# Tertiary Mathematics and Content Connections in the Development of Mathematical Knowledge for Teaching 

by<br>Vanessa E. Radzimski<br>B.Sc., Florida State University, 2012<br>M.Sc., University of British Columbia, 2014<br>\title{ A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF<br><br>Doctor of Philosophy<br><br>in<br><br>THE FACULTY OF GRADUATE AND POSTDOCTORAL STUDIES }<br>(Curriculum Studies)<br>The University of British Columbia<br>(Vancouver)<br>April 2020<br>(c) Vanessa E. Radzimski, 2020

The following individuals certify that they have read, and recommend to the Faculty of Graduate and Postdoctoral Studies for acceptance, the thesis entitled:

## Tertiary Mathematics and Content Connections in the Development of Mathematical Knowledge for Teaching <br> submitted by Vanessa E. Radzimski in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Curriculum Studies.

## Examining Committee:

Cynthia Nicol, Curriculum Studies
Supervisor
Ann Anderson, Curriculum Studies
Co-Supervisor
Alejandro Adem, Mathematics
Supervisory Committee Member
Leah Edelstein-Keshet, Mathematics
University Examiner
Marina Milner-Bolotin, Curriculum Studies
University Examiner

## Abstract

Emphasis on the importance of subject matter expertise in teaching secondary mathematics is found in the research literature and in policy. In the United States, for instance, the No Child Left Behind Act, calls for secondary teachers to be certified in a subject specialization. In Canada, admission to secondary teacher education programs requires extensive subject-specific university coursework. However, it is unclear if or how extensive subject matter expertise impacts the practices of teachers in a secondary classroom.

This study aims to explore how advanced coursework in mathematics, beyond the scope of the high school curriculum, impacts the ways prospective teachers understand and teach secondary content. Using a qualitative case study methodology, five prospective secondary mathematics teachers participated, with data obtained through document analysis and semi-structured task-based interviews. Participants engaged with classroom-relevant tasks and were explicitly asked how they could draw upon advanced mathematics to inform their teaching. Participants also detailed their perceptions of the role advanced mathematics plays in their development as teachers.

Results from this study reveal that participants saw little value in the content of advanced mathematics to their teaching, but expressed value towards the beliefs and values gained through advanced mathematics, such as problem solving and rigour. Some participants demonstrated misconceptions at the secondary level, which had direct connections to content from their post-secondary mathematics coursework. For example, all participants made the false claim that a real-valued
polynomial can be factored if and only if it has a root.

Results extend the literature through rich empirical data which illuminates how prospective secondary mathematics teachers perceive and use advanced mathematics in understanding the secondary curriculum. While participants held content knowledge beyond the secondary curriculum, this knowledge was not integrated in a way that impacted their understanding of secondary mathematics. An understanding of post-secondary mathematics has the potential to be of value to secondary teachers in the classroom, but this potential needs a space to be unlocked. I argue that mathematicians and teacher educators need to work together to build opportunities for prospective teachers to build connections between the mathematics they know and the mathematics they need to teach.

## Lay Summary

I explore the role of advanced mathematics knowledge in the pedagogy of future secondary mathematics teachers. This study utilized a qualitative case study methodology to understand what five teacher candidates perceived as the role of their advanced mathematics expertise, as well as connections they built between secondary and post-secondary content. Results revealed that participants did not view the content from their post-secondary degrees as being relevant to classroom practice. This was supported through participants' engagement in task-based interviews, where they expressed limited connections between secondary and postsecondary mathematics, as well as content misconceptions at both levels. This study extends the literature in suggesting that advanced mathematical coursework may play a very limited role in impacting the practice of future teachers. Results suggest a need for further investigation into the ways mathematicians and teacher educators support the integration of post-secondary mathematics knowledge into the mathematical knowledge for teaching of future teachers.

## Preface

This dissertation is an original intellectual product of the author, V. Radzimski. The research reported in Chapters 4-8 was covered by UBC Human Ethics Certificate ID H17-01767.

## Table of Contents

Abstract ..... iii
Lay Summary ..... v
Preface ..... vi
Table of Contents ..... vii
List of Tables ..... xi
List of Figures ..... xii
Glossary ..... xiv
Acknowledgments ..... XV
Dedication ..... xvi
1 Introduction ..... 1
1.1 Background ..... 1
1.2 Research Aims ..... 3
1.3 Significance ..... 4
1.4 Methods ..... 6
1.5 Theoretical Framework ..... 7
1.6 Personal Statement ..... 9
1.7 Organization ..... 11
2 Literature Review and Theoretical Framework ..... 13
2.1 Teachers' Content Knowledge: Beginnings ..... 15
2.2 Mathematical Knowledge for Teaching ..... 19
2.3 Advanced Mathematics Knowledge in Teaching ..... 24
2.4 Studies on Teacher Knowledge ..... 27
2.5 Frameworks for Teacher Knowledge ..... 32
2.5.1 Framework of Silverman and Thompson ..... 36
2.5.2 Mathematical Knowledge for Teaching ..... 41
2.6 Summary ..... 46
3 Methodology ..... 51
3.1 Research Methodology: Case Study and Interview ..... 51
3.1.1 Case Study ..... 52
3.1.2 Interviews as Data ..... 54
3.1.3 Variations on the Interview ..... 56
3.1.4 Debates on the Interview as Data ..... 57
3.1.5 Summary ..... 60
3.2 Setting ..... 61
3.2.1 Participants ..... 62
3.3 Data Collection ..... 65
3.3.1 Experiences in Post-Secondary Mathematics ..... 65
3.3.2 Connections Between Secondary and Post-Secondary Math- ematics ..... 66
3.3.3 Secondary Mathematics Instrument ..... 71
3.4 Data Analysis ..... 72
3.5 Validity and Reliability ..... 79
3.6 Summary ..... 81
4 Perceptions of the Role of Advanced Mathematics in Pedagogical Development ..... 82
4.1 Perceptions of Mathematics ..... 83
4.2 Role of Advanced Knowledge for Teachers ..... 88
4.3 Advanced Knowledge in Teacher Education ..... 94
4.4 Summary ..... 99
5 The Overextension of Familiar Mathematical Ideas: A Case of Poly- nomials ..... 103
5.1 Mathematical Background ..... 104
5.2 Participant Understandings ..... 111
5.3 Post-secondary Connections ..... 127
5.4 An Experience of Abstract Algebra ..... 133
5.5 Summary ..... 136
6 The Role of Limits, Infinity, and Formal Definitions in Secondary Mathematics ..... 139
6.1 Introduction ..... 139
6.2 Inverse Functions ..... 140
6.2.1 Mathematical Background ..... 140
6.2.2 Participant Understandings ..... 143
6.2.3 Post-Secondary Connections ..... 153
6.3 Limits ..... 157
6.3.1 Mathematical Background ..... 157
6.3.2 Participant Understandings ..... 161
6.3.3 Post-Secondary Connections ..... 164
6.4 Exponentials ..... 169
6.4.1 Mathematical Background ..... 170
6.4.2 Participant Understandings ..... 171
6.4.3 Post-Secondary Connections ..... 179
6.5 Summary ..... 186
7 The Tensions of Proof and Applications Observed Through Geomet- ric Tasks ..... 189
7.1 The Square Root of Two ..... 189
7.1.1 Mathematical Background ..... 190
7.1.2 Participant Understandings and Post-Secondary Connections ..... S192
7.2 Symmetry ..... 198
7.2.1 Mathematical Background ..... 199
7.2.2 Participant Understandings ..... 202
7.2.3 Post-Secondary Connections ..... 204
7.3 Summary ..... 208
8 Conclusion and Implications ..... 210
8.1 Conclusions ..... 210
8.1.1 The Role of University Mathematics in Teacher Development 21
8.1.2 Content Connections Between University and Secondary Mathematics ..... 214
8.1.3 Limitations ..... 221
8.2 Implications ..... 222
8.2.1 Implications in Research ..... 222
8.2.2 Implications in Practice ..... 224
8.3 Closing Remarks ..... 228
References ..... 231
A Interview Materials ..... 243
B Recruitment emails and documents ..... 247

## List of Tables

Table 3.1 Summary of participant backgrounds ..... 64
Table 3.2 Tasks chosen by participants ..... 71
Table 3.3 Interview transcript codes and themes ..... 75
Table 5.1 Partial fraction decomposition guidelines ..... 107
Table 7.1 $\quad$ Truth table for logical equivalence of $P$ and $\neg P \Longrightarrow Q \wedge \neg Q$ ..... 194

## List of Figures

Figure 2.1 Mathematical Knowledge for Teaching (MKT) ..... 21
Figure 2.2 Assessment item for fraction as quantity (Simon, 2006, p.368) ..... 38
Figure 5.1 Casey's work on quadratics and air travel ..... 114
Figure 5.2 Bailey's factoring of $x^{5}-1$ ..... 120
Figure 5.3 Taylor's factoring of $x^{5}-1$ ..... 120
Figure 5.4 Adrian's factoring approach ..... 122
Figure 5.5 Casey's factoring of $x^{4}+1$ ..... 127
Figure 5.6 Casey's factoring of cubics ..... 131
Figure 6.1 Casey's work for the "inverse" of $y=x^{2}$ ..... 150
Figure 6.2 Casey's written work for the inverse of $\sin (x)$ ..... 151
Figure 6.3 Casey's work for circles and inverses ..... 153
Figure 6.4 Adrian's work for the value of $e$ ..... 155
Figure 6.5 Taylor's work for $0.999 \ldots=1$ ..... 162
Figure 6.6 Adrian's work for $0.999 \ldots=1$ ..... 165
Figure 6.7 Bailey's work for the limiting behaviour of 0.999 ..... 166
Figure 6.8 Bailey's work explaining rational exponents ..... 173
Figure 6.9 Adrian's work finding a value of $2^{\sqrt{3}}$ ..... 174
Figure 6.10 Taylor's work for work defining $2^{\sqrt{3}}$ ..... 175
Figure 6.11 Adrian's work extending exponents to irrationals ..... 177
Figure 6.12 Taylor's work on exponents and logarithms ..... 185
Figure 7.1 Taylor's proof that $\sqrt{2}$ is irrational ..... 193
Figure 7.2 Bailey's picture of "zooming in" to $\sqrt{2}$ ..... 197

Figure 7.3 Casey's work on symmetries of an equilateral triangle . . . . 206

## Glossary

Advanced Mathematical Knowledge, a framework used specifically for knowledge of mathematics beyond the secondary curriculum.

Horizon Content Knowledge, a component of Mathematical Knowledge for Teaching defined as the knowledge of the mathematics beyond the curriculum being taught.

Key Developmental Understandings, knowledge that allows a learner to work with a particular concept in new ways and settings.

Mathematical Knowledge for Teaching, a framework for understanding the knowledge used in the teaching of mathematics.

Pedagogical Content Knowledge, a framework which extends the notion of content knowledge to content knowledge in teaching.

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I would be amiss to not thank my mom. You have been my biggest cheerleader in whatever I've done in my life, from pointe shoes and tutus to textbooks and grad school. Your creative spirit has always inspired me to look at the world differently. Thank you for the arts-based environment you worked so hard to foster at home and know that it played a fundamental role in growing into the woman I am today.

To my dearest Paul, you've dealt with me in times where I don't know how you have. Your love, kindness, and encouragement have kept me going in times of extreme self-doubt. Through good times and bad, you continue to love and support me in reaching my goals. I love you more than I can put into words.

A final thanks to the Killam Trust and their generous fellowship that supported my studies. It was a great honour to receive their support for the work I was so excited to embark on.

## Dedication

To my father, Michael Jerome Radzimski, January 21, 1951 - October 1, 2018

Finishing this thesis with everything that transpired over the last year was a challenge, to say the least, but it would not have been possible without your everlasting love and support. I love you and I miss you everyday.

Forever, your little Vee.

## Chapter 1

## Introduction

### 1.1 Background

Admission requirements for most secondary mathematics teacher education programs include a degree in mathematics or a related major. Consequently, university students enter such mathematics education programs with courses in Abstract and Linear Algebra, Number Theory, and Real and Complex Analysis, among others. This requirement does not exist without reason. Much of the content in secondary mathematics curricula is deeply connected to these advanced university courses, despite being well outside the content teachers would be expected to teach secondary students. Some researchers claim that advanced subject matter experience influences teachers to conceptualize the secondary mathematics curriculum at a deeper level and provide a richer learning experience for students, leading to higher student achievement (Paige, 2002). However, there is competing evidence that suggests no positive correlation exists between advanced mathematics coursework taken by teachers and their students' achievement (Monk, 1994).

The post-secondary education of prospective secondary teachers could be conceptualized as two islands: mathematical content and mathematics pedagogy. During a prospective secondary mathematics teachers' mathematics education, mathematics departments and professors are responsible for building the mathematical knowledge of these future teachers, while education departments support future teachers with subject-specific courses in pedagogy. Research in secondary mathematics teacher education follows a similar pattern, with studies focusing on either future teachers' mathematical content knowledge (Almeida et al., 2016; Cofer, 2015; Even, 1993; Knuth, 2002; Leung et al., 2016) or mathematics pedagogy (Dede, 2015; Fernández et al., 2016; Zazkis and Leikin, 2010).

This divide between mathematics content and mathematics pedagogy remains within the structure of the university, as emphasized by Ted Eisenberg, who "lamented profoundly the growing divide between the mathematics community and the mathematics education community" (Fried, 2014, p.3). In the Mathematical Association of America's (MAA) A Call for Change, they state that "the mathematical preparation of teachers must provide experiences in which they develop an understanding of the interrelationships within mathematics and an appreciation of its unity" (Leitzel, 1991, p.3). Unfortunately, the divide between mathematics and education - in teacher education, the research literature, and the university - may not be conducive to such preparation. Do prospective teachers have the opportunity to develop interrelationships between these two fields? That is, are they able to build connections between their advanced university mathematical knowledge and pedagogical practice in secondary mathematics?

This is precisely where the research in this thesis becomes of importance. Rather than studying the islands of university mathematics content and mathematics pedagogy individually, I studied the bridge between the two and sought to uncover the role that university mathematics coursework plays in secondary mathematics teacher education. To date, there appears to be no empirical study which examines how prospective teachers bridge these two islands of knowledge. Wasserman's recent work (Wasserman, 2016; Wasserman et al., 2017) bridges the two islands, but is theoretical in nature and indicates a need for empirical research in the domain. This current study begins to shed light on the effect a post-secondary degree in mathematics has on the ways prospective secondary teachers perceive and understand secondary school concepts, as well as to what extent they draw upon connections between these two bodies of knowledge to inform their pedagogy. Moreover, the investigation has implications for ways in which mathematics teacher educators might support future teachers' MKT. Indeed, fostering opportunities for the development of connections which integrate secondary and post-secondary mathematics may be transformative to prospective teachers' understanding of the secondary curriculum and the ways in which it is taught.

### 1.2 Research Aims

This study aims to observe, understand, and interpret the ways in which prospective secondary mathematics teachers draw upon knowledge from their post-secondary degrees to enhance secondary mathematics learning. Despite the existence of deep connections between secondary and post-secondary mathematics content (Cofer,

2015; Wasserman and Stockton, 2013), research suggests that teachers do not perceive their advanced mathematical knowledge as playing an important role in pedagogical practice (Zazkis and Leikin, 2010). With the majority of studies on secondary teachers' mathematical knowledge focusing explicitly on content in the secondary curriculum, no such study exists that aims to understand the bridge that connects secondary and post-secondary mathematics content. The primary research questions are:

1. What do prospective secondary mathematics teachers perceive as the role of their advanced mathematics knowledge in their development as teachers?
2. In what ways do prospective secondary mathematics teachers relate advanced mathematics knowledge to a mathematics concept in the secondary curriculum?

From the results of this question, I will shed light on the role of advanced mathematical coursework in the development of future teachers, and point to ways in which mathematicians and mathematics teacher educators might foster the development of MKT for future teachers.

### 1.3 Significance

This investigation will provide important information on how a post-secondary degree in Mathematics affects the way prospective secondary teachers understand instruction of secondary school mathematics concepts and how the connections they have between these two bodies of knowledge influence their pedagogy. Indeed, as mentioned above, many secondary mathematics teacher education programs require extensive mathematical coursework at the university level. However, this
knowledge is set aside once teacher education candidates enter their pedagogical study. As Wasserman et al. (2017) argues, advanced mathematical knowledge has the potential to be transformative to a teacher's pedagogy by connecting advanced mathematical knowledge to teaching practices. This study serves as a first step towards understanding the ways prospective teachers have connected their advanced mathematical knowledge to secondary content, without the intervention of mathematics teacher educators.

Moreover, this investigation has implications for ways in which mathematics teacher educators might explicitly support future teachers' understanding of secondary mathematics concepts and mathematics at the post-secondary level. Indeed, the interview topics discussed in this study could constitute rich mathematical discussion in a mathematics pedagogy course, recognizing that teachers have advanced content knowledge that is relevant in a secondary school context and may better inform their pedagogy. As a consequence, the development of mathematical content knowledge that is directly related to secondary mathematics instruction could enrich the mathematical learning environment for secondary students. Such an improvement in mathematics teacher education could result in teachers who are able to engage with secondary mathematics content with more depth and breadth, and in turn, provide secondary students with a rich mathematical learning experience. This shift in mathematics education at the secondary level could improve students' mathematics upon entry to university and in turn, increase entry into STEM fields for post-secondary education.

Finally, this study may encourage mathematics departments to reconsider the
ways in which they support the mathematical development of future teachers. How much are future teachers learning in their advanced mathematics courses? What are they learning and do they find it relevant to their future work? I hope that the results of this study act as an impetus for mathematicians to interrogate the intent, goals, and pedagogies of advanced mathematics courses. In the same vein, but a different context, I hope that my research contributes to conversations about the ways advanced mathematics expertise is drawn upon in teacher education. With prospective teachers needing over 30 credits of mathematics coursework at the senior level, to me, it appears to be a missed opportunity to not draw upon, develop, and encourage reflection on the role of such courses by and for future teachers.

### 1.4 Methods

In an effort to investigate the research question, I utilized a qualitative research methodology. More specifically, the following research is a case study (Yin, 2013). Data collection and analysis was executed through one-on-one interviews, taskbased interviews (Goldin, 2000) and document analysis. Sources of data included transcriptions of interviews, thematic coding of interview transcripts, participants' written work, and participants' academic transcripts. Coding of the transcripts was done through a grounded theory approach, where codes were assigned to statements throughout the interview. Similar codes were then gathered into themes. Although a quantitative data focused methodology could provide insight into future teachers' mathematical knowledge for teaching, it would not provide the depth necessary to understand participants' perceived role of advanced mathematics knowledge in teacher education. The qualitative case study approach was chosen to be
appropriate for this work, since case study allows the researcher to provide rich description and provide insight into the knowledge, experience, and beliefs of the future secondary mathematics teachers in this study.

Participants in this study were recruited on a volunteer basis from the secondary teacher education program at my institution. Since all participants were enrolled in the program with mathematics as a teachable subject, each of them met the mathematics coursework requirement for entry into the teacher education. In total, five teacher candidates enrolled as participants in the study: Taylor, Jaime, Bailey, Adrian, and Casey. Each of the names assigned are pseudonyms and are gender neutral. Throughout the thesis, each participant will be referred to under the pronoun "they," since gender was not part of my analysis of the interviews. As a thank you for volunteering their time and experience in the study, each participant was gifted a $\$ 25$ VISA gift card.

### 1.5 Theoretical Framework

As will be detailed in Chapter 2, teacher knowledge can be studied in many different forms. Much of the empirical research on teacher knowledge falls either the domain of content knowledge, pedagogical knowledge, or the intersection of these two conceptualizations. This intersection, defined as Pedagogical Content Knowledge (PCK), was spearheaded by the work of Shulman (1986). Pedagogical content knowledge is defined as "pedagogical knowledge that goes beyond knowledge of subject matter per se to the dimension of subject matter knowledge for teaching" (Shulman, 1986, p.9). This work was general to the context of teaching
and paved the way for future, subject specific conceptualizations. One such notable example is that of Mathematical Knowledge for Teaching (MKT) (Ball and Bass, 2002; Ball et al., 2008; Hill et al., 2005, 2008). Deborah Ball and her colleagues have been pioneers in the study and measurement of mathematics teachers' MKT, specifically at the elementary level.

However, their framework for MKT may not be appropriate or transferable to the context of secondary mathematics teachers, who have extensive mathematical expertise beyond the school curriculum. As such, the MKT framework of Silverman and Thompson (2008)was utilized to understand the development of MKT for prospective teachers in this study. The use of this framework will be justified in Chapter 2. This particular framework utilizes the work of Piaget and his conceptualization of reflective abstraction (Piaget, 1970, 1985).

Given that my chosen MKT framework is built upon the work of Piaget, it is important to note that my work situates itself within a constructivist theory of learning. The constructivist paradigm foundations itself on the assumption that knowledge is constructed through experience and the processes of assimilation and accommodation (Piaget, 1970). With much of Piaget's later work landing in the domain of mathematical knowledge, the distinction between physical knowledge based on experience and non-physical knowledge (i.e. logical and mathematical structures) was of particular importance. He questioned how one's knowledge of the abstract, that which cannot be directly experienced, is derived. Piaget suggested that knowledge of abstract logical-mathematical structures are acquired through simple abstraction and reflective abstraction (Piaget, 1970).

The constructivist paradigm being situated within experience aligns itself well with my chosen qualitative, case study methodology. Through engaging in one-on-one interviews, I wanted to understand the experience and knowledge of my participants by living an experience along side them. Moreover, the beliefs and values which I personally hold about teaching and learning land in the domain of a constructivist theory. As such, I hold the belief that teachers of mathematics should be "guides on the side," facilitating rich discussions catered to the individual needs of their students.

### 1.6 Personal Statement

Throughout elementary and secondary school, I did not consider mathematics to be a subject of personal interest. Growing up with both of my parents as artists, I valued creativity and imagination; I loved painting, dancing, choreographing, and creating music, all of which challenged my creativity and imagination. My mathematics learning, on the other hand, seemed to be devoid of these characteristics. The emphasis was on speed, not understanding; memorization, rather than creation. I can recall asking my teachers why we followed certain rules and algorithms, only to be told "that is just the way you do it." These responses left me dissatisfied and uninterested, but my parents continued to emphasize the importance of a strong mathematics education. So, I stuck through it, followed the norms of rote memorization, associating my mathematics education with a pathway to success.

At seventeen, I enrolled in courses at the community college. I had spent the last eight years of my life pursuing a career in classical ballet, but my parents con-
tinued to emphasize the importance of an education, particularly if I wanted to pursue a career in dance. The program I was enrolled in counted my community college courses towards a high school diploma, as well as an Associate's Degree from the college. In not knowing what I wanted to study, I took a mixed bag of courses; psychology, art history, foreign language, science, and mathematics. The mathematics course I took was Pre-Calculus. It was a class of about twenty-five students with a professor who taught both at the community college and at the local public university. Being the keen student that I was, I sat myself in the front row of the classroom, assuming that my old ways of doing mathematics would get me through. However, I was surprised by the cultural shift in learning mathematics that I experienced. Thinking mathematically was transformed from being about the destination to being about the journey; creativity and imagination were now valued in learning and doing new mathematics. Some of my questions were answered with responses well beyond the context of the course, but it brought me to see mathematics as something so much bigger and more beautiful than what I perceived it to be for so many years.

This shift in learning and doing mathematics acted as the impetus for pursuing my Bachelor's and Master's degrees in mathematics. When I taught my first course in the fall of 2013, I was brought back to sitting in my Pre-Calculus class years ago and wondering what that professor did to help me become so interested in mathematics. I had years of advanced mathematical coursework under my belt, so I wondered how I could make that accessible to my students so that I might spark their curiosity, as my pre-calculus professor did for me. The culture in my classroom was one of inquiry, and my students would often ask questions beyond the
horizon of the course. However, I did not want to leave them with a reply similar to the ones I received in high school. I was challenged to think beyond the content of the course to my advanced study, and think of how I could use that knowledge to help them better understand the content of our course. This reflective process brought me to have a deeper understanding of the mathematics I was teaching, as well as building a classroom where creativity and imagination were of value.

My interest in the relationship between school and advanced mathematical knowledge stems from these experiences. It is my hope that my current research will begin to bridge this gap and create more dynamic relationships between faculties of education and departments of mathematics. Both of these groups have opportunities to learn and grow with each other. My hope is that one day, mathematics educators could collaborate with mathematics departments to transform advanced mathematics courses to better serve future teachers, while mathematicians could work with teacher educators to transform mathematics methods courses which draw upon and extend the mathematical expertise of future teachers. Together, we can work to enrich the mathematical knowledge of future teachers, and in turn, the mathematical education of students in both elementary and secondary school.

### 1.7 Organization

The front matter of this paper, chapters 1,2 , and 3 , shed light on the theoretical and methodological considerations of the study. In chapter 2, I discuss existing literature on teacher education. I examine the history of teacher education and certification, as well empirical studies measuring teachers' mathematical knowledge for
teaching (MKT). Through this analysis, I elaborate on the gap of an accepted theoretical framework for analyzing and measuring MKT, particularly in the context of secondary mathematics teachers. I provide justification for my chosen theoretical framework of MKT (Silverman and Thompson, 2008).

Chapter 3 examines the methodological considerations of this research, including extended background on all five participants. In Chapters 4, 5, 6, and 7, I examine results of the task-based interviews, through the categorization of mathematically related tasks. At the beginning of each of these chapters, I begin with an overview of some of the connections I have made in my own studies between secondary and post-secondary mathematics. Finally, Chapter 8 discusses the results from the previous chapters by interpreting them within the context of my theoretical framework. More specifically, this chapter will examine the development of participants' MKT in the context of their studies in mathematics and teacher education. The chapter concludes with a discussion on limitations, extensions for the future, and a discussion of the impact of this work for departments of mathematics and faculties of education.

## Chapter 2

## Literature Review and

## Theoretical Framework

Teacher certification exams dating back to the mid-1800s reveal that there has been a long-term interest in teacher knowledge and quality (Shulman, 1986). What knowledge should teachers have? How deeply should they understand the content they are to teach? How would a teacher handle a particular situation in the classroom - pedagogical or otherwise? These are questions that are debated in the teacher education literature to this day. Although there is a great deal of interest in teachers' general pedagogical knowledge, the past three decades have seen a growing body of literature specifically dedicated to teachers of mathematics. In particular, researchers have found themselves concerned with the mathematical knowledge used in teaching mathematics. To this end, the following chapter aims to provide a comprehensive review of this subject: mathematics teachers' Mathematical Knowledge for Teaching (МКТ).

Through this chapter, we will explore the existing literature on secondary mathematics teachers' knowledge. As will be described, early studies focused on mathematics knowledge, but more recent research has focused on frameworks for understanding teachers' knowledge. I will begin by going back centuries to understand and look at the historical development of teacher knowledge in general, noting the pendulum effect through the years; that is, concerns in teachers' knowledge have continually swung back and forth between content knowledge and pedagogical knowledge, creating confusion as to what knowledge is valuable for teachers to have. Next, I will look at the combination of the two in Shulman's (1986) concept of Pedagogical Content Knowledge (РСК).

Finally, I will turn my attention to the knowledge of mathematics teachers, a hot topic among education researchers due to concerns of declining North American mathematics scores at the international level (No Child Left Behind Act of 2001, 2019; Richards, 2014). I will examine various frameworks for "mathematical knowledge for teaching" (MKT), developed as an extension of Shulman's PCK for mathematics teachers. However, as will be argued, the integration of advanced mathematical knowledge is not embedded in this theory, which has lead to alternative theories of MKT, such as that of Silverman and Thompson (2008). I will describe three alternative frameworks for secondary mathematics teachers' knowledge and justify the choice of my framework for this study.

### 2.1 Teachers' Content Knowledge: Beginnings

The historical discussion that follows is drawn from Shulman (1986). In his work Ramus, Method and the Decay of Dialogue, Ong (1958) describes the importance of pedagogy within the medieval university. The environment was one in which "content and pedagogy were part of one indistinguishable body of understanding" (Shulman, 1986, p. 3). Ong asserts that the defining characteristic of rich subject matter understanding was indicated by a students' ability to teach via lecture and discussion. To this day, in order to receive the academic title of "doctor" or "master," one must demonstrate their ability to lead a lecture and discussion during their defense. Even a millennia ago, Aristotle stated the following regarding the nature of knowledge:

Broadly speaking, what distinguishes the man who knows from the ignorant man is an ability to teach, and this is why we hold that art and not experience has the character of genuine knowledge (episteme) - namely, that artists can teach and others (i.e. those who have not acquired an art by study but have merely picked up some skill empirically) cannot. (Shulman, 1986, p. 4)

In the mid-1800s, the majority of examinations for school teachers focused on subject matter knowledge (Shulman, 1986, p. 2). Excellence in teaching was defined by a teacher's mastery of content, while pedagogical knowledge was a secondary concern. Shinkfield and Stufflebeam (2012) provide a detailed account of teacher evaluation in the first half of the 20th century, remarking that very few schools engaged in the formal evaluation of their teachers. Despite limited developments in evaluation, emergent teacher education policies of the 1980s were in
stark contrast to those of the 1870s. A teacher's capacity to teach was then defined by their knowledge of pedagogical practices and basic subject matter knowledge. Shulman argues that the transition towards valuing pedagogical practice over subject matter was partly due to policymakers' decisions being based on educational studies, that themselves ignored subject matter (Shulman, 1986, p. 3). Thus, the evaluation of content knowledge receded, being overtaken by the evaluation of effective teaching practices as defined in various "process-product studies" (Shulman, 1986, p. 3).

The 1983 document A Nation at Risk painted a rather doom and gloom picture of the existing status of the United States education system, claiming that "if an unfriendly foreign power had attempted to impose on America the mediocre educational performance that exists today, we might well have viewed it as an act of war" (National Commission on Excellence in Education, 1983, p.5). The document included an overview of the risks that the U.S. education system faced, evidence from various sources including declining SAT scores, as well as five recommendations for improving the existing system. One of these pertained to improvement in the quality of teacher preparation, with a substantial focus on subject matter knowledge. This recommendation stemmed from a criticism that "the teacher preparation curriculum is weighted heavily with courses in "educational methods" at the expense of courses in subjects to be taught" (National Commission on Excellence in Education, 1983, p.22). With the release of Nation at Risk came reforms such as the National Science Teachers Association's 1984 recommendation that all secondary school science teachers have a minimum of 50 credit hours of science course work at the university level (Weiss, 1987, p.76). Almost two decades later, the No Child

Left Behind Act of 2001 called for teachers to be fully certified in their subject specialization (No Child Left Behind Act of 2001, 2019).

Canada has also experienced pressure in reforms of teacher education (Russell and Dillon, 2015; Sheehan and Fullan, 1995). Although teacher certification is controlled by individual provincial governments, each province requires that secondary teachers have an undergraduate degree in a teachable subject. In Ontario, the 1960 s brought forth a shift in which teacher education was moved from teacher colleges to Faculties of Education, initiating the present notion of secondary subject specialization (Kitchen and Petrarca, 2013). More recently, the Ontario government announced that certified teachers will require both an undergraduate degree and a 4-term teacher education program (Kitchen and Petrarca, 2013), further emphasizing the value they place on pedagogical knowledge. In Quebec, the government mandates that candidates for a teaching diploma in general education at the secondary level have at least 45 credit hours of university coursework in a basic school subject, as well as a 4-year Bachelor of Education program with over 700 hours of practicum (Gouvernement du Québec, 2011; Russell and Dillon, 2015). All of these policies encourage secondary teachers to have subject specific content knowledge; however, once a teacher is hired by a school, they may be allowed to teach other subjects. This is particularly the case in British Columbia where teaching certificates do not signify grade level or subject specialization (British Columbia Ministry of Education, 2016). Nonetheless, all of these reforms point to sustained political pressure to have teachers more educated in their subject area.

Shulman's (1986) notion of pedagogical content knowledge (PCK) acted as a
medium for reconciling concerns about teachers' content knowledge and concerns of teachers' pedagogical practices. For Shulman, the construct of PCK extends content knowledge to content knowledge for teaching. Content knowledge is still at the forefront, but is placed in the context of how teachers navigate such knowledge throughout their teaching. Shulman (1986) emphasized that there was a lack of research focusing on the relationship between content and pedagogy and suggested that teacher education researchers should begin to explore this newly defined terrain. Following Shulman's work, an entire body of work dedicated to PCK (Grossman, 1990; Wilson et al., 1987), extensions to particular subject areas, and more recently the notion of technological pedagogical content knowledge (TPCK) has emerged (Koehler and Mishra, 2014). This general work in teacher knowledge laid the groundwork for mathematics specific research on the relationship between content and pedagogy (Depaepe et al., 2013).

In particular, Ernest (1989) worked to extend Shulman's notion of PCK to a more detailed framework for mathematics teachers. Similar to Shulman, Ernest (1989) argues that mathematics teachers should have both a curricular and pedagogical understanding of mathematics. These constructs mirror that of Shulman's notions of curriculum knowledge and PCK, respectively. What distinguishes Ernest's framework from Shulman's is the attention to the beliefs and attitudes of teachers and the impact on their practice in the classroom. Ernest argues that this system is most likely unique to individual teachers and exists as a product of the individual's "view or conception of the nature of mathematics, model or view of the nature of mathematics teaching," and "model or view of the process of learning mathematics" (Ernest, 1989, p.250). In the context of secondary mathematics
teachers, one might ask the following question: what do prospective secondary mathematics teachers perceive as the role of their advanced mathematics knowledge in their development as teachers? This is a question we will investigate through this study.

### 2.2 Mathematical Knowledge for Teaching

Nearly a decade before Shulman initiated work on PCK, Begle (1979) examined secondary mathematics teachers' mathematical knowledge. His work was undertaken to provide "guidance to those interested in conducting comprehensive reviews of the factual information which exists about the effects of various variables on student learning of mathematics" (Begle, 1979, p. xv). In his review of the empirical literature on mathematics teachers between 1960-1976, Begle (1979) suggested that there was no direct correlation between student success and the number of mathematics courses taken by their teachers. In an effort to support this, Monk (1994) examined secondary mathematics teachers and the effect that various university coursework had on their pupils' improvement in mathematics. Using quantitative measures, Monk found a minor positive relationship between the number of mathematics courses taken and student improvement (Monk, 1994, p.130). Perhaps more interestingly, he also found that the number of courses in mathematics pedagogy had a more positive effect on student learning than increased undergraduate coursework in mathematics (Monk, 1994, p.130). Adding further murkiness to the water, the National Centre for Research on Teacher Education (NCRTE) claimed that teachers with undergraduate majors in the subject they teach did not outperform other teachers in their explanations of fundamental concepts (National Center for Research on Teacher Education, 1987). In their study of prospective
teachers' mathematical content knowledge, Kahan et al. (2003) recognized that the most effective lesson plans from the prospective mathematics teachers in their study were not necessarily those with the highest grade points averages in mathematical coursework.

Indeed, the four decades of work following Begle (1979) have brought forth no common consensus as to what extent university mathematics coursework affects pupils' learning. These inconsistencies have brought researchers to look beyond subject matter knowledge and consider the interplay between content and pedagogy in the mathematics classroom. In an effort to address this question, Deborah Ball and her colleagues extend Shulman's notion of PCK to the teaching of mathematics (Ball, 1988, 1990; Ball and McDiarmid, 1990). Deborah Ball is arguably the pioneer in research on the ways in which teachers of mathematics must know, understand, and teach the mathematical knowledge at stake in the school curriculum (Ball et al., 2005, 2001; Hill et al., 2008). Along with her colleagues, Ball has conducted numerous studies in an attempt to describe what "teachers do in teaching mathematics" (Ball et al., 2005, p.17). In their later work, they define Mathematical Knowledge for Teaching (MKT) as mathematical knowledge that is used in teaching mathematics (Ball and Bass, 2002, p.5).

