \xi} \psi \\
&\quad + \sum_{j=1}^{k} \left\{ (-(-\Delta)^s \eta_{j,R}, -(\Delta)^s \bar{\phi}_j + \bar{\phi}_j (-(-\Delta)^s - \partial_t) \eta_{j,R} \right\} \\
&\quad + \tilde{N}_{\mu, \xi}(\bar{\phi}) + S_{\text{out}} \quad \text{in } \Omega \times (t_0, \infty), \\
\psi &= -u_{\mu, \xi} \quad \text{in } (\mathbb{R}^n \setminus \Omega) \times (t_0, \infty), \\
\psi(\cdot, t_0) &= \psi_0 \quad \text{in } \mathbb{R}^n,
\end{aligned}
\]

(4.44)

with a smooth and sufficiently small initial condition $\psi_0$.

The model problem

To solve problem (4.44), we first consider the linear problem

\[
\begin{aligned}
\partial_t \psi &= -(-\Delta)^s \psi + V_{\mu, \xi} \psi + f(x, t) \quad \text{in } \Omega \times (t_0, \infty), \\
\psi &= g \quad \text{in } (\mathbb{R}^n \setminus \Omega) \times (t_0, \infty), \\
\psi(\cdot, t_0) &= h \quad \text{in } \mathbb{R}^n,
\end{aligned}
\]

(4.45)

where $f(x, t)$, $g(x, t)$ and $h(x)$ are given smooth functions and $V_{\mu, \xi}$ is defined in (4.36). Furthermore, we assume $f$ satisfies

\[
|f(x, t)| \leq M \sum_{j=1}^{k} \frac{\mu_j^{-2s} t^{-\beta}}{1 + |y_j|^{2s + \alpha}}, \quad y_j = \frac{|x - \xi_j|}{\mu_j}
\]

(4.46)

for $\alpha, \beta > 0$. Denote the least $M > 0$ in (4.46) by $\|f\|_{s, \beta, 2s + \alpha}$.

In the rest of this section, we use the symbol $a \lesssim b$ to denote $a \leq Cb$ for a positive constant $C$ independent of $t$ and $t_0$. Then we have the following a priori estimate for the model problem (4.45).

**Lemma 4.1.3.** Suppose that $\|f\|_{s, \beta, 2s + \alpha} < +\infty$ for some $\alpha, \beta > 0, 0 < \alpha \ll 1, \|h\|_{L^\infty(\mathbb{R}^n)} < +\infty$ and $\|\tau^\beta g(x, \tau)\|_{L^\infty((\mathbb{R}^n \setminus \Omega) \times (t_0, \infty))} < +\infty$. Let $\phi = \psi[f, g, h]$ be the
solution of problem (4.45). Then there exists \( \delta = \delta(\Omega) > 0 \) small such that for all \((x,t)\) we have

\[
|\psi(x,t)| \lesssim \|f\|_{s,\beta,2s+\alpha} \left( \sum_{j=1}^{k} \frac{t^{-\beta}}{1 + |y_j|^{\alpha}} \right) 
+ e^{-\delta(t-t_0)} \|h\|_{L^\infty(\mathbb{R}^n)} + t^{-\beta} \|\tau^\beta g(x,\tau)\|_{L^\infty((\mathbb{R}^n \setminus \Omega) \times (t_0,\infty))},
\]

for all \((x,t)\) and \(y_j = \frac{x-\xi_j}{\mu_j}\). Moreover, the following Hölder estimate

\[
[\psi(\cdot,t)]_{\eta,B_{\mu_j}(\xi_j)} \lesssim \|f\|_{s,\beta,2s+\alpha} \left( \sum_{j=1}^{k} \frac{\mu_j^{-\eta} t^{-\beta}}{1 + |y_j|^{\alpha+\eta}} \right)
\]  

holds for some \(\eta \in (0,1)\) and \(|y_j| \leq 2R\). Here

\[
[\psi(\cdot,t)]_{\eta,B_{\mu_j}(\xi_j)} := \sup_{x,y \in B_{\mu_j}(\xi_j)} \frac{|\psi(x,t) - \psi(y,t)|}{|x-y|^{\eta}}
\]

is the Hölder seminorm.

**Proof.** Let \(\psi_0[g,h]\) be the solution of the fractional heat equation

\[
\begin{cases}
\partial_t \psi_0 = -(-\Delta)^s \psi_0 & \text{in } \Omega \times (t_0,\infty), \\
\psi_0 = g & \text{in } (\mathbb{R}^n \setminus \Omega) \times (t_0,\infty), \\
\psi_0(\cdot,t_0) = h & \text{in } \mathbb{R}^n.
\end{cases}
\]

Let \(v(x)\) be the bounded solution of \(-(-\Delta)^s v + 1 = 0\) in \(\Omega\) with \(v = 1\) on \(\mathbb{R}^n \setminus \Omega\). Then \(v \geq 1\) in \(\Omega\) and by direct computations, the function

\[
\psi(x,t) = \left( e^{-\delta(t-t_0)} \|h\|_{L^\infty(\mathbb{R}^n)} + t^{-\beta} \|\tau^\beta g(x,\tau)\|_{L^\infty((\mathbb{R}^n \setminus \Omega) \times (t_0,\infty))} \right) v(x)
\]

is a supersolution to (4.49) if \(\delta = \delta(\Omega) > 0\) is sufficiently small. Then \(|\psi_0| \leq \bar{\psi}\) by the maximum principle (see, for example, [17, 18]). Thus, it suffices to prove the estimates (4.47) and (4.48) for the case \(g = 0, h = 0\).

Let \(p(|z|)\) be the radial positive solution of the equation

\[-(-\Delta)^s p + 4q = 0\] in \(\mathbb{R}^n\)
with \( q(|z|) = \frac{1}{1+|z|^{2s}} \). Then by Riesz kernel, we get \( p(z) \sim \frac{1}{1 + |z|^{2s}} \). For a given sufficiently small \( \delta > 0 \), we have

\[-(-\Delta)^s p + \frac{\delta}{1 + |z|^{2s}} p + 2q \leq 0 \text{ in } \mathbb{R}^n.\]

Thus \( \tilde{p}(x) := \sum_{j=1}^k p \left( \frac{x - \xi_j}{\mu_j} \right) \) satisfies

\[-(-\Delta)^s \tilde{p} + \left( \sum_{j=1}^k \mu_j^{-2s} \frac{\delta}{1 + |x - \xi_j|^{2s}} \right) \tilde{p} + \frac{3}{2} \bar{q} \leq 0 \text{ in } \mathbb{R}^n\]

with \( \bar{q} := \sum_{j=1}^k \mu_j^{-2s} q \left( \frac{x - \xi_j}{\mu_j} \right) \). From the definition of \( V_{\mu, \xi} \), we have

\[ |V_{\mu, \xi}| \leq \sum_{j=1}^k \mu_j^{-2s} \frac{R^{-2s}}{1 + |y_j|^{2s}}. \quad (4.50)\]

For a given number \( \beta > 0 \), it is easy to see that \( \tilde{\psi}(x,t) = 2t^{-\beta} \tilde{p} \) is a positive supersolution to

\[ \partial_t \tilde{\psi} = -(-\Delta)^s \tilde{\psi} + V_{\mu, \xi} \tilde{\psi} + t^{-\beta} \bar{q}, \]

i.e.,

\[ \partial_t \psi \geq -(-\Delta)^s \psi + V_{\mu, \xi} \psi + t^{-\beta} \bar{q} \]

for \( t > t_0 \) and \( t_0 \) is sufficiently large. Therefore, one has

\[ |\psi(x,t)| \leq t^{-\beta} \|f\|_{\ast, \beta, 2s + \alpha} \sum_{j=1}^k \frac{1}{1 + |y_j|^{\alpha}}, \quad (4.51)\]

and (4.47) is proved. To prove (4.48), we let

\[ \psi(x,t) := \tilde{\psi} \left( \frac{x - \xi_j}{\mu_j}, \tau(t) \right), \]

where \( \tau(t) = \mu_j^{-2s}(t) \), namely \( \tau(t) \sim t^{\frac{n - 2s}{n - 4s}} \). Without loss of generality, we assume
Consider \( \tau(t_0) \geq 2 \) by fixing \( t_0 \). Then \( \tilde{\psi} \) satisfies
\[
\begin{align*}
\partial_t \psi &= -(-\Delta)\psi + a(z,t) \cdot \nabla \psi + b(z,t) \psi + \tilde{f}(z, \tau) \quad (4.52)
\end{align*}
\]
for \(|z| \leq \delta \mu_0^{-1}\) and \( \tilde{f}(z, \tau) = \mu^2 f(\xi_j + \mu_j z, t(\tau)) \). The uniformly small coefficients \( a(z,t) \) and \( b(z,t) \) in (4.52) are given by
\[
\begin{align*}
a(z,t) &:= \mu_j^{2s-1} \mu_j z + \mu_j^{2s-1} \xi_j, \\
b(z,t) &:= \mu_j^{2s} \mu_j (\xi_j + \mu_j z).
\end{align*}
\]
Then from assumption (4.46) and (4.51) we have
\[
\begin{align*}
|\tilde{f}(z, \tau)| &\lesssim t(\tau)^{-\beta} \frac{\|f\|_{*,\beta,2s+\alpha}}{1 + |z|^{2s+\alpha}}, \\
|\tilde{\psi}(z, \tau)| &\lesssim t(\tau)^{-\beta} \frac{\|f\|_{*,\beta,2s+\alpha}}{1 + |z|^\alpha}.
\end{align*}
\]
Now fix \( 0 < \eta < 1 \), from the regularity estimates for parabolic integro-differential equations (see [172]), for \( \tau_1 \geq \tau(t_0) + 2 \), we have
\[
\tilde{\psi}(|t, \tau_1|) \leq t(\tau_1)^{-\beta} \frac{\|f\|_{*,\beta,2s+\alpha}}{1 + |z|^\alpha}.
\]
Therefore, choosing an appropriate constant \( c_n \) such that for any \( t \geq c_n t_0 \) we have
\[
(R\mu_j)^\eta [\tilde{\psi}(|t, \tau_1|)]_{\eta,B_{t_0}(\xi_j)} \lesssim t^{-\beta} \|f\|_{*,\beta,2s+\alpha}. \quad (4.53)
\]
By the same token, the estimate (4.53) also holds for \( t_0 \leq t \leq c_n t_0 \). Thus, (4.48) holds for any \( t \geq t_0 \). The proof is completed.

**Solvability of the outer problem.**

Now we fix \( \sigma \) satisfying
\[
0 < \sigma < \bar{\sigma} \text{ where } \bar{\sigma} \leq \frac{n - 2s}{2s} \bar{\sigma}_j b_j^{2 - 2s}, \quad j = 1, \ldots, k, \quad (4.54)
\]
and $\sigma_j$ and $b_j$ are defined in (4.33) and (4.24) respectively. Given $h(t) : (t_0, \infty) \to \mathbb{R}^k$ and $\delta > 0$, we define the weighted $L^\infty$ norm as

$$||h||_\delta := ||\mu_0(t)^{-\delta}h(t)||_{L^\infty([t_0, \infty))}.$$ 

In the rest of this section, we always assume that $a$ is a positive constant satisfying $a > 2s$ and $a - 2s$ is sufficiently small. We also assume the parameters $\lambda$, $\xi$, $\dot{\lambda}$, $\dot{\xi}$ satisfy the following two constraints,

$$\|\dot{\lambda}(t)\|_{n-4s+1+\sigma} + \|\dot{\xi}(t)\|_{n-4s+1+\sigma} \leq \frac{c}{R^{a-2s}}, \quad (4.55)$$

$$\|\lambda(t)\|_{1+\sigma} + \|\xi(t) - q\|_{1+\sigma} \leq \frac{c}{R^{a-2s}}, \quad (4.56)$$

where $c$ is a positive constant independent of $t$, $t_0$ and $R$.

Denote

$$\|\phi\|_{n-2s+\sigma,a} = \max_{j=1,...,k}\|\phi_j\|_{n-2s+\sigma,a},$$

where $\|\phi_j\|_{n-2s+\sigma,a}$ is defined as the least number $M$ such that

$$(1 + |y|)|\nabla \phi_j(y,t)|_{L^\infty(B_{2R}(0))} + |\phi_j(y,t)| \leq M \frac{t^{n-2s+\sigma}}{1 + |y|^{\sigma}}, \quad j = 1, \ldots, k \quad (4.57)$$

holds. Suppose $\phi = (\phi_1, \ldots, \phi_k)$ satisfies

$$\|\phi\|_{n-2s+\sigma,a} \leq ct_0^{-\epsilon} \quad (4.58)$$

for some small $\epsilon > 0$. Then we have the following proposition.

**Proposition 4.1.1.** Assume $\lambda$, $\xi$, $\dot{\lambda}$, $\dot{\xi}$ satisfy (4.55) and (4.56), $\phi = (\phi_1, \ldots, \phi_k)$ satisfies (4.58), $\psi_0 \in C^2(\mathbb{R}^n)$ and we have

$$\|\psi_0\|_{L^\infty(\mathbb{R}^n)} + \|\nabla \psi_0\|_{L^\infty(\mathbb{R}^n)} \leq \frac{t_0^{-\epsilon}}{R^{a-2s}}.$$ 

Then there exists $t_0$ sufficiently large such that the outer problem (4.44) has a unique solution $\psi = \Psi[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi]$. Moreover, there exists $\sigma$ satisfying (4.54)
and \( \varepsilon > 0 \) small such that, for \( y_j = \frac{x - \xi_j}{\mu_0} \),

\[
|\psi(x,t)| \lesssim \frac{t_0^{-\varepsilon}}{R^{a-2s}} \sum_{j=1}^{k} \frac{\mu_0^{\frac{n-2s}{2} + \sigma}(t)}{1 + |y_j|^{a-2s}} + e^{-\delta(t-t_0)} \|\psi_0\|_{L^-(\mathbb{R}^n)},
\]

(4.59)

\[
|\psi(x,t)|_{\eta, \mathcal{B}_{\mu_0}(\xi)} \lesssim \frac{t_0^{-\varepsilon}}{R^{a-2s}} \sum_{j=1}^{k} \frac{\mu_j^{-\eta} \mu_0^{\frac{n-2s}{2} + \sigma}(t)}{1 + |y_j|^{a-2s+\eta}} \text{ for } |y_j| \leq 2R,
\]

(4.60)

where \( R, \rho \) are defined in (4.34) and \( \eta \in (0, 1) \).

Proposition 4.1.1 states that for any small initial conditions \( \psi_0 \), a solution \( \psi \) to (4.44) exists. Moreover, it clarifies the dependence of \( \Psi[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi] \) in the parameters \( \lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi \), which is proved by estimating, for example,

\[
\partial_{\phi} \Psi[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi] |_{s=0} = \partial_s \Psi[\lambda, \xi, \dot{\lambda}, \dot{\xi} + s\phi] = 0
\]

as a linear operator between parameter Banach spaces. For simplicity, we denote the above operator by \( \partial_{\phi} \Psi[\phi] \). Similarly, we have \( \partial_{\lambda} \Psi[\lambda], \partial_{\xi} \Psi[\xi], \partial_{\dot{\lambda}} \Psi[\dot{\lambda}], \partial_{\dot{\xi}} \Psi[\dot{\xi}] \).

**Proof.** Lemma 4.1.3 defines a linear operator \( T \) which associates the solution \( \psi = T(f, g, h) \) of problem (4.45) to any given functions \( f(x,t), g(x,t) \) and \( h(x) \). Denote \( \psi_1(x,t) := T(0, -u_{\mu, \xi}(x,t), \psi_0) \). From (4.28), (4.9) and (4.27), for any \( x \in \mathbb{R}^n \setminus \Omega \) we have

\[
|u_{\mu, \xi}(x,t)| \lesssim \mu_0^{\frac{n+2s}{2}}(t).
\]

(4.61)

By Lemma 4.1.3, we have

\[
|\psi_1| \lesssim e^{-\delta(t-t_0)} \|\psi_0\|_{L^-(\mathbb{R}^n)} + t^{-\beta} \mu_0(t_0)^{2s-\sigma} \text{ where } \beta = \frac{n-2s}{2(n-4s)} + \frac{\sigma}{n-4s}.
\]

Therefore, the function \( \psi + \psi_1 \) is a solution to (4.44) if \( \psi \) is a fixed point for the operator

\[
\mathcal{A}(\psi) := T(f(\psi), 0, 0),
\]

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where
\[ f(\psi) = \sum_{j=1}^{k} \left\{ \left[ -(-\Delta)^{\frac{1}{2}} \eta_{j,R}, -(-\Delta)^{\frac{1}{2}} \tilde{\phi}_j \right] + \tilde{\phi}_j \left( -(-\Delta)^{s} - \partial_t \right) \eta_{j,R} \right\} \]
\[ + \tilde{N}_{\mu, \xi}(\tilde{\phi}) + S_{out}. \tag{4.62} \]

By the Contraction Mapping Theorem, we will prove the existence of a fixed point \( \psi \) for \( \mathcal{A} \) in the following function space
\[ \| \psi \|_{**, \beta, a} \]
is bounded with
\[ \beta = \frac{n - 2s}{2(n - 4s)} + \frac{\sigma}{n - 4s}. \]

Here \( \| \psi \|_{**, \beta, a} \) is the least \( M > 0 \) such that the following inequality holds
\[ |\psi(x,t)| \leq M \sum_{j=1}^{k} \frac{t^{-\beta}}{1 + |y_j|^{a-2s}}, \quad y_j = \frac{|x - \xi_j|}{\mu_j}. \]

As a first step, we establish the following estimates.

1. Estimate for \( S_{out}(x,t) \):
\[ |S_{out}(x,t)| \lesssim \frac{t^{-\epsilon}}{R^{a-2s}} \sum_{j=1}^{k} \frac{\mu_j^{-2s} \mu_0^{-\frac{n-2s}{2}} + \sigma(t)}{1 + |y_j|^a}. \tag{4.63} \]

2. Estimate for \( \sum_{j=1}^{k} \left\{ \left[ -(-\Delta)^{\frac{1}{2}} \eta_{j,R}, -(-\Delta)^{\frac{1}{2}} \tilde{\phi}_j \right] + \tilde{\phi}_j \left( -(-\Delta)^{s} - \partial_t \right) \eta_{j,R} \right\} \):
\[ \left| \sum_{j=1}^{k} \left\{ \left[ -(-\Delta)^{\frac{1}{2}} \eta_{j,R}, -(-\Delta)^{\frac{1}{2}} \tilde{\phi}_j \right] + \tilde{\phi}_j \left( -(-\Delta)^{s} - \partial_t \right) \eta_{j,R} \right\} \right| \lesssim \frac{1}{R^{a-2s}} \| \phi \|_{a-2s + \sigma, a} \sum_{j=1}^{k} \frac{\mu_j^{-2s} \mu_0^{-\frac{n-2s}{2}} + \sigma(t)}{1 + |y_j|^a}. \tag{4.64} \]
(3) Estimate for $\tilde{N}_{\mu,\xi}(\tilde{\phi})$:

$$
\tilde{N}_{\mu,\xi}(\tilde{\phi}) \lesssim \begin{cases} 
  t_0^{-e/2}(\|\phi\|^2_{a-2s+\sigma,a} + \|\psi\|^2_{s,\beta,a}) \frac{1}{R^{a-2s}} \sum_{j=1}^{k} \frac{\mu_j^{-2s} \mu_0^{-2s} + \sigma}{1 + |y_j|^a}, & \text{when } 6s \geq n, \\
  t_0^{-e/2}(\|\phi\|_p^{p}_{a-2s+\sigma,a} + \|\psi\|_p^{p}_{s,\beta,a}) \frac{1}{R^{a-2s}} \sum_{j=1}^{k} \frac{\mu_j^{-2s} \mu_0^{-2s} + \sigma}{1 + |y_j|^a}, & \text{when } 6s < n.
\end{cases}
$$

(4.65)

**Proof of (4.63).** Recall from (4.38) that

$$
S_{out} = S^{(2)}_{\mu,\xi} + \sum_{j=1}^{k} (1 - \eta_{j,R}) S_{\mu,\xi,j}.
$$

By (4.29) and Lemma 4.1.2 in the region $|x - q_j| > \delta$ with $\delta > 0$ small, $S_{out}$ can be estimated for all $j$ as

$$
|S_{out}(x,t)| \lesssim \mu_0^{-2s} (\mu_0^{-2s} + \mu_0^{-4s}) \lesssim \mu_0^{\min(n-4s,2s)-(a-2s)-\sigma} (t_0) \sum_{j=1}^{k} \frac{\mu_j^{-2s} \mu_0^{-2s} + \sigma}{1 + |y_j|^a}.
$$

(4.66)

Now we consider the region $|x - q_j| \leq \delta$ with $\delta > 0$ small, where $j \in \{1, \ldots, k\}$ is fixed. Lemma 4.1.2 implies that

$$
|S^{(2)}_{\mu,\xi}(x,t)| \lesssim \mu_0^{-\frac{a+2s}{a-2s+2}} \frac{\mu_0^{-2s}}{1 + |y_j|^{4s-2}} \lesssim \mu_0^{2s-(a-2s)-\sigma} (t_0) \sum_{j=1}^{k} \frac{\mu_j^{-2s} \mu_0^{-2s} + \sigma}{1 + |y_j|^a}.
$$

(4.67)

From the definition of $\eta_{j,R}$, $(1 - \eta_{j,R}) \neq 0$ if $|x - \xi_j| > \mu_0 R$. Therefore, in the region $|x - q_j| < \delta$,

$$
|(1 - \eta_{j,R}) S_{\mu,\xi,j}| \lesssim \left( \frac{1}{R^{a-2s-a}} + \frac{1}{R^{4s-a}} \right) \frac{1}{R^{a-2s}} \sum_{j=1}^{k} \frac{\mu_j^{-2s} \mu_0^{-2s} + \sigma}{1 + |y_j|^a}.
$$

(4.68)

Here we have used the decaying assumptions (4.55) and (4.56) for $\lambda$ and $\xi$, respectively. Thus, (4.63) is valid.

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From (4.69)-(4.72), we get (4.64). Recall that \( \tilde{\phi}_j(x, t) := \mu_{0j}^{-\frac{n-2s}{2}} \phi_j \left( \frac{x - \xi_j}{\mu_0}, t \right) \). From the assumptions (4.57) and (4.58), we obtain

\[
\left| \left[ -\left( -\Delta \right)^{\frac{s}{2}} \eta_{j,R}, -\left( -\Delta \right)^{\frac{s}{2}} \tilde{\phi}_j \right] \right| (x, t) \\
\lesssim \left[ \int_{\mathbb{R}^n} \left( \frac{\eta_{j,R}(x) - \eta_{j,R}(y)}{|x - y|^{\frac{n}{2} + s}} \right)^2 dy \right]^{\frac{1}{2}} \left[ \int_{\mathbb{R}^n} \left( \tilde{\phi}_j(x) - \tilde{\phi}_j(y) \right)^2 dy \right]^{\frac{1}{2}} \\
\lesssim \frac{1}{R^{n-2s} \| \phi \|_{n-2s+\sigma,a}} \sum_{j=1}^k \mu_j^{-2s} \mu_0^{-\frac{n-2s}{2}+\sigma}(t) \\
\lesssim \frac{1}{R^{n-2s} \| \phi \|_{n-2s+\sigma,a}} \sum_{j=1}^k \mu_j^{-2s} \mu_0^{-\frac{n-2s}{2}+\sigma}(t) \\
\.
\]

(4.69)

Now let us consider the second term \( \tilde{\phi}_j \left( -\left( -\Delta \right)^{\frac{s}{2}} - \partial_t \right) \eta_{j,R}. \) From direct computations, we have

\[
\left| \tilde{\phi}_j \left( -\left( -\Delta \right)^{\frac{s}{2}} - \partial_t \right) \eta_{j,R} \right| \lesssim \frac{-\left( -\Delta \right)^{\frac{s}{2}} \eta \left( \frac{|x - \xi_j|}{R\mu_0} \right)}{R^{2s} \mu_{0j}^2} \| \phi \|_{n-2s+\sigma} \sum_{j=1}^k \mu_j^{-2s} \mu_0^{-\frac{n-2s}{2}+\sigma}(t) \\
= \frac{-\left( -\Delta \right)^{\frac{s}{2}} \eta \left( \frac{|x - \xi_j|}{R\mu_0} \right)}{R^{2s} \mu_{0j}^2} \| \phi \|_{n-2s+\sigma} \sum_{j=1}^k \mu_j^{-2s} \mu_0^{-\frac{n-2s}{2}+\sigma}(t) \\
\lesssim \frac{1}{R^{n-2s} \| \phi \|_{n-2s+\sigma,a}} \sum_{j=1}^k \mu_j^{-2s} \mu_0^{-\frac{n-2s}{2}+\sigma}(t) \\
\.
\]

(4.70)

For the first term in the right hand side of (4.70), by the definition of \( \tilde{\phi}_j \), we obtain

\[
\left| -\left( -\Delta \right)^{\frac{s}{2}} \eta \left( \frac{|x - \xi_j|}{R\mu_0} \right) \right| \mu_0^{-\frac{n-2s}{2}} \| \phi \|_{n-2s+\sigma} \sum_{j=1}^k \mu_j^{-2s} \mu_0^{-\frac{n-2s}{2}+\sigma}(t) \\
\lesssim \frac{1}{R^{n-2s} \| \phi \|_{n-2s+\sigma,a}} \sum_{j=1}^k \mu_j^{-2s} \mu_0^{-\frac{n-2s}{2}+\sigma}(t) \\
\.
\]

(4.71)

where we have used the fact that \( -\left( -\Delta \right)^{\frac{s}{2}} \eta \left( \frac{|x - \xi_j|}{R\mu_0} \right) \sim \frac{1}{1+|x|^2} \). From (4.23) and (4.55), the second term in the right hand side of (4.70) can be estimated as

\[
\left| \eta' \left( \frac{|x - \xi_j|}{R\mu_0} \right) \left( \frac{|x - \xi_j|}{R\mu_0} \left( \mu_0 + \mu_0 \xi \right) \right) \mu_0^{-\frac{n-2s}{2}} \| \phi \|_{n-2s+\sigma,a} \sum_{j=1}^k \mu_j^{-2s} \mu_0^{-\frac{n-2s}{2}+\sigma}(t) \\
\leq \frac{1}{R^{n-2s} \| \phi \|_{n-2s+\sigma,a}} \sum_{j=1}^k \mu_j^{-2s} \mu_0^{-\frac{n-2s}{2}+\sigma}(t) \\
\.
\]

(4.72)

From (4.69)-(4.72), we get (4.64).
Proof of (4.65). Since $p - 2 \geq 0$ gives $6s \geq n$, we have

$$
\tilde{N}_{\mu, \xi}(\psi + \psi_1 + \sum_{j=1}^{k} \eta_{j,R} \tilde{\phi}_j) \lesssim \\
\begin{cases}
(u_{\mu, \xi}^*)^{p-2} \left[|\psi|^2 + |\psi_1|^2 + \sum_{j=1}^{k} |\eta_{j,R} \tilde{\phi}_j|^2\right], & \text{when } 6s \geq n, \\
|\psi|^p + |\psi_1|^p + \sum_{j=1}^{k} |\eta_{j,R} \tilde{\phi}_j|^p, & \text{when } 6s < n.
\end{cases}
$$

(4.73)

When $6s \geq n$, we have

$$
\left| (u_{\mu, \xi}^*)^{p-2} (\eta_{j,R} \tilde{\phi}_j)^2 \right| \lesssim \mu_0^{n-2s+\sigma R^{a-2s}} \|\phi\|_{\frac{n}{2} - 2s + \sigma, a}^2 \frac{1}{R^{a-2s}} \sum_{j=1}^{k} \frac{\mu_j^{2s} \mu_0^{\frac{n-2s}{2}}} {1 + |y_j|^a},
$$

$$
\left| (u_{\mu, \xi}^*)^{p-2} \psi^2 \right| \lesssim R^{a-2s} \mu_0^{n-4s+\sigma + a-2s} \|\psi\|_{\frac{n}{2} - 2s + \sigma, a}^2 \frac{1}{R^{a-2s}} \sum_{j=1}^{k} \frac{\mu_j^{2s} t^{-\beta}} {1 + |y_j|^a}.
$$

When $6s < n$, we have

$$
|\eta_{j,R} \tilde{\phi}_j|^p \lesssim \mu_0^{2s + (p-1)\sigma} R^{a-2s} \mu_0^{2s} \|\phi\|_{\frac{n}{2} - 2s + \sigma, a}^p \frac{1}{R^{a-2s}} \sum_{j=1}^{k} \frac{\mu_j^{2s} \mu_0^{\frac{n-2s}{2} + \sigma}} {1 + |y_j|^a},
$$

$$
|\psi|^p \lesssim \mu_0^{4s(1 + \frac{a}{n-2s}) + p(a-2s) - a} R^{a-2s} \|\psi\|_{\frac{n}{2} - 2s + \sigma, a}^p \frac{1}{R^{a-2s}} \sum_{j=1}^{k} \frac{\mu_j^{2s} \mu_0^{\frac{n-2s}{2} + \sigma}} {1 + |y_j|^a}.
$$

The estimates for $\psi_1$ are similar. Hence we have (4.65).

Now we apply the Contraction Mapping Theorem to prove the existence of a fixed point $\psi$ for $\mathcal{A}$. First, set

$$
\mathcal{B} = \left\{ \psi : \|\psi\|_{\frac{n}{2} + \sigma, a} \leq M \frac{t^0 - \epsilon}{R^{a-2s}} \right\}
$$

with $\beta = \frac{n-2s}{2(n-4s)} + \frac{a}{n-4s}$ and $a, \epsilon$ are fixed as above. Here the positive large constant $M$ is independent of $t$ and $t_0$. For any $\psi \in \mathcal{B}$, $\mathcal{A}(\psi) \in \mathcal{B}$ as a consequence of (4.62).
and the estimates (4.63)-(4.65). We claim that for any $\psi^{(1)}, \psi^{(2)} \in \mathcal{B}$,

$$\|\mathcal{A}(\psi^{(1)}) - \mathcal{A}(\psi^{(2)})\|_{\ast, \beta, a} \leq C\|\psi^{(1)} - \psi^{(2)}\|_{\ast, \beta, a},$$

where $C < 1$ is a constant depending on $t_0$ which is chosen sufficiently large. Indeed,

$$\mathcal{A}(\psi^{(1)}) - \mathcal{A}(\psi^{(2)}) = T\left(\tilde{N}_{\mu, \xi}(\psi^{(1)} + \psi_1 + \phi^{in}) - \tilde{N}_{\mu, \xi}(\psi^{(2)} + \psi_1 + \phi^{in})\right),$$

where

$$\tilde{N}_{\mu, \xi}(\psi^{(1)} + \psi_1 + \phi^{in}) - \tilde{N}_{\mu, \xi}(\psi^{(2)} + \psi_1 + \phi^{in}) =$$

$$\left(u_{\mu, \xi}^{\ast} + \psi^{(1)} + \psi_1 + \phi^{in}\right)^{p} - \left(u_{\mu, \xi}^{\ast} + \psi^{(1)} + \psi_1 + \phi^{in}\right)^{p} - p\left(u_{\mu, \xi}^{\ast}\right)^{p-1}\left[\psi^{(1)} - \psi^{(2)}\right].$$

Similar to (4.73), we have

$$\left|\tilde{N}_{\mu, \xi}(\psi^{(1)} + \psi_1 + \phi^{in}) - \tilde{N}_{\mu, \xi}(\psi^{(2)} + \psi_1 + \phi^{in})\right| \lessapprox$$

$$\begin{cases}
(u_{\mu, \xi}^{\ast})^{p-2}|\phi^{in}|\|\psi^{(1)} - \psi^{(2)}|, & \text{when } 6s \geq n, \\
|\phi^{in}|^{p-1}\|\psi^{(1)} - \psi^{(2)}|, & \text{when } 6s < n.
\end{cases}$$

When $6s \geq n$,

$$\left|\tilde{N}_{\mu, \xi}(\psi^{(1)} + \psi_1 + \phi^{in}) - \tilde{N}_{\mu, \xi}(\psi^{(2)} + \psi_1 + \phi^{in})\right| \lessapprox \|\phi\|_{n-2s+\sigma, a}\|\psi^{(1)} - \psi^{(2)}\|_{\ast, \beta, a} R^{a-2s} \mu_{0}^{\frac{s}{2} + s + \sigma}\left|t_{0}\right|\frac{1}{R^{a-2s}} \sum_{j=1}^{k} \frac{\mu_{j}^{-2s}t^{-\beta}}{1 + |y_{j}|^{a}}.$$

while in the case of $6s < n$,

$$\left|\tilde{N}_{\mu, \xi}(\psi^{(1)} + \psi_1 + \phi^{in}) - \tilde{N}_{\mu, \xi}(\psi^{(2)} + \psi_1 + \phi^{in})\right| \lessapprox \|\phi\|_{n-2s+\sigma, a}^{p-1}\|\psi^{(1)} - \psi^{(2)}\|_{\ast, \beta, a} R^{a-2s} \mu_{0}^{\frac{s}{2} + \sigma + \sigma}\left|t_{0}\right|\frac{1}{R^{a-2s}} \sum_{j=1}^{k} \frac{\mu_{j}^{-2s}t^{-\beta}}{1 + |y_{j}|^{a}}.$$
Hence there exists a choice of \( R \) in the form (4.34) such that

\[
\| \mathscr{A}(\psi^{(1)}) - \mathscr{A}(\psi^{(2)}) \|_{\ast, \beta, a} \leq C \| \psi^{(1)} - \psi^{(2)} \|_{\ast, \beta, a}
\]

holds with \( C < 1 \), provided \( t_0 \) is sufficiently large. Therefore, if \( t_0 \) is fixed sufficiently large, \( \mathscr{A} \) is a contraction map in \( \mathcal{B} \). The validity of (4.60) follows directly from (4.48). The proof is completed. \( \square \)

Properties of the solution \( \psi \).

**Proposition 4.1.2.** Under the assumptions in Proposition 4.1.1, \( \psi \) depends smoothly on the parameters \( \lambda, \xi, \bar{\lambda}, \bar{\xi}, \phi \), for \( y_j = \frac{x-j}{\mu_0} \), we have

\[
| \partial_\lambda \psi[\lambda, \xi, \bar{\lambda}, \bar{\xi}, \phi][\bar{\lambda}, \bar{\xi}](x, t) | \lesssim \frac{t_0^{-\varepsilon}}{R^{a-2s}} \| \bar{\lambda}(t) \|_{1+\sigma} \left( \sum_{j=1}^{k} \frac{\mu_0^{-\frac{a-4s-1+\sigma}{2}}(t)}{1 + |y_j|^{a-2s}} \right), \tag{4.74}
\]

\[
| \partial_\xi \psi[\lambda, \xi, \bar{\lambda}, \bar{\xi}, \phi][\bar{\xi}](x, t) | \lesssim \frac{t_0^{-\varepsilon}}{R^{a-2s}} \| \bar{\xi}(t) \|_{1+\sigma} \left( \sum_{j=1}^{k} \frac{\mu_0^{-\frac{a-4s+1+\sigma}{2}}(t)}{1 + |y_j|^{a-2s}} \right), \tag{4.75}
\]

\[
| \partial_{\bar{\xi}} \psi[\lambda, \xi, \bar{\lambda}, \bar{\xi}, \phi][\bar{\xi}](x, t) | \lesssim \frac{t_0^{-\varepsilon}}{R^{a-2s}} \| \bar{\xi}(t) \|_{a-4s+1+\sigma} \left( \sum_{j=1}^{k} \frac{\mu_0^{-\frac{a-6s-1+\sigma}{2}}(t)}{1 + |y_j|^{a-2s}} \right), \tag{4.76}
\]

\[
| \partial_{\bar{\lambda}} \psi[\lambda, \xi, \bar{\lambda}, \bar{\xi}, \phi][\bar{\lambda}](x, t) | \lesssim \frac{t_0^{-\varepsilon}}{R^{a-2s}} \| \bar{\lambda}(t) \|_{a-4s+1+\sigma} \left( \sum_{j=1}^{k} \frac{\mu_0^{-\frac{a-6s+1+\sigma}{2}}(t)}{1 + |y_j|^{a-2s}} \right), \tag{4.77}
\]

\[
| \partial_\phi \psi[\lambda, \xi, \bar{\lambda}, \bar{\xi}, \phi][\phi](x, t) | \lesssim \frac{1}{R^{a-2s}} \| \phi(t) \|_{n-2s+\sigma,a} \left( \sum_{j=1}^{k} \frac{\mu_0^{-\frac{a-\sigma}{2}}(t)}{1 + |y_j|^{a-2s}} \right). \tag{4.78}
\]

**Proof:** **Step 1.** Proof of (4.74) and (4.75).

We fix \( j = 1 \). \( \psi[\lambda_1] \) is a solution to problem (4.44) for all \( \lambda_1 \) satisfying (4.56). Differentiating problem (4.44) with respect to \( \lambda_1 \) gives us a nonlinear equation. From the Implicit Function Theorem, the solutions are given by \( \partial_{\lambda_1} \psi[\lambda_1](x, t) \).

Decompose \( \partial_{\lambda_1} \psi[\lambda_1](x, t) = Z_1 + Z \) with \( Z_1 = T(0, -\partial_{\lambda_1} \psi[\lambda_1](0, t)), \) where \( T \) is
defined in Lemma 4.1.3. Then $Z$ is a solution of the following nonlinear problem

$$
\begin{cases}
\frac{\partial Z}{\partial t} = -(\Delta)^s Z + V_{\mu, \xi} Z + (\partial_{\lambda_1} V_{\mu, \xi}) [\bar{\lambda}_1] \psi + \partial_{\lambda_1} [\bar{N}_{\mu, \xi} (\psi + \phi^{in})] [\bar{\lambda}_1] \\
\quad + \partial_{\lambda_1} S_{out} [\bar{\lambda}_1] \quad \text{in } \Omega \times (t_0, \infty), \\
Z = 0 \quad \text{in } (\mathbb{R}^n \setminus \Omega) \times (t_0, \infty), \\
Z(\cdot, t_0) = 0 \quad \text{in } \mathbb{R}^n.
\end{cases}
$$

(4.79)

By definition, $\sum_{j=1}^k \{ - (\Delta)^s \eta_{j,R} - (\Delta)^s \tilde{\phi}_j \} + \tilde{\phi}_j (\Delta)^s \eta_{j,R}$ is independent of $\lambda_1$. Then for any $x \in \mathbb{R}^n \setminus \Omega$,

$$
\left| \partial_{\lambda_1} u^s_{\mu, \xi}(x, t) \right| \lesssim \mu_0^{\frac{n-2s}{2}} (t) |\bar{\lambda}_1(t)|.
$$

(4.80)

From (4.80) and Lemma 4.1.3, we obtain

$$
|Z_1(x, t)| \lesssim \frac{t_0^{-\varepsilon}}{R^{s-n}} ||\bar{\lambda}_1||_{1+\sigma} \left( \sum_{j=1}^k \frac{\mu_0^{\frac{n-2s}{2}}(t)}{1 + |y_j|^{a-2s}} \right).
$$

For problem (4.79), we compute

$$
\partial_{\lambda_1} [\bar{N}_{\mu, \xi} (\psi + \phi^{in})] [\bar{\lambda}_1] = p \left( (u^s_{\mu, \xi} + \psi + \phi^{in})^{p-1} - (u^s_{\mu, \xi})^{p-1} \right) (Z + Z_1) \\
\quad + p(p-1) (u^s_{\mu, \xi})^{p-2} (\psi + \phi^{in}) \partial_{\lambda_1} u^s_{\mu, \xi} [\bar{\lambda}_1].
$$

Therefore, $Z$ is a fixed point of the operator

$$
\mathcal{A}_1(Z) = T(f + p \left( (u^s_{\mu, \xi} + \psi + \phi^{in})^{p-1} - (u^s_{\mu, \xi})^{p-1} \right) Z, 0, 0),
$$

(4.81)

where

$$
f = \partial_{\lambda_1} S_{out} [\bar{\lambda}_1] + (\partial_{\lambda_1} V_{\mu, \xi}) [\bar{\lambda}_1] \psi + p \left( (u^s_{\mu, \xi} + \psi + \phi^{in})^{p-1} - (u^s_{\mu, \xi})^{p-1} \right) Z_1 \\
\quad + p(p-1) (u^s_{\mu, \xi})^{p-2} (\psi + \phi^{in}) \partial_{\lambda_1} u^s_{\mu, \xi} [\bar{\lambda}_1].
$$

(4.82)
We claim that

\[
|f(x,t)| \lesssim \frac{t_0^{-\varepsilon}}{R^{a-2s}} \|\bar{\lambda}_1\|_{1+\sigma} \sum_{j=1}^{k} \mu_j^{-2s} \mu_0^{\frac{n-2s}{2}} \frac{\|u\|_{1+\sigma}^{1+\sigma}}{1 + |y_j|^a}.
\]  

(4.83)

To prove (4.83), we first estimate \(\partial_{\lambda_1} S_{\text{out}}[\bar{\lambda}_1]\). In the region \(|x-q_i| > \delta\) \((i = 1, \ldots, k)\), we have the following estimate for \(\partial_{\lambda_1} S(u_{\mu,\xi})\) by (4.29), (4.55) and (4.56)

\[
\partial_{\lambda_1} S(u_{\mu,\xi})[\bar{\lambda}_1](x,t) = \mu_0^{\frac{n-2s}{2}} f(x,\mu_0^{-1}\mu,\xi) \bar{\lambda}_1(t),
\]

where the smooth and bounded function \(f\) depends on \((x,\mu_0^{-1}\mu,\xi)\). Now we fix \(j\) and consider the region \(|x-q_j| \leq \delta\). From (4.31), we have

\[
\partial_{\lambda_1} S(u_{\mu,\xi})[\bar{\lambda}_1](x,t) = \partial_{\lambda_1} S(u_{\mu,\xi})[\bar{\lambda}_1](x,t)(1 + \mu_0 f(x,\mu_0^{-1}\mu,\xi, t)),
\]

where the smooth and bounded function \(f\) depends on \((x,\mu_0^{-1}\mu,\xi, t)\). Differentiating (4.10) with respect to \(\bar{\lambda}_1\), we obtain

\[
\partial_{\lambda_1} S(u_{\mu,\xi})(\bar{\lambda}_1)(x,t) = -\left(\frac{n-2s}{2} + 1\right) \mu_1^{-\frac{n-2s}{2} - 2} \left[ \mu_1 Z_{n+1}(y_1) + \xi_1 \cdot \nabla U(y_1) \right. \\
- \mu_1^{-\frac{n-2s}{2} - 1} \left[ \xi_1 D^2 U(y_1) + \mu_1 \nabla Z_{n+1}(y_1) \right] \cdot \frac{x-q_1}{\mu_1^2} \bar{\lambda}_1(t) \\
+ p \left( \sum_{i=1}^{k} \mu_1^{-\frac{n-2s}{2}} U(y_i) - \mu_1^{-\frac{n-2s}{2}} H(x,q_i) \right)^{p-1} \\
\times \partial_{\lambda_1} \left[ \mu_1^{-\frac{n-2s}{2}} U(y_1) - \mu_1^{-\frac{n-2s}{2}} H(x,q_1) \right] \bar{\lambda}_1(t) \\
- p \left( \mu_1^{-\frac{n-2s}{2}} U(y_1) \right)^{p-1} \partial_{\lambda_1} \left[ \mu_1^{-\frac{n-2s}{2}} U(y_1) \right] \bar{\lambda}_1(t).
\]

From (4.55) and (4.56), we have

\[
\left| \partial_{\lambda_1} S(u_{\mu,\xi})(\bar{\lambda}_1)(x,t) \right| \lesssim \frac{t_0^{-\varepsilon}}{R^{a-2s}} \|\bar{\lambda}_1\|_{1+\sigma} \sum_{j=1}^{k} \mu_j^{-2s} \mu_0^{\frac{n-2s}{2} - 1} \frac{\|u\|_{1+\sigma}^{1+\sigma}}{1 + |y_j|^a}.
\]  

(4.84)
Therefore, by the definition of $S_{\text{out}}$ together with (4.84), we obtain

$$|\partial_t S_{\text{out}}[\tilde{\lambda}_1](x,t)| \lesssim \frac{t_0^{-e}}{R^{\alpha - 2s}} \|\tilde{\lambda}_1\|^{1+\sigma} \sum_{j=1}^k \frac{\mu_j^{-2s} \mu_0^{-\frac{n-2s}{2}}}{1 + |y_j|^a}.$$ 

Next, we estimate the remainders in $f$. Direct computations imply that

$$(\partial_t V_{\mu, \xi})[\tilde{\lambda}_1](x,t) = p(p-1) \left[ (u^{*, \xi}_{\mu, \xi})^{p-2} \partial_{\lambda_1} u^{*, \xi}_{\mu, \xi}[\tilde{\lambda}_1] \right.$$ 

$$\left. - \eta_{1,R}(\mu_1 \frac{n-2s}{2} U(y_1))^{p-2} \partial_{\lambda_1}(\mu_1 \frac{n-2s}{2} U(y_1))[\tilde{\lambda}_1]\right].$$

Since $|\partial_{\lambda_1} \left( \mu_1 \frac{n-2s}{2} U(y_1) \right)| \lesssim \mu_0^{-1} \|\tilde{\lambda}_1\|^{1+\sigma}$ and $\beta = \frac{n-2s}{2(\eta_2 - \eta_1)} + \frac{\sigma}{n-\alpha}$, we have

$$|\partial_{\lambda_1} V_{\mu, \xi}|[\tilde{\lambda}_1] \lesssim \|\psi\|_{s, \beta, \alpha} \frac{t_0^{-e}}{R^{\alpha - 2s}} \|\tilde{\lambda}_1\|^{1+\sigma} \sum_{j=1}^k \frac{\mu_j^{-2s} \mu_0^{-\frac{n-2s}{2}}}{1 + |y_j|^a}.$$ 

By the same token, we can deal with $p(p-1)(u^{*, \xi}_{\mu, \xi})^{p-2}(\psi + \phi^{in}) (\partial_{\lambda_1} u^{*, \xi}_{\mu, \xi}[\tilde{\lambda}_1])$ in (4.82) and obtain

$$p(p-1)(u^{*, \xi}_{\mu, \xi})^{p-2}(\psi + \phi^{in}) \partial_{\lambda_1} u^{*, \xi}_{\mu, \xi}[\tilde{\lambda}_1] \lesssim \frac{t_0^{-e}}{R^{\alpha - 2s}} \|\tilde{\lambda}_1\|^{1+\sigma} \sum_{j=1}^k \frac{\mu_j^{-2s} \mu_0^{-\frac{n-2s}{2}}}{1 + |y_j|^a}.$$ 

Analogously, we can estimate the last term $p \left[ (u^{*, \xi}_{\mu, \xi} + \phi^{in})^{p-1} - (u^{*, \xi}_{\mu, \xi})^{p-1} \right] Z_1$. Therefore, we conclude the validity of (4.83).

