# Combinatorial Properties of Maps on Finite Posets 

by<br>Brian Tianyao Chan<br>B. Sc., Pure Mathematics, University of Calgary, 2014<br>M. Sc., Pure Mathematics, University of Calgary, 2016

# A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF 

 Doctor of Philosophyin
THE FACULTY OF GRADUATE AND POSTDOCTORAL STUDIES (Mathematics)

The University of British Columbia (Vancouver)

July 2020
(c) Brian Tianyao Chan, 2020

The following individuals certify that they have read, and recommend to the Faculty of Graduate and Postdoctoral Studies for acceptance, the dissertation entitled:

Combinatorial Properties of Maps on Finite Posets
submitted by Brian Tianyao Chan in partial fulfillment of the requirements for
the degree of Doctor of Philosophy
in Mathematics

## Examining Committee:

Stephanie van Willigenburg, Mathematics
Supervisor
Andrew Rechnitzer, Mathematics
Supervisory Committee Member
Julia Gordon, Mathematics
University Examiner
Bruce Shepherd, Computer Science
University Examiner

## Additional Supervisory Committee Members:

József Solymosi, Mathematics
Supervisory Committee Member

## Abstract

In this thesis, we make progress on the problem of enumerating tableaux on non-classical shapes by introducing a general family of $P$-partitions that we call periodic $P$-partitions. Such a family of $P$-partitions generalizes the parallelogramic shapes, which were analysed by López, Martínez, Pérez, Pérez, Basova, Sun, Tewari, and van Willigenburg, and certain truncated shifted shapes, where truncated shifted shapes were investigated by Adin, King, Roichman, and Panova. By introducing a separation property for posets and by proving a relationship between this property and $P$-partitions, we prove that periodic $P$-partitions can be enumerated with a homogeneous first-order matrix difference equation.

Afterwards, we consider families of finite sets that we call shellable and that have been characterized by Chang and by Hirst and Hughes as being the families of sets that admit unique solutions to Hall's marriage problem. By introducing constructions on families of sets that satisfy Hall's Marriage Condition, and by using a combinatorial analogue of a shelling order, we prove that shellable families can be characterized by using a generalized notion of hook-lengths. Then, we introduce a natural generalization of standard skew tableaux and Edelman and Greene's balanced tableaux, then prove an existence result about such a generalization using our characterization of shellable families.

## Lay Summary

In combinatorics, counting the number of items in a collection, or the number of objects that satisfy a certain property, can be very difficult and often represents the limits of what is currently known. Such problems pose interesting challenges to researchers. In this thesis, we count the number of ways in which certain objects that exhibit a fixed repeating pattern can be labelled with ordered sequences of numbers.

In discrete maths, there are ways of grouping objects so that every group of objects can be assigned a suitable representative. Also, in discrete maths, there are labelled arrangements of numbers where there are restrictions on where the numbers that fill the arrangement are positioned. In this thesis, we prove some structural results that give a relationship between the above grouping of objects and the above arrangement of numbers.

## Preface

Chapters 3 to 5 of my thesis originated as a project to generalize enumerative results for certain tableaux from Tewari and van Willigenburg's paper [49]. I was successful in accomplishing the orignal goals relating to this project and generalized them to $P$-partitions in the above chapters. I chose this project under the guidance of my supervisor, Stephanie van Willigenburg. Moreover, I am responsible for all aspects of the above work and I plan to submit it for publication.

Chapters 6 to 7 of my thesis originated from a suggestion made by my supervisor to further develop an earlier result that I derived for skew tableaux. I was successful in developing my original result, leading to the above two chapters. Furthermore, I have generalized the results in Chapters 6 and 7 and submitted them for publication. I am responsible for all aspects of the above work.

## Table of Contents

Abstract ..... iii
Lay Summary ..... iv
Preface ..... $\mathbf{V}$
Table of Contents ..... vi
List of Figures ..... viii
List of Symbols ..... ix
Acknowledgments ..... xii
1 Introduction ..... 1
2 Preliminaries ..... 7
3 The $P$-partition enumeration problem ..... 16
4 Periodic ( $P, \omega$ )-partitions ..... 21
5 The matrix difference equation ..... 45
6 Results relating to the marriage condition ..... 70
7 Applications to skew tableaux ..... 80
8 Conclusion ..... 91
Bibliography ..... 96

## List of Figures

Figure 4.1 A three-dimensional analogue of Example 4.8. Here, $X=\{(i, j, 0): 1 \leq$ $i \leq 3$ and $1 \leq j \leq 3\}$ and $\Delta=(1,1,1) . \ldots \ldots 37$

Figure 4.2 A more exotic example of a periodic quadruple system. . . . . . . . . . 43

## List of Symbols

| $a_{*}$ | term of a sequence |
| :--- | :--- |
| $d$ | dimension |
| $f, g, h, f_{*}, g_{*}$ | functions |
| $h_{*}$ | hook-length |
| $i, j, k, \ell, m, n_{*}, k_{*}, \ell_{*}$ | integers |
| $n$ | index associated with periodic quadruple <br> systems, $n \rightarrow \infty,[n], n!, \lambda \vdash n$ |
| $p, q, p_{*}, q_{*}$ | elements of a poset |
| $r, r_{*}$ | element of a set |
| $\mathrm{s}(\cdot)$ | successor function for $\mathbb{Z}$ |
| $u, v$ | vectors |
| $v(i)$ | $i^{\text {th }}$ entry of column vector $v$ |
| $w$ | period of a periodic sequence |
| $A, B, C$ | components of a connected triple |


| $H_{*}$ | hook |
| :---: | :---: |
| $L$ | labellings of $\mathrm{Tb}(S, \omega)$ |
| $R$ | inverse of $L$ |
| M | square matrix |
| $M(i, j)$ | entry in the $i^{\text {th }}$ row and the $j^{\text {th }}$ column of $M$ |
| $N$ | $\|S, \omega\|$, number of rows of a square matrix |
| $\mathbb{N}$ | set of positive integers |
| $\mathbb{N}_{0}$ | set of non-negative integers |
| $P, Q, P_{*}, Q_{*}$ | posets |
| $\mathbb{R}_{>0}$ | set of positive real numbers |
| $S, S^{\prime}$ | subsets of $Z$ |
| $T, T_{*}$ | elements of $\operatorname{Tb}(Q, \omega)$ |
| $U, U_{*}$ | elements of $\mathscr{A}(Q, \omega)$ |
| X, $Y$ | sets |
| Z | countably infinite and locally finite poset |
| $Z(X, \Delta)$ | certain subposet of $\mathbb{Z}^{\text {d }}$ |
| $\mathscr{A}(Q, \omega)$ | a set of $Q$-partitions |
| $\mathscr{F}, \mathscr{F}^{\prime}$ | set partitions, families of sets |
| $\mathrm{Tb}(Q, \omega)$ | $\mathscr{A}(Q, \omega) / \equiv_{Q, \omega}$ |
| $\alpha$ | translation map on $\mathbb{Z}$ |
| $\theta$ | component of a periodic quadruple system |
| $\kappa$ | order embedding from $[n-1]$ to $[n]$ |
| $\lambda$ | integer partition |
| $\sigma, \sigma_{*}$ | permutations |


| $\omega, \omega_{*}$ | poset labellings |
| :--- | :--- |
| $\Delta$ | translation vector |
| $\Re$ | binary relation |
| $\|Q, \omega\|$ | cardinality of $\mathrm{Tb}(Q, \omega)$ |
| $[\cdot, \cdot],[\cdot, \infty)$ | intervals within the set of integers |
| $\sim$ | is asymptotic to |
| $\leq, \leq_{*}$ | partial orders |
| $\equiv, \equiv_{*}$ | equivalence relations |
| $\\|$ | incomparability relation |
| $\emptyset$ | empty set, empty partition |

## Acknowledgments

I would like to thank my parents for their enduring and enthusiatic support throughout my journey.

I am also appreciative of József Solymosi and Andrew Rechnitzer for their encouragement and guidance.

Moreover, I would like to acknowledge that my research was supported in part by NSERC and UBC fellowships.

Lastly, I would like to thank Stephanie van Willigenburg for her mentorship and encouragement. Our frequent meetings, her invaluable guidance in my mathematical development, and her unwavering support through the years have all had a very positive impact on me. I am grateful for her dedication to my academic success.

## Chapter 1

## Introduction

$P$-partitions were first considered by MacMahon in [31]. Later on, the theory of $P$-partitions was developed by Gessel and Stanley [18, 46]. This theory is known to have applications to, for instance, quasisymmetric functions as $P$-partitions are essential for the theory of quasisymmetric functions [18].

In this thesis, we investigate the problem of enumerating tableaux on non-classical shapes by introducing a general class of $P$-partitions that we call periodic $P$-partitions. We first introduce the notion of a connected triple for a poset and prove that periodic $P$-partitions generalize many known tableau on non-classical shapes considered in the literature. Afterwards, we consider the problem of counting the number of periodic $P$-partitions by defining collections of numbers that sum to this number. We prove a structural property of connected triples to prove that the aforementioned collections of numbers, represented as column vectors, can be enumerated with a homogeneous first-order matrix difference equation in which the entries of the matrix have natural combinatorial descriptions.

Our approach towards $P$-partitions does not provide closed-form formulas or product formulas for enumeration. On the other hand, our approach with $P$-partitions is not limited to the bijective fillings of non-classical shapes that are usually analysed in the literature [13, 27, 38, 47, 49]. In Chapter 8 , we briefly outline how our results imply that the $P$-partitions we are interested in satisfy constant coefficient linear recurrences, generalizing tableau enumeration results for non-classical shapes from López, Martínez, Pérez, Pérez, and Basova [27], from Sun [47], and from Tewari and van Willigenburg [49] that we will describe later in this section. We now give an overview of the research that our results on $P$-partitions can be applied to.

Counting the number of $P$-partitions is a generalization of the problem of counting the number of linear extensions of a poset. Counting the number of linear extensions of a poset is in general an interesting problem. It has been considered ([44], p. 258) to be very important for measuring the complexity of a poset. A well-known special case of this is the problem of enumerating standard Young tableaux of various shapes [2]. Among the standard Young tableaux on non-classical shapes that the results of this thesis applies to are $d$-dimensional tableaux of certain shapes for any $d \geq 3$, and standard Young tableaux on many of the truncated shifted shapes.

A class of standard Young tableaux that is of recent interest are standard Young tableaux of truncated shifted shapes. Certain truncated shifted shapes, known as truncated shifted staircase shapes, are known to enumerate the number of geodescics between distinguished pairs of antipodes in the flip graph of triangle-free triangulations [3]. Special cases of enumerating standard Young tableaux on truncated shifted shapes have been established. Adin, King, and Roichman [1] enumerated such tableaux for shifted staircase shapes truncated by a square, or a square minus a single cell in the south-west corner, and Panova [38] indepen-
dently proved, using different methods, the special case of this problem for shifted staircase shapes truncated by a single cell.

Hardin and Heinz conjectured ([43], A181196) constant coefficient recurrence relations for counting the number of standard Young tableaux on shifted strips with constant width up to when the width is seven. Standard Young tableaux on shifted strips with constant width are known to correspond to quasisymmetric functions known as the canonical basis, which is a newly discovered basis of quasisymmetric functions, via descent sets of standard reverse composition tableaux and fundamental quasisymmetric functions [49]. These tableaux were also shown to be connected to the representation theory of the 0-Hecke algebra [49]. Moreover, standard Young tableaux on shifted strips with constant width are also known to be connected to Higman's conjecture, which is concerned with enumerating the number of conjugacy classes in the group of upper unitriangular $n$ by $n$ matrices over $\mathbb{F}_{q}$ [27].

Later research established that Hardin and Heinz's conjectures are correct. Tewari and van Willigenburg [49] proved Hardin and Heinz's conjecture when the constant width is 3, Sun [47] proved Hardin and Heinz's conjecture when the constant width was 4 and 5, and López, Martínez, Pérez, Pérez, and Basova proved all of Hardin and Heinz’s recurrences and established a generalization of these recurrences when the width $k \in \mathbb{N}$ is arbitrary [27].

Lastly, another class of $P$-partitions are semistandard tableaux. Semistandard tableaux are fundamental to constructing Schur functions [45] and they are deeply connected to the Specht modules of the symmetric group [41]. A very well-known class of numbers are the Kostka numbers, which are the numbers of semistandard tableaux on partition shapes [41]. However, very little is known about this number [45].

The work of this thesis implies that semistandard tableaux on certain shapes, such as the
parallelogramic shapes considered by López et.al., Sun, and Tewari and van Willigenburg, can also be enumerated with a matrix difference equation as described at the beginning of this section.

Hall's Marriage Theorem is a combinatorial theorem that characterises when a finite family of sets has a system of distinct representatives, which is also called a transversal. Hall [21] proved that such a family has a system of distinct representatives if and only if this family satisfies the marriage condition. This theorem is known to be equivalent to at least six other theorems [40] which include Dilworth's Theorem, Menger's Theorem, and the Max-Flow Min-Cut Theorem.

Hall Jr. proved [22] that Hall's Marriage Theorem also holds for arbitrary families of finite sets. Afterwards, Chang [10] noted how Hall Jr.'s work in [22] can be used to characterize marriage problems with unique solutions. Specifically, the families of sets that admit marriage problems with unique solutions were characterized [10]. Later on, Hirst and Hughes proved that such a characterization of marriage problems with unique solutions can be derived by only using a subsystem of second order arithmetic known as $R C A_{0}$ [24], and they showed that their work in [24] can also be extended to marriage problems with a fixed finite number of solutions [23]. In this thesis, we call the families of finite sets that admit marriage problems with unique solutions shellable and give a new characterization of these families of sets by generalizing the notion of standard Young tableaux and Edelman and Greene's balanced tableaux.

Standard skew tableaux are well-known and intensively studied in algebraic combinatorics, for example [25, 35, 36, 45]. Moreover, another class of tableaux was introduced by Edelman and Greene in [13, 14], where they defined balanced tableaux on partition shapes. In investigating the number of maximal chains in the weak Bruhat order of the symmetric group, Edel-
man and Greene proved [13, 14] that the number of balanced tableaux of a given partition shape equals the number of standard Young tableaux of that shape. Since then, connections to random sorting networks [5], the Lascoux-Schützenberger tree [28], and a generalization of balanced tableaux pertaining to Schubert polynomials [15] have been explored.

Lastly, properties of products of hook-lengths have recently enjoyed some attention by Pak et.al. [34, 39] and by Swanson [48]. In particular, an inequality between products of hooklengths and products of dual hook-lengths was derived [34, 39, 48]. We introduce a generalization of standard Young tableaux and balanced tableaux for skew shapes, show, using our characterization of marriage problems with unique solutions, that the number of such generalizations that can exist is given by a product of hook-lengths, and show, as a consequence, that the average number of tableaux that belongs to such a generalization is given by the hook-length formula. We then discuss extensions and possible applications of our characterization in Chapter 8.

This thesis is structured as follows. In Chapter 2, we give an overview of the preliminaries and describe the conventions that we will follow. In Chapter 3, we describe in detail the $P$-partition enumeration problem we are considering. In Chapter 4, we introduce the notion of a connected triple, formally define periodic $P$-partitions, give illuminating examples, and prove that this definition includes many tableaux on certain $d$-dimensional non-classical shapes for all $d \geq 3$. In Chapter 5, we consider collections of $P$-partitions, then prove a structural property of connected triples to prove that these collections satisfy a matrix difference equation.

In Chapter 6, we introduce a stronger form of the marriage condition and characterize it using generalized hook-lengths. Moreover, in Chapter 7, we explain how to apply our results to a generalization of standard Young tableaux and balanced tableaux for skew tableaux and
breifly indicate ways in which we can extend our approach. Lastly, in Chapter 8, we give an outline of future directions for this research.

## Chapter 2

## Preliminaries

In this chapter, we give the preliminaries that will be needed for this thesis. Throughout this thesis, let $\mathbb{N}$ denote the set of positive integers and let $\mathbb{N}_{0}$ denote the set of non-negative integers.

For all positive integers $n$, define $[n]=\{1,2, \ldots, n\}$. Moreover, for all positive integers $n_{1}$ and $n_{2}$ such that $n_{1} \leq n_{2}$, define $\left[n_{1}, n_{2}\right]=\left\{k \in \mathbb{N}: n_{1} \leq k \leq n_{2}\right\}$. In particular, $[n, n]=\{n\}$, $[n, n+1]=\{n, n+1\},[n, n+2]=\{n, n+1, n+2\}$, and so on. Furthermore, for all $n \in \mathbb{Z}$, define $[n, \infty)=\{k \in \mathbb{Z}: k \geq n\}$.

Let $X$ and $Y$ be sets. Define $X \backslash Y=\{r \in X: r \notin Y\}$. If $X$ is a subset of $Y$, then write $X \subseteq Y$. Moreover, if $X$ is a proper subset of $Y$, then write $X \subset Y$. Lastly, if $X$ is not a subset of $Y$, then write $X \nsubseteq Y$. If $n \in \mathbb{N}$ and if $X_{1}, X_{2}, \ldots, X_{n}$ are sets, then the Cartesian product $X_{1} \times X_{2} \times \cdots \times X_{n}$ of $X_{1}, X_{2}, \ldots$, and $X_{n}$ is the set of ordered $n$-tuples $\left\{\left(r_{1}, r_{2}, \ldots, r_{n}\right): \forall i \in\right.$ $\left.[n], r_{i} \in X_{i}\right\}$. If $X_{1}=X_{2}=\cdots=X_{n}$ and $X=X_{1}$, then write $X^{n}=X_{1} \times X_{2} \times \cdots \times X_{n}$. We let $\mathbb{Z}^{d}$ denote $X^{n}$ if $X=\mathbb{Z}$ and $d=n$. Moreover, let $\emptyset$ denote the empty set. Furthermore, if $X$
is a set, then let $|X|$ denote the cardinality of $X$.

We denote any sequence $a_{1}, a_{2}, \ldots$ by $\left(a_{n}\right)_{n=1,2, \ldots}$. Lastly, let $w \in \mathbb{N}$. A sequence $\left(a_{n}\right)_{n=1,2, \ldots}$ is periodic with period $w$ if $a_{n}=a_{n+w}$ for all $n \in \mathbb{N}$.

A binary relation on a set $X$ is a subset $\Re$ of $X \times X$. We write $r_{1} \Re r_{2}$ if $\left(r_{1}, r_{2}\right) \in \Re$. A binary relation $\mathfrak{R}$ on $X$ is reflexive if $r \Re r$ for all $r \in X$, symmetric if, for all $r_{1}, r_{2} \in X$, $r_{1} \Re r_{2}$ implies that $r_{2} \Re r_{1}$, antisymmetric if, for all $r_{1}, r_{2} \in X, r_{1} \Re r_{2}$ and $r_{2} \Re r_{1}$ implies that $r_{1}=r_{2}$, and transitive if, for all $r_{1}, r_{2}, r_{3} \in X, r_{1} \Re r_{2}$ and $r_{2} \Re r_{3}$ implies that $r_{1} \Re r_{3}$. An equivalence relation $\equiv$ on $X$ is a binary relation on $X$ that is reflexive, symmetric and transitive. Moreover, a partial order $\leq$ on $X$ is a binary relation on $X$ that is reflexive, antisymmetric, and transitive.

Let $X$ be a set, and let $\equiv$ be an equivalence relation on $X$. Then for all $r \in X$, the equivalence class of $r$ in $X$ with respect to $\equiv$ is the set $\left\{r_{1} \in X: r \equiv r_{1}\right\}$. An equivalence class in $X$ with respect to $\equiv$ is an equivalence class of $r$ in $X$ with respect to $\equiv$ for some $r \in X$. Lastly, let $X / \equiv$ denote the set of all equivalence classes in $X$ with respect to $\equiv$.

Example 2.1. Let $X=\{1,2,3\}$, and let $\equiv$ be the set $\{(1,1),(2,2),(3,3),(1,2),(2,1)\}$. Then $1 \equiv 1,2 \equiv 2,3 \equiv 3,1 \equiv 2$, and $2 \equiv 1$. Moreover, the equivalence classes of $X$ with respect to $\equiv$ are $\{1,2\}$ and $\{3\}$. Hence, $X / \equiv$ equals $\{\{1,2\},\{3\}\}$.

In order to clarify the conventions that we will follow when describing posets, we briefly introduce posets and related notions below. More details can be found in [12, 42, 44]. A set $P$ equipped with a partial order $\leq$ on $P$ is called a poset.

When describing posets, we will usually say "let $P$ be a poset", "if $P$ is a poset", etc. without explicitly mentioning the partial order $\leq$ on $P$. Moreover, an element $p$ of a poset $P$ with
partial order $\leq$ is an element of the set $P$ and we write $p \in P$. Similarly, a subset $X$ of a poset $P$ with partial order $\leq$ is a subset of the set $P$ and we write $X \subseteq P$. In particular, $P \subseteq P$. We will also write $p \geq q$ if $q \leq p, p<q$ if $p \leq q$ and $p \neq q, p>q$ if $q<p, p \not \leq q$ if $p \leq q$ is false, $p \nsupseteq q$ if $q \not \leq p, p \nless q$ if $p<q$ is false, and $p \ngtr q$ if $q \nless p$. Moreover, if $P$ is a poset with partial order $\leq$ and if $p, q \in P$, then we write $p \| q$, and say that $p$ and $q$ are incomparable, if $p \not \leq q$ and $q \not \leq p$. A partial order $\leq$ on a poset $P$ is a total order on $P$ if, for all $p, q \in P$, $p \leq q$ or $q \leq p$.

We will indicate which poset we are referring to if we want to clarify which partial order we are using. For instance, we will say " $x \leq y$ in $P$ " to mean that $x \leq y$, where $x, y \in P$ and $\leq$ is the partial order on $P$, " $x>y>z$ in $P$ " to mean that $z \leq y, y \leq x, x \neq y$, and $y \neq z$ where $x, y, z \in P$ and $\leq$ is the partial order on $P$, and " $x \| y$ in $P$ " to mean that $x \leq y$ is false and $y \leq x$ is false, where $x, y \in P$ and $\leq$ is the partial order on $P$.

We will assume the following when describing subsets of posets. In this paragraph, we will use subscripts to indicate which partial order we are referring to. Let $P$ be a poset with partial order $\leq_{P}$. Then a subposet of $P$ is a subset $Q$ of $P$ equipped with the partial order $\leq_{Q}$ on $Q$ defined, for all $p, q \in Q$, by $p \leq_{Q} q$ if $p \leq_{P} q$. In particular, all subposets are posets. For convenience, we will regard a subset of a poset $P$ as the subposet of $P$ that corresponds to that subset, and we will regard a subposet of a poset $P$ as the subset of $P$ that corresponds to that subposet. For instance, if $P$ is a poset and if $Q$ is a subset of $P$, then when we say things such as " $Q$ is order isomorphic to a five element poset", we are assuming that $Q$ is a subposet of $P$ in the above sense. Moreover, we will, when defining subposets $Q$ of posets $P$, say "let $Q$ be a subset of a poset $P$ ", "let $P$ be a poset and let $Q \subseteq P$ ", and so on.

Definition 2.2 (Folklore, cf. [2]). We will regard $\mathbb{Z}$ as a poset with total order defined by $\cdots<-1<0<1<\cdots$. Moreover, for any $d \in \mathbb{N}$, we will regard $\mathbb{Z}^{d}$ as a poset with partial
order defined by $\left(k_{1}, k_{2}, \ldots, k_{d}\right) \leq\left(\ell_{1}, \ell_{2}, \ldots, \ell_{d}\right)$ if $k_{i} \leq \ell_{i}$ for all $1 \leq i \leq d$.

If $d \in \mathbb{N}$ and if $X \subseteq \mathbb{Z}^{d}$, then, as explained in the paragraph above Definition 2.2, $X$ is a subposet of $\mathbb{Z}^{d}$. Furthermore, if $u, v \in \mathbb{Z}^{d}, u=\left(k_{1}, k_{2}, \ldots, k_{d}\right)$, and $v=\left(\ell_{1}, \ell_{2}, \ldots, \ell_{d}\right)$, then write $u+v=\left(k_{1}+\ell_{1}, k_{2}+\ell_{2}, \ldots, k_{d}+\ell_{d}\right)$, write $k u=\left(k k_{1}, k k_{2}, \ldots, k k_{d}\right)$ for all $k \in \mathbb{Z}$, and write $v-u=v+k u$ if $k=-1$. Lastly, for any $X \subseteq \mathbb{Z}^{d}$ and $u \in \mathbb{Z}^{d}$, write $X+u=\{v+u: v \in$ $X\}$ and write $X-u=\{v-u: v \in X\}$.

If $P$ is a poset, then an element $p \in P$ is a minimal element of $P$ if for all $q \in P, q \geq p$ in $P$ or $q \| p$ in $P$. Moreover, a subset $X \subseteq P$ is an antichain of $P$ if, for all $p, q \in X, p \| q$ in $P$. A poset $P$ is finite if $P$ has a finite number of elements; that is, if $|P|$ is finite. Similarly, a poset $P$ is countable if $|P|$ is countable, and countably infinite if $|P|$ is countably infinite. Moreover, a poset $P$ is locally finite if for all $p, q \in P$ such that $p \leq q$, the number of elements $p_{1} \in P$ such that $p \leq p_{1} \leq q$ in $P$ is finite.

We use the terms function and map interchangeably. Moreover, we define functions on posets as follows. Assume that $R_{1}$ and $R_{2}$ are such that $R_{1}$ and $R_{2}$ are posets, $R_{1}$ is a poset and $R_{2}$ is a set, or $R_{1}$ is a set and $R_{2}$ is a poset. Then a function $f: R_{1} \rightarrow R_{2}$ from $R_{1}$ to $R_{2}$ is a function $f_{0}$ from the set of elements of $R_{1}$ to the set of elements of $R_{2}$ and we write $f(r)=f_{0}(r)$ for all $r \in R_{1}$. Let $f$ and $f_{0}$ be as in the previous sentence. Then $f$ is injective if $f_{0}$ is injective, $f$ is surjective if $f_{0}$ is surjective, and $f$ is bijective if $f_{0}$ is bijective. We also call a function $f: R_{1} \rightarrow R_{2}$ a map from $R_{1}$ to $R_{2}$.

If $R_{1}$ and $R_{2}$ are such that $R_{1}$ and $R_{2}$ are sets, $R_{1}$ and $R_{2}$ are posets, $R_{1}$ is a poset and $R_{2}$ is a set, or $R_{1}$ is a set and $R_{2}$ is a poset, then define the following. Let $f: R_{1} \rightarrow R_{2}$ be a function. Then for all $X \subseteq R_{1}$, let $f(X)=\{f(r): r \in X\}$, and, for all $Y \subseteq R_{2}$, let $f^{-1}(Y)=\left\{r \in R_{1}\right.$ : $f(r) \in Y\}$. If $f$ is injective, then write $f^{-1}(r)=f^{-1}(\{r\})$ for all $r \in f\left(R_{1}\right)$. For all $X \subseteq R_{1}$,
let the restriction of $f$ to $X$, which we denote by $\left.f\right|_{X}$, be the function $g: X \rightarrow R_{2}$ defined by $g(r)=f(r)$ for all $r \in X$. Moreover, assume that $R_{3}$ is a set or a poset. If $f: R_{1} \rightarrow R_{2}$ and $g: R_{2} \rightarrow R_{3}$ are functions, then let $g \circ f$ denote the function $h: R_{1} \rightarrow R_{3}$ defined by $h(r)=g(f(r))$ for all $r \in R_{1}$. Moreover, if $f: R_{1} \rightarrow R_{1}$ is a function, then write $f^{1}=f$ and, for all $n \in \mathbb{N}$, write $f^{n+1}=f \circ f^{n}$. Furthermore, if $f: R_{1} \rightarrow R_{1}$ is a bijection, then let $f^{0}$ be the identity map on $X$ and, for all $n \in \mathbb{Z}$, let $f^{n+1}=f \circ f^{n}$.