In their most detailed description of MKT, Ball et al. (2008) take the categories of PCK, content knowledge, and curricular knowledge as defined by Shulman (1986) and subdivide them into more well-defined subcategories (see Figure 2.1). They conceptualize MKT as having two dimensions: Subject matter knowledge and PCK. These dimensions break down further in their framework. Subject


Figure 2.1: Mathematical Knowledge for Teaching (MKT)
matter knowledge is subdivided into Specialized Content Knowledge (SCK), Common Content Knowledge (CCK), and Horizon Content Knowledge (HCK). The authors define CCK as mathematical knowledge possessed by the average adult. This type of mathematical knowledge may be held by parents of students to help them determine whether or not their child has solved the problem correctly. SCK, on the other hand, is mathematical knowledge held by those beyond the average adult. This knowledge is independent of pedagogy. In order to have SCK, one must have a deeper understanding of mathematics so that they may make modifications or build connections between content.

Finally, the notion of Horizon Content Knowledge (НСк) is a component of teachers' knowledge that recognizes how mathematical topics are interrelated across the mathematics curriculum. Since HCK is a domain of MKT, HCK could be considered an awareness of mathematics beyond the horizon of the curriculum that
impacts teachers' practice (Jakobsen et al., 2013). Advanced mathematics is certainly beyond the horizon of the secondary curriculum, so does taking a course in abstract algebra contribute to a teacher's HCK?

Taking a course in abstract algebra certainly increases a teacher's mathematical content knowledge. After taking such a course, students learn about the mathematical structures of groups and rings, Galois Theory, and extend the notion of linear algebra to arbitrary fields, among other concepts. However, for this knowledge to be included in the domain of HCK, the teacher should be able to recognize the relationships between the concepts in abstract algebra and the secondary curriculum in a way that influences their pedagogy. By recognizing the relationships between advanced subject matter and their previously constructed knowledge of school mathematics, the teacher has made a conceptual advance in their understanding of the elementary concept. For example, consider the case of irreducible polynomials. In high school, students learn that real-valued polynomials can factor into linear and quadratic polynomials. In abstract algebra, however, students learn that these are irreducible elements in the polynomial ring $\mathbb{R}[x]$ and are then able to work with them in a new, more abstract setting. This notion of a conceptual advance that allows one to see a concept through a new lens is what Simon (2006) defines as a key developmental understanding Key Developmental Understandings (KDU). We return to this construct later in this chapter.

As with PCK, the MKT framework takes into account mathematical content knowledge, as well as pedagogical knowledge, and seeks to gain insight into the mathematical work of teachers. Their framework for MKT has been widely used
in empirical studies with elementary mathematics teachers, since the framework itself emerged from practice-based research with elementary teachers. At the elementary level, Hill et al. (2005) found that teachers' knowledge of teaching mathematics was the strongest predictor of student achievement in grades one and three. With these results in mind, Matthews et al. (2010) examined the effect of specialized courses for teaching elementary mathematics to a group of pre-service elementary teachers. Results showed that the teachers in the specialized courses had higher mathematical content knowledge than those who took standard mathematics courses (Matthews et al., 2010, p.7). These results demonstrate that understanding teachers' MKT is both a fruitful and practical endeavour.

The work of Deborah Ball is exceptional in its focus and reach. Her research has transformed the landscape of literature and progress in mathematics teacher knowledge. Having an understanding of what is "in" MKT has served as a launch pad for numerous professional development initiatives for mathematics teachers (Clarke, 2007), as well as inspiring change in the pre-service education of future mathematics teachers (Simon, 2008). Her substantial impact can be seen through the large number of citations of her work; one article, Ball et al. (2008), has been referenced nearly 6,000 times. While the work was initially done in the context of elementary mathematics, it has reached beyond the elementary setting to secondary and post-secondary mathematics (Artzt et al., 2012; Goos, 2013; Tchoshanov et al., 2017). One research direction which has stemmed from Ball's MKT work is that of Advanced Mathematics Knowledge in Teaching, which we explore in the following section.

### 2.3 Advanced Mathematics Knowledge in Teaching

Research on advanced mathematics knowledge of secondary teachers is a new and upcoming field of study. Before we begin a discussion on advanced mathematical knowledge, it is important to define what is meant by it. I borrow the definition of Advanced Mathematical Knowledge (AMK) from Zazkis and Leikin (2010), who define AMK as " knowledge of the subject matter acquired in mathematics courses taken as part of a degree from a university or college" (Zazkis and Leikin, 2010, p.264). However, the roots of relationships between elementary and advanced mathematics date back to over a half a century ago. In his 1939 work Elementary Mathematics from an Advanced Standpoint, mathematician Felix Klein wrote to the teacher who found themselves teaching in the "time honoured way" and whose "university studies [in mathematics] remained only more or less pleasant memory which had no influence upon his teaching" (Klein, 2004). The purpose of the book was to explore elementary mathematics from the school curriculum with the assumption that the reader has extensive post-secondary mathematics expertise.

Mathematicians and non-mathematicians align themselves with the perspective that advanced mathematics knowledge is of value for practicing secondary teachers. In their 2002 report, the U.S. Department of Education made the bold claim that advanced subject matter experience influences teachers to conceptualize the secondary mathematics curriculum at a deeper level and provide a richer learning experience for students, leading to higher student achievement (Paige, 2002). A decade later, this claim still held strong, when the Conference Board of the Mathematical Sciences claimed that the knowledge of secondary mathematics teachers
should be well beyond the scope of the school curriculum and recommended that all secondary mathematics teachers have coursework in single and multi-variable calculus, introduction to linear algebra, statistics and probability, introduction to proof, abstract algebra, real analysis, modelling, differential equations, group theory, number theory, history of mathematics, geometry, complex analysis, and discrete mathematics (Conference Board of the Mathematical Sciences, 2012). This document provides detailed rationale for the inclusion of each of these topics in the mathematics education of future teachers. This list is almost exhaustive of some university's mathematics curriculum and suggests all secondary mathematics teachers should have a full major in mathematics.

Teacher education programs have taken strides to align themselves with such recommendations. Teacher education programs in Canada require that applicants who wish to specialize in secondary mathematics have at least 30 credits of mathematics coursework at the upper level. Even though advanced mathematics coursework is seen to be essential by mathematicians, researchers, and professional organizations, practicing teachers do not share the same sentiment. In their study of mathematics teachers' perceptions of AMK in teaching, Zazkis and Leikin (2010) found that teachers saw benefits from the skills learned in their undergraduate degrees, but saw limited value in content specifics. That is, teachers in the study valued their undergraduate mathematics experience for building their persistence in problem solving, building connections within the curriculum, and overall confidence. However, they perceived content connections between AMK and secondary mathematics as being very limited and non-essential to their teaching. Through this study, they conclude with a call for continued studies on the relationship between

AMK and MKT, as well as a more well-defined relationship between the two ( $\mathrm{Za}-$ zkis and Leikin, 2010).

In very recent work Wasserman and colleagues have been exploring the role of AMK and HCK of secondary mathematics teachers, with the belief that knowledge of mathematics at the horizon can be "impetus for additions or alterations to the teachers' instructional plans" (Wasserman and Stockton, 2013, p.22). In their initial work, they share two vignettes that show how AMK and HCK influence a teachers' pedagogy. For example, the second vignette demonstrates how extended knowledge of abstract algebra and group axioms could influence a teachers' lesson on linear equations. In particular, this knowledge may support lesson design so that students have time to reflect on important mathematics such as the existence of a particular identity or inverse, depending on the operation in question. Wasserman et al. (2017) follow up on this through an effort to make real analysis relevant to teachers, with the hope that such a model for teaching could help future teachers in "developing knowledge that is situated in professional practices and that they will understand as valuable and be able to use in their daily work with students" (Wasserman et al., 2017, 574).

As this section elaborates, the domain of understanding the role of AMK in secondary teachers' practice is a new and developing field. In the section that follows, we examine less recent literature on teacher knowledge, where the focus is on content knowledge at the secondary or post-secondary level, not the intersection of the two.

### 2.4 Studies on Teacher Knowledge

The vast majority of research on teachers' MKT exists at the elementary level. This is problematic since educational backgrounds of secondary and elementary teachers typically differ substantially. Secondary mathematics teachers are often required to take advanced mathematics coursework during their post-secondary degrees, while this is not a requirement of elementary teachers. To this day, there are few studies on teachers' MKT that investigate how this advanced mathematical knowledge is used in their practice. One of the earliest such studies was that of Even (1993), where she examined prospective secondary teachers' understanding of the function concept. The participants in her study were 162 pre-service secondary mathematics teachers who had completed the majority of the mathematical coursework required in their program. A questionnaire was distributed to the participants that included a variety of questions addressing subject matter knowledge of functions, as well as pedagogically focused questions on functions. Furthermore, a subset of the participants engaged in interviews regarding the function concept.

Results revealed that the prospective teachers in Even's study possessed a very limited conception of function. For example, seven out of the ten subjects who participated in the interview phase stated that all functions can be represented by a single symbolic formula, claiming that functions and equations "are the same thing" (Even, 1993, p.105). After her analysis, Even boldly remarks that the results of her study reveal "a situation in which secondary teachers at the end of the 20th century have a limited concept image of function similar to the one of the 18th
century" (Even, 1993, p. 112). Even concludes her work with a call for an emphasis on subject matter preparation in teacher education programs (Even, 1993, p. 113). She states that prospective teachers need an environment that fosters powerful mathematical understandings that can be useful in the teaching of mathematics. In a response to Even (1993), Wilson (1994) conducted a case study of a single prospective secondary mathematics teacher and the impact that a ten week course emphasizing mathematical content and pedagogy had on her understanding of the function concept. Throughout the study, the participant's understandings of the function concept saw significant development, suggesting perhaps unsurprisingly that courses which integrate content and pedagogy could be useful to prospective secondary mathematics teachers. Although it is concerning that some teachers have such limited understanding of secondary mathematics concepts, it is encouraging that significant improvement is attainable.

Following the work of Even (1993), Stump (1999) investigated prospective secondary teachers' understanding of slope. Slope is a fundamental concept in the secondary curriculum and "challenges the distinction between ratio and rate" (Stump, 1999, p.125), requiring that students have a solid understanding of proportional reasoning. Stump (1999) questioned whether the secondary teachers (prospective and in-service) in her study understood the complexities of the slope concept and whether secondary students' difficulties with slope (Barr, 1980, 1981) were present in these teachers. The study revealed that the teachers in question had a limited understanding of the slope concept. Both the pre-service and in-service teachers demonstrated misconceptions surrounding the concept of slope, were unable to answer questions relevant to the secondary curriculum, and lacked connections be-
tween various representations of slope. The results of Stump (1999) bring one to question how secondary mathematics teachers are to provide rich learning experiences for students if they do not have a rich understanding of the mathematics themselves. If mathematics teachers have difficulty with concepts such as functions and slope, how do they fare when it comes to more advanced mathematics, such as those involving proof? Such concerns motivate my own interest in a well developed framework for assessing MKT.

Knuth (2002) sought to answer this question in his article examining secondary school mathematics teachers' conceptions of proof. Through semistructured interviews and proof-focused tasks, Knuth (2002) explored in-service secondary mathematics teachers' conceptions of proof. Through his analysis, he uncovered that the participating teachers recognized and acknowledged the importance of proof in mathematics, but not in mathematical pedagogy. Knuth (2002) argued that these conceptions may exist due to teachers' previous experiences with proof at the secondary and tertiary level. In response to both instances, Knuth (2002) claimed that proof is a mere tool for verification and yields no personal meaning for students (Knuth, 2002, p.400). Furthermore, Knuth (2002) observed that many of the teachers in the study did not have a solid understanding of what constitutes a valid proof (Knuth, 2002, p.401); that is, the teachers in his study were unable to recognize what features distinguish a correct proof from an incorrect one. With teachers having such limited conceptions of proof, he suggests that "university mathematics professors perhaps play the more significant role in shaping teachers' conceptions of proof" (Knuth, 2002, p. 403). Although it may be reasonable to state that university mathematics courses and professors play a significant role in
the development of teachers' conceptions of proof, is it reasonable to assume that mathematics professors see "proof as a meaningful tool for studying and learning mathematics?" (Knuth, 2002, p. 403) What do university mathematics professors see as the role of proof in their classrooms and in their pedagogy? Regardless, the issue of in-service teachers having limited conceptions of proof that Knuth (2002) unveils reiterates the potential post-secondary courses in mathematics could have for secondary teachers' practice.

Following the work of Knuth (2002), Schwarz et al. (2008) conducted a comparative case study on prospective secondary mathematics teachers' knowledge of proof in Germany, Hong Kong, and Australia. Within this study, the researchers were concerned with future teachers' "professional competencies" in the domain of argumentation and proof (Schwarz et al., 2008, p. 792). Similar to Knuth (2002), these researchers were not only interested in participants' ability to execute proofs requiring only secondary level mathematics, but they also probed participants' positionality on the role of proofs in mathematics lessons at the secondary level (Schwarz et al., 2008, p. 793). An open-ended questionnaire was used to examine the various facets and connections among participants' knowledge and interviews were conducted with selected volunteer student teachers afterwards. Overall, 186 prospective teachers from the three countries completed the questionnaire.

Results from the questionnaire revealed that the majority of prospective teachers from all three countries were unable to produce formal proofs from the secondary curriculum and did not succeed at recognizing whether a given mathematical proof was valid or not (Schwarz et al., 2008, p. 807). Their analysis of
questions pertaining to beliefs about the nature of proof revealed that the majority of participants had a high affinity towards utilizing proof to understand more advanced mathematical content. These results directly contradict the finding from Knuth (2002), that the teachers in his study did not mention proof having a significant role in promoting mathematical understanding (Knuth, 2002, p.400). This brings one to question how each of these authors defines "affinity." If the teachers in Knuth's study had participated in the study of Schwarz et al. (2008), would we see this same result? These contradictory results could be due in part to inconsistencies in the theoretical framings of these studies, an issue which we turn our attention to in the next section.

In an effort to understand the dynamic between advanced mathematical knowledge and the school curriculum, Cofer (2015) examined prospective secondary mathematics teachers' understanding of abstract algebra concepts which implicitly appear in the secondary curriculum. Cofer (2015) utilized Ball and colleagues' theoretical conceptualization of MKT in order to identify the mathematical content knowledge of her participants and how that knowledge affected their pedagogical choices. She found that many of the prospective teachers in her study were unable to make meaningful connections between school algebra and university algebra. For example, when asked questions regarding even numbers, participants were unable to make any connections between the abstract definition of even numbers being a subgroup of the integers and the elementary school definition of even number. In fact, there were multiple participants who were unable to provide any accurate definition for an even number. The results from $\operatorname{Cofer}$ (2015) indicate that there is a need for research that examines the connections between school and ter-
tiary mathematics and how that knowledge is visited in teacher education. Indeed, Suominen (2015) argues that undergraduate texts in abstract algebra are lacking in explicit connections between secondary and university mathematics. Students enrolled in abstract algebra courses often exit the course unable to comprehend the concepts studied and find themselves unable to connect the concepts within their existing mathematical understandings (Zazkis and Leikin, 2010). In order to move beyond this issue, Suominen contends that abstract algebra "can no longer be considered simply as the generalization of school algebra but rather it should be regarded as an extension of previous mathematical knowledge from algebra and geometry" (Suominen, 2015, p.79).

### 2.5 Frameworks for Teacher Knowledge

Although each of the studies outlined above examine the mathematical knowledge of prospective secondary teachers, they are primarily concerned with mathematical content knowledge. The role that content knowledge plays in pedagogy is a secondary concern. To this end, there is very little empirical research. While most of these studies make mention of Shulman's theoretical construct of PCK, only one utilizes Ball's framework of MKT. Why is this the case? One answer might arise from the how this particular MKT framework was developed. As elaborated upon in Section 2.2, Ball and colleagues' framework originated in the context of elementary school teachers' mathematical work in practice. Based on this research, a distinction was made between common content knowledge (CCK) and specialized content knowledge (SCK). Ball et al. (2008) define CCK as the mathematical "knowledge of a kind used in a wide variety of settings - in other words not unique
to teaching; these are not specialized understandings but are questions that typically would be answerable by others who know mathematics" (Ball et al., 2008, p. 399). Furthermore, they define SCK as "the mathematical knowledge 'entailed by teaching' - in other words, mathematical knowledge needed to perform the recurrent tasks of teaching mathematics to students" (Ball et al., 2008, p. 399). These definitions have been widely used and accepted by the elementary mathematics education community and continue to influence a growing body of research (Austin, 2015; Copur-Gencturk and Lubienski, 2013; Ottmar et al., 2015).

It is important to ask whether these constructs are transferable to the context of secondary mathematics teachers. Do these descriptions of content knowledge represent the knowledge of secondary mathematics teachers as well as they do for elementary teachers? Is it reasonable to generalize the MKT framework from one context to another? The actions of unpacking and connecting mathematical concepts are necessary tasks for mathematics teachers, but are not necessary for the average mathematically literate person. For example, elementary mathematics teachers need to know of various methods to motivate the idea of place value in arithmetic computations; this is knowledge that the average person does not have. Thus, within the MKT framework, such work falls under the umbrella of SCK. However, if one considers a secondary teacher who has a degree in mathematics, the work of connecting and unpacking mathematics may no longer be considered unique to teaching. That is, the definition of SCK may need to be modified to account for the mathematical experiences of secondary teachers. The work of Ball and her colleagues is valuable because it extends Shulman's framework of PCK to the context of mathematics teachers and links pedagogy, mathematical knowledge,
and student success. However, extending this framework to secondary teachers may require more care than is initially evident.

There have been a number of efforts to evaluate MKT at the secondary level, with theoretical frameworks borrowing heavily from Ball et al. (2008). Among a number of such efforts are the Knowledge of Algebra for Teaching (KAT) project (McCrory et al., 2012), the SimCalc rate and proportionality teaching survey (Shechtman et al., 2006), and the High School Mathematics from an Advanced Standpoint (HSMPAS) Project (Usiskin et al. 2001). Although each of these projects seeks to broaden the understanding of MKT at the secondary level, the intent of each project differs. The KAT project seeks to refine the MKT framework of Ball and colleagues, with KAT's original purpose being to unravel the knowledge used in teaching algebra. Eventually, the KAT project evolved into an instrument used to evaluate secondary teachers' knowledge of teaching algebra (McCrory et al., 2012, p. 591). The SimCalc project is concerned with understanding the ways in which teachers integrate technology to help their students understand conceptually demanding mathematics. In contrast to the KAT project, the SimCalc project is not concerned with the development of theory. Instead, it assumes Ball's MKT framework and uses it as a basis to evaluate the effects of technology integration on teachers' understanding of the mathematics they teach. Finally, the HSMPAS focuses on curriculum development for prospective secondary teachers. Similar to SimCalc, HSMPAS uses Ball's conceptualization of MKT as given, but assumes that advanced mathematical knowledge from post-secondary degrees should have an impact on secondary teachers' mathematical knowledge for teaching.

All of these projects recognize that the teaching of mathematics is a form of "mathematical work" (Ball and Bass, 2002, p.13) and that teachers' knowledge a multi-dimensional, complex network interweaving content and pedagogy. The large number of distinct projects that specifically examine secondary MKT reveals an extensive interest in the phenomena. Unfortunately, with all of these distinct projects comes a disconnection with respect to theory:

It is in the arena of underlying theory, however, that these instruments differ most. Despite claiming to cover roughly the same terrain, these projects have strikingly different approaches to specifying domains of measurement - in essence, different approaches to organizing what is "in" mathematical knowledge for teaching. (Hill et al., 2007, p.131)

The disparity among these theories makes it difficult to link various results in the research literature. If the goal among researchers is to pursue progress in the domain of secondary MKT, I believe that one of the first steps should be to have a theoretical framework for which researchers can agree upon. A more ambitious goal would be the development of an MKT framework which is accepted not just within the secondary context, but in the elementary and tertiary as well. One such framework that might fit these credentials is that of Silverman and Thompson (2008) and their framework for MKT.

### 2.5.1 Framework of Silverman and Thompson

The MKT conceptualization of Silverman and Thompson (2008) was developed while bearing in mind that while a great deal of research exists on the knowledge of mathematics teachers, there is not a "commonly accepted theoretical framework for research in mathematics teacher education" (Silverman and Thompson, 2008, p.501).

In particular, existing frameworks focus on what is "in" MKT. Silverman and Thompson (2008) distinguish their framework from others in its focus. Rather than understanding and quantifying the "what" of MKT, Silverman and Thompson (2008) interrogate how such knowledge is developed and what experiences are necessary to develop understandings that can be pedagogically powerful. The primary inquiry of their work asks what cognitive processes and understandings contribute to the development of MKT. The authors position their work and personal epistemologies in a constructivist paradigm. As a foundation, they utilize Simon's notion of a key developmental understanding (KDU) (Simon, 2006), as well as Piaget's concept of reflective abstraction (Piaget, 1985). Both of these are theoretical frameworks that, on their own, are disciplinary orientations for research on the construction of mathematical understanding. Since Silverman and Thompson's framework is not independent of other frames, it is necessary to have an understanding of KDUs and reflective abstraction in order to fully understand this conceptualization of MKT.

## Key Developmental Understandings

Modern changes to the mathematics curriculum in Canada and the United States have brought forth an increased focus to the conceptual nature of mathematics and towards building student understanding of these concepts. However, the development of such understandings is difficult for students and for the teachers working to foster them. Simon (2006) elaborates that the KDU construct is meant to serve as a way to identify "critical transitions that are essential for students' mathematical development" (Simon, 2006, p. 360). In an effort to distinguish these transitions as "critical," Simon remarks that KDUs have two major characteristics. First, they must involve the students making a "conceptual advance" in their understanding. That is, once a KDU of a particular concept is constructed, students are then able to make more mathematical connections and think about the concept in ways they have not before. Simon provides the example of students transitioning from understanding a fraction, such as $\frac{1}{5}$, as a piece of a whole, to being an independent quantity; this is a "conceptual advance" which allows students to work with fractions in a new context. The second characteristic of a KDU is that students do not acquire it from another person's explanation. The construction of a KDU is through the internal process of the learner whereby the student reflects on their own activity and experience.

The KDU construct can play a major role in mathematics pedagogy, both in instruction and assessment. If fostering the development of KDUs becomes a priority for a teacher, they could plan their lessons in a way that encourages students to make the necessary conceptual advances. These lessons should be interactive,
inductive, and allow students the space to make their own realizations about the mathematical knowledge at stake. Since one must coordinate assessment with instruction, how might one assess whether a student has achieved particular KDU? Turning once again to the KDU of fraction as quantity, Simon presents Figure 2.2 as an example of a question that assesses whether or not a student has made this conceptual advance. This diagram could be associated to a question in which a student must identify if the shaded quantity represents $\frac{1}{4}$ of the whole.


Figure 2.2: Assessment item for fraction as quantity (Simon, 2006, p.368)

The notion of a KDU in the context of AMK is an evolving domain of research. Wasserman and colleagues have been examining the role of such personally powerful understandings in the MKT of prospective and in-service mathematics teachers (Wasserman, 2016; Wasserman et al., 2017). To give the reader a better sense of what a KDU in this domain looks like, I present the following example. Exponential functions are a fundamental concept in the upper secondary curriculum, due to their value in modelling of various real-world phenomena. The notion of an exponent is first studied as repeated multiplication, when the exponent is a whole number and then easily extended to the context of rational exponents. However, the extension to irrational exponents is never explicitly discussed until a univer-
sity course in real analysis, while the use of irrational exponents is used in the secondary curriculum. In order to fully understand how to work with irrational exponents, a learner must have an understanding of irrational numbers beyond the definition of "not rational." Indeed, irrational numbers can be thought of as the limit of a sequence of rationals. Such an understanding and conceptualization of irrational numbers would allow the learner to work with irrational exponents beyond mere approximations.

## Reflective Abstraction

Jean Piaget's work in explicating the possible mechanisms behind children's thinking often found itself grounded in mathematically oriented tasks (Beth and Piaget, 2013). In his later work on the theory of genetic epistemology, Piaget's focus was on understanding the formation of logical structures in children (Piaget, 1970). The distinction between physical knowledge based on experience and non-physical knowledge (i.e. logical and mathematical structures) was of particular importance to Piaget, as he questioned how one's knowledge of the abstract, that which cannot be directly experienced, is derived. Piaget suggested that knowledge of abstract logical-mathematical structures are acquired through simple abstraction and reflective abstraction (Piaget, 1970).

Piaget defined simple abstraction, which he later termed empirical abstraction, as generalizations "drawn directly from external objects" (Piaget, 1980, p. 89). For example, a child may abstract a relationship between weight and size by holding a different object in each hand. They may realize that larger objects imply heav-
ier weight, but may also realize that a larger object need not be heavier. These abstractions are based solely on experience with physical objects, and thus represents an empirical abstraction. Although some mathematical knowledge may be constructed in this way, Piaget argued that the majority of logical-mathematical knowledge is not constructed through experience with physical objects. Rather, it is derived through the coordination of actions performed on physical objects. Piaget defines abstract knowledge constructed in this manner to be a product of reflective abstraction.

Piaget (1980) considered reflective abstraction to be the mental process in which humans construct new knowledge without having a direct interaction with physical objects. The "reflective" aspect of this abstraction has two-dimensions, both based off separate meanings of the word "reflection." Firstly, through reflective abstraction, the knower is projecting their knowledge at one level to a level of increased abstraction, just as light projects off a mirror. The second dimension is the reorganization of knowledge from reflexive thought. Since the knowledge projected originates from a lower level, the knower must reconstruct the abstractions from the lower level so that their knowledge connects within the structure of the higher level of abstraction.

Numerous scholars have recognized the importance of reflective abstraction in the study of mathematics teaching and learning. Although the notion of reflective abstraction was developed in the context of the logical development in children, Piaget himself observed that reflective abstraction could be the logical mechanism that has influenced the historical development of mathematics as a field (Piaget,
1985). Ed Dubinsky bases his theoretical framework for mathematics learning at the postsecondary level on reflective abstraction, arguing that "Piaget's ideas can be extended and reorganized to form a general theory of mathematical knowledge and its acquisition which is applicable to those mathematical ideas that begin to appear at the postsecondary level" (Dubinsky, 2002, p.96). In his thesis on intellectual development in mathematics education and instruction, Brun (1975) remarked that the primary goal of instruction in mathematics should be the fostering of opportunities for reflective abstraction. Despite the significance of reflective abstraction and its accompanying constructivist paradigm in literature on mathematics teaching and learning, critiques do exist. Those who align themselves more with the cultural psychology of Vygotsky (Kozulin, 1990) claim that reflective abstraction and the constructivist paradigm provides a limited view of mathematics learning (Cobb et al., 1992), disregards intersubjectivity (Lerman, 1996), and does not take into account sociocultural theories of teaching and learning (Lerman, 2000).

### 2.5.2 Mathematical Knowledge for Teaching

Silverman and Thompson's conceptualization of MKT is based on the argument that, although a great deal of research exists on the knowledge of mathematics teachers, there is not a "commonly accepted theoretical framework for research in mathematics teacher education" (Silverman and Thompson, 2008, p. 501). The authors contend that the majority of research surrounding mathematics teachers has been centred around what mathematics teachers need to know to teach mathematics (Silverman and Thompson, 2008, p.500). As such, the previous frameworks for MKT focus on the mathematical knowledge that allows teachers to interact
with both students and mathematics on a meaningful level. Although Silverman and Thompson (2008) agree that it is valuable to recognize the attributes of exemplary teaching, they question how teachers develop such knowledge. Thus, in their work, Silverman and Thompson (2008) transfer the focus of "mathematical reasoning, insight, understanding, and skill needed in teaching mathematics" (Silverman and Thompson, 2008, p.500) towards offering experiences that could lead to the transformation of a mathematical understanding "having pedagogical potential to an understanding that does have pedagogical power" (Silverman and Thompson, 2008, p.502).

The work in Silverman and Thompson (2008) situates itself in elementary and secondary mathematics. Contrary to the argument made in Section 2.5 of other MKT frameworks, the developmental MKT framework presented by Silverman and Thompson (2008) can be extended to post-secondary mathematical knowledge. In the context of my study, the goal is to understand how advanced mathematical knowledge contributes to MKT. The post-secondary mathematics curriculum is grounded in abstraction and generality, which as the literature suggests, can be a source of confusion for many students (Suominen, 2015). While the perspective taken in these courses is abstract and general, this does not imply that the content is irrelevant in the context of secondary mathematics. Indeed, Suominen (2015) argues that the content of post-secondary abstract algebra is an extension of secondary school algebra.

However, building connections between the abstract generalizations of postsecondary courses to the concrete context of secondary mathematics requires that
the learner develop their own, personal understandings between the content areas. For mathematics majors intending to continue into teacher education, it is possible that the content knowledge developed in advanced mathematics classes could be constructed in the lofty domain of abstraction and generalization, without grounding in existing content knowledge. If this is the case, the future teacher may be unable to communicate the relevance of this content in secondary mathematics. If connections are made between the content of advanced mathematics courses to the secondary curriculum, this knowledge has the potential to impact the way a teacher approaches particular topics in secondary mathematics. That is, the knowledge constructed may have the potential to have a powerful impact on pedagogical practice.

To give the reader context of how advanced mathematical knowledge could impact the pedagogical practice of future teachers, I present the following example. The notion of a Euclidean Domain is central to the study of rings in abstract algebra. Put simply, a Euclidean Domain is a structure where one can do division, with the familiar constructs of quotients and remainders. I refer the reader to Aluffi (2009) for a detailed description of Euclidean Domains, but in short, the structure of the division algorithm for the integers $(\mathbb{Z})$ and the ring of polynomials with coefficients in the real numbers $(\mathbb{R}[x])$ is identical. That is, dividing integers, like $786 \div 37$, is similar in process to the division of polynomials, such as $\left(x^{3}+\sqrt{2} x^{2}-4 x+\frac{3}{2}\right) \div\left(x^{2}+1\right)$.

Division of integers and the long division algorithm is a topic in the elementary curriculum, while the division of polynomials is a topic in upper secondary math-
ematics. If a secondary teacher has this understanding of the relationship between division algorithms in $\mathbb{Z}$ and $\mathbb{R}[x]$, it may impact their approach to teaching polynomial division. Indeed, one could rethink of 786 and 37 as $7 \cdot 10^{2}+8 \cdot 10^{1}+6$ and $3 \cdot 10+7$, respectively. This could provide a nice context for discussing the procedure of the division algorithm of polynomials, by taking into account what students already know about the division algorithm for integers. In this example, a teacher has personally powerful understanding (KDU) of the division of polynomials, which relates to their prior knowledge of integer long division, which in turn, could potentially impact their pedagogical practice.

The MKT framework of Silverman and Thompson (2008) presents how mathematical understandings with pedagogical potential (KDUs) transform into understandings with pedagogical power via reflective abstraction. This framework continually emphasizes "transformation" and "development," with the intention of understanding how teachers develop the exemplary teaching practices noted in Ball and Bass (2002) and Kahan et al. (2003). Rather than identifying the "what" in teaching, Silverman and Thompson hope to lead other researchers towards interrogating how prospective and in-service teachers develop MKT throughout their careers. In turn, their framework of MKT is intended to encourage and guide teacher educators towards designing teacher education programs that encourage teachers, regardless of whether they are in their first or thirtieth year of teaching, to think critically about their MKT. Thus, the development of MKT becomes the development of habits which examine one's mathematical knowledge, pedagogical practices, and the interplay between them.

## Cognitive Reliance?

Although Silverman and Thompson (2008) rely on the cognitive perspectives of Piaget (1985), I argue that their framework also offers space for sociocultural perspectives of mathematics instruction. Their developmental MKT framework offers teachers room to create their own classroom environments that may explore alternative ways of knowing and allow students to develop rich mathematical understandings. This is indicated by the intended classroom being a "dynamical space, one that will be propitious for individual growth in some intended direction, but will also allow for a variety of understandings that will fit with where individual students are at that moment of time" (Silverman and Thompson, 2008, p.507). In Simon's definition of key developmental understandings (KDU), he states that although KDUs and reflective abstraction derive from a cognitive perspective, "they do not conflict with social constructs such as negotiation of meaning and social and sociomathematical norms" (Simon, 2006, p. 364). He argues that the KDU construct coordinates cognitive and social perspectives of learning, so that research progress might be made on problematic questions such as internalization (Bereiter, 1985). Since Silverman and Thompson base their MKT framework on Simon's notion of a KDU, the case for their MKT framework leaving space for social theories is further justified.

However, one should be cautious in the coordination of social and cognitive theories of mathematics learning. One of the first assumptions in Piagetian theory is that language is a product of thought. Piaget (1970) himself argued that thought and logical structures exist in those who are without language. He used
the case of children who could not hear or speak to justify that there exists "welldeveloped logical thinking in these children even without language" (Piaget, 1970, p.46). However, Vygotskian theorists would argue the exact opposite; to them, language is the mechanism which forms thought. Indeed, Vygotsky believed that "a word is the microcosm of human consciousness" (Vygotsky, 1986, p. 256). The role of language in thought is central to both theories, but they stand on completely opposite grounds in regards to what role it plays. If one is to develop a philosophy of learning mathematics that coordinates these two perspectives, as many researchers have done (Burr, 2015; Confrey, 2002; Ernest, 1998), one must be aware of the underlying theoretical assumptions of each. Indeed, attempts at integrating these two theories together may bring researchers to face theoretical roadblocks similar to the ones that physicists face in their attempts to combine quantum theory and general relativity. Although each of these theories are extremely powerful in their particular contexts, there are major obstacles when attempting to integrate one into the other (Lerman, 1996). However, with so many opposing arguments on this issue, I find it impossible to fully refute the integration of a social element into Piagetian constructivism, nor can I claim it as an unviable framework for the learning of mathematics. Thus, I am still able to justify Thompson and Silverman (2008) as a viable theoretical framework for my intended research.

### 2.6 Summary

Despite common goals, mathematicians and mathematics educators divide themselves not only within the confines of the university, but in research endeavours as well (Fried, 2014). In the Mathematical Association of America's (MAA) A Call
for Change, they state that "the mathematical preparation of teachers must provide experiences in which they develop an understanding of the interrelationships within mathematics and an appreciation of its unity" (Leitzel, 1991, p.3). Unfortunately, the divide between mathematics and education may not be conducive to such preparation. Mathematics departments and professors are responsible for building the mathematical knowledge of future teachers, while education departments support future teachers with subject-specific courses in pedagogy. When, how and where can prospective teachers build connections between their advanced mathematical knowledge and the secondary curriculum?

In a desire to bring these fields together, my research of the mathematics teacher education literature has revealed that there are a number of studies examining secondary teachers' knowledge of secondary mathematics, as well as studies on their knowledge of post-secondary mathematics. However, there is no such study which examines the way that a post-secondary degree in mathematics influences a teacher's understanding of the secondary mathematics curriculum. To fill this gap, I plan to employ the MKT framework of Silverman and Thompson (2008) to determine how prospective secondary mathematics teachers come to develop their MKT through reflective abstraction and key developmental understandings constructed during their post-secondary mathematics degrees. My work aims to answer this question and understand the role that advanced coursework plays in mathematics teacher education by placing advanced coursework in the context of secondary mathematics pedagogy.

Although there is a significant body of research pertaining to mathematical
knowledge for teaching (MKT), the terrain is difficult to navigate. Researchers in the field each have their own definition of what is entailed by MKT and thus different ways to examine and evaluate it. As mentioned above, the terrain of elementary teachers' MKT is a single piece of land with its foundation in Ball and colleagues' conceptualization of MKT, which has allowed researchers to make significant developments in the field. Unfortunately, no such common theory exists for secondary MKT, causing the field's development to progress at a pace that is slower than research at the elementary level. Even though there have been a number of significant findings on secondary teachers' knowledge in teaching mathematics, the theoretical bases of each result are different. Indeed, the lack of a commonly accepted framework is causing the research community to miss opportunities to better understand the knowledge used in teaching mathematics and make connections within the literature as a whole. It is my personal hope that an accepted framework for secondary MKT comes to the forefront in the coming years, so that progress can be made in the future. With my own work concerning secondary MKT, I look forward to being a part of this conversation and making strides to improve the education of secondary mathematics teachers in the future.