Now we consider the fixed point problem (4.81). Then the operator $\mathcal{A}_1$ has a fixed point in the set of functions satisfying

$$|Z(x,t)| \leq M \frac{t_0^{-e}}{R^{\alpha - 2s}} \|\tilde{\lambda}_1\|^{1+\sigma} \sum_{j=1}^k \frac{\mu_j^{-2s} \mu_0^{-\frac{n-2s}{2}}}{1 + |y_j|^a}.$$ 

with the large constant $M$ fixed. In fact, $\mathcal{A}_1$ is a contraction map when $R$ is chosen properly large in terms of $t_0$. Therefore, the estimate (4.74) for $\partial_{\xi} \Psi[\tilde{\lambda}_1]$ holds. The estimate (4.75) for $\partial_{\xi} \Psi[\tilde{\xi}]$ can be verified in a similar way. Here we omit the
Step 2. Proof of (4.76) and (4.77). We fix \( j = 1 \). From the discussions above, the function \( \Psi(\tilde{\lambda}_1) \) is a solution to (4.44) for all \( \tilde{\lambda}_1 \) satisfying (4.56). Then we differentiate problem (4.44) with respect to \( \tilde{\lambda}_1 \) and obtain a nonlinear equation. From the Implicit Function Theorem, the solutions are given by \( \partial_{\tilde{\lambda}_1} \Psi(\tilde{\lambda}_1)(x,t) \). Denote \( Z(x,t) = \partial_{\tilde{\lambda}_1} \Psi(\tilde{\lambda}_1)(x,t) \). Then \( Z \) is a solution to the following nonlinear problem

\[
\begin{align*}
\partial_t Z &= \left\{-(-\Delta)^s Z + V_{\mu,\xi} Z + \partial_{\tilde{\lambda}_1} \tilde{N}_{\mu,\xi}(\Psi + \phi^{in}) \right\} \tilde{\lambda}_1 + \partial_{\tilde{\lambda}_1} S_{out} \tilde{\lambda}_1 \quad \text{in } \Omega \times (t_0, \infty), \\
Z(x,t) &= 0 \quad \text{in } (\mathbb{R}^n \setminus \Omega) \times (t_0, \infty), \\
Z(\cdot, t_0) &= 0 \quad \text{in } \mathbb{R}^n.
\end{align*}
\]

From the definition of \( \tilde{N}_{\mu,\xi}(\Psi + \phi^{in}) \), we have

\[
\partial_{\tilde{\lambda}_1} \tilde{N}_{\mu,\xi}(\Psi + \phi^{in}) \tilde{\lambda}_1 = p \left\{ (u_{\mu,\xi}^* + \Psi + \phi^{in})^{p-1} - (u_{\mu,\xi}^*)^{p-1} \right\} Z(x,t).
\]

Therefore, \( Z \) is a fixed point for the operator

\[
\mathcal{A}_1(Z) = T \left( \partial_{\tilde{\lambda}_1} S_{out} \tilde{\lambda}_1 + p \left\{ (u_{\mu,\xi}^* + \Psi + \phi^{in})^{p-1} - (u_{\mu,\xi}^*)^{p-1} \right\} Z, 0, 0 \right). \quad (4.85)
\]

Now we differentiate \( S(u_{\mu,\xi}^*) \) with respect to \( \tilde{\lambda}_1 \) in (4.30) directly and obtain

\[
\partial_{\tilde{\lambda}_1} S(u_{\mu,\xi}^*)[\tilde{\lambda}_1](x,t) = \mu_1 \frac{n-2s}{2} \left[ Z_{n+1}(y_1) + \frac{n-2s}{2} \mu_1^{n-2s} H(x,q_1) \right] \tilde{\lambda}_1(t) \\
+ \mu_j \frac{n-s}{2} \Phi_1(y_1, t) + y_1 \cdot \nabla_\Phi \Phi_1 \tilde{\lambda}_1(t).
\]

Hence

\[
\left| \partial_{\tilde{\lambda}_1} S(u_{\mu,\xi}^*)[\tilde{\lambda}_1](x,t) \right| \lesssim \frac{t_0^{-\varepsilon}}{R^{n-2s}} \| \tilde{\lambda}_1(t) \|_{n-4s+1+\sigma} \left( \sum_{j=1}^{k} \mu_j^{2s}(t) \mu_0^{\frac{n-s}{2} - 1} \right).
\]

Now we consider the fixed point problem (4.85). Similar to Step 1, \( \mathcal{A}_1 \) has a
fixed point in the set of functions satisfying

\[ |Z(x,t)| \lesssim \frac{t_0^{-E}}{R^{a-2s}} \| \hat{\lambda}_1(t) \|_{n-4s+1+\sigma} \sum_{j=1}^{k} \mu_0^{-\frac{n}{2s}+1} \frac{1}{1+|y_j|^a}. \]

Thus estimate (4.76) holds.

On the other hand, we observe that

\[ \partial_t \xi_1 S(u_{\mu,\xi})(\xi_1)(x,t) = \mu_1^{-\frac{n}{2s}} \left[ \nabla U(y_1) + \nabla \Phi_1(y_1,t) \right] \xi_1(t). \quad (4.86) \]

From (4.86) we have

\[ \left| \partial_t \xi_1 S(u_{\mu,\xi})(\xi_1)(x,t) \right| \lesssim \frac{t_0^{-E}}{R^{a-2s}} \| \hat{\lambda}_1(t) \|_{n-4s+1+\sigma} \left( \sum_{j=1}^{k} \mu_j^{-\frac{n}{2s}} \frac{1}{1+|y_j|^a} \right). \]

Therefore, we have (4.77).

**Step 3.** Proof of (4.78). Define \( Z(x,t) = \partial_\phi \psi \hat{\phi}(x,t) \) with \( \hat{\phi} \) satisfying (4.58).

Therefore, \( Z \) is a solution to

\[
\begin{cases}
\partial_t Z = -(-\Delta)^s Z + V_{\mu,\xi} Z \\
\quad + \sum_{j=1}^{k} \left\{ [ -(-\Delta)^s \eta_{j,R}, -(-\Delta)^s \hat{\phi}_j ] + \hat{\phi}_j ( -(-\Delta)^s - \partial_t ) \eta_{j,R} \right\} 
\quad + p \left[ (u_{\mu,\xi} + \psi + \phi^{in})^{p-1} - (u_{\mu,\xi})^{p-1} \right] \hat{\phi} \quad \text{in } \Omega \times (t_0, \infty), \\
Z = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega \times (t_0, \infty), \\
Z(\cdot, t_0) = 0 \quad \text{in } \mathbb{R}^n,
\end{cases}
\]

where \( \hat{\phi} = \mu_0^{-\frac{n}{2s}} \hat{\phi}_j \left( \frac{x-x_j}{\mu_0}, t \right) \). As in Step 1 and Step 2, we have

\[
\left| \sum_{j=1}^{k} \left\{ [ -(-\Delta)^s \eta_{j,R}, -(-\Delta)^s \hat{\phi}_j ] + \hat{\phi}_j ( -(-\Delta)^s - \partial_t ) \eta_{j,R} \right\} \right| 
\lesssim \frac{1}{R^{a-2s}} \| \hat{\phi} \|_{n-2s+\sigma} \sum_{j=1}^{k} \mu_j^{-\frac{n}{2s}} \frac{1}{1+|y_j|^a}.
\]
\[ p \left[ \left( u^*_{\mu, \xi} + \psi + \phi^{in} \right)^{p-1} - \left( u^*_{\mu, \xi} \right)^{p-1} \right] \phi \]
\[ \lesssim \frac{1}{R^{n-2s}} \| \phi \|_{n-2s+\sigma, a} \left[ \| \psi \|_{n, \beta, a}^{p-1} + \| \phi^{in} \|_{n-2s+\sigma, a}^{p-1} \right] \sum_{j=1}^{k} \frac{\mu_{j}^{2s} \mu_{0}^{-\frac{n-2s}{n}}} {1 + |y_j|^a}. \]

From Lemma 4.1.3, we conclude the validity of (4.78). \hfill \Box

### 4.1.5 The inner problem

Substituting the solution \( \psi = \Psi[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi] \) of the outer problem given by proposition 4.1.1 into the inner problem (4.41), the full problem is reduced to the following system for \( y \in \mathbb{R}^n, t \geq t_0 \)

\[ \mu_{0j}^{2s} \partial_{\tau} \phi_j = -(-\Delta)^{s} \phi_j + pU^{p-1}(y)\phi_j + H_j[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi](y, t) \quad (4.87) \]

for \( j = 1, \ldots, k \), where

\[ H_j[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi] := \left\{ \begin{array}{c} \frac{\mu_{0j}^{2s}}{\mu_{j}^{2s}} S_{\mu, \xi, j}(\xi_j + \mu_{0j} y, t) + B_j[\phi_j] + B_{0j}[\phi_j] \\ + p\mu_{0j}^{\frac{n-2s}{n}} U^{p-1} \left( \frac{\mu_{0j}}{\mu_j} y \right) \psi(\xi_j + \mu_{0j} y, t) \right\} \chi_{B_{2R}(0)}(y), \]

(4.88)

and \( B_j[\phi_j], B_{0j}[\phi_j] \) are defined in (4.42), (4.43) respectively.

After the change of variables

\[ t = t(\tau), \quad \frac{dt}{d\tau} = \mu_{0j}^{2s}(t), \]

(4.87) is reduced to

\[ \partial_{\tau} \phi_j = -(-\Delta)^{s} \phi_j + pU^{p-1}(y)\phi_j + H_j[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi](y, t(\tau)) \quad (4.89) \]

for \( y \in \mathbb{R}^n, \tau \geq \tau_0 \) with \( \tau_0 \) the unique positive number satisfying \( t(\tau_0) = t_0 \).
We will find a solution $\phi = (\phi_1, \ldots, \phi_k)$ to the system
\[
\begin{aligned}
\partial_\tau \phi_j &= -(-\Delta)^s \phi_j + pU^{p-1}(y)\phi_j + H_j[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi](y, t(\tau)), \quad y \in \mathbb{R}^n, \quad \tau \geq \tau_0, \\
\phi_j(y, \tau_0) &= e_{0j}Z_0(y), \quad y \in \mathbb{R}^n,
\end{aligned}
\]
(4.90)
for a constant $e_{0j}$ and all $j = 1, \ldots, k$. Here $Z_0$ is a radially symmetric eigenfunction associated to the unique negative eigenvalue $\lambda_0$ of the eigenvalue problem
\[L_0(\phi) + \lambda \phi = 0, \quad \phi \in L^\infty(\mathbb{R}^n).\]
Note that $\lambda_0$ is simple and $Z_0$ satisfies
\[Z_0(y) \sim |y|^{-n-2s} \text{ as } |y| \to \infty,
\]
see, for example, [84]. We will prove that (4.90) is solvable in the function space of those $\phi_j$’s satisfying (4.58), provided $\xi$ and $\lambda$ are chosen so that $H_j[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi]$ satisfies the orthogonality conditions
\[
\int_{B_{2R}} H_j[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi](y, t(\tau))Z_l(y)dy = 0,
\]
(4.91)
for all $\tau \geq \tau_0, \; j = 1, \ldots, k$ and $l = 1, 2, \ldots, n + 1$. We first develop a linear theory.

**The linear theory.**

In this section, for $R > 0$ fixed large, we find a solution to the nonlocal initial value problem
\[
\begin{aligned}
\partial_\tau \phi &= -(-\Delta)^s \phi + pU^{p-1}(y)\phi + h(y, \tau), \quad y \in \mathbb{R}^n, \quad \tau \geq \tau_0, \\
\phi(y, \tau_0) &= e_0Z_0(y), \quad y \in \mathbb{R}^n.
\end{aligned}
\]
(4.92)
Let $\nu = 1 + \frac{\sigma}{n-2s}$, then $\mu_0^{n-2s+\sigma} \sim \tau^{-\nu}$. Define
\[
\|h\|_{a, \nu, \eta} := \sup_{\tau > \tau_0} \sup_{y \in B_{2R}} \tau^\nu (1 + |y|^a)(|h(y, \tau)| + (1 + |y|^\eta)\chi_{B_{2R}}(y)[h(\cdot, \tau)]_{\eta, B_1(0)}).
\]
In the following, we always assume that \( h = h(y, \tau) \) is a function defined in the whole space \( \mathbb{R}^n \) which is zero outside \( B_{2R}(0) \) for all \( \tau > \tau_0 \).

**Proposition 4.1.3.** Suppose \( a \in (2s, n-2s) \), \( \nu > 0 \), \( \|h\|_{2s+a, \nu, \eta} < +\infty \) and

\[
\int_{B_{2R}} h(y, \tau) Z_j(y) dy = 0 \text{ for all } \tau \in (\tau_0, \infty), \ j = 1, \ldots, n + 1.
\]

For sufficiently large \( R \), there exist \( \phi = \phi[h](y, \tau) \) and \( e_0 = e_0[h](\tau) \) \( (\tau \in (\tau_0, +\infty), \ y \in \mathbb{R}^n) \) satisfying (4.92) and

\[
(1 + |y|)|\nabla_y \phi(y, \tau)| \chi_{B_{2R}(0)}(y) + |\phi(y, \tau)| \lesssim \tau^{-\nu}(1 + |\nu|)^{-a} \|h\|_{2s+a, \nu, \eta}, \ \tau \in (\tau_0, +\infty), \ y \in \mathbb{R}^n,
\]

\[
|e_0[h]| \lesssim \|h\|_{2s+a, \nu, \eta}.
\]

**Lemma 4.1.4.** Suppose \( a \in (2s, n-2s) \), \( \nu > 0 \), \( \|h\|_{2s+a, \nu, \eta} < +\infty \) and

\[
\int_{\mathbb{R}^n} h(y, \tau) Z_j(y) dy = 0 \text{ for all } \tau \in (\tau_0, \infty), \ j = 1, \ldots, n + 1.
\]

For any sufficiently large \( \tau_1 > 0 \), the solution \( (\phi(y, \tau), c(\tau)) \) of the problem

\[
\begin{cases}
\partial_\tau \phi = -(-\Delta)^s \phi + pU^{p-1}(y)\phi + h(y, \tau) - c(\tau)Z_0(y), \ y \in \mathbb{R}^n, \ \tau \geq \tau_0, \\
\int_{\mathbb{R}^n} \phi(y, \tau) Z_0(y) dy = 0 \text{ for all } \tau \in (\tau_0, +\infty), \\
\phi(y, \tau_0) = 0, \ y \in \mathbb{R}^n,
\end{cases}
\]

satisfies the estimates

\[
\|\phi(y, \tau)\|_{a, \tau_1} \lesssim \|h\|_{2s+a, \tau_1},
\]

\[
|c(\tau)| \lesssim \tau^{-\nu} R^a \|h\|_{2s+a, \tau_1} \text{ for } \tau \in (\tau_0, \tau_1).
\]

Here \( \|h\|_{b, \tau_1} := \sup_{\tau \in (\tau_0, \tau_1)} \tau^\nu \|1 + |y|^b \|_{L^\infty(\mathbb{R}^n)} \).

**Proof.** Note that (4.95) is equivalent to

\[
\begin{cases}
\partial_\tau \phi = -(-\Delta)^s \phi + pU^{p-1}(y)\phi + h(y, \tau) - c(\tau)Z_0(y), \ y \in \mathbb{R}^n, \ \tau \geq \tau_0, \\
\phi(y, \tau_0) = 0, \ y \in \mathbb{R}^n,
\end{cases}
\]

\[
(4.97)
\]
for \( c(\tau) \) given by the relation

\[
c(\tau) \int_{\mathbb{R}^n} |Z_0(y)|^2 dy = \int_{\mathbb{R}^n} h(y, \tau) Z_0(y) dy.
\]

It is easy to see that

\[
|c(\tau)| \lesssim \tau^{-\nu} R^a \|h\|_{2\nu+a, \tau_1}
\]

holds for \( \tau \in (\tau_0, \tau_1) \). So we only need to prove (4.96) for the solution \( \phi \) of (4.97).

Inspired by Lemma 4.5 of [55] and the linear theory of [174], we will use the blow-up argument.

First, we claim that, given \( \tau_1 > \tau_0 \), we have \( \|\phi\|_{a, \tau_1} < +\infty \). Indeed, by the fractional parabolic theory (see [117]), given \( R_0 > 0 \) there is a \( K = K(R_0, \tau_1) \) such that

\[
|\phi(y, \tau)| \leq K \text{ in } B_{R_0}(0) \times (\tau_0, \tau_1).
\]

Fix \( R_0 \) large and take \( K_1 \) sufficiently large, \( K_1 \rho^{-a} \) is a supersolution for (4.97) when \( \rho > R_0 \). Hence \( |\phi| \leq 2K_1 \rho^{-a} \) and \( \|\phi\|_{a, \tau_1} < +\infty \) for any \( \tau_1 > 0 \). Next, we claim that the following identities hold,

\[
\int_{\mathbb{R}^n} \phi(y, \tau) \cdot Z_j(y) dy = 0 \text{ for all } \tau \in (\tau_0, \tau_1), j = 0, 1, \ldots, n+1. \quad (4.99)
\]

Indeed, from the definition of \( c(\tau) \), we have

\[
\int_{\mathbb{R}^n} \phi(y, \tau) \cdot Z_0(y) dy = 0.
\]

Testing (4.97) with \( Z_j \eta \), where \( \eta(y) = \eta_0(|y|/R_1), j = 1, \ldots, n+1, R_1 \) is an arbitrary positive constant and the smooth cut-off function \( \eta_0 \) is defined as

\[
\eta_0(r) = \begin{cases} 
1, & \text{for } r < 1, \\
0, & \text{for } r > 2,
\end{cases}
\]

we get

\[
\int_{\mathbb{R}^n} \phi(\cdot, \tau) \cdot Z_j \eta = \int_0^\tau ds \int_{\mathbb{R}^n} (\phi(\cdot, s) \cdot L_0[\eta Z_j] + h \eta Z_j + c(s) Z_0 Z_j \eta).
\]
Furthermore, it holds that

\[
\int_{\mathbb{R}^n} \left( \phi \cdot L_0[\eta Z_j] + h Z_j \eta - c(s) Z_0 Z_j \eta \right) \\
= \int_{\mathbb{R}^n} \phi \cdot \left( Z_j (-(-\Delta)^s) \eta + \left[ -(-\Delta)^{\frac{s}{2}} \eta, -(-\Delta)^{\frac{s}{2}} Z_j \right] \right) \\
- h \cdot Z_j (1 - \eta) + c(s) Z_0 Z_j (1 - \eta) \\
= O(R_1^{-\varepsilon})
\]

for some small positive number \( \varepsilon \) uniformly on \( \tau \in (\tau_0, \tau_1) \). Then (4.99) hold by letting \( R_1 \to +\infty \). Finally, we claim that for all \( \tau_1 > 0 \) large enough, any \( \phi \) with \( \|\phi\|_{a,\tau_1} < +\infty \) solving (4.97) and satisfying (4.99), we have

\[
\|\phi\|_{a,\tau_1} \lesssim \|h\|_{2s+a,\tau_1}.
\]  

Hence (4.96) holds. To prove (4.100), we use the contradiction argument. Suppose that there exist sequences \( \tau_k^1 \to +\infty \) and \( \phi_k, h_k, c_k \) satisfying

\[
\begin{cases}
\partial_\tau \phi_k = -(-\Delta)^s \phi_k + pU^{p-1}(y) \phi_k + h_k - c_k(\tau) Z_0(y), \ y \in \mathbb{R}^n, \ \tau \geq \tau_0, \\
\int_{\mathbb{R}^n} \phi_k(y, \tau) \cdot Z_j(y) \, dy = 0 \text{ for all } \tau \in (\tau_0, \tau_k^1), \ j = 0, 1, \ldots, n + 1, \\
\phi_k(y, \tau_0) = 0, \ y \in \mathbb{R}^n,
\end{cases}
\]

and

\[
\|\phi_k\|_{a,\tau_k^1} = 1, \ \|h_k\|_{2s+a,\tau_k^1} \to 0.
\]  

By (4.98), we have \( \sup_{\tau \in (\tau_0, \tau_k^1)} \tau^\nu c_k(\tau) \to 0 \). First, we claim that

\[
\sup_{\tau_0 < \tau < \tau_k^1} \tau^\nu |\phi_k(y, \tau)| \to 0
\]  

holds uniformly on compact subsets of \( \mathbb{R}^n \). Indeed, if for some \( |y_k| \leq M \) and \( \tau_0 < \tau_2^k < \tau_1^k \),

\[
(\tau_2^k)^\nu |\phi_k(y_k, \tau_2^k)| \geq \frac{1}{2},
\]

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then it is easy to see that $\tau_2^k \to +\infty$. Now, we define

$$\tilde{\phi}_n(y, \tau) = (\tau_2^k)^\nu \phi_n(y, \tau_2^k + \tau).$$

Then

$$\partial_\tau \tilde{\phi}_k = L_0[\tilde{\phi}_k] + \tilde{h}_k - \tilde{c}_k(\tau)Z_0(y) \text{ in } \mathbb{R}^n \times (\tau_0 - \tau_2^k, 0],$$

with $\tilde{h}_k \to 0$, $\tilde{c}_k \to 0$ uniformly on compact subsets of $\mathbb{R}^n \times (-\infty, 0]$ and

$$|\tilde{\phi}(y, \tau)| \leq \frac{1}{1 + |y|^a} \text{ in } \mathbb{R}^n \times (\tau_0 - \tau_2^k, 0].$$

Using the fact that $a \in (2s, n - 2s)$ and the dominant convergence theorem, we have $\tilde{\phi}_k \to \tilde{\phi}$ uniformly on compact subsets of $\mathbb{R}^n \times (-\infty, 0]$ with $\tilde{\phi} \neq 0$ and

$$\begin{cases}
\partial_\tau \tilde{\phi} = -(-\Delta)^s \tilde{\phi} + pU^{p-1}(y) \tilde{\phi} \text{ in } \mathbb{R}^n \times (-\infty, 0], \\
\int_{\mathbb{R}^n} \tilde{\phi}(y, \tau) \cdot Z_j(y)dy = 0 \text{ for all } \tau \in (-\infty, 0], j = 0, 1, \ldots, n + 1, \\
|\tilde{\phi}(y, \tau)| \leq \frac{1}{1 + |y|^a} \text{ in } \mathbb{R}^n \times (-\infty, 0], \\
\tilde{\phi}(y, \tau_0) = 0, y \in \mathbb{R}^n.
\end{cases} \quad (4.103)$$

We claim that $\tilde{\phi} = 0$, which is a contradiction. By fractional parabolic regularity (see [117]), $\tilde{\phi}(y, \tau)$ is smooth. A scaling argument shows

$$(1 + |y|^s)|(-\Delta)^s \tilde{\phi}| + |\partial_\tau \tilde{\phi}| + |(-\Delta)^s \tilde{\phi}| \lesssim (1 + |y|)^{-2s-a}.$$

Differentiating (4.103), we get $\partial_\tau \tilde{\phi}_\tau = -(-\Delta)^s \tilde{\phi}_\tau + pU^{p-1}(y) \tilde{\phi}_\tau$ and

$$(1 + |y|^s)|(-\Delta)^s \tilde{\phi}_\tau| + |\partial_\tau \tilde{\phi}_\tau| + |(-\Delta)^s \tilde{\phi}_\tau| \lesssim (1 + |y|)^{-4s-a}.$$

Moreover, it holds that

$$\frac{1}{2} \partial_\tau \int_{\mathbb{R}^n} |\tilde{\phi}_\tau|^2 + B(\tilde{\phi}_\tau, \tilde{\phi}_\tau) = 0,$$

where

$$B(\tilde{\phi}, \tilde{\phi}) = \int_{\mathbb{R}^n} \left[ |(-\Delta)^s \tilde{\phi}|^2 - pU^{p-1}(y) |\tilde{\phi}|^2 \right] dy.$$
Hence we may assume \( \tilde{\phi} \) and 

\[
\int_{\mathbb{R}^n} \phi(y, \tau) \cdot Z_j(y) \, dy = 0 \quad \text{for all} \quad \tau \in (-\infty, 0), \; j = 0, 1, \ldots, n + 1, \; B(\tilde{\phi}, \tilde{\phi}) \geq 0.
\]

Also, we have \( \int_{\mathbb{R}^n} |\phi_\tau|^2 = -\frac{1}{2} \partial_\tau B(\tilde{\phi}, \tilde{\phi}) \). From these relations, \( \partial_\tau \int_{\mathbb{R}^n} |\phi_\tau|^2 \leq 0 \), \( \int_{-\infty}^{0} d\tau \int_{\mathbb{R}^n} |\phi_\tau|^2 < +\infty \). Hence \( \phi_\tau = 0 \). So \( \tilde{\phi} \) is independent of \( \tau \) and \( L_0[\tilde{\phi}] = 0 \).

Since \( \tilde{\phi} \) is bounded, by the nondegeneracy of \( L_0 \) (see, \([47]\)), \( \tilde{\phi} \) is a linear combination of \( Z_j, \; j = 1, \ldots, n + 1 \). But \( \int_{\mathbb{R}^n} \tilde{\phi} \cdot Z_j = 0, \; j = 1, \ldots, n, \; \tilde{\phi} = 0 \), a contradiction. Thus (4.102) holds.

From (4.101), there exists a certain \( y_k \) with \( |y_k| \to +\infty \) such that

\[
(\tau_2^k)^v (1 + |y_k|^\alpha |\phi_k(y_k, \tau_2^k)|) \geq \frac{1}{2}.
\]

Let

\[
\tilde{\phi}_k(z, \tau) := (\tau_2^k)^v |y_k|^\alpha \phi_k(y_k + |y_k|z, |y_k|^{2s} \tau + \tau_2^k),
\]

then

\[
\partial_\tau \tilde{\phi}_k = -(-\Delta)^s \tilde{\phi}_k + a_k \phi_k + \tilde{h}_k(z, \tau),
\]

where

\[
\tilde{h}_k(z, \tau) = (\tau_2^k)^v |y_k|^{2s+\alpha} h_k(y_k + |y_k|z, |y_k|^{2s} \tau + \tau_2^k).
\]

By the assumption on \( h_k \), one has

\[
|\tilde{h}_k(z, \tau)| \lesssim \alpha \mu |\tilde{\phi}_k + z|^{-2s-a} (|\tau_2^k|^{-1} |y_k|^{2s} \tau + 1)^{-v}
\]

with \( \tilde{\phi}_k \to \phi \). Thus \( \tilde{h}_k(z, \tau) \to 0 \) uniformly on compact subsets of \( \mathbb{R}^n \setminus \{\hat{e}\} \times (-\infty, 0) \) and \( a_k \) has the same property. Moreover, \( |\tilde{\phi}_k(0, \tau_0)| \geq \frac{1}{2} \) and

\[
|\tilde{\phi}_k(z, \tau)| \lesssim |\tilde{\phi}_k + z|^{-a} (|\tau_2^k|^{-1} |y_k|^{2s} \tau + 1)^{-v}.
\]

Hence we may assume \( \tilde{\phi}_k \to \tilde{\phi} \neq 0 \) uniformly on compact subsets of \( \mathbb{R}^n \setminus \{\hat{e}\} \times (-\infty, 0) \) with \( \tilde{\phi} \) satisfying

\[
\tilde{\phi}_k = -(-\Delta)^s \tilde{\phi} \quad \text{in} \quad \mathbb{R}^n \setminus \{\hat{e}\} \times (-\infty, 0) \quad (4.104)
\]

and

\[
|\tilde{\phi}(z, \tau)| \leq |z - \hat{e}|^{-a} \quad \text{in} \quad \mathbb{R}^n \setminus \{\hat{e}\} \times (-\infty, 0]. \quad (4.105)
\]
For problem (4.104)-(4.105), we now prove a Liouville type result $\tilde{\phi} \equiv 0$. Indeed, without loss of generality, we consider

$$
\begin{cases}
\tilde{\phi}_t = -(-\Delta)^s \tilde{\phi} & \text{in } (\mathbb{R}^n \setminus \{0\}) \times (-\infty, 0], \\
|\tilde{\phi}| \leq |z|^{-a} & \text{in } (\mathbb{R}^n \setminus \{0\}) \times (-\infty, 0].
\end{cases}
$$

(4.106)

Inspired by [16], we shall construct a supersolution to problem (4.106). Let $\delta > 0$ be an arbitrary fixed constant and

$$u_0(x) := \begin{cases}
|x|^{-a}, & |x| \geq \varepsilon, \\
\varepsilon^{-a}, & |x| \leq \varepsilon.
\end{cases}$$

Here $\varepsilon > 0$ is small enough such that $\delta > \varepsilon^{a-2s-a}$. Define

$$\bar{u}(x,t) := \int_{\mathbb{R}^n} K_s(x-y,t)u_0(y)dy + \frac{\delta}{|x|^{n-2s}},$$

where $K_s(x,t)$ is the heat kernel to the fractional heat operator $\partial_t + (-\Delta)^s$:

$$K_s(x,t) \asymp \frac{t}{(t^2 + |x|^2)^{n+2s}}.$$

It is clear that $\bar{u}(r^2,0) \geq r^{-a}$. Then for all $M > 0$, $\bar{u}(r^2,\tau + M)$ is a supersolution to

$$
\begin{cases}
\tilde{\phi}_t = -(-\Delta)^s \tilde{\phi} & \text{in } \mathbb{R}^n \setminus \{0\} \times [-M,0], \\
|\tilde{\phi}| \leq |z|^{-a} & \text{in } \mathbb{R}^n \setminus \{0\} \times [-M,0].
\end{cases}
$$

Now for $t > 0$ large, we estimate

$$|\bar{u}(x,t)| \lesssim \int_{\mathbb{R}^n} K_s(x-y,t)u_0(y)dy + \frac{\delta}{|x|^{n-2s}}.$$
Direct computations yield that for any fixed $x \neq 0$,

$$
\left| \int_{\mathbb{R}^n} K_s(x-y,t)u_0(y)dy \right| \leq \int_{\mathbb{R}^n} \frac{t}{(t^{\frac{1}{s}} + |x-y|^2)^{\frac{n+2}{2}}} \frac{1}{|y|^a} dy \\
\leq \left( \int_{B_{2|R|}} + \int_{\mathbb{R}^n\setminus B_{2|R|}} \right) \frac{t}{(t^{\frac{1}{s}} + |x-y|^2)^{\frac{n+2}{2}}} \frac{1}{|y|^a} dy \\
\leq t^{-\frac{a}{n}}|x|^{n-a} + t^{-\frac{n}{2}}.
$$

For any fixed $(x, \tau) \in \mathbb{R}^n \times (-\infty, 0]$, we have that

$$
|\tilde{\phi}(x, \tau)| \lesssim (\tau + M)^{-\frac{n}{2}}|x|^{n-a} + (\tau + M)^{-\frac{n}{2}} + \frac{\delta}{|x|^{n-2s}}, \quad \forall M > 0.
$$

Letting $M \to +\infty$, we obtain that $|\tilde{\phi}(x, \tau)| \lesssim \frac{\delta}{|x|^{n-2s}}$. Since $\delta > 0$ is arbitrary, it holds that $\tilde{\phi}(x, \tau) = 0$. The proof is complete.

**Proof of Proposition 4.1.3** First, we consider the problem

$$\begin{cases}
\partial_\tau \phi = -(-\Delta)^s \phi + pU^{p-1}(y)\phi + h(y, \tau) - c(\tau)Z_0, \quad y \in \mathbb{R}^n, \quad \tau \geq \tau_0, \\
\phi(y, \tau_0) = 0, \quad y \in \mathbb{R}^n.
\end{cases}$$

Let $(\phi(y, \tau), c(\tau))$ be the unique solution of the nonlocal initial value problem (4.95). From Lemma 4.1.4, for any $\tau_1 > \tau_0$, we have

$$
|\phi(y, \tau)| \lesssim \tau^{-V}(1 + |y|)^{-a} \|h\|_{2s+a, \tau_1} \text{ for all } \tau \in (\tau_0, \tau_1), \quad y \in \mathbb{R}^n,
$$

$$
|c(\tau)| \leq \tau^{-V}R^\alpha \|h\|_{2s+a, \tau_1} \text{ for all } \tau \in (\tau_0, \tau_1).
$$

By assumption, $\|h\|_{2s+a, \tau_1} < +\infty$ and $\|h\|_{2s+a, \tau_1} \leq \|h\|_{2s+a, \tau_1}$ for an arbitrary $\tau_1$. It follows that

$$
|\phi(y, \tau)| \lesssim \tau^{-V}(1 + |y|)^{-a} \|h\|_{2s+a, \tau_1} \text{ for all } \tau \in (\tau_0, \tau_1), \quad y \in \mathbb{R}^n,
$$

$$
|c(\tau)| \leq \tau^{-V}R^\alpha \|h\|_{2s+a, \tau_1} \text{ for all } \tau \in (\tau_0, \tau_1).
$$
By the arbitrariness of \( \tau_1 \),
\[
|\phi(y, \tau)| \lesssim \tau^{-\gamma} (1 + |y|)^{-\alpha} \|h\|_{2x+a,v, \eta} \text{ for all } \tau \in (\tau_0, +\infty), \ y \in \mathbb{R}^n,
\]
\[
|c(\tau)| \lesssim \tau^{-\gamma} R^a \|h\|_{2x+a,v, \eta} \text{ for all } \tau \in (\tau_0, +\infty).
\]
From the regularity result of [171] and a scaling argument, we get the validity of (4.93) and (4.94).

\[\square\]

The solvability conditions: choice of the parameters \( \lambda \) and \( \xi \).

Denote
\[
\lambda(t) = \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \\ \vdots \\ \lambda_k(t) \end{pmatrix}, \quad \xi(t) = \begin{pmatrix} \xi_1(t) \\ \xi_2(t) \\ \vdots \\ \xi_k(t) \end{pmatrix}, \quad \dot{\lambda}(t) = \begin{pmatrix} \dot{\lambda}_1(t) \\ \dot{\lambda}_2(t) \\ \vdots \\ \dot{\lambda}_k(t) \end{pmatrix}, \quad \dot{\xi}(t) = \begin{pmatrix} \dot{\xi}_1(t) \\ \dot{\xi}_2(t) \\ \vdots \\ \dot{\xi}_k(t) \end{pmatrix}, \quad q = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_k \end{pmatrix}.
\]

First we consider (4.91) in the case \( l = n + 1 \).

**Lemma 4.1.5.** When \( l = n + 1 \), (4.91) is equivalent to
\[
\dot{\lambda}_j + \frac{1}{t} \left( \lambda^T \text{diag} \left( \frac{(2s - 1)\sigma_j b_j^2 - 2s + 1}{n - 4s} \right) P \lambda \right)_j = \Pi_1[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi](t), \quad (4.107)
\]
where the matrix \( P \), the numbers \( \sigma_j > 0 \) and \( b_j > 0 \) are defined in Section 4.1.2.

The right hand side term can be expressed as
\[
\Pi_1[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi](t) = \frac{t_0^{-\gamma}}{R^{a-2s}} \mu_0^{n+1-4s+\sigma}(t) f(t) + \frac{t_0^{-\gamma}}{R^{a-2s}} \Theta \left[ \dot{\lambda}, \dot{\xi}, \mu_0^{n-4s}(t) \lambda, \mu_0^{n-4s}(\xi - q), \mu_0^{n+1-4s+\sigma} \phi \right](t), \quad (4.108)
\]
where \( f(t) \) and \( \Theta \left[ \dot{\lambda}, \dot{\xi}, \mu_0^{n-4s}(t) \lambda, \mu_0^{n-4s}(\xi - q), \mu_0^{n+1-4s+\sigma} \phi \right](t) \) are smooth and bounded functions for \( t \in [t_0, \infty) \). Further, the following estimates hold,
\[
|\Theta[\dot{\lambda}_1]|(t) - |\Theta[\dot{\lambda}_2]|(t) | \lesssim \frac{t_0^{-\gamma}}{R^{a-2s}} |\dot{\lambda}_1(t) - \dot{\lambda}_2(t)|,
\]
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\[
|\Theta[\xi_1](t) - \Theta[\xi_2](t)| \leq \frac{t_0^{-\varepsilon}}{R^{a-2s}} |\xi_1(t) - \xi_2(t)|,
\]
\[
|\Theta[\mu_0^{-4s} \lambda_1](t) - \Theta[\mu_0^{-4s} \lambda_2](t)| \leq \frac{t_0^{-\varepsilon}}{R^{a-2s}} |\lambda_1(t) - \lambda_2(t)|,
\]
\[
|\Theta[\mu_0^{-4s}(\xi_1 - q)](t) - \Theta[\mu_0^{-4s}(\xi_2 - q)](t)| \leq \frac{t_0^{-\varepsilon}}{R^{a-2s}} |\xi_1(t) - \xi_2(t)|,
\]
\[
|\Theta[\mu_0^{-1-4s+\sigma} \phi_1](t) - \Theta[\mu_0^{-1-4s+\sigma} \phi_2](t)| \leq \frac{t_0^{-\varepsilon}}{R^{a-2s}} \|\phi_1(t) - \phi_2(t)\|_{\alpha}.
\]

(4.109)

**Proof.** Suppose \( \phi \) satisfies (4.58). For a fixed \( j \in \{1, \ldots, k\} \), we compute

\[
\int_{B_{2R}} H_j[\phi, \lambda, \xi, \hat{\lambda}, \hat{\xi}](y, t(\tau))Z_{n+1}(y)dy,
\]

where \( H_j \) is given by (4.88). Decompose

\[
\mu_{0,j}^{n+2s} S_{\mu, \xi, j}(\xi_j + \mu_0 y, t)
\]
\[
= \left( \frac{\mu_0 j}{\mu_j} \right)^{n+2s} \left[ \mu_0 j S_1(z, t) + \lambda_j b_j^{2s-1} S_2(z, t) + \mu_j S_3(z, t) \right]_{z=\xi_j + \mu_0 y}
\]
\[
+ \left( \frac{\mu_0 j}{\mu_j} \right)^{n+2s} \mu_0 j [S_1(\xi_j + \mu_0 y, t) - S_1(\xi_j + \mu_j y, t)]
\]
\[
+ \left( \frac{\mu_0 j}{\mu_j} \right)^{n+2s} \lambda_j b_j^{2s-1} [S_2(\xi_j + \mu_0 y, t) - S_2(\xi_j + \mu_j y, t)]
\]
\[
+ \left( \frac{\mu_0 j}{\mu_j} \right)^{n+2s} \mu_j [S_3(\xi_j + \mu_0 y, t) - S_3(\xi_j + \mu_j y, t)],
\]

where

\[
S_1(z) = (b_j \mu_0)^{2s-2} \hat{\lambda}_j
\]
\[
\times \left( Z_{n+1} \left( \frac{z - \xi_j}{\mu_j} \right) + \frac{n - 2s}{2} \alpha_{n,s} \frac{1}{\left( 1 + \left| \frac{z - \xi_j}{\mu_j} \right|^2 \right)^{\frac{n-2s}{2}}} - 2sApU \left( \frac{z - \xi_j}{\mu_j} \right)^{p-1} \right)
\]

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\[-\mu_0^{n-2s-2}pU(y_j)^{n-1}\sum_{i=1}^{k} M_{ij}\lambda_i,\]

\[S_2(z) = (2s - 1)\mu_0^{2s-2}\mu_0 \left( \frac{z - \xi_j}{\mu_j} \right) + \frac{n - 2s}{2} \alpha_{n,s} \frac{1}{\left(1 + \frac{|z - \xi_j|}{\mu_j} \right)^{\frac{n-2s}{2}}} \]

\[+ pU \left( \frac{z - \xi_j}{\mu_j} \right)^{n-2s-1} \]

\[\times \left( -b_j^{n-4s} H(q_j, q_j) + \sum_{i \neq j} b_j \mu_i^{n-2s} G(q_j, q_i) \right) + (2s - 1)B,\]

\[S_3(z) = \mu_j^{2s-2} \alpha_{n,s}(n - 2s) \frac{\xi_j}{\mu_j} \left(1 + \frac{|z - \xi_j|}{\mu_j} \right)^{\frac{n-2s}{2} + 1} + pU \left( \frac{z - \xi_j}{\mu_j} \right)^{n-1} \]

\[\times \left( -\mu_j^{n-2s} \nabla H(q_j, q_j) + \sum_{i \neq j} \mu_{ij}^{n-2s} \mu_i^{n-2s} \nabla G(q_j, q_i) \right) \cdot \left( \frac{z - \xi_j}{\mu_j} \right).\]

By direct computations, we have

\[
\int_{B_{2r}} S_1(\xi_j + \mu_j y) Z_{n+1}(y) dy = (2sA c_1 + c_2)(1 + O(R^{4s-n})) \lambda_j (b_j \mu_0)^{2s-2} \]

\[+ c_1 (1 + O(R^{-2s})) \mu_0^{n-2s-2} \sum_{i=1}^{k} M_{ij}\lambda_i,\]

\[
\int_{B_{2r}} S_2(\xi_j + \mu_j y) Z_{n+1}(y) dy \]

\[= -(2s - 2)\mu_0^{n-2s-1} \frac{2s A c_1 + c_2}{(n-4s) c_{n,s}^{n-4s}} + O(R^{4s-n} + R^{-2s}) \mu_0^{n-2s-1} \]

\[= -(2s - 2)\mu_0^{n-2s-1} \frac{2sc_1}{(n-2s)} + O(R^{4s-n} + R^{-2s}) \mu_0^{n-2s-1},\]
\[
\int_{B_{2R}} S_3(\xi_j + \mu_j y)Z_{n+1}(y)dy = 0 \text{ (by symmetry)}.
\]

Since \(\frac{\mu_0}{\mu} = (1 + \frac{\lambda}{\mu_0})^{-1}\), for any \(l = 1, 2, 3\), we have

\[
\int_{B_{2R}} [S_l(\xi_j + \mu_0 y, t) - S_l(\xi_j + \mu_j y, t)]Z_{n+1}(y)dy
\]
\[
= g(t, \frac{\lambda}{\mu_0})\mu_0^{2s-2}\lambda_j + g(t, \frac{\lambda}{\mu_0})\mu_0^{2s-2}\xi_j + g(t, \frac{\lambda}{\mu_0})\sum_l \mu_0^{-2s-2}\lambda_l + \mu_0^{-2s-1+\sigma} f(t),
\]

where \(f, g\) are smooth and bounded functions such that \(g(\cdot, s) \sim s\) as \(s \to 0\). Thus

\[
c \left( \frac{\mu_j}{\mu_0} \right)^{\frac{n-2s}{2}} \mu_0^{1-2s} \int_{B_{2R}} \frac{n-2s}{2} R_{\Psi_j} \left( \frac{n-2s}{2} - 2s \right) \left( P^{\lambda} \right)_{j+1}^{} \]
\[
+ \int_{B_{2R}} \frac{n-2s}{2} R_{\Psi_j} \left( g(t, \frac{\lambda}{\mu_0})\lambda_j + \xi_j \right) + \int_{B_{2R}} \frac{n-2s}{2} R_{\Psi_j} \left( g(t, \frac{\lambda}{\mu_0})\xi_j \right),
\]

where \(c\) is a positive number, the function \(g\) is smooth, bounded and \(g(\cdot, s) \sim s\) as \(s \to 0\).

