DONALDSON–THOMAS THEORY OF QUANTUM FERMAT QUINTIC THREEFOLDS

by

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Donaldson–Thomas theory of quantum Fermat quintic threefolds

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Abstract

In this thesis, we study non-commutative projective schemes whose associated graded algebras are finite over their centers. We construct symmetric obstruction theories for their moduli spaces of stable sheaves in the Calabi–Yau-3 case. This allows us to define Donaldson–Thomas (DT) type deformation invariants.

As an application, we study the quantum Fermat quintic threefold which is the quintic threefold in a quantum projective space. We give an explicit description of its local models in terms of quivers with potential. We then give a full computation of its degree zero DT invariants.

Lay Summary

Enumerative geometry is a study of counting numbers of geometric objects. One of the important subjects is Donaldson–Thomas theory, which deals with counting curves on certain (Calabi–Yau) three-dimensional spaces. The most well-known example is that there are precisely 2875 lines on a general quintic threefold.

Our goal is to develop and study Donaldson–Thomas theory for "non-commutative" spaces. Classically, algebraic geometry study zeros of multivariate polynomials, and spaces are considered to be "commutative" since variables of a polynomial are commutative (i.e., $xy = yx$). What we call non-commutative spaces are zeros of "quantized" polynomials, where variables only commute up to a scalar (i.e., $xy = qyx$ for some $q \in \mathbb{C}^*$).

In this thesis, we focus on the simplest and only known example of non-commutative Calabi–Yau space, called the quantum Fermat quintic threefold. We define Donaldson–Thomas theory on this space and give some explicit computation.

Preface

This thesis is original, unpublished, independent work of the author Yu-Hsiang Liu, with the guidance of the author's advisor Prof. Kai Behrend.

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Acknowledgments

First I would like to express my deepest gratitude to my advisor Kai Behrend for his guidance and support. Without his profound mathematical insight, this thesis would not have been possible.

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Lastly, I would like to thank my parents for their endless support throughout my life.

Chapter 1

Introduction

Donaldson–Thomas (DT) invariants, first introduced by Thomas [33], are integer-valued deformation invariants on a compact Calabi–Yau 3-fold. They provide a virtual count of curves embedded into a Calabi–Yau 3-fold, and are conjecturally equivalent to other enumerative invariants, such as Gromov–Witten invariants [25], Pandharipande–Thomas invariants [29], and Gopakumar–Vafa invariants [24]. All of which are closely related to BPS invariants, which are motivated by counting BPS states in the string theory.

DT invariants were originally defined by constructing virtual fundamental classes on moduli spaces of stable sheaves over a 3-fold. One obtains a deformation invariant by integrating the virtual fundamental class whenever the moduli space is proper. Later, Behrend [5] showed that these DT invariants can be expressed as the Euler characteristics of the moduli spaces weighted by the Behrend constructible function. This discovery not only gives a definition of DT invariants for non-compact Calabi–Yau 3-fold, it also provides a motivic point of view of DT invariants, which has been further developed by many authors.

In [19], Joyce and Song used Hall algebra techniques to define generalized DT invariants in the presence of strictly semistable objects. Around the same time, Kontsevich and Soibelman [21] proposed a refinement of DT invariants, called motivic DT invariants, which are conjecturally defined for any Calabi–Yau-3 (CY3) category. One main class of examples is given by quivers with potential, which in some sense are local models for any CY3 category. More

importantly, the technical difficulties in defining motivic DT invariants disappear in this situation. For these reasons, quivers with potential have been a central object in the study of non-commutative DT theory. Motivic DT invariants of various examples have been computed ([27], [13], [11], ...). They also have been studied further within other contexts (such as [28] and [12]).

The motivation of this thesis begins with a special example of CY3 category called the *quantum Fermat quintic threefold* introduced in [20]. Roughly speaking, the quantum Fermat quintic threefold is the Fermat quintic hypersurface in the quantum projective 4-space, using the language of non-commutative projective schemes developed by Artin and Zhang [2]. The quantum Fermat quintic threefold has two crucial features. First, it is truly non-commutative, as it is not equivalent to any Calabi–Yau 3-fold. Second, it is projective, contrary to quivers with potential which are non-commutative analogues of local (affine) Calabi–Yau 3-folds. Here by "projective", we mean its moduli spaces are expected to projective. This allows us to define DT invariants via integrating the virtual fundamental class, which is essential for the deformation invariance of DT invariants.

To this end, the quantum Fermat quintic threefold and, more generally, non-commutative Calabi–Yau projective schemes should provide an interesting class of examples in non-commutative DT theory. However, there are some difficulties. The existence of moduli spaces of semistable sheaves on non-commutative projective schemes is unknown. Even in the case where the moduli spaces have been constructed (for example, Artin and Zhang proved the representability of Hilbert schemes [3]), the projectivity remains open in general. At the same time, the quantum Fermat quintic threefold is the only known example of (non-trivial) non-commutative Calabi–Yau projective schemes. Therefore we will restrict ourselves to this particular example.

In this thesis, we will first define DT invariants of the quantum Fermat quintic threefold. Then we will give an explicit description of its local models, which is given by certain quivers with potential. Finally, we will use these local models to compute of degree zero DT invariants on the quantum Fermat quintic threefold.

Overview of the thesis

In chapter 2, we go over various background material and set up some notation that will be used throughout the whole thesis.

In chapter 3, we start with a brief introduction to non-commutative projective schemes. The key observation is that if a graded algebra A is finite over its center, then A is naturally associated to a smooth projective variety X with a coherent sheaf $\mathcal{A}$ of non-commutative $\mathcal{O}_X$-algebras on X, and the non-commutative projective scheme $\mathsf{qgr}(A)$ defined by A is equivalent to the category $\mathrm{Coh}(\mathcal{A})$ of coherent $\mathcal{A}$-modules on X. Moduli spaces of $\mathcal{A}$-modules on X have been studied by Simpson [31] in the context of Higgs bundles. We also study the general properties of the category $\mathrm{Coh}(\mathcal{A})$. In particular, we give an alternative proof that the quantum Fermat quintic threefold is Calabi–Yau.

In chapter 4, we consider a general pair $(X, \mathcal{A})$ as above and the Simpson moduli space M of stable $\mathcal{A}$-modules on X. We first follow the method in [17] to construct an obstruction theory

$$\mathbb{E} := \Big(R\pi_{M*}R\mathcal{H}\mathrm{om}_{\mathcal{A}_M}(\mathcal{F}, \mathcal{F})\Big)^{\vee}[-1] \to \mathbb{L}_M$$

for the moduli space M. Then we show that if the category $\mathrm{Coh}(\mathcal{A})$ is CY3, the truncated complex $\tau^{[-1,0]}\mathbb{E}$ is a symmetric obstruction theory for M. Furthermore, under suitable conditions, the Hilbert scheme $\mathrm{Hilb}^h(\mathcal{A})$ can be embedded into M as an open subscheme. This allows us to define DT invariants via integrating the virtual fundamental class on $\mathrm{Hilb}^h(\mathcal{A})$, which equals to the weighted Euler characteristic:

$$\mathrm{DT}^h(\mathcal{A}) = \int_{[\mathrm{Hilb}^h(\mathcal{A})]_{\mathrm{vir}}} 1 = \chi_{\mathrm{vir}}\big(\mathrm{Hilb}^h(\mathcal{A})\big).$$

We will focus on Hilbert schemes of points, and denote by

$$Z^{\mathcal{A}}(t) = \sum_{n=0}^{\infty} \chi_{\mathrm{vir}}\big(\mathrm{Hilb}^n(\mathcal{A})\big)\, t^n$$

the generating function of degree zero DT invariants on the quantum Fermat

quintic threefold.

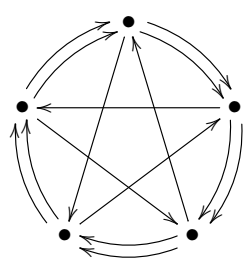
In chapter 5, we give an explicit description of local models of the quantum Fermat quintic threefold $(X, \mathcal{A})$.

Theorem A. There exist a stratification $X = X_{(0)} \coprod \dots \coprod X_{(3)}$ of X and coherent sheaves $\mathcal{J}_{(i)}$ of non-commutative algebras on $\mathbb{C}^3$ such that for any point $p \in X_{(i)}$, there is an ***analytic*** local chart $U \to \mathbb{C}^3$ of p with a (non-unique) isomorphism

$$\mathcal{A}|_U \cong \mathcal{J}_{(i)}|_U$$

of sheaves of non-commutative algebras. These sheaves $\mathcal{J}_{(i)}$'s of algebras are (up to Morita equivalence) Jacobi algebras of quivers with potential.

More specifically, $\mathcal{J}_{(i)}$'s are (again, up to Morita equivalence) just copies of $\mathbb{C}[x, y, z]$ for $i \neq 0$, whose DT invariants are well-studied. The Jacobi algebra $\mathcal{J}_{(0)}$ is defined by the quiver Q



with certain potential W. Its DT invariants will be discussed later.

In chapter 6, we use the local models $\mathcal{J}_{(i)}$'s to compute the generating function $Z^{\mathcal{A}}(t)$. The idea is that Hilbert schemes of points can be stratified with respect to the stratification of X.

Theorem B. We have

$$Z^{\mathcal{A}}(t) = \prod_{i=0}^{3} \left(\sum_{n=0}^{\infty} \chi_{\mathrm{vir}}\big(\mathrm{Hilb}^n(\mathcal{J}_{(i)}), \mathrm{Hilb}^n(\mathcal{J}_{(i)})_0\big)\, t^n \right),$$

where $\mathrm{Hilb}^n(\mathcal{J}_{(i)})_0$ can be regarded as an analogue of punctual Hilbert scheme of points.

Combing with the known result on DT invariants of $\mathbb{C}^3$, it eventually leads to

$$Z^{\mathcal{A}}(t) = Z^{Q,W}(t)^{10} \cdot M(-t^5)^{-50},$$

where $Z^{Q,W}(t)$ is the generating function of DT invariants of the quiver (Q, W) with potential, and $M(t)$ is the MacMahon function.

In chapter 7, we give a computation of the DT invariants $Z^{Q,W}(t)$. One might notice that our quiver Q is the McKay quiver of the μ_5-action on $\mathbb{C}^3$ with weight $(1, 1, 3)$, which is associated to an orbifold $[\mathbb{C}^3/\mu_5]$. A computation of DT invariants on an orbifold $[\mathbb{C}^3/\mu_5]$ was given in [9] using colored plane partitions [35]. However, our DT invariants $Z^{Q,W}$ use a different framing vector (stability condition) than the orbifold. We introduce the notion of *Q-multi-colored plane partitions* associated to a quiver Q. Each Q-multi-colored plane partition has an associated dimension vector $\mathbf{d}$, and we denote by $n_{\mathbf{d}}(Q)$ the number of Q-multi-colored plane partitions.

Theorem C. We have

$$Z^{Q,W}(t) = \sum_{n=1}^{\infty} \left(\sum_{|\mathbf{d}|=n} (-1)^{|\mathbf{d}|+\langle \mathbf{d},\mathbf{d}\rangle_Q} n_{\mathbf{d}}(Q) \right) t^n,$$

and the numbers $n_{\mathbf{d}}(Q)$ can be computed from $\mu_5(1, 1, 3)$-colored plane partitions.

Unfortunately, we do not have a closed formula for the generating function $Z^{Q,W}$, which is probably expected since there is still no closed formula for the DT invariants on the orbifold $[\mathbb{C}^3/\mu_5]$.

In chapter 8, we will discuss several subjects related to our work, and propose future research directions.

Notations

In this thesis, we work over the field $\mathbb{C}$ of complex numbers. All schemes or algebras are separated and noetherian over $\mathbb{C}$. All (sheaves of) algebras are associative and unital.

By "non-commutative", we mean not necessarily commutative, and we assume that non-commutative rings are both left and right noetherian. For a (sheaf of) non-commutative ring A, an A-module is always a left A-module. We will use refer right A-modules to A^{op}-modules. All rings without specified non-commutative are commutative.

We are particularly interested in a special class of non-commutative rings, ***quantum polynomial rings***. They are polynomial rings with variables only commute up to a non-zero scalar. We will use the notation

$$\mathbb{C}\langle x_1, \ldots, x_n \rangle_{(q_{ij})} := \mathbb{C}\langle x_1, \ldots, x_n \rangle \big/ \left(x_i x_j - q_{ij} x_j x_i \right),$$

where $q_{ij} \in \mathbb{C}^*$ and $q_{ii} = q_{ij} q_{ji} = 1$ for all i, j.

For any scheme X of finite type (over $\mathbb{C}$), we write $\chi(X)$ for the ***topological*** Euler characteristic of X with ***analytic*** topology. The topological Euler characteristic has the excision property: let $Z \subset X$ be an ***algebraic*** closed subscheme and U its open complement. Then $\chi(X) = \chi(Z) + \chi(U)$.

We will study various of ***generating functions***. We will use the notation $Z(t_1, \ldots, t_n)$, which is an element in the formal power series $\mathbb{Z}[\![t_1, \ldots, t_n]\!]$. The constant term $Z(0, \ldots, 0)$ will always be 1, hence divisions between generating functions make sense.

Chapter 2

Preliminaries

In this chapter, we go over various background material and set up some notation that will be used throughout the whole thesis. All results in this chapter are taken from the literature.

2.1 Symmetric obstruction theory and the Behrend function

Let X be a scheme of finite type. We consider $\mathcal{D}(X) = \mathcal{D}(\mathcal{O}_{X_{\text{ét}}})$ the derived category of coherent sheaves on the small étale site of X. Let $\mathbb{L}_X \in \mathcal{D}(X)$ be the cotangent complex of X. We will only consider complexes $\mathbb{E}$ concentrated in negative degrees, that is, $h^i(\mathbb{E}) = 0$ for all $i > 0$.

Definition 2.1.1 ([6]). An *obstruction theory* for X is a morphism $\phi : \mathbb{E} \to \mathbb{L}_X$ in $\mathcal{D}(X)$ such that $h^0(\phi)$ is an isomorphism and $h^{-1}(\phi)$ is surjective.

We will often omit the morphism ϕ and simply call $\mathbb{E}$ an obstruction theory for X. The following is the main criterion and one might take it as a more intuitive definition of an obstruction theory.

Theorem 2.1.2 ([6, Theorem 9.7]). *Let $\phi : \mathbb{E} \to \mathbb{L}_X$ be a morphism. The following statements are equivalent:*

(a) $\phi : \mathbb{E} \to \mathbb{L}_X$ *is an obstruction theory.*

(b) *For any scheme S, a square-zero extension $\overline{S}$ of S, a morphism $f : S \to X$, there is an extension $\overline{f} : \overline{S} \to X$ of f if and only if the class*

$$\left(f^*\mathbb{E} \xrightarrow{f^*\phi} f^*\mathbb{L}_X \to \mathbb{L}_S \xrightarrow{\kappa(\overline{S})} I[1] \right) \in \mathrm{Ext}^1_S(f^*\mathbb{E}, I)$$

vanishes, in which case the extensions form a torsor over $\mathrm{Ext}^0(f^*\mathbb{E}, I)$.

Here $\kappa(\overline{S})$ is the Kodaira–Spencer class of the square-zero extension $\overline{S}$. One may see [18] for more about this class (and contangent complexes).

Obstruction theories play a crucial role in enumerative geometry as they are used to construct *virtual fundamental classes.* This is the main result of [6] and we won't discuss further. What we are interested in is a special type of obstruction theories.

Definition 2.1.3 ([5]). A *symmetric obstruction theory* for X is an obstruction theory $\phi : \mathbb{E} \to \mathbb{L}_X$ with a morphism $\theta : \mathbb{E} \to \mathbb{E}^\vee[1]$ such that

(i) the obstruction theory $\phi : \mathbb{E} \to \mathbb{L}_X$ is perfect, i.e., $h^i(\mathbb{E}) = 0$ except $i = 0, 1$.

(ii) θ is a non-degenerate symmetric bilinear form. To be more precise, the bilinear form induced by θ

$$\mathbb{E} \overset{L}{\otimes} \mathbb{E} \to \mathcal{O}_X[1]$$

is non-degenerate and symmetric.

Roughly speaking, an obstruction theory is symmetric when there are (canonical) isomorphisms between "deformation spaces" ($h^0(\mathbb{E})$) and "obstruction spaces" ($h^{-1}(\mathbb{E})$). Any scheme X carrying a symmetric obstruction theory admits a virtual fundamental class $[X]_{\mathrm{vir}} \in A_0(X)$ of virtual dimension 0.

Theorem 2.1.4 ([5]). *If X carries a symmetric obstruction theory and is proper, then*

$$\int_{[X]_{\mathrm{vir}}} 1 = \chi(X, \nu_X) := \sum_{c \in \mathbb{Z}} c \cdot \chi\big(\nu_X^{-1}(c)\big),$$

where $\nu_X : X \to \mathbb{C}$ is the Behrend constructible function.

The construction function ν_X is defined for any scheme of finite type (over $\mathbb{C}$). One important consequence is that the number $\chi(X, \nu_X)$ is intrinsic to the scheme X and does not depend on the choice of symmetric obstruction theory.

We list few properties of the Behrend function ν_X.

(1) If X is smooth, then $\nu_X = (-1)^{\dim X}$ is a constant function.

(2) Let X, Y be schemes of finite type. Then $\nu_{X \times Y}(x, y) = \nu_X(x)\nu_Y(y)$.

(3) If $f : X \to Y$ is étale, then $f \circ \nu_Y = \nu_X$. In particular, $\nu_X(x)$ only depends on the singularity at $x \in X$.

(4) If $X \subset M$ is a critical locus of a smooth function f on a smooth scheme M, then
$$\nu_X(x) = (-1)^{\dim M}\big(1 - \mathrm{MF}(x)\big),$$
where $\mathrm{MF}(x)$ is the Milnor fiber of f at x.

(5) If X carries a symmetric obstruction theory, then ν_X only depends on the analytic topology of X ([5, Proposition 4.22]).

For any scheme X of finite type, we will write

$$\chi_{\mathrm{vir}}(X) = \chi(X, \nu_X).$$

More generally, if $Z \subset X$ is a locally closed subscheme, then

$$\chi_{\mathrm{vir}}(X, Z) = \chi(Z, \nu_X).$$

2.2 Quivers with potential

A quiver Q consists of a pair of finite sets Q_0 and Q_1 with a pair of maps $s, t : Q_1 \to Q_0$. We call Q_0 the set of vertices, Q_1 the set of arrows, and s and t taking an arrow to its source and target respectively.

A path in Q is a sequence of arrows $a_n \cdots a_1$ such that $s(a_{i+1}) = t(a_i)$ for all i, and n is called the length of this path. We also allow the path of length 0

at each vertex $i \in Q_0$, which we denote by e_i. The path algebra $\mathbb{C}Q$ is the free algebra generated by all paths in Q, where the product of two paths p and q is defined to be pq if $s(p) = t(q)$, and 0 otherwise.

A representation V of Q is given by a set of finite-dimensional vector spaces $\{V_i\}_{i \in Q_0}$ and linear maps $\{T_a : V_{s(a)} \to V_{t(a)}\}_{a \in Q_1}$. Alternatively, a representation of Q is a finite-dimensional left $\mathbb{C}Q$-module V (take $V_i = e_i V$). We write $\underline{\dim}(V) = (\dim V_i)_{i \in Q_0}$, called the dimension vector of V.

We will denote $N_Q := \mathbb{Z}^{\oplus Q_0}$ the free abelian group of dimension vectors, and $N_Q^+ := \mathbb{Z}_{\geq 0}^{\oplus Q_0}$. There is a bilinear form on N defined by

$$\langle \mathbf{d}, \mathbf{d}' \rangle_Q = \sum_{i \in Q_0} d_i d_i' - \sum_{a \in Q_1} d_{s(a)} d'_{t(a)}.$$

The Euler pairing is given by

$$\chi_Q(\mathbf{d}, \mathbf{d}') = \langle \mathbf{d}, \mathbf{d}' \rangle_Q - \langle \mathbf{d}', \mathbf{d} \rangle_Q.$$

For a dimension vector $\mathbf{d} = (d_i)_i \in N_Q^+$, we consider the affine space

$$\mathbb{A}^{\mathbf{d}}(Q) = \prod_{a \in Q_1} \mathrm{Hom}_{\mathbb{C}}\big(\mathbb{C}^{d_{s(a)}}, \mathbb{C}^{d_{t(a)}}\big)$$

and the gauge group

$$\mathrm{GL}_{\mathbf{d}} = \prod_{i \in Q_0} \mathrm{GL}_{d_i}(\mathbb{C}),$$

which acts on $\mathbb{A}^{\mathbf{d}}(Q)$ by conjugation. The quotient stack

$$\mathfrak{M}^{\mathbf{d}}(Q) = [\mathbb{A}^{\mathbf{d}}(Q)/\mathrm{GL}_{\mathbf{d}}]$$

is the moduli stack of representations of Q with dimension vector $\mathbf{d}$.

More generally, let $\boldsymbol{f} \in N_Q^+$ be a framing vector, $\boldsymbol{f} \neq 0$. A $\boldsymbol{f}$-framed representation of Q is a representation $V = (V_i, T_a)_{i \in Q_0, a \in Q_1}$ of Q with vectors $v_i^1, \ldots, v_i^{f_i}$ in V_i for each i which generates V (as a left $\mathbb{C}Q$-module). We consider

$$\mathbb{A}^{\boldsymbol{f},\mathbf{d}}(Q) = \mathbb{A}^{\mathbf{d}}(Q) \times \prod_{i \in Q_0} (\mathbb{C}^{d_i})^{f_i}$$

with a natural $\mathrm{GL}_{\mathbf{d}}$-action.

Theorem 2.2.1 ([32]). *There is a linearization of* $\mathrm{GL}_{\mathbf{d}}$ *on* $\mathbb{A}^{\boldsymbol{f},\mathbf{d}}(Q)$ *such that semistable points are precisely* $\boldsymbol{f}$*-framed representations of* Q*. In particular, the GIT quotient*

$$M^{\boldsymbol{f},\mathbf{d}}(Q) := \mathbb{A}^{\boldsymbol{f},\mathbf{d}}(Q) \mathbin{/\!\!/} \mathrm{GL}_{\mathbf{d}}$$

is a quasi-projective smooth variety and is a fine moduli space of $\boldsymbol{f}$*-framed representations of* Q*.*

Let W be a potential of Q, i.e., a linear combination of cyclic paths. We define the Jacobi algebra

$$\mathrm{Jac}(Q,W) = \mathbb{C}Q \big/ \left(\partial_a W\right)_{a\in Q_1},$$

where ∂_a is the non-commutative derivative defined by

$$\partial_a(a') = \begin{cases} 1, & a = a', \\ 0, & a \neq a', \end{cases}$$

and

$$\partial_a(a_1 \cdots a_n) = \sum_{i=1}^{n} \partial_a(a_i) a_{i+1} \cdots a_n a_1 \cdots a_{i-1}.$$

We define a representation of (Q,W) to be a finite-dimensional left $\mathrm{Jac}(Q,W)$-module, and a framed representation of (Q,W) similarly.

The potential W associates a smooth function

$$\mathrm{Tr}(W) : M^{\boldsymbol{f},\mathbf{d}}(Q) \to \mathbb{C}.$$

The critical locus

$$M^{\boldsymbol{f},\mathbf{d}}(Q,W) := \big(d\mathrm{Tr}(W) = 0\big) \subset M^{\boldsymbol{f},\mathbf{d}}(Q)$$

is the fine moduli space of $\boldsymbol{f}$-framed representations of (Q,W). Therefore it

makes sense to define DT invariants via weighted Euler characteristics. Let

$$Z^{Q,W,\boldsymbol{f}}(\boldsymbol{t}) = \sum_{\mathbf{d}\in N_Q^+} \chi_{\mathrm{vir}}\big(M^{\boldsymbol{f},\mathbf{d}}(Q,W)\big)\,\boldsymbol{t}^{\mathbf{d}}$$

be the generating function of DT invariants of (Q,W) with framing vector $\boldsymbol{f}$, where $\boldsymbol{t} = (t_i)_{i\in Q_0}$ and $\boldsymbol{t}^{\mathbf{d}} = \prod_{i\in Q_0} t_i^{d_i}$. The framing vector $\boldsymbol{f}$ should be interpreted as a choice of stability condition.

If the framing vector $\boldsymbol{f} = (1,\dots,1)$, we will simply write $Z^{Q,W}$ for $Z^{Q,W,(1,\dots,1)}$. In this case, $(1,\dots,1)$-framed representations are finite-dimensional *cyclic* $\mathrm{Jac}(Q,W)$-modules. We write

$$\mathrm{Hilb}^{\mathbf{d}}(Q,W) := M^{(1,\dots,1),\mathbf{d}}(Q,W)$$

which can be viewed as Hilbert schemes of points on the non-commutative affine space $\mathrm{Jac}(Q,W)$. For integer n, let

$$\mathrm{Hilb}^n(Q,W) = \prod_{|\mathbf{d}|=n} \mathrm{Hilb}^{\mathbf{d}}(Q,W)$$

We will abuse the notation and write

$$Z^{Q,W}(t) := Z^{Q,W}(t,\dots,t) = \sum_{n=0}^{\infty} \chi_{\mathrm{vir}}\big(\mathrm{Hilb}^n(Q,W)\big)\,t^n.$$

2.3 Plane partitions and Hilbert schemes of points

A plane partition is a finite subset π of $\mathbb{Z}_{\geq 0}^{\oplus 3}$ such that if any of $(i+1,j,k)$, $(i,j+1,k)$, $(i,j,k+1)$ are in π, then so is (i,j,k). We will refer points in a plane partition as "boxes", and a plane partition can be viewed as a pile of boxes stacked in the positive octant. The size $|\pi|$ is the number of boxes.