The presented analysis of Silverman and Thompson (2008) argues that their framework provides researchers with a useful lens for understanding the development of MKT, while also leaving space for sociocultural perspectives on mathematics education. In light of my extended research, I maintain my stance on this facet of Silverman and Thompson (2008), but argue further that their conceptualization for MKT could be the bridge for unifying the research community's understanding of the knowledge of elementary teachers, secondary teachers, and even university
professors. Their use of key developmental understandings (KDUs) and reflective abstraction provide a basis for understanding teachers' knowledge in a way that transfers the focus from "mathematical reasoning, insight, understanding, and skill needed in teaching mathematics" (Silverman and Thompson, 2008, p.500) to a transformation of mathematical understandings "having pedagogical potential to an understanding that does have pedagogical power" (Silverman and Thompson, 2008, p.502). Transitioning to this perspective allows researchers to move away from the specifics of teacher knowledge and towards building "professional practices that would support teachers' ability to continually develop MKT throughout their careers" (Silverman and Thompson, 2008, p.509). In turn, this is a transition from a framework that is dependent on the level being taught, to one which focuses on practices that are conducive to the continued development of MKT for teachers at any level.

As reviewed in this chapter, many educational movements have recognized the important role that mathematics teachers play in supporting a quality education in school mathematics. Beyond the issue of theoretical framing, we have seen that there is a divide between the mathematical knowledge of teachers and how that knowledge impacts their pedagogy. Despite requiring prospective teachers to have advanced mathematical coursework at the tertiary level, it is unclear as to what role AMK plays in their MKT. It is my hope that this study will be able to begin to bridge this gap and create more dynamic relationships between faculties of education and departments of mathematics.

This review has revealed gaps in the literature, which point to a need for a study
which examines what prospective secondary mathematics teachers perceive as the role of their advanced mathematics knowledge and the ways in which they relate their advanced mathematics knowledge to concepts in the secondary curriculum. In the following chapter, I detail the methodological considerations for my study which aims to answer these important research questions.

## Chapter 3

## Methodology

This chapter describes the methodological considerations of this study. I begin with a justification of my chosen research methodology (case study), with respect to the literature on social science research, as well as research in mathematics education. Following this, I describe the setting of the study and the backgrounds of the individual participants who contributed to the study. I then will detail the data that was collected and provide rationale for obtaining such data. The chapter concludes with details on how data collected through the study was analyzed.

### 3.1 Research Methodology: Case Study and Interview

Unlike many studies of secondary mathematics teachers' mathematical knowledge, this study was not intended to examine participants mathematical knowledge of secondary or post-secondary mathematics through standard qualitative tests. Rather, this study aims to explore the connections participants make between secondary and post-secondary mathematics. These connections are dependent on the individual experiences of each participant, which vary due to coursework taken at
the university, teaching experience, and perceptions of mathematics. In order to investigate these connections, a qualitative approach to data collection and analysis had to be employed to explore the mathematical knowledge, connections, and experiences of the prospective teachers in the study. Since the study would be drawing data from a small number of prospective secondary mathematics teachers with an array of experiences in mathematics and pedagogy, a case study was deemed the most appropriate methodology for investigating the complexities of the role advanced mathematical knowledge plays in mathematical knowledge for teaching. In terms of data collection methods, the semi-structured research interview (Kvale, 1996) was chosen as the primary source of data collection. The following sections will argue for the use of case study and the interview as research methods for this study.

### 3.1.1 Case Study

Case study is a flexible research methodology that provides researchers with a rich picture of a phenomena and can be "characterized as being particularistic, descriptive, and heuristic" (Merriam, 1998, p. 29). The definition of what necessarily constitutes a case study varies from author to author (Stoecker, 1991), but some key features remain, including that the heart of any case study is the case. Miles and Huberman (1994) define the case as existing within the bounds of a particular context; that is, the case is a real-life phenomenon, which manifests itself in a restricted domain. For example, studying the general population's understanding of fractions does not constitute a valid case. Firstly, there is no limit in terms of who can participate. The sample would need to be condensed to a specific person or group of people with a unique characteristic (i.e. grade 4 mathematics teachers).

Secondly, the topic of fractions is far too broad; the subject should be condensed to a more concrete problem - such as division of fractions - for it to align with the particularistic nature of case study research.

After defining the case, the flexibility of case study allows the researcher to engage with multiple modes of data collection and to cross reference data from the various sources. As Merriam (1998) remarks, most qualitative studies in education utilize one, at best two, of the three widely used methods of qualitative data collection - interview, observation, and document analysis. Researchers employing a case study methodology, however, often engage with all three of these modes of data collection and triangulate resultant data to converge on research results and theoretical propositions (Yin, 2013).

The case study methodology has been widely used in the field of mathematics education research, particularly in the context of prospective secondary mathematics teachers. Conner et al. (2011) conducted a case study with six prospective secondary mathematics teachers and investigated the change in their beliefs about mathematical reasoning and proof over a two-semester course sequence. Their case study utilized survey, interview, observation, and written work to come to a rich description of changes in the student teachers' belief systems. Kinuthia et al. (2010) investigated pre-service teachers' development of the use of technology in the mathematics classroom by conducting a qualitative case study in which they triangulated focus group interviews and various reflections by the student teachers over the course of a technology integration class. On a more quantitative note, Buchholtz et al. (2008) examined prospective secondary mathematics teachers' ad-
vanced mathematics knowledge at universities in Germany, Hong Kong, China (Hangzhou), and South Korea through a case study. Their study was quantitative in nature, but based off their data, they suggest that prospective teachers are unable to connect advanced knowledge to the secondary curriculum.

In qualitative education research, where many research studies are used as justification for intervention and changes of existing education programs, the credibility of research must be examined in some way. Unlike quantitative research where credibility is associated with the appropriate use of statistical tests, the nature of qualitative research calls for different criteria to judge trustworthiness, credibility, confirmability, and data dependability (Yin, 2013, p. 45); namely external validity, internal validity, and reliability. The issues of internal validity, external validity, and reliability in qualitative research are key criteria in evaluating the rigour and trustworthiness of a case study. These constructs will be addressed within the context of this study in Section 3.5

### 3.1.2 Interviews as Data

How might one come to understand another's thought, experience, story, or culture? To interview is to question. It is the process of asking questions, aimed at a deeper understanding of the interviewee and the topic at hand. How one goes about asking questions, acquiring and interpreting answers is a question of method and may take a wide variety of forms. The flexibility that the interview offers as a mode of inquiry positions it as one of the hallmark research methods within social science research. Holstein and Gubrium (2000) approximate that nearly ninety percent of published social science research utilizes the interview in some form. The technol-
ogy of the twenty-first century has made it easier than ever for researchers to utilize interview data. These technologies include: audio and video recorders which allow researchers to revisit their conversations; transcription machines which greatly reduce days of labour into a mere hours; and coding programs that are able to efficiently manage and analyze massive amounts of data.

Despite these technological advances, the interview itself remains the same; it is a conversation that leads to an "inner view" of the respondent (Chirban, 1996; Kvale, 1996; Kvale and Brinkmann, 2009). Interviews serve as a method to understand the other: the "hows," "whats," and "whys" of their lives. How social science research interviewers come to understand these constructs, as we will see, is nonuniform. The methods and practices of researchers who utilize the interview as a research method depends greatly on epistemological commitments, the research question, and context. Thus, as a researcher engaging with interview as a potential research method, I found myself engulfed in a massive body of methodological literature.

As Kvale (1996) asks in the opening of his book, "if you want to know how people understand their world and their life, why not talk with them?" (Kvale, 1996, p. 1) The research interview is a conversation (Burgess, 2003; Lofland and Lofland, 1984) where participants elaborate on their life experiences, in their own words. For the researcher, the purpose of the interview is to inquire about the perspectives, or views, of an individual. Kvale and Brinkmann (2009) provide two metaphors for how a researcher might conduct an interview: that of a miner or a traveler. Within the traveler metaphor, the interview is a journey where the researcher wanders,
engages in conversation, and perhaps follows a method through their exploration. In contrast, the miner arrives to the interview with a defined goal in mind. The conversation in the interview is directed toward uncovering knowledge which is embedded within the participant.

### 3.1.3 Variations on the Interview

Variations on the ways in which interviews are conducted vary across disciplines and epistemological positions. Stemming from modernist social science tradition, the structured interview is an interview in which all questions are predetermined by the researcher, both in terms of wording and order; multiple respondents will receive the same questions in the same order (Clifford et al., 2016). Fontana and Prokos (2007) remark that the structured interview requires that the interviewer "play a neutral role, never interjecting their opinion of a respondent's answer" (Fontana and Prokos, 2007, p. 20). Thus, structured interviews are intended for obtaining an objective account of another's experience, borrowing from the rigorous practices of the scientific method. On the opposite end of the spectrum is the unstructured interview. With origins in ethnographic methods (Bruner, 1986), the unstructured interview sees the interviewee as a narrator of their experience and life history (Sandelowski, 1991). This type of interview gives the interviewee the ability to adjust the direction of the interview, elaborating on points that are of significance to them through stories that need not follow a particular progression (Denzin, 2001).

Someplace in between these two lies the semistructured interview, which is arguably the most common variation of the interview among researchers in the
social sciences (Kvale and Brinkmann, 2009). Similar to the structured interview, some questions may be prepared and tested ahead of implementation, but unlike the structured interview, there is the freedom for the interviewer to probe and explore responses. The semistructured interview is subjective; different respondents will provide the interviewer with different responses, thus altering the overall course of the interview. Many authors see the subjective nature as a benefit, rather than obstacle (Ginsburg, 1997; Kvale and Brinkmann, 2009; Qu and Dumay, 2011), arguing that variations in responses help one to better understand the complexities of a particular phenomena.

### 3.1.4 Debates on the Interview as Data

Researchers who associate themselves to a postmodern school of thought often dismiss the structured interview, claiming it as an ineffective research tool. For example, in his critique of the research interview from a postmodern standpoint, Scheurich (1997) states the following:

The researcher uses the dead, decontextualized monads of meaning, the tightly boundaried containers, the numbing objectifications, to construct generalizations which are, in the modernist dream, used to predict, control and reform, as in educational practice. (Scheurich, 1997, p. 64)

For Scheurich (1997), the acknowledgement of subjectivity should be at the forefront of research interviewing, not the prospect of generalizability. Proponents of the structured interview have a similar distaste for the unstructured interview of the postmodern tradition. Critics of postmodernism remark that it does not lead to
any "true" understanding (Spiro, 1996) and that its effects are "relativism; nihilism; solipsism; fragmentation, pathos, hopelessness" (Hill et al., 2002, p. 5). Since the postmodern interview does not provide the scientific certainty modernists desire in order to justify changes in policy and practice, it is viewed as a fruitless research tool.

The critiques of those who thoroughly oppose unstructured interviews and those who oppose structured interviews are, in fact, very similar. Each group dismisses research on the basis of an asserted a priori philosophical position. Criticism of these methods and their underlying philosophies are based off characteristics that the philosophies under scrutiny simply do not have. As Rosenau (1991) remarks, the anti-theoretical position that postmodernism foundations itself on, is in fact a theoretical stand. Further, Eagleton (1996) observes that "for all its vaunted openness to the Other, postmodernism can be quite as exclusive and as censorious as the orthodoxies is opposes" (Eagleton, 1996, p. 26). Philosophical beliefs aside, there are obvious benefits to utilizing structured interviews in social science research. Similarly, there is a time, a place, and a purpose for postmodern research interviewing. What is troubling is the explicit rejection of valuable research based on epistemological beliefs. I do not reject the postmodernist interview, nor do I reject the structured survey interview. Rather, I perceive these variations on the interview as having their place in the social science literature and research community. The decision of which variant to use should be dependent on the investigator's research question and what they wish to uncover in their work.

These criticisms are from researchers who are proponents of the interview as a
research method. Despite its widespread use in social science research, criticisms of the interview as a research method still ensue. Critics often claim they are "not scientific, but only [reflect] common sense" (Kvale, 1996, p. 285). I concur with Kvale (1996) that perhaps one must carefully define "science," before defining something as "not science." Merriam Webster has multiple definitions of science; "the state of knowing: knowledge as distinguished from ignorance or misunderstanding," "a department of systematized knowledge as an object of study," "a system of knowledge concerned with the physical world and its phenomena" (Science, 2019). The definition we take to mean "scientific" will change whether the qualitative research interview is "scientific" or not. Unfortunately, just as we have multiple definitions of science, terms such as "knowledge" and "systematized" may also have alternative meanings. With so many variations on what constitutes science, it does not seem as though one can state that the qualitative interview is not science. As Kvale (1996) puts it so well, "the automatic rejection of qualitative research as unscientific reflects a specific, limited conception of science, instead of seeing science as the topic of continual clarification and discussion" (Kvale, 1996, p. 61). The qualitative research interview is capable of providing systematized knowledge, provided that the researcher has a rigorous understanding of the interview as a research method, taking into consideration the issues of validity and reliability.

Scientific practices, assumptions, inferences, and mathematics must be based on solid arguments which are logically sound. The research community of the physical sciences has an understanding of what constitutes "good science." Namely, that it is valid, reliable, and generalizable. The shift of this holy trinity of scien-
tific research to social science research, particularly to the method of interviewing in qualitative social science research, is one of the largest battle grounds for researchers in the field. Some qualitative researchers completely dismiss the constructs of validity, reliability, and generalizability as outdated modernist constructs that are irrelevant to the study of human experience and psychology (Constas, 1998; Scheurich, 1997), while others argue that there exists a definitive reality, thus valuing these constructs (Denzin and Lincoln, 2000). However, these arguments are once again based on asserted a priori philosophical beliefs and definitions of the constructs in question. A characteristic of research that these two camps can agree on, however, is the mutual hope that their research will build the understanding of the phenomenon of study; that their research is trusted and will inform future research. Regardless of your philosophical stance, there are ways to ensure that qualitative interview research responds to validity, reliability, and generalizability.

### 3.1.5 Summary

Regardless of the methodological and philosophical debates surrounding the interview as a research method, the interview still maintains its position as a hallmark research method in the social sciences. The research interview combines structure with flexibility (Legard et al., 2003), positioning it as a method that is available to any researcher, regardless of their epistemological position. Whether one's research relies on descriptive statistics and mathematical analysis (Schwarz et al., 2008) or on narrative and poetic responses (Richardson, 2000), there exists a variation of the interview that will both complement and enhance one's research. Through interviews, we can come to discover not only the phenomena in question,
but connections related to that phenomena in various contexts.

This thesis explores the impact that post-secondary mathematical content knowledge has on prospective secondary mathematics teachers' understanding of secondary mathematics concepts. Although it will be valuable to understand what mathematical content knowledge the participants in my study have, my interests lie in how teachers relate their advanced mathematical knowledge to the secondary curriculum. I hope to go beyond mere content, and explore the meaning that my participants have constructed. Each participant will have unique educational experiences in mathematics and pedagogy, resulting in mathematical knowledge for teaching that is specific to them. For this research, the semi-structured interview offers itself as a methodological tool that will allow me to tackle and explore the subtleties behind mathematical understanding, to treat each participant as an individual with their own unique experience, and as Spradley (1979) noted, to be a learner; to listen and learn of pedagogically powerful mathematics that I myself, may not have thought of before.

### 3.2 Setting

Data for this study was collected from five prospective secondary mathematics teachers. At the time of data collection, each participant was enrolled in the Bachelor of Education program in Secondary Mathematics at the Vancouver campus of the University of British Columbia. As per the requirements of the Secondary Mathematics B.Ed. program at UBC, all participants obtained degrees in mathematics, or a related subject. More specifically, the UBC Teacher Education Office states that potential teacher candidates have at least 30 credits of mathematics
coursework, 18 of which must have been at the senior level (3rd year or higher). Furthermore, there is a "breadth requirement" for potential teacher candidates, stating that a candidate must have at least three credits from at least three of five "core" topics, those being algebra, probability/statistics, geometry, discrete math, and number theory. The participants that were interviewed in this study had mathematical coursework and experience well beyond the secondary curriculum.

Each participant provided their undergraduate transcripts with coursework taken and grades obtained. The subject GPA of each participant was calculated out of 100. This was done by multiplying the number of credits for the courses taken by the grade obtained and summing over all mathematics courses taken. This number was then divided by the total number of credits taken, yielding a score out of 100 .

### 3.2.1 Participants

In this section, I will outline the backgrounds and experiences of each participant in the study. The names used in this thesis are pseudonyms, to protect the identities of the participants. All participants completed their undergraduate mathematical coursework at large, Canadian research institutions.

Taylor had their Bachelor of Science in Mathematics. They completed their BSc in 2017 and transitioned immediately to the Bachelor of Education program in Secondary Mathematics at UBC in September of 2017. In their BSc, Taylor took 62 credits of mathematics coursework, with a final subject GPA of 79.6.

Jaime had their Bachelor of Engineering in Engineering Physics. Their BEng
was completed in 2000, making them the participant farthest removed from postsecondary mathematics coursework. They also completed a Master's of Business Administration, prior to their entrance into the UBC BEd program. Jamie took 28 credits in mathematics coursework, with a subject GPA of 72.1.

Bailey completed their Bachelor of Arts with a double major in Mathematics and English in 2015. Their degree included 45 credits of mathematics coursework, with a subject GPA of 88.7. Bailey had extensive TA experience in mathematics and computer science, prior to entering the teacher education program. Bailey had English and Mathematics as subject specializations in their BEd.

Adrian completed a Bachelor of Science with Honours in Mathematical Physics in 2015 and a Master's degree in Theoretical Physics in 2017, before entering the Bachelor of Education program, with teachable subjects of mathematics and physics. In their BSc, Adrian completed 39 credits of mathematics coursework, with a GPA of 95 . Even before starting their BSc, Adrian was committed to the idea of a career in academia as a research physicist. However, throughout their studies, Adrian was involved in a number of activities involving teaching and learning of mathematics and science, including extensive tutoring, TA experience, and summer science camps for youth. It was during their Master's degree that Adrian decided to pursue a career in education, after realizing their passion was in teaching and learning, rather than academic physics research. Adrian hopes to teach mathematics and physics for International Baccalaureate (IB) students, and was in the IB cohort throughout their BEd studies.

Casey completed their Bachelor of Arts in Mathematics in 2013, prior to entering the Bachelor of Education program. In their degree, Casey completed 61 credits of mathematics coursework, with a GPA of 72.2. Additionally, Casey audited some graduate level mathematics courses offered through their institution. Casey stated an interest and specialization in algebraic structures, which they hoped to bring into their teaching.

The experience of the participants in this study is summarized in Table 3.1.

| Participant | Degrees | Math Credits | Math GPA |
| :---: | :---: | :---: | :---: |
| Taylor | BSc Math | 62 | 80 |
| Jaime | BEng Physics | 28 | 72 |
| Bailey | BA Math, English | 45 | 89 |
| Adrian | BSc Mathematical Physics <br> MSc Physics | 39 | 95 |
| Casey | BA Math | 61 | 72 |

Table 3.1: Summary of participant backgrounds

### 3.3 Data Collection

For the data collection, all five participants provided the researcher with their undergraduate academic transcripts. These transcripts provided the researcher with background knowledge of the participants' mathematical coursework. Additionally, each participant filled out a short survey outlining their relevant work experience. Once the researcher obtained this information, one-on-one semi-structured interviews (Seidman, 2013) with the participants and the researcher were scheduled. The interviews were conducted in a conference room in the Faculty of Education building, with the interviewer and participant sitting next to each other. Scrap paper, pens, and pencils were provided to each participant. Interviews audio recorded and any written work produced was scanned. The interviews consisted of three parts, each of which will be elaborated below.

### 3.3.1 Experiences in Post-Secondary Mathematics

To begin the interview, participants were asked to reflect on their post-secondary mathematics courses and the role this knowledge plays in their pedagogical practice (Zazkis and Leikin, 2010). In this portion of the data collection, I was interested in understanding whether participants perceived their advanced courses in mathematics to have an impact on their practice as secondary mathematics teachers. Leading questions in this portion are included below and were presented orally to the participant:

- How do you conceptualize mathematics as a field of study?
- Do you think it is important for secondary mathematics teachers to know advanced mathematics?
- What roles do you see those four years of learning playing in your next year of study?
- Do you see your post-secondary degree in mathematics having an impact on your teaching?


### 3.3.2 Connections Between Secondary and Post-Secondary Mathematics

The second portion of the interview was a task-based, semi-structured interview (Goldin, 2000), intended to explore the ways in which participants connected their advanced mathematical knowledge to problems in the secondary curriculum. Participants were provided with a list of seven tasks written on a sheet of paper. Participants were then asked to choose four of the tasks to discuss with the researcher.

The tasks I chose to engage participants with are inspired by previous studies of secondary mathematics teachers' PCK and MKT which have drawn from advanced mathematical knowledge. In taking Silverman and Thompson (2008) as a framework for the development of MKT, finding tasks that had the potential to reveal KDUs were chosen. As a gentle reminder for the reader, recall that a KDU is defined to be an understanding which transforms the way a learner understands a particular mathematical concept, allowing them to work with the concept in ways unfamiliar to them previously. In the context of this study, I hoped for participants to share post-secondary mathematical understandings that transformed their understanding of the secondary curriculum. The tasks used in this study needed to have the potential to reveal participants' understandings that connected secondary mathematical knowledge to advanced mathematical knowledge. Furthermore, I
wanted to use tasks which had been used or examined in previous research studies on teachers' secondary and/or post-secondary mathematics content knowledge. Thus, I converged on problems that readily bridged these two content levels.

For example, Task C on the factorization of polynomials was chosen because polynomials are a major concept of the secondary curriculum, as well as the study of abstract algebra in post-secondary mathematics. This task had the potential to reveal KDUs which bridged post-secondary abstract algebra and secondary school algebra. A similar argument can be made for Task A. Indeed, exponents and exponential functions are a major topic of discussion in secondary mathematics, but the proof that this notion can be extended to any "type" of power is not examined until a course in real analysis. Such a construction requires an understanding of irrational numbers as a limit of a sequence of rationals, an understanding that allows the learner to work with irrational numbers in ways they were unable to when irrationals were simply "not rational."

In an effort to dig deeper into the understandings and mathematical knowledge for teaching of the participants, leading questions for each of the tasks are included. Note that many of the questions begin with "what advanced mathematics is relevant here?" This is to recognize whether the participants have developed a personally powerful understanding (KDU) of the particular secondary mathematics concept. Then, the question of "how could you make this relevant/accessible to your students?" explored the participants' efforts to make a KDU pedagogically powerful; that is, to reflect on their advanced mathematical knowledge and to recognize its relevance in the secondary curriculum. This is precisely the process of reflective
abstraction: to reflect on and connect knowledge constructed at various levels of abstraction. Initially, participants were asked how they would respond to the situation. If the response did not clearly make a connection to their post-secondary mathematics work, teachers were asked, quite explicitly, to make connections between advanced mathematics and the secondary curriculum. This approach was chosen to ensure that participants were clear in the types of mathematical connections that I was interested in. Below is the list of mathematical tasks provided to the participants to choose from, along with follow-up questions. The reader may note that each task was inspired by previous work in the field. In comparison to the past work, the focus of the discussion was on the connections participants made between secondary and post-secondary mathematics in the context of the given tasks.

- Task A: Your students are confused as to why we can define and calculate $2^{\sqrt{3}}$.
- How would you respond to your students? What mathematics is relevant here? How can you make it accessible?
- Inspired by work from Wasserman et al. (2017)
- Task B: A student is working through a problem and asks if $0.999 \ldots=1$.
- How would you respond? What knowledge from your post-secondary math classes could you use to explain it? How could you make your response more accessible to your student?
- Inspired by work from Krauss et al. (2008)
- Task C: You are teaching a week on factoring polynomials and you have found that your students are struggling to recognize when they should stop trying to factor.
- How would you respond? Can you think of anything you learned in your post-secondary mathematics courses that might help your students? How could you make it accessible? Is there anything from an advanced course that might provide motivation for this topic?
- Inspired from the researcher's teaching experience.
- Task D: Your students are learning about inverse functions. What would you include in your lesson plan?
- What knowledge from your post-secondary mathematics work might be relevant in this context? How could you make it accessible/useful in your pedagogy?
- Inspired by work from Leung et al. (2016) and Zazkis and Kontorovich (2016)
- Task E: You are teaching a week on symmetry to your students. What would you include in your lesson plan?
- What knowledge from your post-secondary mathematics courses might be relevant here? How could you make it accessible?
- Inspired by Sultan and Artzt (2010)
- Task F: You have been teaching a unit on quadratic functions for a few weeks and one of your students asks you why they need to know about them.
- How would you respond? Did you talk about quadratic functions in any of your university mathematics courses? Could you make it accessible/useful in your pedagogy of this topic?
- Inspired by the researcher's own teaching experience.
- Task G: A student is confused as to whether $\sqrt{2}$ is an irrational or rational number, especially after realizing it is the length of the diagonal of a square of side length 1 .
- How would you respond? In what contexts did rational and irrational numbers appear in your university mathematics courses? How would you use that knowledge in your teaching?
- Inspired by Sirotic and Zazkis (2007)

I have summarized the tasks which participants chose to engage with in Table 3.2. As noted above, participants were asked to choose four tasks with which to engage. However, Adrian, Bailey, and Taylor engaged with five of the tasks. In the case of Adrian, this was due to their desire to continue the conversation, whereas with Bailey and Taylor, the discussion naturally emerged from the discussion in Task G.

| Participant | Task A <br> $\left(2^{\sqrt{3}}\right)$ | Task B <br> $(0.999 \ldots)$ | Task C <br> $($ Factoring $)$ | Task D <br> $($ Inverses $)$ | Task E <br> $($ (Symmetry $)$ | Task F <br> $($ Quadratics) | Task G <br> $(\sqrt{2})$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Taylor | $\checkmark$ | $\checkmark$ | $\checkmark$ |  |  |  |  |
| Jaime |  | $\checkmark$ | $\checkmark$ |  | $\checkmark$ | $\checkmark$ |  |
| Bailey | $\checkmark$ | $\checkmark$ | $\checkmark$ |  | $\checkmark$ | $\checkmark$ |  |
| Adrian | $\checkmark$ | $\checkmark$ | $\checkmark$ | $\checkmark$ |  |  |  |
| Casey |  |  | $\checkmark$ | $\checkmark$ | $\checkmark$ | $\checkmark$ | $\checkmark$ |

Table 3.2: Tasks chosen by participants

### 3.3.3 Secondary Mathematics Instrument

The third, and final portion of the interview consisted of a mathematical instrument containing problems drawn from the BC curriculum which correspond to tasks from the second portion of the interview. As the literature suggests a degree in mathematics does not necessarily imply a thorough understanding of secondary mathematics (Cofer, 2015; Even, 1993; Stump, 1999), this portion of this interview was intended to respond to such claims. With only a few exceptions, participants successfully responded to all questions in the instrument. Bearing this in mind and after reviewing participants' responses to these problems, I concluded that their responses did not substantially contribute to the emergent themes of connections between secondary and post-secondary mathematics observed through the taskbased interviews. Thus, this portion of the data was withdrawn from my analysis.

### 3.4 Data Analysis

The primary mode of data analysis was the transcription of the audio-recorded interviews. After each interview had been completed, the audio-recordings were transcribed. Once all interviews were complete, the portions of the transcripts associated to each task were grouped. The grouped transcripts were analyzed through an emergent coding process. Key phrases, adjectives, and statements from the participants were underlined and associated to a code which summarized the theme of the underlined portion of the transcript. In coding, I was particularly interested in statements that shed light on what the participants understood mathematically in this task, both at the secondary and post secondary level. I was also concerned with participants' remarks that connected mathematical content to pedagogical choices in their future teaching. For example, in the task exploring inverse functions, a participant's statement on the importance of domain restrictions in teaching inverse functions would be associated to the code domain restrictions, while a participant's comments on the general nature of inverses being dependent on operations would be given the code operations. This coding procedure was completed for all the tasks, as well as the initial interview on perceptions.

Common codes were grouped together into emergent themes that are representative of participants' understandings in relation to the relevant task or question. This was done for both the one-on-one interview exploring participants' perceptions of their degrees and for the task-based interviews. Since many participants chose to write out some mathematics during their task-based interviews, the written work produced was used to support the dialogue from the task-based interviews.

Reference to relevant written work was documented in the coded transcripts. Finally, by utilizing the academic transcripts provided by participants, statements were cross-matched to the advanced mathematical coursework they had taken. This cross-matching provided context for the origin of mathematical concepts discussed.

The focus of this study is on the links made between secondary and postsecondary mathematics content. In Chapters 1 and 2 , I argue that both university mathematics courses and mathematics methods courses play a role in developing such knowledge. With respect to data analysis, the MKT framework of Silverman and Thompson (2008) and associated framework of KDUs in Simon (2006) was used to focus my analysis on expressed understandings of the participants, particularly ones which drew upon knowledge from their advanced degrees. Silverman and Thompson (2008) centre the development of MKT in terms of mathematical understandings with pedagogical potential, becoming ones with pedagogical power.

In my coding and development of emergent themes, I was particularly interested in how participants' mathematical understandings bridged secondary and post-secondary content and pedagogical practice. Such comprehension would be examples of personally powerful understandings with the power to change the way they teach secondary mathematics. In Chapter 2, I explain that reflective abstraction may be the mechanism for the development of such knowledge. This theory allowed me to contextualize the expressed experiences of participants' own mathematics education and whether these experiences were conducive to the integration of advanced knowledge into their MKT.

The emergent themes are explored in more detail in the following chapters, but Table 3.3 on the following page summarizes the emergent codes and themes from the tasks. In this table, I have provided all codes that emerged from the transcripts, as well as the themes which were generated from the codes. The rich descriptions of the emergent themes of participants' understandings and mathematical knowledge for teaching will be examined and detailed in Chapters 4, 5, 6, and 7.

| Task | Codes | Themes |
| :---: | :---: | :---: |
| Perceptions | procedural, theory, problem solving, reasoning, facts, skepticism, connections, application, elegant, content, concepts, pedagogy, real world, proof, practical, pragmatic, support, enrichment, exposure, rote, understand, rigour, pure, applied, abstract | two facets of mathematics, content versus beliefs and values problem solving skills pragmatic teacher education disconnect of post-secondary |
| Task A <br> (Value of $2^{\sqrt{3}}$ ) | rational, rigour, faith, continuity, exact, series, infinity, inverse, concrete, define | complexity of exponents, approximation vs. exact, existence versus definition |
| Task B (Value of $0.999 \ldots$. $)$ | memorized, thirds, rational, sequence, convergence, intuition, approaching, limit, concrete, asymptote, faith, cardinality | reliance on memorized proof, concrete conceptualizations limits and infinity |
| Task C (Factoring) | recipe, approach, graphs, factoring, quadratic, roots, complex, cubics, proof, application, Galois, zero, abstract algebra, calculus | relating graphing \& factoring, extend quadratic/cubic behaviour, complex factors \& roots, disconnect from abstract algebra |
| Task D (Inverse functions) | reciprocal, operation, domain, sine, opposite, exponents, application, restriction, test, function, mapping, reflection | Inverse w.r.t operation, domain restrictions reflection vs. functional inverse, "undoing" of functions |
| Task E (Symmetry) | picture, nature, reflection, rotation, physics, tactile, visual, graphs, geometry, groups, triangle, inverse | symmetry, nature and art, <br> types of symmetry, geometric symmetry, example driven |
| Task F (Quadratics) | application, curve, formula factor, basic, complex, vertex, roots, interest, memorize, velocity, extrema, concrete, meaning, visual, graph, bridge, physics, model, accessible | relating graphing \& factoring, beginning access point for learning, limited applications impact interest |
| Task G <br> (Irrationality of $\sqrt{2}$ ) | memorize, proof, contradiction, logic, confidence, intuition, definition, zoom, limit numerically | proof in secondary math, procedural logic, decimal versus exact |

Table 3.3: Interview transcript codes and themes

As stated above, common codes were grouped into themes. However, codes for a particular task may be grouped into multiple themes. To help the reader better understand the coding and thematic generation, consider the following quote from Adrian, in response to Task D:

Adrian: Yeah, these are reciprocal functions, which is not what we are talking about, but some people call them inverses because you are taking the, technically this is the multiplicative inverse of the function, but that's not what we are talking about. What we are talking about is inverse functions, which is taking the opposite of the function.

The codes reciprocal, operation, and opposite were used to interpret this quote. While both codes of reciprocal and opposite explicitly appeared in this quote, the code operation was used to refer to Adrian's mention of multiplicative inverses, versus functional ones. These codes were grouped together into the relevant theme of recognizing inverses with respect to operations, as well as the theme of "undoing" of a function, which Adrian refers to as the "opposite."

As a second example of the coding process, consider the following transcript between the interviewer and Bailey:

Interviewer: Can you think of a limit that would be equal to one but also have something to do with $0.999 \ldots$ ?

Bailey: Like we would want an equation that is approaching one from the
bottom, so that would be a 1 over x plus something. Here's my graph. I'm rusty on all my graphs look like. It would just be 1 over x plus 1. Would that be right? Yeah.

Interviewer: And so as we take this limit we are getting one, but what does this have to do with point 999 ?

Bailey:
Because 1 minus a tiny tiny piece is .9999 , right? So that asymptote is approaching it.

The codes limits, approaching, and asymptote were used to understand how Bailey understood the equality of $0.999 \ldots=1$. The appearance of these codes in Bailey's execution of this task revealed that the connections and understandings Bailey had made between secondary and post-secondary mathematical knowledge existed in the content domain of Calculus and Real Analysis. However, this transcript also reveals that Bailey may not have a personally powerful understanding (KDU) of limits in the context of number, since the use of a continuous function is not appropriate for the justification of $0.999 \ldots=1$. This transcript suggests that Bailey may have benefited from further learning experiences that would help build a more robust understanding of number by drawing upon their advanced content knowledge and engaging in the process of reflective abstraction.

Further, a single code may fit into multiple themes. This can be seen through Bailey's quote from Task C, where they stated:

Bailey: You could probably factor if the coefficients were complex because you can have $i^{2}$ and you can get some negative ones. So my whole argument of being able to get zero becomes untrue. $x^{2}+1$ has no roots in the real plane, but it does in the complex plane.

The codes zero, roots, and complex emerged from this quote and contributed to the theme of relating graphing and factoring, as well as the relationship between complex factors and real-valued intercepts. All of the themes which are related to the tasks are representative of the expressed understandings and mathematical knowledge for teaching of the participants, as individuals and as a group.

A second phase of study was performed on the emergent themes. Since the goal of this research is to better understand the role of advanced mathematics knowledge for future teachers, I wanted to have a better understanding of how these themes fit in the framework of the academic journey to become a secondary teacher. Common themes from all tasks were gathered together and associated to either "advanced mathematics" or "teacher education." The intent of this coding was to help better understand where the development of MKT could be supported. Rather than placing the focus on the knowledge, understanding, and experience of the participants, the study of the themes in the context of a teacher's academic journey will help put into perspective what mathematicians and mathematics teacher educators can do to enhance and build upon the mathematical understandings of their students. We examine this in Chapter 8 .

### 3.5 Validity and Reliability

Particularly in qualitative education research, where many research studies are used as justification for intervention and changes of existing education programs, the credibility of research must be examined in some way. Unlike quantitative research where credibility is associated with the appropriate use of statistical tests, the nature of qualitative research calls for different criteria to judge trustworthiness, credibility, confirmability, and data dependability (Yin, 2013, p. 45); namely external validity, internal validity, and reliability. The issues of internal validity, external validity, and reliability in qualitative research are key criteria in evaluating the rigour and trustworthiness of a case study.

The concept of external validity aligns with generalizability; that is, to what extent can the results of the study be applied to other studies? The issue of generalizability has "plagued qualitative researchers for some time" (Merriam, 1998, p. 207). To reconcile this matter, I find it useful to reiterate the purpose of case study. For Merriam (1998) and myself, the priority of case study is the case. My intent as a case study researcher is to understand my case in depth, not necessarily to generalize to what is true of many. Although I research "the particular" in case study, if research of a similar case is conducted, the results obtained from my work may serve as a flag for themes in their data.