Let $P$ and $Q$ be posets and let $f: P \rightarrow Q$ be a function. Then $f$ is order preserving if for all $p, q \in P, p \leq q$ implies that $f(p) \leq f(q)$, order reversing if for all $p, q \in P, p \leq q$ implies that $f(p) \geq f(q)$, an order embedding if $f$ is injective and if, for all $p, q \in P, p \leq q$ if and only if $f(p) \leq f(q)$, and an order isomorphism if $f$ is a surjective order embedding. Moreover, if $P$ is a poset and if $f: P \rightarrow P$ is a function, then $f$ is an order automorphism if $f$ is an order isomorphism.

We will define Young diagrams in the following way (cf. [2], also cf. [30, 41, 44, 45].) A Young diagram is the empty set or a finite subset $X$ of $\mathbb{N}^{2}$ such that for some $i, j \in \mathbb{N}$, $(1, j) \in X$ and $(i, 1) \in X$. We call the elements of a Young diagram $X$ the cells of $X$. Lastly, given a Young diagram $X$, define, for all $i \in \mathbb{N}$, row $i$ of $X$ to be the following subset of cells

$$
\{r \in X: \exists j \in \mathbb{N} \text { such that } r=(i, j)\}
$$

and, for all $j \in \mathbb{N}$, define column $j$ of $X$ to be the following subset of cells

$$
\{r \in X: \exists i \in \mathbb{N} \text { such that } r=(i, j)\}
$$

Sometimes, when we mention a cell $r=(i, j)$ in a Young diagram, we write $(i, j)$ instead of $r$.

In order for us to follow the conventions used in the literature [2, 30, 41, 44, 45], we will always depict Young diagrams by using an array of boxes where each such box has unit area and where each such box contains an element of $\mathbb{N}^{2}$ at its centre. Moreover, we also follow conventions in the literature by doing the following. We will, when depicting a Young diagram $X$, always draw row $i+1$ of $X$ beneath row $i$ and we will always draw column $j+1$ to the right of column $j$.

Example 2.3. If $X_{1}=\{(1,1),(1,2),(1,3)\}, X_{2}=\{(1,1),(1,3),(1,6),(1,7)\}$, and $X_{3}=$ $\{(1,1),(1,3),(2,2),(3,1),(3,3)\}$, then the Young diagram $X_{1}$ is depicted by

the Young diagram $X_{2}$ is depicted by

and the Young diagram $X_{3}$ is depicted by


Moreover, row 1 of $X_{1}$ is $X_{1}$, column $j$ of $X_{1}$, where $1 \leq j \leq 3$, is $\{(1, j)\}$, row 1 of $X_{2}$ is $X_{2}$, column $j$ of $X_{2}$, where $j \in\{1,3,6,7\}$, is $\{(1, j)\}$, row 1 of $X_{3}$ is $\{(1,1),(1,3)\}$, row 2 of $X_{3}$ is $\{(2,2)\}$, row 3 of $X_{3}$ is $\{(3,1),(3,3)\}$, column 1 of $X_{3}$ is $\{(1,1),(3,1)\}$, column 2 of $X_{3}$ is $\{(2,2)\}$, and column 3 of $X_{3}$ is $\{(1,3),(3,3)\}$.

Let $P$ be a poset, and let $p, q \in P$. Then $q$ covers $p$, or $p$ is covered by $q$, if $p<q$ in $P$ and no element $p^{\prime} \in P$ satisfies $p<p^{\prime}<q$ in $P$. If $P$ is a finite poset, then a Hasse diagram of $P$ is a
visual representation of $P$ such that the elements of $P$ are denoted by small non-intersecting circles and, for all $p, q \in P$ such that $q$ covers $p, q$ is drawn above $p$ and a line segment, or arc, is drawn between $p$ to $q$. We call the non-intersecting circles nodes. Sometimes, when we want to emphasize certain elements in a finite poset, we will replace the nodes corresponding to those elements with the elements themselves.

Example 2.4. Let $P$ be the finite poset with elements $p_{1}, p_{2}, p_{3}, p_{4}$ and partial order on $P$ defined by $p_{1}<p_{2}, p_{2}>p_{3}$, and $p_{3}<p_{4}$. Then a Hasse diagram of $P$ is depicted below.


If we want to emphasize the elements of $P$, then we write


We will depict posets using Hasse diagrams. Moreover, if $X$ is a Young diagram, then, as $X \subset \mathbb{Z}^{2}$, we will interpret $X$ as a poset with partial order as described earlier in this chapter. So if $P$ is a finite poset that is order isomorphic to a poset $Q$ such that $Q$ is a Young diagram, then we will sometimes depict $P$ with the Young diagram $Q$. The orientation of the elements depicted in a Young diagram is different from the orientation of elements depicted in a Hasse diagram. In a Hasse diagram, $p<q$ implies that $q$ is positioned above $p$, but in a Young diagram, $p<q$ implies that $q$ cannot be positioned to the left of $p$ and $q$ cannot be positioned above $p$. We illustrate this with an example.

Example 2.5. If $P$ is the poset depicted by the following Hasse diagram

then any of the Young diagrams shown below can also be used to depict $P$.


If $M$ is a matrix, then let $M(i, j)$ denote the entry in the $i^{t h}$ row and $j^{t h}$ column of $M$. A column vector is a matrix with one column, and a row rector is a matrix with one row. An $n_{1}$ by $n_{2}$ matrix is a matrix with $n_{1}$ rows and $n_{2}$ columns. If $v$ is a column vector with $N$ rows, then for all $1 \leq i \leq N$, let $v(i)$ denote the entry in the $i^{t h}$ row of $v$.

Consider the finite set $[n]$. A set partition of $[n]$ is a set $\mathscr{F}$ of non-empty subsets of $[n]$ such that every element of $[n]$ is contained in exactly one element of $\mathscr{F}$. A family $\mathscr{F}$ of sets is a surjective function $h: I \rightarrow X$ where $I$ and $X$ are sets and where every element of $X$ is a set. A subfamily $\mathscr{F}^{\prime}$ of a family $\mathscr{F}$ of sets is the restricion of a family $h: I \rightarrow X$ to some subset $I^{\prime}$ of $I$.

Let $\mathscr{F}$ be a family $h: I \rightarrow X$ of sets. Then we define a member $F$ of $\mathscr{F}$ to be an ordered pair $(i, h(i))$ where $i \in I$. When we use subscripts to describe the members $F$ of a family $h: I \rightarrow X$ of sets, the subscripts do not necessarily have to be elements of I. If we want to indicate that $F$ is a member of $\mathscr{F}$, then we write $F \in \mathscr{F}$. We write $F_{1}, F_{2}, \cdots \in \mathscr{F}$ if $F_{k} \in \mathscr{F}$ for all $k$.

Moreover, two members $F_{1}=\left(i_{1}, h\left(i_{1}\right)\right)$ and $F_{2}=\left(i_{2}, h\left(i_{2}\right)\right)$ of $\mathscr{F}$ are different if $i_{1} \neq i_{2}$. If $F=(i, h(i))$ is a member of $\mathscr{F}$, then we write $r \in F$ if $r \in h(i)$. We write $r_{1}, r_{2}, \cdots \in F$ if $r_{k} \in F$ for all $k$. For any set $Y$, define a function $f: \mathscr{F} \rightarrow Y$ from $\mathscr{F}$ to $Y$ to be a function $g: I \rightarrow Y$, and for all members $F \in \mathscr{F}$, write $f(F)=g(i)$ if $i \in I$ satisfies $F=(i, h(i))$. We also call $f: \mathscr{F} \rightarrow Y$ a map from $\mathscr{F}$ to $Y$. Such a function $f: \mathscr{F} \rightarrow Y$ is injective if $F_{1}, F_{2} \in \mathscr{F}$ and $f\left(F_{1}\right)=f\left(F_{2}\right)$ implies that $F_{1}$ and $F_{2}$ are not different.

Let $\mathscr{F}$ be a family $h: I \rightarrow X$ of sets. When describing the members $F=(i, h(i))$ of such families, we will write $h(i)$ instead of the ordered pair $(i, h(i))$. We will also use set-theoretic notation to describe families of sets by writing $\mathscr{F}=\{F: F \in \mathscr{F}\}$. For instance, if $\mathscr{F}$ is the family of sets defined by $h:\{1,2\} \rightarrow\{\{1\}\}$, then we write $\mathscr{F}=\{\{1\},\{1\}\}$, where the members $(1,\{1\})$ and $(2,\{1\})$ of $\mathscr{F}$ are both denoted by $\{1\}$. Moreover, we write $|\mathscr{F}|=|I|$, and say that $|\mathscr{F}|$ is the number of members of $\mathscr{F}$. A family $\mathscr{F}$ of sets is finite if $|\mathscr{F}|$ is finite. Lastly, if $\mathscr{F}$ is a family $h: I \rightarrow X$ of sets, then write $\bigcup_{F \in \mathscr{F}} F=\bigcup_{r \in I} h(r)$.

For all $n \in \mathbb{N}$, a partition $\lambda$ of $n$, written $\lambda \vdash n$, is a weakly decreasing sequence of positive integers whose sum is $n$. We write $\lambda=\left(\lambda_{1}, \lambda_{2}, \ldots, \lambda_{\ell}\right)$ to denote such a partition, where $\lambda_{i} \in \mathbb{N}$ for all $1 \leq i \leq \ell$ and $\lambda_{1}+\lambda_{2}+\cdots+\lambda_{\ell}=n$. For instance, $(3,2,2)$ is a partition of 7 and $(3,2,1)$ is a partition of 6 . We also let $\emptyset$ denote the empty partition, which we define to be the only partition of 0 . Whether the symbol $\emptyset$ refers to the empty set or to the empty partition can be easily determined from context.

## Chapter 3

## The $P$-partition enumeration problem

In this chapter, we define some essential terminology and describe the notation that we will use. Afterwards, we describe the $P$-partition enumeration problem that we will be focusing on in this thesis.

If $X$ is a finite set, then a labelling of $X$ is a bijection $f: X \rightarrow[k]$ where $k=|X|$. Next, we introduce terminology for $(P, \omega)$-partitions from [44], but extend labellings to countably infinite posets and define $\mathscr{A}(Q, \omega)$ in a non-standard way. If $P$ is a finite poset, then a labelling of $P$ is a bijection $\omega: P \rightarrow[k]$ where $k=|P|$. Moreover, if $P$ is a countably infinite poset, then a labelling of $P$ is a bijection $\omega: P \rightarrow \mathbb{Z}$. Recall from Chapter 2 that a map $f: P \rightarrow$ $Q$, where $P$ and $Q$ are posets, is order-reversing if $p_{1} \leq p_{2}$ in $P$ implies $f\left(p_{2}\right) \leq f\left(p_{1}\right)$ in $Q$. If $P$ is a countable poset, then a labelling $\omega$ of $P$ is natural if $\omega$ is an order preserving map and dual natural if $\omega$ is an order reversing map. If $P$ is a finite poset and if $\omega$ is a labelling of $P$, then a $(P, \omega)$-partition is an order-reversing map $f: P \rightarrow \mathbb{N}_{0}$ such that $f(x)>f(y)$ if $x<y$ and $\omega(x)>\omega(y)$. We sometimes call a $(P, \omega)$-partition a $P$-partition. Lastly, if $P$ is a
countable poset, if $Q$ is a finite poset such that $Q \subseteq P$, and if $\omega$ is a labelling of $P$, then let $\mathscr{A}(Q, \omega)$ denote the set of order-reversing maps $U: Q \rightarrow \mathbb{N}_{0}$ such that $U(x)>U(y)$ if $x<y$ and $\omega(x)>\omega(y)$. If $Q_{1} \subseteq Q$, then we let $\left.U\right|_{Q_{1}}$ denote the restriction of the function $U$ to $Q_{1}$.

If $P$ is a poset, then we will depict a labelling or an order-reversing map on $P$ with a Hasse diagram (or a Young diagram) whose nodes (or cells) are filled with integers. Specifically, if $P$ is a poset, if $p \in P$, if $X \subseteq \mathbb{Z}$, and if $f: P \rightarrow X$ satisfies $f(p)=k$, then, when depicting $f$ with a diagram, replace the node (or fill in the cell) corresponding to $p$ with $k$.

Example 3.1. If $(P, \leq)$ is the poset depicted by the left-most diagram shown below, where $P=\left\{p_{1}, p_{2}, p_{3}, p_{4}\right\}$, and if $f: P \rightarrow \mathbb{Z}$ is a function such that $f\left(p_{1}\right)=4, f\left(p_{2}\right)=2, f\left(p_{3}\right)=$ -1 , and $f\left(p_{4}\right)=3$, then we will depict $f$ with the right-most diagram shown below.


Example 3.2. If $(P, \leq)$ is the poset depicted by the left-most diagram shown below, where $P=\left\{p_{1}, p_{2}, p_{3}\right\}$, and if $f: P \rightarrow\{1,2,3\}$ is a function such that $f\left(p_{1}\right)=3, f\left(p_{2}\right)=2$, and $f\left(p_{3}\right)=2$, then we will depict $f$ with the right-most diagram shown below.

$$
\begin{array}{|r|r|r|}
\hline p_{1} \\
\cline { 2 - 2 } & p_{3} \\
\hline
\end{array} \quad \begin{array}{|r|r|}
\hline 2 & 2 \\
\hline
\end{array}
$$

If $P$ is a finite poset and if $\omega$ is a labelling of $P$, then we would like to say that two elements $U_{1}, U_{2} \in \mathscr{A}(P, \omega)$ are the same if the relative orderings of their entries are the same. Hence, we define the following equivalence relation on order preserving maps.

Definition 3.3. Let $P$ be a poset, let $S_{1}$ and $S_{2}$ be subsets of $\mathbb{Z}$, and let $f_{1}: P \rightarrow S_{1}$ and $f_{2}: P \rightarrow S_{2}$ be maps. Then $f_{1}$ is order equivalent to $f_{2}$ if there exists an order isomorphism
$g: f_{1}(P) \rightarrow f_{2}(P)$ such that $f_{2}=g \circ f_{1}$. Lastly, we write $f_{1} \equiv f_{2}$ if $f_{1}$ is order equivalent to $f_{2}$.

Example 3.4. Let $f_{1}: P \rightarrow \mathbb{Z}$ and $f_{2}: P \rightarrow \mathbb{Z}$ be depicted by

$$
\begin{array}{|l|l|l|}
\hline 5 & 2 & 2 \\
\hline & 6 & 7 \\
&
\end{array} \quad \text { and } \quad \begin{array}{|c|c|c|}
\hline 8 & -1 & -1 \\
\hline 9 & 15 \\
\hline
\end{array}
$$

respectively. Then $f_{1}(P)=\{2,5,6,7\}, f_{2}(P)=\{-1,8,9,15\}$, and $g: f_{1}(P) \rightarrow f_{2}(P)$ is defined by $g(2)=-1, g(5)=8, g(6)=9$, and $g(7)=15$. Since $f_{2}=g \circ f_{1}$, and since $g$ is an order isomorphism, $f_{1}$ is order equivalent to $f_{2}$.

Remark 3.5. The definition of $\mathscr{A}(Q, \omega)$ we are using is essentially the same as the standard definition of $\mathscr{A}(Q, \omega)$ given in [44]. Let $P$ and $Q$ be posets such that $Q \subseteq P$, and let $\omega$ be a labelling of P. Moreover, let $\omega_{Q}$ be the labelling of $Q$ such that $\left.\omega_{Q} \equiv \omega\right|_{Q}$. Then, $\mathscr{A}(Q, \omega)=\mathscr{A}\left(Q, \omega_{Q}\right)$.

We now formally define the $(P, \omega)$-partitions that we will count in this thesis.

Definition 3.6. Let $P$ and $Q$ be posets such that $Q$ is finite and $Q \subseteq P$. Moreover, let $\omega$ be a labelling of $P$ and let $\equiv_{Q, \omega}$ denote the equivalence relation on $\mathscr{A}(Q, \omega)$ defined by $f \equiv_{Q, \omega} g$ if $f$ is order equivalent to $g$. Then define

$$
\operatorname{Tb}(Q, \omega)=\mathscr{A}(Q, \omega) / \equiv_{Q, \omega}
$$

and define

$$
|Q, \omega|=|\mathrm{Tb}(Q, \omega)| .
$$

Moreover, if $Q_{1} \subseteq Q$ and if $T \in \operatorname{Tb}(Q, \omega)$, then let $\left.T\right|_{Q_{1}}$ be the element $T^{\prime}$ of $\operatorname{Tb}\left(Q_{1}, \omega\right)$ such that, for all $U \in \mathscr{A}(Q, \omega)$ satisfying $U \in T,\left.U\right|_{Q_{1}} \in T^{\prime}$.

Example 3.7. Let $P$ be depicted by the left-most six cell diagram shown below, let $Q \subseteq P$ consist of the cells of the left-most diagram that are filled with bullets, and let $\omega: P \rightarrow$ $\{1,2,3,4,5,6\}$ be depicted by the right-most diagram shown below.


$$
\begin{array}{|l|l|l|l|}
\hline 2 & 1 & 3 & \\
\hline & 6 & 4 & 5 \\
\cline { 2 - 4 } & &
\end{array}
$$

For all $1 \leq k \leq 6$, let $p_{k}=\omega^{-1}(k)$. Then $\mathscr{A}(P, \omega)$ consists of the order-preserving maps $f: P \rightarrow \mathbb{N}_{0}$ such that $f\left(p_{2}\right)>f\left(p_{1}\right), f\left(p_{1}\right) \geq f\left(p_{3}\right), f\left(p_{1}\right) \geq f\left(p_{6}\right), f\left(p_{3}\right) \geq f\left(p_{4}\right), f\left(p_{6}\right)>$ $f\left(p_{4}\right)$, and $f\left(p_{4}\right) \geq f\left(p_{5}\right)$. Three of the elements in $\mathscr{A}(Q, \omega)$ are depicted as follows.

| 3 | 2 | 2 |
| :--- | :--- | :--- |
|  | 2 | 1 |
|  |  |  |$\quad$| 5 | 4 | 3 |
| :--- | :--- | :--- |
|  | 2 | 1 |

The element of $\mathscr{A}(Q, \omega)$ depicted by the left-most diagram shown above is order equivalent to the element of $\mathscr{A}(Q, \omega)$ depicted by the right-most diagram shown above. However, the element of $\mathscr{A}(Q, \omega)$ depicted by the middle diagram shown above is not order equivalent to either of the other two elements just mentioned.

An example of an element of $\operatorname{Tb}(Q, \omega)$ is the following subset of $\mathscr{A}(Q, \omega)$.

$$
\begin{array}{|l|l|l|}
\hline 3 & 2 & 2 \\
\hline & 2 & 1 \\
\hline
\end{array}, \quad \begin{array}{|l|l|l|}
\hline 8 & 5 & 5 \\
\hline
\end{array}, \quad \begin{array}{|l|l|l|}
\hline 10 & 1 & 1 \\
\hline
\end{array}, \quad \ldots
$$

Remark 3.8. If $T \in \operatorname{Tb}(Q, \omega)$, then we will depict $T$ with one of the elements of $T$. For example, if $T$ is the element of $\operatorname{Tb}(Q, \omega)$ whose elements are depicted below

$$
\begin{array}{|l|l|l|}
\hline 3 & 2 & 2 \\
\hline & 2 & 1 \\
\hline
\end{array}, \quad \begin{array}{|l|l|l|}
\hline 8 & 5 & 5 \\
\hline
\end{array}, \quad \begin{array}{|l|l|l|}
\hline 10 & 1 & 1 \\
\hline
\end{array}, \quad \ldots
$$

then we will simply depict $T$ with any one of the above diagrams. For instance, we may say
that $T$ is depicted by

$$
\begin{array}{|l|l|l|}
\hline 8 & 5 & 5 \\
\hline & 5 & 4 \\
\cline { 2 - 3 }
\end{array} .
$$

## Chapter 4

## Periodic $(P, \omega)$-partitions

In this chapter, we define periodic $P$-partitions and investigate classes of periodic $P$-partitions that exist as follows. We first explain why periodic $P$-partitions include parallelogramic shapes and certain truncated shifted shapes. Then, we prove that periodic $P$-partitions also include certain $d$-dimensional, for $d \geq 3$, analogues of parallelogramic shapes. Lastly, we construct an example that differs very much from a parallelogramic shape or a truncated shifted shape.

In this thesis, we will count the quantity $|P, \omega|$ by defining a notion of separation for the poset $P$.

Definition 4.1. Let $P$ be a poset. Then a connected triple $(A, B, C)$ of $P$ is an ordered triple $(A, B, C)$ that satisfies the following two properties.

1. $A, B$, and $C$ are non-empty subsets of $P, A \cup B \cup C=P$, and $A \cap B=B \cap C=A \cap C=\emptyset$.
2. For all $p_{1} \in A$ and for all $p_{2} \in C$, there exists an element $p \in B$ such that $p_{1}<p<p_{2}$
in $P$.

Moreover, if $B$ is a non-empty subset of $P$, then $B$ connects $P$ if there exist non-empty subsets $A$ and $C$ of $P$ such that $(A, B, C)$ is a connected triple of $P$.

Example 4.2. A parallelogramic shape [27] is a Young diagram $X$ such that for some $n, k \in$ $\mathbb{N}$,

$$
X=\bigcup_{i=1}^{n} \bigcup_{j=1}^{k}\{(i, j+i-1)\} .
$$

For instance, if $n=3$ and $k=5$, then the corresponding parallelogramic shape $X$ is the following.


Such shapes are also called shifted strips where the parameter $k$ is called the width of such a strip [47], and they were investigated in [27], [47], and [49]. In this thesis, we will interpret such shapes as posets in the way specified in Chapter 2.

We make the following observation which can be generalized to any parallelogramic shape. If $P$ is the poset corresponding to the parallelogramic shape with $k=4$ and $n=4$ depicted below, then if A is the subset of $P$ that is depicted by the blank cells, if $B$ is the the subset of $P$ that is depicted by the cells filled with bullets, and if $C$ is the subset of $P$ that is depicted by the cells filled with asterisks, then $(A, B, C)$ is a connected triple of $P$.


We extend the usual definition of the successor function from the natural numbers to the
integers. Namely, let $\mathrm{s}: \mathbb{Z} \rightarrow \mathbb{Z}$ be the successor function defined by

$$
\mathrm{s}(n)=n+1
$$

for all $n \in \mathbb{Z}$. With this function, we define periodic quadruple systems.

Definition 4.3. Let $Z$ be a countably infinite poset, and let $\omega$ be a labelling of $Z$. Moreover, let $\pi: Z \rightarrow \mathbb{Z}$ be a surjective order preserving map, and let $\theta: Z \rightarrow Z$ be an order automorphism on $Z$. Then the ordered quadruple $(Z, \omega, \pi, \theta)$ is a periodic quadruple system if the following three properties hold.

1. There exists an order automorphism $\alpha: \mathbb{Z} \rightarrow \mathbb{Z}$ such that the following diagram commutes.

2. For all $n \in \mathbb{Z}, \pi^{-1}(\{n\})$ is finite.
3. There exists a finite subset $S$ of $Z$ such that $S$ connects $Z$.

Remark 4.4. A map $\alpha: \mathbb{Z} \rightarrow \mathbb{Z}$ is an order automorphism on $\mathbb{Z}$ if and only if there exists an integer $w \in \mathbb{Z}$ such that $\alpha(n)=n+w$ for all $n \in \mathbb{Z}$. So in the rest of the thesis, we will describe the order automorphism $\alpha: \mathbb{Z} \rightarrow \mathbb{Z}$ as a map on $\mathbb{Z}$ defined by $\alpha(n)=n+w$. Moreover, in this thesis, we will use the notational conventions for functions that we described in Chapter 2 for the functions $\omega, \pi$, and $\theta$ in Definition 4.3 and for similarly defined functions.

Remark 4.5. In Definition 4.3 $\pi^{-1}(\{n\})$ can be informally thought of as the subposet of $Z$ that is the $n^{\text {th }}$ copy of the subposet $\pi^{-1}(\{0\})$ of $Z$ because $\pi^{-1}(\{n\})=\theta^{n}\left(\pi^{-1}(\{0\})\right)$ by Property 1 of Definition 4.3 and because $\theta$ is an order automorphism on Z. By Property 2 of Definition 4.3, $\pi^{-1}(\{0\})$ is a finite poset and $Z$ is locally finite. Moreover, by Property 1 of Definition 4.3. $Z$ is the pairwise disjoint union of the copies of the subposet $\pi^{-1}(\{0\})$ in $Z$. Hence, we will depict $Z$ with a Hasse diagram of $\pi^{-1}\left(\left[n_{1}, n_{2}\right]\right)$, where $n_{1}, n_{2} \in \mathbb{Z}$ and $n_{2}-n_{1}$ is sufficiently large, and visually indicate how this Hasse diagram is contained in $Z$.

Example 4.6. Let $Z$ be the poset depicted by the left-most diagram shown below, and let $\omega$ be the labelling depicted by the right-most diagram shown below. The labelling $\omega$ is a dual natural labelling of $Z$. Define $\pi: Z \rightarrow \mathbb{Z}$ so that $\omega^{-1}(\{-10,-8,-5,-3\})=\pi^{-1}(\{1\})$, $\omega^{-1}(\{-6,-4,-1,1\})=\pi^{-1}(\{0\}), \omega^{-1}(\{-2,0,3,5\})=\pi^{-1}(\{-1\}), \omega^{-1}(\{2,4,7,9\})=$ $\pi^{-1}(\{-2\})$, and so on. Furthermore, let $\theta: Z \rightarrow Z$ be the order automorphism on $Z$ such that for all $n \in \mathbb{Z}, \theta\left(\omega^{-1}(n)\right)=\omega^{-1}(n-4)$.


To see that $(Z, \omega, \pi, \theta)$ satisfies Property 1 of Definition 4.3 let $\alpha: \mathbb{Z} \rightarrow \mathbb{Z}$ be defined by $\alpha(n)=n-4$ for all $n \in \mathbb{Z}$. Then for all $p \in Z, \mathrm{~s}(\pi(p))=\pi(\theta(p))$ and $\alpha(\omega(p))=$
$\omega(\theta(p))$. For instance, if $p=\omega^{-1}(3)$, then $\mathrm{s}(\pi(p))=\mathrm{s}(-1)=0, \pi(\theta(p))=0, \alpha(\omega(p))=$ $\alpha(3)=-1$, and $\omega(\theta(p))=-1$, implying that $\mathrm{s}(\pi(p))=0=\pi(\theta(p))$ and $\alpha(\omega(p))=-1=$ $\omega(\theta(p))$. Moreover, by how $\pi$ is defined in this example, $\left|\pi^{-1}(n)\right|=4$ for all $n \in \mathbb{Z}$. So $(Z, \omega, \pi, \theta)$ satisfies Property 2 of Definition 4.3. To see that $(Z, \omega, \pi, \theta)$ satisfies Property 3 of Definition 4.3. let $S \subset Z$ be defined by $S=\omega^{-1}(\{-6,-4,-2,-1\})$. The set $S$ is finite, and $S$ connects $Z$. Hence, $(Z, \omega, \pi, \theta)$ is a periodic quadruple system.

Remark 4.7. In Example 4.6 removing $\omega^{-1}(-2)$ from the index shape $S=\omega^{-1}(\{-6,-4$, $-2,-1\})$ results in a subset

$$
S^{\prime}=\omega^{-1}(\{-6,-4,-1\})
$$

of $Z$ that does not connect $Z$. To see this, suppose that $\left(A, S^{\prime}, C\right)$ is a connected triple of $Z$ for some subsets $A \subset Z$ and $C \subset Z$. We first make the following observation.