We denote by $\mathcal{P}$ the set of plane partitions. Let

$$Z_{\mathrm{PL}}(t) = \sum_{\pi\in\mathcal{P}} t^{|\pi|}$$

be the generating function of plane partitions. It can be expressed as

$$Z_{\mathrm{PL}}(t) = M(t) := \prod_{n=1}^{\infty} \frac{1}{(1-t^n)^n},$$

which is now called the MacMahon function.

Plane partitions are linked with topological Euler characteristic of Hilbert schemes of points on $\mathbb{C}^3$ in the following way.

We consider the standard torus $T = (\mathbb{C}^*)^3$ action on $\mathbb{C}^3$. It induces a natural T-action on the Hilbert scheme $\mathrm{Hilb}^n(\mathbb{C}^3)$. Then T-fixed points of $\mathrm{Hilb}^n(\mathbb{C}^3)$ are corresponding to $\mathbb{C}^3$-invariant closed subschemes, which are given by T-invariant ideals of $\mathbb{C}[x, y, z]$. It is well-known that T-invariant ideals are generated by monomials.

Lemma 2.3.1. *There is a one-to-one correspondence between monomial ideals $I \subset \mathbb{C}[x, y, z]$ of index n and plane partitions of size n.*

Proof. The correspondence is given by sending a monomial I to the plane partition

$$\pi = \left\{ (i, j, k) : x^i y^j z^k \notin I \right\}.$$

□

Thus $\mathrm{Hilb}^n(\mathbb{C}^3)$ has finitely many T-fixed points and

$$\chi\big(\mathrm{Hilb}^n(\mathbb{C}^3)\big) = \#(T\text{-fixed points}) = \#(\text{plane partitions of size } n).$$

The generating function

$$Z_{\mathrm{top}}^{\mathbb{C}^3}(t) = \sum_{n=0}^{\infty} \chi\big(\mathrm{Hilb}^n(\mathbb{C}^3)\big)\, t^n = M(t)$$

is given by the MacMahon function.

Chapter 3

Quantum Fermat quintic threefolds

3.1 Non-commutative projective schemes

We first review the notion of non-commutative projective schemes defined by Artin and Zhang ([2]).

Let A be a locally finite $\mathbb{Z}_{\geq 0}$-graded $\mathbb{C}$-algebra, by which we mean $A = \oplus_{i\geq 0} A_i$ and each A_i is finite-dimensional. We define

$$\mathsf{qgr}(A) = \mathsf{gr}(A)/\mathsf{tor}(A)$$

to be the quotient abelian category, where $\mathsf{gr}(A)$ is the category of finitely-generated graded A-modules, and $\mathsf{tor}(A)$ is its Serre subcategory consisting of torsion modules, here a graded A-module M is said to be torsion if each element $m \in M$ is annihilated by $A_{\geq n}$ for some n.

Definition 3.1.1 ([2]). The *non-commutative projective scheme* defined by A is a triple

$$(\mathsf{qgr}(A), \mathbf{A}, [1]),$$

where $\mathbf{A}$ is the object in $\mathsf{qgr}(A)$ corresponding to A as an A-module, and $[1]$ is the functor induced by sending any graded module M to $M[1]$ defined by

$(M[1])_i = M_{i+1}$.

More generally, we consider a triple $(\mathsf{C}, \mathbf{O}, s)$ of an abelian category C, an object $\mathbf{O}$, and a natural equivalence $s : \mathsf{C} \to \mathsf{C}$. A morphism $(\mathsf{C}_1, \mathbf{O}_1, s_1) \to (\mathsf{C}_2, \mathbf{O}_2, s_2)$ between two such triples consists of a functor $F : \mathsf{C}_1 \to \mathsf{C}_2$, an isomorphism $F(\mathbf{O}_1) \cong \mathbf{O}_2$, and a natural isomorphism $s_2 \circ F \cong F \circ s_1$. This morphism is an isomorphism if F is an equivalence of categories.

Definition 3.1.2 ([2]). A non-commutative projective scheme is a triple $(\mathsf{C}, \mathbf{O}, s)$ which is isomorphic to $(\mathsf{qgr}(A), \mathbf{A}, [1])$ for some graded algebra A.

For example, if A is commutative and generated by degree 1 elements, then by Serre's theorem, $(\mathsf{qgr}(A), \mathbf{A}, [1])$ is isomorphic to the triple $(X, \mathcal{O}_X, - \otimes \mathcal{O}_X(1))$, where $X = \mathrm{Proj}(A)$.

More generally, let X be a (smooth) projective variety with a polarization $\mathcal{O}_X(1)$, and $\mathcal{A}$ a coherent sheaf of non-commutative $\mathcal{O}_X$-algebras on X. We may consider the *homogeneous coordinate ring* of $(X, \mathcal{A})$

$$A = \bigoplus_{n=0}^{\infty} H^0(X, \mathcal{A}(n)),$$

which is naturally a graded algebra via the multiplication $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$.

Proposition 3.1.3. *The triple* $(\mathrm{Coh}(\mathcal{A}), \mathcal{A}, - \otimes \mathcal{O}_X(1))$ *is isomorphic to the non-commutative projective scheme defined by the graded algebra* A. *In particular, there is an equivalence of (abelian) categories* $\mathrm{Coh}(\mathcal{A}) \cong \mathsf{qgr}(A)$.

Proof. This follows directly from [2, Theorem 4.5]. □

If the graded algebra A is given by such $(X, \mathcal{A})$, then the homogeneous coordinate ring $B := \oplus_n H^0(X, \mathcal{O}_X(n))$ of X is a graded subalgebra of A, which is contained in the center $Z(A)$. Since $\mathcal{A}$ is coherent, $\mathcal{A}(n)$ is generated by global sections for sufficiently large n. This implies that A is a finite B-module.

Algebras finite over their centers

Let A be a non-commutative graded algebra. For simplicity, we assume both A and $Z(A)$ are finitely-generated. Suppose there exists a graded subalgebra $B \subset Z(A)$ of A such that A is a finite B-module. We consider $X = \mathrm{Proj}(B)$ and

$$\mathcal{A} = \widetilde{{}_B A}$$

the coherent sheaf on X corresponding to the graded B-module ${}_B A$. Then $\mathcal{A}$ is naturally a sheaf of non-commutative $\mathcal{O}_X$-algebras.

From now on we assume that A is of finite global dimension. In fact, this forces A to be an Artin–Schelter regular algebra [2] since A is finite over its center $Z(A)$. We omit the details and only mention two important consequences that A satisfies the technical χ condition in the study of non-commutative projective schemes, and the category $\mathsf{qgr}(A)$ has finite cohomological dimension (which equals to the global dimension of A minus 1).

Proposition 3.1.4. *Suppose A is generated by degree one elements. Then there is an equivalence of (abelian) categories* $\mathsf{qgr}(A) \cong \mathrm{Coh}(\mathcal{A})$.

Proof. If the chosen B is also generated by degree one elements, then we have

$$\mathrm{Hom}_{\mathsf{qgr}(A)}({}_A A, M) = \mathrm{Hom}_{\mathsf{qgr}(B)}({}_B B, {}_B M) = H^0(X, \widetilde{{}_B M})$$

for any graded A-module M. Thus [2], Theorem 4.5(2) states that there is a morphism

$$A \to A' := \bigoplus_{n=0}^{\infty} \mathrm{Hom}_{\mathsf{qgr}(A)}({}_A A, {}_A A[n]) = \bigoplus_{n=0}^{\infty} H^0(X, \mathcal{A}(n))$$

of graded algebras which is an isomorphism at sufficiently large degrees. This then implies that $\mathsf{qgr}(A) \cong \mathsf{qgr}(A') \cong \mathrm{Coh}(\mathcal{A})$.

In general, we choose k such that the graded algebra $B^{(k)} := \oplus_i B_{ki}$ is generated by degree one elements. Then it is well-known that $X = \mathrm{Proj}(B) = \mathrm{Proj}(B^{(k)})$, and ${}_B A$ and ${}_{B^{(k)}} A^{(k)}$ define the same coherent sheaf on X. In fact, the same result also holds non-commutative projective schemes ([2, Proposition

5.10]). Therefore we have equivalences of categories

$$\mathsf{qgr}(A) \cong \mathsf{qgr}(A^{(k)}) \cong \mathrm{Coh}(\mathcal{A}).$$

□

Remark 3.1.5. In [3], Artin and Zhang define the notion of (flat) families of objects in any abelian category. One may check that the equivalence $\mathsf{qgr}(A) \cong \mathrm{Coh}(\mathcal{A})$ in Proposition 3.1.4 induces equivalence between families, which also preserves flatness. This implies that the Hilbert schemes $\mathrm{Hilb}^h(A)$ constructed in [3] agree with Simpson's Hilbert schemes $\mathrm{Hilb}^h(X, \mathcal{A})$ (see 3.3.2). In particular, this proves the projectivity of $\mathrm{Hilb}^h(A)$ under the assumption of Proposition 3.1.4.

Remark 3.1.6. If A is not generated by degree one elements, one may take the *stacky* Proj

$$\mathfrak{X} = \mathfrak{Proj}(B) = \big[(\mathrm{Spec}(B) \setminus \{0\})/\mathbb{C}^*\big],$$

where $\mathbb{C}^*$ acts on $\mathrm{Spec}(B)$ via the grading. Then $\mathfrak{X}$ is a Deligne–Mumford stack with a projective coarse moduli scheme $X = \mathrm{Proj}(B)$. The graded algebra A also defines a sheaf $\mathcal{A}$ of $\mathcal{O}_{\mathfrak{X}}$-algebras on $\mathfrak{X}$. Then there is also an equivalence of categories $\mathrm{Coh}(\mathcal{A}) \cong \mathsf{qgr}(A)$ by the same argument.

Finally, since $Z(A)$ is a finitely-generated commutative algebra, by Noether normalization lemma, there exists a regular subalgebra $B \subset Z(A)$ such that $Z(A)$ is finite over B, hence A is finite over B. Thus we can always choose B so that $X = \mathrm{Proj}(B)$ is smooth (in fact, a projective space).

Example 3.1.7. Consider quantum projective spaces which are non-commutative projective schemes defined by quantum polynomial rings

$$A = \mathbb{C}\langle x_0, \ldots, x_n\rangle_{(q_{ij})}.$$

If q_{ij}'s are roots of unity, then A is finite over its center $Z(A)$.

3.2 Sheaves of non-commutative algebras

Motivated by the previous section, we consider a smooth variety X and a coherent sheaf $\mathcal{A}$ of non-commutative $\mathcal{O}_X$-algebras. We may view $(X, \mathcal{A})$ as a ringed space, and $\pi : (X, \mathcal{A}) \to (X, \mathcal{O}_X)$ is a morphism of ringed spaces defined by the unit map $\mathcal{O}_X \to \mathcal{A}$. We denote $\mathrm{Coh}(\mathcal{A})$ the category of coherent $\mathcal{A}$-modules, that is, coherent sheaves $\mathcal{F}$ on X with a left $\mathcal{A}$-actions $\mathcal{A} \otimes \mathcal{F} \to \mathcal{F}$.

We begin with a few facts about ringed spaces. There are adjoint functors $\pi^* \dashv \pi_* \dashv \pi^!$, where $\pi^* = \mathcal{A} \otimes - : \mathrm{Coh}(X) \to \mathrm{Coh}(\mathcal{A})$, $\pi_* : \mathrm{Coh}(\mathcal{A}) \to \mathrm{Coh}(X)$ is the forgetful functor, and $\pi^! = \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{A}, -) : \mathrm{Coh}(X) \to \mathrm{Coh}(\mathcal{A})$. More generally, we have natural isomorphisms

$$\mathrm{Hom}_{\mathcal{A}}(\mathcal{F} \otimes \mathcal{G}, \mathcal{H}) \cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G}, \mathcal{H}\mathrm{om}_{\mathcal{A}}(\mathcal{F}, \mathcal{H})),$$

for coherent $\mathcal{A}$-modules $\mathcal{F}, \mathcal{H}$ and $\mathcal{O}_X$-module $\mathcal{G}$, and

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G} \otimes_{\mathcal{A}} \mathcal{F}, \mathcal{H}) \cong \mathrm{Hom}_{\mathcal{A}}(\mathcal{F}, \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{G}, \mathcal{H}))$$

for coherent $\mathcal{A}$-module $\mathcal{F}$, $\mathcal{A}^{\mathrm{op}}$-module $\mathcal{G}$, and $\mathcal{O}_X$-module $\mathcal{H}$.

Global dimension

One defines the global dimension of $\mathcal{A}$ similarly to the case of algebras.

Definition 3.2.1. An $\mathcal{A}$-module $\mathcal{F}$ is *locally projective* if for any $x \in X$, there exists an affine open set $U \subset X$ containing x such that $\mathcal{F}|_U$ is a projective $\mathcal{A}|_U$-module.

For any locally free sheaf $\mathcal{P}$ on X, $\mathcal{A} \otimes \mathcal{P}$ is naturally a locally projective $\mathcal{A}$-module. Thus any coherent $\mathcal{A}$-module $\mathcal{F}$ admits a resolution by locally projective $\mathcal{A}$-modules. We define the projective dimension $\mathrm{pd}(\mathcal{F})$ of $\mathcal{F}$ to be the shortest length of a projective resolution. Then it is a standard fact in homological algebra to show that

Proposition 3.2.2. *The following two numbers (possibly ∞) are the same:*

1. *the supremum of* $\mathrm{pd}(\mathcal{F})$ *for all coherent $\mathcal{A}$-modules $\mathcal{F}$;*

2. *the (co)homological dimension of the category* $\mathrm{Coh}(\mathcal{A})$*, that is, the supremum of* $n \in \mathbb{N}$ *such that* $\mathrm{Ext}^n_{\mathcal{A}}(\mathcal{F}, \mathcal{G}) \neq 0$ *for some coherent* $\mathcal{A}$*-modules* $\mathcal{F}$ *and* $\mathcal{G}$*.*

We call this number the global dimension of $\mathcal{A}$*, and denote it by* $\dim(\mathcal{A})$*.*

We say $\mathcal{A}$ is smooth if $\dim(\mathcal{A}) < \infty$. Note that it does not automatically imply $\dim(\mathcal{A}) = \dim(X)$.

For simplicity, from now on we will assume the sheaf $\mathcal{A}$ is locally free on X. One immediate consequence is that any locally projective (injective) $\mathcal{A}$-module is locally free (injective) over $\mathcal{O}_X$. In particular, we have $\dim(\mathcal{A}) \geq \dim(X)$.

Using the local-to-global spectral sequence with some basic properties of non-commutative rings (see for example, [26, Theorem 4.4]), we see that the dimension of $\mathcal{A}$ can be computed locally using the following lemma.

Lemma 3.2.3. *The dimension of* $\mathcal{A}$ *is equal to the supremum of the global dimensions of algebras* $\mathcal{A}_x$ *for all (closed) points* $x \in X$*.*

Serre duality

Since X is a smooth variety, the derived category $\mathcal{D}(\mathrm{Coh}(X))$ admits a Serre functor $(-) \otimes \omega_X[n]$, where ω_X is the dualizing sheaf and $n = \dim(X)$.

Proposition 3.2.4. *If* $\mathcal{A}$ *is smooth, then the derived category* $\mathcal{D}(\mathrm{Coh}(\mathcal{A}))$ *admits a Serre functor* $\omega_{\mathcal{A}} \overset{L}{\otimes}_{\mathcal{A}} (-)[n]$*, where* $\omega_{\mathcal{A}} = \pi^! \omega_X = \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{A}, \omega_X)$ *is the dualizing* $\mathcal{A}$*-bimodule.*

Proof. For any perfect complexes $\mathcal{F}$ and $\mathcal{G}$ of $\mathcal{A}$-modules, we have natural isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{A}}(\mathcal{F}, \mathcal{G}) &= \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, R\mathcal{H}\mathrm{om}_{\mathcal{A}}(\mathcal{F}, \mathcal{G})) \\
&\cong \mathrm{Hom}_{\mathcal{O}_X}(R\mathcal{H}\mathrm{om}_{\mathcal{A}}(\mathcal{F}, \mathcal{G}), \omega_X[n])^* \\
&\cong \mathrm{Hom}_{\mathcal{O}_X}(R\mathcal{H}\mathrm{om}_{\mathcal{A}}(\mathcal{F}, \mathcal{A}) \otimes_{\mathcal{A}} \mathcal{G}, \omega_X[n])^* \\
&\cong \mathrm{Hom}_{\mathcal{A}}(\mathcal{G}, R\mathcal{H}\mathrm{om}_{\mathcal{A}}(\mathcal{F}, \mathcal{A})^\vee \otimes \omega_X[n])^* \\
&\cong \mathrm{Hom}_{\mathcal{A}}(\mathcal{G}, \mathcal{A}^\vee \otimes_{\mathcal{A}} \mathcal{F} \otimes \omega_X[n])^* \\
&= \mathrm{Hom}_{\mathcal{A}}(\mathcal{G}, \omega_{\mathcal{A}} \otimes_{\mathcal{A}} \mathcal{F}[n])^*.
\end{aligned}$$

□

Definition 3.2.5. We say $\mathcal{A}$ is Calabi–Yau of dimension n if the derived category $\mathcal{D}(\mathrm{Coh}(\mathcal{A}))$ is a Calabi–Yau-n category, i.e., the Serre functor is equivalent to $(-)[n]$.

In particular if $\mathcal{A}$ is Calabi–Yau, then $\mathcal{A}$ is smooth and $\dim(\mathcal{A}) = \dim(X)$.

Proposition 3.2.6. *Suppose $\mathcal{A}$ is smooth, then the followings are equivalent:*

1. *$\mathcal{A}$ is Calabi–Yau;*
2. *There is an isomorphism $\mathcal{A} \to \omega_{\mathcal{A}}$ of $\mathcal{A}$-bimodules;*
3. *There is a non-degenerate bilinear form*

$$\sigma : \mathcal{A} \otimes \mathcal{A} \to \omega_X$$

of $\mathcal{O}_X$-modules such that σ is symmetric and the diagram

$$\begin{tikzcd} \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \arrow[r, "m\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes m"'] & \mathcal{A} \otimes \mathcal{A} \arrow[d, "\sigma"] \\ \mathcal{A} \otimes \mathcal{A} \arrow[r, "\sigma"'] & \omega_X \end{tikzcd}$$

commutes. In other words, $(\mathcal{A}, \sigma)$ is a family of symmetric Frobenius algebras over X.

Proof. It is clear that (1) and (2) are equivalent. For (2) and (3), a bilinear form $\sigma : \mathcal{A} \otimes \mathcal{A} \to \omega_X$ is non-degenerate if and only if it induces an isomorphism

$$\mathcal{A} \to \mathcal{A}^\vee \otimes \omega_X \cong \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{A}, \omega_X) = \omega_{\mathcal{A}}$$

of $\mathcal{O}_X$-modules. To check it is a morphism of $\mathcal{A}$-bimodules, we may reduce to affine open sets $\mathrm{Spec}(R)$, where R is regular. Then it is straightforward to verify that two statements are equivalent. □

Example 3.2.7. Let X be a Calabi–Yau variety. Then any Azumaya algebra $\mathcal{A}$ on X is Calabi–Yau.

3.3 Simpson moduli spaces

From now on we assume that X is projective and fix a polarization $\mathcal{O}_X(1)$. Stability conditions for coherent $\mathcal{A}$-modules and their moduli spaces have been studied by Simpson ([31]) for general sheaf $\mathcal{A}$ of non-commutative algebras on X. We recall their definitions and main results.

For a coherent $\mathcal{A}$-module, we define its Hilbert polynomial, rank, slope, and support to be the same as its underlying coherent sheaf on X (with respect to $\mathcal{O}_X(1)$). In particular, we say a coherent $\mathcal{A}$-module is pure (of dimension d) if its underlying coherent sheaf is so.

Definition 3.3.1 ([31]). A coherent $\mathcal{A}$-module $\mathcal{F}$ is (semi)stable if it is pure, and for any non-trivial $\mathcal{A}$-submodule $\mathcal{G} \subset \mathcal{F}$,

$$\frac{p_X(\mathcal{G})(m)}{r(\mathcal{G})} \ {}_{(\leq)} \ \frac{p_X(\mathcal{F})(m)}{r(\mathcal{F})},$$

for sufficiently large m, where p_X is the Hilbert polynomial, and r is the rank.

It was shown in [31] that all standard facts for semistable sheaves (cf. [16]) are also true for semistable $\mathcal{A}$-modules, such as

- Any pure coherent $\mathcal{A}$-module $\mathcal{F}$ has a unique filtration, called the *Harder–Narasimhan filtration*,

  $$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_k = \mathcal{F}$$

  of coherent $\mathcal{A}$-modules such that the quotients $\mathcal{F}_i/\mathcal{F}_{i-1}$'s are semistable $\mathcal{A}$-modules with strictly decreasing reduced Hilbert polynomials.

- Any semistable $\mathcal{A}$-module $\mathcal{F}$ has a filtration, called a *Jordan–Hölder filtration*,

  $$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_k = \mathcal{F}$$

  of coherent $\mathcal{A}$-modules such that the quotients $\mathrm{JH}_i := \mathcal{F}_i/\mathcal{F}_{i-1}$'s are stable $\mathcal{A}$-modules with the same reduced Hilbert polynomial (as $\mathcal{F}$). Furthermore, the polystable $\mathcal{A}$-module $\mathrm{JH}(\mathcal{F}) := \oplus_i \mathrm{JH}_i$ does not depend on

the filtration (up to isomorphic). We say two semistable $\mathcal{A}$-modules $\mathcal{F}$ and $\mathcal{G}$ are *S-equivalent* if $\mathrm{JH}(\mathcal{F}) \cong \mathrm{JH}(\mathcal{G})$ as $\mathcal{A}$-modules.

- If $\mathcal{A}$ is a stable $\mathcal{A}$-module, then $\mathrm{Hom}_{\mathcal{A}}(\mathcal{F}, \mathcal{F}) = \mathbb{C}$.

We fix a polynomial h.

Proposition 3.3.2 ([31]). *The Hilbert scheme* $\mathrm{Hilb}^h(X, \mathcal{A})$ *parameterizing quotients* $\mathcal{A} \to \mathcal{F}$ *as coherent* $\mathcal{A}$*-modules with* $p_X(\mathcal{F}) = h$ *is representable by a projective scheme. In fact, it is the closed subscheme of the Quot scheme* $\mathrm{Quot}^p_X(\mathcal{A})$ *(who parameterizes* $\mathcal{O}_X$*-module quotients of* $\mathcal{A}$*) given by the locus that the universal quotient is a morphism of* $\mathcal{A}$*-modules.*

The moduli spaces of (semi)stable $\mathcal{A}$-modules were constructed in the same way as the ones for (semi)stable sheaves, which is via GIT quotient on certain Hilbert (Quot) schemes. We omit the details and state the main results.

Theorem 3.3.3 ([31]). *Let* $\mathfrak{M}^{(s)s,h}(X, \mathcal{A})$ *be the moduli stack of (semi)stable* $\mathcal{A}$*-modules with Hilbert polynomial* h*. Then*

(a) The moduli stack $\mathfrak{M}^{(s)s,h}(X, \mathcal{A})$ *is an Artin stack of finite type, and admits a good moduli space* $M^{(s)s,h}(X, \mathcal{A})$*.*

(b) The coarse moduli scheme $M^{ss,h}(X, \mathcal{A})$ *is projective, whose points are in one-to-one correspondence with S-equivalent classes of semistable* $\mathcal{A}$*-modules.*

(c) The morphism $\mathfrak{M}^{s,h}(X, \mathcal{A}) \to M^{s,h}(X, \mathcal{A})$ *is a* $\mathbb{C}^*$*-gerbe, and* $M^{s,h}(X, \mathcal{A})$ *is the open subscheme of* $M^{ss,h}(X, \mathcal{A})$ *whose points corresponds to isomorphism classes of stable* $\mathcal{A}$*-modules.*

We will often simply write $\mathrm{Hilb}^h(\mathcal{A})$ and $\mathfrak{M}^{ss,h}(\mathcal{A})$ if the base space X is clear.

3.4 Quantum Fermat quintic threefolds

In this section, we will define the main subject of this thesis — quantum Fermat quintic threefolds introduced in [20].

Definition 3.4.1. A *quantum Fermat quintic threefold* is a non-commutative projective scheme associated to a graded algebra

$$A = \mathbb{C}\langle t_0, \ldots, t_4 \rangle_{(q_{ij})} \Big/ \left(\sum_{k=0}^{4} t_k^5 \right),$$

with $\deg(t_i) = 1$ for all i and q_{ij}'s satisfying

(i) q_{ij} is a 5-th root of unity.

(ii) $q_{ii} = q_{ij}q_{ij} = 1$ for all i, j.

(iii) $\prod_j q_{ij}$ is independent of i.

The condition (iii) is equivalent to the category $\mathsf{qgr}(A)$ being CY3 [20, Theorem 2.1].

In this thesis, we will add an additional condition:

(iv) $q_{ij}q_{jk} \neq q_{ik}$ for all distinct i, j, k.

This is to ensure that the non-commutative projective scheme has the "maximal non-commutativity". To see this, we recall Zhang's twisted graded algebras [36]:

Let $A = \oplus_i A_i$ be a graded algebra. For any automorphism $\sigma : A \to A$ of graded algebras, Zhang defines a new multiplication on A by

$$x * y = x \cdot \sigma^m(y)$$

for any $x \in A_m$ and $y \in A_n$. The graded algebra with the new multiplication $*$ is called a twisted graded algebra of A, denote by A^σ. Then it is shown that non-commutative projective schemes associated to A and any twisted algebra A^σ are isomorphic.