Guba (1981) suggests the analog of external validity in a qualitative paradigm may be considered to be "transferability." He offers thick description as a mode to "permit comparison of this context to other possible contexts to which transfer
might be contemplated" (Guba, 1981, p. 86). The utilization of multiple sources of data in this study is conducive to thick description, as each of my data sources contributes uniquely to the individual participants. Furthermore, my sampling of participants from the teacher education program at UBC is meant to maximize resultant data of the study. Despite representing only a small fraction of their cohort, their varied experience in advanced mathematical coursework yields multiple perspectives for the ways in which future teachers make connections between secondary and post-secondary mathematics.

Validity is not a construct only relevant to the final results or method, but a construct that consistently informs the research interviewer. During the interview portion of the proposed study, the researcher should be attentive to whether the questions and responses in the interview are guiding the respondent in a particular direction. To address this, I followed Ginsburg (1997) and his multi-phase approach in clinical interviews. He advises researchers to return to the same question at various moments in the interview, phrasing the question in a different way each time. In this sense, the researcher can cross-check statements made by the respondent, strengthening the viability of any conclusions made by the researcher. This technique was employed throughout the interview process, to ensure that I was not misinterpreting participants' remarks.

Thomas Schwandt defines "triangulation" as a means of "checking the integrity of the inferences one draws" (Schwandt, 2007, p.298). Triangulation will be the primary mode of increasing validity and reliability in this study, and will be done through cross referencing data from the various interview transcripts The utiliza-
tion of triangulation, as well as detailed accounts of methods throughout the research process are ways in which the I can assure that results align with the data collected. By implementing multiple research methods, I will obtain more data to compare and contrast, to support inferences, and to enrich the research.

### 3.6 Summary

In this chapter, I have detailed the methodological considerations of this study. I discussed the details of the study's setting, participants, data collection and analysis. Finally, I described the ways in which I handled the issues of generalizability, validity, and reliability. After the completion of this chapter, the groundwork for my study is complete. As a reminder to the reader, the work that follows aims to answer the questions:

1. What do prospective secondary mathematics teachers perceive as the role of their advanced mathematics knowledge in their development as teachers?
2. In what ways do prospective secondary mathematics teachers relate advanced mathematics knowledge to a mathematics concept in the secondary curriculum?

In the following chapter, I begin the presentation of results through participants' perceptions of the role of advanced mathematics knowledge in their growth as a teacher.

## Chapter 4

## Perceptions of the Role of

## Advanced Mathematics in

## Pedagogical Development

In this chapter, we discuss the portion of the interview exploring participants' perceptions of the role of post-secondary mathematics education in their growth as secondary mathematics teachers. The primary prompts for this portion of the interviews included:

1. How do you perceive mathematics as a field of study?
2. Is a degree in mathematics needed to teach secondary mathematics?
3. What do you perceive as the role of advanced mathematics knowledge in your growth as a teacher?
4. In what ways is your advanced knowledge being drawn upon in your teacher
education?

### 4.1 Perceptions of Mathematics

To begin the interview with the participants, I wanted to gather a sense of the perceptions and values participants held about mathematics as a field of study. Indeed, as remarked in the literature, teachers' perceptions of mathematics can play a significant role in the pedagogy of teachers. This first interview question of "how do you perceive mathematics as a field of study?" was intended to connect participants' responses to later interview questions and to further understand their future goals in teaching mathematics. Overall, participants' responses to this question revealed two distinct conceptions of mathematics: as a tool for understanding the world and as a pure, abstract, self-contained knowledge system. This distinction was expressed by all participants. Additionally, the relationship between mathematics and problem solving was a common theme expressed by four out of five of the participants.

All participants expressed mathematics as having two facets. This was expressed succinctly by Jaime:

Jaime: I see math as kind of having two sides to it. Part of it is definitely a way of explaining the world. And definitely on the science side, it's a way to explain how things work and kind of simplify them to build a model. The other side though is more pure, abstract math, where it's the realm of a lot of logic. This idea will lead to this idea which may not have an obvious or maybe any direct
relevance to the actual world. So I kind of see those two sides of it.

Jaime and Bailey both expressed mathematics having a characteristic of beauty. This beauty was conceptualized as a product of the abstract and independent nature of mathematics. As Bailey stated, mathematics can be conceptualized as "a system we have constructed that explains things outside of itself" and even with this characteristic, it additionally "has perfection within the system." Beauty was conceptualized by Jaime through a relationship between mathematics and art. Jaime viewed the act of doing mathematics as an artistic endeavour, where creativity was a key component to success. However, they noted the complex nature of conceptualizing mathematics as art, in that mathematicians are governed by "rules" different than that of a painter. Jaime later shifted to conceptualizing creativity in mathematics as similar to creativity in music, in the sense that "everyone can respect someone who can freely improvise and be creative on the piano, but it's a lot of work to get there."

Jaime came to appreciate this relationship later in their undergraduate work, remarking that through their education in engineering, "it got to the point in my education where physics and math were inseparable because everything in physics was explained through mathematics." Jaime later conceptualized mathematics not just as a way of describing, but as a self-contained puzzle, where various theories and structures were brought together to uncover new ideas and solve unknown questions. Casey brought forth a similar conception of mathematics as a way of building new structures and ideas out of existing ones, through the process of rigour and proof.

Although Taylor saw a distinction between applied and pure mathematics, they remarked that one can not exist without the other. Indeed, Taylor mentioned in their interview that "mathematics is about understanding what is in the real world but also in the abstract. Abstractly, you have to theorize everything and whether that theory also fits in the physical world." This was a unique statement from Taylor, with respect to other participants' responses and points to the value of understanding both facets of mathematics. However, Taylor followed up this comment with another, stating that sometime mathematics may exist without application to the physical world and might exist as a "brain exercise." This conception is in line with other participants' perspectives on mathematics as a self-contained knowledge system. Indeed, as Adrian remarked, "we have created rules in a space and we want to see what those rules produce. It's almost like a little game that we've played, but with incredible, far reaching consequence with what we can do with it."

The use of mathematics as a tool to understand the physical world was valued by Jaime and Adrian, who both had their undergraduate work in physics. Although both of these participants saw value in the abstract side, their expertise brought them to value the applied side and looked forward to bringing this into their future teaching. Adrian noted that "with my focus in physics, I looked a lot at math as a way of describing things quantitatively and drawing out patterns and sort of seeing the world in a very structured way." Adrian looked forward to bringing their scientific and mathematical expertise into the classroom, so that future students could see value in mathematics for solving applied problems and understanding why such problems and questions are important. They hoped that in doing this, their number
one goal of making class "interesting, relevant, and engaging" for students could be achieved.

Regardless of these two facets, all participants remarked on the relationship between mathematics and problem solving. Participants saw value in an education in mathematics for building skills in problem solving:

Taylor: $\quad$ Mathematics is the study of the thinking process of logical thinking.

Bailey: Math is a way of working, with problems as a way of learning, rather than problems as a way to reinforce learning.

With their undergraduate mathematics being focused in pure mathematics, Taylor, Bailey and Casey each saw value in mathematics as a way of building and developing critical thinking skills through problem solving. As Casey noted, pure mathematics distinguishes itself with a "purification in proof and theory." Each of these participants saw an intrinsic value in studying mathematics for the sake of mathematics for their future students. Their hope was that the study of mathematics in a self-contained system would help their students learn "how to logically solve problems and work through things" even if it is not their intended field of future study, according to Bailey. This sentiment was echoed by Taylor who viewed the learning of mathematics as an opportunity to develop reasoning and "sophisticated thinking skills" inside and outside of mathematics. With an increased focus on critical thinking in modern curricula, these remarks are not without merit.

Similar to their conceptions of mathematics, Adrian and Jaime took a more application based approach to their values of mathematics and problem solving, viewing mathematics as a powerful tool to explain how the world works. Adrian mentioned the "Math Matters" movement and how powerful mathematics could be in helping students understand local and global issues affecting society today. Casey took a more research based approach to their response, viewing the power of mathematics to "uncover or discover a problem that hasn't been researched before and find a way to progress that problem up to a certain point in our field." This comment aligned well with Adrian's hope of bringing in research and extension projects for their advanced students, so that they might be able to have an idea of current research questions in science.

Overall, participants remarked that their undergraduate experience in mathematics changed their perspective on what constitutes mathematics. Bailey mentioned that mathematics in high school seemed to be a "series of things" with an end and no purpose. Bailey felt that their university experience changed this perspective and that they looked forward to bringing it into their future teaching. They hoped that their university experience would help students see that "if you think you don't like math, maybe you don't like one part of math because there is so much to it." Taylor shared their struggle in shifting from thinking of mathematics as a tool to thinking of it as a form of argumentation. This shift was difficult for Taylor in first and second year mathematics, reflecting on a feeling of "why can't I just understand this?" As remarked above, Jaime experienced a similar shift, now viewing mathematics in an artistic light.

These remarks from participants are an appropriate segue to the following section on participants' perceptions of the role advanced mathematics knowledge plays in their identity as a future secondary teacher. Even though all participants perceived intrinsic value in learning mathematics, these sentiments are not echoed as strongly in the questions that follow.

### 4.2 Role of Advanced Knowledge for Teachers

In this section, we will explore participants' responses to questions 2 and 3; that is, what do participants perceive as the role of their advanced knowledge as a teacher and do they think advanced mathematics knowledge is important for secondary teachers to have?

Participants answered these questions with varied responses and degrees of strength in their beliefs. Overall, all participants expressed that a major degree in mathematics is not necessary to teach secondary mathematics. The view of what extent of post-secondary mathematics training is necessary varied from participant to participant.

Four participants expressed the value of post-secondary content expertise being of value in the classroom. In particular, Bailey, Taylor, Adrian, and Casey stated the importance of having content knowledge beyond their students'. As Bailey mentioned, "you need more experience than where your grade 12 students are going to be." Adrian agreed with this remark from Bailey, stating "I think I could get by with first-year university knowledge." Adrian perceived this as important so
that "you're more advanced than your students and have a perspective on where it can take them in an academic sense."

The view of advanced knowledge being of value for building connections was expressed by all participants, except for Jaime. Adrian, Taylor, and Casey all saw value in post-secondary mathematics degrees. They each expressed that this advanced knowledge is important for being able to field students' questions, answering students' questions in different ways, having an understanding of conceptual background, and for being able provide context for what mathematics is on the horizon.

Taylor expressed a unique perspective on the value of a mathematics degree for secondary teachers, in that it helped them to learn to think like a mathematician.

Taylor: How are you sure that this statement is true? The humbleness of seeing the nature and making sophisticated thinking skills to how much we do not know about the world in general. That's what mathematics taught me and I want students to know that sort of aspect of mathematics. Mathematicians don't make random statements about things. They try to formulate a right question and try to develop in a certain way that the question they pose is helping the big question they originally posed.

Taylor also mentioned the value of their mathematics education in building skepticism, reasoning skills, and knowing connections between different fields of
math. Similar to Taylor, Adrian saw great value in the ability to provide opportunities for enrichment. Adrian remarked that university specialization is of value in building "research and extension projects" and giving advice to students who are genuinely interested in mathematics and university studies.

These remarks focus primarily on skills and practices learned through postsecondary studies in mathematics, as compared to content. When content was the focus of the conversation, Jaime and Bailey expressed seeing little value in advanced content. For them, the notion of advanced content was "too distant" and they saw "limited connection threads" between secondary and post-secondary mathematics. Jaime held the strongest view on this position:

Jaime: I think that a lot of what I did in university math was so distant from what I did in high school, I don't think it was essential. We are doing stuff in three dimensions and all this weird stuff. It's so far away from what high school kids are doing. I think there's a downside of taking a lot of advanced math, that you go pretty deep down the rabbit hole and then you can get out of touch.

Jaime supported this by noting their belief that pedagogical skills are separate from content knowledge and that content knowledge isn't necessary to be an excellent teacher and that advanced degrees might just be "screening tools" for becoming teachers. They shared during the interview that "if someone knew high school math well, they could turn around and teach that well."

Bailey was in partial agreement with Jaime on this, mentioning that "pragmatically, we need more math teachers." Bailey elaborated on this, remarking that "having a love, understanding, and interest" in mathematics is more important than extensive university coursework. They did, however, mention a collection of courses of which it would be useful for a secondary teacher to have. Bailey thought that coursework in calculus, linear algebra, proofs, number theory, and geometry could act as alternative lenses to view the secondary mathematics curriculum through and could offer fun problem solving opportunities.

Even though all of the participants saw at least some value in teachers having post-secondary degrees in mathematics, the participants perceived their degrees as having value to them, personally. The recurring codes in participants' responses were connections, problem solving, and application.

The skill to "build connections" between concepts was seen as valuable to all the participants. Adrian, Casey, and Jaime saw their advanced knowledge having great value in being able to say where content goes later in the curriculum. Jaime summarized this well in saying that their advanced knowledge "gives me a sense of what all this can mean in the end." Adrian took a more research oriented approach to their response, remarking that their advanced degree allowed them to "appreciate the immensity of knowledge that is out there in terms of math and physics and having an idea of what is actually being researched." Pragmatically, Casey viewed that a teacher with a mathematics degree might have conceptual understanding of a particular area of expertise, which they might be able to bring into their classroom for enrichment.

Taylor found value in their advanced degree expanding the scope on what constitutes mathematics. They noted that they did not have a good idea of what mathematics was in high school, but that "in university, I started to realize that it's something very different from what I learned in high school." Since Taylor went through a personal revolution of their views of mathematics, Taylor wanted to share this with their students, with the hope that students could see "that computation is not everything mathematics does, but more about why certain things work the way they do." Similarly, Bailey remarked on how extended content knowledge yields a "bigger sense of how things fit together" and followed this in saying it would help them include some "fun math little tidbits." However, they were unsure on how to do this within the curriculum, as university mathematics "really is diverging from what is taught in high school math."

Concerns of the restrictions of the curriculum were common in participants' views of bringing their advanced content knowledge into their teaching; Adrian elaborated that the role of their advanced degree in their teaching is different in theory and in practice. They remarked on how, even though they see personal value in it, that it might not make a difference in practice:

Adrian: It feels very much like this is what you need to know and it's our job [as a teacher] to get you [the students] to know it. As opposed to this is an interesting field of study, we want to explore it, what kinds of questions can we ask, and leading them on this
inquiry process where we get them to explore ideas beyond their conventional grade level. That just seems like not at all what's happening.

Taylor mirrored this concern, that even if they wanted to include content beyond the curriculum, teaching secondary school is not completely autonomous. They worried that their hopes and goals in teaching might not be achievable in a real classroom.

Experience in problem solving was viewed as a benefit of advanced mathematical coursework by Taylor, Jaime, Adrian, and Bailey. For Taylor and Bailey in particular, experience in proof was viewed as a benefit they both wanted to bring into their classrooms. After stating that their perception of what constituted mathematics changed from high school to university, Taylor remarked that their development to think logically and prove rigourously was a contributing factor. They reflected on the role of computational thinking being heavy in high school and that they did not have a conception of proof. In their university mathematics courses, they admitted that their professors "didn't really explain to me" what it meant to prove something and they struggled in courses where proof was a component. Taylor noted that they wanted to share this experience and university knowledge with their students so that they might not be "totally embarrassed when they go to university."

Bailey also saw immense value of proof in secondary mathematics education, remarking that the habit of teaching mathematics in a "do this, do that" manner
"really seems to kill math." They worried that this mentality makes mathematics "an exercise of just doing steps instead of solving problems." Their hope is that introducing proof techniques, through they might be beyond the curriculum, would be a valuable tool to bring context and explanation to the question of why particular techniques and strategies work or are used.

As Jaime did not have coursework in proof, the value they saw in problem solving was experience in learning how to approach a problem. Jaime remarked on the value of understanding limitations, assumptions, and context of problems, particularly with respect to modelling in problem solving. They viewed their advanced coursework as helping them understand that "creating a model is useful because it tells us something, but we have to remember that it doesn't tell us everything. That's one of the things that I got through my degree is limitations on things that you do and do not know." Adrian saw similar value in problem solving, particularly in applying mathematics to "real world" problems and understanding why these models are useful. However, they feared that their ideas for extensions might not be well-received by students, in which case "everything after first or second year undergrad gets thrown away as not very important."

### 4.3 Advanced Knowledge in Teacher Education

As elaborated above, all of the participants saw some value in their own advanced coursework experience for their future work as teachers, as well as benefits of advanced mathematics courses for secondary teachers in general. However, participants felt that their extended expertise in mathematics was not being drawn upon
in their mathematics teacher education. This concern was succinctly summarized by Bailey, who remarked: "I need my math to teach, but the teacher education program isn't requiring me to have any knowledge of math."

Jaime, Adrian, and Bailey each wanted a more practical and pragmatic approach to their teacher education. Each of these participants expressed a desire for more content focused education with respect to what they perceived as the goals of their methods courses in mathematics:

Jaime: I found them a little bit scattered. I think we are seeing a lot of bits and pieces of here are some neat little ideas, but I find it really hard to pull them together. I think I would have liked to do more or at least see more ideas of how to specifically bring this into teaching the curriculum. I'm more interested in the teaching side. Like how would you introduce a concept? I have found that we got a few neat ideas. One week we did a math art project. And yeah, that's cool for all of us because we like it, but it might not be so relevant for a math teacher.

Adrian: My math methods feel nebulous in terms of what the focus is. It's more like where are some nice connections in math, here are some nice ideas of what a math teacher should be, here are some mathematical related activities, here are some projects, some papers, some analyses. I think it would have been really nice to focus on how am I going to teach this [concept]?

Bailey: We've read a lot of like, theory, theoretical papers, about methods
of teaching math, but not in any concrete way. We didn't go into it with enough meat to do anything with it. It's just this grab basket of oh, you can teach through movement or you can teach math through art. But you can't really. There is a whole curriculum. Yeah you can add that in, but there is a whole curriculum that you need to find a better way to teach.

Adrian and Bailey brought forth interesting perspectives on this end, since they both had two teachable subjects and were taking methods courses in those subject areas.

Bailey: In my other methods courses, we do a lot of really applicable stuff. Like, we look at a paper someone wrote about teaching critical theory about Shakespeare and we look at the actual teaching methods for teaching that topic and teaching different types of writing and book suggestions and like what you teach and all this stuff. And I think it's really building on both our knowledge of English as a discipline and like actually giving us practical ways to teach it. In my math methods, you can do all them knowing literally grade 10 math.

Adrian: I really like the way my other course was structured. It was very much structured around showing cool experiments and how to connect it to the curriculum and give you a chance to teach. In math, I think having a focus on "how am I going to use this?" would have been nice. How do I make all of these teaching ideas and concepts effective and relevant? How do I design an effective
activity that hits all these points, and is engaging, and assess the curriculum?

Adrian and Bailey found that they were building subject expertise through their other methods courses, but found this to be lacking in their mathematics methods courses. They continued to express a desire for "practical, actual math in the classroom" and worry that after their teacher education "we are going to teach math the way we probably would have before. I've come out with no concrete examples for secondary math."

With the exploration of concrete ways to teach content, comes the question of whether or not it is possible in methods courses. Adrian enjoyed opportunities to explore teaching math through social justice, but felt as though "our professors give us an idea that is not enough, but don't really follow up with how to make it enough [in the classroom]," while Jaime remarked that "from grade to grade, the content is going to be different and it's unrealistic to try to cover all that. I'm more interested in the teaching side and how you would introduce a concept." Bailey suggested a pragmatic approach:

Bailey: I think even learning the process of looking at how something is often taught, thinking about how it's taught, thinking of new ideas, doing that for a few topics will probably help you practice doing it for other topics. In my other teachable, I have all these touch points of jumping off new ideas and ways of teaching that I don't feel like I really have from math.

To navigate these struggles, Taylor and Adrian felt that they needed to prompt
themselves through self-guided reflection on how they could use their advanced degree experience, but this was not being prompted through their courses. As Adrian lamented, "I've done it on my own because it's something I'm interested in and something I'm good at, but not something that the instructors have encouraged." Taylor expressed similar experience on their end, remarking that "I think of the materials that I learned in university when I see the material here in the education program and how I can advance that material." However, the feeling of being able to integrate post-secondary expertise into secondary teaching was not held by all participants.

Bailey shared that they felt as though it may not be possible to build subject expertise in their courses, saying that in math "you really are diverging from what is taught in high school math. So there is not use in talking about group theory in an education program for high school math because that's never gonna come, it's a different discipline almost." This is in contrast to their other methods courses, where they said "you're becoming more proficient in writing, which is what your students are doing." Is there such a great divide between secondary and post-secondary mathematics content?

Taylor and Casey perceived some degree of "doing mathematics" in their mathematics methods courses. Interestingly, both Taylor and Casey made mention of the history of mathematics as an example. Taylor took a course in the history of mathematics during their mathematics degree and some of this material was explored in a course in their teacher education program. They saw this extended knowledge being useful in providing alternative proofs and ways of understanding
the Pythagorean Theorem. Casey expressed concern, however, that the content in their teacher education course might not have been as relevant as it could have been for teaching of the secondary curriculum:

Casey: We did cover a lot about the history of zero and the history of one, the stepping stones of math. And I didn't know about Babylonian tablets. But I felt we stopped off at around the year 400 AD . We didn't cover anywhere from the year 1000 to 1900 mathematics. And then I thought what would be relevant for high school mathematics.

The relevance of the history of mathematics was the only direct relationship between the secondary and post-secondary curricula brought forth by the participants. Although this connection existed in their teacher education courses, the participants' responses suggest that this connection may be limited. Indeed, the history explored in the teacher education context is more in line with an ethnomathematical perspective on mathematics education, while the courses offered in mathematics departments tend to be more centred on European perspectives. Regardless, one must ask how this fits in to participants' concerns of "practical and pragmatic" responses to teaching existing mathematics curricula.

### 4.4 Summary

In this chapter, we discussed participants' responses to interview questions which explored their perceptions of the role of advanced mathematics knowledge in their growth as teachers. Through this discussion, a disconnect was observed between
the content versus the beliefs and values learned in post-secondary mathematics coursework. Furthermore, an even greater disconnect was expressed by participants in regard to the ways in which their mathematical expertise was being drawn upon during their teacher education.

All participants expressed a love and affinity for mathematics as a subject and looked forward to bringing this into their teaching. Many of the participants viewed their education in post-secondary mathematics having a personal impact on what they understood to count as mathematics, which they did not experience in high school. Through the skills and values gained in their advanced mathematics courses - such as proof, logic, rigour, and application - participants hoped they would be able to give their future students an opportunity to see the "two-faceted" nature of mathematics as both a way to read the world and as a self-contained knowledge system. The inclusion of problem solving was mentioned by all participants as a skill learned through their post-secondary coursework which they hoped to extend to their secondary teaching to develop critical thinking skills and contextualize mathematical content.

The attributes of their post-secondary mathematics education which participants were excited to bring forward in their classroom were primarily skill based, rather than content based. When the focus of the conversation became about mathematics content, participants did not see much value. Indeed, many participants expressed a great disconnect between the secondary and post-secondary curriculum, considering that university mathematics is too far removed from what students learn in secondary school. While some advanced courses such as number theory
were mentioned as being useful for enrichment, overall, participants felt that the connections between secondary and post-secondary mathematics dropped off after second year university mathematics.

Following this line of thought, participants unanimously agreed that extensive post-secondary mathematics coursework (beyond the second year) need not be a pre-requisite to teach mathematics. While some participants took a more pragmatic approach to this question, addressing the demand for more mathematics teachers, others expressed that advanced content knowledge may not imply better teaching of the curriculum. Participants did not see value in their own content expertise for their teaching, and in turn, did not see the value in requiring such content expertise for other mathematics teachers.

These opinions may have been exacerbated by participants' perceptions of the ways in which their content expertise was being drawn upon in their teacher education, as well as their desire for a more pragmatic teacher education program. In sum, participants did not feel as though they needed the extensive mathematics coursework that was necessary for entrance into their teacher education program. They perceived the amount of mathematics they were doing in their program to be minimal and expressed a desire for critically examining the content of the curriculum. This is in contrast to what they believed to be the focus of their methods courses, which was bringing in fun and interesting connections between mathematics and other disciplines, such as art or social studies. Although participants saw value in this, they expressed concern in not having the expertise to critically examine the existing curricula, current ways of teaching it, and finding better ways to do
so. Overall, participants expressed a disconnect between "the curriculum they have to teach" and the techniques they were learning in their methods courses. Of all the participants, Taylor and Adrian expressed some value of advanced coursework beyond the second year, but their remarks were more based in enrichment for the curious student.

Some participants who had chosen two teaching specialities shared very different experiences in their other methods courses, where they felt as though they were building upon their content expertise to enhance their classroom pedagogy. They felt that their advanced content expertise was of importance, while being drawn upon and extended in these courses.

The initial remarks from participants in this first portion of the interview suggest that the participants do not perceive post-secondary mathematics as an extension of the secondary curriculum. In the following chapters, we examine the aforementioned claim more closely in the context of mathematics questions that do have extensions to the post-secondary curriculum, while also examining content knowledge in more depth.

## Chapter 5

## The Overextension of Familiar

## Mathematical Ideas: A Case of

## Polynomials

The following chapter elaborates on participants' engagement with tasks C and F :

Task C: You are teaching a week on factoring polynomials and you have found that your students are struggling to recognize when they should stop trying to factor. How would you respond?

Task F: You have been teaching a unit on quadratic functions for a few weeks and one of your students asks you why they need to know about them. How would you respond?

Since the mathematics of these tasks are intimately related, the analysis of responses to these tasks will be combined in this chapter. As will be the structure
for the results chapters which follow, I begin with a "mathematical background," which will set the mathematical context for participants' responses. This background is in no way comprehensive, but covers some of the major connections which I have made in my own studies, as well as connections mentioned by participants who engaged with tasks C and F. Next, I examine participants' responses to the tasks and the higher level connections made to advanced mathematics content.

### 5.1 Mathematical Background

Polynomials are a fundamental concept in the secondary mathematics curriculum. Linear graphs are some of the first graphs that students encounter in their mathematical studies and are often used as one of the first examples for the study of functions. Polynomial functions are widely used in many fields outside mathematics in modelling various social and physical phenomena.

Generally, a polynomial of degree $n$ over $R$ is a function which may be written as $f(x)=a_{n} x^{n}+a_{n-1} x^{n-1}+\ldots+a_{1} x+a_{0}$, where $a_{n}, a_{n-1}, \ldots, a_{0}$ are elements in a set $R$ and $x$ is a variable that takes values over $R$.

In this portion of the study, I was interested in exploring the understandings participants' held in regards to the factoring of polynomials. Knowledge of factoring and finding zeros of polynomial functions are key skills in the secondary curriculum, as well as in university mathematics courses. In the following section, I outline some of the major places where polynomial functions appear in the secondary and post-secondary mathematics curricula, along with how and why they
are studied.

In the BC curriculum, students first encounter the notion of lines in grade 9 and continue working with polynomials up to and including Calculus 12. The standard progression of study begins with linear functions (degree one polynomials), quadratics (degree two), and higher degree polynomials. Typically, factoring of polynomials is explored first, before moving to the graphing of polynomial functions, where the relationship between graphing and zeros is particularly useful. Quadratic functions act as a first introduction to a "curved" function, after students have gained confidence working with linear functions. Quadratic functions can be used for modelling phenomena that obtain extreme values. Examples include, but are not limited to: modelling revenue and/or profit, maximizing areas, object trajectories, and scenarios involving time, distance and velocity. Even if quadratics do not precisely describe a particular phenomenon, they act as a welcoming entry point to modelling with functions.

In university, polynomials are central to helping students build an understanding of calculus, both in differential and integral calculus. In differential calculus, polynomial functions are often utilized as examples in nearly all topics, because students are familiar with them. In particular, students' pre-existing knowledge of the existence of roots and knowledge of finding zeros is central to using them as common examples throughout the course. In integral calculus, the use of polynomials is central to the concept of Taylor approximations and Taylor series, which constitutes the latter half of most integral calculus courses. Furthermore, determining anti-derivatives involving polynomial and rational functions constitute a large
portion of the examples students encounter in integral calculus. The most prominent appearance of polynomials in determining anti-derivatives is through the concept of partial fraction decomposition. In this technique, when encountered with a rational function $R(x)=\frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials, students decompose the single rational function into a sum of simpler, rational functions.

The key in this technique is decomposing the denominator into irreducible pieces. For a simpler example, consider $\frac{1}{6}$. When factoring natural numbers, the irreducible components are prime numbers, so we factor 6 as $2 \cdot 3$. And indeed, $\frac{1}{6}$ can be decomposed as $\frac{1}{2}-\frac{1}{3}$. So the question is: What are the irreducible components in the context of polynomials?

Consider the following example of

$$
\int_{0}^{1} \frac{x+1}{x^{2}+5 x+6} d x .
$$

The denominator of the integrand $\frac{x+1}{x^{2}+5 x+6}$, can be factored as $(x+3)(x+2)$. Since the denominator decomposed as a product of two linear functions, we need to find constants $A$ and $B$ such that

$$
\frac{x+1}{x^{2}+5 x+6}=\frac{A}{x+2}+\frac{B}{x+3} .
$$

After a bit of algebra, one can determine that $A=-1$ and $B=2$. Thus, the integral can be rewritten as

$$
\int_{0}^{1} \frac{-1}{x+2}+\frac{2}{x+3} d x
$$

Now, an antiderivative of the integrand is $-\ln |x+2|+2 \ln |x+3|$, so, the Fundamental Theorem of Calculus may be applied to determine the area under the
curve.

These problems become more technical as the degree of the integrand's denominator increases. If the degree of the denominator is 2 , one of two things happens: one, the denominator decomposes as two linear terms, in which case partial fraction decomposition as it was done above is the technique of choice. Or two, the denominator does not factor, remains as an irreducible quadratic (such as $x^{2}+1$ ), so that the anti-derivative may involve the inverse tangent function $\arctan (x)$. Students often encounter a table such as Table 5.1 when learning about the technique of partial fraction decomposition in Integral Calculus.

| Type of factor | Example | Decomposition |
| :---: | :---: | :---: |
| Linear factor | $x-a$ | $\frac{A}{x-a}$ |
| Repeated linear factor | $(x-a)^{n}$ | $\frac{A_{1}}{x-a}+\frac{A_{2}}{(x-a)^{2}}+\cdots+\frac{A_{n}}{(x-a)^{n}}$ |
| Irreducible quadratic factor | $x^{2}+b x+c$ | $\frac{A x+B}{x^{2}+b x+c}$ |
| Repeated irreducible quadratic | $\left(x^{2}+b x+c\right)^{n}$ | $\frac{A_{1} x+B_{1}}{x^{2}+b x+c}+\frac{A_{2} x+B_{2}}{\left(x^{2}+b x+c\right)^{2}}+\cdots+\frac{A_{n} x+B_{n}}{\left(x^{2}+b x+c\right)^{n}}$ |

Table 5.1: Partial fraction decomposition guidelines

This table is dependent on the fact that all real-valued polynomials can be decomposed into a product of linear and irreducible quadratic terms. Indeed, linear terms of the form $a x+b$ and irreducible quadratics of the form $a x^{2}+b x+c$ are the
irreducible, non-factorable pieces in the context of decomposing polynomials, just as prime numbers are the irreducible elements in the context of factoring natural numbers.

Another course in which the factoring of polynomials is of great importance is in Linear Algebra. At its core, Linear Algebra is the study of linear functions and vectors in multidimensional space. Vectors, which have magnitude and direction, are added and multiplied by scalars, while linear functions take vectors as inputs and abide to the rules of vector addition. Matricies, which are the core of study in Linear Algebra, are a way to organize information about linear functions. The majority of introductory Linear Algebra courses in post-secondary restrict their study of vector spaces to "real-valued" space, that is, $\mathbb{R}^{n}$.

The study of Linear Algebra in such courses often culminates with the study of eigenvalues, eigenvectors, and eigenspaces. Many real-world phenomena may be modelled with linear functions and eigenvalues make solving such problems much easier. A classic example of the use of eigenvalues and eigenvectors is the predatorprey phenomenon over time $t$. Suppose that species $x$, wolves, are a predator of species $y$, bunnies. A higher population of wolves will result in a lower the population of bunnies that are available to reproduce. At the same time though, a smaller bunny population means that the wolves have less food available. So, a smaller bunny population will affect the reproduction rate of the wolves. Then, a smaller wolf population makes reproduction easier for the bunnies. This cycle continues with these two populations directly affecting one another and can be represented by a differential equation of the form:

$$
\begin{aligned}
& \frac{d x}{d t}=a x+b y \\
& \frac{d y}{d t}=c x+d y
\end{aligned}
$$

As is, this is a complicated system of differential equations to solve, since the system is "coupled". That is, the two pieces are dependent on each other. However, using linear algebra allows one to "de-couple" the system into independent pieces that can be studied independently. Determining what these independent pieces are is the goal of the study of eigenvectors and eigenvalues.

So what do polynomials have to do with this? The answer lies in the determination of eigenvalues of a given matrix $A$. By definition, $x$ is an eigenvector of $A$ if $A x=\lambda x$. Bringing everything to one side yields $(A-\lambda I) x=0$, where $I$ is the identity matrix. If a non-zero solution to this equation exists, this means that $(A-\lambda I)$ is not invertible. So, the determinant of the matrix $A-\lambda I$ must be zero. The solutions to the characteristic equation $\operatorname{det}(A-\lambda I)=0$ represent the eigenvalues of the matrix $A$. Since the characteristic equation is a polynomial, students must apply their understanding of factoring and polynomial solutions in order to determine eigenvalues of various matrices.

These ideas are expanded upon in-depth if and when students take a course in Abstract Algebra. This course is normally only taken by mathematics majors, with some students interested in theoretical physics enrolling in the course. Abstract algebra is concerned with the study of general algebraic structures. The notions of grade-school arithmetic and high school algebra are placed within a larger, more general theoretical structure. The structures studied abide to pre-defined axioms
and are studied generally. Most universities offer a two course sequence in abstract algebra: one on groups and another on rings and fields. Groups, rings, and fields constitute three categories of algebraic structures with far reaching extensions in mathematics, with groups having relevance in chemistry and physics.

Before trying to understand what a group, ring, or field is abstractly, students often encounter sets that satisfy the axioms of one or more structures. Often, the first example of a group might be the integers, $\mathbb{Z}$, under the addition operation, while the first example of a ring might be the real numbers, $\mathbb{R}$, under the operations of multiplication and addition. These familiar structures are extended to more dynamic ones, including various polynomial rings.

The types of questions one might ask about various rings and field include: What are the irreducible elements? That is, what are the elements of the ring or field in question that cannot be decomposed into less "complicated" pieces? For $\mathbb{R}[x]$, the ring of polynomials whose coefficients are real, the irreducible elements are linear and irreducible quadratics. When one extends $\mathbb{R}$ to include the complex number $i=\sqrt{-1}$, the polynomial ring $\mathbb{C}[x]$ has only linear polynomials as its irreducible elements. This implies the famous Fundamental Theorem of Algebra, which states that all complex valued polynomials in $\mathbb{C}[x]$ may be factored into a product of linear polynomials. Another question might be: How can we create new rings from existing ones? To answer this question, the answer to the previous must first be understood so that the notion of a "prime ideal" may be understood. Then, the notion of a "quotient ring" may be constructed as a way to generate new rings with new algebras. Finally, one might ask whether it is possible to understand the
algebra of one ring by understanding the algebra of another, more familiar one? The answer to this question is yes. One such example is that of polynomial rings whose coefficients belong to a field (such as $\mathbb{R}$ or $\mathbb{C}$ ). In this case, all of the familiar theorems that are true in $\mathbb{Z}$ have analogs in the polynomial ring, including the Division Algorithm, Euclidean Algorithm, and Unique Factorization. Even more information may be gathered if two rings happen to be "isomorphic" to one another, meaning that a perfect bijection exists between the two rings.