For all $p \in S^{\prime}, \pi(p)=0$. Moreover, $\pi$ is order preserving. Hence, Definition 4.1 implies that

$$
\{p \in Z: \pi(p)<0\} \subseteq A \quad \text { and } \quad\{p \in Z: \pi(p)>0\} \subseteq C
$$

for the following reason. Suppose without loss of generality that for some $p \in Z$ satisfying $\pi(p)>0, p \in A$. By Property 1 of Definition 4.1 C is non-empty. So there exists an element $q \in C$. By Property 2 of Definition 4.1, there exists an element $p^{\prime} \in S^{\prime}$ such that $p<p^{\prime}<q$ in Z. But then, as $\pi$ is order preserving,

$$
0<\pi(p)<\pi\left(p^{\prime}\right)=0
$$

which is impossible.

With this observation, we continue as follows. Let $p_{0}=\omega^{-1}(-2)$ and let $q_{0}=\omega^{-1}(-3)$.

Since $\pi\left(p_{0}\right)=-1$ and $\pi\left(q_{0}\right)=1$, the above observation implies that $p_{0} \in A$ and $q_{0} \in C$. But then, by Property 2 of Definition 4.1, there exists an element $p^{\prime} \in S^{\prime}$ such that $p_{0}<p^{\prime}<q_{0}$ in $Z$. But that is impossible because $p_{0} \| q_{0}$ in $Z$. Hence, $S^{\prime}$ does not connect $Z$.

Example 4.8. Let $Z$ be the poset depicted by the left-most diagram shown below, and let $\omega$ be the labelling depicted by the right-most diagram shown below. The labelling $\omega$ is an infinite analogue of certain labellings on skew shapes that are known as Schur labellings [45]. Define $\pi: Z \rightarrow \mathbb{Z}$ so that $\omega^{-1}(\{-5,0,5,10\})=\pi^{-1}(\{1\}), \omega^{-1}(\{-9,-4,1,6\})=\pi^{-1}(\{0\})$, $\omega^{-1}(\{-13,-8,-3,2\})=\pi^{-1}(\{-1\}), \omega^{-1}(\{-17,-12,-7,-2\})=\pi^{-1}(\{-2\})$, and so on. Furthermore, let $\theta: Z \rightarrow Z$ be the order automorphism on $Z$ defined by $\theta\left(\omega^{-1}(n)\right)=$ $\omega^{-1}(n+4)$.


To see that $(Z, \omega, \pi, \theta)$ satisfies Property 1 of Definition 4.3 let $\alpha: \mathbb{Z} \rightarrow \mathbb{Z}$ be defined by $\alpha(n)=n+4$ for all $n \in \mathbb{Z}$. Then for all $p \in Z, \mathrm{~s}(\pi(p))=\pi(\theta(p))$ and $\alpha(\omega(p))=\omega(\theta(p))$. For instance, if $p=\omega^{-1}(-8)$, then $\mathrm{s}(\pi(p))=\mathrm{s}(-1)=0, \pi(\theta(p))=0, \alpha(\omega(p))=\alpha(-8)=$ -4 , and $\omega(\theta(p))=-4$. So $\mathrm{s}(\pi(p))=0=\pi(\theta(p))$ and $\alpha(\omega(p))=-4=\omega(\theta(p))$. Moreover, by how $\pi$ is defined in this example, $\left|\pi^{-1}(n)\right|=4$ for all $n \in \mathbb{Z}$. So $(Z, \omega, \pi, \theta)$ satisfies

Property 2 of Definition 4.3. To see that $(Z, \omega, \pi, \theta)$ satisfies Property 3 of Definition 4.3 let $S \subset Z$ be defined by $S=\omega^{-1}(\{-4,1,2,6\})$. The set $S$ is finite, and $S$ connects $Z$. Hence, $(Z, \omega, \pi, \theta)$ is a periodic quadruple system.

We now formally define the $P$-partitions that we have informally been calling the $P$-partitions that exhibit a certain repeating pattern.

Definition 4.9. Let $(Z, \omega, \pi, \theta)$ be a periodic quadruple system and let $n \in \mathbb{N}$. Then a length $n$ periodic $(P, \omega)$-partition derived from $(Z, \omega, \pi, \theta)$ is a $(P, \omega)$-partition $(P, \omega)$ such that $P=\pi^{-1}([n])$ and $\left.\omega \equiv \omega\right|_{P}$. Moreover, a periodic $(P, \omega)$-partition derived from $(Z, \omega, \pi, \theta)$ is a length n periodic $(P, \omega)$-partition derived from $(Z, \omega, \pi, \theta)$ for some $n \in \mathbb{N}$.

Remark 4.10. Since $P=\pi^{-1}([n])=\bigcup_{k=1}^{n} \pi^{-1}(\{k\})$, $P$ can be informally thought of as the poset that results from pasting together $n$ copies $\pi^{-1}(\{1\}), \pi^{-1}(\{2\}), \ldots$, and $\pi^{-1}(\{n\})$ of the poset $\pi^{-1}(\{0\})$ where the pasting is determined by the periodic quadruple system $(Z, \omega, \pi, \theta)$.

Enumeration formulas for counting certain combinatorial objects that we now describe was established in [27] with special cases being established in [47, 49]. Let $X$ be a parallelogramic shape with $n$ rows and $k$ cells in each row as described in Example 4.2. Then a standard Young tableau of shape $X$ is a bijective filling of the cells of $X$ with integers from $[n k]$ such that the entries increase along every row from left to right and the entries increase along every column from top to bottom, and we call such a standard Young tableau a standard Young tableau of parallelogramic shape. For instance, the following is a standard Young tableau of parallelogramic shape

| 1 | 2 | 4 | 6 |  |
| :--- | :--- | :--- | :--- | :--- |
|  |  |  |  |  |
|  | 3 | 5 | 7 | 10 |
|  |  |  |  |  |
|  |  | 8 | 9 | 11 |
|  |  | 12 |  |  |

as the entries 3,5,7, and 10 increase from left to right along row 2 of the above Young diagram, the entries 4,5 , and 8 increase from top to bottom along column 3 of the above Young diagram, and so on.

Our analysis of $P$-partitions applies the above paragraph for the following reason. Consider a periodic quadruple system $(Z, \omega, \pi, \theta)$. The problem of enumerating the sequence $\left(\left|P_{n}, \omega\right|\right)_{n=1,2, \ldots}$, where $P_{n}=\pi^{-1}([n])$ for all $n$, generalizes the problem of counting the number of standard Young tableaux on parallelogramic shape, with $n$ rows and $k$ cells in each row, when $k$ is fixed. The case when $k=4$ is detailed in the following example.

Example 4.11. Let $Z, \omega, \pi$, and $\theta$ be as in Example 4.6. Then the length n periodic $(P, \omega)$ partitions derived from $(Z, \omega, \pi, \theta)$ correspond to the standard Young tableaux of parallelogramic shape with $n$ rows and four cells in each row. For instance, three length 3 periodic $(P, \omega)$-partitions derived from $(Z, \omega, \pi, \theta)$ are as follows, where the left-most and the middle $(P, \omega)$-partitions depicted below are order equivalent.


In particular, replacing every entry $k$ in the left-most diagram depicted above with 12 $k+1$ gives an example of standard Young tableaux of parallelogramic shape, and replacing every entry $k$ in the middle or right-most diagram depicted above with $12-k+2$ gives two examples of standard Young tableaux of parallelogramic shape.

Next, we explain how we can also apply our results to semistandard tableaux. If $X$ is a parallelogramic shape, then define a semistandard tableau of shape $X$ to be a function $f$ : $X \rightarrow \mathbb{Z}$ where $f\left(i_{1}, j\right)<f\left(i_{2}, j\right)$ if $\left(i_{1}, j\right),\left(i_{2}, j\right) \in X$ and $i_{1}<i_{2}$, and where $f\left(i, j_{1}\right) \leq f\left(i, j_{2}\right)$
if $\left(i, j_{1}\right),\left(i, j_{2}\right) \in X$ and $j_{1}<j_{2}$. That is, fill the cells of $X$ so that the entries weakly increase along every row from left to right, and the entries increase along every column from top to bottom. Moreover, if $f$ is a semistandard tableau of shape $X$, then define a semistandard tableau class on $X$ to be the set $F$ of semistandard tableau of shape $X$ that are order equivalent to $f$. For example, if $X$ is a parallelogramic shape with $n=3$ and $k=4$, then a semistandard tableau class on $X$ can be depicted by

| 1 | 1 | 2 | 2 |  |
| :--- | :--- | :--- | :--- | :--- |
|  |  |  |  |  |
|  | 2 | 3 | 5 | 5 |
|  |  |  |  |  |
|  |  | 5 | 6 | 7 |.

The periodic $P$-partitions that we introduced in Definition 4.9 can be regarded as a generalization of the semistandard tableau of parallelogramic shape $X$ if the number of cells in each row of $X$ is fixed. This is explained in the following example when the number of cells in each row is four.

Example 4.12. Let $Z, \omega, \pi$, and $\theta$ be as in Example 4.8. Then the length $n$ periodic $(P, \omega)$ partitions derived from $(Z, \omega, \pi, \theta)$ correspond to the semistandard tableaux of parallelogramic shape with $n$ rows and four cells in each row. For instance, three length 3 periodic $(P, \omega)$-partitions derived from $(Z, \omega, \pi, \theta)$ are as follows, where the left-most and the middle $(P, \omega)$-partitions depicted below are order equivalent.

| 6 | 6 | 6 | 6 |  |
| :--- | :--- | :--- | :--- | :--- |
|  | 5 | 4 | 3 | 2 |
|  |  |  |  |  |
|  |  | 3 | 2 | 1 |
|  |  | 1 | 1 |  |
|  |  |  |  |  |



| 9 | 9 | 8 | 8 |  |
| :--- | :--- | :--- | :--- | :--- |
|  | 7 | 6 | 6 | 6 |
|  |  |  |  |  |
|  |  | 5 | 5 | 5 |

In particular, replacing every entry $k$ in each of the three diagrams depicted above with $12-k+1$ gives depictions of three semistandard tableaux of shape $X$, where $X$ is the parallelogramic shape with 3 rows and 4 cells in each row.

Truncated shifted shapes are non-classical shapes that are a generalization of the parallelogramic shapes. For the following definition, note that if $X_{1}, X_{2}, \ldots$ is a sequence of sets and if $n_{1}, n_{2} \in \mathbb{N}$ are such that $n_{2}<n_{1}$, then $\bigcup_{i=n_{1}}^{n_{2}} X_{i}=\emptyset$.

Definition 4.13. (cf. [2.49]) Let $n \in \mathbb{N}$ and let $k_{1}, k_{2}, \ldots, k_{n}$ be a sequence of positive integers such that for some $i \in[n], k_{1} \leq k_{2} \leq \cdots \leq k_{i}$ and $k_{i} \geq k_{i+1} \geq \cdots \geq k_{n}$. Then a truncated shifted shape is the Young diagram $X$ defined by

$$
X=\bigcup_{i=1}^{n} \bigcup_{j=i}^{k_{i}}\{(i, j)\}
$$

Remark 4.14. In Definition 4.13, if $i=1$, then $k_{1} \geq k_{2} \geq \cdots \geq k_{n}$, and if $i=n$, then $k_{1} \leq$ $k_{2} \leq \cdots \leq k_{n}$.

Example 4.15. Let $n=3$, let $k_{1}=3$, let $k_{2}=4$, and let $k_{3}=3$. By setting $i=2$, we see that $k_{1} \leq k_{2} \leq \cdots \leq k_{i}$ and $k_{i} \geq k_{i+1} \geq \cdots \geq k_{n}$ because $k_{1} \leq k_{2}$ and $k_{2} \geq k_{3}$. Moreover, the truncated shifted shape $\bigcup_{i=1}^{n} \bigcup_{j=i}^{k_{i}}\{(i, j)\}$ is depicted below.


Definition 4.16. (cf. [2] 49]) Let $X$ be a truncated shifted shape that consists of $n$ cells. Then a standard Young tableau of shape $X$ is a bijective filling of the cells of $X$ with the elements of $[n]$ such that entries increase from left to right along every row of $X$ and entries increase from top to bottom along every column of $X$.

Example 4.17. Consider the truncated shifted shape $X$ in Example 4.15. The four standard Young tableaux of shape $X$ are depicted below.

| 1 | 2 | 3 |  |
| :--- | :--- | :--- | :--- |
|  | 4 | 5 | 6 |
|  |  | 7 |  |
|  |  |  |  |



Periodic $P$-partitions can also be regarded as a generalization of the standard Young tableaux on certain truncated shifted shapes. Fix a number $w \in \mathbb{N}$, assume that $\left(a_{n}\right)_{n=1,2, \ldots}$ is a sequence of positive integers that is periodic with period $w$, and assume that $a_{n} \geq 2$ for all $n \in \mathbb{N}$. For all $m \in \mathbb{N}$, let $Y_{m}$ be a truncated shifted shape with $m w$ rows such that for all $1 \leq i \leq m w$, row $i$ of $Y_{m}$ consists of $a_{i}$ cells. Consider a periodic quadruple system $(Z, \omega, \pi, \theta)$. The problem of enumerating the sequence $\left(\left|P_{n}, \omega\right|\right)_{n=1,2, \ldots}$, where $P_{n}=\pi^{-1}([n])$ for all $n$, is a generalization of the problem of counting the number of standard Young tableaux of shape $Y_{m}$, where the sequence $\left(Y_{m}\right)_{m=1,2, \ldots}$ is a defined above. This is illustrated in the following example.

Example 4.18. Let $\left(a_{n}\right)_{n=1,2, \ldots}$ be a periodic sequence, with period $w=2$, that is defined by $4,5,4,5, \ldots$ Then the sequence of truncated shifted shapes corresponding to this periodic sequence is depicted below


A periodic quadruple system $(Z, \omega, \pi, \theta)$ that we can use to enumerate the number of standard Young tableaux on the above shapes is as follows. Let $Z$ be depicted by the left-most diagram shown below, and let $\omega$ be depicted by the right-most diagram shown below.


Next, let

$$
\pi^{-1}(\{0\})=\omega^{-1}([9])=\omega^{-1}(\{1,2,3,4,5,6,7,8,9\})
$$

let

$$
\pi^{-1}(\{1\})=\omega^{-1}([-8,0])=\omega^{-1}(\{-8,-7,-6,-5,-4,-3,-2,-1,0\})
$$

and so on. Moreover, let $\theta: Z \rightarrow Z$ be defined by $\theta\left(\omega^{-1}(n)\right)=\omega^{-1}(n-9)$ for all $n \in \mathbb{Z}$. Then the map $\alpha: \mathbb{Z} \rightarrow \mathbb{Z}$ defined by $\alpha(n)=n-9$ for all $n \in \mathbb{Z}$ satisfies $\alpha(\omega(p))=\omega(\theta(p))$ for all $p \in Z$. Moreover, for all $p \in Z, \mathrm{~s}(\pi(p))=\pi(\theta(p))$. Hence, the quadruple $(Z, \omega, \pi, \theta)$ satisfies Property 1 of Definition 4.3. Moreover, $\omega^{-1}(\{2,3,4,6,7,8\})$ is a finite subset of $Z$ that connects $Z$. So $(Z, \omega, \pi, \theta)$ also satisfies Property 2 of Definition 4.3. It follows that the quadruple defined is a periodic quadruple system.

A natural question is to ask whether there exist periodic quadruple systems that are $d$ dimensional analogues of the periodic quadruple systems considered in Example 4.11, Example 4.12, and Example 4.18 for $d \geq 3$. To that end, we prove that, for $d \geq 3$, there exist
periodic quadruple systems that are $d$-dimensional analogues of Example 4.6 and Example 4.8

We construct a family of periodic quadruple systems $(Z, \omega, \pi, \theta)$, which generalize Example 4.8 and Example 4.6, such that $Z$ is a subposet of $\mathbb{Z}^{d}$.

Definition 4.19. Fix a positive integer $d \geq 2$, and consider $\mathbb{Z}^{d}$. Then let $n_{1}, n_{2}, \ldots, n_{d-1} \in \mathbb{N}$ and $n_{1}^{\prime}, n_{2}^{\prime}, \ldots, n_{d-1}^{\prime} \in \mathbb{N}$ satisfy $n_{i}>n_{i}^{\prime}$ for all $1 \leq i \leq d-1$. Next, let $X \subset \mathbb{Z}^{d}$ be defined by

$$
X=\left[n_{1}\right] \times\left[n_{2}\right] \times \cdots \times\left[n_{d-1}\right] \times\{0\}
$$

and define $\Delta \in \mathbb{Z}^{d}$ by

$$
\Delta=\left(n_{1}^{\prime}, n_{2}^{\prime}, \ldots, n_{d-1}^{\prime}, 1\right)
$$

where $0<n_{i}^{\prime}<n_{i}$ for all $1 \leq i \leq d-1$. Lastly, define

$$
Z(X, \Delta)=\bigcup_{n \in \mathbb{Z}}(X+n \Delta)
$$

Informally, $Z(X, \Delta)$ is a union of translates of a $(d-1)$-dimensional parallelogram where $X$ is the parallelogram and $\Delta$ is the translate. Moreover, recall that we will regard $\mathbb{Z}^{d}$ as a poset as described in Definition 2.2. So as $Z(X, \Delta) \subset \mathbb{Z}^{d}$ we will regard $Z(X, \Delta)$ as a subposet of $\mathbb{Z}^{d}$. Because $\mathbb{Z}^{d}$ is a countably infinite and locally finite poset, $Z(X, \Delta)$ is a countably infinite and locally finite poset. An example of such a poset $Z(X, \Delta)$ when $d=3, n_{1}=n_{2}=3$, and $n_{1}^{\prime}=n_{2}^{\prime}=1$ is depicted in the left-most diagram of Figure 4.1. Moreover, the poset $Z$ in Example 4.8 and Example 4.6 satisfies $Z=Z(X, \Delta)$ where $X=\{(1,0),(2,0),(3,0),(4,0)\}$ and $\Delta=(1,1)$.

Informally, we can think of $Z=Z(X, \Delta)$ as a pair consisting of a set of points and a shift.

Definition 4.20. Let $Z(X, \Delta)$ be as described in Definition 4.19. Then define $\pi_{X}^{\Delta}: Z(X, \Delta) \rightarrow$ $\mathbb{Z}$ by $\pi(p)=n$ for all $n \in \mathbb{Z}$ and for all $p \in X+n \Delta$, and define $\theta_{X}^{\Delta}: Z(X, \Delta) \rightarrow Z$ by $\theta(p)=$ $p+\Delta$ for all $p \in Z(X, \Delta)$.

Informally, $\pi_{X}^{\Delta}$ indicates which copy of $X$ we have, and $\theta_{X}^{\Delta}$ indicates where the next copy of a point is.

Example 4.21. Let d $=2, X=\{(1,0),(2,0)\}$, and let $\Delta=(1,1)$. Moreover, let $\left(Z_{1}, \omega_{1}, \pi_{1}, \theta_{1}\right)$ be the periodic quadruple system from Example 4.6 and let $\left(Z_{1}, \omega_{2}, \pi_{2}, \theta_{2}\right)$ be the periodic quadruple system from Example 4.8 Then

$$
Z_{1}=Z_{2}=Z(X, \Delta), \pi_{1}=\pi_{2}=\pi_{X}^{\Delta}, \text { and } \theta_{1}=\theta_{2}=\theta_{X}^{\Delta}
$$

Example 4.22. Consider the labelling $\omega$ of $Z(X, \Delta)$ depicted by the right-most diagram in Figure 4.1, and assume that $\omega(X)=[-1,7]$. Then $\pi_{X}^{\Delta}(\{[-10,-2]\})=-1, \pi_{X}^{\Delta}([-1,7])=0$, $\pi_{X}^{\Delta}([8,16])=1$, and so on. Moreover, $\theta_{X}^{\Delta}\left(\omega^{-1}(5)\right)=\omega^{-1}(14), \theta_{X}^{\Delta}\left(\omega^{-1}(-7)\right)=\omega^{-1}(2)$, and so on.

The dual natural labelling $\omega$ in Example 4.6 can be generalized. For we can let $\omega$ be a dual natural labelling of $Z(X, \Delta)$ such that $\omega(p)>\omega(q)$ for all $p, q \in Z(X, \Delta)$ and $\omega\left(p^{\prime}+\Delta\right)=$ $\omega\left(p^{\prime}\right)-|X|$ for all $p^{\prime} \in Z(X, \Delta)$. Moreover, note that such a natural labelling $\omega$ has the property that $\left(Z(X, \Delta), \omega, \pi_{X}^{\Delta}, \theta_{X}^{\Delta}\right)$ satisfies Property 1 of Definition 4.3.

Define the following family of hyperplanes of $\mathbb{Z}^{d}$. For all $i \in[d]$ and $k \in \mathbb{Z}$, let

$$
H_{i, k}=\left\{\left(k_{1}, k_{2}, \ldots, k_{d}\right) \in \mathbb{Z}^{d}: k_{i}=k\right\} .
$$

Informally, $H_{i, k}$ is the hyperplane of $\mathbb{Z}^{d}$ that consists of all points whose $i^{\text {th }}$ coordinate equals to $k$.

Moreover, for brevity, call a subset $S \subseteq Z(X, \Delta)$ a 0 -subset of $Z(X, \Delta)$ if $S=Z(X, \Delta)$, and, if $k \in[d]$, define a $k$-subset of $Z(X, \Delta)$ to be a proper subset $S \subset Z(X, \Delta)$ such that $S \neq \emptyset$ and

$$
S=Z(X, \Delta) \cap \bigcap_{i=1}^{k} H_{d-i+1, n_{i}}
$$

for some $n_{1}, n_{2}, \ldots, n_{k} \in \mathbb{Z}$. Because $Z(X, \Delta) \cap H_{d, n}=X+n \Delta$ for all $n \in \mathbb{Z}$, any 1 -subset of $Z(X, \Delta)$ is finite. It follows that if $k \geq 1$, then any $k$-subset of $Z(X, \Delta)$ is finite.

Example 4.23. Consider the set $X$, the poset $Z(X, \Delta)$, and the labelling $\omega$ of $Z(X, \Delta)$ in Example 4.22 Moreover, refer to Figure 4.1. An example of a 1 -subset of $Z(X, \Delta)$ is $X$, which satisfies $\omega(X)=[-1,7]$. This is because, by letting $n_{3}=0$, we have

$$
X=Z(X, \Delta) \cap H_{3,0}
$$

An example of a 2-subset of $Z(X, \Delta)$ is the subset $S^{(2)}$ of $X$ satisfying $\omega\left(S^{(2)}\right)=\{5,6,7\}$. This is because, by letting $n_{3}=0$ and $n_{2}=0$, we have

$$
S^{(2)}=Z(X, \Delta) \cap H_{3,0} \cap H_{2,0} .
$$

Lastly, an example of a 3-subset of $Z(X, \Delta)$ is the subset $S^{(3)}$ of $S^{(2)}$ satisfying $\omega\left(S^{(3)}\right)=\{6\}$.
This is because, by letting $n_{3}=0, n_{2}=0$, and $n_{1}=1$, we have

$$
S^{(3)}=Z(X, \Delta) \cap H_{3,0} \cap H_{2,0} \cap H_{1,1}
$$

With the notion of a $k$-subset, we can define a generalization of Schur labellings as follows.
Definition 4.24. For all $k \in[d-1]$, let $u_{k}$ be the element of $\mathbb{Z}^{d}$ such that for all $1 \leq i \leq d$, the $i^{\text {th }}$ coordinate of $u_{k}$ is 0 if $i \neq k$ and 1 if $i=k$. Moreover, define $u_{d}=\Delta$. Then $\omega_{X}^{\Delta}$ is the
labelling of $Z(X, \Delta)$ such that $0 \in \omega_{X}^{\Delta}(X)$ and the following holds. If $1 \leq k \leq d, p \in Z(X, \Delta)$, and $S$ is the $(d-k+1)$-subset of $Z(X, \Delta)$ that contains $p$, then

$$
\omega_{X}^{\Delta}\left(p+u_{k}\right)=\omega_{X}^{\Delta}(p)+(-1)^{k-1}|S|
$$

Example 4.25. Consider the set $X$, the poset $Z(X, \Delta)$, and the labelling $\omega$ of $Z(X, \Delta)$ in Example 4.22. Moreover, refer to Figure 4.1. The labelling $\omega$ equals to $\omega_{X}^{\Delta}$. Firstly, by the definition of $\omega$ in Example 4.22, $0 \in \omega(X)$. Secondly, to illustrate how $\omega$ satisfies all conditions of Definition 4.24. let $p \in Z(X, \Delta)$ be defined by $p=\omega^{-1}(6)$. Since $u_{1}=(1,0,0)$, $u_{2}=(0,1,0)$, and $u_{3}=\Delta=(1,1,1)$, we have that $p+u_{1}=\omega^{-1}(7), p+u_{2}=\omega^{-1}(3)$, and $p+u_{3}=\omega^{-1}(15)$. Moreover, let $S^{(1)}=X$, let $S^{(2)}$ be as in Example 4.23 and let $S^{(3)}$ be as in Example 4.23 These three sets are such that $S^{(1)}$ is the 1 -subset of $Z(X, \Delta)$ that contains p, $S^{(2)}$ is the 2-subset of $Z(X, \Delta)$ that contains $p$, and $S^{(3)}$ is the 3-subset of $Z(X, \Delta)$ that contains p. Moreover, $\left|S^{(1)}\right|=|X|=9,\left|S^{(2)}\right|=3$, and $\left|S^{(3)}\right|=1$. Hence,

$$
\begin{gathered}
\omega\left(p+u_{1}\right)=7=6+1=\omega(p)+(-1)^{0} \cdot 1=\omega(p)+(-1)^{0}\left|S^{(3)}\right| \\
\qquad \omega\left(p+u_{2}\right)=3=6-3=\omega(p)+(-1)^{1} \cdot 3=\omega(p)+(-1)^{1}\left|S^{(2)}\right| \\
\text { and } \omega\left(p+u_{3}\right)=15=6+9=\omega(p)+(-1)^{2} \cdot 9=\omega(p)+(-1)^{2}\left|S^{(1)}\right| .
\end{gathered}
$$

The generalized Schur labelling $\omega_{X}^{\Delta}$ of $Z(X, \Delta)$ satisfies

$$
\omega_{X}^{\Delta}(p+\Delta)=\omega_{X}^{\Delta}(p)+(-1)^{d-1}|X|
$$

for all $p \in Z(X, \Delta)$. So, from the above definitions of $\pi_{X}^{\Delta}$ and $\theta_{X}^{\Delta}$, the quadruple $\left(Z(X, \Delta), \omega_{X}^{\Delta}\right.$, $\left.\pi_{X}^{\Delta}, \theta_{X}^{\Delta}\right)$ satisfies Property 1 of Definition 4.3 .


Figure 4.1: A three-dimensional analogue of Example 4.8. Here, $X=\{(i, j, 0): 1 \leq$ $i \leq 3$ and $1 \leq j \leq 3\}$ and $\Delta=(1,1,1)$.

Let $\omega$ be a labelling of $Z(X, \Delta)$ such that $\left(Z(X, \Delta), \omega, \pi_{X}^{\Delta}, \theta_{X}^{\Delta}\right)$ satisfies Property 1 of Definition 4.3. Because $X$ is finite, we see that $\left(Z(X, \Delta), \omega, \pi_{X}^{\Delta}, \theta_{X}^{\Delta}\right)$ satisfies Property 2 of Definition 4.3. We show that $\left(Z(X, \Delta), \omega, \pi_{X}^{\Delta}, \theta_{X}^{\Delta}\right)$ also satisfies Property 3 of Definition 4.3.

Recall that, informally, $Z(X, \Delta)$ is a union of translates of a $(d-1)$-dimensional parallelogram where $X$ is the parallelogram and $\Delta$ is the translate. Moreover, recall that, informally, $H_{i, k}$ is the hyperplane of $\mathbb{Z}^{d}$ that consists of all points whose $i^{t h}$ coordinate equals $k$.