Next we compute \(p\left( \frac{n-2s}{2} R_{\Psi_j} \left( 1 + \frac{\lambda}{\mu_0} \right)^{-2s} \int_{B_{2R}} U^{p-1}(\frac{\mu_j}{\mu_0} y)\psi(\xi_j + \mu_0 y, t)Z_{n+1}(y)dy.\)

The principal part is

\[
I := \int_{B_{2R}} U^{p-1}(y)\psi(\xi_j + \mu_0 y, t)Z_{n+1}(y)dy.
\]

Recall

\[
\psi = \psi(\lambda, \xi, \dot{\xi}, \ddot{\xi}, \phi)(y, t).
\]

We have

\[
I = \psi(0, q, 0, 0, 0)(q_j, t) \int_{B_{2R}} U^{p-1}(y)Z_{n+1}(y)dy
\]
\[
+ \int_{B_{2R}} U^{p-1}(y)Z_{n+1}(y)(\psi(0, q, 0, 0, 0)(\xi_j + \mu_0 y, t) - \psi(0, q, 0, 0, 0)(q_j, t))dy
\]
\[
+ \int_{B_{2R}} U^{p-1}(y)Z_{n+1}(y)(\psi(\lambda, \xi, \dot{\xi}, \ddot{\xi}, \phi)(\xi_j + \mu_0 y, t) - \psi(0, q, 0, 0, 0)(\xi_j + \mu_0 y, t))dy
\]
\[
= I_1 + I_2 + I_3.
\]

By (4.59), \(I_1 = \frac{\mu_0}{\mu_0} \mu_0^{n-2s+\sigma} f(t)\) with \(f\) smooth and bounded. By (4.60), \(I_2 =\)
for a smooth and bounded function $g$ satisfying $g(\cdot, s, \cdot) \sim s$ and $g(\cdot, t, \cdot) \sim s$ as $s \to 0$. From the mean value theorem again, we have

$$I_3 = \int_{B_{2R}} U^{n-1}(y) Z_{n+1}(y) \left[ \partial_y \psi[0, q, 0, 0, 0][s^*](\xi_j + \mu_0 y, t) + \partial_y \psi[0, q, 0, 0, 0][s^*](\xi_j + \mu_0 y, t) \right] dy$$

for some $s \in (0, 1)$. Using Proposition 4.1.2, $I_3$ is the sum of terms like

$$\mu_0 \frac{n - a - 1 - e}{2} f(t) \left( \lambda + \xi \right) F[\lambda, \xi, \xi, \psi, \phi](t)$$

and

$$\mu_0 \frac{n - a - 1 - e}{2} f(t) \left( \lambda + \xi \right) F[\lambda, \xi, \xi, \psi, \phi](t),$$

where $f$ is a smooth, bounded function and $F$ is a nonlocal operator satisfying $F[0, q, 0, 0, 0](t)$ bounded.

Now, we consider the terms $B_j[\phi_j], B_j^0[\phi_j]$ and obtain that

$$\int_{B_{2R}} B_j[\phi_j](y, t) Z_{n+1}(y) dy = \frac{t_0^{-e}}{R^{a-2s}} \mu_0^{n+1-4s+\sigma}(t) \ell[\phi](t) + \xi_j \ell[\phi](t),$$

$$\int_{B_{2R}} B_j^0[\phi_j](y, t) Z_{n+1}(y) dy = \frac{t_0^{-e}}{R^{a-2s}} \mu_0^{n-2s-1} g \left( \frac{\lambda}{\mu_0} \right) \ell[\phi](t)$$

do for a smooth function $g(s)$ satisfying $g(s) \sim s$ as $s \to 0$, $\ell[\phi](t)$ is smooth and bounded in $t$. Combining the above estimates, we conclude the result. □

Similarly, we compute

$$\int_{B_{2R}} H_j[\lambda, \xi, \lambda, \xi, \phi](y, t(\tau)) Z(t) dy$$

for any $j = 1, \ldots, k, l = 1, \ldots, n$. We have

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Lemma 4.1.6. For $j = 1, \ldots, k$, $l = 1, \ldots, n$, (4.91) is equivalent to

$$\dot{\xi}_j = \Pi_2, j[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi](t),$$

(4.111)

$$\Pi_2, j[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi](t) = \mu_0^{n-4s+2} c \left[ b_j^{n-2s} \nabla H(q_j, q_j) - \sum_{i \neq j} b_j^{n-2s} b_i^{n-2s} \nabla G(q_j, q_i) \right] + \mu_0^{n-4s+2+\sigma} (t) f_j(t)$$

$$+ \frac{\rho}{R^{a-2s}} \Theta[\lambda, \dot{\lambda}, \dot{\xi}, \mu_0^{n-2s-2}(t) \lambda, \mu_0^{n-2s-1}(\xi - q), \mu_0^{n+1-4s+\sigma} \phi](t),$$

where $c = \int_{\mathbb{R}^n} U^{p-1} \frac{\partial U}{\partial y_1} dy$, $f_j(t)$ is an $n$ dimensional vector function which is smooth and bounded for $t \in [t_0, \infty)$. The function $\Theta$ has the same properties as in Lemma 4.1.5.

The proof of Lemma 4.1.6 is similar to that of Lemma 4.1.5 so we omit it.

From Lemma 4.1.5 and Lemma 4.1.6, we know that the orthogonality conditions

$$\int_{B_{2R}} H_l[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi](y, t(\tau)) Z_l(y) dy \text{ for } j = 1, \ldots, k \text{ and } l = 1, \ldots, n + 1,$$

are equivalent to the system of ODEs for $\lambda$ and $\xi$

$$\begin{cases}
\dot{\lambda}_j + \frac{1}{t} \left( P^T \text{diag} \left( \frac{n-2s}{2s} \sigma, b_r^{2-2s} + 1 \right) P \right) \lambda_j = \Pi_1[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi](t), \\
\dot{\xi}_j = \Pi_2, j[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi](t), \quad j = 1, \ldots, k.
\end{cases}$$

System (4.112) is solvable for parameters $\lambda$ and $\xi$ satisfying (4.55) and (4.56). Indeed, we have

Proposition 4.1.4. There exists a solution $\lambda = \lambda[\phi](t), \xi = \xi[\phi](t)$ to (4.112) satisfying (4.55) and (4.56). For $t \in (t_0, \infty)$, it holds that

$$\mu_0^{-(1+\sigma)}(t) \left| \lambda[\phi_1](t) - \lambda[\phi_2](t) \right| \lesssim \frac{\rho}{R^{a-2s}} \| \phi_1 - \phi_2 \|_{n-2s+\sigma, a},$$

(4.113)
\[ \mu_0^{-1}\sigma(t)|\xi[\phi_1](t) - \xi[\phi_2](t)| \lesssim \frac{t_0^{-\epsilon}}{R^{n-2s}} \| \phi_1 - \phi_2 \|_{n-2s+\sigma,a}. \]  

**Proof.** Let \( h \) be a vector function with \( \| h \|_{n+1-4s+\sigma} \lesssim \frac{1}{R^{n-2s}} \). The solution to

\[
\hat{\lambda}_j + \frac{1}{t} \left( P^T \text{diag} \left( \frac{n-2s}{2s} \bar{\sigma}_j b_r^{2-2s} + 1 \right) P \Lambda \right)_j = h(t)_j \tag{4.115}
\]

can be expressed as

\[
\lambda(t) = P^T v(t), \quad v(t) = (v_1(t), v_2(t), \ldots, v_k(t))^T, \quad v_j(t) = t^{-\frac{n-2s}{n-4s}} \left[ d_j + \int_0^t \frac{1}{\tau} \left( \frac{n-2s}{n-4s} (Ph)_j(\tau) d\tau \right) \right], \tag{4.116}
\]

where \( d_j, j = 1, \ldots, k \) are arbitrary constants. Then, for \( 0 \leq d := \max_{i=1,\ldots,k} |d_i| \), we have

\[
\| t^{\frac{\sigma}{n-4s}} \lambda(t) \|_{L^\infty(t_0,\infty)} \lesssim t_0^{-\frac{\sigma}{n-4s}} d + \| h \|_{n+1-4s+\sigma},
\]

\[
\| \dot{\lambda}(t) \|_{n+1-4s+\sigma} \lesssim t_0^{-\frac{\sigma}{n-4s}} d + \| h \|_{n+1-4s+\sigma}.
\]

Let \( \Lambda(t) = \hat{\lambda}(t) \), then

\[
\Lambda + \frac{1}{t} \left( P^T \text{diag} \left( \frac{n-2s}{2s} \bar{\sigma}_j b_r^{2-2s} + 1 \right) P \int_t^\infty \Lambda(s) ds = h(t), \tag{4.117}
\]

which defines a linear operator \( \mathcal{L}_1 : h \rightarrow \Lambda \) associating to any \( h \) with \( \| h \|_{n+1-4s+\sigma} \) bounded the solution \( \Lambda \). \( \mathcal{L}_1 \) is continuous between the spaces \( L^\infty(t_0,\infty)^k \) with the \( \| \cdot \|_{n+1-4s+\sigma} \) topology.

For any \( h : [t_0,\infty) \rightarrow \mathbb{R}^k \) with \( \| h \|_{n+1-4s+\sigma} \) bounded, the solution to

\[
\dot{x}_j = \mu_0^{n-4s+2} c \left[ b_j^{n-2s} \nabla H(q_j, q_j) - \sum_{i \neq j} b_j^{n-2s} b_i^{n-2s} \nabla G(q_j, q_i) \right] + h(t) \tag{4.118}
\]

is given by

\[
x_j(t) = x_j(0) + \int_t^\infty h(s) ds, \tag{4.119}
\]

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where
\[ \dot{\xi}_j(t) = q_j + c \left[ -b_j^{n-2} \nabla H(q_j, q_j) + \sum_{i \neq j} b_{ij}^{n-2} b_i^{n-2} \nabla G(q_j, q_i) \right] \int_0^\infty \mu_0^{n-4s+2}(s) ds. \]

Then we have
\[ |\dot{\xi}_j(t) - q_j| \lesssim t^{-\frac{2}{n-2}} + t^{-\frac{1+\sigma}{n-2}} \|h\|_{n+1-4s+\sigma}, \]
\[ \|\dot{\xi}_j - \dot{\xi}_0\|_{n+1-4s+\sigma} \lesssim \|h\|_{n+1-4s+\sigma}. \]

Let \( \Xi(t) = \dot{\xi}(t) - \dot{\xi}^0 \) which is a vector function, then (4.119) defines a linear operator \( \mathcal{L}_2 : h \rightarrow \Xi \) which is continuous in the \( \| \cdot \|_{n+1-4s+\sigma} \)-topology.

Observe that \( (\lambda, \xi) \) is a solution of (4.112) if \( (\Lambda = \dot{\lambda}, \Xi = \dot{\xi} - \dot{\xi}^0) \) is a fixed point for the problem
\[ (\Lambda, \Xi) = \mathcal{A}(\Lambda, \Xi), \tag{4.120} \]
where
\[ \mathcal{A} := (\mathcal{L}_1(\dot{\Pi}_1[\Lambda, \Xi, \phi], \mathcal{L}_2(\dot{\Pi}_2[\Lambda, \Xi, \phi])), (\tilde{A}_1(\Lambda, \Xi), \tilde{A}_2(\Lambda, \Xi))) \]
with
\[ \dot{\Pi}_1[\Lambda, \Xi, \phi] := \Pi_1 \left[ \int_t^\infty \Lambda, q + \int_t^\infty \Xi, \Lambda, \Xi, \phi \right], \]
\[ \dot{\Pi}_2[\Lambda, \Xi, \phi] := \Pi_2 \left[ \int_t^\infty \Lambda, q + \int_t^\infty \Xi, \Lambda, \Xi, \phi \right]. \]

Let
\[ K := R^{2s-2s} \max \{ \|f\|_{n+1-4s+\sigma}, \|f_1\|_{n+1-4s+\sigma}, \ldots, \|f_k\|_{n+1-4s+\sigma} \}, \]
where \( f, f_1, \ldots, f_k \) are defined in Lemma 4.1.5 and Lemma 4.1.6. Now, we show that problem (4.120) has a fixed point \( (\Lambda, \Xi) \) in the following space
\[ \mathcal{B} = \left\{ (\Lambda, \Xi) \in L^\infty(t_0, \infty) \times L^\infty(t_0, \infty) : \right. \]
\[ \left. \|\Lambda\|_{n-2s-1+(2s-1)\sigma} + \|\Xi\|_{n-2s-1+(2s-1)\sigma} \leq \frac{cK}{R^{s-2s}} \right\} \]

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for suitable \( c > 0 \). Indeed, from (4.108) we have

\[
\left| t^{\frac{n+1-4s+4\sigma}{n-4s}} \bar{A}_1(\Lambda, \Xi) \right| \lesssim t_0^{\frac{\sigma - \sigma_d}{n-4s}} d + \frac{1}{R^{a-2s}} \| \phi \|_n \| -2s + \sigma, \mu + \frac{K}{R^{a-2s}} \\
+ \frac{t_0^{-\varepsilon}}{R^{a-2s}} \| \Lambda \|_{n+1-4s+\sigma} + \frac{t_0^{-\varepsilon}}{R^{a-2s}} \| \Xi \|_{n+1-4s+\sigma}.
\]

Thus, for \( d \) satisfying \( t_0^{\frac{\sigma - \sigma_d}{n-4s}} d < \frac{K}{R^{a-2s}} \) and the constant \( c \) chosen sufficiently large, \( \mathcal{A}(B) \subset B \). As for the Lipschitz property of \( \mathcal{A} \), we have

\[
t^{\frac{n+1-4s+4\sigma}{n-4s}} | \bar{A}_1(\Lambda_1, \Xi) - \bar{A}_1(\Lambda_2, \Xi) | \\
= t^{\frac{n+1-4s+4\sigma}{n-4s}} | \mathcal{L}_1(\bar{\Pi}_1[\Lambda_1, \Xi, \phi] - \bar{\Pi}_1[\Lambda_2, \Xi, \phi]) | \\
\leq t^{\frac{n+1-4s+4\sigma}{n-4s}} t_0^{-\varepsilon} | \mathcal{L}_1(\bar{\Theta}_1(\Lambda_1, \Xi) - \bar{\Theta}_1(\Lambda_2, \Xi)) | \\
+ t^{\frac{n+1-4s+4\sigma}{n-4s}} t_0^{-\varepsilon} | \mathcal{L}_1(\bar{\mu}^{n-2s-2n}_0(\Lambda_1, \Xi) - \bar{\mu}^{n-2s-2n}_0(\Lambda_2, \Xi)) | \\
\leq t_0^{-\varepsilon} \| \Lambda_1 - \Lambda_2 \|_{n+1-4s+\sigma}.
\]

The same estimate holds for \( | \bar{A}_1(\Lambda_1, \Xi) - \bar{A}_1(\Lambda, \Xi) | \). Thus, we have

\[
\| \mathcal{A}(\Xi_1) - \mathcal{A}(\Xi_2) \|_{n+1-4s+\sigma} \leq t_0^{-\varepsilon} \| \Lambda_1 - \Lambda_2 \|_{n+1-4s+\sigma}.
\]

Since \( t_0^{-\varepsilon} < 1 \) when \( t_0 \) is large enough, \( \mathcal{A} \) is a contraction map. Hence, from the Contraction Mapping Theorem, there exists a solution to system (4.112) with \( \lambda, \xi \) satisfying (4.55) and (4.56).

To prove (4.113) and (4.114), we observe that \( \dot{\lambda} = \lambda [ \phi_1 ] - \lambda [ \phi_2 ] \) and \( \dot{\xi} = \xi [ \phi_1 ] - \xi [ \phi_2 ] \) satisfy

\[
\dot{\lambda} + \frac{1}{t} \left( P^T \text{diag} \left( \frac{n-2s}{2s} \sigma D_s^2 + 1 \right) P \right) \lambda = \bar{\Pi}_1(t), \quad \dot{\xi}_j = \bar{\Pi}_{2,j}(t), \; j = 1, \ldots, k,
\]

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where

\[
(\bar{\Pi}_1(t))_j = cp\mu_j^{\frac{n-2s}{s}} \mu_0^{1-2s} \int_{B_{2r}} U^{p-1} \left( \frac{\mu_0}{\mu_j} \right) [\psi[\phi] - \psi[\phi_2]] (\xi_j + \mu_{0j}y, t) Z_{n+1}(y) dy
\]

Then (4.113) and (4.114) follow from (4.109). This completes the proof.

4.1.6 Gluing: Proof of Theorem 4.1.1

After we have chosen parameters \( \lambda = \lambda [\phi] \) and \( \xi = \xi [\phi] \) such that the orthogonality conditions (4.91) hold, we only need to solve problem (4.89) in the class of functions with \( \|\phi\|_{a,v} \) (or equivalently \( \|\phi\|_{n-2s+\sigma,a} \)) bounded. With the chosen parameters, we can apply Proposition 4.1.3 which states that there exists a linear operator \( T \) associating any function \( h(y, \tau) \) with \( \|h\|_{2s+a,v} \)-bounded to (4.92). Thus problem (4.89) is reduced to a fixed point problem

\[
\phi = (\phi_1, \ldots, \phi_k) = \mathcal{A}(\phi) := (\mathcal{T}(H_1[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi]), \ldots, \mathcal{T}(H_k[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \phi])).
\] (4.121)
We claim that, for each $j = 1, \ldots, k$, there hold

\[
(1 + |y|^n) \left[ H[\lambda, \xi, \hat{\lambda}, \hat{\xi}, \phi](\cdot, t) \right] \eta_{B_i(0)} \chi_{B_{2n}(0)}(y) + \left| H[\lambda, \xi, \hat{\lambda}, \hat{\xi}, \phi](y, t) \right| 
\lesssim t_0^{-c} \frac{\mu_{0}^{n-2s+\sigma}}{1 + |y|^{2s+\sigma}},
\]

(4.122)

(1 + |y|^n) \left[ H[\phi^{(1)}](\cdot, t) - H[\phi^{(2)}](\cdot, t) \right] \eta_{B_i(0)} \chi_{B_{2n}(0)}(y) + \left| H[\phi^{(1)}] - H[\phi^{(2)}] \right|(y, t)
\lesssim t_0^{-c} \|\phi^{(1)} - \phi^{(2)}\|_{n-2s+\sigma, a},
\]

(4.123)

From (4.122) and (4.123), $\mathcal{J}$ has a fixed point $\phi$ within the set of functions

\[
\|\phi\|_{n-2s+\sigma, a} \leq ct_0^{-c}
\]

for some large positive constant $c$. This proves the existence part of Theorem 4.1.1.

Estimate (4.122) is obtained from the definition of $H_j$, Lemma 4.1.2 and (4.59). As for (4.123), from (4.113) and (4.114), we have

\[
\mu_{0j}^{\frac{n-2s}{2}} \left| S_{\mu_j, \xi_j, \mu_j, \xi_j}(\xi_j, (\xi_j, 1 + \mu_{0j}y, t) - S_{\mu_{j2}, \xi_{j2}, \mu_{j2}, \xi_{j2}}(\xi_{j2}, (\xi_{j2}, 1 + \mu_{0j}y, t) \right|
\lesssim t_0^{-c} \frac{\mu_{0}^{n-2s+\sigma}(t)}{1 + |y|^{2s+\sigma}} \|\phi^{(1)} - \phi^{(2)}\|_{n-2s+\sigma, a},
\]

where

\[
\mu_i = \mu[\phi^{(i)}], \quad \xi_i = \xi[\phi^{(i)}], \quad \xi_{j,i} = \xi_j[\phi^{(i)}], \quad i = 1, 2.
\]

By Proposition 4.1.2, it holds that

\[
p_{\mu_{0j}}^{\frac{n-2s}{2}} \left| \frac{\mu_{0j}^{\frac{n-2s}{2}}}{\mu_{j1}^{\frac{n-2s}{2}}} U^{p-1} \left( \frac{\mu_{0j}^{\frac{n-2s}{2}}}{\mu_{j1}^{\frac{n-2s}{2}}} \right) \psi[\phi^{(1)}](\xi_{j1}, (\xi_{j1}, 1 + \mu_{0j}y, t) \right|
- \frac{\mu_{0j}^{\frac{n-2s}{2}}}{\mu_{j2}^{\frac{n-2s}{2}}} U^{p-1} \left( \frac{\mu_{0j}^{\frac{n-2s}{2}}}{\mu_{j2}^{\frac{n-2s}{2}}} \right) \psi[\phi^{(2)}](\xi_{j2}, (\xi_{j2}, 1 + \mu_{0j}y, t) \right|
\lesssim t_0^{-c} \frac{\mu_{0j}^{n-2s+\sigma}(t)}{1 + |y|^{2s+\sigma}} \|\phi^{(1)} - \phi^{(2)}\|_{n-2s+\sigma, a},
\]

where

\[
\mu_{j,i} = \mu_j[\phi^{(i)}], \quad \psi[\phi^{(i)}] = \Psi[\lambda_i, \xi_i, \hat{\lambda}_i, \hat{\xi}_i, \phi^{(i)}], \quad i = 1, 2.
\]
Finally, from the definitions (4.42) and (4.43) in Section 4.1.3,
\[ \left| B_j[\phi_j^{(1)}] - B_j[\phi_j^{(2)}] \right| \lesssim t_0^{-\frac{n-2s+\sigma(t)}{1+|y|^{2s+a}}} \| \phi^{(1)} - \phi^{(2)} \| \| n_{-2s+\sigma,a} \| \]
\[ \left| B^0_j[\phi_j^{(1)}] - B^0_j[\phi_j^{(2)}] \right| \lesssim t_0^{-\frac{n-2s+\sigma(t)}{1+|y|^{2s+a}}} \| \phi^{(1)} - \phi^{(2)} \| \| n_{-2s+\sigma,a} \| \]
hold. This proves the estimate (4.123).

The stability part of Theorem 4.1.1 is the same as [43], so we omit it. \(\square\)

4.2 Finite time blow-up for the critical heat equation with fractional Laplacian

4.2.1 Introduction

In this section, we consider the fractional heat equation with the critical exponent
\[ \begin{cases} 
  u_t + (-\Delta)^s u = u \frac{\alpha_n}{n+2s} & \text{in } \mathbb{R}^n \times (0, T), \\
  u(x, 0) = u_0(x) & \text{in } \mathbb{R}^n. 
\end{cases} \tag{4.124} \]

Throughout this section, we assume that \(4s < n < 6s\), \(0 < s < 1\) and a function \(Z_0^* \in C^\infty_0(\mathbb{R}^n)\) is chosen such that \(Z_0^*(q_j) < 0\), where \(q_j (j = 1, \ldots, k)\) are distinct \(k\) points. It is well known that
\[ U(y) = \alpha_{n,s} \left( \frac{1}{1+|y|^2} \right)^{\frac{n-2s}{2}} \]
is the bubble solving the fractional Yamabe problem
\[ (-\Delta)^s U = U^{\frac{n+2s}{n-2s}} \text{ in } \mathbb{R}^n, \]
where \(\alpha_{n,s}\) is a constant depending only on \(n\) and \(s\). See, for instance [1, 29, 129].

The scaled bubble is defined as
\[ U_{\lambda, \xi} := \lambda^{-\frac{n-2s}{2}}(t) U \left( \frac{x - \xi(t)}{\lambda(t)} \right). \tag{4.125} \]
We show the existence of finite time blow-up for the fractional critical heat equation (4.124).

**Theorem 4.2.1.** Assume that $4s < n < 6s$, $0 < s < 1$ and $Z_0^*(q_j) < 0$ for $k$ distinct points $q_j$ ($j = 1, \ldots, k$). For $T$ sufficiently small, there exists an initial condition $u_0$ such that the solution $u(x, t)$ to problem (4.124) blows up at $q_1, \ldots, q_k$ at finite time $T$. Furthermore, the solution takes the form

$$u(x, t) = \sum_{j=1}^{k} U_{\lambda_j(t), \xi_j(t)}(x) + Z_0^*(x) + \Theta(x, t),$$

where $U_{\lambda_j(t), \xi_j(t)}(x)$ is the scaled bubble defined in (4.125),

$$\lambda_j(t) \to 0, \quad \xi_j(t) \to q_j \text{ as } t \to T,$$

$$\|\Theta\|_{L^\infty} \leq T^c \text{ for some constant } c > 0.$$ More precisely,

$$\lambda_j(t) = \kappa_j(T - t)^{\frac{2}{n-2s}} (1 + o(1))$$

for some positive constants $\kappa_j > 0$, $j = 1, \ldots, k$.

**Remark 4.2.1.** Theorem 4.2.1 implies that

- For $n = 4$, finite time blow-up takes place for $s \in (2/3, 1)$

- For $n = 5$, finite time blow-up takes place for $s \in (5/6, 1)$

which is a continuation of the local cases $n = 4$, $s = 1$ in [70, 167] and $n = 5$, $s = 1$ in [63, 104]. Also, our construction suggests that no finite time blow-up of this type should exist in higher dimension case $n \geq 6$, $s \in (0, 1)$. Note that type II blow-up for the case $n = 6$, $s = 1$ has recently been constructed in [105].

The proof of Theorem 4.2.1 is close in spirit to [63], which is mainly based on inner-outer gluing method. However, in a central step that the linear theory for the associated linear problem of the inner problem is required, the ODE techniques are no longer applicable in the fractional setting. Instead, we shall use the fractional linear theory developed in Proposition 4.1.3 by using a blow-up argument.
By our construction, finite time blow-up also exists on the bounded domain \( \Omega \subset \mathbb{R}^n \). Suppose a smooth function \( Z^* \in L^\infty(\Omega) \) satisfies \( Z^*(q_j) < 0 \) for given \( k \) distinct points \( q_1, \ldots, q_k \). For the fractional critical heat equation on bounded domain \( \Omega \subset \mathbb{R}^n \)

\[
\begin{align*}
  u_t + (-\Delta)^s u &= u^{\frac{n+2s}{n-2s}} \quad \text{in } \Omega \times (0, T), \\
  u(x, 0) &= u_0(x) \quad \text{in } \Omega,
\end{align*}
\]

(4.126)

finite time blow-up exists and we have the following Theorem.

**Theorem 4.2.2.** Assume that \( 4s < n < 6s, 0 < s < 1 \) and \( Z^*(q_j) < 0 \) for \( k \) distinct points \( q_j \) \( (j = 1, \ldots, k) \). For \( T \) sufficiently small, there exists an initial condition \( u_0 \) such that the solution \( u(x,t) \) to problem (4.126) blows up at \( q_1, \ldots, q_k \) at finite time \( T \). Moreover, at main order, the solution takes the form

\[
u(x, t) = \sum_{j=1}^{k} U_{\lambda_j(t), \xi_j(t)}(x) + Z^*(x) + \Upsilon(x, t),\]

where \( U_{\lambda_j(t), \xi_j(t)}(x) \) is the scaled bubble defined in (4.125),

\[
\lambda_j(t) \to 0, \quad \xi_j(t) \to q_j \quad \text{as } t \to T,
\]

\[
\|\Upsilon\|_{L^\infty} \leq T^c \quad \text{for some constant } c > 0. \]

More precisely,

\[
\lambda_j(t) = v_j(T - t)^{\frac{2}{n-s}} (1 + o(1))
\]

for some positive constants \( v_j > 0, j = 1, \ldots, k. \)

The proof of Theorem 4.2.2 can be carried out similarly as that of Theorem 4.2.1. So we shall only prove Theorem 4.2.1 in this section.

This section is organized as follows. In Section 4.2.2, we construct an approximate solution and compute its error. In Section 4.2.3, the main parts of the parameters \( \lambda \) and \( \xi \) are given. In Section 4.2.4, we develop linear theory for the inner and outer problems. Finally, we shall prove Theorem 4.2.1 in Section 4.2.5. In the sequel, we shall use the symbol “ \( \lesssim \) ” to denote “ \( \leq C \) ” for a positive constant \( C \) independent of \( t \) and \( T \), and \( C \) may change from line to line.
4.2.2 Approximate solution and error estimate

In this section, we shall choose the approximate solution and compute its error. For simplicity, we consider one bubble case. The multiple-bubble case is similar up to some minor modifications which we will point out if necessary.

Our first approximate solution is

\[ w = U_{\lambda, \xi} + Z^*, \]

where \( U_{\lambda, \xi} \) is defined in (4.125), and \( Z^* \) is the solution to the fractional heat equation

\[
\begin{aligned}
Z_t^* + (-\Delta)^s Z^* &= 0 \quad \text{in } \mathbb{R}^n \times (0,T), \\
Z^*(x,0) &= Z_0^*(x) \quad \text{in } \mathbb{R}^n.
\end{aligned}
\]

Here,

\[
\lambda(t) = \lambda_0(t) + \lambda_1(t), \quad \xi(t) = \xi_0(t) + \xi_1(t),
\]

where \( \lambda_0(t) \) and \( \xi_0(t) \) are the main order terms of \( \lambda(t) \) and \( \xi(t) \) respectively, and \( \lambda_1(t) \) and \( \xi_1(t) \) are the reminder terms which are comparatively smaller than \( \lambda_0(t) \) and \( \xi_0(t) \) respectively. Define the error

\[
S(u) = -u_t - (-\Delta)^s u + u^p,
\]

where \( p := \frac{n+2s}{n-2s} \). Direct computations imply that

\[
S(w) = \lambda - \frac{n+2s}{n-2s} \lambda Z_{n+1}(y) + \lambda \frac{n+2s}{n-2s} \xi \cdot \nabla U(y) + (U_{\lambda, \xi} + Z^*)^p - U_{\lambda, \xi}^p,
\]

where \( y = \frac{x-\xi}{\lambda} \), \( Z_{n+1} = \frac{n-2s}{2s} U(y) + y \cdot \nabla U(y) \).

We look for a solution of the following form

\[ u = U_{\lambda, \xi} + Z^* + \varphi. \]

Then \( S(u) = 0 \) yields that

\[
S(u) = -\varphi_t - (-\Delta)^s \varphi + pu^p_{\lambda, \xi} (\varphi + Z^*) + \lambda \frac{n+2s}{n-2s} \varphi + N(\varphi + Z^*) = 0,
\]

(4.128)
where
\[ E(y,t) = \lambda^{2s-1} \dot{\lambda} Z_{n+1}(y) + \lambda^{2s-1} \xi \cdot \nabla U, \quad (4.129) \]
\[ N(\varphi + Z^*) = (U_{\lambda^s \xi^s}^p + \varphi + Z^*) - U_{\lambda^s \xi^s}^{p-1} (\varphi + Z^*). \quad (4.130) \]

We look for perturbation consisting of inner and outer parts
\[ \varphi(x,t) = \lambda^{-\frac{n-2s}{2}} (t) \eta_R(y) \varphi(y,t) + \psi(x,t), \quad (4.131) \]
where \( R > 0, \eta \) is a smooth cut-off function such that
\[ \eta(s) = \begin{cases} 1, & s < 1, \\ 0, & s > 2, \end{cases} \]
and \( \eta_R = \eta(|y|/R) \). Then we can express \((4.128)\) in terms of \( \varphi \) and \( \psi \)
\[ -\psi_t - (-\Delta)^s \varphi + p \lambda^{-2} (1 - \eta_R) U^{p-1}(y)(\psi + Z^*) + \mathcal{E}(\varphi) \]
\[ + \mathcal{B}(\varphi) + \lambda^{-\frac{n-2-s}{2}} \mathcal{E}(1 - \eta_R) + N(\varphi + Z^*) \]
\[ + \eta_R \lambda^{-\frac{n-2-s}{2}} \left( -\lambda^s \varphi_t - (-\Delta)^s \varphi + p U^{p-1}(y)(\varphi + \lambda \frac{n-2}{2} (\psi + Z^*)) + \mathcal{E} \right) = 0, \]
where
\[ \mathcal{E}(\varphi) := \lambda^{-\frac{n-2-s}{2}} \left[ (-(-\Delta)^s - \partial_t) \eta_R(y) \varphi + \left( -(-\Delta)^{s/2} \eta_R(y), -(\Delta)^{s/2} \varphi(y) \right) \right], \quad (4.132) \]
\[ \mathcal{B}(\varphi) := \lambda^{-\frac{n-2-s}{2}} \left[ \eta_R \lambda \left( \frac{n-2s}{2} \varphi + y \cdot \nabla y \varphi \right) \right] \]
\[ + \eta_R \xi \cdot \nabla y \varphi + \varphi \left( \lambda y \cdot \nabla y \eta_R + \xi \cdot \nabla y \eta_R \right). \quad (4.133) \]

Here
\[ \left[ -(-\Delta)^{s/2} f(x), -(-\Delta)^{s/2} g(x) \right] := c_\eta, s P.V. \int_{\mathbb{R}^n} \frac{[f(y) - f(x)][g(x) - g(y)]}{|x-y|^{n+2s}} dy \quad (4.134) \]
with \( c_{n,s} = \frac{2^s \cdot \Gamma \left( \frac{n+2}{2} \right)}{\Gamma(1-s) \pi^2} \). Therefore,

\[
u = U_\lambda, \xi + Z^* + \lambda^{-\frac{n+2}{2}} \eta_R(y) \phi(y,t) + \psi(x,t)
\]
solves (4.124) if \((\phi(y,t), \psi(x,t))\) solves the inner-outer gluing system

\[
\lambda^{2s} \phi_t = -(-\Delta)^s \phi + pU^{p-1}(y)\phi + \mathcal{H}(\phi, \psi, \lambda, \xi) \quad \text{in} \quad B_{2R}(0) \times (0,T),
\]

\[
\begin{cases}
\psi_t = -(-\Delta)^{\frac{s}{2}} \psi + \mathcal{G}(\phi, \psi, \lambda, \xi) & \text{in} \quad \mathbb{R}^n \times (0,T), \\
\psi(x,0) = 0 & \text{in} \quad \mathbb{R}^n,
\end{cases}
\]

where

\[
\mathcal{H}(\phi, \psi, \lambda, \xi) := \lambda^{\frac{n+2}{2}} pU^{p-1}(y) \left( \psi(\lambda y + \xi, t) + Z^*(\lambda y + \xi, t) \right) + \mathcal{E}(y,t),
\]

\[
\mathcal{G}(\phi, \psi, \lambda, \xi) := p\lambda^{-2s}(1 - \eta_R)U^{p-1}(y) \left( \psi + Z^* \right) + \mathcal{E}(\phi) + \mathcal{H}(\phi)
\]

\[
+ \lambda^{-\frac{n+2}{2}} \mathcal{E}(1 - \eta_R) + N(\phi + Z^*).
\]

### 4.2.3 Choices of \( \lambda_0(t) \) and \( \xi_0(t) \)

We shall choose \( \lambda_0(t) \) and \( \xi_0(t) \) defined in (4.127) in this section. Basically, the inner problem (4.135) will determine the parameter functions \( \lambda \) and \( \xi \) at main order. By the fractional linear theory developed in Section 4.2.4, the inner problem (4.135) will be solved under the orthogonality conditions

\[
\int_{B_{3R}} \mathcal{H}(\phi, \psi, \lambda, \xi) Z_j(y) dy = 0 \quad \text{for all} \quad t \in (0,T), \ j = 1, \ldots, n+1,
\]

where \( R \) is fixed sufficiently large and

\[
Z_j(y) = \partial_y U(y), \quad j = 1, \ldots, n, \quad Z_{n+1}(y) = \frac{n-2}{2} U(y) + y \cdot \nabla U(y).
\]
The leading term of $\mathcal{H}$ is

$$h[\lambda_0, \xi_0] = \lambda_0^{2s-1} \lambda_0 Z_{n+1}(y) + \lambda_0^{2s-1} \dot{\xi}_0 \cdot \nabla U + \lambda_0^{\frac{n-2s}{2}} p U^{p-1}(y) Z_0^*(q) + \lambda_0^{\frac{n-2s}{2} + 1} p U^{p-1}(y) \nabla Z_0^*(q) \cdot y.$$ 

It is reasonable to choose $\lambda_0(t)$, $\xi_0(t)$ such that

$$\int_{\mathbb{R}^n} h[\lambda_0, \xi_0] Z_j(y) dy = 0 \text{ for all } t \in (0, T), \ j = 1, \ldots, n + 1$$

are satisfied. For $j = n + 1$, we have

$$\int_{\mathbb{R}^n} h[\lambda_0, \xi_0] Z_{n+1}(y) dy = \lambda_0^{2s-1} \lambda_0 \int_{\mathbb{R}^n} Z_{n+1}^2(y) dy + \lambda_0^{\frac{n-2s}{2}} p Z_0^*(q) \int_{\mathbb{R}^n} U^{p-1}(y) Z_{n+1}(y) dy$$

and thus

$$c_0 \dot{\lambda}_0 + c_1 Z_0^*(q) \lambda_0^{\frac{n-d+2}{2}} = 0,$$

where $c_0 = \int_{\mathbb{R}^n} Z_{n+1}^2(y) dy$, $c_1 = p \int_{\mathbb{R}^n} U^{p-1}(y) Z_{n+1}(y) dy$. Observer that $c_0$ is well-defined thanks to the assumption $n > 4s$ and $c_2 < 0$. Therefore, in order that $\lambda(t) \to 0$ as $t \to T$, we suppose $Z_0^*(q) < 0$ and then we obtain the main order

$$\lambda_0(t) = a(T - t)^{\frac{2}{4-s}} \quad (4.140)$$

with $a = \left( \frac{2c_0}{c_1 Z_0^*(q)(4s-n)} \right)^{\frac{2}{n-2s}}$.

Similarly, we consider the case $j = 1, \ldots, n$ and get

$$\int_{\mathbb{R}^n} h[\lambda_0, \xi_0] Z_j(y) dy = \lambda_0^{2s-1} \int_{\mathbb{R}^n} \dot{\xi}_0 \cdot \nabla U(y) Z_j(y) dy + \lambda_0^{\frac{n-2s}{2} + 1} p \int_{\mathbb{R}^n} U^{p-1}(y) Z_j(y) \nabla Z_0^*(q) \cdot y dy.$$ 

So we can write $\dot{\xi}_0(t) = \lambda_0^{\frac{n-2s}{2} + 2} v$ for some vector $v$. Hence, by (4.140), we obtain

$$\dot{\xi}_0(t) = q + O(T - t)^{\frac{4}{4-s}} v$$

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for some vector \( \vec{v} \). The remainders \( \lambda_1(t) \) and \( \xi_1(t) \) defined in (4.127) will be chosen when we finally solve the inner-outer gluing system in Section 4.2.5.

### 4.2.4 The linear problems

In this section, we shall get proper a priori estimates of the associated linear problems of the outer problem (4.136) and the inner problem (4.135).

We start from the linear outer problem. We consider the linear problem

\[
\begin{aligned}
\partial_t \psi(x, t) &= -(\Delta)^s \psi(x, t) + f(x, t) & \text{in } \mathbb{R}^n \times (0, T), \\
\psi(x, 0) &= 0 & \text{in } \mathbb{R}^n.
\end{aligned}
\] (4.141)

A solution to the Cauchy problem (4.141) is guaranteed. Recall that the heat kernel to the fractional heat operator \( \partial_t + (-\Delta)^s \) is given by

\[
K_s(x, t) \sim \frac{t}{(t^2 + |x|^2)^{\frac{n+2s}{2}}}. \tag{4.142}
\]

Then by Duhamel’s formula

\[
\psi(x, t) = \int_0^t \int_{\mathbb{R}^n} K_s(x - z, t - r) f(z, r) dz dr
\]
is the solution to (4.141).

Define the norms

\[
\| \psi \|_* := \sup_{(x, t) \in \mathbb{R}^n \times (0, T)} \frac{|\psi(x, t)|}{\rho_*}, \tag{4.143}
\]

\[
\| f \|_{**} := \sup_{(x, t) \in \mathbb{R}^n \times (0, T)} \frac{|f(x, t)|}{\rho_{**}}, \tag{4.144}
\]

where

\[
\rho_* := 1 + \frac{1}{1 + \frac{t - q}{\lambda_0(t)}} \frac{n-4s}{n-2s} \quad \text{and} \quad \rho_{**} := 1 + \frac{\lambda_0^{-2s}(t)}{1 + \frac{x - q}{\lambda_0(t)} \frac{n-4s}{n-2s}}.
\]

We have the following lemma.

**Lemma 4.2.1.** Assume that \( \| f \|_{**} < +\infty \). For sufficiently small \( T > 0 \), the solution
\( \psi \) to problem (4.141) satisfies

\[ \| \psi \|_* \lesssim \| f \|_*^\ast . \]

Proof. It is direct to check by the Duhamel’s formula that \(|\psi(x,t)|\) is bounded as \(|x| \to +\infty\). Now we build a supersolution to problem (4.141) with \( \| f \|_*^\ast < +\infty \).

Let \( p(y) \) be the solution to

\[ (-\Delta)^s p(y) = \frac{1}{1 + |y|^{n-2s}}, \]

where \( y = \frac{x-q}{\lambda_0(t)} \). By the Riesz potential, it is direct to see that

\[ p(y) \sim \frac{1}{1 + |y|^{n-4s}} \text{ as } |y| \to \infty. \]

We let \( \psi_1(x,t) = 2\| f \|_*^\ast p(y) \) and compute

\[ \partial_t \psi_1 + (-\Delta)^s \psi_1 = -2\| f \|_*^\ast \lambda_0 \lambda_0^{-1} y \cdot \nabla p(y) + \frac{2\| f \|_*^\ast \lambda_0^{-2s}}{1 + |y|^{n-2s}} \]

\[ \geq |f| + \frac{\| f \|_*^\ast \lambda_0^{-2s}}{1 + |y|^{n-2s}} - 2\| f \|_*^\ast \lambda_0 \lambda_0^{-1} y \cdot \nabla p(y) - \| f \|_*^\ast . \]

Observe that for some constants \( \gamma, c > 0 \) independent of \( T \) and \( t \)

\[ 2\| f \|_*^\ast \lambda_0 \lambda_0^{-1} y \cdot \nabla p(y) - \frac{\| f \|_*^\ast \lambda_0^{-2s}}{1 + |y|^{n-2s}} \leq \begin{cases} 0 & \text{for } |x-q| \leq c(T-t)^{\frac{1}{\beta}}, \\ \gamma(T-t)^{\beta-1} & \text{for } |x-q| \geq c(T-t)^{\frac{1}{\beta}}, \end{cases} \]

where \( \beta = \frac{(n-2s)(n-4s)}{2s(n-s)} \). We take

\[ \psi_2 = C\| f \|_*^\ast t + \gamma \beta^{-1} [T^\beta - (T-t)^\beta], \quad \psi = \psi_1 + \psi_2, \]

where \( C \) is a sufficiently large constant. Combining (4.145)-(4.147), we conclude that \( \bar{\psi} \) is a supersolution to problem (4.141) and the estimate

\[ \| \psi \|_* \lesssim \| f \|_*^\ast \]
follows immediately from the definitions of the norms (4.143) and (4.144).

**Remark 4.2.2.** For arbitrary $T' < T$, we have

$$
\| f \|_\infty \lesssim \lambda_0^{-2s} (T') \| f \|_{**}.
$$

Then fractional parabolic estimates (see [117] and the references therein) imply the following Hölder estimate

$$
[\psi]_{\alpha, T'} \lesssim \lambda_0^{-2s} (T') \| f \|_{**}
$$

for some $0 < \alpha < 1$. Here the space-time Hölder semi-norm is defined as

$$
[\psi]_{\alpha, T'} := \sup_{x_1, x_2 \in \mathbb{R}^n \atop t_1, t_2 \in [0, T']}
\frac{|\psi(x_1, t_1) - \psi(x_2, t_2)|}{|x_1 - x_2|^\alpha + |t_1 - t_2|^{\alpha/2}}.
$$

In order to solve the inner problem (4.135), we consider the associated linear problem

$$
\lambda^{2s} \phi = -(-\Delta)^s \phi + pU_{p-1}(y) \phi + h(y, t) \text{ in } B_{2R}(0) \times (0, T).
$$

(4.149)

Recall that the linearized operator

$$
L_0 := -(-\Delta)^s + pU_{p-1}
$$

has a only positive eigenvalue $\mu_0$ such that

$$
L_0(Z_0) = \mu_0 Z_0, \ Z_0 \in L^\infty(\mathbb{R}^n),
$$

where the corresponding eigenfunction $Z_0$ is radially symmetric with the asymptotic behavior

$$
Z_0(y) \sim |y|^{-n-2s} \text{ as } |y| \to +\infty,
$$

(4.150)

see [84] for instance. Multiplying equation (4.149) by $Z_0$ and integrating over $\mathbb{R}^n$, we obtain that

$$
\lambda^2(t) \dot{p}(t) - \mu_0 p(t) = q(t),
$$

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where \( p(t) = \int_{\mathbb{R}^n} \phi(y, t)Z_0(y)dy \) and \( q(t) = \int_{\mathbb{R}^n} h(y, t)Z_0(y)dy \). Then we have

\[
p(t) = e^{\int_0^t \mu_\lambda^{-2s} dr} \left( p(0) + \int_0^t q(\eta) \lambda^{-2s}(\eta) e^{-\int_0^\eta \mu_\lambda^{-2s}(r) dr} d\eta \right).
\]

In order to get a decaying solution, the initial condition

\[
p(0) = -\int_0^T q(\eta) \lambda^{-2s}(\eta) e^{-\int_0^\eta \mu_\lambda^{-2s}(r) dr} d\eta
\]

is required. The above formal argument suggests that a linear constraint should be imposed on the initial value \( \phi(y, 0) \). Therefore, we consider the associated linear Cauchy problem of the inner problem (4.135)

\[
\begin{cases}
\phi_\tau = -(-\Delta)^s \phi + pU^{p-1}(y) \phi + h(y, \tau) & \text{in } B_{2R}(0) \times (\tau_0, \infty), \\
\phi(y, \tau_0) = e_0Z_0(y) & \text{in } B_{2R}(0),
\end{cases}
\]

(4.151)

where \( R > 0 \) is fixed sufficiently large, and we have performed the change of variables \( \frac{dT}{dt} = \lambda^{-2s}(t) \). Let \( \nu > 0, \ a > 0 \) such that \( \tau - \nu \sim \lambda^{\frac{n-2s}{2}} \). Define

\[
\|h\|_{a, \nu, \eta} := \sup_{y \in B_{2R}} \tau^{\nu} (1 + |y|^a) (|h(y, \tau)| + (1 + |y|^{\eta}) \chi_{B_{2R}}[h(\cdot, \tau)]_{\eta, B_{2R}}),
\]

(4.152)

where the Hölder semi-norm is defined by

\[
[h(\cdot, \tau)]_{\eta, B_{2R}} := \sup_{x, y \in B_{2R}} \frac{|h(x, \tau) - h(y, \tau)|}{|x - y|^{\eta}}
\]

for \( 0 < \eta < 1 \). In the sequel, we consider \( h = h(y, \tau) \) as a function in the whole space \( \mathbb{R}^n \) with zero extension outside of \( B_{2R} \) for all \( \tau > \tau_0 \). We have the following proposition whose proof is given in Section 4.1.5.

