Negative curves in blowups of weighted projective planes

by

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Abstract

We study the Mori dream space property for blowups at a general point of weighted projective planes or, more generally, of toric surfaces with Picard number one. Such a variety is a Mori dream space if and only if it contains two irreducible disjoint curves; one of them necessarily having non-positive self-intersection. We call such a curve a "negative curve". A significant part of this thesis is dedicated to the study of such negative curves, as they largely govern the Mori dream space property for these varieties.

Our study begins by constructing two one-parameter families of negative curves and subsequently a larger two-parameter class of negative curves having the previous two families as boundary cases.

Once such a variety is known to contain a negative curve, we determine if it contains a disjoint curve by using different procedures. For example, prime characteristic and cohomological methods. Furthermore, we introduce an independent technique that applies to a broader class of cases. As a result, for each of the negative curves constructed we provide examples and non-examples of Mori dream spaces containing the curve.

Lay Summary

Algebraic Geometry is the area of mathematics concerned with the study of geometric shapes defined by so-called polynomial equations. Think, for example, of a parabola or a sphere. These shapes can be patched together to form more complicated objects, just like one can sew together pieces of cloth to create a garment.

If we are given a particularly complicated piece of clothing it might be really hard to understand its shape. However, a very fruitful tool to do this is to pick individual threads and sort them in terms of how many times they cross other threads. For weighted projective planes (garments), this procedure singles out one special thread which we call a "negative curve". This thesis is concerned with the classification of weighted projective planes in terms of the properties of the negative curve they contain.

Preface

- Chapter 1, the introduction, presents a brief history and motivation for the problem we study.
- Chapters 2 and 3 together constitute and extended version of our published work with José Luis González and Kalle Karu in [11, 12]. Rather than presenting our results in both publications as separate chapters, we give a unified exposition to avoid redundancies. In Chapter 2 we develop the machinery that is later used in Chapter 3 to prove the main theorems.

  The paper *On a family of negative curves* [12] was published in the Journal of Algebra in 2019. Research was conducted as an equal collaboration, while I did most of the manuscript preparation.

  The paper *Constructing non-Mori Dream Spaces from negative curves* [11] was published in the Journal of Pure and Applied Algebra in 2019. Research was conducted as an equal collaboration during a three weeks long visit of J.L. González to UBC during the summer of 2018. The ideas presented in it are the result of daily working sessions (M–F) during this period.
- Chapter 4 is adapted from the preprint *Curves generating extremal rays in blowups of weighted projective planes*, by José Luis González, Kalle Karu and myself. Research was conducted as an equal collaboration, mainly taking place during a visit of J.L. to UBC during the summer of 2019. The ideas presented in it are the result of daily working sessions (M–F) during this period. The preprint has been submitted for publication.

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Chapter 1

Introduction

Let X be a normal, $\mathbb{Q}$-factorial projective variety over a field K. Roughly speaking, the Cox ring of X is

$$\operatorname{Cox}(X) = \bigoplus_{[D]\in \operatorname{Pic}(X)} H^0(X, \mathcal{O}_X(D)).$$

Such an X said to be a Mori dream space (MDS) if its Cox ring is a finitely generated K-algebra.

Determining whether a given variety is a MDS is a difficult and nuanced question. There are many kinds of varieties that are MDS, such as toric, flag and Fano varieties [1]. However, even an otherwise well-behaved procedure like blowing up a point can destroy the MDS property. For example, in [3] the authors present a formula quantifying how many points can be blown up in $\mathbb{P}^n_{\mathbb{C}}$ before the resulting variety is not a MDS.

One of the simplest nontrivial examples to study this phenomenon is the blow-up of weighted projective planes at a generic point. Let $T = (K^\times)^2 \subseteq \mathbb{P}(a, b, c)$ be the open algebraic torus and $t_0 \in T$ its identity element. We study the finite generation of $\operatorname{Cox}(X)$ for

$$X = \operatorname{Bl}_{t_0} \mathbb{P}(a, b, c).$$

This problem dates back to the early eighties, when it was studied by the commutative algebra community as part of a program to better understand Hilbert's 14th problem, which had already been disproven by Nagata. Their aim was to determine the finite generation of the symbolic Rees algebra of the weighted polynomial ring $S = k[x_1, x_2, x_3]$ and the kernel of the map $S \to k[t]$ given by $x_i \mapsto t^{\operatorname{wt}(x_i)}$. In modern terminology, the finite generation of this symbolic Rees algebra is equivalent to that one of the Cox ring of the blowup $\operatorname{Bl}_{pt} \operatorname{Proj} S$ at the point corresponding the the kernel described above.

Most of these efforts seem to have culminated with Cutkosky's 1991 paper [8], in which he fully geometrized the problem and used the machinery of birational geometry to extend most (if not all) of the previously known results. His work, for example, recovers some important previous results by Huneke [18]. Nonetheless, it wasn't until 1994 that the first non-MDS examples were found using a prime characteristic reduction method by Goto, Nishida and Watanabe [14]. See [3] for a more detailed account on the history of the problem.

Recently the problem has received renewed interest because of the key role these varieties play in showing that the moduli space of pointed rational curves, $\overline{\mathcal{M}}_{0,n}$, is not a MDS for $n > 9$, see [4, 10, 16].

More generally, we study the blow-up X of toric surfaces with Picard number $\rho(X) = 1$ at the torus identity t_0. This class of surfaces include weighted projective planes, but in general are quotients of these by finite subgroups of the algebraic torus.

What makes such an X amenable to study is that it has Picard number 2. Its cone of curves is a two-dimensional cone in $N_1(X) \cong \mathbb{R}^2$ with boundary rays generated by the exceptional curve E and another curve class γ of non-positive self-intersection. Its dual, the nef cone $Nef(X)$, is generated by the hyperplane class H and the class of another curve D, orthogonal to γ. Cutkosky [8] showed that X is a MDS if and only if γ is generated by an irreducible curve C and D is semi-ample, this is, $C \cap D = \emptyset$. We call a curve C as above a negative curve.

Therefore, we can divide the problem of determining if X is a MDS into two parts: first find the negative curve C, then find the disjoint curve D.

Our approach is to construct the varieties $X = \mathrm{Bl}_{t_0} X_\Delta$ as follows. Let $T = (K^\times)^2$ be the two-dimensional torus and let $C^0 \subset T$ be an irreducible curve that has multiplicity m at the point $t_0 = (1, 1)$. We compactify T to the toric variety X_Δ by choosing a rational triangle $\Delta \subset \mathbb{R}^2$ that contains the Newton polygon of the curve C^0. Such a triangle fully determines X_Δ. For example, if we let u, v and $w \in \mathbb{Z}^2$ be the three outward-pointing normal vectors to Δ, then the weights a, b and c are the smallest positive integers satisfying $au + bv + cw = 0$.

Call $\overline{C^0}$ the closure of C^0 in X_Δ and let C be the strict transform of $\overline{C^0}$ in the blowup $X = \mathrm{Bl}_{t_0} X_\Delta$. Then, the divisor class of C is $\pi^*H - mE$ and

$$C \cdot C = H \cdot H + m^2 E \cdot E = 2\,\mathrm{Area}(\Delta) - m^2.$$

Thus, C is a negative curve if the area of Δ is $\leq \frac{m^2}{2}$. A significant part of this thesis deals with constructing negative curves using the procedure we just described. As a result we obtain examples of varieties X that are indexed by the negative curves that they contain. It then remains to study whether these varieties $X = \mathrm{Bl}_{t_0} X_\Delta$ are a MDS or not. This is, does there exists a curve $D \subset X$ disjoint from the negative curve? In this thesis we explore two different ideas to tackle this problem: Kurano and Nishida's prime characteristic method and our technique from [13], which is characteristic independent.

Our road-map is as follows. In Chapter 3 we construct two one-parameter families of non-isomorphic negative curves and use the prime characteristic method of Kurano and Nishida from Chapter 2 to determine if the spaces under consideration are Mori dream spaces. In Chapter 4 we construct two two-parameter families which contain the aforementioned families as boundary cases. Moreover, we recover the results from previous chapters while avoiding positive characteristic methods and higher cohomology.

Chapter 2

Cutkosky's criterion and the Huneke condition

In this chapter and Chapter 3 we study blowups of weighted projective planes at a general point and, more generally, blowups of toric surfaces of Picard number one. Based on the positive characteristic methods of Kurano and Nishida [20], we give a general method for constructing examples of Mori Dream Spaces and non-Mori Dream Spaces among such blowups. We consider examples of X that contain a negative curve in the class $\pi^*H - mE$, where H is the class of a divisor pulled back from the toric variety X_Δ and E is the class of the exceptional curve. For any $m \geq 1$ we construct examples where the Cox ring is finitely generated and examples where it is not. Before our examples no negative curves with higher values of m were known. Furthermore, our results generalize those of Srinivasan [21] and Kurano-Nishida [20].

The two chapters are divided as follows: In the present chapter we prove a generalized version of Cutkosky's criterion and present the method of Kurano and Nishida in geometric terms. This method is deeply connected with the cohomology of X, so in the last couple of sections we study the sheaf cohomology of these spaces and their relation with lattice point interpolation. In Chapter 3 we use the tools presented here to construct examples and non-examples of Mori dream spaces.

2.1 Preliminaries about toric varieties and Cox rings.

Let X be a normal $\mathbb{Q}$-factorial surface. The class group $\mathrm{Cl}(X)$ is the group of Weil divisors modulo linear equivalence and $N_1(X)$ is the $\mathbb{R}$-vector space of numerical equivalence classes of curves in X. Every curve in X has a class in $\mathrm{Cl}(X)$ and its image in $N_1(X)$. We denote by $C \cdot D$ the intersection product between curves. The nef cone of X is the cone in $N_1(X)$ generated by classes of nef divisors. Its dual cone (also in $N_1(X)$ via the intersection pairing) is the closure of the cone of effective curves of X, also called the Mori cone.

2.1.1 Weil divisors and rational triangles.

Let X be a normal variety and D an integral Weil divisor in X. Recall that the sheaf $\mathcal{O}_X(D)$ is defined so that its sections s on an open set $U \subseteq X$ are rational functions $f \in K(X)$ such that $\mathrm{div}(f) + D$ is effective on U. We define the vanishing locus $V(s)$ of the section s as the support of the divisor $\mathrm{div}(f) + D$. Notice that the sheaf $\mathcal{O}_X(D)$ is invertible away from the support of D, and more generally, away from $V(s)$ for any global section s. If two Weil divisors are linearly equivalent (that means, they have the same class in $\mathrm{Cl}(X)$), then the corresponding sheaves are isomorphic.

Let X_Δ be a toric surface defined by a rational triangle. Recall that the class group $\mathrm{Cl}(X_\Delta)$ is generated by the torus-invariant divisors. Indeed, the coordinate ring of the algebraic torus $T = (K^\times)^2$ is a UFD, so any divisor which intersects $T \subseteq X_\Delta$ is linearly equivalent to one that doesn't.

When X_Δ is a toric variety defined by a rational triangle Δ, an ample T-invariant $\mathbb{Q}$-Weil divisor H corresponds to a rational triangle Δ_H with sides parallel to the sides of Δ. Such $\mathbb{Q}$-Weil divisor H is Weil if and only if the three lines containing the edges of Δ_H contain lattice points. Two such divisors are linearly equivalent if and only if their triangles differ by an integral translation. The divisors have the same numerical equivalence class if their triangles differ by a rational translation. In particular, the class group may have torsion.

Let H be a divisor with class corresponding to a triangle Δ_H. Then, the space of global sections of $\mathcal{O}_{X_\Delta}(H)$ is isomorphic to the space of Laurent polynomials with Newton polygon contained in Δ_H.

Let $P \in X_\Delta$ be a T-fixed point, corresponding to a vertex of the triangle Δ. The divisor given by a global section of $\mathcal{O}_{X_\Delta}(H)$ does not pass through the point P if and only if the corresponding vertex of Δ_D is integral and the monomial corresponding to this vertex occurs with nonzero coefficient in the global section.

2.2 Cutkosky's criterion and a generalization

For the rest of this chapter $X = \mathrm{Bl}_{t_0} X_\Delta$, where X_Δ is the toric variety defined by a triangle Δ with rational vertices.

Cutkosky [8] showed that when $X_\Delta = \mathbb{P}(a, b, c)$, such an X is a MDS if and only if it contains two disjoint irreducible curves different from the exceptional divisor. If these curves exist, one of them must have non-positive self-intersection. We call such a curve a negative curve.

Let us generalize this result from $\mathrm{Bl}_{t_0} \mathbb{P}(a, b, c)$ to the general case $X = \mathrm{Bl}_{t_0} X_\Delta$, where X_Δ is any toric variety defined by a rational triangle Δ. The proof is the same as in [8].

Proposition 2.2.1 (Cutkosky's criterion)**.** *The variety X is a MDS if and only if*

1. *X contains an irreducible curve C such that $C \cdot C \leq 0$.*

2. *X contains a curve D disjoint from C.*

Furthermore, if $\mathrm{char}\, K > 0$ *the first condition implies the second one.*

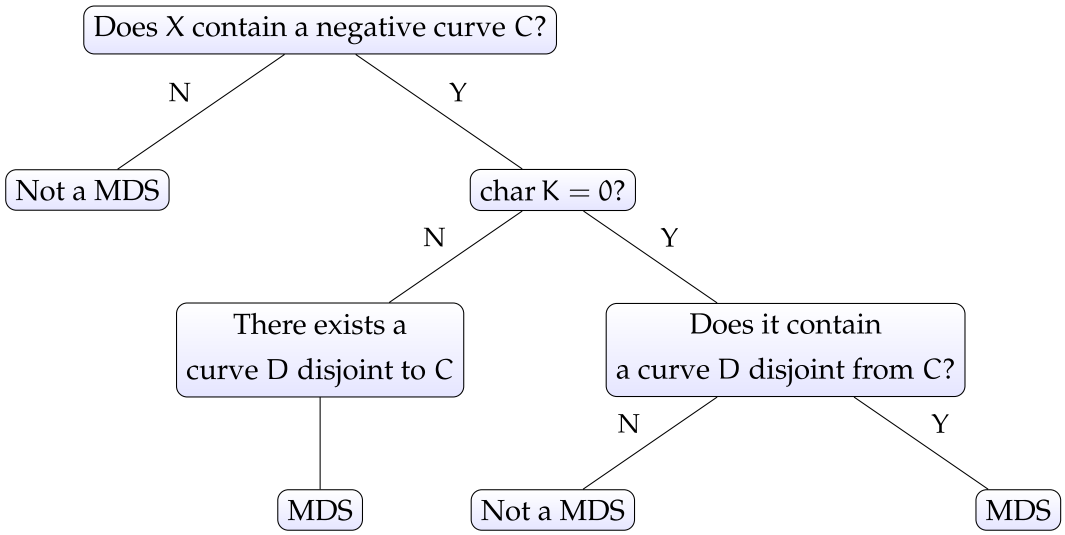


Figure 2.1: Diagrammatic representation of Cutkosky's criterion (Proposition 2.2.1).

A curve C *as above is called a negative curve*[1]*. See Figure 2.1 for a visual representation of the result.*

Proof. We only show the first part of the Proposition. For the second claim Cutkosky's proof carries over to the cases at hand, see [8, Proposition 3].

The cone $\overline{NE(X)}$ has extremal rays $E\mathbb{R}_{\geq 0}$ and $\gamma\mathbb{R}_{\geq 0}$ for some class $\gamma = \pi^*H - cE$, $c > 0$. Here $\gamma \cdot \gamma \leq 0$ because otherwise γ would satisfy Kleiman's criterion for ampleness.

Assume that X is a MDS. Then $NE(X) = \overline{NE(X)}$, hence the ray $\gamma\mathbb{R}_{\geq 0}$ is generated by the class of a curve C, $C \cdot C \leq 0$. We may replace C with a component of C and assume that C is irreducible. This shows that X contains a negative curve C different from E.

Consider the extremal ray $\delta\mathbb{R}_{\geq 0}$ of $Nef(X)$ such that $\delta \cdot \gamma = 0$. Since X is a MDS, every nef divisor is semi-ample. By the lemma below, simpleness's of δ is equivalent to the existence of a curve D, with class on the ray $\delta\mathbb{R}_{\geq 0}$, such that D is disjoint from C.

Conversely, assume that the two conditions are satisfied. Again by the lemma below, D is semi-ample. Lemma 3 in [8] states that if H and D are semi-ample on a surface X, then the ring

$$\bigoplus_{m,n \geq 0} \mathcal{O}_X(mH + nD)$$

is finitely generated. This implies that the Veronese subring of $Cox(X)$, with degrees in $Nef(X)$, is finitely generated. Any other effective divisor F intersects either E or C negatively, hence it contains a multiple of E or C, and so either $F - lE$ or $F - lC$ is effective for some positive integer l. Thus, the defining equations of E and C to the generators of the Veronese subring gives the generators of $Cox(X)$. □

[1] This naming convention is rather unfortunate since it encompasses the zero self-intersection case. We adhere to this notation to align with the existing literature.

Lemma 2.2.2. *Let* C *be a negative curve in* X *with class*

$$[C] = \pi^*H - mE \in N_1(X).$$

Consider the unique class

$$\delta = \pi^*H' - mE \in N_1(X)$$

such that $\delta \cdot C = 0$*. Then* δ *is semi-ample if and only if there exists a curve* D *of class* $n\delta$ *for some* $n > 0$*, disjoint from* C*.*

Proof. Assume that δ is semi-ample. Then there exists an effective curve D in class $n\delta$ that does not have C as a component. Since $D \cdot C = 0$, this D is disjoint from C.

Conversely, assume that such a D exists. We need to show that the stable base locus of the divisor D is empty. If D has nonempty stable base locus, then this base locus must be contained in D.

The divisor $D - nC = \pi^*(nH' - nH)$ is the pullback of a semi-ample divisor in X_Δ, and hence is semi-ample. Since $D - nC$ has empty stable base locus, the stable base locus of D must be contained in C. However, $C \cap D$ is empty, hence D is semi-ample. □

2.3 Bounding the degree of D

In this section we consider the scenario in which X contains a fixed negative curve C and then give a condition to determine whether there exist a curve D disjoint from C. Kurano and Nishida [20] show how to determine the divisor class in which such a D must lie. We give geometric versions of their definitions and generalize some of their results; all our proofs are based on the ideas of their algebraic proofs.

Recall that a curve in X is said to be negative if it's irreducible and has non-positive self-intersection. In the remainder of this section we fix the following notation and assumptions:

- There exists a negative curve $C_{\mathbb{Z}} \subset X_{\mathbb{Z}}$ defined over the integers. Its class in $\mathrm{Cl}(X_{\mathbb{Z}})$ is

  $$[C_{\mathbb{Z}}] = \pi^*H - mE,$$

  for some ample divisor $H \in \mathrm{Cl}(X_{\Delta,\mathbb{Z}})$ and positive integer $m \geq 1$.

- Let $\pi : X \to X_\Delta$ denote the blowup map. Assume $\pi(C_{\mathbb{Z}})$ passes through a T-fixed point $P \in X_{\Delta,\mathbb{Z}}$. (This condition is not strictly necessary, but it simplifies some proofs below.)

- Let $D_{0,\mathbb{Z}}$ be a divisor in the class $\delta = \pi^*H' - nE \in \mathrm{Cl}(X_{\mathbb{Z}})$ such that $C_{\mathbb{Z}} \cdot \delta = 0$. Here $H' \in \mathrm{Cl}(X_{\Delta,\mathbb{Z}})$ is an ample class and n a positive integer.

Note that all these properties are preserved by base changing to a field K provided that $\mathrm{char}(K) = 0$ or $\mathrm{char}(K) = p$ with $p \gg 1$.

Definition 2.3.1.

$$\mathrm{HC}_K = \{l \in \mathbb{Z}_{>0} \mid X_K \text{ contains a divisor } D_K \in |l\delta| \text{ such that } C_K \cap D_K = \emptyset\}.$$

The abbreviation HC stands for Huneke's condition and the subscript K indicates that we consider X defined over K.

We list some properties of HC_K that follow from the definition:

1. HC_K is nonempty if and only if X is a MDS over K.

2. HC_K is closed under addition, hence a subsemigroup of $(\mathbb{Z}_{>0}, +)$. This implies that there exists an integer $l_0 \geq 0$ such that $\mathrm{HC}_K \subseteq l_0\mathbb{Z}_{>0}$ and $\mathrm{HC}_K = l_0\mathbb{Z}_{>0}$ for large numbers, this is, there exist an integer N such that $\mathrm{HC}_K \cap \mathbb{Z}_{>N} = l_0\mathbb{Z} \cap \mathbb{Z}_{>N}$.

3. Since $C_K \cdot D_K = 0$ and C_K is irreducible, the condition $C_K \cap D_K = \emptyset$ is equivalent to $C_K \not\subseteq D_K$. In the examples below we fix a point P in C_K and check that $P \notin D_K$. We choose for P a T-fixed point in X_Δ corresponding to a vertex of Δ. Then, $P \in C_K$ if and only if the vertex does not lie in the Newton polytope of the polynomial defining C_K. A similar condition holds for $P \in D_K$. It follows that checking if a fixed l lies in HC_K is a finite dimensional linear algebra problem. We look for a polynomial that vanishes to order lm' at t_0. The Newton polytope of the polynomial must lie in $l\Delta'$ and include the vertex corresponding to P.

4. If $l \in \mathrm{HC}_K$, then $\mathcal{O}_X(lD)$ is invertible near C because it has a global section that does not vanish at any point of C.

Our goal now is to determine whether $\mathrm{HC}_\mathbb{C}$ is empty by studying $\mathrm{HC}_{\overline{\mathbb{F}}_p}$ for $p \gg 1$.