Theorem 4.26. Let $Z(X, \Delta)$ be as described in Definition 4.19. and let $\pi_{X}^{\Delta}$, and $\theta_{X}^{\Delta}$ be
as described in Definition 4.20. If $\omega$ is a labelling of $Z(X, \Delta)$ such that the quadruple $\left(Z(X, \Delta), \omega, \pi_{X}^{\Delta}, \theta_{X}^{\Delta}\right)$ satisfies Property 1 of Definition 4.3 then $\left(Z(X, \Delta), \omega, \pi_{X}^{\Delta}, \theta_{X}^{\Delta}\right)$ is a periodic quadruple system. In particular, the quadruple $\left(Z(X, \Delta), \omega_{X}^{\Delta}, \pi_{X}^{\Delta}, \theta_{X}^{\Delta}\right)$ is a periodic quadruple system, there are periodic quadruple systems $\left(Z(X, \Delta), \omega, \pi_{X}^{\Delta}, \theta_{X}^{\Delta}\right)$ where $\omega$ is natural, and there are periodic quadruple systems $\left(Z(X, \Delta), \omega, \pi_{X}^{\Delta}, \theta_{X}^{\Delta}\right)$ where $\omega$ is dual natural.

We roughly indicate how we will prove Theorem 4.26 in the following example.

Example 4.27. Consider the poset $Z(X, \Delta)$ where $X$ is

$$
\{(1,1,0),(1,2,0),(1,3,0),(2,1,0),(2,2,0),(2,3,0),(3,1,0),(3,2,0),(3,3,0)\}
$$

and where $\Delta=(1,1,1)$. Moreover, let $\pi=\pi_{X}^{\Delta}$, where $\pi_{X}^{\Delta}$ is as described in Definition 4.20 In each of the three diagrams in this example, the point $(1,1,0)$ is highlighted in blue, the point $(1,2,0)$ is highlighted in red, and the point $(1,3,0)$ is highlighted in green.

The intersection $H_{1,2} \cap Z(X, \Delta)$ consists of nine elements and is depicted by the nine elements in the left-most diagram shown below that are filled with bullets. Moreover, the intersection $H_{2,2} \cap Z(X, \Delta)$ consists of nine elements and is depicted by the nine elements in the right-most diagram shown below that are filled with bullets or that are highlighted in red.


Lastly, the intersection $H_{3,0} \cap Z(X, \Delta)$ consists of the nine elements depicted below that are filled with bullets, highlighted in blue, highlighted in red, or highlighted in green.


From the depictions of $H_{1,2} \cap Z(X, \Delta), H_{2,2} \cap Z(X, \Delta)$, and $H_{3,0} \cap Z(X, \Delta)$, we see that for all $p^{\prime}, q \in Z(X, \Delta)$ such that $\pi(q) \geq 2$ and $\pi\left(p^{\prime}\right)=-2, p^{\prime}<q$ because $H_{1,2}$ lies in between $p^{\prime}$ and $q, H_{2,2}$ lies in between $p^{\prime}$ and $q$, and $H_{3,0}$ lies in between $p^{\prime}$ and $q$. Similarly, we see that for all $p, p^{\prime} \in Z(X, \Delta)$ such that $\pi\left(p^{\prime}\right)=-2$ and $\pi(p) \leq-6, p<p^{\prime}$. Hence, the finite set $\pi^{-1}([-5,1])$, which has $7 \cdot 9=63$ elements, connects $Z(X, \Delta)$.

Now, we prove Theorem 4.26.

Proof. If $\omega$ is a labelling of $Z(X, \Delta)$ such that the quadruple $\left(Z(X, \Delta), \omega, \pi_{X}^{\Delta}, \theta_{X}^{\Delta}\right)$ satisfies Property 1 of Definition 4.3, then the following can be said. Assume that there is a finite
subset $S$ of $Z(X, \Delta)$ such that $S$ connects $Z(X, \Delta)$, then the quadruple $\left(Z(X, \Delta), \omega, \pi_{X}^{\Delta}, \theta_{X}^{\Delta}\right)$ satisfies Property 3 of Definition 4.3. Moreover, by Definition 4.19 and Definition 4.20, $\left(Z(X, \Delta), \omega, \pi_{X}^{\Delta}, \theta_{X}^{\Delta}\right)$ satisfies Property 2 of Definition 4.3. This implies, by Definition 4.3. that the quadruple $\left(Z(X, \Delta), \omega, \pi_{X}^{\Delta}, \theta_{X}^{\Delta}\right)$ is a periodic quadruple system.

Hence, to prove the theorem it is enough to prove that there is a finite subset $S$ of $Z(X, \Delta)$ such that $S$ connects $Z(X, \Delta)$. Let $\pi=\pi_{X}^{\Delta}$, and let $d$ be the positive integer corresponding to $X, \Delta$, and $Z(X, \Delta)$ as described in Definition 4.19 .

We first show that for all $i \in[d]$ and $k \in \mathbb{Z}, Z(X, \Delta) \cap H_{i, k}$ is finite. Suppose that $Z(X, \Delta) \cap H_{i, k}$ is infinite for some $i \in[d]$ and $k \in \mathbb{Z}$. Then consider the subsets $(X+n \Delta) \cap H_{i, k}$ for all $n \in \mathbb{Z}$. Since $Z(X, \Delta) \cap H_{i, k}$ is infinite, since $X+n \Delta$ is finite for all $n \in \mathbb{Z}$, and since

$$
Z(X, \Delta)=\bigcup_{n \in \mathbb{Z}} X+n \Delta
$$

there is an infinite subset $Y \subseteq \mathbb{Z}$ such that $(X+n \Delta) \cap H_{i, k} \neq \emptyset$ for all $n \in Y$. Recall that $\Delta=\left(n_{1}^{\prime}, n_{2}^{\prime}, \ldots, n_{d-1}^{\prime}, 1\right)$, and write $n_{d}^{\prime}=1$. Since $n_{j}^{\prime}>0$ for all $j \in[d]$, the definition of $H_{i, k}$ implies that the following holds for all $p \in X$

$$
\left|\{p+n \Delta: n \in \mathbb{Z}\} \cap H_{i, k}\right| \leq 1
$$

So for all distinct $n_{1}, n_{2} \in Y$,

$$
\left\{p \in X: p+n_{1} \Delta \in H_{i, k}\right\} \cap\left\{p \in X: p+n_{2} \Delta \in H_{i, k}\right\}=\emptyset .
$$

But then, as $Y$ is infinite and as

$$
\bigcup_{n \in Y}\left\{p \in X: p+n \Delta \in H_{i, k}\right\} \subseteq X
$$

it follows that $X$ is infinite, contradicting the assumption that $X$ is finite. So $Z(X, \Delta) \cap H_{i, k}$ is finite for all $i \in[d]$ and $k \in \mathbb{Z}$.

Since $\Delta=\left(n_{1}^{\prime}, n_{2}^{\prime}, \ldots, n_{d}^{\prime}\right)$ and, for all $j \in[d], n_{j}^{\prime}>0$, it follows from the definition of $H_{i, k}$ that for all $i \in[d]$ and $k \in \mathbb{Z}$,

$$
H_{i, k} \cap Z(X, \Delta) \neq \emptyset .
$$

Hence, as $Z(X, \Delta) \cap H_{i, k}$ is finite for all $i \in[d]$ and $k \in \mathbb{Z}$, there is a positive integer $m$ and a sequence of integers $k_{1}^{*}, k_{2}^{*}, \ldots, k_{d}^{*}$, such that $Z \cap H_{i, k_{i}^{*}} \neq \emptyset$ and $Z(X, \Delta) \cap H_{i, k_{i}^{*}} \subseteq \pi^{-1}([m])$ for all $i \in[d]$. Now, observe that the definition of the partial order on $\mathbb{Z}^{d}$ implies the following. If $v_{1}, v_{2} \in \mathbb{Z}^{d}$, and if, for all $i \in[d]$, there are integers $n_{i}^{\prime}, n_{i}^{\prime \prime} \in \mathbb{Z}$ such that $n_{i}^{\prime}<n_{i}^{\prime \prime}, v_{1} \in H_{i, n_{i}^{\prime}}$, and $v_{2} \in H_{i, k_{i}^{\prime \prime}}$, then $v_{1}<v_{2}$ in $\mathbb{Z}^{d}$. Hence, if $p, q \in Z(X, \Delta)$ are such that $\pi(p)<1$ and $\pi(q)>m$, then $p<q$ in $Z(X, \Delta)$. Repeating this argument for $\pi^{-1}([m+2,2 m+1])$ instead of $\pi^{-1}([m])$, we see that if $p, q \in Z(X, \Delta)$ are such that $\pi(p)<1$ and $\pi(q)>2 m+1$, then there is an element $p^{\prime} \in \pi^{-1}(\{m+1\})$ such that $p<p^{\prime}<q$ in $Z(X, \Delta)$. Therefore, the finite subset $S \subset Z(X, \Delta)$ defined by $S=\pi^{-1}([2 m+1])$ connects $Z(X, \Delta)$. From this, the theorem follows.

Hence, there are $d$-dimensional analogues of Example 4.8 and Example 4.6 for all $d \geq 3$.

Remark 4.28. The proof of Theorem 4.26 can be used to generalize Theorem 4.26 to include examples such as Example 4.18. This is based on the fact that the proof of Theorem 4.26 does


Figure 4.2: A more exotic example of a periodic quadruple system.
not entirely depend on Definition 4.19

Periodic quadruple systems, and their corresponding periodic $(P, \omega)$-partitions, can be very different from the examples we have considered so far. The following is a simple example of such a system.

Example 4.29. Let $Z$ be the poset depicted in Figure 4.2. And let $p_{1}, q_{1}, p_{0}, q_{0}, p_{-1}$, and $q_{-1}$ be the six elements of $Z$ that are as specified in Figure 4.2. Let $\omega: Z \rightarrow \mathbb{Z}$ be defined by $\omega\left(p_{0}\right)=0, \omega\left(q_{0}\right)=1$, and, for all $p \in Z, \omega(\theta(p))=\omega(p)+2$, let $\pi: Z \rightarrow \mathbb{Z}$ satisfy $\pi\left(p_{1}\right)=\pi\left(q_{1}\right)=1, \pi\left(p_{0}\right)=\pi\left(q_{0}\right)=0, \pi\left(p_{-1}\right)=\pi\left(q_{-1}\right)=-1$, and so on, and let $\theta: Z \rightarrow Z$
satisfy $\theta\left(p_{-1}\right)=p_{0}, \theta\left(q_{-1}\right)=q_{0}, \theta\left(p_{0}\right)=p_{1}, \theta\left(q_{0}\right)=q_{1}$, and so on. To check that the quadruple $(Z, \omega, \pi, \theta)$ satisfies Property 1 of Definition 4.3 let $\alpha: \mathbb{Z} \rightarrow \mathbb{Z}$ be defined by $\alpha(n)=n+2$ for all $n \in \mathbb{Z}$. Then for all $p \in Z, \mathrm{~s}(\pi(p))=\pi(\theta(p))$ and $\alpha(\omega(p))=\omega(\theta(p))$.

To see that $(Z, \omega, \pi, \theta)$ satisfies Property 3 of Definition 4.3. let $P_{n}=\pi^{-1}([n])$ for all $n \in \mathbb{N}$. We show that $P_{15}$ connects $Z$. Consider the element $q_{0} \in Z$. We have the inequalities,

$$
\begin{gathered}
q_{0}<\theta^{4}\left(p_{0}\right)<\theta^{7}\left(p_{0}\right)<\theta^{10}\left(p_{0}\right)<\theta^{13}\left(p_{0}\right), \\
q_{0}<\theta^{4}\left(p_{0}\right)<\theta^{4}\left(q_{0}\right)<\theta^{8}\left(p_{0}\right)<\theta^{11}\left(p_{0}\right)<\theta^{14}\left(p_{0}\right),
\end{gathered}
$$

and

$$
q_{0}<\theta^{4}\left(p_{0}\right)<\theta^{4}\left(q_{0}\right)<\theta^{8}\left(p_{0}\right)<\theta^{8}\left(q_{0}\right)<\theta^{12}\left(p_{0}\right)
$$

Moreover, $\left\{\theta^{14}\left(p_{0}\right), \theta^{13}\left(p_{0}\right), \theta^{12}\left(p_{0}\right)\right\}$ is the set of minimal elements of $\pi^{-1}([12, \infty))$. So as $\left\{\theta^{14}\left(p_{0}\right), \theta^{13}\left(p_{0}\right), \theta^{12}\left(p_{0}\right)\right\} \subset P_{15}$, it follows that for all $p \in Z$ satisfying $\pi(p) \geq 16$, there is an element $q \in P_{15}$ such that $q_{0}<q<p$. Since $\theta$ is an order automorphism on $Z$, the same conclusions also hold for $\theta^{-1}\left(q_{0}\right)$ and $\theta^{-1}\left(P_{15}\right)=\pi^{-1}([0,14])$, and for $\theta^{-2}\left(q_{0}\right)$ and $\theta^{-2}\left(P_{15}\right)=\pi^{-1}([-1,13])$. So as $\left\{q_{0}, \theta^{-1}\left(q_{0}\right), \theta^{-2}\left(q_{0}\right)\right\}$ is the set of maximal elements of $\pi^{-1}(\mathbb{Z} \backslash \mathbb{N})$, it follows that $P_{15}$ connects $Z$. Hence, $(Z, \omega, \pi, \theta)$ is a periodic quadruple system.

## Chapter 5

## The matrix difference equation

In this chapter, we enumerate the number of periodic $P$-partitions as follows. We define collections of numbers that sum to the number we are interested in. Each member of this collection counts periodic $P$-partitions that satisfy certain additional restrictions. Afterwards, we introduce tableau transfer matrices, whose entries count certain $P$-partitions, and prove the following. We first prove a structural property of the connected triples we introduced in the previous chapter. Then, using this structural property, we prove that the aforementioned collections of numbers, represented as column vectors, can be enumerated with a homogeneous first-order matrix difference equation in which the matrix is a tableau transfer matrix.

Building from the previous chapter, we define the following notion.

Definition 5.1. Let $(Z, \omega, \pi, \theta)$ be a periodic quadruple system. Then an index shape of $(Z, \omega, \pi, \theta)$ is a finite subset $S$ of $\pi^{-1}(\mathbb{Z} \backslash \mathbb{N})$ that satisfies the following two properties.

1. $\left(\pi^{-1}(\mathbb{Z} \backslash \mathbb{N}) \backslash S, S, \pi^{-1}(\mathbb{N})\right)$ is a connected triple of $Z$.
2. $\theta(S) \subseteq S \cup \pi^{-1}(\{1\})$.

Remark 5.2. If $S$ is an index shape of a periodic quadruple system $(Z, \omega, \pi, \theta)$, then $S$ connects $Z$. This fact, and the fact that $S$ is finite, will become important later on in this chapter.

Using Example 4.6 as a guide, we start our running example.

Example 5.3. Let $(Z, \omega, \pi, \theta)$ and $S$ be as defined in Example 4.6 Since in Example 4.6 we $\operatorname{saw} \theta\left(\omega^{-1}(n)\right)=\omega^{-1}(n-4)$,

$$
S=\omega^{-1}(\{-6,-4,-2,-1\}),
$$

$\left(\pi^{-1}(\mathbb{Z} \backslash \mathbb{N}) \backslash S, S, \pi^{-1}(\mathbb{N})\right)$ is a connected triple of $Z$. Hence, $S$ satisfies Property 1 of Definition 5.1. Since

$$
\begin{gathered}
\theta(S)=\omega^{-1}(\{-10,-8,-6,-5\}) \text { and } \pi^{-1}(\{1\})=\omega^{-1}(\{-10,-8,-5,-3\}), \\
\quad \theta(S)=\omega^{-1}(\{-10,-8,-6,-5\}) \\
\subseteq \omega^{-1}(\{-10,-8,-6,-5,-4,-3,-2,-1\})=S \cup \pi^{-1}(\{1\}) .
\end{gathered}
$$

Hence, $\theta(S) \subseteq S \cup \pi^{-1}(\{1\})$, implying that $S$ satisfies Property 2 of Definition 5.1 It follows that $S$ is an index shape of $(Z, \omega, \pi, \theta)$.

The following definition is an analogue of the notion of order equivalence that depends on the order automorphism $\theta$ on $Z$ in a periodic quadruple system.

Definition 5.4. Let $(Z, \omega, \pi, \theta)$ be a periodic quadruple system, and let $S_{1}$ and $S_{2}$ be subsets of $Z$. And assume that $S_{2}=\theta^{k}\left(S_{1}\right)$ for some $k \in \mathbb{Z}$. Then for all $T_{1} \in \operatorname{Tb}\left(S_{1}, \omega\right)$ and $T_{2} \in$
$\mathrm{Tb}\left(S_{2}, \omega\right)$, write $T_{1} \equiv_{\theta} T_{2}$ if for all $U_{1} \in T_{1}$ and $U_{2} \in T_{2}$, there is an order isomorphism $g: U_{1}\left(S_{1}\right) \rightarrow U_{2}\left(S_{2}\right)$ such that $g \circ U_{1}=U_{2} \circ \theta^{k}$.

Informally, the above definition says that $T_{1} \equiv{ }_{\theta} T_{2}$ if the relative ordering of the entries in $T_{1}$ is the same as the relative ordering of the entries in $T_{2}$.

Example 5.5. Let $(Z, \omega, \pi, \theta)$ and $S$ be as described in Example 5.3. First note from Example 5.3 that since $\theta\left(\omega^{-1}(n)\right)=\omega^{-1}(n-4), \theta^{-1}\left(\omega^{-1}(n)\right)=\omega^{-1}(n+4)$. The subposet $S$ of $Z$ can be depicted using the left-most Young diagram shown below, the subposet $S \cup \theta^{-1}(S) \cup \pi^{-1}(\{0\})$ of $Z$ can be depicted using the right-most Young diagram shown below that consists of eight cells. In the right-most Young diagram, the cells filled with circles or asterisks represents $\theta^{-1}(S)$ and the cells filled with asterisks or bullets represents $S$.


Let $T$ be the element of $\operatorname{Tb}\left(S \cup \theta^{-1}(S) \cup \pi^{-1}(\{0\})\right.$, $\left.\omega\right)$ that is depicted by the left-most diagram shown below.

|  | 7 |  |
| :--- | :--- | :--- |
| 8 | 6 | 3 |
|  |  |  |
| 5 | 4 | 2 | 1.1.



Now, let $S_{1}=\theta^{-1}(S)$, let $S_{2}=S$, let $T_{1}=\left.T\right|_{S_{1}}$, let $T_{2}=\left.T\right|_{S_{2}}$, and let $k=1$. Firstly, $S_{2}=$ $\theta^{k}\left(S_{1}\right)$. Secondly, for all $U_{1} \in T_{1}$ and $U_{2} \in T_{2}$, there is an order isomorphism $g: U_{1}\left(S_{1}\right) \rightarrow$ $U_{2}\left(S_{2}\right)$ such that $g \circ U_{1}=U_{2} \circ \theta^{k}$. For instance, if $U_{1}$ is depicted by the middle diagram shown above and if $U_{2}$ is depicted by the right-most diagram shown above, then define $g$ : $\{3,6,7,8\} \rightarrow\{1,2,3,4\}$ by $g(3)=1, g(6)=2, g(7)=3$, and $g(8)=4$. The map $g$ is an
order isomorphism from $\{3,6,7,8\}$ to $\{1,2,3,4\}$ and $g$ satisfies $g \circ U_{1}=U_{2} \circ \theta^{k}$. From this, we see that $T_{1} \equiv_{\theta} T_{2}$.

We now use index shapes to define the following family of square matrices. Informally, we are defining matrices that are built from index shapes and that allow us to enumerate many different $(P, \omega)$-partitions at once. Recall that we write $M(i, j)$ to denote the entry in row $i$ and column $j$ of a matrix $M$. Lastly, in the following definition, note that

$$
S \subseteq \theta^{-1}(S) \cup \pi^{-1}(\{0\})
$$

due to Property 2 of Definition 5.1.
Definition 5.6. Let $(Z, \omega, \pi, \theta)$ be a periodic quadruple system, and let $S$ be an index shape of $(Z, \omega, \pi, \theta)$. Moreover, let $L$ be an indexing of $\operatorname{Tb}(S, \omega)$, let $N=|S, \omega|$, and let $R:[N] \rightarrow$ $\mathrm{Tb}(S, \omega)$ be the inverse of $L$. Then the tableau transfer matrix $M$ derived from $(Z, \omega, \pi, \theta)$, $S$, and $L$ is the $N$ by $N$ matrix $M$ such that for all $1 \leq i \leq N$ and $1 \leq j \leq N, M(i, j)$ is the number of elements $T$ of

$$
\operatorname{Tb}\left(\theta^{-1}(S) \cup \pi^{-1}(\{0\}), \omega\right)
$$

such that

$$
\left.T\right|_{S} \equiv_{\theta} R(i) \quad \text { and }\left.\quad T\right|_{\theta^{-1}(S)} \equiv_{\theta} R(j)
$$

When $S$ and $L$ are not specified, we will call $M$ a tableau transfer matrix derived from $(Z, \omega, \pi, \theta)$.

Example 5.7. Consider the periodic quadruple system $(Z, \omega, \pi, \theta)$ and the index shape $S$ from Example 5.5. There are two elements of $\operatorname{Tb}(S, \omega)$, and they are depicted by the below diagrams.

|  | 4 | 4 |
| :--- | :--- | :--- | :--- | :--- |
| 3 | 2 | 1 |$\quad$| 4 | 2 | 1 |
| :--- | :--- | :--- |

In particular, $N=2$. Next, define $L: \operatorname{Tb}(S, \omega) \rightarrow\{1,2\}$ so that $L$ sends the element of $\mathrm{Tb}(S, \omega)$ depicted by the left-most diagram shown above to 1 and $L$ sends the element of $\mathrm{Tb}(S, \omega)$ depicted by the right-most diagram shown above to 2 . Lastly, let $R=L^{-1}$ be the inverse of L so that

$$
R(1)=\begin{array}{|l|l|}
\hline & 4 \\
3 & 2
\end{array} 1 . \quad \text { and } \quad R(2)=\begin{array}{|l|l|l}
\hline 4 & 3 & \\
\hline 4 & 2 & 1 \\
\hline
\end{array}
$$

The tableau transfer matrix $M$ derived from $(Z, \omega, \pi, \theta), S$, and $L$ is equal to

$$
\left[\begin{array}{ll}
3 & 4 \\
2 & 3
\end{array}\right] .
$$

To see how to construct $M$, we calculate $M(2,1)$. The number $M(2,1)$ is the number of elements $T \in \operatorname{Tb}\left(S \cup \theta^{-1}(S) \cup \pi^{-1}(\{0\}), \omega\right)$ such that $\left.T\right|_{S}=R(2)$ and $\left.T\right|_{\theta^{-1}(S)} \equiv{ }_{\theta} R(1)$. Consider the below diagram, which depicts $S$ as the set of cells filled with bullets or asterisks, which depicts $\theta^{-1}(S)$ as the set of cells filled with circles or asterisks, and which depicts $\pi^{-1}(\{0\})$ as the set of cells filled with bullets or are empty.


By an exhaustive search in which we use the left-most diagram shown below as a reference, it can be checked that the two elements $T$ of $\operatorname{Tb}\left(S \cup \theta^{-1}(S) \cup \pi^{-1}(\{0\}), \omega\right)$ such that $\left.T\right|_{S}=R(2)$ and $\left.T\right|_{\theta^{-1}(S)} \equiv_{\theta} R(1)$ are the elements that are depicted by the left-most diagram shown below or the right-most diagram shown below.

| 8 |  |  |  | 8 |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 7 | 6 | 3 |  | 7 | 5 | 3 |  |
| 5 | 4 | 2 | 1 | 6 | 4 | 2 | 1 |

To complement Definition 5.6, we define the following. Informally, we are defining the column vectors that correspond to the tableau transfer matrices. Recall that we write $v(i)$ to denote the entry in row $i$ of a column vector $v$.

Definition 5.8. Let $(Z, \omega, \pi, \theta)$ be a periodic quadruple system, let $S$ be an index shape of $(Z, \omega, \pi, \theta)$, and let $L$ be an indexing of $\operatorname{Tb}(S, \omega)$. Moreover, let $R=L^{-1}$ be the inverse of $L$. Then an admissible number for $(Z, \omega, \pi, \theta)$ and $S$ is an integer $n^{\prime}$ such that $n^{\prime} \leq$ 0 and $S \subseteq \pi^{-1}\left(\left[n^{\prime}, 0\right]\right)$. Next, let $n^{\prime}$ be an admissible number for $(Z, \omega, \pi, \theta)$ and $S$, let $T_{0} \in \operatorname{Tb}\left(\pi^{-1}\left(\left[n^{\prime}, 0\right]\right), \omega\right)$, let $n \in \mathbb{N}$, let $Q_{n}=\pi^{-1}\left(\left[n^{\prime}, n\right]\right)$, let $P_{n}=\pi^{-1}([n])$, and let $P^{\prime}=$ $\pi^{-1}\left(\left[n^{\prime}, 0\right]\right)$.

Define the $n^{\text {th }}$ set derived from $(Z, \omega, \pi, \theta), S$, and $T_{0}$ to be the set $X_{n}\left(T_{0}\right)$ of elements $T \in$ $\mathrm{Tb}\left(Q_{n}, \omega\right)$ such that $\left.T\right|_{P^{\prime}}=T_{0}$ and the following condition holds. If $U \in T, p \in P^{\prime}$, and $q \in P_{n}$, then $U(p)>U(q)$.

Moreover, for all $1 \leq i \leq|S, \omega|$, let the $i^{t h}$ part of the $n^{\text {th }}$ set derived from $(Z, \omega, \pi, \theta), S, L$, and $T_{0}$ be the set $X_{n, i}\left(T_{0}\right)$ of elements $T \in X_{n}\left(T_{0}\right)$ such that $\left.T\right|_{\theta^{n}(S)} \equiv_{\theta} R(i)$. Lastly, define the $n^{\text {th }}$ vector derived from $(Z, \omega, \pi, \theta), S, L$, and $T_{0}$ to be the column vector $v_{n}$ with $|S, \omega|$ entries such that, for all $1 \leq i \leq|S, \omega|, v_{n}(i)=\left|X_{n, i}\left(T_{0}\right)\right|$.

Informally, the above definition describes the following. The $n^{\text {th }}$ sets as given in the above definition are a collection of modified periodic $(P, \omega)$-partitions that will allow us to enumerate the periodic $(P, \omega)$-partitions themselves. Moreover, the $i^{t h}$ parts of such sets, as given in the above definition, provides us with a set partition of such collections of modified periodic
$(P, \omega)$-partitions that we will use later on in the chapter, and the $n^{t h}$ vectors give the cardinalities of these $i^{t h}$ parts. Lastly, admissible numbers enable us to effectively use such modified periodic $(P, \omega)$-partitions.

Example 5.9. Let $(Z, \omega, \pi, \theta), S$, and $L$ be as in Example 5.7 and let $R=L^{-1}$ be the inverse of $L$. In particular, from Example 5.7, $S=\omega^{-1}(\{-6,-4,-2,-1\})$. The number $n^{\prime}=-1$ is an admissible number for $(Z, \omega, \pi, \theta)$ and $S$ because

$$
S \subseteq \pi^{-1}([-1,0]),
$$

and from Example 4.6

$$
\pi^{-1}(\{0\})=\omega^{-1}(\{-6,-4,-1,1\}) \text { and } \pi^{-1}(\{-1\})=\omega^{-1}(\{-2,0,3,5\})
$$

This is depicted below where the eight cells represent $\pi^{-1}([-1,0])$ and the cells filled with bullets represent $S$.