Now, if $q_{ij}q_{jk} = q_{ik}$ for some distinct i, j, k, then we take the automorphism $\sigma : A \to A$ defined by $x_j \mapsto q_{ij}x_j$ and $x_k \mapsto q_{ik}x_k$. The twisted algebra A^σ has quantum parameters $q_{ij} = q_{jk} = q_{ik} = 1$, which means the non-commutative projective scheme $\mathsf{qgr}(A)$ contains a commutative closed subscheme of positive dimension.

Theorem 3.4.2. *Up to a possible change of the primitive root $q \in \mu_5$, all quantum Fermat quintic threefolds (satisfying (i)–(iv)) are isomorphic as non-commutative projective schemes.*

Proof. We fix a primitive 5-th root q of unity. Any quantum Fermat quintic threefold is determined by a skew-symmetric matrix $N = (n_{ij})_{i,j} \in M_5(\mathbb{Z}/5\mathbb{Z})$ such that $(1,1,1,1,1)^\intercal$ is an eigenvector of N. We denote by A_N the graded algebra with quantum parameters $q_{ij} = q^{n_{ij}}$.

Consider the following actions on the set of above matrices N's:

(a) Change of the primitive root q: For any element $a \in (\mathbb{Z}/5\mathbb{Z})^\times$, changing q to q^a is equivalent to multiple all elements n_{ij} in N by a.

(b) Change of variables t_i's: For any permutation $\sigma \in S_5$, we consider the change of variables $\tilde{t}_i = t_{\sigma(i)}$. Then the new graded algebra has quantum parameters given by $\tilde{n}_{ij} = n_{\sigma(i),\sigma(j)}$.

(c) Twisted graded algebras: For any $(a_0, \ldots, a_4) \in (\mathbb{Z}/5\mathbb{Z})^5$, let $\sigma : A_N \to A_N$ be the automorphism defined by $\sigma(x_i) = q^{a_i} x_i$. Then the twisted algebra $(A_N)^\sigma$ has quantum parameters given by

$$N^\sigma := (n_{ij} + a_i - a_j)_{i,j}.$$

The proof is done with the aid of computer. There are precisely 3000 choices of N's. All of them are equivalent under the three actions above. □

For computation purpose, we will fix a particular choice of quantum parameters

$$(q_{ij})_{i,j} = \begin{pmatrix} 1 & q & q^{-1} & q & q^{-1} \\ q^{-1} & 1 & q & q^{-1} & q \\ q & q^{-1} & 1 & q & q^{-1} \\ q^{-1} & q & q^{-1} & 1 & q \\ q & q^{-1} & q & q^{-1} & 1 \end{pmatrix} \tag{3.1}$$

where $q \in \mu_5$ is a (fixed) primitive 5-th root of unity, and we will call it *the* quantum Fermat quintic threefold.

To associate a pair $(X, \mathcal{A})$, we take

$$B = \mathbb{C}[x_0, \ldots, x_4] / \left(\sum_{k=0}^{4} x_k \right),$$

where $x_i = t_i^5$ for each i, and $X = \mathrm{Proj}(B) \cong \mathbb{P}^3$. Since A is a graded-free B-module, the sheaf $\mathcal{A}$ of non-commutative $\mathcal{O}_X$-algebras induced by A is locally free. In fact, A is a graded free B-module, and we can write

$$\mathcal{A} = \mathcal{O}_X \oplus \mathcal{O}_X(-1)^{\oplus 121} \oplus \mathcal{O}_X(-2)^{\oplus 381} \oplus \mathcal{O}_X(-3)^{\oplus 121} \oplus \mathcal{O}_X(-4)$$

as a $\mathcal{O}_X$-module.

It is shown in [20] that the graded algebra A is of finite global dimension, so $\mathcal{A}$ also has finite global dimension. Here we give an alternative proof that $(X, \mathcal{A})$ is Calabi–Yau.

Lemma 3.4.3. *The sheaf $\mathcal{A}$ of $\mathcal{O}_X$-algebras is Fobenius via*

$$(-,-) : \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{A} \to \mathcal{A} \to \omega_X,$$

where the first arrow is the multiplication map, and the second arrow is the projection to the component $\mathcal{O}_X(-4) \cong \omega_X$. If $\prod_j q_{ij} = 1$ for all i, then the pairing is symmetric.

Proof. We write down the multiplication maps of $\mathcal{A}$ explicitly. Consider

$$I = \Big\{ \mathbf{a} = (a_0, a_1, \ldots, a_4) \in \{0, 1, \ldots, 4\}^5, a_0 + a_1 + \ldots + a_4 \text{ is a multiple of } 5 \Big\},$$

a basis of $A^{(5)}$ over $B = B^{(5)}$. For simplicity, we will write $a_0 + a_1 + \ldots + a_4 = 5\,|\mathbf{a}|$. Note that I is naturally an abelian group as a subgroup of $(\mathbb{Z}/5\mathbb{Z})^5$ (but the function $|-|$ is not linear). Then as a $\mathcal{O}_X$-module, we may write

$$\mathcal{A} = \bigoplus_{\mathbf{a} \in I} \mathcal{O}_X(-|\mathbf{a}|).$$

We denote the multiplication map $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ on each component by

$$\mathcal{O}_X(-|\mathbf{a}|) \otimes \mathcal{O}_X(-|\mathbf{b}|) \xrightarrow{\phi_{\mathbf{a},\mathbf{b}}} \mathcal{O}_X(-|\mathbf{a}+\mathbf{b}|),$$

where

$$\phi_{\mathbf{a},\mathbf{b}} = q_{\mathbf{a},\mathbf{b}} x_0^{c_0} x_1^{c_1} x_2^{c_2} x_3^{c_3} x_4^{c_4}$$

is the section in $H^0\big(X, \mathcal{O}_X(|\mathbf{a}| + |\mathbf{b}| - |\mathbf{a}+\mathbf{b}|)\big)$ given by

$$q_{\mathbf{a},\mathbf{b}} = \prod_{i>j} q_{ij}^{a_i b_j}, \quad c_i = \begin{cases} 5, & a_i + b_i \geq 5; \\ 0, & a_i + b_i < 5. \end{cases}$$

Write $\mathbf{4} = (4,4,4,4,4) \in I$, $\mathcal{O}_X(-|\mathbf{4}|) = \mathcal{O}_X(-4)$ is the component corresponding to ω_X. Since for each component $\mathcal{O}_X(-|\mathbf{a}|)$, there is a unique component $\mathcal{O}_X(-|\mathbf{4}-\mathbf{a}|)$ such that the multiplication map

$$\mathcal{O}_X(-|\mathbf{a}|) \otimes \mathcal{O}_X(-|\mathbf{4}-\mathbf{a}|) \to \mathcal{O}_X(-|\mathbf{4}|) \cong \omega_X$$

is an isomorphism, the induced map $\mathcal{A} \to \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{A}, \omega_X)$ is an isomorphism of $\mathcal{O}_X$-modules.

The pairing $(-,-)$ is symmetric if and only if $q_{\mathbf{a},\mathbf{4}-\mathbf{a}} = q_{\mathbf{4}-\mathbf{a},\mathbf{a}}$ for all $\mathbf{a} \in I$. That is,

$$\prod_{i>j} q_{ij}^{a_i(4-a_j)} = \prod_{i>j} q_{ij}^{(4-a_i)a_j} \iff \prod_{i>j} q_{ij}^{a_i - a_j} = 1,$$

which is equivalent to that $\prod_j q_{ij} = 1$ for all i. $\square$

Chapter 4

Donaldson–Thomas invariants of $(X, \mathcal{A})$

The purpose of this chapter is to define DT invariants on a CY3 pair $(X, \mathcal{A})$. We begin with a study of deformation-obstruction theory of $\mathcal{A}$-modules and construct an obstruction theory for the moduli space of stable $\mathcal{A}$-modules. Then we will use it to construct a symmetric obstruction theory and define DT invariants using the Hilbert schemes of $(X, \mathcal{A})$.

4.1 Obstruction theories for $\mathcal{A}$-modules

Let X be a smooth projective variety and $\mathcal{A}$ a locally free sheaf of non-commutative $\mathcal{O}_X$-algebras. Let S be a scheme, and $\mathcal{F}$ a coherent $\mathcal{A}_S$-module on $X \times S$, flat over S. Suppose $S \subset \overline{S}$ is a square-zero extension with ideal sheaf I.

Definition 4.1.1. An $\mathcal{A}$-module extension of $\mathcal{F}$ over $\overline{S}$ is a coherent $\mathcal{A}_{\overline{S}}$-module $\overline{\mathcal{F}}$ on $X \times \overline{S}$, flat over $\overline{S}$, such that $\overline{\mathcal{F}}|_{X \times S} \cong \mathcal{F}$ as $\mathcal{A}_S$-modules.

It is a general fact ([23]) that existence of such extensions must be governed by an obstruction class in $\mathrm{Ext}^2_{\mathcal{A}_S}(\mathcal{F}, \mathcal{F} \otimes \pi_S^* I)$. However, to obtain an obstruction theory on the moduli spaces, it requires a more explicit description of the obstruction class. We generalize the result in [17], showing that the obstruction class is the product of Atiyah and Kodiara–Spencer classes.

Theorem 4.1.2. *There exists a natural class*

$$\mathrm{at}_{\mathcal{A}}(\mathcal{F}) \in \mathrm{Ext}^1_{\mathcal{A}_S}(\mathcal{F}, \mathcal{F} \otimes \pi_S^* \mathbb{L}_S),$$

called the Atiyah class, such that for any square-zero extension $S \subset \overline{S}$ with ideal sheaf I, an $\mathcal{A}$-module extension of $\mathcal{F}$ over $\overline{S}$ exists if and only if the obstruction class

$$\mathrm{ob} = \left(\mathcal{F} \xrightarrow{\mathrm{at}_{\mathcal{A}}(\mathcal{F})} \mathcal{F} \otimes \pi_S^* \mathbb{L}_S[1] \xrightarrow{\mathrm{id}_{\mathcal{F}} \otimes \pi_S^* \kappa(S/\overline{S})[1]} \mathcal{F} \otimes \pi_S^* I[2]\right) \in \mathrm{Ext}^2_{\mathcal{A}_S}(\mathcal{F}, \mathcal{F} \otimes \pi_S^* I)$$

vanishes, where $\kappa(S/\overline{S}) \in \mathrm{Ext}^1_S(S, I)$ is the Kodiara–Spencer class for the extension $S \subset \overline{S}$. Moreover, if an extension of $\mathcal{F}$ over $\overline{S}$ exists, then all (equivalence classes of) extensions form an affine space over $\mathrm{Ext}^1_{\mathcal{A}}(\mathcal{F}, \mathcal{F} \otimes \pi_S^* I)$.

We follow closely the method in [17]. The key idea is that the obstruction classes are given universally by a morphism of Fourier–Mukai transforms. The proof will be given in the next section. We set up some notations for this chapter.

For any morphism $f : S \to T$ of schemes, we will abuse the notation and write f also for the induced morphism $\mathrm{id}_X \times f : X \times S \to X \times T$. So there are natural functors

$$f^* : \mathrm{Coh}(\mathcal{A}_T) \to \mathrm{Coh}(\mathcal{A}_S), \quad f_* : \mathrm{Coh}(\mathcal{A}_S) \to \mathrm{Coh}(\mathcal{A}_T),$$

where the latter one is induced by the natural morphism $\mathcal{A}_T = \mathcal{A} \boxtimes \mathcal{O}_T \to f_* \mathcal{A}_S = \mathcal{A} \boxtimes f_* \mathcal{O}_S$. We denote by $\mathcal{D}^{(b)}(\mathcal{A}_S)$ the (bounded) derived category of $\mathrm{Coh}(\mathcal{A}_S)$. In this section, tensor products $\otimes$ will always be derived tensor products over $\mathcal{O}$ unless stated otherwise.

We briefly recall the definition of Fourier–Mukai transforms. For any schemes S and T, and a complex $\mathcal{P} \in \mathcal{D}(\mathcal{O}_{S \times T})$ of coherent $\mathcal{O}_{S \times T}$-modules, we may define the Fourier–Mukai functor

$$\Phi_{\mathcal{P}} : \mathcal{D}(\mathcal{A}_S) \xrightarrow{\pi_S^*} \mathcal{D}(\mathcal{A}_{S \times T}) \xrightarrow{- \otimes \mathcal{P}} \mathcal{D}(\mathcal{A}_{S \times T}) \xrightarrow{R\pi_{T*}} \mathcal{D}(\mathcal{A}_T),$$

where $\mathcal{P}$ is called the Fourier–Mukai kernel. Given any morphism $\phi : \mathcal{P}_1 \to \mathcal{P}_2$ in $\mathcal{D}(\mathcal{O}_{S \times T})$, it induces a natural transformation $\Phi_{\mathcal{P}_1} \to \Phi_{\mathcal{P}_2}$ between functors.

For any object $\mathcal{F} \in \mathcal{D}(\mathcal{A}_S)$, we write

$$\phi(\mathcal{F}) : \Phi_{\mathcal{P}_1}(\mathcal{F}) \to \Phi_{\mathcal{P}_2}(\mathcal{F})$$

for the induced morphism in $\mathcal{D}(\mathcal{A}_T)$.

Now, we use the Atiyah class to construct an obstruction theory on the moduli space. Let S be any scheme, and $\mathcal{F}$ be a coherent $\mathcal{A}_S$-module, flat over S. Let α be a class in $\mathrm{Ext}^1_{\mathcal{A}_S}(\mathcal{F}, \mathcal{F} \otimes \pi_S^* \mathbb{L}_S)$. Then α defines a morphism

$$\alpha : \mathcal{F} \to \mathcal{F} \otimes \pi_S^* \mathbb{L}_S[1] \quad \text{in } \mathcal{D}(\mathcal{A}_S).$$

Since $\mathcal{F}$ is flat over S, $\mathcal{F}$ is perfect. The natural map $\mathcal{F} \to \mathcal{F}^{\vee\vee}$ is an isomorphism in $\mathcal{D}(\mathcal{A}_S)$. Thus there is an isomorphism

$$\mathcal{F} \otimes \pi_S^* \mathbb{L}_S \cong R\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}^\vee, \pi_S^* \mathbb{L}_S) \quad \text{in } \mathcal{D}(\mathcal{A}_S)$$

By adjunction, α defines a morphism

$$\mathcal{F}^\vee \otimes_{\mathcal{A}_S} \mathcal{F} \to \pi_S^* \mathbb{L}_S[1] \quad \text{in } \mathcal{D}(\mathcal{O}_{X \times S}).$$

We then apply Verdier duality, it yields a morphism

$$R\pi_{S*}\Big((\mathcal{F}^\vee \otimes_{\mathcal{A}_S} \mathcal{F}) \otimes \pi_X^* \omega_X\Big)[n-1] \to \mathbb{L}_S \quad \text{in } \mathcal{D}(\mathcal{O}_S).$$

Lemma 4.1.3. *For any perfect complexes $\mathcal{F}$ and $\mathcal{G}$ of coherent $\mathcal{A}_M$-modules, there is a canonical isomorphism*

$$\Big(R\pi_{M*} R\mathcal{H}\mathrm{om}_{\mathcal{A}_M}(\mathcal{F}, \mathcal{G})\Big)^\vee \cong R\pi_{M*}\left(\mathcal{G}^\vee \otimes_{\mathcal{A}_M} \mathcal{F} \otimes \pi_X^* \omega_X\right)[n].$$

Proof. Use the same argument as above, with $\pi_S^* \mathbb{L}_S$ replaced by $\pi_S^* \mathcal{O}_S = \mathcal{O}_{X \times S}$. □

We conclude that any class $\alpha \in \mathrm{Ext}^1_{\mathcal{A}_S}(\mathcal{F}, \mathcal{F} \otimes \pi_S^* \mathbb{L}_S)$ defines a morphism

$$\Big(R\pi_{S*} R\mathcal{H}\mathrm{om}_{\mathcal{A}_S}(\mathcal{F}, \mathcal{F})\Big)^\vee [-1] \to \mathbb{L}_S. \tag{4.1}$$

It is almost by definition to see that if $S = M$ is a *fine* moduli space and $\mathcal{F}$ is the universal family of coherent $\mathcal{A}$-modules, then the morphism (4.1) induced by a class α is an obstruction theory for M if and only if α is the Atiyah class.

In general, the moduli space $M := M_X^{s,p}(\mathcal{A})$ is not a fine moduli space because the $\mathbb{C}^*$-gerbe $\mathfrak{M} \to M$ is not trivial so the universal family on $\mathfrak{M}$ does not descend to M. We use the fact that any $\mathbb{C}^*$-gerbe is étale locally trivial, that is, there is an étale cover $U \to X$ with a section

$$\begin{tikzcd} & \mathfrak{M} \arrow[d] \\ U \arrow[ur] \arrow[r] & M. \end{tikzcd}$$

We denote by $\mathcal{F}$ the pullback of the universal family of stable $\mathcal{A}$-modules to $X \times U$.

Consider the natural transformation

$$\Phi : \mathrm{Hom}_{\mathsf{Sch}}(-, U) \to \mathcal{M},$$

where $\mathcal{M}$ is the moduli functor for stable $\mathcal{A}$-modules on X, and Φ sends any morphism $f : S \to U$ to the $\mathcal{A}_T$-module $(\mathrm{id}_X \times f)^*\mathcal{F}$ on $X \times S$.

Lemma 4.1.4. *The natural transformation Φ satisfies*

1. $\Phi(\mathrm{Spec}\mathbb{C})$ *is surjective;*
2. *For any $f : S \to U$ and any square-zero extension $S \subset \overline{S}$, the map $\Phi(\overline{S})$ induces a bijection between subsets*

$$\left\{\overline{f} : \overline{S} \to U : \overline{f}|_S = f\right\} \subset \mathrm{Hom}_{\mathsf{Sch}}(\overline{S}, U)$$

and

$$\left\{\mathcal{A}\text{-module extensions of } \mathcal{F} \text{ over } \overline{S}\right\} \subset \mathcal{M}(\overline{S}).$$

In other words, U has the same deformation-obstruction theory as a fine moduli space.

Proof. This is essentially the definition of the morphism $U \to \mathfrak{M}$ being étale. □

Theorem 4.1.5. *Let U and $\mathcal{F}$ be described as above. Then the Atiyah class $\mathrm{at}_{\mathcal{A}}(\mathcal{F})$ defines an obstruction theory*

$$\mathbb{E} := \Big(R\pi_{U*}R\mathcal{H}\mathrm{om}_{\mathcal{A}_U}(\mathcal{F},\mathcal{F})\Big)^{\vee}[-1] \to \mathbb{L}_U \tag{4.2}$$

for U.

Proof. Let $f : S \to U$ be a morphism, and $S \subset \overline{S}$ be a square-zero extension with ideal sheaf I. We consider the class

$$o = \Big(Lf^*\mathbb{E} \to Lf^*\mathbb{L}_U \to \mathbb{L}_S \xrightarrow{\kappa(S/\overline{S})} I[1]\Big) \in \mathrm{Ext}^1_{\mathcal{O}_S}(Lf^*\mathbb{E}, I).$$

Observe that there are natural isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{O}_S}(Lf^*\mathbb{E},\mathbb{L}_S) &\cong \mathrm{Hom}_{\mathcal{O}_U}(\mathbb{E}, Rf_*\mathbb{L}_S)\\
&\cong \mathrm{Hom}_{\mathcal{O}_{X\times U}}(\mathcal{F}^{\vee}\otimes_{\mathcal{A}_U}\mathcal{F}, \pi_U^*Rf_*\mathbb{L}_S[1])\\
&\cong \mathrm{Hom}_{\mathcal{O}_{X\times U}}(\mathcal{F}^{\vee}\otimes_{\mathcal{A}_U}\mathcal{F}, Rg_*\pi_S^*\mathbb{L}_S[1])\\
&\cong \mathrm{Hom}_{\mathcal{O}_{X\times S}}(g^*(\mathcal{F}^{\vee}\otimes_{\mathcal{A}_U}\mathcal{F}), \pi_S^*\mathbb{L}_S[1])\\
&\cong \mathrm{Hom}_{\mathcal{O}_{X\times S}}(g^*\mathcal{F}^{\vee}\otimes_{g^*\mathcal{A}_U} g^*\mathcal{F}, \pi_S^*\mathbb{L}_S)\\
&\cong \mathrm{Hom}_{\mathcal{O}_{X\times S}}((g^*\mathcal{F})^{\vee}\otimes_{\mathcal{A}_S}(g^*\mathcal{F}), \pi_S^*\mathbb{L}_S)\\
&\cong \mathrm{Hom}_{\mathcal{A}_S}(g^*\mathcal{F}, g^*\mathcal{F}\otimes\pi_S^*\mathbb{L}_S)
\end{aligned}$$

where $g = \mathrm{id}_X \times f : X \times S \to X \times U$. By the functoriality of Atiyah classes, it sends the composition $(Lf^*\mathbb{E} \to Lf^*\mathbb{L}_U \to \mathbb{L}_S)$ to the Atiyah class $\mathrm{at}_{\mathcal{A}}(g^*\mathcal{F})$ on $X \times S$. Therefore the class $o \in \mathrm{Ext}^1_{\mathcal{O}_S}(Lf^*\mathbb{E}, I)$ corresponds to the obstruction class $\mathrm{ob} \in \mathrm{Ext}^2_{\mathcal{A}_S}(g^*\mathcal{F}, g^*\mathcal{F}\otimes\pi_S^*\mathbb{L}_S)$ for the coherent $\mathcal{A}_S$-module $g^*\mathcal{F}$ on $X \times S$.

The rest of the proof follows from Lemma 4.1.4: there exists an extension of $f : S \to U$ to $\overline{S}$ if and only if there exists an $\mathcal{A}$-module deformation of $g^*\mathcal{F}$ over $\overline{S}$. □

The coherent $\mathcal{A}_U$-module $\mathcal{F}$ on $X \times U$ is pulled back from the $\mathbb{C}^*$-gerbe $\mathfrak{M} \to M$, so it can be regarded as a *twisted* $\mathcal{A}_M$-module on $X \times M$.

Lemma 4.1.6. *The complex $R\mathcal{H}\mathrm{om}_{\mathcal{A}_U}(\mathcal{F},\mathcal{F})$ on $X\times U$ descends to a complex in $\mathcal{D}_{\acute{e}t}(\mathcal{O}_{X\times M})$, which we denote by $R\mathcal{H}\mathrm{om}_{\mathcal{A}_M}(\mathcal{F},\mathcal{F})$.*

Proof. Note that for any coherent $\mathcal{A}$-module $\mathcal{G}$, $\mathcal{H}\mathrm{om}_{\mathcal{A}}(\mathcal{G},\mathcal{G})$ is the equalizer of

$$\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{G},\mathcal{G}) \rightrightarrows \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{A}\otimes\mathcal{G},\mathcal{G}).$$

Then the proof follows from the fact that for any twisted sheaf $\mathcal{F}$, $\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F},\mathcal{F})$ is a (untwisted) coherent sheaf (see for instance [10]). □

By functoriality of the Atiyah class, the obstruction theory $\mathbb{E}\to\mathbb{L}_U$ also descends to a morphism in $\mathcal{D}_{\text{ét}}(\mathcal{O}_M)$

$$\Big(R\pi_{M*}R\mathcal{H}\mathrm{om}_{\mathcal{A}_M}(\mathcal{F},\mathcal{F})\Big)^{\vee}[-1]\to\mathbb{L}_M \quad \text{in } \mathcal{D}_{\text{ét}}(\mathcal{O}_M). \tag{4.3}$$

Since being an obstruction theory is an étale local property, we conclude that

Corollary 4.1.7. *The morphism (4.3) is an obstruction theory for the moduli space M.*

4.2 Proof of Theorem 4.1.2

We fix a scheme S and a square-zero extension $i: S\subset\overline{S}$ with ideal sheaf I. Let $\mathcal{F}$ be a coherent $\mathcal{A}_S$-module on $X\times S$, flat over S. Suppose $\overline{\mathcal{F}}$ is an $\mathcal{A}$-module extension of $\mathcal{F}$ over $\overline{S}$. The isomorphism $Li^*\overline{\mathcal{F}}=i^*\overline{\mathcal{F}}\to\mathcal{F}$ induces an exact triangle

$$Ri_*(\mathcal{F}\otimes\pi_S^*I)\to\overline{\mathcal{F}}\to Ri_*\mathcal{F}\xrightarrow{e}Ri_*(\mathcal{F}\otimes\pi_S^*I)[1],$$

which gives a class $e\in\mathrm{Ext}^1_{\mathcal{A}_{\overline{S}}}(Ri_*\mathcal{F},Ri_*(\mathcal{F}\otimes\pi_S^*I))$. For any $\mathcal{F}$, we have an exact triangle

$$Q_{\mathcal{F}}\to Li^*Ri_*\mathcal{F}\to\mathcal{F} \tag{4.4}$$

in $\mathcal{D}(\mathcal{A}_S)$ given by adjunction.

Lemma 4.2.1. *A class $e\in\mathrm{Ext}^1_{\mathcal{A}_{\overline{S}}}(Ri_*\mathcal{F},Ri_*(\mathcal{F}\otimes\pi_S^*I))$ is given by an $\mathcal{A}$-module deformations of $\mathcal{F}$ over $\overline{S}$ if and only if the composition*

$$\Phi_e: Q_{\mathcal{F}}\to Li^*Ri_*\mathcal{F}\xrightarrow{Li^*e}Li^*Ri_*(\mathcal{F}\otimes\pi_S^*I)$$

is an isomorphism in $\mathcal{D}(\mathcal{A}_S)$.

