Aspects of quantum information in quantum field theory and quantum gravity

by

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Abstract

In this thesis we discuss applications of quantum information theoretic concepts to quantum gravity and the low-energy regime of quantum field theories.

The first part of this thesis is concerned with how quantum information spreads in four-dimensional scattering experiments for theories coupled to quantum electrodynamics or perturbative quantum gravity. In these cases, every scattering process is accompanied by the emission of an infinite number of soft photons or gravitons, which cause infrared divergences in the calculation of scattering probabilities. There are two methods to deal with IR divergences: the inclusive and dressed formalisms. We demonstrate that in the late-time limit, independent of the method, the hard outgoing particles are entangled with soft particles in such a way that the reduced density matrix of the hard particles is essentially completely decohered. Furthermore, we show that the inclusive formalism is ill-suited to describe scattering of wavepackets, requiring the use of the dressed formalism. We construct the Hilbert space for QED in the dressed formalism as a representation of the canonical commutation relations of the photon creation/annihilation algebra, and argue that it splits into superselection sectors which correspond to eigenspaces of the generators of large gauge transformations.

In the second part of this thesis, we turn to applications of quantum information theoretic concepts in the AdS/CFT correspondence. In pure AdS, we find an explicit formula for the Ryu-Takayanagi (RT) surface for special subregions in the dual conformal field theory, whose entangling surface lie on a light cone. The explicit form of the RT surface is used to give a holographic proof of Markovicity of the CFT vacuum on a light cone. Relative entropy of a state on such special subregions is dual to a novel measure of energy associated with a timelike vector flow between the causal and entanglement wedge. Positivity and monotonicity of relative entropy imply positivity and monotonicity of this energy, which yields a consistency conditions for solutions to quantum gravity.

Lay Summary

Quantum information theory, the theory of how information is processed in quantum systems, plays an important role in deepening our understanding of quantum gravity, a theory which seeks to unify quantum and gravitational physics. In this thesis we apply quantum information theoretic concepts in two contexts.

First, we investigate the quantum information carried away by radiation produced after particles interact gravitationally or through the electromagnetic interaction. In such interactions, an infinite number of very low-energy particles are produced; these particles carry away a large amount of information about the particles undergoing the interaction. We formulate methods of calculation which allow investigation of the information spread due to the production of these low-energy particles.

Second, we translate quantum information theoretic inequalities into inequalities in quantum gravity. This supplements the equations of gravitational physics with additional constraints that must be obeyed in a consistent theory of quantum gravity.

Preface

A large part of the body of this thesis has been published elsewhere and is included verbatim. The ordering of author names is alphabetical.

Most of chapter 4 is an adapted version of D. Carney, L. Chaurette, D. Neuenfeld and G. Semenoff, *Infrared quantum information*, Phys.Rev.Lett. 119 (2017) no.18, 180502 [1]. Like the two following papers, this publication is a result of many discussions and close collaboration between all authors. My main contributions were towards the identification of the currents and the the proof of their relation to the decoherence condition. The manuscript was drafted by D. Carney and edited by all authors. Chapter 4.5 is unpublished, original work. I thank L. Chaurette for discussions at an early stage.

A version of chapter 5 has appeared as D. Carney, L. Chaurette, D. Neuenfeld and G. Semenoff, *Dressed infrared quantum information*, Phys.Rev. D97 (2018) no.2, 025007 [2]. The calculation which lead to equation (5.9) was carried out by D. Carney and L. Chaurette. The generalization to multi-particle states and the proof of the finiteness of the reduced density matrix was joint work between all authors. Furthermore I contributed to chapters 5.4 and 5.5 which discuss the physical interpretation of dressed states and the relation to black hole information. A first draft of the manuscript was prepared by D. Carney and L. Chaurette and edited by all authors.

Chapter 6 contains a version of D. Carney, L. Chaurette, D. Neuenfeld and G. Semenoff, *On the need for soft dressing*, J. High Energ. Phys. (2018) 2018:121 [3]. Most of the preliminary calculations were work shared between L. Chaurette and myself. I contributed the findings on the inconsistency of scattering of normalized wave packets in the inclusive formalism, chapter 6.4, and a first draft of the manuscript, which was edited by all authors. Versions of chapters 4 - 6 have also appeared in [4].

A version of chapter 7 was uploaded to the Arxiv as *Infrared-safe scattering without photon vacuum transitions and time-dependent decoherence* [5]. I am the sole author of this work, which has greatly benefited from discussions with D. Carney, L. Chaurette and G. Semenoff.

Chapter 9 has been published as D. Neuenfeld, K. Saraswat and M. Van Raamsdonk, *Positive gravitational subsystem energies from CFT cone relative entropies*, J. High Energ. Phys. (2018) 2018:50, [6]. The paper is a result of close collaboration between the authors. Calcuations were shared work between K. Saraswat and myself, while drafting the manuscript was shared work between all authors. Related material also appeared in [7].

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Acknowledgments

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Chapter 1

Quantum information in fundamental physics

1.1 Black hole entropy and the quest for quantum gravity

Based on the requirement that the second law of thermodynamics should hold even in the presence of black holes, Bekenstein [8] conjectured that black holes should possess entropy. If this were not the case, one could drop a system with non-zero entropy into a black hole and thus – at least operationally – violate the second law of thermodynamics. Bekenstein conjectured the entropy of a black hole to be proportional to the area of its event horizon, A_{BH}, divided by Newton's constant G_N. Consequently, to save the second law of thermodynamics, the concept of entropy should be replaced with a generalized entropy

$$S_{\mathrm{gen}} = \frac{A_{\mathrm{BH}}}{4G_N} + S_{\mathrm{out}}, \tag{1.1}$$

which does not decrease; here, S_{out} denotes the entropy of matter outside the black hole horizon. The conjecture that black holes have entropy and thus should be seen as thermodynamical systems was subsequently supported by Hawking [9], who demonstrated that black holes radiate at a temperature proportional to their surface gravity. The results were in line with the predicted scaling of entropy with horizon area, and made black hole thermodynamics consistent. In thermodynamics, statistical physics, and information theory, entropy is a measure of the lack of knowledge about the microstate of a system, assuming we know its macroscopic properties. At least in string theory, this interpretation also applies to the entropy of certain black holes, as can be shown by microstate counting.

The relation between area and entropy indicates that certain quantities in quantum gravity can be understood in information theoretic terms. If, as is widely believed, quantum gravity is a true quantum theory, it thus seems reasonable that progress can be made by using concepts from *quantum* information theory in the study of quantum gravity.

1.2 Quantum information theory in fundamental physics

In the past decades, the application of quantum information theory has been at the center of various important discoveries in fundamental physics. One of the most renowned discoveries is the black hole information paradox. The radiation emitted by black holes, as calculated by Hawking, was found to be completely random. If this were to remain true in the full quantum theory, the evaporation of a black hole would erase all information about what has fallen into it, thereby violating the basic premise of quantum theory that time-evolution is unitary, i.e., information conserving [9]. A version of the black hole information paradox [10] can be explained in terms of quantum information theoretic quantities. Excited modes close to the black hole horizon have to be strongly entangled with modes behind the horizon in order to give a smooth geometry and thus allow for the equivalence principle of general relativity to hold, which states that a freely falling observer should not note anything out of the ordinary when they cross the horizon. On the other hand, at least at late times, modes close to the horizon must also be strongly entangled with early-time modes of the Hawking radiation if unitarity is to be preserved [11]. A property referred to as *monogamy of entanglement* prohibits strong entanglement with two disparate subsystems, thus posing a paradox: under certain additional physically-motivated assumptions, either the geometry at the horizon is not smooth and the equivalence principle fails, or black hole evaporation is not unitary.

Concepts from quantum information theory have also played an important role in understanding how spacetime emerges in the AdS/CFT correspondence. In its simplest form, the AdS/CFT correspondence [12][1] is a proposed duality between a superconformal field theory in d dimensions and string theory in an asymptotically anti-de Sitter spacetime in $d + 1$ dimensions. The conformal field theory can be thought of as living at the conformal boundary of the anti-de Sitter spacetime. The entanglement entropy of a subregion of the field theory can be computed in the gravitational theory as the area of a special surface anchored on this boundary [13, 14]. This suggests that in a holographic theory, spacetime in the gravitational picture is intimately linked to entropy in the field theory [15, 16].

The use of quantum information theoretic quantities has lead to new conjectures and proofs in semiclassical gravity and quantum field theory; see e.g., [17–19]. Moreover, concepts from quantum information theory have been used to obtain a better understanding of the dynamics of black holes [20], and to find discrete toy models [21] and explain properties of the AdS/CFT correspondence such as subregion duality [22].

The success of quantum information theoretic ideas in black hole physics and quantum gravity motivates furthering those investigations, and applying these methods to other problems such as scattering theory [23, 24].

1.3 The roadmap

The first part of this thesis analyzes the impact of infrared (IR) divergences on quantum information theoretic quantities. We will investigate how the presence of IR divergences affects the information carried away by unobserved particles in scattering. The surprising result is that QED and perturbative quantum gravity both predict that unobservable radiation carries away an essentially maximal amount of information and leaves the observed particles in a mixed state; this is independent of which method is used to render IR divergences finite. However, closer investigation shows that the typical prescription for removing IR divergences, while applicable to the scattering of momentum eigenstates, cannot be used to study the scattering of wavepackets. This hints at a rich structure of the Hilbert spaces of QED and perturbative quantum gravity, which split into superselection sectors corresponding

[1] see also chapter 8

to representations of the canonical commutation relations. The so-called *dressed formalism* takes into account this structure, and can be used to define approximate finite-time scattering amplitudes, which allow for the calculation of decoherence rates.

There has also been a recent resurgence of interest in the infrared structure of gauge theories and gravity coming from a seemingly different perspective.[2] It has been shown that certain theorems involving soft bosons can be understood as Ward identities of asymptotic symmetries; they can be thought of as gauge transformations that extend to infinity [27–29]. This has lead to speculations about how black holes store information [30–33]. We will see below that in four dimensions, infrared divergences, decoherence and large gauge transformations are intimately linked. We will use this relation to comment on the role of the infrared in solutions to the black hole information paradox.

In the second part of this thesis, we will briefly introduce the AdS/CFT correspondence. There exists a large body of work which links information-theoretic inequalities in the CFT to geometric constraints in gravitational theories, i.e., [34–39]. Here, we extend results that posit an equivalence between the relative entropy of ball-shaped subregions of a holographic CFT and a measure of energy defined on a subregion of its holographic dual, broadening these results to a more general class of subregions. We obtain explicit expressions for extremal surfaces in pure AdS and use them to give straightforward holographic proofs of the Markov property for the vacuum state of a ball-shaped region.

In the next chapter, we give a brief review of concepts from quantum information theory which are relevant for the rest of this thesis. Reviews of infrared divergences and ways of dealing with them, as well as concepts relevant to the AdS/CFT correspondence, can be found in the introductions to parts I and II of this thesis, respectively.

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[2] For a review, see [25]. For earlier work, see [26].

Chapter 2

A very short introduction to quantum information

This chapter will give a brief introduction to the quantum information theoretic quantities which appear in this thesis. Sections 2.1 and 2.2 are relevant for both parts of the thesis, whereas sections 2.3 to 2.5 are only relevant for the second part. More detailed introductions can be found in [40, 41].

2.1 Quantum mechanics

In quantum mechanics, the state of a physical system is described by a unit normalized vector in Hilbert space, up to a phase. Given two physical systems A and B with Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, any state of the joint system is described by a vector in the product Hilbert space $\mathcal{H}_{A\cup B} = \mathcal{H}_A \otimes \mathcal{H}_B$. In particular, the system can be in a superposition

$$|\psi\rangle = \cos(\alpha)\,|0\rangle_A \otimes |0\rangle_B + \sin(\alpha)\,|1\rangle_A \otimes |1\rangle_B\,. \tag{2.1}$$

Here, the states $|0\rangle_{A/B}$ and $|1\rangle_{A/B}$ are two orthogonal states of the Hilbert space $\mathcal{H}_{A/B}$. For generic values of α, it is impossible to write the total state as the product of the states of the systems A and B,

$$|\psi\rangle \neq |\phi\rangle_A \otimes |\eta\rangle_B\,, \tag{2.2}$$

and thus the states of subsystems A and B are correlated.

Measurements on quantum systems are represented as Hermitean operators, $\mathcal{O}^\dagger = \mathcal{O}$. The real eigenvalues of $\mathcal{O}$ give the allowed measurement outcomes, and the average outcome of a measurement of $\mathcal{O}$ in the state $|\psi\rangle$ is given by the inner product $\langle\psi|\,\mathcal{O}\,|\psi\rangle$. We can equivalently describe the state $|\psi\rangle$ by a density matrix

$$\rho = |\psi\rangle\,\langle\psi|\,, \tag{2.3}$$

such that the expectation value is given by

$$\langle\psi|\,\mathcal{O}\,|\psi\rangle = \mathrm{tr}(\mathcal{O}\rho). \tag{2.4}$$

If ρ is constructed from a state as shown in equation (2.3), it is called *pure*.

In the case of a multi-partite system $\mathcal{H}_{A\cup B}$, we can imagine operations which only act on one subsystem, say subsystem A. Such measurements are represented by operators $\mathcal{O}_A = \tilde{\mathcal{O}}_A \otimes \mathbb{1}_B$. If the multipartite system is in a product state, for example $|\psi\rangle = |1\rangle \otimes |0\rangle$, the expectation value of $\mathcal{O}_A$ (and all composite operators) can be calculated by ignoring $|0\rangle_B$,

$$(\langle 1| \otimes \langle 0|)(\tilde{\mathcal{O}}_A \otimes \mathbb{1}_B)(|1\rangle \otimes |0\rangle) = \langle 1|\,\tilde{\mathcal{O}}_A\,|1\rangle\,\langle 0|0\rangle_B = \langle 1|\,\tilde{\mathcal{O}}_A\,|1\rangle\,. \tag{2.5}$$

It can be shown that for operations which only act on subsystem A, $|1\rangle_A$ is a complete description.

However, if the system is in an entangled state, such as equation (2.1), a description of A in terms of a state in the Hilbert space $\mathcal{H}_A$ is not available anymore. Instead, a complete description of the quantum state for operators which only act on the A subsystem is given by the reduced density matrix

$$\rho_A = \mathrm{tr}_B(\rho), \tag{2.6}$$

where ρ is the density matrix which describes the system and tr_B traces over all states in $\mathcal{H}_B$. Unless the system was in a product state, the trace will turn a previously pure state into a *mixed* one with

$$\rho^{\text{mixed}} = \sum_{i}^{N} c_i \left|\eta_i\right\rangle \left\langle\eta_i\right| . \tag{2.7}$$

The quantum system described by this density matrix describes a classical ensemble of pure states $|\eta_i\rangle$. The probability to find the system in the state $|\eta_i\rangle$ is c_i.

2.2 Entanglement entropy

To quantify the lack of knowledge of how the reduced density matrix ρ_A was purified by subsystem B we can use the *von Neumann entropy* of the reduced density matrix [42],

$$S(\rho_A) = -\text{tr}(\rho_A \log \rho_A). \tag{2.8}$$

The von Neumann entropy $S(\rho)$ vanishes if ρ is a pure state and is maximal if ρ is maximally mixed, i.e., proportional to the identity matrix. The von Neumann entropy of a reduced density matrix is oftentimes called *entanglement entropy*, which indicates that non-zero von Neumann entropy can result from entanglement with another system. However, note that a non-zero von Neumann entropy also measures classical uncertainty, for example if the whole system is described by a thermal ensemble. The reason is that if ρ is not pure, $S(\rho)$ also obtains a contribution due to the c_i in equation (2.7); this counts the statistical entropy of the ensemble of pure states. In the following, we will use the terms *entanglement entropy* and *von Neumann entropy* interchangeably.

2.3 Relative entropy

We can define a measure for the distinguishability of two states of our quantum system, called *relative entropy*,

$$S(\rho\|\sigma) = \text{tr}(\rho \log \rho) - \text{tr}(\rho \log \sigma). \tag{2.9}$$

Relative entropy is positive definite, i.e., it is positive (or infinite) except when $\rho = \sigma$, for which it vanishes. It is also monotonic, meaning

$$S(\rho_A \| \sigma_A) \leq S(\rho_{AB} \| \sigma_{AB}). \tag{2.10}$$

Furthermore, positivity of relative entropy can be used to show certain properties of entanglement entropy, such as subadditivity, $S(\rho_A) + S(\rho_B) \geq S(\rho_{AB})$.

We can get some intuition for relative entropy by considering the case where ρ and σ are simultaneously diagonalizable, such that the trace in (2.9) reduces to a sum over eigenvalues ρ_i and σ_i. If ρ and σ describe orthogonal pure states, then $\mathrm{tr}(\rho \log \sigma) = \sum_i \rho_i \log \sigma_i$ contains a term where ρ_i is positive but σ_i vanishes. Thus $S(\rho\|\sigma) = \infty$ and the two states are perfectly distinguishable. This is also true if ρ is maximally mixed and σ is pure. On the other hand, if ρ is pure and σ is maximally mixed, the relative entropy will be finite. Thus, relative entropy is not symmetric. Roughly speaking, relative entropy measures how easy it is to disprove the hypothesis that a system is described by σ, given that its actual state is given by ρ.

2.4 Markovicity of quantum states

If three random variables X, Y, Z have conditional probabilities that satisfy $p(X|Y,Z) = p(X|Y)$, they are said to form a *Markov chain*. Using the definition of conditional probability, $p(A|B) \equiv p(A,B)/p(B)$, one can show that this is equivalent to

$$\tilde{S}(XYZ) + \tilde{S}(Y) = \tilde{S}(XY) + \tilde{S}(YZ), \tag{2.11}$$

where $\tilde{S} = -\sum_i p_i \log p_i$ is the Shannon entropy. A "quantum version" of this equation is

$$S(A \cup B) + S(A \cap B) = S(A) + S(B), \tag{2.12}$$

which uses the von Neumann entropy and where we have identified subsystem A with the random variables X and Y and subsystem B with the variables Y and Z. A state of the joint system $A \cup B$ which obeys equation (2.12) is called a *Markov* *state*. In fact, obeying the Markov condition is equivalent to saturating strong subadditivity,

$$S(A \cup B) + S(A \cap B) \leq S(A) + S(B), \tag{2.13}$$

which generally holds for entanglement entropies.

2.5 Quantum information in quantum field theories

In this thesis, we see the above concepts applied to states in quantum field theories. The subsystems under consideration will either be subsystems in momentum space or position space. For a discussion in momentum space we want to define the trace on Hilbert space. If the Hilbert space is non-separable, it is in general not clear how such a definition would look like and we thus want to require that our Hilbert space is separable. This is generally the case in free field theories with massive particles, and we will see that this requirement has implications for the Hilbert space structure of theories with IR divergences.

If we are interested in subsystems in position space, the situation is more complicated, since the Hilbert space does not factorize into a product of Hilbert spaces of subregions. Instead of considering the Hilbert space of a subregion, we should consider the von Neumann algebra of operators associated with a subregion. It is then possible to define relative entropy in terms of the von Neumann algebra. Entanglement entropy of a subregion is an ill-defined concept since it is always divergent due to an infinite amount of entanglement in high energy modes across the boundary of that region. Nonetheless, if suitably regularized, the naive treatment of entanglement entropy works for all practical purposes and thus in this thesis we will be taking on this naive picture. Alternatively, oftentimes one can study UV divergence-free quantum information theoretic quantities such as relative entropy.

For a more detailed review see, e.g., [43]. Defining the modular Hamiltonian associated to a subregion as the negative logarithm of the reduced density matrix on that subregion,

$$H_\rho = -\log \rho, \tag{2.14}$$

we can bring relative entropy into a form which will be useful in chapter 9,

$$\begin{aligned} S(\rho||\sigma) &= \mathrm{tr}(\rho \log \rho) - \mathrm{tr}(\rho \log \sigma) \\ &= \mathrm{tr}(\rho \log \rho) - \mathrm{tr}(\sigma \log \sigma) + \mathrm{tr}(\sigma \log \sigma) - \mathrm{tr}(\rho \log \sigma) \\ &= \Delta S - \Delta \langle H_\sigma \rangle , \end{aligned} \tag{2.15}$$

where ΔS is the difference in von Neumann entropies of the states ρ and σ, and $\Delta \langle H_\sigma \rangle$ is the difference of expectation values of H_σ in those states. The modular Hamiltonian generally is a non-local operator. However, in some special cases, H_ρ can be written as an integral over local operators. For example, this is the case if ρ is the reduced vacuum density matrix of a half-space in a local quantum field theory, or if ρ is the reduced density matrix of the vacuum state on a ball-shaped region in a conformal field theory.

Part I

Quantum information in the infrared

Chapter 3

Infrared divergences in quantum field theory

In chapter 1.1 we briefly reviewed how quantum information theory has lead to important insights in fundamental physics. In the cases discussed, the applications of concepts from quantum information theory to quantum gravity are mostly based on an analysis in position space. For example, quantities of interest are relative or entanglement entropies of subregions. A natural extension of these ideas is to investigate whether quantum information theory in momentum space can yield equally interesting insights. The question which motivate the research in this part of the thesis is: *How does quantum information spread in scattering?*

In the following we will investigate quantum information theoretic aspects of scattering in four dimensions in the presence of long range forces such as gravity and electromagnetism. In such situations, scattering amplitudes are plagued by *infrared divergences* (IR divergences), which occur beyond leading order in the calculations of Feynman diagrams. Their appearance sets almost all scattering amplitudes to zero. We will be concerned with how to define information theoretic quantities in the presence of IR divergences and what IR divergences teach us about the Hilbert space structure. This will lay the foundation of a framework in which the spread of entanglement in scattering can be determined, even in the presence of IR divergences.

There are many more motivations to better understand information theory and dynamics in the infrared. Apart from the importance of infrared physics for the understanding of confinement, long wavelength modes seem to play an important role in the quest for a theory of quantum gravity. They are important for understanding non-locality [44], soft hair is proposed to capture black hole microstates [33][3] and several solutions of the black hole information paradox [10, 47] which are based on low-energy physics have been proposed, e.g., [48–51]. Since black hole formation and evaporation can be understood as a scattering problem, it seems worthwhile investigating the fate of information in scattering. Lastly, if the lessons learned so far from the AdS/CFT correspondence are correct, the bulk and the boundary theory should share the same Hilbert space and a better understanding of the Hilbert space of flat-space perturbative quantum gravity might yield hints towards the structure of the correct dual theory.

More concretely, the methods developed here are useful for investigating various questions related to quantum gravity. It has been argued [30–33] that information about what has fallen into a black hole can be stored in and retrieved from low-energy or *soft* field modes. A detailed understanding of the spread of information in scattering would enable us to quantify how much information can be carried by different parts of the spectrum. This question is also potentially relevant for experiments testing quantum mechanics or quantum gravity in the laboratory. As we will see below, almost all processes are accompanied by the emission of soft radiation which potentially destroys quantum coherence. While the tools developed in this thesis enable a thorough analysis of the above questions, answering them is beyond the scope of this thesis and will be deferred to possible future work.

The present chapter gives a review of infrared divergences in quantum field theory, before we give the main results in the subsequent chapters. Most parts of chapters 4 to 6 are heavily based on work which previously appeared in [1–3], and chapter 7 is a redacted version of a preprint [5]. Section 4.5 is original work which has not been published before.

[3] See also the older proposals [30, 31] and criticism thereof [45, 46].

3.1 Scattering and the asymptotic Hilbert space

To begin, let us briefly review the standard method of how scattering amplitudes are calculated in quantum field theory (see, e.g., [52, 53]). Physical states of a quantum field theory are represented as vectors in a Hilbert space $\mathcal{H}$. Time evolution is implemented by a unitary operator e^{-iHt} which acts on states in the Hilbert space (*Schrödinger picture*) or evolves operators in time (*Heisenberg picture*).

To motivate the definition of the S-matrix we imagine an idealized experiment. An experimentalist sets up a set of well-separated particles at some early time and is interested in the amplitude[4] with which the system turns into some set of well-separated particles at very late times. The S-matrix captures this information, and if we express the states in the Heisenberg picture, it is defined as

$$S_{\beta,\alpha} = {}_{\text{out,H}}\langle\beta|\alpha\rangle_{\text{in,H}}, \tag{3.1}$$

where $|\alpha\rangle_{\text{in,H}}$ and $|\beta\rangle_{\text{out,H}}$ are Heisenberg states which correspond to well-separated particles if measured at early or late times, respectively. In order to calculate quantum information theoretic quantities before and after scattering, we need the density matrices which describe incoming and outgoing states,

$$\rho^{\text{in}}(\alpha) = |\alpha\rangle_{\text{in}} \langle\alpha| , \qquad\qquad \rho^{\text{out}}(\beta) = |\beta\rangle_{\text{out}} \langle\beta| . \tag{3.2}$$

Since the particle content of $|\alpha\rangle_{\text{in,H}}, |\beta\rangle_{\text{out,H}}$ as measured at early and late times, respectively, is well-separated, the particles can be described as approximately non-interacting. This means that at early and late times we should be able to describe the system by a free theory with a Hamiltonian H_0 with the same spectrum as the full Hamiltonian H. In other words, if we use Schrödinger picture state $|\alpha, t_i\rangle_{\text{in,S}}$ to make the time-dependence explicit, there are states $|\alpha, t_i\rangle_{\text{in,0}}$ which evolve with the free Hamiltonian and approximate the Schrödinger picture states at early times t_i, $t_i - t < 0$,

$$e^{-iH(t-t_i)} |\alpha, t_i\rangle_{\text{in,S}} \sim e^{-iH_0(t-t_i)} |\alpha, t_i\rangle_{\text{in,0}} , \tag{3.3}$$

[4]Technically, she is interested in the probability which can be obtained from the amplitude.

and similarly for $|\beta\rangle_{\text{out}}$ at late times. These states are called *asymptotic states*. Consequently, we can write the S-matrix in the Schrödinger picture as

$$S_{\beta,\alpha} = \lim_{t'/t''\to\mp\infty} {}_{\text{out}}\langle\beta|\, e^{iH_0(t''-t_f)} e^{-iH(t''-t')} e^{-iH_0(t'-t_i)}\, |\alpha\rangle_{\text{in}}\,. \tag{3.4}$$

In this expression, we have dropped the zero subscript and will do so for the rest of this thesis. We have furthermore defined fixed times $t_{i/f}$ at which the states $|\alpha\rangle_{\text{in}}\,/|\beta\rangle_{\text{out}}$ in the Heisenberg and Schrödinger picture agree. H_0 is the free Hamiltonian in which the mass parameter takes its physical value. At a mathematical level, the role of the terms including H_0 is that they ensure convergence of the above expression [54]. We could remove the dependence on $t_{i/f}$ by redefining the S-matrix $S \to e^{iH_0(t_f-t_i)}S$.[5]

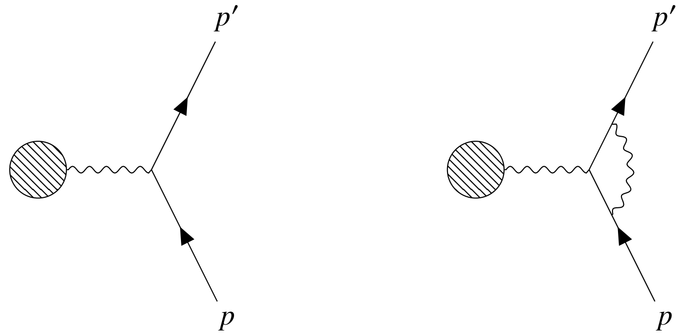
Going to the *interaction picture* in which operators evolve with the free Hamiltonian, while states evolve with the interaction Hamiltonian H_{int} allows us to rewrite the S-matrix in the well-known form [53],

$$S = \mathcal{T} e^{-i\int_{-\infty}^{\infty} dt\, H_{\text{int}}(t)}. \tag{3.5}$$

The time-dependence in the interaction Hamiltonian comes from the interaction picture fields and possibly some explicit time dependence. The interaction Hamiltonian is controlled by a small coupling constant which typically allows us to expand the S-matrix order by order in the coupling.

The Hilbert space of asymptotic states is usually constructed by expanding the fields of the theory in terms of creation and annihilation operators $a^\dagger(\mathbf{k})$, $a(\mathbf{k})$ and constructing the Fock representation of the canonical commutation relations $[a(\mathbf{k}), a^\dagger(\mathbf{k}')] = (2\pi)^3 2E_{\mathbf{k}}\delta^{(3)}(\mathbf{k}-\mathbf{k}')$. The Fock representation are all normalizable states which can be constructed by acting with creation operators which are convoluted with square-integrable functions on a vacuum state $|0\rangle$, which is annihilated by all annihilation operators, i.e., $a(\mathbf{k}')\,|0\rangle = 0$. We will see below that this choice of representation for asymptotic scattering states is problematic, if long-range forces are present. The reason is that in this case, even at very early or late times, the fields

[5] Oftentimes one chooses the convention that $t_f = t_i = T$, i.e., the incoming and outgoing particles are defined on the same, arbitrary timeslice.



(a) **(b)**

Figure 3.1: **(a)** Tree-level diagram for a fermion scattering off of a potential represented by the shaded blob. **(b)** The IR divergent one-loop correction to the process (a).

cannot be treated as approximately free.

3.2 Infrared divergences in S-matrix scattering

Following the standard prescription for calculating scattering amplitudes in theories with massless bosons in four dimensions between Fock space states, we encounter infrared divergences. For example, consider a scattering process in QED in which an electron with momentum $\mathbf{p}$ scatters off of a potential while transferring a momentum $\mathbf{q} = \mathbf{p} - \mathbf{p}'$, figure 3.1. The correction to the tree-level diagram in Feynman gauge is given by

$$(ie)^3(-i)^3 \int \frac{d^4k}{(2\pi)^4} \frac{\left[\gamma_\nu(-\not{p}' - \not{k} + m)\gamma^\mu(-\not{p} - \not{k} + m)\gamma^\nu\right]}{((p'+k)^2 + m^2 - i\epsilon)((p+k)^2 + m^2 - i\epsilon)(k^2 - i\epsilon)}, \tag{3.6}$$

where we have, like in the rest of this chapter, followed the notation of [55]. If the fermion propagators are almost on-shell, the integrand scales like $\frac{1}{k}$. This suggests a logarithmic divergence as $|\mathbf{k}|, k^0 \to 0$. This expectation is indeed correct and is a general feature of any non-trivial scattering process in four-dimensional electrodynamics [56, 57].

For a treatment of the general case, let us consider a matrix element $\mathcal{M}$ of an arbitrary scattering process, figure 3.2a. As argued above, IR divergences are

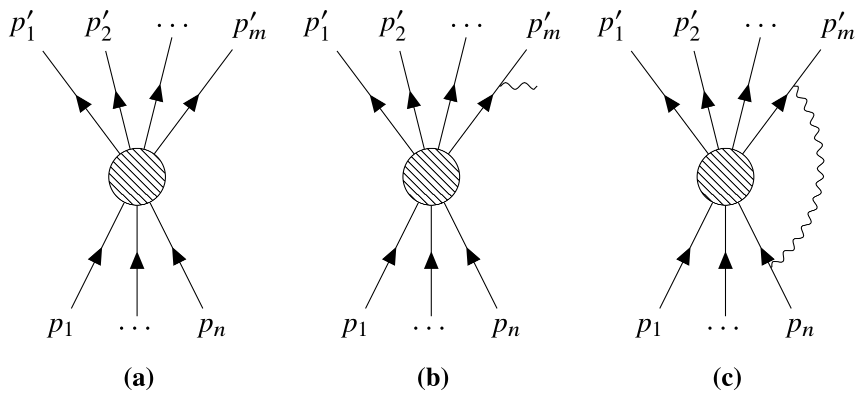


Figure 3.2: Construction of loops on the external legs. **(a)** An arbitrary scattering process which involves n incoming and m outgoing particles. **(b)** A vertex can be added to any external leg which emits a soft photon. **(c)** Multiple vertices can be connected by photon propagators to yield (soft) photon loops.

expected to appear as propagators go on-shell while emitting or absorbing a virtual photon of long wavelength. This can happen if incoming or outgoing legs emit or absorb virtual photons. The emission of a (virtual) soft photon from an outgoing leg requires us to add a vertex to the amplitude, for example, figure 3.2b,

$$\overline{u}_s(\mathbf{p})\mathcal{M}(p) \to \overline{u}_s(\mathbf{p})(ie\gamma^\mu)(-i)\frac{(-\not{p} - \not{k} + m)}{(p+k)^2 + m^2 - i\epsilon} \times \mathcal{M}(p+k). \tag{3.7}$$

Performing a similar replacement on a different leg, connecting the vertices with a photon propagator and integrating over the loop momentum gives us an IR divergent loop correction to the amplitude. To extract the divergence we split the loop integral into an integral over *soft* ($0 < \lambda \leq |\mathbf{k}| < \Lambda$) and *hard* ($\Lambda < |\mathbf{k}|$) momenta,

$$\int \frac{d^4k}{(2\pi)^4} \to \int_\lambda^\Lambda \frac{d^4k}{(2\pi)^4} + \int_\Lambda^\infty \frac{d^4k}{(2\pi)^4}. \tag{3.8}$$

The integral with $|\mathbf{k}| > \Lambda$ is UV divergent and needs to be renormalized. We will implicitly include contributions from hard momenta into $\mathcal{M}$ and write $\mathcal{M}^\Lambda$ to indicate that these contributions depend on Λ. The scale Λ should be much smaller than the electron mass and other relevant energy scales. We have also introduced a cutoff λ to regulate the IR divergence. At the end of the calculation λ has to be taken to zero.

Only the divergent parts of the integral with $|\mathbf{k}| \leq \Lambda$ are relevant to the discussion of IR effects, which is what we will be focussing on. Using the explicit form of the spinors [55] equation (3.7) can be written as

$$\mathcal{M} = \overline{u}_s(\mathbf{p})\widetilde{\mathcal{M}}(p) \to \left(\frac{ep^\mu}{p \cdot k - i\epsilon}\right) \times \overline{u}_s(\mathbf{p})\widetilde{\mathcal{M}}(p) + \text{(non-divergent)}. \tag{3.9}$$

In the general case, a similar argument shows that leading order in the inverse boson momentum, a vertex that emits a (virtual) photon of momentum k^μ is added by multiplying the matrix element with

$$\frac{\eta_n e_n p_n^\mu}{p_n \cdot k - i\eta_n \epsilon} + \mathcal{O}(1). \tag{3.10}$$

The factors p_n and e_n are the momentum and charge carried along the n-th leg. η_n takes values $+1$ or -1 if the n-th leg is outgoing or incoming, respectively. To leading order the matrix element $\mathcal{M}$ is independent of k and the contributions from soft photon loops factorize. The one-loop contribution coming from a soft loop between the n-th and m-th leg is $e_n e_m \eta_n \eta_m J_{mn}$ with

$$J_{mn} \equiv (-i) \int_\lambda^\Lambda \frac{d^4k}{(2\pi)^4} \frac{p_n \cdot p_m}{(k^2 - i\epsilon)(p_n \cdot k - i\eta_n\epsilon)(-p_m \cdot k - i\eta_m\epsilon)}. \tag{3.11}$$

The k^0 and $|\mathbf{k}|$ integrals can be performed and evaluate to

$$\begin{aligned} J_{mn} = &-\frac{1}{2(2\pi)^3} \int d\Omega \frac{v_n \cdot v_m}{(1 - \hat{\mathbf{k}} \cdot \mathbf{v}_n)(1 - \hat{\mathbf{k}} \cdot \mathbf{v}_m)} \log\left(\frac{\Lambda}{\lambda}\right) \\ &- \frac{i}{4\pi\beta_{nm}} \frac{(1 + \eta_n\eta_m)}{2} \log\left(\frac{\Lambda}{\lambda}\right), \end{aligned} \tag{3.12}$$

with $v^\mu = p^\mu/p^0$. The remaining integral over the unit vector $\hat{\mathbf{k}}$ yields

$$J_{mn} = \frac{1}{8\pi^2} \frac{1}{\beta_{nm}} \log\left(\frac{1 + \beta_{nm}}{1 - \beta_{nm}}\right) \log\left(\frac{\Lambda}{\lambda}\right) - \frac{i}{4\pi\beta_{nm}} \frac{(1 + \eta_n\eta_m)}{2} \log\left(\frac{\Lambda}{\lambda}\right). \tag{3.13}$$

We will postpone the physical interpretation of these terms to section 3.3. The imaginary part of J_{mn}, called *Coulomb phase*, only contributes to loops which connect two outgoing or two incoming legs. The factor of β_{nm} is the relative velocity of particles n and m in either rest frame and is given by

$$\beta_{nm} \equiv \sqrt{1 - \frac{m_n^2 m_m^2}{(p_n \cdot p_m)^2}}. \tag{3.14}$$

As we take the IR cutoff λ to zero we see that equation (3.13) – and thus the one loop correction which is proportional to J_{mn} – diverges. This indicates that for small enough IR cutoff perturbation theory breaks down and we need to resum the result to all orders. Luckily, the structure of IR divergent contributions in the infrared is simple enough to do this.