**Proposition 4.2.1.** Assume \( 2s < a < n - 2s, \ \nu > 0, \ \|h\|_{2s+a, \nu, \eta} < +\infty \) and

\[
\int_{B_{2R}} h(y, \tau)Z_j(y)dy = 0, \ \forall \tau \in (\tau_0, \infty), \ j = 1, \ldots, n + 1.
\]

For sufficiently large \( R \), there exist \( \phi = \phi[h](y, \tau) \) and \( e_0 = e_0[h](\tau) \) solving (4.151).
with

\[(1 + |y|^s) \left( \int_{\mathbb{R}^n} \frac{[\phi(y, \tau) - \phi(x, \tau)]^2}{|y - x|^{n + 2s}} \, dx \right)^{\frac{1}{2}} + |\phi(y, \tau)| \lesssim \tau^{-\nu} (1 + |y|)^{-a} \|h\|_{2s + a, \nu, \eta}, \tag{4.153} \]

\[|e_0[h]| \lesssim \|h\|_{2s + a, \nu, \eta}, \tag{4.154} \]

for \((y, \tau) \in \mathbb{R}^n \times (\tau_0, \infty)\).

**Remark 4.2.3.** In the inner problem (4.135), \(\mathcal{H}\) behaves like

\[\mathcal{H} \lesssim \lambda^{\frac{n-2s}{2}} \left( \frac{1}{1 + |y|^{n-2s}} + \frac{1}{1 + |y|^{4s}} \right).\]

Recall that \(4s < n < 6s\). For \(a > 2s\), we get

\[\mathcal{H} \lesssim \frac{\lambda^{\frac{n-2s}{2}}}{1 + |y|^{a+2s}} (\mathcal{R}^{a+4s-n} + \mathcal{R}^{a-2s}) \lesssim \frac{\lambda^{\frac{n-2s}{2}}}{1 + |y|^{a+2s}} \mathcal{R}^{a+4s-n}.\]

Define the \(\| \cdot \|_B\) norm as

\[\|\phi\|_B := \sup_{\tau \in (0, T), y \in B_{2r}} \left\{ \lambda^{-\frac{n-2s}{2}} (1 + |y|^a) \left[ |\phi(y, t)| \right. \right. \right. \]

\[\left. + \left. (1 + |y|^s) \left( \int_{\mathbb{R}^n} \frac{[\phi(y, t) - \phi(x, t)]^2}{|y - x|^{n + 2s}} \, dx \right)^{\frac{1}{2}} \right] \right\}. \tag{4.155} \]

Then by Proposition 4.2.1, we obtain

\[\|\phi\|_B \lesssim \|\mathcal{H}\|_{n-2s, \nu, \eta}.\]

### 4.2.5 Solving the inner-outer gluing system

In this section, we shall solve the inner-outer gluing system (4.135)-(4.136). We shall solve the inner-outer gluing system as a fixed point problem for \(\vec{p} = (\phi, \psi, \lambda, \xi)\) in a proper Banach space.
We define
\[ c_j[\mathcal{H}(\phi, \psi, \lambda, \xi)] := \frac{\int_{B_2R} \mathcal{H}(\phi, \psi, \lambda, \xi) Z_j(y) dy}{\int_{B_2R} |Z_j(y)|^2 dy} \]
and
\[ \bar{\mathcal{H}}(\phi, \psi, \lambda, \xi) := \mathcal{H}(\phi, \psi, \lambda, \xi) - \sum_{j=1}^{n+1} c_j[\mathcal{H}(\phi, \psi, \lambda, \xi)] Z_j. \]

Then the linear theory is automatically applicable to the following problem
\[
\begin{cases}
\lambda^2 \phi_t = -(-\Delta)^s \phi + pU^{p-1}(y)\phi + \bar{\mathcal{H}}(\phi, \psi, \lambda, \xi) & \text{in } B_{2R} \times (0, T), \\
\phi(x, 0) = eZ_0(x) & \text{in } B_{2R}.
\end{cases}
\]
(4.156)

Problem (4.156) can be formulated as the following fixed point problem
\[ \phi = \mathcal{P}_1^{in}[\bar{\mathcal{H}}(\phi, \psi, \lambda, \xi)] := \mathcal{P}_1(\phi, \psi, \lambda, \xi). \]  
(4.157)

If in addition we have
\[ c_j[\mathcal{H}(\phi, \psi, \lambda, \xi)] = 0 \quad \text{for all } j = 1, \ldots, n+1, \]  
(4.158)
we get a true solution to the real inner problem. Similarly, for the outer problem, we look for a fixed point of
\[ \psi = \mathcal{P}_2^{out}[\mathcal{G}(\phi, \psi, \lambda, \xi)] := \mathcal{P}_2(\phi, \psi, \lambda, \xi). \]  
(4.159)

Therefore, the inner-outer gluing system is now reduced to the system (4.157)–(4.159). We shall solve the system by Leray-Schauder degree theory. For \( \theta \in [0, 1] \), we define the homotopy class
\[ \mathcal{H}_\theta(\psi, \lambda, \xi)(y, t) = \lambda^{2s-1} \lambda Z_{n+1}(y) + \lambda^{2s-1} \sum_{j=1}^{n} \hat{\xi}_j Z_j(y) + \lambda^{\frac{n-2s}{2}} pU^{p-1}(y)Z_0(q) + \lambda^{\frac{n-2s}{2}} pU^{p-1}(y)\nabla Z_0(q) \cdot y + \theta \lambda^{\frac{n-2s}{2}} pU^{p-1}(y)[Z'(\lambda y + \xi, t) - Z_0(q)] - \lambda y \cdot \nabla Z_0(q) + \psi(\lambda y + \xi, t). \]
Consider the following system

\[
\begin{cases}
\phi = \mathcal{T}^\text{in}_\lambda \left[ \mathcal{K}_\theta(\phi, \psi, \lambda, \xi) - \sum_{j=1}^{n+1} c_j \mathcal{K}_\theta(\phi, \psi, \lambda, \xi)Z_j \right], \\
c_j \mathcal{K}_\theta(\phi, \psi, \lambda, \xi) = 0 \quad \text{for all } j = 1, \ldots, n+1, \\
\psi = \mathcal{T}^\text{out}[\theta \mathcal{G}(\phi, \psi, \lambda, \xi)].
\end{cases}
\] (4.160)

The case \( \theta = 1 \) corresponds to the original system that we need to solve.

We write

\[
\begin{align*}
\lambda(t) &= \lambda_0(t) + \lambda_1(t), \\
\xi(t) &= q + \xi_1(t), \quad t \in [0, T],
\end{align*}
\]

where \( \lambda_0(t) \) is defined in (4.140) and \( \lambda_1(T) = 0, \xi_1(T) = 0. \)

Suppose that we have a solution \( (\phi, \psi, \lambda_1, \xi_1) \) to system (4.160) satisfying the constraints

\[
|\dot{\lambda}_1(t)| + |\dot{\xi}_1(t)| \leq \delta_0, \quad \|\phi\|_B + \|\psi\|_\infty \leq \delta_1,
\] (4.161)

where \( \delta_0 \) and \( \delta_1 \) are small positive constants to be determined later and the norm \( \|\cdot\|_B \) is defined in (4.155). We also assume that \( \|Z^*_\|_\infty \ll 1 \) independent of \( T \).

From section 4.2.3, the function \( \lambda_0(t) \) solves the equation

\[
\lambda_0(t) \int_{\mathbb{R}^n} Z_{n+1}^2 dy + \lambda_0(t) \frac{n}{n+1} Z_0^*(q) \int_{\mathbb{R}^n} pU^p Z_{n+1} dy = 0. \] (4.162)

The equation

\[
c_{n+1}(H_\theta(\psi, \lambda_0 + \lambda_1, \xi))(t) = 0, \quad t \in [0, T),
\] (4.163)

which corresponds to

\[
0 = \dot{\lambda}(t) \left( \int_{B_{2\sigma}} Z_{n+1}^2 dy \right) + \lambda(t) \frac{n}{n+1} Z_0^*(q) \int_{B_{2\sigma}} pU^p Z_{n+1} dy \\
+ \theta \lambda(t) \frac{n}{n+1} \int_{B_{2\sigma}} pU(y)U^p \left( Z_0^*(\xi + \lambda y, t) - Z_0^*(q) \right) \\
- \lambda t \cdot \nabla Z_0^*(q) + \psi(\xi + \lambda y, t) \right) Z_{n+1}^1(y) dy,
\]

and it can be written as

\[
\dot{\lambda}(t) + \beta \lambda(t) \frac{n}{n+1} = \lambda(t) \frac{n}{n+1} (\delta_\theta + \theta \pi(\psi, \xi, \lambda_1)). \] (4.164)
for a suitable number $\beta > 0$, $\delta_R = O(R^{-2s})$ and the operator $\pi$ satisfies, for some absolute constant $C$,
\[
|\pi(\psi, \xi, \lambda_1)| \lesssim T + \|\psi\|_\infty.
\]
By (4.162), for a suitable $\gamma > 0$, equation (4.164) can be written in the linearized form
\[
\dot{\lambda}_1 + \frac{\gamma}{T-t} \lambda_1 = \left( T - t \right)^{\frac{2}{\pi-\sigma} - 1} g_0(\psi, \lambda_1, \xi, \theta)
\]
with
\[
|g_0(\psi, \lambda_1, \xi, \theta)(t)| \lesssim \|\psi\|_\infty + T + R^{-2s}.
\]
The linear problem
\[
\dot{\lambda}_1 + \frac{\gamma}{T-t} \lambda_1 = \left( T - t \right)^{\frac{2}{\pi-\sigma} - 1} g_0(t), \quad \lambda_1(T) = 0
\]
can be uniquely solved by the operator in $g_0$,
\[
\lambda_1(t) = \mathcal{F}(0)[g_0](t) := -(T - t)^\gamma \int_t^T (T - s)^{\frac{2}{\pi-\sigma} - 1} g_0(s)ds.
\]
It defines a linear operator on $g_0$ with estimates
\[
\|(T - t)^{\frac{2}{\pi-\sigma} + 1} \dot{\lambda}_1\|_\infty + \|(T - t)^{\frac{2}{\pi-\sigma}} \lambda_1\|_\infty \lesssim \|g_0\|_\infty.
\]
Equation (4.163) then becomes
\[
\lambda_1(t) = \mathcal{F}(0)[g_0(\psi, \lambda_1, \xi, \theta)](t) \quad \text{for all} \ t \in [0, T),
\]
and we get
\[
\|(T - t)^{\frac{2}{\pi-\sigma} + 1} \dot{\lambda}_1\|_\infty + \|(T - t)^{\frac{2}{\pi-\sigma}} \lambda_1\|_\infty \lesssim \|\psi\|_\infty + T + R^{-2s} (4.165)
\]
Similarly, equations
\[
c_j[\mathcal{H}(\psi, \lambda, \xi)] = 0 \quad \text{for all} \ j = 1, \ldots, n,
\]

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can be written in vector form as
\[ \xi_1(t) = \mathcal{F}^{(1)}[g_1(\psi, \lambda_1, \xi_1, \theta)](t) \text{ for all } t \in [0, T), \] (4.166)
where
\[ \mathcal{F}^{(1)}[g_1] := \int_t^T (T-s)^{\frac{2}{n-2}} g_1(s) ds \]
and
\[ |g_1(\psi, \lambda_1, \xi_1, \theta)(t)| \lesssim \|\psi\|_\infty + T. \]

From equation (4.166), we have
\[ \|(T-t)^{\frac{2}{n-2}} \xi_1\|_\infty + \|(T-t)^{\frac{2}{n-2}} \dot{\xi}_1\|_\infty \lesssim \|\psi\|_\infty + T. \] (4.167)

On the other hand, we have
\[ |\mathcal{H}(\phi, \psi, \lambda, \xi)(y,t)| \lesssim \lambda_0(t)^{\frac{n+2s}{2}} \left( \|\psi\|_\infty + \|Z^*\|_\infty \right) + \frac{\lambda^{2s-1} \dot{\lambda}}{1 + |y|^{n-2s}} + \frac{\lambda^{2s-1} \dot{\xi}_1}{1 + |y|^{n-2s+1}}, \]
hence for \(a > 2s\), we have
\[ |\mathcal{H}(\phi, \psi, \lambda, \xi)(y,t)| \lesssim \lambda_0(t)^{\frac{n+2s}{2}} R^{a - 4s - n} \left( \|\psi\|_\infty + \|Z^*\|_\infty \right). \]

From (4.160) and Proposition 4.2.1 we obtain that
\[ \|\phi\|_B \lesssim \|\psi\|_\infty + \|Z^*\|_\infty, \] (4.168)
where the norm \(\|\cdot\|_B\) is defined in (4.155). Next we consider the outer problem (4.160) and estimate \(\mathcal{G}(\phi, \psi, \lambda, \xi)\) term by term. It is direct to see that
\[ |p \lambda^{-2s}(1 - \eta_R)(y)\psi + Z^*)| \lesssim \frac{\lambda^{-2s}}{R^{2s-\sigma}} \frac{1}{1 + |y|^{2s+\sigma}} (\|Z^*\|_\infty + \|\psi\|_\infty), \] (4.169)
and
\[ |\lambda^{-\frac{n+2s}{2}} \mathcal{E}(1 - \eta_R)| \lesssim \frac{1}{\lambda^{2s}} \left[ \frac{1}{1 + |y|^{n-2s}} \lambda^{-\frac{n+2s}{2}} (|\lambda^{2s-1} \dot{\lambda}| + |\lambda^{2s-1} \dot{\xi}_1|) \right] \left|_{|y| > 2R} \right. \]
Let us now estimate the term $C(\phi)$. Let us choose $0 < \sigma < n - 4s$. Then we have

$$\lambda^{-\frac{n-2s}{2}} \left| \left[ \left[ \eta_R - \eta_R \right] \right] (x, t) \right| \lesssim \lambda^{-\frac{n-2s}{2}} \left[ \int_{\mathbb{R}^n} \left( \frac{\eta_R(x) - \eta_R(y)}{|x-y|^{\frac{n}{2}+s}} \right)^2 dy \right]^{\frac{1}{2}} \left[ \int_{\mathbb{R}^n} \left( \frac{\phi(x) - \phi(y)}{|x-y|^{\frac{n}{2}+s}} \right)^2 dy \right]^{\frac{1}{2}} \lesssim \lambda^{-2s} R^{\sigma+4s-n} \frac{1}{1 + |y|^{2s+\sigma}} \|\phi\|_B.$$

Similar computations yield that

$$|\mathcal{C}(\phi)| \lesssim \lambda^{-2s} R^{\sigma+4s-n} \frac{1}{1 + |y|^{2s+\sigma}} \|\phi\|_B,$$  \hspace{1cm} (4.171)

$$|\mathcal{R}(\phi)| \lesssim \lambda^{-2s} R^{\sigma+4s-n} \frac{1}{1 + |y|^{2s+\sigma}} \|\phi\|_B.$$  \hspace{1cm} (4.172)

Now for some $\sigma \in (0, n - 4s)$, we have

$$|N(Z^* + \lambda^{-\frac{n-2s}{2}} \eta_R \phi + \psi)| \lesssim \left( \lambda^{-\frac{n-2s}{2}} \|\eta_R \phi\|_\infty \right)^p + (Z^* + \psi)^p \lesssim \lambda^{-2s} R^{\sigma+6s-n} \lambda^{\sigma+2s+\sigma} \|\phi\|_B^p + (\|Z^*\|_\infty + \|\psi\|_\infty)^p.$$  \hspace{1cm} (4.173)

Collecting the above estimates (4.169)–(4.173), we get by using Lemma 4.2.1 that

$$\|\psi\|_\infty \lesssim T^{\sigma'} \|Z^*\|_\infty + R^{-\sigma'} \|\phi\|_B$$  \hspace{1cm} (4.174)

for some positive constant $\sigma'$. By (4.165)–(4.168) and (4.174), we obtain

\[
\begin{align*}
\|\psi\|_\infty &\lesssim T^{\sigma'} \|Z^*\|_\infty, \\
\|\phi\|_B &\lesssim \|Z^*\|_\infty, \\
\|T^{-\frac{n}{2s}+1} \hat{\lambda}_1\|_\infty + \|T^{-\frac{n}{2s}} \hat{\lambda}_1\|_\infty \lesssim T^{\sigma'} (\|Z^*\|_\infty + 1), \\
\|T^{-\frac{n}{2s}+1} \hat{\xi}_1\|_\infty + \|T^{-\frac{n}{2s}} \hat{\xi}_1\|_\infty \lesssim T^{\sigma'} (\|Z^*\|_\infty + 1) + R^{-2s},
\end{align*}
\]  \hspace{1cm} (4.175)
Then the inner-outer gluing system (4.160) can be written in the form

\[
\begin{aligned}
\phi &= \mathcal{F}_{\lambda}^{in}[\mathcal{H}_{\theta}([\mathcal{F}_{out}[\Theta_{G}(\phi, \psi, \lambda, \xi)], \lambda, \xi])], \\
\psi &= \mathcal{F}_{out}[\Theta_{G}(\phi, \psi, \lambda, \xi)], \\
\lambda_1 &= \mathcal{F}^{(0)}[\tilde{g}_0(\psi, \lambda_1, \xi_1, \theta)], \\
\xi_1 &= \mathcal{F}^{(1)}[\tilde{g}_1(\psi, \lambda_1, \xi_1, \theta)],
\end{aligned}
\]

where \(\tilde{g}_0\) and \(\tilde{g}_1\) can be expressed as

\[
\begin{aligned}
\tilde{g}_0(\psi, \lambda_1, \xi_1, \theta) &= c_R^0 \int_{B_{2R}} \mathcal{H}_{\theta}([\mathcal{F}_{out}[\Theta_{G}(\phi, \psi, \lambda, \xi)], \lambda, \xi])Z_{n+1}(y)dy, \\
\tilde{g}_1(\psi, \lambda_1, \xi_1, \theta) &= c_R^2 \int_{B_{2R}} \mathcal{H}_{\theta}([\mathcal{F}_{out}[\Theta_{G}(\phi, \psi, \lambda, \xi)], \lambda, \xi])U(y)dy
\end{aligned}
\]

for suitable positive constants \(c_R^0\) and \(c_R^2\). For \(\varepsilon > 0\) fixed sufficiently small, we consider the following problem defined only up to time \(t = T - \varepsilon\)

\[
\begin{aligned}
\phi &= \mathcal{F}_{\lambda}^{in}[\mathcal{H}_{\theta}([\mathcal{F}_{out}[\Theta_{G}(\phi, \psi, \lambda, \xi)], \lambda, \xi]), \quad (y, t) \in B_{2R} \times [0, T - \varepsilon], \\
\psi &= \mathcal{F}_{out}[\Theta_{G}(\phi, \psi, \lambda, \xi)], \quad (x, t) \in \mathbb{R}^n \times [0, T - \varepsilon], \\
\lambda_1 &= \mathcal{F}^{(0)}_{\varepsilon}[\tilde{g}_0(\psi, \lambda_1, \xi_1, \theta)], \quad t \in [0, T - \varepsilon], \\
\xi_1 &= \mathcal{F}^{(1)}_{\varepsilon}[\tilde{g}_1(\psi, \lambda_1, \xi_1, \theta)], \quad t \in [0, T - \varepsilon],
\end{aligned}
\]

(4.176)

where

\[
\mathcal{F}^{(0)}_{\varepsilon}[g](t) := -(T - t)^{\gamma} \int_{t}^{T - \varepsilon} (T - s)^{\frac{2}{n-\gamma} - 1} g(s) ds,
\]

\[
\mathcal{F}^{(1)}_{\varepsilon}[g] := \int_{t}^{T - \varepsilon} (T - s)^{\frac{2}{n-\gamma} - 1} g(s) ds.
\]

We consider problem (4.176) in the space of functions

\[(\phi, \psi, \lambda_1, \xi_1) \in X_1 \times X_2 \times X_3 \times X_4,\]

where \(X_\ell (\ell = 1, \ldots, 4)\) and corresponding norms are defined as

\[X_1 = \{ \phi : \phi \in C(B_{2R} \times [0, T - \varepsilon]), \nabla^s_\phi \in C(B_{2R} \times [0, T - \varepsilon]) \}, \]

\[\| \phi \|_{X_1} = \| \phi \|_{\infty} + \| \nabla^s_\phi \|_{\infty},\]

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\[ X^2 = \{ \psi : \phi \in C(R^n \times [0, T - \varepsilon]) \}, \quad \| \psi \|_{X^2} = \| \psi \|_{\infty}, \]
\[ X^3 = \{ \lambda_1 : \lambda_1 \in C^1[0, T - \varepsilon] \}, \quad \| \lambda_1 \|_{X^3} = \| \lambda_1 \|_{\infty} + \| \dot{\lambda}_1 \|_{\infty}, \]
\[ X^4 = \{ \xi_1 : \xi_1 \in C^1[0, T - \varepsilon] \}, \quad \| \xi_1 \|_{X^4} = \| \xi_1 \|_{\infty} + \| \dot{\xi}_1 \|_{\infty}, \]

where \( \nabla_j^s \phi := \left( \int_{R^n} \frac{[\phi(y,t) - \phi(x,t)]^2}{|y - x|^{n+2}} \, dx \right)^{1/2} \). As a direct consequence of Arzelà–Ascoli’s theorem, compactness on bounded sets of all the operators involved in the above expression (4.176) follows from the Hölder estimate (4.148) for the operator \( T^{\text{out}} \). On the other hand, the a priori estimates we obtained for \( \varepsilon = 0 \) hold equally well, uniformly for arbitrary small \( \varepsilon > 0 \), and for a solution of (4.176).

We now apply Leray–Schauder degree theory in a suitable ball \( B \) containing the origin which is slightly larger than the one defined by relations (4.175), which amounts to a choice of the parameters \( \delta_0 \) and \( \delta_1 \) in (4.161). The homotopy connects with the identity at \( \theta = 0 \), and hence the total degree in the region defined by relations (4.175) is equal to 1. Hence we have the existence of a solution to the approximate problem satisfying bounds (4.175). Finally, by a standard diagonal argument, we find a solution to the original problem for \( k = 1 \) with the desired size.

The multiple-bubble case of \( k \) distinct points \( q_1, \ldots, q_k \) is actually identical. In this case, we have \( k \) inner problems and one outer problem with similar properties. We want to find a solution of the form

\[ u(x,t) = \sum_{j=1}^{k} U_{\lambda_j, y_j}(x) + Z^*(x,t) + \lambda_j^{-\frac{n+2}{2}} \phi(y_j,t) \eta_k(y_j) + \psi(x,t), \quad (4.177) \]

where \( y_j = \frac{x - \xi_j}{\lambda_j} \), \( Z^* \) solves heat equation with initial condition \( Z_0^* \) which is chosen such that \( Z_0^*(q_j) < 0 \), and \( \xi_j(T) = q_j, \lambda_j(T) = 0 \) for \( j = 1, \ldots, k \). Then by solving a series of fixed point problems similar as the one bubble case, we obtain a solution of form (4.177). Hence we omit the details here. \( \square \)
Chapter 5

New gluing methods for the nematic liquid crystal flows

5.1 Introduction

In this Chapter, we consider the following initial-boundary value problem of nematic liquid crystal flow in a bounded, smooth domain $\Omega$ in $\mathbb{R}^2$, and $T > 0$

\[
\begin{cases}
\partial_t v + v \cdot \nabla v + \nabla P = \Delta v - \varepsilon_0 \nabla \cdot \left( \nabla u \otimes \nabla u - \frac{1}{2} |\nabla u|^2 I \right) & \text{in } \Omega \times (0, T), \\
\nabla \cdot v = 0 & \text{in } \Omega \times (0, T), \\
\partial_t u + v \cdot \nabla u = \Delta u + |\nabla u|^2 u & \text{in } \Omega \times (0, T),
\end{cases}
\]

with initial condition

\[
(v, u) \big|_{t=0} = (v_0, u_0) \text{ in } \Omega,
\]

and boundary condition

\[
\begin{align*}
v &= 0 \text{ on } \partial \Omega \times (0, T), \\
u &= u_0 \text{ on } \partial \Omega \times (0, T),
\end{align*}
\]
where $v : \Omega \times [0, T) \to \mathbb{R}^2$ is the fluid velocity field, $P : \Omega \times [0, T) \to \mathbb{R}$ is the fluid pressure function, $u : \Omega \times [0, T) \to S^2$ stands for the orientation field of nematic liquid crystal molecules, $\nabla \cdot$ denotes the divergence operator, $\nabla u \odot \nabla u$ denotes the $2 \times 2$ matrix given by $(\nabla u \odot \nabla u)_{ij} = \nabla_i u \cdot \nabla_j u$, and $I_2$ is the identity matrix on $\mathbb{R}^2$. The parameter $\varepsilon_0 > 0$ represents the competition between kinetic energy and elastic energy. $(v_0, u_0) : \Omega \to \mathbb{R}^2 \times S^2$ is a given initial data such that $\nabla \cdot v_0 = 0$.

The system (5.1) can be viewed as a coupling between the incompressible Navier–Stokes equations and the equations of heat flow of harmonic maps. Both the incompressible Navier–Stokes equations and the equations of harmonic map heat flow have been studied extensively. For the incompressible Navier–Stokes equations, the existence of global weak solutions to the initial value problem has been well-known since the fundamental works of Leray [127] and Hopf [112]. A more comprehensive theory on the Navier–Stokes equation can be found in classical books of Temam [185], Lions [140], see also [87, 126, 169, 186] and the references therein. The fundamental solution of the Stokes system, which is a linearized Navier–Stokes equation, was established by Solonnikov in [175], together with estimates of weak solutions to the Cauchy problem. Solonnikov also derived similar estimates of the initial-boundary value problem of the Stokes system in [176–178], and these sharp estimates would be very important in our construction. For the harmonic map heat flow, Struwe [180] and Chang [25] established the existence of a unique global weak solution in dimension two, which has at most finitely many singular points. In higher dimensions, the existence of a global weak solution has been proved by Chen and Struwe in [34] (see also Chen and Lin [33]). Examples of finite time blow-up solutions have been constructed in dimension $n \geq 3$ in [32, 42], see also [96, 97]. In dimension two, Chang, Ding and Ye [26] established the first example of finite time singularities by a sub-super solution method for axially symmetric maps into the standard sphere. Angenent, Hulshof and Matano [4] analyzed a 1-corotational blow-up solution in a disk with profile

$$u(x, t) = W \left( \frac{x}{\Lambda(t)} \right) + O(1),$$
where \( W \) is the least energy harmonic map (of degree one)

\[
W(y) = \frac{1}{1 + |y|^2} \left[ \frac{2y}{|y|^2 - 1} \right], \quad y \in \mathbb{R}^2,
\]

\( O(1) \) denotes a term that is bounded in \( H^1 \)-norm, and \( 0 < \lambda(t) \to 0 \) as \( t \to T \). They obtained an estimation of the blow-up rate as \( \lambda(t) = o(T-t) \). Using matched asymptotics formal analysis, van den Berg, Hulshof and King [187] showed that this rate should be given by

\[
\lambda(t) \sim \kappa \frac{T-t}{\log(T-t)^2}
\]

for some \( \kappa > 0 \). Raphaël and Schweyer succeeded in constructing an entire 1-corotational solution with this blow-up rate rigorously [164]. Recently, Dávila, del Pino and Wei [55] constructed a non-symmetric solution that exhibits finite time blow-up at multiple points and studied its stability by using the inner–outer gluing method. More precisely, for any given finite set of points in \( \Omega \), they constructed solution blowing up exactly at those points simultaneously under suitable initial and boundary conditions. In another aspect, for higher-degree corotational harmonic map heat flow, global existence and blow-up have been investigated in a series of works [99–102] and the references therein. For the general analysis of the bubbling phenomena and regularity results of the harmonic map heat flow, we refer the readers to the book [132].

The model equations for the nematic liquid crystal flow (5.1) that will be studied in this Chapter are proposed in [137], and it is a simplified version of the Ericksen–Leslie system for the hydrodynamics flow of nematic liquid crystal material established by Ericksen [76] and Leslie [128]. The existence and uniqueness of solutions to (5.1) has attracted a lot of interests in recent years. In an earlier work [138], Lin and Liu considered the Ericksen-Leslie system with variable degree of orientations, and established a global existence of weak and classical solutions in dimensions three and two. There is also a partial regularity theorem for suitable weak solutions of approximate systems for (5.1), see [139], similar to those for the Navier–Stokes equation established by Caffarelli–Kohn–Nirenberg in [21].
Later in [136], a global existence of Leray–Hopf–Struwe type weak solutions of (5.1) in two dimensions is proved (see also [110, 111, 113, 125, 188, 189]). More importantly, the uniqueness of such weak solution in dimension two can also be shown [133]. For the case of dimension three, much less is known. Lin and Wang [135] proved a global existence of (suitable) weak solutions satisfying the global energy inequality under a restrictive assumption that the initial orientation field \( u_0(\Omega) \subset S^2_+ \). There are also blow up criteria for finite time singularities for local strong solutions of (5.1) in both dimensions two and three, for instance, Huang and Wang [114]. We should also point out a recent interesting work by Chen and Yu [30]. They constructed global \( m \)-equivariant solutions in \( \mathbb{R}^2 \) where the orientation field blows up logarithmically as \( t \to +\infty \). See also [31] for the infinite time blow-up in \( \mathbb{R}^3 \). For a survey of some recent important developments of mathematical analysis of nematic liquid crystals we refer to [134].

The main concern of this Chapter is the existence of classical solutions to the nematic liquid crystal flow (5.1), that develop finite time singularities. In dimension three, the work [115] has provided two examples of finite time singularity of (5.1). The first example is an axisymmetric finite time blow-up solution constructed in a cylindrical domain (as remarked in [115] Remark 1.2(c), this blow-up example does not satisfy the no-slip boundary condition). The second example is constructed in a ball for any generic initial data \((v_0, u_0)\) that has small enough energy, and \( u_0 \) has a non-zero Hopf-degree.

In this Chapter, we consider the two-dimensional nematic liquid crystal flow (5.1), where the velocity field satisfies no-slip boundary condition, i.e., \( v = 0 \) on \( \partial \Omega \). We wish to point out that if \( v \equiv 0 \) in (5.1), then \( u \) is not only a solution of the harmonic map heat flow, it also satisfies the compatibility condition \( \nabla \cdot (\nabla u \circ \nabla u - \frac{1}{2} |\nabla u|^2 I) = \nabla P \) for a scalar function \( P \). In fact, one can check that for the blow-up solution \( u \) to the harmonic map heat flow constructed by [26], as it is axisymmetric, \((u, 0)\) is also a blow-up solution to (5.1). On the other hand, the blow-up solutions \( u \) to the harmonic map heat flow in [55] can not satisfy (5.1) with \( v \equiv 0 \), whenever the number of blow up points \( k > 1 \).

Using the inner–outer gluing method for both \( u \) and \( v \), we construct a solution \((v, u)\) to problem (5.1) exhibiting finite time singularity when the parameter \( \varepsilon_0 \) is sufficiently small. More precisely, we have
**Theorem 5.1.1.** There exists a sufficiently small \( \varepsilon_0 > 0 \) such that given \( k \) distinct points \( q_1, \ldots, q_k \in \Omega \), if \( T > 0 \) is sufficiently small, then there exists a smooth initial data \( (v_0, u_0) \) such that the short time smooth solution \( (v, u) \) to the system (5.1) blows up exactly at those \( k \) points as \( t \to T \). More precisely, there exist numbers \( \kappa_j^* > 0 \), \( \omega_j^* \) and \( u_s \in H^1(\Omega) \cap C(\bar{\Omega}) \) such that

\[
 u(x, t) - u_s(x) - \sum_{j=1}^k Q_{\omega_j}^1 Q_{\alpha_j}^2 Q_{\beta_j}^3 \left[ W \left( \frac{x - q_j}{\lambda_j(t)} \right) - W(\infty) \right] \to 0 \quad \text{as} \quad t \to T,
\]

in \( H^1(\Omega) \cap L^\infty(\Omega) \), where the blow-up rate and angles satisfy

\[
 \lambda_j(t) = \kappa_j^* \frac{T - t}{\log(T - t)^2 \log(1 + o(1))} \quad \text{as} \quad t \to T,
\]

\[
 \omega_j \to \omega_j^*, \quad \alpha_j \to 0, \quad \beta_j \to 0 \quad \text{as} \quad t \to T,
\]

and \( Q_{\omega_j}^1, Q_{\alpha_j}^2 \) and \( Q_{\beta_j}^3 \) are rotation matrices defined in (5.7). In particular, it holds that

\[
 |\nabla u(\cdot, t)|^2 dx \to |\nabla u_s|^2 dx + 8\pi \sum_{j=1}^k \delta_{q_j} \quad \text{as} \quad t \to T,
\]

as convergence of Radon measures. Furthermore, the velocity field satisfies

\[
 |v(x, t)| \leq c \sum_{j=1}^k \frac{\lambda_j^{\nu_j - 1}(t)}{1 + \frac{|x - q_j|}{\lambda_j(t)}} \quad 0 < t < T,
\]

for some \( c > 0 \) and \( 0 < \nu_j < 1 \), \( j = 1, \ldots, k \).

**Remark 5.1.1.**

- At each blow-up point \( q_j \in \Omega, 1 \leq j \leq k \), the behavior of the velocity field \( v \) is precisely

\[
 |v(x, t)| \leq c \lambda_j^{\nu_j - 1}(t) + o(1) \quad \text{for} \quad \nu_j \in (0, 1).
\]

Theorem 5.1.1 suggests that \( v \) might also blow up in finite time. In fact we conjecture that \( ||v(\cdot, t)||_{L^\infty} \sim |\log(T - t)| \) as \( t \to T \). The singularity formation of the velocity field is driven by the Ericksen stress tensor \( \nabla \cdot (\nabla u \circ \nabla u - \ldots \ldots) \)
\( \frac{1}{2} |\nabla u|^2 I_2 \), which is induced by the liquid crystal orientation field \( u(x,t) \). Namely, \( u(x,t) \) plays a role on generating the singular forcing in the incompressible Navier–Stokes equation. For results of the Navier–Stokes equation with singular forcing in dimension two, we refer to [41].

• It is well-known that the pressure \( P \) can be recovered from the velocity field \( v \) and the forcing. See for instance [87] and [186].

• The proof of Theorem 5.1.1 actually yields, on one hand, that the small constant \( \varepsilon_0 \) can be chosen to be a universal constant, that is independent of the domain \( \Omega \), blow-up points \( q_1, \ldots, q_k \), and time \( T \). On the other hand, no matter how small \( \varepsilon_0 \) would be, the two systems are fully coupled, because of the following scaling invariance:

\[
(v_\lambda(x,t), P_\lambda(x,t), u_\lambda(x,t)) = (\lambda v(\lambda x, \lambda^2 t), \lambda^2 P(\lambda x, \lambda^2 t), u(\lambda x, \lambda^2 t)).
\]

In addition, this nonlinear coupling property is also preserved in the linearized inner problem:

\[
\begin{align*}
\partial_t v + \nabla P &= \Delta v - \varepsilon_0 \nabla \cdot (\nabla W \odot \nabla \phi), \\
\nabla \cdot v &= 0, \\
\phi \tau + v \cdot \nabla \phi &= \Delta \phi + |\nabla W|^2 \phi + 2(\nabla W \cdot \nabla \phi)W.
\end{align*}
\]

**Remark 5.1.2.** While, in order to carry out fixed point argument in the inner–outer gluing procedure, we need to assume \( \varepsilon_0 > 0 \) in (5.1) to be sufficiently small, Theorem 5.1.1 does cover the relevant physical cases of the hydrodynamics of nematic liquid crystals where the fluid tends to have a large viscous effect. More precisely, instead of (5.1), if we consider

\[
\begin{align*}
\partial_v + v \cdot \nabla v + \nabla P &= \mu \Delta v - \tilde{\lambda} \nabla \cdot (\nabla u \odot \nabla u - \frac{1}{2} |\nabla u|^2 I_2) \quad \text{in } \Omega \times (0,T), \\
\nabla \cdot v &= 0 \quad \text{in } \Omega \times (0,T), \\
\partial_t u + v \cdot \nabla u &= \tilde{\gamma} (\Delta u + |\nabla u|^2 u) \quad \text{in } \Omega \times (0,T),
\end{align*}
\]

(5.4)

where \( \mu > 0, \tilde{\lambda} > 0, \) and \( \tilde{\gamma} > 0 \) represents the fluid viscosity, the competition pa-
rameter between the kinetic energy of fluid and the elastic energy of the liquid crystal orientation field, and the macroscopic relaxation time parameter respectively. Assume that $\frac{\mu}{\lambda} \gg 1$ and $\frac{\tilde{\gamma}}{\mu} \approx 1$. If we set $(\tilde{v}, \tilde{u}, \tilde{P})(x,t) = \left(\frac{1}{\mu} v, u, \frac{1}{\mu^2} P\right)(x, \frac{t}{\mu})$, then it follows from direct calculations that $(\tilde{v}, \tilde{u}, \tilde{P})$ solves (5.1) with the parameter $\varepsilon_0 = \frac{1}{\mu} \ll 1$.

The nematic liquid crystal flow (5.1) is a strongly coupled system of the incompressible Navier–Stokes equation and the transported harmonic map heat flow. In this Chapter, the construction of the finite time blow-up solution is close in spirit to the singularity formation of the standard two dimensional harmonic map heat flow

$$\begin{cases} \partial_t u = \Delta u + |\nabla u|^2 u & \text{in } \Omega \times (0,T), \\ u = u_0 & \text{on } \partial\Omega \times (0,T), \\ u(\cdot,0) = u_0 & \text{in } \Omega. \end{cases} \quad (5.5)$$

In [55], by the inner–outer gluing method, Dávila, del Pino and Wei successfully constructed type II finite time blow-up for the harmonic map heat flow (5.5). More precisely, the solution constructed in [55] takes the bubbling form

$$|\nabla u(\cdot,t)|^2 \rightharpoonup |\nabla u_*|^2 + 8\pi \sum_{j=1}^k \delta_{q_j} \quad \text{as } t \to T,$$

where $u_* \in H^1(\Omega) \cap C(\bar{\Omega})$, $(q_1, \ldots, q_k) \in \Omega^k$ are given $k$ points, and $\delta_{q_j}$ denotes the unit Dirac mass at $q_j$ for $j = 1, \ldots, k$. The construction in [55] consists of finding a good approximate solution based on the 1-corotational harmonic maps and then looking for the inner and outer profiles of the small perturbations. Basically, the inner problem is the linearization around the harmonic map which captures the heart of the singularity formation, while the outer problem is a heat equation coupled with the inner problem.

Our construction of a finite time blow-up solution to the nematic liquid crystal flow (5.1)–(5.3) relies crucially on the delicate analysis carried out in [55]. However, because of the strong coupling between the Navier–Stokes equation with forcing for $v$ and the transported harmonic map heat flow equation for $u$, we have to develop several new ingredients in our inner–outer gluing procedure for the system.
(5.1)–(5.3):

- Although the advection term $v \cdot \nabla v$ can be realized as a small perturbation in the Stokes system with forcing, the transported term $v \cdot \nabla u$ in the equation for the orientation field $u$ can only be realized as a small perturbation of the outer problem for $u$, but not of the inner problem for $u$ where the singularity occurs. In fact, since the system (5.1) is invariant under the following parabolic scalings:

\[
(v_\lambda(x,t), P_\lambda(x,t), u_\lambda(x,t)) = (\lambda v(\lambda x, \lambda^2 t), \lambda^2 P(\lambda x, \lambda^2 t), u(\lambda x, \lambda^2 t))
\]

\forall \lambda > 0 in the self-similar variable $(y, \tau)$ near a singular point $(q, T)$, roughly speaking, the order of $v[\phi] \cdot \nabla U$ is the same as that of $h$ in the inner-linearized equation:

\[
\partial_\tau \phi + v[\phi] \cdot \nabla y W = L_W[\phi] + h,
\]

where $L_W$ is the linearization of harmonic map equation around $W$ given by (5.8). See Section 5.4 for more details.

- In [55], the parameter functions $\lambda(t)$, $\xi(t)$, $\omega(t)$, which correspond to the dilation, translation in the domain, and rotation about $z$-axis in the target space, respectively, were introduced to adjust certain orthogonality conditions to guarantee the existence of desired solutions to the harmonic map heat flow. However, to find the desired solution of the nematic liquid crystal flow (5.1) as stated in Theorem 5.1.1, we need to introduce two new parameter functions $\alpha(t)$ and $\beta(t)$ associated to the rotations about $x$ and $y$ axes in the target space, respectively. The reasons behind this are:

  i) Heuristically, in the inner problem of $u$, the velocity $v$ may exhibit a logarithmic singularity induced by the off-diagonal effect of the Oseen-kernel $S_{ij}$ (see (5.79)). The addition of these new parameter functions $\alpha(t)$ and $\beta(t)$ can balance such a logarithmic singularity off.

  ii) We need to solve the inner linearized problem of $u$ to get a solution with space and time decay rates faster than that by [55], since we need to control the stress-tensor $\nabla \cdot (\nabla u \odot \nabla u - \frac{1}{2} |\nabla u|^2 I_2)$ appearing in the
equation for velocity field $v$ of (5.1). For this purpose, we introduce two new parameter functions $\alpha(t)$ and $\beta(t)$ associated to the rotations about $x$ and $y$ axes in the target space, respectively, to adjust the orthogonality conditions at mode $-1$. See Section 5.2 for details.

iii) After the adjustment by suitable $\alpha(t)$ and $\beta(t)$, the smallness of parameter $\epsilon_0$ can reduce $v \cdot \nabla u$ into a truly small perturbation in the inner problem of $u$.

- We also need to develop a new linear theory for the Stokes system with some novel weighted $L^\infty$ estimates, which shall have its own interest. The construction of desired velocity field $v$ shall be carried out by another new inner–outer gluing procedure, since the forcing term $\nabla \cdot \left( \nabla u \odot \nabla u - \frac{1}{2} |\nabla u|^2 I_2 \right)$ in (5.1) is concentrated near the blow-up points. See Section 5.3 for details.

The following picture roughly describes the above process.

\[ -\epsilon_0 \nabla \cdot \left( \nabla u \odot \nabla u - \frac{1}{2} |\nabla u|^2 I_2 \right) \text{ in } (5.1)_1: \xrightarrow{(1)} v \text{ in } (5.1)_3 \]

Mode $k$ in the inner problem of $u$: $\phi_k \xleftarrow{(3)} v \cdot \nabla u \text{ in } (5.1)_3$

(1) Solve the incompressible Navier–Stokes equation with forcing coupled from the orientation $u$.

(2) The velocity $v$ provides transported effect in the harmonic map heat flow.

(3) The transported term $v \cdot \nabla u$ is coupled in a nontrivial way through the inner problem at mode $k$ since the velocity $v = v[\phi_k]$ carries the information of $\phi_k$ in step (1).

(4) Faster spatial and time decay of $\phi_k$ yields better forcing term in (5.1)1, ensuring the implementation of the whole loop.

This Chapter is organized as follows. In Section 5.2 we will develop a new inner–outer gluing method for the harmonic map heat flow in order to handle the
difficulties arising from the coupling effects of (5.1). In Section 5.3, we develop the linear theory for the Stokes system. In Section 5.4, using the newly developed inner–outer gluing method, we construct a finite time blow-up solution to the nematic liquid crystal flow by the fixed point argument.

Notation. Throughout this Chapter, we shall use the symbol “≤C” to denote “≤C” for a positive constant C independent of t and T. Here C might be different from line to line.