Lemma 2.3.2. $l \in \mathrm{HC}_k$ *if and only if* $l \in \mathrm{HC}_{\overline{\mathbb{F}}_p}$ *for all* $p \gg 1$.

Proof. Let the class H' correspond to a triangle Δ'. Global sections of $\mathcal{O}_X(lD_0)$ can then be given as polynomials in $K[x^{\pm 1}, y^{\pm 1}]$ that vanish to order ln at t_0 and have their Newton polygon contained in $l\Delta'$. The condition of vanishing at t_0 is equivalent to the vanishing of partial derivatives up to degree $ln - 1$ at t_0. This holds when $\mathrm{char}(K) = 0$ or $\mathrm{char}(K) = p \gg 1$ (with l and n fixed). It follows that the space of global sections of $\mathcal{O}_X(lD_0)$ is the kernel of an integer matrix M, where we view the matrix as a linear map $K^r \to K^s$. The matrix can be put in the Smith normal form (the matrix becomes diagonal), and then it is clear that its kernel over $K = \mathbb{C}$ or $K = \overline{\mathbb{F}}_p$ for $p \gg 1$ is equal to the kernel over $\mathbb{Z}$ tensored with K. In particular, the kernel has the same dimension over any such field.

Similarly, the condition that every global section of $\mathcal{O}_X(lD_0)$ vanishes at the T-fixed point P is independent of the field K. It is equivalent to the condition that all global sections defined over $\mathbb{Z}$ vanish at P. □

Proposition 2.3.3. *Suppose that* $l, l+\mu \in HC_K$ *and* $H^1(X, \mathcal{O}_X(\mu D_0 - \nu C)) = 0$ *for some* $l, \mu, \nu \in \mathbb{Z}_{>0}$. *Then,* $\mu \in HC_K$.

Proof. Let $\xi \in H^0(\mathcal{O}_X(C))$ define C, and let $\zeta \in H^0(\mathcal{O}_X(lD_0))$ define D that gives $l \in HC_K$. Since $C \cap D = \emptyset$ we have a short exact sequence

$$0 \longrightarrow \mathcal{O}_X(\mu D_0 - \nu C) \xrightarrow{(\zeta, -\xi^\nu)} \mathcal{O}_X((l+\mu)D_0 - \nu C) \oplus \mathcal{O}_X(\mu D_0) \xrightarrow{\cdot(\xi^\nu, \zeta)} \mathcal{O}_X((l+\mu)D_0) \longrightarrow 0.$$

Indeed, the exactness on the left and the middle are straightforward, and the exactness on the right is easily verified by restricting separately to the complement of C and the complement of D.

By the assumption that $H^1(X, \mathcal{O}_X(\mu D_0 - \nu C)) = 0$ we have a surjective homomorphism

$$H^0(\mathcal{O}_X((l+\mu)D_0 - \nu C)) \oplus H^0(\mathcal{O}_X(\mu D_0)) \longrightarrow H^0(\mathcal{O}_X((l+\mu)D_0)).$$

Let $\gamma \in H^0(\mathcal{O}_X((l+\mu)D_0))$ be a section giving $l+\mu \in HC_K$. Then, $\gamma = f\xi^\nu + g\zeta$ for some $f \in H^0(\mathcal{O}_X((l+\mu)D_0 - \nu C))$ and some $g \in H^0(\mathcal{O}_X(\mu D_0))$. We claim that g does not vanish at any point of C, hence giving $\mu \in HC_K$. Let us check the equivalent condition that g does not vanish along C. In the equation $\gamma = f\xi^\nu + g\zeta$ we know that ξ vanishes along C and γ does not, hence g does not vanish along C. □

Corollary 2.3.4. *Let the classes of* C *and* D_0 *be as above.*

(a) Then, $m \in HC_K$ *if* $l, l+m \in HC_K$ *for some* $l, m > 0$.

(b) Let $\mu, \nu \in \mathbb{Z}_{>0}$ *be such that* $\mu n - \nu m = -1$. *Then,* $\mu \in HC_K$ *if* $l, l+\mu \in HC_K$ *for some* $l > 0$.

Proof. The claims follow from Proposition 2.3.3 by choosing appropriate values of μ and ν.

For (a) let $\mu = m$, so that $\mu D_0 - \nu C = \pi^*(mH' - \nu H)$. The condition in part (b) implies $\mu D_0 - \nu C = \pi^*(\mu H' - \nu H) + E$. By Corollary 2.3.12 (a), $H^1(\mathcal{O}_X(\mu D_0 - \nu C)) = 0$ in both cases. □

Proposition 2.3.5. *Assume that for all* $p \gg 1$ *there exists* $n_p \in \mathbb{Z}_{\geq 0}$ *such that* $p^{n_p} \in HC_{\overline{\mathbb{F}}_p}$. *Then,* $HC_{\mathbb{C}}$ *is not empty if and only if* $\mu \in HC_{\mathbb{C}}$ *or, equivalently,* $m \in HC_{\mathbb{C}}$, *where* $\mu, \nu \in \mathbb{Z}_{>0}$ *are solutions to* $\mu n - \nu m = -1$.

Proof. By Lemma 2.3.2, a fixed l lies in $HC_{\mathbb{C}}$ if and only if l lies in $HC_{\overline{\mathbb{F}}_p}$ for all $p \gg 1$. Suppose that $HC_{\mathbb{C}}$ is not empty and fix $l_0 \in HC_{\mathbb{C}}$. Then, $l_0 \in HC_{\overline{\mathbb{F}}_p}$ for $p \gg 1$. Since $HC_{\overline{\mathbb{F}}_p}$ is a subsemigroup of $\mathbb{Z}_{>0}$, there exist $l_p, N_p \in \mathbb{Z}_{>0}$ such that $HC_{\overline{\mathbb{F}}_p} \subseteq l_p\mathbb{Z}$ and $HC_{\overline{\mathbb{F}}_p} \cap \mathbb{Z}_{>N_p} = l_p\mathbb{Z} \cap \mathbb{Z}_{>N_p}$. From $l_0, p^{n_p} \in HC_{\overline{\mathbb{F}}_p}$, we deduce that $l_p = 1$ for all $p \gg 1$. Then, by Corollary 2.3.4 (a) and (b) we get $\mu, m \in HC_{\overline{\mathbb{F}}_p}$ for all $p \gg 1$. Thus, $\mu, m \in HC_{\mathbb{C}}$ □

Remark 2.3.6. It's always possible to choose $\mu < m$ in Corollary 2.3.4 and Proposition 2.3.5. Furthermore, in some cases it's possible to improve over these values. However, this is contingent on the vanishing of $H^1(\mathcal{O}_X(\mu' D - \nu' C))$ for some carefully chosen $\mu', \nu' > 0$. This problem is equivalent to a lattice interpolation problem, as detailed in Subsection 2.3.1.

2.3.1 Cohomology and lattice point interpolation

Let Δ be a triangle with rational vertices and such that the line through any two of its vertices passes through a lattice point. Consider the toric variety X_Δ associated to Δ and denote its corresponding Weil divisor in X_Δ by H. As before $\pi : X \to X_\Delta$ is the blowup map at the torus identity t_0.

The goal of this section is to prove the following:

Proposition 2.3.7. *There is a vector space isomorphism between* $H^1(\mathcal{O}_X(\pi^* H - mE))$ *and the vector space of polynomials* $P(u, v) \in K[u, v]$ *of total degree less than* m *and such that* $P(i, j) = 0$ *for all* $(i, j) \in \Delta$.

The connection between linear systems in blowups of weighted projective planes and lattice point interpolation is well-known, see for example [2, 9, 17]. However, to the best of our knowledge, its relation with higher cohomology hasn't been explicitly written elsewhere.

Let I_{t_0} be the ideal sheaf of the point $t_0 = (1, 1) \in T \subset X_\Delta$ and consider the short exact sequence

$$H^0(\mathcal{O}_{X_\Delta}(H) \otimes I^m_{t_0}) \hookrightarrow H^0(\mathcal{O}_{X_\Delta}(H)) \xrightarrow{A} H^0(\mathcal{O}_{X_\Delta}/I^m_{t_0}) \twoheadrightarrow H^1(\mathcal{O}_{X_\Delta}(H) \otimes I^m_{t_0}).$$

Fix the bases $\{x^i y^j : (i, j) \in \Delta\}$ and $\{(1 - x)^a (1 - y)^b : 0 \le a + b < m\}$ for the second and third vector spaces in this sequence, respectively. In terms of these, the components of the map A are

$$A_{(a,b),(i,j)} = a!\binom{i}{a} b!\binom{j}{b} = (i)_a (j)_b,$$

where $(x)_n = x(x - 1) \cdots (x - n + 1)$ is the falling factorial.

Lemma 2.3.8. *[2, Lemma 8.2] The matrix* A *is row equivalent to the Vandermonde-type matrix* B*, where* $B_{(a,b),(i,j)} = i^a j^b$.

Proof. For a fixed $m > 0$ the polynomials $(x)_n \in K[x]$ with $n < m$ define a basis for $K[x]_{<m}$. Indeed, the Stirling numbers of the second kind are defined so that for any $0 < N$:

$$\sum_{n=0}^{N} S(N, n)(x)_n = x^N.$$

Define the square matrix U of size $\binom{m+1}{2}$ as

$$U_{(c,d),(a,b)} = \begin{cases} S(c, a)S(d, b), & \text{if } a \le c \text{ and } b \le d; \\ 0 & \text{otherwise.} \end{cases}$$

Under the lexicographical order of the tuples this matrix is lower triangular with all its diagonal entries equal to 1. Then,

$$\begin{aligned}(UA)_{(c,d),(i,j)} = \sum_{0\le a+b<m} U_{(c,d),(a,b)}A_{(a,b),(i,j)} &= \sum_{0\le a+b<m} S(c,a)S(d,b)(i)_a(j)_b \\ &= \sum_{b=0}^{d}\sum_{a=0}^{c} S(c,a)S(d,b)(i)_a(j)_b \\ &= \sum_{b=0}^{d} S(d,b)(j)_b \cdot \sum_{a=0}^{c} S(c,a)(i)_a \\ &= i^c j^d \\ &= B_{(c,d),(i,j)}.\end{aligned}$$

Therefore, $B = UA$ is row equivalent to A. ☐

We are now ready to show Proposition 2.3.7.

Proof of Proposition 2.3.7. Consider the vector space $K[u,v]_{<m}$ of polynomials with total degree less than m. Let ∂_x and ∂_y denote the usual partial derivatives with respect to x and y. There is a perfect pairing

$$\begin{aligned} H^0(\mathcal{O}_{X_\Delta}/I_{t_0}^m) \times K[u,v]_{<m} &\longrightarrow K \\ (f + I^m, P(u,v)) &\longmapsto P(\partial_x,\partial_y)(f)|_{t_0}. \end{aligned}$$

Because of this and Lemma 2.3.8 there is a commutative diagram

$$\begin{array}{ccccccc} H^0(\mathcal{O}_{X_\Delta}(H)\otimes I_{t_0}^m) & \hookrightarrow & H^0(\mathcal{O}_{X_\Delta}(H)) & \xrightarrow{B} & H^0(\mathcal{O}_{X_\Delta}/I_{t_0}^m) & \twoheadrightarrow & H^1(\mathcal{O}_{X_\Delta}(H)\otimes I_{t_0}^m) \\ \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\ H^0(\mathcal{O}_{X_\Delta}(H)\otimes I_{t_0}^m)^* & \twoheadleftarrow & H^0(\mathcal{O}_{X_\Delta}(H))^* & \xleftarrow[B^T]{} & K[u,v]_{<m} & \hookleftarrow & H^1(\mathcal{O}_{X_\Delta}(H)\otimes I_{t_0}^m)^* \end{array}$$

In particular, the map B^T is explicitly given by

$$B^T : P(u,v) \longmapsto (P(i,j) : (i,j)\in\Delta) \in H^0(\mathcal{O}_{X_\Delta}(H))^*.$$

Therefore, $\ker B^T = H^1(\mathcal{O}_{X_\Delta}(H)\otimes I_{t_0}^m)^*$ is the space of polynomials interpolating the points of Δ. The result then follows from Proposition 2.3.11. ☐

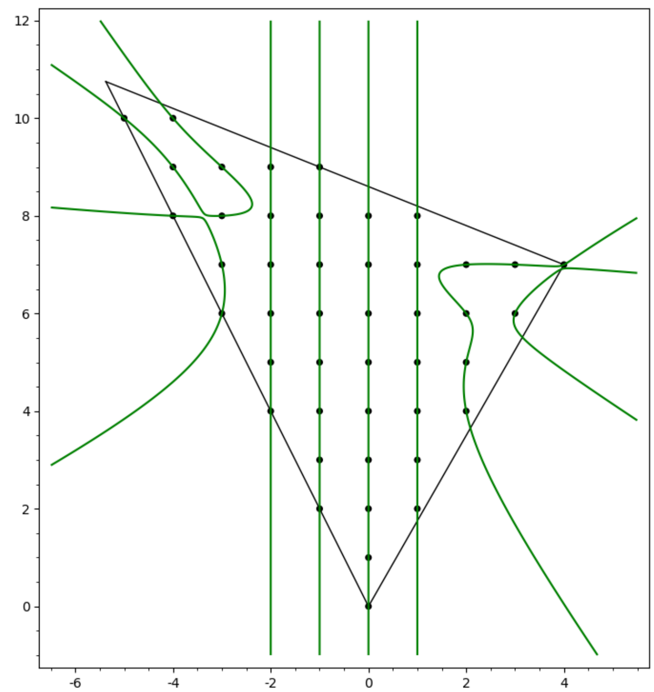
Corollary 2.3.9. *Consider* $l \in \mathbb{Z}$ *such that* $\dim H^0(\mathcal{O}_{X_\Delta}(H)) = \binom{m+1}{2} + l$. *Then,*

$$\dim H^0(\mathcal{O}_X(\pi^*H - mE)) = \dim H^1(\mathcal{O}_X(\pi^*H - mE)) + l.$$

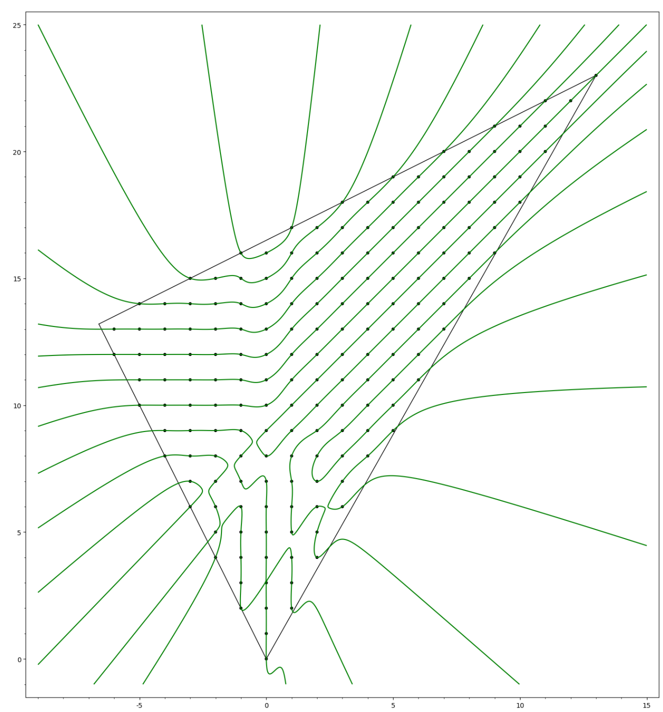
Proof. By Proposition 2.3.11 we have $\dim H^i(\mathcal{O}_{X_\Delta}(H)\otimes I_{t_0}^m) = \dim H^i(\mathcal{O}_X(\pi^*H - mE))$ for $i = 0, 1$. The result then follows by the equality $\dim H^0(\mathcal{O}_{X_\Delta}/I_{t_0}^m) = \binom{m+1}{2}$. ☐

If a negative curve $C \subset X$ exists in the class $\pi^*H - mE \in Cl(X)$, then it is unique. This means that $H^0(\mathcal{O}_X(\pi^*H - mE))$ has dimension 1. By virtue of the previous corollary, $1 - l$ is the number of algebraic relations that the lattice points in Δ must satisfy for a negative curve to exist. In particular, negative curves can only exists when $l \leq 1$.

Negative curves with $l = 1$ are notable by themselves. However, Kurano and Matsuoka [19] found two examples of negative curves in the case $l = 0$. We call a negative curve for which $l \leq 0$ a *super-exceptional curve*. During the revision process of this thesis we were able to construct an infinite class of super-exceptional curves such that $l \to -\infty$. However, more work needs to be done to better understand the combinatorial nature of these curves. Expanding on the ideas of [5] we've also been able to see that there are no super-exceptional curves with order of vanishing $m < 8$, yet one of the examples of Kurano and Matsuoka has $m = 9$, see Figure 2.2.



(a) Polynomial of degree 8 interpolating $45 = \binom{9+1}{2}$ integral points. It corresponds to the super-exceptional curve with $m = 9$ in $\mathrm{Bl}_{t_0}\,\mathbb{P}(8, 15, 43)$.



(b) Polynomial of degree 17 interpolating $171 = \binom{18+1}{2}$ integral points. It corresponds to the super-exceptional curve with $m = 18$ in $\mathrm{Bl}_{t_0}\,\mathbb{P}(5, 33, 49)$.

Figure 2.2: The two super-exceptional curves of Kurano and Matsuoka.

2.3.2 Some cohomological lemmas

Lemma 2.3.10. *Let X_Δ be a toric variety defined by a rational triangle Δ, and let A be a Weil divisor on X_Δ. Then*

$$H^1(X_\Delta, \mathcal{O}_{X_\Delta}(A)) = 0.$$

Proof. Since X_Δ has Picard-number one, either A is nef or $-A$ is nef. In either case, we conclude that $H^1(X_\Delta, \mathcal{O}_{X_\Delta}(A)) = 0$ by the Demazure and Batyrev-Borisov vanishing theorems in [7, Theorem 9.3.5]. □

Proposition 2.3.11. *Consider the blowup $\pi : X \to Y$ of a surface Y at a smooth closed point t_0, and the sheaf $\mathcal{F} = \mathcal{O}_X(\pi^*A - mE)$, where A is a Weil divisor on Y and E is the exceptional divisor. Then:*

(a) $\pi_*\mathcal{F} = \mathcal{O}_Y(A) \otimes \pi_*\mathcal{O}_X(-mE) = \begin{cases} \mathcal{O}_Y(A), & \text{if } m \leq 0; \\ \mathcal{O}_Y(A) \otimes I_{t_0}^m, & \text{if } m > 0. \end{cases}$

(b) $R^1\pi_*\mathcal{F} = 0, \quad \text{if } m \geq -1.$

(c) $H^1(X, \mathcal{F}) = \begin{cases} H^1(Y, \mathcal{O}_Y(A)), & \text{if } m = -1 \text{ or } m = 0; \\ H^1(Y, \mathcal{O}_Y(A) \otimes I_{t_0}^m), & \text{if } m > 0. \end{cases}$

Here I_{t_0} is the ideal sheaf of the point t_0.

Proof. Part (c) follows directly from (a) and (b). To prove (a) and (b) we use that the problem is local in Y.

For (a), consider the map $\phi : \mathcal{O}_Y(A) \otimes \pi_*\mathcal{O}_X(mE) \to \pi_*(\pi^*\mathcal{O}_Y(A) \otimes \mathcal{O}_X(mE))$, coming from the adjunction $\pi^* \dashv \pi_*$ and the natural map $\pi^*(\mathcal{O}_Y(A) \otimes \pi_*\mathcal{O}_X(mE)) \to \pi^*\mathcal{O}_Y(A) \otimes \mathcal{O}_X(mE)$. The map ϕ is an isomorphism over any open subset where either π is an isomorphism or A is Cartier (by the projection formula). We can cover Y with two open subsets where one of these cases applies.

In (b), replacing Y with a small affine neighbourhood of t_0, we may assume that $A = 0$ and there exists a fibre square

$$\begin{array}{ccc} X & \xrightarrow{\psi} & \mathrm{Bl}_0\,\mathbb{A}^2 \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\rho} \\ Y & \xrightarrow{\phi} & \mathbb{A}^2, \end{array}$$

where the morphism ϕ is étale. By [15, Proposition III.9.3] we have

$$R^1\pi_*\mathcal{F} = R^1\pi_*\psi^*\mathcal{O}_{\mathrm{Bl}_0\,\mathbb{A}^2}(-mE) \cong \phi^*R^1\rho_*\mathcal{O}_{\mathrm{Bl}_0\,\mathbb{A}^2}(-mE).$$

Here we have denoted by E also the exceptional curve in $\mathrm{Bl}_0\,\mathbb{A}^2$. We may thus replace the blowup of Y with the blowup of $\mathbb{A}^2$ at the origin. This last morphism is toric and we can use toric vanishing theorems. For $m \geq 0$, the divisor $-mE$ is nef on $\mathrm{Bl}_0\,\mathbb{A}^2$ and hence its higher cohomology vanishes. For $m = -1$, the divisor E on $\mathrm{Bl}_0 \mathbb{A}^2$ can be written as the round-down of a nef $\mathbb{Q}$-divisor D, for example

$$\mathcal{O}_{\mathrm{Bl}_0 \mathbb{A}^2}(E) = \mathcal{O}_{\mathrm{Bl}_0 \mathbb{A}^2}(\frac{1}{2}\rho^* D_1 + \frac{1}{2}\rho^* D_2)$$

where D_1, D_2 are the toric irreducible divisors on $\mathbb{A}^2$. Now the $\mathbb{Q}$-divisor D is nef, hence $\mathcal{O}_{\mathrm{Bl}_0 \mathbb{A}^2}(D)$ has no higher cohomology [7, Theorem 9.3.5]. □

Corollary 2.3.12. *Let* $\pi : X \to X_\Delta$ *be the blowup of the toric variety* X_Δ *associated to a rational triangle* Δ*, at the point* $t_0 = (1, 1)$*. Consider any toric Weil divisor* A *in* X_Δ *and the sheaf* $\mathcal{F} = \mathcal{O}_X(\pi^* A - mE)$*.*

(a) If $m = -1$ *or* $m = 0$*, then* $H^1(X, \mathcal{F}) = 0$*.*

(b) If $m > 0$*, then* $H^1(X, \mathcal{F}) = 0$ *if and only if the evaluation map* $H^0(\mathcal{O}_{X_\Delta}(A)) \to H^0(\mathcal{O}_{X_\Delta}(A) \otimes \mathcal{O}_{X_\Delta}/I_{t_0}^m)$ *is surjective.*

(c) If $m = 1$*, then* $H^1(X, \mathcal{F}) = 0$ *if and only if* $H^0(\mathcal{O}_{X_\Delta}(A)) \neq 0$*.*

Remark 2.3.13. See Proposition 2.3.7 for a generalization of (c) to $m \geq 1$.