If we resum the contribution from soft loops to all orders, we need an expression which takes multiple photon emissions per leg into account. At leading order, adding a second vertex which emits momentum k_1 to the n-th leg, which already emits momentum k_2, yields a factor of

$$\left(\frac{\eta_n e_n p_n^\mu}{p_n \cdot k_1 - i\eta_n \epsilon}\right)\left(\frac{\eta_n e_n p_n^\nu}{p \cdot (k_1 + k_2) - i\eta_n \epsilon}\right) \tag{3.15}$$

in front of the matrix element, which corresponds to the case where k_2 is emitted before k_1. In addition, there will be a term which is obtained by swapping k_1 and k_2 corresponding to the case where k_1 is emitted before k_2,

$$\left(\frac{\eta_n e_n p_n^\mu}{p_n \cdot k_2 - i\eta_n \epsilon}\right)\left(\frac{\eta_n e_n p_n^\nu}{p \cdot (k_1 + k_2) - i\eta_n \epsilon}\right). \tag{3.16}$$

Summing both terms, we obtain

$$\left(\frac{\eta_n e_n p_n^\mu}{p \cdot k_1 - i\eta_n \epsilon}\right)\left(\frac{\eta_n e_n p_n^\nu}{p \cdot k_2 - i\eta_n \epsilon}\right). \tag{3.17}$$

Note that one would, in principle, expect a contribution of $\mathcal{O}(k^{-1})$ that could also be IR divergent. However, it turns out that all the divergences factorize [58]. This suggests the following rule, which can be proved by induction: To leading order in low momenta, we can account for the emission of M (virtual) soft bosons from the

n-th leg of a Feynman diagram by multiplying the amplitude with

$$\mathcal{M} \to \left(\prod_i^M \frac{\eta_n e_n p_n^{\mu_i}}{p_n \cdot k_i - i\eta_n \epsilon} \right) \times \mathcal{M}. \tag{3.18}$$

These can be connected by photon propagators as before.

Since the soft contributions to loop integrals factorize, the leading part of the N-th order correction is proportional to the N-th power of equation (3.11). More precisely, for the scattering between states $|\alpha\rangle$ and $|\beta\rangle$ it is

$$\mathcal{M}^\Lambda \times \sum_N \frac{1}{2^N N!} \left(\sum_{n,m \in \alpha,\beta} \eta_n \eta_m e_n e_m J_{mn} \right)^N = \mathcal{M}^\Lambda \times \left(\frac{\lambda}{\Lambda} \right)^{A_{\beta,\alpha}/2}, \tag{3.19}$$

with

$$A_{\beta,\alpha} = - \sum_{n,m \in \alpha,\beta} \frac{e_n e_m \eta_n \eta_m}{8\pi^2} \beta_{nm}^{-1} \log \left(\frac{1 + \beta_{nm}}{1 - \beta_{nm}} \right) + \text{(phase factor)}. \tag{3.20}$$

The phase factor is given in (3.13). The factor of 2^{-N} in equation (3.19) makes sure we do not count twice diagrams which only differ by the orientation of the photon line, while the factor of $N!$ corrects for overcounting different permutations of the photon lines. The function $A_{\beta,\alpha}$ is positive. Trivial terms in the S-matrix have $A_{\alpha,\alpha} = 0$ and thus can also be multiplied be the prefactor $\left(\frac{\lambda}{\Lambda}\right)^{A_{\beta,\alpha}/2}$. Thus, the factor which multiplies the whole S-matrix is the same as that for the matrix element. The scattering probability is

$$p(\alpha \to \beta) = \left(\frac{\lambda}{\Lambda} \right)^{A_{\beta,\alpha}} \left| S_{\beta,\alpha}^\Lambda \right|^2, \tag{3.21}$$

where the superscript on the S-matrix reminds us that loop diagrams are calculated with a cutoff Λ.

The prefactor damps the amplitude such that it vanishes in the limit $\lambda \to 0$. As we will see in the remainder of this thesis, this is not merely some technical problem. Quantum electrodynamics and perturbative quantum gravity in fact correctly predict that transition amplitudes between Fock space states vanish.

As can be seen from equation (3.11), the occurrence of IR divergences is tied to the number of spacetime dimensions and the structure of the propagators of the involved particles, most notably the absence of a regulating mass term in the boson propagator. It is thus not surprising that analogous divergences appear in four dimensions whenever massless bosons are exchanged. One example of particular importance is gravity. A similar argument to the discussion above shows that soft graviton loops contribute an infrared divergence of the form [56]

$$\mathcal{M} = \mathcal{M}^{\Lambda} \times \left(\frac{\lambda}{\Lambda}\right)^{B_{\beta,\alpha}/2}, \tag{3.22}$$

with the positive coefficient

$$B_{\beta,\alpha} = \sum_{n,m\in\alpha,\beta} \frac{m_n m_m \eta_n \eta_m}{16\pi^2 M_p^2} \frac{1+\beta_{nm}^2}{\beta_{nm}\sqrt{1-\beta_{nm}^2}} \log\left(\frac{1+\beta_{nm}}{1-\beta_{nm}}\right). \tag{3.23}$$

Another large class of theories with IR divergences are four-dimensional Yang-Mills theories. While in a non-perturbative treatment confinement might make sure that no IR divergences appear, in perturbative calculations they do appear in the fashion outlined above as soft divergences and need to be treated as well.

It turns out that the preceding discussion does not cover all possibilities for how IR divergences can appear in quantum field theory. Another source of IR divergences are collinear emissions which appear when massless particles emit other massless particles along their direction of propagation. Apart from Yang-Mills theory, this effect also appears in massless QED and gravity at high energies. In this thesis, we will not be interested in effects of collinear divergences, and only refer the reader to the existing literature [59, 60].

3.3 A semiclassical analysis

In order to make predictions that can be compared to experiment, one needs to eliminate infrared divergences. Approaches which accomplish this are based on the observation that, during a scattering process, Bremsstrahlung is produced. The produced radiation carries a finite amount of energy at arbitrarily long wavelengths, such that the number of photons of arbitrarily long wavelengths must diverge, since

$$\lim_{\omega\to 0} N(\omega) = \lim_{\omega\to 0} \frac{E(\omega)}{\hbar\omega} \to \infty. \tag{3.24}$$

The Fock space representation does not allow for infinite occupation numbers, which explains why generically, Fock space states cannot be used as scattering out-states, and consequently why the S-matrix elements between those states must vanish. In this section, we will use a semiclassical argument to derive the form of the expected photon out-state.

Assume we have a charged particle with momentum p^μ, which is scattered at the origin. After scattering, it has momentum p'^μ. The current for this particle is given by

$$j^\mu(x) = e\int_0^\infty d\tau \frac{p'^\mu}{m}\delta^{(4)}\left(x^\mu - \frac{p'^\mu}{m}\tau\right) + e\int_{-\infty}^0 d\tau \frac{p^\mu}{m}\delta^{(4)}\left(x^\mu - \frac{p^\mu}{m}\tau\right). \tag{3.25}$$

We now want to investigate the corresponding classical radiation field. To this end, we Fourier transform the above expression after introducing convergence factors $i\epsilon$,

$$j^\mu(k) = \int d^4x e^{-ikx} j^\mu_{t_0}(x) = -ie\left(\frac{p'^\mu}{p'\cdot k - i\epsilon} - \frac{p^\mu}{p\cdot k + i\epsilon}\right). \tag{3.26}$$

In Lorenz gauge, the solution to Maxwell's equations can be written as

$$A^\mu(x) = -ie\int \frac{d^4k}{(2\pi)^4} e^{ikx}\frac{1}{k^2}\left(\frac{p'^\mu}{p'\cdot k - i\epsilon} - \frac{p^\mu}{p\cdot k + i\epsilon}\right). \tag{3.27}$$

The term $\frac{1}{k^2}$ is the Green's function for the vector potential in Lorenz gauge.

To obtain the radiation produced in the scattering, we need to choose the Green's function to be the difference between the retarded and advanced Green's function.[6] The outgoing radiation can then be obtained by closing the k^0 contour in the lower

[6]In order to describe the full outgoing vector potential we would consider only the retarded Green's function. However, the conclusion we will be drawing is the same in both cases.

half-plane and is given by

$$A^{\mu}_{\rm cl,out}(x) = e\int \frac{d^3\mathbf{k}}{2|\mathbf{k}|(2\pi)^3}\left(\frac{p'^{\mu}}{k\cdot p'} - \frac{p^{\mu}}{k\cdot p}\right)\left(e^{ik\cdot x} + e^{-ik\cdot x}\right)\Big|_{k^0=|\mathbf{k}|}. \tag{3.28}$$

The quantum field theoretical description of a classical field is given in terms of a coherent state. We can formally write the coherent state which corresponds to equation (3.28) as a state in the photon Fock space,

$$|A^{\mu}_{\rm cl,out}\rangle = \mathcal{N}\exp\left(\int \frac{d^3\mathbf{k}}{(2\pi)^3 2|\mathbf{k}|} f^{\mu}(\mathbf{p},\mathbf{p}',\mathbf{k})a^{\dagger}_{\mu}(\mathbf{k})\right)|0\rangle\,, \tag{3.29}$$

with

$$f^{\mu}(\mathbf{p},\mathbf{p}',\mathbf{k}) = e\left(\frac{p'^{\mu}}{p'\cdot k} - \frac{p^{\mu}}{p\cdot k}\right), \tag{3.30}$$

and the normalization

$$\mathcal{N} = \exp\left(-\frac{1}{2}\int \frac{d^3\mathbf{k}}{(2\pi)^3 2|\mathbf{k}|}|f(\mathbf{p},\mathbf{p}',\mathbf{k})|^2\right). \tag{3.31}$$

The expectation value of the number operator in a thin shell in momentum space around momentum $\mathbf{k}$, $N_{\mathbf{k}} = a^{\dagger}_{\mu}(\mathbf{k})a^{\mu}(\mathbf{k})d^3\mathbf{k}$, is

$$\langle A^{\mu}_{\rm cl,out}|N_{\mathbf{k}}|A^{\mu}_{\rm cl,out}\rangle \sim \frac{d^3\mathbf{k}}{2|\mathbf{k}|}\left(\frac{v'^{\mu}}{k\cdot v'} - \frac{v^{\mu}}{k\cdot v}\right)^2 \sim \frac{d|\mathbf{k}|}{|\mathbf{k}|}. \tag{3.32}$$

This clearly shows that in the quantum mechanical description the infrared contains an infinite number of photons. The logarithm of the normalization $\mathcal{N}$ of the coherent state $|A^{\mu}_{\rm cl,out}\rangle$ is also proportional to equation (3.32) and thus divergent, and we see that the state which represents the classical Bremsstrahlung is not part of the Fock space represention.

The above argument explains why all S-matrix elements vanish. Quantum electrodynamics should reproduce classical physics at long distances. However, as we have seen, the expected out-state has a vanishing norm. Moreover, the overlap of Fock states with coherent states of the above form vanishes, which implies that the S-matrix maps Fock space states into a vector space orthogonal to Fock space. In conclusion, the IR divergences coming from loop corrections should be understood as a physical prediction. It sets all scattering probabilities between different Fock states to zero, simply because an infinite number of soft modes will be created.

The terms which appear in the normalization are reminiscent of the real part of equations (3.11) and (3.13). However, as noted around equation (3.13), there is also a divergent phase factor. It appears if more than one particle is present in the in- or out-state. Physically, it comes from the potential energy of a charged particle in the field created by a second particle, which can be seen by considering the non-relativistic case. The energy of a non-relativistic outgoing particle in the field of a different outgoing particle is given by

$$E(t) = m + \frac{m}{2}v_1^2 + \frac{q_1 q_2}{4\pi(r_0 + v_1 t)}. \tag{3.33}$$

Thus, at very late times, the phase of the corresponding state goes like

$$-i\int^t dt' E(t') \sim -iE_0 t - i\frac{q_1 q_2}{4\pi v_1}\log(t). \tag{3.34}$$

The treatment of outgoing particles as free Fock states only accounts for the first term, $-iE_0 t$, and the mismatch between the time evolution as a free state and equation (3.33) gives rise to the divergent phase factor in equation (3.13).

3.4 Dealing with infrared divergences

The prescriptions used to cancel the IR divergences can be classified into *inclusive* and *dressed* formalisms. The philosophy behind the inclusive formalisms is that one should not ask questions one cannot experimentally answer. We cannot build a detector that measures photons of arbitrarily low energies and therefore we should not ask how likely it is to scatter from a certain in-state to an out-state without any additional photons that might have escaped detection. Instead, we should ask for inclusive probabilities, i.e., the probability to scatter from $|\alpha\rangle$ to $|\beta\rangle$ plus any possible configuration of photons which escape detection. This implies a treatment in which amplitudes are regulator-dependent and cannot be assigned a physical interpretation. In light of more fundamental questions, however, this approach is unsatisfactory. For example, in the AdS/CFT correspondence, gravity is a quantum theory whose states live in a Hilbert space. If anything like this should be true in flat or de Sitter space, there must be a way of assigning regulator-independent quantum states to the out-state of a scattering experiment. Moreover, questions about the unitarity of time evolution can only be answered at the level of amplitudes.

A different approach is followed by dressed formalisms. These formalisms are built on the assumption that asymptotic states are not correctly modeled by Fock space states. Instead, physical states, such as electrons or generally massive particles, are accompanied by a certain photon/graviton field configuration called *dressing*. These dressings resemble the coherent states of the previous subsection. Amplitudes between dressed fields are finite, since the excitations contained within the dressing cancel IR divergences order by order.[7] In the following we will briefly summarize the inclusive and dressed formalisms.

3.4.1 The inclusive formalism

The objects of interest in scattering calculations are typically not scattering amplitudes, but scattering probabilities or scattering cross-sections, since these are physical and can be determined in experiment. As we have seen above, non-trivial scattering processes produce asymptotic states with an infinite number of soft bosons. Due to the limited volume of any apparatus and the limited duration of any experiment, it is clear that some of these bosons will escape undetected. To predict a detector response, we need to sum over all possible outcomes of our experiment which are consistent with our measurement, i.e., we need to sum over all possible soft boson emissions, where the energy of the boson is below some detection threshold. Consequently, the probability to scatter a state $|\alpha\rangle$ to a final state $|\beta\rangle$ should be calculated as

$$p^{\text{incl}}(\alpha \to \beta) = \sum_{\text{unobs.}b} |\langle \beta, b|\alpha\rangle|^2, \tag{3.35}$$

where the sum indicates that we consider the addition of unobservable soft bosons b in the final state. The so-obtained probability $p^{\text{incl}}(\alpha \to \beta)$ is called the inclusive

[7]Sometimes dressing is used to describe the process of adding photon/graviton field excitations to a state in order to make it gauge invariant, an idea pioneered in [61]. The IR part of such a dressing can in principle also be chosen to cancel IR divergences, but a priori both concepts are independent.

transition probability between states $|\alpha\rangle$ and $|\beta\rangle$.

Calculating inclusive probabilities as opposed to the naive probabilities $p = |\langle\beta|\alpha\rangle|^2$ is the textbook way of dealing with IR divergences [53, 57, 58], first established for QED [58, 62] and subsequently expanded to include the case of soft gravitons [56].

In section 3.2 we already discussed that the emission of soft (virtual) photons from incoming or outgoing legs requires multiplying the matrix element by a soft factor, equation (3.10). If a real photon is emitted, the same soft factor appears, the momentum k has to be put on-shell and the free index of the vertex needs to be contracted with a polarization vector, $\varepsilon_\ell(\mathbf{k})$ for outgoing and $\varepsilon^*_\ell(\mathbf{k})$ for incoming photons. For on-shell photons, the notion of *soft* is controlled by an additional threshold energy scale E_T which is smaller than all relevant energy scales of the experiment. This includes scales associated with the experiment's dimensions.

The addition of on-shell soft factors directly leads to Weinberg's soft theorems. To leading order in the inverse photon momentum, the S-matrix element for scattering between two asymptotic states $|\alpha, a\rangle_{\text{in}}$, $_{\text{out}}\langle\beta, b|$, with hard particles α, β and soft bosons a, b can be written as

$$S_{\beta b,\alpha a} = \prod_{i\in a,b}\left(\sum_{n\in\alpha,\beta}\frac{\eta_n e_n \varepsilon^{\ell_i}_\mu(\mathbf{k}_i) p_n{}^\mu}{p_n\cdot k_i}\right)\times S_{\beta,\alpha}, \tag{3.36}$$

where $\mathbf{k}_i$ and ℓ_i are the momentum and helicity of the i-th photon and $k^0 = |\mathbf{k}|$. The momentum $\mathbf{k}_i$ is taken to be outgoing from the vertex. This formula and the equivalent formula for soft gravitons

$$S_{\beta b,\alpha a} = \prod_{i\in a,b}\left(\sum_{n\in\alpha,\beta}\frac{1}{M_p}\frac{\eta_n \epsilon^{\ell_i}_{\mu\nu}(\mathbf{k}_i) p^\mu_n p^\nu_n}{p_n\cdot k_i}\right)\times S_{\beta,\alpha} \tag{3.37}$$

are known as Weinberg's soft theorems. Here the index n runs over all the incoming and outgoing hard particles, i runs over the outgoing soft bosons; $\eta_n = -1$ for an incoming and $+1$ for an outgoing hard particle. The e_n are electric charges and $M_p = (8\pi G_N)^{-1/2}$ is the Planck mass, and the ϵ's are polarization vectors or tensors for outgoing soft photons and gravitons, respectively. Recently, it was shown that these soft theorems can be understood as the Ward identities of asymptotic symmetries [25, 29, 63]. We will briefly discuss this in section 3.5.

We can use the soft theorems to show that inclusive transition probabilities are finite. Consider equation (3.35). The sum over soft bosons is implemented by integrating the momenta of all possible soft photon emissions up to some scale E_i and summing over all photon helicities. The sum of all photon energies is constrained to be less that E_T. For the leading order contribution, the emission of a single soft photon with unknown helicity we obtain

$$p^{\text{incl},(1)}(\alpha \to \beta) = \sum_{\ell=\pm} \int_{\lambda}^{E_T} \frac{d^3\mathbf{k}}{2|\mathbf{k}|(2\pi)^3} S_{\beta\mathbf{k},\alpha} S^*_{\beta\mathbf{k},\alpha} = C \times S_{\beta,\alpha} S^*_{\beta,\alpha}, \tag{3.38}$$

where again the boson's momentum is on-shell and

$$C = -\sum_{n,m\in\alpha,\beta} \sum_{\ell=\pm} \int_{\lambda}^{E_T} \frac{d^3\mathbf{k}}{(2\pi)^3 2|\mathbf{k}|} \left(\frac{\eta_n e_n \varepsilon^{\ell}_{\mu}(\mathbf{k}) p^{\mu}_n}{p_n \cdot k}\right) \left(\frac{\eta_m e_m \varepsilon^{\ell}_{\nu}(\mathbf{k}) p^{\nu}_m}{p_m \cdot k}\right). \tag{3.39}$$

We can simplify this expression by using that

$$\begin{aligned} \sum_{\ell=\pm} \varepsilon^{\ell *}_{\mu}(\mathbf{k}) \varepsilon^{\ell}_{\nu}(\mathbf{k}) &= g_{\mu\nu} - k_{\mu} c_{\nu} - c_{\mu} k_{\nu}, \\ k^{\mu} = |\mathbf{k}| \begin{pmatrix} 1 \\ \hat{\mathbf{k}} \end{pmatrix}^{\mu}, &\qquad c^{\mu} = \frac{1}{2|\mathbf{k}|} \begin{pmatrix} -1 \\ \hat{\mathbf{k}} \end{pmatrix}^{\mu}. \end{aligned} \tag{3.40}$$

The terms proportional to k_{μ} vanish upon contraction in equation (3.39). The integral over angles is precisely the same integral we have already encountered in equation (3.13) and we are left with

$$\begin{aligned} &\left(\sum_{n,m\in\alpha,\beta} \eta_n \eta_m e_n e_m \int_{\lambda}^{E} \frac{d|\mathbf{k}|}{(2\pi)^3 2|\mathbf{k}|} \int d\Omega \left(\frac{v_n \cdot v_m}{(1-\mathbf{v}_n\cdot\hat{\mathbf{k}})(1-\mathbf{v}_m\cdot\hat{\mathbf{k}})}\right)\right) \\ &= A_{\alpha,\beta} \int_{\lambda}^{E} \frac{d|\mathbf{k}|}{|\mathbf{k}|}, \end{aligned} \tag{3.41}$$

where $A_{\alpha,\beta}$ was given in equation (3.20). For the emission of N bosons with total energy below the threshold, we consider N factors of the form (3.39) whose total energy is constrained to be less than E_T. Summing over all possible emissions yields

$$\sum_{N=0}^{\infty} \frac{A^N}{N!} \prod_{i=1}^{N} \left(\int_{\lambda}^{E_i} \frac{d^3\mathbf{k}_i}{|\mathbf{k}_i|} \right) \theta(E_T - \sum_i E_i). \tag{3.42}$$

Here, we have made sure not to overcount identical photon emissions by introducing a factor of $\frac{1}{N!}$. The Heaviside theta function can be rewritten as

$$\theta(E_T - \sum_i E_i) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin(E_T u)}{u} \exp\left(iu \sum_i E_i \right). \tag{3.43}$$

In the following we will assume that all $E_i = E$. With this, the inclusive probability becomes

$$p^{\text{incl}}(\alpha \to \beta) = \mathcal{F}(E/E_T, A_{\alpha,\beta}) \left(\frac{E}{\lambda} \right)^{A_{\alpha,\beta}} p(\alpha \to \beta), \tag{3.44}$$

where $\mathcal{F}$ comes from evaluating the integral in equation (3.43) and $p(\alpha \to \beta)$ is the hard scattering probability. The function $\mathcal{F}$ is given by

$$\mathcal{F}(x, A) = \frac{1}{\pi} \int_{-\infty}^{\infty} du \frac{\sin(u)}{u} \exp\left(A \int_0^x \frac{d\omega}{\omega} (e^{i\omega u} - 1) \right). \tag{3.45}$$

The parameters E and E_T can be chosen such that if $A \ll 1$, $\mathcal{F}$ is close to one, e.g., $\mathcal{F}(1, A) \approx 1 - \frac{1}{12}\pi^2 A^2$. Note that, due to the positivity of $A_{\alpha,\beta}$, the prefactor in equation (3.44) diverges in the limit $\lambda \to \infty$. The dependence on λ is just right to cancel against the λ dependence which makes the loop-corrected amplitude vanish, equation (3.21). Thus, the inclusive scattering probability is free of λ dependences and IR finite,

$$p^{\text{incl}}(\alpha \to \beta) = \left(\frac{E}{\Lambda} \right)^{A_{\alpha,\beta}} \left| S^{\Lambda}_{\beta,\alpha} \right|^2 \mathcal{F}(E/E_T, A_{\alpha,\beta}). \tag{3.46}$$

For gravity, the situation seems more complicated. Since gravitons are themselves a source of stress-energy, they can set off a cascade of softer gravitons which might spoil the simple form of the expression for soft photon emission. However, we are fortunate as the coupling to gravitons is proportional to the energy of a particle. Consequently, such terms are subleading in momentum and do not contribute to divergences. Similarly, this also explains why loop-corrected graviton loops do not play any role: the above argument goes through and we end up with the same expression, equation (3.46) with $A_{\alpha,\beta}$ replaced by $B_{\alpha,\beta}$.

For Yang-Mills theories this argument does not work, since the coupling is not proportional to the momenta of the involved particles. Furthermore, apart from the soft divergences discussed, the appearance of collinear divergences causes additional problems. However, in these cases the KLN theorem [59, 60] guarantees that a modified prescription also produces scattering probabilities free of IR divergences. The modification consists of also including a sum over incoming soft particles.

3.4.2 Dressed formalisms

The inclusive formalism outlined above gives up the notion of scattering amplitudes. Dressed formalisms are an alternative approach with which finite amplitudes can be calculated. The underlying idea is to add additional soft radiation to incoming and outgoing states whose emission and absorption cancels IR divergences. The added radiation takes the form of the coherent states of section 3.3. In this section we will give a rough outline of the idea, following early work by Chung [64], which is sufficient until chapter 7. There, we will take a closer look at the more elaborate dressed formalisms of Faddeev and Kulish [65] and investigate the Hilbert space structure.

Dressed formalisms propose to replace momentum eigenstates by *dressed states*,

$$|\mathbf{p}_1, \ldots, \mathbf{p}_n\rangle \to \|\mathbf{p}_1, \ldots, \mathbf{p}_n\rangle\!\rangle. \tag{3.47}$$

In the case of a one-particle momentum eigenstate, the corresponding dressed state is defined as

$$\|\mathbf{p}\rangle\!\rangle = W_\lambda[f_\ell(\mathbf{p}, \mathbf{k})]\,|\mathbf{p}\rangle\,, \tag{3.48}$$

where $W_\lambda[f_\ell(\mathbf{p},\mathbf{k})]$ is an operator that creates a coherent state

$$W_\lambda[f_\ell] \equiv \exp\left(\int_\lambda^E \frac{d^3\mathbf{k}}{(2\pi)^3 2|\mathbf{k}|} \sum_{\ell=\pm} \left(f_\ell(\mathbf{p},\mathbf{k}) a_\ell^\dagger(\mathbf{k}) - h.c.\right)\right) \tag{3.49}$$

and

$$f_\ell(\mathbf{p},\mathbf{k}) = -e\frac{p\cdot\epsilon_\ell}{p\cdot k}\phi(\mathbf{p},\mathbf{k}). \tag{3.50}$$

Here, p, k are on-shell four-vectors, and $\phi(\mathbf{p},\mathbf{k})$ can be any function that goes to 0 as $|\mathbf{k}| \to 0$. The dressed state depends on an IR cutoff λ through the coherent state operator. This IR cutoff ensures that the normalization of the state created by $W[f_\ell]$ is finite, compare to the discussion around equation (3.31). The extension to the multi-particle case is straight forward and will be discussed in section 5.3. These states can be used to calculate scattering amplitudes,

$$\mathbb{S}_{\beta,\alpha} \equiv \langle\langle\beta\|S\|\alpha\rangle\rangle = \langle\beta| W_\beta^\dagger S W_\alpha |\alpha\rangle\,, \tag{3.51}$$

which are finite as $\lambda \to 0$. We call $\mathbb{S}$ the *dressed S-matrix*.

To see how the dressing removes IR divergences, consider the scattering of a dressed electron with momentum $\mathbf{p}$ to a dressed electron with momentum $\mathbf{p}'$. The cancellation of IR divergences takes place order by order, and we will show the first non-trivial order, $\mathcal{O}(e^2)$. We need to replace $|\mathbf{p}\rangle$ by $\|\mathbf{p}\rangle\rangle$ which at leading order reads

$$\begin{aligned}\|\mathbf{p}\rangle\rangle = &\left(1 - \frac{1}{2}\sum_{\ell=\pm}\int_\lambda^E \frac{d^3\mathbf{k}}{(2\pi)^3 2|\mathbf{k}|}|f_\ell(\mathbf{p},\mathbf{k})|^2\right) \\ &\times\left(1 + \sum_{\ell=\pm}\int_\lambda^E \frac{d^3\mathbf{k}}{(2\pi)^3 2|\mathbf{k}|} f_\ell(\mathbf{p},\mathbf{k}) a_\ell^\dagger(\mathbf{k})\right)|\mathbf{p}\rangle\,,\end{aligned} \tag{3.52}$$

where $f \sim \mathcal{O}(e)$. Note that the dressing only needs to be expanded to order e, since the absorption or emission of a photon from the dressing is also of order e, yielding a term of order e^2. Dressed S-matrix elements $\mathbb{S}_{\mathbf{p}',\mathbf{p}}$ equal bare S-matrix elements,

$S_{\mathbf{p}',\mathbf{p}}$, multiplied by a correction,

$$\left(1+\int_\lambda^E \frac{d^3\mathbf{k}}{(2\pi)^3 2|\mathbf{k}|}\sum_{\ell=\pm}\left(f_\ell^*(\mathbf{p}',\mathbf{k})f_\ell(\mathbf{p},\mathbf{k})-\frac{1}{2}|f_\ell(\mathbf{p},\mathbf{k})|^2-\frac{1}{2}|f_\ell(\mathbf{p}',\mathbf{k})|^2\right)\right)S_{\mathbf{p}',\mathbf{p}}$$
$$+\int_\lambda^E \frac{d^3\mathbf{k}}{(2\pi)^3 2|\mathbf{k}|}\sum_{\ell=\pm}\left(f_\ell(\mathbf{p},\mathbf{k})S_{\mathbf{p}',\mathbf{p}\mathbf{k}}+f_\ell^*(\mathbf{p}',\mathbf{k})S_{\mathbf{p}'\mathbf{k},\mathbf{p}}\right). \tag{3.53}$$

The first line comes from the process where the photon does not interact with the scattered particles at all and the change in normalization of the in- and out-going state. The second line consists of terms which appear since the dressing of the incoming and outgoing state interacts with the scattering process.

The second line can be rewritten using the soft theorem (3.36) as

$$\int_\lambda^E \frac{d^3\mathbf{k}}{(2\pi)^3 2|\mathbf{k}|}\sum_{\ell=\pm}\left(f_\ell(\mathbf{p},\mathbf{k})\sum_{n\in\{\mathbf{p},\mathbf{p}'\}}\eta_n f_\ell^*(\mathbf{p}_n,\mathbf{k})-f_\ell^*(\mathbf{p}',\mathbf{k})\sum_{n\in\{\mathbf{p},\mathbf{p}'\}}\eta_n f_\ell(\mathbf{p}_n,\mathbf{k})\right)S_{\mathbf{p}',\mathbf{p}}. \tag{3.54}$$

In summary, the total correction is

$$\mathbb{S}_{\mathbf{p}',\mathbf{p}}=S_{\mathbf{p}',\mathbf{p}}\left(1+\int_\lambda^E \frac{d^3\mathbf{k}}{(2\pi)^3 2|\mathbf{k}|}\sum_{\ell=\pm}\left(-f_\ell^*(\mathbf{p}',\mathbf{k})f_\ell(\mathbf{p},\mathbf{k})+\frac{1}{2}|f_\ell(\mathbf{p},\mathbf{k})|^2+\frac{1}{2}|f_\ell(\mathbf{p}',\mathbf{k})|^2\right)\right). \tag{3.55}$$

Now recall from equations (3.12) and (3.19) that we can split off the IR divergence coming from loops in the calculation of the S-matrix as

$$S_{\mathbf{p}',\mathbf{p}}=S^{\Lambda}_{\mathbf{p}',\mathbf{p}}\left(1-\frac{1}{2}\sum_{\ell=\pm}\sum_{n,m}\int_\lambda^\Lambda \frac{d^3\mathbf{k}}{(2\pi)^3 2|\mathbf{k}|}\left(\eta_n\eta_m f_\ell(\mathbf{p}_n,\mathbf{k})f_\ell(\mathbf{p}_n,\mathbf{k})\right)\right). \tag{3.56}$$

Using equation (3.40) it is easy to show that the corrections coming from the dressing, equation (3.55), and soft loops, equation (3.56), exactly cancel to order $\mathcal{O}(e^2)$. This argument can be extended to all orders [64].

The reason this procedure works can be understood from the semi-classical analysis in section 3.3. We can see from equation (3.28) that the IR radiation produced in a scattering process consists of two terms, one which depends on the incoming and one which depends on the outgoing momenta. The dressed states discussed here correspond to a case where we send fine-tuned radiation into the scattering region which only depends on the incoming hard particles and cancels the part of the outgoing radiation which depends on the incoming hard momenta. We will see in chapter 6 how this can be generalized.

The proposal reviewed here has several shortcomings. First, although we have well-defined amplitudes, we have to introduce an IR regulator into the states. As we send it to zero, the states become non-normalizable. As a consequence of this, the structure of the Hilbert space is unclear in the limit of vanishing regulator. Second, in the form presented here, it is not clear whether the divergence associated with the Coulomb phase, equation (3.34), still persists.

Steps to ameliorate these problems were taken in a series of papers by Kibble, who modified the procedure to take into account the divergent Coulomb phase and proposed to use a von Neumann space as the Hilbert space of dressed states without IR cutoff [66–69]. The proposed Hilbert space is non-separable, i.e., it does not have a countable basis, and the S-matrix maps states between different separable subspaces. This proposal was developed further by Faddeev and Kulish [65] who gave a derivation of the dressing from first principles and identified a subspace of Kibble's Hilbert space which is separable and stable under the action of the S-matrix. Their derivation of the dressing from first principles will be reviewed in chapter 7.

Another slightly different dressed formalism was proposed by Bagan, Lavelle and McMullan [70, 71]. However, the only difference between their approach and that of Faddeev-Kulish is that instead of dressing asymptotic states, they dress operators. For example the dressed operator A^μ creates modes on top of a classical radiation background, equation (3.28). Thus, with slight modifications, all statements made in this thesis also apply to their dressed formalism.

In [72] a dressed formalism for gravity was proposed. In this case, the dressing is again given by (3.49). This time, however, a and $a^\dagger$ are the graviton annihilation and creation operators and the functions $f(\mathbf{k}, \mathbf{p})$ depend on the polarization of the graviton $\epsilon_{\mu\nu}$,

$$f_\ell^{\rm gr}(\mathbf{k},\mathbf{p}) = \frac{p_\mu \epsilon_\ell^{\mu\nu} p_\nu}{k \cdot p} \phi(\mathbf{k},\mathbf{p}). \tag{3.57}$$

3.5 An infinity of conserved charges

Recently, interest in the IR behavior of gauge theories and gravity was revived from a different perspective. The work initiated in [28, 63, 73] demonstrated that *soft theorems* and *asymptotic symmetries* can be understood in a unified way. Moreover, these findings were used to suggest new ways of how black holes can store information [30, 31]. Dressed states also arise naturally in the recent discussions of asymptotic gauge symmetries [25, 28–30, 74, 75], which imply the existence of selection sectors [76–79]. See also [80, 81] for work on soft charges and dressing in holography. Throughout this thesis we will comment on the relation of our findings to asymptotic symmetries: *large gauge transformations* [73] in the case of QED and *BMS transformations* [82] in the case of gravity. The next subsections reviews the relevant aspects of the connection between asymptotic symmetries and Weinberg's soft theorems in the case of QED. A more complete review, also covering the case of non-abelian gauge theories and gravity can be found in [25].

3.5.1 Anti-podal matching and conserved charges

At leading order, solutions to Maxwell's equations obey an anti-podal matching condition at light-like infinity $\mathcal{I}^\pm$, c.f. figure 3.3. This is easy to see for the Liénard-Wiechert field of a point particle with charge e moving at constant velocity $\mathbf{v}$,

$$F_{rt}(\mathbf{x},t) = \frac{e}{4\pi} \frac{\gamma_{\mathbf{v}}(r - t\hat{\mathbf{x}} \cdot \mathbf{v})}{|\gamma_{\mathbf{v}}^2(t - r\hat{\mathbf{x}} \cdot \mathbf{v})^2 - t^2 + r^2|^{3/2}}. \tag{3.58}$$

Here, $\hat{\mathbf{x}} \cdot r$ is the three vector at which the field is evaluated at time t and $\gamma_{\mathbf{v}}$ is the relativistic gamma factor. We are interested in the electric field at light-like infinity $\mathcal{I}^\pm$. To obtain an expression on $\mathcal{I}^+$ we change coordinates to $(u = t - r, r, \hat{\mathbf{x}})$ and take

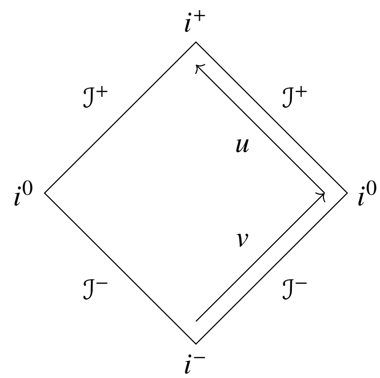


Figure 3.3: This figure shows the Penrose diagram of Minkowski space. Infinity is conformally mapped to a finite distance, thus distances are not faithfully represented, however, the causal structure is. Light runs at 45° angles. Lightlike future and past infinity, $\mathcal{I}^\pm$, are a good Cauchy slices for massless particles, while massive particles start and end at $i^\pm$. Spacelike infinity is denoted by i^0.

the limit of $r \to \infty$. while keeping u and $\hat{\mathbf{x}}$ constant. The result is

$$F_{rt}(\mathbf{x}, t)\Big|_{\mathcal{I}^+} = \frac{e}{4\pi r^2} \frac{1}{\gamma_{\mathbf{v}}^2 (1 - \hat{\mathbf{x}} \cdot \mathbf{v})^2}. \tag{3.59}$$

Using coordinates $(v = t + r, r, \hat{\mathbf{x}})$ we can take the limit of F_{rt} to $\mathcal{I}^-$ and find

$$F_{rt}(\mathbf{x}, t)\Big|_{\mathcal{I}^-} = \frac{e}{4\pi r^2} \frac{1}{\gamma_{\mathbf{v}}^2 (1 + \hat{\mathbf{x}} \cdot \mathbf{v})^2}. \tag{3.60}$$

Equations (3.59) and (3.60) are related to each other by $\hat{\mathbf{x}} \to -\hat{\mathbf{x}}$.