### 5.2 Participant Understandings

All five participants engaged with the researcher with discussion of the factorization of polynomials, while all but Bailey engaged with the tasks on quadratics. The popularity of participants choosing these tasks should not be a surprise, due to the importance that is placed on the study of polynomials, both in secondary and postsecondary mathematics courses.

Among the participants, all had taken calculus and linear algebra, with Taylor, Bailey, and Casey having taken courses in Abstract Algebra. Taylor and Bailey took two algebra courses (one on groups and another on rings and fields) at their respective universities. Casey took one course officially, as well as auditing two courses without credit. Adrian admitted to knowing very little about Abstract Algebra, having acquired an understanding of its utility through brief mention in some theoretical physics courses. As an engineering major, Jaime had no exposure to the concepts of Abstract Algebra in their mathematics courses, but did have numerous courses in differential equations, where the techniques of determining eigenvectors
and eigenvalues are of significance.

In their responses to justifying the study of quadratics in secondary school, many participants viewed the study as an access point to studying curved functions and modelling progressively complex phenomena:

Taylor: Quadratic functions incorporate the concept of a curved function and are the basis for how a lot of mechanisms work.

Jaime: $\quad$ For modelling, you would be using lots of more complicated functions and all the work with quadratics is setting up for that, so you can get them into a form where you can do things meaningfully, or so you can find where the zero points are.

Adrian: A lot of things can be modelled by quadratics, as it turns out. They are a pretty good basis for a lot of modelling. Also, it's accessible, it builds on stuff they already know.

Although each of these participants expressed value in teaching quadratic functions as an entry point to modelling complex phenomena, Adrian and Taylor expressed concern about making the concept sufficiently interesting to students. Adrian was concerned that "polynomials can describe certain systems, but outside of science and math, I don't think it's super useful," while Taylor lamented that "my knowledge of applications of quadratics is very limited and I need to find out how I can make students more interested." Taylor expressed that they appreciated math for the sake of math and that a focus on "real world application" takes away from studying mathematics independent of application.

On the other hand, Jaime viewed applications as important to "get the mystique" out of quadratics. They noted that "a lot of the time we just see these equations and they don't really mean anything." Jaime's concern in teaching quadratics was to put meaning to "a formula that just exists almost for the sake of existing." Graphing was considered to be a valuable entry point for building meaning, since "you can just look and see behaviour."

While Taylor, Jaime, and Adrian's responses focused more broadly on the features of quadratics, such as zeros, vertex/extrema, and applications to physics, Casey's responses were example driven:

Casey: You could talk about quadratics and engineering the construction of a bridge. Quadratics have a very symmetric property to it, because the arch is the most economically sound method of connecting to points across a body of water. Speaking of speed equals distance over time, you could talk about quadratics with that and flying aircraft.

When prompted to explain how the second example related to quadratics, Casey emphasized a skill based focus to this problem. Casey stated they would use rational functions as a means to teach the importance of quadratics. However, in their response, it was evident that quadratics were used as a means to solve the problem, rather than motivating the concept independently:


$$
(s-2)(s+2)
$$

$$
\frac{24(s+2)+24(s-2)}{(s-2)(s+2)}
$$

Figure 5.1: Casey's work on quadratics and air travel

Casey: We say a plane flies, the way I learned it was Glasgow to Halifax, with umm, it goes faster going from Glasgow to Halifax, so the airwind is a couple more kilometres per minute, but coming back we get delayed, presuming there is the same wind, because of course, presuming the plane is just pushing you in the same direction, you go faster from G to H than from H to G . So this is the distance, that's some kind of time and it's delayed by two hours and this is two hours. But to solve this quadratic here, what would you get? You would get a $48 s$ and these two would cancel out, but then you would have this problem here and calculate the average speed of the plane. That's what I would explain why is it useful, the arc of a bridge kind of problem and also the plane problem. But I might start with the plane one first.

It is unclear as to why Casey considered this to be a valuable introduction to the use of quadratic functions. While quadratics are useful in modelling phenomenon
with distance, rate, and time, using them in conjunction with rational functions, factoring, and physics might be a difficult entry point for many students. As we will see below, the factorization of polynomials is not as trivial as some may think. Indeed, all participants in the study demonstrated misconceptions regarding the factoring of polynomials, in both their secondary and post-secondary knowledge. Of all the participants, Jaime was the least confident in discussing the factoring of polynomials. Immediately, however, they recognized the importance of this type of open ended question for students.

Jaime: We are developing habits of how do you approach a problem. And part of this is that we have these different techniques we have used and it's going to be a mental checklist of does it look like this or that? You may need to use one technique, two, maybe you can't use any.

Although Jaime recognized the complexity of this problem, when asked to work through some problems, Jaime struggled.

Interviewer: So if we started off with a simple case of factoring a quadratic, how would you respond?

Jaime: It's been a long time since I factored quadratics, but I remember this type, the question and having this in my mind and being not quite sure if I'm done or not.

Interviewer: So what could it look like?

Jaime: I have no idea. It's been a long time. I'm not sure where you are going with that.

Interviewer: What about something like $x^{2}+2 x+1$ versus $x^{2}+1$ ? For these two, how would you approach factoring these? If you can factor or if you can't?

Jaime: I feel really embarrassed. I know there are specific things and patterns, but it's been so long I can't remember what they were. Sorry, I have a vague recollection of where this goes, but I don't know.

Interviewer: Let's talk about this one $\left(x^{2}+1\right)$. Is there a way to graph this function that would tell you what it looks like? Do you remember a way to relate factoring to graphing?

Jaime: So we are dealing with parabolas and shifting. Yeah, I remember the graphing side of it. I like the idea of bringing that in. How that was factored out though, I don't know. It's been so very long. The way I look at this, the idea of shifting it, being symmetric about $x$, these are the things I would tap into. I honestly have no idea about factoring or the point of it, other than finding a difference of squares. So yeah, I see that your zero is going to be here (at $y=1$ ).

Interviewer: So where is your zero in this case?

Jaime: It's at 1 . So that's the stuff I remember, but in terms of factoring, I don't know. But I chose it (this task) because I remember that
feeling of not knowing. There is an uncertainty and I remember it getting worse in university.

This dialogue with Jaime brings up two major concerns. The first being a limited understanding of a fundamental concept from the secondary curriculum. Jaime admitted earlier in the interview that they were a bit rusty on their mathematics and needed to review, but the concept of factoring quadratics is not an advanced secondary concept. However, at the time of the interview, Jaime was half way through their teacher education program and preparing to go on practicum, where they would be teaching in a classroom. If tasked with teaching this fundamental concept in their practicum, Jaime would not only be working on developing the pedagogical aspects of their teaching, but the mathematical as well.

Secondly, the difficulties Jaime experienced in this task raise questions about the degree of review Jaime would have to do before teaching such a concept. The two examples posed to Jaime are two of the simplest examples for quadratic factorization, but the extent to which Jaime was able to talk about these two quadratics was limited. One should question whether the depth of mathematical knowledge for teaching developed through such a review would include HCK and to what extent KDUs would be developed. As a new teacher entering into a career in education, how much time will Jaime have to dedicate to reviewing mathematics to a depth and breadth beyond the context of the prescribed curriculum and text?

Of the remaining four participants, a common thread was shared. This common thread, which points to a major misconception of participants' understanding
of polynomial functions, factoring, and roots, was explicitly observed with each individual participant. At some point of engaging with the task, each participant stated a variation of the following:

## A real-valued polynomial can be factored if and only if it has a root.

Although this is true when the polynomial in question is of degree two, this assumption fails for any polynomial of a higher degree. However, participants each had strongly held misconceptions to this end. Consider the following dialogue with Bailey:

Interviewer: So with the idea of roots and factoring, if a polynomial doesn't have any roots, does that mean it can't be factored?

Bailey: Yes. Yes. And if it doesn't have any nice roots, it can't be factored nicely.

Interviewer: What would you encourage your students to do in factoring $x^{5}-$ 1 ?

Bailey: I would have them graph it. I can't think now of what it looks like. It's going to be a squiggly-ish kind of thing. It's gonna have one root. You know that $1^{5}-1=0$, right? So you can find that.

Interviewer: $\quad$ So what would be a factor of $x^{5}-1$ ?

Bailey: $\quad x-1$ would be a factor

Bailey was correct in their description that if a polynomial has a root at $x=a$, then $x-a$ is a factor of the polynomial. Indeed, connecting graphing of polynomials to factoring could help students build deeper connections between geometric
and algebraic representations of polynomials. However, Bailey later generalized this logic in the opposite direction:

Interviewer: And what would be leftover? It doesn't have to be exact.

Bailey: Let's do polynomial division. Wow, I don't remember how. We are multiplying the $x^{4}$, gonna multiply it in, ummm, $x^{4}-1 \mathrm{mi}-$ nus.........it's gonna be $x^{4}+x^{3}+x^{2}+x+1 \ldots$ something like that. Maybe there will be a negative somewhere.

Interviewer: So can this be factored?

Bailey:
No, no. So since we have a one at the end, things get nice and I think you could do a fun, you could talk about powers of 1 and how if it's negative, if it's an odd power, we could have a negative one, but it's even. We have an $x^{3}$ and an $x$, so if we have negative numbers. We have $1-1+1-1+1$ and we have two of the same thing. This is always going to cancel itself out, so we can't possibly get zero.


Figure 5.2: Bailey's factoring of $x^{5}-1$

Bailey seems to suggest here that in order to determine whether a polynomial can be factored, one just needs to see if it has a root. As mentioned above, this claim is false. Bailey has generalized the logical implication that a root implies factoring to factoring implies a root. Taylor chose the same polynomial of $x^{5}-1$ as Bailey to work with. After finding the factor of $x-1$ and performing long division, Taylor claimed that the leftover quartic factor could not be decomposed any further, as is seen in Figure 5.3

$$
\begin{aligned}
x^{5}-1 & =\left(x^{4}+x^{3}+x^{2}+x\right)(x-1) \\
& =\left(x^{4}+x^{3}+x^{2}+x+1\right)(x-1)
\end{aligned}
$$

Figure 5.3: Taylor's factoring of $x^{5}-1$

Adrian demonstrated a similar misconception and was very explicit in their understanding. Adrian demonstrated a robust understanding in relating the graphs of
quadratic and cubic functions to the existence of roots. The written work associated to this dialogue excerpt may be seen in Figure 5.4:

Adrian: $\quad$ My first instinct is always drawing a picture. If I draw this picture, and I'm going to assume they have learned vertex form, but at the least I can plug it into Desmos. Maybe they don't understand why it looks like that, but you can get it. So this one (quadratic with two roots) looks like this (draws graph). And what we are doing is that we are able to break it down into these two points. These are the points that go to zero and allow us to break it up (factor). The fact that this crosses the x axis at these points allows us to factor it like this. But for this one $\left(x^{2}+1\right)$ it looks like this (above the $x$-axis). It does not have those pieces that we can break it up into, so we are not able to factor it. These are the two possibilities. Well, no. I guess the other possibility is that you have it where it just touches the $x$-axis, in which case, there are two, it's gonna look something like $(x-a)^{2}$. The point being that there are three different possibilities. It touches once, it touches twice, it doesn't touch at all. And that tells us it's going to look like this, can't factor it and it will look like this (above the $x$-axis)

Interviewer: And what about a cubic? What are the possibilities in breaking it down into cases if you had a cubic as well?

Adrian: Well, there is going to be at least one solution guaranteed, since the mean value theorem says there will be one solution. There is a case where you get one solution, two solutions where it will
just touch like that, and a case where you have three solutions, and that's it.

Interviewer: And so for factoring, what would that entail?

Adrian:
I think for this one (a single root) you're going to end up with an $x-1$ and a quadratic without any real roots. This one (two roots)

I think you're gonna end up with one root and a double root on the other side, and this one, you'll have three roots and three linear factors, something like that.


Figure 5.4: Adrian's factoring approach

In this portion of the interview, Adrian demonstrated a deep understanding of the relationship between graphing and roots of polynomial, in the context of cubic and quadratic functions. Adrian was able to connect understandings of the general algebraic form to the impact on the shape and location of the graph in the Cartesian plane and did not need explicit examples in order to demonstrate this understanding. However, when extending the polynomial to a higher degree, Adrian generalized their understanding of irreducible quadratics to polynomials of higher degree.

Interviewer: So we are seeing a relationship between roots and being able to factor. Does that apply generally?

Adrian: I suppose, yeah.

Interviewer: If we move to something like a quartic, something like $x^{4}+1$, what about that? Can this be factored?

Adrian:
No, because it doesn't have any $x$-intercepts. So it's just going to sit like that. The factors link to whether there is a solution to it being equal to zero, because if you let $x$ be one of those values, zero times something that isn't zero is still zero. Whole thing is zero. These are connected to this idea of where does it touch the $x$-axis. So if your function touches the $x$-axis and it's a polynomial, usually, in almost any, I can't think of a case where you can't break it up like this.

Here, Adrian explains why $x^{4}+1$ cannot be factored, by connecting it to the shape of the graph. Indeed, the graph of $y=x^{4}+1$ is a translation of the graph of
$y=x^{4}$ up the $x$-axis by one unit. Thus, the graph of $y=x^{4}+1$ does not have any $x$-intercepts. That is, there are no values of $x$ for which $x^{4}+1=0$. However, this does not imply that $x^{4}+1$ cannot be factored. The roots do tell us something about its factorization, but it does not tell us everything. Since the curve does not have any real roots, we know that it will not have any linear factors of the form $x-a$. Indeed, if it did, we would have that:

$$
\begin{equation*}
y=x^{4}+1=(x-a)\left(x^{3}+b x^{2}+c x+d\right) \tag{5.1}
\end{equation*}
$$

According to Adrian's earlier explanation of cubic polynomials, the polynomial of $x^{3}+b x^{2}+c x+d$ must have at least one root, so $x^{4}+1$ would then have two roots, with $x=a$ being the second (substituting $x=a$ into Equation 5.1 yields an output of zero). So, we could have that $x^{4}+1$ factors into two irreducible quadratics. We can write $x^{4}+1$ as a difference of squares by observing that

$$
\begin{aligned}
x^{4}+1 & =x^{4}+2 x^{2}+1-2 x^{2} \\
& =\left(x^{2}+1\right)^{2}-(\sqrt{2} x)^{2} \\
& =\left(x^{2}+\sqrt{2} x+1\right)\left(x^{2}-\sqrt{2} x+1\right)
\end{aligned}
$$

Indeed, the quadratics of $x^{2}+\sqrt{2} x+1$ and $x^{2}-\sqrt{2} x+1$ are both irreducible as their graphs lie above the $x$-axis and are concave up, as per Adrian's earlier justifications for the irreducibility of quadratics.

Adrian reconfirmed the misconception again later in the interview:

Adrian: If you can graph that (the polynomial) and it doesn't touch the $x$-axis, you are done factoring.

Interviewer: And so we are saying that if we are up at degree five (for a polynomial), we don't....

Adrian: We know there is one (root).

Interviewer: Right, we know there is one root. But we don't necessarily know we can factor the rest? Is that true?

Adrian: Yes.

Casey held multiple misconceptions about the factoring of polynomials and roots at the secondary level. Casey began their discussion with mention of the evaluation of the discriminant of a quadratic.

Casey: I would start with saying well what is the value of $2 a$ ? And we can get different examples of $2 a$ and they can plug it into their calculator. And then we can kind of work and reverse engineer a little and say that ok, well, for any given $a$ here, and we do this separately on a page and do the top, the top bit here (the discriminant of $b^{2}-4 a c$ ). So I would start, I don't even know. 36 and 4 and the 6 and then 9 .

Interviewer: Let's not think of particular examples. In what ways is this going to factor?

Casey: Well, it should look like $x$ minus number, $x$ minus number.
Interviewer: So is it (the polynomial) always going to look like this?

Casey: Yeah, basically.

In this excerpt, Casey claims that all quadratics can be factored into two linear terms. While this is true of some quadratics, it is not true for all. This was somewhat contradicted later in the interview when discussing $x^{4}+1$ and through their description of the relationship between roots and factoring of polynomials.

Interviewer: Let's say something like $x^{4}+1$. Can this be factored?

Casey: No.
Interviewer: Ok, why not?
Casey: Well, because, ok, well, if we could break it down here into say $x^{2}+1$. Then what I would do is make the substitution $y=x^{2}$ and say that's $y^{2}+1$ and then square root of negative one. They (the students) understand square root of negative one. That's what I would do, I would go back to grade 9 kind of material and explain that.

Interviewer: What would you say is the relationship between roots and factoring?

Casey: Well, factoring is just like a visual representation of what the polynomial looks like. And we know that. Umm, I mean, that's how I would start. That is really the very definition of what factoring really becomes.

$$
x^{4}+1=\left(x^{2}\right)^{2}+1
$$

Figure 5.5: Casey's factoring of $x^{4}+1$

In this excerpt, Casey seems to combine the technique of substitution for the reduction of quartics to quadratics and claims that because the quartic reduces to an irreducible quadratic, that the quartic does not factor. Their work can be seen in Figure 5.5

### 5.3 Post-secondary Connections

When asked to discuss polynomials in a post-secondary context, Adrian, Bailey, and Taylor were able to speak to the behaviour of polynomials when complex numbers are considered. Each of these participants was familiar with and mentioned the Fundamental Theorem of Algebra, in one way or another.

Theorem 1. The Fundamental Theorem of Algebra: Let $f(z)$ be a degree $n$ polynomial with coefficients in $\mathbb{C}$. Then, there are exactly $n+1$ complex numbers $w_{0}, w_{1}, \ldots, w_{n}$ (not necessarily distinct) such that

$$
f(z)=w_{0}\left(z-w_{1}\right)\left(z-w_{2}\right) \cdots\left(z-w_{n}\right) .
$$

That is, every polynomial function over $\mathbb{C}$ can be factored into linear factors over $\mathbb{C}$.

Below are excerpts from these participants regarding the extension of polynomials to include complex roots.

Bailey: You could probably factor if the coefficients were complex because you can have $i^{2}$ and you can get some negative ones. So
my whole argument of being able to get zero becomes untrue. $x^{2}+1$ has no roots in the real plane, but it does in the complex plane.

Adrian:

Taylor:

There are $n$ factors and they will either be real or complex. If you want to know how many real ones, graph it and see how many times it touches the $x$-axis, that's about it. I mean, there is always $n$ solutions to it, they are just usually complex. You can talk about these (polynomials) having complex roots and that means different things in different situations. Especially in quantum mechanics and any sort of periodic, ahh, what's it called, Fourier Analysis. Anything with periodic functions, Fourier decompositions, all that sort of stuff, complex numbers and roots are really important.

A quartic polynomial can be factored, but we would have to incorporate the complex number, the imaginary number. It's a Fundamental Theorem of Algebra that when you have a degree of 4 , degree of $n$, there are $n$ solutions including the solution for complex numbers, but then it doesn't tell us how we can write its roots. The proof wasn't very easy to understand, but now that I see those number of roots, how many roots can be there, that they let me know that there are four solutions, four complex solutions to the polynomial equation, not necessarily telling us how to do it, but telling us there are four solutions.

These participants knew the extension of this problem to the context of com-
plex analysis. Indeed, some of the concepts in complex analysis are deeply rooted in problems in the secondary curriculum. However, based on the dialogue in the interviews, their understanding of the significance of this statement may not have been fully developed. The power of the above theorem is that any polynomial may be factored into linear terms that are dependent on its roots. While participants knew the statement of the theorem and recognized its significance in the context of complex numbers and the factorization of polynomials, their dialogue suggests that their understanding on the matter was restricted to the course they took in complex analysis or abstract algebra. That is, participants held limited links between the complex analysis and secondary mathematics content. In what ways could the Fundamental Theorem of Algebra be motivated so that it builds their understanding of Complex Analysis, while simultaneously drawing upon their knowledge (and misconceptions) of related concepts in the context of real numbers?

Casey frequently connected their understanding of post-secondary content. However, the depth in which they were able to do so was limited. In particular, Casey made frequent mention of concepts from Galois Theory, but it seems as though their understanding of these advanced concepts may have negatively impacted their understanding of the material in the secondary context. When discussing the factorization of cubic polynomials, these misconceptions became evident.

Interviewer: Ok, so let's say we have a cubic then, let's start out with $a x^{3}+$ $b x^{2}+c x+d$. If you were to factor this, what are the possibilities? What would it look like?

Casey: Well that would be something that goes back to here (to the alter-
nating group).

Interviewer: And what do you mean by that?

Casey: If it's the 1 then you're going to get the whole number here, if it's $A_{3}$, it's the rotational groupa, if it's the $S_{3}$ it's the, well if it's $A_{3}$ it's all three rational numbers and if it's the $S_{3}$ one then you've got that and two imaginary numbers.

Interviewer: And are you always going to be able to factor this?

Casey: No.

Interviewer: And why not?

Casey: Because you can't. Because you just have to make some examples that end up being something that, well, the one that I had in this lesson plan was like, I guess going back to your question, it was a really really, I guess if you want to call it squirly cubic, but anyway, ultimately one of the $x$ 's happened to be something like this and another $x$ happened to be that. It was something, but I graphed that but then I graphed the inverse.

Interviewer: So focusing on this, so you're saying that it is possible that a cubic cannot be factored?

Casey: Very much so. And in how to factor the $n^{\text {th }}$ polynomial, and sort of how beyond the cubic, it's not always umm, well no, beyond the quartic, it's not always possible. Like we talked about $a x^{3}$. This is an odd polynomial and it might not be factorable. There
are specific cases depending on what relation $a, b$, and $c$ have, but if it works out that this relation happens, it's a lot more likely that this is factorable.

$$
a x^{3}+b x^{2}+c x+d=\rightarrow \text { may not tactor }
$$

Figure 5.6: Casey's factoring of cubics

In discussing the factorization of cubics, Casey tries to relate the cubic polynomials to their Galois groups of the alternating and symmetric groups of three elements ( $A_{3}$ and $S_{3}$, respectively). Indeed, one of the major points of study in a course in Galois Theory is the factorization of cubics and the existence of a "quadratic formula equivalent" for higher degree polynomials, which is more commonly known as "solvability by radicals". In the early 1800s, Paolo Ruffini of Italy and Niels Henrik Abel of Norway proved that, given a polynomial of degree five, there is no algebraic formula to solve for its roots. Evariste Galois later refined these ideas in what was later defined as Galois Theory.

The precise statement to which Casey was probably referring, with respect to the factorization of cubics, is the following:

Theorem 2. Let $f(X)$ be a separable, irreducible cubic in $\mathbb{Q}[X]$ with discriminant $\Delta$. If $\Delta$ is a perfect square in $\mathbb{Q}$ then the Galois group of $f(X)$ over $\mathbb{Q}$ is $A_{3}$. If $\Delta$ is not a perfect square in $\mathbb{Q}$ then the Galois group of $f(X)$ over $\mathbb{Q}$ is $S_{3}$.

Casey was trying to draw upon their post-secondary understanding of the factorization of cubics and corresponding Galois groups, but it appears as though the
exposure to these concepts may have confused their understanding of them at the secondary level. The connection between roots and splitting fields (i.e. where a polynomial can be fully factored) is detailed in the following theorem.

Theorem 3. Let $f(X) \in \mathbb{Q}[X]$ be a separable cubic with discriminant $\Delta$. If $r$ is one root of $f(X)$ then a splitting field of $f(X)$ over $\mathbb{Q}$ is $K(r, \sqrt{\Delta})$. In particular, if $f(X)$ is a reducible cubic then its splitting field over $\mathbb{Q}$ is $\mathbb{Q}(\sqrt{\Delta})$.

With all of the claims that exposure to advanced mathematics content helps deepen future teachers' understanding of the secondary curriculum, this excerpt with Casey suggests an instance of the opposite occurring. Casey also makes the claim that beyond the quartic, you may not be able to factor. Once again, this is an instance of mixing theorems from Abstract Algebra with content from the secondary curriculum. Indeed, it is true that there is no "quadratic formula equivalent" for polynomials above degree 5 and Taylor described this in their interview through the following excerpt:

Taylor: $\quad$ Quadratics pop up quite frequently in many, many areas. Calculus for sure and Galois theory, as well. Quadratic function has formula for its solution. So what we know of as quadratic formula. Even cubic equation has a solution, based on the coefficients of that cubic function, but once you increase the degree of the function to, from five and bigger, you no longer have a formula expressible by coefficients, and isn't it amazing that you don't have the formula out of this coefficients? Because we have a solution for quadratic, cubic, and fourth degree, but why not fifth? And how do we know? How did mathematicians know
this and how did they figure it out? Yeah. But then again, maybe quadratic equation is the most famous for having, deriving the solution, finding the root of that equation, it's really the most basic one. Why does it have it while the fifth degree function doesn't have it?

### 5.4 An Experience of Abstract Algebra

As laid out in Chapter 4, many of the participants did not see their advanced mathematics courses as having a significant impact on their mathematical knowledge for teaching, besides the development of their problem solving skills and understanding of the importance of proof and rigour.

With respect to abstract algebra and the factoring of polynomials in this particular task, Bailey shared a profound reflection of some of their advanced, abstract mathematics courses. Recall that Bailey was one year out of their undergraduate degree at the time of this interview.

Interviewer: In thinking about all the courses that you've taken, when did the study of polynomials come up?

Bailey: Calculus. In calculus, you talk a ton about polynomials and you need to understand how they work. Math like really does move away from numbers or functions. You're really just talking about concepts and ideas. Especially in something like abstract algebra. You're really out there....

Interviewer: Did you talk polynomials in abstract algebra at all?

Bailey: Ummm, I feel like no, it's been a long time. That course is a blur.

Interviewer: Why was it a blur?

Bailey: I had Professor X. Do you know him? That was my professor. I barely survived. So, I took group theory with him and I also took honours linear algebra with him in my second year. And he is obviously a genius but he doesn't teach in a way that's accessible. Once you lose the train of the class you're just gone for the rest of it. So you kind of have to teach yourself the whole thing, especially if you're not able to keep up with the class, which I really was not. So, you can tell he's a genius, but in terms of teaching, I think it was really good for the couple of students that were also geniuses and could keep up with that.

Interviewer: And did you take another abstract algebra course after that?

Bailey: Yeah, I took ring theory after.

Interviewer: And did you, what did you talk about in that class?

Bailey: I really forget ring theory, I gotta say. Yeah, it's a total blur. It's interesting because I was supposed to have someone who was supposed to be very good and then she got sick and had to leave. So I had a sub kind of thing, a grad student, and he was not very good either. But it moved at a slower pace and I could follow it more. I feel like there are some polynomials in ring theory. I really, I actually don't remember those classes. Which is strange.

It was only 2 years ago. I never applied it to anything else. They were just isolated things and I never looked at them at all.

Interviewer: When you took those courses, who did you think the professors were seeing as the target audience?

Bailey: Like, people who were going to do math research, probably.
Interviewer: Do you think that's what the, who the target audience should be?

Bailey: Umm, I don't know. It's interesting at that level. Because that math is pretty advanced and it's applicable, but not really beyond math. So I guess it's fair that's who the audience is.

The examination of most abstract algebra texts would yield significant content associated to polynomials. One could argue that claiming polynomials were not a topic of study in abstract algebra would be equivalent to never examining the Renaissance in an art history course. The study of these structures occupies a significant portion of many courses in this realm.

Rather than focusing on the fact that Bailey did not recall learning this content in their courses, I turn the reader's attention to the experience Bailey had as a student in these courses. Bailey shares that their experience in this course may not have been the most conducive for learning. Recall that Bailey was a strong student, who graduated with an $89 \%$ average in their university mathematics courses. This leads one to question what advanced university mathematics courses are offering future teachers, or more broadly, students who do not plan to take careers in academia? I will return to this theme in the discussion in Chapter 8 .

### 5.5 Summary

In this chapter, I examined participants' engagement with tasks $C$ and $F$, both of which explored the content domain of polynomial functions. All five participants engaged with task C , with Bailey being the only participant who did not engage with task F. The analysis of the dialogue from these tasks revealed that participants, overall, did not hold conceptualizations of polynomial functions that bridged secondary and post-secondary mathematics. Indeed, as was elaborated in Section 5.4, Bailey perceived the content of abstract algebra to be very "out there" and not relevant to the content of the secondary curriculum. While Bailey was the only participant to explicitly state such a disconnect, the lack of connections expressed by other participants prompts further questioning. Even if participants do recognize content connections between post-secondary and secondary mathematics, these connections may not be substantial enough to impact their practice. Indeed, Taylor recognized that the work in Galois theory was related to Task C, but they did not perceive it as having potential in their pedagogical practice with the task.

I argue that the study of polynomials in abstract algebra does have potential to impact a teacher's mathematical knowledge for teaching. All participants who responded to the task recognized a substantial connection between graphing, the existence of roots, and factoring. As many participants expressed, the graphing of a quadratic can be easily extended to understand the factorization of the polynomial in terms of its roots. This is also true in the case of cubics. However, these understandings were overextended to all polynomials, with the claim that " a
polynomial can be factored if and only if it has a root." This misconception could be adjusted with exposure to the ideas of irreducible polynomials, as studied in a course in abstract algebra. However, the participants who did have this coursework experience held the same misconception. The only secondary to post-secondary connection that was expressed by multiple participants was the connection of factoring to complex valued roots and the Fundamental Theorem of Algebra. That is, if a quadratic has no $x$-intercepts, it has a complex root. However, this conceptualization was extended to quartics and higher-order polynomials.

Even though the behaviour and mathematics of quadratics was overextended to all polynomials, the ideas and concepts of quadratics are a central component of the secondary curriculum. Many participants expressed the study of quadratics as a beginning access point for ideas such as graphing, factoring, and mathematical modelling. Participants expressed significant value towards problem solving, application, and mathematical modelling in Chapter 4 and viewed quadratic functions as an accessible context for exploration and the development of problem solving skills. At the time, however, participants also expressed concerns regarding the limitations of applications involving quadratic functions. While participants were aware of some commonly used applications, such as kinematics, they were unsure on how to make the concept meaningful and interesting for a wide variety of students. They were further concerned that repeating the same types of applications of quadratics might actually hinder student's interest and perpetuate the notion that mathematics has limited "real-world" applications.

Participants' engagement with these two tasks is a valuable first look into the
connections future teacher's have constructed between their advanced mathematics knowledge and the content they are to teach. The forthcoming chapters will further the exploration of such connections in other content domains.

## Chapter 6

## The Role of Limits, Infinity, and <br> Formal Definitions in Secondary

## Mathematics

### 6.1 Introduction

Many degrees in pure mathematics are characterized by two courses: Abstract Algebra and Real Analysis. These courses constitute the basis for more advanced coursework, graduate studies, and mathematical research. Prior to taking these courses, many undergraduates enrol in a course on Mathematical Proof, in which the principles of proof, abstraction, and common definitions in many mathematical domains are studied. In Chapter 5, I discussed Abstract Algebra at length, with respect to polynomial functions and factoring. In this chapter, I will examine participants' engagement with the tasks that relate to courses in Real Analysis,

Calculus, and the introduction of advanced mathematics. The interview tasks that draw on these subjects include the tasks on inverse functions, limits, and exponents.

### 6.2 Inverse Functions

I remind the reader that the task regarding inverse functions was the following:

You are teaching a unit on inverse functions. What would you include in your
lesson plan?

This task was partially inspired by the work of Zazkis and Kontorovich (2016) and Leung et al. (2016), where they examined teachers' understanding of inverses and their associated notations. Among the five participants, Adrian and Casey engaged in this task. Before exploring their understandings of this topic, let us dive into some of the mathematics of inverse functions to provide context for the interview data in Sections 6.2.2 and 6.2.3. The responses from these two participants vary drastically in their appropriateness and depth of understanding.

### 6.2.1 Mathematical Background

Among all the concepts covered in this study, the notion of inverses has the earliest appearance in the school curriculum. The term "inverse" is very general, mathematically. Indeed, the first place in which an inverse is studied in school is with respect to addition and subtraction. When the idea of 0 is presented to students, you could see questions like "if you have two cookies and I take two away, how many cookies do you have?" The student would reply that they have no cookies and that we denote the idea of "none" by the number 0 . Similarly, one could ask "if you have two cookies and I share none with you, how many do you have?" The
number 0 is known as the "additive identity" in the real numbers. That is, if you take a real number $a$ and add 0 , you still have $a$. Then, one would say that $-a$ is the additive inverse of $a$, since $a+(-a)=0$.

A similar idea is explored when students gain the key developmental understanding that a fraction is a kind of number. Initially, when fractions are learned in school. they are understood as a piece of a whole. For example, $\frac{1}{2}$ would be viewed as one half of a whole of something, whether that something be the area of a square, a pizza, or a collection of cookies. Later on, students learn that you can treat $\frac{1}{2}$ as a number. Just as you can add 2 to 1 , you can add $\frac{1}{2}$ to 1 through the properties of fraction addition, so that the mixed number $1 \frac{1}{2}$ is equivalent to $\frac{3}{2}$.

Furthermore, fractions can be multiplied, just as whole numbers may be multiplied, once additional properties of fraction multiplication are explored. Then, students can come to realize that $\frac{1}{2} \cdot 2=1, \frac{3}{2} \cdot \frac{2}{3}=1$, and $\frac{1}{2} \cdot 1=\frac{1}{2}$. By "flipping" a number and finding its "reciprocal," the product of the original number and the reciprocal equals 1. That is, given a real number $a, a \cdot \frac{1}{a}=1$. In formal language, we say 1 is the multiplicative identify element of the reals and that $\frac{1}{a}$ is the multiplicative inverse of $a$.

The notion of inverse is then generalized to concepts outside of numbers, the first being the notion of an inverse function. Just as additive and multiplicative inverses return a number to the identity, functional inverses undo the output of a function and return an identity element. The identify element in the realm of functions is the function that returns exactly what was put in: that is, $f(x)=x$.

However, in the previous examples, the identity was followed by an operation. In the instance of functions, the operation is function composition. The following two definitions are that of the identity function and inverse functions. In these cases, we take $X$ to be an arbitrary set of elements.

Definition 1. The function $f: X \rightarrow X$ defined by $f(x)=x$ for all $x \in X$ is the identity function on $X$. We use $I_{x}$ to denote the identity function on $X$.

Definition 2. Let $f$ and $g$ be functions. $f$ and $g$ are inverse functions if and only if $f \circ g=I_{D_{g}}$ and $g \circ f=I_{D_{f}}$, where $D_{f}$ and $D_{g}$ are the domains of $f$ and $g$, respectively. We say that a function $f$ is invertible if and only if an inverse exists.

This is the final appearance of inverses in the school curriculum, but not in the post-secondary curriculum. Any student who takes a course in Linear Algebra will see the notion of inverses for matricies. Similar to the earlier instances of inverses and identity elements, these must be defined in the case of matricies. Special "matrix multiplication" and "matrix addition" are defined in the first week of any linear algebra course, along with "additive" and "multiplicative" identity matricies.

The notion of inverses is fundamental in all fields of mathematics. These concepts are studied generally in courses such as an introduction to proof, abstract algebra, real and complex analysis, topology, differential equations, among many other courses. Essentially, all mathematics majors will encounter the study of inverse functions multiple times through the course of their mathematical studies.

### 6.2.2 Participant Understandings

As mentioned above, Adrian and Casey were the two participants to engage with this task. The responses given were very different between these two participants, with Adrian demonstrating a depth of understanding of inverse functions, while Casey demonstrated multiple misconceptions at the secondary and post-secondary levels.

Adrian's initial response to the question was immediately indicative of their depth of understanding:

Adrian: I did a little bit about this awhile ago. So when you say inverse functions, you mean inverse, not reciprocal functions? Like arcsin, arccos, log?

Interviewer: Well, what's the difference?

Adrian: Well there is a huge difference!

Interviewer: Tell me about it!