In particular, as $n^{\prime}=-1, P^{\prime}=\pi^{-1}([-1,0])$ and $\operatorname{Tb}\left(\pi^{-1}\left(\left[n^{\prime}, 0\right]\right), \omega\right)=\operatorname{Tb}\left(\pi^{-1}([-1,0]), \omega\right)$.

Depicted below are the first three terms of $\left(Q_{n}\right)_{n=1,2, \ldots}$. The first three terms of $\left(P_{n}\right)_{n=1,2, \ldots}$ are depicted by the cells filled will bullets (note that $P_{n} \subseteq Q_{n}$ for all $n=1,2, \ldots$ ). Moreover, in each of the three diagrams, the eight blank cells depict $P^{\prime}=\pi^{-1}([-1,0])$.


Next, assume that $T_{0}$ is the following element of $\operatorname{Tb}\left(\pi^{-1}([-1,0]), \omega\right)$.

| 8 | 7 | 6 | 4 |
| :--- | :--- | :--- | :--- |

We illustrate $X_{n}\left(T_{0}\right), X_{n, i}\left(T_{0}\right)$, and $v_{n}$ when $n=1$. As described in Example 5.7, $\mathrm{Tb}(S, \omega)$ has two elements. Hence, by Definition 3.6, $|S, \omega|=2$ and it follows that the range for the index $i$ is $1 \leq i \leq 2$. The $1^{\text {st }}$ set $X_{1}\left(T_{0}\right)$ derived from $(Z, \omega, \pi, \theta)$, $S$, and $T_{0}$ can be determined as follows.

Consider the element $T \in \operatorname{Tb}\left(\pi^{-1}([-1,1]), \omega\right)$ defined by

| 1211 | 10 | 8 |  |  |
| :---: | :---: | :---: | :---: | :---: |
| 9 | 7 | 6 | 5 |  |
|  | 4 | 3 | 2 | 1 |

To see that $T \in X_{1}\left(T_{0}\right)$, we note the following. Since $\left.T\right|_{P^{\prime}}$ is depicted below,

$$
\begin{array}{|l|c|c|c|c|}
\hline 12 & 11 & 10 & 8 & \\
\hline 9 & 7 & 6 & 5 \\
\hline
\end{array}
$$

$\left.T\right|_{P^{\prime}}=T_{0}$ by definition. Lastly, note that for all $U \in T, p \in P^{\prime}$, and $q \in P_{1}, U(p)>U(q)$. For instance, let $U \in T$ be defined by

\[

\]

let $p \in P^{\prime}$ be depicted below by the cell filled with a circle, and let $q \in P_{1}$ be depicted below by the cell filled with a bullet.


Then,

$$
U(p)=11>4=U(q)
$$

Any element $T^{\prime}$ of $\mathrm{Tb}\left(\pi^{-1}([-1,1], \omega)\right.$ that is not $T$ does not satisfy the condition $\left.T^{\prime}\right|_{P^{\prime}}=T_{0}$ or does not satisfy the condition $U(p)>U(q)$ for some $U \in T^{\prime}, p \in P^{\prime}$, and $q \in P_{1}$. Hence, the $1^{\text {st }}$ set $X_{1}\left(T_{0}\right)$ derived from $(Z, \omega, \pi, \theta), S$, and $T_{0}$ is

$$
X_{1}\left(T_{0}\right)=\{T\}
$$

Let $T$ be the element of $X_{1}\left(T_{0}\right)$. Since $\theta(S)$ is depicted by the four cells filled with bullets, where the twelve cell diagram below depicts $Q_{1}$

it follows that $\left.T\right|_{\theta(S)}$ is depicted by the following from the definition of $T$ on the previous page.

\[

\]

It follows that $\left.T\right|_{\theta(S)} \equiv_{\theta} R(1)$, hence the $1^{\text {st }}$ part, $X_{1,1}\left(T_{0}\right)$, of the $1^{\text {st }}$ set derived from $(Z, \omega, \pi, \theta), S, L$, and $T_{0}$, is $X_{1}\left(T_{0}\right)$. Moreover, the $2^{\text {nd }}$ part, $X_{1,2}\left(T_{0}\right)$, of the $1^{\text {st }}$ set derived from $(Z, \omega, \pi, \theta), S, L$, and $T_{0}$, is the empty set because the only element of $X_{1}\left(T_{0}\right)$ is $T,\left.T\right|_{\theta(S)} \equiv{ }_{\theta} R(1)$, and $R(2) \neq R(1)$.

Therefore, as $\left|X_{1,1}\left(T_{0}\right)\right|=1$ and $\left|X_{1,2}\left(T_{0}\right)\right|=0$, the $1^{\text {st }}$ vector derived from $(Z, \omega, \pi, \theta), S, L$, and $T_{0}$ is the following column vector.

$$
v_{1}=\left[\begin{array}{l}
1 \\
0
\end{array}\right]
$$

The sum of the entries in the column vectors defined in Definition 5.8 gives the number of periodic $(P, \omega)$-partitions.

Proposition 5.10. Let $(Z, \omega, \pi, \theta)$ be a periodic quadruple system. Moreover, let $S$ be an index shape of $(Z, \omega, \pi, \theta)$, let $L$ be an indexing of $\mathrm{Tb}(S, \omega)$, let $R=L^{-1}$ be the inverse of $L$, let $P_{n}=\pi^{-1}([n])$ for all $n \in \mathbb{N}$, let $n^{\prime}$ be an admissible number for $(Z, \omega, \pi, \theta)$ and $S$, let $T_{0} \in \operatorname{Tb}\left(\pi^{-1}\left(\left[n^{\prime}, 0\right]\right), \omega\right)$, and let $N=|S, \omega|$. Lastly, let $v_{n}$ be the $n^{\text {th }}$ vector derived from $(Z, \omega, \pi, \theta), S, L$, and $T_{0}$. Then for all $n \in \mathbb{N}$,

$$
\left|P_{n}, \omega\right|=\sum_{i=1}^{N} v_{n}(i)
$$

Before proving this proposition, we illustrate this proposition with an example.
Example 5.11. Let $(Z, \omega, \pi, \theta), S$, and $L$ be as in Example 5.7. Moreover, let $R=L^{-1}$ be the inverse of L. By Example 5.9 an admissible number $n^{\prime}$ for $(Z, \omega, \pi, \theta)$ is $n^{\prime}=-1$. So let $T_{0} \in \operatorname{Tb}\left(\pi^{-1}\left(\left[n^{\prime}, 0\right]\right), \omega\right)$ be depicted by the following diagram as calculated in Example 5.9

| 8 | 7 | 6 | 4 |
| :--- | :--- | :--- | :--- |
|  |  |  |  |
|  | 5 | 3 | 2 |

One can calculate that the $2^{\text {nd }}$ vector $v_{2}$ derived from $(Z, \omega, \pi, \theta), S, L$, and $T_{0}$ is equal to

$$
\left[\begin{array}{l}
3 \\
2
\end{array}\right] .
$$

To see how to calculate $v_{2}$, we note the following. Let $P^{\prime}=\pi^{-1}([-1,0])$ be as in Example 5.9 and set $n=2$. Then, as $n^{\prime}=-1, Q_{n}=Q_{2}$ is depicted by the twelve cell diagram below, $P_{n}=P_{2}$ is depicted by the cells filled with bullets, and $P^{\prime}$ is depicted by the blank cells as calculated in Example 5.9


In the same way that we calculated the $1^{\text {st }}$ set $X_{1}\left(T_{0}\right)$ derived from $(Z, \omega, \pi, \theta), S$, and $T_{0}$, we can calculate that the $2^{\text {nd }}$ set $X_{2}\left(T_{0}\right)$ derived from $(Z, \omega, \pi, \theta), S$, and $T_{0}$ consists of the following five elements.


| 16 | 15 | 14 | 12 |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | 13 | 11 | 10 |  | 9 |  |  |
|  |  | 8 | 7 |  | 5 | 4 |  |
|  |  |  | 6 |  | 3 | 2 | 1 |


| 16 | 15 | 14 | 12 |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 13 | 11 | 10 | 9 |  |  |  |
|  | 8 | 7 | 6 | 4 |  |  |
|  |  | 5 | 3 | 2 | 1 |  |
|  |  |  |  |  |  |  |

To check that the above diagrams depict elements of $X_{2}\left(T_{0}\right)$, note that each element $T \in$ $X_{2}\left(T_{0}\right)$ depicted above satisfies $\left.T\right|_{P^{\prime}} \equiv_{\theta} T_{0}$ because $\left.T\right|_{P^{\prime}}$ is depicted below.

$$
\begin{array}{|l|l|l|l|l}
\hline 16 & 15 & 14 & 12 & \\
\hline 13 & 11 & 10 & 9 \\
\hline
\end{array}
$$

Moreover, as in Example 5.9. it can be seen that for all $T \in X_{2}\left(T_{0}\right)$ depicted above, for all $U \in T$, for all $p \in P^{\prime}$, and for all $q \in P_{2}, U(p)>U(q)$.

Next, note that, $\theta^{2}(S)$ is depicted by the four cells filled with bullets, where the sixteen cell diagram below depicts $Q_{2}$ as calculated in Example 5.9.


Hence, the following holds. The $1^{\text {st }}$ part, $X_{2,1}\left(T_{0}\right)$, of the $2^{\text {nd }}$ set derived from $(Z, \omega, \pi, \theta), S$, $L$, and $T_{0}$ consists of the elements $T \in X_{2}\left(T_{0}\right)$ such that $T_{\theta^{2}(S)} \equiv_{\theta} R(1)$. It can be checked, by checking the five elements earlier in this example and comparing the entries that are in $\theta^{2}(S)$ with $R(1)$ as given in Example 5.9 that the elements of $X_{2,1}\left(T_{0}\right)$ are depicted below.


| 1615 | 14 | 12 |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 13 | 11 | 10 | 9 |  |  |
|  | 8 | 7 | 6 | 4 |  |
|  |  | 5 | 3 | 2 | 1 |

In particular,

$$
v_{2}(1)=\left|X_{2,1}\left(T_{0}\right)\right|=3 .
$$

The $2^{\text {nd }}$ part, $X_{2,2}\left(T_{0}\right)$, of the $2^{\text {nd }}$ set derived from $(Z, \omega, \pi, \theta), S, L$, and $T_{0}$ consists of the elements $T \in X_{2}\left(T_{0}\right)$ such that $T_{\theta^{2}(S)} \equiv_{\theta} R(2)$. It can be checked similarly to the calculation of $X_{2,1}\left(T_{0}\right)$ above that the elements of $X_{2,1}\left(T_{0}\right)$ are depicted below.


| 1615 | 14 | 12 |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 13 | 11 | 10 |  | 9 |  |  |
|  | 8 | 7 |  | 5 | 3 |  |
|  |  | 6 |  | 4 | 2 | 1 |

In particular,

$$
v_{2}(2)=\left|X_{2,2}\left(T_{0}\right)\right|=2 .
$$

The sum of the entries of $v_{2}$ is $v_{2}(1)+v_{2}(2)=3+2=5$. Moreover, $\operatorname{Tb}\left(P_{2}, \omega\right)$ consists of the following five elements.


In particular, $\left|P_{2}, \omega\right|=5$, which equals to the sum of the entries of $v_{2}$.

Proof. Let $n \in \mathbb{N}$, let $X_{n, i}\left(T_{0}\right)$ be the $i^{\text {th }}$ part of the $n^{\text {th }}$ set derived from $(Z, \omega, \pi, \theta), S, L$, and $T_{0}$ for all $1 \leq i \leq N$, and let $X_{n}\left(T_{0}\right)$ be the $n^{t h}$ set derived from $(Z, \omega, \pi, \theta), S$, and $T_{0}$. Recall that, by Definition 5.8, $X_{n, i}\left(T_{0}\right)$ is the set of elements $T \in X_{n}\left(T_{0}\right)$ such that

$$
\left.T\right|_{\theta^{n}(S)} \equiv_{\theta} R(i)
$$

for all $1 \leq i \leq N$. Moreover, for all $T \in X_{n}\left(T_{0}\right)$, there is exactly one number $1 \leq i \leq N$ such that $\left.T\right|_{\theta^{n}(S)} \equiv_{\theta} R(i)$ because $R\left(i_{1}\right) \neq R\left(i_{2}\right)$ for all $1 \leq i_{1} \leq N$ and $1 \leq i_{2} \leq N$ satisfying $i_{1} \neq i_{2}$. Hence, we have

$$
X_{n}\left(T_{0}\right)=\bigcup_{i=1}^{N} X_{n, i}\left(T_{0}\right)
$$

where the union is pairwise disjoint. Moreover, $\left|X_{n}\left(T_{0}\right)\right|=\left|P_{n}, \omega\right|$ for the following reason. Define the map $f: X_{n}\left(T_{0}\right) \rightarrow \mathrm{Tb}\left(P_{n}, \omega\right)$ by

$$
f(T)=\left.T\right|_{P_{n}}
$$

By Definition 5.8, $\left.T\right|_{P^{\prime}}=\left.T^{\prime}\right|_{P^{\prime}}=T_{0}$ for all $T, T^{\prime} \in X_{n}\left(T_{0}\right)$, where $P^{\prime}=\pi^{-1}\left(\left[n^{\prime}, 0\right]\right)$. Furthermore, for all $T \in X_{n}\left(T_{0}\right)$, for all $U \in T$, for all $p \in P^{\prime}$, and for all $q \in P_{n}, U(p)>U(q)$.

Hence, for all $T, T^{\prime} \in X_{n}\left(T_{0}\right), T \neq T^{\prime}$ if and only if $\left.T\right|_{P_{n}} \neq\left. T^{\prime}\right|_{P_{n}}$. So as $f(T)=\left.T\right|_{P_{n}}$ for all $T \in \operatorname{Tb}\left(P_{n}, \omega\right), f$ is injective.

To see that the map $f$ is surjective, let $T \in \operatorname{Tb}\left(P_{n}, \omega\right)$ and let $U \in T$. Set $Q_{n}=\pi^{-1}\left(\left[n^{\prime}, n\right]\right)$ and let $U_{0} \in T_{0}$ be such that for all $p \in P^{\prime}$ and $q \in P_{n}$,

$$
U_{0}(p)>U(q)
$$

Then, define $U^{\prime} \in \mathscr{A}\left(Q_{n}, \omega\right)$ by

$$
U^{\prime}(p)= \begin{cases}U_{0}(p) & \text { if } p \in P^{\prime} \\ U(p) & \text { if } p \in P_{n}\end{cases}
$$

Lastly, let $T^{\prime}$ be the element in $\operatorname{Tb}\left(Q_{n}, \omega\right)$ that satisfies $U^{\prime} \in T^{\prime}$. Then,

$$
f\left(T^{\prime}\right)=\left.T^{\prime}\right|_{P_{n}}=T
$$

Hence, as the choice of $T$ was arbitrary, $f$ is surjective.

It follows that $f$ is a bijection and, as $f$ is a bijection, $\left|X_{n}\left(T_{0}\right)\right|=\left|P_{n}, \omega\right|$. Lastly, by Definition 5.8, $v_{n}(i)=\left|X_{n, i}\left(T_{0}\right)\right|$ for all $1 \leq i \leq N$. Therefore,

$$
\left|P_{n}, \omega\right|=\left|X_{n}\left(T_{0}\right)\right|=\sum_{i=1}^{N}\left|X_{n, i}\left(T_{0}\right)\right|=\sum_{i=1}^{N} v_{n}(i) .
$$

Next, we prove a structural property of connected triples. It is a crucial lemma that will allow us to prove the main result of this chapter. Informally, the following lemma states that for
connected triples of finite posets, we can define a notion of union for two $P$-partitions in a well-defined manner.

For the proof of the following lemma, recall that if $U_{1}, U_{2} \in \mathscr{A}(P, \omega)$ for some finite poset $P$ and labelling $\omega$ of $P$, then $U_{1}$ and $U_{2}$ are order equivalent if $U_{1} \equiv U_{2}$.

Lemma 5.12. Let $Q$ be a finite poset, let $(A, B, C)$ be a connected triple of $Q$, and let $\omega_{Q}$ be a labelling of $Q$. Then for all $T^{\prime} \in \operatorname{Tb}\left(A \cup B, \omega_{Q}\right)$ and $T^{\prime \prime} \in \operatorname{Tb}\left(B \cup C, \omega_{Q}\right)$ such that $\left.T^{\prime}\right|_{B}=\left.T^{\prime \prime}\right|_{B}$, there is a unique element $T \in \operatorname{Tb}\left(Q, \omega_{Q}\right)$ such that $\left.T\right|_{A \cup B}=T^{\prime}$ and $\left.T\right|_{B \cup C}=T^{\prime \prime}$.

Example 5.13. Let $Q$ be the twelve element poset depicted by the below diagram. Moreover, let $A$ be the subposet of $Q$ depicted by the cells that are filled with asterisks, let $B$ be the subposet of $Q$ depicted by the cells that are filled with bullets, and let $C$ be the subposet of $Q$ depicted by the blank cells. Moreover, let $\omega_{Q}$ be a dual natural labelling of $Q$, where in terms of tableaux the entries in the rows decrease from left to right, and the entries in the columns decrease from top to bottom.


From how A, B, and C are defined, $(A, B, C)$ is a connected triple of $Q$ by Definition 4.1. Let $T^{\prime}$ be the element of $\operatorname{Tb}\left(A \cup B, \omega_{Q}\right)$ that is depicted by the left-most diagram below, and let $T^{\prime \prime}$ be the element of $\operatorname{Tb}\left(B \cup C, \omega_{Q}\right)$ that is depicted by the right-most diagram below.


| 8 7 <br>   <br> 8 3 <br>   <br> 6 4 | 1 |
| :--- | :--- | :--- |

Then $\left.T^{\prime}\right|_{B}=\left.T^{\prime \prime}\right|_{B}$ for the following reason. The element $\left.T^{\prime}\right|_{B}$ of $\operatorname{Tb}(B, \omega)$ is depicted by
the left-most diagram shown below and the element $\left.T^{\prime \prime}\right|_{B}$ of $\operatorname{Tb}(B, \omega)$ is depicted by the right-most diagram shown below.

|  | 3 |  |
| :--- | :--- | :--- |
| 4 | 2 | 1 |


|  | 7 |  |
| :--- | :--- | :--- |
| 8 | 5 | 3 |

Hence, $\left.T^{\prime}\right|_{B}=\left.T^{\prime \prime}\right|_{B}$. So by Lemma 5.12 there exists a unique element $T \in \operatorname{Tb}\left(Q, \omega_{Q}\right)$ that satisfies $\left.T\right|_{A \cup B}=T^{\prime}$ and $\left.T\right|_{B \cup C}=T^{\prime \prime}$. This unique element $T$ is depicted by the following diagram.

\[

\]

The reason that $\left.T\right|_{A \cup B}=T^{\prime}$ and $\left.T\right|_{B \cup C}=T^{\prime \prime}$ is because $\left.T\right|_{A \cup B}$ is depicted by the left-most diagram shown below, $\left.T\right|_{B \cup C}$ is depicted by the right-most diagram shown below, and the left-most diagram shown below also depicts $T^{\prime}$.

| 12 | 11 | 10 | 7 |
| :---: | :---: | :---: | :---: |
|  | 9 | 8 | 5 |
|  |  | 3 |  |


|  | 7 |  |
| :--- | :--- | :--- |

Proof. It is enough to prove the following. Assume that there are elements $U_{1,1}, U_{1,2} \in$ $\mathscr{A}\left(A \cup B, \omega_{Q}\right)$ and elements $U_{2,1}, U_{2,2} \in \mathscr{A}\left(B \cup C, \omega_{Q}\right)$ such that $U_{1,1} \equiv U_{1,2}, U_{2,1} \equiv U_{2,2}$, $\left.\left.U_{1,1}\right|_{B} \equiv U_{2,1}\right|_{B}$, and $\left.\left.U_{1,2}\right|_{B} \equiv U_{2,2}\right|_{B}$. Then the following two statements hold.

1. There exist elements $U_{1} \in \mathscr{A}\left(Q, \omega_{Q}\right)$ and $U_{2} \in \mathscr{A}\left(Q, \omega_{Q}\right)$ such that $\left.U_{1}\right|_{A \cup B} \equiv U_{1,1}$, $\left.U_{1}\right|_{B \cup C} \equiv U_{2,1},\left.U_{2}\right|_{A \cup B} \equiv U_{1,2}$, and $\left.U_{2}\right|_{B \cup C} \equiv U_{2,2}$.
2. If $U_{1}^{\prime} \in \mathscr{A}\left(Q, \omega_{Q}\right)$ and $U_{2}^{\prime} \in \mathscr{A}\left(Q, \omega_{Q}\right)$ satisfy $\left.U_{1}^{\prime}\right|_{A \cup B} \equiv U_{1,1},\left.U_{1}^{\prime}\right|_{B \cup C} \equiv U_{2,1},\left.U_{2}^{\prime}\right|_{A \cup B} \equiv$ $U_{1,2}$, and $\left.U_{2}^{\prime}\right|_{B \cup C} \equiv U_{2,2}$, then $U_{1}^{\prime} \equiv U_{2}^{\prime}$.

We first prove Statement 1. Since $\left.\left.U_{1, j}\right|_{B} \equiv U_{2, j}\right|_{B}$ for all $1 \leq j \leq 2$, there are order embeddings $g_{1, j}: U_{1, j}(A \cup B) \rightarrow \mathbb{N}_{0}$ and $g_{2, j}: U_{2, j}(B \cup C) \rightarrow \mathbb{N}_{0}$ such that, for all $p \in B$,

$$
g_{1, j}\left(U_{1, j}(p)\right)=g_{2, j}\left(U_{2, j}(p)\right)
$$

So for all $1 \leq j \leq 2$, define $U_{j}: Q \rightarrow \mathbb{N}_{0}$ by

$$
U_{j}(p)= \begin{cases}g_{1, j}\left(U_{1, j}(p)\right) & \text { if } p \in A \cup B \\ g_{2, j}\left(U_{2, j}(p)\right) & \text { if } p \in B \cup C\end{cases}
$$

For all $1 \leq j \leq 2$, the above map $U_{j}$ is well-defined because of the definition of $g_{1, j}$ and $g_{2, j}$. Moreover, for all $1 \leq j \leq 2$, the map $U_{j}$ satisfies

$$
\left.U_{j}\right|_{A \cup B}=g_{1, j} \circ U_{1, j} \equiv U_{1, j} \text { and }\left.U_{j}\right|_{B \cup C}=g_{2, j} \circ U_{2, j} \equiv U_{2, j} .
$$

Hence, to prove Statement 1 , it is enough to prove that $U_{j} \in \mathscr{A}\left(Q, \omega_{Q}\right)$ for all $1 \leq j \leq 2$. So let $j \in\{1,2\}$. To see that $U_{j}$ is order reversing as required by the definition of $\mathscr{A}\left(Q, \omega_{Q}\right)$ in Chapter 3, suppose otherwise.

Because $\left.U_{j}\right|_{A \cup B} \in \mathscr{A}\left(A \cup B, \omega_{Q}\right)$ and $\left.U_{j}\right|_{B \cup C} \in \mathscr{A}\left(B \cup C, \omega_{Q}\right),\left.U_{j}\right|_{A \cup B}$ and $\left.U_{j}\right|_{B \cup C}$ are order reversing.

So, as we are supposing that $U_{j}$ is not order reversing, there are elements $p \in A$ and $q \in C$ such that $p<q$ but $U_{j}(p)<U_{j}(q)$. By Property 2 of Definition 4.1, there exists an element $p^{\prime} \in B$ such that $p<p^{\prime}<q$ in $Q$. As $p, p^{\prime} \in A \cup B$, as $p<p^{\prime}$ in $A \cup B$, and as $\left.U_{j}\right|_{A \cup B}$ is order reversing, it follows that $U_{j}(p) \geq U_{j}\left(p^{\prime}\right)$. Moreover, as $p^{\prime}, q \in B \cup C$, as $p^{\prime}<q$ in $B \cup C$, and as $\left.U_{j}\right|_{B \cup C}$ is order reversing, it follows that $U_{j}\left(p^{\prime}\right) \geq U_{j}(q)$. But then, $U_{j}(p) \geq U_{j}\left(p^{\prime}\right) \geq U_{j}(q)$,
which is contrary to the assumption that $U_{j}(p)<U_{j}(q)$.

So $U_{j}$ is order reversing. Suppose that $U_{j} \notin \mathscr{A}\left(Q, \omega_{Q}\right)$. Then there are elements $p, q \in Q$ such that $p<q$ in $Q, \omega_{Q}(p)>\omega_{Q}(q)$, and $U_{j}(p)=U_{j}(q)$. Because $\left.U_{j}\right|_{A \cup B} \in \mathscr{A}\left(A \cup B, \omega_{Q}\right)$ and $\left.U_{j}\right|_{B \cup C} \in \mathscr{A}\left(B \cup C, \omega_{Q}\right)$, it follows that $p \in A$ and $q \in C$. Since $p \in A$ and $q \in C$, Property 2 of Definition 4.1 implies that there exists an element $p^{\prime} \in B$ such that $p<p^{\prime}<q$ in $Q$.

If $\omega_{Q}(p) \leq \omega_{Q}\left(p^{\prime}\right)$ and $\omega_{Q}\left(p^{\prime}\right) \leq \omega_{Q}(q)$, then $\omega_{Q}(p) \leq \omega_{Q}\left(p^{\prime}\right) \leq \omega_{Q}(q)$, implying that $\omega_{Q}(p) \leq \omega_{Q}(q)$. But that is contrary to the assumption that $\omega_{Q}(p)>\omega_{Q}(q)$. Hence, $\omega_{Q}(p)>$ $\omega_{Q}\left(p^{\prime}\right)$ or $\omega_{Q}\left(p^{\prime}\right)>\omega_{Q}(q)$. So as $p, p^{\prime} \in A \cup B, p<p^{\prime}$ in $A \cup B, p^{\prime}, q \in B \cup C, p^{\prime}<q$ in $B \cup C$, $\left.U_{j}\right|_{A \cup B} \in \mathscr{A}\left(A \cup B, \omega_{Q}\right)$, and $\left.U_{j}\right|_{B \cup C} \in \mathscr{A}\left(B \cup C, \omega_{Q}\right)$, it follows that

$$
U_{j}(p)>U_{j}\left(p^{\prime}\right) \geq U_{j}(q) \quad \text { or } \quad U_{j}(p) \geq U_{j}\left(p^{\prime}\right)>U_{j}(q)
$$

But then, $U_{j}(p)>U_{j}(q)$, which is contrary to the assumption that $U_{j}(p)=U_{j}(q)$.

Hence, $U_{i} \in \mathscr{A}\left(Q, \omega_{Q}\right)$, and Statement 1 follows.

To prove Statement 2, let $U_{1,1}, U_{1,2}, U_{1}^{\prime}, U_{1,2}, U_{2,2}$, and $U_{2}^{\prime}$ be as described in the beginning of the proof, and suppose that $U_{1}^{\prime}$ is not order equivalent to $U_{2}^{\prime}$. Because $\left.U_{1}^{\prime}\right|_{A \cup B} \equiv U_{1,1} \equiv$ $\left.U_{1,2} \equiv U_{2}^{\prime}\right|_{A \cup B}$ and because $\left.\left.U_{1}^{\prime}\right|_{B \cup C} \equiv U_{2,1} \equiv U_{2,2} \equiv U_{2}^{\prime}\right|_{B \cup C}$, we have that $\left.\left.U_{1}^{\prime}\right|_{A \cup B} \equiv U_{2}^{\prime}\right|_{A \cup B}$ and $\left.\left.U_{1}^{\prime}\right|_{B \cup C} \equiv U_{2}^{\prime}\right|_{B \cup C}$. So there are elements $p, q \in Q$ such that $p \in A, q \in C$, and exactly one of the following holds.