Light-like infinity has the topology of a cylinder $\mathbb{R} \times S^2$, where $\mathbb{R}$ is parametrized by u or v and $\hat{\mathbf{x}}$ parametrizes the S^2. To make the resulting equations simpler, one conventionally changes coordinates on the S^2 to complex coordinates $(z, \overline{z})$ such that the coordinates on the sphere of future infinity $\mathcal{I}^+$ are related to those on the sphere of past light-like infinity $\mathcal{I}^-$ by $(z, \overline{z}) \to (-z, -\overline{z})$. This way, a light ray which enters on $\mathcal{I}^-$ through the point $(z, \overline{z})$ exits at $\mathcal{I}^+$ at an angle given by the same coordinates.

With these conventions, the field strength tensor $F_{\mu\nu}$ obeys the matching condition

$$F^{(2)}_{ru}(z,\bar{z})\Big|_{\mathcal{I}^+_-} = F^{(2)}_{rv}(z,\bar{z})\Big|_{\mathcal{I}^-_+}, \tag{3.61}$$

where $F^{(n)}_{\mu\nu}$ denotes the coefficient of the r^{-n} term in a large-r expansion of $F_{\mu\nu}$. Since equations (3.59) and (3.60) are u and v independent, we have decided to evaluate F on $\mathcal{I}^+_-$ and $\mathcal{I}^-_+$. The two-sphere $\mathcal{I}^+_-$ is located on $\mathcal{I}^+$ at $u \to -\infty$ and similarly $\mathcal{I}^-_+$ is located on $\mathcal{I}^-$ at $v \to \infty$. Thanks to the matching condition, there exists an infinite number of trivially conserved charges

$$\mathcal{Q}^+_\varepsilon \equiv \int_{\mathcal{I}^+_-} \varepsilon(z,\bar{z}) \star F = \int_{\mathcal{I}^-_+} \varepsilon(z,\bar{z}) \star F \equiv \mathcal{Q}^-_\varepsilon, \tag{3.62}$$

where $\star$ is the Hodge star operator. This expression is true for any function $\varepsilon(z,\bar{z})$ defined such that $\varepsilon(z,\bar{z})|_{\mathcal{I}^+_-} = \varepsilon(z,\bar{z})|_{\mathcal{I}^-_+}$. For constant ε the conserved charge is simply the electric charge.

These charges are the generators of *large gauge transformations*, i.e., gauge transformations which do no vanish at infinity but reduce to transformations which are only functions of the coordinates z and $\bar{z}$ at infinity.

3.5.2 Hard and soft charges

Using Maxwell's equations

$$d \star F = \star j, \tag{3.63}$$

the charges can be rewritten as

$$\begin{aligned} \mathcal{Q}^+_\varepsilon &= \int_{\mathcal{I}^+_-} \varepsilon \star F = \int_{\mathcal{I}^+} d(\varepsilon \star F) + \int_{\mathcal{I}^+_+} \varepsilon \star F \\ &= \underbrace{\int_{\mathcal{I}^+} d\varepsilon \wedge \star F}_{\mathcal{Q}^+_{\varepsilon,S}} + \underbrace{\int_{\mathcal{I}^+_+} \varepsilon \star F}_{\mathcal{Q}^+_{\varepsilon,H}}, \end{aligned} \tag{3.64}$$

where we have used that $\int_{\mathcal{I}^+} \varepsilon \star j = 0$, since in QED there are no charges leaving Minkowski space through light-like infinity. The first term in equation (3.64) only depends on the behavior of the transverse electric field at future light-like infinity and is called the *soft charge*, $\mathcal{Q}^+_{\varepsilon,S}$. The second term depends on the longitudinal part of the electric field, weighted by ε, and is called the *hard charge*, $\mathcal{Q}^+_{\varepsilon,H}$. The soft charge is a measure of soft radiation, while the hard charge is a measure of the long-wavelength part of the longitudinal fields of charged matter particles. The same argument can be used to show that $\mathcal{Q}^-_{\varepsilon}$ also splits into a soft and hard part.

It can be shown [29, 74] that, independent of their photon content, out-states of definite momentum are eigenstates of $\mathcal{Q}^{\pm}_{\varepsilon,H}$. Similarly, dressed states are eigenstates of $\mathcal{Q}^{\pm}_{\varepsilon,S}$. Their eigenvalue is proportional to an integral, whose integrand depends on the residue of the dressing function $f_h(\mathbf{p},\mathbf{k})$ as $|\mathbf{k}|$ goes to zero.

3.5.3 Weinberg's soft theorems

Conservation of $\mathcal{Q}^{\pm}_{\epsilon}$ implies that the operator commutes with the Hamiltonian and thus in particular with the S-matrix,

$$0 = \langle\beta|\,[\mathcal{Q}_{\varepsilon}, S]\,|\alpha\rangle = \langle\beta|\,(\mathcal{Q}^+_{\varepsilon}S - S\mathcal{Q}^-_{\varepsilon})\,|\alpha\rangle\,. \tag{3.65}$$

The presence of IR divergences can be related to the conservation of the charges $\mathcal{Q}_{\epsilon}$ [29]. Calling the eigenvalues with respect to the soft charges $\mathcal{Q}^{\pm}_{\varepsilon,S}$, N_{out} and N_{in}, respectively, we find

$$(N_{\text{out}} - N_{\text{in}})\,\langle\beta|\,S\,|\alpha\rangle = \sum_{n\in\alpha,\beta} \frac{\sqrt{2}}{1+z\bar{z}} \frac{e_n\eta_n\epsilon^+\cdot p_n}{p_n\cdot k}\,\langle\beta|\,S\,|\alpha\rangle\,. \tag{3.66}$$

For the states in the Fock space representation one can check explicitly that the eigenvalue of the soft charge operator is zero, i.e., the left hand side of equation (3.66) vanishes. Hence, the only way equation (3.66) can hold is if the factor that multiplies the amplitude on the right-hand side vanishes or the amplitude itself is zero. For any non-trival scattering process, the prefactor is non-zero, so the amplitude must vanish. Moreover, it can be shown that equation (3.66) is simply a coordinate-transformed version of Weinberg's soft theorem [25].

The case of gravity is completely analogous, with the electric field in equation (3.62) replaced by the gravitational field.

Chapter 4

Infrared quantum information

This chapter is a redacted version of [1].

4.1 Introduction

We have seen in the previous chapter that in the standard treatment of scattering the S-matrix becomes ill-defined due to divergences coming from low-energy virtual bosons. The usual solution to this problem is to use the inclusive formalism, i.e., to argue that an infinite number of low-energy bosons are radiated away during a scattering event; this leads to divergences which cancels the divergences from the virtual states, and physical predictions in terms of infrared-finite inclusive transition probabilities.

In this chapter, we study quantum information-theoretic aspects of this proposal. Since each photon and graviton has two polarization states and three momentum degrees of freedom, one might suspect that the low-energy radiation produced during scattering could carry a huge amount of information. Here we demonstrate that, according to the methodology of [56, 58, 62], which was summarized in section 3.4.1, if the initial state is an incoming n-particle momentum eigenstate, the soft bosonic divergences can lead to complete decoherence of the outgoing hard particles, with the momentum eigenstates as the pointer basis [83]. This decoherence is avoided only for superpositions of pairs of outgoing states for which an infinite set of angle-dependent currents match, see equation (4.9). In simple examples like QED, this will be enough to get complete decoherence of all momentum superpositions. In less simple cases, one is still left with an extremely sparse density matrix dominated by its diagonal elements. See [84–86] for related work.

Having traced the radiation in this fashion, we obtain an infrared-finite, mixed reduced density matrix for the hard particles. In the simple cases when we get a completely diagonal matrix, we compute the entanglement entropy carried by the soft gauge bosons. The answer is finite and scales like the logarithm of the energy resolution E of a hypothetical soft boson detector.

While the tracing out of the soft radiation can be viewed as a physical statement about the energy resolution of a real detector, in this formalism, the trace is also forced on us by mathematical consistency: it is the only way to get well-defined transition probabilities from the infrared-divergent S-matrix.

Recently, the infrared structure of gauge theories has become a topic of much interest due to the proposal that soft radiation may encode information about the history of formation of a black hole [30, 31, 45]. We also hope that this work can make the discussion more quantitatively grounded; we comment on black holes at the end of this chapter. More generally, it is of interest to understand the information-theoretic nature of the infrared sector of quantum field theories, and this work is intended to make some first steps in this direction.

4.2 Decoherence of the hard particles

Fix a single-particle energy resolution E. We define soft bosons as those with energy less than E, and hard particles as anything else. Consider an incoming state $|\alpha\rangle_{\text{in}}$ consisting of hard particles, charged or otherwise, of definite momenta.[8] The S-matrix evolves this into a coherent superposition of states with hard particles β and soft bosons $b = \gamma, h$ (photons γ and gravitons h),

$$|\alpha\rangle_{\text{in}} = \sum_{\beta b} S_{\beta b,\alpha} \, |\beta b\rangle_{\text{out}} \,. \tag{4.1}$$

[8] Labels like α, β, b mean a list of free-particle quantum numbers, e.g., $|\alpha\rangle_{\text{in}} = |\mathbf{p}_1 \ell_1, \ldots\rangle_{\text{in}}$ listing momenta and spin of the incoming particles.

Hereafter we drop the subscript on kets, which will always be out-states. Tracing out the bosons $|b\rangle$, the reduced density matrix for the outgoing hard particles is

$$\rho = \sum_{\beta\beta' b} S_{\beta b,\alpha} S^*_{\beta' b,\alpha} |\beta\rangle \langle\beta'| \,. \tag{4.2}$$

Using the usual soft factorization theorems, equations (3.36) and (3.37), we can write the amplitudes in terms of the amplitudes for $\alpha \to \beta$ multiplied by soft factors, one for each boson. By an argument identical to the one employed in the last chapter, and assuming we can neglect the total lost energy E_T compared to the energy of the hard particles, we can use this factorization to perform the sum over soft bosons in (4.2), and we find that

$$\begin{aligned}\sum_b S_{\beta b,\alpha} S^*_{\beta' b,\alpha} = S_{\beta,\alpha} S^*_{\beta',\alpha} \left(\frac{E}{\lambda}\right)^{\tilde{A}_{\beta\beta',\alpha}} \left(\frac{E}{\lambda}\right)^{\tilde{B}_{\beta\beta',\alpha}} \\ \times \mathcal{F}\left(\frac{E}{E_T}, \tilde{A}_{\beta\beta',\alpha}\right) \mathcal{F}\left(\frac{E}{E_T}, \tilde{B}_{\beta\beta',\alpha}\right).\end{aligned} \tag{4.3}$$

Here $\lambda \ll E$ is an infrared regulator used to cut off momentum integrals which we will send to zero later; one can think of λ as a mass for the photon and graviton. The exponents are

$$\begin{aligned}\tilde{A}_{\beta\beta',\alpha} &= -\sum_{\substack{n\in\alpha,\beta \\ n'\in\alpha,\beta'}} \frac{e_n e_{n'} \eta_n \eta_{n'}}{8\pi^2} \beta_{nn'}^{-1} \log\left[\frac{1+\beta_{nn'}}{1-\beta_{nn'}}\right] \\ \tilde{B}_{\beta\beta',\alpha} &= \sum_{\substack{n\in\alpha,\beta \\ n'\in\alpha,\beta'}} \frac{m_n m_{n'} \eta_n \eta_{n'}}{16\pi^2 M_p^2} \frac{1+\beta_{nn'}^2}{\beta_{nn'}\sqrt{1-\beta_{nn'}^2}} \log\left[\frac{1+\beta_{nn'}}{1-\beta_{nn'}}\right],\end{aligned} \tag{4.4}$$

and $\mathcal{F}$ is given in equation (3.45). In these formulas, $\beta_{nn'}$ is the relative velocity between particles n and n', given in (3.14). For future use, we note that $0 \leq \beta \leq 1$, and both of the dimensionless functions of β appearing in (4.4) run over $[2, \infty)$ as β runs from 0 to 1. We have $\beta_{nm} = 0$ if and only if $p_n = p_m$.

The divergences as $\lambda \to 0$ in (4.3) come from summing over an infinite number of radiated, on-shell bosons. There are also infrared divergences inherent to the transition amplitude $S_{\beta,\alpha}$ itself coming from virtual bosons. We can add these divergences up, and we have that

$$S_{\beta,\alpha} = S^{\Lambda}_{\beta,\alpha} \left(\frac{\lambda}{\Lambda}\right)^{A_{\beta,\alpha}/2} \left(\frac{\lambda}{\Lambda}\right)^{B_{\beta,\alpha}/2}, \tag{4.5}$$

where now $S^{\Lambda}_{\beta,\alpha}$ means the amplitude computed using only virtual bosons of energy above Λ, and $A_{\beta,\alpha}$ and $B_{\beta,\alpha}$ were given in equations (3.20) and (3.23) and are repeated here for convenience,

$$\begin{aligned} A_{\beta,\alpha} &= -\sum_{n,m\in\alpha,\beta} \frac{e_n e_m \eta_n \eta_m}{8\pi^2} \beta_{nm}^{-1} \log\left[\frac{1+\beta_{nm}}{1-\beta_{nm}}\right] \\ B_{\beta,\alpha} &= \sum_{n,m\in\alpha,\beta} \frac{m_n m_m \eta_n \eta_m}{16\pi^2 M_p^2} \frac{1+\beta_{nm}^2}{\beta_{nm}\sqrt{1-\beta_{nm}^2}} \log\left[\frac{1+\beta_{nm}}{1-\beta_{nm}}\right]. \end{aligned} \tag{4.6}$$

The infrared-divergent Coulomb phase from equation (3.20) is suppressed in (4.5). We will see shortly that this phase cancels out of all the relevant density matrix elements.

Putting the above results together, we find that the reduced density matrix coefficient for $|\beta\rangle\langle\beta'|$ is given by

$$\begin{aligned} \rho_{\beta\beta'} = {} & S^{\Lambda}_{\beta,\alpha} S^{\Lambda *}_{\beta',\alpha} \left(\frac{E}{\lambda}\right)^{\tilde{A}_{\alpha,\beta\beta'}} \left(\frac{\lambda}{\Lambda}\right)^{A_{\beta,\alpha}/2 + A_{\beta',\alpha}/2} \\ & \times \left(\frac{E}{\lambda}\right)^{\tilde{B}_{\alpha,\beta\beta'}} \left(\frac{\lambda}{\Lambda}\right)^{B_{\beta,\alpha}/2 + B_{\beta',\alpha}/2} \mathcal{F}(\tilde{A}_{\beta\beta',\alpha}) \mathcal{F}(\tilde{B}_{\beta\beta',\alpha}). \end{aligned} \tag{4.7}$$

The question is how this behaves in the limit of vanishing infrared regulator, $\lambda \to 0$. The coefficient scales as $\lambda^{\Delta A + \Delta B}$, where

$$\begin{aligned} \Delta A_{\beta\beta',\alpha} &= \frac{A_{\beta,\alpha}}{2} + \frac{A_{\beta',\alpha}}{2} - \tilde{A}_{\beta\beta',\alpha} \\ \Delta B_{\beta\beta',\alpha} &= \frac{B_{\beta,\alpha}}{2} + \frac{B_{\beta',\alpha}}{2} - \tilde{B}_{\beta\beta',\alpha}. \end{aligned} \tag{4.8}$$

In appendix A, we prove that both of these exponents are positive-definite, $\Delta A_{\beta\beta',\alpha} \geq 0$ and $\Delta B_{\beta\beta',\alpha} \geq 0$. The density matrix components (4.7) which survive as the regulator $\lambda \to 0$ are those for which $\Delta A = \Delta B = 0$; all other density matrix elements will vanish.

To give necessary and sufficient conditions for $\Delta A = \Delta B = 0$, we define two currents for each spatial velocity vector $\mathbf{v}$. We assume for simplicity that only massive particles carry electric charge. For massive particles, there are electromagnetic and gravitational currents defined as

$$\begin{aligned} j^{\mathrm{em}}_{\mathbf{v}} &= \sum_i e^i a^{i\dagger}(\mathbf{p}_i(\mathbf{v})) a^i(\mathbf{p}_i(\mathbf{v})), \\ j^{\mathrm{gr}}_{\mathbf{v}} &= \sum_i E_i(\mathbf{v}) a^{i\dagger}(\mathbf{p}_i(\mathbf{v})) a^i(\mathbf{p}_i(\mathbf{v})). \end{aligned} \tag{4.9}$$

Here i labels particle species, e^i their charges and m^i their masses; the kinematic quantities $\mathbf{p}_i(\mathbf{v}) = m_i \mathbf{v}/\sqrt{1-\mathbf{v}^2}$ and $E_i(\mathbf{v}) = m_i/\sqrt{1-\mathbf{v}^2}$ are the momentum and energy of species i when it has velocity $\mathbf{v}$. For lightlike particles we have to separately define the gravitational current, since a velocity and species does not uniquely determine a momentum:

$$j^{\mathrm{gr,m=0}}_{\mathbf{v}} = \sum_i \int_0^\infty d\omega\, \omega a^{i\dagger}(\omega \mathbf{v}) a^i(\omega \mathbf{v}). \tag{4.10}$$

Momentum eigenstates of any number of particles are obviously eigenstates of these currents and we denote their eigenvalues $j_{\mathbf{v}} |\alpha\rangle = j_{\mathbf{v}}(\alpha) |\alpha\rangle$.

The photonic exponent $\Delta A_{\beta\beta',\alpha}$ is zero if and only if the charged currents in β are the same as those in β'; the gravitational exponent $\Delta B_{\beta\beta',\alpha}$ is zero if and only if *all* the hard gravitational currents in β are the same as those in β'. This is demonstrated in detail in appendix A. For any such pair of outgoing states $|\beta\rangle, |\beta'\rangle$, (4.7) becomes independent of the IR regulator λ and is thus finite as $\lambda \to 0$,

$$\rho_{\beta\beta'} = S^{\Lambda *}_{\beta',\alpha} S^{\Lambda}_{\beta,\alpha} \mathcal{G}_{\beta\alpha}\left(E, E_T, \Lambda\right), \tag{4.11}$$

where

$$\mathcal{G}_{\beta\alpha} = \mathcal{F}\left(\frac{E}{E_T}, A_{\beta,\alpha}\right) \mathcal{F}\left(\frac{E}{E_T}, B_{\beta,\alpha}\right) \left(\frac{E}{\Lambda}\right)^{A_{\beta\alpha}+B_{\beta\alpha}}. \tag{4.12}$$

This is the case in particular for diagonal density matrix elements $\beta = \beta'$, for which we obtain the standard transition probabilities

$$\rho_{\beta\beta} = \left| S^{\Lambda}_{\beta,\alpha} \right|^2 \mathcal{G}_{\beta\alpha}\left(E, E_T, \Lambda\right). \tag{4.13}$$

On the other hand, if there is even a single $\mathbf{v}$ for which one of the currents (4.9) or (4.10) does not have the same eigenvalue in $|\beta\rangle$ and $|\beta'\rangle$, then the density matrix coefficient decays as $\lambda^{\Delta A + \Delta B} \to 0$ as the regulator $\lambda \to 0$. We see that the unobserved soft bosons have almost completely decohered the momentum state of the hard particles. Only a very sparse subset of superpositions survive, in which the currents agree for all velocities $\mathbf{v}$,

$$j_{\mathbf{v}}(\beta) = j_{\mathbf{v}}(\beta'). \tag{4.14}$$

4.3 Examples

To get a feel for the results presented in the previous section, we consider a few examples. First, consider any scattering with a single incoming and outgoing charged particle, like potential or single particle Compton scattering. Let the incoming momentum be $\alpha = \mathbf{p}$ and the outgoing momenta of the two branches $\beta = \mathbf{q}, \beta' = \mathbf{q}'$. We have either directly from the definition (4.8) or the theorem (A.1) that

$$\Delta A_{\mathbf{q}\mathbf{q}',\mathbf{p}} = -\frac{e^2}{8\pi^2}\left[2 - \gamma_{\mathbf{q}\mathbf{q}'}\right], \tag{4.15}$$

where $\gamma_{\mathbf{q}\mathbf{q}'} = \beta^{-1}_{\mathbf{q}\mathbf{q}'} \log(1 + \beta_{\mathbf{q}\mathbf{q}'})/(1 - \beta_{\mathbf{q}\mathbf{q}'})$. This ΔA is easily seen to equal zero if and only if $\mathbf{q} = \mathbf{q}'$. Thus other than the spin degree of freedom, the resulting density matrix for the charge is exactly diagonal in momentum space.

To see an example where the current-matching condition is non-trivially fulfilled, consider a theory with two charged particle species of charge e and $e/2$ and the same mass. Then we can get an outgoing superposition of a state $\beta = (e, \mathbf{q})$ and one with two half-charges $\beta' = (e/2, \mathbf{q}'_1) + (e/2, i\mathbf{q}'_2)$. The differential exponent for such a superposition is

$$\Delta A_{\beta\beta',\mathbf{p}} = -\frac{e^2}{8\pi^2}\left[3 + \frac{1}{2}\gamma_{\mathbf{q}_1\mathbf{q}_2} - \gamma_{\mathbf{q}\mathbf{q}_1} - \gamma_{\mathbf{q}\mathbf{q}_2}\right], \tag{4.16}$$

which is zero if $\mathbf{q} = \mathbf{q}_1 = \mathbf{q}_2$. In other words, the currents (4.9) cannot distinguish between a full charge of momentum $\mathbf{q}$ and two half-charges of the same momentum.

4.4 Entropy of the soft bosons

We have seen that the reduced density matrix for the outgoing hard particles is very nearly diagonal in the momentum basis. In a simple example like a theory with various scalar fields ϕ_i of different, non-zero masses m_i, the soft graviton emission causes *complete* decoherence into a diagonal momentum-space reduced density matrix for the hard particles. More generally, we may have a sparse set of superpositions, and in any case spin and other internal degrees of freedom are unaffected by the soft emission.

In a simple example with a purely diagonal reduced density matrix, it is straightforward to compute the entanglement entropy of the soft emitted bosons. The total hard + soft system is in a bipartite pure state, with the partition being between the hard particles and soft bosons, so the entanglement entropy of the bosons is the same as that of the hard particles. Following the calculation in [23, 24, 87], we can simply write down the entropy:

$$S = \sum_{\beta} \left| S^{\Lambda}_{\beta,\alpha} \right|^2 \mathcal{G}_{\beta\alpha} \log \left[\left| S^{\Lambda}_{\beta,\alpha} \right|^2 \mathcal{G}_{\beta\alpha} \right] . \tag{4.17}$$

This sum is infrared-finite; again, $\mathcal{G}$ is given in (4.12), and the superscript Λ means the naive S-matrix computed with virtual bosons only of energies greater than Λ. Given the explicit form of $\mathcal{G}$, we see that the entropy scales like the log of the infrared detector resolution E.

4.5 Relation to large gauge symmetries

The decoherence condition (4.14) can be rephrased in the language of large gauge transformations. The condition that given two momentum eigenstates $|\beta\rangle$ and $|\beta'\rangle$, the density matrix element $\rho_{\beta\beta'}$ vanishes unless the same amount of charge is carried with the same velocity vector in both states, is equivalent to the condition that the hard charges $\mathcal{Q}^{+}_{\varepsilon,H}$ agree on $|\beta\rangle$ and $|\beta'\rangle$ for all $\varepsilon(z,\bar{z})$.

To prove this, we start by showing that if condition (4.14) holds for momentum eigenstates $|\beta\rangle$ and $|\beta'\rangle$, it follows that the eigenvalues of $\mathcal{Q}^+_{\varepsilon,H}$ also agree. Let $|\beta\rangle$, $|\beta'\rangle$ be two momentum eigenstates which contain a finite number of charged particles which carry electric charge $Q(\mathbf{v})$ (and $Q'(\mathbf{v})$, respectively) with velocity $\mathbf{v}$, alongside with a number of uncharged particles which we will ignore. For example, if two different particles carry charge e along $\mathbf{v} = \mathbf{v}_0$, then $Q(\mathbf{v}_0) = 2e$. If equation (4.14) holds,then

$$Q(\mathbf{v}) = Q'(\mathbf{v}) \tag{4.18}$$

for every $\mathbf{v}$. The eigenvalues of the out-states with respect to the hard charges are given by

$$Q^+_{\varepsilon,H}\,|\beta\rangle = \int_{\mathcal{I}^+_+} d^2z\sqrt{\gamma}\varepsilon(z,\bar{z})F^{(2),\beta}_{rt}(z,\bar{z})\,|\beta\rangle \tag{4.19}$$

where

$$F^{(2),\beta}_{rt}(z,\bar{z}) = \sum_i \frac{1}{4\pi\gamma_i^2}\frac{Q(\mathbf{v}_i)}{(1-\hat{\mathbf{x}}\cdot\mathbf{v}_i)^2} \tag{4.20}$$

for $|\beta\rangle$ and the sum runs over the (finite) number of velocity vectors along which charge is carried. For $|\beta'\rangle$ we get the same expression where we have to replace $Q \to Q'$. However, since $Q(\mathbf{v}) = Q'(\mathbf{v})$, the same amount of charge is carried with the same velocity and $F^{(2)}_{rt}(z,\bar{z})$ is the same on $|\beta\rangle$ and $|\beta'\rangle$. Therefore the eigenvalues of the hard charges agree.

Conversely, we will now show that equal eigenvalues with respect to $\mathcal{Q}^+_{\varepsilon,H}$ for two momentum eigenstates $|\beta\rangle$, $|\beta'\rangle$ imply that the same amount of charge is carried along the same velocity in both states. That is for either state we can construct functions $Q(\mathbf{v})$ and $Q'(\mathbf{v})$, respectively, which represents the charge carried along velocity vectors $\mathbf{v}$, with $Q(\mathbf{v}) = Q'(\mathbf{v})$. Since we know that these functions are in one-to-one correspondence with eigenvalues of the operators $j_{\mathbf{v}}$ we can conclude that also the eigenvalues $j_{\mathbf{v}}(\beta)$ and $j_{\mathbf{v}}(\beta')$ agree. Consider two states $|\beta\rangle$, $|\beta'\rangle$ with $Q^+_{\varepsilon,H}$ eigenvalues q^ε_β and $q^\varepsilon_{\beta'}$. We assume that the eigenvalues agree for any choice of ε and in particular we can choose

$$\varepsilon \propto \delta^{(2)}(z-w). \tag{4.21}$$

Then the condition $q^{\varepsilon}_{\beta} = q^{\varepsilon}_{\beta'}$ translates to a pointwise equality for the functions $F^{(2),\beta}_{rt}$ and $F^{(2),\beta'}_{rt}$,

$$\sum_{n\in\beta} \frac{1}{4\pi\gamma_n^2} \frac{e_n}{(1-\hat{\mathbf{x}}\cdot\mathbf{v}_n)^2} = \sum_{m\in\beta'} \frac{1}{4\pi\gamma_m'^2} \frac{e_m}{(1-\hat{\mathbf{x}}\cdot\mathbf{v}'_m)^2}. \tag{4.22}$$

It is clear that by combining terms this can be rewritten as

$$\sum_{i\in\mathcal{V}} \frac{1}{4\pi\gamma_i^2} \frac{Q(\mathbf{v}_i)}{(1-\hat{\mathbf{x}}\cdot\mathbf{v}_i)^2} = \sum_{i\in\mathcal{V}} \frac{1}{4\pi\gamma_i^2} \frac{Q'(\mathbf{v}_i)}{(1-\hat{\mathbf{x}}\cdot\mathbf{v}_i)^2}. \tag{4.23}$$

The set $\mathcal{V}$ contains all velocities along which charge is carried in either β or β'. We now assume that equation (4.23) holds but $Q(\mathbf{v}_i)$ disagrees with $Q'(\mathbf{v}_i)$ and show that this leads to a contradiction. We solve for one of the terms in disagreement, whose associated velocity we denote by $\mathbf{v}_0$. This leaves us with

$$\frac{Q(\mathbf{v}_0)-Q'(\mathbf{v}_0)}{\gamma_0^2(1-\hat{\mathbf{x}}\cdot\mathbf{v}_0)^2} = \sum_{i\in\mathcal{V}\setminus\{0\}} \frac{Q'(\mathbf{v}_i)-Q(\mathbf{v}_i)}{\gamma_n^2(1-\hat{\mathbf{x}}\cdot\mathbf{v}_i)^2}. \tag{4.24}$$

The sum on the right hand side runs over all velocities except $\mathbf{v}_0$. Multiplying by all denominators and defining $\Delta Q(\mathbf{v}_i) = Q(\mathbf{v}_i) - Q'(\mathbf{v}_i)$ we find

$$\prod_{i\in\mathcal{V}\setminus\{0\}} (1-\hat{\mathbf{x}}\cdot\mathbf{v}_i)^2 = -\frac{\gamma_0^2(1-\hat{\mathbf{x}}\cdot\mathbf{v}_0)^2}{\Delta Q(\mathbf{v}_0)} \left(\sum_{i\in\mathcal{V}\setminus\{0\}} \frac{\Delta Q(\mathbf{v}_i)}{\gamma_i^2} \prod_{j\in\mathcal{V}\setminus\{0,i\}} (1-\hat{\mathbf{x}}\cdot\mathbf{v}_j)^2 \right). \tag{4.25}$$

Treated as functions of $\hat{\mathbf{x}}$, both sides are polynomials on S^2. Since the ring of polynomials with real coefficients on the sphere is a unique factorization domain, the factorization of both sides in factors of the form $(1+v_1x+v_2y+v_3z)$ is unique with two factors being identical if and only if all v_i agree. Since the right hand side contains a factor of $(1-\hat{\mathbf{x}}\cdot\mathbf{v}_0)^2$ it follows that such a factor must also appear on the left hand side of the equation, but we have assumed that a term containing $\mathbf{v}_0$ is not included in the product. This contradicts our initial assumption and hence we have shown that $Q(v_i) = Q'(v_i)$. It then also follows that the eigenvalues of the currents $j_{\mathbf{v}}$ acting on $|\beta\rangle$ and $|\beta'\rangle$ must agree.

4.6 Discussion

According to the solution of the infrared catastrophe advocated in [56, 58, 62], an infinite number of very low-energy photons and gravitons are produced during scattering events. We have shown that if taken seriously, considering this radiation as lost to the environment completely decoheres almost any momentum state of the outgoing hard particles. The basic idea is simple: the radiation is essentially classical, so any two scattering events are easy to distinguish by their radiation.

The physical content of this result is somewhat unclear. A conservative view is that the methodology of [56, 58, 62] is ill-suited to finding outgoing density matrices. As remarked earlier, in this formalism, one *must* trace the radiation to get well-defined transition probabilities. An alternative would be to use the infrared-finite S-matrix program [64–69, 72], in which no trace over radiation is needed at all. But then we need to understand where the physical low-energy radiation is within that formalism–since after all, a photon that is lost to the environment certainly does decohere the system. We will turn to this in the next chapters.

The decoherence found here is for the momentum states of the particles: at lowest order in their momenta, soft bosons do not lead to decoherence of spin degrees of freedom. However, the sub-leading soft theorems [27, 88, 89] do involve the spin of the hard particles, so going to the next order in the soft particles would be interesting.[9] We would also like to understand to what extent our answers depend on the infinite-time approximation used in the S-matrix approach.

To end, we comment on potential applications to the black hole information paradox. The idea advocated in [30, 31] is that correlations between the hard and soft particles mean that information about the black hole state can be encoded into soft radiation. In [45, 46, 76], the dressed-state formalism and soft factorization has been used to argue that the soft particles simply factor out of the S-matrix and thus contain no such information. In the approach used here, it is manifest that

[9] We understand that Strominger has confirmed this. (Private communication)

the outgoing hard state and outgoing soft state are highly correlated, leading to the decoherence of the hard state. The outgoing density matrix for the hard particles, while not completely thermal, has been mixed in momentum as much as possible while retaining consistency with standard QED/perturbative gravity predictions. It is tempting to conjecture that this generalizes to all asymptotically measurable quantum numbers.

Chapter 5

Dressed infrared quantum information

This chapter is a redacted version of [2].

5.1 Introduction

In the inclusive formalism, one is forced to trace out soft photons to get finite answers. In the previous chapter, we have seen that this leads to an almost completely decohered density matrix for the outgoing state after a scattering event. This chapter analyses the situation in dressed state formalisms, in which no trace over IR photons is needed to obtain a finite outgoing state. However, consider the measurement of an observable sensitive only to electronic and high-energy photonic degrees of freedom. We show that for such observables, there will be a loss of coherence identical to that obtained in the inclusive probability method. Quantum information is lost to the low-energy bremsstrahlung photons created in the scattering process.

The primary goal of this chapter is to give concrete calculations exhibiting the dressed formalism and how it leads to decoherence. To this end, we work with the formulas from the papers of Chung [64] and Faddeev-Kulish [65]. The result of this calculation should carry over identically to any of the existing refinements of Chung's formalism. In section 5.4, we make a number of remarks on possible refinements to the basic dressing formalism, give an expanded physical interpretation of our results, and relate our work to literature in mathematical physics on QED superselection rules. In section 5.5 we make remarks on how this work fits into the recent literature on the black hole information paradox; in brief, we believe that our results are consistent with the recent proposal of Strominger [32], but not the original proposal of Hawking, Perry and Strominger [30, 31].

5.2 IR-safe S-matrix formalism

Following Chung, we study an electron with incoming momentum **p** scattering off a weak external potential. This $1 \to 1$ process is simple and sufficient to understand the basic point; at the end of the next section, we show how to generalize our results to n-particle scattering. The electron spin will be unimportant for us and we supress it in what follows. The standard free-field Fock state $|\mathbf{p}\rangle$ for the electron is promoted to a dressed state $\|\mathbf{p}\rangle\!\rangle$ as discussed in section 3.4.2,

$$\|\mathbf{p}\rangle\!\rangle = W_{\mathbf{p}} \, |\mathbf{p}\rangle \equiv W_{\lambda}[f_{\ell}(\mathbf{k}, \mathbf{p})]. \tag{5.1}$$

This consists of the electron and a coherent state of on-shell, transversely-polarized photons.

We introduce an IR regulator ("photon mass") λ and an upper infrared cutoff $E > \lambda$, which can be thought of as the energy resolution of a single-photon detector in our experiment. Here and in the following all momentum-space integrals are evaluated in the shell $\lambda < |\mathbf{k}| < E$.

Consider now an incoming dressed electron scattering into a superposition of outgoing dressed electron states. The outgoing state is, to lowest order in perturbation theory in the electric charge,

$$|\psi\rangle = \int d^3\mathbf{q} \mathbb{S}_{\mathbf{qp}} \|\mathbf{q}\rangle\!\rangle. \tag{5.2}$$

At higher orders there will be additional photons in the outgoing state; as explained in the next section, these will not affect the infrared behavior studied here, so we ignore them for now. Here the S-matrix is just the standard Feynman-Dyson time evolution operator, evaluated between dressed states. That is,

$$\mathbb{S}_{\mathbf{qp}} = \langle\langle \mathbf{q} \| S \| \mathbf{p} \rangle\rangle, \tag{5.3}$$

with $S = \text{‘}T \exp\left(-i\int_{-\infty}^{\infty} V(t)dt\right)$ as usual. As calculated by Chung, the dressed $1 \to 1$ elements of this matrix are independent of the IR regulator λ and thus infrared-finite as we send $\lambda \to 0$. We can write the matrix element

$$\mathbb{S}_{\mathbf{qp}} = \left(\frac{E}{\Lambda}\right)^A S^{\Lambda}_{\mathbf{qp}} \tag{5.4}$$

where

$$A = -\frac{e^2}{8\pi^2}\beta^{-1}\log\left[\frac{1+\beta}{1-\beta}\right], \quad \beta = \sqrt{1 - \frac{m^4}{(p \cdot q)^2}}. \tag{5.5}$$

As discussed in section 3.4, the undressed S-matrix element on the right side means the amplitude computed by Feynman diagrams with photon loops evaluated only with photon energies above Λ and evaluated between undressed electron states, that is, with no external soft photons. By definition, this quantity is infrared-finite and the dependence on the scale Λ cancels between the prefactor and S^{Λ}.