Proof. Observe the diagram

$$\begin{tikzcd}
 & & Q_{\mathcal{F}} \arrow[d] \arrow[dr, "\Phi_e"] & \\
Li^*Ri_*(\mathcal{F}\otimes\pi_S^*I) \arrow[r] & Li^*\overline{\mathcal{F}} \arrow[r] \arrow[dr, "r"'] & Li^*Ri_*\mathcal{F} \arrow[r] \arrow[d] & Li^*Ri_*(\mathcal{F}\otimes\pi_S^*I)[1] \\
 & & \mathcal{F} &
\end{tikzcd} \tag{4.5}$$

in $\mathcal{D}(\mathcal{A}_S)$. The morphism $r : Li^*\overline{\mathcal{F}} \to \mathcal{F}$ is an isomorphism if and only if Φ_e is. Then $\overline{\mathcal{F}}$ is an $\mathcal{A}$-module deformation of $\mathcal{F}$. □

Now, we apply $\mathrm{Hom}_{\mathcal{A}_S}(-, \mathcal{F}\otimes\pi_S^*I)$ to the exact triangle (4.4), it yields

$$\begin{tikzcd}
 & \mathrm{Ext}^1_{\mathcal{A}_{\overline{S}}}(Ri_*\mathcal{F}, Ri_*(\mathcal{F}\otimes\pi_S^*I)) \arrow[d, equal] & & \\
\mathrm{Ext}^1_{\mathcal{A}_S}(\mathcal{F}, \mathcal{F}\otimes\pi_S^*I) \arrow[r] & \mathrm{Ext}^1_{\mathcal{A}_S}(Li^*Ri_*\mathcal{F}, \mathcal{F}\otimes\pi_S^*I) \arrow[r] & \mathrm{Ext}^1_{\mathcal{A}_S}(Q_{\mathcal{F}}, \mathcal{F}\otimes\pi_S^*I) \arrow[r, "\delta"] & \mathrm{Ext}^2_{\mathcal{A}_S}(\mathcal{F}, \mathcal{F}\otimes\pi_S^*I)
\end{tikzcd}$$

The second arrow sends a class e to the morphism

$$\Psi_e : Q_{\mathcal{F}} \xrightarrow{\Phi_e} Li^*Ri_*(\mathcal{F}\otimes\pi_S^*I)[1] \to \mathcal{F}\otimes\pi_S^*I[1],$$

where the second map is the adjunction map. The proof consists of following steps.

(a) There exists a class $\pi_{\mathcal{F}} \in \mathrm{Ext}^1_{\mathcal{A}_S}(Q_{\mathcal{F}}, \mathcal{F}\otimes\pi_S^*I_S)$ such that $\Psi_e = \pi_{\mathcal{F}}$ for any class e given by a deformation.

(b) The obstruction class $\delta(\pi_{\mathcal{F}}) \in \mathrm{Ext}^2_{\mathcal{A}_S}(\mathcal{F}, \mathcal{F}\otimes\pi_S^*I)$ is the product of Atiyah class and Kodaira–Spencer class.

(c) If the obstruction class $\delta(\pi_{\mathcal{F}})$ vanishes, then there exists a class e given by a deformation such that $\Psi_e = \pi_{\mathcal{F}}$.

(d) Suppose e and e' are two classes such that $\Psi_e = \Psi_{e'}$. If a class e is given by a deformation, then so is e'.

Construction of $\pi_{\mathcal{F}}$

We first consider the trivial case $\mathcal{A} = \mathcal{O}_X$, $\mathcal{F} = \mathcal{O}_{X\times S}$, and the class e given by the deformation $\mathcal{O}_{X\times\overline{S}}$. Then the morphism Ψ_e is determined by the square-zero extension $S \subset \overline{S}$, which we denote by

$$\pi : Q_{\mathcal{O}_{X\times S}} \to Li^* Ri_* \pi_S^* I[1] \to \pi_S^* I[1]. \tag{4.6}$$

Note that it is the pull-back of the morphism $Q_{\mathcal{O}_S} \to I[1]$ in $\mathcal{D}(\mathcal{O}_S)$ defined in the same way.

Assume that an $\mathcal{A}$-module deformation $\overline{\mathcal{F}}$ of $\mathcal{F}$ over $\overline{S}$ exists. Then for any coherent sheaf $\mathcal{P}$ on $X \times S$, there are canonical isomorphisms

$$\mathcal{F} \otimes Li^* Ri_* \mathcal{P} \cong Li^*(\overline{\mathcal{F}} \otimes Ri_* \mathcal{P}) \cong Li^* Ri_*(\mathcal{F} \otimes \mathcal{P}).$$

This implies that Ψ_e is equal to $\pi_{\mathcal{F}} := \mathcal{F} \otimes \pi$, up to a canonical isomorphism $Q_{\mathcal{F}} \cong \mathcal{F} \otimes Q_{\mathcal{O}_{X\times S}}$.

The obstruction class $\delta(\pi_{\mathcal{F}})$

Observe that $Q_{\mathcal{F}}$ and $\mathcal{F} \otimes Q_{\mathcal{O}_{X\times S}}$ may not be isomorphic for general coherent $\mathcal{A}$-module $\mathcal{F}$, so we need an alternative definition of $\pi_{\mathcal{F}}$. To define $\pi_{\mathcal{F}}$ and study the obstruction class $\delta(\pi_{\mathcal{F}})$, we recall several facts proved in [17].

(i) There exists an exact triangle

$$\mathcal{Q} \to \mathcal{H} \to \Delta_* \mathcal{O}_S \xrightarrow{\delta_0} \mathcal{Q}[1] \quad \text{in } \mathcal{D}(\mathcal{O}_{S\times S}),$$

where $\Delta : S \to S \times S$ is the diagonal map, such that the exact triangle (4.4) is given by Fourier–Mukai transforms

$$\Phi_{\pi_{S\times S}^*\mathcal{Q}}(\mathcal{F}) \to \Phi_{\pi_{S\times S}^*\mathcal{H}}(\mathcal{F}) \to \mathcal{F} \xrightarrow{(\pi_{S\times S}^*\delta_0)(\mathcal{F})} \Phi_{\pi_{S\times S}^*\mathcal{Q}}(\mathcal{F})[1].$$

In particular, the map $\delta : \mathrm{Ext}^1_{\mathcal{A}}(Q_{\mathcal{F}}, \mathcal{F} \otimes \pi_S^* I) \to \mathrm{Ext}^2_{\mathcal{A}}(\mathcal{F}, \mathcal{F} \otimes \pi_S^* I)$ is the composition with $(\pi_{S\times S}^*\delta_0)(\mathcal{F})$.

(ii) There exists a natural morphism

$$\pi_0 : \mathcal{Q} \to \Delta_* I[1] \quad \text{in } \mathcal{D}(\mathcal{O}_{S\times S})$$

such that $(\pi^*_{S\times S}\pi_0)(\mathcal{F}) = \pi_{\mathcal{F}}$ for any $\mathcal{F}$ admitting a deformation. Therefore we define $\pi_{\mathcal{F}} = (\pi^*_{S\times S}\pi_0)(\mathcal{F})$ for general coherent $\mathcal{A}_S$-module $\mathcal{F}$.

(iii) The ***universal obstruction class***

$$\omega_0 := \pi_0[1] \circ \delta_0 : \Delta_*\mathcal{O}_S \to \Delta_* I[2] \quad \text{in } \mathcal{D}(\mathcal{O}_{S\times S})$$

decomposes into

$$\omega_0 : \Delta_*\mathcal{O}_S \xrightarrow{\alpha_S} \Delta_*\mathbb{L}_S[1] \xrightarrow{\Delta_*\kappa(S/\overline{S})[1]} \Delta_* I[2],$$

where $\alpha_S \in \mathrm{Ext}^1_{S\times S}(\Delta_*\mathcal{O}_S, \Delta_*\mathbb{L}_S)$ is the ***(truncated) universal Atiyah class***, which is intrinsic and functorial to S, and $\kappa(S/\overline{S}) \in \mathrm{Ext}^1_S(\mathbb{L}_S, I)$ is the (truncated) Kodaira–Spencer class of the square-zero extension $S \subset \overline{S}$. (See [17], Definition 2.3 and 2.7 for details.)

Definition 4.2.2. We define the Atiyah class for a coherent $\mathcal{A}_S$-module $\mathcal{F}$ to be

$$(\pi^*_{S\times S}\alpha_S)(\mathcal{F}) \in \mathrm{Ext}^1_{\mathcal{A}_S}(\mathcal{F}, \mathcal{F} \otimes \pi^*_S\mathbb{L}_S).$$

The proofs can be found in [17, Section 2.5 and Section 3.1]. We remark that while [17] only consider coherent sheaves, the proofs are all done at the level of Fourier–Mukai kernels. Therefore it also works for any coherent $\mathcal{A}$-modules.

Existence of deformations

We will show that if the obstruction class $\delta(\pi_{\mathcal{F}}) = (\pi^*_{S\times S}\omega_0)(\mathcal{F})$ vanishes, then there exists an $\mathcal{A}$-module extension of $\mathcal{F}$ over $\overline{S}$.

First we remark that we may assume both X and S are affine as in [17], Section 3.3. Although the existence of deformations is ***not*** a local property, for any given class $e \in \mathrm{Ext}^1_{\mathcal{A}_S}(Li^*Ri_*\mathcal{F}, \mathcal{F} \otimes \pi^*_S I)$, the conditions that Φ_e being an isomorphism and $\Psi_e = \pi_{\mathcal{F}}$ can be checked locally. In other words, if $\delta(\pi_{\mathcal{F}}) = 0$

but there is no deformation of $\mathcal{F}$, then there is a class e such that $\Psi_e = \pi_{\mathcal{F}}$ but e is not given by a deformation. Then we can find an affine open set U such that $\Phi_e|_U$ is not an isomorphism but $\Psi_e|_U = \pi_{\mathcal{F}}|_U$, which is a contraction.

Therefore we may assume that $X = \mathrm{Spec}(B)$, $\mathcal{A} = A$ is an R-algebra, $S = \mathrm{Spec}(R)$, and $\overline{S} = \mathrm{Spec}(\overline{R})$. Let M be a (left) $A_R := A \otimes_{\mathbb{C}} R$-module, flat over R. For convenience, a tensor product $\otimes$ without subscript is over R. We first recall the standard obstruction theory for modules (cf. [22]).

Choose a (possibly-infinite) free resolution of M

$$\ldots \to A_R^{\oplus n_{-3}} \xrightarrow{d_{-3}} A_R^{\oplus n_{-2}} \xrightarrow{d_{-2}} A_R^{\oplus n_{-1}} \to M \to 0.$$

Consider the trivial deformation $A_{\overline{R}}^{\oplus n_\bullet}$ of $A_R^{\oplus n_\bullet}$. Then we choose an arbitrary lifting $d'_\bullet : A_{\overline{R}}^{\oplus n_\bullet} \to A_{\overline{R}}^{\oplus n_{\bullet+1}}$ of $d_\bullet$. Since $(d'_{\bullet+1} \circ d'_\bullet)|_R = d_{\bullet+1} \circ d_\bullet = 0$, the map $d'_{\bullet+1} \circ d'_\bullet$ factors into

$$d'_{\bullet+1} \circ d'_\bullet : A_{\overline{R}}^{\oplus n_\bullet} \to A_R^{\oplus n_\bullet} \xrightarrow{\mathrm{ob}_\bullet} A_R^{\oplus n_{\bullet+2}} \otimes I \to A_{\overline{R}}^{\oplus n_{\bullet+2}},$$

where the first and third arrows are given by the extension $0 \to I \to \overline{R} \to R \to 0$. It is well-known that the class

$$\{\mathrm{ob}_\bullet\} \in \mathrm{Hom}_{A_R}(A_R^{\oplus n_\bullet}, A_R^{\oplus n_{\bullet+2}} \otimes I) \tag{4.7}$$

is a 2-cocycle defining a class in $\mathrm{Ext}^2_{A_R}(M, M)$ which is independent of the choice of the resolution $(A_R^{\oplus n_\bullet}, d_\bullet)$ and lifting $d'_\bullet$. We will show that this obstruction class is the same as $\delta(\pi_{\mathcal{F}})$.

Recall that the universal obstruction class $\omega_0 \in \mathrm{Ext}^2_{S\times S}(\Delta_*\mathcal{O}_S, \Delta_* I)$ is represented by the 2-extension

$$0 \to \Delta_* I \to \mathcal{J}|_{S\times S} \to \mathcal{O}_{S\times S} \to \Delta_*\mathcal{O}_S \to 0,$$

where $\mathcal{J}$ is the ideal sheaf defining $\overline{S} \subset \overline{S} \times \overline{S}$. We omit the details but a crucial consequence is that if $M = A_R^{\oplus n}$ is free, then the obstruction class $(\pi^*_{S\times S}\omega_0)(M) \in \mathrm{Ext}^2_{A_R}(M, M \otimes I)$ is represented by the 2-extension (of A_R-

modules)

$$0 \to A_R^{\oplus n} \otimes I \to K^{\oplus n}|_R \to \Gamma^{\oplus n}|_R \to A_R^{\oplus n} \to 0,$$

where the restriction $-|_R$ is the tensor product $- \otimes_{\overline{R}} R$, $\Gamma = A_{\overline{R}} \otimes_{\mathbb{C}} R$ is the free $A_{\overline{R}}$-module, the arrow

$$\Gamma = A_{\overline{R}} \otimes_{\mathbb{C}} R = A \otimes_{\mathbb{C}} \overline{R} \otimes_{\mathbb{C}} R \to A_R = A \otimes_{\mathbb{C}} R$$

is induced by the $\overline{R}$-linear evaluation map $\overline{R} \otimes_{\mathbb{C}} R \to R$ via $\overline{R} \to R$, and $K = \ker(\Gamma \to A_R)$ is the kernel. Since $\overline{R} \otimes_{\mathbb{C}} R$ is a free $\overline{R}$-module, we may choose a (non-canonical) splitting

$$\overline{R} \otimes_{\mathbb{C}} R \cong L \oplus \overline{R}$$

such that the evaluation map $\overline{R} \otimes_{\mathbb{C}} R$ is given by the short exact sequence

$$0 \to L \oplus I \to L \oplus \overline{R} \to R \to 0.$$

Then $\Gamma \cong N \oplus A_{\overline{R}}$, $K \cong N \oplus (A_R \otimes I)$, where $N = A \otimes_{\mathbb{C}} L$ is a free $A_{\overline{R}}$-module.

Let $A_R^{\oplus n_\bullet} \to M$ be a free resolution. Then it associates a 2-extension

$$0 \to A_R^{\oplus n_\bullet} \otimes I \to K^{\oplus n_\bullet}|_R \to \Gamma^{\oplus n_\bullet}|_R \to A_R^{\oplus n_\bullet} \to 0 \tag{4.8}$$

of A_R-modules, and a short exact sequence

$$0 \to K^{\oplus n_\bullet} \to \Gamma^{\oplus n_\bullet} \to A_R^{\oplus n_\bullet} \to 0$$

of complexes of $A_{\overline{R}}$-modules, where the differentials are arbitrarily chosen lifting of the differentials in (4.8). We write down the differentials explicitly with respect to the splitting $\Gamma \cong N \oplus A_{\overline{R}}$ and $K \cong N \oplus (A_R \otimes I)$:

$$\begin{tikzcd}
0 \arrow[r] & N^{\oplus n_\bullet} \oplus (A_R \otimes I)^{\oplus n_\bullet} \arrow[r] \arrow[d, "\left(\begin{smallmatrix} * & * \\ \eta_\bullet & * \end{smallmatrix}\right)"'] & N^{\oplus n_\bullet} \oplus A_{\overline{R}}^{\oplus n_\bullet} \arrow[r] \arrow[d, "\left(\begin{smallmatrix} * & \gamma_\bullet \\ \eta_\bullet & d'_\bullet \end{smallmatrix}\right)"'] & A_R^{\oplus n_\bullet} \arrow[r] \arrow[d, "d_\bullet"] & 0 \\
0 \arrow[r] & N^{\oplus n_{\bullet+1}} \oplus (A_R \otimes I)^{\oplus n_{\bullet+1}} \arrow[r] & N^{\oplus n_{\bullet+1}} \oplus A_{\overline{R}}^{\oplus n_{\bullet+1}} \arrow[r] & A_R^{\oplus n_{\bullet+1}} \arrow[r] & 0
\end{tikzcd}$$

and

$$\begin{tikzcd}
0 \arrow[r] & (A_R \otimes I)^{\oplus n_\bullet} \arrow[r] \arrow[d] & N^{\oplus n_\bullet}|_R \oplus (A_R \otimes I)^{\oplus n_\bullet} \arrow[r] \arrow[d, "\begin{pmatrix} * & * \\ \sigma_\bullet & * \end{pmatrix}"'] & N^{\oplus n_\bullet}|_R \oplus A_R^{\oplus n_\bullet} \arrow[r] \arrow[d, "\begin{pmatrix} * & \beta_\bullet \\ * & d_\bullet \end{pmatrix}"'] & A_R^{\oplus n_\bullet} \arrow[r] \arrow[d, "d_\bullet"] & 0 \\
0 \arrow[r] & (A_R \otimes I)^{\oplus n_{\bullet+1}} \arrow[r] & N^{\oplus n_{\bullet+1}}|_R \oplus (A_R \otimes I)^{\oplus n_{\bullet+1}} \arrow[r] & N^{\oplus n_{\bullet+1}}|_R \oplus A_R^{\oplus n_{\bullet+1}} \arrow[r] & A_R^{\oplus n_{\bullet+1}} \arrow[r] & 0
\end{tikzcd}$$

Observe that

$$\left\{ A_R^{\oplus n_\bullet} \xrightarrow{\beta_\bullet} N^{\oplus n_{\bullet+1}}|_R \xrightarrow{\sigma_{\bullet+1}} (A_R \otimes I)^{\oplus n_{\bullet+2}} \right\} \in \mathrm{Hom}_{A_R}(A_R^{\oplus n_\bullet}, A_R^{\oplus n_{\bullet+2}} \otimes I)$$

defines a 2-cocycle which corresponds to the class of the 2-extension (4.8) in $\mathrm{Ext}^2_{A_R}(M, M \otimes I)$, i.e., the obstruction class $\delta(\pi_{\mathcal{F}})$.

On the other hand, $d'_\bullet : A_{\overline{R}}^{\oplus n_\bullet} \to A_{\overline{R}}^{\oplus n_{\bullet+1}}$ is a lifting of $d_\bullet$. The differentials on $\Gamma^{\oplus n_\bullet}$ implies that

$$-d'_{\bullet+1} \circ d'_\bullet = \eta_{\bullet+1} \circ \gamma_\bullet : A_{\overline{R}}^{\oplus n_\bullet} \to N^{\oplus n_{\bullet+1}} \to A_{\overline{R}}^{\oplus n_{\bullet+2}}.$$

Since $\eta_{\bullet+1}$ factors through $N^{\oplus n_{\bullet+1}} \to (A_R \otimes I)^{\oplus n_{\bullet+2}} \to A_{\overline{R}}^{\oplus n_{\bullet+2}}$, the composition can be decomposed into

$$\eta_{\bullet+1} \circ \gamma_\bullet : A_{\overline{R}}^{\oplus n_\bullet} \to A_R^{\oplus n_\bullet} \xrightarrow{\gamma_\bullet|_R} N^{\oplus n_{\bullet+1}}|_R \xrightarrow{\eta_{\bullet+1}|_R} (A_R \otimes I)^{\oplus n_{\bullet+2}} \to A_{\overline{R}}^{\oplus n_{\bullet+2}}.$$

By definition, $\gamma_\bullet|_R = \beta_\bullet$ and $\eta_{\bullet+1}|_R = \sigma_{\bullet+1}$. This shows that the classical obstruction class (4.7) defined by the lifting $d'_\bullet$ is the class $-\delta(\pi_{\mathcal{F}})$.

Finally, suppose e is the class corresponding to a deformation $(A_{\overline{R}}^{\oplus n_\bullet}, d'_\bullet)$, and e' is another class such that $\Psi_e = \Psi_{e'}$. Then $e - e'$ is in the image of a 1-cocycle

$$\{f_\bullet\} \in \mathrm{Hom}_{A_R}(A_R^{\oplus n_\bullet}, A_R^{\oplus n_{\bullet+1}} \otimes I).$$

Then it is a standard fact that $(A_{\overline{R}}^{\oplus n_\bullet}, d'_\bullet + \tilde{f}_\bullet)$ also defines a deformation, where

$$\tilde{f}_\bullet : A_{\overline{R}}^{\oplus n_\bullet} \to A_R^{\oplus n_\bullet} \xrightarrow{f_\bullet} A_R^{\oplus n_{\bullet+1}} \to A_{\overline{R}}^{\oplus n_{\bullet+1}},$$

which then corresponds to the class e'.

4.3 Donaldson–Thomas invariants for $\mathrm{Coh}(\mathcal{A})$

We now restrict our attention to $(X, \mathcal{A})$ being CY3. Let M be a quasi-projective coarse moduli scheme of stable $\mathcal{A}$-modules with a universal twisted $\mathcal{A}_M$-module $\mathcal{F}$ on $X \times M$.

First we construct a symmetric bilinear form

$$\theta : R\pi_{M*}R\mathcal{H}\mathrm{om}_{\mathcal{A}_M}(\mathcal{F}, \mathcal{F}) \to \Big(R\pi_{M*}R\mathcal{H}\mathrm{om}_{\mathcal{A}_M}(\mathcal{F}, \mathcal{F})\Big)^{\vee}[1].$$

We write $\mathbb{F} = R\mathcal{H}\mathrm{om}_{\mathcal{A}_M}(\mathcal{F}, \mathcal{F})$. For a given isomorphism $\omega_{\mathcal{A}} \to \mathcal{A}$ of $\mathcal{A}$-bimodules, it induces an isomorphism

$$R\mathcal{H}\mathrm{om}_{\mathcal{A}_M}(\mathcal{F}, \omega_{\mathcal{A}} \otimes_{\mathcal{A}} \mathcal{F}) \to \mathbb{F} \quad \text{in } \mathcal{D}(\mathcal{O}_{X\times M}).$$

Taking $(-)^{\vee}[-1]$ on both sides, it gives an isomorphism

$$(R\pi_{M*}\mathbb{F})^{\vee}[-1] \to \Big(R\pi_{M*}R\mathcal{H}\mathrm{om}_{\mathcal{A}_M}(\mathcal{F}, \omega_{\mathcal{A}} \otimes_{\mathcal{A}} \mathcal{F})\Big)^{\vee}[-1] \quad \text{in } \mathcal{D}(\mathcal{O}_M), \tag{4.9}$$

and the right hand side is isomorphic to

$$R\pi_{M*}\Big(\big(\pi_X^*\omega_{\mathcal{A}} \otimes_{\mathcal{A}_M} \mathcal{F}\big)^{\vee} \otimes_{\mathcal{A}_M} \mathcal{F} \otimes \pi_X^*\omega_X\Big)[2].$$

Note that

$$\begin{aligned}
\big(\pi_X^*\omega_{\mathcal{A}} \otimes_{\mathcal{A}_M} \mathcal{F}\big)^{\vee} &= R\mathcal{H}\mathrm{om}_{\mathcal{O}_{X\times M}}(\pi_X^*\omega_{\mathcal{A}} \otimes_{\mathcal{A}_M} \mathcal{F}, \mathcal{O}_{X\times M}) \\
&\cong R\mathcal{H}\mathrm{om}_{\mathcal{A}_M}(\mathcal{F}, R\mathcal{H}\mathrm{om}_{\mathcal{O}_{X\times M}}(\pi_X^*\omega_{\mathcal{A}}, \mathcal{O}_{X\times M})) \\
&\cong R\mathcal{H}\mathrm{om}_{\mathcal{A}_M}(\mathcal{F}, \mathcal{A}_M \otimes \pi_X^*\omega_X^{\vee}) \qquad \text{in } \mathcal{D}(\mathcal{A}_M^{\mathrm{op}}),
\end{aligned}$$

where the last isomorphism is given by $\pi_X^*\omega_{\mathcal{A}} \cong \mathcal{H}\mathrm{om}_{\mathcal{O}_{X\times M}}(\pi_X^*\mathcal{A}, \pi_X^*\omega_X) \cong \mathcal{A}_M^{\vee} \otimes \pi_X^*\omega_X$. Therefore

$$\big(\pi_X^*\omega_{\mathcal{A}} \otimes_{\mathcal{A}_M} \mathcal{F}\big)^{\vee} \otimes_{\mathcal{A}_M} \big(\mathcal{F} \otimes \pi_X^*\omega_X\big) \cong R\mathcal{H}\mathrm{om}_{\mathcal{A}_M}(\mathcal{F}, (\mathcal{F} \otimes \pi_X^*\omega_X) \otimes \pi_X^*\omega_X^{\vee}) \cong \mathbb{F}.$$

Thus (4.9) is an isomorphism

$$\theta : (R\pi_{M*}\mathbb{F})^\vee[-1] \to R\pi_{M*}\mathbb{F}\,[2] = \Big((R\pi_{M*}\mathbb{F})^\vee[-1]\Big)^\vee[1].$$

In fact, we have $\theta^\vee[1] = \theta$, so θ is symmetric.

Theorem 4.3.1. *The moduli scheme M carries a symmetric obstruction theory*

$$\Big(\tau^{[1,2]}R\pi_{M*}R\mathcal{H}\mathrm{om}_{\mathcal{A}_M}(\mathcal{F},\mathcal{F})\Big)^\vee[-1] \to \mathbb{L}_M$$

which in particular is a perfect obstruction theory.

Proof. Consider the map $\sigma : \mathcal{O}_M \to R\pi_{M*}\mathbb{F}$ induced by the scalar map $R\pi_M^*\mathcal{O}_M = \mathcal{O}_{X\times M} \to \mathbb{F}$. For any point $m \in M$, this map induces the scalar map $\sigma_m : \mathbb{C} \to \mathrm{Hom}_{\mathcal{A}}(\mathcal{F}_m, \mathcal{F}_m)$, which is an isomorphism since $\mathcal{F}_m$ is stable. Thus the cone of σ is the truncated complex $\tau^{\geq 1}R\pi_{M*}\mathbb{E}$.