Proof. Part (a) follows from Proposition 2.3.11 (c) and Lemma 2.3.10.

The vanishing in part (b) is by Proposition 2.3.11 (c) equivalent to the vanishing of $H^1(X_\Delta, \mathcal{O}_X(A) \otimes I_{t_0}^k)$. The conclusion now follows by considering the exact sequence

$$H^0(\mathcal{O}_{X_\Delta}(A)) \to H^0(\mathcal{O}_{X_\Delta}(A) \otimes \mathcal{O}_{X_\Delta}/I_{t_0}^m) \to H^1(\mathcal{O}_{X_\Delta}(A) \otimes I_{t_0}^m) \to H^1(\mathcal{O}_{X_\Delta}(A)) = 0.$$

For (c), notice that for $m = 1$ the surjectivity of the evaluation map in (b) is equivalent to $H^0(\mathcal{O}_{X_\Delta}(A)) \neq 0$. Indeed, $H^0(\mathcal{O}_{X_\Delta}(A) \otimes \mathcal{O}_{t_0})$ is a one dimensional vector space and the image of a section $\chi^u \in H^0(\mathcal{O}_{X_\Delta}(A))$ is nonzero. □

Chapter 3

The first two families of negative curves

In this chapter we make use of the machinery developed in the previous one to prove the main theorems in [11, 12]. We give a unified presentation of the results in both papers with some additions and variations.

Our work in [11] subsumes that one in [12]; however, the ideas and techniques used in each one of these papers have their own beauty and merit, so we present both in the present chapter. Furthermore, we hope that the techniques we present in here can continue to be useful in the future as new families of negative curves are discovered.

This is joint work with José Luis González and Kalle Karu.

3.1 Introduction

We are interested in studying the finite generation of the Cox ring of surfaces $X = \mathrm{Bl}_{t_0} X_\Delta$, where X_Δ is a toric surface defined by a triangle with rational vertices in $\mathbb{R}^2$ and t_0 is the identity point of the torus $T \subset X_\Delta$.

By Cutkosky's criterion (Proposition 2.2.1), such a variety is a MDS if and only if it contains two disjoint irreducible curves different from the exceptional divisor. This is equivalent to the existence of a negative curve $C \subset X$ and a curve D disjoint curve from it.

In what follows we construct the varieties $X = \mathrm{Bl}_{t_0} X_\Delta$ as follows. Let $T = (K^\times)^2$ be the two-dimensional torus and let $C^\circ \subset T$ be an irreducible curve that has multiplicity m at the point $t_0 = (1,1)$. We compactify T to the toric variety X_Δ by choosing a rational triangle $\Delta \subset \mathbb{R}^2$ that contains the Newton polygon of the curve C°. Let $\overline{C^\circ}$ be the closure of C° in X_Δ and C be the strict transform of $\overline{C^\circ}$ in the blowup $X = \mathrm{Bl}_{t_0} X_\Delta$. Then, the divisor class of C is $\pi^* H - mE$ and

$$C \cdot C = H \cdot H + m^2 E \cdot E = 2\,\mathrm{Area}(\Delta) - m^2.$$

Thus, C is a negative curve if the area of Δ is $\leq \frac{m^2}{2}$.

Recall that a bivariate polynomial is said to be supported in a polygon if its Newton polygon is contained in the former.

Theorem 3.1.1. *For every integer $m \geq 1$ there exist two unique polynomials*

$$\xi_m,\ \eta_m \in \mathbb{Z}[x, y]$$

that vanish to order m *at* t_0*, have constant term equal to* 1 *and are irreducible as elements of* $K[x,y]$ *for any field* K*.*

Furthermore, the Newton polygon of ξ_m *is the triangle* $\Delta_1^0(m)$ *with vertices* $(0,0)$, $(m-1,0)$, $(m, m+1)$*. Similarly,* η_m *is supported in the triangle* $\Delta_2^0(m)$ *with vertices* $(0,0)$, $\left(\frac{2m-1}{4}, 0\right)$, $(m, 2m+1)$*.*

The family of polynomials ξ_m was first introduced and studied in [12], while the polynomials η_m only appeared implicitly in [11]. In the spirit of [12], in the next section we give a constructive proof of the previous theorem. For an indirect proof of their existence we refer the reader to Chapter 4, where these two families of polynomials are shown to be just boundary cases of a more general 2-parameter class of polynomials.

Next we put a triangle Δ around the Newton polygons of ξ_m and η_m and consider $X = \mathrm{Bl}_{t_0} X_\Delta$. Consider the case $\Delta = \Delta_1^0(m)$. Theorem 3.1.1 tells us that the polynomial ξ_m defines an irreducible curve $C \subset X$ in the class $\pi^*H - mE$, where H is the divisor class in X_Δ corresponding to Δ. One readily checks that this C is a negative curve:

$$C \cdot C = 2\,\mathrm{Area}(\Delta_1^0) - m^2 = -1 < 0.$$

Similarly, if $\Delta = \Delta_2^0(m)$ the polynomials η_m define a negative curve $C \subset X$ in the class $\pi^*H - mE$ and with self-intersection $C \cdot C = -\frac{1}{4}$.

In general, any rational triangle Δ containing the support of either ξ_m or η_m and with area $< \frac{m^2}{2}$ defines a variety $X = \mathrm{Bl}_{t_0} X_\Delta$ containing a negative curve C coming from ξ_m or η_m. We determine the existence of a curve D disjoint from C as we deform Δ:

Theorem 3.1.2. *Consider* $m \geq 1$ *and let* $\Delta_1 = \Delta_1(m)$ *be the triangle with vertices*

$$(-\alpha, 0), (m-1+\beta, 0) \ \textit{and} \ (m, m+1),$$

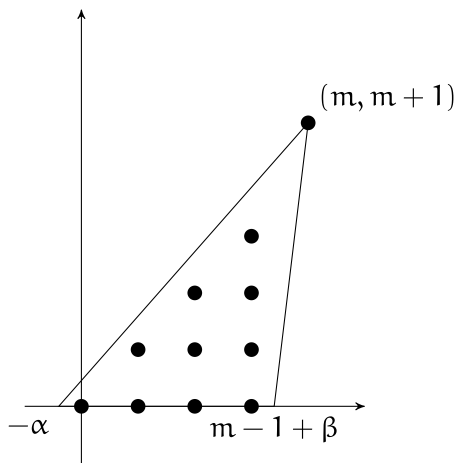
where α, β *are non-negative rational numbers. See Figure 3.1a. Then, if* $\alpha + \beta < \frac{1}{m+1}$ *the polynomial* ξ_m *defines a negative curve* C *in* $X = \mathrm{Bl}_{t_0} X_{\Delta_1}$ *with class* $\pi^*H - mE \in \mathrm{Cl}(X)$*. In this case the variety* X *is a MDS if and only* $0 \leq \alpha, \beta \leq \frac{1}{m+2}$*.*

Theorem 3.1.3. *Consider* $m \geq 1$ *and let* $\Delta_2 = \Delta_2(m)$ *be the triangle with vertices*

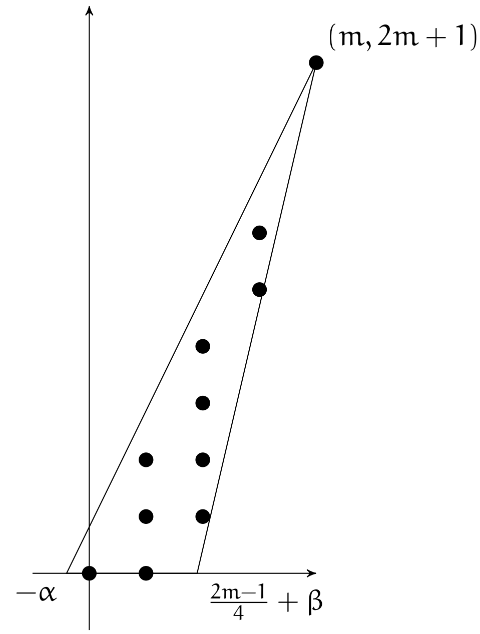
$$(-\alpha, 0), \left(\frac{2m-1}{4} + \beta, 0\right) \ \textit{and} \ (m, 2m+1),$$

where α, β *are non-negative rational numbers. See Figure 3.1b. Then, if* $\alpha + \beta < \frac{1}{2m+1}$ *the polynomial* η_m *defines a negative curve* C *in* $X = \mathrm{Bl}_{t_0} X_{\Delta_2}$ *with class* $\pi^*H - mE \in \mathrm{Cl}(X)$*. In this case the variety* X *is a MDS if and only* $\alpha = 0$ *or* $\beta = 0$*.*

By Proposition 2.3.5, we prove Theorems 3.1.2 and 3.1.3 by showing that for all $p \gg 1$, there exist some $n_p \in \mathbb{Z}$ giving $p^{n_p} \in \mathrm{HC}_{\overline{\mathbb{F}}_p}$, and then determining the values of α and β for which $m \in \mathrm{HC}_{\mathbb{C}}$. However, in each case we use different methods to show the prime characteristic part. For the first family of polynomials we exploit a symmetry of the triangles Δ_1^0 to construct the polynomials giving $p \in \mathrm{HC}_{\overline{\mathbb{F}}_p}$. Such an argument is not viable for the second family. In that case we give a more general proof using cohomological methods.



(a) The triangle $\Delta_1(4)$.



(b) The triangle $\Delta_2(4)$.

Figure 3.1: Triangles in Theorems 3.1.2-3.1.3 for $m = 4$.

3.2 Two families of negative curves: A proof of Theorem 3.1.1

Consider the following polynomials in $\mathbb{Z}[x, y]$:

$$f = 1 - xy, \quad g = 1 - xy^2, \quad h = 1 - y.$$

Notice that f, g, h vanish at $t_0 = (1, 1)$.

Proposition 3.2.1. *There exist unique polynomials $\xi_m \in \mathbb{Z}[x, y]$ for all integers $m \geq 1$ such that*

1. *(a)* $\xi_1 = g$.

   (b) $\xi_{m+1} = f\xi_m + x^m h^{m+1}$.

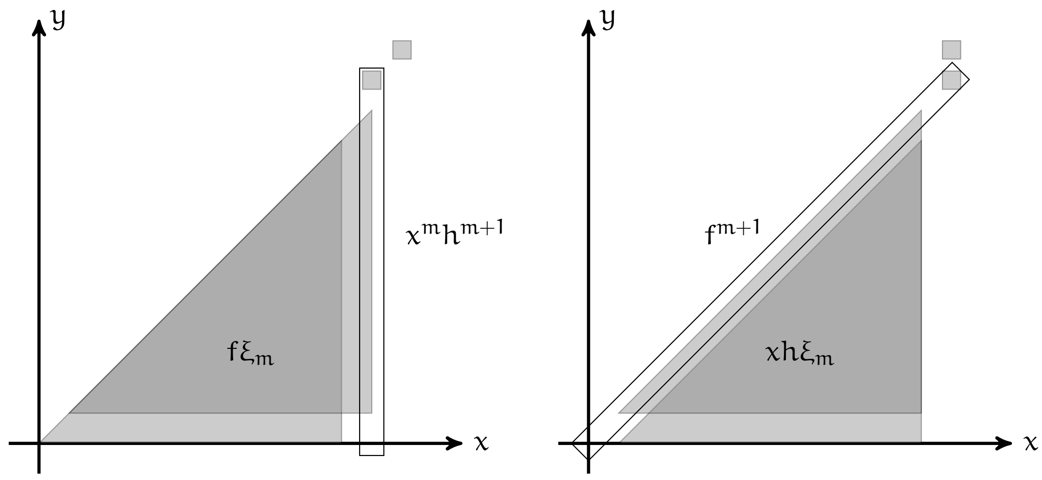
   (c) $\xi_{m+1} = xh\xi_m + f^{m+1}$.

2. *ξ_m vanishes to order m at $t_0 = (1, 1)$.*

3. *The Newton polygon of ξ_m is the triangle with vertices $(0, 0)$, $(m-1, 0)$, $(m, m+1)$. Call this triangle $\Delta_1^0(m)$. In addition, the monomials in ξ_m corresponding to these lattice points have coefficients 1, 1 and $(-1)^m$, respectively.*

4. *ξ_m is irreducible in $K[x, y]$, where K is any field.*

Proof of 3.2.1. For each $m \geq 1$ define two families of polynomials, ξ_m and ξ'_m, using the recurrence relations (1.b) and (1.c), respectively, and setting $\xi_1 = \xi'_1 = g$. We claim both families



(a) $\xi_{m+1} = f\xi_m + x^m h^{m+1}$ (b) $\xi_{m+1} = xh\xi_m + f^{m+1}$

Figure 3.2: Depiction of the supports of all polynomials involved in the recurrence relations (1.b) and (1.c).

satisfy (2), (3) and (4) above. We will show this claim for the polynomials ξ_m only, since the proof for the ξ'_m is almost identical. To conclude we use the negativity of the curves defined by both families to show that in fact $\xi_m = \xi'_m$ for all m. Clearly $\xi_1 = g$ satisfies (2), (3) and (4).

(3) Assume ξ_m satisfies (3). Consider the polynomial $f\xi_m$. Its nonzero monomials corresponding to lattice points in the shaded area in Figure 3.2a and its Newton polygon is the convex hull with vertices $(0,0)$, $(m-1,0)$, $(m,1)$, $(m+1,m+2)$ and $(m,m+1)$. Furthermore, the coefficients of the monomials corresponding to the points $(0,0)$, $(m+1,m+2)$ and $(m,m+1)$ are precisely 1, $(-1)^{m+1}$ and $(-1)^m$, respectively.

On the other hand, the Newton polygon of $x^m h^{m+1}$ consists only of the segment joining the points $(m,0)$ and $(m,m+1)$ (the boxed region in Figure 3.2a, with the corresponding monomials having coefficients 1 and $(-1)^{m+1}$, respectively.

From these considerations it follows that that the monomial corresponding to $(m,m+1)$ in the sum $\xi_{m+1} = f\xi_m + x^m h^{m+1}$ has coefficient $(-1)^m + (-1)^{m+1} = 0$. Hence, the Newton polygon of ξ_{m+1} is the triangle with vertices $(0,0)$, $(m,0)$ and $(m+1,m+2)$ with coefficients 1, 1 and $(-1)^{m+1}$, respectively. This shows ξ_{m+1} satisfies (3).

(2) We argue by induction on m. Assume ξ_m satisfies (2) and note that f and h vanish to order 1 at t_0. Now, the relation (1.b) implies that ξ_{m+1} vanishes to order at least $m+1$ at t_0. To see that it vanishes exactly to order $m+1$ note that the derivative $\partial_x^{m+1}\xi_{m+1} = \partial_x^{m+1}(-1)^m x^{m+1} y^{m+2}$ is nonzero at t_0 in characteristic 0. In characteristic $p \neq 0$ the derivative needs to be replaced with the Hasse-Weil derivative, and the result still holds.

(4) By (3), ξ_m has a nonzero constant term, thus $x, y \nmid \xi_m$ for any $m \geq 1$. Then, it is irreducible in $K[x, y]$ if and only if it is irreducible in $K[x^{\pm 1}, y^{\pm 1}]$. We perform a change of coordinates in $K[x^{\pm 1}, y^{\pm 1}]$ and consider the new polynomial

$$\tilde{\xi}_m = x\xi_m(x, y/x).$$

Then the Newton polygon of $\tilde{\xi}_m$ is the triangle with vertices $(1, 0)$, $(m, 0)$ and $(0, m+1)$. In particular, $\tilde{\xi}_m$ lies in $K[x, y]$ and is irreducible in $K[x, y]$ if and only if it is irreducible in $K[x^{\pm 1}, y^{\pm 1}]$. Now $\tilde{\xi}_m$ can be written as

$$\tilde{\xi}_m = (-1)^m y^{m+1} + a_m(x)y^m + \cdots + a_0(x),$$

where x divides $a_i(x)$ for $i = 1, \ldots, m$ and x^2 does not divide $a_0(x) = x$. Hence $\tilde{\xi}_m$ is irreducible in $K[x, y]$ by Eisenstein's criterion.

Let Δ be the triangle with vertices defined by the points in (3) and H its corresponding class in $\mathrm{Cl}(X_\Delta)$. Then, for every m, the polynomials ξ_m and ξ'_m define two negative curves in $X = \mathrm{Bl}_{t_0} X_\Delta$ in the class $\pi^*H - mE \in \mathrm{Cl}(X)$. Indeed, the irreducibility follows from (4) and their self-intersection is:

$$(\pi^*H - mE) \cdot (\pi^*H - mE) = 2\mathrm{Area}(\Delta) - m^2 = (m-1)(m+1) - m^2 = -1 < 0.$$

Thus, both curves are actually the same and we can conclude that $\xi_m = c \cdot \xi'_m$ for some constant c. In particular $\xi_m(0, 0) = c \cdot \xi'_m(0, 0)$, which implies $c = 1$ by (3). □

Remark 3.2.2. It is not hard to find a transformation of the lattice mapping each $\Delta_1^0(m)$ to the isosceles triangle with vertices $(-1, -1)$, $(m, 0)$ and $(0, m)$. Let $\tilde{\xi}_m$ be the image of ξ_m under this transformation. The existence of the two recurrence relations (1.b) and (1.c) says that the additional symmetry of this triangle lifts to the level of polynomials. More precisely, the existence of the two relations is equivalent to the symmetry $\tilde{\xi}_m(x, y) = \tilde{\xi}_m(y, x)$.

Proposition 3.2.3. *There exist unique polynomials $\eta_m \in \mathbb{Z}[x, y]$ for all integers $m \geq 1$ such that*

1. *(a)* $\eta_1 = 1 - xy^3$.
   (b) $\eta_2 = 1 + xy - 3xy^2 + x^2y^5$.
   (c) $\eta_{m+2} = g\,\eta_{m+1} + xh^2\,\eta_m$.
2. η_m *vanishes to order* m *at* $t_0 = (1, 1)$.
3. *The Newton polygon of* ξ_m *is contained in the triangle with vertices* $(0, 0)$, $\left(\frac{2m-1}{4}, 0\right)$, $(m, 2m+1)$. *Denote this triangle by* $\Delta_2^0(m)$. *Furthermore, the constant of term of* η_m *is* 1 *and the monomial corresponding to the lattice point* $(m, 2m+1)$ *in* η_m *has coefficient* $(-1)^m$.
4. η_m *is irreducible in* $K[x, y]$, *where* K *is any field.*

Proof of 3.2.3. For each $m \geq 1$ define family of polynomials η_m by the recurrence relation (1.c). Note that η_2 is the same as ξ_2 up to an automorphism of the torus. Thus, η_1 and η_2 satisfy (2), (3) and (4).

(3) Assume η_m and η_{m+1} satisfy (3). We claim that, with the exception of $x^{m+1}y^{2m+3}$, all the monomials of $g\eta_{m+1}$ and $xh^2\eta_m$ are contained in $\Delta_2^0(m+2)$. This immediately implies the result. Indeed, the coefficient of this monomial in $g\eta_{m+1}$ is $(-1)^{m+1}$ and it is $(-1)^m$ in $xh^2\eta_m$, so their contributions cancel out. Then, since $xh^2\eta_m$ has no constant term and doesn't contain the monomial $x^{m+2}y^{2m+5}$, the claim follows.

First consider $g\eta_{m+1}$. Its Newton polygon is contained in the Minkowski sum of $\Delta_2^0(m+1)$ and the segment joining the origin with the point $(1,2)$. This polygon has vertices $(0,0)$, $(m+1, 2m+3)$, $(m+2, 2m+5)$, $\left(\frac{2m+5}{4}, 2\right)$ and $\left(\frac{2m+1}{4}, 0\right)$. It is immediate to verify that all these points, except $(m+1, 2m+3)$, lie in the boundary of $\Delta_2^0(m+2)$. Then, since the vectors $(m+2, 2m+5)$ and $(m+1, 2m+3)$ form a basis for the lattice in $\mathbb{R}^2$, no lattice point can lie in the parallelogram they span. Thus, all other monomials of $g\eta_{m+1}$ are supported in $\Delta_2^0(m+2)$.

Similarly, the Newton polygon of $xh^2\eta_m$ is contained in the convex hull of the points $(1,0), (1,2), (m+1, 2m+3)$ and $\left(\frac{2m+3}{4}, 0\right)$. The previous analysis shows that all of its monomials other than $x^{m+1}y^{2m+3}$ lie in $\Delta_2^0(m+2)$.

(2,4) The proofs of (2) and (4) are the same as the corresponding ones in Proposition 3.2.1 with minor modifications.