5.3 Soft radiation and decoherence

The state (5.2) is a coherent superposition of states, each containing a bare electron and its corresponding photonic dressing. The presence of hard photons in the outgoing state will not change our conclusions below, so for simplicity we ignore them. In particular, the density matrix formed from this state has off-diagonal elements of the form

$$\mathbb{S}^*_{\mathbf{q'p}}\mathbb{S}_{\mathbf{qp}} \| \mathbf{q} \rangle\rangle \langle\langle \mathbf{q'} \|. \tag{5.6}$$

These states have highly non-trivial photon content. However, if one is doing a measurement involving only the electron degree of freedom, then these photons are unobserved, and we can make predictions with the reduced density matrix of the electron, obtained by tracing the photons out. The resulting electron density matrix has coefficients damped by a factor involving the overlap of the photon states, namely

$$\rho_{\text{electron}} = \int d^3\mathbf{q}d^3\mathbf{q}'\mathbb{S}^*_{\mathbf{q}'\mathbf{p}}\mathbb{S}_{\mathbf{q}\mathbf{p}}D_{\mathbf{q}\mathbf{q}'}\left|\mathbf{q}\right\rangle\left\langle\mathbf{q}'\right| \tag{5.7}$$

where the dampening factor is given by the photon-vacuum expectation value

$$D_{\mathbf{q}\mathbf{q}'} = \langle 0|W^\dagger_{\mathbf{q}'}W_{\mathbf{q}}|0\rangle\,. \tag{5.8}$$

Straightforward computation gives this factor as

$$\begin{aligned} D_{\mathbf{q}\mathbf{q}'} &= \exp\left\{-\frac{e^2}{2}\sum_{\ell=\pm}\int\frac{d^3\mathbf{k}}{(2\pi)^3 2|\mathbf{k}|}\left|f_\ell(\mathbf{q}) - f^*_\ell(\mathbf{q}')\right|^2\right\} \\ &= \exp\left\{-e^2\int\frac{d^3\mathbf{k}}{(2\pi)^3 2|\mathbf{k}|}\frac{(q-q')^2}{(q\cdot k)(q'\cdot k)}\right\}. \end{aligned} \tag{5.9}$$

In this integrand, since q and q' are two timelike vectors with the same temporal component, we have that the numerator is positive definite and the denominator is positive. It is therefore manifest that we have $D = 1$ if $q = q'$ and $D = 0$ otherwise, since the integral over $d^3\mathbf{k}$ diverges in its lower limit. Thus, tracing the photons leads to an electron density matrix that is completely diagonalized in momentum space.

It is noteworthy that the factor (5.9) depends only on properties of the outgoing superposition; it has no dependence on the initial state. This may seem surprising since we are tracing over outgoing radiation, the production of which depends on both the initial and final electron state. The point is that the damping factor measures the distinguishability of the radiation fields given the processes $\mathbf{p} \to \mathbf{q}$ and $\mathbf{p} \to \mathbf{q}'$. The radiation field for a scattering process consists of two pieces added together: a term $A_\mu \sim p_\mu/p\cdot k$ peaked in the direction of the incoming electron and a term $A_\mu \sim q_\mu/q\cdot k$ peaked in the direction of the outgoing electron. The outgoing radiation fields with outgoing electrons q, q' are then only distinguishable by the second terms here, since both radiation fields will have the same pole in the incoming direction.

The damping factor (5.9) is precisely what was found in the previous chapter, reduced to the problem of $1 \to 1$ scattering. The mechanism is the same: physical, low-energy photon bremsstrahlung is emitted in the scattering. These photons are highly correlated with the electron state and thus, if one does not observe them jointly with the electron, one will measure a highly-decohered electron density matrix. The only difference is bookkeeping: in the dressed formalism, the bremsstrahlung photons are folded into the dressed electron states (the incoming/outgoing parts of the bremsstrahlung in the incoming/outgoing dressing, respectively). However, referring to "an electron" as a state including these soft photons is an abuse of semantics. In an actual measurement of the electron momentum, one does not measure these soft photons.

The dressed states are not energy eigenstates, and in fact contain states of arbitrarily high total energy. This should be contrasted with the inclusive-probability treatment used by Weinberg, which has a cutoff on both the single-photon energy E and the total outgoing energy contained by all the photons $E_T \geq E$ in the outgoing state [56]. This additional parameter, however, appears only in the ratio E_T/E in Weinberg's probability formulas, and one finds that the dependence on E_T vanishes as $E_T \to \infty$. This can be understood because what is important is the very low-energy behavior of the photons, so moving an upper cutoff has limited impact.

We note that (5.2) does not include effects from the bremsstrahlung of additional soft photons beyond those in the dressing. There is no kinematic reason to exclude such photons, so the outgoing state should properly be written as

$$|\psi\rangle = \sum_{n=0}^{\infty} \sum_{\{\ell\}} \int d^3\mathbf{q} d^{3n}\mathbf{k} \mathbb{S}_{\mathbf{q}\{\mathbf{k}\ell\};\mathbf{p}} \|\mathbf{q}\rangle\!\rangle. \tag{5.10}$$

Here $\{\mathbf{k}\ell\} = \{\mathbf{k}_1\ell_1, \ldots, \mathbf{k}_n\ell_n\}$ is a list of n photon momenta and polarizations. By the dressed version of the soft photon factorization theorem (see appendix B), we have that

$$\mathbb{S}_{\mathbf{qk}\ell;\mathbf{p}} = \mathbb{S}_{\mathbf{qp}} \times e\mathcal{O}\left(|\mathbf{k}|^0\right), \tag{5.11}$$

or in other words $\lim_{|\mathbf{k}|\to 0} |\mathbf{k}| \mathbb{S}_{\mathbf{qk}\ell;\mathbf{p}} = 0$. Thus, when we take a trace over n-photon dressed states in (5.10), we obtain a sum of additional decoherence factors of the form

$$
\begin{aligned}
D^{nm}_{\mathbf{q}\mathbf{q}'} = e^{n+m}\mathcal{O}\left(|\mathbf{k}|^0\right) \times \sum_{\ell_1,\dots,\ell_n}\sum_{\ell'_1,\dots,\ell'_m}\int d^{3n}\mathbf{k}\, d^{3m}\mathbf{k}' \\
\langle 0|a_{\ell'_m}(\mathbf{k}'_m)\cdots a_{\ell'_1}(\mathbf{k}'_1) W^\dagger_{\mathbf{q}'} W_{\mathbf{q}} a^\dagger_{\ell_1}(\mathbf{k}_1)\cdots a^\dagger_{\ell_n}(\mathbf{k}_n)|0\rangle\,.
\end{aligned}
\tag{5.12}
$$

Evaluating the inner product one finds

$$
D^{nm}_{\mathbf{q}\mathbf{q}'} \sim \left[\sum_{\ell=\pm}\int d^3\mathbf{k}\ \mathrm{Re}\left(f_\ell(\mathbf{q}) - f_\ell(\mathbf{q}')\right)\right]^{n+m}, \tag{5.13}
$$

which is infrared-finite. Summing these contributions, which exponentiate, will not change the conclusion that (5.9) leads to vanishing off-diagonal electron density matrix elements.

Finally, we explain the generalization to n-electron states. We will find that the same decoherence is found in the dressed formalism as in the inclusive formalism. Following Faddeev-Kulish [65], we define the multi-particle dressing operator by replacing

$$
f_\ell(\mathbf{p},\mathbf{k}) \to \int \frac{d^3\mathbf{p}}{(2\pi)^3} f_\ell(\mathbf{p},\mathbf{k})\rho(\mathbf{p}), \tag{5.14}
$$

in the definition of $W_\lambda[f_\ell]$. Here, we have introduced an operator which counts charged particles with momentum $\mathbf{p}$.

$$
\rho(\mathbf{p}) = \sum_s \left(b^\dagger_{\mathbf{p},s} b_{\mathbf{p},s} - d^\dagger_{\mathbf{p},s} d_{\mathbf{p},s}\right), \tag{5.15}
$$

and the b and d are electron and positron operators, respectively.[10] As in the one-particle case, additional photons do not affect the IR behaviour of scattering amplitudes. Hence, we will ignore them and only consider the case where the out-state is a linear superposition of dressed electron states. In that case we have to replace the outgoing momentum by list of momenta, $\mathbf{q} \to \beta = \{\mathbf{q}_1, \mathbf{q}_2, \dots\}$ and

[10] Note that in the multi-particle case there is an infinite phase factor which needs to be included in the definition of the S-matrix. Since this phase factor does not affect our discussion, we ignore it in the following.

similarly $\mathbf{q}' \to \beta' = \{\mathbf{q}'_1, \mathbf{q}'_2, \ldots\}$. This results in a replacement in (5.9) of

$$\begin{aligned} f_\ell(\mathbf{q}) &\to \sum_{n\in\beta} f_\ell(\mathbf{q}_n) \\ f^*_\ell(\mathbf{q}') &\to \sum_{m\in\beta'} f^*_\ell(\mathbf{q}'_m). \end{aligned} \tag{5.16}$$

Using the explicit form of F in the limit $\mathbf{k} \to 0$, the damping factor (5.9) then then becomes

$$D_{\beta\beta'} = \exp\left[-e^2 \int \frac{d^3\mathbf{k}}{(2\pi)^3 2|\mathbf{k}|} \sum_{m,n\in\beta,\beta'} \frac{\eta_m \eta_n p_m \cdot q_n}{(q_m \cdot k)(q_n \cdot k)}\right]. \tag{5.17}$$

In this equation the labels m, n both run over the full set $\beta \cup \beta'$, and $\eta_n = +1$ if $n \in \beta$ while $\eta_n = -1$ if $n \in \beta'$. This is precisely the quantity $\Delta A_{\beta\beta',\alpha}$ defined in the previous section, so we see that the results carry over to the dressed formalisms used here.

5.4 Physical interpretation

Dressed-state formalisms are engineered to provide infrared-finite transition amplitudes, as opposed to inclusive probabilities constructed in the traditional approach studied in the previous section. In the dressed formalism, the outgoing state (5.2) is a coherent superposition of states $\|\mathbf{p}\rangle\rangle$ consisting of electrons plus dressing photons. However, if one does a measurement of an observable sensitive only to the electron state, the measurement will exhibit decoherence because the unobserved dressing photons are highly correlated with the electron state. We have given a concrete calculation showing that the damping factor (5.17) is identical in either the dressed or undressed formalism.

The physical relevance of this calculation rests on the idea that the basic observable is a simple electron operator in Fock space. What would be much better would be to use a dressed LSZ reduction formula to understand the asymptotic limits of electron correlation functions [90, 91]. Nevertheless, the basic physical picture seems clear: in a scattering experiment, one does not measure an electron plus a finely-tuned shockwave of outgoing bremsstrahlung photons, just the electron on its own. This is responsible for well-measured phenomena like radiation damping.

QED has a complicated asymptotic Hilbert space structure which is still somewhat poorly understood. For example, although Faddeev-Kulish try to define a single, separable Hilbert space $\mathcal{H}_{as}$ [65, 91] other authors have argued that one needs an uncountable set of separable Hilbert spaces [66, 90]. Formally, this is related to the fact that the dressing operator does not converge on the usual Fock space. We will discuss this in chapter 7. A related idea is that one can argue that QED has an infinite set of superselection rules based on the asymptotic charges

$$Q(\Omega) = \lim_{r\to\infty} r^2 E_r(r,\Omega) \tag{5.18}$$

defined by the radial electric field at infinity [92, 93]. We believe that the calculations presented here and in chapter 4 demonstrate the physical mechanism for enforcing such a superselection rule. The charges (5.18), the currents defined in the previous chapter, and the large-$U(1)$ charges defined in [29, 74] are presumably closely related, and working out the precise relations is an interesting line of inquiry.

5.5 Black hole information

Let us again comment on the proposal of Hawking, Perry and Strominger suggesting that information apparently lost in the process of black hole formation and evolution could be encoded in soft radiation [30, 31]. The original proposal was that there are symmetries which relate hard scattering (like the black hole formation or evaporation process) to soft scattering and thus led to constraints on the S-matrix. As emphasized by a number of authors, this is not true in the dressed state approach [45, 46, 76, 94]. As we review in appendix B, soft modes decouple from the dressed hard scattering event at lowest order, in the sense that $\lim_{\omega\to 0}[a_\omega, S_{\text{dressed}}] = 0$. Dropping a soft boson into the black hole will not yield any information about the black hole formation and evaporation process.

However, a more recent proposal due to Strominger is to simply posit that outgoing soft radiation purifies the outgoing Hawking radiation [32]. That is, the state after the black hole has evaporated is of the form $|\psi\rangle = \sum_a c_a |a\rangle_{\text{Hawking}} |a\rangle_{\text{soft}}$, such that the Hawking radiation is described by a thermal density matrix, i.e., $\rho_{\text{Hawking}} = \text{tr}_{\text{soft}} |\psi\rangle\langle\psi| \approx \rho_{\text{thermal}}$. We believe that both the results presented here and those in our previous work are consistent with this proposal. In either the inclusive or dressed formalism, the final state of any scattering process contains soft radiation which is highly correlated with the hard particles because the radiation is created due to accelerations in the hard process. The open issue is to explain why the hard density matrix coefficients behave thermally, which likely relies on details of the black hole S-matrix.

5.6 Conclusions

When charged particles scatter, they experience acceleration, causing them to radiate low-energy photons. If one waits an infinitely long time (as mandated by an S-matrix description), these photons cause severe decoherence of the charged particle momentum state. This was demonstrated in the preceding chapter in the standard formulation of QED involving IR-finite inclusive cross section, and here we have shown the same conclusion holds in IR-safe, dressed formalisms of QED; they should carry over in a simple way to perturbative quantum gravity. These results constitute a sharp and robust connection between the infrared catastrophe and quantum information theory, and should provide guidance in problems related to the infrared structure of gauge theories.

Chapter 6

On the need for soft dressing

This chapter is a redacted version of [3].

6.1 Introduction

Both, the dressed and inclusive formalisms, are designed to give the same predictions for the probability of scattering from an incoming set of momenta $\mathbf{p}_1, \ldots, \mathbf{p}_n$ into an outgoing set of momenta $\mathbf{p}'_1, \ldots, \mathbf{p}'_m$. The measurement of observables which only depend on the hard particles should be predictable from the reduced density matrix obtained by tracing over soft bosons, which are invisible to a finite size detector. Given an incoming momentum eigenstate, we have argued in the previous two chapters that the two formalisms agree. Thus, one might naively think for calculating cross-sections it does not matter which formalism one chooses. We show in this chapter that this is not the case: the two approaches differ in their treatment of incoming superpositions.

Consider a simple superposition of two momentum eigenstates for a single charged particle

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\mathbf{p}\rangle + |\mathbf{q}\rangle), \tag{6.1}$$

scattering off of a classical potential. We expect the out-state to be described by a density matrix of the form

$$\rho = \frac{1}{2} S \left(|\mathbf{p}\rangle \langle \mathbf{p}| + |\mathbf{p}\rangle \langle \mathbf{q}| + |\mathbf{q}\rangle \langle \mathbf{p}| + |\mathbf{q}\rangle \langle \mathbf{q}| \right) S^{\dagger}. \tag{6.2}$$

Here S is the scattering operator and we have performed a trace over the soft radiation, hence ρ is the density matrix for the hard particles. If $|\mathbf{p}\rangle, |\mathbf{q}\rangle$ are dressed states, this expectation is indeed correct. In the inclusive formalism, however, where $|\mathbf{p}\rangle$, $|\mathbf{q}\rangle$ are Fock space momentum eigenstates, there is no interference between the different momenta as opposed to the diagonal terms of (6.2). We find that the diagonal entries of the density matrix which encode the cross-sections are of the form

$$\sigma_{\psi \to \text{out}} \propto \langle \text{out}| \rho^{\text{incl}} |\text{out}\rangle = \frac{1}{2} \langle \text{out}| \; S \left(|\mathbf{p}\rangle \langle \mathbf{p}| + |\mathbf{q}\rangle \langle \mathbf{q}| \right) S^{\dagger} |\text{out}\rangle . \tag{6.3}$$

In other words, the cross-section behaves as if we had started with a classical ensemble of states with momenta $\mathbf{p}$ and $\mathbf{q}$. The entire scattering history is decohered by the loss of the soft radiation. This appears to contrast starkly with any realistic experiment.

Moreover, as we will show, repeating the analysis for wavepackets, e.g., $|\psi\rangle = \int d\mathbf{p} f(\mathbf{p}) |\mathbf{p}\rangle$, leads to the nonsensical conclusion that a wave-packet is not observed to scatter at all. However, in the dressed state formalism of Faddeev-Kulish the interference appears as in equation (6.2). This strongly suggests that scattering theory in quantum electrodynamics and perturbative quantum gravity should really not be formulated in terms of standard Fock states of charged particles. Formulating the theories using dressed states seems to be a good alternative.

Our findings have a nice interpretation in the language of asymptotic symmetries: only superpositions of states within the same selection sector, defined using the charges that generate the symmetries, can interfere. This explains the failure of the undressed approach. In the inclusive formalism, essentially any pair of momentum eigenstates live in different charge sectors. In contrast, the Faddeev-Kulish formalism is designed so that all of the dressed states live within the same charge sector.

Our results can also be viewed in the context of the black hole information problem [10, 47]. In particular, Hawking, Perry, and Strominger [30] and Strominger [32] have recently suggested that black hole information may be encoded in soft radiation. In black hole thought experiments, one typically imagines preparing an initial state of wavepackets organized to scatter with high probability to form an intermediate black hole. Our results suggest then that one needs to use dressed initial states to study this problem. See also [45, 46] for some remarks on the use of dressed or inclusive formalisms for studying black hole information.

The rest of the chapter is organized as follows. We start by presenting the calculations showing that the dressed and undressed formalisms disagree in section 6.2 for discrete superpositions and in section 6.3 for wavepackets. The discussion and interpretation of the results takes place in section 6.4. There, we will argue why our findings imply that dressed states are better suited to describe scattering than the inclusive Fock-space formalism. We will give a new very short argument for the known result of [78] that the dressing operators and the S-matrix weakly commute and argue for a more general form of dressing beyond Faddeev-Kulish. We will then interpret our results in terms of asymptotic symmetries and selection sectors before concluding in section 6.5. Appendix C contains proofs of certain statements in sections 6.2 and 6.3.

6.2 Scattering of discrete superpositions

In this and the next section we generalize the results of chapters 4 and 5 to the case of incoming superpositions of momentum eigenstates. We begin in this section by studying discrete superpositions $|\psi\rangle = |\alpha_1\rangle + \cdots + |\alpha_N\rangle$ of states with various momenta $\alpha = \mathbf{p}_1, \mathbf{p}_2, \ldots$. We will see that the dressed and inclusive formalisms give vastly different predictions for the probability distribution of the outgoing momenta: dressed states will exhibit interference between the α_i whereas undressed states do not.

6.2.1 Inclusive formalism

Consider scattering of an incoming superposition of charged momentum eigenstates

$$|\text{in}\rangle = \sum_i^N f_i \,|\alpha_i\rangle\,, \tag{6.4}$$

with $\sum_i |f_i|^2 = 1$. The outgoing density matrix vanishes due to IR divergences in virtual photon loops. However, as before, we can obtain a finite result if we trace over outgoing radiation [1, 56, 58, 62]. The resulting reduced density matrix of the hard particles takes the form

$$\rho = \sum_b \sum_{i,j}^N \iint d\beta\; d\beta' f_i f_j^* S_{\beta b,\alpha_i} S^*_{\beta' b,\alpha_j} \,|\beta\rangle\,\langle\beta'|\,, \tag{6.5}$$

where β and β' are lists of the momenta of hard particles in the outgoing state, and the sum over b denotes the trace over soft bosons. We will be interested in the effect of infrared divergences on this expression.

The sum over external soft boson states b produces IR divergences which cancel those coming from virtual boson loops. We can regulate these divergences by introducing an IR cutoff (e.g., a soft boson mass λ). Following the standard soft photon resummation techniques [56], one finds that the total effect of these divergences yields reduced density matrix elements of the form

$$\rho_{\beta\beta'} = \sum_{i,j}^N f_i f_j^* S^\Lambda_{\beta,\alpha_i} S^{\Lambda *}_{\beta',\alpha_j} \lambda^{\Delta A_{\beta\beta',\alpha_i\alpha_j} + \Delta B_{\beta\beta',\alpha_i\alpha_j}} \mathcal{G}_{\beta\beta',\alpha_i\alpha_j}(E, E_T, \Lambda). \tag{6.6}$$

Here we have introduced "UV" cutoffs Λ, E on the virtual and real soft boson energies, so S^Λ are S-matrix elements with the soft boson loops cut off below Λ and we only trace over outgoing bosons with individual energies up to E and total energy E_T. The explicit form of the Sudakov rescaling function $\mathcal{G}$ defined analogously to (4.12). What concerns us here is the behavior of this expression in the limit where we remove the IR regulator $\lambda \longrightarrow 0$, which is controlled by the exponents

$$\begin{aligned}
\Delta A_{\beta\beta',\alpha\alpha'} &= -\frac{1}{2} \sum_{n,n' \in \alpha, \bar{\alpha}', \beta, \bar{\beta}'} \frac{e_n e_{n'} \eta_n \eta_{n'}}{8\pi^2} \beta_{nn'}^{-1} \log \left[\frac{1+\beta_{nn'}}{1-\beta_{nn'}} \right], \\
\Delta B_{\beta\beta',\alpha\alpha'} &= -\frac{1}{2} \sum_{n,n' \in \alpha, \bar{\alpha}', \beta, \bar{\beta}'} \frac{m_n m_{n'} \eta_n \eta_{n'}}{16\pi^2 M_p^2} \beta_{nn'}^{-1} \frac{1+\beta_{nn'}^2}{\sqrt{1-\beta_{nn'}^2}} \log \left[\frac{1+\beta_{nn'}}{1-\beta_{nn'}} \right].
\end{aligned} \tag{6.7}$$

The factor η_n is defined as $+1$ (-1) if particle n is incoming (outgoing). The quantities $\beta_{nn'}$ are the relative velocities between pairs of particles given in equation (3.14) and a bar interchanges incoming states for outgoing and vice versa. The expressions for ΔA and ΔB come from contributions of soft photons and gravitons, respectively. The question now is which terms survive.

The special case of no superposition, $\alpha_i = \alpha_j = \alpha$, was discussed in chapter 4. There it was shown that $\Delta A_{\beta\beta',\alpha\alpha} \geq 0$ and $\Delta B_{\beta\beta',\alpha\alpha} \geq 0$, so that in the limit $\lambda \longrightarrow 0$, all of the terms in the sum except those with $\Delta A = \Delta B = 0$ will vanish. The equality holds if and only if the out states β and β' contain particles such that the amount of electrical charge and mass carried with any choice of velocity agrees for β and β'. This can be phrased in terms of an infinite set of operators which measure charges flowing along a velocity $\mathbf{v}$, defined in equations (4.9) and (4.10). Momentum eigenstates are eigenstates of these operators. Using them, the equality of currents reads

$$j_{\mathbf{v}} |\beta\rangle \sim j_{\mathbf{v}} |\beta'\rangle , \tag{6.8}$$

where the tilde means that the eigenvalues of the states are the same on both sides for all velocities. In appendix C.1, we show that the more general exponents $\Delta A_{\beta\beta',\alpha\alpha'}$ and $\Delta B_{\beta\beta',\alpha\alpha'}$ are positive. Similarly to the argument in the previous chapters, one can show that ΔA and ΔB are non-zero if and only if

$$j_{\mathbf{v}} |\alpha_i\rangle + j_{\mathbf{v}} |\beta'\rangle \sim j_{\mathbf{v}} |\alpha_j\rangle + j_{\mathbf{v}} |\beta\rangle , \tag{6.9}$$

that is if the list of hard currents in states $|\alpha\rangle$ and $|\beta'\rangle$ is the same as the list of hard currents in states $|\alpha'\rangle$ and $|\beta\rangle$. An easy way to understand the form of equation (6.9) is by looking at equation (6.7). There, the bar over α' (which corresponds to

α_j) indicates that it should be treated as an outgoing particle, i.e., similarly to β. On the other hand $\bar{\beta}'$ should be treated similarly to α. Hence, we obtain equation (6.9) from (6.8) by replacing $\beta' \to \alpha_i + \beta'$ and $\beta \to \alpha_j + \beta$. On the other hand it is clear that in the case of $|\alpha_i\rangle = |\alpha_j\rangle = |\alpha\rangle$ equation (6.9) reduces to equation (6.8).

Armed with these results, we can calculate the cross-sections given an incoming superposition. These are proportional to the diagonal elements $\beta = \beta'$ of the density matrix; for simplicity we ignore forward scattering terms. The diagonal terms of the density matrix (6.6) are proportional to $\lambda^{\Delta A+\Delta B}$. This factor reduces to unity if $j_{\mathbf{v}}|\alpha_i\rangle \sim j_{\mathbf{v}}|\alpha_j\rangle$ for all of the currents (4.9) and (4.10) and is zero otherwise. For a generic superposition, this implies that only terms with $i = j$ contribute and we find

$$\sigma_{\text{in}\to\beta} \propto \rho_{\beta\beta} = \sum_{i,j}^{N} f_i f_j^* \mathcal{G}_{\beta\beta,\alpha_i\alpha_j} S^{\Lambda}_{\beta\alpha_i} S^{\Lambda *}_{\beta\alpha_j} \delta_{\alpha_i\alpha_j} = \sum_{i}^{N} |f_i|^2 |S^{\Lambda}_{\beta,\alpha_i}|^2 \mathcal{G}_{\beta\beta,\alpha_i\alpha_i}. \tag{6.10}$$

As we see, no interference terms between incoming states are present. Instead, the total cross-section is calculated as if the incoming states were part of a classical ensemble with probabilities $|f_i|^2$. The reason is that in the inclusive approach the information about the interference is carried away by unobservable soft radiation. To define the scattering cross-section, however, we need to trace out the soft radiation and we obtain the above prediction, which is at odds with the naive expectation, equation (6.2).

6.2.2 Dressed formalism

The calculation above was done using the usual, undressed Fock states of hard charges, which required to calculate inclusive cross-sections. The alternative approach we will now turn to is to consider transitions between dressed states. For concreteness, we will follow the dressing approach of Chung and Faddeev-Kulish[11], which contains charged particles accompanied by a cloud of real bosons which radiate out to lightlike infinity [64, 65, 72]. For a given set of momenta $\alpha = \mathbf{p}_1, \mathbf{p}_2, \ldots$,

[11]Recently, a generalization of Faddeev-Kulish states was suggested [77]. We will extend our discussion to those states in section 6.4.

we write the dressed state as

$$\|\alpha\rangle\!\rangle \equiv W_\alpha \, |\alpha\rangle \equiv W_\lambda[f_\ell(\mathbf{k},\alpha)], \tag{6.11}$$

where multi-particle dressed states are introduced as discussed in the previous chapter,

$$f_\ell(\mathbf{k},\alpha) = \sum_{\mathbf{p}\in\alpha} \frac{\epsilon_\ell \cdot p}{k \cdot p} \phi(\mathbf{k},\mathbf{p}) \tag{6.12}$$

The operator W_α equips the state $|\alpha\rangle$ with a cloud of photons/gravitons. For QED, W_α with a finite cutoff λ is a unitary operator Letting W_α act on Fock space states for $\lambda = 0$ gives states with vanishing normalization, hence in the strict $\lambda \to 0$ limit W_α is no good operator on Fock space. Thus, as before, we will do calculations with finite λ and only at the end will we take $\lambda \to 0$.[12]

Consider now an incoming state consisting of a discrete superposition of such dressed states,

$$\|\text{in}\rangle\!\rangle = \sum_i f_i \|\alpha_i\rangle\!\rangle. \tag{6.13}$$

The outgoing density matrix is then

$$\rho = \sum_{i,j} \iint d\beta d\beta' \, f_i f_j^* \mathbb{S}_{\beta\alpha_i} \mathbb{S}^*_{\beta'\alpha_j} \|\beta\rangle\!\rangle\langle\!\langle\beta'\|. \tag{6.14}$$

However, every experiment should be able to ignore soft radiation. Following chapter 5, we treat the soft modes as unobservable and trace them out. This yields the reduced density matrix for the outgoing hard particles,

$$\rho^{\text{hard}}_{\beta\beta'} = \sum_{i,j} f_i f_j^* \mathbb{S}_{\beta\alpha_i} \mathbb{S}^*_{\beta'\alpha_j} \, \langle 0| W_\beta^\dagger W_{\beta'} |0\rangle \,. \tag{6.15}$$

The last term is the photon vacuum expectation value of the out-state dressing operators. This factor reduces to one or zero as shown in chapter 4 and 5; one

[12]Note that as argued in [65], a proper definition of W in the limit $\lambda \to 0$ should be possible on a von Neumann space.

if $j(\beta) \sim j(\beta')$ and zero otherwise. This is responsible for the decay of most off-diagonal elements in (6.15). However, if we are interested in the cross-section for a particular outgoing state β, this is again given by a diagonal density matrix element,

$$\sigma_{\text{in}\to\beta} \propto \rho_{\beta\beta} = \sum_{i,j} f_i f_j^* \mathbb{S}_{\beta,\alpha_i} \mathbb{S}^*_{\beta,\alpha_j} . \tag{6.16}$$

In stark contrast to the result obtained in the inclusive formalism, equation (6.10), this cross-section exhibits the usual interference between the various incoming states, like expected in equation (6.2). The reason for this is that in the dressed formalism, the outgoing radiation is described by the dressing which only depends on the out-state and not on the in-state. We will discuss this in more detail in section 6.4. This establishes that the inclusive and dressed formalism are not equivalent but yield different predictions for cross-sections of finite superpositions.

6.3 Wavepackets

We will now proceed to look at scattering of wavepackets and find that the result is even more disturbing. After tracing out infrared radiation in the undressed formalism, no indication of scattering is left in the hard system. On the contrary, once again we will see that with dressed states, one gets the expected scattering out-state.

6.3.1 Inclusive formalism

We consider incoming wavepackets of the form

$$|\text{in}\rangle = \int d\alpha f(\alpha) |\alpha\rangle , \tag{6.17}$$

normalized such that $\int d\alpha |f(\alpha)|^2 = 1$. The full analysis of the preceding section still applies, provided we replace $\sum_{\alpha_i} \to \int d\alpha$, $\alpha_i \to \alpha$, $f_i \to f(\alpha)$ and similarly for $a_j \to \alpha'$. The only notable exception is the calculation of single matrix elements as in equation (6.10), which now reads

$$\rho_{\beta\beta} = \iint d\alpha d\alpha' f(\alpha) f^*(\alpha') S^{\Lambda}_{\beta,\alpha} S^{\Lambda *}_{\beta,\alpha'} \delta_{\alpha\alpha'} \mathcal{G}_{\beta\beta,\alpha\alpha'}(E, E_T, \Lambda). \tag{6.18}$$

Note that here, by the same argument as before, the λ-dependent factor is turned into a Kronecker delta, which now reduces the integrand to a measure zero subset on the domain of integration. The only term that survives the integration is the initial state, which is acted on with the usual Dirac delta $\delta(\alpha - \beta)$, i.e., the "1" term in $S = 1 - 2\pi i \mathcal{M}$. The detailed argument can be found in appendix C.2. Thus we conclude that

$$\rho^{\text{out}}_{\beta\beta'} = f(\beta) f^*(\beta') = \rho^{\text{in}}_{\beta\beta'}. \tag{6.19}$$

The hard particles show no sign of a scattering event.

6.3.2 Dressed wavepackets

The dressed formalism has perfectly reasonable scattering behavior. Consider wavepackets built from dressed states

$$\|\text{in}\rangle\!\rangle = \int d\alpha \; f(\alpha) \|\alpha\rangle\!\rangle, \tag{6.20}$$

with $\|\alpha\rangle\!\rangle$ a dressed state in the same notation as in equation (6.11). The S-matrix applied on dressed states is infrared-finite and the outgoing density matrix can be expressed as

$$\rho = \iint d\beta d\beta' \iint d\alpha d\alpha' f(\alpha) f^*(\alpha') \mathbb{S}_{\beta,\alpha} \mathbb{S}^*_{\beta',\alpha'} \|\beta\rangle\!\rangle \langle\!\langle \beta' \|. \tag{6.21}$$

Tracing over soft modes, we find

$$\rho_{\beta\beta'} = \iint d\alpha d\alpha' f(\alpha) f^*(\alpha') \mathbb{S}_{\beta,\alpha} \mathbb{S}^*_{\beta',\alpha'} \left\langle W^\dagger_\beta W_{\beta'} \right\rangle. \tag{6.22}$$

Again the expectation value is taken in the photon vacuum. The crucial point here is that this factor is independent of the initial states α. Upon sending the IR cutoff λ to zero, the expectation value for $W^\dagger W$ takes only the values 1 or 0, leading to decoherence in the outgoing state, but the cross-sections still exhibit all the usual interference between components of the incoming wavefunction,

$$\rho_{\beta\beta} = \iint d\alpha d\alpha' f(\alpha) f^*(\alpha') \mathbb{S}_{\beta,\alpha} \mathbb{S}^*_{\beta,\alpha'}, \tag{6.23}$$

unlike in the inclusive formalism.

6.4 Implications

In this section we will discuss the implications of our results and generalize and re-interpret our findings in particular in view of asymptotic gauge symmetries in QED and perturbative quantum gravity.

6.4.1 Physical interpretation

The reason for the different predictions of the inclusive and dressed formalism is the IR radiation produced in the scattering process. The key idea is that accelerated charges produce radiation fields made from soft bosons. In the far infrared, the radiation spectrum has poles as the photon frequency $k^0 \to 0$ of the form $p_i / p_i \cdot k$, where p_i are the hard momenta. These poles reflect the fact that the radiation states are essentially classical and are completely distinguishable for different sets of asymptotic currents $j_{\mathbf{v}}$.

In the inclusive formalism, we imagine incoming states with no radiation, and so the outgoing radiation state has poles from both the incoming hard particles α and the outgoing hard particles β. In the dressed formalism, the incoming part of the radiation is instead folded into the dressed state $\|\alpha\rangle\rangle$, which is designed precisely so that the outgoing radiation field *only* includes the poles from the outgoing hard particles. Thus if we scatter undressed Fock space states, a measurement of the radiation field at late times would determine the dynamical history at long wavelengths of the process $\alpha \to \beta$, leading to the classical answer (6.10). If we instead scatter dressed states, the outgoing radiation has incomplete information about the incoming charged state, which is why the various incoming states still interfere in (6.16). Given that this type of interference is observed all the time in nature, this seems to strongly suggest that the dressed formalism is correct for any problem involving incoming superpositions of momenta.

Based on the result of section 6.2, one might argue that equation (6.10) perhaps is the correct answer and one would have to test experimentally whether or not interference terms appear if we give a scattering process enough time so that the decoherence becomes sizable. After all, the inclusive and dressed approach to calculating cross-sections are at least in principle distinguishable, although maybe not in practice due to very long decoherence times. However, we have demonstrated in section 6.3 that the inclusive formalism predicts an even more problematic result for continuous superpositions, namely that no scattering is observed at all. We thus propose that using the dressed formalism is the most conservative and physically sensible solution to the problem of vanishing interference presented in this chapter.