5.2 Singularity formation for the harmonic map heat flow in dimension two

Closely related to the harmonic map heat flow in dimension two, the equation for the orientation field u can be regarded as a transported version with drift term. In this Section, we consider the two dimensional harmonic map heat flow $u : \Omega \times [0, T) \rightarrow \mathbb{S}^2$:

$$\begin{align*}
\partial_t u &= \Delta u + |\nabla u|^2 u & \text{in } \Omega \times (0, T), \\
u &= u_0 & \text{on } \partial \Omega \times (0, T), \\
u(\cdot, 0) &= u_0 & \text{in } \Omega.
\end{align*}$$

While following closely the general strategy of the construction developed by [55], we will establish several new estimates that are needed for the system (5.1). More precisely,

- A new linear theory at mode $-1$: This procedure consists of the following steps
  - Step 1: New corrections are added at mode $-1$ to cancel out the leading order of slow decaying error corresponding to the rotations around x and y axes in the target space (see Section 5.2.2).
  - Step 2: New orthogonality conditions are imposed at mode $-1$ which determine the dynamics of the new parameters $\alpha(t)$ and $\beta(t)$ (see Section 5.2.4).
  - Step 3: Under the orthogonality conditions at mode $-1$, the new linear theory at mode $-1$ is developed (see Section 5.2.5).
Higher order estimates for inner and outer solutions are obtained in order to handle the forcing 
\(-\epsilon_0 \nabla \cdot (\nabla u \odot \nabla u - \frac{1}{2} |\nabla u|^2 I_2)\) in the equation for velocity (5.1) \(_1\) (see Sections 5.2.5–5.2.6).

We first introduce some notations and preliminaries.

5.2.1 Stationary problem: the equation of harmonic maps and its linearization

The equation of harmonic maps for \(U : \mathbb{R}^2 \rightarrow S^2\) is the quasilinear elliptic system

\[
\Delta U + |\nabla U|^2 U = 0 \text{ in } \mathbb{R}^2.
\]  

(5.6)

For \(\lambda > 0, \xi \in \mathbb{R}^2, \omega, \alpha, \beta \in \mathbb{R}\), we consider the family of solutions to (5.6) given by the following 1-corotational harmonic maps

\[
U_{\lambda, \xi, \omega, \alpha, \beta}(x) = Q_{\omega}^1 Q_{\alpha}^2 Q_{\beta}^3 W \left( \frac{x - \xi}{\lambda} \right), \quad x \in \mathbb{R}^2,
\]

where

\[
Q_{\omega}^1 := \begin{bmatrix}
\cos \omega & -\sin \omega & 0 \\
\sin \omega & \cos \omega & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad Q_{\alpha}^2 := \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \alpha & -\sin \alpha \\
0 & \sin \alpha & \cos \alpha
\end{bmatrix}, \quad Q_{\beta}^3 := \begin{bmatrix}
\cos \beta & 0 & \sin \beta \\
0 & 1 & 0 \\
-\sin \beta & 0 & \cos \beta
\end{bmatrix}
\]

(5.7)

are the rotation matrices about \(z, x\) and \(y\) axes in the target space, respectively, and \(W\) is the least energy harmonic map

\[
W(y) = \frac{1}{1 + |y|^2} \begin{bmatrix}
2y \\
|y|^2 - 1
\end{bmatrix}, \quad y \in \mathbb{R}^2.
\]
In the polar coordinates $y = \rho e^{i\theta}$, $W(y)$ can be represented as

$$W(y) = \begin{bmatrix} e^{i\theta} \sin w(\rho) \\ \cos w(\rho) \end{bmatrix}, \quad w(\rho) = \pi - 2\arctan(\rho),$$

and we have

$$w_\rho = -\frac{2}{\rho^2 + 1}, \quad \sin w = -\rho w_\rho = \frac{2\rho}{\rho^2 + 1}, \quad \cos w = \frac{\rho^2 - 1}{\rho^2 + 1}.$$ 

For simplicity, we write $Q_{\omega, \alpha, \beta} := \Omega_\alpha^1 \Omega_\alpha^2 \Omega_\beta^3$.

The linearization of the harmonic map operator around $W$ is the elliptic operator

$$L_W[\phi] = \Delta_x \phi + |\nabla W(y)|^2 \phi + 2(\nabla W(y) \cdot \nabla \phi) W(y), \quad (5.8)$$

whose kernel functions are given by

$$Z_{0,1}(y) = \rho w_\rho(\rho) E_1(y),$$
$$Z_{0,2}(y) = \rho w_\rho(\rho) E_2(y),$$
$$Z_{1,1}(y) = w_\rho(\rho)[\cos \theta E_1(y) + \sin \theta E_2(y)],$$
$$Z_{1,2}(y) = w_\rho(\rho)[\sin \theta E_1(y) - \cos \theta E_2(y)],$$
$$Z_{-1,1}(y) = \rho^2 w_\rho(\rho)[\cos \theta E_1(y) - \sin \theta E_2(y)],$$
$$Z_{-1,2}(y) = \rho^2 w_\rho(\rho)[\sin \theta E_1(y) + \cos \theta E_2(y)], \quad (5.9)$$

where the vectors

$$E_1(y) = \begin{bmatrix} e^{i\theta} \cos w(\rho) \\ -\sin w(\rho) \end{bmatrix}, \quad E_2(y) = \begin{bmatrix} ie^{i\theta} \\ 0 \end{bmatrix}$$

form an orthonormal basis of the tangent space $T_{W(y)}S^2$. We see that

$$L_W[Z_{i,j}] = 0 \quad \text{for } i = \pm 1, 0, \ j = 1, 2.$$
Note that
\[ L_U[\varphi] = \lambda^{-2} Q_{\omega,\alpha,\beta} L_W[\varphi], \quad \varphi(x) = \phi(y), \quad y = \frac{x - \xi}{\lambda}. \]

In the sequel, it is of significance to compute the action of \( L_U \) on functions whose value is orthogonal to \( U \) pointwisely. Define
\[ \Pi_{U\perp} \varphi := \varphi - (\varphi \cdot U)U. \]

We invoke several useful formulas proved in [55, Section 3]:
\[ L_U[\Pi_{U\perp} \Phi] = \Pi_{U\perp} \Delta \Phi + \tilde{L}_U[\Phi], \]
where
\[ \tilde{L}_U[\Phi] := |\nabla U|^2 \Pi_{U\perp} \Phi - 2\nabla(\Phi \cdot U) \nabla U, \quad (5.10) \]
with \( \nabla(\Phi \cdot U) \nabla U = \partial_j(\Phi \cdot U) \partial_j U \). In the polar coordinates \( \Phi(x) = \Phi(r, \theta), \) \( x = \xi + re^{i\theta}, (5.10) \) can be expressed as (see [55, Section 3])
\[ \tilde{L}_U[\Phi] = -\frac{2}{\lambda} w_\rho(\rho) \left[ (\Phi_r \cdot U) Q_{\omega,\alpha,\beta} E_1 - \frac{1}{r} (\Phi_\theta \cdot U) Q_{\omega,\alpha,\beta} E_2 \right], \quad r = \lambda \rho. \]

Assume that \( \Phi(x) : \Omega \to \mathbb{C} \times \mathbb{R} \) is a \( C^1 \) function in the form
\[ \Phi(x) = \begin{bmatrix} \varphi_1(x) + i\varphi_2(x) \\ \varphi_3(x) \end{bmatrix}. \quad (5.11) \]

If we write
\[ \varphi = \varphi_1 + i\varphi_2, \quad \bar{\varphi} = \varphi_1 - i\varphi_2, \]
\[ \text{div} \varphi = \partial_1 \varphi_1 + \partial_2 \varphi_2, \quad \text{curl} \varphi = \partial_1 \varphi_2 - \partial_2 \varphi_1, \]
then we have the following formula (see [55, Section 3])
\[ \tilde{L}_U[\Phi] = \tilde{L}_U[\Phi]_0[\Phi] + \tilde{L}_U[\Phi]_1[\Phi] + \tilde{L}_U[\Phi]_2[\Phi], \quad (5.12) \]
where

\[
\begin{align*}
[L_U]_0[\Phi] &= \lambda^{-1} \rho w^2 \frac{\partial}{\partial \rho} \left( \text{div} (e^{-i\omega \varphi}) Q_{\omega, \alpha, \beta} E_1 \right) + \text{curl} (e^{-i\omega \varphi}) Q_{\omega, \alpha, \beta} E_2], \\
[L_U]_1[\Phi] &= -2\lambda^{-1} \rho w \cos \theta \left[ (\partial_1 \varphi) \cos \theta + (\partial_2 \varphi) \sin \theta \right] Q_{\omega, \alpha, \beta} E_1 \\
&\quad + 2\lambda^{-1} \rho w \cos \theta \left[ (\partial_1 \varphi) \sin \theta - (\partial_2 \varphi) \cos \theta \right] Q_{\omega, \alpha, \beta} E_2, \\
[L_U]_2[\Phi] &= \lambda^{-1} \rho w^2 \left( \text{div} (e^{i\omega \varphi}) \cos 2\theta - \text{curl} (e^{i\omega \varphi}) \sin 2\theta \right) Q_{\omega, \alpha, \beta} E_1 \\
&\quad + \lambda^{-1} \rho w^2 \left( \text{div} (e^{i\omega \varphi}) \sin 2\theta + \text{curl} (e^{i\omega \varphi}) \cos 2\theta \right) Q_{\omega, \alpha, \beta} E_2.
\end{align*}
\]

(5.13)

If we assume

\[\Phi(x) = \begin{bmatrix} \phi(r) e^{i\theta} \\ 0 \end{bmatrix}, \quad x = \xi + re^{i\theta}, \quad r = \lambda \rho,\]

where \(\phi(r)\) is complex-valued, then we have the following formula

\[\tilde{L}_U[\Phi] = 2\lambda^{-1} \rho w^2 \left( \text{Re} (e^{-i\omega \partial_\rho \phi(r)}) Q_{\omega, \alpha, \beta} E_1 + \frac{1}{r} \text{Im} (e^{-i\omega \phi(r)}) Q_{\omega, \alpha, \beta} E_2 \right) .\]

If \(\Phi\) is of the form

\[\Phi(x) = \varphi_1(\rho, \theta) Q_{\omega, \alpha, \beta} E_1 + \varphi_2(\rho, \theta) Q_{\omega, \alpha, \beta} E_2, \quad x = \xi + \lambda \rho e^{i\theta}\]

in the polar coordinates, then the linearized operator \(L_U\) acting on \(\Phi\) can be expressed as (see [55, Section 3])

\[
L_U[\Phi] = \lambda^{-2} \left( \frac{\partial_{\rho \rho} \varphi_1}{\rho} + \frac{\partial_\rho \varphi_1}{\rho^2} + \frac{\partial_{\theta \theta} \varphi_1}{\rho^2} + (2w^2 - \frac{1}{\rho^2}) \varphi_1 - \frac{2}{\rho^2} \partial_\theta \varphi_2 \cos \omega \right) Q_{\omega, \alpha, \beta} E_1
\]
\[\quad + \lambda^{-2} \left( \frac{\partial_{\rho \rho} \varphi_2}{\rho} + \frac{\partial_\rho \varphi_2}{\rho^2} + \frac{\partial_{\theta \theta} \varphi_2}{\rho^2} + (2w^2 - \frac{1}{\rho^2}) \varphi_2 + \frac{2}{\rho^2} \partial_\theta \varphi_1 \cos \omega \right) Q_{\omega, \alpha, \beta} E_2.
\]

In next section, we shall find proper approximate solutions to the harmonic map heat flow based on the 1-corotational harmonic maps, and evaluate the error.
5.2.2 Approximate solution and error estimates

We now consider the harmonic map heat flow

$$\begin{cases}
\partial_t u = \Delta u + |\nabla u|^2 u & \text{in } \Omega \times (0, T), \\
u = u_0 & \text{on } \partial \Omega \times (0, T), \\
u(\cdot, 0) = u_0 & \text{in } \Omega, 
\end{cases}$$

(5.14)

where $u : \tilde{\Omega} \times (0, T) \to S^2$, and $u_0 : \tilde{\Omega} \to S^2$ is a given smooth map. For notational simplicity, we shall only carry out the construction in the single bubble case $k = 1$ and mention the minor changes for the general case when needed. We define the error operator

$$S[u] = -\partial_t u + \Delta u + |\nabla u|^2 u.$$

We shall look for solution $u(x, t)$ to problem (5.14) which at leading order takes the form

$$U(x, t) := U_{\lambda(t), \xi(t), \omega(t), \alpha(t), \beta(t)} = Q_{\omega(t), \alpha(t), \beta(t)} W \left( \frac{x - \xi(t)}{\lambda(t)} \right).$$

(5.15)

Here $\lambda(t)$, $\xi(t)$, $\omega(t)$, $\alpha(t)$ and $\beta(t)$ are parameter functions of class $C^1((0, T))$ to be determined later. To get a desired blow-up solution, we assume

$$\lambda(t) \to 0, \ \xi(t) \to q \ \text{as } t \to T,$$

where $q$ is a given point in $\Omega$.

A useful observation is that as long as the constraint $|u| = 1$ is kept for all $t \in (0, T)$ and $u = U + \tilde{w}$ where the perturbation $\tilde{w}$ is uniformly small, say, $|\tilde{w}| \leq \frac{1}{2}$, then for $u$ to solve (5.14), it suffices that

$$S(U + \tilde{w}) = b(x, t) U$$

(5.16)

for some scalar function $b$. Indeed, since $|u| = 1$, we get

$$b(U \cdot u) = S(u) \cdot u = -\frac{1}{2} \frac{d}{dt} |u|^2 + \frac{1}{2} |u|^2 = 0.$$
Thus $b \equiv 0$ follows from $U \cdot u \geq \frac{1}{2}$.

We look for the small perturbation $\tilde{w}(x,t)$ with $|U + \tilde{w}| = 1$ in the form

$$\tilde{w} = \Pi_{U \perp} \varphi + a(\Pi_{U \perp} \varphi)U,$$

where $\varphi$ is an arbitrarily small perturbation with values in $\mathbb{R}^3$, and

$$\Pi_{U \perp} \varphi := \varphi - (\varphi \cdot U)U, \quad a(\zeta) = \sqrt{1 - |\zeta|^2} - 1.$$  

By $\Delta U + |\nabla U|^2 U = 0$, we compute

$$S(U + \Pi_{U \perp} \varphi + aU) = -U_t - \partial_i \Pi_{U \perp} \varphi + L_U(\Pi_{U \perp} \varphi) + N_U(\Pi_{U \perp} \varphi) + c(\Pi_{U \perp} \varphi)U,$$

where for $\zeta = \Pi_{U \perp} \varphi$, $a = a(\zeta)$,

$$L_U(\zeta) = \Delta \zeta + |\nabla U|^2 \zeta + 2(\nabla U \cdot \nabla \zeta)U,$$

$$N_U(\zeta) = \left[2\nabla(aU) \cdot \nabla(U + \zeta) + 2\nabla U \cdot \nabla \zeta + |\nabla \zeta|^2 + |\nabla(aU)|^2 \right] \zeta$$

$$- aU_t + 2\nabla a \cdot \nabla U,$$

$$c(\zeta) = \Delta a - a_t + (|\nabla(U + \zeta + aU)|^2 - |\nabla U|^2)(1 + a) - 2\nabla U \cdot \nabla \zeta.$$

Since we just need to have an equation in the form (5.16) satisfied, we obtain that

$$u = U + \Pi_{U \perp} \varphi + a(\Pi_{U \perp} \varphi)U$$  \hspace{1cm} (5.17)

solves (5.14) if $\varphi$ satisfies

$$-U_t - \partial_i \Pi_{U \perp} \varphi + L_U(\Pi_{U \perp} \varphi) + N_U(\Pi_{U \perp} \varphi) + b(x,t)U = 0$$  \hspace{1cm} (5.18)

for some scalar function $b(x,t)$. The strategy for constructing $\varphi$ is based on the inner–outer gluing method. We decompose $\varphi$ in (5.17) into inner and outer profiles

$$\varphi = \varphi_{in} + \varphi_{out},$$

where $\varphi_{in}$, $\varphi_{out}$ solve the inner and outer problems we shall describe below. In
terms of $\varphi_{in}$ and $\varphi_{out}$, equation (5.18) is reduced to

$$\begin{align*}
- \partial_t \varphi_{in} + L_U[\varphi_{in}] + \tilde{L}_U[\varphi_{out}] - \Pi_U \left[ \partial_t \varphi_{out} - \Delta \varphi_{out} + U_t \right] \\
+ N_U (\varphi_{in} + \Pi_U \varphi_{out}) + (\varphi_{out} \cdot U) U_t + bU = 0.
\end{align*}$$

(5.19)

The inner solution $\varphi_{in}$ will be assumed to be supported only near $x = \xi(t)$ and better expressed in the scaled variable $y = \frac{x - \xi(t)}{\lambda(t)}$ with zero initial condition and $\varphi_{in} \cdot U = 0$ so that $\Pi_U \varphi_{in} = \varphi_{in}$, while the outer solution $\varphi_{out}$ will consist of several parts whose role is essentially to satisfy (5.19) in the region away from the concentration point $x = \xi(t)$.

For the outer problem, since we want the size of the error to be small, we shall add three corrections $\Phi^0$, $\Phi^\alpha$ and $\Phi^\beta$ which depend on the parameter functions $\lambda(t)$, $\xi(t)$, $\omega(t)$, $\alpha(t)$, $\beta(t)$ such that

$$\Pi_U \left[ \partial_t (\Phi^0 + \Phi^\alpha + \Phi^\beta) - \Delta (\Phi^0 + \Phi^\alpha + \Phi^\beta) + U_t \right]$$

gets concentrated near $x = \xi(t)$ by eliminating the leading orders in the first error $U_t$ associated to the dilation and rotations about $x$, $y$ and $z$ axes. We write

$$\varphi_{out}(x,t) = \Psi^*(x,t) + \Phi^0(x,t) + \Phi^\alpha(x,t) + \Phi^\beta(x,t),$$

where

$$\Psi^* = \psi + Z^*$$

with $Z^* : \Omega \times (0,\infty) \rightarrow \mathbb{R}^3$ satisfying

$$\begin{cases}
\partial_t Z^* = \Delta Z^* & \text{in } \Omega \times (0,\infty), \\
Z^*(\cdot,t) = 0 & \text{on } \partial \Omega \times (0,\infty), \\
Z^*(\cdot,0) = Z^*_0 & \text{in } \Omega.
\end{cases}$$

For the inner problem, we define

$$\varphi_{in}(x,t) = \eta_R Q_{\omega,\alpha,\beta} \phi(y,t)$$

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with

\[ \eta_R(x, t) = \eta \left( \frac{|x - \xi(t)|}{\lambda(t)R(t)} \right), \quad y = \frac{x - \xi(t)}{\lambda(t)}, \quad \eta(s) = \begin{cases} 1 & \text{for } s < 1, \\ 0 & \text{for } s > 2, \end{cases} \]

where \( \phi(y, t) \) satisfies \( \phi(\cdot, 0) = 0 \) and \( \phi(\cdot, t) \cdot W = 0 \), and \( R(t) > 0 \) is determined later. Then equation (5.18) becomes

\[ 0 = \lambda^{-2} \eta R Q_{\omega, \alpha, \beta} [-\lambda^2 \phi_t + L_W [\phi] + \lambda^2 Q^{-1}_{\omega, \alpha, \beta} \tilde{L}_U [\Psi^*]] + \eta R Q_{\omega, \alpha, \beta} (\lambda^{-1} \lambda_y \cdot \nabla_y \phi + \lambda^{-1} \xi \cdot \nabla_y \phi - (Q^{-1}_{\omega, \alpha, \beta} \frac{d}{dt} Q_{\omega, \alpha, \beta}) \phi) + \tilde{L}_U [\Phi^0 + \Phi^\alpha + \Phi^\beta] - \Pi U [\partial_t (\Phi^0 + \Phi^\alpha + \Phi^\beta) - \Delta_t (\Phi^0 + \Phi^\alpha + \Phi^\beta) + U_t] \]

\[ - \partial_t \Psi^* + \Delta \Psi^* + (1 - \eta R) \tilde{L}_U [\Psi^*] + Q_{\omega, \alpha, \beta} [(\Delta_t \eta R) \phi + 2 \nabla \eta R \nabla \phi - (\partial_t \eta R) \phi] + N_U (\eta R Q_{\omega, \alpha, \beta} \phi + \Pi U [\Phi^0 + \Phi^\alpha + \Phi^\beta + \Psi^*]) + ((\Psi^* + \Phi^0 + \Phi^\alpha + \Phi^\beta) \cdot U) U_t + bU. \]

We now give the precise definitions of \( \Phi^0, \Phi^\alpha, \Phi^\beta \), and estimate the error

\[ \tilde{L}_U [\Phi^0 + \Phi^\alpha + \Phi^\beta] - \Pi U [\partial_t (\Phi^0 + \Phi^\alpha + \Phi^\beta) - \Delta_t (\Phi^0 + \Phi^\alpha + \Phi^\beta) + U_t]. \]

We shall choose \( \Phi^0, \Phi^\alpha, \Phi^\beta \) in a way such that

\[ \partial_t (\Phi^0 + \Phi^\alpha + \Phi^\beta) - \Delta_t (\Phi^0 + \Phi^\alpha + \Phi^\beta) + U_t \approx 0 \quad \text{for} \quad |x - \xi| \gg \lambda \]

so that the error in the outer problem is of smaller order.

The error of the approximate solution defined in (5.15) is

\[ \mathcal{J}[U] = -\partial_t U = \left[ -\tilde{\lambda} \partial_x U + \omega \partial_\omega U + \tilde{\xi} \partial_\xi U + \dot{\alpha} \partial_\alpha U + \dot{\beta} \partial_\beta U \right], \]

\[ \vdots = \tilde{\delta}_0 \quad \vdots = \tilde{\delta}_1 \]

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where

\[
\begin{align*}
\partial_x U(x) &= \lambda^{-1} Q_{\omega, \alpha, \beta} Z_{0,1}(y), \\
\partial_\omega U(x) &= Q_{\omega, \alpha, \beta} Z_{0,2}(y) + Q_{\omega, \alpha, \beta} (A_{\alpha, \beta} - J_1) W(y), \\
\partial_{\xi_1} U(x) &= \lambda^{-1} Q_{\omega, \alpha, \beta} Z_{1,1}(y), \\
\partial_{\xi_2} U(x) &= \lambda^{-1} Q_{\omega, \alpha, \beta} Z_{1,2}(y), \\
\partial_\alpha U(x) &= \frac{1}{2} Q_{\omega, \alpha, \beta} [Z_{-1,2}(y) + Z_{1,2}(y)] + Q_{\omega, \alpha, \beta} (A_\beta - J_2) W(y), \\
\partial_\beta U(x) &= -\frac{1}{2} Q_{\omega, \alpha, \beta} [Z_{-1,1}(y) + Z_{1,1}(y)],
\end{align*}
\]

with \(Z_{i,j}\) defined in (5.9) for \(i = 0, \pm 1, j = 1, 2,\)

\[
A_{\alpha, \beta} = \begin{bmatrix} 0 & -\cos \alpha \cos \beta & \sin \alpha \\ \cos \alpha \cos \beta & 0 & \cos \alpha \sin \beta \\ -\sin \alpha & -\cos \alpha \sin \beta & 0 \end{bmatrix}, \quad J_1 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (5.21)
\]

\[
A_\beta = \begin{bmatrix} 0 & -\sin \beta & 0 \\ \sin \beta & 0 & -\cos \beta \\ 0 & \cos \beta & 0 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.
\]

It is worth mentioning that \(A_{\alpha, \beta} - J_1 = o(1)\) and \(A_\beta - J_2 = o(1)\) as \(\alpha, \beta \ll 1.\)

Writing \(y = \frac{x - \tilde{\xi}}{\lambda} = \rho e^{i\theta},\) we have

\[
\begin{align*}
\delta_0(x,t) &= -Q_{\omega, \alpha, \beta} \left[ \dot{\lambda} \lambda^{-1} \rho w_p(\rho) E_1(y) + \dot{\omega} \rho w_p(\rho) E_2(y) \right], \\
\delta_1(x,t) &= -\tilde{\xi}_1 \lambda^{-1} \rho w_p(\rho) Q_{\omega, \alpha, \beta} \left[ \cos \theta E_1(y) + \sin \theta E_2(y) \right] \\
&\quad - \tilde{\xi}_2 \lambda^{-1} \rho w_p(\rho) Q_{\omega, \alpha, \beta} \left[ \sin \theta E_1(y) - \cos \theta E_2(y) \right].
\end{align*}
\]

Notice that the slow decaying part of the error \(\mathcal{S}[U]\) consists of

\[
\begin{align*}
\delta_0(x,t) &= -\frac{2r}{r^2 + \lambda^2} \left( \dot{\lambda} Q_{\omega, \alpha, \beta} E_1 + \dot{\omega} Q_{\omega, \alpha, \beta} E_2 \right) \\
&\approx -\frac{2r}{r^2 + \lambda^2} \begin{bmatrix} (\dot{\lambda} + i\lambda \dot{\omega}) e^{i(\theta + \omega)} \\ 0 \end{bmatrix}
\end{align*}
\]

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and
\[ E_{-1}(x,t) = Q_{x,y} \left[ \frac{\alpha}{2} [Z_{-1,2}(y) + Z_{1,2}(y)] + \dot{\alpha}(A_x - J_z)W \right. \]
\[ \left. - \frac{\beta}{2} [Z_{-1,1}(y) + Z_{1,1}(y)] \right] \]
\[ := E_{-1,2} + E_{-1,1}, \]
where
\[ E_{-1,2} = Q_{x,y} \frac{\dot{\alpha}}{1 + \rho^2} \left[ \begin{array}{c} -2\rho \sin \beta \sin \theta \\ 2\rho \sin \beta \cos \theta - (\rho^2 - 1) \cos \beta \\ 2\rho \cos \beta \sin \theta \end{array} \right], \]
\[ E_{-1,1} = Q_{x,y} \frac{\beta}{1 + \rho^2} \left[ \begin{array}{c} \rho^2 - 1 \\ 0 \\ -2\rho \cos \theta \end{array} \right]. \]
In the sequel, we write
\[ p(t) = \lambda(t)e^{i\omega(t)}. \]
Then
\[ -\frac{2r}{r^2 + \lambda^2} \left[ \begin{array}{c} (\dot{\lambda} + i\lambda \omega)e^{i(\theta + \omega)} \\ 0 \end{array} \right] = -\frac{2r}{r^2 + \lambda^2} \left[ \begin{array}{c} \dot{\lambda}(t)e^{i\theta} \\ 0 \end{array} \right] := \tilde{E}_0(x,t). \]
To reduce the size of \( \mathcal{A}[U] \), we add corrections
\[ \Phi^0[p, \xi] := \left[ \begin{array}{c} \varphi^0(r,t)e^{i\theta} \\ 0 \end{array} \right], \Phi^\alpha = Q_{x,y} \alpha(t) \left[ \begin{array}{c} 0 \\ \alpha(t) \end{array} \right], \Phi^\beta = Q_{x,y} \beta \left[ \begin{array}{c} 0 \\ 1 \end{array} \right], \quad (5.22) \]
where
\[ \varphi^0(r,t) = -\int_{-T}^t \dot{r} \dot{p}(s)k(z(r),t-s)ds \]
with \( z(r) = \sqrt{r^2 + \lambda^2}, \quad k(z,t) = 2\frac{1 - e^{-\frac{z^2}{\sigma^2}}}{\sigma^2} \). By direct computations, the new error
produced by $\Phi^0$ is

$$
\Phi_t^0 - \Delta_c \Phi^0 + \tilde{e}_0 = \mathcal{R}_0 + \mathcal{R}_1, \quad \mathcal{R}_0 = \begin{bmatrix} \mathcal{R}_0 \\ 0 \end{bmatrix}, \quad \mathcal{R}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},
$$

where

$$
\mathcal{R}_0 := -re^{i\theta} \frac{\lambda^2}{\varepsilon^4} \int_{-T}^{t} \dot{p}(s)(z_k - z^2 k_{zz})(z(r), t - s) ds,
$$

$$
\mathcal{R}_1 := -e^{i\theta} \text{Re}(e^{-i\theta} \dot{\xi}(t)) \int_{-T}^{t} \dot{p}(s) k(z(r), t - s) ds
+ \frac{r}{z^2} e^{i\theta} (\lambda \dot{\lambda}(t) - \text{Re}(e^{i\theta} \dot{\xi}(t))) \int_{-T}^{t} \dot{p}(s) z_k(z(r), t - s) ds.
$$

Observe that $\mathcal{R}_1$ is of smaller order. Moreover, we can evaluate

$$
L_U[\Phi^0] + \Pi_{U^\perp}[-U_t + \Delta \Phi^0 - \Phi^0]
= L_U[\Phi^0] - \tilde{e}_1 + \Pi_{U^\perp}[\tilde{e}_0] - \tilde{e}_0 - \Pi_{U^\perp}[\mathcal{R}_0] - \Pi_{U^\perp}[\mathcal{R}_1] - \tilde{e}_1
= \mathcal{K}_0[p, \xi] + \mathcal{K}_1[p, \xi] - \Pi_{U^\perp}[\mathcal{R}_1] - \tilde{e}_1,
$$

where

$$
\mathcal{K}_0[p, \xi] = \mathcal{K}_0[p, \xi] + \mathcal{K}_2[p, \xi]
$$

with

$$
\mathcal{K}_0[p, \xi] := -\frac{2}{\lambda} \rho w^2 \int_{-T}^{t} \text{Re}(\dot{p}(s)e^{-i\omega(t)}) Q_{\omega, \alpha, \beta} E_1
+ \text{Im}(\dot{p}(s)e^{-i\omega(t)}) Q_{\omega, \alpha, \beta} E_2 \cdot k(z, t - s) ds,
$$

$$
\mathcal{K}_2[p, \xi]
:= \frac{1}{\lambda} \rho w^2 \left[ \hat{\lambda} - \int_{-T}^{t} \text{Re}(\dot{p}(s)e^{-i\omega(t)}) r k_z(z, t - s) ds \right] Q_{\omega, \alpha, \beta} E_1
- \frac{1}{4\lambda} \rho w^2 \cos \omega \left[ \int_{-T}^{t} \text{Re}(\dot{p}(s)e^{-i\omega(t)}) (zk_z - z^2 k_{zz})(z, t - s) ds \right] Q_{\omega, \alpha, \beta} E_1
$$
$$\kappa_1[p, \xi] := \frac{1}{\lambda} \omega \rho \left[ \cos \beta - \beta \cos \cos \beta \right]$$

Next we consider the new error estimates produced by \( \Phi^\alpha \) and \( \Phi^\beta \). It is obvious that \( \tilde{L}_U[\Phi^\alpha] = 0 \) and \( \tilde{L}_U[\Phi^\beta] = 0 \). Direct computations show that

\[
Q_{\omega, \alpha, \beta}^{-1} \left( \frac{d}{dt} Q_{\omega, \alpha, \beta} \right) \begin{bmatrix} 0 \\ \alpha \\ 0 \end{bmatrix} = \begin{bmatrix} -\omega \alpha \cos \beta - \alpha \dot{\alpha} \sin \beta \\ 0 \\ \alpha \alpha \cos \beta - \omega \alpha \cos \sin \beta \end{bmatrix},
\]

\[
Q_{\omega, \alpha, \beta}^{-1} \left( \frac{d}{dt} Q_{\omega, \alpha, \beta} \right) \begin{bmatrix} -\beta \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \dot{\omega} \sin \alpha + \beta \\ \omega (\cos \alpha \sin \beta - \beta \cos \cos \beta) - \alpha (\beta \sin \beta + \cos \beta) \\ \omega \beta \sin \alpha + \beta \beta \end{bmatrix},
\]

and thus

\[
-\partial_t \Phi^\alpha + \Delta \Phi^\alpha - \mathcal{E}_{-1,2} = Q_{\omega, \alpha, \beta} \begin{bmatrix} \dot{\omega} \alpha \cos \beta + \alpha \sin \beta \left( \alpha + \frac{2p}{1+p^2} \sin \theta \right) \\ -\dot{\alpha} \left( 1 - \frac{\rho^2}{1+p^2} \sin \beta \cos \cos \beta \right) \\ \omega \alpha \cos \sin \beta - \alpha \cos \beta \left( \alpha + \frac{2p}{1+p^2} \sin \theta \right) \end{bmatrix}
\]

\[
= \mathcal{R}_{-1,2}[\alpha, \beta]
\]

and

\[
-\partial_t \Phi^\beta + \Delta \Phi^\beta - \mathcal{E}_{-1,1} = Q_{\omega, \alpha, \beta} \begin{bmatrix} \dot{\omega} \beta - \omega \sin \alpha - \beta \\ \frac{2p}{1+p^2} \beta - \omega \sin \alpha - \beta \\ -\dot{\omega} \beta \sin \alpha - \beta \left( \beta - \frac{2p}{1+p^2} \cos \theta \right) \end{bmatrix}
\]

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Consequently, we obtain

\[-\partial_t (\Phi^\alpha + \Phi^\beta) + \Delta (\Phi^\alpha + \Phi^\beta) - \mathcal{E}_{-1} = R_{-1} [\alpha, \beta],\]

where

\[R_{-1} [\alpha, \beta] := R_{-1,1} [\alpha, \beta] + R_{-1,2} [\alpha, \beta].\] (5.28)

### 5.2.3 Inner–outer gluing system

Collecting the error estimates in the previous section, we will get a solution solving (5.20) if the pair \((\phi, \Psi^*)\) solves the inner–outer gluing system

\[
\begin{cases}
\lambda^2 \partial_t \phi = L_W [\phi] + \lambda^2 Q_{\omega, \alpha, \beta}^{-1} \left[ \bar{L}_U [\Psi^*] + \mathcal{K}_0 [p, \xi] + \mathcal{K}_1 [p, \xi] \right] + \Pi_{U^\perp \left[ R_{-1} [\alpha, \beta] \right]} & \text{in } \mathcal{D}_{2R}, \\
\phi (\cdot, 0) = 0 & \text{in } B_{R(0)}, \\
\phi \cdot W = 0 & \text{in } \mathcal{D}_{2R},
\end{cases}
\] (5.29)

\[\partial_t \Psi^* = \Delta_x \Psi^* + \mathcal{G} [p, \xi, \Psi^*, \alpha, \beta, \phi] \text{ in } \Omega \times (0, T), \] (5.30)

where

\[\mathcal{G} [p, \xi, \Psi^*, \alpha, \beta, \phi] := (1 - \eta_R) \bar{L}_U [\Psi^*] + (\Psi^* \cdot U) U_t + Q_{\omega, \alpha, \beta} (\phi \Delta_x \eta_R + 2 \nabla_x \eta_R \cdot \nabla_x \phi - \phi \partial_y \eta_R) + \eta_R Q_{\omega, \alpha, \beta} (-(Q_{\omega, \alpha, \beta})^{-1} \frac{d}{dt} Q_{\omega, \alpha, \beta}) \phi + \lambda^{-1} \lambda_Y \cdot \nabla_y \phi + \lambda^{-1} \lambda_T \cdot \nabla_y \phi + (1 - \eta_R) (\mathcal{K}_0 [p, \xi] + \mathcal{K}_1 [p, \xi] + \Pi_{U^\perp \left[ R_{-1} [\alpha, \beta] \right]} - \Pi_{U^\perp \left[ R_{1} \right]} + N_U [\eta_R Q_{\omega, \alpha, \beta} \phi + \Pi_{U^\perp \left[ \Phi^0 + \Phi^\alpha + \Phi^\beta + \Psi^* \right]} + (\Phi^0 + \Phi^\alpha + \Phi^\beta) \cdot U) U_t, \]
the linearization $L_W[\phi]$ is defined in (5.8), and

$$\mathcal{D}_{2R} := \{(y, t) : y \in B_{2R(t)}, t \in (0, T)\}$$

with the radius

$$R = R(t) = \lambda_s(t)^{-\gamma_s} \text{ with } \lambda_s(t) = \frac{|\log T|(T - t)}{|\log(T - t)|^2} \text{ and } \gamma_s \in (0, 1/2). \quad (5.31)$$

The reason for choosing such $R(t)$ and $\lambda_s(t)$ will be made clear later on. If the pair $(\phi, \Psi^*)$ solves the inner–outer gluing system (5.29)–(5.30), then we get a desired solution

$$u(x, t) = U + \Pi_U^{-1}[\eta_R Q_{\omega, \alpha, \beta} \phi + \Psi^r + \Phi^0 + \Phi^\alpha + \Phi^\beta]$$

$$+ a(\Pi_U^{-1}[\eta_R Q_{\omega, \alpha, \beta} \phi + \Psi^r + \Phi^0 + \Phi^\alpha + \Phi^\beta])U,$$

which solves problem (5.14). We take the boundary condition $u|_{\partial \Omega} = e_3 := \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, which amounts to

$$\Pi_U^{-1}[\Psi^r + \Phi^0 + \Phi^\alpha + \Phi^\beta] + a(\Pi_U^{-1}[U + \Psi^r + \Phi^0 + \Phi^\alpha + \Phi^\beta])U$$

$$= e_3 - U \text{ on } \partial \Omega \times (0, T).$$

So it suffices to take the boundary condition for the outer problem (5.30) as

$$\Psi^r|_{\partial \Omega} = e_3 - U - \Phi^0 - \Phi^\alpha - \Phi^\beta.$$

### 5.2.4 Reduced equations for parameter functions

In this section, we will derive the parameter functions $\lambda(t)$, $\xi(t)$, $\omega(t)$, $\alpha(t)$ and $\beta(t)$ at leading order as $t \to T$.
The inner problem (5.29) has the form

\[
\begin{cases}
\lambda^2 \phi_t = L_W[\phi] + h[p, \xi, \alpha, \beta, \Psi^*](y, t) & \text{in } D_{2R}, \\
\phi \cdot W = 0 & \text{in } D_{2R}, \\
\phi(\cdot, 0) = 0 & \text{in } B_{2R(0)}.
\end{cases}
\] (5.32)

Here we recall that we write \( p(t) = \lambda(t)e^{i\omega(t)} \). For convenience, we assume that \( h(y, t) \) is defined for all \( y \in \mathbb{R}^2 \) extending outside \( D_{2R} \) as

\[
h[p, \xi, \alpha, \beta, \Psi^*] = \lambda^2 Q_{\omega, \alpha, \beta}^{-1} \chi_{D_{2R}}[L_W[\Psi^*] + \mathcal{K}_0[p, \xi] + \mathcal{K}_1[p, \xi] + \Pi U_1]\chi_{D_{2R}}[\alpha, \beta],
\]

where \( \chi_A \) denotes the characteristic function of a set \( A \), \( \mathcal{K}_0 \) is defined in (5.23), (5.24), \( \mathcal{K}_1 \) in (5.25) and \( \mathcal{R}_1 \) in (5.28). If \( \lambda(t) \) has a relatively smooth vanishing as \( t \to T \), it is then natural that the term \( \lambda^2 \phi_t \) is of smaller order and the equation (5.32) is approximated by the elliptic problem

\[
L_W[\phi] + h[p, \xi, \alpha, \beta, \Psi^*] = 0, \quad \phi \cdot W = 0 \quad \text{in } B_{2R}. \quad (5.33)
\]

We consider the kernel functions \( Z_{l,j}(y) \) defined in (5.9), which satisfy \( L_W[Z_{l,j}] = 0 \) for \( l = 0, \pm 1, j = 1, 2 \). If there is a solution \( \phi(y, t) \) to (5.33) with sufficient decay, then necessarily

\[
\int_{B_{2R}} h[p, \xi, \alpha, \beta, \Psi^*](y, t) \cdot Z_{l,j}(y) \, dy = 0 \quad \text{for all } t \in (0, T), \quad (5.34)
\]

for \( l = 0, \pm 1, j = 1, 2 \). These orthogonality conditions (5.34) amount to an integro-differential system of equations for \( p(t) \), \( \xi(t) \), \( \alpha(t) \), \( \beta(t) \), which, as a matter of fact, determine the correct values of the parameter functions so that the solution pair \( (\phi, \Psi^*) \) with appropriate asymptotics exists.