□

3.3 A proof of Theorem 3.1.2

Recall that for each $m \geq 1$ the triangle $\Delta_1(m)$ is the triangle with vertices $(-\alpha, 0)$, $(m-1+\beta, 0)$ and $(m, m+1)$, where α and β are non-negative rational numbers.

For the rest of the section we fix a value of $m \geq 1$ and define $\Delta = \Delta_1(m)$.

Let X_Δ be the corresponding toric variety equipped with an ample line bundle $H \in \mathrm{Cl}(X_\Delta)$, and let $X = \mathrm{Bl}_{t_0} X_\Delta$. Define $C = \mathrm{div}(\xi_m)$, considering ξ_m as a global section of $\pi^*H - mE \in \mathrm{Cl}(X)$.

The irreducibility of ξ_m is equivalent to that one of C, while the condition $\alpha + \beta < \frac{1}{m+1}$ is equivalent to $C \cdot C < 0$. Indeed,

$$C \cdot C = (m+1)(m-1+\alpha+\beta) - m^2 = -1 + (m+1)(\alpha+\beta),$$

where $m-1+\alpha+\beta$ is the base of Δ.

Consider the triangle Δ' with edges parallel to those of Δ and whose base is the interval $[0, m]$. Let H' be its corresponding divisor class in $\mathrm{Cl}(X_\Delta)$. Then, the class $\delta = H' - (m+1)E \in \mathrm{Cl}(X)$

satisfies $C \cdot \delta = 0$. Indeed, H' is linearly equivalent to $\frac{m}{\text{base}(\Delta)} H$, so that

$$C \cdot \delta = \frac{m}{\text{base}(\Delta)} \cdot 2\,\text{Area}(\Delta) - m(m+1) = 0.$$

Lemma 3.3.1. $X = \text{Bl}_{t_0} X_{\Delta_1}$ *is a Mori dream space if* $\alpha, \beta \leq \frac{1}{m+2}$ *and* $\alpha + \beta < \frac{1}{m+1}$.

Proof. We already saw that ξ_m defines a negative curve C in X as long as $\alpha + \beta < \frac{1}{m+1}$ and a straightforward computation shows that $D = \text{div}(\xi_{m+1})$ is a Weil divisor in the class of the triangle Δ' if and only if $\alpha, \beta \leq \frac{1}{m+2}$.

All the vertices of Δ are in the support of ξ_m, so the curve C doesn't pass through any of the torus-fixed points in X. On the other hand, D passes through the torus-fixed point P corresponding to the top vertex because this vertex is not a lattice point.

Now, since $C \cdot D = 0$, either one curve contains the other or they are disjoint. However, C and D are both irreducible and D contains the point $P \notin C$. Therefore, $C \cap D = \emptyset$ and the result follows by Cutkosky's criterion. □

To prove the remaining cases of Theorem 3.1.2 we show that $p \in HC_{\overline{\mathbb{F}}_p}$ for all $p \gg 1$ and $m \notin HC_{\mathbb{C}}$ when α or β are greater than $\frac{1}{m+2}$. The result then follows from Proposition 2.3.5.

Let's first make some reductions. As a consequence of the symmetry of the polynomials discussed in Remark 3.2.2, we can assume without loss of generality that $0 \leq \alpha$ and $\frac{1}{m+2} < \beta$. Indeed, the shear transformation of the lattice defined by $(i, j) \mapsto (i - j, j)$ maps the triangles Δ to a reflection of itself along the vertical axis, exchanging the roles of α and β. Furthermore, it's enough to show this for α and β like this and sufficiently small:

Lemma 3.3.2 (Propagation lemma)**.** *Let* Δ *and* $\tilde{\Delta}$ *be triangles as in Theorem 3.1.2 defined by parameters* α, β *and* $\tilde{\alpha}, \tilde{\beta}$, *respectively. Assume that* $0 < \alpha \leq \tilde{\alpha}$ *and* $0 < \beta \leq \tilde{\beta}$, *so that* $\Delta_1^0 \subsetneq \Delta \subseteq \tilde{\Delta}$. *If* $\tilde{X} = \text{Bl}_{t_0} X_{\tilde{\Delta}}$ *is a MDS, then* $X = \text{Bl}_{t_0} X_{\Delta}$ *is a MDS.*

Proof. Since $\alpha, \beta > 0$, the negative curves in both spaces pass through the two torus fixed points corresponding to the vertices in the bottom edge of the triangles. Given that $\tilde{\Delta}$ is a MDS, there is a positive integer μ and a curve $\tilde{D}$ in $\tilde{X}$ with class $\mu\delta$ which is disjoint from C. Then, there is a Laurent polynomial $\zeta(x, y)$ supported in $\tilde{\Delta}'$, vanishing at $(1, 1)$ to order $\mu(m+1)$ and containing the points $(0, 0)$ and $(m, 0)$ in its support.

Observe that $\tilde{\Delta}' \subseteq \Delta'$. Hence, $\zeta(x, y)$ is a Laurent polynomial supported in Δ' vanishing at $(1, 1)$ to order $\mu(m+1)$ and containing the points $(0, 0)$ and $(m, 0)$ in its support.

This Laurent polynomial defines a curve D in X with class $\mu\delta$ which is disjoint from C and the claim follows. □

Remark 3.3.3. The propagation lemma works in more general situations with an almost identical proof. For instance, it applies to all triangles of the form (†) as in Section 4.2 of Chapter 4.

Lemma 3.3.4. *Let* C, δ *as described above. Then, for* $0 < \alpha$ *and* $\frac{1}{m+2} < \beta$ *sufficiently small* $p \in HC_{\overline{\mathbb{F}}_p}$ *for all* $p \gg 1$.

Proof. For any $p \gg 1$ we need to find a polynomial $\zeta = \zeta_p$ that lies in degree $[0, pm]$, vanishes to order $p(m+1)$ at t_0, and has nonzero constant term. To construct it, begin by considering the relations (1.b) and (1.c) in Proposition 3.2.1. Write $p = (m+1)k + l$, for some $k \in \mathbb{Z}_{\geq 0}$ and $0 \leq l \leq m$. Then from $f\xi_m = \xi_{m+1} - x^m h^{m+1}$ we get

$$(\xi_{m+1} - x^m h^{m+1})^p = (f\xi_m)^p.$$

Using the equality $f^{m+1} = \xi_{m+1} - xh\xi_m$, this becomes

$$\xi_{m+1}^p + (-x^m h^{m+1})^p = (f\xi_m)^{(m+1)k+l} = (f^{m+1})^k f^l \xi_m^p = (\xi_{m+1} - xh\xi_m)^k f^l \xi_m^p,$$

so that

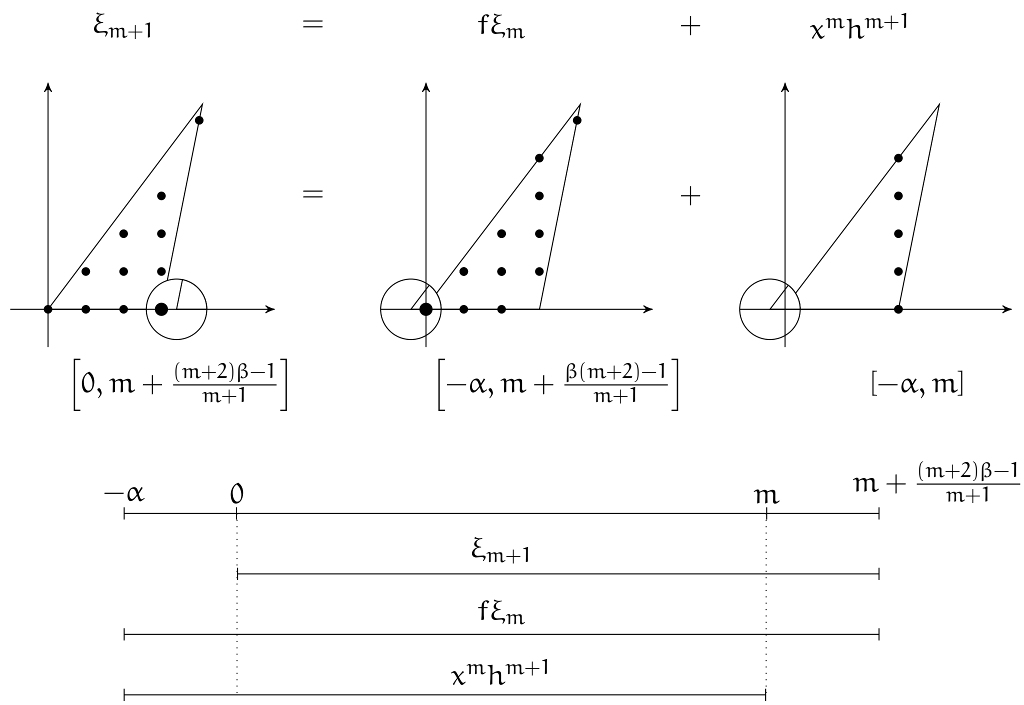
$$\xi_{m+1}^p + (-x^m h^{m+1})^p = \sum_{i=0}^{k} \binom{k}{i} (-1)^i \xi_{m+1}^{k-i} (xh\xi_m)^i \cdot f^l \xi_m^p. \tag{3.3.1}$$

Then, ζ_p is constructed by redistributing the terms in equation (3.3.1). More precisely, we claim that for $p \gg 1$, there exists $0 \leq j \leq k$ such that the left hand side of

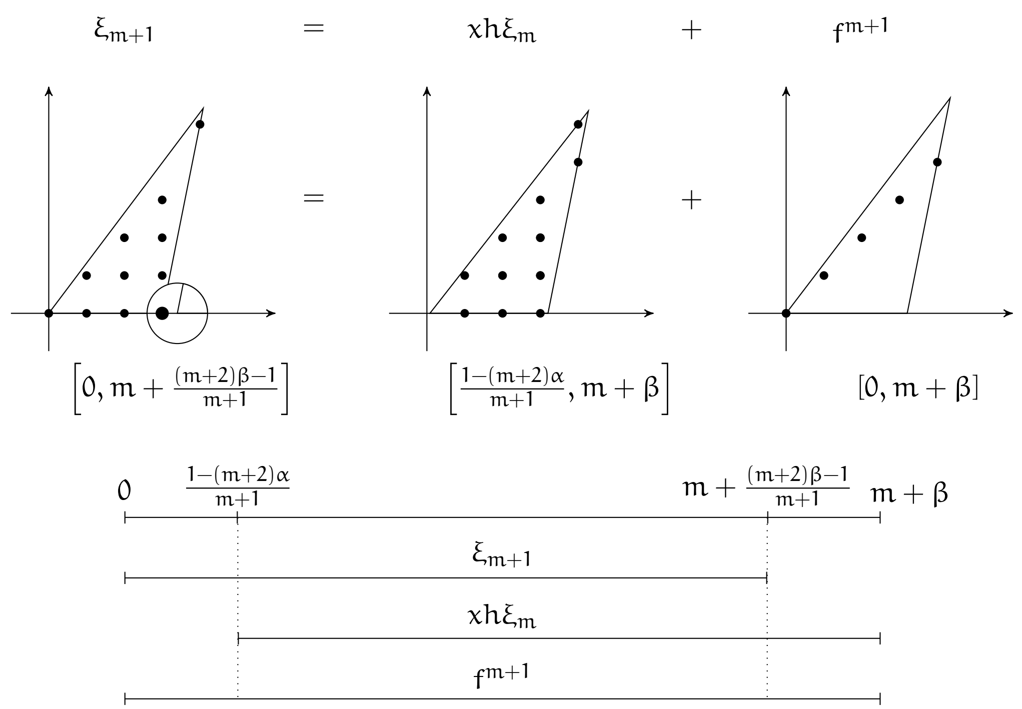
$$\begin{aligned}\zeta_p := \xi_{m+1}^p - \sum_{i=j+1}^{k} \binom{k}{i} (-1)^i \xi_{m+1}^{k-i} (xh\xi_m)^i \cdot f^l \xi_m^p \\ = (-1)^{p+1} (x^m h^{m+1})^p + \sum_{i=0}^{j} \binom{k}{i} (-1)^i \xi_{m+1}^{k-i} (xh\xi_m)^i \cdot f^l \xi_m^p.\end{aligned}$$

lies in degree $[0, B]$ and the right hand side lies in degree $[-A, pm]$ for some $A, B > 0$. Then ζ_p lies in the intersection of these degrees, $[0, pm]$, as required. The other required properties of ζ_p now follow easily. Every term in the definition of ζ_p vanishes to order $p(m+1)$ at t_0, hence so does ζ_p. Moreover, since $0 \leq j$, the constant term of $\zeta_p = \xi_{m+1}^p - \sum(\cdots)$ is equal to the constant term of ξ_{m+1}^p, which is 1 by Proposition 3.2.1. This proves that $p \in HC_{\overline{\mathbb{F}}_p}$.

To compute the degrees of the left and right hand side of equation (3.3.1), we start by computing the degrees in which the terms of the relations (1.b) and (1.c) of Proposition 3.2.1 lie. These degrees are shown in Figure 3.3. For example, to find the degree of ξ_{m+1}, we find the smallest triangle with sides parallel to Δ that contains the Newton polygon of ξ_{m+1}. The left side of this triangle passes through $(0, 0)$ and the right side passes through $(m+1, m+2)$. The base of the triangle is then the interval $\left[0, m + \frac{(m+2)\beta - 1}{m+1}\right]$. We have assumed here that $0 < \alpha < \frac{1}{m+2}$ and $\frac{1}{m+2} < \beta < 1$, otherwise the degrees would not be as pictured. We will sharpen these bounds as we move forward.



(a) Identity (1.b) in Proposition 3.2.1 together with the degrees in which all its terms lie.



(b) Identity (1.c) in Proposition 3.2.1 together with the degrees in which all its terms lie.

Figure 3.3: Visual representation of the identities in Proposition 3.2.1 for $m = 3$, $\alpha = \frac{1}{22}$ and $\beta = \frac{101}{500}$. The circles represent a zoom in into the area of the figures they cover.

We can now find the degrees in which the terms of equation (3.3.1). The terms on the left hand side lie in degrees

$$\begin{aligned} \xi_{m+1}^p &: p \cdot \left[0, m + \frac{(m+2)\beta - 1}{m+1}\right] = \left[0, pm + p \cdot \frac{(m+2)\beta - 1}{m+1}\right], \\ (x^m h^{m+1})^p &: p \cdot [-\alpha, m] = [-p\alpha, pm]. \end{aligned}$$

To find the degree of each summand on the right hand side, we find the degrees

$$\begin{aligned} \xi_{m+1}^{k-i} &: (k-i) \cdot \left[0, m + \frac{(m+2)\beta - 1}{m+1}\right] = \left[0, (k-i)m + (k-i) \cdot \frac{(m+2)\beta - 1}{m+1}\right], \\ (xh\xi_m)^i &: i \cdot \left[-\alpha + \frac{1-\alpha}{m+1}, (m-1+\beta)+1\right] = \left[i \cdot \frac{1-(m+2)\alpha}{m+1}, im + i\beta\right], \\ f^l &: \left[0, l \cdot \frac{m+\beta}{m+1}\right], \\ \xi_m^p &: p \cdot [-\alpha, m-1+\beta] = [-p\alpha, p(m-1) + p\beta]. \end{aligned}$$

Thus, the degree in which the i-th term in the right hand side of (3.3.1) lies is computed by adding up the all these degrees, which after some algebra can be simplified to

$$\left[-p\alpha + i \cdot \frac{1-(m+2)\alpha}{m+1}, pm + p \cdot \frac{(m+2)\beta - 1}{m+1} - (k-i) \cdot \frac{1-\beta}{m+1}\right].$$

Figure 3.4 summarizes the previous calculations.

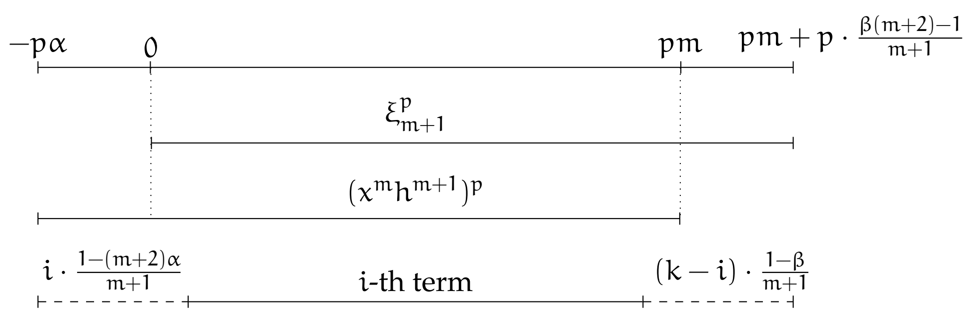


Figure 3.4: All the relevant degrees in equation 3.3.1.

Now we are ready to show that for $p \gg 1$ there exists $0 \leq j \leq k$ such that the i-th term of the sum lies in the same degree as $(x^m h^{m+1})^p$ for $i = 0, \ldots, j$ and in the same degree as ξ_{m+1}^p for $i = j+1, \ldots, k$. This is equivalent to:

$$p\alpha \leq (j+1) \cdot \frac{1-(m+2)\alpha}{m+1} \qquad \text{and} \qquad p \cdot \frac{(m+2)\beta - 1}{m+1} \leq (k-j) \cdot \frac{1-\beta}{m+1}.$$

After solving for j in these equations and substituting $p = (m+1)k + l$, these inequalities become

$$\left(\frac{(m+1)^2\alpha}{1-(m+2)\alpha}\right) k + (\cdots) \leq j \leq \left(\frac{(m+2) - ((m+1)(m+2)+1)\beta}{1-\beta}\right) k + (\cdots),$$

where the terms $(\cdots)$ depend on l, m, α and β but do not grow with k. The existence of j for large p (that means, large k) now follows from the strict inequality

$$\frac{(m+1)^2\alpha}{1-(m+2)\alpha} < \frac{(m+2)-((m+1)(m+2)+1)\beta}{1-\beta}$$

For $\alpha = 0$ the LHS of this inequality is zero and increasing, while the RHS is equal to 1 for $\beta = \frac{1}{m+2}$. Hence, for $0 < \alpha$, $\frac{1}{m+2} < \beta$ sufficiently small the inequality holds and both sides are positive. We conclude that $p \in HC_{\overline{\mathbb{F}}_p}$ for $p \gg 1$. □

Lemma 3.3.5. *Consider parameters such that* $0 < \alpha$, $\frac{1}{m+2} < \beta$ *and* $\alpha + \beta < \frac{1}{m+1}$*. Then,* $m \notin HC_{\mathbb{C}}$*.*

Proof. Assume $m \in HC_k$, so that there exists a $\zeta \in H^0(\mathcal{O}_X(mD))$ that lies in degree $[0, m^2]$, vanishes to order $m(m+1)$ at t_0 and has nonzero constant term. Note that ξ^m_{m+1} also vanishes to order $m(m+1)$ at t_0 and has nonzero constant term. We show the existence of a triangle $\overline{\Delta}$ such that it contains the Newton polygons of both ζ and ξ^m_{m+1}, and has area less than $\frac{m^2(m+1)^2}{2}$. This implies that both ζ and ξ^m_{m+1} define negative curves in $\mathrm{Bl}_{t_0} X_{\overline{\Delta}}$ of the same divisor class. Since ξ_{m+1} is irreducible, $\zeta = c \cdot \xi^m_{m+1}$ for some constant c. This gives a contradiction as follows.

Let Δ_1 be the triangle of mH' containing the Newton polygon of ζ, and let Δ_2 be the Newton polygon of ξ^m_{m+1}. Observe that Δ_2 contains a lattice point at the top vertex that does not lie in Δ_1, see Figure 3.5. This contradicts the equality $\zeta = c \cdot \xi^m_{m+1}$.

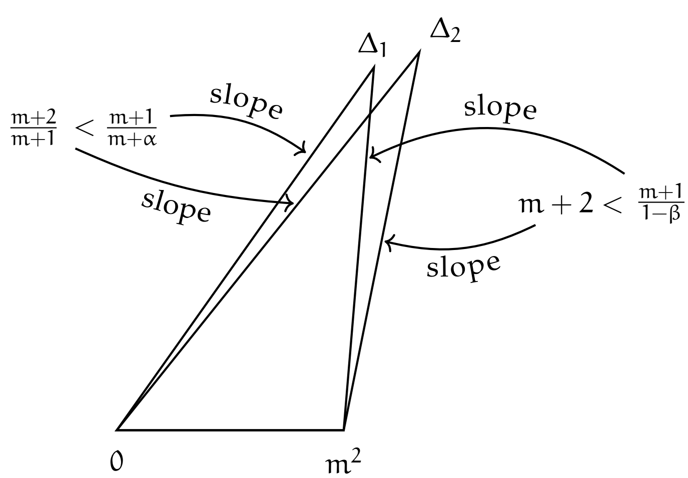


Figure 3.5: The triangle Δ_1 is associated to ζ and is similar to the triangle Δ; the triangle Δ_2 is the Newton polygon of ξ^m_{m+1} and its upper vertex is at $m(m+1, m+2)$. Note that given the slopes of the triangles their relative position is as shown in the picture for $\alpha < \frac{1}{m+2} < \beta$.

We proceed to show the existence of $\overline{\Delta}$.