6.4.2 Allowed dressings

Dressing operators weakly commute with the S-matrix

It was conjectured in [77] and proven in [78] that the far IR part of the dressing weakly commutes with the S-matrix to leading order in the energy of the bosons contained in the dressing. In particular, this means that the amplitudes

$$\langle\beta|\, W_\beta^\dagger S W_\alpha \,|\alpha\rangle \sim \langle\beta|\, W_\beta^\dagger W_\alpha S \,|\alpha\rangle \sim \langle\beta|\, S W_\beta^\dagger W_\alpha \,|\alpha\rangle \tag{6.24}$$

are all IR finite, while they might differ by a finite amount. A short proof of this in QED, complementary to [78], can be given as follows (the gravitational case follows analogously). Recall that Weinberg's soft theorem for QED states that to lowest order in the soft photon momentum $\mathbf{q}$ of outgoing soft photons

$$\langle \epsilon_{\ell_1} a_{\mathbf{q}_1}^{\ell_1} \ldots \epsilon_{\ell_N} a_{\mathbf{q}_N}^{\ell_N} S\rangle \sim \prod_{i=1}^{N} \left(\sum_{j}^{M} \eta_j e_j \frac{\epsilon_{\ell_i} \cdot p_j}{q_i \cdot p_j} \right) \langle S \rangle \,. \tag{6.25}$$

A similar argument holds for incoming photons. For incoming photons with momentum $\mathbf{q}$ we find that

$$\langle S\epsilon^*_{\ell_1} a^{\ell_1\dagger}_{\mathbf{q}_1} \dots \epsilon^*_{\ell_N} a^{\ell_N\dagger}_{\mathbf{q}_N}\rangle \sim \prod_{i=1}^{N}\left(-\sum_j^M \eta_j e_j \frac{\epsilon^*_{\ell_i}\cdot p_j}{q_i\cdot p_j}\right)\langle S\rangle\,. \tag{6.26}$$

The reason for the relative minus sign is that incoming photons add energy-momentum to lines in the diagram instead of removing it. That means that the momentum in the denominator of the propagator changes $(p-q)^2+m^2 \to (p+q)+m^2$ and vice versa. For small momentum, the denominator becomes $-2pq \to 2pq$. From this it directly follows that for general dressings at leading order in the IR divergences,

$$\begin{aligned}\langle SW\rangle = \langle Se^{\int d^3k(f_\ell(\mathbf{k})a^{\ell\dagger}_{\mathbf{k}} - f^*_\ell(\mathbf{k})a^\ell_{\mathbf{k}})}\rangle &\sim \mathcal{N}\,\langle Se^{\int d^3k f_\ell(\mathbf{k})a^{\ell\dagger}_{\mathbf{k}}}\rangle \\ &\sim \mathcal{N}\,\langle e^{-\int d^3k f^*_\ell(\mathbf{k})a^\ell_{\mathbf{k}}}S\rangle \\ &\sim \langle e^{\int d^3k(f_\ell(\mathbf{k})a^{\ell\dagger}_{\mathbf{k}} - f^*_\ell(\mathbf{k})a^\ell_{\mathbf{k}})}S\rangle = \langle WS\rangle\,.\end{aligned} \tag{6.27}$$

Here, we have suppressed a factor of $((2\pi)^3 2|\mathbf{k}|)^{-1}$ and the sum over polarizations in the integrals. In the first and third step we have split the exponential using the Baker-Campbell-Hausdorff formula ($\mathcal{N}$ is the normalization which is finite for finite λ) and in the second equality we have used Weinberg's soft theorem for outgoing and incoming particles.

Dressings cannot be arbitrarily moved between in- and out-states

This opens up the question about the most general structure of a consistent Faddeev-Kulish-like dressing. For example, one could ask whether one can consistently define S-matrix elements with the dressing only acting on the out-state. To answer this question, we assume that the dressing of the out-state has the same IR structure as equation (3.49), but is more general in that it may also include the momenta of (some) particles of the in-state, i.e., $W_\beta \to W_\beta W_{\tilde{\alpha}}$ or any other momenta which might not even appear in the process, $W_\beta W_{\tilde{\alpha}} \to W_\beta W_{\tilde{\alpha}} W_\zeta$. The IR structure of the in-dressing is then fixed by the requirement that the S-matrix element is finite. In addition to the requirement of IR-finiteness we ask that the so-defined S-matrix elements give rise to the correct rules for superposition and the correct scattering for wavepackets, even after tracing out soft radiation.

Applying the logic of the previous sections and 5, one finds that tracing over the soft bosons yields for a diagonal matrix element $\rho_{\beta\beta}$

$$\rho_{\beta\beta}^{\text{hard}} = \sum_{i,j} f_i f_j^* \mathbb{S}_{\beta\alpha_i} \mathbb{S}^*_{\beta'\alpha'_j} \langle 0| W^\dagger_{\tilde{\alpha}'} W_{\tilde{\alpha}} |0\rangle \tag{6.28}$$

and

$$\rho_{\beta\beta}^{\text{hard}} = \iint d\alpha d\alpha' f(\alpha) f^*(\alpha') \mathbb{S}_{\beta\alpha} \mathbb{S}^*_{\beta'\alpha'} \langle 0| W^\dagger_{\tilde{\alpha}'} W_{\tilde{\alpha}} |0\rangle \tag{6.29}$$

for finite and continuous superpositions, respectively. Here, we have used that

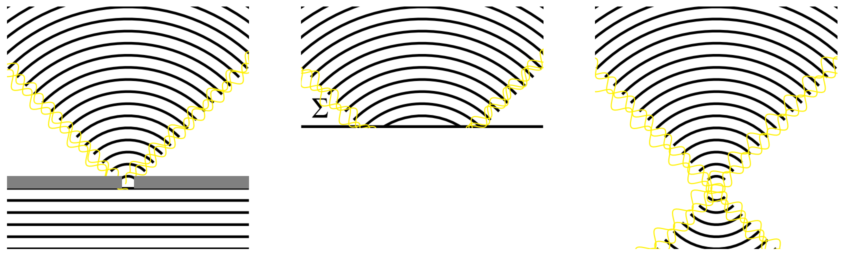
$$\langle W^\dagger_{\tilde{\alpha}'} W^\dagger_{\beta'} W_\beta W_{\tilde{\alpha}} \rangle \Big|_{\beta=\beta'} = \langle W^\dagger_{\tilde{\alpha}'} W_{\tilde{\alpha}} \rangle \,. \tag{6.30}$$

The expectation value is taken in the soft boson Fock space. The expression in the case of $\tilde{\alpha} = \alpha$ and $\tilde{\alpha}' = \alpha'$ was already encountered in sections 6.2 and 6.3 in the context of inclusive calculations, where it was responsible for the unphysical form of the cross-sections. By the same logic it follows that even in the case where $\tilde{\alpha}$ is a proper subset of α, we will obtain a Kronecker delta which sets $\tilde{\alpha} = \tilde{\alpha}'$ and we again do not obtain the expected form of the cross-section. Instead, particles from the subset $\tilde{\alpha}$ will cease to interfere. We thus conclude that the dressing of the out-states must be independent of the in-states and it is not consistent to build superposition of states which are dressed differently. This means that building superpositions from hard and charged Fock space states is not meaningful. In particular, we cannot use undressed states to span the in-state space by simply moving all dressings to the out-state.

Generalized Faddeev-Kulish states

However, it would be consistent to define dressed states by acting with a constant dressing operator W_ζ for fixed ζ on states $\|\alpha\rangle\!\rangle$,

$$\|\alpha\rangle\!\rangle_\zeta \equiv W^\dagger_\zeta W_\alpha |\alpha\rangle \,. \tag{6.31}$$



(a) **(b)** **(c)**

Figure 6.1: **(a)** A plane wave goes through a single slit and emerges as a localized wavepacket. The scattering of the incoming wavepacket results in the production of Bremsstrahlung. **(b)** We can also define some Cauchy slice Σ and create the state by an appropriate initial condition. **(c)** Evolving this state backwards in time while forgetting about the slit results in an incoming localized particle which is accompanied by a radiation shockwave.

Physically this corresponds to defining all asymptotic states on a fixed, coherent soft boson background, defined by some momenta ζ. This state does not affect the physics since soft modes decouple from Faddeev-Kulish amplitudes [45] and thus this additional cloud of soft photons will just pass through the scattering process. The difference between the Faddeev-Kulish dressed state $\|\alpha\rangle\!\rangle$ and the generalized states of the form $\|\alpha\rangle\!\rangle_\zeta$ is that the state $\|\zeta\rangle\!\rangle_\zeta = W_\zeta^\dagger W_\zeta \left|\zeta\right\rangle = \left|\zeta\right\rangle$ does not contain additional photons. This also explains why QED calculations using momentum eigenstates without any additional dressing give the correct cross-sections once we trace over soft radiation. Such a calculation can be interpreted as happening in a set of dressed states defined by

$$\|\alpha\rangle\!\rangle_{\text{in}} = W_{\text{in}}^\dagger W_\alpha \left|\alpha\right\rangle , \tag{6.32}$$

such that the in-state $\|\text{in}\rangle\!\rangle_{\text{in}}$ does not contain photons and looks like a standard Fock-space state.

Localized particles are accompanied by radiation

We also conclude from the previous sections that there are no charged, normalizable states which do not contain radiation. The reason is that within each selection sector there is at most one delta-function normalizable state which does not contain radiation. Thus building a superposition to obtain a normalizable state will necessarily include dressed states which by definition contain soft bosons. A nice argument which makes this behavior plausible was given by Gervais and Zwanziger [92], see figure 6.1.

6.4.3 Selection sectors

Everything said so far has a nice interpretation in terms of the charges $\mathcal{Q}^{\pm}_{\varepsilon}$ of large gauge transformations (LGT) for QED and supertranslations for perturbative quantum gravity.

It turns out that also our generalized version of Faddeev-Kulish states $\|\alpha\rangle\!\rangle_{\zeta}$, equation (6.31), are eigenstates of the generators $\mathcal{Q}^{\pm}_{\varepsilon}$ with eigenvalues which depend on ζ. To see this note that [76]

$$[\mathcal{Q}^{\pm}_{\varepsilon}, W^{\dagger}_{\zeta}] = [\mathcal{Q}^{\pm}_{\varepsilon,S}, W^{\dagger}_{\zeta}] \propto \int_{S^2} d^2z\sqrt{\gamma}\,\frac{\zeta^2}{\zeta\cdot\hat{q}}\,\varepsilon(z,\bar{z}), \tag{6.33}$$

and similarly for gravity [78]. Thus the generalized Faddeev-Kulish states span a space of states which splits into selection sectors parametrized by ζ. The statement that we can build physically reasonable superpositions using generalized Faddeev-Kulish states translates into the statement that superpositions can be taken within a selection sector of the LGT and supertranslation charges $\mathcal{Q}^{\pm}_{\varepsilon}$.

6.5 Conclusions

Calculating cross-sections in standard QED and perturbative quantum gravity forces us to deal with IR divergences. Tracing out unobservable soft modes seems to be a physically well-motivated approach which has successfully been employed for plane-wave scattering. However, as we have shown this approach fails in more generic examples. For finite superpositions it does not reproduce interference terms which are expected; for wavepackets it predicts that no scattering is observed. We have demonstrated in this chapter that dressed states à la Faddeev-Kulish (and certain generalizations) resolve this issue, although it is not clear if the inclusive and dressed formalism are the only possible resolutions. Importantly, we have shown that predictions of different resolutions can disagree, making them distinguishable.

Superpositions must be taken within a set of states with most of the states dressed by soft bosons. The corresponding dressing operators are only well-defined on Fock space if we use an IR-regulator which we only remove at the end of the day. In the strict $\lambda \to 0$ limit, the states are not in Fock space but rather in the much larger von Neumann space which allows for any photon content, including uncountable sets of photons [66, 90]. This suggests an interesting picture which seems worth investigating. The Hilbert space of QED is non-separable but has separable subspaces which are stable under action of the S-matrix and form selection sectors. These subspaces are not the usual Fock spaces but look like the state spaces defined by Faddeev and Kulish [65], in which almost all charged states are accompanied by soft radiation. In the next chapter, we will make these statements more precise.

Our results also raise doubt on whether physical observables exist which can take a state from one selection sector into another. If they did we could use them to create a superpositions of states from different sectors. But as we have seen above, in this case interference would not happen, which is in conflict with basic postulates of quantum mechanics.

Our results may have implications for the black hole information loss problem. Virtually all discussions of information loss in the black hole context rely on the possibility of localizing particles – from throwing a particle into a black hole to keeping information localized. We argued above that normalizable (and in particular localized) states are necessarily accompanied by soft radiation. It is well known that the absorption cross-section of radiation with frequency ω vanishes as $\omega \to 0$ and therefore it seems plausible that, whenever a localized particle is thrown into a black hole, the soft part of its state which is strongly correlated with the hard part remains outside the black hole. If this is true a black hole geometry is always in a mixed state which is purified by radiation outside the horizon.

Chapter 7

An infrared-safe Hilbert space for QED

This chapter is a redacted version of [5].

7.1 Introduction

The dressed formalisms discussed previously remove the IR divergences by including the radiation as coherent states in incoming and/or outgoing states. However, due to the infinite number of soft-modes, the dressed states are not Fock space states. Instead, as we will discuss in section 7.2, they live in representations of the photon canonical commutation relations (CCR) which are different from the standard Fock representation. Physically speaking, one could either say that states in different CCR representations differ by an infinite number of low-energy excitations, or that they represent states which are expanded around classical backgrounds which differ at arbitrarily long wavelengths. Since the radiation produced in scattering depends on the momenta of incoming and outgoing charges, a state which contains a charged particle with momentum $\mathbf{p}$ will generally be in a different CCR representation than a state containing a charged particle with momentum $\mathbf{q} \neq \mathbf{p}$.

In this chapter we will restrict our attention to the case of QED. The infrared structure of perturbative quantum gravity shares many qualitative features with the structure of QED at low energies. Thus, a first step towards a detailed analysis of

IR physics of gravity can be taken by investigating the IR dynamics and kinematics of QED.

The fact that generic out-states consist of superpositions of states in different CCR representations becomes an issue if one wants to ask questions about the information content or the dynamics of low energy modes, since a meaningful comparison of the photon content between different states in different representations is impossible. A related problem recently mentioned in [95] is that the entirety of dressed states is non-separable [65], i.e., they do not have a countable basis, and thus existing dressed formalisms do not allow for the definition of a trace. And in fact, when using an IR cutoff to make the trace over IR modes well-defined, the reduced density matrix of the hard modes again essentially complete decoheres once the cutoff is removed, see chapter 5.

The soft photon production which is responsible for the IR divergences is well approximated by a classical process, but a classical analysis suggests the number of zero-modes should stay constant: although the radiation fields which are classically produced during scattering modify the vector potential at arbitrarily long wavelengths, this change is compensated by the change of the Liénard-Wiechert potentials sourced by the charges. Hence, taking the off-shell modes of the classical field into account, the dynamics of the zero-modes become completely trivial and in the deep IR, the field remains constant in all physical processes.

In this chapter we will see that this picture is accurate even at the quantum level. We develop a new dressed formalism for QED in which the asymptotic Hilbert spaces carry only a single representation of the canonical commutation relations. In other words, all relevant photon states only differ by a finite amount of excited modes. Moreover, the representations for in- and out-states are unitarily equivalent. This implies that the S-matrix is a manifestly unitary operator. Our proposal is a modification of the dressed state formalism of [65]. In addition to coherent states describing radiation, we also incorporate off-shell modes into the definition of states and approximate the time-evolution at late times. The outgoing density matrix of any scattering is IR finite and tracing-out IR modes of the field is well-defined and does not completely decohere the density matrix at finite times. This allows for an IR safe investigation of scattering at late but finite times and enables us to discuss information theoretic properties of quantum states, e.g., time evolution of

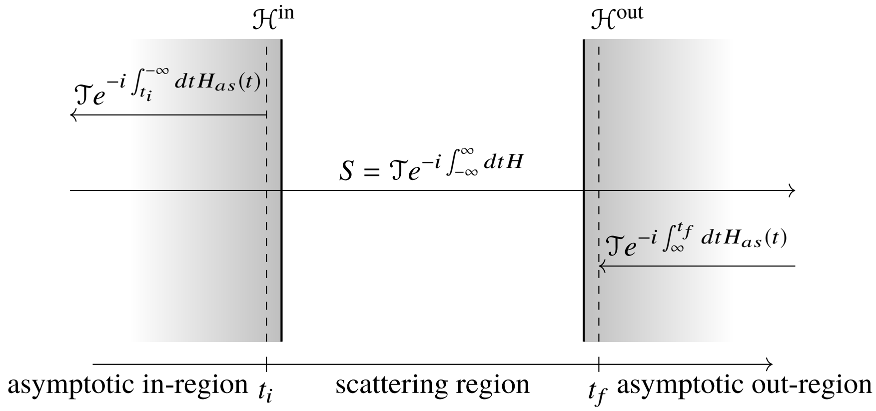


Figure 7.1: The asymptotic Hilbert spaces $\mathcal{H}^{\text{in/out}}$ are defined at finite times t_i and t_f. We assume the particles to be well-separated before and after t_i and t_f, respectively (shaded regions). The time evolution of theories with long range forces is not given by the free Hamiltonian H_0, but approximated by the asymptotic Hamiltonian H_{as} which takes the coupling to very long wavelength modes of the gauge field into account. Charged eigenstates of the free Hamiltonian are replaced by states dressed with transverse off-shell photons which reproduce the correct Liénard-Wiechert potential at long wavelengths. The dressed S-matrix $\mathbb{S}$ evolves a state from $t = t_i$ to $t = -\infty$ with the asymptotic Hamiltonian, which removes the off-shell modes. It is then evolved by the standard S-matrix S to $t = \infty$ and mapped onto $\mathcal{H}^{\text{out}}$ by another asymptotic time-evolution, dressing it with the correct Liénard-Wiechert modes. The states $\mathcal{H}^{\text{in/out}}$ are related by a unitary transformation.

entanglement.

7.1.1 Summary of results

At times earlier than some initial time t_i or later than some final time t_f, well separated states of the full theory are well approximated by states in an asymptotic Hilbert space. The dynamics relevant at long wavelengths are captured by time-evolution with an asymptotic Hamiltonian, which differs from the free Hamiltonian. This is summarized in figure 7.1. The asymptotic Hilbert spaces of QED are of the form

$$\mathcal{H}^{\text{in/out}} = \mathcal{H}_m \otimes \mathcal{H}_\otimes(f_\ell), \tag{7.1}$$

where $\mathcal{H}_m$ is the free fermion Fock space and $\mathcal{H}_\otimes(f_\ell)$ is an incomplete direct product space (IDPS) (which despite the name is a Hilbert space and in particular complete) with a single representation of the photon canonical commutation relations. The precise definition is discussed in section 7.4. The choice of representation depends on a function f_ℓ, which generally is different for different incoming particles. $\mathcal{H}_\otimes(f_\ell)$ can be understood as the image of Fock space under a coherent state operator and the function f_ℓ as specifying the low energy modes of the classical background. States in this Hilbert space are dressed and take the form

$$\|\mathbf{p},\mathbf{k}\rangle\!\rangle_{\{\tilde{f}_\ell\}} = |\mathbf{p}\rangle \otimes W[\tilde{f}_\ell(\mathbf{p},\dots)]\,|\mathbf{k}\rangle\,, \tag{7.2}$$

where $W[\tilde{f}_\ell]$ are operator valued functionals which create coherent states of transverse modes whose wavefunction is given by $\tilde{f}_\ell$ with polarization ℓ. The constraint on $\tilde{f}_\ell$ is that for small photon momenta it agrees with f_ℓ appearing in equation (7.1).[13] This guarantees that it is a state in $\mathcal{H}_\otimes(f_\ell)$. The coherent state generally contains transverse off-shell excitations which ensure that at low energies, the expectation value of the photon field agrees with the classical expectation value. It contains additional on-shell radiation which makes sure that the bosonic part of the dressed state lives in $\mathcal{H}_\otimes(f_\ell)$. The dressed S-matrix is defined as

$$\mathbb{S} = \left(\mathcal{T}e^{-i\int_\infty^{t_f} dt H_{as}(t)}\right) S \left(\mathcal{T}e^{-i\int_{-\infty}^{t_i} dt H_{as}(t)}\right)^\dagger \tag{7.3}$$

and is a unitary operator on $\mathcal{H}_\otimes(f_\ell)$ for any f_ℓ. The first and last terms in the definition of the S-matrix remove off-shell modes from the states. This leaves states dressed with on-shell photons which are scattered by the standard S-matrix, similar to the proposal of [65].

This framework can be used to investigate the correlation between charged particles and IR modes. Each $\mathcal{H}_\otimes(f_\ell)$ inherits the trace operation from Fock space.

[13]Note that, unlike in [65], the IR profile of soft modes in the state $\|\mathbf{p},\mathbf{k}\rangle\!\rangle_\alpha$ does not depend on $\mathbf{p}$ but only on α.

Tracing the density matrix of a superposition of dressed states over soft modes with wavelengths above some scale Λ yields time-dependent decoherence in the momentum eigenbasis. At late times, off-diagonal density matrix elements are proportional to

$$\rho^{\text{reduced}}_{\text{off-diagonal}} \propto (t\Lambda)^{-A_1} e^{A_2(t,\Lambda)}. \tag{7.4}$$

The precise form of the exponents is discussed around equation (7.87). The exponents are proportional to a dimensionless coupling and depend on the relative velocities of the charged matter. The factor A_1 is the same one found in [56] and whose role for decoherence was discussed in chapter 4. The dependence on time and energy scale has been found in [95] through a heuristic argument. The new factor A_2 suppresses decoherence relative to $(t\Lambda)^{-A_1}$. The only information stored in the zero-momentum modes is the information about the CCR representation and decoherence is caused by modes with non-zero momentum. As time passes, these modes become strongly entangled with the hard charges.

In the following, we assume that QED is quantized in Coulomb gauge, since this makes the physical interpretation of our construction more obvious. Section 7.2 reviews the construction of different representations of the CCR which are important for our purposes. Section 7.3 derives the asymptotic Hamiltonian and the dressed S-matrix in Coulomb gauge. The construction of the asymptotic Hilbert space is explained in section 7.4. Section 7.5 contains a proof of the unitarity of the S-matrix. In section 7.6 we explicitly calculate the S-matrix in the presence of a classical current and investigate the correlation between IR modes and charged particles. The density matrix of superpositions of the fields of classical currents, reduced over IR modes, decoheres with time. The conclusions comment on further directions.

7.2 Representations of the canonical commutation relations

7.2.1 Inequivalent CCR representations

Theories with massless particles allow for different representations of the CCR algebra which are not unitarily equivalent. This can easily be seen in a toy model [96]. Consider the Hamiltonian

$$H = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2|\mathbf{k}|} |\mathbf{k}| a^\dagger(\mathbf{k}) a(\mathbf{k}) - \int \frac{d^3\mathbf{k}}{(2\pi)^3 2|\mathbf{k}|} j(\mathbf{k}, t)(a^\dagger(\mathbf{k}) + a(-\mathbf{k})), \tag{7.5}$$

where $j(x)$ is a real source. The Hamiltonian can be diagonalized using a canonical transformation

$$a(\mathbf{k}) \to b(\mathbf{k}) = a(\mathbf{k}) + \frac{j(\mathbf{k})}{|\mathbf{k}|} \qquad a^\dagger(\mathbf{k}) \to b^\dagger(\mathbf{k}) = a^\dagger(\mathbf{k}) + \frac{j^*(\mathbf{k})}{|\mathbf{k}|}, \tag{7.6}$$

so that the commutation relations agree for $b(\mathbf{k}), b^\dagger(\mathbf{k})$ and $a(\mathbf{k}), a^\dagger(\mathbf{k})$. The diagonalized Hamiltonian is given by

$$\tilde{H} = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2|\mathbf{k}|} |\mathbf{k}| b^\dagger(\mathbf{k}) b(\mathbf{k}) + \frac{1}{2} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{|j(\mathbf{k})|^2}{|\mathbf{k}|^2}. \tag{7.7}$$

We will assume that $\lim_{|\mathbf{k}|\to 0} j(\mathbf{k}) = \mathcal{O}(1)$. In this case and with appropriate falloff conditions at large momenta, $\tilde{H}$ is bounded from below. We will assume this in the following. The formally unitary transformation which implements the transformation in equation (7.6) takes the form

$$W \equiv e^F = \exp\left(\int \frac{d^3\mathbf{k}}{(2\pi)^3 2|\mathbf{k}|} \left(\frac{j(-\mathbf{k})}{|\mathbf{k}|} a^\dagger(\mathbf{k}) - h.c. \right) \right). \tag{7.8}$$

However, W is not a good operator on the representation of the $a(\mathbf{k}), a^\dagger(\mathbf{k})$ CCR, since for example

$$\| F \left| 0 \right\rangle \|^2 = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2|\mathbf{k}|^3} |j(\mathbf{k})|^2 = \infty. \tag{7.9}$$

This argument shows that generally, representations of the CCR of a massless field in 3+1 dimensions coupled to different currents will be unitarily inequivalent, which is exactly the problem discussed in the introduction. The choice of representation of the commutation relations of the photon field will generally depend on the presence of charged particles. Before we discuss how to deal with this in the case of QED, we first need to develop some formalism.

7.2.2 Von Neumann space

Formally unitary operators like the one in (7.8) can be given a meaning as operators on a complete direct product space [97], henceforth *von Neumann space* $\mathcal{H}_\otimes$. The non-separable von Neumann space splits into an infinite number of separable *incomplete direct product spaces* (IDPS) on each of which one can define an irreducible representation of the canonical commutation relations [98]. Let us review this construction in this and the next subsection.

Given a countably infinite set of separable Hilbert spaces $\mathcal{H}_n$, we define the infinite tensor product space $\mathcal{H}'_\otimes$ as

$$\mathcal{H}'_\otimes \equiv \bigotimes_n \mathcal{H}_n. \tag{7.10}$$

Vectors $|\psi\rangle \in \mathcal{H}'_\otimes$ of this space are product vectors built from sequences $|\psi_n\rangle$ of normalized vectors in $\mathcal{H}_n$,

$$|\phi\rangle = \bigotimes_n |\psi_n\rangle \, . \tag{7.11}$$

Two such vectors are called equivalent, $|\psi\rangle \sim |\phi\rangle$, if and only if

$$\sum_n |1 - \langle\psi_n|\phi_n\rangle \,| < \infty. \tag{7.12}$$

If the vectors are equivalent their inner product is defined via

$$\langle\psi|\phi\rangle = \prod_n \langle\psi_n|\phi_n\rangle \, . \tag{7.13}$$

If two vectors are inequivalent, their inner product is set to zero by definition. The von Neumann space $\mathcal{H}_\otimes$ is then defined as the space obtained by extending the definition to all finite linear combinations of the vectors in $\mathcal{H}'_\otimes$ and subsequent completion of the resulting space. In order to make the inner product definite, we also require that two states are equal if their difference has zero inner product with any state in $\mathcal{H}_\otimes$. The so-obtained space is non-separable, but splits into separable Hilbert spaces $\mathcal{H}_\otimes(\psi)$ called incomplete direct product spaces (IDPS). $\mathcal{H}_\otimes(\psi)$ consists of all vectors equivalent to some $|\psi\rangle$.

Given a unitary operator $\mathcal{U}_n$ on each $\mathcal{H}_n$ we can define a unitary operator $\mathcal{U}_\otimes$ on $\mathcal{H}_\otimes$ through

$$\mathcal{U}_\otimes \bigotimes_n |\psi_n\rangle \equiv \bigotimes_n \mathcal{U}_n |\psi_n\rangle \tag{7.14}$$

and extend its definition to all states in $\mathcal{H}_\otimes$ by linearity. Clearly, this is not the set of all possible unitary operators on $\mathcal{H}_\otimes$. Multiplication and inverse of such operators is defined through multiplication and inverse of the $\mathcal{U}_n$. It can then be shown that these unitary operators map different IDPS onto each other, i.e., $\mathcal{U}_\otimes \mathcal{H}_\otimes(\psi) \sim \mathcal{H}_\otimes(\psi')$ with $\mathcal{U}_\otimes |\psi\rangle = |\psi'\rangle$. An operator $\mathcal{U}_\otimes$ is a unitary operator on $\mathcal{H}_\otimes(\psi)$ if $\mathcal{U}_\otimes |\psi\rangle \sim |\psi\rangle$.

In a quantum mechanical Hilbert space physical states are only identified with vectors up to a phase. In order to make this precise in a von Neumann space we define a generalized phase. Given a set of real numbers $\lambda = \{\lambda_1, \lambda_2, \ldots\}$ we define the generalized phase operator $\mathcal{V}_\otimes(\lambda)$ as a unitary operator with $\mathcal{V}_n = e^{i\lambda_n}$. If $\sum_n \lambda_n$ converges absolutely, $\mathcal{V}_\otimes(\lambda) = e^{i\sum_n \lambda_n}$. Two vectors which differ by a generalized phase represent the same physical state. States are called weakly equivalent $|\psi\rangle \sim_w |\phi\rangle$, if and only if there exists a $\mathcal{V}_\otimes(\lambda)$ such that

$$\mathcal{V}_\otimes(\lambda) |\psi\rangle \sim |\phi\rangle \, . \tag{7.15}$$

7.2.3 Unitarily inequivalent representations on IDPS

Given the notion of a unitary operator on a von Neumann space, we can find representations of the photon CCR [66]. Let us define the Hilbert space $\mathcal{H}_\gamma$ of photon wavefunctions $f_\ell(\mathbf{k})$ which obey

$$\sum_{\ell=\pm}\int \frac{d^3\mathbf{k}}{(2\pi)^3 2|\mathbf{k}|}|f_\ell(\mathbf{k})|^2 < \infty. \tag{7.16}$$

The inner product is given by

$$\langle g|f\rangle = \sum_{\ell=\pm}\int \frac{d^3\mathbf{k}}{(2\pi)^3 2|\mathbf{k}|} g_\ell^*(\mathbf{k}) f_\ell(\mathbf{k}). \tag{7.17}$$

We are only interested in a special class of CCR representations discussed in [66]. We define the coherent state operator[14]

$$W[f_\ell] \equiv \exp\left(\int \frac{d^3\mathbf{k}}{(2\pi)^3 2|\mathbf{k}|}\left[\sum_{\ell=\pm} f_\ell(\ldots,\mathbf{k},t) a_\ell^\dagger(\mathbf{k}) - h.c.\right]\right) \tag{7.18}$$

which formally obeys

$$W[f_\ell]W[g_\ell] = \exp\left(\sum_{\ell=\pm}\int \frac{d^3\mathbf{k}}{(2\pi)^3 2|\mathbf{k}|}\left(g_\ell^* f_\ell - f_\ell^* g_\ell\right)\right) W[g_\ell]W[f_\ell]. \tag{7.19}$$

Note that this is the same definition as equation (3.49), but with vanishing IR cutoff, $\lambda \to 0$. By functionally differentiating this equation with respect to f_ℓ and g_ℓ^* at $f_\ell = g_\ell^* = 0$ we see that the operators $a_\ell^\dagger(\mathbf{k})$ and $a_\ell(\mathbf{k})$ obey the standard CCR. If f_ℓ, g_ℓ are elements of $\mathcal{H}_\gamma$ the integrals in equation (7.19) converge and we obtain a representation on $\mathcal{H}_\otimes(0)$ which consists of all states equivalent to the photon vacuum $|0\rangle = \bigotimes_n |0_n\rangle$. This is the standard Fock representation. It is clear that any operator of the form $W[h_\ell]$ with $h_\ell \in \mathcal{H}_\gamma$ is a unitary operator on Fock space.

To obtain other representations we need to find operators which obey equation (7.19) on an IDPS $\mathcal{H}_\otimes(\psi)$ which is not weakly equivalent to Fock space $\mathcal{H}_\otimes(0)$. (It was shown in [98] that commutation relation representations on weakly equivalent

[14] To make contact with the previous definition in terms of modes n, we need to expand f_ℓ in a basis e_n of the space of wavefunctions and define $a_n \sim \int d^3\mathbf{k} e_n(\mathbf{k}) a_\ell(\mathbf{k})$ to be the annihilation operator on $\mathcal{H}_n$.

IDPS are unitarily equivalent.) Consider the space of functions $\mathcal{A}_\gamma$ defined by

$$\sum_\ell \int \frac{d^3\mathbf{k}}{(2\pi)^3 2|\mathbf{k}|} \frac{|\mathbf{k}|+1}{|\mathbf{k}|} |f_\ell(\mathbf{k})|^2 < \infty. \tag{7.20}$$

Functions which obey this inequality are still dense in $\mathcal{H}_\gamma$. The dual vector space $\mathcal{A}_\gamma^*$, taken with respect to the inner product, equation (7.17), consists of functions for which

$$\sum_\ell \int \frac{d^3\mathbf{k}}{(2\pi)^3 2|\mathbf{k}|} \frac{|\mathbf{k}|}{|\mathbf{k}|+1} |f_\ell(\mathbf{k})|^2 < \infty \tag{7.21}$$

and $\langle g|f\rangle$ is well defined for all $g \in \mathcal{A}_\gamma^*$ and $f \in \mathcal{A}_\gamma$. Let us define the state $|h\rangle = W[h_\ell]\,|0\rangle$, where h_ℓ lies in $\mathcal{A}_\gamma^*$, but not in $\mathcal{A}_\gamma$. Since $W[h_\ell]$ formally diverges, the state $|h\rangle$ is inequivalent to the photon vacuum $|0\rangle$ (even weakly). This time, operators $W[f_\ell]$ with $f_\ell \in \mathcal{H}_\gamma$ do not yield a representation of the CCR on $\mathcal{H}_\otimes(h)$, since

$$\langle h|\,W[f_\ell]\,|h\rangle = \exp\left(-\frac{1}{2}\int \frac{d^3\mathbf{k}}{(2\pi)^3 2|\mathbf{k}|}|f_\ell|^2\right)\exp\left(\int \frac{d^3\mathbf{k}}{(2\pi)^3 2|\mathbf{k}|}\left(h_\ell^* f_\ell - f_\ell^* h_\ell\right)\right) \tag{7.22}$$

and the integral in the argument of the second exponential will generally diverge. Here, we left the sum over ℓ implicit. However, if we choose $f_\ell \in \mathcal{A}_\gamma$, the phase converges and we obtain a representation, this time on the separable space $\mathcal{H}_\otimes(h)$ which can be obtained from Fock space by the formally unitary operator $W[h]$. These are the representations we will need in the following.

7.3 Asymptotic time-evolution and definition of the S-matrix

7.3.1 The naive S-matrix

In the standard treatment of scattering in quantum field theory, one defines the S-matrix as

$$S_{\beta,\alpha} \simeq \lim_{t'/t'' \to \mp\infty} \langle\beta| \, e^{-iH(t''-t')} \, |\alpha\rangle \, . \tag{7.23}$$

However, already in free theory it is clear that the limits $t' \to -\infty$ and $t'' \to \infty$ do not exist due to the oscillating phase at large times. More carefully we take the states $|\alpha\rangle_{\text{in}} / |\beta\rangle_{\text{out}}$ at some fixed times $t_{i/f}$ and define the S-matrix as

$$S_{\beta,\alpha} = \lim_{t'/t'' \to \mp\infty} {}_{\text{out}}\langle\beta| \, e^{iH_0(t''-t_f)} e^{-iH(t''-t')} e^{-iH_0(t'-t_i)} \, |\alpha\rangle_{\text{in}} \, . \tag{7.24}$$

H_0 is the free Hamiltonian in which the mass parameter takes its physical value. At times later (earlier) than t_f (t_i) we assume that all particles are well separated such that their time-evolution can approximately be described by the free Hamiltonian. The contribution to phase factors coming from the renormalized Hamiltonian $H = H_0 + H_{\text{int}}$ cancels the one coming from the free evolution as $t', t'' \to \mp\infty$.

However, it is well known that the free-field approximation is not valid for QED even at late times, since the interaction falls off too slowly. Mathematically, the problem is that the expression for the S-matrix, equation (7.24), does not converge [54]. Physically, the issue is that massless bosons given rise to a conserved charge (e.g., electric charge in QED or ADM mass in gravity) which can be measured at infinity as an integral over the long range fields. Turning off the coupling completely at early and late times, no field is created. In this chapter we use canonically quantized QED in Coulomb gauge. One might argue that the conserved charge is already taken into account by the solution to the constraint equation, which creates a Coulomb field around the source. However, for all but stationary particles, this is not the correct field configuration. Well-separated particles with non-vanishing velocity should be accompanied by the correct Liénard-Wiechert field which differs from the Coulomb field by transverse off-shell modes. Again, these modes can only be excited if the coupling is not turned off completely.

7.3.2 The asymptotic Hamiltonian

In order to understand which terms of the full Hamiltonian remain important at early and late times, let us approximate how the states evolve if they do not interact strongly for a long time. We ignore all UV issues, which are dealt with by using renormalization, and consider the normal ordered version of the interaction Hamiltonian,

$$H_{\text{int}} \sim -e \int d^3\mathbf{x} : \bar{\psi}\gamma_i\psi : (\mathbf{x}) \cdot \mathbf{A}^i(\mathbf{x}) + \iint d^3\mathbf{x} d^3\mathbf{y} \frac{: \psi^\dagger\psi(\mathbf{x})\psi^\dagger\psi(\mathbf{y}) :}{4\pi|\mathbf{x}-\mathbf{y}|}. \quad (7.25)$$

In the asymptotic regions it is then assumed that the fields, masses and couplings take their physical values instead of the bare ones. In [65] it was shown that at late times coupling to long-wavelength photon modes still remain important. Here we will take a slightly different route to arrive at the exact same expression for the *asymptotic Hamiltonian*, i.e., the Hamiltonian which approximates time evolution at very early and late times.