For the reduced equations of \( p(t) \) and \( \xi(t) \) which correspond to mode \( l = 0 \) and mode \( l = 1 \), respectively, we invoke some useful expressions and results in [55, Section 5]. Let

\[
\mathcal{B}_0[p](t)
\]
\[ := \frac{\lambda}{2\pi} \int_{\mathbb{R}^2} Q^{-1}_{\omega, \alpha, \beta}[\mathcal{K}_0[p, \xi] + \mathcal{K}_1[p, \xi] + \Pi_U'[\mathcal{K}^{-1}_1[\alpha, \beta]]] \cdot Z_{0,j}(y) \, dy, \quad j = 1, 2. \]

From (5.28), (5.27) and (5.26), direct computations yield

\[ \int_{B_{2R}} Q^{-1}_{\omega, \alpha, \beta}[\mathcal{K}_0[p, \xi] + \mathcal{K}_1[p, \xi] + \Pi_U'[\mathcal{K}^{-1}_1[\alpha, \beta]]] \cdot Z_{0,1}(y) \, dy = \pi \left( -\frac{16R^2}{4R^2 + 1} + 4\log(4R^2 + 1) \right) (\dot{\omega}\alpha \cos \beta - \alpha \ddot{\alpha} \sin \beta - \dot{\omega}\beta \sin \alpha + \beta \ddot{\beta}), \]

(5.35)

\[ \int_{B_{3R}} Q^{-1}_{\omega, \alpha, \beta}[\mathcal{K}_0[p, \xi] + \mathcal{K}_1[p, \xi] + \Pi_U'[\mathcal{K}^{-1}_1[\alpha, \beta]]] \cdot Z_{0,2}(y) \, dy = \pi \left( -\frac{16R^2}{4R^2 + 1} + 4\log(4R^2 + 1) \right) \dot{\alpha} \sin \beta. \]

(5.36)

Combining (5.23), (5.24) with (5.35) and (5.36), the following expressions for \( B_{01} \) and \( B_{02} \) are readily obtained by similar computations as in [55, Section 5]

\[ B_{01}[p](t) = \int_{-T}^{t} \text{Re} (\hat{p}(s)e^{-i\omega(t)}) \Gamma_1 \left( \frac{\lambda(t)^2}{t - s} \right) \frac{ds}{t - s} - 2\dot{\lambda}(t) + o(1), \]

\[ B_{02}[p](t) = \int_{-T}^{t} \text{Im} (\hat{p}(s)e^{-i\omega(t)}) \Gamma_2 \left( \frac{\lambda(t)^2}{t - s} \right) \frac{ds}{t - s}, \]

where \( o(1) \to 0 \) as \( t \to T \), and \( \Gamma_j(\tau) \) are smooth functions defined as

\[ \Gamma_1(\tau) = -\int_0^{\rho} \rho^3 w_\rho^3 \left[ K(\xi) + 2\xi K_\xi(\xi) \frac{\rho^2}{1 + \rho^2} - 4\cos(w) \xi^2 K_{\xi\xi}(\xi) \right]_{\xi = \tau(1 + \rho^2)} d\rho, \]

\[ \Gamma_2(\tau) = -\int_0^{\rho} \rho^3 w_\rho^3 \left[ K(\xi) - \xi^2 K_{\xi\xi}(\xi) \right]_{\xi = \tau(1 + \rho^2)} d\rho, \]

where \( K(\xi) = 2^{1-\xi} \xi^2 \). Using the expressions of \( \Gamma_j(\tau) \), we get

\[ \left\{ \begin{array}{ll}
|\Gamma_j(\tau) - 1| \leq C\tau(1 + |\log \tau|) & \text{for } \tau < 1, \\
|\Gamma_j(\tau)| \leq \frac{C}{\tau} & \text{for } \tau > 1.
\end{array} \right. \]
We thus obtain that the four conditions (5.34) for $B$

Similarly, we let

Directly using the expressions (5.28), (5.27) and (5.26), we have

Therefore, by (5.25), (5.9) and the fact that $\int_0^\infty \rho w^2 d\rho = 2$, we obtain

At last, we let

We thus obtain that the four conditions (5.34) for $l = 0, 1$ are reduced to the system

Define

$$\mathcal{B}_0[p] := \frac{1}{2} e^{i\omega(t)} (\mathcal{B}_{01}[p] + i\mathcal{B}_{02}[p]),$$

(5.37)

$$a_0[p, \xi, \alpha, \beta, \Psi^*] := -\frac{\lambda}{2\pi} \int_{B_{3R}} Q_{\omega, \alpha, \beta}^{-1} \hat{L}_U[\Psi^*] \cdot Z_{0,j}(y) \, dy, \quad j = 1, 2,$$

$$a_0[p, \xi, \alpha, \beta, \Psi^*] := \frac{1}{2} e^{i\omega(t)} (a_{01}[p, \xi, \alpha, \beta, \Psi^*] + ia_{02}[p, \xi, \alpha, \beta, \Psi^*]).$$

Similarly, we let

$$\mathcal{B}_1[\xi](t) := \frac{\lambda}{2\pi} \int_{R^2} Q_{\omega, \alpha, \beta}^{-1} [\mathcal{X}_0[p, \xi] + \mathcal{X}_1[p, \xi] + \mathcal{R}_{-1}[\alpha, \beta]] \cdot Z_{1,j}(y) \, dy, \quad j = 1, 2,$$

$$\mathcal{B}_1[\xi](t) := \mathcal{B}_{11}[\xi](t) + i\mathcal{B}_{12}[\xi](t).$$

Directly using the expressions (5.28), (5.27) and (5.26), we have

$$\int_{B_{3R}} Q_{\omega, \alpha, \beta}^{-1} \hat{\Pi}_U[\mathcal{R}_{-1}[\alpha, \beta]] \cdot Z_{1,1}(y) \, dy$$

$$= \frac{8\pi R^2}{4R^2 + 1} (\omega \alpha \cos \alpha \cos \beta + \alpha \alpha \sin \beta - \omega \sin \alpha + \beta),$$

$$\int_{B_{3R}} Q_{\omega, \alpha, \beta}^{-1} \hat{\Pi}_U[\mathcal{R}_{-1}[\alpha, \beta]] \cdot Z_{1,2}(y) \, dy$$

$$= -\frac{8\pi R^2}{4R^2 + 1} (\alpha - \omega \beta \cos \alpha \cos \beta - \alpha \beta \sin \beta + \omega \cos \alpha \sin \beta).$$

Therefore, by (5.25), (5.9) and the fact that $\int_0^\infty \rho w^2 d\rho = 2$, we obtain

$$\mathcal{B}_1[\xi](t) = 2[\xi_1(t) + i\xi_2(t) + o(1)] \text{ as } t \to T.$$
of two complex equations

\[
\mathcal{B}_0[p] = a_0[p, \xi, \alpha, \beta, \Psi^*], \quad (5.38)
\]

\[
\mathcal{B}_1[\xi] = a_1[p, \xi, \alpha, \beta, \Psi^*]. \quad (5.39)
\]

We observe that

\[
\mathcal{B}_0[p] = \int_{t - \lambda}^{t} \frac{\dot{p}(s)}{t - s} ds + O(\|p\|_\infty) + o(1) \quad \text{as} \quad t \to T.
\]

To get an approximation for \(a_0\), we need to analyze the operator \(\tilde{L}_U\) in \(a_0\). To this end, we write

\[
\Psi^* = \begin{bmatrix} \psi^* \\ \psi_3^* \end{bmatrix}, \quad \psi^* = \psi_1^* + i\psi_2^*.
\]

From (5.12) and (5.13), we have

\[
\tilde{L}_U[\Psi^*](y, t) = [\tilde{L}_U]_0[\Psi^*] + [\tilde{L}_U]_1[\Psi^*] + [\tilde{L}_U]_2[\Psi^*],
\]

where

\[
\begin{cases}
[\tilde{L}_U]_0[\Psi^*] = \lambda^{-1} Q_{\omega, \alpha, \beta} \rho w_p^2 \left[ \div (e^{-i\omega \Psi^*})E_1 + \curl (e^{-i\omega \Psi^*})E_2 \right], \\
[\tilde{L}_U]_1[\Psi^*] = -2\lambda^{-1} Q_{\omega, \alpha, \beta} \rho w_p \cos \theta \left[ (\partial_{\xi_1} \psi_3^*) \cos \theta + (\partial_{\xi_2} \psi_3^*) \sin \theta \right]E_1 \\
-2\lambda^{-1} Q_{\omega, \alpha, \beta} \rho \cos \theta \left[ (\partial_{\xi_1} \psi_3^*) \sin \theta - (\partial_{\xi_2} \psi_3^*) \cos \theta \right]E_2, \\
[\tilde{L}_U]_2[\Psi^*] = \lambda^{-1} Q_{\omega, \alpha, \beta} \rho w_p^2 \left[ \div (e^{i\omega \Psi^*}) \cos 2\theta - \curl (e^{i\omega \Psi^*}) \sin 2\theta \right]E_1 \\
+ \lambda^{-1} Q_{\omega, \alpha, \beta} \rho w_p^2 \left[ \div (e^{i\omega \Psi^*}) \sin 2\theta + \curl (e^{i\omega \Psi^*}) \cos 2\theta \right]E_2,
\end{cases}
\]

and the differential operators in \(\Psi^*\) on the right hand sides are evaluated at \((x, t)\) with \(x = \xi(t) + \lambda(t)y\), \(y = \rho e^{i\theta}\) while \(E_j = E_j(y)\) for \(j = 1, 2\). From the above decomposition, assuming that \(\Psi^*\) is of class \(C^1\) in the space variable, we then get

\[
a_0[p, \xi, \alpha, \beta, \Psi^*] = [\div \psi^* + i\curl \psi^*](\xi, t) + o(1) \quad \text{as} \quad t \to T.
\]

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Similarly, since $\int_{0}^{\infty} w_{\rho}^{2} \cos w \rho \, d\rho = 0$, we get

$$a_1[p, \xi, \alpha, \beta, \Psi] = 2(\partial_{\alpha} \xi_{3} + i \partial_{\beta} \xi_{3})(\xi, t) \int_{0}^{\infty} \cos w w_{\rho}^{2} \rho \, d\rho + o(1)$$

$$= o(1) \text{ as } t \to T.$$  

We now simplify the system (5.38)–(5.39) in the form

$$\int_{-T}^{t-\lambda^{2}(t)} \frac{\dot{p}(s)}{t-s} \, ds = [\text{div} \Psi^* + i \text{curl} \Psi^*](\xi(t), t) + o(1) + O(||\dot{p}||_{\infty}),$$

$$\dot{\xi}(t) = o(1) \text{ as } t \to T. \quad (5.40)$$

For the moment, we assume that the function $\Psi^*(x, t)$ is fixed and sufficiently regular, and we regard $T$ as a parameter that will always be taken smaller if necessary. We recall that we need $\xi(T) = q$ where $q \in \Omega$ is given, and $\lambda(T) = 0$. Equation (5.40) immediately suggests us to take $\xi(t) \equiv q$ as the first approximation. Neglecting lower order terms, $p(t) = \lambda(t)e^{i\omega(t)}$ satisfies the following integro-differential system

$$\int_{-T}^{t-\lambda^{2}(t)} \frac{\dot{p}(s)}{t-s} \, ds = \text{div} \Psi^*(q, 0) + i \text{curl} \Psi^*(q, 0) =: a_0^*.$$  

(5.41)

At this point, we make the following assumption

$$\text{div} \Psi^*(q, 0) < 0,$$  

(5.42)

which implies that $a_0^* = -|a_0^*|e^{i\omega_0}$ for a unique $\omega_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Let us take

$$\omega(t) \equiv \omega_0.$$  

Equation (5.41) then becomes

$$\int_{-T}^{t-\lambda^{2}(t)} \frac{\dot{\lambda}(s)}{t-s} \, ds = -|a_0^*|. \quad (5.43)$$
We claim that a good approximate solution to (5.43) as $t \to T$ is given by
\[ \dot{\lambda}(t) = -\frac{\kappa}{\log^2(T-t)} \]
for a suitable $\kappa > 0$. Indeed, we have
\[
\int_{-T}^{t} \frac{\dot{\lambda}(s)}{t-s} \, ds = \int_{-T}^{t-(T-t)} \frac{\dot{\lambda}(s)}{t-s} \, ds + \dot{\lambda}(t) \left[ \log(T-t) - 2\log(\lambda(t)) \right]
\]
\[
+ \int_{t-(T-t)}^{t} \frac{\dot{\lambda}(s) - \dot{\lambda}(t)}{t-s} \, ds
\]
\[
\approx \int_{-T}^{t} \frac{\dot{\lambda}(s)}{T-s} \, ds - \dot{\lambda}(t) \log(T-t) := \Upsilon(t)
\]
as $t \to T$. We see that
\[ \log(T-t) \frac{d\Upsilon(t)}{dt} = \frac{d}{dt} \left( \log^2(T-t) \dot{\lambda}(t) \right) = 0 \]
from the explicit form of $\dot{\lambda}(t)$. Thus $\Upsilon(t)$ is a constant. As a consequence, equation (5.43) is approximately satisfied if $\kappa$ is such that
\[ \kappa \int_{-T}^{T} \frac{\dot{\lambda}(s)}{T-s} \, ds = -|a_0^*|, \]
which finally gives us the approximate expression
\[ \dot{\lambda}(t) = -|\text{div} \, \psi^*(q,0) + i \text{curl} \, \psi^*(q,0)| \dot{\lambda}_s(t), \]
where
\[ \dot{\lambda}_s(t) = -\frac{\log T}{\log^2(T-t)}. \]
Naturally, imposing $\lambda_s(T) = 0$, we then have
\[ \dot{\lambda}_s(t) = \frac{\log T}{\log^2(T-t)} \frac{1}{(1 + o(1))} \text{ as } t \to T. \quad (5.44) \]
Next, we consider (5.34) for the case of mode $l = -1$, which gives the reduced
equations of $\alpha(t)$ and $\beta(t)$. By (5.28), (5.27) and (5.26), we evaluate
\begin{align*}
\int_{B_{2R}}^{} Q^{-1}_{\omega, \alpha, \beta} \Pi_U - [\mathcal{R}_{-1}[\alpha, \beta]] \cdot Z_{-1,1}(y) \, dy \\
= 4\pi \left( -\frac{4R^2(2R^2+1)}{4R^2+1} + \log(4R^2+1) \right) \\
\times (-\dot{\beta} - \dot{\omega} \sin \alpha + \dot{\omega} \alpha \cos \alpha \cos \beta + \dot{\alpha} \alpha \sin \beta) \\
= 8\pi \left[ (R^2 - \log R) \dot{\beta} (1 + o(1)) \right],
\end{align*}
\begin{align*}
\int_{B_{2R}}^{} Q^{-1}_{\omega, \alpha, \beta} \Pi_U - [\mathcal{R}_{-1}[\alpha, \beta]] \cdot Z_{-1,2}(y) \, dy \\
= 4\pi \left( \frac{4R^2(2R^2+1)}{4R^2+1} - \log(4R^2+1) \right) \\
\times (\dot{\alpha} (1 - \beta \sin \beta - 2 \cos \beta) + \dot{\omega} \cos \alpha (\sin \beta - \beta \cos \beta)) \\
= 8\pi \left[ (-R^2 + \log R) \dot{\alpha} (1 + o(1)) \right],
\end{align*}
where we recall that $\omega(t) \equiv \omega_0$. Since
\begin{align*}
\int_{B_{2R}}^{} \lambda^2 Q^{-1}_{\omega, \alpha, \beta} \left[ \tilde{L}_U \Psi^r + \mathcal{K}_0 + \mathcal{K}_1 \right] \cdot Z_{-1,j}(y) \, dy = c_j \lambda
\end{align*}
for some $c_j \in \mathbb{R}$, for $j = 1, 2$, the orthogonality condition (5.34) with $l = -1$ gives
\begin{align*}
8\pi \lambda^2 (-R^2 + \log R) \dot{\beta} (1 + o(1)) = c_1 \lambda, \\
8\pi \lambda^2 (R^2 - \log R) \dot{\alpha} (1 + o(1)) = c_2 \lambda.
\end{align*}
Thus, by (5.44) and the definition of $R = R(t)$ in (5.31), good choices for $\alpha(t)$ and $\beta(t)$ at leading orders are
\begin{align*}
\alpha(t) = c_\alpha (T - t)^{\delta_1} (1 + o(1)), \quad \beta(t) = c_\beta (T - t)^{\delta_2} (1 + o(1)) \quad \text{as} \quad t \to T
\end{align*}
for some $\delta_1, \delta_2 > 0$ and $c_\alpha, c_\beta \in \mathbb{R}$.

### 5.2.5 Linear theory for the inner problem

To capture the heart of the singularity formation, a linear theory of the inner problem (5.29) is required. In contrast with that by [55], it turns out that we will have
to establish a decay estimate of second order derivative of \( \phi \) in order to handle the coupling effects between the inner–outer problem of \( u \) and that of \( v \) below. We consider

\[
\begin{align*}
\lambda^2 \partial_t \phi &= L_W[\phi] + h(y,t) & \text{in} & \quad \mathcal{D}_{2R}, \\
\phi(\cdot,0) &= 0 & \text{in} & \quad B_{2R(0)}, \\
\phi \cdot W &= 0 & \text{in} & \quad \mathcal{D}_{2R},
\end{align*}
\]

where we recall from (5.31) that

\[
R = R(t) = \lambda_\ast(t)^{-\gamma} \quad \text{with} \quad \lambda_\ast(t) = \frac{|\log T|(T-t)}{|\log(T-t)|^2} \quad \text{and} \quad \gamma \in (0, 1/2).
\]

We regard \( h(y,t) \) as a function defined in \( \mathbb{R}^2 \times (0,T) \) with compact support, and assume that \( h(y,t) \) has the space-time decay of the following type

\[
|h(y,t)| \lesssim \frac{\lambda_\ast^\nu(t)}{1 + |y|^a}, \quad h \cdot W = 0,
\]

where \( \nu > 0 \) and \( a \in (2, 3) \). Define the norm

\[
\|h\|_{\nu,a} := \sup_{(y,t) \in \mathbb{R}^2 \times (0,T)} \lambda_\ast^{-\nu}(t)(1 + |y|^a)|h(y,t)|.
\]

In the polar coordinates, \( h(y,t) \) can be written as

\[
h(y,t) = h^1(\rho, \theta, t)E_1(y) + h^2(\rho, \theta, t)E_2(y), \quad y = \rho e^{i\theta}
\]

since \( h \cdot W = 0 \). Expanding in the Fourier series, we write

\[
\tilde{h}(\rho, \theta, t) := h^1 + ih^2 = \sum_{k=-\infty}^{\infty} \tilde{h}_k(\rho, t)e^{ik\theta}, \quad \tilde{h}_k = \tilde{h}_{k1} + i\tilde{h}_{k2}
\]

such that

\[
h(y,t) = \sum_{k=-\infty}^{\infty} h_k(y,t) := h_0(y,t) + h_1(y,t) + h_{-1}(y,t) + h_\perp(y,t)
\]
with

\[ h_k(y,t) = \text{Re}(\tilde{h}_k(p,t)e^{ik\theta})E_1 + \text{Im}(\tilde{h}_k(p,t)e^{ik\theta})E_2, \quad k \in \mathbb{Z}. \quad (5.48) \]

We consider the kernel functions \( Z_{k,j} \) defined in (5.9), and define

\[ \tilde{h}_k(y,t) := \sum_{j=1}^{2} \frac{\chi Z_{k,j}(y)}{\int_{\mathbb{R}^2} \chi |Z_{k,j}|^2} \int_{\mathbb{R}^2} h(z,t) \cdot Z_{k,j}(z) dz, \quad k = 0, \pm 1, j = 1, 2, \quad (5.49) \]

where

\[ \chi(y,t) = \begin{cases} \frac{w^2(|y|)}{R^2} & \text{if } |y| < 2R(t), \\ 0 & \text{if } |y| \geq 2R(t). \end{cases} \]

**Proposition 5.2.1.** Assume that \( a \in (2,3), \nu > 0, \delta \in (0,1) \) and \( \|h\|_{v,a} < +\infty. \)

Let us write \( h = h_0 + h_1 + h_{-1} + h_{-\perp} \) with \( h_{\perp} = \sum_{k \neq 0, \pm 1} h_k. \) Then there exists a solution \( \phi[h] \) of problem (5.45), which defines a linear operator of \( h, \) and satisfies the following estimate in \( \mathcal{D}_{2R} \)

\[
|\phi(y,t)| + (1 + |y|)|\nabla \phi(y,t)| + (1 + |y|)^2 |\nabla^2 \phi(y,t)| \lesssim \lambda^v(t) \min \left\{ \frac{R^{\delta(5-a)}(t)}{1 + |y|^3}, \frac{1}{1 + |y|^{a-2}} \right\} \|h_0 - \tilde{h}_0\|_{v,a} + \frac{\lambda^v(t)R^2(t)}{1 + |y|} \|\tilde{h}_0\|_{v,a} \\
+ \frac{\lambda^v(t)}{1 + |y|^{a-2}} \|h_1 - \tilde{h}_1\|_{v,a} + \frac{\lambda^v(t)R^4(t)}{1 + |y|^2} \|\tilde{h}_1\|_{v,a} + \lambda^v(t) \|h_{-1} - \tilde{h}_{-1}\|_{v,a} \\
+ \lambda^v(t) \log R(t) \|\tilde{h}_{-\perp}\|_{v,a} + \frac{\lambda^v(t)}{1 + |y|^{a-2}} \|h_{\perp}\|_{v,a}.
\]

The construction of the solution \( \phi \) to problem (5.45) will be carried out in each Fourier mode. Write

\[ \phi = \sum_{k=-\infty}^{\infty} \phi_k, \quad \phi_k(y,t) = \text{Re}(\phi_k(p,t)e^{ik\theta})E_1 + \text{Im}(\phi_k(p,t)e^{ik\theta})E_2. \]

In each mode \( k, \) the pair \((\phi_k, h_k)\) satisfies

\[
\begin{align*}
\lambda^2 \partial_t \phi_k &= L_W[\phi_k] + h_k(y,t) & \text{in } \mathcal{D}_{4R}, \\
\phi_k(y,0) &= 0 & \text{in } B_{4R(0)},
\end{align*}
\]

(5.50)
which is equivalent to the following problem

\[
\begin{aligned}
\lambda^2 \partial_t \phi_k &= \mathcal{L}_k[\phi_k] + \tilde{h}_k(\rho, t) \quad \text{in } \mathcal{D}_4R, \\
\phi_k(\rho, 0) &= 0 \quad \text{in } (0, 4R(0)),
\end{aligned}
\]

where \(\mathcal{D}_4R = \{(\rho, t) : t \in (0, T), \rho \in (0, 4R(t))\}\), and

\[
\mathcal{L}_k[\phi_k] := \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} \phi_k + \frac{\partial}{\partial \rho} \phi_k - \left( k^2 + 2k \cos \omega + \cos(2\omega) \right) \phi_k / \rho^2.
\]

It is direct to see that the kernel functions for \(\mathcal{L}_k\) such that \(\mathcal{L}_k[Z_k] = 0\) at modes \(k = 0, \pm 1\) are given by

\[
Z_0(\rho) = \frac{\rho}{1 + \rho^2}, \quad Z_1(\rho) = \frac{1}{1 + \rho^2}, \quad Z_{-1}(\rho) = \frac{2\rho^2}{1 + \rho^2}.
\]

We have the following lemma proved in [55, Section 7].

**Lemma 5.2.1 ([55]).** Suppose \(\nu > 0\), \(0 < a < 3\), \(a \neq 1, 2\) and \(\|h_k(y, t)\|_{v,a} < +\infty\). Then problem (5.50) has a unique solution which takes the form

\[
\phi_k(y, t) = \text{Re}(\phi_k(\rho, t)e^{ik\theta})E_1 + \text{Im}(\phi_k(\rho, t)e^{ik\theta})E_2
\]

and satisfies the boundary condition

\[
\phi_k(y, t) = 0, \quad y \in \partial B_{4R(t)}(0), \quad \forall t \in (0, T).
\]

Moreover, the following estimates hold

\[
|\phi_k(y, t)| \lesssim \lambda^\nu k^{-2} \|h\|_{v,a} \begin{cases} R^{2-a} & \text{for } a < 2 \\
(1 + \rho)^{2-a} & \text{for } a > 2
\end{cases} \text{ for } k \geq 2,
\]

\[
|\phi_{-1}(y, t)| \lesssim \lambda^\nu \|h\|_{v,a} \begin{cases} R^{2-a} & \text{for } a < 2 \\
\log R & \text{for } a > 2
\end{cases}.
\]

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The higher regularity estimates for solutions constructed in Lemma 5.2.1 are given by the following lemma. Before we state the lemma, we first introduce the Hölder semi-norm, which is better expressed in the \((y, \tau)\)-variable. Define

\[
\tau(t) = \int_0^t \frac{ds}{\lambda^2(s)}
\]

so that

\[
\begin{cases}
\partial_\tau \phi = L_W[\phi] + h(y, \tau) & \text{in } \mathcal{D}_{4\gamma R}, \\
\phi(\cdot, 0) = 0 & \text{in } B_{4\gamma R(0)}.
\end{cases}
\]

We denote the parabolic ball

\[
\mathcal{B}_\ell(y, \tau) = \{(y', \tau') : |y - y'|^2 + |\tau - \tau'| < \ell^2\},
\]

and also introduce the Hölder semi-norm

\[
[g]_{\alpha, A} := \sup_{(y, \tau), (y', \tau') \in A} \frac{|g(y, \tau) - g(y', \tau')|}{|y - y'|^\alpha + |\tau - \tau'|^{\alpha/2}}
\]

for \(\alpha \in (0, 1)\) and a set \(A\). We denote \(C^{\alpha, \alpha/2}(A)\) by the set of functions on \(A\) such that \([g]_{\alpha, A} < +\infty\), endowed with the norm

\[
\|g\|_{C^{\alpha, \alpha/2}(A)} = \|g\|_{L^\infty(A)} + [g]_{\alpha, A}.
\]

**Lemma 5.2.2.** Let \(\phi\) be a solution to

\[
\begin{cases}
\lambda^2 \partial_t \phi = L_W[\phi] + h(y, t) & \text{in } \mathcal{D}_{4\gamma R}, \\
\phi(\cdot, 0) = 0 & \text{in } B_{4\gamma R(0)},
\end{cases}
\]

where \(h(y, t) \in C^{\alpha, \alpha/2}(\mathcal{B}_\ell(y, \tau) \cap \mathcal{D}_{4\gamma R})\) for some \(\alpha > 0\) and \(\ell = \frac{|y|}{4} + 1\). If for some
in $B$ where the coefficients $A$, $\tau$, $\lambda$ and $\rho \tau$ are uniformly bounded by $O((1 + \rho)^{-2})$ in $B_1(0) \times (-1, 0)$ and $h(z, s) = \rho^2 h(y_1 + \rho z, \tau_1 + \rho^2 s)$. Let $b' > 0$ such that $\tau^{-b'} \sim \lambda^b(t)$ from (5.52). By the facts $\rho \leq CR(\tau_1)$ and $R^2(\tau_1) \ll \tau_1$ for $\tau_1$ large, we have

$$C_1 \tau^{-b'} \leq (\tau_1 + \rho^2 s)^{-b'} \leq C_2 \tau^{-b'}$$

for the case $\tau_1 < \rho^2$, $\tilde{\phi}(z, s)$ satisfies the following equation

$$\partial_s \tilde{\phi} = \Delta_z \tilde{\phi} + A(z, s) \cdot \nabla_z \tilde{\phi} + B(z, s) \tilde{\phi} + \tilde{h}(z, s) \quad \text{in } B_1(0) \times (-1, 0],$$

where the coefficients $A(z, s)$ and $B(z, s)$ are uniformly bounded by $O((1 + \rho)^{-2})$ in $B_1(0) \times (-1, 0]$ and $\tilde{h}(z, s) = \rho^2 h(y_1 + \rho z, \tau_1 + \rho^2 s)$. Let $b' > 0$ such that $\tau^{-b'} \sim \lambda^b(t)$ from (5.52). By the facts $\rho \leq CR(\tau_1)$ and $R^2(\tau_1) \ll \tau_1$ for $\tau_1$ large, we have

$$C_1 \tau^{-b'} \leq (\tau_1 + \rho^2 s)^{-b'} \leq C_2 \tau^{-b'}$$

Proof. In the $(y, \tau)$-variable with $\tau$ given by (5.52), problem (5.53) reads as

$$\begin{aligned}
\partial_\tau \phi &= L_W [\phi] + h(y, \tau) \quad \text{in } \mathcal{D}_{4\gamma R}, \\
\phi (\cdot, 0) &= 0 \quad \text{in } B_{4\gamma R(0)}.
\end{aligned}$$

Let $\tau_1 > 0$ and $y_1 \in B_{3\gamma R(\tau_1)}(0)$. Let $\rho = \frac{|y_1|}{4} + 1$ so that $B_\rho(y_1) \subset B_{4\gamma R(\tau_1)}(0)$. We prove (5.55) by the scaling argument. Define

$$\tilde{\phi}(z, s) = \phi(y_1 + \rho z, \tau_1 + \rho^2 s), \quad z \in B_1(0), \quad s > -\frac{\tau_1}{\rho^2}.$$
for some positive constants $C_1, C_2$ independent of $\tau_1$. Then standard interior gradient estimates together with the assumption (5.54) imply

$$
\|\nabla z\tilde{\phi}\|_{L^\infty(B_{1/4}(0)\times(1,2))} \lesssim \|\tilde{\phi}\|_{L^\infty(B_{1/2}(0)\times(0,2))} + \|\tilde{h}\|_{L^\infty(B_{1/2}(0)\times(0,2))} \lesssim \tau_1^{-b'} \rho^{2-a},
$$

which in particular gives

$$
\rho |\nabla_y \phi(y_1, \tau_1)| = |\nabla_z \tilde{\phi}(0,1)| \lesssim \tau_1^{-b'} \rho^{2-a}.
$$

On the other hand, from interior parabolic Schauder estimates and (5.54), we have

$$
\|\nabla^2 z\tilde{\phi}\|_{L^\infty(B_{1/4}(0)\times(1,2))} \lesssim \|\tilde{\phi}\|_{L^\infty(B_{1/2}(0)\times(0,2))} + \|\tilde{h}\|_{C^{a/2}(B_{1/2}(0)\times(0,2))} \lesssim \tau_1^{-b'} \rho^{2-a},
$$

and in particular

$$
\rho^2 |\nabla^2_y \phi(y_1, \tau_1)| = |\nabla^2_z \tilde{\phi}(0,1)| \lesssim \tau_1^{-b'} \rho^{2-a}.
$$

For the case $\tau_1 \geq \rho^2$ the argument is similar. In this case $\tilde{\phi}$ satisfies the equation in $B_1(0) \times (-\frac{n}{\rho^2},0]$ and it has initial condition 0 at $s = -\frac{n}{\rho^2}$. Then similarly by the standard boundary estimate, we get the desired bound. Finally, translating the above bounds into $(y,t)$-variable, we conclude the validity of (5.55).

As we can see from Lemma 5.2.1, the estimates at modes $k = 0, \pm 1$ are worse than high modes $k \geq 2$. In fact, if certain orthogonality conditions are imposed on $h(y,t)$, better estimates of $\phi$ can be obtained at modes $k = 0, \pm 1$. In the sequel, we omit the subscript for each mode if there is no confusion.

**Mode $k = 0$**

We consider

$$
\begin{cases}
\lambda^2 \partial_t \varphi = L_W[\varphi] + h(y,t) + \sum_{j=1,2} \tilde{c}_0 j Z_0 j w_0^2 & \text{in } D_{2R}, \\
\varphi \cdot W = 0 & \text{in } D_{2R}, \\
\varphi = 0 & \text{on } \partial B_{2R} \times (0,T), \\
\varphi(\cdot,0) = 0 & \text{in } B_{2R(0)},
\end{cases}
$$

(5.56)

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at mode 0. By carrying out another inner–gluing scheme for mode 0, the following Lemma was proved in [55, Proposition 7.2].

**Lemma 5.2.3** ([55]). Let $\delta \in (0, 1)$, $\nu > 0$ and $a \in (2, 3)$. Assume $\|h\|_{\nu, a} < +\infty$. Then there exists a solution $(\phi, \tilde{c}_{0j})$ of problem (5.56) which defines a linear operator in $h(y, t)$ such that

$$
|\phi(y, t)| + (1 + |y|)|\nabla_y \phi(y, t)| \lessapprox \lambda^\nu(t)\|h\|_{\nu, a} \begin{cases}
R^{\delta(5-a)} \frac{(1 + |y|)^3}{1 + |y|^{a-2}} & \text{for } |y| \leq R^\delta,
\frac{1}{1 + |y|^{a-2}} & \text{for } 2R^\delta \leq |y| \leq 2R,
\end{cases}
$$

$$
\tilde{c}_{0j} = -\frac{\int_{\mathbb{R}^2} hZ_{0, j}}{\int_{\mathbb{R}^2} w^2_0 |Z_{0, j}|^2} - G[h],
$$

where $G$ is linear in $h$ satisfying

$$
|G[h]| \lessapprox \lambda^\nu(t) R^{-\delta}\sigma'|\|h\|_{\nu, a}
$$

for $\sigma' \in (0, a - 2)$.

**Mode $k = -1$**

We consider problem (5.50) for $k = -1$ and the kernel functions defined in (5.9). We first state a result proved in [55, Lemma 7.5].

**Lemma 5.2.4** ([55]). Let $a \in (2, 3)$, $\nu > 0$ and $k = -1$. If $h_{-1}$ in (5.50) satisfies $\|h_{-1}\|_{\nu, a} < \infty$ and

$$
\int_{\mathbb{R}^2} h_{-1}(y, t)Z_{-1, j}(y)dy = 0 \text{ for } j = 1, 2, \forall \ t \in (0, T),
$$

then there exists a solution $\phi_{-1}$ to problem (5.50) at mode $-1$ which defines a linear operator of $h_{-1}$, and $\phi_{-1}$ satisfies

$$
|\phi_{-1}(y, t)| \lessapprox \lambda^\nu(t)\|h_{-1}\|_{\nu, a} \min \left\{ \log R, \frac{R^{1-a}}{1 + |y|^2} \right\}.
$$

Since the incompressible Navier–Stokes equation is essentially coupled with the transported harmonic map heat flow through the inner problem, the linear the-
ory required for mode $k = -1$ should be very refined, and Lemma 5.2.4 cannot be applied to gain contraction when we finally show the existence of desired blow-up solution. Instead, we shall develop a new linear theory at mode $-1$. The main result for mode $-1$ is stated as follows.

**Lemma 5.2.5.** Let $a \in (2, 3)$, $\nu > 0$ and $k = -1$. If $h_{-1}$ in (5.50) satisfies $\|h_{-1}\|_{\nu, a} < \infty$ and 

$$
\int_{\mathbb{R}^2} h_{-1}(y, t) Z_{-1, j}(y) dy = 0 \quad \text{for} \quad j = 1, 2, \forall \, t \in (0, T),
$$

then there exists a solution $\phi_{-1}$ to problem (5.50) at mode $-1$ which defines a linear operator of $h_{-1}$, and $\phi_{-1}$ satisfies

$$
|\phi_{-1}(y, t)| \lesssim \lambda_{\nu}^{-}(t) \|h_{-1}\|_{\nu, a}.
$$

**Proof.** For convenience, we change variable (5.52) and consider

$$
\partial_\tau \varphi_{-1} = \mathcal{L}_{-1}[\varphi_{-1}] + \bar{h}_{-1}.
$$

By letting $\varphi_{-1}(\rho, \tau) = Z_{-1}(\rho) f_{-1}(\rho, \tau)$ and using $\mathcal{L}_{-1}[Z_{-1}] = 0$, we obtain

$$
\partial_\tau f_{-1} = \frac{1}{Z_{-1}^2} \text{div}(Z_{-1}^2 \nabla f_{-1}) + \frac{\bar{h}_{-1}}{Z_{-1}},
$$

where $Z_{-1}(\rho)$ is defined in (5.51). We first solve

$$
\text{div}(Z_{-1}^2 \nabla f_{0}) = \bar{h}_{-1} Z_{-1}.
$$

By the orthogonality condition $\int_{\mathbb{R}^2} h_{-1}(y, t) Z_{-1, j}(y) dy = 0$, we get

$$
|\nabla f_{0}| \lesssim \frac{\tau^{-\nu'}}{1 + |y|^{a-1}} \|h_{-1}\|_{\nu, a},
$$

where $\nu' > 0$ is the number such that $\lambda_{\nu}^{+} \sim \tau^{-\nu'}$ under the change of variable (5.52). Thus, by (5.58), the problem (5.57) becomes

$$
\partial_\tau f_{-1} = \frac{1}{Z_{-1}^2} \text{div}(Z_{-1}^2 \nabla f_{-1}) + \frac{1}{Z_{-1}^2} \text{div}(Z_{-1}^2 \nabla f_{0}).
$$

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In order to estimate $f_{-1}$, we need to estimate the fundamental solution $S$ to the problem
\[
\begin{align*}
\partial_\tau S &= \frac{1}{Z_{-1}^2} \text{div}(Z_{-1}^2 \nabla S), \\
S|_{\tau=0} &= \delta_0,
\end{align*}
\]
where $\delta_0$ is the Dirac delta function at the origin. We consider
\[
\begin{align*}
\partial_\tau S^\varepsilon &= \frac{1}{Z_{-1}^2} \text{div}(Z_{-1}^2 \nabla S^\varepsilon), \\
S^\varepsilon|_{\tau=0} &= \frac{1}{2\pi \varepsilon^2} e^{-|\frac{x}{\varepsilon}|^2}.
\end{align*}
\]
We note that as $\varepsilon \to 0$, $S^\varepsilon|_{\tau=0} dx \to \delta_0$. Let $V^\varepsilon = S^\varepsilon_\rho$. Then differentiating the above equation with respect to $\rho$, we obtain
\[
\begin{align*}
\partial_\tau V^\varepsilon &= \frac{1}{Z_{-1}^2} \text{div}(Z_{-1}^2 \nabla V^\varepsilon) + \partial_\rho \rho (\log Z_{-1}^2) V^\varepsilon, \\
V^\varepsilon|_{\tau=0} &= -\frac{|x|}{2\pi \varepsilon^4} e^{-|\frac{|x|^2}{\varepsilon}|}.
\end{align*}
\]
We claim that $V^\varepsilon < 0$. Indeed, we can easily check that $\partial_\rho \rho (\log Z_{-1}^2) < 0$. Therefore, by $V^\varepsilon|_{\tau=0} = -\frac{|x|}{2\pi \varepsilon^4} e^{-|\frac{|x|^2}{\varepsilon}|} < 0$ and the maximum principle, we have $V^\varepsilon < 0$. Then we can write
\[
\int_0^\infty |S^\varepsilon_\rho(s, \rho)| ds = -\int_0^\infty V^\varepsilon(s, \rho) ds := -M^\varepsilon(\rho).
\]
Integrating equation (5.60) over $\tau$ from 0 to $\infty$, we get
\[
\frac{1}{Z_{-1}^2} \text{div}(Z_{-1}^2 \nabla M^\varepsilon) + \partial_\rho \rho (\log Z_{-1}^2) M^\varepsilon = -\frac{|x|}{2\pi \varepsilon^4} e^{-|\frac{|x|^2}{\varepsilon}|}.
\]
Let $M^\varepsilon = \partial_\rho G^\varepsilon$, where $G^\varepsilon$ satisfies
\[
\frac{1}{Z_{-1}^2} \text{div}(Z_{-1}^2 \nabla G^\varepsilon) = \frac{1}{2\pi \varepsilon^2} e^{-|\frac{|x|^2}{\varepsilon}|}.
\]
By $Z_{-1}(\rho) = \frac{2\rho^2}{\rho^2 + 1}$, we write
\[
\frac{1}{Z_{-1}} \text{div}(Z_{-1}^2 \nabla G^e) = \frac{1}{Z_{-1}(\rho)} \rho \frac{\partial (Z_{-1}^2(\rho) \rho)}{\partial \rho} \partial \rho G^e
\]
\[
= \partial \rho G^e + \frac{\rho^2 + 5}{\rho(\rho^2 + 1)} \partial \rho G^e. \tag{5.62}
\]

From (5.61) and (5.62), we obtain
\[
\int_0^\infty |S^e(\rho, \rho)| ds = -M^e(\rho) = -\partial \rho G^e(\rho)
\]
\[
= \frac{1}{2\pi \varepsilon^2} \frac{(1+\rho^2)^2}{\rho^5} \int_\rho^\infty \frac{r^5}{(1+r^2)^2} e^{-\frac{r^2}{2\varepsilon^2}} dr
\]
\[
\leq \frac{1}{2\pi} \frac{1+\rho^4}{\rho^5}.
\]

Therefore, by letting $\varepsilon \to 0$, we obtain
\[
\int_0^\infty |S^e(\rho, \rho)| ds \lesssim \frac{1+\rho^4}{\rho^5}. \tag{5.63}
\]

Duhamel’s formula gives
\[
f_{-1}(0, \tau) = \int_\tau^\infty \int_0^\infty S^e(\rho, \rho) \nabla f_0 Z_{-1}^2(\rho) \rho d\rho ds
\]
\[
\lesssim \int_0^\infty \left( \int_\tau^\infty |S^e(\rho, \rho)| ds \right) |\nabla f_0| Z_{-1}^2(\rho) \rho d\rho.
\]

By (5.59) and (5.63), we conclude $|f_{-1}(0, \tau)| \lesssim \tau^{-\nu_1}$. In the original time variable $t$, we get $|f_{-1}(0, t)| \lesssim \lambda_\nu^*(t)$, and parabolic regularity theory readily yields $|f_{-1}(\rho, t)| \lesssim \lambda_\nu^*(t)$. Therefore, we obtain $|\phi_{-1}(y, t)| \lesssim \lambda_\nu^*(t) \|h_{-1}\|_{\nu, a}$ as desired. \hfill \Box

Mode $k = 1$

We assume that $h_1(y, t)$ is defined in the entire space $\mathbb{R}^2 \times (0, T)$ such that
\[
h_1(y, t) = \text{div}_y G(y, t) \tag{5.64}
\]
with
\[ |G(y,t)| \lesssim \frac{\lambda^\nu(t)}{1 + |y|^{a-1}}, \quad (y,t) \in \mathbb{R}^2 \times (0,T) \] (5.65)

for \( \nu > 0 \) and \( a \in (2,3) \). By the blow-up argument, the following lemma was proved in [55, Lemma 7.6].

**Lemma 5.2.6 ([55])**. Assume that \( \nu > 0 \), \( a \in (2,3) \) and \( h_1 \) takes the form (5.64) such that (5.65) holds and

\[
\int_{\mathbb{R}^2} h_1(y,t)Z_{1,j}(y)dy = 0 \quad \text{for all} \quad t \in (0,T)
\]

for \( j = 1,2 \). Then there exists a solution \( \phi_1(y,t) \) to problem (5.50) for \( k = 1 \) which defines a linear operator of \( h_1(y,t) \), and \( \phi_1(y,t) \) satisfies

\[
|\phi_1(y,t)| \lesssim \frac{\lambda^\nu(t)}{1 + |y|^{a-2}} \|h_1\|_{\nu,a} \quad \text{in} \quad \mathcal{D}_3R.
\]

A direct consequence of Lemma 5.2.6 is the following

**Lemma 5.2.7 ([55])**. Assume \( \nu > 0 \), \( a \in (2,3) \) and

\[
\int_{B_2R} h_1(y,t)Z_{1,j}(y)dy = 0 \quad \text{for all} \quad t \in (0,T)
\]

for \( j = 1,2 \). Then there exists a solution \( \phi_1(y,t) \) to problem (5.50) with \( k = 1 \) which defines a linear operator of \( h_1(y,t) \), and \( \phi_1(y,t) \) satisfies

\[
|\phi_1(y,t)| \lesssim \frac{\lambda^\nu(t)}{1 + |y|^{a-2}} \|h_1\|_{\nu,a}.
\]

By the construction in each mode, now we prove Proposition 5.2.1.

**Proof of Proposition 5.2.1**. Let \( h \) be defined in \( \mathcal{D}_{2R} \) with \( \|h\|_{\nu,a} < +\infty \). We consider

\[
\begin{aligned}
\lambda^2 \partial_t \phi &= L_W[\phi] + h \quad \text{in} \quad \mathcal{D}_{AR}, \\
\phi(\cdot,0) &= 0 \quad \text{in} \quad B_{4R(0)}.
\end{aligned}
\]
Let $\phi_k$ be the solution estimated in Lemma 5.2.1 to

\[
\begin{cases}
\lambda^2 \partial_t \phi_k = L_W[\phi_k] + h_k & \text{in } \mathcal{D}_{4R}, \\ 
\phi_k(\cdot, t) = 0 & \text{on } \partial B_{4R} \times (0, T), \\ 
\phi_k(\cdot, 0) = 0 & \text{in } B_{4R(0)}.
\end{cases}
\]

In addition, we let $\phi_{0,1}, \phi_{1,1}, \phi_{-1,1}$ solve

\[
\begin{cases}
\lambda^2 \partial_t \phi_{k,1} = L_W[\phi_{k,1}] + \bar{h}_k & \text{in } \mathcal{D}_{4R}, \\ 
\phi_{k,1}(\cdot, t) = 0 & \text{on } \partial B_{4R} \times (0, T), \\ 
\phi_{k,1}(\cdot, 0) = 0 & \text{in } B_{4R(0)},
\end{cases}
\]

for $k = 0, \pm 1$, where $\bar{h}_k$ is defined in (5.49). Consider the functions $\phi_{0,2}$ constructed in Lemma 5.2.3, $\phi_{-1,2}$ constructed in Lemma 5.2.5, and $\phi_{1,2}$ constructed in Lemma 5.2.6 that solve for $k = 0, \pm 1$

\[
\begin{cases}
\lambda^2 \partial_t \phi_{k,2} = L_W[\phi_{k,2}] + h_k - \bar{h}_k & \text{in } \mathcal{D}_{3R}, \\ 
\phi_{k,2}(\cdot, 0) = 0 & \text{in } B_{3R(0)},
\end{cases}
\]

Define

\[
\phi := \sum_{k=0, \pm 1} (\phi_{k,1} + \phi_{k,2}) + \sum_{k \neq 0, \pm 1} \phi_k
\]

which is a bounded solution to the following equation

\[
\lambda^2 \partial_t \phi = L_W[\phi] + h(y, t) \quad \text{in } \mathcal{D}_{3R}.
\]

Moreover, it defines a linear operator of $h$. Applying the estimates for the components in Lemmas 5.2.1, 5.2.3, 5.2.5, and 5.2.6, we obtain

\[
|\phi(y, t)| \lesssim \lambda_y^\nu(t) \min \left\{ \frac{R^{(5-a)}(t)}{1 + |y|^3}, \frac{1}{1 + |y|^{a-2}} \right\} \|h_0 - \bar{h}_0\|_{v,a} + \frac{\lambda_y^\nu(t) R^2(t)}{1 + |y|} \|\bar{h}_0\|_{v,a}
\]

\[
+ \frac{\lambda_y^\nu(t)}{1 + |y|^{a-2}} \|h_1 - \bar{h}_1\|_{v,a} + \frac{\lambda_y^\nu(t) R^4(t)}{1 + |y|^2} \|\bar{h}_1\|_{v,a}
\]

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Finally, Lemma 5.2.2 yields that the same bound holds for \((1 + |y|)|\nabla_y \phi|\) and \((1 + |y|)^2 |\nabla_y^2 \phi|\) in \(\mathcal{D}_{2R}\). The function \(\phi \mid_{\mathcal{D}_{2R}}\) solves equation (5.45), and it defines a linear operator of \(h\) satisfying the desired estimates. The proof is complete.

5.2.6 Linear theory for the outer problem

In order to solve the outer problem (5.30), we need to develop a linear theory to the associated linear problem of (5.30), which is basically a heat equation. However, we will have to establish a decay estimate of second order derivative of \(\psi\) in order to handle the coupling effects between the inner–outer problem of \(u\) and that of \(v\) below.

For \(q \in \Omega\) and \(T > 0\) sufficiently small, we consider the problem

\[
\begin{cases}
\psi_t = \Delta_x \psi + f(x,t) & \text{in } \Omega \times (0,T), \\
\psi = 0 & \text{on } \partial \Omega \times (0,T), \\
\psi(x,0) = 0 & \text{in } \Omega.
\end{cases}
\] (5.66)

The right hand side of (5.66) is assumed to be bounded with respect to some weights that appear in the outer problem (5.30). Thus we define the weights

\[
\begin{cases}
\rho_1 := \lambda^\Theta_\star \left( \lambda_\star R \right)^{-1} \chi_{\{ r \leq 3 \lambda_\star R \}}, \\
\rho_2 := T^{-\sigma_0} \lambda^1 \frac{1-\sigma_0}{r^2} \chi_{\{ r \geq \lambda_\star R \}}, \\
\rho_3 := T^{-\sigma_0},
\end{cases}
\] (5.67)

where \(r = |x-q|\), \(\Theta > 0\) and \(\sigma_0 > 0\) is small. For a function \(f(x,t)\) we define the \(L^\infty\)-weighted norm

\[
\|f\|_{**} := \sup_{\Omega \times (0,T)} \left( 1 + \sum_{i=1}^{3} \rho_i(x,t) \right)^{-1} |f(x,t)|.
\] (5.68)
The factor $T^\theta$ in front of $\rho_2$ and $\rho_3$ is a simple way to have parts of the error small in the outer problem. Also, we define the $L^\infty$-weighted norm for $\psi$

\[
\|\psi\|_{\sharp,T,\gamma} := \frac{1}{|\log T|} \left\| \psi \right\|_{L^\infty(\Omega \times (0,T))} + \lambda_*^{-\gamma}(0) \|\nabla_x \psi\|_{L^\infty(\Omega \times (0,T))} + \sup_{\Omega \times (0,T)} \frac{1}{|\log (T-t)|} |\psi(x,t) - \psi(x,T)| \\
+ \sup_{\Omega \times (0,T)} \left| \nabla_x \psi(x,t) - \nabla_x \psi(x,T) \right| + \|\nabla^2 \psi\|_{L^\infty(\Omega \times (0,T))} + \sup_{\Omega \times (0,T)} \frac{1}{|\log (T-t)|} |\lambda_*^{-\gamma}(0) R(t) |^{2\gamma} \|
abla_x \psi(x,t) - \nabla_x \psi(x',t')\| \right\|_{\Omega \times (0,T)}^{2\gamma} (|x-x'|^2 + |t-t'|^2)^\gamma, 
\]

where $\Theta > 0$, $\gamma \in (0, \frac{1}{2})$, and the last supremum is taken in the region

$x, x' \in \Omega, \ t, t' \in (0, T), \ |x-x'| \leq 2\lambda_* R(t), \ |t-t'| < \frac{1}{4}(T-t)$.