Let us start with a triangle $\tilde{\Delta}$ that contains both Δ_1 and Δ_2. Namely, let $\tilde{\Delta}$ have base $[0, m^2]$, the left slope $\frac{m+1}{m}$ and the right slope $m+2$. This triangle however is not the smallest one containing Δ_1 and Δ_2 because the left slope of Δ_1 is $\frac{m+1}{m+\alpha}$ which is strictly smaller than the left

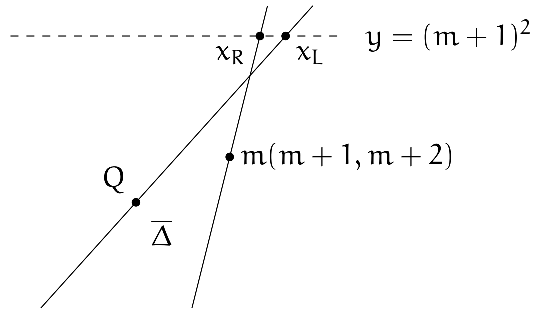


Figure 3.7: The top vertex of $\overline{\Delta}$ always lies below the line $y = (m+1)^2$.

slope of $\tilde{\Delta}$. Thus, no nonzero lattice point on the left edge of $\tilde{\Delta}$ lies in Δ_1. We may now decrease the triangle $\tilde{\Delta}$ by lowering the slope of its left edge until the edge hits the first lattice point in $\tilde{\Delta}$ and take this smaller triangle as $\overline{\Delta}$. Then $\overline{\Delta}$ contains all lattice points from Δ_1 and Δ_2. When we lower the left edge of $\tilde{\Delta}$, the first lattice point it meets is $Q = (m^2+1, m(m+1)+1)$, see Figure 3.6.

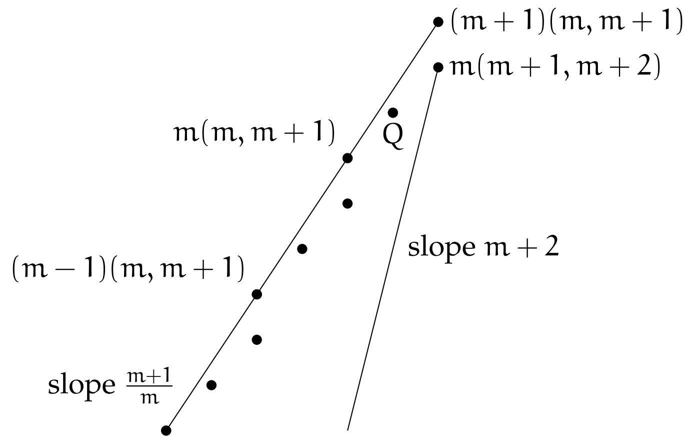


Figure 3.6: The top part of $\overline{\Delta}$ for $m = 2$.

To show that the curves defined by ξ^m_{m+1} and ζ in $\mathrm{Bl}_{t_0}(X_{\overline{\Delta}})$ are negative we need the area of the triangle $\overline{\Delta}$ to be smaller than $\frac{m^2(m+1)^2}{2}$, i.e., its height must be less than $(m+1)^2$.

The line spanned by the left edge of $\overline{\Delta}$ reaches the height $(m+1)^2$ at the x-coordinate $x_L = m(m+1) + \frac{m+1}{m(m+1)+1}$, while the right edge does it at $x_R = m(m+1) + \frac{1}{m+2}$. The height of the triangle is then smaller than $(m+1)^2$ whenever $x_R < x_L$ (see Figure 3.7), that is, when

$$m(m+1) + \frac{1}{m+2} < m(m+1) + \frac{m+1}{m(m+1)+1} \quad \Longleftrightarrow \quad \frac{1}{m+2} < \frac{m+1}{m(m+1)+1},$$

which always holds. □

3.4 A proof of Theorem 3.1.3

Recall that for each $m \geq 1$ the triangle $\Delta_2(m)$ is the triangle with vertices $(-\alpha, 0)$, $(\frac{2m-1}{4} + \beta, 0)$ and $(m, 2m+1)$, where α and β are non-negative rational numbers.

For the rest of the section we fix a value of $m \geq 1$ and define $\Delta = \Delta_2(m)$.

As in the previous section, define $C = \operatorname{div}(\xi_m)$, considering η_m as a global section of $\pi^*H - mE \in \operatorname{Cl}(X)$. The irreducibility of η_m is equivalent to that one of C, while the condition $\alpha + \beta < \frac{1}{4(2m+1)}$ is equivalent to $C \cdot C < 0$. Indeed,

$$C \cdot C = 2\,\operatorname{Area}(\Delta) - m^2 = -\frac{1}{4} + (2m+1)(\alpha + \beta),$$

Consider the triangle Δ' with edges parallel to those of Δ and whose base is the interval $[0, m]$. Let H' be its corresponding divisor class in $\operatorname{Cl}(X_\Delta)$. Then, the class $\delta = H' - (2m+1)E \in \operatorname{Cl}(X)$ satisfies $C \cdot \delta = 0$. Indeed, H' is linearly equivalent to $\frac{m}{\operatorname{base}(\Delta)} H$, so that

$$C \cdot \delta = \frac{m}{\operatorname{base}(\Delta)} \cdot 2\,\operatorname{Area}(\Delta) - m(2m+1) = 0.$$

Lemma 3.4.1. *$X = \operatorname{Bl}_{t_0} X_{\Delta_1}$ is a Mori dream space if $\alpha + \beta < \frac{1}{4(2m+1)}$ and $\alpha = 0$ or $\beta = 0$.*

Proof. The first condition means that η_m defines a negative curve C on X. Direct computations show that if $\alpha = 0$, then the curve $D = \operatorname{div}(x^m(1-y)^{2m+1})$ is a Weil divisor in the class of Δ'. The same is true for $D = \operatorname{div}(\eta_m \eta_{m+1})$ if $\beta = 0$.[2]

The top vertex of Δ is in the support of η_m, so the curve doesn't pass through the corresponding torus-fixed point. On the other hand, both of the previous possibilities of D pass through this torus-fixed point as this vertex of Δ' is not a lattice point. Similarly, both D as above don't pass through point corresponding to the right-hand vertex, while C does because that's a rational vertex of Δ.

It follows that $C \cap D = \emptyset$ because $C \cdot D = 0$ and each curve contains a point the other one doesn't. The result now follows by Cutkosky's criterion. □

To prove the remaining cases of Theorem 3.1.2 we show that $p \in HC_{\overline{\mathbb{F}}_p}$ for all $p \gg 1$ and $m \notin HC_{\mathbb{C}}$ when $0 < \alpha, \beta$. The result then follows from Proposition 2.3.5.

Lemma 3.4.2. *For every prime p there exists an $n_p \geq 0$ such that $p^{n_p} \in HC_{\overline{\mathbb{F}}_p}$.*

Proof. We show the existence of polynomials ζ giving $p^l \in HC_{\overline{\mathbb{F}}_p}$ for all $l \gg 1$.

Consider the polynomial $x^m(1-y)^{2m+1} \in \overline{\mathbb{F}}_p[x, y]$, with Newton polytope containing all the integral points $(0, 0), \ldots, (0, 2m+1)$ and vanishing to order $2m+1$ at t_0. If $\alpha = 0$, the polynomial $x^m(1-y)^{2m+1}$ is supported in Δ' and yields $1 \in HC_{\overline{\mathbb{F}}_p}$, so the result follows. Now assume $\alpha > 0$.

[2] Originally these polynomials were found by trial and error, however, they are not arbitrary at all. Their existence and structure is a consequence of a beautiful relation between a bigger class of negative curves and the MDS property for the spaces they live in. This is part of our results in Chapter 4.

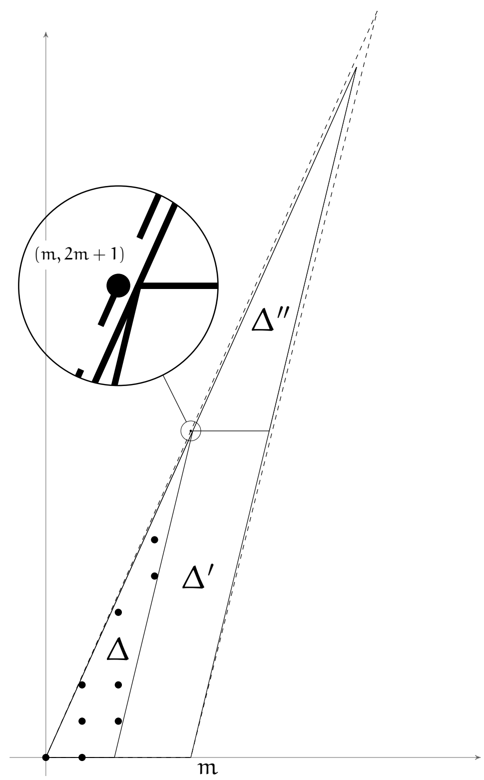


Figure 3.8: The triangles Δ, Δ' and Δ'' for $m = 4$. For $\alpha > 0$ the triangles Δ and Δ'' intersect in a non-integral point. The lattice points displayed are the monomials in the support of η_m.

Thus, with exception of its highest degree term, the polynomial $(x^m(1-y)^{2m+1})^{p^l}$ is supported in the triangle $p^l\Delta'$.

To show the result we will prove that for $l \gg 1$ there exists a polynomial F that vanishes to order $p^l(2m+1)$ at t_0 and is supported in $p^l\Delta'$, except for its constant term, which has a nonzero coefficient. Then, multiplying F by a scalar if necessary, $\zeta = (x^m(1-y)^{2m+1})^{p^l} + F$ is supported in $p^l\Delta'$ and vanishes to order $p^l(2m+1)$ at t_0. Furthermore, we construct F so that its Newton polytope does not include the bottom right-hand vertex of $p^l\Delta'$. Since this vertex lies in the support of $(x^m(1-y)^{2m+1})^{p^l}$, it follows that ζ does not vanish at the corresponding torus-fixed point. Then, the same argument as the one in the proof of Lemma 3.4.1 shows that $D = \operatorname{div}(\zeta)$ and C are disjoint.

Consider the triangle Δ'' as shown in Figure 3.8. Its left-hand edge lies in a line passing through the origin with slope $\frac{2m+1}{m+\alpha}$, while the right-hand one in the one through $(m, 0)$. Thus, this triangle corresponds to the class of a Weil divisor H'' in X_Δ. Since Δ' is the Minkowski sum $\Delta + \Delta''$, we get $H' = H + H''$ in $N_1(X_\Delta)$. (Note that this Δ is a rational translate of the one originally considered, so this equality is only true for numerical classes.)

We look for the polynomial F of the form $F = (\xi_m)^{p^l}(1+g)$, with g supported in the triangle $p^l\Delta''$. Such F would be supported in Δ', except for its $x^m y^{2m+1}$ monomial. The polynomial F

vanishes to order $p^l(2m+1)$ at t_0 if $1+g$ vanishes to order $p^l(m+1)$ at t_0. In other words, we are looking for a polynomial g whose restriction to the $p^l(m+1)$-st order infinitesimal neighbourhood of t_0 coincides with the function -1 on that neighbourhood. The existence of such a g follows if we can prove more generally that for any function on the infinitesimal neighbourhood there exists a g whose restriction to the neighbourhood agrees with the given function. Thus, we want surjectivity of the morphism

$$H^0(\mathcal{O}_{X_\Delta}(p^l H'')) \longrightarrow H^0(\mathcal{O}_{X_\Delta}(p^l H'')/I_{t_0}^{p^l(m+1)}).$$

Let us denote the right-hand space by $H^0(\mathcal{O}_{p^l(m+1)t_0})$. Then, the morphism fits into the long exact sequence

$$0 \longrightarrow H^0(\mathcal{O}_{X_\Delta}(p^l H'') \otimes I_{t_0}^{p^l(m+1)}) \longrightarrow H^0(\mathcal{O}_{X_\Delta}(p^l H'')) \longrightarrow H^0(\mathcal{O}_{p^l(m+1)t_0}) \longrightarrow H^1(\mathcal{O}_{X_\Delta}(p^l H'') \otimes I_{t_0}^{p^l(m+1)}) \longrightarrow 0.$$

To guarantee the existence of g it is enough to show that $H^1(\mathcal{O}_{X_\Delta}(p^l H'') \otimes I_{t_0}^{p^l(m+1)}) = 0$ for $l \gg 1$. By Proposition 2.3.11, this is equivalent to the vanishing $H^1(\mathcal{O}_X(p^l(H''-nE))) = 0$, where $n = m+1$. We claim that $H''-nE$ is ample on X. Indeed, from $[D_0] = [C] + H'' - nE$ we get

$$(H''-nE)\cdot C = (D_0 - C)\cdot C = -C\cdot C > 0,$$
$$(H''-nE)\cdot E = n > 0.$$

Thus, by Kleiman's criterion $H''-nE$ is ample. It follows that its higher cohomology groups vanish for big enough multiples, in particular $H^1(\mathcal{O}_X(p^l(H''-nE))) = 0$ for all $l \gg 1$. □

We will now prove that $m \notin HC_{\mathbb{C}}$ if $0 < \alpha, \beta$.

Assume by contradiction that $m \in HC_{\mathbb{C}}$, given by a polynomial ζ that defines the curve D in class $m\delta = [mD_0]$. The idea of the proof is as follows. The polytope $m\Delta'$ is not the convex hull of its lattice points. We may thus decrease the size of $m\Delta'$ so that it still supports ζ. In fact, we will construct a new triangle $\tilde{\Delta}$ satisfying:

(a) The polynomial $1-y$ defines a negative curve $\tilde{C}$ in $\tilde{X} = \mathrm{Bl}_{t_0} X_{\tilde{\Delta}}$.

(b) The polynomial ζ defines a curve $\tilde{D}$ in $\tilde{X}$ such that $\tilde{C}\cdot\tilde{D} < 0$.

These properties give a contradiction to the existence of ζ as follow. Since $\tilde{C}$ is a negative curve in $\tilde{X}$ that intersects $\tilde{D}$ negatively, it follows that $\tilde{C}$ is a component of $\tilde{D}$, in other words, $1-y$ divides ζ. This implies that the left vertex of $m\Delta'$ cannot lie in the support of ζ and D passes through the corresponding torus-fixed point. However, C also passes through that point because $\alpha > 0$, a contradiction.

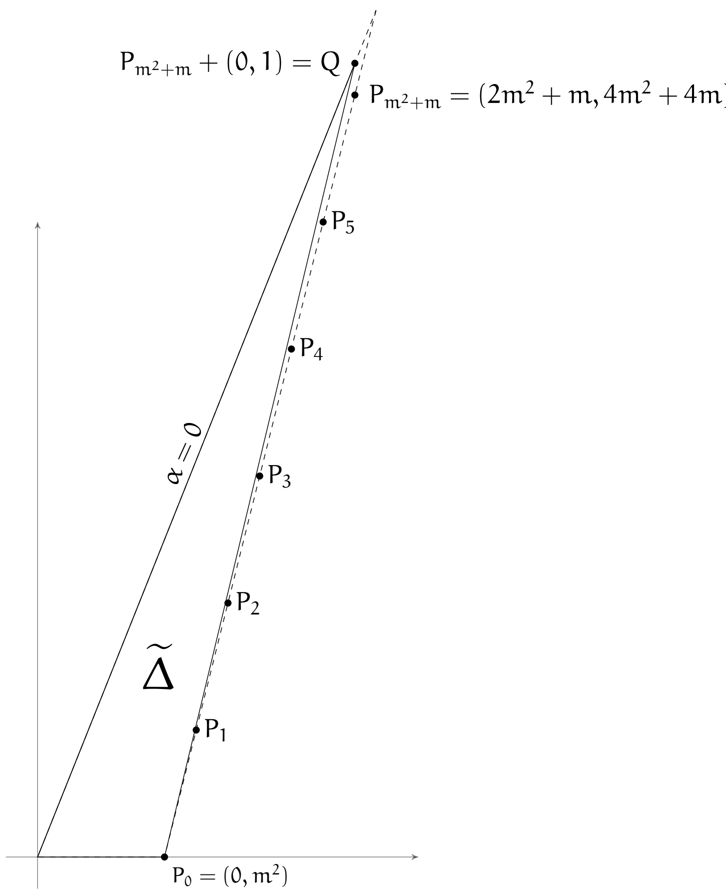


Figure 3.9: Triangle $\tilde{\Delta}$ for $m = 2$ in the second family.

Consider first the triangle Δ_2^0 (recall that this is the triangle Δ with $\alpha = \beta = 0$). We modify the slope of its right-hand edge and ask how large does this slope have to be so that $1 - y$ defines a negative curve in the blowup of the resulting toric variety. If $\overline{\Delta}$ is the new triangle, then the correct condition is that the height of $\overline{\Delta}$ is greater than its width. Here the height is measured vertically from the bottom right-hand vertex to the left-hand edge. A simple calculation shows that the slope needs to be greater than $4 + \frac{1}{m(m+1)}$.

Let us now construct the triangle $\tilde{\Delta}$. We start with the triangle $m\Delta'$. In the first step we pivot the left-hand edge around the origin so that we are in the case $\alpha = 0$. This step makes the triangle larger, hence the new triangle still supports ζ. In the second step we pivot the right-hand edge of the triangle about the bottom right-hand vertex. We make the slope as steep as possible so that the triangle still supports ζ. Call the resulting triangle $\tilde{\Delta}$. See Figure 3.9. Let us consider the two families separately.

We start with the right-hand edge having slope 4 and pivoting it about the right-hand vertex of the triangle. When the slope is equal to 4, the lattice points on the right-hand edge are

$$P_i = (m^2, 0) + i(1,4), \qquad i = 0, 1, \ldots, m^2 + m,$$

where P_0 is the bottom right-hand vertex of $m\Delta'$ and $P_{m^2+m} = (2m^2+m, 4m^2+4m)$ is the highest lattice point on the edge. To see this we use that the slopes of the P_i must be smaller than that one of the left-hand edge:

$$\frac{4i}{m^2+i} \leq 2 + \frac{1}{m} \iff i \leq \frac{(2m+1)m^2}{2m-1} = m^2 + m + \frac{1}{2(2m-1)} + \frac{1}{2},$$

where $m \geq 1$.

Then, as we pivot the right-hand edge the first lattice point in $m\Delta'$ that the edge hits is

$$Q = P_{m^2+m} + (0,1) = (2m^2+m, 4m^2+4m+1).$$

If we let the right-hand edge go through the vertex and the point Q, then it has slope equal to $4 + \frac{1}{m(m+1)}$. However, the point Q isn't in the support of ζ because it lies on the left-hand edge of $m\Delta'$ when $\alpha = 0$. Thus, we can pivot the left edge a bit more to make its slope greater than $4 + \frac{1}{m(m+1)}$. Thus, $1 - y$ defines a negative curve in $\tilde{X}$.

With $\tilde{\Delta}$ defined, let us now check that $\tilde{C} \cdot \tilde{D} < 0$. Since ζ is supported in $\tilde{\Delta}$, it defines the curve $\tilde{D}$ with class

$$[\tilde{D}] = \pi^*\tilde{H} - m(2m+1)E,$$

where $\tilde{H}$ is the class in $\mathrm{Cl}(X_{\tilde{\Delta}})$ corresponding to the triangle $\tilde{\Delta}$. The curve $\tilde{C}$ has class

$$[\tilde{C}] = \frac{1}{h}\tilde{H} - E,$$

where h is the height of $\tilde{\Delta}$, measured vertically from the bottom right-hand vertex to the left-hand edge. The intersection number is now

$$\tilde{C} \cdot \tilde{D} = \frac{1}{h}\tilde{H}^2 - m(2m+1) = \frac{wh}{h} - m(2m+1),$$

where w is the horizontal width of $\tilde{\Delta}$. Thus, we need to prove that $w < m(2m+1)$. However, we have

$$w < h = m(2m+1).$$

This finishes the proof of the main theorem.

Remark 3.4.3. The same argument we just used works verbatim for the first family as soon as we consider $\alpha > 0$ and $\beta > \frac{1}{m+2}$ or vice versa. Here we omit this case to make the exposition more clear; the interested reader may find the complete argument in [11].

It is also important to note that it's not clear how to make the same argument work when it's not possible to make the right-hand slope bigger due to the existence of additional points in the support of ζ. This is the case in most of the families constructed in Chapter 4. In those cases we determine whether the varieties are MDS or not using a different technique that doesn't require prime characteristic methods or higher cohomology.

Chapter 4

Two-parameter families of negative curves

In this chapter we present our work with José Luis González and Kalle Karu in [13].

We give a unifying construction of negative curves on the surfaces $X = \mathrm{Bl}_{t_0} X_\Delta$. In particular, the families presented in Chapter 3 appear as boundary cases of this. The classification consists of two classes of said curves, each depending on two parameters. Every curve in these two classes is algebraically related to other curves in both classes; this allows us to find their defining equations inductively. The key observation is as follows: Let X be a MDS, C its (non-exceptional) negative curve and D a disjoint curve from C. Then, it's possible to find another compactification of the two-torus, say $X' = \mathrm{Bl}_{t_0} X_{\Delta'}$, where the defining equation of D yields a negative curve and that one of C gives a curve disjoint from D in X'.

For each curve in our classification, we consider a family of blowups in which the curve defines an extremal class in the effective cone. We give a complete classification of these blowups into Mori Dream Spaces and non-Mori Dream Spaces; in particular, we recover Theorems 3.1.2 and 3.1.3.

Remarkably, not only the results in this chapter subsume those in the previous chapter, but the approach taken in here greatly simplifies previous proofs; avoiding positive characteristic methods and higher cohomology altogether.

4.1 Introduction

We work over an algebraically closed field k of characteristic zero.