The normal ordered current in the interaction picture in momentum space is given by

$$\begin{aligned} : j^\mu(\mathbf{x}) :\sim e \sum_{s,t} \iint \frac{d^3\mathbf{p} d^3\mathbf{q}}{(2\pi)^6 4E_\mathbf{p}E_\mathbf{q}} \Big(& b_s^\dagger(\mathbf{p}) b_t(\mathbf{q}) \overline{u}_s(\mathbf{p})\gamma^\mu u_t(\mathbf{q}) e^{-i(p-q)x} \\ & -d_t^\dagger(\mathbf{q}) d_s(\mathbf{p}) \overline{v}_s(\mathbf{p})\gamma^\mu v_t(\mathbf{q}) e^{i(p-q)x} + \dots \Big), \end{aligned} \quad (7.26)$$

where we have omitted terms proportional to $b_s^\dagger(\mathbf{p}) d_t^\dagger(\mathbf{q})$ and $b_t(\mathbf{q}) d_s(\mathbf{p})$. They correspond to pair creation or annihilation with the emission or absorption of a high energetic photons. In the asymptotic regions it should be a reasonable assumption to ignore these effects. Generally, we do not want external momenta to strongly couple to the current. Thus we restrict the integral over $\mathbf{q}$ to a small shell around $\mathbf{p}$ and set $\mathbf{p} = \mathbf{q}$ everywhere except in the phases. After a Fourier transform and keeping only leading order terms in $|\mathbf{k}|$ we obtain the *asymptotic current*,

$$\begin{aligned} : j^{\mu}_{\text{as}}(\mathbf{k},t) : &\sim e \sum_s \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_{\mathbf{p}}} \frac{p^{\mu}}{E_{\mathbf{p}}} \left(b^{\dagger}_s(\mathbf{p}) b_s(\mathbf{p}) - d^{\dagger}_s(\mathbf{p}) d_s(\mathbf{p}) \right) e^{-i\mathbf{v}_{\mathbf{p}}\mathbf{k}t} \\ &\sim e \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_{\mathbf{p}}} \frac{p^{\mu}}{E_{\mathbf{p}}} \rho(\mathbf{p}) e^{-i\mathbf{v}_{\mathbf{p}}\mathbf{k}t}, \end{aligned} \tag{7.27}$$

where we have defined $\rho(\mathbf{p}) = \sum_s \left(b^{\dagger}_s(\mathbf{p}) b_s(\mathbf{p}) - d^{\dagger}_s(\mathbf{p}) d_s(\mathbf{p}) \right)$ and $\mathbf{v}_{\mathbf{p}} = \mathbf{p}/E_{\mathbf{p}}$. At late and early times, the free Hamiltonian in equation (7.24) should thus be replaced by the time-dependent *asymptotic Hamiltonian*,

$$H_{\text{as}}(t) = H_0 + V_{\text{as}}(t), \tag{7.28}$$

which is obtained by replacing the current with the asymptotic current. The interaction potential $V_{\text{as}}(t)$ which replaces the interaction Hamiltonian is given in the interaction picture by

$$V_{\text{as}}(t) = -\int_{\text{IR}} \frac{d^3\mathbf{k}}{(2\pi)^3} \left(: \mathbf{j}^i(-\mathbf{k},t) : \mathbf{A}^i(\mathbf{k},t) - \frac{1}{2|\mathbf{k}|^2} : j^0(\mathbf{k},t) j^0(-\mathbf{k},t) : \right). \tag{7.29}$$

The domain of integration is restricted to soft momenta. The first term describes the coupling of transverse photon degrees of freedom to the transverse current,

$$V^{(1)}_{\text{as}}(t) = -\int_{\text{IR}} \frac{d^3\mathbf{k}}{(2\pi)^3 2|\mathbf{k}|} \mathbf{j}^i(\mathbf{k},t) \left[\varepsilon^{*i}_{\ell}(-\mathbf{k}) a_{\ell}(-\mathbf{k}) e^{-i|\mathbf{k}|t} + \varepsilon^{i}_{\ell}(\mathbf{k}) a^{\dagger}_{\ell}(\mathbf{k}) e^{i|\mathbf{k}|t} \right], \tag{7.30}$$

with a sum over the spatial directions i implied. The second term,

$$V^{(2)}_{\text{as}}(t) = \frac{e^2}{2} \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_{\mathbf{p}}} \int_{\text{IR}} \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{|\mathbf{k}|^2} : \rho(\mathbf{p}) j^0(\mathbf{q},t) : e^{-i\mathbf{v}_{\mathbf{p}}\mathbf{k}t}, \tag{7.31}$$

gives the energy of a charge in a Coulomb field created by a second charge.

7.3.3 The dressed S-matrix

In the spirit of equation (7.24) we define the *dressed S-matrix* as an operator which maps the asymptotic Hilbert space of incoming states $\mathcal{H}^{\text{in}}$ to the asymptotic Hilbert space of outgoing states $\mathcal{H}^{\text{out}}$,

$$\mathbb{S} = \lim_{t'/t'' \to \mp\infty} \mathcal{T} e^{-i\int_{t''}^{t_f} dt H_{\text{as}}(t)} e^{-iH(t''-t')} \mathcal{T} e^{-i\int_{t_i}^{t'} dt H_{\text{as}}(t)}, \tag{7.32}$$

where $\mathcal{T}$ denotes time-ordering. It seems plausible that in the case of QED this expression has improved convergence over equation (7.23), since H_{as} takes into account the asymptotic behavior of H.[15] In order to simplify the expression for the S-matrix and relate it to the standard expression, we insert the identity, $\mathbb{1} = e^{-iH_0(t''-t_f)} e^{iH_0(t''-t_f)}$ and $\mathbb{1} = e^{-iH_0(t'-t_i)} e^{iH_0(t'-t_i)}$, between the time ordered exponentials and the full time evolution. We then obtain

$$\mathbb{S} = \lim_{t'/t'' \to \mp\infty} U(t_f, t'')\, S\, U(t', t_i), \tag{7.33}$$

where $S = e^{iH_0(t''-t_f)} e^{-iH(t''-t')} e^{-iH_0(t'-t_i)}$ reduces to the usual S-matrix in non-dressed formalisms once the limits are taken. The unitaries $U(t_1, t_0)$ obey the differental equation

$$i \frac{\partial}{\partial t_1} U(t_1, t_0) = V_{\text{as}}(t_1) U(t_1, t_0), \tag{7.34}$$

where V_{as} is in the interaction picture and given by equation (7.29). The solution to this is standard[16]

$$U(t_1, t_0) = \mathcal{T} e^{-i\int_{t_0}^{t_1} dt V_{\text{as}}(t)}. \tag{7.35}$$

[15]It has been conjectured in [91] that a similar expression in the context of the Nelson model converges. However, other work [99] indicates that there might be subleading divergences coming from current-current interactions.

[16]See, e.g., chapter 4.2 of [53].

We can bring this into an even more convenient form [65] by splitting $U(t_1, t_0)$ in the following way,

$$\begin{aligned} U(t_i, t_0) &= \mathcal{T} e^{-i\left(\int_{t_i-\epsilon}^{t_i} + \dots + \int_{t_0}^{t_0+\epsilon}\right) dt V_{\text{as}}(t)} \\ &= \mathcal{T} e^{-i\int_{t_i-\epsilon}^{t_i} dt V_{\text{as}}(t)} \dots e^{-i\int_{t_0}^{t_0+\epsilon} dt V_{\text{as}}(t)} \\ &= \mathcal{T} e^{-i\int_{t_i-\epsilon}^{t_i} dt V_{\text{as}}(t)} \dots \mathcal{T} e^{-i\int_{t_0}^{t_0+\epsilon} dt V_{\text{as}}(t)}. \end{aligned} \tag{7.36}$$

In the limit $\epsilon \longrightarrow 0$ we can remove the time-ordering symbols. Since $[V_{\text{as}}(t), V_{\text{as}}(t')]$ only depends on $\rho(\mathbf{p})$ which commutes with all operators we can use the Baker-Campbell-Hausdorff formula $e^A e^B = e^{A+B} e^{1/2[A,B]}$ to combine the exponentals into

$$U(t_i, t_0) = e^{-i\int_{t_0}^{t_i} dt V_{\text{as}}(t)} e^{-\frac{1}{2}\int_{t_0}^{t_i} dt \int_{t_0}^{t} dt' [V_{\text{as}}(t), V_{\text{as}}(t')]}. \tag{7.37}$$

The first factor couples currents to the transverse electromagnetic potential and also contains the charge-charge interaction given in equation (7.31). The second factor makes sure that $U(t_2, t_1)U(t_1, t_0) = U(t_2, t_0)$. We are interested in the limit where $t_0 \longrightarrow -\infty$. In this case the second factor can be calculated as follows. Since the density $\rho(\mathbf{p})$ commutes with all operators present in the asymptotic potential, the only relevant contributions to the commutator come from the photon annihilation and creation operators. The unequal-time commutator of the asymptotic potential with itself is given by

$$[V_{\text{as}}(t), V_{\text{as}}(t')] = \int_{\text{IR}} \frac{d^3\mathbf{k}}{(2\pi)^3 2|\mathbf{k}|} \mathbf{j}^{\perp}_{\text{as}}(-\mathbf{k}, t) \mathbf{j}^{\perp}_{\text{as}}(\mathbf{k}, t') \left(e^{i|\mathbf{k}|(t'-t)} - e^{-i|\mathbf{k}|(t-t')}\right), \tag{7.38}$$

with the transverse current $\mathbf{j}^{\perp,i}(\mathbf{k}, t) = \sum_\ell \varepsilon^{i*}_\ell(\mathbf{k}) \varepsilon^j_\ell(\mathbf{k}) \mathbf{j}^j(\mathbf{k}, t)$. We can now perform the integral over t' and drop the boundary conditions as $t = -\infty$ knowing that in any final calculation they will be canceled by the corresponding term coming from the full Hamiltonian. The result is

$$
\begin{aligned}
H_c^{\perp}(t) &= -\frac{1}{2}\int_{-\infty}^{t} dt'[V(t),V(t')] \\
&= \frac{i}{2}\int_{\mathrm{IR}} \frac{d^3\mathbf{k}}{(2\pi)^3 2|\mathbf{k}|}\int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_{\mathbf{p}}} \frac{\mathbf{v_p}-\frac{\mathbf{k}(\mathbf{k}\cdot\mathbf{v_p})}{|\mathbf{k}|^2}}{|\mathbf{k}|-\mathbf{k}\cdot\mathbf{v_p}} \\
&\qquad \times\Big[:\rho(\mathbf{p})\mathbf{j}_{\mathrm{as}}(-\mathbf{k},t): e^{-i\mathbf{k}\mathbf{v_p}t} + h.c.\Big],
\end{aligned}
\tag{7.39}
$$

where we have used that $\rho(\mathbf{p}) : j_{\mathrm{as}}(-\mathbf{k},t) :=: \rho(\mathbf{p}) j_{\mathrm{as}}(-\mathbf{k},t) :$ up to terms that are renormalized away [65]. This corrects the phase due to the Coulomb energy, equation (7.31), to

$$
e^{i\Phi(t)} \equiv e^{i\int_{-\infty}^{t} dt'(H_c(t')+H_c^{\perp}(t'))}, \tag{7.40}
$$

which gives the phase due to the energy of a charge in the Liénard-Wiechert field of another charge. The total asymptotic time evolution takes the form

$$
U(-\infty,t_i) = e^{i\Phi(t)} e^{i\int_{-\infty}^{t} dt' V_{\mathrm{as}}^{(1)}(t')}. \tag{7.41}
$$

An analogous expression follows for $U(t_f,\infty)$, where we have to drop the boundary terms at $t=\infty$.

7.4 Construction of the asymptotic Hilbert space

7.4.1 The asymptotic Hilbert space

We can finally discuss the asymptotic Hilbert space. For now, we will ignore free photons and moreover focus on a single particle. The generalization to many particles and the inclusion of free photons is straight forward and will be done later. We require that our asymptotic states evolve with the asymptotic Hamiltonian instead of the free one. Naively, we might be tempted to think that our asymptotic particle agrees with a free field excitation at some time t. However, as discussed in the previous section, if our field couples to a massless boson this will generally not be correct. Given a charged excitation of momentum $\mathbf{p}$ we define

$$\begin{aligned} \|\mathbf{p}\rangle\!\rangle^{\text{in}}_{\mathbf{p}} &\equiv U(t_i,-\infty)(|\mathbf{p}\rangle^{\text{in}} \otimes |0\rangle) \\ &\equiv |\mathbf{p}\rangle^{\text{in}} \otimes W[f_\ell^{\text{in}}(\mathbf{p},\mathbf{k},t)]\,|0\rangle\,. \end{aligned} \tag{7.42}$$

The state $|\mathbf{p}\rangle^{\text{in}}$ is a free field fermion Fock space state defined at time t_i and $|0\rangle$ is the photon Fock space vacuum. $U(t_i,-\infty)$ was given in equation (7.41) and does not change the matter component of the state. We can therefore write its action as an operator on the photon Hilbert space, $W[f_\ell^{\text{in}}]$, with $W[\,\cdot\,]$ given in equation (7.18). In (7.41), we have dropped the boundary term at $-\infty$. This is analogous to the standard procedure one uses to get the electric field of a current at a time t from the retarded correlator. The subscript in equation (7.42) indicates that the asymptotic Hilbert space containing the state $\|\mathbf{p}\rangle\!\rangle^{\text{in}}_{\mathbf{p}}$ is

$$\mathcal{H}_{\text{as}} = \mathcal{H}_{\text{m}} \otimes \mathcal{H}_{\otimes}(f_\ell^{\text{in}}(\mathbf{p},\mathbf{k},t_i)), \tag{7.43}$$

where $\mathcal{H}_{\text{m}}$ is the standard free fermion Fock space and $H_{\otimes}(f_\ell^{\text{in}}(\mathbf{p},\mathbf{k},t_i))$ is an incomplete direct product space which carries a representation of the canonical commutation relations for the photon as explained in the previous subsection. Performing the integral in $U(t_i,-\infty)$, we can determine $f_\ell^{\text{in}}(\mathbf{p},\mathbf{k},t_i)$ to be

$$f_\ell^{\text{in}}(\mathbf{p},\mathbf{k},t) = -e\frac{p\cdot\epsilon_\ell(\mathbf{k})}{p\cdot k}\theta(k^{\text{max}}-|\mathbf{k}|)e^{-iv\cdot k t_i}. \tag{7.44}$$

Here, p^μ and k^μ are on-shell and $v^\mu = p^\mu/E_{\mathbf{p}}$. The Heaviside function makes sure that only modes with wave number smaller than k^{max} are contained in the dressing. Analogously, we can construct out-states as

$$\begin{aligned} \|\mathbf{p}\rangle\!\rangle^{\text{out}}_{\mathbf{p}} &\equiv U(t_f,\infty)(|\mathbf{p}\rangle^{\text{out}} \otimes |0\rangle) \\ &\equiv |\mathbf{p}\rangle^{\text{out}} \otimes W[f_\ell^{\text{out}}(\mathbf{p},\mathbf{k},t)]\,|0\rangle\,, \end{aligned} \tag{7.45}$$

and

$$f_\ell^{\text{out}}(\mathbf{p},\mathbf{k},t_f) = -e\frac{p\cdot\epsilon_\ell(\mathbf{k})}{p\cdot k}\theta(k^{\text{max}}-|\mathbf{k}|)e^{-iv\cdot k t_f} = f_\ell^{\text{in}}(\mathbf{p},\mathbf{k},t_f). \tag{7.46}$$

In the following, we will leave the sum over ℓ and the dependence of $f_\ell(\mathbf{p},\mathbf{k},t)$ on $\mathbf{k}$ and t implicit. It can be checked by power counting that the exponent of

$$\langle 0|\, W[f_\ell^{\rm in}(\mathbf{p})]\,|0\rangle = \exp\left(-\frac{1}{2}\sum_\ell \int \frac{d^3\mathbf{k}}{(2\pi)^3 2|\mathbf{k}|}|f_\ell^{\rm in}(\mathbf{p})|^2\right) \tag{7.47}$$

is IR divergent, so $W[f_\ell^{\rm in}(\mathbf{p})]$ is not a unitary operator on Fock space. It can also be checked that $W[f_\ell^{\rm in}(\mathbf{p})]$ obeys equation (7.21) so that the commutation relation representation is inequivalent to the Fock space representation. On the other hand $W[f_\ell^{\rm out}(\mathbf{p}) - f_\ell^{\rm in}(\mathbf{p})]$ is a unitary operator on any representation since its argument is in $\mathcal{A}_\gamma$, defined through equation (7.20). This operator maps in-states to out-states and it follows that $\mathcal{H}_\otimes(f(\mathbf{p})_\ell^{\rm out}) = \mathcal{H}_\otimes(f(\mathbf{p})_\ell^{\rm in})$. Since the Hilbert spaces are related by unitary time-evolution using the asymptotic Hamiltonian, in the following we will oftentimes drop the in and out labels on the states. Equivalently we can set $t_i = t_f = T$ without affecting any argument in the following.

The coherent state of transverse modes in equation (7.42) which accompanies the matter field $|\mathbf{p}\rangle^{\rm in}$ is *not* a cloud of on-shell photons. The reason is that the time-dependence of $f_\ell^{\rm in}(\mathbf{p})$ modifies the dispersion relation of the modes created by this coherent state from $E_\mathbf{k} = |\mathbf{k}|$ to $E_\mathbf{k} = \mathbf{k}\mathbf{v}$. To understand the role of these modes consider the expectation values of the four-potential in such a dressed state,

$$\langle\!\langle \mathbf{p}\|A^0\|\mathbf{p}\rangle\!\rangle = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{|\mathbf{k}|^2}\langle\!\langle \mathbf{p}\|j^0(\mathbf{k},t)\|\mathbf{p}\rangle\!\rangle e^{i\mathbf{k}\mathbf{x}}, \tag{7.48}$$

$$\langle\!\langle \mathbf{p}\|\mathbf{A}\|\mathbf{p}\rangle\!\rangle = e\int_0^{k^{\max}} \frac{d^3\mathbf{k}}{(2\pi)^3 2|\mathbf{k}|}\frac{\mathbf{v_p} - \frac{\mathbf{k}(\mathbf{k}\cdot\mathbf{v_p})}{|\mathbf{k}|^2}}{|\mathbf{k}| - \mathbf{k}\cdot\mathbf{v_p}}\left[e^{i\mathbf{k}(\mathbf{x}-\mathbf{v_p}t)} + h.c.\right]\langle\!\langle \mathbf{p}\|\mathbf{p}\rangle\!\rangle. \tag{7.49}$$

The expectation value of $\mathbf{A}$ agrees with the classical 3-vector potential of a point charge moving in a straight line with velocity $\mathbf{v_p}$ at long wavelength which passes through $\mathbf{x} = 0$ at $t = 0$,

$$j^\mu(\mathbf{k},t) = ev^\mu e^{-i\mathbf{v_p}\mathbf{k}t}. \tag{7.50}$$

In other words, the dressed state constructed above obeys Ehrenfest's theorem at long wavelengths. If we had not dressed the state, we would have found $\langle\!\langle \mathbf{p}\|\mathbf{A}\|\mathbf{p}\rangle\!\rangle = 0$

and the corresponding electric field would have been only the Coulomb field of a static charge.[17]

Given two momenta $\mathbf{p} \neq \mathbf{q}$, the Hilbert spaces $\mathcal{H}_\otimes(f_\ell^{\text{in}}(\mathbf{p}))$ and $\mathcal{H}_\otimes(f_\ell^{\text{out}}(\mathbf{q}))$ are inequivalent. To see this, note that $\tilde{W} \equiv W[f_\ell^{\text{out}}(\mathbf{q})]W^\dagger[f_\ell^{\text{in}}(\mathbf{p})]$ maps $\mathcal{H}_\otimes(f_\ell^{\text{in}}(\mathbf{p}))$ to $\mathcal{H}_\otimes(f_\ell^{\text{in}}(\mathbf{q}))$ and up to a phase equals $\tilde{W} = W[f_\ell^{\text{in}}(\mathbf{q}) - f_\ell^{\text{in}}(\mathbf{p})]$. If the Hilbert spaces were equivalent, $\tilde{W}$ would have to be a unitary operator on $\mathcal{H}_\otimes(f_\ell^{\text{in}}(\mathbf{p}))$. However, it is easy to see that $f_\ell^{\text{in}}(\mathbf{q}) - f_\ell^{\text{in}}(\mathbf{p})$ does not obey (7.20) and thus the two Hilbert spaces cannot be equivalent.

Since we have started with the claim, that we want all in- and out-states to be elements of the Hilbert space (7.43), it seems our program has failed. However, this is too naive. Assume we scatter an initial state $\|\mathbf{p}\rangle\!\rangle_{\mathbf{p}}$ off of a classical potential. Our outgoing state will be a superposition of different momentum eigenstates. However, the state $\|\mathbf{q}\rangle\!\rangle_{\mathbf{q}}$ will not be part of this superposition. A scattering process produces an infinite number of long-wavelength photons as bremsstrahlung, but $\|\mathbf{q}\rangle\!\rangle_{\mathbf{q}}$ contains no such radiation. The IR part of the classical radiation field produced during scattering from momentum $\mathbf{p}$ to $\mathbf{q}$ is created by a coherent state operator

$$\begin{aligned} R(\mathbf{p}, \bar{\mathbf{q}}) &\equiv W[f_\ell^{\text{rad}}(\mathbf{p}, \mathbf{k}, t) - f_\ell^{\text{rad}}(\mathbf{q}, \mathbf{k}, t)] \\ &= W[f_\ell^{\text{rad}}(\mathbf{p}, \mathbf{k}, t)]W^\dagger[f_\ell^{\text{rad}}(\mathbf{q}, \mathbf{k}, t)] \end{aligned} \tag{7.51}$$

with

$$f_\ell^{\text{rad}}(\mathbf{p}, \mathbf{k}) = \frac{ep \cdot \epsilon_\ell(\mathbf{k})}{p \cdot k} g(|\mathbf{k}|) \approx -f_\ell^{\text{in}}(\mathbf{p}, \mathbf{k}, 0). \tag{7.52}$$

The bar in the definition of $R(\mathbf{p}, \bar{\mathbf{q}})$ denotes that the terms containing $\mathbf{q}$ come with a relative minus sign. Here, $g(|\mathbf{k}|)$ is a function which goes to 1 as $|\mathbf{k}| \to 0$ and can be chosen at will otherwise. Thus the state which is obtained by scattering an excitation with momentum $\mathbf{p}$ into an excitation with momentum $\mathbf{q}$ plus the long wavelength part of the corresponding bremsstrahlung is given by

$$\|\mathbf{q}\rangle\!\rangle_{\mathbf{p}} \equiv |\mathbf{q}\rangle \otimes W[f_\ell^{\text{in}}(\mathbf{q})]R(\mathbf{q}, \bar{\mathbf{p}})\,|0\rangle \tag{7.53}$$

[17] In the case of a plane wave the charge distribution is smeared over all of space.

up to a finite number of photons. This state contains the field of the state $\|\mathbf{q}\rangle\!\rangle_{\mathbf{q}}$ as well as the radiation produced by scattering the state $\|\mathbf{p}\rangle\!\rangle_{\mathbf{p}}$ to momentum $\mathbf{q}$ at long wavelengths.

The operator $W[f^{\text{in}}(\mathbf{q})]R(\mathbf{p}, \bar{\mathbf{q}})$ again is not a unitary operator on any CCR representation. However, the combination

$$W[f_\ell^{\text{out}}(\mathbf{q})]R(\mathbf{q}, \bar{\mathbf{p}})W^\dagger[f_\ell^{\text{in}}(\mathbf{p})] \tag{7.54}$$

converges on Fock space. The convergence up to phase is easy to see since up to a phase, equation (7.54) equals $W[f_\ell^{\text{out}}(\mathbf{q}) + f_\ell^{\text{rad}}(\mathbf{q}) - f_\ell^{\text{in}}(\mathbf{p}) - f_\ell^{\text{rad}}(\mathbf{p})]$ and since the function in the argument vanishes as $|\mathbf{k}| \to 0$ it clearly satisfies equation (7.20). It is an easy exercise to prove that the phase is also convergent. We will give an example below. This shows that the states $\|\mathbf{p}\rangle\!\rangle_{\mathbf{p}}$ and $\|\mathbf{q}\rangle\!\rangle_{\mathbf{p}}$ live in the same subspace $\mathcal{H}_\otimes(f_\ell^{\text{in}}(\mathbf{p}))$. Moreover, all states which are physically accessible from $\|\mathbf{p}\rangle\!\rangle_{\mathbf{p}}$ must contain radiation. States of the form $\|\mathbf{q}\rangle\!\rangle_{\mathbf{p}}$ are constructed to precisely contain the IR tail of the classical radiation. Hence, all single fermion states which are physically accessible take the form of equation (7.53) up to a finite number of photons and thus live in the same separable IDPS. With the appropriate dressing, also multi-fermion states and thus all physically accessible states live in this subspace. Note that this structure is different to existing constructions [65, 66, 95], where an out-state is generally a superposition of vectors from inequivalent subspaces of $\mathcal{H}_\otimes$.

7.4.2 Multiple particles and classical radiation backgrounds

The generalization to multiple particles is straight forward. Given a state which contains multiple charges with momenta $\mathbf{p}_1, \mathbf{p}_2, \ldots$, the operator $U^\dagger(t_i, -\infty)$ acts on the photon state as[18]

$$W\left[\sum_i f_\ell^{\text{in}}(\mathbf{p}_i)\right] \tag{7.55}$$

and maps the Fock space vacuum into a a different separable Hilbert space, $\mathcal{H}_\otimes(\sum_i f_\ell^{\text{in}}(\mathbf{p}_i))$, which acts as our asymptotic photon Hilbert space. Similarly,

[18]In the case of multiple particle species with different charges, we should replace $e \to e_i$ in the definition of $f_\ell^{\text{in}}(\mathbf{p})$.

we can define a coherent state operator

$$R(\mathbf{p}_1, \mathbf{p}_2, \ldots; \overline{\mathbf{q}_1}, \overline{\mathbf{q}_2}, \ldots) \tag{7.56}$$

which lets us define states

$$\|\mathbf{q}_1, \mathbf{q}_2, \ldots\rangle\!\rangle_{\{\mathbf{p}_1, \mathbf{p}_2, \ldots\}} \in \mathcal{H}_\otimes\left(\sum_i f_\ell^{\text{in}}(\mathbf{p}_i)\right), \tag{7.57}$$

which contain particles with momenta $\mathbf{q}_1, \mathbf{q}_2, \ldots$ and the appropriate bremsstrahlung produced by scattering charged particles of momenta $\{\mathbf{p}_1, \mathbf{p}_2, \ldots\}$ to charged particles of momenta $\{\mathbf{q}_1, \mathbf{q}_2, \ldots\}$. Up to a finite number of additional photons all out states will be of this form.

We can also incorporate classical background radiation described by

$$A^0 = 0 \tag{7.58}$$

$$\mathbf{A} = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2|\mathbf{k}|}\left[h_\ell(\mathbf{k})e^{ikx} + h.c.\right] \tag{7.59}$$

with $\lim_{|\mathbf{k}|\to 0} |\mathbf{k}| h_\ell(\mathbf{k}) = \mathcal{O}(1)$, i.e., backgrounds which contain an infinite number of additional infrared photons. In the presence of charged particles with momenta $\mathbf{p}_1, \mathbf{p}_2, \ldots$ the corresponding asymptotic Hilbert space is $\mathcal{H}_\otimes(h_\ell + \sum_i f_\ell^{\text{in}}(\mathbf{p}_i))$.

7.4.3 Comments on the Hilbert space

The construction presented in this chapter has a number of properties which are known to be realized in theories with long range forces in 3 + 1 dimensions.

Existence of selection sectors

The existence of selection sectors in four-dimensional QED and gravity is well established [26, 100] and has recently been rediscovered [29]. In the present construction, the choice of selection sector corresponds to a choice of representation of the canonical commutation relations on a separable Hilbert space $\mathcal{H}_\otimes(\psi) \subset \mathcal{H}_\otimes$. That these are indeed selection sectors will be shown in the next section where we prove that $\mathbb{S}$ is unitary.

Charged particles as infraparticles

It was shown in [96, 100, 101] that there are no states in QED (or more generally in theories with long range forces), which sit exactly on the mass-shell $p^2 = -m^2$. Our construction reproduces this behavior. Although $P \cdot P \|\mathbf{p}\rangle\!\rangle_{\mathbf{p}} = -m^2 \|\mathbf{p}\rangle\!\rangle_{\mathbf{p}}$, the state is not non-normalizable.[19] A normalizable state must be built from a superposition of different states $\|\mathbf{q}\rangle\!\rangle_{\mathbf{p}}$. However, any other state in $\mathcal{H}_\otimes(f_\ell^{\text{in}}(\mathbf{p}))$ contains extra photons and thus cannot be on the mass-shell $p^2 = -m^2$. Also note that in [3] it was argued that consistent scattering of wavepackets in theories with long range forces in four dimensions requires to take superpositions of particle states including photons.

Spontaneous breaking of Lorentz invariance

The spontaneous breaking of Lorentz invariance in QED has already been noted in [100, 102] (see also [103]). In our construction, there is an infinite number of possible $\mathcal{H}_\otimes(\psi)$ one can choose from. This choice spontaneously breaks Lorentz invariance. The states $\|\mathbf{p}\rangle\!\rangle_{\mathbf{p}}$ and $\|\mathbf{q}\rangle\!\rangle_{\mathbf{q}}$ describe boosted versions of the same configuration, namely a charged particle in the absence of radiation. However, as shown above they live in inequivalent representations. Thus, a Lorentz transformation cannot be implemented as a unitary operator on $\mathcal{H}_\otimes(f_\ell^{\text{in}}(\mathbf{p}))$. An analogous argument applies for any configuration of charged particles $\mathbf{p}_1, \mathbf{p}_2, \ldots$.

7.5 Unitarity of the S-matrix

The form of the S-matrix follows from equation (7.32),

$$\mathbb{S} = U(t_f, \infty)\, S\, U^\dagger(t_i, -\infty), \tag{7.60}$$

with $U(t_1, t_0)$ given in equation (7.37). The operator S is the textbook S-matrix. Comparing to equation (7.42) we see that the role of the operators $U(t_f, \infty)$, $U^\dagger(t_i, -\infty)$ is to remove the part of the dressing which corresponds to the classical field. Thus, the off-shell dressing $U(t_i, -\infty)$ in the definition of the asymptotic states, equation (7.53), can be ignored whenever we are calculating S-matrix ele-

[19] P is the 4-momentum operator.

ments.

Consider the action of the dressed S-matrix on $\|\mathbf{p}_1, \mathbf{p}_2, \ldots\rangle\!\rangle_{\{f_\ell\}} \in \mathcal{H}_\otimes(f_\ell)$. We establish unitarity on $\mathcal{H}_\otimes(f_\ell)$ by showing that dressed S-matrix elements between states with given f_ℓ are finite, as well as that dressed S-matrix elements between states of different separable subspaces, i.e., various $f_\ell, \tilde{f}_\ell$ with different IR asymptotics vanish. Unitarity then follows from unitarity of U in the von Neumann space sense and unitarity of S.

For the sake of clarity we will neglect the possibility of a classical background radiation field in the following. Taking this possibility into account corresponds to acting with some coherent state operator $\tilde{R}$ on the Fock space vacuum and does not affect the proof. We take an otherwise arbitrary, dressed in-state

$$\|\text{in}\rangle\!\rangle = |\mathbf{p}_1, \ldots\rangle \otimes W[f_\ell^{\text{in}}(\mathbf{p}_1) + \ldots] R(\mathbf{p}_1, \ldots; \overline{\mathbf{q}_1}, \ldots) \, |\mathbf{k}_1, \ldots\rangle \tag{7.61}$$

and similarly define a general out-state

$$\|\text{out}\rangle\!\rangle = |\mathbf{p}_1', \ldots\rangle \otimes W[f_\ell^{\text{out}}(\mathbf{p}_1') + \ldots] R(\mathbf{p}_1', \ldots; \overline{\mathbf{q}_1}, \ldots) \, |\mathbf{k}_1', \ldots\rangle \,. \tag{7.62}$$

Both states are elements of $\mathcal{H}_\otimes(\sum_i f_\ell(\mathbf{q}_i))$. For ease of notation, we will omit the ellipses ... and indices in the following. The S-matrix elements take the form

$$\begin{aligned} \mathbb{S}_{\text{out,in}} &= \langle\!\langle \text{out} \| U(t_f, \infty) S U^\dagger(t_i, -\infty) \| \text{in} \rangle\!\rangle \\ &= \Big(\langle \mathbf{p}' | \otimes \langle \mathbf{k}' | \, R^\dagger(\mathbf{p}; \overline{\mathbf{q}}) \Big) \, S \, \Big(|\mathbf{p}\rangle \otimes R(\mathbf{p}; \overline{\mathbf{q}}) \, |\mathbf{k}_1\rangle \Big). \end{aligned} \tag{7.63}$$

It was conjectured in [77] and shown in [78] (see also [3]) that we can move dressings through the S-matrix without jeopardizing the IR-finiteness. We can therefore move all $\mathbf{q}_i$ dependent terms on one side and obtain

$$\begin{aligned} \langle\!\langle \text{out} \| R^\dagger(\mathbf{p}'; \overline{\mathbf{q}}) S R(\mathbf{p}; \overline{\mathbf{q}}) \| \text{in} \rangle\!\rangle &= \langle\!\langle \text{out} \| R(\mathbf{q}; \overline{\mathbf{p}'}) S R(\mathbf{p}; \overline{\mathbf{q}}) \| \text{in} \rangle\!\rangle \\ &= \langle\!\langle \text{out} \| R(0; \overline{\mathbf{p}'}) S R(\mathbf{p}; 0) \| \text{in} \rangle\!\rangle + (\text{finite}). \end{aligned} \tag{7.64}$$

Hence, the divergence structure of the matrix element is the same as the one of

$$\mathbb{S}_{\text{out,in}} \sim \Big(\langle \mathbf{p}_1', \dots | \otimes \langle \mathbf{k}_1', \dots | \, R^\dagger(\mathbf{p}_1', \dots; 0) \Big) \, S \, \Big(|\mathbf{p}_1, \dots\rangle \otimes R(\mathbf{p}_1, \dots; 0) \, |\mathbf{k}_1, \dots\rangle \Big). \tag{7.65}$$

However, these are just Faddeev-Kulish amplitudes which are known to be IR finite [65].

Let us now show that if $\|\mathbf{p}_1, \dots\rangle\rangle_{\mathbf{q}_1,\dots}$ and $\|\mathbf{p}_1', \dots\rangle\rangle_{\mathbf{q}_1',\dots}$ live in inequivalent representations, the matrix element vanishes. We again omit the ellipses and indices. Consider

$$\begin{aligned} \mathbb{S}_{\text{out}',\text{in}} &= \langle\langle \text{out} \| U(t_f, \infty) S U^\dagger(t_i, -\infty) \| \text{in} \rangle\rangle & (7.66) \\ &= \Big(\langle \mathbf{p}' | \otimes \langle \mathbf{k}' | \, R^\dagger(\mathbf{p}'; \overline{\mathbf{q}'}) \Big) \, S \, \Big(|\mathbf{p}\rangle \otimes R(\mathbf{p}; \overline{\mathbf{q}}) \, |\mathbf{k}\rangle \Big). & (7.67) \end{aligned}$$

Moving the dressing through the S-matrix, we find that up to finite terms

$$\mathbb{S}_{\text{out}',\text{in}} \sim \langle \text{out}' | \, R(\mathbf{q}', \overline{\mathbf{q}}) R^\dagger(\mathbf{p}'; 0) S R(\mathbf{p}; 0) \, |\text{in}\rangle \, . \tag{7.68}$$

The previous proof showed that $R^\dagger(\mathbf{p}'; 0) S R(\mathbf{p}; 0)$ is a unitary operator on Fock space. Further, it can be shown that $R(\mathbf{q}', \overline{\mathbf{q}})$ vanishes on Fock space if $\mathbf{q}_1, \dots \neq \mathbf{q}_1', \dots$ [1]. Therefore we can conclude that the S-matrix element vanishes and have shown that the S-matrix is a stabilizer of the asymptotic Hilbert spaces defined in section 7.4.