On the other hand, we may consider

$$R\pi_{M*}\mathbb{F} \xrightarrow{\theta^{-1}[-2]} (R\pi_{M*}\mathbb{F})^\vee[-3] \xrightarrow{\sigma^\vee[-3]} \mathcal{O}_M[-3].$$

The induced map $\mathrm{Ext}^3_{\mathcal{A}}(\mathcal{F}_m, \mathcal{F}_m) \to \mathbb{C}$ on each point $m \in M$ is dual to the scalar map, which is an isomorphism. The cone of the composition $\tau^{\geq 1}R\pi_{M*}\mathbb{F} \to \mathcal{O}_M[-3]$ is the truncated complex $\tau^{[1,2]}R\pi_{M*}\mathbb{F}$.

Therefore, $\tau^{[1,2]}R\pi_{M*}\mathbb{F}$ is perfect of amplitude in degree 1 and 2, and the obstruction theory $(R\pi_{M*}\mathbb{F})^\vee[-1] \to \mathbb{L}_M$ induces a morphism

$$(\tau^{[1,2]}R\pi_{M*}\mathbb{F})^\vee[-1] \to \mathbb{L}_M.$$

Furthermore, by the construction, θ induces a symmetric bilinear form

$$\theta : (\tau^{[1,2]}R\pi_{M*}\mathbb{F})^\vee[-1] \to \big((\tau^{[1,2]}R\pi_{M*}\mathbb{F})^\vee[-1]\big)^\vee[1].$$

□

If the moduli space $M = M^{s,h}(X, \mathcal{A})$ is projective (for example, when the Hilbert polynomial h has coprime coefficients), then we may define the DT in-

variant via integration

$$\int_{[M]_{\mathrm{vir}}} 1,$$

which equals to the Behrend function weighted Euler characteristic $\chi(M, \nu_M)$ by [5]. In particular, this invariant depends only on the scheme structure of the moduli space M, which depends only on the abelian category $\mathrm{Coh}(\mathcal{A})$ with a chosen stability condition.

Donaldson–Thomas invariants

Recall that in classical Donaldson–Thomas theory, the Hilbert schemes (with a fixed curve class) are isomorphic to the moduli spaces of stable sheaves with fixed determinant. Therefore one may study Donaldson–Thomas invariants on Hilbert schemes. We do not have the notion of determinant for $\mathcal{A}$-modules, but we can still define Donaldson–Thomas type invariants on Hilbert schemes under some suitable assumptions.

Lemma 4.3.2. *Let h be a polynomial of degree ≤ 1. Suppose the stalk $\mathcal{A}_\eta$ at generic point $\eta \in X$ is a division algebra, then there is a natural morphism*

$$\mathrm{Hilb}^h(\mathcal{A}) \to M^{s,p-h}(\mathcal{A}), \tag{4.10}$$

sending any quotient $\mathcal{A} \to \mathcal{F}$ to its kernel, where p is the Hilbert polynomial of $\mathcal{A}$.

Proof. This is the analogue of the classical result that all torsion-free rank 1 sheaves are stable. Since $\mathcal{A}_\eta$ is a division algebra, any $\mathcal{A}$-submodule of $\mathcal{A}$ has the same rank as $\mathcal{A}$. □

Proposition 4.3.3. *Under the assumptions in Lemma 4.3.2, if $H^1(X, \mathcal{A}) = 0$, then the natural morphism (4.10) is an open immersion.*

Proof. Let $0 \to \mathcal{I} \to \mathcal{A} \to \mathcal{F} \to 0$ be a quotient in $\mathrm{Hilb}^h(\mathcal{A})$. Since $\mathcal{F}$ has codimension ≥ 2 support, the map $\mathcal{I} \to \mathcal{A}$ induces an isomorphism

$$\mathcal{I}^{\vee\vee} \to \mathcal{A}^{\vee\vee} \cong \mathcal{A}$$

of $\mathcal{A}$-modules. This shows that the quotient $\mathcal{A} \to \mathcal{F}$ is uniquely determined by its kernel $\mathcal{I}$, which means that the morphism (4.10) is injective.

Next, we show the morphism is étale. Let S be a scheme and $S \subset \overline{S}$ a square-zero extension. Fix a flat family of $\mathcal{A}$-module quotient

$$0 \to \mathcal{I} \to \mathcal{A}_S \to \mathcal{F} \to 0$$

on $X \times S$. Suppose there exists an $\mathcal{A}$-module extension $\overline{\mathcal{I}}$ of $\mathcal{I}$ over $\overline{S}$, then $\overline{\mathcal{I}}$ induces a short exact sequence

$$0 \to \overline{\mathcal{I}} \to \overline{\mathcal{I}}^{\vee\vee} \to \overline{\mathcal{F}} \to 0$$

of $\mathcal{A}_{\overline{S}}$-modules. Since $\overline{\mathcal{I}}^{\vee\vee}|_S = \mathcal{I}^{\vee\vee} \cong \mathcal{A}_S$, $\overline{\mathcal{I}}^{\vee\vee}$ is a deformation of $\mathcal{A}_S$ over $\overline{S}$, which by our assumption, it must be $\mathcal{A}_{\overline{S}}$. Note that taking double dual $(-)^{\vee\vee}$ in general does not preserve flatness, but it is flat over an open subset. So in our situation, $\mathcal{I}^{\vee\vee}$ is flat over S, which implies it is flat over $\overline{S}$. Thus $\mathcal{A}_{\overline{S}} \to \overline{F}$ is a deformation (as $\mathcal{A}$ modules) of the quotient $\mathcal{A}_S \to \mathcal{F}$. This is easy to see that the converse is also true: any deformation of the quotient $\mathcal{A}_S \to \mathcal{F}$ gives a deformation of $\mathcal{I}$. The proof is completed. $\square$

Consequently, the Hilbert scheme $\mathrm{Hilb}^h(\mathcal{A})$ for polynomial h with $\deg(h) \leq 1$ carries a symmetric obstruction theory. We define the DT invariant

$$\mathrm{DT}^h(\mathcal{A}) = \int_{[\mathrm{Hilb}^h(\mathcal{A})]_{\mathrm{vir}}} 1.$$

which is equal to the weighted Euler characteristic $\chi_{\mathrm{vir}}(\mathrm{Hilb}^h(\mathcal{A})$.

Example 4.3.4. Recall that any Azumaya algebra on a Calabi–Yau variety is again Calabi–Yau. Via the correspondence between Azumaya algebras and gerbes, this gives a definition of Donaldson–Thomas invariants on gerbes $\mathfrak{X} \to X$ over Calabi–Yau threefolds, which has been studied by [15].

Remark 4.3.5. While we only construct a perfect obstruction theory when $\mathcal{A}$ is Calabi–Yau, it is expected that (some truncation of) the obstruction theory (4.2) is perfect for more general $(X, \mathcal{A})$. For example, if $\mathcal{A}$ is of global dimension

≤ 2, then $\mathrm{Ext}^3_{\mathcal{A}}(\mathcal{F},\mathcal{F}) = 0$ for all $\mathcal{F}$, thus the obstruction theory is automatically perfect. This gives an analogue of Donaldson invariants for certain non-commutative surfaces.

Another example is if $\mathcal{A}$ is of global dimension 3 and $H^i(\mathcal{A}) = 0$ for all $i > 0$, then it is easy to see that $\mathrm{Ext}^3_{\mathcal{A}}(\mathcal{I},\mathcal{I}) = 0$ for any ideal sheaf $\mathcal{I}$ in $\mathrm{Hilb}^h(\mathcal{A})$ (see [25, Lemma 2]). This implies that $\mathrm{Hilb}^h(\mathcal{A})$ carries a perfect obstruction theory via Proposition 4.3.3.

Example 4.3.6. We consider the case $X = \mathrm{Spec}(\mathbb{C})$ is a point with $\mathcal{A} = A$ a finite-dimensional algebra of finite global dimension. Then (stable) coherent $\mathcal{A}$-modules are exactly finite-dimensional (irreducible) representations of A. These DT invariants give virtual counts of irreducible representations. Unfortunately, the moduli space $M^{s,n}(A)$ is in general not projective unless $n = 1$.

Our main interest is the pair $(X, \mathcal{A})$ given by the quantum Fermat quintic threefold described in 3.4. We verify that $\mathcal{A}$ satisfies the assumptions in Proposition 4.3.3. Since the graded algebra A is a domain, the stalk $\mathcal{A}_\eta$ at the generic point $\eta \in X$ is also a domain. Combining with the fact that $\mathcal{A}_\eta$ is a finite-dimensional algebra over the function field $k(X)$, $\mathcal{A}_\eta$ is a division algebra. Also $X \cong \mathbb{P}^3$ and $\mathcal{A}$ is a sum of line bundles, thus $H^1(X, \mathcal{A}) = 0$.

We will focus on Hilbert schemes of points on the quantum Fermat quintic threefold. That is, the Hilbert polynomial h is a constant. We will write

$$Z^{\mathcal{A}}(t) = \sum_{n=0}^{\infty} \chi_{\mathrm{vir}}\big(\mathrm{Hilb}^n(\mathcal{A})\big)\, t^n$$

for the generating function of degree zero DT invariants of the quantum Fermat quintic threefold.

Chapter 5

Local models of $(X, \mathcal{A})$

In this chapter, we study zero-dimensional coherent $\mathcal{A}$-modules, which we will simply call finite-dimensional $\mathcal{A}$-modules. Let $\mathrm{Coh}(\mathcal{A})_{\mathrm{fd}}$ be the category of finite-dimensional $\mathcal{A}$-modules.

5.1 Finite-dimensional $\mathcal{A}$-modules

We first note that the category $\mathrm{Coh}(\mathcal{A})_{\mathrm{fd}}$ can be studied locally. Any finite-dimensional $\mathcal{A}$-module is supported (as a coherent sheaf on X) at finitely many points.

Lemma 5.1.1. *For any finite-dimensional $\mathcal{A}$-module $\mathcal{F}$,*

$$\mathcal{F} = \bigoplus_{x \in \mathrm{supp}(\mathcal{F})} \mathcal{F}_x.$$

Proof. We have $\mathcal{F} = \bigoplus_{x \in \mathrm{supp}(\mathcal{F})} \mathcal{F}_x$ as coherent sheaves on X. Since the $\mathcal{A}$-action on $\mathcal{F}$ is defined locally, $\mathcal{F}_x$ is naturally an $\mathcal{A}$-submodule for each x. Furthermore, it is clear that the projection $\mathcal{F} \to \mathcal{F}_x$ is a morphism of $\mathcal{A}$-modules. We see that $\mathcal{F} = \bigoplus_{x \in \mathrm{supp}(\mathcal{F})} \mathcal{F}_x$ in $\mathrm{Coh}(\mathcal{A})_{\mathrm{fd}}$. $\square$

If we choose an affine open cover $\{U_i\}$ of X, then for each U_i, $\mathcal{A}|_{U_i}$ is given by a non-commutative algebra, and $\mathrm{Coh}(\mathcal{A}|_{U_i})$ is the category $\mathrm{Mod}(\mathcal{A}|_{U_i})$ of honest finite-dimensional modules. Then the categories $\{\mathrm{Coh}(\mathcal{A}|_{U_i})\}_i$ can be regarded as an affine open cover of $\mathrm{Coh}(\mathcal{A})$.

For the quantum Fermat quintic threefold $(X, \mathcal{A})$,

$$X = \mathrm{Proj}\big(\mathbb{C}[x_0, \ldots, x_4]/(x_0 + \ldots + x_4)\big)$$

is a hyperplane in $\mathbb{P}^4$. Let $\{U_{ij}\}_{i \neq j}$ be the affine open cover of X defined by $U_{ij} = (x_i x_j \neq 0)$.

Lemma 5.1.2. *For each $i \neq j$, there is an isomorphism $f : U_{01} \to U_{ij}$ such that $f^*(\mathcal{A}|_{U_{ij}})$ is isomorphic to A, up to a possible change of primitive root $q \in \mu_5$.*

Proof. This is proved by an explicit computation. For each $i \neq j$, let $\sigma \in S_5$ be a permutation mapping $\{i, j\}$ to $\{0, 1\}$. Then σ defines a change of variables and induces an automorphism $f : U_{01} \to U_{ij}$. In this case we can compute the non-commutative algebra $f^*(\mathcal{A}|_{U_{ij}})$ as below. We see that there exists a permutation σ so that $f^*(\mathcal{A}|_{U_{ij}})$ is equal to A, after a possible change of primitive root $q \in \mu_5$. $\square$

Therefore it is sufficient to study the non-commutative algebra $A := \mathcal{A}|_{U_{01}}$.

Now we write down the non-commutative algebra A explicitly. By definition, A is the degree zero part of the graded algebra

$$\left(\mathbb{C}\langle t_0, \ldots, t_4\rangle \Big/ \Big(\sum_k t_k^5, t_i t_j - q_{ij} t_j t_i\Big)\right)\left[\frac{1}{t_0^5}, \frac{1}{t_1^5}\right].$$

Then we can write

$$A = \mathbb{C}\Big\langle u_1, \frac{1}{u_1}, u_2, u_3, u_4\Big\rangle \Big/ \left(1 + \sum_{k=1}^{4} u_k^5, u_i u_j - \overline{q}_{ij} u_j u_i\right), \tag{5.1}$$

where $u_i = (t_i t_0^4)/(t_0^5)$ and

$$(\overline{q}_{ij}) = \begin{pmatrix} 1 & q^3 & q^4 & q^3 \\ q^2 & 1 & q^4 & q^4 \\ q & q & 1 & q^3 \\ q^2 & q & q^2 & 1 \end{pmatrix}. \tag{5.2}$$

5.2 Simple $\mathcal{A}$-modules and the Ext-quiver

As seen in [21], semistable objects in a Calabi–Yau-3 category should be locally given by representations of quivers with potential, and the quivers are obtained by the Ext-quivers of stable objects. See also [34] for the case of the category of coherent sheaves on a Calabi–Yau threefold.

For the category $\mathrm{Coh}(\mathcal{A})_{\mathrm{fd}}$, a finite-dimensional $\mathcal{A}$-module is always semistable, and it is stable if and only if it is simple. We consider the natural forgetful map

$$\mathrm{Hilb}^n(\mathcal{A}) \to M^n(\mathcal{A}) := M^{\mathrm{ss},n}(\mathcal{A}).$$

This is the analogue of Hilbert–Chow map. The closed points of the coarse moduli scheme $M^n(\mathcal{A})$ correspond to polystable $\mathcal{A}$-modules, that is, semisimple $\mathcal{A}$-modules.

As shown in the previous section, we only need to consider finite-dimensional A-modules.

Lemma 5.2.1. *All simple A-modules have dimension 1 or a multiple of 5. Furthermore, there are exactly 5 one-dimensional A-modules given by*

$$(u_1, u_2, u_3, u_4) = (\xi, 0, 0, 0),$$

where $\xi \in \mathbb{C}$ and $1 + \xi^5 = 0$.

Proof. Let V be a d-dimensional simple $\mathcal{A}$-module. We abuse the notation and write $u_i \in \mathrm{End}_{\mathbb{C}}(V)$ for the action of $u_i \in A$. The relations $u_i u_j = q_{ij} u_j u_i$ imply that for each i, $\ker(u_i) \subset V$ is an invariant subspace. Thus for each i, u_i is either 0 or invertible. Next, taking determinants of the relations yields

$$\det(u_i)\det(u_j) = q_{ij}^d \det(u_j)\det(u_i).$$

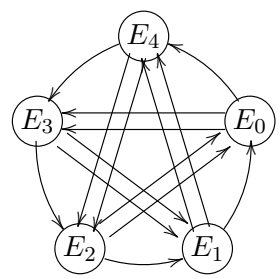
If d is not a multiple of 5, then $q_{ij}^d \neq 1$. Since u_0 is invertible, $\det(u_i) = 0$ for all $i \neq 0$ and thus $u_i = 0$. We conclude that $d = 1$ and u_0 acts as a scalar $\xi \in \mathbb{C}$ with $1 + \xi^5 = 0$. $\square$

We observe that the 5 one-dimensional simple A-modules are supported at the point $p_{01} = [1 : -1 : 0 : 0 : 0] \in X$, and they are all simple A-modules

supported at p_{01}. We denote these 5 simple modules by E_i's, which corresponds to $\xi = -q^i$, for $i \in \mathbb{Z}/5$.

Definition 5.2.2. The Ext-quiver Q associated to $\{E_i\}_i$ is the quiver whose vertex set $Q_0 = \{E_i\}_i$ and the number of arrows from E_i to E_j is equal to the dimension of $\mathrm{Ext}^1_A(E_i, E_j)$.

Proposition 5.2.3. *The* Ext*-quiver Q associated to $(E_0, E_1, \ldots, E_4)$ is*



(5.3)

Proof. The group $\mathrm{Ext}^1_A(E_i, E_j)$ is classified by extensions $0 \to E_j \to F \to E_i \to 0$. The u_i-action on F is of the form

$$u_0 = \begin{pmatrix} -q^j & 0 \\ 0 & -q^i \end{pmatrix} \text{ and } u_k = \begin{pmatrix} 0 & a_k \\ 0 & 0 \end{pmatrix} \text{ for } k > 0.$$

The relations implies that $a_k(\overline{q}_{1k}q^i - q^j) = 0$. Thus

$$\dim \mathrm{Ext}^1_A(E_i, E_j) = \text{the number of } k\text{'s such that } \overline{q}_{1k} = q^{j-i}.$$

□

We denote the arrows of Q by $a_i, c_i : E_i \to E_{i+3}$ and $b_i : E_i \to E_{i+4}$. From the construction of the Ext-quiver Q, we see that the arrows a_i's, b_i's, and c_i's correspond to the actions of u_2, u_3, and u_4 respectively. Then the q-commuting relations translate into

$$\begin{aligned} a_i b_{i+1} - q^4 b_{i+4} a_{i+1} &= 0, \\ b_i c_{i+2} - q^3 c_{i+1} b_{i+2} &= 0, \\ a_i c_{i+2} - q^4 c_i a_{i+2} &= 0. \end{aligned}$$

As expected, these relations can be patched into a potential

$$W = \boldsymbol{bac} - q^4 \boldsymbol{bca},$$

where

$$\begin{aligned}
\boldsymbol{a} &= a_0 + a_1 + a_2 + a_3 + a_4, \\
\boldsymbol{b} &= q^4 b_0 + b_1 + q b_2 + q^2 b_3 + q^3 b_4, \\
\boldsymbol{c} &= c_0 + c_1 + c_2 + c_3 + c_4.
\end{aligned}$$

In the next section, we will show that the quiver (Q, W) with potential in fact gives a local model of $\mathcal{A}$ near the point $p_{01} \in X$.

5.3 Local models of $(X, \mathcal{A})$

Let (Q, W) be the quiver with potential from the previous section.

Lemma 5.3.1. *The Jacobi algebra* $\mathrm{Jac}(Q, W)$ *is isomorphic to*

$$\mathbb{C}\langle e, u, v, w\rangle_{(\overline{q}_{ij})} \big/ (e^5 - 1),$$

where $(\overline{q}_{ij})$ *is the quantum parameters* (5.2).

Proof. Consider the element $e = e_0 + qe_1 + q^2 e_2 + q^3 e_3 + q^4 e_4 \in \mathbb{C}Q$, where e_i is the idempotent corresponding to the vertex i, and $u = a_0 + a_1 + a_2 + a_3 + a_4$, $v = b_0 + b_1 + b_2 + b_3 + b_4$, and $w = c_0 + c_1 + c_2 + c_3 + c_4$. It is clear that $e^5 = 1$ and the elements e, u, v, w satisfy the appropriated q-commuting relations. Therefore $\mathbb{C}\langle e, u, v, w\rangle_{(\overline{q}_{ij})} \big/ (e^5 - 1)$ is naturally a subalgebra of $\mathrm{Jac}(Q, W)$.

To show that they are isomorphic, it is sufficient to show that the element e generates e_i for all i. For each k, we have $e^k = e_0 + q^k e_1 + q^{2k} e_2 + q^{3k} e_3 + q^{4k} e_4$. So

$$\begin{pmatrix} 1 \\ e \\ e^2 \\ e^3 \\ e^4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & q & q^2 & q^3 & q^4 \\ 1 & q^2 & q^4 & q & q^3 \\ 1 & q^3 & q & q^4 & q^2 \\ 1 & q^4 & q^3 & q^2 & q \end{pmatrix} \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix}.$$

The matrix in the middle is a Vandermonde matrix, which is invertible. This completes the proof. □

Remark 5.3.2. One alternative description of the Jacobi algebra $\mathrm{Jac}(Q, W)$ is as follows. Consider the algebra

$$\mathbb{C}\langle u, v, w\rangle_q := \mathbb{C}\langle u, v, w\rangle / \left(uv - q^4 vu, vw - q^4 wv, wu - q^4 wu\right),$$

which is the Jacobi algebra of a quantized affine 3-space ([11]). Let $G = \mu_5$ with an action on $\mathbb{C}\langle u, v, w\rangle_q$ defined by

$$q \cdot (u, v, w) = (q^3 u, q^4 v, q^3 w).$$

Then the Jacobi algebra $\mathrm{Jac}(Q, W)$ is isomorphic to the crossed product $\mathbb{C}\langle u, v, w\rangle_q \rtimes \mu_5$, note that here the v corresponds to $q^4 b_0 + b_1 + q b_2 + q^2 b_3 + q^3 b_4$. From this perspective, the Jacobi algebra $\mathrm{Jac}(Q, W)$ is a quantization of the orbifold $[\mathbb{C}^3/\mu_5]$.

The Jacobi algebra $\mathrm{Jac}(Q, W)$ contains a subalgebra

$$\mathbb{C}[x, y, z] := \mathbb{C}[u^5, v^5, w^5] \subset Z(\mathrm{Jac}(Q, W))$$

and is a finite $\mathbb{C}[x, y, z]$-module. Thus $\mathrm{Jac}(Q, W)$ can be regarded as a sheaf $\mathcal{J}$ of non-commutative algebras on $\mathbb{C}^3 = \operatorname{Spec} \mathbb{C}[x, y, z]$. There is a canonical embedding

$$U_{01} \hookrightarrow \mathbb{C}^3, \left(\frac{x_2}{x_0}, \frac{x_3}{x_0}, \frac{x_4}{x_0}\right) = (x, y, z) \tag{5.4}$$

which maps the special point p_{01} to the origin. For any subset $U \subset U_{01}$, we will simply identify it with its image in $\mathbb{C}^3$ without further comment.

Theorem 5.3.3. *For any point $p \in U_{01}$, there is an* analytic *open neighborhood $U \subset U_{01}$ of p such that there is a (non-unique) isomorphism*

$$\mathcal{A}|_U \cong \mathcal{J}|_U$$

of sheaves of non-commutative algebras on U.

Proof. Both sheaves $\mathcal{A}$ and $\mathcal{J}$ are locally free of rank 625, and we can write them down explicitly,

$$\mathcal{A}|_U = \mathcal{O}_U\langle u_1, u_2, u_3, u_4\rangle_{(\overline{q}_{ij})} \Big/ \left(1 + \sum_{k=1}^{4} u_k^5, u_2^5 - x, u_3^5 - y, u_4^5 - z\right)$$

and

$$\mathcal{J}|_U = \mathcal{O}_U\langle e, u, v, w\rangle_{(\overline{q}_{ij})} \Big/ \left(e^5 - 1, u^5 - x, v^5 - y, w^5 - z\right),$$

where $x, y, z \in H^0(U, \mathcal{O}_U)$ are (holomorphic) functions corresponding to the coordinates of $U \subset \mathbb{C}^3$.

Suppose $U \subset \mathbb{C}^3$ is an analytic open subset such that 5-th roots of the holomorphic function $-1 - x - y - z$ are well-defined, that is, there exists an element

$$f(x, y, z) \in H^0(U, \mathcal{O}_U) \text{ such that } f(x, y, z)^5 = -1 - x - y - z.$$

Then we define a morphism of sheaves of non-commutative algebras

$$\mathcal{A}|_U \to \mathcal{J}|_U, (u_1, u_2, u_3, u_4) \mapsto \big(f(x, y, z)e, u, v, w\big).$$

This defines an isomorphism since $f(x, y, z)$ is non-vanishing on U and thus is invertible.

Finally, U_{01} is the open subset of $\mathbb{C}^3$ defined by $x + y + z \neq -1$. Therefore it can be covered by analytic open subsets satisfying the property above. $\square$

Next, we analyze the Jacobi algebra $\mathrm{Jac}(Q, W)$ in more detail.

Proposition 5.3.4. *Let $p = (x_0, x_1, x_2) \in \mathbb{C}^3$ with $x_0 \neq 0$. Then there is an analytic open neighborhood $U \subset \mathbb{C}^3$ of p such that*

$$\mathcal{J}|_U \cong M_5(\mathbb{C}) \underset{\mathbb{C}}{\otimes} \Big(\mathcal{O}_U[v, w] \big/ \left(v^5 - y, w^5 - x\right)\Big),$$

where $M_5(\mathbb{C})$ is the ring of 5-by-5 matrices. Similar results also hold for points with $y_0 \neq 0$ or $z_0 \neq 0$.

Proof. Recall that

$$\mathcal{J}|_U = \mathcal{O}_U\langle e, u, v, w\rangle_{(\overline{q}_{ij})} / \left(e^5 - 1, u^5 - x, v^5 - y, w^5 - z\right),$$

Since $x_0 \neq 0$, we can choose an analytic open neighborhood U of p such that there exists a holomorphic function $f(x, y, z)$ on U such that $f(x, y, z)^5 = x$. We consider a change of coordinates

$$e_1 = e, \quad e_2 = \frac{u}{f}, \quad \overline{v} = (e_1^3 e_2^2)v, \quad \overline{w} = (e_1^4 e_2^3)w.$$

Then $\mathcal{J}|_U$ is generated by $e_1, e_2, \overline{v}, \overline{w}$ since $e_2^5 = 1$. A straightforward computation shows that the elements $\overline{v}, \overline{w}$ are, in fact, lying in the center $Z(\mathcal{J}|_U)$.