We shall measure the solution $\psi$ to the problem (5.66) in the norm $\|\|_{\sharp,T,\gamma}$ defined in (5.69) where $\gamma \in (0, \frac{1}{2})$, and we require that $\Theta$ and $\gamma_*$ (recall that $R = \lambda_*^{-\gamma}$ in (5.31)) satisfy

\[
\gamma_* \in \left(0, \frac{1}{2}\right), \quad \Theta \in (0, \gamma_*). 
\]

(5.70)

The condition $\gamma_* \in (0, \frac{1}{2})$ is a basic assumption to have the singularity appear inside the self-similar region. We invoke a useful proposition proved in [55, Appendix A]:

**Proposition 5.2.2.** Assume (5.70) holds. For $T > 0$ sufficiently small, there is a linear operator that maps a function $f : \Omega \times (0, T) \to \mathbb{R}^3$ with $\|f\|_{**} < \infty$ into $\psi$ which solves problem (5.66). Moreover, the following estimate holds

\[
\|\psi\|_{\sharp,T,\gamma} \leq C \|f\|_{**},
\]

where $\gamma \in (0, \frac{1}{2})$. 

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The proof of Proposition 5.2.2 was achieved in [55] by considering

$$
\begin{align*}
\psi_t &= \Delta \psi + f \quad \text{in } \Omega \times (0, T), \\
\psi(x, 0) &= 0 \quad x \in \Omega, \\
\psi(x, t) &= 0 \quad x \in \partial \Omega \times (0, T),
\end{align*}
$$

(5.71)

and decomposing the equation into three parts corresponding to the weights of the right hand side defined in (5.67).

**Remark 5.2.1.** We note that the estimates for \( |\nabla^2_x \psi(x, t)| \) in Proposition 5.2.2 are achieved by writing the original equation (5.71) in the self-similar variables \((y, \tau)\):

$$
\psi(x, t) = \tilde{\psi}\left(\frac{x - \xi}{\lambda}, \tau(t)\right),
$$

where \( y = \frac{x - \xi}{\lambda} \) and \( \tau \) is defined in (5.52). Then \( \tilde{\psi}(y, \tau) \) satisfies the equation

$$
\partial_\tau \tilde{\psi} = \Delta_y \tilde{\psi} + (\lambda \dot{\xi} + \dot{\lambda} \lambda y) \cdot \nabla_y \tilde{\psi} + \lambda^2 f(\lambda y + \xi, \tau).$$

By similar argument as in the proof of Lemma 5.2.2, we can show the boundedness of \( |\nabla^2_x \psi(x, t)| \) by the scaling argument and parabolic regularity estimates, which is sufficient for the final gluing procedure in Section 5.4 to work.

### 5.3 Model problem: Stokes system

In order to solve the incompressible Navier–Stokes equation in (5.1), a linear theory of certain linearized problem is required. In this section, we consider the Stokes system

$$
\begin{align*}
\partial_t v + \nabla P &= \Delta v + \nabla \cdot F \quad \text{in } \Omega \times (0, T), \\
\nabla \cdot v &= 0 \quad \text{in } \Omega \times (0, T), \\
v &= 0 \quad \text{on } \partial \Omega \times (0, T), \\
v(\cdot, 0) &= v_0 \quad \text{in } \Omega,
\end{align*}
$$

(5.72)

which is the linearized problem of the incompressible Navier–Stokes equation in (5.1). The idea is the following. Apriori we assume that the nonlinearity \( v \cdot \nabla v \)
is a perturbation under certain topology. Then we develop a linear theory for the Stokes system under which we shall see that \( v \cdot \nabla v \) is indeed a smaller perturbation under some assumptions in Section 5.4.

Our aim is to find a velocity field \( v \) solving (5.72) with proper decay ensuring the inner–outer gluing scheme to be carried out. Suppose that \( F(\mathbf{x}, t) \) in (5.72) has the space-time decay of the type

\[
|F(\mathbf{x}, t)| \leq C \lambda_{\nu}^{\nu - 2} \left( \frac{\lambda_{\nu}(t)}{|\mathbf{x} - q|} \right)^{a+1}, \quad |\nabla_x F(\mathbf{x}, t)| \leq C \lambda_{\nu}^{\nu - 3} \left( \frac{\lambda_{\nu}(t)}{|\mathbf{x} - q|} \right)^{a+2}
\]

(5.73)

for \( \nu > 0 \) and \( a > 1 \). Here \( q \in \Omega \) is the singular point for the orientation field \( u(x, t) \) and

\[
\lambda_{\nu}(t) = \frac{|\log T|(T - t)}{|\log(T - t)|^2}.
\]

We define the norm

\[
\|F\|_{S, \nu - 2, a + 1} := \sup_{(\mathbf{x}, t) \in \Omega \times (0, T)} \lambda_{\nu}^{2 - \nu}(t) \left( 1 + \left| \frac{x - q}{\lambda_{\nu}(t)} \right|^{a+1} \right) |F(\mathbf{x}, t)| + \sup_{(\mathbf{x}, t) \in \Omega \times (0, T)} \lambda_{\nu}^{3 - \nu}(t) \left( 1 + \left| \frac{x - q}{\lambda_{\nu}(t)} \right|^{a+2} \right) |\nabla_x F(\mathbf{x}, t)|.
\]

(5.74)

**Proposition 5.3.1.** Assume that \( \|F\|_{S, \nu - 2, a + 1} < +\infty \) with \( \nu > 0, a > 1 \), \( \|v_0\|_{B^{\nu - 2/p}_{p, p}} < +\infty \), where the Besov norm \( \| \cdot \|_{B^{\nu - 2/p}_{p, p}} \) is defined by (5.109). Then there exists a solution \((v, P)\) to the Stokes system (5.72) satisfying

- in the region near \( q \): \( B_{2\delta}(q) = \{ \mathbf{x} \in \Omega : |\mathbf{x} - q| < 2\delta \} \) for \( \delta > 0 \) fixed and small,

\[
|v(\mathbf{x}, t)| \lesssim \|F\|_{S, \nu - 2, a + 1} \lambda_{\nu}^{\nu - 1}(t) \left( 1 + \left| \frac{x - q}{\lambda_{\nu}(t)} \right|^{a+1} \right),
\]

\[
|P(\mathbf{x}, t)| \lesssim \|F\|_{S, \nu - 2, a + 1} \left( \frac{\lambda_{\nu}^{\nu}(t)}{|\mathbf{x} - q|^2} + \frac{\lambda_{\nu}^{\nu - 2}(t)}{1 + \left| \frac{x - q}{\lambda_{\nu}(t)} \right|^{a+1}} \right).
\]
• in the region away from \( q \): \( \Omega \setminus B_\delta(q) \)

\[
\|v\|_{W^2,1_p(\Omega \setminus B_\delta(q) \times (0,T))} + \|\nabla P\|_{L^p(\Omega \setminus B_\delta(q) \times (0,T))} \lesssim \|F\|_{S,v^{-2,a+1} + B_2^{2-2/p}}
\]

for \((v - 1)p + 1 > 0\). Moreover, if \( v > 1/2 \), then

\[
\|v\|_{C^{a/2}(\Omega \setminus B_\delta(q) \times (0,T))} \lesssim \|F\|_{S,v^{-2,a+1} + B_2^{2-2/p}}
\]

for \( 0 < \alpha \leq 2 - 4/p \).

To prove Proposition 5.3.1, we decompose the solution \( v(x,t) \) to problem (5.72) into inner and outer profiles

\[
v(x,t) = \eta_\delta v_{in}(x,t) + v_{out}(x,t),
\]

where the smooth cut-off function

\[
\eta_\delta(x) = \begin{cases} 
1 & \text{for } |x - q| < \delta, \\
0 & \text{for } |x - q| > 2\delta,
\end{cases}
\]

with \( \delta > 0 \) fixed and sufficiently small such that \( \text{dist}(q, \partial \Omega) > 2\delta \). We denote

\[
B_{2\delta}(q) = \{x \in \Omega : |x - q| < 2\delta\}.
\]

It is direct to see that a solution to problem (5.72) is found if \( v_{in} \) and \( v_{out} \) satisfy

\[
\begin{cases}
\partial_t v_{in} + \nabla P = \Delta v_{in} + \nabla \cdot F_{in} & \text{in } \mathbb{R}^2 \times (0,T), \\
\nabla \cdot v_{in} = 0 & \text{in } \mathbb{R}^2 \times (0,T), \\
v_{in}(\cdot,0) = 0 & \text{in } \mathbb{R}^2,
\end{cases}
\]  

(5.75)
\begin{align}
\frac{\partial v_{\text{out}}}{\partial t} + \nabla (P - \eta \delta P_1) &= \Delta v_{\text{out}} + (1 - \eta \delta) \nabla \cdot F + 2 \nabla \eta \delta \cdot \nabla v_{\text{in}} \\
&\quad + (\Delta \eta \delta) v_{\text{in}} - P_1 \nabla \eta \delta \quad \text{in} \quad \Omega \times (0, T), \\
\nabla \cdot v_{\text{out}} &= -\nabla \eta \delta \cdot v_{\text{in}} \quad \text{in} \quad \Omega \times (0, T), \\
v_{\text{out}} &= 0 \quad \text{on} \quad \partial \Omega \times (0, T), \\
v_{\text{out}}(\cdot, 0) &= v_0 \quad \text{in} \quad \Omega,
\end{align}

(5.76)

where \( F_{\text{in}} = F \chi_{(B_2(q) \times (0, T))} \). The estimate of the inner part (5.75) is achieved by the representation formula in the entire space, while the outer part (5.76) is done by \( W^2_1 \)-theory of the Stokes system.

**Lemma 5.3.1.** For \( \|F\|_{S,v-2,a+1} < +\infty \), the solution \((v_{\text{in}}, P_1)\) of the system (5.75) satisfies

\[
|v_{\text{in}}(x, t)| \lesssim \|F\|_{S,v-2,a+1} \frac{\lambda^v(t)}{1 + \left|\frac{x-q}{\lambda^v(t)}\right|},
\]

(5.77)

\[
|P_1(x, t)| \lesssim \|F\|_{S,v-2,a+1} \left( \frac{\lambda^v(t)}{|x-q|^2} + \frac{\lambda^{v-2}(t)}{1 + \left|\frac{x-q}{\lambda^v(t)}\right|^{a+1}} \right).
\]

(5.78)

**Proof.** For simplicity, we shall write \( v_{\text{in}} \) as \( v \) in the following proof. Denote \( v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \). The estimate (5.77) is obtained by the well-known representation formula in the entire space

\[
v_i(x, t) = \int_{\mathbb{R}^2} S_{ij}(x - z, t)(v(\cdot, 0))_j(z) \, dz - \int_0^t \int_{\mathbb{R}^2} \partial_x S_{ij}(x - z, t - s) F_{jk}(z, s) \, dz \, ds,
\]

where \( S_{ij} \) is the Oseen tensor, which is the fundamental solution of the non-stationary Stokes system derived by Oseen [159], defined by

\[
S_{ij}(x, t) = G(x, t) \delta_{ij} - \frac{1}{2\pi} \frac{\partial^2}{\partial x_i \partial x_j} \int_{\mathbb{R}^2} G(y, t) \log |x-y| \, dy
\]

(5.79)

with \( G(x, t) = \frac{|x|^2}{4\pi t^3} \), and \( F = (F_{jk})_{2 \times 2} \). It is well known (see [175] for instance) that

\[
|D^l_i D^k_j S(x, t)| \leq C_{k,l} \frac{1}{(|x|^2 + t)^{k + \frac{2\alpha}{2\alpha - 1}}},
\]

(5.80)
Since \( v(\cdot, 0) = 0 \), we then get for \( i = 1, 2, \)
\[
|v_i(x, t)| \lesssim \|F\|_{S, v^{-2, a+1}} \int_0^t \int_{\mathbb{R}^2} \frac{1}{(|x-z| + \sqrt{t-s})^3} \frac{\lambda_v^{-2}(s)}{1 + \left| \frac{z-q}{\lambda_v(s)} \right|^{a+1}} dz ds
\]
\[
:= \|F\|_{S, v^{-2, a+1}} (I_1 + I_2),
\]
where we decompose
\[
I_1 = \int_0^{t-(T-t)^2} \int_{\mathbb{R}^2} \frac{1}{(|x-z| + \sqrt{t-s})^3} \frac{\lambda_v^{-2}(s)}{1 + \left| \frac{z-q}{\lambda_v(s)} \right|^{a+1}} dz ds,
\]
\[
I_2 = \int_{t-(T-t)^2}^t \int_{\mathbb{R}^2} \frac{1}{(|x-z| + \sqrt{t-s})^3} \frac{\lambda_v^{-2}(s)}{1 + \left| \frac{z-q}{\lambda_v(s)} \right|^{a+1}} dz ds.
\]
To estimate \( I_1 \), we evaluate
\[
I_1 \lesssim \int_0^{t-(T-t)^2} \int_{\mathbb{R}^2} \frac{1}{(|x-z|^2 + (t-s))^{3/2}} \frac{\lambda_v^{a-1}(s)}{\lambda_v(s)^{a+1} + |z-q|^{a+1}} dz ds
\]
\[
\lesssim \lambda_v^{a-1}(t) \int_{\mathbb{R}^2} \frac{1}{\lambda_v^{a+1}(t) + |z-q|^{a+1} (|x-z|^2 + (T-t)^2)^{1/2}} dz
\]
\[
\lesssim \lambda_v^{a-1}(t) \left( \int_{D_1(x)} + \int_{D_2(x)} + \int_{D_3(x)} \right) \frac{1}{\lambda_v^{a+1}(t) + |z-q|^{a+1}}
\]
\[
\times \frac{1}{(|x-z|^2 + \lambda_v^2(t))^{1/2}} dz,
\]
where
\[
D_1(x) := \left\{ z \in \mathbb{R}^2 : |z-q| \leq \frac{|x-q|}{2} \right\},
\]
\[
D_2(x) := \left\{ z \in \mathbb{R}^2 : \frac{|x-q|}{2} \leq |z-q| \leq 2|x-q| \right\},
\]
\[
D_3(x) := \left\{ z \in \mathbb{R}^2 : |z-q| \geq 2|x-q| \right\}.
\]
We first compute
\[
\int_{D_{1}(s)} \frac{1}{\lambda_{s}^{a+1}(t) + |z - q|^{a+1} \left( |x - z|^2 + \lambda_{s}^{2}(t) \right)^{1/2}} \ dz
\]
\[
\lesssim \frac{1}{|x - q| + \lambda_{s}(t)} \int_{0}^{\frac{|x - q|}{\lambda_{s}(t)}} \frac{r}{\lambda_{s}^{a+1}(t) + r^{a+1}} \ dr
\]
\[
\lesssim \frac{1}{|x - q| + \lambda_{s}(t)}. \tag{5.86}
\]

Similarly, we have
\[
\int_{D_{2}(s)} \frac{1}{\lambda_{s}^{a+1}(t) + |z - q|^{a+1} \left( |x - z|^2 + \lambda_{s}^{2}(t) \right)^{1/2}} \ dz
\]
\[
\lesssim \frac{1}{\lambda_{s}^{a+1}(t) + |x - q|^{a+1}} \int_{0}^{\frac{|x - q|}{\lambda_{s}(t)}} \frac{r}{r + \lambda_{s}(t)} \ dr
\]
\[
\lesssim \frac{1}{|x - q|^{a} + \lambda_{s}^{a}(t)}. \tag{5.87}
\]

\[
\int_{D_{3}(s)} \frac{1}{\lambda_{s}^{a+1}(t) + |z - q|^{a+1} \left( |x - z|^2 + \lambda_{s}^{2}(t) \right)^{1/2}} \ dz
\]
\[
\lesssim \frac{1}{|x - q| + \lambda_{s}(t)} \int_{|x - q|}^{\infty} \frac{r}{\lambda_{s}^{a+1}(t) + r^{a+1}} \ dr
\]
\[
\lesssim \frac{1}{|x - q|^{a} + \lambda_{s}^{a}(t)}. \tag{5.88}
\]

Collecting (5.82), (5.86), (5.87) and (5.88), we obtain
\[
I_{1} \lesssim \frac{\lambda_{s}^{(y-1)}(t)}{1 + |y|}, \tag{5.89}
\]

where we write \( y = \frac{x - z}{\lambda_{s}(t)} \) for simplicity.

To estimate \( I_{2} \), we change variable \( \tilde{s} = \frac{|x - z|}{(t - \tau)^{1/2}} \), and thus
\[
I_{2} \lesssim \int_{\mathbb{R}^{2}} \int_{|\tilde{s}|}^{\infty} \frac{1}{(1 + \tilde{s})^{3} |x - z|} \frac{\lambda_{s}^{(y + a - 1)}(t)}{\lambda_{s}^{a+1}(t) + |z - q|^{a+1}} \ d\tilde{s} \ dz
\]

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where \( D_1(x), D_2(x) \) and \( D_3(x) \) are defined in (5.83), (5.84) and (5.85), respectively.

For the above integral, we consider the following two cases.

- **Case 1:** \( |x - q| \leq \lambda_s(t) \). We have

\[
\int_{D_1(x)} \frac{1}{\lambda_s^{a+1}(t) + |z - q|^{a+1}} \frac{1}{|x - z|} \frac{1}{\lambda_s^2(t) + |x - z|^2} \, dz \\
\leq \lambda_s^{a+1}(t) \int_{D_1(x)} \frac{1}{|x - q|^{a+1}} \frac{1}{|x - z|} \frac{1}{\lambda_s^2(t) + |x - z|^2} \, dz \\
\leq \lambda_s^{a+1}(t) \int_{0}^{r} r \, dr \\
\leq \lambda_s^{a-2}(t),
\]

**Case 2:** \( |x - q| \geq \lambda_s(t) \). We have

\[
\int_{D_2(x)} \frac{1}{\lambda_s^{a+1}(t) + |z - q|^{a+1}} \frac{1}{|x - z|} \frac{1}{\lambda_s^2(t) + |x - z|^2} \, dz \\
\leq \lambda_s^{a+1}(t) \int_{D_2(x)} \frac{1}{|x - q|^{a+1}} \frac{1}{|x - z|} \frac{1}{\lambda_s^2(t) + |x - z|^2} \, dz \\
\leq \lambda_s^{a-2}(t),
\]

Observe that in this case \( |x - q| \leq \lambda_s(t) \) we have \( 1 \leq \frac{1}{1 + |y|^2} \) for \( \varepsilon > 0 \), where \( y = \frac{x - q}{\lambda_s(t)} \). Therefore, for the case \( |x - q| \leq \lambda_s(t) \), we conclude

\[
I_2 \leq \frac{\lambda_s^{v-1}(t)}{1 + |y|^2},
\]

(5.94)
by (5.90)–(5.93).

- Case 2: $|x - q| \geq \lambda_s(t)$. In this case, we compute

$$
\int_{D_1(x)} \frac{1}{\lambda_s^{a+1}(t) + |z - q|^{a+1}} \frac{1}{|x - \bar{z}|} \frac{1}{|x - z|^2} dz 
\lesssim \frac{1}{\lambda_s^2(t) + |x - q|^2} \frac{1}{|x - q|} \int_0^{\frac{|x-q|}{r}} \frac{r}{\lambda_s^{a+1}(t) + r^{a+1}} dr 
\lesssim \frac{\lambda_s^{-a-2}(t)}{1 + |y|^2},
$$

(5.95)

$$
\int_{D_2(x)} \frac{1}{\lambda_s^{a+1}(t) + |z - q|^{a+1}} \frac{1}{|x - \bar{z}|} \frac{1}{|x - z|^2} dz 
\lesssim \frac{1}{\lambda_s^{a+1}(t) + |x - q|^{a+1}} \int_0^{\frac{|x-q|}{r}} \frac{r}{\lambda_s^2(t) + r^2} dr 
\lesssim \frac{\lambda_s^{-a-2}(t)}{1 + |y|^{a+1}},
$$

(5.96)

$$
\int_{D_3(x)} \frac{1}{\lambda_s^{a+1}(t) + |z - q|^{a+1}} \frac{1}{|x - \bar{z}|} \frac{1}{|x - z|^2} dz 
\lesssim \frac{1}{\lambda_s^2(t) + |x - q|^2} \frac{1}{\lambda_s(t)} \int_0^{\frac{|x-q|}{r}} \frac{r}{\lambda_s^{a+1}(t) + r^{a+1}} dr 
\lesssim \frac{\lambda_s^{-a-2}(t)}{1 + |y|^{a+1}}.
$$

(5.97)

From (5.90), (5.95), (5.96) and (5.97), one has

$$
I_2 \lesssim \frac{\lambda_s^{-a-2}(t)}{1 + |y|^2}
$$

(5.98)

for the case $|x - q| \geq \lambda_s(t)$.

In conclusion, we get

$$
|v_{in}(x,t)| \lesssim \|F\|_{S,v-2,a+1} \frac{\lambda_s^{v-1}(t)}{1 + |y|}
$$

from (5.81), (5.89), (5.94) and (5.98).

We now derive the estimate (5.78) for $P_1$. Recall the representation formula for $P_1$:

$$
P_1(x,t) = \int_0^t \int_{\mathbb{R}^2} Q_j(x - z, t - s) \partial_z F_{jk}(z,s) dz ds,
$$

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where \( Q_j \) is given by
\[
Q_j(x,t) = \frac{\delta(t)}{2\pi} \frac{x_j}{|x|^2}.
\]

Thus,
\[
P_1(x,t) = \int_{\mathbb{R}^2} \frac{1}{2\pi} \frac{x_j - z_j}{|x - z|^2} \partial_{\bar{z}} F_{jk}(z,t) \, dz
\]
\[
= \int_{D_1(x)} + \int_{D_2(x)} + \int_{D_3(x)} \frac{1}{2\pi} \frac{x_j - z_j}{|x - z|^2} \partial_{\bar{z}} F_{jk}(z,t) \, dz
\]
\[
:= I + II + III,
\]
where \( D_1(x), D_2(x), \) and \( D_3(x) \) are defined in (5.83), (5.84), and (5.85), respectively. We perform integration by parts to estimate I. In fact, one has
\[
I \lesssim \| F \|_{S,v - 2,a + 1} \left( \int_{D_1(x)} \frac{1}{|x - z|^2} \frac{\lambda_s^{v-2}(t)}{1 + \frac{|z - q|}{\lambda_s(t)}} a + 1 \, dz + \int_{\partial D_1(x)} \frac{1}{|x - z|} \frac{\lambda_s^{v-2}(t)}{1 + \frac{|z - q|}{\lambda_s(t)}} a + 1 \, dz \right)
\]
\[
\lesssim \| F \|_{S,v - 2,a + 1} \left( \frac{\lambda_s^{v-2}(t)}{|x - q|^2} \int_0^{|x - q|} \frac{\lambda_s^{a+1}(t)}{\lambda_s^{a+1}(t) + r} r \, dr \right)
\]
\[
\lesssim \| F \|_{S,v - 2,a + 1} \left( \frac{\lambda_s^v(t)}{|x - q|^2} + \frac{\lambda_s^{v-2}}{1 + \frac{|x - q|}{\lambda_s(t)}} a + 1 \right).
\]
(5.99)

The way to estimate II and III is straightforward. More specifically, we have
\[
II = \frac{1}{2\pi} \int_{D_2(x)} \frac{x_j - z_j}{|x - z|^2} \partial_{\bar{z}} F_{jk}(z,t) \, dz
\]
\[
\lesssim \| F \|_{S,v - 2,a + 1} \int_{D_2(x)} \frac{1}{|x - z|^2} \frac{\lambda_s^{v-3}(t)}{1 + \frac{|z - q|}{\lambda_s(t)}} a + 2 \, dz
\]
\[
\lesssim \| F \|_{S,v - 2,a + 1} \left( \frac{\lambda_s^{v-3}(t)}{1 + \frac{|x - q|}{\lambda_s(t)}} a + 2 \right) \int_0^{|x - q|} \frac{r}{r} \, dr
\]

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Collecting (5.99)–(5.101), we obtain (5.78), and the proof is complete.

Lemma 5.3.2. Under the assumptions of Lemma 5.3.1, the following estimates hold

\[ \| \nabla v_{in}(x,t) \| \lesssim \| F \|_{S,\nu-2,a+1} \frac{\lambda_s^{\nu-2}(t)}{1 + |\frac{x-q}{\lambda_s(t)}|^{a+1}}, \]  

(5.102)

\[ \| \partial_t (v_{in} \cdot \nabla \eta_\delta) \|_{L^p(B_{2\delta}(q) \setminus B_{\delta}(q)) \times (0,T)} \lesssim \| F \|_{S,\nu-2,a+1} \]  

(5.103)

for \((\nu - 1)p + 1 > 0\).  

In order to apply \(W_{2,1}^p\)-theory of the Stokes system to the outer part (5.76), the estimates for \(\nabla v_{in}\) and \(\partial_t (v_{in} \cdot \nabla \eta_\delta)\) are further needed. We have the following lemma.

Lemma 5.3.2. Under the assumptions of Lemma 5.3.1, the following estimates hold

\[ \| \nabla v_{in}(x,t) \| \lesssim \| F \|_{S,\nu-2,a+1} \frac{\lambda_s^{\nu-2}(t)}{1 + |\frac{x-q}{\lambda_s(t)}|^{a+1}}, \]  

(5.102)

\[ \| \partial_t (v_{in} \cdot \nabla \eta_\delta) \|_{L^p(B_{2\delta}(q) \setminus B_{\delta}(q)) \times (0,T)} \lesssim \| F \|_{S,\nu-2,a+1} \]  

(5.103)

for \((\nu - 1)p + 1 > 0\).  

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Proof. Since we impose zero initial condition on \( v_{m} \), we have

\[
|\partial_{t} v_{1}(x,t)| \lesssim \|F\|_{S,v-2,a+1} \int_{t}^{t^{'}} \int_{\mathbb{R}^{2}} \frac{1}{(|x-z|^{2} + (t-s))^{3/2}} \frac{\lambda_{s}^{\nu-3}(s)}{1 + \left| \frac{z-q}{\lambda_{s}(s)} \right|^{a+2}} dzds,
\]

where we have used (5.80). We decompose the above integral and first estimate

\[
\int_{0}^{t^{'}} \int_{\mathbb{R}^{2}} \frac{1}{(|x-z|^{2} + (t-s))^{3/2}} \frac{\lambda_{s}^{\nu-3}(s)}{1 + \left| \frac{z-q}{\lambda_{s}(s)} \right|^{a+2}} dzds
\]

\[
\lesssim \lambda_{x}^{\nu+a-1}(t) \left( \int_{D_{1}(x)} \int_{D_{2}(x)} + \int_{D_{3}(x)} \right) \frac{1}{|x-q|^{a+2} + \lambda_{s}(t)} dz,
\]

where \( D_{1}(x), D_{2}(x) \) and \( D_{3}(x) \) are defined in (5.83), (5.84) and (5.85), respectively. Then we can easily check the following

\[
\int_{D_{1}(x)} \frac{1}{\lambda_{s}^{a+2}(t) + |x-q|^{a+2} + \lambda_{s}(t)} dz \lesssim \frac{\lambda_{s}^{\nu-3}(t)}{|x-q|^{a+2} + \lambda_{s}(t)},
\]

\[
\int_{D_{2}(x)} \frac{1}{\lambda_{s}^{a+2}(t) + |x-q|^{a+2} + \lambda_{s}(t)} dz \lesssim \frac{1}{|x-q|^{a+1} + \lambda_{s}(t)},
\]

\[
\int_{D_{3}(x)} \frac{1}{\lambda_{s}^{a+2}(t) + |x-q|^{a+2} + \lambda_{s}(t)} dz \lesssim \frac{1}{|x-q|^{a+1} + \lambda_{s}(t)},
\]

and thus

\[
\int_{0}^{t^{'}} \int_{\mathbb{R}^{2}} \frac{1}{(|x-z|^{2} + (t-s))^{3/2}} \frac{\lambda_{s}^{\nu-3}(s)}{1 + \left| \frac{z-q}{\lambda_{s}(s)} \right|^{a+2}} dzds \lesssim \lambda_{x}^{\nu-2}(t) \frac{1}{|y|},
\]

where we write \( y = \frac{x-q}{\lambda_{s}} \). For the other part, we have

\[
\int_{t}^{t^{'}} \int_{\mathbb{R}^{2}} \frac{1}{(|x-z|^{2} + (t-s))^{3/2}} \frac{\lambda_{s}^{\nu-3}(s)}{1 + \left| \frac{z-q}{\lambda_{s}(s)} \right|^{a+2}} dzds
\]

\[
\lesssim \int_{\mathbb{R}^{2}} \int_{0}^{\infty} \frac{1}{(1+\delta)^{3} |x-z|} \lambda_{s}^{a+2}(t) + |x-q|^{a+2} d\delta dz
\]

\[
\lesssim \lambda_{x}^{\nu+a+1}(t) \int_{\mathbb{R}^{2}} \frac{1}{\lambda_{s}^{a+2}(t) + |x-q|^{a+2} + \lambda_{s}(t) + |x-z|^{2}} dz,
\]

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where we have changed variable $\tilde{s} = \frac{|x - z|}{\sqrt{t - s}}$. Similar to the proof of Lemma 5.3.1, the following bound holds

$$\int_{t-(T-t)^2}^{t} \int_{\mathbb{R}^2} \frac{1}{(|x - z|^2 + (t - s))^{3/2}} \frac{\lambda_x^{-3}(s)}{1 + \left| \frac{z - q}{\lambda_x(s)} \right|^{a+2}} dz ds \lesssim \frac{\lambda_x^{v-2}(t)}{1 + |y|^2}.$$  

Collecting the above estimates, we conclude the validity of (5.102).

Next we prove (5.103). Multiplying equation (5.75) by $\nabla \eta_\delta$, we obtain that $v_\in \cdot \nabla \eta_\delta$ satisfies the equation

$$\partial_t (v_\in \cdot \nabla \eta_\delta) = \Delta (v_\in \cdot \nabla \eta_\delta) - \Delta (\nabla \eta_\delta) \cdot v_\in - 2\nabla^2 \eta_\delta \cdot \nabla v_\in - \nabla P_1 \cdot \nabla \eta_\delta + (\nabla \cdot F_\in) \cdot \nabla \eta_\delta.$$  

Thanks to the cut-off function $\eta_\delta$, standard $W^{2,1}_p$-theory for parabolic equation yields

$$\| \partial_t (v_\in \cdot \nabla \eta_\delta) \|_{L^p((B_{2\delta}(q) \setminus B_\delta(q)) \times (0,T))} \lesssim \| v_\in \|_{L^p((B_{2\delta}(q) \setminus B_\delta(q)) \times (0,T))} + \| \nabla v_\in \|_{L^p((B_{2\delta}(q) \setminus B_\delta(q)) \times (0,T))}$$

$$+ \| \nabla P_1 \|_{L^p((B_{2\delta}(q) \setminus B_\delta(q)) \times (0,T))} + \| \nabla \cdot F \|_{L^p((B_{2\delta}(q) \setminus B_\delta(q)) \times (0,T))}. \quad (5.104)$$

Using the $W^{2,1}_p$-theory for the Stokes system (see [179] for instance), we readily see that

$$\| \nabla P_1 \|_{L^p((B_{2\delta}(q) \setminus B_\delta(q)) \times (0,T))} \lesssim \| \nabla \cdot F \|_{L^p((B_{2\delta}(q) \setminus B_\delta(q)) \times (0,T))}. \quad (5.105)$$

From (5.104), (5.105), (5.77), (5.102) and the assumption $\| F \|_{S, v-2, a+1} < +\infty$, we obtain

$$\| \partial_t (v_\in \cdot \nabla \eta_\delta) \|_{L^p((B_{2\delta}(q) \setminus B_\delta(q)) \times (0,T))} \lesssim \| F \|_{S, v-2, a+1}$$

provided $(v - 1)p + 1 > 0$. The proof is complete. \hfill \box

We are ready to estimate the outer part (5.76).

**Lemma 5.3.** For $\| F \|_{S, v-2, a+1} < +\infty$ and $\| v_0 \|_{B^{2,1}_p} < +\infty$, the solution $(v_{out}, P)$
of the system (5.76) satisfies

$$\|v_{out}\|_{W^{2,1}_p(\Omega \times (0,T))} + \|\nabla (P - \eta_\delta P_i)\|_{L^p(\Omega \times (0,T))} \lesssim \|F\|_{S,v-2,a+1} + \|v_0\|_{B^{2-2/p}_{p,p}}$$

(5.106)

for \((v-1)p + 1 > 0\). If we further assume \(v \in (1/2,1)\), then we have

$$\|v_{out}\|_{C^{\alpha/2}(\Omega \times (0,T))} \lesssim \|F\|_{S,v-2,a+1} + \|v_0\|_{B^{2-2/p}_{p,p}}$$

(5.107)

for \(0 < \alpha \leq 2 - 4/p\).

**Proof:** The \(W^{2,1}_p\) estimate of solutions to Stokes system with non-zero divergence derived in [179, Theorem 3.1] shows that

$$\|v_{out}\|_{W^{2,1}_p(\Omega \times (0,T))} + \|\nabla (P - \eta_\delta P_i)\|_{L^p(\Omega \times (0,T))} \lesssim \|(1 - \eta_\delta)\nabla \cdot F + 2\nabla \eta_\delta \cdot \nabla v_{in} + (\Delta \eta_\delta)v_{in} - P_1\nabla \eta_\delta\|_{L^p(\Omega \times (0,T))} + \|\nabla \eta_\delta \cdot v_{in}\|_{L^p(0,T;W^{2,1}_p(\Omega))} + \|\partial_t (\nabla \eta_\delta \cdot v_{in})\|_{L^p(0,T;W^{-1,1}_p(\Omega))} + \|v_0\|_{B^{2-2/p}_{p,p}},$$

(5.108)

where \(\| \cdot \|_{B^{2-2/p}_{p,p}}\) is the Besov norm defined in (5.109). Thanks to the cut-off function \(\eta_\delta\), we get

$$\|(1 - \eta_\delta)\nabla \cdot F\| \lesssim \|F\|_{S,v-2,a+1}\lambda^v_{a+1},$$

and from (5.77), (5.78), (5.102) and (5.103), one has

$$\|\nabla \eta_\delta \cdot \nabla v_{in}\| + \|(\Delta \eta_\delta)v_{in}\| + |P_1\nabla \eta_\delta| \lesssim \|F\|_{S,v-2,a+1}\lambda^v_a,$$

and also

$$\|\nabla \eta_\delta \cdot v_{in}\| \lesssim \|F\|_{S,v-2,a+1}\lambda^v_a,$$

$$\|\partial_t (\nabla \eta_\delta \cdot v_{in})\|_{L^p(0,T;W^{-1,1}_p(\Omega))} \lesssim \|F\|_{S,v-2,a+1}.$$

It is worth noting that \(\| \cdot \|_{L^p(0,T;W^{2,1}_p(\Omega))} \lesssim \| \cdot \|_{L^p(0,T,L^p(\Omega))}\) (see [2] for instance). Therefore, estimate (5.108) together with the above bounds imply (5.106) for \((v-1)p + 1 > 0\). The Hölder estimate (5.107) then follows from a standard Morrey type inequality (see [131] for instance). The proof is complete. \qed
The proof of Proposition 5.3.1 is a direct consequence of Lemma 5.3.1 and Lemma 5.3.3.

For the behavior of the velocity field $v$, we further make several remarks:

**Remark 5.3.1.**

- From (5.74), Proposition 5.3.1 implies
  \[ \|v\|_{S^{\nu-1,1}} \lesssim \|F\|_{S^{\nu-2,\alpha+1}}. \]

- Since $v$ is divergence-free, we can write $v \cdot \nabla v = \nabla \cdot (v \otimes v)$, where $\otimes$ is the tensor product defined by $(v \otimes w)_{ij} = v_i w_j$. If we solve $v$ in the class $\|v\|_{S^{\nu-1,1}} < \infty$, then the nonlinearity in the Navier–Stokes equation
  \[ |v \cdot \nabla v| \leq \frac{\lambda_s^{2\nu-3}(t)}{1 + \left| \frac{x-q}{\lambda_s(t)} \right|^\beta} \]
  is indeed a perturbation compared to $\nabla \cdot F$, which enables us to solve $v$ by the fixed point argument in Section 5.4.

- The initial velocity $v_0$ in the outer problem (5.76) can be chosen arbitrarily in the Besov space $B_{p,p}^{2-2/p}$, with $(\nu - 1)p + 1 > 0$, in which the norm is defined by
  \[ \|v_0\|_{B_{p,p}^{2-2/p}} := \left( \int_{|z|<1} |z|^{-2p} \int_{\Omega(z)} \left| v_0(x+2z) - 2v_0(x+z) + v_0(x) \right|^p dx dz \right)^{1/p}. \]
  (5.109)
  where $\Omega(z) = \{ x \in \Omega : x + tz \in \Omega, t \in [0,1] \}$, as long as it agrees with zero at the boundary and satisfies the condition
  \[ \nabla \cdot v_0 = -\nabla \eta_\delta \cdot v_{in}(x,0) = 0. \]
5.4 Solving the nematic liquid crystal flow

In this section, we shall apply the linear theories developed in Section 5.2 and Section 5.3 to show the existence of the desired blow-up solution to (5.1)–(5.3) by means of the fixed point argument. Apriori we need some assumptions on the behavior of the parameter functions $p(t) = \lambda(t)e^{i\omega(t)}$ and $\dot{\xi}(t)$

$$c_1|\lambda_*(t)| \leq |p(t)| \leq c_2|\lambda_*(t)| \text{ for all } t \in (0, T),$$

$$|\dot{\xi}(t)| \leq \lambda_*^\sigma(t) \text{ for all } t \in (0, T),$$

where $c_1, c_2$ and $\sigma$ are some positive constants independent of $T$. We recall that

$$R = R(t) = \lambda_*^{-\gamma_*}(t) \text{ with } \lambda_* (t) = \frac{|\log T|(T-t)}{|\log(T-t)|^2} \text{ and } \gamma_* \in (0, 1/2).$$

Similar to the harmonic map heat flow, we look for solution $u$ solving problem (5.1) in the form

$$u = U + \Pi_{U^\perp}\varphi + a(\Pi_{U^\perp}\varphi)U,$$

with

$$\varphi = \eta RQ_{0, \alpha, \beta} \phi(y, t) + \Psi^\sigma(x, t) + \Phi^0(x, t) + \Phi^\alpha(x, t) + \Phi^\beta(x, t),$$

where we decompose $\Psi^\sigma$ into

$$\Psi^\sigma = Z^* + \psi.$$

Here $Z^*$ satisfies

$$\begin{cases}
\partial_t Z^* = \Delta Z^* & \text{in } \Omega \times (0, \infty), \\
Z^*(\cdot, t) = 0 & \text{on } \partial\Omega \times (0, \infty), \\
Z^*(\cdot, 0) = Z^*_0 & \text{in } \Omega.
\end{cases}$$

For the same technical reasons as shown in [55], we make some assumptions on $Z^*_0(x)$ as follows. Let us write

$$Z^*_0(x) = \begin{bmatrix}
z^*_0(x) \\
z^{*3}_0(x)
\end{bmatrix} , \quad z^*_0(x) = z^*_0(x) + iz^{*2}_0(x).$$

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Consistent with (5.42), the first condition that we need is \( \text{div} \varphi_0(q) < 0 \). In addition, we require that \( Z_0'(q) \approx 0 \) in a non-degenerate way.

We will get a desired solution \((v, u)\) to problem (5.1) if \((v, \phi, \Psi^*, p, \xi, \alpha, \beta)\) solves the following inner–outer gluing system

\[
\begin{align*}
\partial_t v + v \cdot \nabla v + \nabla p &= \Delta v - \varepsilon(t) \nabla v \cdot \mathcal{F}[p, \xi, \alpha, \beta, \Psi^*, \phi, v] \quad \text{in} \quad \Omega \times (0, T), \\
\nabla \cdot v &= 0 \quad \text{in} \quad \Omega \times (0, T), \\
v &= 0 \quad \text{on} \quad \partial \Omega \times (0, T), \\
v(\cdot, 0) &= v_0 \quad \text{in} \quad \Omega,
\end{align*}
\]

(5.110)

\[
\begin{align*}
\lambda^2 \partial_t \phi &= L_W[\phi] + \mathcal{H}[p, \xi, \alpha, \beta, \Psi^*, \phi, v] \quad \text{in} \quad \mathcal{D}_2\mathbb{R}, \\
\phi(\cdot, 0) &= 0 \quad \text{in} \quad B_{2R(0)}, \\
\phi \cdot W &= 0 \quad \text{in} \quad \mathcal{D}_2\mathbb{R}, \\
\partial_t \Psi^* &= \Delta \Psi^* + \mathcal{G}[p, \xi, \alpha, \beta, \Psi^*, \phi, v] \quad \text{in} \quad \Omega \times (0, T), \\
\Psi^* &= e_3 - U - \Phi^0 - \Phi^\alpha - \Phi^\beta \quad \text{in} \quad \partial \Omega \times (0, T), \\
\Psi^*(\cdot, 0) &= (1 - \chi) \left( e_3 - U - \Phi^0 - \Phi^\alpha - \Phi^\beta \right) \quad \text{in} \quad \Omega,
\end{align*}
\]

(5.111)

(5.112)

where

\[
\mathcal{F}[p, \xi, \alpha, \beta, \Psi^*, \phi, v] = \left( \nabla u \odot \nabla u - \frac{1}{2} |\nabla u|^2_2 \right),
\]

(5.113)

with

\[
u = U + \Pi_{U^*} \left[ \eta R Q_{\omega, \alpha, \beta} \phi + \Psi^* + \Phi^0 + \Phi^\alpha + \Phi^\beta \right]
+ a(\Pi_{U^*} \left[ \eta R Q_{\omega, \alpha, \beta} \phi + \Psi^* + \Phi^0 + \Phi^\alpha + \Phi^\beta \right]) U,
\]

\[
\mathcal{H}[p, \xi, \alpha, \beta, \Psi^*, \phi, v]
= \lambda^2 Q_{\omega, \alpha, \beta}^{-1} \left[ L_U[\Psi^*] + \mathcal{K}_0[p, \xi] + \mathcal{K}_1[p, \xi] + \Pi_{U^*} \mathcal{R}_{-1} - \lambda^{-1} \Pi_{U^*} (v \cdot \nabla) U \right]
- \lambda^{-1} \Pi_{U^*} \left( v \cdot \nabla \left( \Pi_{U^*} \left[ \eta R Q_{\omega, \alpha, \beta} \phi + \Psi^* + \Phi^0 + \Phi^\alpha + \Phi^\beta \right] \right) \right)
- \lambda^{-1} \Pi_{U^*} \left( v \cdot \nabla \left( a(\Pi_{U^*} \left[ \eta R Q_{\omega, \alpha, \beta} \phi + \Psi^* + \Phi^0 + \Phi^\alpha + \Phi^\beta \right]) U \right) \right),
\]

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\[ \mathcal{H}[p, \xi, \alpha, \beta, \Psi^*, \phi, v] := (1 - \eta_R)\tilde{L}_U[\Psi^*] + (\Psi^* \cdot U)U_t + Q_{o,a,\beta}(\phi\Delta, \eta_R + 2\nabla, \eta_R \cdot \nabla, \phi - \phi \partial_t \eta_R) \\
+ \eta_R Q_{o,a,\beta}\left(-\left(Q_{o,a,\beta}\frac{d}{dt}Q_{o,a,\beta}\right)\phi + \lambda^{-1}\lambda y \cdot \nabla, \phi + \lambda^{-1}\xi \cdot \nabla, \phi\right) \\
+ (1 - \eta_R)(\mathcal{X}_0[p, \xi] + \mathcal{X}_1[p, \xi] + \Pi_U[\mathcal{R}_1]) - \Pi_U[\tilde{\mathcal{R}_1}] \\
+ N_U[\eta_R Q_{o,a,\beta}\phi + \Pi_U[\Phi^0 + \Phi^\alpha + \Phi^\beta + \Psi^*]] \\
+ \left((\Phi^0 + \Phi^\alpha + \Phi^\beta) \cdot U\right)U_t - (1 - \eta_R) v \cdot \nabla U \\
- (1 - \eta_R) v \cdot \nabla \left(\Pi_U[\eta_R Q_{o,a,\beta}\phi + \Psi^* + \Phi^0 + \Phi^\alpha + \Phi^\beta]\right) \\
- (1 - \eta_R) v \cdot \nabla \left(a \left(\Pi_U[\eta_R Q_{o,a,\beta}\phi + \Psi^* + \Phi^0 + \Phi^\alpha + \Phi^\beta]\right)U\right). \]

Here \( \chi \) in (5.112) is a smooth cut-off function which is supported near a fixed neighborhood of \( q \) independent of \( T \).