7.6 Example: Classical current

7.6.1 Calculation of the dressed S-matrix

The formalism devised in the preceding sections can be used to investigate the time dependence of decoherence in scattering processes. A simple example can be given by considering QED coupled to a classical current $j^\mu(x)$. The current enters with momentum $\mathbf{p}$ and at $x^\mu = x_0^\mu$ is deflected to a momentum $\mathbf{p}'$,

$$\begin{aligned} j^\mu(x) = e \int_0^\infty d\tau \frac{p'^\mu}{m} \delta^{(4)} \left(x^\mu - x_0^\mu - \frac{p'^\mu}{m}\tau \right) \\ + e \int_{-\infty}^0 d\tau \frac{p^\mu}{m} \delta^{(4)} \left(x^\mu - x_0^\mu - \frac{p^\mu}{m}\tau \right). \end{aligned} \tag{7.69}$$

We assume that initially no radiation is present and the current is carried by an infinitely heavy particle. The initial state of the transverse field excitations is not the Fock vacuum but $\|\text{in}\rangle\!\rangle = W[f_\ell^{\text{in}}(\mathbf{p})]\,|0\rangle$, which is the vacuum of the CCR representation $\mathcal{H}_\otimes(f_\ell^{\text{in}}(\mathbf{p}))$. This state represents a situation in which the classical field of the current j^μ is present at wavelengths longer than the inverse mass. Since we deal with an infinitely massiv source, the integrals are taken over all of values of $\mathbf{k}$. The IR divergent Fock space S-matrix in the presence of a current can be calculated explicitly, see e.g., [52], and is given by

$$S = R(\mathbf{q}, \overline{\mathbf{p}}) = W[f_\ell^{\text{rad}}(\mathbf{q}, \mathbf{k}) - f_\ell^{\text{rad}}(\mathbf{p}, \mathbf{k})]. \tag{7.70}$$

According to our prescription, the dressed S-matrix is given by

$$\mathbb{S} = W[f_\ell^{\text{out}}(\mathbf{q}, \mathbf{k}, t_f)]\; S\; W^\dagger[f_\ell^{\text{in}}(\mathbf{p}, \mathbf{k}, t_i)]. \tag{7.71}$$

The out state is given by $\|\text{out}\rangle\!\rangle = \mathbb{S}\,\|\text{in}\rangle\!\rangle$ and contains the radiation field produced by the acceleration as well as a correction to the Coulomb field which depends on the outgoing current. Combining everything, the dressed S-matrix becomes

$$\mathbb{S} = W[f_\ell^S(\mathbf{p}, \mathbf{q}, \mathbf{k}, t_i, t_f)] \exp\left(ie^2 \int \frac{d^3\mathbf{k}}{(2\pi)^3 2|\mathbf{k}|} \Phi(\mathbf{k}, \mathbf{q}, \mathbf{p}) \right) \tag{7.72}$$

with

$$
\begin{aligned}
f_\ell^S(\mathbf{p},\mathbf{q},\mathbf{k},t) =& e\left(\frac{q\cdot\varepsilon_\ell(\mathbf{k})}{q\cdot k}(1-e^{iv_\mathbf{q}\cdot kt_f})-\frac{p\cdot\varepsilon_\ell(\mathbf{k})}{p\cdot k}(1-e^{iv_\mathbf{p}\cdot kt_i})\right),\\
\Phi(\mathbf{k},\mathbf{q},\mathbf{p}) =& \left(\frac{\mathbf{q}^\perp}{q\cdot k}-\frac{\mathbf{p}^\perp}{p\cdot k}\right)\left(\frac{\mathbf{q}^\perp}{q\cdot k}\sin(v_\mathbf{q}\cdot kt_f)+\frac{\mathbf{p}^\perp}{p\cdot k}\sin(v_\mathbf{p}\cdot kt_i)\right)\\
&+\frac{\mathbf{q}^\perp}{q\cdot k}\frac{\mathbf{p}^\perp}{p\cdot k}\sin\left((t_f v_\mathbf{q}-t_i v_\mathbf{p})\cdot k\right)
\end{aligned}
\tag{7.73}
$$

The superscripts on the momentum vectors $\mathbf{p}^\perp \equiv P^\perp(\hat{\mathbf{k}})\mathbf{p}$ denote the part of $\mathbf{p}$ which is perpendicular to $\mathbf{k}$. The projection operator $P^\perp(\hat{\mathbf{k}})$ arises from the sum over polarizations, $P^\perp(\hat{\mathbf{k}}) = \sum_{\ell=\pm}\varepsilon_\ell^*(\mathbf{k})\varepsilon_\ell(\mathbf{k})$. From here it is easy to see that as $|\mathbf{k}|\to 0$, f_ℓ^S has no poles and Φ only goes like $|\mathbf{k}|^{-1}$. Therefore, $\mathbb{S}$ is a well defined unitary operator.

7.6.2 Tracing out long-wavelength modes

A big advantage of formulating scattering in terms of the dressed states introduced above is that it allows an IR divergence free definition of the trace operation the on asymptotic Hilbert space. The trace operation is inherited from Fock space. For example, a basis for the Hilbert space of photon excitations in $\mathcal{H}_\otimes(f_\ell^{\text{in}}(\mathbf{p}))$ is given by

$$
\begin{aligned}
&W[-f_\ell^{\text{rad}}(\mathbf{p})]\,|0\rangle\,, W[-f_\ell^{\text{rad}}(\mathbf{p})]a_{\ell'}^\dagger(\mathbf{k})\,|0\rangle\,,\\
&\ldots, W[-f_\ell^{\text{rad}}(\mathbf{p})]\frac{1}{\sqrt{n!}}\left(a_{\ell'}^\dagger(\mathbf{k})\right)^n|0\rangle
\end{aligned}
\tag{7.74}
$$

We could have chosen any other $\tilde{f}_\ell(\mathbf{p},\mathbf{k},t)$ in place of f_ℓ^{rad} as long as $\lim_{\mathbf{k}\to 0}|\mathbf{k}|f_\ell^{\text{in}}(\mathbf{p},\mathbf{k},t_i) = \lim_{\mathbf{k}\to 0}|\mathbf{k}|\tilde{f}_\ell(\mathbf{p},\mathbf{k},t)$. For example we could have chosen $\tilde{f}_\ell(\mathbf{p},\mathbf{k},t) = f_\ell^{\text{out}}(\mathbf{p},\mathbf{k},t_f)$, since the trace is invariant under a change of basis.

As an example, let us consider a superposition of fields created by classical currents, i.e., the outgoing state is

$$
\|\text{out}\rangle\!\rangle = \frac{1}{\sqrt{2N}}\left(W_{\mathbf{q}_1}+W_{\mathbf{q}_2}\right)|0\rangle\,,
\tag{7.75}
$$

where

$$W_{\mathbf{q}_i} \equiv W[f_\ell^{\text{out}}(\mathbf{q}_i, \mathbf{k}, t)]\, W[f_\ell^{\text{rad}}(\mathbf{q}_i, \mathbf{k}) - f_\ell^{\text{rad}}(\mathbf{p}, \mathbf{k})] \tag{7.76}$$

and N is given by

$$N = 1 + \text{Re}\Big(\langle 0|\, W_{\mathbf{q}_1}^\dagger W_{\mathbf{q}_2} \,|0\rangle \Big). \tag{7.77}$$

In order to calculate the reduced density matrix we split the dressing $W_{\mathbf{q}_i} = W_{\mathbf{q}_i}^{\text{IR}} + W_{\mathbf{q}_i}^{\text{UV}}$ into a part we will trace over (IR) and the complement (UV). The "IR" part contains all modes with wavelength longer than some cutoff Λ, which is smaller than $k^{\max}$. The reduced density matrix obtained by tracing over "IR" then becomes

$$\rho^{\text{UV}} = \frac{1}{N}\Big(W_{\mathbf{q}_1}^{\text{UV}} |0\rangle \langle 0|\, W_{\mathbf{q}_1}^{\text{UV}\dagger} + \langle 0|\, W_{\mathbf{q}_2}^{\text{IR}\dagger} W_{\mathbf{q}_1}^{\text{IR}} \,|0\rangle \quad W_{\mathbf{q}_1}^{\text{UV}} |0\rangle \langle 0|\, W_{\mathbf{q}_2}^{\text{UV}\dagger} \tag{7.78}$$

$$+ (\mathbf{q}_1 \leftrightarrow \mathbf{q}_2) \Big). \tag{7.79}$$

We see that the off-diagonal elements are multiplied by a factor of $\langle 0|\, W_{\mathbf{q}_2}^{\text{IR}\dagger} W_{\mathbf{q}_1}^{\text{IR}} \,|0\rangle$ which is responsible for decoherence. A similar *dampening factor* already appeared in chapter 5. There, the calculation was done for Faddeev-Kulish dressed states and it was shown that the dampening factor has an IR divergence in its exponent which makes it vanish, unless $\mathbf{q}_1 = \mathbf{q}_2$. As we will see, using the dressing devised in this chapter, the dampening factor is IR finite for finite times.

The magnitude of the dampening factor is simply the normal-ordering constant of $W_{\mathbf{q}_2}^{\text{IR}\dagger} W_{\mathbf{q}_1}^{\text{IR}}$ which is given by

$$\exp\left(-\frac{1}{2} \int \frac{d^3\mathbf{k}}{(2\pi)^3 2|\mathbf{k}|} \sum_{\ell=\pm} |f_\ell^1 - f_\ell^2|^2 \right) \tag{7.80}$$

with

$$f_\ell^i(\mathbf{q}_i, \mathbf{k}, t) = e\frac{q_i \cdot \varepsilon_\ell(\mathbf{k})}{q_i \cdot k}(1 - e^{-iv\cdot kt}). \tag{7.81}$$

We can rearrange the terms proportional to $|f^i|^2$. We go to spherical polar coordinates and separate the $|\mathbf{k}|$ integral to find

$$\int \frac{d^3k}{2|\mathbf{k}|} \sum_{\ell=\pm} |f_\ell^i|^2 = e^2 \int d^2\Omega \frac{\mathbf{q}_i^\perp \mathbf{q}_i^\perp}{(q_i \cdot k)^2} \int_0^\Lambda \frac{d|\mathbf{k}|}{|\mathbf{k}|} \sin^2\left(|\mathbf{k}| \frac{(-v_i \cdot \hat{k})}{2} t\right) \tag{7.82}$$

The $|\mathbf{k}|$ integral can be performed and the result can be expressed in terms of logarithms and cosine integral functions $\mathrm{Ci}(x)$.

$$\begin{aligned} &\int_0^\Lambda \frac{d|\mathbf{k}|}{|\mathbf{k}|} \sin^2\left(|\mathbf{k}| \frac{(-v_i \cdot \hat{k})}{2} t\right) \\ &\qquad = \frac{1}{2}\left(\log(\Lambda t) + \gamma + \log(|v_i \cdot \hat{k}|) - \mathrm{Ci}(\Lambda t |v_i \cdot \hat{k}|)\right). \end{aligned} \tag{7.83}$$

Here, γ is the Euler-Mascheroni constant. Using $\mathrm{Ci}(x) \sim \gamma + \log(x) + \mathcal{O}(x^2)$ for small x, we see that at $\Lambda, t = 0$ the exponent vanishes. The $|\mathbf{k}|$ integral for the cross-term involving f_ℓ^1 and f_ℓ^2 is only slightly more complicated and can also be performed. One finds

$$\begin{aligned} &\int \frac{d^3\mathbf{k}}{2|\mathbf{k}|} \sum_{\ell=\pm} \mathrm{Re}(f_\ell^{1*} f_\ell^2) \\ &\qquad = 2e^2 \int d^2\Omega \frac{\mathbf{q}_1^\perp \mathbf{q}_2^\perp}{(q_1 \cdot \hat{k})(q_2 \cdot \hat{k})} \int_0^\Lambda \frac{d|\mathbf{k}|}{|\mathbf{k}|} \sin\left(|\mathbf{k}| \frac{(-v_1 \cdot \hat{k})}{2} t\right) \times \\ &\qquad\quad \sin\left(|\mathbf{k}| \frac{(-v_2 \cdot \hat{k})}{2} t\right) \cos\left(|\mathbf{k}| \frac{(-(v_1 - v_2) \cdot \hat{k})}{2} t\right) \end{aligned} \tag{7.84}$$

The integral evaluates to

$$\begin{aligned} &\frac{1}{4}\Big(2\log(\Lambda t) + \gamma + \log(|v_1 \cdot \hat{k}|) + \log(|v_2 \cdot \hat{k}|) - \log(\Lambda t |(v_1 - v_2) \cdot \hat{k}|) \\ &\qquad -\mathrm{Ci}(\Lambda t |v_1 \cdot \hat{k}|) - \mathrm{Ci}(\Lambda t |v_2 \cdot \hat{k}|) + \mathrm{Ci}(\Lambda t |(v_1 - v_2) \cdot \hat{k}|)\Big). \end{aligned} \tag{7.85}$$

Clearly, as $t \to 0$ the dampening factor becomes zero and no decoherence takes place. This is sensible is the example at hand, where we have assumed that the current changes direction at $t = 0$. Different to the situation in [1], the density matrix is well defined even without an IR cutoff. In any real experiment we measure the field at very late times after the scattering process has happened and all wavelengths shorter than those that will be traced out had enough time to be produced, i.e., $\Lambda t \gg 1$. In this limit, the integrals are dominated by the logarithms. Furthermore, we need to keep the term which contains $\mathrm{Ci}(\Lambda t|(v_1 - v_2)\cdot \hat{k}|) - \gamma$, since the cosine integral diverges as $v_1 \to v_2$ and $\hat{\mathbf{k}} \perp \mathbf{v}_1, \mathbf{v}_2$.

Similarly, the phases of the off-diagonal terms in the density matrix can be calculated. Since we only have a single charge present, the Coulomb interactions $H_c + H_c^{\perp}$ does not contribute anything to the phase. The only contributions come from the normal ordering of the coherent state operators. After some cancellations and performing the integration over $|\mathbf{k}|$ we obtain

$$\exp\left(i\frac{e^2}{2(2\pi)^3}\int d^2\Omega \frac{\mathbf{q}_1^{\perp}\mathbf{q}_2^{\perp}}{(q_1\cdot\hat{k})(q_2\cdot\hat{k})}\mathrm{Si}(\Lambda t(v_1 - v_2)\hat{k})\right). \tag{7.86}$$

Thus, at late times, the dampening factor becomes

$$\langle 0| W_{\mathbf{q}_2}^{\mathrm{IR}\dagger} W_{\mathbf{q}_1}^{\mathrm{IR}} |0\rangle = (\Lambda t)^{-A_1} e^{A_2(\Lambda,t)} \tag{7.87}$$

with

$$A_1 = \frac{e^2}{2(2\pi)^3}\int d^2\Omega \left(\frac{\mathbf{q}_1^{\perp}}{q_1\cdot\hat{k}} - \frac{\mathbf{q}_2^{\perp}}{q_2\cdot\hat{k}}\right)\left(\frac{\mathbf{q}_1^{\perp}}{q_1\cdot\hat{k}} - \frac{\mathbf{q}_2^{\perp}}{q_2\cdot\hat{k}}\right) \tag{7.88}$$

$$\begin{aligned} A_2(t,\Lambda) = -\frac{e^2}{2(2\pi)^3}\int d^2\Omega \frac{\mathbf{q}_1^{\perp}\mathbf{q}_2^{\perp}}{(q_1\cdot\hat{k})(q_2\cdot\hat{k})}\Big(&\mathrm{Ci}(\Lambda t|(v_1 - v_2)\cdot\hat{k}|) \\ &- i\mathrm{Si}(\Lambda t(v_1 - v_2)\cdot\hat{k}) - \gamma - \log(\Lambda t|(v_1 - v_2)\cdot\hat{k}|)\Big). \end{aligned} \tag{7.89}$$

This is consistent with earlier results obtained in [85, 95]. The appearance of the factor A_2 makes the decoherence rate for particles milder than suggested by the term which only depends on A_1. The qualitative behavior at infinite times, however, reproduces exactly what has been found before based on calculations which only take the emitted radiation into account, namely that any reduced density matrix decoheres in the infinite time limit.

7.7 Conclusions

In this chapter we presented a construction of an infinite class of asymptotic Hilbert spaces which are stable under S-matrix scattering with a unitary, dressed S-matrix. The major improvement over existing work is that all asymptotic states live in the same separable Hilbert space with a single representation of the photon canonical commutation relations. Our construction relied on the fact that transverse IR modes of the Liénard-Wiechert field are included in the definition of the asymptotic states. This should be a good approximation if the included wavelengths are smaller than any other scale in the problem. The construction enables an analysis of the information content of IR modes in the late-time density matrix. As an example, we studied a density matrix which describes a superposition of the field of two classical currents. The reduced density matrix decoheres as a power law with time. The increase of decoherence with time shows that the entanglement of charged particles with infrared modes increases over time. The physical reason for the decoherence is that at times $t \sim \frac{1}{\Lambda}$ we can tell apart on- and off-shell modes with wavelengths larger than $\lambda \sim \frac{1}{\Lambda}$. Since charged matter is accompanied by a cloud of off-shell modes creating the correct momentum dependend electric field, this allows to identify the momenta of the involved particles. One might argue that this is incompatible with the picture of conserved charges from large gauge transformations (LGT) (for a recent review see [25]). There it is argued that a photon vacuum transition must happen since the soft charge generally changes during a scattering process. However, in our approach we take into account off-shell excitations which contribute to the hard charge. The increase of decoherence with time can be understood as learning to tell apart soft and hard charges as time goes on. Hence, in flat space scattering, no information is stored in the LGT charges, but in the way the charge splits between the hard and soft part.

This work leaves open some interesting questions. We have seen that near-zero energy modes decohere the outgoing density matrix in the momentum basis. Unlike in chapter 4, this decoherence happens although the scattering is fundamentally IR finite. Furthermore, the decoherence cannot be avoided by chosing an appropriate dressing, since we can only add radiation, i.e., on-shell modes, as additional dressing. At zero energy there is no difference between on and off-shell modes, however, at finite times those can be distinguished which leads to decoherence. This opens up the possibility that a similar mechanism at subleading order in the asymptotic current could also decohere additional quantum numbers like spin. Moreover, although we have constructed dressed states, we have not discussed how they can be obtained by an LSZ-like formalism from operators. Due to the presence of long wavelength modes of classical fields and radiation, the correct operators must be non-local. Presumably there should be an infinite family of operators, similar to the situation in [70, 71], for each Hilbert space which must contain radiative modes in their definition. Filling in the details is left for future work.

Lastly, as motivated in the introduction, an extension of the presented ideas to gravity would be desirable for a variety of reasons. While one might expect that a generalization to linearized gravity should be fairly straight forward, an extension beyond linear order will presumably more difficult. The discussion in the context of gravity could be interesting in the context of the black hole information paradox: We have seen that in our construction no information is stored in the zero-energy excitations. This agrees with statements made in [45, 46]. However, by waiting long enough, charged matter can be arbitrarily strong correlated with near zero-energy modes and those modes might store information. Tracing out the matter thus leaves one with a completely mixed density matrix of soft modes, which might be related to the ideas presented in [33]. The fact that "softness" is an observer-dependent notion might aid arguments in favor of complementarity. Clearly, more work is required to make these arguments more precise.

Part II

Quantum information in quantum gravity

Chapter 8

The AdS/CFT correspondence

8.1 Holography in string theory

While a complete understanding of quantum gravity in a four-dimensional de Sitter universe, such as the one we live in, does not seem to be in reach, considerable progress has been made in quantum gravity in anti-de Sitter spacetime. Based on the early developments outlined in section 1.1, it was proposed that gravity is holographic [104, 105], i.e., that the true degrees of freedom within a volume of spacetime can be thought of as being encoded on a hypersurface of one dimension less. The AdS/CFT correspondence is a duality between a gravitational theory in an anti-de Sitter (AdS) spacetime and a conformal field theory (CFT) in one dimension less, and therefore a concrete realization of the holographic principle.

8.1.1 AdS/CFT

In its generic form, the AdS/CFT duality relates a d-dimensional conformal field theory CFT_d to a gravitational theory on AdS_{d+1}.[20] One of the most prominent examples of this duality is the conjecture that $\mathcal{N} = 4$ Super Yang-Mills (SYM) theory on four-dimensional Minkowski space is dual to String theory on an $\mathrm{AdS}_5 \times S^5$ background. This can be motivated by the following argument first posited in [12].

[20]The stringy origin of the gravitational side enables one to rewrite the gravitational theory as a theory on $\mathrm{AdS}_{d+1} \times X$ where X is some compact internal space. The dimensions of X and the anti-de Sitter space add up to 10 or 11.

Ten dimensional type IIB string theory contains non-perturbative, $3+1$ dimensional hypersurfaces called D-branes. At weak coupling, the dynamics of these objects can be described by open strings, whose ends are restricted to lie on the branes. Consider a stack of N such D-branes. At low energies, the dynamics of this system are described by two sectors: supergravity in ten-dimensional flat space and four-dimensional $\mathcal{N} = 4$ SYM theory with gauge group $SU(N)$, which describes the brane dynamics. At very low energies, these two sectors decouple.

At strong coupling, the branes backreact on the geometry and their low energy description is a p-brane solution of ten-dimensional supergravity. In the low energy limit, the dynamics of string theory on that background split into a sector away from the branes which effectively lives in flat space and string theory close to the horizon of the backreacted solution. Again, these sectors decouple.

The AdS/CFT conjecture identifies the theories at strong and weak coupling. Both theories contain a decoupled sector of low energy supergravity in flat space. The non-trivial statement is that the other sectors, namely $\mathcal{N} = 4$ SYM theory and string theory in the near-horizon region of the p-branes, should also be identified. They are different descriptions of the same theory at different couplings.

The regime of validity of either description can be extracted from the above argument. The Yang-Mills coupling constant in the gauge theory is given in terms of the string coupling by $g_{YM}^2 = 2\pi g_s$, while Newton's constant on the AdS side is given by $G_N = \alpha'^4 g_s^2$. The constant α' is related to the string length via $\alpha' = l_s^2$. Lastly, the tension of the brane stack is given by $\frac{N}{g_s \alpha'^2}$. The characteristic length scale we can build from these quantities is $R^4 = N g_s \alpha'^2$, which is proportional to the curvature scale of the p-brane background, $R_{\text{AdS}}^4 = 4\pi N g_s \alpha'^2$.

The gravity description should be a good approximation at low curvature, i.e., if the radius of curvature is much bigger than the string length, $4\pi N g_s \alpha'^2 \gg \alpha'^2$. Moreover, the description of the gravitational theory as strings moving on AdS space requires that the string scale is bigger than the Planck scale, i.e., $l_p^2 \equiv g_s l_s^2 < l_s^2$. This tells us that the gravitational description should be valid if

$$N > N g_s \gg 1. \tag{8.1}$$

The gauge theory description is valid in the opposite regime, $N g_s < 1$.

8.1.2 The dictionary

More precisely, the duality states that the partition function of string theory on a negatively curved space equals the partition function of a conformal theory in lower dimensions which can be thought of as being located on the conformal boundary of the AdS space [106, 107]. The partition functions of both theories agree,

$$\langle \exp\left(\int \phi_0 \mathcal{O}\right)\rangle_{\mathrm{CFT}} = Z_{\mathrm{Grav}}(\phi_0). \tag{8.2}$$

Here, ϕ_0 schematically denote the asymptotic value of fields ϕ in the gravity theory with partition function $Z_{\mathrm{Grav}}(\phi_0)$. The left-hand side is the partition function of the dual conformal theory in which the ϕ_0 play the role of sources for operators $\mathcal{O}$. The operator $\mathcal{O}$ which multiplies ϕ_0 is called the dual operator to the field ϕ. The equivalence of the partition functions allows one to translate quantities in the conformal field theory to quantities in the dual gravitational theory. By taking functional derivaties with respect to the sources we can express CFT correlation functions in terms of derivatives of the gravity partition function.[21] Alternatively, we can choose the following prescription, known as the *extrapolate dictionary*. Close to the boundary of AdS, a scalar field can be expanded as

$$\phi = az^{d-\Delta} + \ldots + bz^{\Delta} + \ldots, \tag{8.3}$$

where the boundary sits at $z = 0$. The value of ϕ_0 is given by the coefficient a, which defines a non-normalizable solution to the Klein-Gordon equation. The normalizable solution with highest power z^{Δ} has a leading coefficient b which is related to the expectation value of the CFT operator dual to ϕ.

Similar arguments can be made for fields of higher spin. For example, the CFT stress-energy tensor is the operator dual to the metric. One can choose coordinates close to the boundary such that the metric takes the Fefferman-Graham form,

$$ds^2 = \frac{1}{z^2}\left(-dt^2 + dz^2 + dx^\mu dx_\mu + z^d \Gamma^{(d)}_{\mu\nu}(x,z)dx^\mu dx^\nu\right). \tag{8.4}$$

[21] In order to obtain well-defined expressions free of divergences, *holographic renormalization* [108] needs to be employed.

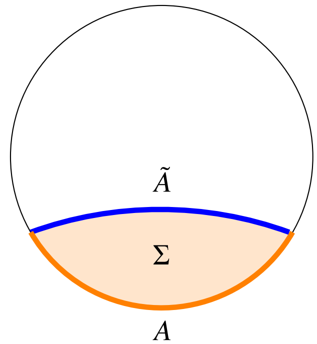


Figure 8.1: The relation between extremal surface and boundary region. This image shows a time-slice of AdS_3. The dual conformal field theory can be thought of as living on the boundary (the black circle). The orange part of the boundary is the subregion A of the CFT. Its entanglement entropy is dual to the length of an extremal codimension two surface $\tilde{A}$ (blue line) in the gravitational theory. The region Σ between the boundary and the extremal surface (shaded orange) is a slice of the entanglement wedge.

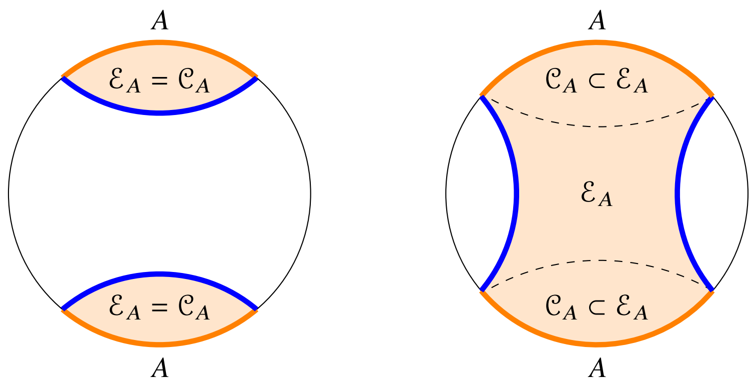
In this gauge, the extrapolate dictionary gives the expectation value of the stress-energy tensor as

$$\langle T_{\mu\nu} \rangle = \frac{d}{16\pi G_N} \Gamma^{(d)}_{\mu\nu}(x, 0). \tag{8.5}$$

8.1.3 Holographic entanglement entropy

The entanglement entropy of a subsystem A, equation (2.8), equals the area of an extremal bulk $\tilde{A}$ surface which is homologous to the subsystem A [13, 14, 109–111]. Roughly speaking, this means that it can be smoothly deformed onto A. More precisely, the Ryu-Takayanagi formula or its covariant formulation, the Hubeny-Rangamani-Takayanagi prescription, states that the von Neumann entropy of the reduced density matrix on A is proportional to, at leading order in N, the area of the extremal surface $\tilde{A}$,

$$S = \frac{\text{Area}(\tilde{A})}{4G_N}, \tag{8.6}$$



(a) **(b)**

Figure 8.2: **a)** A disconnected region on the boundary of a timeslice of AdS_3 (orange) is dual to a a disconnected region in the bulk bounded by two separate RT surfaces (blue). **b)** If the boundary region is bigger than half of the total boundary, the dual region becomes connected. It is still bounded by the corresponding RT surface; however, not all points in the bulk region can causally communicate with $\mathcal{D}_A$. The boundary of the causal wedge is indicated by dashed lines. For a detailed discussion, refer to the main text.

see figure 8.1. In the rest of this theses we will use the acronym HRRT for this expression. If there are many extremal surfaces $\tilde{A}$, the one with smallest area must be chosen.

As mentioned in section 2.5, entanglement entropy of subregions is an ill-defined concept in the quantum field theory because of UV divergences close to the boundary of that region. The divergences can be regulated using a UV cutoff. Similarly, the extremal surface defined above has divergent area. The divergence arises from the fact that the boundary of AdS is infinitely far away from every point in the interior. This is an example of the UV/IR connection of AdS/CFT: UV divergences in the CFT are related to divergences which appear as a consequence of the infinite size of the AdS spacetime [112].

8.1.4 Causal wedge vs entanglement wedge

There are two natural subregions in the bulk one can construct given a boundary subregion. On the conformal boundary, a spatial subregion A has an associated *causal diamond* or *domain of dependence*, $\mathcal{D}_A$. A point to the future of A lies within $\mathcal{D}_A$ if all past-directed timelike curves through p intersect A. Similarly, a point to the past of A lies within $\mathcal{D}_A$ if all future-directed curves through that point intersect A. In other words, $\mathcal{D}_A$ is the spacetime region whose time evolution is uniquely determined by specifying initial conditions on A.

The *causal wedge* $\mathcal{C}_A$ is the intersection of the causal future and past of $\mathcal{D}_A$ in the bulk, i.e., all points which can send and receive lightlike signals from and to $\mathcal{D}_A$. The information contained in the reduced density matrix ρ_A associated with subregion A captures the physics at least in the causal diamond $\mathcal{C}_A$. If this was not the case, we could e.g., place a small mirror inside the causal wedge without changing the density matrix and thus change the boundary conditions of fields in the bulk. Via the AdS/CFT dictionary, this would affect expectation values in the CFT, which leads to a contradiction [113].

The *entanglement wedge* $\mathcal{E}_A$ is the domain of dependence of a bulk Cauchy slice bounded by the boundary and the bulk extremal surface. Figure 8.2a shows a $t = 0$ slice of pure AdS_3 with two boundary regions and their associated RT surfaces. Both the causal and the entanglement wedge agree and are bounded by the HRRT surface.

However, the entanglement wedge and the causal wedge are generally not the same. Figure 8.2b shows slightly bigger regions and their RT surfaces, which have undergone a phase transition. The boundary of the causal wedge are given by the dashed lines, while the entanglement wedge corresponds to the shaded region. Thus we see that in this case the causal and entanglement wedge are different. That the causal and entanglement wedge are different is the generic situation away from AdS vacuum, even if the boundary region is connected.

While it seems natural that the boundary subregion contains information about the associated causal wedge, the HRRT formula shows that in fact it must contain information about the entanglement wedge. Since the area of the HRRT surface can be computed from the reduced density matrix, the density matrix must contain information about the geometry close to the HRRT surface. An in fact, the density matrix of a subregion of a holographic CFT can be used to reconstruct bulk physics in the associated entanglement wedge [22].

Chapter 9

Positive gravitational subsystem energies from CFT cone relative entropies

This chapter is a redacted version of [6].

9.1 Introduction

Via the AdS/CFT correspondence, it is believed that any consistent quantum theory of gravity defined for asymptotically AdS spacetimes with some fixed boundary geometry $\mathcal{B}$ corresponds to a dual conformal field theory defined on $\mathcal{B}$. Recently, it has been understood that many natural quantum information theoretic quantities in the CFT correspond to natural gravitational observables (see, for example [13], or [114, 115] for a review). Through this correspondence, properties which hold true for the quantum information theoretic quantities can be translated to statements about gravitational physics. In this way, we can obtain an alternative/deeper understanding of some known properties of gravitational systems, but also discover novel properties that must hold in consistent theories of gravity. A particularly interesting quantum information theoretic quantity to consider is relative entropy [116]. As we have seen in chapter 2, for a general state $|\Psi\rangle$ of the CFT, we can associate a reduced density matrix ρ_A to a spatial region A by tracing out the degrees of freedom outside of A and relative entropy $S(\rho_A||\rho_A^0)$ quantifies how different this state is from the vacuum density matrix ρ_A^0 reduced on the same region. Relative entropy is typically UV-finite, always positive, and has the property that it increases as we increase the size of the region A (known as the monotonicity property). According to the AdS/CFT correspondence, this should correspond to some quantity in the gravitational theory which also obeys these positivity and monotonicity properties.

As we review in section 9.2, by making use of the holographic formula relating CFT entanglement entropies to bulk extremal surface areas (the "HRRT formula" [13, 14]), it is possible to explicitly write down the gravitational quantity corresponding to relative entropy as long as the vacuum modular Hamiltonian ($H_A^0 = -\log \rho_A^0$) for the region A is *local*, that is, it can be written as a linear combination of local operators in the CFT. Until recently, such a local form was only known for the modular Hamiltonian of ball-shaped regions [110]. For these regions, relative entropy has been shown to correspond to an energy that can be associated with the bulk entanglement wedge corresponding to this ball [37, 117]. The positivity of relative entropy then implies an infinite family of positive energy constraints (reviewed below) [39].

Ball-shaped regions (of Minkowski space) have the property that their boundary lies on the past lightcone of a point p and the future lightcone of some other point q. In the recent work [118], it has been shown that the vacuum modular Hamiltonian for a region A has a local expression so long as the boundary ∂A of A lies on the past lightcone of a point p *or* the future lightcone of a point q.[22] Thus, we have a much more general class of regions for which the relative entropy and its properties can be interpreted gravitationally. The main goal of the present chapter is to explain this interpretation.

In the general case, we denote by $\hat{A}$ the region of the lightcone bounded by ∂A, as shown in figure 9.1. The modular Hamiltonian can then be written as

$$H_A^0 = \int_{\hat{A}} \zeta_A^\mu(x) T_{\mu\nu}(x) \epsilon^\nu , \tag{9.1}$$

where $T_{\mu\nu}$ is the CFT stress-energy tensor, ϵ^μ is a volume form defined in section

[22]The existence of such a region depends on the relativistic nature of the theory under consideration, which guarantees the existence of a codimension-0 domain of dependence.

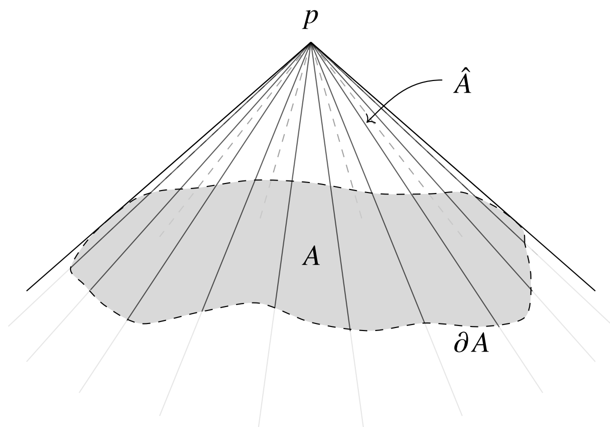


Figure 9.1: Subregion A of the CFT whose boundary ∂A is on the past light-cone of the point p. $\hat{A}$ denotes the surface of the cone bounded by ∂A.

9.2, and $\zeta_A^\mu(x)$ is a vector field on $\hat{A}$ directed towards the tip of the cone and vanishing at the tip of the cone and on ∂A.

To describe the gravitational interpretation of the relative entropy for region A, we consider any codimension one spacelike surface Σ in the dual geometry such that Σ intersects the AdS boundary at $\hat{A}$ and is bounded in the bulk by the HRRT surface $\tilde{A}$ (the minimal area extremal surface homologous to A). This is illustrated in figure 9.2. Next, we define a timelike vector field ξ in a neighborhood of Σ with the properties that ξ approaches ζ_A at the AdS boundary and behaves near the extremal surface $\tilde{A}$ like a Killing vector associated with the local Rindler horizon at $\tilde{A}$. The timelike vector field ξ represents a particular choice of time on the surface Σ and we can define an energy H_ξ associated with this. While generally there are many choices for the surface Σ and the vector field ξ, we can show that all of them lead to the same value for the energy H_ξ. It is this quantity that corresponds to the CFT relative entropy $S(\rho_A||\rho_A^0)$.[23]

The independence of H_ξ on the surface Σ used to define it can be understood as a bulk conservation law for this notion of energy. In the case of a ball-shaped

[23]In this work, we focus on the leading contribution to the CFT relative entropy at large N and make use of the classical HRRT formula. More generally, we expect that the bulk quantity will be corrected by a term $-\Delta S_\Sigma$ measuring the vacuum-subtracted bulk entanglement of the region Σ.