Consequently, we can write

$$\begin{aligned}\mathcal{J}|_U &= \mathcal{O}_U\langle e_1, e_2\rangle_{(q_{12})}[\overline{v}, \overline{w}] / \left(e_1^5 - 1, e_2^5 - 1, \overline{v}^5 - y, \overline{w}^5 - z\right) \\ &\cong \left(\mathbb{C}\langle e_1, e_2\rangle_{(q_{12})} / (e_1^5 - 1, e_2^5 - 1)\right) \underset{\mathbb{C}}{\otimes} \left(\mathcal{O}_U[\overline{v}, \overline{w}] / (\overline{v}^5 - y, \overline{w}^5 - z)\right).\end{aligned}$$

To see that the finite-dimensional Frobenius algebra

$$\mathbb{C}\langle e_1, e_2\rangle / \left(e_1 e_2 - q^3 e_2 e_1, e_1^5 - 1, e^2 - 1\right)$$

is isomorphic to $M_5(\mathbb{C})$, one may verify that the morphism defined by

$$e_1 \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & q & 0 & 0 & 0 \\ 0 & 0 & q^2 & 0 & 0 \\ 0 & 0 & 0 & q^3 & 0 \\ 0 & 0 & 0 & 0 & q^4 \end{pmatrix}, \quad e_2 \mapsto \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

is an isomorphism. □

Remark 5.3.5. We may consider the finite covering $\pi : \mathbb{C}^3 \to \mathbb{C}^3$, $\pi(x, y, z) =$

(x, y^5, z^5). Then for any $U \subset \mathbb{C}^3$, we have

$$\begin{aligned}
&M_5(\mathbb{C}) \underset{\mathbb{C}}{\otimes} \Big(\mathcal{O}_U[v,w]/(v^5 - y, w^5 - z)\Big) \\
\cong\, &M_5(\mathbb{C}) \underset{\mathbb{C}}{\otimes} \pi_* \mathcal{O}_{\pi^{-1}(U)} = \pi_* \Big(M_5(\mathbb{C}) \underset{\mathbb{C}}{\otimes} \mathcal{O}_{\pi^{-1}(U)}\Big) \\
=\, &\pi_* \Big(M_5(\mathcal{O}_{\mathbb{C}^3})|_{\pi^{-1}(U)}\Big).
\end{aligned}$$

This gives an equivalence between coherent $\mathcal{J}$-modules on U and coherent $M_5(\mathcal{O}_{\mathbb{C}^3})$-modules on $\pi^{-1}(U)$.

Now we consider a stratification

$$\mathbb{C}^3 = \mathbb{C}^3_{(0)} \coprod \mathbb{C}^3_{(1)} \coprod \mathbb{C}^3_{(2)} \coprod \mathbb{C}^3_{(3)},$$

where $\mathbb{C}^3_{(i)}$ consists of points with exactly i coordinates being non-zero. Let $p(x_0, y_0, z_0) \in \mathbb{C}^3_{(2)}$. For simplicity we assume $z_0 = 0$, then Proposition 5.3.4 shows that there is an analytic neighborhood U of p such that

$$\mathcal{J}|_U \cong \Big(M_5(\mathbb{C})^{\oplus 5}\Big) \underset{\mathbb{C}}{\otimes} \Big(\mathcal{O}_U[w]/\left(w^5 - z\right)\Big).$$

Similarly, for point $p \in \mathbb{C}^3_{(3)}$, there is an analytic neighborhood U of p such that

$$\mathcal{J}|_U \cong \Big(M_5(\mathbb{C})^{\oplus 25}\Big) \underset{\mathbb{C}}{\otimes} \mathcal{O}_U.$$

Finally, we recall that there is an open cover $\{U_{ij}\}_{i \neq j}$ such that all sheaves $\mathcal{A}|_{U_{ij}}$ of non-commutative algebras are (canonically) isomorphic. There is a natural stratification

$$X = X^3_{(0)} \coprod X_{(1)} \coprod X_{(2)} \coprod X_{(3)}, \tag{5.5}$$

where $X_{(i)}$ consists of points with exactly $i + 2$ coordinates (of $\mathbb{P}^4$) being non-zero. It is clear that the canonical isomorphism $U_{01} \cong U_{ij}$ and the canonical embedding (5.4) preserve the strata.

Definition 5.3.6. Let $p \in X$. An *analytic chart* U of p is an analytic open

neighborhood $U \subset X$ of p with an embedding $U \to \mathbb{C}^3$ mapping p to the origin.

Putting all above results together, we obtain the following theorem.

Theorem 5.3.7. *We define the sheaves of non-commutative algebras on* $\mathbb{C}^3$

$$\begin{aligned}
\mathcal{J}_{(0)} &= \mathcal{J}, \\
\mathcal{J}_{(1)} &= M_5(\mathbb{C}) \underset{\mathbb{C}}{\otimes} \Big(\mathcal{O}_{\mathbb{C}^3}[v,w] / \left(v^5 - y, w^5 - z\right) \Big), \\
\mathcal{J}_{(2)} &= \Big(M_5(\mathbb{C})^{\oplus 5} \Big) \underset{\mathbb{C}}{\otimes} \Big(\mathcal{O}_{\mathbb{C}^3}[w] / \left(w^5 - z\right) \Big), \\
\mathcal{J}_{(3)} &= \Big(M_5(\mathbb{C})^{\oplus 25} \Big) \underset{\mathbb{C}}{\otimes} \mathcal{O}_{\mathbb{C}^3}.
\end{aligned}$$

Then for all i and any point $p \in X_{(i)}$, there is an analytic chart $U \to \mathbb{C}^3$ of p such that

$$\mathcal{A}|_U \cong \mathcal{J}_{(i)}|_U.$$

Moreover, if $p \in X_{(1)}$, then the chart $U \to \mathbb{C}^3$ maps $U \cap X_{(1)}$ to the locus $(y = z = 0)$; and if $p \in X_{(2)}$, then the chart $U \to \mathbb{C}^3$ maps $U \cap X_{(2)}$ to the locus $(z = 0)$.

Remark 5.3.8. From the construction of the chart $U \to \mathbb{C}^3$, we see that if $p \in X_{(1)}$, then the chart maps $U \cap X_{(1)}$ to the locus $(y = z = 0)$. Similarly, if $p \in X_{(2)}$, then the chart maps $U \cap X_{(2)}$ to the locus $(z = 0)$. Besides, since the charts are constructed to make 5-th roots of certain holomorphic function well-defined, we can choose finitely many charts to cover $X_{(i)}$'s for each i.

In particular, if $p \in X_{(i)}$, then the category $\mathrm{Coh}(\mathcal{A})_p$ of coherent $\mathcal{A}$-modules supported at p is equivalent to the category of $\mathcal{J}_{(i)}$-modules supported at the origin.

Corollary 5.3.9. *Let $p \in X_{(i)}$, $i \neq 0$. There are 5^{i-1} simple $\mathcal{A}$-modules supported at p and all of them are of dimension 5. Furthermore, the Ext-quiver associated to these simple modules is the quiver consisting of 5^{i-1} vertices, and three loops at each vertex.*

Remark 5.3.10. For $i \neq 0$, the sheaves $\mathcal{J}_{(i)}$ of algebras on $\mathbb{C}^3$ are given by the

non-commutative $\mathbb{C}[x, y, z]$-algebras

$$M_5(\mathbb{C}[x, y, z])^{\oplus 5^{i-1}},$$

which are Morita equivalent to the commutative algebras $\mathbb{C}[x, y, z]^{\oplus 5^{i-1}}$. Moreover, these algebras are Jacobi algebras of a quiver with potential. Therefore, one may interpret Theorem 5.3.7 as an explicit (analytic) local model of $(X, \mathcal{A})$ by quivers with potential.

Chapter 6

DT invariants of the quantum Fermat quintic threefold

In this chapter, we aim to give a computation of the generating function

$$Z^{\mathcal{A}}(t) = \sum_{n=0}^{\infty} \chi_{\mathrm{vir}}\big(\mathrm{Hilb}^n(\mathcal{A})\big)\, t^n$$

of degree zero DT invariants of the quantum Fermat quintic threefold.

6.1 Analytification and weighted Euler characteristics

For technical reasons, we consider the category $\mathcal{C}$ of *analytic* schemes carrying an *algebraic* constructible function. Objects in $\mathcal{C}$ are pairs (U, ν) such that U is an analytic open subset of an algebraic scheme X and ν is a function $U \to \mathbb{Z}$ that extends to an algebraic constructible function $\overline{\nu} : X \to \mathbb{Z}$. Morphisms from (U_1, ν_1) to (U_2, ν_2) are analytic morphisms $f : U_1 \to U_2$ such that $\nu_2 \circ f = \nu_1$. Also, the product exists in the category $\mathcal{C}$ given by

$$(U_1, \nu_1) \times (U_2, \nu_2) = (U_1 \times U_2, \nu_1 \times \nu_2),$$

where $(\nu_1 \times \nu_2)(x_1, x_2) = \nu_1(x_1)\nu_2(x_2)$.

It is clear that if (U_1, ν_1) and (U_2, ν_2) are isomorphic in $\mathcal{C}$, then $\chi(U_1, \nu_1) =$

$\chi(U_2, \nu_2)$.

Lemma 6.1.1. *Let X and Y be schemes. If X and Y are analytic local isomorphic, that is, there exist analytic open covers $\{U_\alpha\}_\alpha$ of X and $\{V_\alpha\}_\alpha$ of Y such that for each α, there is an analytic isomorphism $f_\alpha : U_\alpha \to V_\alpha$. Then*

$$\chi_{\mathrm{vir}}(X) = \chi_{\mathrm{vir}}(Y).$$

Proof. Since the Behrend function depends only on the analytic topology of a scheme, the analytic isomorphism f_α induces an isomorphism

$$f_\alpha : (U_\alpha, \nu_{U_\alpha}) \to (V_\alpha, \nu_{V_\alpha})$$

in $\mathcal{C}$. Then the result follows from the fact that Euler characteristics can be computed from an open cover. □

Definition 6.1.2. Let $f : X \to Y$ be a morphism of schemes, F a scheme with constructible functions $\mu : X \to \mathbb{Z}$, $\nu : F \to \mathbb{Z}$. We say

$$f : (X, \mu) \to Y$$

is an *analytic local* fibration with fibre (F, ν) if there is an analytic open cover $\{U_\alpha\}_\alpha$ of Y such that for each α, there is an isomorphism

$$\big(f^{-1}(U_\alpha), \mu\big) \cong (U_\alpha, 1) \times (F, \nu)$$

in $\mathcal{C}$. Note that $f : (X, \mu) \to (Y, 1)$ is generally not a morphism in $\mathcal{C}$.

Lemma 6.1.3. *If $f : (X, \mu) \to Y$ is an analytic local fibration with fibre (F, ν), then*

$$\chi(X, \mu) = \chi(Y) \cdot \chi(F, \nu).$$

Proof. It is easy to see that for any $c \in \mathbb{Z}$, $f : \mu^{-1}(c) \to Y$ is an analytic-local fibration with fibre $\nu^{-1}(c)$, thus

$$\chi(\mu^{-1}(c)) = \chi(Y) \cdot \chi(\nu^{-1}(c)).$$

$\square$

Since our local models of $(X, \mathcal{A})$ are given in analytic topology, it is not obvious that it gives an analytic isomorphism between algebraic moduli spaces.

For the rest of the section, let X be a quasi-projective smooth variety and $\mathcal{A}$ a locally free sheaf of non-commutative algebras on X.

Definition 6.1.4. The Hilbert–Chow map is the composition

$$\mathrm{Hilb}^n(\mathcal{A}) \to M^n(\mathcal{A}) \to M^n(X) \cong \mathrm{Sym}^n(X),$$

where the middle morphism sends a finite-dimension $\mathcal{A}$-module to its underlying coherent sheaves on X, which has zero-dimensional support and is of length n.

For any analytic or algebraic subset $S \subset X$, we define the fiber product

$$\begin{tikzcd} \mathrm{Hilb}^n(\mathcal{A})_S \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{Hilb}^n(\mathcal{A}) \arrow[d] \\ \mathrm{Sym}^n(S) \arrow[r, hook] & \mathrm{Sym}^n(X) \end{tikzcd}$$

in the appropriate category. In other words, $\mathrm{Hilb}^n(X, \mathcal{A})_S$ parameterizes $\mathcal{A}$-module quotients supported in S. Clearly if S is analytic open in X, then $\mathrm{Hilb}^n(\mathcal{A})$ is also analytic open in $\mathrm{Hilb}^n(\mathcal{A})$. Also if S is a locally closed subscheme of X, then $\mathrm{Hilb}^n(\mathcal{A})$ is also a locally closed subscheme in $\mathrm{Hilb}^n(\mathcal{A})$

We first prove the equivalence between algebraic and analytic finite-dimensional $\mathcal{A}$-modules.

Lemma 6.1.5. *The analytification defines an equivalence*

$$\mathrm{Coh}(\mathcal{A})_{\mathrm{fd}} \cong \mathrm{Coh}(\mathcal{A}^{\mathrm{an}})_{\mathrm{fd}}$$

of categories, where $\mathrm{Coh}(\mathcal{A}^{\mathrm{an}})_{\mathrm{fd}}$ *is the category of analytic coherent sheaves on* X *with an* $\mathcal{A}^{\mathrm{an}}$*-action.*

Proof. As mentioned before, this statement is Zariski local on X. We may assume $X = \mathrm{Spec}(R)$ is affine, and then $\mathcal{A}$ is a non-commutative R-algebra.

First, we may choose a compactification $X \subset \overline{X}$ so we can apply GAGA theorem [30]. Since analytification preserves supports of coherent sheaves, it induces an equivalence between $\mathrm{Coh}(\mathcal{O}_X)_{\mathrm{fd}}$ and $\mathrm{Coh}(\mathcal{O}_X^{\mathrm{an}})_{\mathrm{fd}}$.

While $\mathcal{O}_X^{\mathrm{an}}$ and $\mathcal{A}^{\mathrm{an}}$ are not in the category $\mathrm{Coh}(\mathcal{O}_X^{\mathrm{an}})_{\mathrm{fd}}$, for any $\mathcal{F}^{\mathrm{an}}$ in $\mathrm{Coh}(\mathcal{O}_X^{\mathrm{an}})_{\mathrm{fd}}$, the ring $\mathrm{Hom}_{\mathcal{O}_X^{\mathrm{an}}}(\mathcal{F}^{\mathrm{an}}, \mathcal{F}^{\mathrm{an}})$ is naturally an R-algebra. By definition, an $\mathcal{A}^{\mathrm{an}}$-module is given by an analytic coherent sheaf $\mathcal{F}^{\mathrm{an}}$ with a morphism

$$A \to \mathrm{Hom}_{\mathcal{O}_X^{\mathrm{an}}}(\mathcal{F}^{\mathrm{an}}, \mathcal{F}^{\mathrm{an}})$$

of R-algebras, which must be algebraic. Therefore the analytification

$$\mathrm{Coh}(\mathcal{A})_{\mathrm{fd}} \cong \mathrm{Coh}(\mathcal{A}^{\mathrm{an}})_{\mathrm{fd}}$$

is an equivalence of categories. □

Proposition 6.1.6. *Let U be an analytic open subset of X. Then analytification of $\mathrm{Hilb}^n(\mathcal{A})_U$ is the analytic moduli space parameterizing analytic cyclic $\mathcal{A}^{\mathrm{an}}|_U$-modules.*

Proof. We first note that such analytic moduli space exists. Since $\mathcal{A}^{\mathrm{an}}$ is locally free, there is a Quot space $\mathcal{Q}$ parameterizing quotients of $\mathcal{A}^{\mathrm{an}}$, and we take $\mathcal{M}$ to the closed subspace of $\mathcal{Q}$ consisting of points $[\mathcal{A}^{\mathrm{an}} \to \mathcal{F}^{\mathrm{an}}]$ such that the kernel is $\mathcal{A}^{\mathrm{an}}$-invariant.

Since any family of algebraic $\mathcal{A}$-modules is analytic, there is a canonical morphism

$$\mathrm{Hilb}^n(\mathcal{A})_U \to \mathcal{M}, \tag{6.1}$$

which is bijective from the previous lemma. Note that the previous lemma also works for any family $\mathcal{A}$-modules over a proper scheme (where GAGA applies). It gives an equivalence between infinitesimal deformations of algebraic and analytic $\mathcal{A}$-modules. In other words, (6.1) is étale and hence is an analytic isomorphism. □

Corollary 6.1.7. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two coherent sheaves of non-commutative algebras on X. Suppose there is an analytic open subset $U \subset X$ such that $\mathcal{A}_1|_U \cong \mathcal{A}_2|_U$.*

Then there is an analytic isomorphism

$$\mathrm{Hilb}^n(\mathcal{A}_1)_U \cong \mathrm{Hilb}^n(\mathcal{A}_2)_U.$$

6.2 Stratifications of Hilbert schemes

For any locally closed subset $Z \subset X$, we have $\mathrm{Hilb}^n(\mathcal{A})_Z$, the locally closed subscheme of $\mathrm{Hilb}^n(\mathcal{A})$ parameterizing $\mathcal{A}$-module quotients supported in Z with the induced Hilbert–Chow map $\mathrm{Hilb}^n(\mathcal{A})_Z \to \mathrm{Sym}^n Z$. Let

$$Z_Z^{\mathcal{A}}(t) = \sum_{n=0}^{\infty} \chi_{\mathrm{vir}}\big(\mathrm{Hilb}^n(\mathcal{A}), \mathrm{Hilb}^n(\mathcal{A})_Z\big)\, t^n$$

be the generating function.

Proposition 6.2.1. *Let $X = X_1 \coprod X_2 \coprod \ldots \coprod X_r$ be a stratification of X. Then*

$$Z^{\mathcal{A}}(t) = \prod_{i=1}^{r} Z_{X_i}^{\mathcal{A}}(t).$$

Proof. Write π_n for the set of r-tuples $(n_1, \ldots, n_r)$ of non-negative integers such that $n_1 + \ldots + n_r = n$. The stratification $X = \coprod_i X_i$ induces a stratification

$$\mathrm{Hilb}^n(\mathcal{A}) = \coprod_{(n_1,\ldots,n_r)\in\pi_n} \mathrm{Hilb}^n(\mathcal{A})_{(n_1,\ldots,n_r)},$$

where $\mathrm{Hilb}^n(\mathcal{A})_{(n_1,\ldots,n_r)}$ parameterizes quotients $\mathcal{A} \to \mathcal{F}$ such that $\mathcal{F}|_{X_i}$ is of length n_i for all i. Then

$$\mathrm{Hilb}^n(\mathcal{A})_{(n_1,\ldots,n_r)} \cong \prod_{i=1}^{r} \mathrm{Hilb}^{n_i}(\mathcal{A})_{X_i}. \tag{6.2}$$

It remains to show that the isomorphism (6.2) induces an isomorphism

$$\big(\mathrm{Hilb}^n(\mathcal{A})_{(n_1,\ldots,n_r)}, \nu_n\big) \cong \prod_{i=1}^{r} \big(\mathrm{Hilb}^{n_i}(\mathcal{A})_{X_i}, \nu_{n_i}\big)$$

in $\mathcal{C}$, where ν_n is the Behrend function on $\mathrm{Hilb}^n(\mathcal{A})$. Let $p = [\mathcal{A} \to \mathcal{F}] \in \mathrm{Hilb}^n(\mathcal{A})$ be a closed point. We choose analytic open subsets $U_i \subset X$ such that $\overline{U_i} \cap \overline{U_j} = \varnothing$ and

$$\mathrm{supp}(\mathcal{F}) \cap X_i \subset U_i.$$

Then $\mathrm{Hilb}^n(\mathcal{A})_{\coprod_i U_i}$ is analytic open in $\mathrm{Hilb}^n(\mathcal{A})_{\coprod_i U_i} \subset \mathrm{Hilb}^n(\mathcal{A})$, and

$$\mathrm{Hilb}^n(\mathcal{A})_{\coprod_i U_i} \cong \prod_i \mathrm{Hilb}^{n_i}(\mathcal{A})_{U_i} \subset \prod_i \mathrm{Hilb}^{n_i}(\mathcal{A}).$$

Thus

$$\begin{aligned}\big(\mathrm{Hilb}^n(\mathcal{A})_{\coprod_i U_i}, \nu_n\big) &= \big(\mathrm{Hilb}^n(\mathcal{A})_{\coprod_i U_i}, \nu_{\mathrm{Hilb}^n(\mathcal{A})_{\coprod_i U_i}}\big) \\ &\cong \prod_i \big(\mathrm{Hilb}^{n_i}(\mathcal{A})_{U_i}, \nu_{\mathrm{Hilb}^{n_i}(\mathcal{A})_{U_i}}\big) = \prod_i \big(\mathrm{Hilb}^{n_i}(\mathcal{A})_{U_i}, \nu_{n_i}\big).\end{aligned}$$

which completes the proof. □

As in the classical case, $\mathrm{Hilb}^n(X, \mathcal{A})_S$ has a standard stratification indexed by partitions of n. The Hilbert–Chow map sends the stratum $\mathrm{Hilb}^n(X, \mathcal{A})_{S,(n)}$ to the diagonal $S \hookrightarrow \mathrm{Sym}^n(S)$.

Proposition 6.2.2. *Suppose the induced morphism*

$$\big(\mathrm{Hilb}^n(\mathcal{A})_{S,(n)}, \nu_n\big) \to S$$

is an analytic-local fibration with fibre (F_n, μ_n). *Then*

$$Z_S^{\mathcal{A}}(t) = \left(\sum_{n=0}^{\infty} \chi(F_n, \mu_n)\, t^n\right)^{\chi(S)}.$$

Proof. This is the analogue of [7, Theorem 4.11], but we use analytic open cover instead of étale cover. The main idea is that there is a stratification

$$\mathrm{Hilb}^n(\mathcal{A})_S = \coprod_{\alpha \vdash n} \mathrm{Hilb}^n(\mathcal{A})_{S,\alpha}$$

and for each $\alpha = (\alpha_1 \leq \ldots \leq \alpha_r) \vdash n$,

$$\mathrm{Hilb}^n(\mathcal{A})_{S,\alpha} \subset \prod_{i=1}^{r} \mathrm{Hilb}^{\alpha_i}(\mathcal{A})_{S,(\alpha_i)}.$$

By the same argument in Proposition 6.2.1, we get that

$$\big(\mathrm{Hilb}^n(\mathcal{A})_{S,\alpha}, \nu_n\big) \subset \prod_{i=1}^{r} \big(\mathrm{Hilb}^{\alpha_i}(\mathcal{A})_{S,(\alpha_i)}, \nu_{\alpha_i}\big)$$

is an immersion in $\mathcal{C}$, and by assumption, the right hand side is an analytic-local fibration over S with fibre $(F_{\alpha_i}, \nu_{\alpha_i})$. Then the formula follows from a standard calculation. □

6.3 Calculation of DT invariants

Let $(X, \mathcal{A})$ be the quantum Fermat quintic threefold. We apply Proposition 6.2.1 to the stratification (5.5) of X and obtain

$$Z^{\mathcal{A}}(t) = \prod_{i=0}^{3} Z^{\mathcal{A}}_{X_{(i)}}(t).$$

Theorem 6.3.1. *For each i,*

$$(\mathrm{Hilb}^n(\mathcal{A})_{X_{(i)},(n)}, \nu_n) \to X_{(i)}$$

is an analytic local fibration with fibre $(\mathrm{Hilb}^n\big(\mathbb{C}^3, \mathcal{J}_{(i)})_0, \mu_n\big)$*, where $\mathcal{J}_{(i)}$'s are the sheaves of algebras on $\mathbb{C}^3$ defined in Theorem 5.3.7, and μ_n is the Behrend function of* $\mathrm{Hilb}^n(\mathbb{C}^3, \mathcal{J}_{(i)})$.

Proof. We only prove for the case $i = 1$, as the proofs of the other cases are similar. First, we choose an analytic open cover $\{U_\alpha\}$ of charts. Since $\mathcal{A}|_{U_\alpha} \cong \mathcal{J}_{(1)}|_{U_\alpha}$, we have an analytic isomorphism

$$\mathrm{Hilb}^n(\mathcal{A})_{U_\alpha} \cong \mathrm{Hilb}^n(\mathbb{C}^3, \mathcal{J}_{(1)})_{U_\alpha},$$

by Corollary 6.1.7, which gives an isomorphism

$$\big(\mathrm{Hilb}^n(\mathcal{A})_{U_\alpha}, \nu_n\big) \cong \big(\mathrm{Hilb}^n(\mathbb{C}^3, \mathcal{J}_{(1)})_{U_\alpha}, \mu_n\big) \tag{6.3}$$

in $\mathcal{C}$. Let Z be the locus of $\mathbb{C}^3$ defined by $y = z = 0$. Observe that

$$\mathcal{J}_{(1)} = M_5(\mathbb{C}) \underset{\mathbb{C}}{\otimes} \Big(\mathcal{O}_{\mathbb{C}^3}[v, w] / \big(v^5 - y, w^5 - z\big)\Big)$$

is invariance under the translation of x, the Hilbert–Chow map

$$\big(\mathrm{Hilb}^n(\mathbb{C}^3, \mathcal{J}_{(1)})_{Z,(n)}, \mu_n\big) \to Z$$

is a *Zariski* local fibration with fibre $\big(\mathrm{Hilb}^n(\mathbb{C}^3, \mathcal{J}_{(i)})_0, \mu_n\big)$. Here we use the fact that the Behrend function is constant on orbits of a group action.