Adrian: Well in one case, you have a function, you can talk about the reciprocal of that function which is 1 over the function. For every value, take 1 over that value. And typically that's done in a graphing sense and if I want to graph the reciprocal of that function, basically anywhere it is zero, I'll have an asymptote and anywhere in between, if it's big and positive, it will be small and positive. If it's big and negative, the reciprocal will be small and negative. And you go through this whole sort of, usually what you'll end up with, if you have some sort of polynomial curve,
you're going to end up with something with big dips in places. It's never zero because 1 over something can never be zero. It can be very small, but not 0 , unless it goes to infinity, in which case I guess it would be zero, technically sort of.

Interviewer: So this is reciprocal functions?

Adrian:
Yeah, these are reciprocal functions, which is not what we are talking about, but some people call them inverses because you are taking the, technically this is the multiplicative inverse of the function, but that's not what we are talking about. What we are talking about is inverse functions, which is taking the opposite of the function. So if I have $f(x)=y$, what I am talking about is $f$ inverse of $y$ and basically swapping $x$ and $y$, which is what is taught in school. You're taking the opposite of that function, which is a weird idea in and of itself. And sometimes it's easiest to look at how you would go about doing this in practice. Classically, what usually gets done is that you can get between these two by simply swapping $x$ and $y$, and that's equivalent to reflecting over the line $y=x$ and this usually gets paired after transformations so that they know how to do reflections over a certain axis. Some stuff I like to do is if you have a piece of paper that is sufficiently thin and you can draw a thick black line, we can actually flip it around like this and look through the piece of paper and see what it would look like and physically perform the manipulation. Which is kind of valuable, especially if students
are struggling with the actual mental how do I flip that? And I think beyond that, examples are really awesome. logs and exponents. You can talk about undoing functions. If they have already gone through lessons on exponents, they have probably encountered logs, so the idea that you could undo this, going backwards, doing the opposite of this. sin, arcsin. I don't think there are any other common inverse functions I can think of. You can talk about domain restrictions, so in order for this to still be a function you still need all your function things, which is more of a grade 10 topic, but it's still brought in as to what it means to be an inverse function. And you can define your inverse functions differently, depending on what range you want to talk about.

In this excerpt, Adrian immediately spoke to detailed understandings on the differences between inverses and reciprocals, with respect to functions. Adrian was later able to formalize their understanding of inverse functions via his claim that "if $g(x)$ is the inverse function of $f(x)$, then $f(g(x))=g(f(x))=x$. In their description of inverse functions, Adrian began to speak to domain restrictions, but this was not explored further in the interview.

Adrian brought up an interesting understanding with respect to logarithms. Other participants, when engaging with the task on exponents, conceptualized the logarithmic function as being independent of exponential functions. However, Adrian shared his conceptualization of the logarithmic function as follows:

Interviewer: And so for the case of $y=e^{x}$, how do we get the inverse function

$$
\log ?
$$

Adrian: $\quad$ This reflection, by swapping $x$ and $y$ and defining something. Really, that's the way logarithms are defined, is what is the thing that undoes an exponential? We call it log. This is what it looks like because we know this whole idea that if we have an inverse, we swap the $x$ and $y$ coordinates and we insist that it has these properties and it's logically consistent and now we have logarithms in our mathematical construction.

Adrian shares their understanding that the construction of the logarithm is a consequence of the properties of exponential functions. Historically, the construction of logarithms came well before exponential functions, due to their properties of mapping multiplication to addition. Regardless, the construction that Adrian shares is the one that is most commonly seen in modern school mathematics. Overall, Adrian demonstrated a full-bodied understanding of functional inverses in this portion of the interview, sharing some common misconceptions students may make, as well as in-depth applications, which will be described in the following section on post-secondary understandings.

When Casey started to respond to this task, they immediately conveyed misconceptions about inverse functions.

Casey: You could have something to do with the very fact that these two circles are reflected in the line $y=x$. Now, I mean there is a lot more umm, you know, I guess inherent with inverse functions.

Interviewer: Such as?

Casey: Well, if we want to, it depends because if it's grade 11 then it would be algebraic inverses.

Interviewer: What is an algebraic inverse?

Casey: It's sort of flipping the parabola around (draws graph of $y=$ $\pm \sqrt{x}$.

Interviewer: So the inverse of this (graph of quadratic) is that (graph of $y=$ $\pm \sqrt{x})$ ?

Casey: Yes, but then we start to talk about one-to-one and onto, right? But then we start talking about real number domain and you begin to ask why can't it be the U here and just over here?

In this portion, Casey demonstrates two misconceptions, but also alludes to a detailed understanding of domain restrictions with respect to inverse functions. The first misconception is in their first claim of reflecting circles being an inverse. The notion of "swapping $x$ and $y$ " corresponding to a reflection over the $x$-axis is relevant to functions, but a circle is not a function. Immediately, Casey provides an example that is relevant to a unit on symmetry and reflections, but not with respect to inverse functions.

This same misconception is demonstrated when Casey draws the inverse of a quadratic function in the shape of a sideways "U". Casey begins to clean up this misconception by mentioning domain restrictions. Indeed, if we consider the
function $f(x)=x^{2}$, its inverse function is $g(x)=\sqrt{x}$. The relation $h(x)= \pm \sqrt{x}$, as Casey drew, is not a function (it has two outputs for every input). Even though $y=x^{2}$ is an easy example in most mathematical contexts, it is not a simple example in the case of inverse functions. This is due to the following theorem:

Theorem 4. A function is invertible if and only if it is one-to-one and onto.
Casey had this vocabulary and used it throughout the interview, but also demonstrated misunderstandings while using it:

Casey: It was in mathematical proofs, it was actually the first time I ever saw what an inverse really means. To be one-to-one and onto and if both those criteria are satisfied, then it is a true inverse.

Interviewer: What do you mean by that? If a function is one-to-one and onto, then what does that mean?

Casey: Well, injective and surjective.

Interviewer: Then it is...?

Casey: Then it's like a real inverse, I think

Interviewer: And what does that mean?

Casey: Well, oh I know, it means the vertical line test. In grade 10, you take just a ruler and go across the graph here and of course, just the regular U (parabola) it's all one-to-one but when you turn it 90 degrees, no it's not one to one, there are two points, and it's the inverse of the left hand side of the $y$ graph there. So there's a problem there, it's not one-to-one.

Interviewer: The sideways one is not one-to-one?

Casey: Yeah, the sideways horseshoe.

Interviewer: And what does it mean to be one-to-one exactly?

Casey: $\quad$ That every $x$ coordinate maps to every $y$ coordinate exactly once. There is a unique point. But what I feel an inverse functions, and that's why I chose this task, is there's a real, there's a real problem here, especially with this cubic thing happening here that I found students had trouble with, is that well, the vertical line test, this is the first time they have ever seen an S shape, but the one-toone problem here is they kind of understand taking a ruler and seeing like you just go across the graph and see that vertically every point is one-to-one

Interviewer: So the vertical line test is for showing that something is one to one?

Casey: Yeah, I should call this actually the "C" parabola shape, well, the letter "C", no, you can just stick the ruler right there and you have two points. But the biggest disconnect is realizing that the lower point cannot map to the one that's over there.


Figure 6.1: Casey's work for the "inverse" of $y=x^{2}$

In this dialogue excerpt, Casey starts off well by noting that inverse functions have something to do with being one-to-one and onto. In order to clarify the mathematics here, we say that $f$ is a surjective (onto) function if and only if the range (i.e. the possible outputs of the function) of $f$ is equal to the codomain (i.e. the set $f$ sends its inputs to). For example, $f(x)=x^{3}$, where $f: \mathbb{R} \rightarrow \mathbb{R}$, would be an onto function, since the range of $f(x)$ is all real numbers, which is equal to the codomain of $\mathbb{R}$. However, the function $g(x)=x^{2}$, where $g: \mathbb{R} \rightarrow \mathbb{R}$, is not one-to-one, since the range of $g(x)$ is $\mathbb{R}_{\geq 0}$, which is not equivalent to $\mathbb{R}$. One could construct $g$ to be onto, by changing the codomain. However, this is not customary in practice.

Similarly, a function is one-to-one if every output comes from a single input. For example, $f(x)$ as above would be one-to-one since there are no two inputs that yield the same output. However, $g(x)$ as defined above is not one to one, since every output (other than 0 ) comes from exactly two inputs.

From this, we can observe Casey's understanding of what it means for a func-
tion to be one-to-one is flawed. In fact, one does not need these more advanced definitions to see this. The vertical line test is presented in high school as a test to determine whether or not a graph is a function. However, Casey uses the vertical line test to justify that $f(x)=x^{2}$ is one-to-one (which it's not) and that the graph of $\pm \sqrt{x}$ is not one-to-one. There are two major issues with this second remark. Firstly, $\pm \sqrt{x}$ is not a function, since the various inputs go to two outputs. As a consequence, trying to remark whether this graph is one-to-one function is impossible. The function of $g(x)=x^{2}$ is often utilized as a counterexample to the notions of one-to-one and onto, due to students' familiarity with it. The use of it as a primary example for introducing inverse functions could be more confusing for students, rather than helpful. As a teacher myself, this is an example I would avoid until later in a lesson plan.

When describing inverse trigonometric functions, Casey brought in the notion of one-to-one, but without the misconceptions of earlier:



Figure 6.2: Casey's written work for the inverse of $\sin (x)$

Casey: This would have to be restricted one-to-one, this we can call $\sin (x)$ and you can see that this looks like a [graph of] $\sin (x)$
but we would need to know to stop here at this point and then just make sure it's back at this point to make $\arcsin (x)$.

Interviewer: What do you mean of this point back to that point?

Casey:
Well, to make it one-to-one, so it does not overlap. Because the minute you cross this root here, you're in trouble. You get a repeat with the first one because it's periodic.

In this excerpt, Casey demonstrates a correct conception of one-to-one, with a minor flaw. Casey was able to effectively communicate that in order to define the inverse function $\arcsin (x)$, we needed to restrict the domain of $\sin (x)$. This, however, was the only instance throughout the interview where Casey demonstrated a correct conceptualization. Later on, Casey returned to their generalization of inverses "switching $x$ and $y$ " with respect to their example with circles (see Figure 6.3 ):

Casey: I wanted them to do this concept here [of $(x, y)$ to $(y, x)$ ]. Any old way they wanted to show me, as I walked. around that oh, yeah, $(8,1)$ is now $(1,8)$. If you have the Euclidean, because the inversion here is $y=x$, you just flip the image up here and you can do that.


Figure 6.3: Casey's work for circles and inverses

Casey seems to have generalized the notion of a reflection to that of inverse functions. Although there is a connection between real valued functions and reflections over the line $y=x$, this can not be generalized in the way Casey outlines above. Through this dialogue with Casey, it is evident that they have extensive knowledge of inverse functions, but also hold some misconceptions that could impact their pedagogical practice.

### 6.2.3 Post-Secondary Connections

The analysis above documents participants' understandings of inverses that is relevant to their teaching in a secondary classroom. In the following section, I describe the understandings participants held in the context of post-secondary mathematics, and the ways (if any) they thought this knowledge could be brought into their sec-
ondary classrooms.

Adrian continually mentioned exponential and logarithmic functions throughout their description of inverses. Through their discussion, they thought that discussion of logarithmic scales could be of great value to secondary students:

Adrian: Well, even if they heard of scales that are logarithmic, but have never thought of what it is to be a logarithmic scale. It's weird, even now, doing the mental acrobatics to actually figure out that if it is increased by a factor of 10 , it really is increasing by 1 . It's bizarre, I think, for a lot of students.

In response to the number $e$, I inquired as to why Adrian thought it was so important for students to know and care about $e$ :

Adrian: Honestly, I did not understand why $e$ was so important until I did calculus. If $y=e^{x}$ and I take the derivative of $y$, it's also equal to $e^{x}$. And this is the only function for which it is true. That's kind of neat. This is the only function for which its derivative and all of its subsequent derivatives are equal to the value of the function.

Interviewer: And how do you define $e$ ? When you think of $e$, what is the first definition you go to?

Adrian: $\quad$ This one [the differential equation], $100 \%$.

Interviewer: Any others?

Adrian: I know there is one definition where it is like, $\left(1+\frac{1}{n}\right)^{n}$, limit as $n$ goes to infinity, where this comes back to compounding interest. I guess the other I can think of is just Taylor series, but that's the only other one I can think of.


Figure 6.4: Adrian's work for the value of $e$

In this excerpt, Adrian shares three different definitions of $e$ that could be used in teaching (see Figure 6.4). Each of these definitions was covered in postsecondary mathematics, but as is mentioned at the beginning, Adrian did not really learn the importance of the number $e$ until university. This is an explicit example of advanced mathematics contributing to a secondary teachers' mathematical knowledge for teaching in the secondary classroom.

Casey's connections of inverses to post-secondary mathematics were based in coursework. Casey's immediate mention of inverses was with respect to a course in mathematical proof, which was elaborated in an excerpt in the previous section. When prompted to discuss. inverses in a post-secondary context, Casey continued to talk about mathematics without accurately describing what was being discussed. For example, Casey discussed the field of algebraic topology as being relevant to
inverse functions, but was unable to articulate how: "Sort of the topological sense of these inverse functions go into more in the realm of the hypercube kind of thing. Because it's a lot more topological wise where you can deform a shape to ultimately create an inverse rather than just taking the parabola and going, you need the Euclidean plane."

At this point, Casey was on the verge of developing a very interesting example of inverses, which aligns with the notion of "undoing," as Adrian mentions is important with inverses. Indeed, Casey is referring to functions which will deform one space into another, while the "inverse" will bring us back to where we started. Precisely, what Casey is referring to is known rigourously as a homeomorphism. Abstractly, homeomorphisms are continuous, bijective functions from one space to another, which have a continuous inverse. The classic example of a homeomorphism would be the deformation of a coffee cup into a donut. Indeed, if a coffee cup were made out of moldable clay, one could "deform" and manipulate the clay to transform it into a donut. This can be done without tearing or poking holes into the clay.

While Casey had this example in their mind, they were unable to articulate the ways in which it was connected to the earlier discussions on inverses (through undoing a function). Rather, Casey attempted to relate it to the example of the parabola, which as was discussed earlier, is an inappropriate example for this prompt. Although Casey held this more advanced knowledge, they were unable to link it to what they already knew. Once again, this example is a demonstration of the power of post-secondary mathematics knowledge in providing examples and
contexts for building a deeper understanding of secondary mathematics concepts.

### 6.3 Limits

The task on number sense and limits which participants responded to was the following:

A student is working through a problem and asks you whether $0.99999 \ldots=1$.
How would you respond?

Of the five participants, Adrian, Bailey, Jaime, and Taylor shared their understandings and the pedagogical choices they would make in the classroom. This task interrogates participants' understanding of number, limits, and approximations.

### 6.3.1 Mathematical Background

Students explicitly encounter the notion of a limit if and when they take a math course that examines the notion of asymptotes. Asymptotes are first seen in relationship to rational functions, which may be studied in grades 10 and 11. Asymptotes have significant applications to the physical and social sciences, through the modelling of various phenomena that "level out" over time.

In high school, asymptotes are often introduced as features of graphs. However, the precise definition of an asymptote requires the notion of a limit. These may be observed through the following definitions.

Definition 3. The line $y=L$ is a horizontal asymptote of the graph of $f(x)$ if $\lim _{x \rightarrow \infty} f(x)=L$ or $\lim _{x \rightarrow-\infty} f(x)=L$.

Definition 4. The line $x=a$ is a vertical asymptote of the graph of $f(x)$ if $\lim _{x \rightarrow a^{-}} f(x)=$ $\pm \infty$ or $\lim _{x \rightarrow a^{+}} f(x)= \pm \infty$.

The avoidance of the limit definition of asymptotes is challenging, as it often causes a misconception that asymptotes are lines that graphs "cannot cross," resulting in a less than full-bodied understanding of what a limit represents.

The notion of a limit is the foundation of all calculus. Indeed, the derivative and the integral are both limits, at their core. All courses in Calculus begin by defining the notion of a limit. However, the level of rigour in which the limit is defined varies. In Stewart's Calculus, which is probably the most widely used Calculus text in North America, the definition of the limit is as follows:

Definition 5. We write $\lim _{x \rightarrow a} f(x)=L$ if we can take the values of $f(x)$ arbitrarily close to L (as close to L as we like) by taking x sufficiently close to a (on either side of $a$ ), but not equal to $a$.

This definition uses non-rigourous language to define a limit. Computations of limits are then relegated to a "plug and chug" method, where most of the functions studied are ones which are continuous, have a removable discontinuity at $a$, or have a jump discontinuity at $a$. This restricts the types of functions that students have exposure to in first-year calculus, while simultaneously simplifying the complexity of a limit.

The proper definition of a limit is seen once a student enters a course in Real Analysis. Rudin's Principles of Mathematical Analysis is a widely used text in North American undergraduate real analysis. The definition of limit is given as
follows:

Definition 6. We write $\lim _{x \rightarrow a} f(x)=L$ if there is a point $L$ with the property that for every $\varepsilon>0$ there exists a $\delta>0$ such that $|f(x)-L|<\varepsilon$ for all points $x$ in the domain of f for which $0<|x-a|<\delta$.

This definition provides a mathematical context for the words "sufficiently" and "arbitrarily" that are used in Stewart's definition of limit. Through this use of rigourous language, the family of function in which one can study limits of functions opens up dramatically.

This is not the only instance of the notion of a limit that students run into in their mathematical careers. Indeed, the notion of a limit is observed in the concept of sequences, which is a concept that may be studied in school at a very early age. For example, the Fibonacci numbers are an example of a sequence. This sequence is commonly used with school age children to connect mathematics with nature and art, as this sequence has connections to the organization of pine cones, the nautilus shell, the golden ratio, and other natural phenomenon. Mathematically, the Fibonacci sequence is the sequence of numbers $\left\{F_{n}\right\}_{n \geq 0}=$ $\{1,1,2,3,5,8,11,19,30,49, \ldots\}$. Each new term in the Fibonacci sequence is generated by adding the two terms prior. That is, $F_{n}=F_{n-1}+F_{n-2}$.

When students first encounter sequences in Calculus, the definition of the limit of a sequence is colloquially simplified, as it was for functions. The following definition is seen in Stewart's Calculus:

Definition 7. A sequence $\left\{a_{n}\right\}_{n \geq 0}$ has a limit L if we can take the terms $a_{n}$ to be as
close as we like to $L$ by choosing $n$ to be sufficiently large. Then, we say $\lim _{n \rightarrow \infty} a_{n}=L$ and that the sequence converges to $L$.

The use of colloquial language in this definition may lead the learner to make certain generalizations about the convergence of sequences due to a limited collection of sequences which may be studied. Again, the formal definition of a convergent sequence may be viewed in Körner's Companion to Analysis:

Definition 8. We say that a sequence $a_{1}, a_{2}, \ldots$ tends to a limit $L$ as $n$ tends to infinity if, given any $\varepsilon>0$, we can find an integer $n_{0}(\varepsilon)$ such that $\left|a_{n}-L\right|<\varepsilon$ for all $n \geq n_{0}(\varepsilon)$.

Once again, one can observe the rigour given to the words "sufficiently" and "arbitrarily" when the context is changed from Calculus to Real Analysis. Many learners come into courses in real analysis with pre-existing conceptions of what it means for a limit to exist, based on their experiences and definitions in calculus. In all honesty, I was one of these students. When I entered my first course in real analysis, it was as though I had learned nothing at all in my calculus courses. If anything, I would argue that my studies in calculus hindered my initial learning in real analysis, since I came in with certain ideas of what was the "right" way to deal with limits, which was invalid in this new context.

The final instance of limits which will be discussed in this section, is that of series. A series, at its core, is an infinite sum of a sequence.

Definition 9. The series $\sum_{n \geq 0} a_{n}$ converges if and only if the sequence of partial sums given by $\left\{S_{n}\right\}_{n \geq 0}=\left\{a_{0}, a_{0}+a_{1}, a_{0}+a_{1}+a_{2}, \ldots\right\}$ is a convergent sequence.

The convergence of a series is dependent on the notion of a limit, since we want to show that the sum of the various terms of the sequence have a limit, as $n$ goes to infinity. As will be described in the next section, one participant used their understanding of sums, limits, and series to describe the equivalence of $0.999 \ldots=$ 1 in their interview.

### 6.3.2 Participant Understandings

Upon first glance of this problem, three out of four of the participants went to the same example: the equivalence of $\frac{1}{3}$ and $0.333 \ldots$.

Taylor: $\quad$ Starting with the fraction 1 over 3, cause that leaves us point 333 that's going to be infinitely long and if I multiply by 1 over 3 by 3 , that gives us 1 . But if you are on the decimal side, that's gonna be point 9999 , so yeah. I would say they are equal. Yeah, because if you look at the fraction part, it clearly gives us 1 , but the decimal point gives us infinitely long repeated decimal. Yeah.

Bailey: You can start with a fraction. We know that a third equals 0.333 repeating forever and then from that it's clear that 3 thirds is 1 . Because one third plus one third plus one third is point 999 forever and no matter where we start, we are just gonna get nines. We can take as many threes as we want for the dots, if we take it to infinity, it's just gonna be 9999 forever.

Adrian: $\quad$ This one is an easy one! 1 over 3 is 0.333 dot dot dot. And if we multiply by three, over here we get 1 , but on the other side we get .999 dot dot dot.

$$
\begin{aligned}
& \frac{1}{3}=0.3333 \cdots \\
& \begin{aligned}
\frac{1}{3} \times 3 & =1 \\
& =0.9999 \cdots
\end{aligned}
\end{aligned}
$$

Figure 6.5: Taylor's work for $0.999 \ldots=1$

These quotes are interesting in that they use a seemingly less obvious equivalence to prove another equivalence. Indeed, it seems more intuitive that $0.999 \ldots=$ 1 , rather than $\frac{1}{3}=0.333 \ldots$. Each of these participants used an equivalence that holds the same amount of uncertainty as the initial problem, if not more.

All participants felt the need to "concretize" this problem for students. This was particularly relevant to Jaime, who believed that $0.999 \ldots \neq 1$.

Interviewer: So what would be your mathematical justification to a grade 9 student about this problem?

Jaime: Just figuring out why they are different? I would take the approach of using an analogy. Like $\frac{1}{2}$ and 1 are not the same. But if you are looking at half a centimetre versus one centimetre from a metre away or ten metres away, there is a technical difference. Well, what if we make it .75 and .8 ? Then would you say they are the same? Like, could you come up with a definitive rule on that? And I think that would be the way I angle that, is that any kind of rule of making them the same is arbitrary. I'm thinking again
about analogies with measuring. These two things are a little bit different.

Interviewer: If a student wanted an arithmetic argument as to why they were different, what would you say?

Jaime: Like nine ninety nine pricing. If you have it as two digits, if you pay with coins, you get nothing back, but a bill, you get something. So they are fundamentally different.

Jaime wanted to provide students with a concrete context for the abstract issue of infinity, which is an excellent way to build a more robust understanding. However, it seems that through these concrete examples, Jaime compromised their own understanding of infinity. As Bailey noted, "I would need to see a lot of evidence as to why that's true in different ways. Cause it just feels unnatural" and as Taylor states, "this tells us that certain things are true even though they are not intuitive." Jaime continually discussed the idea of understanding this problem from a "practical point of view" and "eventually having to truncate and make a compromise." However, in compromising and truncating, the conceptual complexity of the problem may be lost and may promote the development of misconceptions in students' understanding of infinity. Adrian captures this challenge in the following excerpt, with an excellent example for building understanding of infinity:

Adrian: I think the biggest issue is infinity. The idea of what it means to be truly infinite. Maybe introducing the idea of the infinite hotel or the fact that there is the same number of even numbers as there are all positive numbers. I think this sort of strikes at the core
of what it means to be infinite, which I think is a really tough problem to wrap your head around. I would have them look at how many numbers there are, different types of infinities, or what it means to be infinite. Not just something that's very large.

### 6.3.3 Post-Secondary Connections

The mention of limits and post-secondary mathematics was prevalent throughout this task. All participants made mention of limits, in one way or another. Through the interview, it became clear that the problem hinged on the conceptualization of a number as a limit. This will be explored further through the analysis of the exponents task, but we first focus our attention on the concept of number as limit through participants' engagement with this task.

Of all the participants, Adrian shared the most robust understanding of this problem in terms of limits:

Interviewer: How could you prove that 0.999 with infinitely many digits is equal to one?

Adrian: Hmm, my first instinct is limits, but I'm not sure how to employ limits here.

Interviewer: So what is your instinct to go with limits?

Adrian: Cause it looks like a limit here. In my mind, whenever I am thinking about infinities or large lists of things, I start thinking about limits. Umm, I mean, I could maybe do a limit of....ok,
how could I set this up like a limit? I could do a magnitude of 1 minus the sum as $i$ goes from 1 to $n 0.9$ times 0.1 . Actually, I'll do this from 0 to $n$ and 0.1 to the power of $i$ and look at the limit as $n$ goes to infinity here. Now, I can set up this as a limit and maybe try to convince myself that this limit should be zero. That the distance between these two numbers should be 0 and then these two numbers would be the same.

$$
\lim _{\infty \rightarrow \infty}\left|1-\sum_{i=0}^{n}(0.9)(0.1)^{i}\right|
$$

Figure 6.6: Adrian's work for $0.999 \ldots=1$

Adrian was able to construct $0.9999 \ldots$ as $\lim _{n \rightarrow \infty} \sum_{i=0}^{n} 0.9(0.1)^{n}$ (see Figure 6.6), by writing the number as the limit of a sum of a sequence. In this case, what Adrian constructed is equivalent to the geometric series $\sum_{i=0}^{\infty} 0.9(0.1)^{n}$. Using concepts from Calculus and Real Analysis, one can prove that $\sum_{i=0}^{\infty} 0.9(0.1)^{n}$ converges to $\frac{0.9}{1-0.1}=1$. Thus, the limit that Adrian constructed proves the equivalence of $0.999 \ldots=1$. This construction is in contrast to the construction Bailey presented, using the concepts of asymptotes of real-valued functions, as observed in Figure 6.7 and the following dialogue:


Figure 6.7: Bailey's work for the limiting behaviour of $0.999 \ldots$

Interviewer: Is there any material from your math degree to help you answer this question.

Bailey: I can't think of a simple answer

Interviewer: And that's ok. You can be as complicated as you want it to be.

Bailey: I guess I can't, it really seems, I don't know. I can't even verbalize why that's true other than it's a system we created and in our system it clearly is true that .99999 ... equals 1 . I assume there is some reason that is much more eloquent than that, that I'm not thinking about. But talking about approaching infinity and infinite number of digits.

Interviewer: So what mathematical ideas do you need for that?

Bailey: Like Calculus, right? Like asymptotes and approaching infinity. Like some stuff from proofs about different sizes of infinity is
good knowledge to back up your understanding of it.

Interviewer: What's a fundamental idea needed to define an asymptote?

Bailey: Approaching infinity? Approaching some number, like limits? Is that what we are looking for?

Interviewer: So how could we define using a limit that .999 is equal to one? What kind of limit could we look at for this?

Bailey: We could just look at a function that, you know...

Interviewer: What kind of function would you look at?

Bailey: Like 1 over x would probably work? Actually, that one wouldn't really work. You could set up a graph that has that asymptote, like plus one. And talk about limits. But that wouldn't help me explain it to a grade 10 student.

Interviewer: Can you think of a limit that would be equal to one but also have something to do with this .999 infinite?

Bailey: Like we would want an equation that is approaching one from the bottom, so that would be a 1 over $x$ plus something. Here's my graph. I'm rusty on all my graphs look like. It would just be 1 over $x$ plus 1. Would that be right? Yeah.

Interviewer: And so as we take this limit we are getting one, but what does this have to do with .9999 ?

Bailey: Because 1 minus a tiny tiny piece is .9999 , right? So that asymptote is approaching it.

Initially, Bailey was unsure of what advanced mathematics could be used to solve this problem, but eventually landed on the notion of asymptote. In the process, Bailey revealed some misconceptions of the asymptote concept. This is evident in the following excerpt:

Interviewer: Let's say you had a really talented student in grade 11. How could you take these ideas and make it accessible to push them beyond what they are doing in the curriculum?

Bailey: Like you could draw a graph for them that has an asymptote approaching one, from the bottom and say, let's look at different values and calculate a table and see that it is clearly going to one. But I don't know if that would really explain it, but that kind of shows that it's never getting to one

Interviewer: And why's that? Could you elaborate?

Bailey: Because an asymptote never gets there, it gets close, but never gets there. So even though with this example, the limit is 1 , it's not, I mean, I think it might be counterintuitive. I don't know if it explains that it's true. It might almost be an argument as to why it's not true. Cause like it's not actually ever getting there.

The primary misconception here is that "an asymptote never gets there." Indeed, many of the examples students see in school mathematics suggest this. How-
ever, if one examines the limit definition of an asymptote, this is not implicit. Indeed, the function $f(x)=1$ has a horizontal asymptote at $y=1$ since $\lim _{x \rightarrow \infty} 1=1$. Similarly, there are functions such as $g(x)=\frac{\sin (x)}{x}$, which cross the asymptote of $y=1$ infinitely many times. This misconception of asymptote is well-documented in the literature (Kajander and Lovric, 2009).

However, this observation is somewhat tangential to the task. The primary observation from these excerpts is that Bailey does not appear to conceptualize $0.999 \ldots$ as a limit. Indeed, this could be inferred from comments such as "Because 1 minus a tiny tiny piece is .9999 , right?" Bailey seems to think of $0.999 \ldots$ as being independent from this problem. Rather, Bailey wants to look at concrete instances of $0.999 \ldots$, that is, instances such as $0 . \underbrace{999 \ldots 9}_{n \text { digits }}$.

### 6.4 Exponentials

The task on numbers which participants responded to was the following:
Your students are confused as to why they can define and compute $2^{\sqrt{3}}$. How would you respond?

At its core, this question attends to participants' understanding of exponential functions and their extension to non-integral powers. Of the five participants, Adrian, Bailey, and Taylor responded to this scenario.

### 6.4.1 Mathematical Background

The notion of an exponent is first seen in school with integral exponents. That is, $a^{n}=\underbrace{a \cdot a \cdots a}_{n \text { times }}$. That is, $a^{n}$ is $a$ times itself $n$ times. We say that $a$ is the base and $n$ is the exponent. In this definition, $n$ must be a counting number. Otherwise, the notation of " $n$ times" does not make much sense. Indeed, what would it mean to multiply something by itself 1.2 times?

This notion is later extended to negative exponents by defining $a^{-n}=\frac{1}{a^{n}}$ and is extended further to rational numbers by examining connections between radicals and exponents, where $\sqrt[n]{a}=a^{1 / n}$ and $(\sqrt[n]{a})^{m}=a^{\frac{m}{n}}$.

Exponents are examined from a mostly computational basis, until the concept of exponential functions is studied in late secondary school. The exponential function with base $a$ is defined as $f(x)=a^{x}$. The graphs of these functions are often motivated by having students create a table of values for a given base $a$ with various inputs of $x$ (normally integers) and connecting the dots to create a continuous line.

However, prior to this "connecting of the dots," learners had only been exposed to evaluating powers which were integral or rational. What about all of those numbers in between (i.e. irrational numbers)? The exponential function is defined for $x$ values which are irrational, but this idea is never explored until a student takes a course in real analysis. How does one define numbers like $3^{\pi}, 2^{\sqrt{3}}$, and $e^{e}$ ?

The theorem for rational exponents is dealt with in detail in the first 10 pages of Rudin's Principles of Mathematical Analysis, with the problem of extending to all real numbers being an exercise in the first chapter. Indeed, the study of the numbers system that provides the basis for the remainder of the study in real analysis is necessary before progressing to complex mathematical notions that depend on this system. Included in this study is the recognition of irrational numbers as the limit of a sequence of rationals. This is an instance of concepts which implicitly appear in secondary mathematics, but cannot be properly dealt with until a learner takes a course in real analysis. Even if this concept cannot be covered in secondary school, the teacher with the awareness of the complexity of extending exponential powers to rational and irrational powers could bring context and take extra care when teaching related material with their learners.

### 6.4.2 Participant Understandings

As mentioned above, the participants who chose to engage with this question were Adrian, Bailey, and Taylor. Overall, all participants were able to engage with this problem at the secondary level, but did not have rigourous mathematical explanations to justify the construction of irrational exponents. Despite this, there were a number of commonalities amongst the responses. In particular, the use of rational exponents and inverse functions.

At the very beginning of the interview, Taylor made the observation that "having the exponent that's not a whole number is just weird to [students], I guess." All participants mentioned the "concrete" nature of whole number exponents: the ability to multiply a number by itself a finite number of times. From this, partici-
pants shifted their attention to rational exponents and square roots, but recognized a large conceptual shift for students:

Taylor: $\quad$ Well, the square root of 2 is actually the same thing as 2 to the power of a half. And this is true by definition, because when you square the square root of 2 , that the student may know or not know, but this is by the definition of square root and you're squaring it again, which means that you're multiplying the number by itself, then you get the square root of 4 which is 2 . That's why it's 1 over 2 because when you have 2 to the power of a half and square it again, you get 2 to the power of 2 over 2 , which is just 2 . So, square root of 2 and 2 to the power of a half are the same thing. So, the exponent doesn't always have to be a whole number.

Bailey: A student I tutored had a lot of trouble conceptualizing what powers to a half are. Because all of the sudden we go from exponents meaning times itself so many times and then something else. Like, negative one doesn't mean we are timesing negative one times, it means something else. I conceptualize it as the inverse functions. Instead of multiplying by itself twice, we had to multiply the previous thing to get 2 .


Figure 6.8: Bailey's work explaining rational exponents

Adrian: You can start talking about fractional powers, like the square root, cube roots, and so on and so forth. I would probably do this from the inverse. If this is, if I have $x=2^{1 / 2}$, I can square both sides and get this idea. Something squared is equal to 2 , what is that something? You're throwing square roots in, which students don't like as it is.

All participants instantly used the relationship between radicals and roots to justify rational exponents. Furthermore, relating the rational exponents (in particular, exponents of the form $\frac{1}{n}$ ) allowed participants to relate non-integral powers back to the more concrete territory of integral powers. However, participants lost this convenience when the conversation shifted to irrational powers:

Taylor: $\quad$ I could approximate it with a rational number. I want to be careful as to not exceed the value of $\sqrt{3}$. But also it has to be smaller than 2 , so tightening the range of what potential value can be. We don't know that this value exists. Where does it lie on that graph [of the exponential function] would be the question to students. I think that narrowing down the range would be able to help them understand the value that it takes.

Bailey: If we think of $2^{\sqrt{3}}$ as 2 to the 1.7 something, I would say it's like 2 to the power of 1 and then some fraction. Obviously we can't write it as a fraction, but we could write 1.75 as 1 and three quarters. Like, we understand what that means. We can understand that this is root four to the power of 3. And from there, I feel like we could elaborate to say that even though we can't put it into a power and that it's very close to this.