As discussed in Section 5.2.5, suitable inner solution with space-time decay can be obtained under certain orthogonality conditions, which will be achieved by adjusting the parameter functions \( p(t), \xi(t), \alpha(t) \) and \( \beta(t) \). In order to solve the inner problem (5.111), we further decompose it based on the Fourier modes

\[ \mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_3 + \mathcal{H}_4, \]

with

\[ \mathcal{H}_1[p, \xi, \alpha, \beta, \Psi^*, \phi, v] = \left(\lambda^2 Q_{o,a,\beta}^{-1}\left(\tilde{L}_U[\Psi^*]_0 + \tilde{L}_U[\Psi^*]_2 + \mathcal{X}_0[p, \xi]\right) + \lambda^2 Q_{o,a,\beta}\right) \mathcal{X}_2, \]

\[ \mathcal{H}_2[p, \xi, \alpha, \beta, \Psi^*, \phi, v] = \left(\lambda^2 Q_{o,a,\beta}^{-1}\left(\tilde{L}_U[\Psi^*]_1 + \mathcal{X}_1[p, \xi]\right) + \lambda^2 Q_{o,a,\beta}\right) \mathcal{X}_3, \]

\[ \mathcal{H}_3[p, \xi, \alpha, \beta, \Psi^*, \phi, v] = \lambda^2 Q_{o,a,\beta}^{-1}\left(\tilde{L}_U[\Psi^*]_1 - \tilde{L}_U[\Psi^*]_0\right) \mathcal{X}_4, \]

\[ \mathcal{H}_4[p, \xi, \alpha, \beta, \Psi^*, \phi, v] = \left(\lambda^2 Q_{o,a,\beta}\left(\Pi_U[\mathcal{R}_1] + \Pi_U[\mathcal{R}_2]\right) + \lambda^2 Q_{o,a,\beta}\right) \mathcal{X}_5. \]
where $[\Pi_{U \cdot (v \cdot \nabla u)}]_0$, $[\Pi_{U \cdot (v \cdot \nabla u)}]_{-1}$, $[\Pi_{U \cdot (v \cdot \nabla u)}]_1$ and $[\Pi_{U \cdot (v \cdot \nabla u)}]_\perp$ correspond respectively to modes 0, −1, 1 and higher modes $k \geq 2$ defined in (5.46)–(5.48), and

\[
\tilde{L}_U[\Phi]^{(0)} = -2\lambda^{-1}w_\rho \cos w \left[ (\varphi_{3}(\xi(t),t)) \cos \theta + (\varphi_{2}(\xi(t),t)) \sin \theta \right] Q_{\omega,\alpha,\beta}E_1
\]

\[
= -2\lambda^{-1}w_\rho \cos w \left[ (\varphi_{3}(\xi(t),t)) \sin \theta - (\varphi_{2}(\xi(t),t)) \cos \theta \right] Q_{\omega,\alpha,\beta}E_2
\]

in the notation (5.11). Then by decomposing $\phi = \phi_1 + \phi_2 + \phi_3 + \phi_4$ in a similar manner as $\mathcal{M}_i$'s, the inner problem (5.111) becomes

\[
\begin{align*}
\lambda^2 \partial_t \phi_1 &= L_{W}[\phi[1]] + \mathcal{M}_1[p, \xi, \alpha, \beta, \Psi^*, \phi, v] \\
&\quad - \sum_{j=1,2} c_{0j}[\mathcal{M}_1[p, \xi, \alpha, \beta, \Psi^*, \phi, v]]w_\rho^2 Z_{0,j} \\
&\quad - \sum_{j=1,2} c_{1j}[\mathcal{M}_1[p, \xi, \alpha, \beta, \Psi^*, \phi, v]]w_\rho^2 Z_{1,j} \quad \text{in } \mathcal{D}_2R, \quad (5.114)
\end{align*}
\]

\[
\phi_1 \cdot W = 0 \quad \text{in } \mathcal{D}_2R, \\
\phi_1(\cdot, 0) = 0 \quad \text{in } B_{2R(0)},
\]

\[
\begin{align*}
\lambda^2 \partial_t \phi_2 &= L_{W}[\phi[2]] + \mathcal{M}_2[p, \xi, \alpha, \beta, \Psi^*, \phi, v] \\
&\quad - \sum_{j=1,2} c_{1j}[\mathcal{M}_2[p, \xi, \alpha, \beta, \Psi^*, \phi, v]]w_\rho^2 Z_{1,j} \quad \text{in } \mathcal{D}_2R, \quad (5.115)
\end{align*}
\]

\[
\phi_2 \cdot W = 0 \quad \text{in } \mathcal{D}_2R, \\
\phi_2(\cdot, 0) = 0 \quad \text{in } B_{2R(0)},
\]

\[
\begin{align*}
\lambda^2 \partial_t \phi_3 &= L_{W}[\phi[3]] + \mathcal{M}_3[p, \xi, \alpha, \beta, \Psi^*, \phi, v] \\
&\quad - \sum_{j=1,2} c_{0j}[\mathcal{M}_3[p, \xi, \alpha, \beta, \Psi^*, \phi, v]]w_\rho^2 Z_{0,j} \\
&\quad + \sum_{j=1,2} c_{0j}[\mathcal{M}_3[p, \xi, \alpha, \beta, \Psi^*, \phi, v]]w_\rho^2 Z_{0,j} \quad \text{in } \mathcal{D}_2R, \quad (5.116)
\end{align*}
\]

\[
\phi_3 \cdot W = 0 \quad \text{in } \mathcal{D}_2R, \\
\phi_3(\cdot, 0) = 0 \quad \text{in } B_{2R(0)}.
\]
\[
\begin{aligned}
\lambda^2 \partial_t \phi_4 &= L_W[\phi_4] + \mathcal{H}_3[p, \xi, \alpha, \beta, \Psi', \phi, v] \\
&\quad - \sum_{j=1,2} c_{-1,j}[\mathcal{H}_3[p, \xi, \alpha, \beta, \Psi', \phi, v]] w_0^2 z_{-1,j} \text{ in } \mathcal{D}_2, \\
\phi_4 \cdot W &= 0 \text{ in } \mathcal{D}_2, \\
\phi_4(\cdot, 0) &= 0 \text{ in } B_{2R(0)},
\end{aligned}
\] (5.117)

\[
c_{0j}(t) - \tilde{c}_{0j}(t) = 0 \text{ for all } t \in (0, T), \quad j = 1, 2,
\]
(5.118)

\[
c_{1j}(t) = 0 \text{ for all } t \in (0, T), \quad j = 1, 2,
\]
(5.119)

\[
c_{-1,j}(t) = 0 \text{ for all } t \in (0, T), \quad j = 1, 2.
\]
(5.120)

Based on the linear theory developed in Section 5.2.5, we shall solve the inner problems (5.114)–(5.117) in the norms below.

- We use the norm \( \| \cdot \|_{v_i, a_i} \) to measure the right hand side \( \mathcal{H}_i \) with \( i = 1, \ldots, 4 \), where

\[
\| h \|_{v_i, a_i} = \sup_{\mathbb{R}^2 \times (0, T)} \frac{|h(y, t)|}{\lambda^{v_i}_e(t)(1 + |y|)^{-a_i}}
\] (5.121)

with \( v_i > 0, a_i \in (2, 3) \) for \( i = 1, 2, 4 \), and \( a_3 \in (1, 3) \).

- We use the norm \( \| \cdot \|_{v_1, a_1, \delta} \) to measure the solution \( \phi_1 \) solving (5.114), where

\[
\| \phi \|_{v_1, a_1, \delta} = \sup_{\mathcal{D}_2} \frac{|\phi(y, t)| + (1 + |y|)|\nabla_y \phi(y, t)| + (1 + |y|)^2|\nabla^2_y \phi(y, t)|}{\lambda^{v_1}_e(t) \max \left\{ \frac{\delta^{(5-a_1)}}{(1+|y|)^2}, \frac{1}{(1+|y|)^{a_1-2}} \right\}}
\]

with \( v_1 \in (0, 1), a_1 \in (2, 3), \delta > 0 \) fixed small.

- We use the norm \( \| \cdot \|_{v_2, a_2-2} \) to measure the solution \( \phi_2 \) solving (5.115), where

\[
\| \phi \|_{v_2, a_2-2} = \sup_{\mathcal{D}_2} \frac{|\phi(y, t)| + (1 + |y|)|\nabla_y \phi(y, t)| + (1 + |y|)^2|\nabla^2_y \phi(y, t)|}{\lambda^{v_2}_e(t)(1 + |y|)^{2-a_2}}
\]

with \( v_2 \in (0, 1), a_2 \in (2, 3) \).
We use the norm \( \| \cdot \|_{\ast \ast, \nu} \) to measure the solution \( \phi_3 \) solving (5.116), where
\[
\| \phi \|_{\ast \ast, \nu_3} = \sup_{\mathcal{D}_2R} \left| \phi(y,t) \right| + (1 + |y|) \left| \nabla_y \phi(y,t) \right| + (1 + |y|)^2 \left| \nabla_y^2 \phi(y,t) \right|/ \lambda_{\nu_3} y^1(t) R^2(t) (1 + |y|)^{-1}
\]
with \( \nu_3 > 0 \).

We use the norm \( \| \cdot \|_{\ast \ast \ast, \nu} \) to measure the solution \( \phi_4 \) solving (5.117), where
\[
\| \phi \|_{\ast \ast \ast, \nu_4} = \sup_{\mathcal{D}_2R} \left| \phi(y,t) \right| + (1 + |y|) \left| \nabla_y \phi(y,t) \right| + (1 + |y|)^2 \left| \nabla_y^2 \phi(y,t) \right|/ \lambda_{\nu_4} y^2(t)
\]
with \( \nu_4 > 0 \).

Based on the linear theory in Section 5.2.6, we shall solve the outer problem (5.112) in the following norms.

We use the norm \( \| \cdot \|_\ast \) defined in (5.68) to measure the right hand side \( G \) in the outer problem (5.112).

We use the norm \( \| \cdot \|_{\Theta, \gamma} \) defined in (5.69) to measure the solution \( \psi \) solving (5.112), where \( \Theta > 0 \) and \( \gamma \in (0, 1/2) \).

Based on the linear theory developed in Section 5.3, we shall solve the incompressible Navier–Stokes equation (5.110) in the following norms.

We use the norm \( \| \cdot \|_{S, \nu-2, a+1} \) defined in (5.74) to measure the forcing \( \mathcal{F} \), where \( \nu > 0 \) and \( a \in (1, 2) \).

We use the norm \( \| \cdot \|_{S, \nu-1, 1} \) defined in (5.74) to measure the velocity field \( v \) solving problem (5.110), where \( \nu > 0 \).

We then define
\[
\tilde{E}_1 = \{ \phi_1 \in L^\infty(\mathcal{D}_2R) : \nabla_y \phi_1 \in L^\infty(\mathcal{D}_2R), \| \phi_1 \|_{\ast \ast, \nu_1, a_1, \delta} < \infty \},
\]
\[
\tilde{E}_2 = \{ \phi_2 \in L^\infty(\mathcal{D}_2R) : \nabla_y \phi_2 \in L^\infty(\mathcal{D}_2R), \| \phi_2 \|_{\text{in}, \nu_2, a_2-2} < \infty \},
\]
\[
\tilde{E}_3 = \{ \phi_3 \in L^\infty(\mathcal{D}_2R) : \nabla_y \phi_3 \in L^\infty(\mathcal{D}_2R), \| \phi_3 \|_{\ast \ast \ast, \nu_3} < \infty \},
\]
\[
\tilde{E}_4 = \{ \phi_4 \in L^\infty(\mathcal{D}_2R) : \nabla_y \phi_4 \in L^\infty(\mathcal{D}_2R), \| \phi_4 \|_{\ast \ast \ast \ast, \nu_4} < \infty \},
\]
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and use the notations
\[ E_\phi = \tilde{E}_1 \times \tilde{E}_2 \times \tilde{E}_3 \times \tilde{E}_4, \quad \Phi = (\phi_1, \phi_2, \phi_3, \phi_4) \in E_\phi, \]
\[ \| \Phi \|_{E_\phi} = \| \phi_1 \| \times_{Y_1, \alpha_1, \delta} + \| \phi_2 \| \times_{Y_2, \alpha_2 - 2} + \| \phi_3 \| \times_{Y_3, \nu_3} + \| \phi_4 \| \times_{Y_4, \nu_4}. \]

We define the closed ball
\[ \mathcal{B} = \{ \Phi \in E_\phi : \| \Phi \|_{E_\phi} \leq 1 \}. \]

For the outer problem (5.112), we shall solve \( \psi \) in the space
\[ E_\psi = \{ \psi \in L^\infty(\Omega \times (0, T)) : \| \psi \|_{\psi, \Theta, \gamma} < \infty \}. \]

For the incompressible Navier–Stokes equation (5.110), we shall solve the velocity field \( v \) in the space
\[ E_v = \{ v \in L^2(\Omega; \mathbb{R}^2) : \nabla \cdot v = 0, \| v \|_{S, \nu, 1} < M \epsilon_0 \}, \quad (5.122) \]
where \( \epsilon_0 > 0 \) is the number in (5.1) which is fixed sufficiently small, and \( M > 0 \) is some fixed number.

To introduce the space for the parameter function \( p(t) \), we recall the integral operator \( \mathcal{B}_0 \) defined in (5.37) of the approximate form
\[ \mathcal{B}_0[p] = \int_{-T}^{T} \frac{\dot{p}(s)}{t-s} ds + O(\| \dot{p} \|_\infty). \]

For \( \Theta \in (0, 1), \alpha \in \mathbb{R} \) and a continuous function \( g : I \rightarrow \mathbb{C} \), we define the norm
\[ \| g \|_{\Theta, \alpha} = \sup_{t \in [-T, T]} (T-t)^{-\Theta} | \log(T-t) | | g(t) |, \]
and for \( \gamma \in (0, 1), m \in (0, \infty), \alpha \in \mathbb{R} \), we define the semi-norm
\[ [g]_{\gamma, m, \alpha} = \sup_{t \in [-T, T]} (T-t)^{-m} | \log(T-t) | | g(t) - g(s) | | t-s |^\gamma, \]
where the supremum is taken over \( s \leq t \) in \([-T, T]\) such that \( t - s \leq \frac{1}{10} (T-t) \).
The following result was proved in [55, Section 8].

**Proposition 5.4.1.** Let \(\alpha, \gamma \in (0, \frac{1}{2})\), \(l \in \mathbb{R}\), \(C_1 > 1\). If \(\alpha_0 \in (0, 1), \Theta \in (0, \alpha_0), m \in (0, \Theta - \gamma], \) and \(a(t) : [0, T] \rightarrow C\) satisfies

\[
\begin{cases}
\frac{1}{C_1} \leq |a(T)| \leq C_1, \\
T^\Theta |\log T|^{1+\sigma-l} |a(\cdot) - a(T)|_{\Theta, l-1} + [a]_{\gamma, m, l-1} \leq C_1,
\end{cases}
\]

for some \(\sigma > 0\), then for \(T > 0\) sufficiently small there exist two operators \(P\) and \(R_0\) so that \(p = P[a] : [-T, T] \rightarrow C\) satisfies

\[
B_0[p](t) = a(t) + R_0[a](t), \quad t \in [0, T]
\]

with

\[
|R_0[a](t)| \leq C \left( T^\sigma + T^\Theta |\log T| \right) \left| a(\cdot) - a(T) \right|_{\Theta, l-1} + [a]_{\gamma, m, l-1} \left( T - t \right)^{m+(1+\alpha)\gamma} \frac{|\log(T-t)|}{|\log(T)|},
\]

for some \(\sigma > 0\).

Proposition 5.4.1 gives an approximate inverse \(P\) of the operator \(B_0\), so that given \(a(t)\) satisfying (5.123), \(p = P[a]\) satisfies

\[
B_0[p] = a + R_0[a] \quad \text{in} \quad [0, T],
\]

for a small remainder \(R_0[a]\). Moreover, the proof of Proposition 5.4.1 in [55] gives the decomposition

\[
P[a] = p_{0,\kappa} + P_1[a]
\]

with

\[
p_{0,\kappa}(t) = \kappa |\log T| \int_t^T \frac{1}{|\log(T-s)|^2} ds, \quad t \leq T,
\]

\(\kappa = \kappa[a] \in C\), and the function \(p_1 = P_1[a]\) has the estimate

\[
\|p_1\|_{\kappa, 3-\sigma} \leq C |\log T|^{1-\sigma} \log^2(|\log T|).
\]
Here the semi-norm $\| \cdot \|_{*; 3-\sigma}$ is defined by

$$\| g \|_{*; 3-\sigma} = \sup_{t \in [-T, T]} |\log(T-t)|^{3-\sigma} |g(t)|,$$

and $\sigma \in (0,1)$. This leads us to define the space

$$X_p := \{ p_1 \in C([-T, T; \mathbb{C}]) \cap C^1([-T, T; \mathbb{C}]) : p_1(T) = 0, \| p_1 \|_{*, 3-\sigma} < \infty \},$$

where we represent $p$ by the pair $(\kappa, p_1)$ in the form $p = p_0, \kappa + p_1$.

We define the space for $\xi(t)$ as

$$X_{\xi} = \{ \xi \in C^1((0, T); \mathbb{R}^2) : \dot{\xi}(T) = 0, \| \xi \|_{X_{\xi}} < \infty \},$$

where

$$\| \xi \|_{X_{\xi}} = \| \xi \|_{L^\infty(0,T)} + \sup_{t \in (0,T)} \lambda_+^{-\sigma}(t) |\dot{\xi}(t)|$$

for some $\sigma \in (0,1)$, and we define the spaces for $\alpha(t), \beta(t)$ as follows

$$X_{\alpha} = \{ \alpha \in C^1((0, T)) : \alpha(T) = 0, \| \alpha \|_{X_{\alpha}} < \infty \},$$

where

$$\| \alpha \|_{X_{\alpha}} = \sup_{t \in (0,T)} \lambda_+^{-\delta_1}(t) |\alpha(t)| + \sup_{t \in (0,T)} \lambda_+^{-\delta_1}(t) |\dot{\alpha}(t)|,$$

$$X_{\beta} = \{ \beta \in C^1((0, T)) : \beta(T) = 0, \| \beta \|_{X_{\beta}} < \infty \},$$

$$\| \beta \|_{X_{\beta}} = \sup_{t \in (0,T)} \lambda_+^{-\delta_2}(t) |\beta(t)| + \sup_{t \in (0,T)} \lambda_+^{-\delta_2}(t) |\dot{\beta}(t)|.$$

Here $\delta_1, \delta_2 \in (0,1)$.

In conclusion, we will solve the inner–outer gluing system (5.110), (5.112), (5.114), (5.115), (5.116), (5.117), (5.118), (5.119) and (5.120) in the space

$$\mathcal{X} = E_v \times E_\psi \times E_\phi \times X_p \times X_{\xi} \times X_{\alpha} \times X_{\beta} \quad (5.124)$$

by means of fixed point argument.
5.4.1 Estimates of the orientation field $u$

The equation for the orientation field $u$ is close in spirit to the harmonic map heat flow (5.14). To get the desired blow-up, we only need to show the drift term $v \cdot \nabla u$ is a small perturbation in the topology chosen above. Then the construction of the orientation field $u$ is a direct consequence of [55] with slight modifications.

**Effect of the drift term $v \cdot \nabla u$ in the outer problem**

In the outer problem (5.112), it is direct to see that the main contribution in the drift term $v \cdot \nabla u$ comes from $v \cdot \nabla U$ since all the other terms are of smaller orders. We get that for some positive constant $\varepsilon$,

$$
|(1 - \eta R)v \cdot \nabla u| \lesssim \lambda^{-1}(t) \frac{\lambda(t)}{1 + \frac{x - q}{\lambda(t)}} \left|\frac{x - q}{2 + \lambda^{-1}(t) \chi_{\{|x - q| \geq \lambda(t) R(t)\}}}ight|
$$

(5.125)

provided $v > m$ with $m \in (1/2, 1)$, where $\rho_2$ is the weight of the $\| \cdot \|_{**}$-norm (see (5.67)) for the right hand side of the outer problem. Therefore, as long as $v$ is chosen sufficiently close to 1, the influence of the drift term $v \cdot \nabla u$ in the outer problem is negligible, and it is indeed a perturbation compared to the rest terms already estimated in the harmonic map heat flow [55, Section 6.6].

**Effect of the drift term $v \cdot \nabla u$ in the inner problem**

Since the inner problem is decomposed into different modes (5.114)--(5.117), a key observation is that the drift term $v \cdot \nabla u$ will get coupled in each mode. In other words, the mode $k$ solved from the velocity equation with forcing $-\varepsilon_0 \nabla \cdot (\nabla U \otimes \nabla \phi_k)$ enters mode $k$ of the inner problem via the drift term $v \cdot \nabla u$. We now analyze the projections of $v \cdot \nabla u$ on different modes. Recall that

$$
v \cdot \nabla u = v \cdot \nabla [U + \phi_{in} \Pi_{U^\perp} \phi_{out} + a(\Pi_{U^\perp} (\phi_{in} + \phi_{out})) U],
$$

where

$$
\phi_{in} = \eta R Q_{0,0,0} (\phi_1 + \phi_2 + \phi_3 + \phi_4), \quad \phi_{out} = \Psi^\varepsilon + \Phi^0 + \Phi^\alpha + \Phi^\beta.
$$
Notice that the leading term in \( v \cdot \nabla u \) is \( v \cdot \nabla U \). Since \( (v \cdot \nabla U, U) = 0 \), we have

\[
\Pi_{U \perp} (v \cdot \nabla U) = v \cdot \nabla U.
\]

Denote \( v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \). We write \( U \) in the polar coordinates

\[
\nabla U = \lambda^{-1} \begin{bmatrix} \cos \theta w_\rho E_1 - \frac{\sin \theta}{\rho} \sin \theta w E_2 \\ \sin \theta w_\rho E_1 + \frac{\cos \theta}{\rho} \sin \theta w E_2 \end{bmatrix}.
\]

Therefore, the projection of \( v \cdot \nabla u \) on mode \( k (k \in \mathbb{Z}) \) is of the following size

\[
|\Pi_{U \perp} (v \cdot \nabla u)_k| \lesssim |\Pi_{U \perp} (v \cdot \nabla U)_k| \\
\lesssim \int_0^{2\pi} (v_1 \cos(k-1)\theta - v_2 \sin(k-1)\theta) d\theta \\
+ i \int_0^{2\pi} (-v_1 \sin(k-1)\theta - v_2 \cos(k-1)\theta) d\theta
\]

from which we obtain

\[
|\lambda |\Pi_{U \perp} (v \cdot \nabla u)_k| \leq \frac{M \varepsilon_0 \lambda^\gamma}{1 + |y|^3}, \quad (5.126)
\]

where \( M \) and \( \varepsilon_0 \) are given in (5.122). Thus, it holds that

\[
\| \lambda |\Pi_{U \perp} (v \cdot \nabla u)_k \|_{v,a} \leq M \varepsilon_0.
\]

Since \( \varepsilon_0 \) is a sufficiently small number, we find that the projection \( \Pi_{U \perp} (v \cdot \nabla u)_k \) can be regarded as a perturbation compared to the rest terms in the right hand sides of the inner problems (5.114)–(5.117).

In summary, the coupling of the drift term \( v \cdot \nabla u \) in the inner and outer problems of the harmonic map heat flow is essentially negligible under the topology chosen above. Therefore, with slight modifications, the fixed point formulation for

\[
\partial_t u + v \cdot \nabla u = \Delta u + |\nabla u|^2 u
\]

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can be carried out in a similar manner as in [55].

For the outer problem (5.112), it was already estimated in [55] that in the space \( \mathcal{X} \) defined in (5.124), it holds that for some \( \varepsilon > 0 \)

\[
\| \mathcal{G}[p, \xi, \alpha, \beta, \Psi^*, \phi, v] - (1 - \eta_R) v \cdot \nabla u \|_{\mathcal{X}} \lesssim T \varepsilon \left( \| \Phi \|_{\mathcal{E}_\phi} + \| \Psi \|_{\mathcal{E}_{\Theta, \gamma}} + \| p \|_{\mathcal{X}_p} + \| \xi \|_{\mathcal{X}_\xi} + \| \alpha \|_{\mathcal{X}_\alpha} + \| \beta \|_{\mathcal{X}_\beta} + 1 \right)
\]

provided

\begin{align*}
\left\{ \begin{array}{l}
0 < \Theta < \min \left\{ \gamma_s, \frac{1}{2} - \gamma_s, v_1 - 1 + \gamma_s (a_1 - 1), v_2 - 1 + \gamma_s (a_2 - 1), \\
v_3 - 1, v_4 - 1 + \gamma_s \right\}, \\
\Theta < \min \{ v_1 - \delta \gamma_s (5 - a_1) - \gamma_s, v_2 - \gamma_s, v_3 - 3 \gamma_s, v_4 - \gamma_s \}, \\
\delta \ll 1.
\end{array} \right.
\end{align*}

(5.127)

On the other hand, from (5.125), we find that

\[
\| (1 - \eta_R) v \cdot \nabla u \|_{\mathcal{X}} \lesssim T^\varepsilon \left( \| v \|_{\mathcal{S}, v - 1, 1} + \| \Phi \|_{\mathcal{E}_\phi} + \| \Psi \|_{\mathcal{E}_{\Theta, \gamma}} + \| p \|_{\mathcal{X}_p} + \| \xi \|_{\mathcal{X}_\xi} + \| \alpha \|_{\mathcal{X}_\alpha} + \| \beta \|_{\mathcal{X}_\beta} + 1 \right)
\]

provided

\[
v > \frac{1}{2}.
\]

(5.128)

Therefore, we conclude the validity of the following proposition by Proposition 5.2.2.

**Proposition 5.4.2.** Assume (5.127) and (5.128) hold. If \( T > 0 \) is sufficiently small, then there exists a solution \( \psi = \Psi(v, \Phi, p, \xi, \alpha, \beta) \) to problem (5.112) with

\[
\| \Psi(v, \Phi, p, \xi, \alpha, \beta) \|_{\mathcal{E}_{\Theta, \gamma}} \lesssim T^\varepsilon \left( \| v \|_{\mathcal{S}, v - 1, 1} + \| \Phi \|_{\mathcal{E}_\phi} + \| p \|_{\mathcal{X}_p} + \| \xi \|_{\mathcal{X}_\xi} + \| \alpha \|_{\mathcal{X}_\alpha} + \| \beta \|_{\mathcal{X}_\beta} + 1 \right),
\]

for some \( \varepsilon > 0 \).

We denote \( \mathcal{T}_\psi \) by the operator which returns \( \psi \) given in Proposition 5.4.2.
For the inner problems (5.114)–(5.117), our next step is to take \( \Phi \in E_\phi \) and substitute
\[
\Psi^*(v, \Phi, p, \xi, \alpha, \beta) = Z^* + \Psi(v, \Phi, p, \xi, \alpha, \beta)
\]
into (5.111). We then write equations (5.114)–(5.117) as the fixed point problem
\[
\Phi = \mathcal{A}(\Phi),
\]
where
\[
\mathcal{A}(\Phi) = (\mathcal{A}_1(\Phi), \mathcal{A}_2(\Phi), \mathcal{A}_3(\Phi), \mathcal{A}_4(\Phi)), \quad \mathcal{A} : \tilde{B}_1 \subset E_\phi \rightarrow E_\phi
\]
with
\[
\mathcal{A}_1(\Phi) = \mathcal{T}_1(\mathcal{H}_1[v, \Psi^*(v, \Phi, p, \xi, \alpha, \beta), p, \xi, \alpha, \beta]),
\]
\[
\mathcal{A}_2(\Phi) = \mathcal{T}_2(\mathcal{H}_2[v, \Psi^*(v, \Phi, p, \xi, \alpha, \beta), p, \xi, \alpha, \beta]),
\]
\[
\mathcal{A}_3(\Phi) = \mathcal{T}_3\left(\mathcal{H}_3[v, \Psi^*(v, \Phi, p, \xi, \alpha, \beta), p, \xi, \alpha, \beta]\right. \\
\left. + \sum_{j=1}^{2} c_0[j][v, \Psi^*(v, \Phi, p, \xi, \alpha, \beta), p, \xi, \alpha, \beta]w_\rho^2 Z_{0,j}\right),
\]
\[
\mathcal{A}_4(\Phi) = \mathcal{T}_4\left(\mathcal{H}_4[v, \Psi^*(v, \Phi, p, \xi, \alpha, \beta), p, \xi, \alpha, \beta]\right).
\]

Neglecting \( \Pi_{U^\perp} (v \cdot \nabla u) \), the contraction for the inner problem was shown in [55, Section 6.7] under the conditions
\[
\begin{align*}
\nu_1 &< 1, \\
\nu_2 &< 1 - \gamma(a_2 - 2), \\
\nu_3 &< \min\left\{1 + \Theta + 2\gamma, \nu_1 + \frac{1}{2}\delta(a_1 - 2)\right\}, \\
\nu_4 &< 1.
\end{align*}
\]

On the other hand, from (5.126), we obtain
\[
\left\| Q_\omega \mathcal{A}^{-1}_{\omega, \alpha, \beta} [\Pi_{U^\perp} (v \cdot \nabla u)]_0 \right\|_{v_1, a_1} \leq M e_0 \lambda^{-\nu_1}_{\alpha}(t),
\]

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\begin{align*}
\left\| \lambda Q_{\phi,\alpha,\beta}^{-1} \left( \Pi_{U \perp} (v \cdot \nabla u) \right) \right\|_{v_2,a_2} & \leq M \varepsilon_0 \lambda^*-v_2(t), \\
\left\| \lambda Q_{\phi,\alpha,\beta}^{-1} \left[ \Pi_{U \perp} (v \cdot \nabla u) \right] \right\|_{v_4,a} & \leq M \varepsilon_0 \lambda^*-v_4(t).
\end{align*}

Recall that the parameter \( \varepsilon_0 > 0 \) in (5.1) is fixed and sufficiently small. Therefore, by letting
\[
\begin{aligned}
v = v_1 = v_2 = v_4, \\
1 < a < 2,
\end{aligned}
(5.132)
\]
the smallness in (5.131) comes from \( \varepsilon_0 \ll 1 \). Applying the linear theory developed in Section 5.2.5 for the inner problems (5.114)–(5.117), we then conclude the following proposition.

**Proposition 5.4.3.** Assume (5.130) and (5.132) hold. If \( T > 0 \) and \( \varepsilon_0 > 0 \) are sufficiently small, then the system of equations (5.129) for \( \Phi = (\phi_1, \phi_2, \phi_3, \phi_4) \) has a solution \( \Phi \in E_\Phi \).

We denote by \( T_p, T_\xi, T_\alpha, T_\beta \) the operators which return the parameter functions \( p(t), \xi(t), \alpha(t), \beta(t) \), respectively. The argument for adjusting the parameter functions such that (5.118)–(5.120) hold is essentially similar to that of [55]. Note that the influence of the coupling \( v \cdot \nabla u \) is negligible as shown in Section 5.4.1. Therefore, the leading orders for the parameter functions \( p(t), \xi(t), \alpha(t), \beta(t) \) are the same as in Section 5.2.4. The reduced problem (5.118) yields an integro-differential equation for \( p(t) \) which can be solved by the same argument as in [55], while the reduced problems (5.119)–(5.120) give relatively simpler equations for \( \xi(t), \alpha(t), \beta(t) \), which can be solved by the fixed point argument. We omit the details.

### 5.4.2 Estimates of the velocity field \( v \)

To solve the incompressible Navier–Stokes equation (5.110), we need to analyze the coupled forcing term
\[
\varepsilon_0 \nabla \cdot (\nabla u \otimes \nabla u - 1/2 |\nabla u|^2_2).
\]
Observe that the main contribution in the forcing comes from \( U + \eta_0 Q_{\omega,\alpha,\beta}(\phi_0 + \phi_1 + \phi_{-1} + \phi_{\perp}) \), where \( \phi_0, \phi_1, \phi_{-1}, \phi_{\perp} \) are in mode 0, 1, \(-1\) and higher modes, respectively. From the linear theory in Section 5.2.5, the dominant terms are \( U \) and \( \phi_0 \). So we next need to evaluate

\[
\nabla \cdot (\nabla U \otimes \nabla U - 1/2 |\nabla U|^2 I_2) \text{ and } \nabla \cdot (\nabla U \otimes \nabla \phi_0 - 1/2 (\nabla U : \nabla \phi_0) I_2),
\]

where \( \nabla U : \nabla \phi_0 = \sum_{ij} \partial_j U_i \partial_i (\phi_0)_j \). Recall

\[
U(y) = \begin{bmatrix} e^{i\theta} \sin w(\rho) \\ \cos w(\rho) \end{bmatrix}, \quad E_1(y) = \begin{bmatrix} e^{i\theta} \cos w(\rho) \\ -\sin w(\rho) \end{bmatrix}, \quad E_2(y) = \begin{bmatrix} ie^{i\theta} \\ 0 \end{bmatrix}
\]

so that

\[
\partial_\rho U = w_\rho E_1, \quad \partial_\theta U = \sin w E_2,
\]

\[
\partial_\rho E_1 = -w_\rho U, \quad \partial_\theta E_1 = \cos w E_2.
\]

Note that

\[
\nabla \cdot (\nabla U \otimes \nabla U - 1/2 |\nabla U|^2 I_2) = \Delta U \cdot \nabla U = -|\nabla U|^2 U \cdot \nabla U = 0.
\]

For \( \nabla \cdot (\nabla U \otimes \nabla \phi_0 - 1/2 (\nabla U : \nabla \phi_0) I_2) \), we express the forcing in the polar coordinates. Since \( \phi_0 = \phi_0 E_1 \) where \( \phi_0 = \phi_0(\rho) \), the first component

\[
(\lambda^3 \nabla \cdot (\nabla U \otimes \nabla \phi_0))_1
= \partial_{y_1} \left( \cos^2 \theta \partial_\rho \phi_0 w_\rho + \frac{\sin^2 \theta}{\rho^2} \phi_0 \sin w \cos w \right)
+ \partial_{y_2} \left( \sin \theta \cos \theta \partial_\rho \phi_0 w_\rho - \frac{\sin \theta \cos \theta}{\rho^2} \phi_0 \sin w \cos w \right).
\]

Changing \( \partial_{y_1} \) and \( \partial_{y_2} \) into \( \partial_\rho \) and \( \partial_\theta \), we obtain

\[
(\lambda^3 \nabla \cdot (\nabla U \otimes \nabla \phi_0))_1
= \cos \theta \partial_\rho \left( \cos^2 \theta \partial_\rho \phi_0 w_\rho + \frac{\sin^2 \theta}{\rho^2} \phi_0 \sin w \cos w \right)
\]
\[-\frac{\sin \theta}{\rho} \partial_{\theta} \left( \cos^2 \theta \partial_{\rho} \phi_0 w_\rho + \frac{\sin^2 \theta}{\rho^2} \phi_0 \sin w \cos \theta \right) \]
\[+ \sin \theta \partial_{\rho} \left( \sin \theta \cos \theta \partial_{\rho} \phi_0 w_\rho - \frac{\sin \theta \cos \theta}{\rho^2} \phi_0 \sin w \cos \theta \right) \]
\[+ \frac{\cos \theta}{\rho} \partial_{\theta} \left( \sin \theta \cos \theta \partial_{\rho} \phi_0 w_\rho - \frac{\sin \theta \cos \theta}{\rho^2} \phi_0 \sin w \cos \theta \right) \]
\[= \cos \theta \left( \partial_{\rho}^2 \phi_0 w_\rho + \partial_{\rho} \phi_0 w_{\rho \rho} + \frac{1}{\rho} \partial_{\rho} \phi_0 w_\rho - \frac{1}{\rho^3} \phi_0 \sin w \cos \theta \right) \]
\[= \cos \theta \left[ \partial_{\rho} \left( \phi_0 w_\rho + \frac{\phi_0 w_{\rho \rho}}{\rho} + \int \phi_0 w_\rho^2 \right) \right]. \]

A similar calculation implies that the second component

\[\langle \lambda^3 \nabla \cdot (\nabla U \odot \nabla \phi_0) \rangle_2 = \sin \theta \left[ \partial_{\rho} \left( \phi_0 w_\rho + \frac{\phi_0 w_{\rho \rho}}{\rho} + \int \phi_0 w_\rho^2 \right) \right]. \]

So \( \nabla \cdot (\nabla U \odot \nabla \phi_0) = \nabla \left[ \lambda^{-3} \left( \partial_{\rho} \phi_0 w_\rho + \frac{\phi_0 w_{\rho \rho}}{\rho} + \int \phi_0 w_\rho^2 \right) \right] \) is a potential. Moreover, it is obvious that \( \nabla \cdot (|\nabla U|^2 \|_2) \) is a potential. Therefore, \( \nabla \cdot (\nabla U \odot \nabla \phi_0 - 1/2 (\nabla U : \nabla \phi_0) \|_2) \) is a potential, which can be absorbed in the pressure \( P \) in problem (5.110).

Therefore, the leading term in \( F \) defined by (5.113) is

\[ |\nabla U \odot \nabla \phi_{-1}| \leq \frac{\lambda^{v_{-1}-2}(t)}{1 + |y|^\tau} \|\phi_{-1}\|_{\|***,v_4},} \tag{5.133} \]

from which we conclude that

\[ \|e_0 \nabla U \odot \nabla \phi_{-1}\|_{5,v-2,\alpha+1} \leq e_0, \]

where we have used (5.132).

**Remark 5.4.1.** As we can see above, using the new linear theory at mode \(-1\) (see Lemma [5.2.5]), the size of the transported term turns out to be of the same order as the right hand side at mode \(-1\). This is the motivation of introducing new parameters \( \alpha(t), \beta(t) \) and developing new linear theory at mode \(-1\), otherwise the transported term would carry extra logarithm growth in time (see Lemma [5.2.4]).

On the other hand, the assumption \( e_0 \ll 1 \) is required here to guarantee the contraction in the fixed point argument.
On the other hand, as mentioned in Remark 5.3.1, the nonlinear term $v \cdot \nabla v$ in (5.110) is of smaller order compared to the forcing $\varepsilon_0 \nabla \cdot F$ if we look for a solution $v$ in the function space $E_v$ defined in (5.122). Indeed, since $v \in E_v$, we have

$$|v \cdot \nabla v| \lesssim \frac{\lambda^2 \nu^{-2}(t)}{1 + |y|^3}$$

so that

$$\|v \cdot \nabla v\|_{S, \nu^{-2}, a+1} \lesssim \lambda^v(t) \ll 1 \quad \text{as} \quad t \to T.$$

Thus, the incompressible Navier–Stokes equation (5.110) can be regarded as a perturbed Stokes system

$$\partial_t v + \nabla P = \Delta v - \varepsilon_0 \nabla \cdot \mathcal{F}_1[p, \xi, \alpha, \beta, \Psi^*, \phi, v]$$

with

$$\mathcal{F}_1[p, \xi, \alpha, \beta, \Psi^*, \phi, v] = \mathcal{F}[p, \xi, \alpha, \beta, \Psi^*, \phi, v] + v \otimes v,$$

where we have used the fact that $v$ is divergence-free so that we can write $v \cdot \nabla v = \nabla \cdot (v \otimes v)$. We denote $\mathcal{T}_v$ by the operator which returns the solution $v$, namely

$$\mathcal{T}_v : E_v \to E_v,$$

$$v \mapsto \mathcal{T}_v(v).$$

By (5.133) and the linear theory for the Stokes system developed in Section 5.3, we obtain

$$\|\mathcal{T}_v(v)\|_{S, \nu^{-1}, 1} \leq C \varepsilon_0(\|v\|_{S, \nu^{-1}, 1} + \|\Phi\|_{E_\phi} + \|\Psi\|_{L, \Theta, \gamma} + \|p\|_{X_\rho} + \|\xi\|_{X_{\xi}} + \|\alpha\|_{X_{\alpha}} + \|\beta\|_{X_\beta} + 1).$$

(5.134)

**5.4.3 Proof of Theorem 5.1.1**

Consider the operator

$$\mathcal{T} = (\mathcal{A}_v, \mathcal{F}_v, \mathcal{F}_p, \mathcal{F}_z, \mathcal{F}_\alpha, \mathcal{F}_\beta)$$

(5.135)

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defined in Section 5.4.1 and Section 5.4.2. To prove Theorem 5.1.1, our strategy is to show that the operator $T$ has a fixed point in $X$ by the Schauder fixed point theorem. Here the function space $X$ is defined in (5.124). The existence of a fixed point in the desired space $X$ follows from a similar manner as in [55].

By collecting Proposition 5.4.1, Proposition 5.4.2, Proposition 5.4.3 and (5.134), we conclude that the operator maps $X$ to itself. On the other hand, the compactness of the operator $T$ can be proved by suitable variants of the estimates. Indeed, if we vary the parameters $\gamma_*, \Theta, \nu, a, v_1, a_1, v_2, v_3, v_4, \delta$ slightly such that all the restrictions in (5.127), (5.128), (5.130) and (5.132) are satisfied, then one can show that the operator $T$ has a compact embedding in the sense that if a sequence is bounded in the new variant norms, then there exists a subsequence which converges in the original norms used in $X$. Thus, the compactness follows directly from a standard diagonal argument by Arzelà–Ascoli’s theorem. Therefore, the existence of the desired solution for the single bubble case $k = 1$ follows from the Schauder fixed point theorem.

The general case of multiple-bubble blow-up is essentially identical. The ansatz is modified as follows: we look for solution $u$ of the form

$$u(x,t) = \sum_{j=1}^{k} U_j + \Pi_{U_j} \varphi_j + a(\Pi_{U_j} \varphi_j) U_j,$$

where

$$U_j = U_{\lambda_j(t), \xi_j(t), \omega_j(t), \alpha_j(t), \beta_j(t)}, \quad \varphi_j = \varphi_j^{in} + \varphi_j^{out},$$

$$\varphi_j^{in} = \eta_R(t) (y_j) Q_{\omega_j(t), \alpha_j(t), \beta_j(t)} \varphi(y_j, t), \quad y_j = \frac{x - \xi_j(t)}{\lambda_j(t)},$$

$$\varphi_j^{out} = \psi(x, t) + Z^*(x, t) + \Phi_0^j + \Phi_\alpha^j + \Phi_\beta^j.$$

Here $\Phi_0^j, \Phi_\alpha^j$ and $\Phi_\beta^j$ are corrections defined in a similar way as in (5.22) with $\lambda, \xi, \omega, \alpha, \beta$ replaced by $\lambda_j, \xi_j, \omega_j, \alpha_j, \beta_j$. We are then led to one outer problem and $k$ inner problems for $u$ together with one Navier–Stokes equation for $v$ with exactly analogous estimates. A string of fixed point problems can be solved in the same manner. We omit the details. \qed
Chapter 6

Conclusion

In this Chapter, we shall summarize the works in Chapters 2-5 and provide possible generalizations related to the problems in this dissertation.

In Chapter 2, we generalize the line bubbling phenomenon in [60] to low dimensions $n = 6, 7$ for the supercritical Lane-Emden-Fowler problem. This requires a refined linear theory for the linearized problem and also careful choices of weighted topologies. The motivation of the methodology and techniques is from the parabolic gluing method recently developed in [43, 55]. In more general settings on manifolds, the method developed in this Chapter can also be applied to reduce the co-dimensions in [49, 72]. On the other hand, a natural question is that whether line bubbling phenomenon exists for lower dimension case $n = 4, 5$. One of the major difficulties arises when dealing with the slow decaying error which shares a similar flavor as that of low dimensional parabolic problems discussed in Chapter 3. As a result, the reduced equation for the scaling parameter function will be nonlocal.

In the first part of Chapter 3, we consider the energy critical heat equation in dimension three and give the first rigorous construction of type II blow-up solutions confirming all the blow-up rates predicted in [80]. In the second part of Chapter 3 we construct type II blow-up for the energy supercritical heat equation in low dimensions $n = 5, 6, 7$, where the solution blows up along a novel shrinking concentration sphere as time evolves. The possible extension of this Chapter is to work
with Fujita equation with different nonlinearities $p = 5, 7/3, 2$ being supercritical in higher dimensions. Those nonlinearities correspond to energy critical heat equations in $\mathbb{R}^3, \mathbb{R}^5, \mathbb{R}^6$ whose constructions of type II blow-ups can be used as building blocks to extend similar results to energy supercritical heat equations. Also, lifting the symmetry assumption on the domain potentially obtaining other blow-up sets is a rather interesting and difficult issue. In another aspect, investigating different non-generic blow-up rates for energy critical and supercritical heat equations might be of interest.

In Chapter 4, we construct both infinite and finite time blow-up solutions for the fractional critical heat equation. Due to the absence of the ODE techniques and monotonicity formula for $\partial_t + (-\Delta)^s$, much less is known for the fractional heat equations. The systematic method we develop in this Chapter is a new fractional gluing method, which might be applied to other nonlinear parabolic problems with fractional Laplacian such as fractional porous medium equations and dissipative quasi-geostrophic equations. Our constructions in this Chapter suggest that no type II blow-up exists in higher dimensions, while we believe that both infinite and finite time blow-ups should be present in lower dimension case $2s < n \leq 4s$. In this case, the treatment of non-local reduced problems and a new linear theory in low dimensions are required. The techniques carried out in Chapter 3 might also provide a guide and insights.

In Chapter 5, we construct finite time blow-up to the nematic liquid crystal flow in the critical dimension $n = 2$. The systematic gluing method we devise in Chapter 5 can be applied to a broader context of strongly coupled systems with non-variational structure. The key parts are to analyze precisely the coupling effects which reflect in different Fourier modes of the linearization and to adjust modulation parameters ensuring the implementation of the iteration scheme. In continuation of this work, generalizations of this Chapter can be in several directions. We plan to investigate

- nematic liquid crystal flows in dimension three where there are abundant and more complicated phenomena.
- nematic liquid crystal flows with natural physical anchoring conditions including certain free boundary conditions.
• compressible nematic liquid crystal flows in physical dimensions $n = 2, 3$.

A more ambitious next step is to study various different models in the rich physical literatures. For example, the starting point might be more general theories such as the Landau-de Gennes theory and the Doi-Onsager theory. Also, the presence of obstacles or droplets in the fluids might be of interest.

In summary, the inner-outer gluing method has applications to many different kinds of nonlinear models from both physical world and pure mathematical perspective.
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