Finally, since the chart $U_\alpha \to \mathbb{C}^3$ map $X_{(1)}$ into Z (see Remark 5.3.8), the isomorphism 6.3 induces an isomorphism

$$\big(\mathrm{Hilb}^n(\mathcal{A})_{U_\alpha \cap X_{(1)}}, \nu_n\big) \cong \big(\mathrm{Hilb}^n(\mathbb{C}^3, \mathcal{J}_{(1)})_{U_\alpha \cap Z}, \mu_n\big)$$

Then the theorem follows. □

Next we deal with the sheaves $\mathcal{J}_{(i)}$ of algebras before we state our main theorem. Recall that for $i \neq 0$, $\mathcal{J}_{(i)}$ can be written as direct sums of $M_5(\mathcal{O}_{\mathbb{C}^3})$ (see Remark 5.3.5).

Lemma 6.3.2. *If n is not a multiple of 5, then* $\mathrm{Hilb}^n\big(\mathbb{C}^3, M_5(\mathcal{O}_{\mathbb{C}^3})\big) = \varnothing$. *For $n = 5k$, there is a canonical isomorphism*

$$\mathrm{Hilb}^{5k}\big(\mathbb{C}^3, M_5(\mathcal{O}_{\mathbb{C}^3})\big) \cong \mathrm{Quot}^k(\mathbb{C}^3, \mathcal{O}_{\mathbb{C}^3}^{\oplus 5}).$$

Proof. The Morita equivalence

$$\mathrm{Coh}(\mathbb{C}^3) \to \mathrm{Coh}(M_5(\mathcal{O}_{\mathbb{C}^3}))$$

is given by $\mathbb{C}^{\oplus 5} \otimes_{\mathbb{C}} -$, where $\mathbb{C}^{\oplus 5}$ is the canonical representation of $M_5(\mathbb{C})$. This implies that the dimension of a $M_5(\mathcal{O}_{\mathbb{C}^3})$-module must be a multiple of 5.

The isomorphism is a direct consequence of the following fact: Let R be a commutative ring, and M be an R-module. For any n, we consider the simple $M_n(R)$-module $R^{\oplus n}$. Then a morphism

$$s : M_n(R) \to R^{\oplus n} \otimes M$$

of $M_n(R)$-modules is surjective if and only if the induced morphism

$$\delta(s) : R^{\oplus n} \hookrightarrow M_n(R) \xrightarrow{s} R^{\oplus n} \otimes M \to M$$

of R-modules is surjective, where the first map is the diagonal map, and the last map is defined by $(r_1, \ldots, r_n) \otimes m \mapsto \sum_i r_i m$. □

Theorem 6.3.3. *We have*

$$Z^{\mathcal{A}}(t) = Z^{Q,W}(t)^{10} \cdot M(-t^5)^{-50},$$

where $M(t)$ is the MacMahon function.

Proof. We use Proposition 6.2.2 with Theorem 6.3.1 to obtain

$$Z^{\mathcal{A}}(t) = \prod_{i=0}^{3} \left(Z_0^{\mathbb{C}^3, \mathcal{J}_{(i)}}(t) \right)^{\chi(X_{(i)})}.$$

For $i \neq 0$, we have

$$\begin{aligned} Z_0^{\mathbb{C}^3, \mathcal{J}_{(i)}}(t) &= Z_0^{\mathbb{C}^3, M_5(\mathcal{O})^{\oplus 5^{i-1}}}(t) \\ &= \sum_{k=0}^{\infty} \chi_{\mathrm{vir}}\Big(\mathrm{Quot}^k\big((\mathbb{C}^3)^{\coprod 5^{i-1}}, \mathcal{O}^{\oplus 5}\big), \mathrm{Quot}^k\big((\mathbb{C}^3)^{\coprod 5^{i-1}}, \mathcal{O}^{\oplus 5}\big)_0 \Big)\, t^{5k} \\ &= \left(\sum_{k=0}^{\infty} \chi_{\mathrm{vir}}\big(\mathrm{Quot}^k(\mathbb{C}^3, \mathcal{O}^{\oplus 5}), \mathrm{Quot}^k(\mathbb{C}^3, \mathcal{O}^{\oplus 5})_0 \big)\, t^{5k} \right)^{5^{i-1}}. \end{aligned}$$

For the first equality, see Remark 5.3.5. The second equality is given by the previous lemma, and the third equality is a standard result about Hilbert schemes of a disjoint union of schemes.

Now, DT invariants of Quot schemes of points are well-known (for example, see [4]).

$$\sum_{k=0}^{\infty} \chi_{\mathrm{vir}}\big(\mathrm{Quot}^k(\mathbb{C}^3, \mathcal{O}^{\oplus 5}), \mathrm{Quot}^k(\mathbb{C}^3, \mathcal{O}^{\oplus 5})_0\big)\, t^k = M(-t)^5.$$

We conclude that

$$Z^{\mathcal{A}}(t) = Z_0^{\mathbb{C}^3,\mathcal{J}}(t)^{\chi(X_{(0)})} \cdot \Big(M(-t^5)^5\Big)^{\chi(X_{(1)})+5\chi(X_{(2)})+25\chi(X_{(3)})},$$

with $\chi(X_{(0)}) = 0$ and $\chi(X_{(1)}) + 5\chi(X_{(2)}) + 25\chi(X_{(3)}) = -10$.

Finally, recall that $\mathcal{J}_{(i)}$'s also are local models of $\mathcal{J}$ on strata $\mathbb{C}^3_{(i)}$ of $\mathbb{C}^3$. We repeat all above arguments to $(\mathbb{C}^3, \mathcal{J})$ which lead to

$$\begin{aligned} Z^{\mathbb{C}^3,\mathcal{J}}(t) &= Z_0^{\mathbb{C}^3,\mathcal{J}}(t)^{\chi(\mathbb{C}^3_{(0)})} \cdot \Big(M(-t^5)^5\Big)^{\chi(\mathbb{C}^3_{(1)})+5\chi(\mathbb{C}^3_{(2)})+25\chi(\mathbb{C}^3_{(3)})} \\ &= Z_0^{\mathbb{C}^3,\mathcal{J}}(t). \end{aligned}$$

since $\chi(\mathbb{C}^3_{(3)}) = \chi(\mathbb{C}^3_{(3)}) = \chi(\mathbb{C}^3_{(3)}) = 0$. The sheaf $\mathcal{J}$ on $\mathbb{C}^3$ is given by the Jacobi algebra $\mathrm{Jac}(Q, W)$, and by definition, finite-dimensional quotients of $\mathcal{J}$ are exactly framed representations of (Q, W) with framing vector $(1, 1, 1, 1, 1)$. Hence by our definition, $Z^{\mathbb{C}^3,\mathcal{J}}(t) = Z^{Q,W}(t)$. □

For the generating function $Z^{Q,W}$, we have seen that (Q, W) can be viewed as a quantization of an orbifold $[\mathbb{C}^3/\mu_5]$, and DT invariants of an orbifold is known to be related to ***colored plane partitions*** [35]. We will discuss more of this in Chapter 7, and show that $Z^{Q,W}$ can be computed using some combinatorics.

As a final remark, we give a possible geometric interpretation of this result. Recall that finite-dimension $\mathcal{A}$-modules are of dimension 1 or 5. We denote M^d_{sp} the moduli space of d-dimensional $\mathcal{A}$-modules. Then Theorem 5.3.7 implies that

(a) There is a morphism $M^1_{\mathrm{sp}} \to X_{(0)}$ which is a μ_5-torsor.

(b) M^5_{sp} is smooth, and there is a (ramified) covering $M^5_{\mathrm{sp}} \to X \setminus X_{(0)}$, which is 5^{i-1}-to-1 on the stratum $X_{(i)}$.

In particular,

$$\chi(M^5_{\mathrm{sp}}) = \chi(X_{(1)}) + 5\chi(X_{(2)}) + 25\chi(X_{(3)}).$$

The factor $M(-t^5)^{-50}$ can be expressed as

$$Z^{M^5_{\mathrm{sp}}, M_5(\mathcal{O})}(t) = \sum_{k=0}^{\infty} \chi_{\mathrm{vir}}\big(\mathrm{Quot}^k(M^5_{\mathrm{sp}}, \mathcal{O}^{\oplus 5})\big)\, t^{5k}$$

On the other hand, the M^1_{sp} consists of 50 points, and if we consider the Ext-quiver $\tilde{Q}$ associated to M^1_{sp}, then $\tilde{Q}$ is the disjoint union of 10 copies of the quiver Q. Therefore we can write

$$Z^{Q,W}(t)^{10} = Z^{\tilde{Q},\tilde{W}}(t)$$

and

$$Z^{\mathcal{A}}(t) = Z^{\tilde{Q},\tilde{W}}(t) \cdot Z^{M^5_{\mathrm{sp}}, M_5(\mathcal{O})}(t)$$

This strongly suggests that the DT invariants of the Calabi–Yau-3 category $\mathrm{Coh}(\mathcal{A})_{\mathrm{fd}}$ can be computed ***directly*** from the moduli space of simple objects in $\mathrm{Coh}(\mathcal{A})_{\mathrm{fd}}$ and the Ext-quivers between simple objects.

Chapter 7

The quiver Q and colored plane partitions

7.1 Colored plane partitions

In this section, we recall the notion of colored plane partitions and their relation with orbifolds. These results are taken from [35], [9] (see also [14]).

Let $G = \mu_r$ be the finite group of r-th roots of unity in $\mathbb{C}$. We consider the μ_r-action on $\mathbb{C}^3$ with weights (a, b, c). We will denote this action $\mu_5(a, b, c)$. We identify $\hat{G}$, the abelian group of characters, with $\mathbb{Z}/r\mathbb{Z}$.

Definition 7.1.1. A $\mu_r(a, b, c)$-colored plane partition is a plane partition $\pi \in \mathcal{P}$ with the coloring $K : \pi \to \mathbb{Z}/r\mathbb{Z}$ defined by

$$K(i, j, k) = ai + bj + ck.$$

For each color i, let $|\pi|_i$ be the number of boxes in π with color i.

We define the generating function of $\mu_r(a, b, c)$-colored plane partitions

$$Z_{\mathrm{PL}}^{\mu_r(a,b,c)}(t_0, \ldots, t_{r-1}) = \sum_{\pi \in \mathcal{P}} t_0^{|\pi|_0} \cdots t_{r-1}^{|\pi|_{r-1}}.$$

For the $\mu_r(a, b, c)$-action, we consider the ***McKay quiver*** $Q_r(a, b, c)$ whose

vertices correspond to irreducible representations of μ_5. Thus the set $Q_r(a,b,c)_0$ of vertices is identified with $\hat{\mu_r} \cong \mathbb{Z}/r\mathbb{Z}$. Arrows of $Q_r(a,b,c)$ are

$$x_i : i \to i+a, \quad y_i : i \to i+b, \quad z_i : i+c$$

for all vertex i. There is a natural potential

$$W = \boldsymbol{yxz} - \boldsymbol{yzx},$$

where $\boldsymbol{x} = \sum_i x_i$, $\boldsymbol{y} = \sum_i y_i$, and $\boldsymbol{z} = \sum_i z_i$.

Given any plane partition π, the $\mu_r(a,b,c)$-coloring defines a dimension vector

$$|\pi| := (|\pi|_0, \ldots, |\pi|_{r-1}) \in \mathbb{Z}^{\oplus Q_r(a,b,c)_0}.$$

On the other hand, the $\mu_r(a,b,c)$-action defines an orbifold $\mathcal{X} = [\mathbb{C}^3/\mu_r]$. For any $\rho \in K_0(\mathrm{Rep}(\mu_r))$, we consider the Hilbert scheme $\mathrm{Hilb}^\rho(\mathcal{X})$ parameterizing μ_5-invariant subschemes $Z \subset \mathbb{C}^3$ such that the induced μ_5-representation on $\mathcal{O}_Z$ is in the class ρ. The group $K_0(\mathrm{Rep}(\mu_r))$ is canonically identified with $\mathbb{Z}^{\oplus Q_r(a,b,c)_0}$. Furthermore, it is well-known that the Hilbert scheme $\mathrm{Hilb}^{\mathbf{d}}(\mathcal{X})$ is isomorphic to the fine moduli space $M^{\mathbf{d},e_0}(Q_r(a,b,c),W)$ of framed representations of the quiver $Q_r(a,b,c)$ with potential W, with the framing vector $e_0 = (1,0,\ldots,0)$. We define the generating function

$$Z^{\mathcal{X}}(t_0,\ldots,t_{r-1}) = \sum_{\mathbf{d}=(d_0,\ldots,d_{r-1})} \chi_{\mathrm{vir}}\big(\mathrm{Hilb}^{\mathbf{d}}(\mathcal{X})\big)\, t_0^{d_0} \cdots t_{r-1}^{d_{r-1}}$$

of DT invariants of the orbifold $\mathcal{X}$, which is equal to the generating function $Z^{Q_r(a,b,c),W,e_0}$ in our notation.

Proposition 7.1.2 ([9]). *The generating function $Z^{\mathcal{X}}$ is, up to signs, given by the generating function $Z_{\mathrm{PL}}^{\mu_r(a,b,c)}$ of colored plane partitions. More specifically, we have*

$$Z^{\mathcal{X}}(t_0,\ldots,t_{r-1}) = \sum_{\pi\in\mathcal{P}} (-1)^{|\pi|_0 + \langle|\pi|,|\pi|\rangle}\, t_0^{|\pi|_0} \cdots t_{r-1}^{|\pi|_{r-1}},$$

where $\langle -,- \rangle$ is the bilinear form associated to the quiver $Q_r(a,b,c)$.

7.2 Multi-colored plane partitions

Let $Q = (Q_0, Q_1)$ be a quiver with a labeling of arrows $\ell : Q_1 \to \{x, y, z\}$. We denote $\mathcal{S}(Q_0)$ be the set of non-empty subsets of Q_0.

Definition 7.2.1. A Q-multi-colored plane partition consists of a plane partition $\pi \in \mathcal{P}$ with a multi-coloring

$$K : \pi \to \mathcal{S}(Q_0)$$

such that for any arrow $a : v \to w$ labeled with x, if $w \in K(i, j, k)$ for some $(i, j, k) \in \pi$, then $v \in K(i-1, j, k)$, and similar conditions hold for arrows labeled with y and z. Note that there are many different Q-multi-colorings on one plane partition π.

Given a Q-multi-colored plane partition $\overline{\pi} = (\pi, K)$, it associates a dimension vector $\mathbf{w}(\overline{\pi}) := \big(w_v(\overline{\pi})\big)_{v \in Q_0}$ defined by

$$\mathbf{w}_v(\overline{\pi}) = \text{number of boxes } (i, j, k) \in \pi \text{ such that the color } v \in K(i, j, k).$$

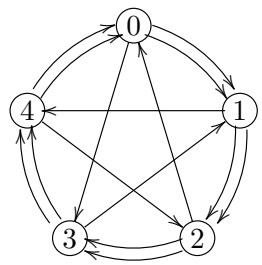
For any dimension vector $\mathbf{d} \in \mathbb{Z}^{\oplus Q_0}$, we denote by $n_Q(\mathbf{d})$ the number of Q-multi-colored plane partitions with dimension vector $\mathbf{d}$. We define the generating function

$$Z^Q_{\mathrm{PL}}(\boldsymbol{t}) = \sum_{\mathbf{d}} n_Q(\mathbf{d})\, \boldsymbol{t}^{\mathbf{d}}$$

of Q-multi-colored plane partitions, where we write $\boldsymbol{t}$ for $(t_v)_{v \in Q_0}$, $\mathbf{d} = (d_v)_{v \in Q_0}$ and $\boldsymbol{t}^{\mathbf{d}} = \prod_{\mathbf{d} \in Q_0} t_v^{d_v}$.

For simplicity, from now on we only consider the quiver (Q, W) (5.3) with potential from the quantum Fermat quintic threefold. We will rearrange the

vertices and arrows, and write



with outer arrows $x_i, y_i : i \to i+1$ and inner arrows $z_i : i+3$. The induced potential is

$$W = \boldsymbol{yxz} - q\boldsymbol{yzx},$$

where $\boldsymbol{x} = \sum x_i$, $\boldsymbol{y} = \sum y_i$, and $\boldsymbol{z} = z_0 + qz_1 + q^2 z_2 + q^3 z_3 + q^4 z_4$.

Our main theorem of the section is to associate DT invariants of (Q, W) with Q-multi-colored plane partitions.

Theorem 7.2.2. *The generating function $Z^{Q,W}$ is, up to signs, given by the generating function Z^Q_{PL} of Q-multi-colored plane partitions. More specifically, we have*

$$Z^{Q,W}(\boldsymbol{t}) = \sum_{\mathbf{d}} (-1)^{|\mathbf{d}| + \langle |\mathbf{d}|, |\mathbf{d}| \rangle} \, n_Q(\mathbf{d}) \, \boldsymbol{t}^{\mathbf{d}},$$

where $|\mathbf{d}| = \sum_i d_i$ and $\langle -, - \rangle$ is the bilinear form associated to the quiver Q.

We leave the proof of Theorem 7.2.2 to the next section.

Observe that this quiver is the same as the McKay quiver of $\mu_5(1,1,3)$. Recall that the vertices of the McKay quiver correspond to the irreducible representations of μ_5, in which there is a distinguished one, the trivial representation. However, the vertices of Q are simple $\mathcal{A}$-modules. There are exactly 5 ways to identify the quiver Q with the McKay quiver of $\mu_5(1,1,3)$, depending on a choice of a vertex in Q_0. In other words, for each $v \in Q_0$, there is a unique bijection $\alpha_v : Q_0 \to \mathbb{Z}/5\mathbb{Z}$ with $\alpha_v(v) = 0$ identifying the quiver Q with the McKay quiver of $\mu_5(1,1,3)$.

Definition 7.2.3. Let $v \in Q_0$ be a vertex. A (Q, v)-colored plane partition is a

plane partition π with the coloring $K_v : \pi \to Q_0$ making $\alpha_v \circ K_v : \pi \to \mathbb{Z}/5\mathbb{Z}$ the $\mu_5(1,1,3)$-coloring of π.

Given five plane partitions π_v indexed by Q_0. We may define a Q-multi-colored plane partition by taking π to be the union of π_v's with multi-coloring

$$K(i,j,k) = \{K_v(i,j,k) : (i,j,k) \in \pi_v\} \subset Q_0.$$

Lemma 7.2.4. *Any Q-multi-colored plane partition is uniquely determined by the (Q,v)-colored plane partitions π_v for $v \in Q_0$.*

Proof. This follows from the fact that Q can be identified with a McKay quiver of a group action on $\mathbb{C}^3$. To be more specific, the quiver Q satisfies the following properties:

(a) For each vertex i, there are exactly 3 arrows starting at i which are labeled with x, y, z; and there are exactly 3 arrows ending at i, also labeled with x, y, z.

(b) For each vertex i, the target of any non-trivial composition of arrows starting at i depends only on the numbers of arrows labeled with x, y, z.

Thus given any plane partition π with a Q-multi-coloring K, we can define π_v to be

$$\pi_v = \big\{(i,j,k) \in \pi : K_v(i,j,k) \in K(i,j,k)\big\}.$$

It is easy to check that π_v is a plane partition and the union of π_v's is π. $\square$

Corollary 7.2.5. *We have*

$$Z^Q_{\mathrm{PL}}(t_0,t_1,t_2,t_3,t_4) = \prod_{i \in \mathbb{Z}/5\mathbb{Z}} Z^{\mu_5(1,1,3)}_{\mathrm{PL}}(t_i,t_{i+1},t_{i+2},t_{i+3},t_{i+4}).$$

This reduces the computation of numbers of Q-multi-colored plane partitions to the ones for $\mu_5(1,1,3)$-colored plane partitions.

Remark 7.2.6. Unfortunately, the signs in the DT invariants $Z^{Q,W}$ and $Z^{[\mathbb{C}^3/\mu_5]}$

do not agree. That is,

$$Z^{Q,W}(t_0,t_1,t_2,t_3,t_4) \neq \prod_{i\in\mathbb{Z}/5\mathbb{Z}} Z^{[\mathbb{C}^3/\mu_5]}(t_i,t_{i+1},t_{i+2},t_{i+3},t_{i+4}),$$

and there is no obvious modification (changes of variables or taking the product in some Poisson algebra) to equalize them.

Remark 7.2.7. Both $Z^{Q,W}$ and $Z^{[\mathbb{C}^3/\mu_5]}$ are DT invariants of the quiver (Q,W) with potential, with different framing vectors. To put it another way, they are DT invariants of the same Calabi–Yau-3 category with different stability conditions. Thus there should be a formula ("wall-crossing") connecting two series $Z^{Q,W}$ and $Z^{[\mathbb{C}^3/\mu_5]}$. This can be achieved by, for example, Joyce–Song's generalized DT invariants [19]. However, it does not reduce to a simple formula (at least not obvious to us) due to the fact that the fact that the Euler pairing $\chi_Q(-,-)$ of the quiver Q is not trivial. To be more precise, the formula in [19, Corollary 7.24] does not hold.

Here we write down the series $Z^{Q,W}$ up to degree 5, where we denote $\boldsymbol{t}^{(a_0,\dots,a_4)} = \sum_i t_i^{a_0}\cdots t_{i+4}^{a_4}$:

$$\begin{aligned}
Z^{Q,W}(\boldsymbol{t}) = 1 &+ \boldsymbol{t}^{(1,0,0,0,0)} + 3\boldsymbol{t}^{(1,1,0,0,0)} - 2\boldsymbol{t}^{(1,0,1,0,0)} \\
&+ 3\boldsymbol{t}^{(1,2,0,0,0)} + \boldsymbol{t}^{(2,0,1,0,0)} - 8\boldsymbol{t}^{(1,1,1,0,0)} + 8\boldsymbol{t}^{(1,1,0,1,0)} \\
&+ \boldsymbol{t}^{(1,3,0,0,0)} + 3\boldsymbol{t}^{(2,1,1,0,0)} - 12\boldsymbol{t}^{(1,2,1,0,0)} + 7\boldsymbol{t}^{(1,1,2,0,0)} \\
&- 12\boldsymbol{t}^{(1,2,0,1,0)} + 5\boldsymbol{t}^{(1,1,0,2,0)} - 34\boldsymbol{t}^{(1,1,1,1,0)} \\
&- 3\boldsymbol{t}^{(2,2,1,0,0)} - 4\boldsymbol{t}^{(2,1,2,0,0)} - 6\boldsymbol{t}^{(1,3,1,0,0)} + 18\boldsymbol{t}^{(1,2,2,0,0)} - 2\boldsymbol{t}^{(1,1,3,0,0)} \\
&+ 8\boldsymbol{t}^{(1,3,0,1,0)} + 10\boldsymbol{t}^{(1,2,0,2,0)} + 20\boldsymbol{t}^{(2,1,1,1,0)} + 56\boldsymbol{t}^{(1,2,1,1,0)} + 35\boldsymbol{t}^{(1,1,2,1,0)} \\
&- 54\boldsymbol{t}^{(1,1,1,2,0)} - 171t_0t_1t_2t_3t_4 + O((t_0,t_1,t_2,t_3,t_4)^6)
\end{aligned}$$

Take $t = t_0 = t_1 = t_2 = t_3 = t_4$, we obtain

$$Z^{Q,W}(t) = 1 + 5t + 5t^2 + 20t^3 - 210t^4 - 131t^5 + O(t^6).$$

We conclude that

$$\begin{aligned} Z^{\mathcal{A}}(t) &= \left(Z^{Q,W}(t)\right)^{10} \cdot M(-t^5)^{-50} \\ &= 1 + 50t + 1175t^2 + 17450t^3 + 184275t^4 + 1450740t^5 + O(t^6). \end{aligned}$$

7.3 Proof of Theorem 7.2.2

We mainly follow the same method in the computation of DT invariants on $\mathbb{C}^3$ [7]. The path algebra $\mathbb{C}Q$ is

$$\mathbb{C}Q = \mathbb{C}\langle e, u, v, w\rangle / (e^5 - 1, eu - que, ev - qve, ew - q^3we).$$

There is a standard $T = (\mathbb{C}^*)^3$-action on $\mathbb{C}Q$ given by

$$(\lambda_1, \lambda_2, \lambda_3) \cdot (u, v, w) = (\lambda_1 u, \lambda_2 v, \lambda_3 w).$$

Let $T_0 \subset T$ be the sub-torus defined by $\lambda_1\lambda_2\lambda_3 = 1$, then T_0 fixes the potential W. Thus it gives a T_0-action on $\mathrm{Jac}(Q, W)$.

The Hilbert schemes $\mathrm{Hilb}^\bullet(Q, W)$ parameterize quotients of $\mathrm{Jac}(Q, W)$, which are equivalent to left ideals of $\mathrm{Jac}(Q, W)$. Therefore the T_0-action on $\mathrm{Jac}(Q, W)$ induces a T_0-action on $\mathrm{Hilb}^\bullet(Q, W)$. The following is a generalization of [7, Lemma 4.1]

Lemma 7.3.1. *For each dimension vector* $\mathbf{d}$*, there is a one-to-one correspondence between* T_0*-fixed points of* $\mathrm{Hilb}^{\mathbf{d}}(Q, W)$ *and* Q*-multi-colored plane partitions with dimension vector* $\mathbf{d}$.

Proof. Since e_i's are idempotent, any ideal in $\mathrm{Jac}(Q, W)$ is generated by polynomials of the form $e_i f(x, y, z)$ for some i and $f(u, v, w)$. We want to show that any T_0-invariant ideal can be generated by monomials. We remark that the proof of [7, Lemma 4.1] uses Hilbert's Nullstellensatz, hence it does not apply directly here. However, $\mathrm{Jac}(Q, W)$ contains a T_0-invariant subring $\mathbb{C}[u^5, v^5, w^5]$. We take $I_0 = I \cap \mathbb{C}[u^5, v^5, w^5]$ is a T_0-invariant ideal in $\mathbb{C}[u^5, v^5, w^5]$. Therefore I_0 is a monomial ideal, and particularly, there exists n such that $u^{5n}, v^{5n}, w^{5n} \in I_0 \subset I$.