Let X be the blowup of a weighted projective plane $\mathbb{P}(a, b, c)$ at a general point e. More generally, we allow X to be the blowup of a projective toric surface X_Δ defined by a triangle Δ:

$$X = \mathrm{Bl}_e X_\Delta.$$

Such an X is a projective variety of Picard number 2. Its Mori cone of curves is 2-dimensional, generated by the class of the exceptional curve E and another class γ, not necessarily rational, of non-positive self-intersection. If γ can be chosen to be the class of an irreducible curve C then we call this uniquely determined curve C a negative curve in X. It is not known if every X contains a negative curve. As an example, Kurano and Matsuoka [19] conjecture that $X = \mathrm{Bl}_e \mathbb{P}(9, 10, 13)$

contains no negative curve. This conjecture, if true, would imply that the Mori cone of X is not rational. The existence and classification of negative curves is the first main topic of this article.

The existence of a negative curve is closely related to the Mori Dream Space (MDS) property of X. Recall that a projective variety is called a MDS if its Cox ring is a finitely generated k-algebra. Cutkosky [8] shows that X is a MDS if and only if it contains a negative curve C and another curve D disjoint from C. There is an extensive literature on proving that certain X is a MDS (see, for example, [6, 8, 18, 21]). There are also many proofs of the non-MDS property (for example, [10–12, 14, 16, 17, 20]). Our second goal is to unify and simplify many of these results by completely classifying if X in a family is a MDS or a non-MDS.

We let $T \cong \mathbb{G}_m^2 \subset X_\Delta$ be the torus, and without loss of generality take $e = (1,1) \in T$. A negative curve in X is defined by a polynomial $f(x,y) \in k[x^{\pm 1}, y^{\pm 1}]$ that vanishes to order m at e, and whose Newton polygon lies in a translation and dilation of the triangle Δ with area $\leq \frac{m^2}{2}$. By abuse of notation, we will define a negative curve to be a curve in the torus T, defined by an irreducible polynomial $f(x,y)$ that vanishes to order m at e and whose Newton polygon fits into *some* triangle of area $\leq \frac{m^2}{2}$. Given such a negative curve in the torus, its strict transform defines a negative curve in X for suitable choices of the triangle Δ.

Having defined negative curves in the torus T, we can now try to classify them. For each $m > 0$ there is, up to an automorphism of T, a finite number of negative curves. Using computer search one can find all these curves when m is small [5]. For $m = 1$ there is a unique curve defined by $f(x,y) = 1 - y$. This curve with $m = 1$ appeared in all examples of [10, 14, 16, 17]. Similarly, for $m = 2$ there is a unique curve defined by $f(x,y) = 3 - x - y - x^{-1}y^{-1}$. This case $m = 2$ appeared in the examples of [20, 21]. For $m = 3$ there are two non-isomorphic negative curves and for $m = 4$ there are four. In [11, 12] we constructed two infinite families of negative curves, giving two non-isomorphic negative curves for each $m \geq 3$.

All examples of negative curves mentioned above have the property that the Newton polygon of $f(x,y)$ lies in a triangle Δ that contains exactly $\binom{m+1}{2} + 1$ lattice points. Since vanishing to order m at e imposes $\binom{m+1}{2}$ conditions, the existence of such $f(x,y)$ supported in the triangle is clear; its irreducibility still needs to be proved. There do exist negative curves whose supporting triangle Δ contains fewer than $\binom{m+1}{2} + 1$ lattice points. Kurano and Matsuoka [19] gave two such examples. These negative curves seem to be very exceptional and we will not have much to say about them in this article. As an example, if $\mathrm{Bl}_e \mathbb{P}(9, 10, 13)$ contains a negative curve, then the curve must be of this exceptional type.

We will only consider negative curves defined by $f(x,y)$ supported in a triangle Δ with at least $\binom{m+1}{2} + 1$ lattice points. It then follows from the uniqueness of the negative curve in $\mathrm{Bl}_e X_\Delta$ that the number of lattice points in Δ must be exactly $\binom{m+1}{2} + 1$. This assumption divides the problem of finding negative curves into two steps: first find a triangle of small area and many lattice points, then prove that the polynomial $f(x,y)$ supported in it is irreducible.

In [12] we gave a criterion for irreducibility of $f(x,y)$ given its Newton polygon. Namely, if a triangle Δ has an edge whose only lattice points are its endpoints, and the Newton polygon of $f(x, y)$ lies in the triangle and contains the two vertices, then $f(x, y)$ is irreducible. We will impose this irreducibility condition on all negative curves that we study.

We can now state the main theorem in two parts.

Theorem 4.1.1.A. *Let* $M, N \geq 0$, $K \geq 3$ *be integers satisfying the equation*

$$(M + N)^2 = KMN + 1. \tag{4.1.1}$$

To each such triple we associate an integral triangle $IT(M, N)$ *and a rational triangle* $RT(M, N)$ *with vertices:*

$$IT(M, N): \quad (0, 0), (M + N, KN), (M, 0),$$
$$RT(M, N): \quad (0, 0), (M, M + N), \left(M - \frac{M + N}{K}, 0\right).$$

Then, each of these triangles supports a polynomial $f(x, y)$ *defining a negative curve that vanishes at* e *to order* $m = M+N$ *in the integral case and* $m = M$ *in the rational case. The negative curves corresponding to* $IT(M, N)$ *for* $M \geq N > 0$ *and* $RT(M, N)$ *for* $M > N > 1$ *are pairwise non-isomorphic.*

We omit K from the notation $IT(M, N)$, $RT(M, N)$. When M and N are positive then K is determined by them. The statement about the curves being non-isomorphic means that there is no automorphism of the torus T that carries one curve to another. An automorphism of the torus is given by an affine linear automorphism of the lattice $\mathbb{Z}^2$.

In Section 4.6.2 below we show that the polynomials $f(x, y)$ defining negative curves in $IT(M, N)$ and $RT(M, N)$ satisfy algebraic relations. Using these relations we can compute the polynomials explicitly.

Theorem 4.1.1.B. *Conversely, if a negative curve is defined by* $f(x, y)$ *vanishing at* e *to order* $m > 0$ *and supported in a triangle* Δ *that satisfies*

(a) Δ *contains at least* $\binom{m+1}{2} + 1$ *lattice points and has area* $\leq \frac{m^2}{2}$,

(b) Δ *has two integral vertices* $(0, 0)$ *and* (m, h) *where* m *and* h *are relatively prime, and a possibly non-integral vertex* (r, s) *with* $0 < r < m$,

then the negative curve is isomorphic to one defined in Theorem 4.1.1.A.

The theorem classifies all negative curves where the triangles Δ satisfy conditions (a) and (b). As explained above, condition (a) removes the negative curves of exceptional kind. Condition (b) is used to prove irreducibility of $f(x, y)$.

All solutions to equation (4.1.1) can be written down explicitly.

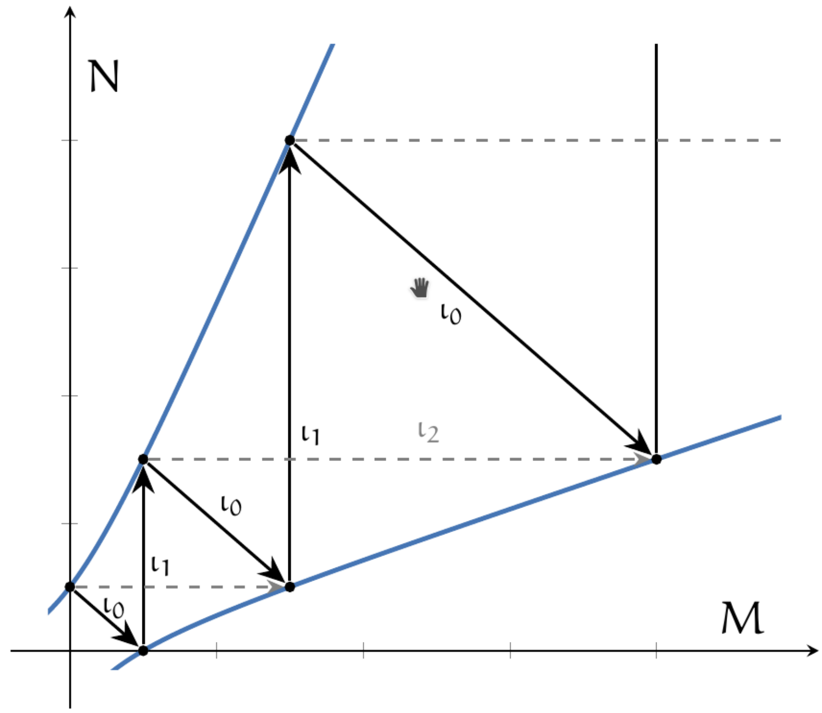


Figure 4.1: Involutions generating all integral solutions to $(M+N)^2 = KMN + 1$ as described in Theorem 4.1.2

Theorem 4.1.2. *Let $K \geq 3$ be fixed. There are three involutions on the set of integral solutions of (4.1.1):*

$$\begin{aligned} \iota_0 &: (M,N) \mapsto (N,M), \\ \iota_1 &: (M,N) \mapsto (M,(K-2)M-N) \\ \iota_2 &: (M,N) \mapsto ((K-2)N-M,N). \end{aligned}$$

These involutions satisfy the relation $\iota_2 = \iota_0\iota_1\iota_0$. See Figure 4.1. All integral non-negative solutions of equation (4.1.1) can be obtained by starting with $(0,1)$ and applying ι_0 and ι_1 alternately:

$$(0,1) \xmapsto{\iota_0} (1,0) \xmapsto{\iota_1} (1,K-2) \xmapsto{\iota_0} (K-2,1) \xmapsto{\iota_1} (K-2,(K-2)^2-1) \xmapsto{\iota_0} \ldots.$$

One can see from the previous theorem that for $K = 3$ there are three solutions $(0,1), (1,0), (1,1)$. For $K \geq 4$ there is an infinite sequence of solutions. The integral and rational cases then give us two 2-parameter families of distinct negative curves.

For $K \geq 4$, the solutions (M,N) can be divided into two sets, $M > N$ and $M < N$. The triangles in the two sets are isomorphic. More precisely, there exist affine linear automorphisms of the lattice $\mathbb{Z}^2$ that induce isomorphisms of triangles:

$$IT(M,N) \cong IT(\iota_0(M,N)),$$

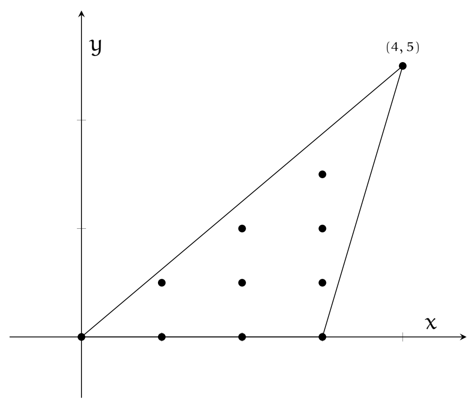
$$\mathrm{RT}(M,N) \cong \mathrm{RT}(\iota_1(M,N)).$$

Example 4.1.3. *1. The triples* $M \geq N = 1$, $K = M+2$ *are solutions of (4.1.1). The corresponding rational triangles have vertices* $(0,0), (M, M+1), (M-1+1/K, 0)$. *This is the family of negative curves studied in [12].*

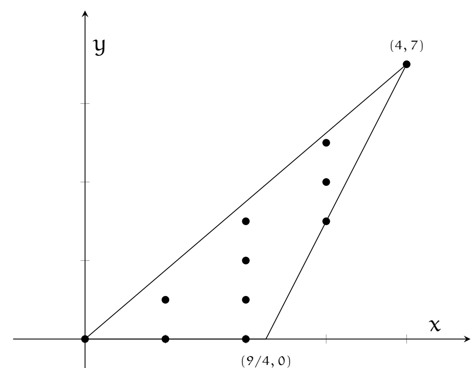
2. *When* $K = 4$ *then* $M \geq 2, N = M-1$ *are solutions of (4.1.1). The corresponding rational triangles have vertices* $(0,0), (M, 2M-1), \left(M - \frac{2M-1}{4}, 0\right)$. *This is the family of negative curves studied in [11].*

3. *The integral triangle corresponding to* $K = 4, M = 3, N = 2$ *is the simplest one that does not appear in either of the two families in [11, 12]. It has vertices* $(0,0), (5,8), (0,3)$ *and* $m = 5$.

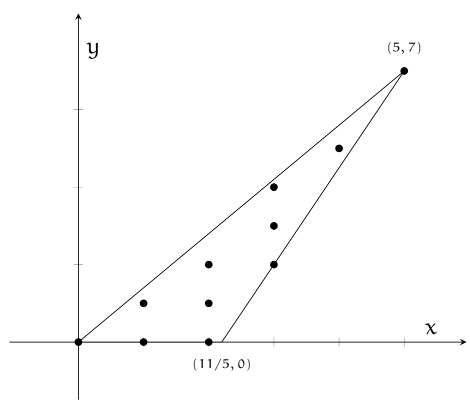
4. *Let us look for negative curves with* $m = 4$. *There is one integral triangle* $\mathrm{IT}(3,1)$, $K = 5$, *and one rational triangle* $\mathrm{RT}(4,3)$, $K = 4$. *There are two more negative curves with* $m = 4$, *see Figure 4.2.*



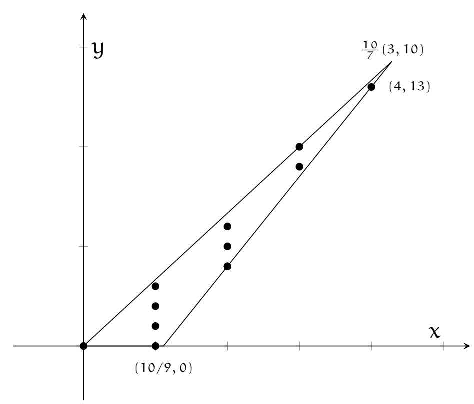
(a) $\mathrm{IT}(3,1)$ with $K = 5$, member of the family from [12] with $C \cdot C = -1$.



(b) $\mathrm{RT}(4,3)$ with $K = 4$, member of the family from [11] with $C \cdot C = -\frac{1}{4}$.



(c) Negative curve in $\mathrm{Bl}_e\, \mathbb{P}(5,7,11)$ with $C \cdot C = -\frac{3}{5}$.



(d) Negative curve in $\mathrm{Bl}_e\, \mathbb{P}(7,9,10)$ with $C \cdot C = -\frac{8}{63}$.

Figure 4.2: Every negative curve with $m = 4$.

In [11, 12] we studied two families of negative curves (Example 4.1.3 (1), (2)) and for each such curve we constructed examples of MDS and non-MDS. We generalize these results to all negative curves described above.

Let Δ be one of the triangles $\mathrm{IT}(M,N)$ or $\mathrm{RT}(M,N)$ described in Theorem 4.1.1.A, and let $C^0 \subseteq T$ be the corresponding negative curve. We consider larger triangles $\Delta_{\alpha,\beta}$ obtained from Δ by increasing its base:

$$\mathrm{IT}(M,N)_{\alpha,\beta}: \quad (-\alpha,0),(M+N,KN),(M+\beta,0),$$
$$\mathrm{RT}(M,N)_{\alpha,\beta}: \quad (-\alpha,0),(M,M+N),\left(M-\frac{M+N}{K}+\beta,0\right).$$

Here $\alpha,\beta \geq 0$. Let $X = \mathrm{Bl}_e X_{\Delta_{\alpha,\beta}}$. Then, the strict transform of C^0 is a negative curve in X if $Area(\Delta_{\alpha,\beta}) \leq \frac{m^2}{2}$. This is equivalent to $\alpha+\beta \leq 1/NK$ in the integral case and $\alpha+\beta \leq 1/K(M+N)$ in the rational case.

Theorem 4.1.4. *Let the variety X be constructed from either $\mathrm{IT}(M,N)$ or $\mathrm{RT}(M,N)$ by choosing $\alpha,\beta \geq 0$ such that the strict transform of C^0 is a negative curve in X.*

1. *If $\alpha = 0$ or $\beta = 0$ then X is a MDS.*
2. *If $\alpha > 0$ and $\beta > 0$ then X is a non-MDS when $N > 1$ in the integral case and $M+N > 1$ in the rational case.*

Remark 4.1.5. The theorem applies with $M > N$ and with $M < N$. Even though these triangles are isomorphic, when choosing $\alpha,\beta \geq 0$ we lengthen different edges of the same triangle.

Remark 4.1.6. The negative curve C is allowed to have self-intersection number zero. The theorem gives many examples of non-MDS where C has zero self-intersection.

Remark 4.1.7. The triangles $\mathrm{IT}(K-2,1)$ and $\mathrm{RT}(K-1,1)$ support the same negative curve: the former is the convex hull of the lattice points in the latter. When choosing a set of non-isomorphic curves in Theorem 4.1.1.A we have discarded the triangles $\mathrm{RT}(K-1,1)$. In contrast, the second part of Theorem 4.1.4 applies to $\mathrm{RT}(K-1,1)$, but not to $\mathrm{IT}(K-2,1)$. These negative curves are studied in more detail in [12].

Remark 4.1.8. For small values of M and N, the possibly degenerate triangles $\mathrm{IT}(0,1)$, $\mathrm{IT}(1,0)$, $\mathrm{RT}(1,0)$, $\mathrm{RT}(1,1)$ support the same negative curve with vanishing order $m = 1$. This case has been widely studied in the literature [10, 14, 16, 17, 20]. The present paper does not say anything more about the $m = 1$ case.

4.2 Existence of negative curves

Recall that a negative curve in $X = \mathrm{Bl}_e X_\Delta$ is an irreducible curve C of non-positive self-intersection, different from the exceptional curve E. If the curve C has strictly negative self-intersection then it is unique in X.

Let us say that a triangle Δ supports a negative curve if Δ has area $\leq \frac{m^2}{2}$ and there exists a polynomial $f(x,y) \in k[x^{\pm 1}, y^{\pm 1}]$ with Newton polygon in Δ that vanishes to order m at the point $e = (1,1)$. Such an $f(x,y)$ defines a curve C in $X = \mathrm{Bl}_e X_\Delta$ with self-intersection number

$$C \cdot C = 2\mathrm{Area}(\Delta) - m^2 \leq 0.$$

If Δ supports a negative curve with strictly negative self-intersection then the polynomial $f(x,y)$ is unique up to a constant multiple. This implies that Δ can contain at most $\binom{m+1}{2} + 1$ lattice points because vanishing at e imposes $\binom{m+1}{2}$ conditions.

Let us define a form of triangles that will appear in the proofs below. The triangles $IT(M,N)$ and $RT(M,N)$ are of this form.

Definition 4.2.1. We say that a triangle Δ is of the form $(\dagger)$ if it has vertices

$$(0,0), (b,0) \text{ and } (m,h),$$

where $m, h > 0$ are relatively prime integers, $0 < b < m$ is a rational number, and the slope of the right edge

$$K = \frac{h}{m-b}$$

is an integer.

The main goal of this section is to give a sufficient condition for such a triangle Δ to support a negative curve.

Proposition 4.2.2. *Let Δ be a triangle of the form $(\dagger)$ with area $\leq \frac{m^2}{2}$ and containing at least $\binom{m+1}{2} + 1$ lattice points. Then Δ supports an irreducible negative curve vanishing to order m at e.*

The triangle Δ clearly supports a polynomial $f(x,y)$ that vanishes to order m at e. We need to prove that this $f(x,y)$ is irreducible. Let us recall an irreducibility criterion proved in [12].

Lemma 4.2.3. *Let Δ be a triangle and $f(x,y)$ a polynomial supported in Δ. Suppose an edge of Δ intersects $\mathbb{Z}^2$ at its endpoints only and these endpoints lie in the support of $f(x,y)$. Then $f(x,y)$ is irreducible.*

The lemma is proved by showing that the Newton polygon of $f(x,y)$ cannot be written as the Minkowski sum of two smaller polygons.

In the case of the triangle Δ of the form $(\dagger)$, we wish to prove that the two vertices $(0,0)$ and (m,h) lie in the Newton polygon of $f(x,y)$. We will see below that the triangle Δ contains exactly $\binom{m+1}{2} + 1$ lattice points. Thus, if one of the vertices does not lie in the Newton polygon of $f(x,y)$, then we are in the exceptional situation where vanishing at e to order m imposes $\binom{m+1}{2}$ conditions on the same number of monomials. It follows that one of these conditions must be trivial. This can be stated in terms of lattice point interpolation:

Lemma 4.2.4. *Let S be a set of $\binom{m+1}{2}$ lattice points on the plane. Then, S supports a Laurent polynomial vanishing to order m at $e = (1, 1)$ if and only if there is a degree $m - 1$ curve interpolating all points in S.*

Proof. A polynomial $f(x, y)$ supported on S vanishes to order m at e if all partial derivatives up to order $m - 1$ vanish when applied to $f(x, y)$ and evaluated at e. The same is true if we replace partial derivatives with logarithmic partial derivatives $p(x\partial_x, y\partial_y) \in k[x\partial_x, y\partial_y]$. When the number of monomials is the same as the number of conditions given by derivatives, then a nontrivial solution $f(x, y)$ exists if and only if one condition is trivial, meaning some logarithmic partial derivative p vanishes on all monomials in S when evaluated at e. Now

$$p(x\partial_x, y\partial_y)(x^a y^b)|_{(x,y)=(1,1)} = p(a, b).$$

This p is a polynomial of degree at most $m - 1$ that vanishes at all lattice points in S. □

The previous lemma is well-known. See for example [2, 9, 17]. Let us use it to prove Proposition 4.2.2 with some additional assumptions.