- $U_{1}^{\prime}(p)<U_{1}^{\prime}(q)$ and $U_{2}^{\prime}(p)>U_{2}^{\prime}(q)$
- $U_{1}^{\prime}(p)>U_{1}^{\prime}(q)$ and $U_{2}^{\prime}(p)<U_{2}^{\prime}(q)$
- $U_{1}^{\prime}(p)=U_{1}^{\prime}(q)$ and $U_{2}^{\prime}(p) \neq U_{2}^{\prime}(q)$
- $U_{1}^{\prime}(p) \neq U_{1}^{\prime}(q)$ and $U_{2}^{\prime}(p)=U_{2}^{\prime}(p)$

Suppose that $U_{1}^{\prime}(p)<U_{1}^{\prime}(q)$ and $U_{2}^{\prime}(p)>U_{2}^{\prime}(q)$. Then, as $U_{1}^{\prime}$ and $U_{2}^{\prime}$ are order reversing maps, it follows that $p \| q$ in $Q$. But as $(A, B, C)$ is a connected triple of $Q$, that violates Property 2 of Definition 4.1. Similarly, if $U_{1}^{\prime}(p)>U_{1}^{\prime}(q)$ and $U_{2}^{\prime}(p)<U_{2}^{\prime}(q)$, then Property 2 of Definition 4.1 would be violated. So without loss of generality, suppose that $U_{1}^{\prime}(p)=$ $U_{1}^{\prime}(q)$ and $U_{2}^{\prime}(p) \neq U_{2}^{\prime}(q)$.

Since $p \in A$ and $q \in C$, Property 2 of Definition 4.1 implies that there exists an element $p^{\prime} \in B$ such that $p<p^{\prime}<q$ in $Q$. Hence, as $U_{1}^{\prime}$ is order reversing and as $U_{1}^{\prime}(p)=U_{1}^{\prime}(q), U_{1}^{\prime}(p)=$ $U_{1}^{\prime}\left(p^{\prime}\right)=U_{1}^{\prime}(q)$. So as $p, p^{\prime} \in A \cup B$ and $\left.\left.U_{1}^{\prime}\right|_{A \cup B} \equiv U_{2}^{\prime}\right|_{A \cup B}$, it follows that $U_{2}^{\prime}(p)=U_{2}^{\prime}\left(p^{\prime}\right)$. Similarly, as $p^{\prime}, q \in B \cup C$ and $\left.\left.U_{1}^{\prime}\right|_{B \cup C} \equiv U_{2}^{\prime}\right|_{B \cup C}$, it follows that $U_{2}^{\prime}\left(p^{\prime}\right)=U_{2}^{\prime}(q)$. But then, $U_{2}^{\prime}(p)=U_{2}^{\prime}(q)$, which is contrary to the assumption that $U_{2}^{\prime}(p) \neq U_{2}^{\prime}(q)$. Hence, Statement 2 follows.

Remark 5.14. Note that the converse of Lemma 5.12 is also true. If $T \in \operatorname{Tb}(Q, \omega)$, then $T$ uniquely determines $\left.T\right|_{A \cup B} \in \operatorname{Tb}(A \cup B, \omega)$ and $\left.T\right|_{B \cup C} \in \operatorname{Tb}(B \cup C, \omega)$.

In preparation for the main result of this chapter, we introduce the following technical definition. It defines a positive integer that depends on the periodic quadruple system being considered.

Definition 5.15. Let $(Z, \omega, \pi, \theta)$ be a periodic quadruple system and let $S$ be an index shape of $(Z, \omega, \pi, \theta)$. Then the minimum number for $(Z, \omega, \pi, \theta)$ and $S$ is the smallest positive integer $m$ such that if $n \in \mathbb{N}$ satisfies $n \geq m$, then for all $p \in Z$ satisfying $\pi(p)=n+1$, there exists an element $q \in \theta^{n}(S)$ such that $q<p$ in $Z$ and $\pi(q) \geq 1$.

Remark 5.16. The minimum number of a periodic quadruple system always exists. Let $(Z, \omega, \pi, \theta)$ be a periodic quadruple system and let $S$ be an index shape of $(Z, \omega, \pi, \theta)$. Since $S$ is finite, there is a positive integer $n$ such that, for all $p \in \theta^{n}(S), \pi(p) \geq 1$. Moreover, $S$ connects $Z$, so as $\theta$ is an order automorphism on $Z, \theta^{n}(S)$ connects $Z$. Hence, by Property 2 of Definition 4.1, it follows that for all $p \in Z$ satisfying $\pi(p)=n+1$ and $p^{\prime} \in Z$ satisfying $\pi\left(p^{\prime}\right) \leq 0$, there exists an element $q \in \theta^{n}(S)$ such that $p^{\prime}<q<p$. But as $\pi(q) \geq 1$ for all $q \in \theta^{n}(S)$, it follows from Definition 5.15 that the minimum number of $(Z, \omega, \pi, \theta)$ exists and is at most $n$.

Example 5.17. If $(Z, \omega, \pi, \theta)$ is a periodic quadruple system built from a truncated shifted shape such as the periodic quadruple system in Example 4.18, then there is an index shape $S$ of $(Z, \omega, \pi, \theta)$ such that the minimum number for $(Z, \omega, \pi, \theta)$ and $S$ is 1 . In particular, there are such index shapes for periodic quadruple systems built from parallelogramic shapes such as the one in Example 4.6 To see how to check that such index shapes exist, we observe the following special case.

Let $(Z, \omega, \pi, \theta)$ and $S$ be as in Example 5.11. We explain why the minimum number $m$ for $(Z, \omega, \pi, \theta)$ and $S$ is 1. Consider the twelve cell diagram below.


The above diagram depicts the poset $\pi^{-1}([0,2])$. Moreover, $\pi^{-1}(\{0\})$ is represented by the four cells in the top row of the diagram, $\theta(S)$ is depicted by the cells filled with an asterisk and the cells filled with circles, and $\pi^{-1}(\{2\})$ is depicted by the cells filled with bullets.

Set $m=1$, and set $n=m$. From the above diagram, it can be seen that if $p \in Z$ satisfies $\pi(p)=n+1=2$, then $p$ is represented by one of the cells filled with bullets. From the same
diagram, it can be seen that there exists an element $q \in \theta^{n}(S)=\theta(S)$, specifically one of the cells filled with a circle, such that $q<p$ in $Z$ and $\pi(q) \geq 1$. Hence, $m=1$ is the minimum number of $(Z, \omega, \pi, \theta)$.

Remark 5.18. Example 5.17 can be generalized to prove the following. If $(Z, \omega, \pi, \theta)$ is a periodic quadruple system as described in Theorem 4.26 then there is an index shape $S$ of $(Z, \omega, \pi, \theta)$ such that the minimum number for $(Z, \omega, \pi, \theta)$ and $S$ is 1.

We now prove the main result of this chapter by proving the existence of a certain matrix difference equation. For the proof, recall the definitions of the sets $X_{n}\left(T_{0}\right)$ and $X_{n, i}\left(T_{0}\right)$ in Definition 5.8

Theorem 5.19. Let $(Z, \omega, \pi, \theta)$ be a periodic quadruple system, let $S$ be an index shape of $(Z, \omega, \pi, \theta)$, and let $L$ be an indexing of $\mathrm{Tb}(S, \omega)$. Moreover, let $n^{\prime}$ be an admissible number for $(Z, \omega, \pi, \theta)$ and $S$, let $T_{0} \in \operatorname{Tb}\left(\pi^{-1}\left(\left[n^{\prime}, 0\right]\right), \omega\right)$, and let $\left(v_{n}\right)_{n=0,1,2, \ldots}$ be a sequence such that for all $n \in \mathbb{N}_{0}, v_{n}$ is the $n^{\text {th }}$ vector derived from $(Z, \omega, \pi, \theta), S, L$, and $T_{0}$. Lastly, let $M$ be the tableau transfer matrix derived from $(Z, \omega, \pi, \theta), S$, and $L$, and let $m$ be the minimum number for $(Z, \omega, \pi, \theta)$ and $S$. Then for all $n \geq m$,

$$
v_{n+1}=M v_{n} .
$$

Example 5.20. Let $(Z, \omega, \pi, \theta), S, L, T_{0}$ be as in Example 5.9 and Example 5.11. By Example 5.17. the minimum number $m$ for $(Z, \omega, \pi, \theta)$ and $S$ is 1 . So consider the $1^{\text {st }}$ vector $v_{1}$ derived from $(Z, \omega, \pi, \theta), S$, and L. As shown in Example 5.9.

$$
v_{1}=\left[\begin{array}{l}
1 \\
0
\end{array}\right]
$$

Next, let $M$ be the tableau transfer matrix from Example 5.7. What this theorem allows us to do is to determine $v_{n}$ from $M$ and $v_{1}$ for any $n \geq 1$. For instance, we showed in Example 5.11 that

$$
v_{2}=\left[\begin{array}{l}
3 \\
2
\end{array}\right]
$$

This vector can also be obtained from $M$ and $v_{1}$ as follows.

$$
v_{2}=M v_{1}=\left[\begin{array}{ll}
3 & 4 \\
2 & 3
\end{array}\right]\left[\begin{array}{l}
1 \\
0
\end{array}\right]=\left[\begin{array}{l}
3 \\
2
\end{array}\right]
$$

Proof. Let $n \in \mathbb{N}$. Define $Q=\pi^{-1}\left(\left[n^{\prime}, n+1\right]\right), C=\pi^{-1}(\{n+1\}), B=\theta^{n}(S)$, and $A=$ $\pi^{-1}\left(\left[n^{\prime}, n\right]\right) \backslash B$. Then $(A, B, C)$ is a connected triple of $Q$. Note that $A \cup B=\pi^{-1}\left(\left[n^{\prime}, n\right]\right)$ and $\theta(B)=\theta^{n+1}(S)$.

If $T \in X_{n+1}\left(T_{0}\right)$, then $\left.T\right|_{A \cup B} \in \operatorname{Tb}(A \cup B, \omega)$ and $\left.T\right|_{B \cup C} \in \operatorname{Tb}(B \cup C, \omega)$. Since $T \in X_{n+1}\left(T_{0}\right)$, Definition 5.8 implies that for all $U \in T, p \in \pi^{-1}\left(\left[n^{\prime}, 0\right]\right)$, and $q \in \pi^{-1}([n+1]), U(p)>U(q)$. Hence, for all $U \in T, p \in \pi^{-1}\left(\left[n^{\prime}, 0\right]\right)$, and $q \in \pi^{-1}([n+1]), U(p)>U(q)$, and it follows that $\left.T\right|_{A \cup B} \in X_{n}\left(T_{0}\right)$ and $\left.T\right|_{B \cup C} \in \operatorname{Tb}(B \cup C, \omega)$. As noted in Remark 5.14, $T \in X_{n+1}\left(T_{0}\right)$ uniquely determines $\left.T\right|_{A \cup B}$ and $\left.T\right|_{B \cup C}$.

Next, define the map

$$
f: X_{n+1}\left(T_{0}\right) \rightarrow\left\{\left(T^{\prime \prime}, T^{\prime}\right) \in \operatorname{Tb}(B \cup C, \omega) \times X_{n}\left(T_{0}\right):\left.T^{\prime \prime}\right|_{B}=\left.T^{\prime}\right|_{B}\right\}
$$

by

$$
f(T)=\left(\left.T\right|_{B \cup C},\left.T\right|_{A \cup B}\right)
$$

for all $T \in X_{n+1}\left(T_{0}\right)$. By what we just showed, $f$ is well-defined and injective. We will prove
that $f$ is also a bijection. To that end, it is enough to show that $f$ is surjective.

Let $T^{\prime} \in X_{n}\left(T_{0}\right)$, let $T^{\prime \prime} \in \operatorname{Tb}(B \cup C, \omega)$, and assume that $\left.T^{\prime}\right|_{B}=\left.T^{\prime \prime}\right|_{B}$. Recall that $\operatorname{Tb}(Q, \omega)=$ $\operatorname{Tb}\left(Q, \omega_{Q}\right)$, where $\omega_{Q}$ is the labelling of $Q$ such that $\left.\omega\right|_{Q} \equiv \omega_{Q}$. Hence, by Lemma 5.12, there is a unique element $T \in \mathrm{~Tb}(Q, \omega)$ such that $\left.T\right|_{A \cup B}=T^{\prime}$ and $\left.T\right|_{B \cup C}=T^{\prime \prime}$. So to prove that $f$ is surjective, it is enough to prove that $T \in X_{n+1}\left(T_{0}\right)$.

Let $P^{\prime}=\pi^{-1}\left(\left[n^{\prime}, 0\right]\right)$ and let $P=\pi^{-1}([n+1])$. Because $\left.T\right|_{A \cup B}=T^{\prime} \in X_{n}\left(T_{0}\right)$ and because $P^{\prime} \subseteq A \cup B$, Definition 5.8 implies that $\left.T\right|_{P^{\prime}}=T_{0}$. Hence, by Definition 5.8, it is enough to show that for all $U \in T, p \in P^{\prime}$, and $q \in P, U(p)>U(q)$. To that end, let $U \in T$, let $p \in P^{\prime}$, and let $q \in P$.

If $\pi(q) \leq n$, then $p, q \in A \cup B$, implying, as $\left.T\right|_{A \cup B} \in X_{n}\left(T_{0}\right)$, that $U(p)>U(q)$ by Definition 5.8 applied to $\left.T\right|_{A \cup B}$. So assume without loss of generality that $\pi(q)=n+1$. Because $n \geq m$, by hypothesis where $m$ is the minimum number for $(Z, \omega, \pi, \theta)$ and $S$, Definition 5.15 implies that there exists an element $p^{\prime} \in \theta^{n}(S)$ such that $p^{\prime}<q$ in $Z$ and $\pi\left(p^{\prime}\right) \geq 1$. In particular, as $p^{\prime}<q$ and $U$ is order reversing, $U\left(p^{\prime}\right) \geq U(q)$.

Because $B=\theta^{n}(S), p^{\prime} \in B$. Moreover, $\left.T\right|_{A \cup B} \in X_{n}\left(T_{0}\right)$ and $p \in P^{\prime}$. Furthermore, by Definition 5.8, $U\left(p^{\prime \prime}\right)>U\left(q^{\prime \prime}\right)$ for all $p^{\prime \prime} \in P^{\prime}$ and $q^{\prime \prime} \in \pi^{-1}([n])$. Lastly, $p^{\prime} \in \pi^{-1}([n])$ because $p^{\prime} \in \theta^{n}(S)$ and $\pi\left(p^{\prime}\right) \geq 1$. So as $p \in P^{\prime}$ and $p^{\prime} \in \pi^{-1}([n])$, we have $U(p)>U\left(p^{\prime}\right)$. Hence,

$$
U(p)>U\left(p^{\prime}\right) \geq U(q)
$$

implying that $U(p)>U(q)$. From this, it follows that $T \in X_{n+1}\left(T_{0}\right)$. Hence, $f$ is a bijection.

Whence, for all $T_{2} \in \operatorname{Tb}(S, \omega)$, the number of elements $T \in X_{n+1}\left(T_{0}\right)$ satisfying $\left.T\right|_{\theta^{n+1}(S)} \equiv_{\theta}$
$T_{2}$ is

$$
\begin{equation*}
\sum_{T_{1} \in \operatorname{Tb}(S, \omega)} M\left(T_{2}, T_{1}\right)\left|\left\{T \in X_{n}\left(T_{0}\right):\left.T\right|_{\theta^{n}(S)} \equiv_{\theta} T_{1}\right\}\right| \tag{5.1}
\end{equation*}
$$

for the following reasons.

Since $B=\theta^{n}(S)$, the fact that $f$ is a bijection implies that the following is true for all $T_{2} \in \operatorname{Tb}(S, \omega)$. If $g$ is the restriction of $f$ to the set of elements $T \in X_{n+1}\left(T_{0}\right)$ satisfying $\left.T\right|_{\theta^{n+1}(S)} \equiv_{\theta} T_{2}$, then $g$ is injective and the range of $g$ is the set of ordered pairs $\left(T^{\prime \prime}, T^{\prime}\right) \in$ $\mathrm{Tb}(B \cup C, \omega) \times X_{n}\left(T_{0}\right)$ satisfying $\left.T^{\prime \prime}\right|_{\theta^{n+1}(S)} \equiv{ }_{\theta} T_{2}$ and $\left.T^{\prime \prime}\right|_{\theta^{n}(S)}=\left.T^{\prime}\right|_{\theta^{n}(S)}$.

Moreover, as $\theta$ is an order automorphism on $Z$, as

$$
B \cup C=\theta^{n}(S) \cup \pi^{-1}(\{n+1\})=\theta^{n+1}\left(\theta^{-1}(S) \cup \pi^{-1}(\{0\})\right),
$$

as $\theta^{n}(S)=\theta^{n+1}\left(\theta^{-1}(S)\right)$, and as $M$ is the tableau transfer matrix derived from $(Z, \omega, \pi, \theta)$, $S$ and $L$, Definition 5.6 implies that for all $T_{1}, T_{2} \in \mathrm{~Tb}(S, \omega)$, the number of elements $T^{\prime \prime} \in$ $\operatorname{Tb}(B \cup C, \omega)$ satisfying $\left.T^{\prime \prime}\right|_{\theta^{n+1}(S)} \equiv_{\theta} T_{2}$ and $\left.T^{\prime \prime}\right|_{\theta^{n}(S)} \equiv{ }_{\theta} T_{1}$ is $M\left(T_{2}, T_{1}\right)$.

Hence, the number of elements $T \in X_{n+1}\left(T_{0}\right)$ satisfying $\left.T\right|_{\theta^{n+1}(S)} \equiv_{\theta} T_{2}$ is given by Expression 5.1 .

Lastly, by Definition 5.8,

$$
\left|X_{n, L\left(T_{1}\right)}\left(T_{0}\right)\right|=\left|\left\{T \in X_{n}\left(T_{0}\right):\left.T\right|_{\theta^{n}(S)} \equiv_{\theta} T_{1}\right\}\right|
$$

for all $T_{1} \in \operatorname{Tb}(S, \omega)$, and

$$
\left|X_{n+1, L\left(T_{2}\right)}\left(T_{0}\right)\right|=\left|\left\{T \in X_{n+1}\left(T_{0}\right):\left.T\right|_{\theta^{n+1}(S)} \equiv_{\theta} T_{2}\right\}\right|
$$

for all $T_{2} \in \mathrm{~Tb}(S, \omega)$.

Therefore, for all $T_{2} \in \mathrm{~Tb}(S, \omega)$, Definition 5.8 implies that

$$
\begin{aligned}
v_{n+1}\left(L\left(T_{2}\right)\right) & =\left|X_{n+1, L\left(T_{2}\right)}\left(T_{0}\right)\right| \\
& =\left|\left\{T \in X_{n+1}\left(T_{0}\right):\left.T\right|_{\theta^{n+1}(S)} \equiv_{\theta} T_{2}\right\}\right| \\
& =\sum_{T_{1} \in \operatorname{Tb}(S, \omega)} M\left(T_{2}, T_{1}\right)\left|\left\{T \in X_{n}\left(T_{0}\right):\left.T\right|_{\theta^{n}(S)} \equiv{ }_{\theta} T_{1}\right\}\right| \\
& =\sum_{T_{1} \in \operatorname{Tb}(S, \omega)} M\left(T_{2}, T_{1}\right)\left|X_{n, L\left(T_{1}\right)}\left(T_{0}\right)\right| \\
& =\sum_{T_{1} \in \operatorname{Tb}(S, \omega)} M\left(T_{2}, T_{1}\right) v_{n}\left(L\left(T_{1}\right)\right) .
\end{aligned}
$$

From this, the theorem follows from the definition of matrix multiplication.

## Chapter 6

## Results relating to the marriage condition

In this chapter, we consider families of sets that satisfy Hall's marriage condition. We introduce a generalized notion of hook-lengths for such families. Then, we establish an existence result based on such generalized hook-lengths that gives a new characterization of marriage problems with unique solutions. Afterwards, we prove a corollary that complements this existence result.

Definition 6.1. (Hall, [21]) Let $n \in \mathbb{N}$, and let $\mathscr{F}$ be a finite family of subsets of $[n]$. Then a transversal of $\mathscr{F}$ is an injective function $t: \mathscr{F} \rightarrow[n]$ such that $t(F) \in F$ for all $F \in \mathscr{F}$.

Informally, a transversal maps each $F$ to one of its elements.

Definition 6.2. (Hall, [21]) Let $n \in \mathbb{N}$, and let $\mathscr{F}$ be a finite family of subsets of $[n]$. Then $\mathscr{F}$ satisfies the marriage condition iffor all subfamilies $\mathscr{F}^{\prime}$ of $\mathscr{F}$,

$$
\left|\mathscr{F}^{\prime}\right| \leq\left|\bigcup_{F \in \mathscr{F}^{\prime}} F\right|
$$

Example 6.3. A simple example illustrating both Definition 6.1 and Definition 6.2 is as follows. Let $n=5$, and let

$$
\mathscr{F}=\{\{1\},\{1,2\},\{1,2,3\},\{1,2,3,4\},\{1,2,3,4,5\}\} .
$$

Then $\mathscr{F}$ satisfies the marriage condition. For example, if $\mathscr{F}^{\prime}$ is the subfamily of $\mathscr{F}$ defined by $\mathscr{F}^{\prime}=\{\{1\},\{1,2,3\}\}$, then $\left|\mathscr{F}^{\prime}\right|=2$ and $|\{1\} \cup\{1,2,3\}|=3$. The map $t: \mathscr{F} \rightarrow\{1,2,3,4,5\}$ defined by $t([k])=k$ for all $1 \leq k \leq 5$ is a transversal of $\mathscr{F}$.

One could interpret the above example as evidence to the possibility that a family of sets of [ $n$ ] has a transversal if and only if it satisfies the marriage condition. It turns out that this is always true. The following is known as Hall's Marriage Theorem.

Theorem 6.4. (Hall, [21]) Let $n \in \mathbb{N}$, and let $\mathscr{F}$ be a family of non-empty subsets of $[n]$. Then $\mathscr{F}$ has a transversal if and only if $\mathscr{F}$ satisfies the marriage condition.

In order to use the families of sets in Hall's Marriage Theorem, we will define more structure on the objects being considered. Definition 6.5 represents the local conditions and generalized hook-lengths mentioned in Chapter 1; how this relates to hook-lengths will become clear in the next chapter. Recall the notation we use for functions in Chapter 2.

Definition 6.5. Let $n \in \mathbb{Z}$, let $\mathscr{F}$ be a family of non-empty subsets of $[n]$, and let $t$ be a transversal of $\mathscr{F}$. Then a configuration $f$ of $t$ is a function $f:[n] \rightarrow \mathbb{N}$ such that for all $F \in \mathscr{F}$,

$$
f(t(F)) \leq|F| .
$$

Moreover, a permutation $\sigma:[n] \rightarrow[n]$ satisfies $f$ if the following holds for all $F \in \mathscr{F}$. The positive integer $\sigma(t(F))$ is the $k^{\text {th }}$ smallest element of $\sigma(F)$, where $k=f(t(F))$.

Example 6.6. Let $n=5$. Moreover let $\mathscr{F}$ and let $t: \mathscr{F} \rightarrow[n]$ be as defined in Example 6.3 Furthermore, let $F_{i}=[i]$ for all $1 \leq i \leq 5$ so $t\left(F_{i}\right)=i$. Lastly, let $f:[n] \rightarrow \mathbb{N}$ be defined by $f(1)=1, f(2)=1, f(3)=2, f(4)=4$, and $f(5)=2$. The map $f$ is a configuration of $t$. For instance, since $F_{3}=\{1,2,3\}, t\left(F_{3}\right)=3, f\left(t\left(F_{3}\right)\right)=2,\left|F_{3}\right|=3$, and $f\left(t\left(F_{3}\right)\right) \leq\left|F_{3}\right|$. Similarly, $f\left(t\left(F_{1}\right)\right)=1 \leq 1=\left|F_{1}\right|, f\left(t\left(F_{2}\right)\right)=1 \leq 2=\left|F_{2}\right|, f\left(t\left(F_{4}\right)\right)=4 \leq 4=\left|F_{4}\right|$, and $f\left(t\left(F_{5}\right)\right)=2 \leq 5=\left|F_{5}\right|$.

Moreover, the permutation $\sigma:[n] \rightarrow[n]$ defined by $\sigma=41352$ satisfies $f$. For example, consider $F_{3}=\{1,2,3\}$. As before, $t\left(F_{3}\right)=3$ and $f\left(t\left(F_{3}\right)\right)=2$, so $k=2$. Moreover, $\sigma\left(t\left(F_{3}\right)\right)=$ $\sigma(3)=$ 3. Lastly, $\sigma\left(F_{3}\right)=\{\sigma(1), \sigma(2), \sigma(3)\}=\{1,3,4\}$, and $\sigma\left(t\left(F_{3}\right)\right)=3$ is the second smallest element of $\sigma\left(F_{3}\right)$. Similarly, for $F_{1}, k=1, \sigma\left(t\left(F_{1}\right)\right)=1$, and $\sigma\left(F_{1}\right)=\{4\}$; for $F_{2}, k=1, \sigma\left(t\left(F_{2}\right)\right)=1$, and $\sigma\left(F_{2}\right)=\{1,4\} ;$ for $F_{4}, k=4, \sigma\left(t\left(F_{4}\right)\right)=5$, and $\sigma\left(F_{4}\right)=$ $\{1,3,4,5\}$; and for $F_{5}, k=2, \sigma\left(t\left(F_{5}\right)\right)=2$, and $\sigma\left(F_{5}\right)=\{1,2,3,4,5\}$.

Configurations satisfy the following property, whose usefulness will become more apparent in the next chapter.

Lemma 6.7. Let $n \in \mathbb{N}$, and let $\mathscr{F}$ be a family of subsets of $[n]$ that has a transversal $t$ : $\mathscr{F} \rightarrow[n]$ such that $t$ is surjective. Then every permutation $\sigma:[n] \rightarrow[n]$ satisfies exactly one configuration $f$ of $t$.

Proof. Let $\sigma:[n] \rightarrow[n]$ be a permutation. Then $\sigma$ satisfies the configuration $f$ of $t$ defined by letting, for all $F \in \mathscr{F}, f(t(F))=k$ where $\sigma(t(F))$ is the $k^{t h}$ smallest element of the set $\sigma(F)$. Now, suppose that $\sigma$ satisfies more than one configuration of $t$. Then, let $f_{1}$ and $f_{2}$ be two distinct configurations of $t$. Because $f_{1} \neq f_{2}$ and because $t$ is surjective, there is an element $F \in \mathscr{F}$ such that $f_{1}(t(F)) \neq f_{2}(t(F))$. So write $k_{1}=f_{1}(t(F))$ and write $k_{2}=f_{2}(t(F))$. Since $\sigma$ satisfies $f_{1}$, Definition 6.5 implies that $\sigma(t(F))$ is the $k_{1}^{t h}$ smallest element of $\sigma(F)$.

Moreover, since $\sigma$ satisfies $f_{2}$, Definition 6.5 implies that $\sigma(t(F))$ is the $k_{2}^{t h}$ smallest element of $\sigma(F)$. However, this is impossible because $k_{1}=f_{1}(t(F)) \neq f_{2}(t(F))=k_{2}$.

Now, we define the following stronger form of the marriage condition that was defined by Chang [10] and Hirst and Hughes in [24].

Definition 6.8. (cf. ([24], Theorem 4)) Let $n \in \mathbb{N}$, let $\mathscr{F}$ be a finite family of subsets of $[n]$, and write $m=|\mathscr{F}|$. Then $\mathscr{F}$ is shellable if there exists a bijection $\sigma_{\mathscr{F}}:[m] \rightarrow \mathscr{F}$ such that for all $k \in[m]$,

$$
\begin{equation*}
\left|\bigcup_{i=1}^{k} \sigma_{\mathscr{F}}(i)\right|=k \tag{6.1}
\end{equation*}
$$

Informally, $\sigma_{\mathscr{F}}$ maps each $k$ to a subset, such that the union of the first $k$ subsets has cardinality $k$.