Adrian: If you can understand 2 to the 1.7, I don't think it's too much to extend it $2^{\sqrt{3}}$. Like, get students to understand that these things can exist on their own. And maybe this is a good way of breaking it down. it's $2^{1}$ times $2^{0.5}$ times all these things multiplied together and you can find each of those separately and estimate them in different ways. Maybe that's a good way of doing it? It's two to the 1.7 ish, we don't really care.

$$
\begin{aligned}
& x=2^{17 .} \rightarrow \\
& 2^{1} \cdot 2^{05} \cdot 2^{0.2} \\
& 2 \cdot \sqrt{2} \cdot \sqrt[5]{2} 2^{x} \\
& 2 \cdot 1.3
\end{aligned}
$$

Figure 6.9: Adrian's work finding a value of $2^{\sqrt{3}}$

The idea of relating an irrational exponent to a rational exponent is a very reasonable approach and is certainly a way to define irrational exponents. However, no participant was familiar with the formal definition and drew upon the notion of approximations and other mathematical ideas (such as continuity) in order to justify the concept of an irrational power:

Taylor: I mean, I would say that the exponent can be continuous that by that I would say that it doesn't have to be a whole number or rational value. But something that falls out of range of that whole number and rational numbers. So irrational, since it lies on that number line from zero to infinity, it lies on that number continuum. I would say that square root of 3 is also somewhere on that, since I know the value, somewhere between 1.7 and 1.8. So you can have the value as an exponent too. And as you saw through the graph of it, the exponent function is continuous, so there's a value for 2 square root of 3 as well.

$$
\begin{aligned}
& \sqrt{2}=2^{\frac{1}{2}} \\
& (\sqrt{2})^{2}=\sqrt{2} \cdot \sqrt{2}=\sqrt{4}=2 \\
& \left(2^{\frac{1}{2}}\right)^{2}=2^{\frac{2}{2}}=2 \\
& 2^{\sqrt{2}}<2^{\sqrt{3}}<2^{\sqrt{4}} \\
& 2^{0.5}<2^{\sqrt{3}}<2^{2} \\
& 2^{1.3}<2^{\sqrt{3}}<2^{2}
\end{aligned}
$$

Figure 6.10: Taylor's work for work defining $2^{\sqrt{3}}$

Bailey: $\quad$ So yeah, I would just take this and make it a fraction over a million, hundred million. So essentially we are taking the hundred millionth root and taking the 7 hundred whatever power. I think it would be pretty easy to show it does exist, but where I think it would be hard would to have some reason as to why it exists. We can come up for something as to what it means for fraction. Like square root, cube root. That has a definition that makes sense. Whereas once it becomes an irrational number, I don't see any kind of concrete definition. I guess you could do something like I'm looking here. If you take 2 to the root three and take it to the root three and then you get 2 to the 3 . Maybe you can get somewhere with that? You've obviously got lots of irrational powers but maybe if you played around with that maybe you could work it to someplace that shows something.

Adrian:
Plug it into Desmos and see what happens. You get this graph. What can you recognize? You recognize two to the 0 is one, two to the 1 is two, two to the 2 is four. You can even recognize the points down here...how about the points in the middle, how do we access these points? And giving them this idea that you can work in between, the idea that there is something between these two numbers, that there is a continuous pattern that exists. Umm, that might be a good way to do it, but it's still not satisfying. It's not as, like the fact that you can plug this into Desmos doesn't prove it's a thing. For some students it will be fine, for some students it
won't be. And as a mathematician, I'm not satisfied with it. And I'm trying to remember how this was actually introduced to me because it's been so long.

$$
\begin{aligned}
& 2^{1} \Rightarrow 2^{2} \rightarrow \\
& 2^{-1} \Rightarrow 2^{-24} \Rightarrow \\
& 2^{1 / 2} \Rightarrow 2^{1 / 3} \Rightarrow 2^{2 / 3} \\
& x=2^{1 / 2} \Rightarrow x^{2}=2 \\
& x=2^{\sqrt{3}} \Rightarrow \sqrt{3} \sqrt{x}=2
\end{aligned}
$$

Figure 6.11: Adrian's work extending exponents to irrationals

Adrian and Taylor had a similar idea of bringing the graph of exponential functions in to help convince students that they could define and compute $2^{\sqrt{3}}$. However, as Adrian notes, this is not sufficient. Both Taylor and Adrian thought they could bring in the notion of continuity in order to justify the existence of $2^{\sqrt{3}}$, but this is not mathematically sound, as Adrian notes. In fact, existence is a necessary condition for continuity.

Definition 10. A function $f(x)$ is continuous at $x=a$ if the following three condiions are all satisfied:

1. $f(a)$ exists
2. $\lim _{x \rightarrow a} f(x)$ exists.
3. $\lim _{x \rightarrow a} f(x)=f(a)$

If continuity is to be used as a justification, existence is necessary. This is an interesting example that demonstrates how advanced mathematical knowledge could negatively impact a teacher's understanding of secondary mathematics. Indeed, the notion of continuity is a concept that can be explained non-rigourously and is often associated to the idea of drawing a line and never lifting your pen up. However, the rigourous definition requires an understanding of what happens at a particular point, where the notion of a point in Euclidean space is abstract in and of itself. As Adrian remarked, "the fact that you can plug this into Desmos doesn't prove it's a thing."

Of all the participants, Bailey came the closest to a rigourous definition:

Interviewer: So how would you conceptualize an irrational power?

Bailey: I feel like we could elaborate to say that even though we can't put it into a power, that it's very close to this. And you could almost do the whole approaching from both sides thing, using rational numbers. But yeah, I don't actually have a good explanation for that and I don't know that there is one. It's gotten so abstract. I don't know of a concrete explanation for how it exists.

Bailey was very focused on concretizing the idea of exponents, seeing as the first conceptualization of integral powers is very concrete. As Bailey notes, even rational powers can be concretized "because we have a concrete representation of
where square roots exist in nature and there are cool origami proofs. And if you can do that, you can convince them [students] that square root two exists." As will be elaborated below, this dialogue brings forward an interesting perspective on the issue of "exactness" and "existence."

### 6.4.3 Post-Secondary Connections

The notions of "defining," "exactness," and "existence" were problems for Taylor during this task, as demonstrated in the following excerpt. As will be seen, Taylor came very close to the proper definition, but their progress may have been hindered by confusing these three notions.

Taylor: I'm pretty sure that there is a way to approximate this...ummm, I think it involves some series. But yeah, I would have to google it. I'm pretty sure there is a way to do it

Interviewer: What would you use the series for? Just kind of roughly?
Taylor: $\quad$ So it's not square root of 3 for the one I remember but finding the value of pi, there is a way to know the exact value of pi using either the Taylor series or the Newton's method. I can't quite recall what the name was, but yeah. But the process is quite beautiful. Yeah. But yeah, so in terms of finding square root of 3, I think there would be a way to find it using Taylor series, which I'm not quite sure now, but yeah.

Here, Taylor wants to use the idea of series to deal with $\sqrt{3}$. This is not far off from what Adrian and Bailey suggested in decomposing $\sqrt{3}$ into rational pieces.

However, the issue of defining the irrational exponent still exists. Taylor's goal at this point it to find the value, rather than justify existence. In order to try to see if Taylor could use this idea to get to the definition, the interviewer hoped Taylor would recognize $\sqrt{3}$ as the limit of a sequence of rationals.

Interviewer: So if you were to associate a series to the square root of three, what kind of series would you associate it to? What would the series be made up of?

Taylor: Mmmmm, I'm not so sure.

Interviewer: No?

Taylor: No.

Interviewer: Could you associate a sequence to it?

Taylor: Ummmm, yeah, I think, you can associate a sequence, it's not very specific but let's say we have a value that is approaching from, I don't even, I don't know if I'm using the right term.

Interviewer: That's okay don't worry.

Taylor:
Approaching from above and approaching from downward then there is some value that is in between. Then finding a correct function of these two would allow us to find the value of square root of 3 .

Taylor shifts attention from defining to finding an "exact" value. This is interesting because $\pi$ and $\sqrt{3}$ are exact values, while other methods, such as a series,
are alternative representations which converge to $\sqrt{3}$. This dialogue continued to a point in which a misconception of irrational numbers was unveiled:

Interviewer: So what you are saying is you would want to define two sequences that would approach square root of three?

Taylor: Right, yeah.

Interviewer: And generally can you think of a....is there an association generally between irrational numbers and sequences?

Taylor: Yeah, umm, I would think so, yeah.

Interviewer: And what kind of relationship would there be?

Taylor: Because infinite series can incorporate the problem of infinite decimal point and sometimes it's not very repeating per se, I mean not repeating as decimals, so when those series are incorporated they would also involve a value of irrational numbers such as square root of two or three.

It is in the last remark from Taylor where they claim that a series representation of an irrational number should involve irrational numbers. This is not the case. Indeed, as mentioned in 6.4.1, all irrational numbers may be the limit of a sequence of rationals. Although Taylor refers to series in their justification, it is not true that series which converge to irrational numbers contain irrational terms. Indeed, the famous series $\sum_{n>1} \frac{1}{n^{2}}=1+\frac{1}{4}+\frac{1}{9}+\frac{1}{25}+\ldots$ is a series with all rational terms converging to $\frac{\pi^{2}}{6}$, which is irrational. This is one example of many.

Overall, from these excerpts comes forward key developmental understanding of exactness and irrational numbers. The "exact" form of $\pi$ is $\pi$. By not writing out the decimal expansion of $\pi$, "exactness" is maintained. The case is similar for $\sqrt{3} ; \sqrt{3}, e, \pi$, and $\sqrt{2}$ are all exact values. At this moment, super computers are searching for the next digits of $\pi$ and this process will never stop. Through a decimal expansion, information about the number is lost, as is "exactness." Taylor's remarks are mirrored in Bailey's lament of wanting "concrete representations and explanations" for irrational numbers. Indeed, one might suggest that irrational numbers are concretely irrational.

One final path that both Taylor and Adrian saw as being relevant for this task was the relationship between exponentials and logarithms. Through this discussion came an interesting conception of the logarithm function. In the high school curriculum, exponentials are introduced first, partly due to the "concrete" nature of the definition of $a^{x}$, when $x$ is a natural number. Much later in the curriculum, the logarithm base $a$ is defined as the inverse of the exponential function $f(x)=a^{x}$. That is, $g(x)=\log _{a}(x)$ is the function such that $(f \circ g)(x)=(g \circ f)(x)=x$, as discussed in 6.2.1

While Adrian tried to bring logarithms into their strategy, they quickly abandoned that route, recognizing that it did not simplify the situation in any way:

Adrian: $\quad$ Something squared is equal to two, what is that something? I guess you can sort of use the same idea here, but it's not as neat.

Interviewer: And why is it not as neat? Could you elaborate on that?

Adrian: I think if you can connect this to fractional powers, they can understand it as a cube root, or even if you have two to the two over three, you can think of it as two squared, cube root. But this one does not have as nice of an interpretation. You're now thinking of this as, yeah.....uhh, you can't think of it in terms of multiplied a certain number of times or taking an integer form. Umm, yeah, and I mean even if you use this argument, this rationale, you can take the square root three root (i.e. $\sqrt[{\sqrt[3]{x}}]{x}$ ) of both sides. Something to the root of square root three $(\sqrt[{\sqrt{3}}]{x})$ equals two, which is probably not helpful and makes it worse.

Although Taylor did not respond to the task which explicitly explores understandings of inverse functions (see 6.2), the dialogue that follows unveils an interesting conception:

Interviewer: Do you have anything you would want to add to this? Any ideas from your math courses that you took that are coming up that come to mind with this problem?

Taylor: $\quad$ Since I am dealing with the exponent function I would introduce the idea of the inverse of two to the power of x .

Interviewer: And how would that be useful?

Taylor: That would be useful because, not really useful, but again how inverse function the exponent function relate.

Interviewer: And how do they relate?

Taylor: $\quad$ They relate because it gives identity function?

Interviewer: Go ahead, you're fine!

Taylor: Let's say this is an exponent function, $y=2^{x}$. To find the inverse function there is logarithmic function: y becomes x and x becomes y , then we are trying to figure out what y is in terms of x here. So I would bring $\log$ on both side to get $\log (x)$ equal to $\log (2)$ to the power of y , which is y times $\log (2)$ so that again would be $\log (x)$ over $\log (2)$ which is why that would imply that by the logarithmic rule, would be $\log _{2}(x)$ which is y . This is an inverse function of $2^{x}$. Then it's inverse, how do you check it? We just, let's say this exponent $2^{x}$ is $f(x)$ and let this be, let the inverse function $\log _{2}(x)$ be $g(x)$. And if we make it a composite function $f \circ g$, would be by definition $f$ at $g(x)$, that means two to the power of, and in the place of x , we substitute $\log$, so $2^{\log _{2}(x)}$, and that gives us a value of x , by definition that proves that the logarithmic function is inverse function of $2^{x}$, because when you composite it gives identity function.


Figure 6.12: Taylor's work on exponents and logarithms

The interest in this excerpt lies in Taylor's use of logic. At the end, Taylor states that "by definition that proves that the logarithmic function is inverse function of $2^{x}$, because when you composite it gives identity function." However, in order to prove that $\log _{2}(x)$ is the inverse function of $2^{x}$, Taylor must use the fact that it is an inverse. This relates to Taylor's previous dialogues in the sense that it brings forward some confusion between what constitutes a definition and/or existence. Indeed, the logarithmic function may be defined as the inverse function to the exponential. One can show that an inverse of $f(x)=2^{x}$ exists (since $2^{x}$ is injective), but this does not tell one what the inverse is.

A post-secondary education in mathematics is where prospective teachers encounter such logical notions such as definitions, axioms, existence, and exactness. The results from this task reveal that mathematics educators might want to reconsider the ways in which these logical pillars are addressed in the post secondary curriculum, so that future mathematics teachers may have a more robust understanding of the mathematics they teach.

### 6.5 Summary

In this chapter, I have explored participants' responses to tasks involving concepts from Calculus and Real Analysis. Results from these tasks reveal that overall, advanced mathematics coursework in calculus and real analysis did not have a significant impact on teachers' mathematical knowledge for teaching, with respect to these tasks.

Participants remarked that their advanced mathematics coursework offered valuable experiences which helped them build a more robust understanding of the nature of infinity. However, while engaging in the interview tasks, some misunderstandings about infinity were revealed. Indeed, with respect to both the task involving $0.999 \ldots$, as well as $2^{\sqrt{3}}$, participants were unable to conceptualize these two numbers as limits. Particularly in the case of $2^{\sqrt{3}}$, this limited participants in their justification of "what" this number was and how to define it. Approximation versus exactness and existence versus the definition were two tensions exhibited by the participants who engaged with this task. In the case of $0.999 \ldots$, all participants who engaged in this task presented a memorized justification, which is arguably less intuitive. Rather than focusing on a limiting notion of this number, participants wished to justify the equivalence concretely, using metaphors and graphs. While metaphors and examples can be excellent tools for developing understanding of abstract concepts, some participants drew upon examples that did not fully capture the concepts they were hoping to concretize.

While only two participants engaged in the task on inverses, analysis of the transcripts revealed that the notions of operation and the "undoing" of these operations were important facets of their mathematical knowledge for teaching in this domain. However, this revealed a tension with respect to what it means to "undo" in a functional sense. Casey revealed an overextension of the idea of inverses with respect to the notion of reflecting the graph of a function over the line $y=x$. This same misconception also brought forth the issue of domain restrictions and the role that these play in the existence and finding of inverse functions. While Casey mentioned that domain restrictions should be studied when teaching about inverse functions, it was clear that Casey held numerous misconceptions regarding the mechanics and use of domain restrictions.

Overall, results from this chapter suggest that participants' understanding of these tasks were primarily held in the domain of secondary content. While some connections were made to notions in calculus and real analysis, these understandings did not appear to impact the ways in which participants would approach them in a classroom. That is, the understandings developed in their university mathematics coursework were not built up from their existing mathematics knowledge. While participants held understandings of limits, sequences, exactness, and continuity, these understandings seem to only exist in the domain of "university math" which, as detailed in Chapter 4, was perceived to be disconnected from secondary mathematics content. Results from this chapter suggest that university courses in calculus and real analysis might benefit from a reconsideration of the ways in which they support the construction of connections between secondary and postsecondary mathematics, so they might better foster the development of MKT for
future teachers. This will be discussed more in-depth in Chapter 8

## Chapter 7

## The Tensions of Proof and

## Applications Observed Through

## Geometric Tasks

### 7.1 The Square Root of Two

In this section, I will discuss participants' responses to Task G, which was the following:

A student is confused as to whether $\sqrt{2}$ is an irrational or rational number, especially after realizing it is the length of the diagonal of a square of side length

1. How would you respond?

The purpose of this task was to explore participants' understandings of the real number system. Of the five participants in the study, Taylor and Bailey chose to engage in this task. Overall, responses to this task revealed a reliance on memorized
proofs, but analysis of the transcripts revealed more general misunderstanding in the context of proofs in secondary and post-secondary mathematics.

### 7.1.1 Mathematical Background

Irrational numbers could be considered one of the most abstract concepts of the secondary curriculum. Without them, the real number system is vastly incomplete, but the theoretical jump from rational to irrational numbers can be challenging for students and teachers alike (Fischbein et al., 1995). These difficulties are due in part to the nature of irrational numbers. One such aspect of their nature that might hinder theoretical understanding is the fact that, despite the rationals being a dense set, they do not cover the entire real number line. That is, any interval of the real line, no matter how small, is guaranteed to have a rational number in it. However, there are "holes" which need to be filled. These numbers are the irrationals. A second piece which may hinder understanding of irrationals, is their relationship to incommensurability. As noted by Fischbein et al. (1995), this difficulty can even be observed through the historical development of understanding irrational numbers, as the discovery of incommensurable segments was a result of early Greek mathematicians, while the fully theory of irrationals was not developed until the nineteenth century.

The challenges associated with irrational numbers are exacerbated by subsets of numbers such as constructible, algebraic, and transcendental numbers. The first definition of irrational numbers that students might encounter is that irrational numbers are decimals which do not terminate and do not repeat. This definition is vague and leaves space for interpretation. Indeed, a number like $0.10100100010000 \ldots$ is
irrational, but it somewhat follows a repeating pattern.

Some irrational numbers are constructible; that is, they can be constructed using a finite number of arithmetic operations (including the square root) on the integers. $\sqrt{2}$ is an example of an irrational constructible number, while irrationals such as $e$ and $\pi$ are not. Although the definition of "constructible" is purely geometric, there is a deep relationship between constructible numbers and field extensions in abstract algebra. I turn the reader to Aluffi (2009) for an extended mathematical discussion. Irrational numbers such as $e$ and $\pi$ are examples of transcendental numbers, who are not roots to any polynomial equation, with integer coefficients. Numbers which are roots to such equations are called algebraic. The relationship between these subsets of irrational numbers grows further, with any constructible number being algebraic.

A course in elementary number theory may be one of the first places undergraduate students begin to rigorously look at the real number system, despite the focus of most elementary number theory classes being the integers. One of the first such exposures would be the proof of $\sqrt{2}$ as irrational. This proof utilizes properties of integers and the rationals, in conjunction with a proof by contradiction. The proof requires some degree of abstraction, but does not involve the use of unfamiliar definitions and constructions. Indeed, the most complicated piece of this proof may be the technique of proof by contradiction, as will be observed by participants' responses in 7.1.2. I present the proof of $\sqrt{2}$ being irrational to provide context for the section that follows:

Proof. Suppose that $\sqrt{2}$ is rational. Then, $\sqrt{2}=\frac{m}{n}$, for some $m, n \in \mathbb{Z}$, where $m$ and $n$ are relatively prime. That is, $m$ and $n$ have no common factors. Squaring both sides, we have that $2=\frac{m^{2}}{n^{2}}$, which is equivalent to $2 n^{2}=m^{2}$. This means that $m^{2}$ is an even number, which implies that $m$ must be even. Thus, we may write $m=2 k$, for some $k \in \mathbb{Z}$. So, $2 n^{2}=4 k^{2}$, which after dividing both sides by 2 , yields $n^{2}=2 k^{2}$. Under the same argument as before, this implies that $n$ is also even. This is a contradiction to our assumption that $m$ and $n$ are relatively prime, since we have just shown that both $m$ and $n$ are even numbers. Thus, $\sqrt{2}$ is not rational.

### 7.1.2 Participant Understandings and Post-Secondary Connections

As mentioned above, of the five participants, Taylor and Bailey engaged in this task. Interestingly, both participants initially responded in an almost identical manner, making reference to the elementary number theoretic proof that the square root of two is irrational, which was outlined in 7.1.1. The interview dialogue was brief, due to both participants immediately drawing from post-secondary mathematics knowledge, so I have combined sections where they have previously been written separately.

Taylor: I would first ask the student what is a rational number. They might say it is a fraction, but by fraction, what do you mean? Well, all fractions have a numerator and denominator which are integers or whole numbers. So ok, I would proceed with saying let's suppose square root of 2 is rational and argue by contradiction.

Bailey: So, first I would talk about rational and irrational numbers, be-
cause it seems the student is confusing what that means. If I had students who understood what that means, I might actually show them the proof of root 2 being irrational. Proof by contradiction is a little confusing, but it's a cool proof.

$$
\begin{aligned}
& \text { rational number }=f \\
& \sqrt{2} \text { is rational } \sqrt{2}=\frac{a}{b}{ }^{2} \text { deno we don't have common } \\
& (\sqrt{2})^{2}=2=\frac{a^{2}}{b^{2}} \\
& \Rightarrow 2 b^{2}=a^{2} \cdots \text { (1) } \\
& \Rightarrow a^{2} \text { is even } a \text { is also even } \\
& \Rightarrow a=2 k \text { for } k \in \mathbb{Z} \\
& \Rightarrow \not 2 b^{2}=(2 k)^{2} \nRightarrow 4 k^{2} \\
& \Rightarrow b^{2}=2 k^{2} \\
& b^{2} \text { is eien } \Rightarrow b \text { iseven } \\
& \Rightarrow b=2 l
\end{aligned}
$$

Figure 7.1: Taylor's proof that $\sqrt{2}$ is irrational

Taylor chose to talk though the entirety of the proof during their interview (see Figure 7.1), while Bailey made reference only to the strategy of using the proof. Although the intent of this task was to explore participants' understanding of the real number system, these remarks opened an opportunity to discuss methods of proof. With this in mind, I was curious to understand how Taylor and Bailey conceptualized the method of proof by contradiction.

Taylor: Assuming from what is absurd, you can derive the truth. That means that if you assume a stupid thing, that you can, then you will in the end derive something that doesn't make sense from what you assumed.

Bailey: I would say that we know that this first thing is true and if we can take steps where we know all of them are valid and we get to something that is clearly untrue, then your first premise had to be flawed by logic.

Taylor and Bailey both demonstrated an understanding of the utility and strategy behind proof by contradiction. That is, if you are trying to prove that a statement $P$ is true, assume that $\neg P$ (read "not $P$ ") is true. From this, a successful proof by contradiction will yield that if $\neg P$ is true, then a statement $Q$ and $\neg Q$ are both true. Since $Q$ and $\neg Q$ are opposite statements, both cannot be true at the same time. This will always be false. The truth or falsity of $P$ is actually equivalent to the truth or falsity of the conditional statement if $P$ then $Q \wedge \neg Q$. This may be observed in Table 7.1 .

| $P$ | $\neg P$ | $Q \wedge \neg Q$ | $\neg P \Longrightarrow Q \wedge \neg Q$ |
| :---: | :---: | :---: | :---: |
| T | F | F | T |
| T | F | F | T |
| F | T | F | F |
| F | T | F | F |

Table 7.1: Truth table for logical equivalence of $P$ and $\neg P \Longrightarrow Q \wedge \neg Q$

Notice that the far-left and far-right columns are equivalent. Thus, proving $P$ is equivalent to proving $\neg P \Longrightarrow Q \wedge \neg Q$. The latter conditional statement may be proven using a direct proof method. When participants were asked why proof by contradiction is a valid method of proof for the problem they wanted to use it for, dialogue revealed that they were able to use it as a tool, but did not have a rigorous justification for its utility:

Taylor: Umm, it works because, umm, I think it works because when you make an argument that is not true, by assuming that certain thing then you see a contradiction because of that assumption. Then we have to meet that the statement doesn't work. So you have to go to the other assumption and start with that. Yeah, uh, the statement. We make a statement that is not true and if you assume that as a true then we get, we might, get a statement that is directly contradicting the statement we assumed.

Bailey:
The deductive logic is hard. Saying that if all these steps are valid and this is untrue then the only option is that the premise was untrue.

From this dialogue, it is unclear whether or not Taylor and Bailey understand the reasoning behind the proof by contradiction strategy. Although both make note of the strategy of $P$ versus $\neg P$, they make reference to the premise of the statement being the fundamental component to the argument, rather than the logical equivalence of $P$ to $\neg P \Longrightarrow Q \wedge \neg Q$.

When prompted to reflect on the logic behind the strategy, Taylor seemed to recognize that they did not fully understand the reasoning behind the method of proof by contradiction:

Taylor: $\quad$ For me, I would say the logic was always what was taken for granted. In my undergrad, it was a rule of law, for making arguments. To make my students understand why it's making sense, I have to study it for myself first.

While Taylor and Bailey were both very familiar with the proof by contradiction strategy, the dialogue suggests that they may have never had the opportunity to develop a personally powerful understanding of this proof method. Taylor recognizes that they used this method regularly in their mathematics studies, but that they never questioned how or why it worked. In order to have a pedagogically powerful understanding, one needs to have a personally powerful understanding - a KDU.

This dialogue brings forth an interesting connection between post-secondary mathematics education and mathematical knowledge for teaching. Both Taylor and Bailey saw value in bringing strategies of proof into their future teaching. Bailey remarked on how they found it strange that proof did not have a more significant role in the curriculum, since it holds such an important position in post-secondary mathematics. However, as Taylor notes, methods of proof may not be intuitive and a deep understanding of why such methods of argumentation are valid requires deep understanding of logic. Even so, Taylor felt as though the methods of proof were taken for granted in their studies. This resulted in Taylor recognizing a tension
in their MKT, stating "it's ironic because I argue an important aspect of mathematics is asking why, but I'm sort of reinforcing students to just accept this process as a legitimate solution."

Taylor chose to conclude this task with the number theoretic proof, since they viewed it as a simple but rigourous argument that would be accessible to high school students. Bailey, on the other hand, presented an alternative method focused on "zooming in" on the number line (see Figure 7.2):

Bailey: I would probably "zoom in." Ok, maybe $\sqrt{2}$ is like 1.41 something. So I would say, well, here's 1.5. And we would zoom in and have a new number line where this is 1.4 and this is 1.5 and ok, it's in between here. And do a few of those to show no matter how far deep we go, we have more, we are closer to the number on either side.


Figure 7.2: Bailey's picture of "zooming in" to $\sqrt{2}$

In this excerpt, Bailey uses the decimal expansion form of $\sqrt{2}$, so I was curious to see how Bailey would respond without this assumption in place:

Interviewer: So numerically, how would you justify the decimal form of $\sqrt{2}$ ?

Bailey: We could do 1 squared is 1 , and 2 squared is 4 . So like, let's try 1.5. That's too much. Let's try 1.3. That's too little. And just get closer and closer to it. It would be like an asymptote of a graph. We are approaching $\sqrt{2}$. But $I$ feel like it might convince my students the opposite. Like, it's so clear that it never gets there, so it makes it seem like the number doesn't exist.

Bailey's remarks point to the understanding of irrational numbers being the limit of a sequence of rationals, as outlined in 6.4.1. However, based on Bailey's dialogue in 6.4.2, it is unclear as to whether or not this understanding was fully developed. Recall that Bailey had difficulty in providing a justification for the equality of $0.999 \ldots=1$ and wanted to relate the number $0.999 \ldots$ to the function $f(x)=\frac{1}{x}+1$. During this task, Bailey was unable to conceptualize $0.999 \ldots$ as a limit of a discrete sequence. While their work with $\sqrt{2}$ points to some conception of number as limit, remarks from 6.4.2 leave room for interpretation.

### 7.2 Symmetry

In this section, I will explore participants' responses to the task on what they would include in a lesson plan unit on symmetry. The task was phrased as follows:

You are teaching a week on symmetry to your students. What would you include in your lesson plan?

Of the five participants, Jaime, Bailey, and Casey engaged with this task. Overall, connections to the post-secondary curriculum were limited, with only Casey bringing forward an explicit example of symmetry in the post-secondary curriculum. The mathematics of symmetry discussed was focused at the secondary or ele-
mentary level. Even for discussion generated at the post-secondary level, the conversation was primarily in the context of particular examples or problems, rather than a general extension to the post-secondary curriculum.

### 7.2.1 Mathematical Background

At its core, symmetry is a notion of balance and proportion. Mathematically, the definition becomes more complicated. For our purposes, I define a geometric object to be symmetric if it is invariant to particular geometric transformations. These transformations include reflection, rotation, scaling, and translation.

Symmetry has the possibility of appearing early in the elementary mathematics curriculum, through the examination of plants, animals, and other symmetric objects in nature. Later in the secondary curriculum, some formality can come to symmetry through the language of functions: Even functions, where $f(x)=f(-x)$ are symmetric across the $y$-axis, odd functions who are symmetric to the line $y=x$ and satisfy $f(-x)=-f(x)$. The absolute value of a function, $|f(x)|$ is an additional context where symmetry can be explored, particularly if all outputs of the function $f(x)$ are negative. Overall, translations and reflections of functions in the plane are a context where the familiar notions of symmetry are made rigourous.

Translations, reflections, and rotations of geometric shapes can be combined to create mathematical works of art, and tessellations can be an exciting way for students to explore various types of symmetry. By definition, a tessellation is a tiling of the Euclidean plane using one or more geometric shapes. The key in constructing a tessellation is that no gaps should exist and no geometric shapes
should overlay. Tessellations have a rich history in ancient architecture, design, and art. One of the most famous examples of bridging tessellations, mathematics, and art is the work of the artist M.C. Escher. As was noted by mathematician Doris Schattschneider, although the relationship to mathematics was evident in his art, the work of Escher was heavily mathematical Schattschneider (2010). The mathematics in his work was non-trivial and required a rich understanding of geometry and symmetry. Such a context could be a rich avenue for students and future teachers of mathematics to extend and expand their mathematical horizons, while drawing upon content they already know.

In the context of advanced mathematics, a significant appearance of symmetry in the post-secondary curriculum would be the symmetric and alternating groups, $S_{3}$ and $A_{3}$, respectively. Initially, one may define these groups in the following way: Consider the set $X=\{1,2,3\}$. We apply a bijective function from $X$ to $X$ that rearranges the elements of $X$. Such a function $f: X \rightarrow X$ may be as follows: $f(1)=2, f(2)=3$, and $f(3)=1$. For short hand, we would represent this as $f=\left(\begin{array}{lll}1 & 2 & 3 \\ 2 & 3 & 1\end{array}\right)$. The set $S_{3}$ is the set of all permutations of these three objects. In total, there are six permutations of the set $X=\{1,2,3\}$. They are:

$$
\begin{aligned}
& f_{1}=\left(\begin{array}{lll}
1 & 2 & 3 \\
1 & 2 & 3
\end{array}\right), f_{2}=\left(\begin{array}{lll}
1 & 2 & 3 \\
1 & 3 & 2
\end{array}\right), f_{3}=\left(\begin{array}{lll}
1 & 2 & 3 \\
2 & 1 & 3
\end{array}\right), \\
& f_{4}=\left(\begin{array}{lll}
1 & 2 & 3 \\
3 & 2 & 1
\end{array}\right), f_{5}=\left(\begin{array}{lll}
1 & 2 & 3 \\
2 & 3 & 1
\end{array}\right), f_{6}=\left(\begin{array}{lll}
1 & 2 & 3 \\
3 & 1 & 2
\end{array}\right)
\end{aligned}
$$

This definition can be extended to construct the symmetric group on $n$ elements, denoted $S_{n}$. Now, with this first introduction to the symmetric group, one might question why it has the name "symmetric" group. Indeed, the construction of this group, at a surface level, does not seem to have anything to do with symmetry, but of permutations. The direct relationship between the symmetric group and geometric symmetry appears when one introduces the notion of a dihedreal group.

A dihedral group is the group of symmetries on a regular polygon. For a regular polygon with $n$ sides, the dihedral group on that $n$-gon would be denoted by $D_{n}$. Given an equilateral triangle, the group of symmetries would be $D_{3}$. How many elements does this group have? That is, how many symmetries are there to an equilateral triangle? The symmetries we consider are rotational and reflectional symmetries and they exist as follows:


Observing the pictures above, note that the element $d_{2}$ is a reflection along the vertical bisector, which flips the vertices 2 and 3 . The element $d_{3}$, on the other hand, is a rotational symmetry of 60 degrees, in the counterclockwise direction. Returning to the example of $S_{3}$ and permutations of the set $\{1,2,3\}$, notice that each element of $S_{3}$ corresponds to an element of $D_{3}$. With a little bit more work, one can actually show that these two groups are in fact isomorphic. That is, the algebra of the dihedral and symmetric groups of three elements are equivalent.

The examples presented above only scratch the surface of possibilities for exploring symmetry in secondary and post-secondary mathematics. The implications of symmetry in physics, chemistry, biology, and art are far-reaching and could take up the entirety of a book. For those looking for extended literature on symmetry, mathematics and applications, please refer to Field and Golubitsky (2009).

### 7.2.2 Participant Understandings

Of the five participants, Jaime, Bailey, and Casey engaged with this task. While Casey immediately jumped to connections in the post-secondary curriculum, Jaime and Bailey kept most of their examples to lie within the secondary context. In their dialogue, the emergent theme was a focus on visual and tactile symmetry.

The relationship between nature and mathematics was expressed as an interesting avenue to explore symmetry, by both Jaime and Bailey. Jaime suggested having students find and suggest places that they see symmetry in their daily lives, while Bailey thought it would be valuable to bring in interesting examples like snowflakes and plant growth to have a more interdisciplinary discussion on the role of symmetry and nature. Although a connection was not explicitly mentioned by Jaime, they remarked that "facial symmetry is very tied to people's perception of beauty." While Jaime did not mention this, their connection between nature and symmetry had the potential to open up an interesting lesson exploring the Golden Ratio, symmetry, art, and biology.

Both Jaime and Bailey recognized that there are two types of symmetry within
the secondary curriculum, rotational and reflective. Jaime and Bailey saw these as being particularly important, since both of these symmetries appear frequently in nature, yielding multiple places of entry for "students to build their own working definition" of symmetry, as Jaime suggested.

When considering the mathematical content of the secondary curriculum, Jaime and Bailey both mentioned graphing and transformations as being a major component of symmetry in the curriculum. Bailey mentioned "the only symmetry we talk about in high school is the graph, like here's the axis of symmetry and that's really straightforward." Jaime mirrored this comment when mentioning the graphing of parabolic functions and axes of symmetry, but went a little further:

Jaime: I think that the idea of symmetry with respect to time is this idea that we are talking about not just folding these papers and seeing a reflection, it's a more general concept. Like, if you were reversing time, the same thing would happen but it would go backwards. We see a ball fall down, but according to Newton's laws of motion, it has the exact same motion coming up.

Jaime used their experience in physics to give context to the symmetric shape of parabolic functions, which extended the notion of symmetry as a visual/tactile concept, to something more abstract. Although all of the examples Bailey suggested were purely visual, they considered bringing in mathematical objects such as fractals, to explore a more abstract conceptualization of symmetry. The key point for Bailey was "bringing in cool visualizations, cool stuff they can see." Bailey thought that students would have difficulty understanding the abstract terms of
symmetry and thought that visual and tactile representations of symmetry would be an optimal "entry way" for students to begin exploring the concept.

Interestingly, none of the participants made mention of the relationship between mathematics, art, and symmetry. This is particularly interesting considering that Jaime and Bailey both made mention of the math and art projects completed in their math methods courses during their teacher education. Both participants found the assignment to be fun and productive, but did mention concern about how to fit these types of projects into the stated curriculum.

### 7.2.3 Post-Secondary Connections

Overall, connections made between the secondary curriculum and post-secondary mathematics by participants were limited. Jaime made no explicit connections to post-secondary content, Casey brought forth one example, and Bailey mentioned some examples at a surface level, but the general view was captured in the following statement:

Bailey: I can't think of ways that my math degree would help them understand the stuff that's in the curriculum, like symmetry of graphs.

As has been the case for the other tasks, connections made between symmetry, secondary, and post-secondary mathematics were primarily example driven rather than conceptually driven. As mentioned above, Casey started the interview by once again discussing the alternating and symmetric groups of three elements ( $A_{3}$ and $S_{3}$, respectively). Although Casey had these groups in mind as a post-secondary
example of symmetry, when discussing how it would be used in their pedagogy, it became clear that Casey's understanding was primarily process driven.

Interviewer: What would you include in a lesson on symmetry?

Casey: Given a 60-60-60 triangle, we could take the bisector and label the corners $\mathrm{A}, \mathrm{B}$, and C . The bottom line is if we flip the triangle, it's the same. But then, also given the rotations of the triangle, we could also explain that it's the same as if you flipped it along one of the bisectors. I would just draw a fancy squiggle line and show that the verticies of the triangle have flipped. It's related to Galois theory.