Lemma 4.2.5. *Let Δ be a triangle of the form (†) with area $\leq \frac{m^2}{2}$ such that the number of lattice points in the $m + 1$ columns of Δ is $1, 1, 2, 3, \ldots, m$ (possibly permuted). Then Δ supports an irreducible negative curve vanishing to order m at e.*

Proof. Notice that the number of lattice points in Δ is $1 + 1 + 2 + \ldots + m = \binom{m+1}{2} + 1$. This implies that there exists a polynomial $f(x, y)$ supported in Δ and vanishing to order m at e. If the Newton polygon of $f(x, y)$ does not contain one of the vertices $(0, 0)$ or (m, h), then there must be a degree $m - 1$ curve through the remaining lattice points that lie in columns of size $1, 2, 3, \ldots, m$. This is not possible by Bezout's theorem: the curve must consist of vertical lines along the columns with $m, m - 1, \ldots, 2$ points, after which there is still one point left over. □

It remains to prove that a triangle as in Proposition 4.2.2 indeed contains the correct number of lattice points in its columns.

Consider Δ of the form (†). Let us divide the triangle Δ into two smaller triangles using the vertical line $x = b$. The two smaller triangles have a new common vertex

$$\left(b, b\frac{h}{m}\right).$$

To the right hand triangle we apply the shear transformation

$$(x, y) \mapsto (x, y - K(x - b)).$$

Let Δ^{sh} be the union of the left triangle and the sheared right triangle. Then Δ^{sh} is again a triangle with vertices

$$(0, 0), (m, 0) \text{ and } \left(b, b\frac{h}{m}\right).$$

The shear transformation maps lattice points to lattice points and preserves columns. Hence, Δ and Δ^{sh} contain the same number of lattice points column by column. See Figure 4.3.

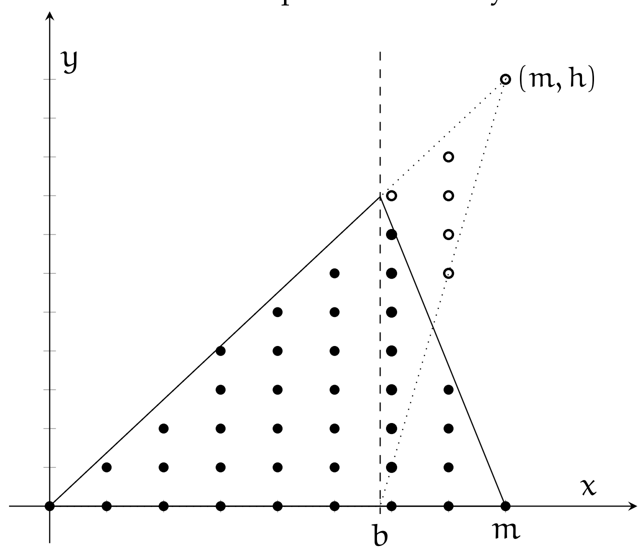


Figure 4.3: Comparison between Δ and Δ^{sh} for $RT(8,3)$ with $K = 5$.

Lemma 4.2.6. *Let $\tilde{\Delta}$ be the triangle with vertices $(0,0),(m,0),(b',m)$, where $m > 0$ is an integer and $0 < b' < m$ is rational. If the only lattice points on the boundary of $\tilde{\Delta}$ are on its lower edge, then the number of lattice points in the columns of $\tilde{\Delta}$ is $1,1,2,3,\dots,m$ (possibly permuted).*

Proof. By assumption, $\tilde{\Delta}$ has both width and height equal to m. Hence the slice of the triangle with y constant is an interval of length $m - y$. Since the only lattice points on the boundary of the triangle are the points on its lower edge, then the number of lattice points in the rows $y = 0,1,\dots,m-1$ is $m+1, m-1, m-2,\dots,1$.

Consider a column of lattice points in $\tilde{\Delta}$, $(c,0),(c,1),\dots,(c,d)$. Let $(c,d+1)$ be the lattice point above the column; we count these points. When we add a row with r lattice points on top of a row with $r+1$ lattice points, we create one such lattice point above a column. This gives the stated number of lattice points in the columns of $\tilde{\Delta}$. $\square$

Proof of Proposition 4.2.2. If the area of Δ is $\frac{m^2}{2}$, then $b\frac{h}{m} = m$ and the conclusion follows by Lemma 4.2.6 applied to Δ^{sh}. To deal with the case where the area is less than $\frac{m^2}{2}$, we compare Δ^{sh} with a larger triangle $\tilde{\Delta}^{sh}$ that has area $\frac{m^2}{2}$. Let $\tilde{\Delta}^{sh}$ have vertices

$$(0,0),(m,0),(b',m)$$

for some rational $0 < b' < m$ such that $\Delta^{sh} \subseteq \tilde{\Delta}^{sh}$. We can choose b' so that $\tilde{\Delta}^{sh}$ has no lattice points on its boundary except on the lower edge. By Lemma 4.2.6, $\tilde{\Delta}^{sh}$ has columns $1,1,2,\dots,m$. If we now know that Δ (and hence also Δ^{sh}) contains at least $\binom{m+1}{2}+1$ lattice points, then

$$\Delta^{sh} \cap \mathbb{Z}^2 = \tilde{\Delta}^{sh} \cap \mathbb{Z}^2$$

and hence both Δ^{sh} and Δ must also have columns $1, 1, 2, \dots, m$. The claim now follows from Lemma 4.2.5. □

4.3 Proofs of Theorem 4.1.1.A and Theorem 4.1.2

We start by proving that the triangles $IT(M, N)$ and $RT(M, N)$ support a negative curve. This follows from Proposition 4.2.2 once we know that the triangles contain enough lattice points because the triangles are of the form (†) defined at the beginning of Section 4.2.

Lemma 4.3.1. *The triangles* $IT(M, N)$ *and* $RT(M, N)$ *contain* $\binom{m+1}{2} + 1$ *lattice points, where* $m = M + N$ *in the integral case and* $m = M$ *in the rational case.*

Proof. We apply the Pick's Theorem to the integral triangle $\Delta = IT(M, N)$:

$$|\Delta \cap \mathbb{Z}^2| = \mathrm{Area}(\Delta) + \frac{P(\Delta)}{2} + 1,$$

where $P(\Delta)$ is the perimeter of Δ. The triangle has base M and height KN, hence

$$2\mathrm{Area}(\Delta) = KMN = (M + N)^2 - 1 = m^2 - 1.$$

The boundary of the triangle contains exactly one lattice point for every integral value of x between 0 and $M + N$, hence $P(\Delta) = M + N + 1 = m + 1$. This gives

$$|\Delta \cap \mathbb{Z}^2| = \frac{m^2 - 1}{2} + \frac{m + 1}{2} + 1 = \binom{m + 1}{2} + 1.$$

Let now $\Delta = RT(M, N)$. We will apply Pick's Theorem to the convex hull of the lattice points in Δ. Let us find the lowest lattice point on the right edge of the triangle. From equation (4.1.1) it follows that K divides $M + N \pm 1$. This implies that the lowest lattice point on the right edge of Δ has y-coordinate 1 or $K - 1$. In either case the convex hull of lattice points in Δ is a 4-gon with area

$$\mathrm{Area}(\Delta) - \frac{1}{2}\left(1 - \frac{1}{K}\right) = \frac{m^2 - 1}{2}.$$

Again, the boundary of Δ contains one lattice point for every integral value of x from 0 to M. Pick's Theorem now says that Δ contains $\binom{m+1}{2} + 1$ lattice points. □

This proves the first part of Theorem 4.1.1.A.

Next we study isomorphisms between these negative curves. Recall that two such curves are isomorphic if there exists an automorphism of the torus T mapping one curve to another. An automorphism of T is given by an integral affine linear automorphism of $\mathbb{Z}^2$. Two negative curves are then isomorphic only if such an integral affine linear automorphism maps the Newton polygon of one $f(x, y)$ to the Newton polygon of another.

Lemma 4.3.2. *The only integral affine linear isomorphisms between the triangles* $\mathrm{IT}(M,N)$ *and* $\mathrm{RT}(M,N)$ *are*

$$\mathrm{IT}(M,N) \cong \mathrm{IT}(\iota_0(M,N)),$$

$$\mathrm{RT}(M,N) \cong \mathrm{RT}(\iota_1(M,N)).$$

Proof. The first isomorphism is given by the transformation

$$(x,y) \mapsto (M+N-x, K(M-x)+y),$$

the second one by

$$(x,y) \mapsto (M-x, K(M-x)+y-(M+N)).$$

These two transformations map the left edge of one triangle to the left edge of the other and exchange the other two edges.

To prove that there are no other isomorphisms we consider the normal fan of the triangle. It has three maximal cones with integer multiplicities m_1, m_2, m_3. This set of multiplicities is preserved under an isomorphism. The multiplicities are NK, K and MK in the integral case and $M+N$, K and $MK-M-N$ in the rational case.

Clearly no integral triangle is isomorphic to a rational triangle. In the integral case the set $\{NK, K, MK\}$ determines K (the GCD of the triple) and the set $\{N, M\}$. Thus, only the triangles that differ by ι_0 have the same triple of multiplicities.

In the rational case, the multiplicity K corresponds to the unique non-integral vertex. If K is known then from the set $\{M+N, MK-(M+N)\}$ we recover M. The only triangles with the same K and M are the two that differ by ι_1. ☐

Let us now prove the second part of Theorem 4.1.1.A, showing that the negative curves are pairwise non-isomorphic. We will restrict to the curves in the statement of the theorem, namely those corresponding to $\mathrm{IT}(M,N)$ for $M \geq N > 0$ and those corresponding to $\mathrm{RT}(M,N)$ for $M > N > 1$. We will also restrict to $K \geq 4$, the case $K=3, M=N=1$ giving the unique negative curve that vanishes to order 2 at e.

To show that these curves are non-isomorphic it suffices to prove that the Newton polygons of the defining polynomials $f(x,y)$ are non-isomorphic. We show that the Newton polygon determines the triangle uniquely, hence by the previous lemma, no two Newton polygons can be isomorphic.

In Lemma 4.6.2 below we show that in the integral case the Newton polygon coincides with the triangle. This shows that no two negative curves supported in the integral triangles are isomorphic to each other or to a negative curve supported in a rational triangle.

For rational triangles the Newton polygons are harder to determine. In general they are not equal to the convex hull of lattice points in the triangle. What follows is a proof that the Newton polygon nevertheless uniquely determines the triangle.

In Lemma 4.6.3 below we show that in addition to the left edge of the triangle, the Newton polygon also contains edges along the other two sides of the triangle. Suppose now that the Newton polygon uniquely determines one of its edges, the one coinciding with the left edge of the triangle. Then, it determines the triangle by extending the two adjacent edges until they meet.

Consider the edge in the Newton polygon that coincides with the left edge of the triangle. We will write down properties of this edge that uniquely determine it among all edges of the Newton polygon. Consider the following properties for an edge in the Newton polygon of $f(x, y)$:

1. There exists a triangle that contains the Newton polygon and shares the same edge.

2. The lattice length of the edge is 1.

3. The coefficients of the two monomials in $f(x, y)$ corresponding to the vertices of the edge have the same absolute value.

4. The triangle constructed from the edge by extending the two adjacent sides must contain $\binom{m+1}{2} + 1$ lattice points for some m.

The Newton polygon corresponding to $RT(M, N)$ satisfies (1)-(4) with respect to the edge from $(0, 0)$ to $(M, M + N)$. Indeed, (1) and (2) are clear, (3) is shown in Section 4.6.2, and (4) is shown in Lemma 4.3.1. In an arbitrary convex polygon that is not a triangle there can be at most two edges satisfying property (1), and these edges have to be adjacent. Aside from the left hand side edge, the only other edge of the Newton polygon of the negative curve of $RT(M, N)$ that satisfies properties (1) and (2) is the edge from $(M, M + N)$ to $(M - 1, M + N - K)$. Indeed, the other possible edge from $(0, 0)$ to $(1, 0)$ does not satisfy property (1): the left edge of the triangle has larger slope than the line segment from $(1, 0)$ to $(M - 1, M + N - K)$.

Now consider the edge from $(M, M + N)$ to $(M - 1, M + N - K)$. Lemma 4.6.3 shows that $f(x, y)$ has the form $f(x, y) = \pm 1 + bx^{M-1}y^{M+NK} + \ldots$, where $b \neq \pm 1$ when $N > K - 2$; hence (3) is not satisfied. When $N = K - 2$ then Lemma 4.6.4 shows that the edge with slope K in the Newton polygon has length $K - 3$, so (2) is not satisfied when $K > 4$. The only case remaining to consider is $K = 4, M = 3$ and $N = 2$. The triangle that we get from this edge contains 9 lattice points; hence (4) is not satisfied. Then, properties (1)-(4) determine the left edge of the triangle in all cases.

This finishes the proof of Theorem 4.1.1.A.

4.3.1 Proof of Theorem 4.1.2.

It is easy to check that the three involutions map integral solutions of equation (4.1.1) to integral solutions and satisfy the given relation. It is also clear from the equation that there is no integral solution with $M < 0, N > 0$.

For $K = 3$ one can find that the only solution is $(M, N) = (1, 1)$. For $K \geq 4$, the solutions lie on the two branches of a hyperbola, one branch with $M > N$ and the other with $M < N$. The three involutions exchange the branches. In particular, starting from the branch where $M < N$ and applying ι_1, we decrease N and leave M the same. Given a positive integer solution (M, N), we apply ι_1 if $M < N$ and we apply ι_0 otherwise. A sequence of these involutions will keep M, N non-negative, and it will decrease the sum $M + N$ until $(M, N) = (0, 1)$.

4.4 Proof of Theorem 4.1.1.B

Let Δ be a triangle as in Theorem 4.1.1.B and let Z be the convex hull of its lattice points. Then Z is an integral polytope with top edge from $(0, 0)$ to (m, h) and at least two edges on the lower boundary. We will assume that Δ is a minimal triangle containing Z, by which we mean that among the sequence of edges on the lower boundary of Z the first and last lie on the edges on Δ.

Our main tool will be Pick's formula applied to Z:

$$C = A + \frac{P}{2} + 1,$$

where P is the lattice perimeter of Z, A is its area, and C is the number of lattice points in Z.

Consider the case where Δ is either $IT(M, N)$ or $RT(M, N)$. In the integral case Z is equal to Δ and its lower boundary consists of two edges with integral slopes 0 and K. In the rational case Z is a 4-gon. Its lower boundary consists of three edges with integral slopes 0, L, K, where $L = 1$ or $L = K - 1$, and the edge with slope L has lattice length 1. Let us prove that these properties of the lower edges of Z characterize the triangles $IT(M, N)$ and $RT(M, N)$.

Lemma 4.4.1. *Let Δ be a triangle as in Theorem 4.1.1.B and Z the convex hull of its lattice points. Assume that Δ is the minimal triangle containing Z.*

1. *If Z is a triangle with lower edges having integral slopes 0 and K, then Δ is equal to $IT(M, N)$ for some M, N satisfying equation (4.1.1).*

2. *If Z has three lower edges with integral slopes 0, L and K, where $L = 1$ or $L = K - 1$ and the edge with slope L has lattice length 1, then Δ is equal to $RT(M, N)$ for some M, N satisfying equation (4.1.1).*

In both cases of the lemma, the perimeter P of Z equals $m + 1$. Let us first apply Pick's formula to this situation.

Lemma 4.4.2. *If a lattice polygon Z contains at least $\binom{m+1}{2} + 1$ lattice points, has area $\leq \frac{m^2}{2}$ and perimeter $m + 1$, then the area of Z is $\frac{m^2-1}{2}$ and it contains $\binom{m+1}{2} + 1$ lattice points.*

Proof. Let the area of Z be $\frac{m^2-\varepsilon}{2}$ for some integer $\varepsilon \geq 0$. Pick's formula applied to Z gives

$$\binom{m+1}{2} + 1 \leq C = \frac{m^2 - \varepsilon + m + 1}{2} + 1 = \binom{m+1}{2} + 1 + \frac{1-\varepsilon}{2},$$

which simplifies to $\varepsilon \leq 1$. Notice that if $\varepsilon = 0$ then C is not an integer, hence $\varepsilon = 1$ is the only possibility. □

Proof of Lemma 4.4.1. In both cases equation (4.1.1) follows from the area of Z being $\frac{m^2-1}{2}$. Let us do the integral case only. Denote by M and N the lattice lengths of the two lower edges of Z, so that $m = M + N$. The height of Z is NK, hence twice its area is MNK. This must be equal to $(M+N)^2 - 1$, giving equation (4.1.1). A similar argument applies in the rational case. □

Let us now return to a general triangle as in Theorem 4.1.1.B. Notice that Z has perimeter $m+1$ if and only if it contains a lattice point on its lower boundary for every $x = 0, 1, \ldots, m$. This is equivalent to all lower edges having integral slopes.

Lemma 4.4.3. *Let Δ be a triangle as in Theorem 4.1.1.B. Then the perimeter P of Z is equal to $m+1$.*

Proof. We know that the perimeter of Z cannot be more than $m+1$ because its top boundary consists of one edge of lattice length 1.

By assumption, the area of Z is $\leq \frac{m^2}{2}$ and its number of lattice points is $\geq \binom{m+1}{2} + 1$. Pick's formula now gives that $P \geq m$.

Let us rule out the case $P = m$. In that case, again from Pick's formula, the area of Z is exactly $\frac{m^2}{2}$. Hence Z is an integral triangle. One of its lower edges must have integral slope, the other edge has rational slope. Using a linear transformation, we may assume that Z has vertices $(0,0)$, (m,h), and $(m-2,0)$. We get that twice the area of Z is $m^2 = (m-2)h$. This gives a contradiction to m and h being relatively prime. □

The previous lemma shows that the lower edges of Z have integral slopes. We can apply a shear transformation $(x,y) \mapsto (x, y+ax)$ so that the first of the lower edges has slope 0. We need to show that these slopes are as in the case of $IT(M,N)$ or $RT(M,N)$.

Lemma 4.4.4. *Let Δ be a triangle as in Theorem 4.1.1.B. Then, after a shear transformation, Z is either a triangle with lower edges having integer slopes 0 and K, or Z is a 4-gon with lower edges having integer slopes $0, L, K$, where $L = 1$ or $L = K-1$ and the edge with slope L has lattice length 1.*

Proof. Assume that Z is not a triangle, and let its lower edges have integer slopes, with the first edge being horizontal. Then Z cannot have more than 3 edges on its lower boundary. Otherwise the triangle Δ contains a lattice point on the x-axis that does not lie in Z. Thus, Z must be a 4-gon with lower edges having integer slopes $0, L, K$.

We know that Z has area $\frac{m^2-1}{2}$ and Δ has area $\leq \frac{m^2}{2}$. This implies that the edge with slope L has lattice length 1, and the area of the small triangle removed from Δ to obtain Z is at most $\frac{1}{2}$. This small triangle has base $1 - L/K$ and height L, giving the inequality

$$\left(1 - \frac{L}{K}\right) L \leq 1.$$

The only integer solutions $K > L > 0$ to this are $K > L = 1$, $L = K - 1 > 0$ and $L = 2, K = 4$. We need to rule out the last case. Here Z has vertices $(0,0)$, $(a,0)$, $(a+1,2)$ and $(m,h) = (a+1+b, 2+4b)$ for some integers $a, b > 0$. Twice the area of Δ is

$$m^2 = \left(a + \frac{1}{2}\right) h.$$

This equality is equivalent to $a = b$. However, in that case $m = 2a+1$ and $h = 4a+2$ are not relatively prime. □

Combining Lemma 4.4.4 with Lemma 4.4.1, we get that the triangle Δ has to be isomorphic to either $IT(M,N)$ or $RT(M,N)$ for some M, N. This finishes the proof of Theorem 4.1.1.B.

4.5 Mori Dream Spaces

In this section we will consider triangles Δ that support a negative curve C. We start with general triangles and then specialize to the case $IT(M,N)$, $RT(M,N)$.

4.5.1 General triangles.

Consider a triangle Δ of the form (†). We assume that Δ supports a negative curve C^0 that vanishes to order m at e. If C^0 is defined by a polynomial ξ, we further assume that the lattice points $(0,0)$ and (m,h) lie in the Newton polygon of ξ. We make Δ larger by increasing its base:

$$\Delta_{\alpha,\beta} : (-\alpha, 0), (b+\beta, 0), (m,h), \quad \alpha, \beta \geq 0.$$

Let $X = \mathrm{Bl}_e X_{\Delta_{\alpha,\beta}}$. We assume that $\alpha, \beta \geq 0$ are small enough so that the strict transform C of C^0 is a negative curve in X. This condition is equivalent to the area of $\Delta_{\alpha,\beta}$ being $\leq \frac{m^2}{2}$, which is the same as

$$\alpha + \beta \leq \frac{m^2}{h} - b.$$

We study curve classes in X. Let D_0 be in the class $H' - hE$ where H' corresponds to the triangle with vertices:

$$\Delta'_{\alpha,\beta} : (0,0), (m,0), \frac{m}{b+\alpha+\beta}(m+\alpha, h).$$

Lemma 4.5.1. $D_0 \cdot C = 0$.

Proof. Let the class of C be $H - mE$. Here H^2 equals twice the area of Δ, which is bh. When we scale the triangle Δ to Δ' with base of length b', then the corresponding classes multiply as

$$H \cdot H' = H \cdot \frac{b'}{b} H = b'h.$$

Now

$$C \cdot D_0 = (H - mE)(H' - hE) = H \cdot H' - mh = 0.$$

□

Cutkosky [8] has proved (see also [12]) that the variety $X = \mathrm{Bl}_e X_{\Delta_{\alpha,\beta}}$ is a MDS if and only if there exists a divisor D in the class μD_0 for some integer $\mu > 0$ such that $C \cap D = \emptyset$. This intersection property is equivalent to D not having C as a component, which can be checked by finding a vertex of the triangle $\Delta_{\alpha,\beta}$ that does not lie in the Newton polygon of the polynomial defining C, but the corresponding vertex of $\Delta'_{\alpha,\beta}$ lies in the Newton polygon of the polynomial defining D. Then C passes through the corresponding T-fixed point, but D does not.