Remark 6.9. Shellable families of sets are connected to Theorem 6.4 Chang ([10], Theorem 1) noted that a simple consequence of Hall Jr.'s work ([22], Theorem 2) is that a finite family $\mathscr{F}$ of subsets of $[n]$ has exactly one transversal if and only if $\mathscr{F}$ is shellable. Later on, Hirst and Hughes showed that this can be proved using a subsystem of second order arithmetic called $R C A_{0}[24]$ and proved an extension of this result involving infinite families of finite sets in the context of reverse mathematics. From the aforementioned characterization of finite families of subsets of $[n]$ that have exactly one transversal, we have, by Theorem 6.4 that all shellable families satisfy the marriage condition.

Remark 6.10. The term shellable is not used in [10], [22], and [24]. However, we use this terminology because Definition 6.8 resembles the definition of a shellable pure $d$-dimensional simplicial complex ([8], Appendix A2.4, Definition A2.4.1). The differences between Definition 6.8 and Definition A2.4.1 are as follows. The sets in Definition 6.8 do not require additional conditions on the cardinalities of the members of $\mathscr{F}$. Also, in Definition A2.4.1,
the requirement of the existence of a bijection $\sigma_{\mathscr{F}}:[m] \rightarrow \mathscr{F}$ as described in Definition 6.8 is relaxed to requiring the existence of a certain bijection from a subset of $[m]$ to a subset of $\mathscr{F}$.

Remark 6.11. When describing the members of a shellable family, we will use a total ordering on the members of that family. Specifically, let $\mathscr{F}$ be a shellable family of subsets of $[n]$ and let $m=|\mathscr{F}|$. By Definition 6.8 there exists a bijection $\sigma_{\mathscr{F}}:[m] \rightarrow \mathscr{F}$ such that Equation 6.1 is satisfied for all $k \in[m]$. From this bijection $\sigma_{\mathscr{F}}$, define a total ordering $<\mathscr{F}$ on the members of $\mathscr{F}$ by defining, for all members $F^{\prime}, F^{\prime \prime} \in \mathscr{F}, F^{\prime}<\mathscr{F} F^{\prime \prime}$ if $\sigma_{\mathscr{F}}^{-1}\left(F^{\prime \prime}\right)<\sigma_{\mathscr{F}}^{-1}\left(F^{\prime}\right)$. The shelling order of a shellable complex from ([8], Appendix A2.4, Definition A2.4.1) is a variant of this total ordering.

Example 6.12. Let $n \in \mathbb{N}$, and define the following finite family of sets.

$$
\mathscr{F}=\{[i]: i \in[n]\}
$$

Then $\mathscr{F}$ is shellable for the following reason. Firstly, $|\mathscr{F}|=n$, so the variable m in Definition 6.8 satisfies $m=n$. Next, define the bijection $\sigma_{\mathscr{F}}:[n] \rightarrow \mathscr{F}$ be letting $\sigma_{\mathscr{F}}(k)=[k]$ for all $k \in[n]$. Then for all $k \in[n]$,

$$
\left|\bigcup_{i=1}^{k} \sigma_{\mathscr{F}}(i)\right|=|[k]|=k
$$

So as $\mathscr{F}$ and $\sigma_{\mathscr{F}}$ satisfy Equation 6.1, $\mathscr{F}$ is shellable.

Example 6.13. If $n \in \mathbb{N}$ and $n \geq 3$, then a family of subsets of [ $n]$ that satisfies the marriage condition but is not shellable is

$$
\mathscr{F}=\{[n] \backslash\{k\}: k \in[n]\} .
$$

This family satisfies the marriage condition because for any subfamily $\mathscr{F}^{\prime}$ of $\mathscr{F}$ with at least
one member,

$$
\left|\bigcup_{F \in \mathscr{F} \mathscr{F}^{\prime}} F\right|= \begin{cases}n-1 & \text { if }\left|\mathscr{F}^{\prime}\right|=1 \\ n & \text { else. }\end{cases}
$$

However, if $\mathscr{F}$ is shellable, where $m=|\mathscr{F}|$, then the following holds. By Definition 6.8 and Equation 6.1 there exists a bijection $\sigma_{\mathscr{F}}:[m] \rightarrow \mathscr{F}$ such that $\left|\sigma_{\mathscr{F}}(1)\right|=1$. So as $\sigma_{\mathscr{F}}(1) \in \mathscr{F}$, it follows that $\mathscr{F}$ has a member whose cardinality is one. However, for all $F \in \mathscr{F},|F|=n-1 \geq 2$. So it follows that $\mathscr{F}$ is not shellable.

Now, we prove the main result of this chapter. It is a partial converse of Lemma 6.7.

Theorem 6.14. Let $n \in \mathbb{N}$. Moreover, let $\mathscr{F}$ be a family of subsets of $[n]$ such that $\mathscr{F}$ satisfies the marriage condition, let t be a transversal of $\mathscr{F}$, and assume that $|\mathscr{F}|=n$. Then $\mathscr{F}$ is shellable if and only if the following holds. For all configurations $f$ of $t$, there exists a permutation $\sigma:[n] \rightarrow[n]$ that satisfies $f$.

Example 6.15. Let $n=3$. Moreover, let $\mathscr{F}=\{\{1,2,3\},\{1,3\}\}$, and let $t: \mathscr{F} \rightarrow[n]$ be defined by $t(\{1,2,3\})=1$ and $t(\{1,3\})=3$. The family $\mathscr{F}$ is not shellable since we cannot find a bijection $\sigma_{\mathscr{F}}:[m] \rightarrow \mathscr{F}$ such that $\left|\sigma_{\mathscr{F}}(1)\right|=1$. Now, let $f:[n] \rightarrow \mathbb{N}$ be the configuration of $t$ defined by $f(1)=1, f(2)=2$, and $f(3)=1$. It is a configuration of $t$ since $f(t(\{1,2,3\}))=f(1)=1 \leq 3=|\{1,2,3\}|$ and $f(t(\{1,3\}))=f(3)=1 \leq 2=|\{1,3\}|$. Then no permutation $\sigma:[n] \rightarrow[n]$ satisfies $f$ as follows.

Suppose that there is a permutation $\sigma_{0}:[n] \rightarrow[n]$ that satisfies $f$. First, consider the element $F_{1}=\{1,2,3\}$ of $\mathscr{F}$. Then $k=f\left(t\left(F_{1}\right)\right)=f(1)=1$. Moreover, $\sigma_{0}\left(F_{1}\right)=\{1,2,3\}$. So as $\sigma_{0}$ satisfies $f, \sigma_{0}\left(t\left(F_{1}\right)\right)=\sigma_{0}(1)$ is the smallest element of $\{1,2,3\}$. Hence, $\sigma_{0}(1)=1$. Next, consider the element $F_{2}=\{1,3\}$ of $\mathscr{F}$. Then $k=f\left(t\left(F_{2}\right)\right)=f(3)=1$. So as $\sigma_{0}$ satisfies $f, \sigma_{0}\left(t\left(F_{2}\right)\right)=\sigma_{0}(3)$ is the smallest element of $\sigma_{0}\left(F_{2}\right)=\left\{\sigma_{0}(1), \sigma_{0}(3)\right\}$. But then,
$\sigma_{0}(3)<\sigma_{0}(1)$, contradicting the fact that $\sigma_{0}(1)=1$.

Proof. First assume that for all configurations $f$ of $t$, there exists a permutation $\sigma:[n] \rightarrow$ $[n]$ that satisfies $f$. If $n=1$, then the only family of $\{1\}$ with a transversal is the family $\mathscr{F}=\{\{1\}\}$, which is shellable. So assume without loss of generality that $n \geq 2$. Consider the configuration $f_{1}$ of $t$ defined by $f_{1}(t(F))=|F|$ for all $F \in \mathscr{F}$. By assumption, there exists a permutation $\sigma^{\prime}:[n] \rightarrow[n]$ that satisfies $f_{1}$. Moreover, let $k \in[n-1]$, and assume that we can fix an ordering $\mathscr{F}=\left\{F_{i}^{\prime}: i \in[n]\right\}$ of $\mathscr{F}$ so that the following holds for all integers $0 \leq j \leq k-1$.

$$
\begin{equation*}
\left|\bigcup_{i=1}^{n-j} F_{i}^{\prime}\right|=n-j \tag{6.2}
\end{equation*}
$$

Note that Equation 6.2 holds if $k=1$ because the fact that $\mathscr{F}$ has a transversal implies that $\bigcup_{F \in \mathscr{F}} F=[n]$.

Next, let $1 \leq s \leq n-k+1$ satisfy

$$
\begin{equation*}
\sigma^{\prime}\left(t\left(F_{s}^{\prime}\right)\right)=\max _{1 \leq j \leq n-k+1} \sigma^{\prime}\left(t\left(F_{j}^{\prime}\right)\right) \tag{6.3}
\end{equation*}
$$

Suppose that there exists an element $j \in[n]$ such that $1 \leq j \leq n-k+1, j \neq s$, and $t\left(F_{s}^{\prime}\right) \in F_{j}^{\prime}$. By Equation 6.3, $\sigma^{\prime}\left(t\left(F_{j}^{\prime}\right)\right) \leq \sigma^{\prime}\left(t\left(F_{s}^{\prime}\right)\right)$. So as $t\left(F_{s}^{\prime}\right) \in F_{j}^{\prime}$ and $t\left(F_{s}^{\prime}\right) \neq t\left(F_{j}^{\prime}\right)$, it follows that for some $1 \leq \ell \leq\left|F_{j}^{\prime}\right|-1, \sigma^{\prime}\left(t\left(F_{j}^{\prime}\right)\right)$ is an $\ell^{t h}$ smallest element of $\sigma^{\prime}\left(F_{j}^{\prime}\right)$. But then, as $f_{1}\left(t\left(F_{j}^{\prime}\right)\right)=\left|F_{j}^{\prime}\right|, \sigma^{\prime}$ does not satisfy $f_{1}$, contradicting the assumption that $\sigma^{\prime}$ satisfies $f_{1}$.

Hence, $t\left(F_{s}^{\prime}\right) \notin F_{i}^{\prime}$ for all $1 \leq i \leq n-k+1$ satisfying $i \neq s$. In particular, fix an ordering $\mathscr{F}=\left\{F_{i}^{\prime \prime}: i \in[n]\right\}$ of $\mathscr{F}$ so that $F_{i}^{\prime \prime}=F_{i}^{\prime}$ if $i>n-k+1$ and $F_{n-k+1}^{\prime \prime}=F_{s}^{\prime}$, where $s$ is as described in the above paragraph. From Equation 6.2, this ordering of the members of $\mathscr{F}$
satisfies the following equation for all integers $0 \leq j \leq k$.

$$
\left|\bigcup_{i=1}^{n-j} F_{i}^{\prime \prime}\right|=n-j
$$

As the choice of $k \in[n-1]$ is arbitrary, it follows that there exists an ordering $\mathscr{F}=\left\{F_{1}, F_{2}, \ldots, F_{n}\right\}$ of $\mathscr{F}$ such that

$$
\left|\bigcup_{i=1}^{k} F_{i}\right|=k
$$

for all $1 \leq k \leq n$. Hence, $\mathscr{F}$ satisfies Equation 1 of Definition 6.8. So, by Definition 6.8, $\mathscr{F}$ is shellable.

Next, assume that $\mathscr{F}$ is shellable. Because $\mathscr{F}$ is shellable, we will use the total ordering as described in Remark 6.11 to describe the members of this family. We proceed by induction on $n$. If $n=1$, then the only family of subsets of $\{1\}$ with a transversal is the family $\mathscr{F}=$ $\{\{1\}\}$. Moreover, with $t$ being the transversal of $\mathscr{F}$ defined by mapping $\{1\}$ to 1 , the only configuration $f$ that satisfies $t$ is the function $f:\{1\} \rightarrow \mathbb{N}$ defined by $f(1)=1$, and any permutation $\sigma:\{1\} \rightarrow\{1\}$ satisfies $f$.

So let $n \geq 2$ and assume that the induction hypothesis holds. Let $t$ be a transversal of $\mathscr{F}$ and let $f$ be a configuration of $t$. Because $\mathscr{F}$ is shellable, Definition 6.8 and Remark 6.11 imply that there is an element $n^{\prime} \in[n]$ such that, for all $F \in \mathscr{F}, n^{\prime} \notin F$ or $t(F)=n^{\prime}$. So without loss of generality, assume that $n^{\prime}=n$. Let $\mathscr{F}^{\prime}$ be the family of sets defined by

$$
\mathscr{F}^{\prime}=\{F \in \mathscr{F}: t(F) \neq n\} .
$$

As $n \notin F$ for all $F \in \mathscr{F}$ such that $t(F) \neq n, \mathscr{F}^{\prime}$ is a family of subsets of $[n-1]$. Next, define $t^{\prime}: \mathscr{F}^{\prime} \rightarrow[n-1]$ by letting $t^{\prime}(F)=t(F)$ for all $F \in \mathscr{F}^{\prime}$. Moreover, because $t$ is a transversal
of $\mathscr{F}, t^{\prime}$ is a transversal of $\mathscr{F}^{\prime}$. By Definition 6.8 and the choice of $n=n^{\prime}, \mathscr{F}^{\prime}$ is shellable for the following reason.

Define the bijection $\sigma_{\mathscr{F}}:[n-1] \rightarrow \mathscr{F}^{\prime}$ by

$$
\sigma_{\mathscr{F}^{\prime}}(k)=\sigma_{\mathscr{F}}(k)
$$

for all $k \in[n-1]$. Because $\sigma_{\mathscr{F}}$ satisfies Equation 6.1 of Definition 6.8, $\sigma_{\mathscr{F} \prime}$ satisfies Equation 6.1 of Definition 6.8. Hence, $\mathscr{F}^{\prime}$ is shellable. So by the induction hypothesis, there exists a permutation $\sigma^{\prime}:[n-1] \rightarrow[n-1]$ that satisfies all configurations $f^{\prime}$ of $t^{\prime}$.

Let $m=f(n)$, and let $F_{\sigma}$ be the element of $\mathscr{F}$ such that $t\left(F_{\sigma}\right)=n$. There is an order embedding $\kappa:[n-1] \rightarrow[n]$ such that the element $k \in[n] \backslash \kappa([n-1])$ is the $m^{\text {th }}$ smallest element of $\kappa\left(F_{\sigma}\right)$. With $\kappa$ defined, define $\sigma:[n] \rightarrow[n]$ as follows. Let $\sigma(n)$ be the element of $[n]$ that is not in $\kappa([n-1])$, and, for all $k \in[n-1]$, let $\sigma(k)=\kappa\left(\sigma^{\prime}(k)\right)$. Because $n=n^{\prime}$ and $n^{\prime} \in F$ for exactly one element $F \in \mathscr{F}, \sigma$ satisfies $f$. From this, the theorem follows.

Remark 6.16. A family $\mathscr{F}$ of subsets of $[n]$ such that $\left|\bigcup_{F \in \mathscr{F}} F\right|=|\mathscr{F}|=n$ is called a critical block in [22]. In [22], Hall Jr. used this notion as a very important ingredient in extending Hall's Marriage Theorem to infinite families of finite sets.

As a corollary, we show the following.

Corollary 6.17. Let $n \in \mathbb{N}$. Moreover, let $\mathscr{F}$ be a family of subsets of $[n]$ that has a transversal, let t be a transversal of $\mathscr{F}$, and assume that $|\mathscr{F}|=n$. Then every configuration $f$ of $t$ is satisfied by some permutation $\sigma:[n] \rightarrow[n]$ if and only if the following holds. The configuration $f_{0}$ of $t$ defined by $f_{0}(t(F))=1$ for all $F \in \mathscr{F}$ is satisfied by some permutation $\sigma_{0}:[n] \rightarrow[n]$.

Example 6.18. The family of sets in Example 6.15 is, as shown in that example, a family where the configuration $f_{0}$ as defined in Corollary 6.17 is not satisfied by any permutation.

Proof. By Theorem 6.4, $\mathscr{F}$ has a transversal if and only if $\mathscr{F}$ satisfies the marriage condition. So by Theorem 6.14, it is enough to prove that $\mathscr{F}$ is shellable if and only if the configuration $f_{0}$ of $t$ as described in the corollary is satisfied by some permutation $\sigma_{0}:[n] \rightarrow[n]$.

We first show that if $\mathscr{F}$ is not shellable, then the configuration $f_{0}$ is not satisfied by any permutation. Let $f_{1}$ be the configuration of $t$ defined by $f_{1}(t(F))=|F|$ for all $F \in \mathscr{F}$. The first part of the proof of Theorem 6.14 proves that if $f_{1}$ is satisfied by some permutation $\sigma:[n] \rightarrow[n]$, then $\mathscr{F}$ is shellable. So as $\mathscr{F}$ is not shellable, $f_{1}$ is not satisfied by any permutation $\sigma:[n] \rightarrow[n]$. Moreover, a permutation $\sigma:[n] \rightarrow[n]$ satisfies $f_{0}$ if and only if the permutation $\sigma^{\prime}:[n] \rightarrow[n]$ defined, for all $k \in[n]$, by

$$
\sigma^{\prime}(k)=n-\sigma(k)+1
$$

satisfies $f_{1}$. Hence, it follows from the above that if $\mathscr{F}$ is not shellable, then $f_{0}$ is not satisfied by any permutation.

So assume that $\mathscr{F}$ is shellable, and use a total order to describe the members of $\mathscr{F}$ by letting $\sigma_{\mathscr{F}}:[n] \rightarrow \mathscr{F}$ be as described in Definition 6.8. Define the permutation $\sigma_{0}:[n] \rightarrow[n]$ by having

$$
\sigma_{0}\left(t\left(\sigma_{\mathscr{F}}(k)\right)=n-k+1\right.
$$

for all $k \in[n]$. This permutation satisfies $f_{0}$ because for all $k \in[n], \sigma_{0}\left(t\left(\sigma_{\mathscr{F}}(k)\right)\right)=n-k+1$ is the smallest element of $\sigma_{0}\left(\sigma_{\mathscr{F}}(k)\right)$. This completes the proof of the corollary.

## Chapter 7

## Applications to skew tableaux

In this chapter, we describe how the results in the previous chapter can be applied to skew shapes. Specifically, we introduce a generalization of standard skew tableaux and Edelman and Greene's balanced tableaux, then prove some existence results about these generalized structures as described in Chapter 1 by using the characterization of the stronger form of the marriage condition. Afterwards, we briefly indicate other ways in which we can apply the results of Chapter 7.

Definition 7.1. (cf. [30], p. 7 and [41], Definition 2.1.1, Definition 3.7.1) Let $\lambda=\left(\lambda_{1}, \lambda_{2}, \ldots, \lambda_{\ell}\right)$ and $\mu=\left(\mu_{1}, \mu_{2}, \ldots, \mu_{\ell^{\prime}}\right)$ be partitions of positive integers such that $\ell^{\prime} \leq \ell$ and $\mu_{i} \leq \lambda_{i}$ for all $1 \leq i \leq \ell^{\prime}$. Moreover, let

$$
X=\bigcup_{i=1}^{\ell} \bigcup_{j=\mu_{i}+1}^{\lambda_{i}}\{(i, j)\}
$$

Lastly, let $X^{\prime} \subset \mathbb{N}^{2}$ be such that $X^{\prime}=X+v$ for some $v \in \mathbb{Z}^{2}, X^{\prime}-(0,1) \nsubseteq \mathbb{N}^{2}$, and $X^{\prime}-(1,0) \nsubseteq$ $\mathbb{N}^{2}$. Then define the skew shape $\lambda / \mu$ to be the Young diagram that is equal to $X^{\prime}$.

If $\mu=\emptyset$ is the empty partition, then for any partition $\lambda$ of a positive integer, define the skew
shape $\lambda / \mu$ to be the Young diagram that is equal to

$$
\bigcup_{i=1}^{\ell} \bigcup_{j=1}^{\lambda_{i}}\{(i, j)\}
$$

and call this Young diagram the Young diagram of $\lambda$. We also call the Young diagram of $\lambda$ a normal shape. Lastly, if $\lambda=\emptyset$ is the empty partition, then we define the Young diagram of $\lambda$ to be the empty set.

Example 7.2. Let $\lambda=(4,2,1,1)$, and consider the Young diagram of $\lambda$. By Definition 7.1. row 1 of this diagram consists of $\lambda_{1}=4$ cells, row 2 of this diagram consists of $\lambda_{2}=2$ cells, row 3 of this diagram consists of $\lambda_{3}=1$ cell, and row 4 of diagram consists of $\lambda_{4}=1$ cell. Hence, the Young diagram is as follows.


Remark 7.3. Let $\lambda$ be a partition of a non-negative integer. Then we will refer to the Young diagram of $\lambda$ as $\lambda$. In particular, we can speak of cells of $\lambda$ or even rows of $\lambda$. Since we will do this, we will say things such as "let $\lambda$ be a normal shape" when mentioning the Young diagram of $\lambda$.

Example 7.4. Let $\lambda=(4,3,2,2)$ and $\mu=(2,2,1)$. Then $\ell=4, \ell^{\prime}=3$, and for all $1 \leq i \leq \ell^{\prime}$, $\mu_{i} \leq \lambda_{i}$. Hence, the skew shape $\lambda / \mu$ is well-defined. The set $X$ as described in Definition 7.1 is obtained from the Young diagram of $\lambda$ by deleting the $\mu_{1}=2$ left-most cells of row 1 of $\lambda$, the $\mu_{2}=2$ left-most cells of row 2 of $\lambda$, and, as $\mu_{3}=1$, the left-most cell of row 3 of $\lambda$. Because this set $X$ satisfies $X-(0,1) \nsubseteq \mathbb{N}^{2}$ and $X-(1,0) \nsubseteq \mathbb{N}^{2}$, it follows that $X^{\prime}=X$.

Hence, by Definition 7.1 the skew shape $\lambda / \mu$ is the following Young diagram.


Remark 7.5. When mentioning skew shapes $\lambda / \mu$, we simply say "let $\lambda / \mu$ be a skew shape" without explicitly mentioning that $\lambda$ and $\mu$ are partitions that satisfy the conditions described in Definition 7.1.

Definition 7.6. (Folklore, cf. [30 41 45]) Let $\lambda / \mu$ be a skew shape consisting of $n$ cells. Then $a$ standard skew tableau of shape $\lambda / \mu$ is a bijective filling of the cells of $\lambda / \mu$ with numbers from $[n]$ such that entries increase along every row from left to right and entries increase along every column from top to bottom. Moreover, a reverse standard skew tableau of shape $\lambda / \mu$ is a bijective filling of the cells of $\lambda / \mu$ such that the entries decrease along every row from left to right and entries decrease along every column from top to bottom. If $\mu=\emptyset$, then a standard skew tableau of shape $\lambda / \mu$ is a standard Young tableau of shape $\lambda$ and a reverse standard skew tableau of shape $\lambda / \mu$ is a standard reverse tableau of shape $\lambda$.

Example 7.7. Consider the skew shape $\lambda / \mu$ from Example 7.4. An example of a standard skew tableau of shape $\lambda / \mu$ is the following.


To see that the above is a standard skew tableau, note that the entries, 1 and 4, in row 1 of this tableau are increasing from left to right, that the entries 5 and 6, in column 2 of this tableau are increasing from top to bottom, and so on. An example of a reverse standard skew
tableau of shape $\lambda / \mu$ is the following.


The above is a reverse standard skew tableau since the entries, 6 and 4, in column 3 of this tableau are decreasing from top to bottom, the entries, 3 and 2, in row 4 of this tableau are decreasing from left to right, and so on.

When describing families of sets, we will replace $[n]$ in the last chapter with the set of cells of $\lambda / \mu$. Moreover, in place of the permutations $\sigma:[n] \rightarrow[n]$, we define generalized standard skew tableaux.

Definition 7.8. (cf. [41], Definition 2.1.3) Let $\lambda / \mu$ be a skew shape with $n$ cells. Then a generalized standard skew tableau of shape $\lambda / \mu$ is a bijective filling of the cells of $\lambda / \mu$ with numbers from $[n]$.

Example 7.9. If $\lambda=(4,3,1)$ and $\mu=(2)$, then an example of a generalized skew tableau of shape $\lambda / \mu$ is as follows.

|  |  | 3 | 5 |
| :--- | :--- | :--- | :--- |
| 6 | 1 | 2 |  |
| 4 |  |  |  |
|  |  |  |  |

Definition 7.10. ([41]) Let $\lambda / \mu$ be a skew shape. For any cell $(i, j)$ in $\lambda / \mu$, define the corresponding hook $H_{(i, j)}$ and hook-length $h_{(i, j)}$ as follows:

- $H_{(i, j)}=\left\{\left(i^{\prime}, j\right) \in \lambda / \mu: i^{\prime} \geq i\right\} \cup\left\{\left(i, j^{\prime}\right) \in \lambda / \mu: j^{\prime} \geq j\right\}$,
- $h_{(i, j)}=\left|H_{(i, j)}\right|$.

Example 7.11. Consider the following skew shape $\lambda / \mu$, where $\lambda=(5,4,3,3)$ and $\mu=$ $(2,2,1)$. Moreover, let $r$ be the cell of $\lambda / \mu$ depicted below that is filled with a bullet. Then $H_{r}$ consists of the cells that are filled with asterisks or bullets, and $h_{r}=4$.


Let $\lambda$ be a normal shape. Then an inner corner of $\lambda$ ([41], Definition 2.8.1) is a cell $r \in \lambda$ such that deleting $r$ from $\lambda$ results in another normal shape. With this definition in mind, let $\lambda / \mu$ be a skew shape with $n$ cells, and consider the family of sets defined by $\mathscr{F}=\left\{H_{r}: r \in\right.$ $\lambda / \mu\}$. Then $\mathscr{F}$ is shellable. To see this, let $r_{1}, r_{2}, \ldots, r_{n}$ be a sequence of cells in $\lambda / \mu$ that is obtained as follows.

- Let $r_{1}$ be an inner corner of $\lambda$.
- If $1 \leq k<n$ and if $r_{1}, r_{2}, \ldots, r_{k}$ have already been defined, then let $\lambda^{(k)}$ be the Young diagram that results from deleting cells $r_{1}, r_{2}, \ldots$, and $r_{k}$ from $\lambda$, and let $r_{k+1}$ be an inner corner of $\lambda^{(k)}$.

Lastly, let $\lambda^{(n)}=\mu$. Define $\sigma_{\mathscr{F}}:[n] \rightarrow \mathscr{F}$ by letting $\sigma_{\mathscr{F}}(k)=H_{r_{k}}$ for all $k \in[n]$. The bijection $\sigma_{\mathscr{F}}$ satisfies Equation 6.1 because, for all $k \in[n], \lambda^{(k)}$ has $n-k$ cells,

$$
\begin{equation*}
\lambda^{(k)} \cup \bigcup_{i=1}^{k} H_{r_{i}}=\lambda / \mu \tag{7.1}
\end{equation*}
$$

and

$$
\begin{equation*}
\lambda^{(k)} \cap \bigcup_{i=1}^{k} H_{r_{i}}=\emptyset \tag{7.2}
\end{equation*}
$$

Hence, $\mathscr{F}$ is shellable by Definition 6.8. In particular, by Remark 6.9, $\mathscr{F}$ has a unique transversal. The unique transversal $t: \mathscr{F} \rightarrow \lambda / \mu$ of $\mathscr{F}$ is given by $t\left(H_{r}\right)=r$ for all $r \in \lambda / \mu$. Example 7.12. Let $\lambda=(4,2,2)$ and let $\mu=(1)$. Next, let $\mathscr{F}=\left\{H_{r}: r \in \lambda / \mu\right\}$. We illustrate why $\mathscr{F}$ is shellable. The normal shape $\lambda$ is depicted below and all inner corners of $\lambda$ are filled with bullets.