Interviewer: Can you elaborate on the relationship to Galois theory?

Casey: Well, we are taking this whole 60 degrees and we are flipping it, but the point is that in the Euclidean plane, it is symmetric, so...

Interviewer: Could you be a bit more specific as to how this relates to Galois theory?

Casey: Well, it's the alternating group. And this one is the symmetric group. Alternating group is how you rotate the triangle, but the symmetric group is not just rotations, but reflections. What I would do is start with a dotted line (bisector) and then it flips and I think that it would ultimately, it would be something like CBA because we, I think we are flipping it here.

Interviewer: And so what are you trying to do exactly?

Casey: Well, we are trying to explain symmetry in a triangle, but this I don't think would be understandable to students.


Figure 7.3: Casey's work on symmetries of an equilateral triangle

Casey recognized that the various symmetries of a triangle would be an advanced connection between symmetry in the secondary curriculum and advanced mathematics content, as is evident by the dialogue and Figure 7.3. However, based on the dialogue, it appears as though the connective threads were limited. Casey tried to walk through the example and provided accurate definitions, but the discussion of these connective threads was limited in depth. At the end of the excerpt, Casey mentions that this concept would not be accessible to secondary students. When prompted as to whether or not it could be made accessible, Casey stated that the point of the triangle was to explain symmetry. Once again, this is an example of advanced mathematical knowledge that has pedagogical potential, but has not been developed enough to impact pedagogy in a classroom setting.

This excerpt of dialogue is particularly interesting. The alternating and symmetric groups are a tactile example of a very abstract concept. Bringing this structure into a unit in the secondary curriculum could be a rich opportunity to engage students with some advanced mathematics beyond the horizon. However, in order to do so, teachers must have a rich understanding of the mathematics in order to make it accessible to a novice. Even though Casey had the content example, the utility of it in the classroom would be limited, due to a somewhat isolated understanding of the groups in question.

As is evident from the quote at the beginning of this section, Bailey was not entirely sure how post-secondary mathematics could be helpful in helping students understand the content of the prescribed, secondary curriculum. This quote mirrors Bailey's concern from 4.2, where they expressed that post-secondary mathematics content was "too distant" from the secondary curriculum.

Despite this view, Bailey did bring forward a number of examples of symmetry in the post-secondary context. In particular, Bailey mentioned symmetric matrices, Euclidean and non-Euclidean geometry. Bailey viewed the concepts from Euclidean geometry being of value, particularly when considering the use of symmetry in various geometric proofs. They remarked that "in the Euclidean geometry course I took, symmetry obviously comes up in the proof of geometric shapes. It's valuable to see that two things are the same, but opposite." In regard to non-Euclidean geometry, Bailey admitted to not knowing an extensive amount, but thought that considering geometry on different types of surfaces, such
as a parabaloid or sphere, could be an interesting avenue to explore how "different symmetries exist" on those and how "we have this one system and there are other totally different systems which have totally different systems in there." They later mentioned that they would enjoy finding some resources on that to bring into the classroom. Once again, this is an example of how teacher educators might have an opportunity to draw upon and develop the extensive mathematical expertise of future teachers.

### 7.3 Summary

In this chapter, I examined participants' responses to the mathematical tasks that were geometric in nature. While participants did draw upon post-secondary mathematics to provide context to how they would approach tasks in the classroom (i.e the proof that $\sqrt{2}$ is irrational), most of the discussion was focused on individual examples, rather than conceptual connections between content areas and levels.

This was particularly evident in the context of symmetry, where connections were example driven and did not contribute to an understanding of symmetry different from a secondary context. Participants who engaged in this task were quick to mention applications of symmetry in art and nature, but the depth of mathematics mentioned did not extend past the secondary level. Indeed, all of the examples brought forth by participants were visual examples of reflectional or rotational symmetry. Casey was the only participant who engaged in this task that brought an explicit connection to post-secondary with dihedral and symmetric groups, but the connections were once again example driven. When prompted to elaborate on their
understanding of these mathematical objects and their role in higher mathematics, Casey did not provide further context beyond a worked example.

The task which discussed the irrationality of $\sqrt{2}$ revealed an unexpected but fruitful result regarding the role of proof in the secondary and post-secondary mathematics classroom. The dialogue revealed that perhaps post-secondary mathematics courses should be more conscious of students' understanding of proof concepts, so that they do not simply become techniques without an understanding behind them. Indeed, there has been major pushback towards technique driven mathematics teaching at the elementary and secondary level and nearly all participants mentioned their hope of bringing their knowledge of proof into their future secondary mathematics classroom. This result leads one to question whether despite a mathematician's desire for students to have deep, conceptual understanding, many techniques may be "taken for granted," as they were for Taylor. Finally, the complicated nature of irrational numbers was mentioned, similar to the task involving $2^{\sqrt{3}}$. Participants mentioned that justifying $\sqrt{2}$ simply as a number with the value of $\sqrt{2}$ could be a challenging conceptualization for many students and turned to decimal representations to help understand "the value" of $\sqrt{2}$.

In this chapter of results, the common thread of limited conceptual connections between secondary and post-secondary mathematics continues to be woven. While some connections were made through explicit proofs and examples, transcripts revealed that these connections did not run deep. As I move into the discussion, I prompt the reader to consider how to facilitate the construction of mathematical knowledge for teaching with depth and breadth in content connections.

## Chapter 8

## Conclusion and Implications

### 8.1 Conclusions

In this study, I sought to examine the ways in which prospective secondary mathematics teachers drew upon advanced mathematics in their practice. My work was motivated partly by my own practice and experience, but primarily by the claims that coursework in advanced mathematics helps build connections to the secondary curriculum that can be transformative to teachers' practice and deepen their understanding of the secondary curriculum. After examining the literature, it was clear to me that a gap existed between empirical studies on teachers' secondary content knowledge and advanced content knowledge. One of the goals of this research was to bridge these two content areas and begin to understand the ways in which prospective secondary mathematics teachers build their own bridges and connective threads.

As a refresher, the research questions for this study were:

1. What do prospective secondary mathematics teachers perceive as the role of their advanced mathematics knowledge in their development as teachers?
2. In what ways do prospective secondary mathematics teachers relate advanced mathematics knowledge to a mathematics concept in the secondary curriculum?

I utilized a qualitative, case study methodology to examine these questions. This methodology allowed me to gain rich descriptions of the understandings, beliefs, and experiences of the participants in my study, as expressed through the one-on-one, task-based interviews with participants. The five participants engaged in their choice of four tasks, from a list of seven pre-chosen tasks which embedded connections between secondary and post-secondary mathematics. The interviews were transcribed and coded, as described in Chapter 3. The themes that emerged from the data suggest that the prospective mathematics teachers in this study had limited opportunities to build content connections between secondary and postsecondary mathematics. In the discussion that follows, I examine the themes in the context of improving the education of future teachers, both in university mathematics courses and teacher education.

### 8.1.1 The Role of University Mathematics in Teacher Development

The first research question stated above aimed to extend the work of Zazkis and Leikin (2010) and understand what prospective teachers perceived as the role of their advanced knowledge in their development as teachers. One of the major themes developed in my analysis of participants' perceptions of the role of ad-
vanced mathematical knowledge (AMK) in their work as a teacher was the notion that much of the content in post-secondary mathematics is disconnected from what is taught in secondary school. Although all participants perceived value in having an advanced degree for reasons such as having experience beyond the students, being able to field questions, and having an increased awareness of what mathematics is, the majority of the benefits mentioned were focused on skills and beliefs, rather than content.

The participants in this study lamented that although admission to the teacher education program required advanced mathematical coursework, their mathematics methods courses did not require them to use their extensive mathematical expertise. Participants with two teachable subjects found this to be unique to their mathematics methods, since their content expertise was being extended and drawn upon in their other methods courses. In particular, in Section 4.3, Bailey remarked on how they felt their expertise in literature was used and extended during their English methods courses, while their mathematics courses required no more than grade 10 mathematics. As a caveat, I must once again remark that I did not observe or obtain syllabi for the methods courses taken by participants. My remarks and analysis are based solely on the shared perceptions of participants in my study.

Bailey's remarks on the differences between their two methods courses connects well with other participants' commentary on the structure of their teacher education program. Many participants remarked that they wanted a more "pragmatic" approach to their mathematics methods courses. Since I am unaware of the precise nature of the methods courses, I am unable to discuss their structure in any
specific way; however, I can speak to one of the sub-themes revealed in the participants' remarks. The desire for a more pragmatic program was often followed by a remark that the mathematics methods courses did not require any knowledge of mathematics.

Indeed, teaching mathematics and building MKT is mathematical work (Ball and Bass, 2002). Generating appropriate activities, understanding where content extends to and develops from within the curriculum, and evaluating where students might be confused in a particular lesson are all elements of MKT and require not only pedagogical expertise, but mathematical expertise. KDUs are understandings that transform the ways in which one understands and works with a concept. In turn, such an understanding could transform and impact MKT, as understood through the framework of developing MKT (Silverman and Thompson, 2008). Although the notion of a KDU was developed to help teachers understand key learning moments in the curriculum, the construct could be used to help future teachers see how their advanced mathematical knowledge is connected to the mathematics they plan to teach. However, in order to develop KDUs, prospective teachers need to experience opportunities for learning that foster the development of such understandings. My research indicates that prospective secondary mathematics teachers perceive that there were few opportunities to draw upon and little need for advanced mathematical knowledge in their mathematics methods courses.

To summarize, participants in this study perceived their expertise as unnecessary in their teacher education program, as well as in their future work as teachers. This view was evidenced by participants' perceived disconnect between sec-
ondary and post-secondary mathematics content. It is entirely possible that despite a perceived disconnect between these content domains, participants may have held transformed understandings of secondary mathematics concepts due to their content expertise. The task-based interviews served as a context for exploring this possibility. As will be elaborated below, the perceived disconnect was explicitly observed through participants' engagement in the tasks.

### 8.1.2 Content Connections Between University and Secondary Mathematics

The second research question above aimed to understand the ways prospective secondary teachers' related advanced mathematics knowledge to secondary mathematics content. Overall, I found that participants had limited content connections between secondary and post-secondary mathematics content. Participants expressed a perceived disconnect between secondary and post-secondary content and this disconnect was observed in discussing connections between post-secondary mathematics and specific problems at the secondary level.

Participants in the study demonstrated multiple misconceptions about the behaviour of real-valued polynomial functions, both at the secondary and post-secondary level (Chapter 5). While depth of understanding of base cases, such as quadratics and cubics, were demonstrated by participants such as Adrian and Taylor, these understandings remained at the secondary level. The content knowledge gained in courses, such as abstract or linear algebra, did not seem to have an impact on their understanding of polynomials, beyond the use of specific examples. Participants'
discussion of number and limits, while accurate in a secondary context, did not go beyond the horizon of the secondary curriculum (Chapter 6). Even participants such as Bailey, Taylor, and Adrian who had taken advanced courses in real analysis did not hold conceptualizations of irrational numbers beyond the definition of "not rational." With respect to examining prospective teachers' connections within geometric tasks, results were somewhat tangential to what was expected (Chapter 7). Indeed, results indicate that participants had a limited understanding of the use of proof methods, particularly proof by contradiction. Taylor, for example, recognized this gap in their understanding and and connected it to their desire to use proof as a means for building understanding in their future classrooms (Chapter 4). Through our discussion of the irrationality of $\sqrt{2}$, Taylor remarked that they may have taken their understanding of proof strategies for granted and that they would need to teach themselves again, before teaching others. Taken together, these results reveal that participants' did not hold personally powerful understandings of post-secondary mathematics that had the potential to impact their pedagogical practice.

The process of reflective abstraction is understood to be the mechanism in which learners build new knowledge, with the construction of new knowledge being based on past knowledge and experience. So, according to Piaget, the construction of new understandings should be built by extending already existing ones. Stakeholders and educators alike have continually emphasized the importance of building mathematical connections, to help build a more robust understanding of the secondary curriculum (Conference Board of the Mathematical Sciences, 2012). Indeed, as elaborated in the review of the literature, this is one of the major reasons
for requiring mathematics teacher to have extensive experience in advanced mathematics. However, taking university mathematics does not necessarily imply that reflective abstraction is happening.

Bailey remarked that university mathematics "really is diverging from what is taught in high school." In the interviews, Bailey expressed a love and passion for mathematics and teaching, but based on their remarks throughout the interview, their university mathematics education may have not offered the meaningful opportunities necessary to engage in reflective abstraction and build KDUs. As noted in the analysis of participants' responses during the polynomials tasks, Bailey expressed that in the abstract algebra courses they took, the content was "out there," far removed from secondary mathematics, and the structure of the course was useful for the students who would pursue research careers in mathematics (Chapter 5).

Suominen (2015) followed on the work of Cofer (2015), who found that "undergraduate abstract algebra students are not recognizing mathematical connections between abstract algebra and secondary school mathematics" (Suominen, 2015, p. 75). Through examining the connections to secondary mathematics explicitly stated in abstract algebra texts, she argues that the teaching of abstract algebra should be reconceptualized as an extension of secondary algebra and geometry, rather than a generalization. The work of Suominen (2015) dove tails with the laments of Bailey. Possibly due to Bailey's experiences in advanced mathematics, they came to the belief that secondary and post-secondary mathematics share limited connection threads. Simon (2006) argues that KDUs are not constructed by seeing examples or being relayed information, but through a personal process of
reflective abstraction. If advanced mathematics courses are taught in the traditional manner, are there explicit opportunities for students to build meaningful KDUs and engage in reflective abstraction? One should consider how the work of Suominen (2015) in the context of abstract algebra extends to other advanced mathematics courses.

One such example, is the conceptualization of number as limit. This conception appeared in the task on $0.999 \ldots=1$, as well as the task on $2^{\sqrt{3}}$. Building a robust understanding of number has been stated to be of immense value and thoroughly relevant content knowledge for mathematics teachers (Conference Board of the Mathematical Sciences, 2012). Past work suggests that real analysis can be a rich context for future teachers to develop their MKT, as it opens opportunities to gain a robust understanding of certain mathematical concepts, such as my study's tasks about $2^{\sqrt{3}}$ and $0.999 \ldots$ (Wasserman et al., 2017). Despite the majority of the participants having coursework in analysis, the results of these tasks suggest missed opportunities to extend the content of the secondary curriculum to a more robust and rigorous post-secondary context. Wasserman et al. (2017) present a framework for teaching real analysis that may be a suitable for mathematicians who are interested in teaching a course that will help future teachers build connections and KDUs to related content of the secondary curriculum.

In their framework, they argue that future teachers would benefit from a course in real analysis that is "building up from practice and stepping down from practice" (Wasserman et al., 2017, p.562), so that fundamental mathematical ideas that are buried in the secondary curriculum resurface in real analysis courses. I argue that
the results of my study further support a need for the utilization of Wasserman's framework, which may prove useful for professors of real analysis. Gauging by the extensive post-secondary mathematical coursework taken by the participants of this study, they appeared to have the mathematical skill necessary to build connections between secondary and post-secondary content. However, based both on the limited connections drawn during the task-based interviews, as well as the perception that advanced courses are more relevant for gaining skills than extending content, their university mathematics coursework may have provided limited opportunities to engage in reflective abstraction and develop KDUs. Employing the framework of Wasserman et al. (2017) might be an appropriate first step in considering how mathematicians can build real-analysis courses that will help build the MKT of future teachers.

While some students, such as Taylor and Adrian, exited their advanced courses with a sense of a bigger picture, others, like Bailey and Jaime, left their advanced courses feeling a disconnect between university mathematics and the curriculum they are to teach. I am in no way arguing that university mathematics pedagogy should be completely transformed or that the traditional pedagogical methods of university mathematics courses should be disposed; indeed, as was noted by many participants, there are benefits to learning how to "think like a mathematician," as Taylor put it. Problem solving skills, rigour, proof, and understanding are fundamental to a quality university mathematics education. Unfortunately, based on the results of this study, a focus on the first three elements may undermine understanding.

Although the traditional definition-theorem-proof approach used in many upperyear mathematics courses (Thurston, 1998) may be working for some, it may leave just as many (if not more) in the dark. University mathematics courses do have the opportunity to provide future secondary mathematics teachers with knowledge that could impact their MKT. With many advanced courses being requirements for future teachers, the results here suggest that advanced courses may not be providing sufficient opportunities for teacher candidates to build connections between the content they know, the content they are learning, and the content they will eventually teach. All of the participants in this work were successful university students. Indeed, many of the participants completed their undergraduate coursework with very high GPAs and course marks. Most courses in advanced mathematics focus heavily on theory and rigour, but the results of this study suggest that such a focus on theory may hinder understanding. Bailey, who was a very successful mathematics student, lamented that some of their classes felt like "a total blur," only two years later. How might mathematicians adjust their pedagogical practice so that successful students, like Bailey, complete their coursework seeing the relevance and connections of advanced coursework to mathematics they already know? However, I feel that it is necessary to note that I am not entirely aware of the lived experiences and lived curriculum of the participants in my study. Indeed, participants may have experienced connections in their coursework, but these experiences were not recalled during interviews in the study.

Additionally, some participants' demonstrated misconceptions of secondary mathematics content, which is in line with previous work in the field (Even, 1993; Leung et al., 2016; Stump, 1999). Indeed, misconceptions at the secondary level
were observed in every task. In the context of polynomials, all participants over extended their knowledge of quadratics and cubics to the cases of higher degree polynomials. With exponential functions, irrational numbers, and decimals, misconceptions were held by multiple participants with respect to approximation versus exactness. The notion of reflection symmetry was overextended in the context of inverses by Casey. While all of these misconceptions exist in the domain of secondary mathematics, they have the potential to be corrected by drawing upon participants' advanced mathematical expertise.

The results of this thesis shed light on the content links future teachers make between secondary and post-secondary mathematics. The links observed in this work were few and did not extend beyond first-year mathematics, for the most part. If there were connections to content beyond the first-year mathematics curriculum, the depth and power of these connections to teaching secondary mathematics was limited to singular examples. Despite extensive mathematics coursework at the post-secondary level, results suggest that their knowledge of advanced mathematics did not transform their understanding of secondary mathematics content. In the following section on implications, I will detail suggested actions educators and researchers may want to take to support the development of MKT that integrates advanced mathematical knowledge into future teachers' understanding of secondary mathematics content.

### 8.1.3 Limitations

As was elaborated in Chapter 3, the qualitative, case study approach was utilized to help provide rich descriptions of the ways participants perceive and draw upon advanced mathematics knowledge to inform their teaching. The sample size of this study, at only five participants, allowed for me to deeply engage with the qualitative data obtained during the one-on-one interviews. However, it is important to note that these results are not generalizable, and were not intended to be. The data, results, and description in this study are unique to the participants of this study. The results do not necessarily extend to all prospective secondary mathematics teachers with similar backgrounds. Rather, the intent of this study was to examine the ways post-secondary mathematics knowledge informs mathematical knowledge for teaching (MKT), to provide initial insights that mathematicians and mathematics educators might consider as they reevaluate the ways they support the development of MKT for future teachers. The results of this study suggest the need for future work which examines faculty members' (in mathematics and mathematics education) perceptions of the role of their courses in the development of future teachers' MKT.

Another limitation of the study comes from the tasks included in the task-based interviews. Although all of the tasks offered to participants were inspired and developed from past literature on MKT, they constitute a very limited collection of tasks that could be used to understand the role of advanced mathematics knowledge (AMK) in developing MKT. It is entirely possible that participants may have been able to draw upon post-secondary mathematics knowledge in other tasks, but
this work would require an additional study where the tasks allow participants to draw upon their mathematical expertise more generally. However, results from the one-on-one interviews about participants' perceptions of the role of AMK in their growth as teachers suggests connections may still be limited, due to the perceived disconnect between content in secondary and post-secondary mathematics.

Regardless of these limitations, the results of this study are of value and have implications for the ways mathematicians, mathematics teacher educators, and secondary mathematics teachers (practicing or pre-service) build connections between secondary and post-secondary mathematics.

### 8.2 Implications

When I entered into this work, I hoped that this study could be a step in helping both mathematicians and teacher educators to consider the ways in which they support connections between advanced mathematics and the development of mathematical knowledge for teaching and key developmental understandings. In the sections that follow, I outline the prospective implications of this work in both research and practice.

### 8.2.1 Implications in Research

The results of this work have opened up a number of new questions and avenues for exploration. The first portion of the study examined prospective teachers' perceptions of the role of advanced content for their teaching, but I am now curious to explore mathematicians' perspectives. In particular, what role do mathematicians perceive advanced mathematics plays in the development of secondary teachers?

In what ways do they support the development of MKT in their courses? When they are teaching advanced courses, who is their target audience? To whom are they teaching? These are all questions that I believe should be explored, so that we have context for the ways advanced mathematics courses are currently taught.

Another extension area for future research into AMK and MKT would be classroom observations in university mathematics courses. These observations would explore the explicit connections being made between secondary content and the content of the course being observed, by both the professor and the students. This work could occur in any advanced mathematics course, but courses in abstract algebra, real analysis, geometry, and proof appear to be important, due to their direct and rich relationship to the secondary curriculum. Furthermore, such observations would provide depth and context to some of the current study's participants' claims that there are limited connections being made in university mathematics classes. In understanding how and where mathematicians already build connections between secondary and post-secondary mathematics, we may gain a sense of how and where to make such opportunities more frequent.

Additionally, an interesting domain to investigate is the analog of the above study, but in teacher education. That is, in what ways do teacher educators and future teachers build connective threads between post-secondary and secondary mathematics content in math methods courses? Again, classroom observations would provide some support or defence to the claims made by participants in this study, such as "not needing math beyond grade 10 " and feeling as though there was not a focus on "how am I going to teach a concept"? Finally, these classroom
observations could provide context on the ways in which mathematics methods courses help build the various facets of MKT for future teachers, including connections between secondary and university mathematics.

### 8.2.2 Implications in Practice

The results of this study suggest that prospective teachers may need more explicit opportunities for reflective abstraction in their advanced mathematics courses. This is an important, but ambitious endeavour, that requires mathematicians to think deeply about the content they teach, where the construction of KDUs might occur, and would involve reconceptualizing some courses as extensions, rather than generalizations (Suominen, 2015). Furthermore, it would require professors to include explicit opportunities inside or outside of class that encourages students to go through the process of reflective abstraction, relate new content to what they already know, and build new meaning and inferences.

The notion of a KDU may be a helpful context for mathematicians to consider the ways they are supporting the development of MKT. Although Simon (2006) focuses on how elementary mathematics teachers could identify critical learning points in the elementary curriculum, this construct could be equally as beneficial for mathematicians to consider in university mathematics. In particular, mathematicians may want to question what moments in the curriculum are a "transformation" of concepts previously studied by students. Once these moments are identified, Simon (2006) argues there must be transformation of the instructor's practice, if the focus is to be on the development of KDUs. Indeed, as mentioned
in Chapter 2, students are the ones who have to go through the process of reflective abstraction to develop their own KDU. Identifying these key moments in the post-secondary mathematics curriculum may provide a basis for the development of connections between what advanced mathematics and the mathematics they already know and plan to teach.

Alternatively, mathematics departments might want to consider the development of a new course that would explicitly examine secondary mathematics from an advanced perspective. Many departments offer mathematics courses for future elementary school teachers, but few offer a mathematics course specifically aimed at future secondary teachers. If it is indeed the case that limited connective threads are being developed in advanced mathematics courses, prospective secondary mathematics teachers might benefit from a course that takes the content from advanced courses - considered to be "disconnected" and "out-there" - and relates it retrospectively through the content of the secondary curriculum. Following a framework similar to Wasserman et al. (2017), such a course could be a transformative course for prospective teachers that changes their perspective on the role of advanced mathematics in the development of MKT. The results of my study could provide a start for this kind of mapping of connected threads and concepts in advanced mathematics courses.

Mathematicians, particularly those in positions of course construction and development, may not be the only ones who should consider the development of opportunities to build connections between secondary and post-secondary mathematics content. Even if prospective teachers exit their undergraduate degrees hold-
ing the perception that their advanced mathematics knowledge is not relevant to their future work as teachers, teacher educators are in a prime position to disrupt this belief. Participants' responses to the task on symmetry is an excellent example. Although most participants' responses landed in the domain of secondary content, there were connective threads to post-secondary mathematics. With the existence of these threads, such as symmetry groups and alternative geometries, comes an opportunity for teacher candidates to build upon and explore these connections with respect to the curriculum. While some may argue that such advanced knowledge isn't necessary, the argument I make in this work is that it has the potential to have a positive effect on future teachers' understanding and pedagogy of secondary mathematics. Teacher educators can provide meaningful, curriculum centred learning moments for future teachers to develop their MKT.

The task on applications of quadratic functions could be another content area for teacher educators to build the MKT of prospective teachers. Despite recognizing the importance of the concept as an "entry point" for more advanced mathematical modelling, some of the participants communicated that they had limited examples for how to motivate this central concept of the secondary curriculum. This dovetails with some participants' concern that they are afraid they are going to "teach the way they were taught," due to the view that they have limited concrete examples to bring into the classroom. These same participants lamented profoundly at the prospect of students loosing interest in mathematics due to limited and contrived examples. The exploration of new applications of concepts (such as quadratic functions) could be an opportunity for teacher educators to develop the pedagogical and content expertise of future teachers. The inclusion of more math-
ematical work in math methods courses may be an interesting avenue for teacher educators to explore.

Adrian mentioned that they tried to reflect on how their advanced content expertise connected the secondary curriculum in their own practice, but did not find it being emphasized in their methods courses. All the participants in this study had extensive mathematical expertise, which was unique to the coursework they had taken in their undergraduate studies. While Adrian had post-secondary expertise in physics and applied mathematics, Casey had more experience in pure mathematics. Both of the participants, being in the shared space of a mathematics methods course, had much to learn from each other.

Indeed, the shared space of a mathematics methods course has the potential to yield opportunities for teacher candidates to share content extensions and mathematical knowledge that could impact their future teaching practice. In order for this to happen, there need to be incentives and opportunities to do so. Teacher candidates are content experts and teacher education is a space to build MKT, of which pedagogical and subject matter knowledge are elements. Even if the focus is on pedagogical knowledge, the results of this study could encourage teacher educators to consider the ways in which they are building, extending, and drawing upon the mathematical content knowledge and expertise of future teachers.

All the above future work remains in the context of prospective secondary mathematics teachers. Another extension of this work would be in the domain of practicing teachers. Zazkis and Leikin (2010) examined practicing teachers'
perceptions of the role of their advanced mathematics knowledge (AMK) in their teaching and found that the participating teachers held many of the same views as participants in this study. In particular, they observed that participants viewed AMK as valuable in building skill and confidence, but not necessarily in regard to content. Combining the insights from Zazkis and Leikin (2010) and the current study leaves me interested to follow-up with the current study's participants to explore whether they maintain the same perceptions of the role AMK in their pedagogy, after several years of teaching experience. Due to limited research in this area, both task-based interviews and/or classroom observations, would prove fruitful.

Overall, I foresee numerous opportunities to develop resources for mathematicians and mathematics teacher educators to help them in developing connections between secondary and post-secondary content. Before doing this, however, it would be prudent to investigate how these connections are already being developed in university mathematics and teacher education. I look forward to the research community engaging with the results of this study through related work that will one day benefit the mathematical learning of future teachers, while simultaneously building meaningful relationships between mathematicians and mathematics educators.

### 8.3 Closing Remarks

This study has examined the role of advanced mathematics knowledge in the mathematical knowledge for teaching of future secondary mathematics teachers. Through a qualitative case study, I examined the ways in which participants explicitly drew
upon advanced knowledge to inform their teaching. This was done through taskbased interviews, which were composed of potential classroom situations where advanced mathematics knowledge could be used to enhance their pedagogy. I supported the results from these task-based interviews with interviews that explored what participants perceived more generally as the role of their advanced mathematics knowledge in their growth as teachers.

Results from this study suggest a perceived disconnect between mathematics studied at the university level and mathematics taught and studied in secondary school, which has been observed in existing literature (McGuffey et al., 2019; Zazkis and Leikin, 2010). This was observed through participants' remarks of the role of their advanced knowledge in their teaching. Although beliefs and values developed in university mathematics were viewed as valuable by many participants, connections between mathematics content were viewed as limited and irrelevant to their future work. This was supported through the task-based interviews, where participants demonstrated content misconceptions at the secondary level, on top of providing a limited number of content connections between secondary and postsecondary mathematics.

Previous literature suggests that advanced mathematics knowledge has the potential to transform teachers' understanding of the secondary curriculum through the expansion of one's horizon content knowledge (Wasserman and Stockton, 2013). However, past literature also suggests that secondary mathematics teachers do not perceive advanced mathematics to play an important role in their teaching (Wasserman et al. 2015; Zazkis and Leikin, 2010). The results of this study are in line with
previous literature and further support the need for building connections between secondary and post-secondary content. My study extends previous literature by explicitly examining prospective secondary teachers' perceptions of their advanced content expertise and the connective threads of this expertise to the content they will eventually teach. The development of such connections to impact pedagogical practice may require a reconceptualization of advanced mathematics courses and mathematics methods courses, so that they more frequently engage students in the process of reflective abstraction (Piaget, 1970) and the construction of key developmental understandings (Simon, 2006) between secondary and post-secondary mathematics.

It is my hope that the results of this study will encourage mathematicians and mathematics educators to find common ground in the domain of secondary teacher education. Both of these parties play fundamental roles in the education and training of future mathematics teachers, who in turn prepare prospective students for university mathematics. This study could inspire further collaboration between mathematicians and mathematics education scholars in the academy, as the results are of importance to both of these academic departments. Although some strides are being made increase cross-departmental research and collaboration (Fried, 2014), I believe that the domain of secondary mathematics teacher education has the potential to be a mutual investment for these two groups to further collaborate in enhancing the mathematical knowledge for teaching of future teachers, and in turn, the mathematics learning of students in secondary school.

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## Appendix A

## Interview Materials

## Please choose four tasks from the following list:

- TASK B: Your students are confused as to why we can define and calculate $2^{\sqrt{3}}$.
- TASK C: A student is working through a problem and asks you if $0.99999 . . .=1$.
- TASK D: You are teaching a week on factoring polynomials and you have found that your students are struggling to recognize when they should stop trying to factor.
- TASK E: Your students are learning about inverse functions. What would you include in your lesson plan?
- TASK F: You are teaching a week on symmetry to your students. What would you include in your lesson plan?
- TASK G: You have been teaching a unit on quadratic functions for a few weeks and one of your students asks you why they need to know about them.
- TASK H: A student is confused as to whether $\sqrt{2}$ is an irrational or rational number, especially after realizing it is the length of the diagonal of a square of side length 1 .

1. One of the factors of $3 x^{2}-16 x+k$ is $x-7$. Determine the value of k . Justify your answer.
2. Which of the following is not a function? Justify your answer.
A.

B.

C.

D.

3. When a polynomial $P(x)$ is divided by $x+3$, the remainder is 2 . Which point must be on the curve $y=P(x)$ ? Justify your answer.
a. $(3,-2)$
b. $(-3,0)$
c. $(-3,2)$
d. $(3,2)$
4. For which of the following functions is $f(f(x))=x$ for all x in the domain of $f(x)$ ? I: $f(x)=x$, II: $f(x)=-x$, III: $f(x)=1 / x$
a. I and II only
b. I and III only
c. II and III only
d. I, II, and III
5. Write $0 . \overline{99998}$ as a fraction. Justify your answer.
6. Find the graph of $y=3^{x}-5$. What is the $x$-intercept? Justify your answers.

- How do you conceptualize mathematics as a field of study?
- Do you think it is important for secondary mathematics teachers to know advanced mathematics? Why or why not?
$\bullet$
- What roles do you see those four years of learning playing in your next year of study? What grade(s) do you hope to teach?
- Do you see your post-secondary degree in mathematics having an impact on your teaching? In what ways?


## Appendix B

## Recruitment emails and

## documents

## Prospective Secondary Mathematics Teachers’ Knowledge for Teaching

Dear Teacher Candidate,
We are writing to invite you to take part in the doctoral dissertation research of Vanessa Radzimski in the Department of Curriculum and Pedagogy at the University of British Columbia, Vancouver Campus. The primary investigator for this study is Dr. Cynthia Nicol and the co-investigator is Vanessa Radzimski. Through this research study, we hope to better understand the relationships between university level mathematics knowledge and knowledge for teaching in the secondary mathematics classroom. As a future secondary mathematics teacher with a post-secondary degree in mathematics you are in an ideal position to provide valuable first-hand information from your own perspective.

As a participant, we invite you to participate in an interview component (one-on-one or focus group). We'll ask you to complete a short survey so that we may assign you to either the one-on-one or focus group interviews. This will ensure that we have a diverse group of participants in each. We ask all participants to provide the researchers with their university transcripts, clearly indicating mathematical courses taken, in order to provide background information for the interviews. For those in the one-on-one interviews, we'll ask you to think about math problems of the type found in the BC Grade 8-12 math curriculum. All interviews will be audio recorded and transcribed. Your responses to the questions and academic transcripts will be kept confidential. You will be assigned a false name for all written reports and publications. Total time for participation will be around $2-2.5$ hours. For those in the focus group interviews, we will discuss your experiences in your mathematics degrees. The interview will be transcribed and you will be assigned a false name for all written reports and publications. Total time for participation in the focus group will be around $1-1.5$ hours.

As a token of appreciation for your time, you will receive a $\$ 25$ VISA gift card as compensation for your participation in the study. Your participation will be a valuable addition to our research and findings could lead to greater public understanding of mathematics teacher education and influence a higher level of communication between departments of mathematics and departments of teacher education at the university. It could also serve as an opportunity for you to think deeply about the ways in which your advanced mathematical knowledge might inform your pedagogy.

If you are willing to participate please contact Vanessa Radzimski by E-mail or phone within ten days of receipt of this letter, suggesting a day and time that suits you and we will do our best to arrange a meeting to your availability. The purpose of this meeting will be to discuss the study in more depth and sign the consent form, if you agree to participate. If you have any questions please do not hesitate to ask.

Thank you again for considering this research opportunity!
Cynthia Nicol and Vanessa Radzimski

## E-mail to Course Instructor for Recruitment

Dear XXX,

Under the supervision of Dr. Cynthia Nicol of the University of British Columbia, I am conducting a study entitled "Prospective Secondary Mathematics Teachers' Knowledge for Teaching." Through this research study, we hope to begin to understand how a post-secondary degree (minor or major) in mathematics influences teachers' pedagogy in the secondary mathematics classroom. As the course instructor for EDCP 342: Mathematics - Secondary: Curriculum and Pedagogy at the University of British Columbia, we would like to request a time to visit to your course so that we can recruit potential participants for this study.

We would like to arrange a time, at your convenience, to visit a session of your class and outline the research to your students. The co-researcher (myself) will provide consent forms to students in the class and return at the end of the course to answer any questions and complete the consent process with any students who wish to consent that day. Additionally, students will be able to email Dr. Nicol, or myself, with an interest to participate within ten days of the date of initial contact. In total, we will only need approximately ten minutes at the beginning of your class.

Precautions will be taken to protect your confidentiality. I am the only one who will know your identity and all identifying information will be masked.

If you have any questions please contact me (by phone or e-mail) or my supervisor, Dr. Cynthia Nicol (by e-mail). Our contact information is provided on the attached consent form. Please respond by e-mail within seven days of the date this e-mail was sent to express your approval of my visit. If you would like me to provide you with a hard-copy of the consent form in advance, please let me know and I will make arrangements.

Best wishes,

Vanessa Radzimski
PhD Candidate
University of British Columbia

## Verification of Payment for Participation in "Prospective Secondary Teachers' Mathematical Knowledge for Teaching"

By signing this document, you are verifying that you, $\qquad$ , received the $\$ 25$ VISA gift card that you would receive for participating in the study entitled "Prospective Secondary Teachers’ Mathematical Knowledge for Teaching" with Cynthia Nicol (primary investigator) and Vanessa Radzimski (co-investigator).

Signature: $\qquad$

Date: $\qquad$