Lemma 4.5.2. *When $\alpha = 0, \beta \geq 0$ then $X = \mathrm{Bl}_e X_{\Delta_{0,\beta}}$ is a MDS.*

Proof. The polynomial $x^m(1-y)^h$ is supported in $\Delta'_{0,\beta}$ and vanishes to order h at e. This defines the divisor D disjoint from C. □

Lemma 4.5.3. *Assume that the slope of the right edge of Δ is an integer K such that*

$$h\left(b + \frac{1}{K}\right) > m^2.$$

Then $X = \mathrm{Bl}_e X_{\Delta_{\alpha,\beta}}$ is not a MDS for any $\alpha, \beta > 0$ such that the curve C has non-positive self-intersection.

Proof. Suppose we have a divisor D in the class μD_0 that is disjoint from C. Let D be defined by a polynomial ζ supported in $\mu\Delta'_{\alpha,\beta}$. Then ζ is also supported in the larger triangle $\mu\Delta'_{0,0}$ where we have set $\alpha = \beta = 0$. The triangle $\mu\Delta'_{0,0}$ supports another polynomial $x^{\mu m}(1-y)^{\mu h}$. We will use the two polynomials to get a contradiction.

Since $D \cdot C = 0$, the sheaf $\mathcal{O}_X(D)$ restricts to the trivial sheaf on C. Hence any two global sections of the sheaf must be constant multiples of each other when restricted to C. Let ξ be the polynomial defining C. Then

$$\zeta \equiv c x^{\mu m}(1-y)^{\mu h} \mod \xi$$

for some constant $c \neq 0$. Let

$$\zeta = c x^{\mu m}(1-y)^{\mu h} + \xi g, \tag{4.5.1}$$

where g is a polynomial that vanishes at e to order at least $\mu h - m$ and is supported in the triangle Δ'' with base $\mu m - b$ and sides parallel to $\Delta_{0,0}$. However, since D was assumed to exist when $\beta > 0$, the right edge of the triangle Δ'' should not intersect the Newton polygon of g. Thus, g is in fact supported in a smaller triangle, still with sides parallel to $\Delta_{0,0}$. Since the right side of $\Delta_{0,0}$ has integer slope K, we can take the smaller triangle by shortening the base of Δ'' by $1/K$. In summary, g is supported in the triangle with base $[0, \mu m - b - 1/K]$ and must vanish to order at least $\mu h - m$ at e.

Consider the terms in Equation (4.5.1) lying on the left edge of $\Delta_{0,0}$. The Newton polygon of ζ intersects the left edge only at $(0,0)$ because $\alpha > 0$. The Newton polygon of $x^{\mu m}(1-y)^{\mu h}$ intersect the left edge at $\mu(m,h)$ only. The Newton polygon of ξ has two lattice points on the left edge, $(0,0)$ and (m,h). Equation (4.5.1) now implies that g is not divisible by ξ. (By the same reason why a polynomial $1 - ax^{\mu}$ is not divisible by $(1 - bx)^2$ when $a, b \neq 0$ and the characteristic is 0.)

To get a contradiction to the existence of D, we now check that $\overline{D} \cdot C < 0$ in $\mathrm{Bl}_e X_{\Delta_{0,0}}$, where $\overline{D}$ is defined by the polynomial g. This implies that ξ must divide g, which is impossible. We have

$$\overline{D} \cdot C \leq \left(\mu m - b - \frac{1}{K}\right) h - (\mu h - m) m = -\left(b + \frac{1}{K}\right) h + m^2.$$

The last quantity is negative by assumption. □

4.5.2 The triangles $\mathrm{IT}(M, N)$ and $\mathrm{RT}(M, N)$.

We now specialize to the case of triangles described in Theorem 4.1.1.A. Let $\xi^{int}_{M,N}$ and $\xi^{rat}_{M,N}$ be the polynomials defining negative curves in the two families of triangles, normalized to have constant term equal to 1.

The triangles $\mathrm{IT}(M, N)$ and $\mathrm{RT}(M, N)$ are of the form (†). In particular, the varieties X are MDS when $\alpha = 0$. Let us prove the same for $\beta = 0$.

Lemma 4.5.4. *Let Δ be either $\mathrm{IT}(M, N)$ with $N > 0$ or $\mathrm{RT}(M, N)$ with $M > 0$, and let X be constructed by choosing $\alpha, \beta \geq 0$. If $\beta = 0$ then X is a MDS.*

Proof. Consider the class $[D_0]$ defined in the previous subsection. In the integral case it has the form $H' - KNE$, where H' corresponds to the triangle with base $[0, M + N]$. The polynomial

$$\left(\xi^{rat}_{N,(K-1)N-(M+N)}\right)^K = \left(\xi^{rat}_{\iota_1\iota_0(M,N)}\right)^K$$

defines a divisor D in the class $[D_0]$, disjoint from C.

In the rational case the class $[D_0]$ has the form $H' - (M + N)E$, where H' corresponds to the triangle with base M. The polynomial $\xi^{int}_{M,N}$ defines a divisor D in the class $[D_0]$ that is disjoint from C. □

Let us now check when Lemma 4.5.3 applies.

Lemma 4.5.5. *Let X be defined by choosing $\alpha, \beta > 0$. Then X is not a MDS when $N > 1$ in the integral case and $M + N > 1$ in the rational case.*

Proof. In the integral case $m = M + N$, $h = NK$ and $b = M$. The inequality in Lemma 4.5.3 becomes

$$NK\left(M + \frac{1}{K}\right) > (M + N)^2 = MNK + 1.$$

This is equivalent to $N > 1$.

In the rational case $m = M$, $h = M + N$ and $b = M - \frac{M+N}{K}$. The inequality now is

$$(M + N)\left(M - \frac{M + N - 1}{K}\right) > M^2.$$

This is equivalent to $M + N > 1$. □

This finishes the proof of Theorem 4.1.4.

4.6 Recurrence relations

In this section we give explicit computations of the pairs (M, N) satisfying equation (4.1.1). We then show how to find the polynomials $\xi^{int}_{M,N}$ and $\xi^{rat}_{M,N}$ inductively.

We will consider $K \geq 4$ and pairs $M > N$. The other pairs can be obtained by switching M and N. Let τ be the transformation $\tau(M, N) = \iota_1\iota_0(M, N) = (N, (K-1)N - (M+N))$. Recall from Theorem 4.1.2 that, for a fixed K, all (M, N) can be obtained by starting with $(1, 0)$ and applying ι_1, ι_0 alternately. Hence all solutions $M > N$ can be reduced to $(1, 0)$ by applying τ:

$$(1,0) \overset{\tau}{\longleftarrow\!\shortmid} (K-2,1) \overset{\tau}{\longleftarrow\!\shortmid} ((K-2)^2-1, K-2) \overset{\tau}{\longleftarrow\!\shortmid} \dots.$$

4.6.1 Finding (M, N).

When $K = 4$ then all solutions $M > N$ have the form $M = N + 1$. We fix $K > 4$. Let

$$(M_0, N_0) = (1, 0), \quad (M_n, N_n) = \tau^{-n}(M_0, N_0), \quad n > 0.$$

This sequence of pairs can be written in terms of a sequence of numbers:

$$(M_n, N_n) = (F_{n+1}, F_n),$$

Where $F_0 = 0, F_1 = 1$ and

$$F_{n+2} = (K-2)F_{n+1} - F_n, \quad n \geq 0.$$

We can solve for F_n using eigenvalues. Let

$$\lambda_\pm = \frac{K - 2 \pm \sqrt{(K-2)^2 - 4}}{2}.$$

Then, for $n \geq 0$,

$$F_n = \frac{\lambda_+^n - \lambda_-^n}{\lambda_+ - \lambda_-}.$$

More concretely, for $n > 0$,

$$F_n = \sum_{i=0}^{n-1} (-1)^i \binom{2n-1-i}{i} K^{n-1-i}.$$

From this, one can easily write polynomial expressions for the vertices of all the triangles $IT(M, N)$ and $RT(M, N)$.

Another way to find M_n, N_n is to use the continued fractions expression of λ_+:

$$\lambda_+ = (K-2) - \frac{1}{(K-2) - \frac{1}{(K-2)-\dots}}.$$

Then M_n/N_n is the truncation of this continued fraction. For example,

$$\frac{M_2}{N_2} = (K-2) - \frac{1}{(K-2)} = \frac{(K-2)^2 - 1}{K-2}.$$

4.6.2 Finding $\xi^{int}_{M,N}$ and $\xi^{rat}_{M,N}$.

Recall that these polynomials are normalized to have constant term 1. Let us denote by $\varepsilon^{int}_{M,N}$ and $\varepsilon^{rat}_{M,N}$ the coefficient of the highest degree term in $\xi^{int}_{M,N}$ and $\xi^{rat}_{M,N}$, respectively. We will see below that these coefficients are equal to ± 1.

Lemma 4.6.1. *The polynomials* $\xi^{int}_{M,N}$ *and* $\xi^{rat}_{M,N}$ *satisfy the relations*

$$\xi^{int}_{M,N} = \xi^{rat}_{M,N}\xi^{rat}_{\tau(M,N)} - \varepsilon^{rat}_{M,N} x^M (y-1)^{M+N},$$
$$\left(\xi^{rat}_{\tau(M,N)}\right)^K = \xi^{int}_{M,N}\xi^{int}_{\tau(M,N)} - \varepsilon^{int}_{M,N} x^{M+N} (y-1)^{KN}.$$

The first equality holds when $M > 0$*, the second one when* $N > 0$*.*

Proof. Consider the rational triangle $\Delta = RT(M,N)$, with $M > 0$. We let $\alpha = \beta = 0$ and consider the class $[D_0] = H' - (M+N)E$, where H' corresponds to the triangle with base $[0, M]$. We know two divisors in the class $[D_0]$, defined by $x^M(1-y)^{M+N}$ from the case $\alpha = 0$ and $\xi^{int}_{M,N}$ from the case $\beta = 0$. These two polynomials must be constant multiples of each other modulo $\xi^{rat}_{M,N}$. Write

$$\xi^{int}_{M,N} = \xi^{rat}_{M,N} g - \varepsilon^{rat}_{M,N} x^M (y-1)^{M+N}$$

for some g with constant term 1, supported in $RT(N, (K-1)N - (M+N)) = RT(\tau(M,N))$ and vanishing to order at least N at e. There is only one such polynomial, $g = \xi^{rat}_{\tau(M,N)}$.

Now consider the integral triangle $\Delta = IT(M,N)$, with $N > 0$. Again let $\alpha = \beta = 0$ and consider the class $[D_0] = H' - KNE$, where H' corresponds to the triangle with base $[0, M+N]$. There are two divisors in the class $[D_0]$, defined by polynomials $x^{M+N}(1-y)^{KN}$ and $\left(\xi^{rat}_{\tau(M,N)}\right)^K$. Write

$$\left(\xi^{rat}_{\tau(M,N)}\right)^K = \xi^{int}_{M,N} g - \varepsilon^{int}_{M,N} x^{M+N} (y-1)^{KN}$$

where g is a polynomial with constant term 1, supported in $IT(\tau(M,N))$, and vanishing to order at least $KN - (M+N)$. Again, g has to be equal to $\xi^{int}_{\tau(M,N)}$. □

One can use the two formulas to compute the polynomials $\xi_{M,N}$ inductively for all $M > N$. Assume that we know the coefficients $\varepsilon_{M,N}$. Start with

$$\xi^{int}_{1,0} = 1 - x, \qquad \xi^{rat}_{1,0} = 1 - xy.$$

Then use the second equation to solve for $\xi^{int}_{M,N}$ from $\xi^{int}_{\tau(M,N)}$, $\xi^{rat}_{\tau(M,N)}$, and the first equation to find $\xi^{rat}_{M,N}$ from $\xi^{int}_{M,N}$, $\xi^{rat}_{\tau(M,N)}$.

One can also determine the coefficients $\varepsilon_{M,N}$ explicitly in the case $M > N$. The two equations give us the relations

$$\varepsilon^{int}_{M,N} = \varepsilon^{rat}_{M,N}\varepsilon^{rat}_{\tau(M,N)},$$
$$\left(\varepsilon^{rat}_{\tau(M,N)}\right)^K = \varepsilon^{int}_{M,N}\varepsilon^{int}_{\tau(M,N)}.$$

Starting with

$$\varepsilon^{\mathrm{int}}_{1,0} = \varepsilon^{\mathrm{rat}}_{1,0} = -1,$$

one can compute these coefficients inductively. Let $(M_n, N_n) = \tau^{-n}(1,0)$. Then for K even, $\varepsilon^{\mathrm{int}}_{M_n,N_n} = -1$ and $\varepsilon^{\mathrm{rat}}_{M_n,N_n} = (-1)^{n+1}$. When K is odd then

$$\varepsilon^{\mathrm{int}}_{M_n,N_n} = \begin{cases} 1 & \text{if } n \equiv 1 \mod 3, \\ -1 & \text{otherwise.} \end{cases} \qquad \varepsilon^{\mathrm{rat}}_{M_n,N_n} = \begin{cases} 1 & \text{if } n \equiv 2 \mod 3, \\ -1 & \text{otherwise.} \end{cases}$$

4.6.3 Newton polygons of $\xi^{\mathrm{int}}_{M,N}$ and $\xi^{\mathrm{rat}}_{M,N}$.

When proving in Section 4.3 that the negative curves in Theorem 4.1.1.A are pairwise not isomorphic, we used some facts about the Newton polygons of their defining equations. Let us prove these facts.

Lemma 4.6.2. *The Newton polygon of* $\xi^{\mathrm{int}}_{M,N}$ *is* $\mathrm{IT}(M,N)$.

Proof. By the discussion above it's enough to show that the monomial x^M appears in $\xi^{\mathrm{int}}_{M,N}$ with a nonzero coefficient. The bases of the triangles $\mathrm{RT}(M,N)$ and $\mathrm{RT}(\tau(M,N))$ add up exactly to M. These two triangles don't have a monomial in their bottom right vertices, so the product of these two polynomials cannot contain the monomial x^M. Then, by Lemma 4.6.1, the only contribution to this monomial in $\xi^{\mathrm{int}}_{M,N}$ comes from $x^M(1-y)^{M+N}$. □

The Newton polygon of $\xi^{\mathrm{rat}}_{M,N}$ may be smaller than the convex hull of lattice points in $\mathrm{RT}(M,N)$. For example, the Newton polygon of $\xi^{\mathrm{rat}}_{8,3}$ does not contain the point $(5,0)$. We know that the Newton polygons contain the left edge of the corresponding triangle. We want to show that the two adjacent edges lie on the other two edges of the triangle. Let us write the polynomials as

$$\xi^{\mathrm{rat}}_{M,N} = 1 + a^{\mathrm{rat}}_{M,N}x + \ldots + b^{\mathrm{rat}}_{M,N}x^{M-1}y^{M+N-K} \pm x^M y^{M+N},$$
$$\xi^{\mathrm{int}}_{M,N} = 1 + a^{\mathrm{int}}_{M,N}x + \ldots + b^{\mathrm{int}}_{M,N}x^{M+N-1}y^{NK-K} \pm x^{M+N} y^{NK}.$$

Lemma 4.6.3. *Let* $(M_n, N_n) = \tau^{-n}(1,0)$.

1. *The coefficients* $a^{\mathrm{int}}_{M_n,N_n}$, $b^{\mathrm{int}}_{M_n,N_n}$ *are nonzero when* $n > 0$. *They are not equal to* ± 1 *when* $n > 1$.
2. *The coefficients* $a^{\mathrm{rat}}_{M_n,N_n}$, $b^{\mathrm{rat}}_{M_n,N_n}$ *are nonzero when* $n > 1$. *They are not equal to* ± 1 *when* $n > 2$.

Proof. Let us start with the coefficients $a^{\mathrm{rat}}_{M,N}$ and $a^{\mathrm{int}}_{M,N}$. The linear terms of the equations in Lemma 4.6.1 give the relations

$$a^{\mathrm{int}}_{M,N} = a^{\mathrm{rat}}_{M,N} + a^{\mathrm{rat}}_{\tau(M,N)},$$
$$K a^{\mathrm{rat}}_{\tau(M,N)} = a^{\mathrm{int}}_{M,N} + a^{\mathrm{int}}_{\tau(M,N)}.$$

Starting with $a^{int}_{1,0} = -1$ and $a^{rat}_{1,0} = 0$, one can inductively find all these coefficients.

The two equations above can be written in the form

$$a^{rat}_{M,N} = (K-1)a^{rat}_{\tau(M,N)} - a^{int}_{\tau(M,N)},$$
$$a^{int}_{M,N} = Ka^{rat}_{\tau(M,N)} - a^{int}_{\tau(M,N)}.$$

Let us take new variables $x_n = a^{rat}_{M_n,N_n}$, $y_n = a^{int}_{M_n,N_n} - a^{rat}_{M_n,N_n}$, where $(M_n, N_n) = \tau^{-n}(1,0)$. Then the equations have the form

$$x_{n+1} = (K-2)x_n - y_n,$$
$$y_{n+1} = x_n,$$

with initial condition $x_0 = 0$, $y_0 = -1$. This is the same recurrence relation that we saw above when computing (M, N). The solutions are $x_n = F_n$, $y_n = F_{n-1}$. Going back to the original variables,

$$a^{rat}_{M_n,N_n} = F_n, \quad a^{int}_{M_n,N_n} = F_n + F_{n-1}.$$

From $F_n > 0$ when $n > 0$ and $F_n > 1$ when $n > 1$ we get the statements about $a^{rat}_{M_n,N_n}$ and $a^{int}_{M_n,N_n}$.

For the coefficients $b^{rat}_{M,N}$ and $b^{int}_{M,N}$ we can use the same argument as above. It is best to normalize the polynomials $\xi^{int}_{M,N}$ and $\xi^{rat}_{M,N}$ so that their highest degree terms have coefficient 1. Then $b^{rat}_{M,N}$, $b^{int}_{M,N}$ satisfy the same recurrence relations as $a^{rat}_{M,N}$, $a^{int}_{M,N}$, but with different initial values. We need to make sure that the last terms in the equations of Lemma 4.6.1 do not affect the relations. This is true when $(M, N) = (M_n, N_n)$ with $n \geq 2$. Thus, given $b^{rat}_{M_1,N_1}$ and $b^{int}_{M_1,N_1}$, we can compute all these numbers for $n \geq 2$. We have $b^{rat}_{M_1,N_1} = 0$ because the right edge of the triangle contains only one lattice point. To find $b^{int}_{M_1,N_1}$, we apply the first equation in Lemma 4.6.1 in the case $(M, N) = (M_1, N_1)$, taking into account the last term of the equation. Since we normalized $\xi^{int}_{M,N}$ so that its highest degree coefficient is 1, we get

$$b^{int}_{M_1,N_1} = -\varepsilon^{int}_{M_1,N_1}\varepsilon^{rat}_{M_1,N_1}(-1)^{M_1+N_1} = -\varepsilon^{rat}_{M_0,N_0}(-1)^{K-1} = (-1)^{K-1}.$$

The same computation as above now gives $b^{rat}_{M_1,N_1} = 0, b^{int}_{M_1,N_1} = (-1)^{K-1}$,

$$b^{rat}_{M_n,N_n} = (-1)^K F_{n-1}, \quad b^{int}_{M_n,N_n} = (-1)^K(F_{n-1} + F_{n-2}), \quad n \geq 2.$$

The statements about $b^{rat}_{M_n,N_n}$ and $b^{int}_{M_n,N_n}$ now follow from the properties of F_n. □

The previous lemma shows that the Newton polygon of $\xi^{rat}_{M_n,N_n}$ when $n > 1$ contains the left edge of the corresponding triangle, and in addition edges of nonzero length along the other two sides of the triangle. For $n = 2$ we can compute the lattice length of the edge with slope K:

Lemma 4.6.4. *The Newton polygon of $\xi^{rat}_{M_2,N_2}$ has an edge of lattice length $K-3$ with vertices (M_2, M_2+N_2) and $(M_2-(K-3), 1)$.*

Proof. We compute $\xi^{\mathrm{rat}}_{M_2,N_2}$ using the equations in Lemma 4.6.1. Let us change scaling and coordinates so that the top vertex of each triangle corresponds to the constant term 1 in all polynomials ξ, and the next lattice point on the edge with slope K corresponds to monomial z. Then, working modulo all monomials that do not lie on right edge of the triangle, we find:

$$\xi^{\mathrm{int}}_{M_1,N_1} \equiv \frac{(\xi^{\mathrm{rat}}_{M_0,N_0})^K \pm x^{M_1+N_1}}{\xi^{\mathrm{int}}_{M_0,N_0}} \equiv \frac{1 \pm z}{1},$$

$$\xi^{\mathrm{int}}_{M_2,N_2} \equiv \frac{(\xi^{\mathrm{rat}}_{M_1,N_1})^K \pm x^{M_2+N_2}}{\xi^{\mathrm{int}}_{M_1,N_1}} \equiv \frac{1 \pm z^{K-1}}{1 \pm z} \equiv 1 \pm z \pm z^2 \pm \ldots \pm z^{K-2},$$

$$\xi^{\mathrm{rat}}_{M_2,N_2} \equiv \frac{\xi^{\mathrm{int}}_{M_2,N_2} \pm x^{M_2}}{\xi^{\mathrm{rat}}_{M_1,N_1}} \equiv \frac{\xi^{\mathrm{int}}_{M_2,N_2} \pm z^{K-2}}{1} \equiv 1 \pm z \pm z^2 \pm \ldots \pm z^{K-3}.$$

Notice that in the numerator on the last line the term z^{K-2} must cancel as it is not supported in the triangle. □

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