Now, I is generated by eigenvectors of T_0, which are polynomials of the form $m(u,v,w)g(uvw)e_i$ for some monomial m, $g \in \mathbb{C}[t]$ with $g(0) \neq 0$. Suppose $m(u,v,w)g(uvw)e_i \in I$. We write

$$g(uvw) = a_0 + a_r(uvw)^r + \dots$$

Then $a_r(uvw)^r g(uvw)$

$$\begin{aligned}&\left(g(uvw) - \frac{a_r}{a_0}(uvw)^r g(uvw)\right) m(u,v,w)e_i \\ &= \big(a_0 + \tilde{a}_{2r}(uvw)^{2r} + \dots\big) m(u,v,w)e_i \in I\end{aligned}$$

Repeating this process, we get $(a_0 + c(uvw)^N + \dots)m(u,v,w)e_i \in I$ for some $N \geq 5n$, then $m(u,v,w)e_i \in I$. This shows that I is a monomial ideal.

For a monomial ideal I in $\mathrm{Jac}(Q,W)$, we associate a Q-multi-colored plane partition π as follows: we define

$$\pi = \big\{(i,j,k) : e_\ell u^i v^j w^k \notin I \text{ for some } \ell\big\}$$

and a Q-multi-coloring

$$K(i,j,k) = \big\{\ell : e_\ell u^i v^j w^k \notin I\big\}.$$

We left the details to the reader to check that this indeed defines a Q-multi-colored plane partitions. Also, we can associate any Q-multi-colored plane partition to a monomial ideal in the obvious way. To compare the dimension vectors, for any (monomial) ideal I, there is a natural decomposition

$$I = e_0 I \oplus e_1 I \oplus e_2 I \oplus e_3 I \oplus e_4 I$$

as vector spaces, and the dimension vector $\mathbf{d}$ of the module induced by I is given by

$$d_i = \dim_{\mathbb{C}} \big(e_i \mathrm{Jac}(Q,W)\big)/\big(e_i I\big),$$

which agrees with the dimension vector of the associated multi-colored plane

partition. $\square$

Corollary 7.3.2. *For any dimension vector* $\mathbf{d}$*, we have*

$$\chi\big(\mathrm{Hilb}^{\mathbf{d}}(Q,W)\big) = n_Q(\mathbf{d}).$$

Now we are ready to finalize the proof.

Proof of Theorem 7.2.2. Recall that $\mathrm{Hilb}^{\mathbf{d}}(Q,W)$ is the critical locus of the function $\mathrm{Tr}(W)$ on the smooth scheme $\mathrm{Hilb}^{\mathbf{d}}(\mathbb{C}Q)$. Since T_0 acts on $\mathbb{C}Q$ and $W \in \mathbb{C}Q$ is T_0-invariant, the torus T_0 acts on $\mathrm{Hilb}^{\mathbf{d}}(\mathbb{C}Q)$ and the function $\mathrm{Tr}(W)$ is T_0-invariant. By [7, Proposition 3.3], the Behrend function ν of $\mathrm{Hilb}^{\mathbf{d}}(Q,W)$ is equal to $(-1)^m$ on T_0-fixed points, where m is the dimension of $\mathrm{Hilb}^{\mathbf{d}}(\mathbb{C}Q)$, and hence

$$\chi_{\mathrm{vir}}\big(\mathrm{Hilb}^{\mathbf{d}}(Q,W)\big) = (-1)^m \chi\big(\mathrm{Hilb}^{\mathbf{d}}(Q,W)\big) = (-1)^m\, n_Q(\mathbf{d}).$$

The Hilbert scheme $\mathrm{Hilb}^{\mathbf{d}}(\mathbb{C}Q)$ is the fine moduli space of framed representations of Q with framing vector $(1,\dots,1)$, that is,

$$\mathrm{Hilb}^{\mathbf{d}}(\mathbb{C}Q) = \Big(\prod_{(a:i\to j)\in Q_1} \mathrm{Hom}(\mathbb{C}^{d_i},\mathbb{C}^{d_j}) \times \prod_i \mathbb{C}^{d_i}\Big) /\!\!/ \prod_{i\in Q_0} \mathrm{GL}_{d_i}(\mathbb{C})$$

which has dimension

$$m = \dim \mathrm{Hilb}^{\mathbf{d}}(\mathbb{C}Q) = \sum_{a:i\to j} d_i d_j + \sum_i d_i - \sum_i d_i^2 = |\mathbf{d}| - \langle \mathbf{d},\mathbf{d}\rangle_Q.$$

$\square$

Chapter 8

Related problems and future directions

In this chapter, we discuss several subjects related to our works and propose future research directions.

8.1 Topological invariants

Our method also gives a computation of topological Euler characteristics of Hilbert schemes of points on the quantum Fermat quintic threefold. We denote the generating function by

$$Z_{\mathrm{top}}^{\mathcal{A}}(t) = \sum_{n=0}^{\infty} \chi\big(\mathrm{Hilb}^n(\mathcal{A})\big)\, t^n.$$

Then Theorem 6.3.3 shows that

$$\begin{aligned} Z_{\mathrm{top}}^{\mathcal{A}}(t) &= Z_{\mathrm{top}}^{Q,W}(t)^{10} \cdot \left(\sum_{n=0}^{\infty} \chi\big(\mathrm{Quot}^n(\mathbb{C}^3)_0\big)\, t^{5n} \right)^{-10} \\ &= Z_{\mathrm{top}}^{Q,W}(t)^{10} \cdot M(t^5)^{-50}. \end{aligned}$$

For the quiver (Q, W) with potential, we use Theorem 7.2.2 and obtain

$$Z_{top}^{Q,W}(t_0, \ldots, t_4) = Z_{\mathrm{PL}}^{Q}(t_0, \ldots, t_4),$$

Then Corollary 7.2.5 states that

$$Z_{\mathrm{PL}}^{Q}(t_0, t_1, t_2, t_3, t_4) = \prod_{i \in \mathbb{Z}/5\mathbb{Z}} Z_{\mathrm{PL}}^{\mu_5(1,1,3)}(t_i, t_{i+1}, t_{i+2}, t_{i+3}, t_{i+4}).$$

If we take $t = t_0 = \ldots = t_4$, then $Z_{\mathrm{PL}}^{\mu_5(1,1,3)}$ simply becomes $M(t)$ as it forgets the coloring. This yields a nice formula

$$Z_{\mathrm{top}}^{\mathcal{A}}(t) = M(t)^{\chi(M_{\mathrm{sp}}^1)} \Big(M(t^5)^5 \Big)^{\chi(M_{\mathrm{sp}}^5)}.$$

One interpretation is that the topological Euler characteristics of moduli spaces of cyclic modules are completely determined by the topological Euler characteristics of moduli spaces of simple modules.

Such statement is certainly true for any commutative smooth algebra A, since all simple modules are 1-dimensional and its moduli space is the affine scheme $\mathrm{Spec}(A)$, and moduli spaces of cyclic modules are precisely Hilbert schemes of points on $\mathrm{Spec}(A)$. On the other hands, it is clearly false in some extreme cases (for example, free algebras).

8.2 Cohomology theories of $(X, \mathcal{A})$

In the study of DT theory, one often fixes some discrete data, such as Chern characters for compact Calabi–Yau 3-folds, or dimension vectors for quivers with potential. One important property for these discrete data is the presence of Euler pairing, which is required in, for example, wall-crossing formula.

Throughout this thesis, we use Hilbert polynomials as the discrete data. However, it is not the "correct" one, as the Euler pairing χ on $K_0(\mathrm{Coh}(\mathcal{A}))_{\mathrm{fd}})$ is non-trivial and does not descend through the induced morphism $K_0(\mathrm{Coh}(\mathcal{A})_{\mathrm{fd}}) \to \mathbb{Z}$.

Question. *Find a (co)homology theory $H^\bullet(X, \mathcal{A})$ for general $(X, \mathcal{A})$ with a mor-*

phism $K_0(\mathrm{Coh}(\mathcal{A})) \to H^\bullet(X, \mathcal{A})$ *so that the Euler pairing descends.*

Natural candidates are some (relative) versions of cyclic (co)homology or Hochschild (co)homology, in which an analogue of Chern character might be defined.

In this section, we will define a specific one that works for $\mathrm{Coh}(\mathcal{A})_{\mathrm{fd}}$, and we will use them later. The idea is that we will define an abelian group N generated by components of the coarse moduli scheme $M^{\mathrm{ss},5}(\mathcal{A})$.

Recall that M^1_{sp} consists of 50 points which can be regarded as the vertices of 10 copies of the quiver Q, and M^5_{sp} is a non-compact connected smooth manifold of dimension 3. The moduli M^5_{sp} is compactified in $M^{\mathrm{ss},5}(\mathcal{A})$ by adding the representation of each quiver Q with dimension vector $(1,1,1,1,1)$. We denote by e_i^k, $i = 0, \ldots, 4$, the vertices of the k-th quiver Q, $k = 1, \ldots, 10$, and e the class of 5-dimensional simple $\mathcal{A}$-modules. Then

$$N = \Big(\text{free abelian group generated by } e_i^k\text{'s and } \boldsymbol{e}\Big) \Big/ \Big(\boldsymbol{e} = \sum_{i=0}^{4} e_i^k \text{ for all } k\Big).$$

The morphism $K_0(\mathrm{Coh}(\mathcal{A})_{\mathrm{fd}}) \to N$ is defined by the semisimplification. The Euler pairing descents to N since the dimension vector $(1,1,1,1,1)$ is in the kernel of χ_Q. Any element in N can be written uniquely in the form

$$n\boldsymbol{e} + \mathbf{d}^1 + \ldots + \mathbf{d}^{10},$$

where $n \in \mathbb{Z}$, and $\mathbf{d}^k$ is a dimension vector of the quiver Q, with $d_i^k = 0$ for some vertex i. The Euler pairing χ is given by $\chi(\boldsymbol{e}, -) = 0$ and

$$\chi(e_i^k, e_j^\ell) = \begin{cases} \chi_Q(e_i, e_j), & k = \ell, \\ 0, & k \neq \ell. \end{cases}$$

All of our results hold when we use N as the discrete data, but the formula will become more complicated.

8.3 Motivic DT invariants

Since our computation is based on a stratification of Hilbert schemes of points, we may follow the idea from [8] to define motivic DT invariants.

We define the motivic DT invariants

$$Z^{\mathcal{A}}_{\mathrm{mot}}(t) = \left(\sum_{n=0}^{\infty} [\mathrm{Hilb}^n(Q,W)_0]_{\mathrm{vir}}\, t^n\right)^{[X_{(0)}]} \left(\sum_{n=0}^{\infty} [\mathrm{Quot}^n(\mathbb{C}^3, \mathcal{O}^{\oplus 5})_0]_{\mathrm{vir}}\, t^{5n}\right)^{[M^5_{\mathrm{sp}}]}.$$

The virtual motivic class $[\mathrm{Hilb}^n(Q,W)_0]_{\mathrm{vir}}$ is defined via $\mathrm{Hilb}^n(Q,W)_0 \subset \mathrm{Hilb}^n(Q,W)$ which is a critical locus, see [8, Section 2.4] for details.

On the other hand, motivic DT invariants of (Q,W) is defined and is *expected* to satisfy the equation

$$Z^{Q,W}_{\mathrm{mot}}(t) = \left(\sum_{n=0}^{\infty} [\mathrm{Hilb}^n(Q,W)_0]_{\mathrm{vir}}\, t^n\right)^{[\mathbb{C}_{(0)}]} \left(\sum_{n=0}^{\infty} [\mathrm{Quot}^n(\mathbb{C}^3, \mathcal{O}^{\oplus 5})_0]_{\mathrm{vir}}\, t^{5n}\right)^{\mathbb{L}^3-1}.$$

Combining these with the known result for $[\mathrm{Quot}^n(\mathbb{C}^3, \mathcal{O}^{\oplus 5})_0]_{\mathrm{vir}}$ (see for example, [4]), we obtain

$$Z^{\mathcal{A}}_{\mathrm{mot}}(t) = Z^{Q,W}_{\mathrm{mot}}(t)^{10} \cdot \mathrm{EXP}\left(\frac{\mathbb{L}^{\frac{5}{2}} - \mathbb{L}^{-\frac{5}{2}}}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}} \cdot \frac{-([M^5_{\mathrm{sp}}] - 10\mathbb{L}^3 + 10)\, t^5}{(1+\mathbb{L}^{\frac{5}{2}} t^5)(1+\mathbb{L}^{-\frac{5}{2}} t^5)}\right),$$

Of course one should use the discrete data N described in the previous section, and a similar result still holds.

While the potential W has a linear factor so we can apply dimension reduction, the full computation of $Z^{Q,W}_{\mathrm{mot}}(t)$ seems very difficult. Here we record the

motivic DT invariants of (Q,W) up to degree 4:

$$\begin{aligned}
Z_{\mathrm{mot}}^{Q,W}(\boldsymbol{t}) = 1 &+ \mathbb{L}^0\boldsymbol{t}^{(1,0,0,0,0)} + \mathbb{L}^{-1}(\mathbb{L}^2+\mathbb{L}+1)\boldsymbol{t}^{(1,1,0,0,0)} + \mathbb{L}^{-\frac{1}{2}}(\mathbb{L}+1)\boldsymbol{t}^{(1,0,1,0,0)} \\
&+ \mathbb{L}^{-1}(\mathbb{L}^2+\mathbb{L}+1)\boldsymbol{t}^{(1,2,0,0,0)} + \mathbb{L}^0\boldsymbol{t}^{(2,0,1,0,0)} + \mathbb{L}^{-\frac{3}{2}}(\mathbb{L}+1)^3\boldsymbol{t}^{(1,1,1,0,0)} \\
&+ \mathbb{L}^{-2}(\mathbb{L}^2+1)(\mathbb{L}+1)^2\boldsymbol{t}^{(1,1,0,1,0)} + \mathbb{L}^0\boldsymbol{t}^{(1,3,0,0,0)} + \mathbb{L}^{-1}(\mathbb{L}^2+\mathbb{L}+1)\boldsymbol{t}^{(2,1,1,0,0)} \\
&+ \mathbb{L}^{-\frac{5}{2}}(\mathbb{L}^2+\mathbb{L}+1)(\mathbb{L}^2+1)(\mathbb{L}+1)\boldsymbol{t}^{(1,2,1,0,0)} + \mathbb{L}^{-1}(2\mathbb{L}^2+3\mathbb{L}+2)\boldsymbol{t}^{(1,1,2,0,0)} \\
&+ \mathbb{L}^{-\frac{5}{2}}(\mathbb{L}^2+\mathbb{L}+1)(\mathbb{L}^2+1)(\mathbb{L}+1)\boldsymbol{t}^{(1,2,0,1,0)} + \mathbb{L}^{-2}(\mathbb{L}^4+\mathbb{L}^3+\mathbb{L}^2+\mathbb{L}+1)\boldsymbol{t}^{(1,1,0,2,0)} \\
&+ \mathbb{L}^{-\frac{5}{2}}(2\mathbb{L}^4+4\mathbb{L}^3+5\mathbb{L}^2+4\mathbb{L}+2)(\mathbb{L}+1)\boldsymbol{t}^{(1,1,1,1,0)} + O(\boldsymbol{t}^5).
\end{aligned}$$

We want to mention a conjecture of Cazzaniga–Morrison–Pym–Szendrői [11], which suggests that motivic DT invariants of (Q,W) should in some sense determined by its simple modules. To be more precise, let

$$\mathcal{Z}^{Q,W}(\boldsymbol{t}) = \sum_{\mathbf{d}} [\mathfrak{M}^{\mathbf{d}}(Q,W)]_{\mathrm{vir}}\, \boldsymbol{t}^{\mathbf{d}}$$

be the generating function of motivic DT invariants of (Q,W) (without framing). Since all simple modules of $\mathrm{Jac}(Q,W)$ have dimension 1 or 5 (with dimension vector $(1,1,1,1,1)$, the conjecture suggests that $\mathcal{Z}^{Q,W}(\boldsymbol{t})$ should be of the form

$$\mathrm{EXP}\left(\frac{1}{\mathbb{L}^{\frac{1}{2}}-\mathbb{L}^{-\frac{1}{2}}}\frac{t_0+t_1+t_2+t_3+t_4}{1-t_0t_1t_2t_3t_4} + \frac{M}{\mathbb{L}^{\frac{1}{2}}-\mathbb{L}^{-\frac{1}{2}}}\frac{t_0t_1t_2t_3t_4}{1-t_0t_1t_2t_3t_4}\right)$$

for some M. This is clearly not true for one simple reason: $\mathcal{Z}^{Q,W}(\boldsymbol{t})$ is not a symmetric function, the coefficient of t_0t_1 is $\mathbb{L}^2(\mathbb{L}-1)^{-2}$ and coefficient of t_0t_2 is $\mathbb{L}^{\frac{3}{2}}(\mathbb{L}-1)^{-2}$.

The main issue is the nontrivialness of the Euler pairing χ_Q. It seems to us that the formula should be taken in some non-commutative products (for example, take $\boldsymbol{t}^{\mathbf{d}_1} * \boldsymbol{t}^{\mathbf{d}_2} = \mathbb{L}^{\chi(\mathbf{d}_1,\mathbf{d}_2)}\boldsymbol{t}^{\mathbf{d}_1+\mathbf{d}_2}$). But in that case it is problematic to define the exponential. One naive (and unsuccessful) attempt is to simply take

$$\mathrm{EXP}\left(\frac{1}{\mathbb{L}^{\frac{1}{2}}-\mathbb{L}^{-\frac{1}{2}}}(t_0+t_1+t_2+t_3+t_4)\right) = \sum_{n=0}^{\infty}\frac{1}{[\mathrm{GL}_n]_{\mathrm{vir}}}(t_0+t_1+t_2+t_3+t_4)^n$$

and expand the right hand side using non-commutative products.

To the best of our knowledge, all quivers with potential whose motivic DT invariants have been successfully computed have trivial Euler pairings, or equivalently, they are symmetric quivers.

8.4 Joyce–Song's generalized DT invariants

To define Joyce–Song's generalized DT invariants, one key ingredient is the *Behrend function identities* [19, Theorem 5.11]. Recall that for any indecomposable $\mathcal{A}$-modules must support at a single point of X. Then our local models (Theorem 5.3.7) implies that it is sufficient to check the Behrend function identities for moduli stacks of quivers with potential, which has been verified in [19, Theorem 7.11].

We will use N (defined in earlier) for discrete data. For any $\gamma \in N$, we write $\overline{\mathrm{DT}}^{\gamma}(\mathcal{A})$ for the generalized DT invariants. For any class γ, the moduli scheme $\mathrm{PI}^{\gamma,n}$ of stable pairs is just the Hilbert scheme $\mathrm{Hilb}^{\gamma}(\mathcal{A})$ of points. Then the wall-crossing formula [19, Theorem 5.27] states that

$$\chi_{\mathrm{vir}}\big(\mathrm{Hilb}^{\gamma}(\mathcal{A})\big) = \sum_{\gamma_1+\ldots+\gamma_\ell=\gamma} \frac{(-1)^{\ell}}{\ell!} \prod_{i=1}^{\ell} (-1)^{\chi(\mathcal{A}-\gamma_1-\ldots-\gamma_{i-1},\gamma_i)} \chi(\mathcal{A}-\gamma_1-\ldots-\gamma_{i-1},\gamma_i)\, \overline{\mathrm{DT}}^{\gamma_i}(\mathcal{A}).$$

For any class γ, $\chi(\mathcal{A},\gamma) = |\gamma|$ is the dimension of a finite-dimensional $\mathcal{A}$-module. We will use this formula to compute some invariants $\overline{\mathrm{DT}}^{\gamma}(\mathcal{A})$.

Recall that any element in N can be written uniquely in the form $n\boldsymbol{e} + \mathbf{d}^1 + \ldots + \mathbf{d}^{10}$, where $\mathbf{d}^k$ is a dimension vector of the quiver Q with $\mathbf{d}^k_i = 0$ for some vertex i. In the case $n = 0$, we have

$$\overline{\mathrm{DT}}^{\mathbf{d}^1+\ldots+\mathbf{d}^{10}}(\mathcal{A}) = \begin{cases} \overline{\mathrm{DT}}^{\mathbf{d}^i}(Q,W), & \text{if } \mathbf{d}^j = 0 \text{ for all } j \neq i \\ 0, & \text{otherwise,} \end{cases}$$

where $\overline{\mathrm{DT}}(Q,W)$ is the generalized DT invariant of the quiver (Q,W) with potential.

For the class $\boldsymbol{e} \in N$, we compare the wall-crossing formula to $\mathrm{Hilb}^{\boldsymbol{e}}(\mathcal{A})$ and

to $\mathrm{Hilb}^{(1,1,1,1,1)}(Q,W)$, which yields

$$\chi_{\mathrm{vir}}(\mathrm{Hilb}^{\boldsymbol{e}}(\mathcal{A}))-10\cdot\chi_{\mathrm{vir}}(\mathrm{Hilb}^{(1,1,1,1,1)}(Q,W)) = 5\Big(\overline{\mathrm{DT}}^{e}(\mathcal{A})-10\cdot\overline{\mathrm{DT}}^{(1,1,1,1,1)}(Q,W)\Big)$$

For the left hand side, we have a stratification of $\mathrm{Hilb}^{e}(\mathcal{A})$ that gives

$$\chi_{\mathrm{vir}}\big(\mathrm{Hilb}^{\boldsymbol{e}}(\mathcal{A})\big) = 10\cdot\chi_{\mathrm{vir}}\big(\mathrm{Hilb}^{(1,1,1,1,1)}(Q,W)\big) + \chi_{\mathrm{vir}}\big(\mathrm{Quot}^{1}(M^{5}_{\mathrm{sp}},\mathcal{O}^{\oplus 5})\big).$$

If we pretend the non-compact 3-fold M^{5}_{sp} is Calabi–Yau, then the wall-crossing formula says

$$\chi_{\mathrm{vir}}\big(\mathrm{Quot}^{1}(M^{5}_{\mathrm{sp}},\mathcal{O}^{\oplus 5})\big) = -(-1)^{\chi([\mathcal{O}^{5}],1)}\chi([\mathcal{O}^{5}],1)\overline{\mathrm{DT}}^{1}(M^{5}_{\mathrm{sp}}).$$

We conclude that

$$\overline{\mathrm{DT}}^{\boldsymbol{e}}(\mathcal{A}) = 10\cdot\overline{\mathrm{DT}}^{(1,1,1,1,1)}(Q,W) + \overline{\mathrm{DT}}^{1}(M^{5}_{\mathrm{sp}}).$$

This result might seem natural from our computation of $Z^{\mathcal{A}}(t)$, which is given by 10 copies of (Q,W) and M^{5}_{sp}. However, due to the nontrivialness of Euler pairing χ, the wall-crossing formula is really complicated so that does not give this result directly. It is not clear to us what would $\mathrm{DT}^{n\boldsymbol{e}}(\mathcal{A})$ be for $n>1$.

8.5 Future directions

Generalization to arbitrary non-commutative projective schemes

This thesis only considers non-commutative projective schemes that are finite over their centers, which is a strong restriction in non-commutative geometry. Assuming good moduli spaces with respect to certain stability condition, the construction of symmetric obstruction theories seems rather formal, and we are surprised that it is not known in the literature already.

Particularly, let C be any abelian category. It is well-known that the deformation-obstruction theory for an object F in C is governed by $\mathrm{Ext}^{1}_{\mathsf{C}}(F,F)$ and $\mathrm{Ext}^{2}_{\mathsf{C}}(F,F)$. However this is not enough to construct an obstruction theory for moduli spaces.

Question. *Does Theorem 4.1.2 (existence of the Atiyah class) hold for any abelian category?*

This would allow us to define numerical DT invariants for CY3 abelian categories, which would be deformation invariants for non-commutative projective schemes.

Deformations of the quantum Fermat quintic threefold

Our quantum Fermat quintic threefold is the Fermat quintic that lies in quantum projective space. Because the Fermat quintic equation needs to be in the center, this puts strong restrictions on the quantum parameters q_{ij} and also on the quintic equations. There is no obvious (non-trivial) deformation of the quantum Fermat quintic threefold.

Unlike commutative projective spaces which are rigid, there are many non-trivial deformations of non-commutative projective spaces (see, for example, [1] for non-commutative projective planes).

Question. *Does there exist non-trivial deformations of the quantum Fermat quintic threefold which lie in other non-commutative projective* 4*-spaces?*

Enumerative geometry for the quantum Fermat quintic threefold

Non-commutative projective schemes should also serve as interesting examples in enumerative geometry. Compared to the usual (commutative) quintic threefolds, the quantum Fermat quintic threefolds seem to have "smaller" moduli spaces with more complicated structures. On the DT side alone, there are already many problems that can be explored. For example,

Question. *Is the moduli space of sheaves on the quantum Fermat threefold a* d*-critical locus? More generally, there is a natural derived enhancement for any non-commutative projective scheme. Does the derived moduli stack of sheaves on the quantum Fermat threefold carry a* (-1)*-shifted symplectic structure?*

Other directions for study include Gromov–Witten invariants, Pandharipande–Thomas invariants and homological mirror symmetry.

Generalization to non-Calabi–Yau cases

Classically, DT invariants are defined for any projective 3-fold. We only managed to construct a perfect obstruction theory in the Calabi–Yau case due to the lack of notions of determinant and trace. Nevertheless, a perfect obstruction theory should exist more generally, for example for Fano pairs $(X, \mathcal{A})$. In particular if $\mathcal{A}$ is of global dimension 2, then the obstruction theory is automatically perfect. This yields an analogue of Donaldson invariants for non-commutative surfaces, which might have applications in the study of non-commutative surfaces.

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