Pick the inner corner $r_{1}=(3,2)$ of $\lambda$. Then the normal shape $\lambda^{(1)}$ is depicted below and all inner corners of $\lambda^{(1)}$ are filled with bullets

and we can pick the inner corner $r_{2}=(1,4)$ of $\lambda^{(1)}$. Continuing in this way, one possibility is the following sequence of cells in $\lambda / \mu$ depicted below.

\[

\]

In particular, $r_{3}=(2,2), r_{4}=(3,1), r_{5}=(1,3), r_{6}=(1,2)$, and $r_{7}=(2,1)$. Now, define $\sigma_{\mathscr{F}}$ : $\{1,2, \ldots, 7\} \rightarrow \mathscr{F}$ so that $\sigma_{\mathscr{F}}(1)=H_{(3,2)}, \sigma_{\mathscr{F}}(2)=H_{(1,4)}, \sigma_{\mathscr{F}}(3)=H_{(2,2)}, \sigma_{\mathscr{F}}(4)=H_{(3,1)}$, $\sigma_{\mathscr{F}}(5)=H_{(1,3)}, \sigma_{\mathscr{F}}(6)=H_{(1,2)}$, and $\sigma_{\mathscr{F}}(7)=H_{(2,1)}$. This bijection satisfies Equation 7.1 and Equation 7.2. Hence, $\mathscr{F}$ is shellable.

Edelman and Greene introduced the following variant of standard Young tableaux.

Definition 7.13. (Edelman and Greene, [13]) Let $\lambda$ be a normal shape containing $n$ cells. Then $a$ balanced tableau of shape $\lambda$ is a bijective filling of the cells of $\lambda$ with numbers from $[n]$ such that if $(i, j) \in \lambda$ and if $i^{\prime}$ is the largest positive integer such that $\left(i^{\prime}, j\right) \in \lambda$, if $k=i^{\prime}-i+1$, and if $S_{i, j}$ is the set of entries $m$ such that $m$ is contained in a cell in $H_{(i, j)}$, then the entry in cell $(i, j)$ of $\lambda$ is the $k^{\text {th }}$ smallest entry of $S_{i, j}$.

Example 7.14. Let $\lambda=(4,3,2)$. Then a balanced tableau of shape $\lambda$ is as follows.

| 4 | 5 | 8 | 3 |
| :--- | :--- | :--- | :--- |
| 6 | 7 | 9 |  |
| 1 | 2 |  |  |
|  |  |  |  |

For instance, let $i=2$ and $j=1$. Then the entry contained in cell $(i, j)$ of $\lambda$ is 6 . Moreover, the largest integer $i^{\prime}$ such that $\left(i^{\prime}, j\right) \in \lambda$ is $3, k=i^{\prime}-i+1=3-2+1=2, H_{(i, j)}=$ $\{(2,1),(2,2),(2,3),(3,1)\}$ and $S_{i, j}$, the set of entries $m$ of this tableau such that $m$ is contained in a cell in $H_{(i, j)}$, equals $\{1,6,7,9\}$. Hence, the $k^{\text {th }}$ smallest entry of $S_{i, j}$ is 6 , which is the entry in cell $(2,1)$ of the above tableau.

In order to generalize standard skew tableaux, reverse standard skew tableaux, and balanced tableaux, we introduce the following special case of configurations from Definition 6.5.

Definition 7.15. Let $\lambda / \mu$ be a skew shape. A configuration of $\lambda / \mu$ is a function $f: \lambda / \mu \rightarrow \mathbb{N}$ from the cells of $\lambda / \mu$ to the positive integers so that if $r \in \lambda / \mu$, then $f(r) \in \mathbb{N}$ and $f(r) \leq h_{r}$. Remark 7.16. We say that $f$ is a configuration of $\lambda / \mu$ rather than say that $f$ is a configuration of the transversal $t$ of the set $\mathscr{F}=\left\{H_{r}: r \in \lambda / \mu\right\}$ defined by $t\left(H_{r}\right)=r$ for all $r \in \lambda / \mu$.

Example 7.17. Consider the skew shape $\lambda / \mu$ where $\lambda=(3,2,1)$ and $\mu=(1)$. We denote configurations $f$ of $\lambda / \mu$ by filling, for all $r \in \lambda / \mu$, cell $r$ with the number $f(r)$. For instance, three configurations of $\lambda / \mu$ are the following.


Now, we define the special case of the notion of satisfaction from Definition 6.5.

Definition 7.18. Let $T$ be a generalized standard skew tableau of shape $\lambda / \mu$ and let $f$ be $a$ configuration of $\lambda / \mu$. Then $T$ satisfies $f$ if for all cells $r \in \lambda / \mu$, the entry in cell $r$ of $T$ is the $k^{\text {th }}$ smallest, where $k=f(r)$, entry in the set of entries of $T$ that are located in the hook $H_{r}$.

In particular, a standard skew tableau of shape $\lambda / \mu$ is precisely a generalized standard skew tableau of shape $\lambda / \mu$ that satisfies the configuration $f_{0}$ of $\lambda / \mu$ defined by $f_{0}(r)=1$ for all $r \in \lambda / \mu$, and a reverse standard skew tableau of shape $\lambda / \mu$ is precisely a generalized standard skew tableau of shape $\lambda / \mu$ that satisfies the configuration $f_{1}$ of $\lambda / \mu$ defined by $f_{1}(r)=h_{r}$ for all $r \in \lambda / \mu$. We will see examples of this in Example 7.19.

Moreover, if $\lambda$ is a normal shape, then let $f$ be the configuration of $\lambda$ such that, for all $(i, j) \in \lambda$, if $i^{\prime}$ is the largest positive integer such that $\left(i^{\prime}, j\right) \in \lambda$, then $f((i, j))=i^{\prime}-i+1$. So any tableau $T$ of shape $\lambda$ is balanced if and only if $T$ satisfies $f$. This characterization of balanced tableaux was used in [13] as the definition of balanced tableaux; the special case of Definition 7.15 for normal shapes also appears in [13] under a different name. Namely, Edelman and Greene call $f(r)$ the hook rank of $r$. However, they only use hook ranks to define balanced tableaux. In this thesis, we have a very different emphasis as we focus on properties of the configurations themselves.

Example 7.19. Consider the skew shape $\lambda / \mu$ from and the three configurations of $\lambda / \mu$ from Example 7.17. The generalized standard skew tableaux that satisfy the leftmost configuration depicted in Example 7.17 are precisely the standard skew tableaux of shape $\lambda / \mu$. Moreover,
the generalized standard skew tableaux that satisfy the rightmost configuration depicted in Example 7.17 are precisely the reverse standard skew tableaux of shape $\lambda / \mu$. Furthermore, four examples of generalized standard skew tableaux that satisfy the middle configuration depicted in Example 7.17 are displayed below.

Definition 7.20. Let $\lambda / \mu$ be a skew shape and $h$ be a configuration of $\lambda / \mu$. Then we write $N(h)$ to denote the number of generalized standard skew tableaux of shape $\lambda / \mu$ that satisfy $h$.

Corollary 7.21. Let $\lambda / \mu$ be a skew shape. Then the number of configurations $f$ of $\lambda / \mu$ such that $N(f)>0$ is

$$
\prod_{r \in \lambda / \mu} h_{r} .
$$

Proof. There are $\prod_{r \in \lambda / \mu} h_{r}$ configurations $f$ of $\lambda / \mu$ since $f(r) \leq h_{r}$ for every $r \in \lambda / \mu$. So, since $\left\{H_{r}: r \in \lambda / \mu\right\}$ is a shellable family of subsets of the set of cells of $\lambda / \mu$ by the discussion after Example 7.11, Theorem 6.14 implies that $N(f)>0$ for all configurations $f$ of $\lambda / \mu$. From this, the corollary follows.

A well-known formula is the hook-length formula, first proved by Frame, Robinson, and Thrall [16]. It is as follows. If $\lambda$ is a normal shape with $n$ cells, then the number of standard Young tableaux of shape $\lambda$ equals

$$
\frac{n!}{\prod_{r \in \lambda} h_{r}}
$$

Moreover, the above formula was also proved by Edelman and Greene to equal to the number of balanced tableaux of shape $\lambda$ [13]. In our context, we will show that the above formula
also has interesting connections to the configurations that we are investigating.

Corollary 7.21 has the following consequence.

Theorem 7.22. Let $\lambda / \mu$ be a skew shape with $n$ cells, and let $X(\lambda / \mu)$ denote the set of configurations of $\lambda / \mu$. Moreover, let $N$ be the number of configurations $f$ of $\lambda / \mu$ such that $N(f)>0$. Then,

$$
\frac{1}{N} \sum_{f \in X(\lambda / \mu)} N(f)=\frac{n!}{\prod_{r \in \lambda / \mu} h_{r}}
$$

Example 7.23. Let $\lambda / \mu=(4,3,2) /(2,1)$. Then

$$
\frac{1}{N} \sum_{f \in X(\lambda / \mu)} N(f)=\frac{n!}{\prod_{r \in \lambda / \mu} h_{r}}=\frac{6!}{1 \cdot 3 \cdot 1 \cdot 3 \cdot 1 \cdot 2}=40
$$

The hook-lengths are represented with the following diagram.


Proof. Every generalized standard skew tableau of shape $\lambda / \mu$ satisfies exactly one configuration of $\lambda / \mu$ by Lemma 6.7, so by Definition 7.20 and the fact that there are $n$ ! generalized standard skew tableaux of shape $\lambda / \mu$,

$$
\sum_{f \in X(\lambda / \mu)} N(f)=n!.
$$

Moreover, by Corollary 7.21,

$$
N=\prod_{r \in \lambda / \mu} h_{r} .
$$

From this, the theorem follows.

Lastly, we note that a special case of our work has also been considered in the literature by Viard. We derived our work independently of Viard.

Remark 7.24. Consider a finite subset $S$ of $\mathbb{N}^{2}$. Next, for all $r=(i, j) \in S$, define $F_{r}=$ $\left\{\left(i_{1}, j\right) \in S: i_{1} \geq i\right\} \cup\left\{\left(i, j_{1}\right): j_{1} \geq j\right\}$, and define $\mathscr{F}=\left\{F_{r}: r \in S\right\}$. This construction is related to the tools we used in Chapter 7 for the following reason. By using the same explanation as the one we gave for why $\left\{H_{r}: r \in \lambda / \mu\right\}$ is shellable, we observe that $\mathscr{F}$ is shellable and that its unique transversal is defined by $t\left(F_{r}\right)=r$ for all $r \in S$.

Let $\mathscr{F}$ and t be as described in the above paragraph. Viard [50 51] considered objects that are equivalent to configurations of t and permutations that satisfy such configurations. Viard [50. 51] asserted that he has established one direction of a special case of Theorem 6.14 by claiming to have proved a statement equivalent to asserting that all configurations $f$ of tare satisfied by at least one permutation $\sigma: S \rightarrow S$. In particular, using his claim, he derives two consequences that imply Corollary 7.21 and Theorem 7.22. There are two versions of his arguments (a less general version in [50] and a more general version in [51]), both versions are different from our proof of Theorem 6.14

## Chapter 8

## Conclusion

In this thesis, we proved that the number of periodic $P$-partitions can be analysed with the homogeneous first-order matrix difference equation in Theorem 5.19.

The case of Theorem 5.19 when the labelling $\omega$ is dual-natural can be applied to the following. As indicated by Example 4.6, Theorem 5.19 can be applied to bijective fillings of the parallelogramic shapes considered by López et.al. [27], by Sun [47], and by Tewari and van Willigenburg [49]. Moreover, as indicated in Example 4.18, Theorem 5.19 can be applied to bijective fillings of certain truncated shifted shapes. This would add to what is known about enumerating bijective fillings on truncated shifted shapes from Adin, King, and Roichman [1, 3] and from Panova [38].

The case of Theorem 5.19 when the labelling $\omega$ is a generalized Schur labelling can be applied to the problem of enumerating semistandard tableaux. The problem of enumerating tableaux known as semi-standard tableaux is, in general, far from solved [45]. For semistandard tableaux on partition shapes, the number of such numbers are called Kostka num-
bers. These numbers are related to Specht modules and have several implications in algebraic combinatorics [41, 45].

Moreover, in this thesis, we gave a new characterization of shellable families in Theorem 6.14 by generalizing the notion of standard Young tableaux and Edelman and Greene's balanced tableaux and proved an existence result for such generalized tableaux on skew shapes in Theorem 7.22. This would add to the properties described in Remark 6.9 that are known about shellable families of sets. Additionally, as shellable families are families of sets that satisfy a stronger form of the marriage condition, Theorem 6.14 adds to what is known about the marriage condition. Further to this, Theorem 6.14 and Theorem 7.22 provide existence results to natural generalizations of balanced tableaux and skew tableaux and establish that the numbers of such tableaux are related to the enumerative formulas for balanced tableaux discovered by Edelman and Greene in [13, 14] and the enumerative formulas for standard Young tableaux discovered by Frame, Robinson, and Thrall in [16].

We now discuss the future directions that we have in mind for this work.

Recall that the Cayley-Hamilton Theorem states that if $M$ is a square matrix and if $p(x)$ is the characteristic polynomial of $M$, then $p(M)=0$. By applying the Cayley-Hamilton Theorem to the square matrix in Theorem 5.19 and by using Lemma 5.10, we plan to prove that the matrix difference equation in Theorem 5.19 can be used to prove that periodic $P$-partitions can be enumerated with constant coefficient linear recurrence relations. In particular, we plan to recover all of the constant coefficient linear recurrence relation results of López et.al. [27], Sun [47], and Tewari and van Willigenburg [49].

In view of the above plan, we aim to analyse six aspects of the above recurrence relations obtained via the Cayley-Hamilton Theorem, which, in this chapter, we call periodic recurrence
relations after the periodic $P$-partitions that they enumerate, and the number of periodic $P$ partitions, which are enumerated by the periodic recurrence relations.

Firstly, we plan to use a known result that express the characteristic polynomial of a square matrix $A$ in terms of its trace $\operatorname{tr} A$ [26] and to give combinatorial descriptions for $\operatorname{tr} M$ when $M$ is the matrix in Theorem 5.19. From this, we can obtain combinatorial descriptions for the coefficients of the periodic recurrence relations. Using the work from ([6], Question 1284192; c.f. [9], Section 3.1) to describe the characteristic polynomial of the square matrix $A$, the above combinatorial descriptions can also be converted into a determinantal formulas for the above coefficients.

Secondly, we plan to analyse the asymptotics of the number of periodic $P$-partitions. Consider the notation $P_{n}$ as given in Proposition 5.10. We plan to prove that $\left|P_{n}, \omega\right|$ is asymptotic to $c r^{n}$ as $n \rightarrow \infty$ for some constants $c>0$ and $r>1$ that depend on the sequence $\left(P_{n}\right)_{n=1,2, \ldots}$. To do this we will use properties of transfer matrices for directed graphs [44] to prove further combinatorial properties of the matrix in Theorem 5.19, then invoke results in PerronFrobenius theory [4, 11, 32].

Thirdly, we plan to analyse the largest and second largest eigenvalues of the matrix in Theorem 5.19. For the largest eigenvalue, we plan to give a combinatorial description and to give bounds. For the bounds, we will use Perron-Frobenius theory and matrix theory [33]. Such an approach can also utilize many known bounds, that have varying levels of complexity, for the largest eigenvalue [7, 17, 20, 32, 33, 37, 52]. For the second largest eigenvalue, we plan to use certain tools in matrix and spectral graph theory [11, 33] to give combinatorial descriptions.

Fourthly, in a related direction, we plan to establish a link between certain finite posets and
certain monic polynomials in $\mathbb{Z}[x]$. Speficially, we plan to prove a relationship between the minimal polynomials of the matrices in Theorem 5.19 and the posets that are the index shapes in Theorem 5.19 by proving the following.

Conjecture 8.1. The condition that one such minimal polynomial divides another such minimal polynomial is equivalent to the condition that one such poset is a subposet of another such poset.

Fifthly, as an application of the connected triple concept we introduced, we plan to prove that many columns of the transfer matrices in Theorem 5.19 are identical. A generalization of such properties was explored by Lundow [29] where symmetry properties of certain recurrence relations were investigated.

Lastly, in the case when the labelling $\omega$ in the periodic quadruple system $(Z, \omega, \pi, \theta)$ is dualnatural, we also plan to adapt López et.al's proof technique from [27], by generalizing it to the level of generality considered in this thesis, to derive exponential upper bounds on the orders of the periodic recurrence relations, and to derive exponential upper bounds on the degrees of the minimal polynomials of the matrices in Theorem 5.19.

For the results relating to generalized balanced tableaux and marriage problems with unique solutions, there are three aspects of these results that we plan to analyse.

In a preprint submitted for publication, we generalize Theorem 6.14 to certain words in $[n]^{m}$, where $m \leq n$ and $m$ is bounded below by a formula that depends on the shellable family $\mathscr{F}$. Moreover, we generalize Theorem 7.22 to shellable families and the aforementioned words in $[n]^{m}$ and the expression in Theorem 7.22 is generalized to include Stirling numbers of the second kind. In discussing the feasibility of such a formula, we use known properties of these Stirling numbers.

There is a known formula in the literature, that is a more complex form of the hook-length formula, for determining the number of standard skew tableaux of shape $\lambda / \mu$ [34]. Moreover, asymptotic properties of such numbers were analysed by Morales, Pak, and Panova in [34]. By combining these results with Theorem 7.22, we plan to investigate the number of configurations $f$ such that $N(f)$ is strictly less than the expression given in Theorem 7.22. Such an investigation appears to be generalizable since there are variants and generalizations of Naruse's formula that are known [19, 36], and since for at least some of these variants, extensions of Morales, Pak, and Panova's asymptotic properties are conjectured to extend to at least some of these variants.

Lastly, we plan to define a natural partial ordering on the possible configurations $f$ of a transversal $t$ of a family of sets. With this partial order, we plan to derive a product formula that would give a general upper bound for $N(f)$, where $N(f)$ is as defined in Theorem 7.22, by utilizing the order-preserving maps in the proof of Theorem 6.14 .

## Bibliography

[1] R. Adin, R. King, and Y. Roichman, Enumeration of standard Young tableaux of certain truncated shapes, Electron. J. Comb. 18(2) (2011). $\rightarrow$ pages 2, 91
[2] R. Adin and Y. Roichman, Standard Young Tableaux Chapter 14 of Handbook of Enumerative Combinatorics $1^{\text {st }}$ Edition, Edited by Miklós Bóna, Chapman and Hall/CRC (2015). $\rightarrow$ pages $2,9,11,12,30$
[3] R. Adin and Y. Roichman, Triangle-Free Triangulations, Hyperplane Arrangements and Shifted Tableaux Electron. J. Comb. 19(3) (2012). $\rightarrow$ pages 2, 91
[4] J. Allouche and J. Shallit, Automatic Sequences, Theory, Applications, Generalizations, Cambridge Univ. Press (2003). $\rightarrow$ page 93
[5] O. Angel, A. Holroyd, D. Romik, and B. Virág, Random sorting networks, Adv. Math. 215 839-868 (2007). $\rightarrow$ page 5
[6] J. Atwood and J. Spolsky, Math Stack Exchange, https://math.stackexchange.com $\rightarrow$ page 93
[7] A. Brauer, The theorems of Ledermann and Ostrowski on positive matrices, Duke Math. J. 24, 265-274 (1957). $\rightarrow$ page 93
[8] A. Björner and F. Brenti, Combinatorics of Coxeter Groups, Grad. Texts in Math. 231, Springer, (2005). $\rightarrow$ pages 73,74
[9] T. Curtright and D. Fairlie, A Galileon Primer, arxiv : 1212.6972v1 (2012). $\rightarrow$ page 93
[10] G. Chang, On the Number of SDR of a $(t, n)$-family European J. Combin.10, 231-234 (1989). $\rightarrow$ pages 4, 73
[11] F. Chung, Laplacians and the Cheeger inequality for directed graphs Ann. Comb. Volume 9, Issue 1, 1-19 (2005). $\rightarrow$ page 93
[12] B. Davey and H. Priestley, Introduction to Lattices and Order, Second Edition Cambridge Univ. Press (2002). $\rightarrow$ page 8
[13] P. Edelman and C. Greene, Balanced Tableaux Adv. Math. 63 42-99 (1987). $\rightarrow$ pages 4, 5, 86, 87, 88, 92
[14] P. Edelman and C. Greene, Combinatorial correspondences for Young tableaux, balanced tableaux, and maximal chains in the weak Bruhat order of $S_{n}$, Contemp. Math. 34 (1984). $\rightarrow$ pages 4, 5, 92
[15] S. Fomin, C. Greene, V. Reiner, M. Shimozono, Balanced Labellings and Schubert Polynomials European J. Combin. 18, 373-389 (1997). $\rightarrow$ page 5
[16] J. Frame, G. Robinson, R. Thrall, The hook graphs of the symmetric group Canad. J. Math. 6, 316-325 (1954). $\rightarrow$ pages 88, 92
[17] K. Garren, Bounds for the eigenvalues of a matrix National Aeronautics and Space Administration, Washington D.C., Langley Research Center, Langley Station, Hampton, Va. (1968). $\rightarrow$ page 93
[18] I. Gessel, Multipartite P-partitions and inner products of skew Schur functions Comb. Algebra, Proc. Conf., Boulder/CO 1983, Contemp. Math. 34, 289-301 (1984). $\rightarrow$ page 1
[19] W. Graham and V. Kreiman, Excited Young diagrams, equivariant K-theory, and Schubert varieties, Trans. Amer. Math. Soc. Vol. 367, No. 9 6597-6645 (2015). $\rightarrow$ page 95
[20] C. Hall and T. Porsching, Bounds for the maximal eigenvalue of a nonnegative irreducible matrix. Duke Math. J. 36, 159-164 (1969). Review by R. Rinehart available on mathsci.net $\rightarrow$ page 93
[21] P. Hall, On Representatives of Subsets, J. London Math. Soc, 10 (1) 26-30 (1935). $\rightarrow$ pages 4, 70, 71
[22] M. Hall Jr, Distinct Representatives of Subsets, Bull. Amer. Math. Soc. Vol. 54, No. 10, 922-926(1948). $\rightarrow$ pages 4, 73, 78
[23] J. Hirst and N. Hughes, Reverse mathematics and marriage problems with finitely many solutions Arch. Math. Logic 55:1015-1024 (2016). $\rightarrow$ page 4
[24] J. Hirst and N. Hughes, Reverse mathematics and marriage problems with unique solutions Arch. Math. Logic 54:49-57(2015). $\rightarrow$ pages 4, 73
[25] M. Konvalinka, A bijective proof of the hook-length formula for skew shapes, arXiv:1703.08414v2 (2018). $\rightarrow$ page 4
[26] M. Lewin, On the coefficients of the characteristic polynomial of a matrix Discrete Math. 125 255-262 (1994). $\rightarrow$ page 93
[27] A. López, L. Martínez, A. Pérez, B. Pérez, O. Basova, Combinatorics related to Higman's conjecture I: Parallelogramic digraphs and dispositions, Linear Alg. Appl. 530 414-444 (2017). $\rightarrow$ pages 2, 3, 22, 27, 91, 92, 94
[28] D. Little, Combinatorial aspects of the Lascoux - Schützenberger tree, Adv. Math. 174 236-253 (2003). $\rightarrow$ page 5
[29] P. Lundow Compression of transfer matrices Discrete Math. 231 321-329, Elsevier (2001). $\rightarrow$ page 94
[30] K. Luoto, S. Mykytiuk, S. van Willigenburg, An introduction to quasisymmetric Schur functions - Hopf algebras, quasisymmetric functions, and Young composition tableaux - Springer (2013). $\rightarrow$ pages 11, 12, 80, 82
[31] P. MacMahon, Combinatory analysis, Chelsea Publishing Co., New York (1960). $\rightarrow$ page 1
[32] C. Meyer, Matrix Analysis and Applied Linear Algebra, First Edition, SIAM (2000). $\rightarrow$ page 93
[33] H. Minc, Nonnegative Matrices, Wiley Interscience Series in Discrete Mathematics and Optimization (1988). $\rightarrow$ page 93
[34] A. Morales, I. Pak, G. Panova, Asymptotics of the number of standard Young tableaux of skew shape, Electron. J. Combin. 70 26-49 (2018). $\rightarrow$ pages 5, 95
[35] A. Morales, I. Pak, G. Panova, Hook formulas for skew shapes I. q-analogues and bijections J. Combin. Theory Ser. A 154 350-405 (2018). $\rightarrow$ page 4
[36] H. Naruse and S. Okada, Skew hook formula for d-complete posets via equivariant K-theory Algebr. Comb. Vol. 2 Issue 4541 - 571 (2019). $\rightarrow$ pages 4, 95
[37] A. Ostrowski, H. Schneider Bounds for the maximal characteristic root of a nonnegative irreducible matrix. Duke Math. J. 27, 547 - 553 (1960). Review by W. Ledermann available on mathsci.net $\rightarrow$ page 93
[38] G. Panova, Tableaux and plane partitions of truncated shapes Adv. Appl. Math. 49196 - 217 (2012). $\rightarrow$ pages 2,91
[39] I. Pak, F. Petrov, and V. Sokolov, Hook Inequalities arXiv:1903.11828v2 (2019). $\rightarrow$ page 5
[40] P. Reichmeider, The Equivalence of Some Combinatorial Matching Theorems, Polygonal Pub House (1985). $\rightarrow$ page 4
[41] B. Sagan, The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions (2nd Edition), Graduate Texts in Mathematics 203, SpringerVerlag New York Inc., (2001). $\rightarrow$ pages 3, 11, 12, 80, 82, 83, 84, 92
[42] B. Schröder, Ordered Sets, An Introduction with Connections from Combinatorics to Topology Second Edition, Birkhäuser, Springer International Publishing (2016). $\rightarrow$ page 8
[43] N. Sloane, The on-line encyclopedia of integer sequences http://oeis.org. $\rightarrow$ page 3
[44] R. Stanley, Enumerative Combinatorics, Volume 1, Second Edition Cambridge Stud. in Adv. Math. 49, Cambridge Univ. Press (2012). $\rightarrow$ pages 2, 8, 11, 12, 16, 18, 93
[45] R. Stanley, Enumerative Combinatorics, Volume 2, First Edition Cambridge Stud. in Adv. Math. 62, Cambridge Univ. Press (1999). $\rightarrow$ pages 3, 4, 11, 12, 26, 82, 91,92
[46] R. Stanley, Ordered structures and partitions Revision of R. Stanley's PhD thesis at Harvard Univ. 1971, Dep. Math. Univ. California Berkley, California $94720 . \rightarrow$ page 1
[47] P. Sun, Enumeration of standard Young tableaux of shifted strips with constant width Electron. J. Combin. 24 1-11, (2017). $\rightarrow$ pages 2, 3, 22, 27, 91,92
[48] J. Swanson, On the Existence of Tableaux with Given Modular Major Index, Algebr. Comb. Vol. 1, No. 1, 3-21 (2018). $\rightarrow$ page 5
[49] V. Tewari and S. van Willigenburg, Modules of the 0-Hecke algebra and quasisymmetric Schur functions Adv. Math. 2851025 - 1065 (2015). $\rightarrow$ pages v, 2, 3, 22, 27, 30, 91, 92
[50] F. Viard, A natural generalization of balanced tableaux, arxiv 1407.6217v2 (2016). $\rightarrow$ page 90
[51] F. Viard, Des graphes orientés aux treillis complets : une nouvelle approche de l'ordre faible sur les groupes de Coxeter. Univ. Claude Bernard Lyon 1 École doctorale InfoMath, ED 512 Spécialité : Mathématiques N. dórdre 232-2015. $\rightarrow$ page 90
[52] J. Zhang, Some bounds for the determinant and eigenvalues of a matrix. Knowledge Practice Math., no. 3, 19-25 (1981). Review by J. Kim available on mathsci.net $\rightarrow$ page 93