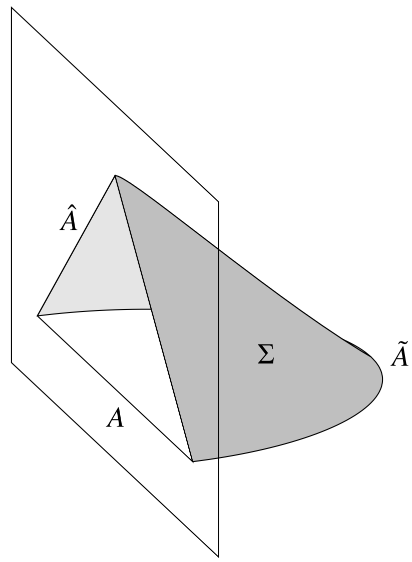


Figure 9.2: CFT relative entropy associated with boundary region A corresponds to a certain energy associated with a gravitational subsystem defined by the domain of dependence of any spatial region Σ bounded by cone region $\hat{A}$ with $\partial\hat{A} = \partial A$ and extremal surface $\tilde{A}$ with $\partial\tilde{A} = \partial A$.

region [39], the energy H_ξ is conserved in a stronger sense (or a bigger volume), since there we are also free to vary the boundary surface $\hat{A}$ to be any spatial surface A' homologous to A in the domain of dependence $\mathcal{D}_A$ of A. In that case, the vector field ζ_A can be defined everywhere in $\mathcal{D}_A$ such that the expression (9.1) for the modular Hamiltonian gives the same result for any surface A'. The bulk vector field ξ can be defined on the full entanglement wedge for A, i.e., the union of spacelike surfaces ending on $\tilde{A}$ and on any A' in $\mathcal{D}_A$, so we can think of the energy H_ξ as being associated with the entire entanglement wedge. In the more general case considered here, the collection of allowed surfaces Σ generally still define a codimension zero region W_A of the bulk spacetime (equivalent to the bulk domain of dependence of any particular Σ), but this region intersects the boundary only on the lightlike surface $\hat{A}$ rather than the whole domain of dependence region $\mathcal{D}_A$.

In section 9.4, we consider the limit where the geometry is a small deformation away from pure AdS. For pure AdS, we show that the extremal surface $\tilde{A}$ associated with a region A whose boundary lies on the lightcone of p always lies on the bulk lightcone of p. Thus, in a limit where perturbations to AdS become small, the wedge W_A collapses to the portion $\hat{A}_{\text{bulk}}$ of this lightcone between p and $\tilde{A}$. We present an analytic expression for the extremal surface $\tilde{A}$ and a canonical choice for the vector field ξ on $\hat{A}_{\text{bulk}}$. In terms of these, we can write an explicit expression for the leading perturbative contribution to the energy H_ξ, which takes the form of an integral over $\hat{A}_{\text{bulk}}$ quadratic in the bulk field perturbations.

In section 9.5, we point out that the explicit form of the extremal surface $\tilde{A}$ in the pure AdS case (in particular, the fact that it lies on the bulk lightcone) leads immediately to a holographic proof of the Markov property for subregions of a CFT in its vacuum state, namely that for two regions A and B the strong subadditivity inequality

$$S(A) + S(B) - S(A \cap B) - S(A \cup B) \geq 0, \tag{9.2}$$

is saturated if their boundaries lie on the past or future lightcone of the same point p. This was shown for general CFTs in [118], so it had to hold in this holographic case. The holographic proof extends easily to cases where the field theory is Lorentz-invariant but non-conformal, for example a CFT deformed by a relevant perturbation. In this case, the statement holds for subregions A, B whose boundaries lie on a null-plane.

We conclude in section 9.6 with a discussion of some possible future directions.

9.2 Background

9.2.1 Relative entropy in conformal field theories

Recall from section 2.3 that we can rewrite the expression for relative entropy as [116]

$$S(\rho\|\sigma) = \Delta\langle H_\sigma\rangle - \Delta S \tag{9.3}$$

where Δ indicates a quantity calculated in the state ρ minus the same quantity calculated in the reference state σ.

For a conformal field theory in the vacuum state, the modular Hamiltonian of a ball-shaped region takes a simple form [110]. For a ball B of radius R centered at x_0 in the spatial slice perpendicular to the unit timelike vector u^μ, the modular

Hamiltonian is

$$H_B = \int_{B'} \zeta_B^\mu T_{\mu\nu} \epsilon^\nu , \tag{9.4}$$

where $\epsilon_\nu = \frac{1}{(d-1)!} \epsilon_{\nu \mu_1 \cdots \mu_{d-1}} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{d-1}}$ is a volume form and ζ_B is the conformal Killing vector

$$\zeta_B^\mu = \frac{\pi}{R} \left\{ [R^2 - (x - x_0)^2] u^\mu + [2u_\nu (x - x_0)^\nu](x - x_0)^\mu \right\} , \tag{9.5}$$

with some four-velocity u^μ. The modular Hamiltonian is the same for any surface B' with the same domain of dependence as B.

Using the expression (9.4) in (9.3), the relative entropy for a state ρ compared with the vacuum state may be expressed entirely in terms of the entanglement entropy and the stress tensor expectation value. For a holographic theory in a state with a classical gravity dual, these quantities can be translated into gravitational language using the HRRT formula (which also implies the usual holographic relation between the CFT stress-energy tensor expectation value and the asymptotic bulk metric [119]). Thus, the CFT relative entropy for a ball-shaped region corresponds to some geometrical quantity in the gravitational theory with positivity and monotonicity properties. In [117] and [39], this quantity was shown to have the interpretation of an energy associated with the gravitational subsystem associated with the interior of the entanglement wedge associated with the ball.

Recently, Casini, Testé, and Torroba have provided an explicit expression for the vacuum modular Hamiltonian of any spatial region A whose boundary lies on the lightcone of a point [118]. To describe this, consider the case where ∂A lies on the past lightcone of a point p and let $\hat{A}$ be the region on the lightcone that forms the future boundary of the domain of dependence of A. For $x \in \hat{A}$, define a function $f(x)$ that represents what fraction of the way x is along the lightlike geodesic from p through x to ∂A (so that $f(p) = 0$ and $f(x) = 1$ for $x \in \partial A$). Now, define a lightlike vector field on $\hat{A}$ by

$$\zeta_A^\mu(x) \equiv 2\pi (f(x) - 1)(p^\mu - x^\mu) . \tag{9.6}$$

Then the modular Hamiltonian can be expressed as

$$H_A = \int_{\hat{A}} \zeta_A^{\mu} T_{\mu\nu} \epsilon^{\nu} \ . \tag{9.7}$$

In general, we cannot extend the vector field away from the surface $\hat{A}$ such that the expression (9.7) remains valid when integrated over an arbitrary surface A' with $\partial A' = \partial A$. In equation (9.38) we give an explicit expression for H_A in a convenient coordinate frame.

Using this expression in (9.3), we can express the relative entropy for the region A in a form that can be translated to a geometrical quantity using the HRRT formula. We would again like to understand the gravitational interpretation for this positive quantity.

9.2.2 Gravity background

We now focus on states in a holographic CFT dual to some asymptotically AdS spacetime with a good classical description. For any spatial subsystem A of the CFT, there is a corresponding region on the boundary of the dual spacetime (which we will also call A). The HRRT formula asserts that the CFT entanglement entropy for the spatial subsystem A in a state $|\Psi\rangle$, at leading order in the $1/N$ expansion, is equal to $1/(4G_N)$ times the area of the minimal area extremal surface $\tilde{A}$ in the dual spacetime which is homologous to the region A on the boundary.

For pure AdS, when the CFT region is a ball B, the spatial region Σ between B and $\tilde{B}$ forms a natural subsystem of the gravitational system, in that there exists a timelike Killing vector ξ_B defined on the domain of dependence $\mathcal{D}_\Sigma$ of Σ and vanishing on $\tilde{B}$. At the boundary of AdS, this reduces to the vector ζ_B appearing in the modular Hamiltonian (9.4) for B. The vector ξ_B gives a notion of time evolution which is confined to $\mathcal{D}_\Sigma$. From the CFT point of view, this time evolution corresponds to evolution by the modular Hamiltonian (9.4) within the domain of dependence of B, which by a conformal transformation can be mapped to hyperbolic space times time.

For states which are small perturbations to the CFT vacuum state, it was shown in [117] that the relative entropy for a ball B at second order in perturbations to the vacuum state corresponds to the perturbative bulk energy associated with the timelike Killing vector ξ_B in $\mathcal{D}_\Sigma$ (known as the *canonical energy* associated with this vector).

This result was extended to general states in [39]. While there are no Killing vectors for general asymptotically AdS geometries, it is always possible to define a vector field ξ_B that behaves near the AdS boundary and near the extremal surface in a similar way to the behavior of the Killing vector ξ_B in pure AdS. Specifically, we impose conditions

$$\begin{aligned} \xi^a|_B &= \zeta^a_B, && (9.8)\\ \nabla^{[a}\xi^{b]}|_{\tilde{B}} &= 2\pi n^{ab}, && (9.9)\\ \xi|_{\tilde{B}} &= 0\,, && (9.10) \end{aligned}$$

where n^{ab} is the binormal to the codimension two extremal surface $\tilde{B}$. Given any such vector field, we can define a diffeomorphism

$$g \to g + \mathcal{L}_\xi g\ . \tag{9.11}$$

This represents a symmetry of the gravitational theory, so we can define a corresponding conserved current and Noether charge. The resulting charge H_ξ turns out to be the same for any vector field satisfying the conditions (9.8) – (9.10). It can be interpreted as an energy associated to the vector field ξ_B or alternatively as the Hamiltonian that generates the flow (9.11) in the phase space formulation of gravity. The main result of [39] is that the CFT relative entropy for a state $|\Psi\rangle$ comparing the reduced density matrix ρ_B to its vacuum counterpart $\rho_B^{(\mathrm{vac})}$ is equal to the difference of this gravitational energy between the spacetime M_ψ dual to $|\Psi\rangle$ and pure AdS,

$$S(\rho_B||\rho_B^{(\mathrm{vac})}) = H_\xi(M_\psi) - H_\xi(\mathrm{AdS})\ . \tag{9.12}$$

We will review the derivation of this identity in the next section when we generalize it to our case.

To write H_ξ explicitly, we start with the Noether current (expressed as a d-form)

$$J_\xi = \theta(\mathcal{L}_\xi g) - \xi \cdot L\ , \tag{9.13}$$

where L is the Lagrangian density and θ is defined by

$$\delta L(g) = d\theta(\delta g) + E(g)\delta g \ . \tag{9.14}$$

Here, $E(g)$ are the equations of motion obtained in the usual way by varying the action. The Noether current is conserved off-shell for Killing vector fields and on-shell for any vector field ξ,

$$dJ_\xi = E(g) \cdot \mathcal{L}_\xi g. \tag{9.15}$$

Then, up to a boundary term, the energy H_ξ is defined in the usual way as the integral of the Noether charge over a spatial surface:

$$H_\xi = \int_\Sigma J_\xi - \int_{\partial\Sigma} \xi \cdot K. \tag{9.16}$$

Here, Σ is any spacelike surface bounded by the HRRT surface $\tilde{B}$ and by a spacelike surface $\Sigma_{\partial M}$ on the AdS boundary with the same domain of dependence as B. For a ball-shaped region B, the quantity H_ξ is independent of both the bulk surface Σ (as a consequence of diffeomorphism invariance) and also the spacelike surface $\Sigma_{\partial M}$ at the boundary of AdS (as a consequence of the fact that ζ_B defines an asymptotic symmetry).

The quantity K in the boundary term is defined so that

$$\delta(\xi \cdot K) = \xi \cdot \theta(\delta g) \quad \text{on } \partial\Sigma \ . \tag{9.17}$$

As explained in [39], this ensures that the difference (9.12) does not depend on the regularization procedure used to calculated the energies and perform the subtraction.

We can rewrite H_ξ completely as a boundary term using the fact that on-shell, J_ξ can be expressed as an exact form [39]

$$J_\xi = dQ_\xi \ . \tag{9.18}$$

Thus, for a background satisfying the gravitational equations, we have

$$H_\xi = \int_{\partial\Sigma} Q_\xi - \int_{\partial\Sigma} \xi \cdot K \ . \tag{9.19}$$

This shows that the definition of H_ξ is independent of the details of the vector field ξ in the interior of Σ. In our derivations below, it will be useful to have a differential version of this expression that gives the change in H_ξ under on-shell variation of the metric. By combining (9.19) with (9.17), we obtain

$$\delta H_\xi = \int_{\partial\Sigma} (\delta Q_\xi - \xi \cdot \theta) \tag{9.20}$$

The interpretation of H_ξ as a Hamiltonian for the phase space transformation associated with (9.11) can be understood by recalling that the symplectic form on this phase space is defined by

$$\Omega(\delta g_1, \delta g_2) = \int_\Sigma \omega(g, \delta g_1, \delta g_2) \tag{9.21}$$

where the d-form ω is defined in terms of θ as

$$\omega(g, \delta_1 g, \delta_2 g) = \delta_1 \theta(g, \delta_2 g) - \delta_2 \theta(g, \delta_1 g) \ . \tag{9.22}$$

In terms of ω we have that for an arbitrary on-shell metric perturbation

$$\delta H_\xi = \Omega(\delta g, \mathcal{L}_\xi g) = \int_\Sigma \omega(g, \delta g, \mathcal{L}_\xi g). \tag{9.23}$$

This amounts to the usual relation $dH = v_H \cdot \Omega$ between a Hamiltonian (in this case H_ξ) and its corresponding vector field (in this case $\mathcal{L}_\xi g$) via the symplectic form Ω.

9.3 Bulk interpretation of relative entropy for general regions bounded on a lightcone

Consider now a more general spacelike CFT subsystem A whose boundary lies on some lightcone. In this case – unless the boundary is a sphere – there is no longer a conformal Killing vector defined on the domain of dependence region $\mathcal{D}_A$ and we cannot write the boundary modular Hamiltonian as in (9.4) where the result is independent of the surface $\hat{B}$. Nevertheless, we have a similar expression (9.7) for the modular Hamiltonian as a weighted integral of the CFT stress tensor over the lightcone region $\hat{A}$ (shown in figure 9.1). Thus, making use of the formula (9.3) for relative entropy, together with the holographic entanglement entropy formula and the holographic dictionary for the stress-energy tensor, we can translate the CFT relative entropy to a gravitational quantity. In this section, we show that this can again be interpreted as an energy difference,

$$S(\rho_A||\rho_A^{\text{vac}}) = H_\xi(M_\psi) - H_\xi(\text{AdS}) \tag{9.24}$$

for an energy H_ξ associated with a bulk spatial region Σ bounded by $\hat{A}$ and the bulk extremal surface $\tilde{A}$.

To begin, we choose a bulk vector field ξ satisfying

$$\xi^a|_{\hat{A}} = \zeta_A^a, \tag{9.25}$$

$$\nabla^{[a}\xi^{b]}|_{\tilde{A}} = 2\pi n^{ab}, \tag{9.26}$$

$$\xi|_{\tilde{A}} = 0. \tag{9.27}$$

The argument that the latter two conditions can be satisfied is the same as in [39], making use of the fact that we can define Gaussian null coordinates near the surface $\tilde{A}$. To enforce the first condition, we will make use of Fefferman-Graham (FG) coordinates for which the near-boundary metric takes the form

$$ds^2 = \frac{1}{z^2}(dz^2 + dx_\mu dx^\mu + z^d \Gamma^{(d)}_{\mu\nu} dx^\mu dx^\nu + \mathcal{O}(z^{d+1})) \tag{9.28}$$

and choose a vector field expressed in these coordinates as

$$\begin{aligned} \xi^\mu &= \zeta_A^\mu + z\xi_1^\mu + z^2\xi_2^\mu + \dots \\ \xi^z &= z\xi_1^z + z^2\xi_2^z + \dots \, . \end{aligned} \tag{9.29}$$

We will now evaluate δH_ξ for this vector field starting from (9.20) and find that it matches with a holographic expression for the change in relative entropy. First, we evaluate the part at the AdS boundary. Explicit calculations in the FG gauge, which are done in appendix D.1, show that

$$\delta Q_\xi - \xi\cdot\theta|_{z\to 0} = \frac{d}{16\pi G_N}\xi^a\delta\Gamma^{(d)}_{ab}\hat{\epsilon}^b|_{z\to 0} = \frac{d}{16\pi G_N}\zeta^\mu_{\hat{A}}\delta\Gamma^{(d)}_{\mu\nu}\epsilon^\mu\,, \tag{9.30}$$

where ϵ was defined in the previous section and

$$\hat{\epsilon}_{a_1\dots a_k} = \frac{\sqrt{-g}}{(d+1-k)!}\epsilon_{a_1\dots a_k b_1\cdots b_{d+1-k}}dx^{b_1}\wedge\cdots\wedge dx^{b_{d+1-k}}. \tag{9.31}$$

Using the standard holographic relation between the asymptotic metric and the CFT stress tensor expectation value, we obtain

$$\int_{\hat{A}}(\delta Q_\xi - \xi\cdot\theta) = \frac{d}{16\pi G_N}\int_{\hat{A}}\zeta^\mu_{\hat{A}}\delta\Gamma^{(d)}_{\mu\nu}\epsilon^\mu = \int_{\hat{A}}\zeta^\mu_{\hat{A}}\delta\left\langle T_{\mu\nu}\right\rangle\epsilon^\mu = \delta\left\langle H_{\hat{A}}\right\rangle\,. \tag{9.32}$$

Here, $H_{\hat{A}}$ is the boundary modular Hamiltonian for the region A, so this term represents the variation in the modular Hamiltonian term in the expression (9.3) for relative entropy.

Next, we look at the part of (9.20) coming from the other boundary of Σ, at the extremal surface. By condition (9.27) we have that ξ vanishes on $\tilde{A}$ and we are left with the integral over δQ_ξ. Q_ξ can be brought into the form $\frac{1}{16\pi}\nabla^a\xi^b\hat{\epsilon}_{ab}$ [120] and by virtue of (9.26) we obtain the entanglement entropy using the HRRT conjecture,

$$\int_{\tilde{A}}\delta Q_\xi = \frac{1}{4G_N}\int_{\tilde{A}} = \delta S. \tag{9.33}$$

Combining both contributions to (9.20), we have that

$$\delta H_\xi = \delta\left\langle H_{\hat{A}}\right\rangle - \delta S, \tag{9.34}$$

where the variation corresponds to an infinitesimal variation of the CFT state. Integrating this from the CFT vacuum state up to the state $|\psi\rangle$, we have that

$$S(\rho_A||\rho_A^{\text{vac}}) = \Delta\left\langle H_{\hat{A}}\right\rangle - \Delta S = \Delta H_\xi. \tag{9.35}$$

Thus, we have established that for a boundary region A with ∂A on a lightcone, the

CFT relative entropy is interpreted in the dual gravity theory as an energy associated with the timelike vector field ξ.

The energy H_ξ is naturally associated with a certain spacetime region of the bulk, foliated by spatial surfaces bounded by the boundary lightcone region $\hat{A}$ and the bulk extremal surface $\tilde{A}$. That such spatial surfaces exist is a consequence of the fact that the extremal surface $\tilde{A}$ always lies outside the causal wedge of the region A (the intersection of the causal past and the causal future of the domain of dependence of A) [121].

9.4 Perturbative expansion of the holographic dual to relative entropy

In this section, we consider the expression for H_ξ in the case where the CFT state is a small perturbation of the vacuum state so that the density matrix can be written perturbatively as $\rho_A = \rho_A^{\text{vac}} + \lambda\rho_1 + \lambda^2\rho_2 + \ldots$. In this case, the CFT state will be dual to a spacetime with metric $g_{\mu\nu}(\lambda) = g^{(0)}_{\mu\nu} + \lambda g^{(1)}_{\mu\nu} + \lambda^2 g^{(2)}_{\mu\nu} + \ldots$.

We recall that relative entropy vanishes up to second order in perturbations; making use of the expression (9.23), we will check that the gravitational expression for relative entropy also vanishes up to second order for general regions A bounded on a light cone. We then further make use of (9.23) to derive a gravitational expression dual to the first non-vanishing contribution to relative entropy, expressing it as a quadratic form in the first order metric perturbation.

9.4.1 Light cone coordinates for AdS

It will be convenient to introduce coordinates for AdS_{d+1} tailored to the light cone on which the boundary of A lies. Starting from standard Poincaré coordinates with metric

$$ds^2 = \frac{1}{z^2}\left(dz^2 - dt^2 + d\vec{x}^2\right), \tag{9.36}$$

we assume that the point p whose light cone contains ∂A is at $\vec{x} = z = 0$ and $t = \rho_0^+$, where ρ_0^+ is an arbitrary constant. On the AdS boundary, we introduce polar coordinates $(t, \rho, \Omega) = (t, \rho, \phi^1, \ldots, \phi^{d-2})$ centered at $\vec{x} = 0$ and define $\rho^\pm = t \pm \rho$.

The surface ∂A is then described by $\rho^+ = \rho_0^+$ and some function $\rho^- = \Lambda(\phi^i)$.

With these coordinates, the vector field (9.6) defining the boundary modular flow takes the form

$$\xi|_{\hat{A}} = \frac{2\pi(\rho_0^+ - \rho^-)(\rho^- - \Lambda(\phi^i))}{\rho_0^+ - \Lambda(\phi^i)}\partial_-. \tag{9.37}$$

and the modular Hamiltonian (9.7) may be written explicitly as

$$H_A = 4\pi \iint_{\Lambda(\phi^i)}^{\rho_0^+} d\rho^- d\Omega \left(\frac{\rho_0^+ - \rho^-}{2}\right)^{d-1} \left[\frac{\rho^- - \Lambda(\phi^i)}{\rho_0^+ - \Lambda(\phi^i)}\right] T_{--}. \tag{9.38}$$

For the choice $\Lambda(\phi^i) = -\rho_0^+$ the region A is a ball of radius ρ_0^+ centered at the origin on the $t = 0$ slice and the expression reduces to the usual expression for a modular Hamiltonian of such a ball-shaped region.

In the bulk, we similarly define polar coordinates $(t, r, \theta, \phi^1, \ldots, \phi^{d-2})$ where $(\rho, z) = r(\cos\theta, \sin\theta)$ and define $r^\pm \equiv t \pm r$ so that the bulk light cone of the point p is $r^+ = \rho_0^+$. We will see below that for pure AdS, the extremal surface $\tilde{A}$ lies on this bulk light cone on a surface that we will parameterize as $r^- = \Lambda(\theta, \phi^i)$, where $\Lambda(\theta = 0, \phi^i)$ is the function that parameterized the surface ∂A.

The AdS$_{d+1}$ line element in these coordinates reads

$$ds^2 = \frac{1}{\sin^2\theta}\left(-\frac{4dr^+dr^-}{(r^+ - r^-)^2} + d\theta^2 + \cos^2\theta g_{ij}^\Omega d\phi^i d\phi^j\right), \tag{9.39}$$

where g_{ij}^Ω is the metric on the unit $d-2$ sphere and only depends on ϕ^i.

9.4.2 HRRT surface in pure AdS

In this section, we derive an analytic expression for the extremal surface $\tilde{A}$ in pure AdS whose boundary is the region ∂A on the lightcone of p. This will be useful in giving more explicit expressions for the relative entropy at leading order in perturbations.

We choose static gauge, parameterizing the surface using the spacetime coordinates θ and ϕ^i and describing its profile in the other directions by $\rho^\pm(\theta, \phi^i)$. The equations which determine its location are

$$\gamma^{ab}\frac{\partial\gamma_{ab}}{\partial r^{\pm}} = -\frac{1}{\sqrt{\gamma}}\partial_a\left(\frac{8\sqrt{\gamma}\gamma^{ab}}{\sin^2\theta(r^+-r^-)^2}\partial_b r^{\mp}\right). \tag{9.40}$$

Let us make the ansatz that even away from the boundary the extremal surface lives on the lightcone $r^+ = \rho_0^+$ and $r^- = \Lambda(\theta,\phi^i)$. The induced metric of this codimension two surface is

$$\gamma_{ab} = \frac{1}{\sin^2\theta}\left(\delta_a^\theta\delta_b^\theta + \cos^2\theta g_{ij}^\Omega\delta_a^i\delta_b^i\right), \tag{9.41}$$

where $a, b \in \{\theta, \phi^1, ..., \phi^{d-2}\}$ and $i, j \in \{\phi^1, ..., \phi^{d-2}\}$. This metric is independent of $r^\pm$; we will see in section 9.5 that this is related to the Markov property of CFT subregions with boundary on a lightcone.

Since the induced metric is independent of $r^\pm$, the left hand side of the equations of motion (9.40) vanishes and we can see from the right hand side that the ansatz $r^+ = \rho_0^+$ solves the equations. The remaining equation for $f(\theta,\phi^i) \equiv \rho_0^+ - \Lambda(\theta,\phi^i)$ reads

$$0 = \partial_a\left(\frac{\sqrt{\gamma}\gamma^{ab}}{\sin^2\theta}\partial_b\frac{1}{f(\theta,\phi^i)}\right). \tag{9.42}$$

The solution which corresponds to ball-shaped entangling surfaces is well known to be located at $\rho^2 + z^2 = \text{const}$. In order to obtain the solution for entangling surfaces of arbitrary shape (but still on a lightcone) we substitute the expression for the induced metric and separate the equation for r^- into

$$\cos^3\theta\tan^{d-1}\theta\partial_\theta\left(\frac{1}{\cos\theta\tan^{d-1}\theta}\partial_\theta\frac{1}{f(\theta,\phi^i)}\right) = -\frac{1}{\sqrt{g^\Omega}}\partial_i\left(\sqrt{g^\Omega}(g^\Omega)^{ij}\partial_j\frac{1}{f(\theta,\phi^i)}\right). \tag{9.43}$$

Here, we followed our conventions and used indices i, j for the angular coordinates ϕ^i. If we write $\frac{1}{f} = h(\theta)\Phi(\phi^i)$ we find that the left hand side can be solved if $\Phi(\phi^i)$ is a spherical harmonic. In $d-2$ dimensions, the eigenvalues of the Laplacian on S^{d-2} are given by $n(3-d-n)$ for the n-th harmonic. Every level n has a corresponding set of degenerate eigenfunctions Φ_n^l with $l = 1, \ldots, \frac{2n+d-3}{n}\binom{n+d-4}{n-1}$ [122]. The left hand side reads

$$\cos^2\theta h''(\theta) - \cot\theta(\cos^2\theta + (d-2))h'(\theta) + n(3-d-n)h(\theta) = 0. \tag{9.44}$$

This differential equation can be solved in terms of hypergeometric functions,

$$\begin{aligned} h(\theta) =& c_1 \cos^{3-d-n}\theta\, {}_2F_1\left(\frac{2-d-n}{2}, \frac{3-d-n}{2}; \frac{5-d-2n}{2}; \cos^2\theta\right) \\ &+ c_2 \cos^n\theta\, {}_2F_1\left(\frac{n-1}{2}, \frac{n}{2}; \frac{d-1+2n}{2}; \cos^2\theta\right). \end{aligned} \tag{9.45}$$

To fix the constants in (9.45) it helps to use intuition from the solutions in the case where the boundary of a subregion is located on a null-plane instead of a lightcone (see appendix B). In that case it is clear that effects from perturbations away from a constant entangling surface on the extremal surface die off as $z \to \infty$. Under a transformation which maps the Rindler result to a ball-shaped region, the distant part of the extremal surface corresponds to $\theta = \pi/2$. Consequently, we require that $h_n(\pi/2) \to 0$ for $n \geq 1$ and $h_n(\pi/2) = 1$ for $n = 0$. At the same time, for $\theta \to 0$ we need that $h_n(\theta)$ is constant and different from zero. These constraints are easily solved with $c_1 = 0, c_2 = 1$. Introducing a normalization factor to ensure that $h_n(0) = 1$, we are left with

$$h_n(\theta) = \cos^n\theta \frac{\Gamma(\frac{d+n}{2})\Gamma(\frac{d-1+n}{2})}{\Gamma(\frac{d-1}{2}+n)\Gamma(\frac{d}{2})}\, {}_2F_1\left(\frac{n-1}{2}, \frac{n}{2}; \frac{d-1+2n}{2}; \cos^2\theta\right). \tag{9.46}$$

In conclusion this shows that extremal surfaces in the bulk are located at $r^+ = \rho_0^+$ and $r^- = \Lambda(\theta, \phi^i)$ with

$$\Lambda(\theta, \phi^i) = \rho_0^+ - \frac{1}{C_0 + \sum_{n=1}^{\infty}\sum_l C_{n,l} h_n(\theta)\Phi_n^l(\phi^i)}. \tag{9.47}$$

Here, n runs over spherical harmonics in $d-2$ dimensions and l over their respective degeneracy. They intersect the boundary at

$$\Lambda(\phi^i) = \rho_0^+ - \frac{1}{C_0 + \sum_{n=1}^{\infty} \sum_l C_{n,l} \Phi_n^l(\phi^i)}. \tag{9.48}$$

Thus, the constants $C_{n,l}$ are determined in terms of the function parameterizing the boundary surface by performing the spherical harmonic expansion

$$\frac{1}{\rho_0^+ - \Lambda(\phi^i)} = C_0 + \sum_{n=1}^{\infty} \sum_l C_{n,l} \Phi_n^l(\phi^i) \,. \tag{9.49}$$

As a simple example, one choice of surface involving only the $n = 1$ harmonics for the AdS_4 case takes the form

$$\rho^+(\phi) = \rho_0^+, \qquad \qquad \rho^-(\phi) = \rho_0^+ - \frac{2\rho_0^+ \sqrt{1-\beta^2}}{1 + \beta \cos\phi}, \tag{9.50}$$

and correspond to ball-shaped regions in a reference frame boosted relative to the original one by velocity β in the x-direction.

9.4.3 The bulk vector field

Our next step is to provide an explicit expression for the vector field on the extremal surface which obeys equations (9.25) – (9.27), such that the quantity H_ξ is dual to relative entropy.

Using (9.37), the explicit form of equation (9.25) is

$$\xi|_{\hat{A}} = \frac{2\pi(\rho_0^+ - \rho^-)(\rho^- - \Lambda(0, \phi^i))}{\rho_0^+ - \Lambda(0, \phi^i)} \partial_-. \tag{9.51}$$

Equation (9.26) requires knowledge of the unit binormal

$$n^{\mu\nu} = n_2^\mu n_1^\nu - n_1^\mu n_2^\nu, \tag{9.52}$$

but thanks to the knowledge about the expression for the extremal surface which we found in the preceding section it is possible to calculate it explicitly. Here, $n_{1,2}$ denote two orthogonal normal vectors to the RT surface. The calculation is delegated to appendix D.3. The non-zero components of the unit binormal read

$$n^{+-} = g^{+-}, \qquad n^{a-} = -\partial^a \Lambda(\theta, \phi^i), \tag{9.53}$$

where a again runs over coordinates (θ, ϕ^i). One possible choice of a vector field satisfying the boundary conditions given by equations (9.26) is:[24]

$$\begin{aligned} \xi = & \frac{2\pi(\rho_0^+ - r^+)(r^+ - \Lambda(\theta,\phi^i))}{\rho_0^+ - \Lambda(\theta,\phi^i)}\partial_+ + \frac{2\pi(\rho_0^+ - r^-)(r^- - \Lambda(\theta,\phi^i))}{\rho_0^+ - \Lambda(\theta,\phi^i)}\partial_- \\ & + \frac{4\pi(\rho_0^+ - r^+)}{\sin^2\theta}\partial^a\left(\frac{1}{\rho_0^+ - \Lambda(\theta,\phi^i)}\right)\partial_a . \end{aligned} \tag{9.54}$$

Here, the ∂_- and ∂_+ components are chosen to match with the expression for the Killing vector ξ in the case when Λ is constant. On the light cone, only the ∂_- component (along the lightcone) is nonzero, and this has the same qualitative behavior as the vector ζ on the boundary lightcone. It is immediately clear that conditions (9.25) and (9.27) are satisfied. It is also straightforward to verify the condition involving the unit binormal using the fact that for a torsion free connection we have $\nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu = \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$.

Calculating the Lie derivative of the metric with respect to this vector field gives zero on the light cone $r^+ = \rho_0^+$ but not away from the light cone. This is in contrast to the case of a ball-shaped region, where the Lie derivative vanished everywhere inside the entanglement wedge.

9.4.4 Perturbative formulae for ΔH_ξ

To write an explicit perturbative expression for ΔH_ξ, we begin with the on-shell result

$$\delta H_\xi = \int_\Sigma \omega\left(g(\lambda), \frac{d}{d\lambda}g, \mathcal{L}_\xi g(\lambda)\right) . \tag{9.55}$$

[24] Upon expanding the sums in equation (9.54) it looks like the ϕ^i components of the vector field diverge as $\theta \to \frac{\pi}{2}$ and for $d > 3$ as $\phi^i \to 0, \pi$ due to the metric on the S^{d-2} sphere. However, these divergences can be shown to be mere coordinate singularities: From equation (9.46) we see that $\partial_i \Lambda \sim \cos\theta$. Hence the ϕ_i components of the vector field go only as $\cos^{-1}\theta$. This happens as a consequence of the coordinate singularity at $\theta = \pi/2$ in polar coordinates which can be removed by going into Poincaré coordinates $(t, z, \vec{x})$. Similar arguments also hold for singularities due to the S^{d-2} metric. Coordinate independent quantities like the norm of the spatial part of the vector field remain finite as can be seen from inspecting the metric.

Here, the symplectic d-form ω is explicitly given by

$$\omega\left(g(\lambda), \frac{d}{d\lambda}g, \mathcal{L}_\xi g(\lambda)\right) = \frac{1}{16\pi G_N}\hat{\epsilon}_\mu P^{\mu\nu\alpha\beta\sigma\rho}\left(\mathcal{L}_\xi g_{\nu\alpha}\nabla_\beta \frac{d}{d\lambda}g_{\sigma\rho} - \frac{d}{d\lambda}g_{\nu\alpha}\nabla_\beta \mathcal{L}_\xi g_{\sigma\rho}\right), \tag{9.56}$$

where

$$P^{\mu\nu\alpha\beta\sigma\rho} = g^{\mu\sigma}g^{\nu\rho}g^{\alpha\beta} - \frac{1}{2}g^{\mu\beta}g^{\nu\sigma}g^{\rho\alpha} - \frac{1}{2}g^{\mu\nu}g^{\alpha\beta}g^{\sigma\rho} - \frac{1}{2}g^{\nu\alpha}g^{\mu\sigma}g^{\beta\rho} + \frac{1}{2}g^{\nu\alpha}g^{\mu\beta}g^{\sigma\rho}. \tag{9.57}$$

Since both $P^{\mu\nu\alpha\beta\sigma\rho}$ and $\hat{\epsilon}_\mu$ depend on the metric they will have a series expansions in λ when we express the metric as a series. Also in this case we will use sub- or superscripts in parenthesis to indicate the order of the term in λ. Here and in the following we will use ∇_μ to denote covariant derivatives with respect to $g_{\mu\nu}(0) = g^{(0)}_{\mu\nu}$.

It will be convenient for us to choose a gauge for the metric perturbations such that the extremal surface stays at the same coordinate location for any variation of the metric. It was shown in [120] that this is always possible. In this case, we have at first order

$$\Delta H^{(1)}_\xi = \int_\Sigma \omega\left(g(\lambda), \frac{d}{d\lambda}g, \mathcal{L}_\xi g(\lambda)\right) . \tag{9.58}$$

We will see in the next section that this vanishes, in accord with the general vanishing of relative entropy at first order (also known as the first law of entanglement).

At second order, we have

$$\frac{d^2}{d\lambda^2}S(g(\lambda)||g_0)|_{\lambda=0} = \int_\Sigma \frac{d}{d\lambda}\omega\left(g(\lambda), \frac{d}{d\lambda}g, \mathcal{L}_\xi g(\lambda)\right)\Bigg|_{\lambda=0}. \tag{9.59}$$

We will calculate this more explicitly in section (9.4.4).

Vanishing of the first order expression

In this section, we demonstrate that our gravitational expression for the relative entropy vanishes for first order perturbations as required. Expanding the first order expression (9.58) for ω yields

$$\int_\Sigma \omega\left(g(\lambda), \frac{d}{d\lambda}g, \mathcal{L}_\xi g(\lambda)\right)\Bigg|_{\lambda=0} = -\frac{1}{32\pi G_N}\int_\Sigma \epsilon_+^{(0)}\left(g_{(0)}^{+-}\right)^2 g_{--}^{(1)} g_{(0)}^{ab}\partial_+\mathcal{L}_\xi g_{ab}^{(0)}, \tag{9.60}$$

where repeated lower case letters a, b imply summation over angular coordinates (θ, ϕ^i). Using the definition of the Lie derivative

$$g_{(0)}^{ab}\partial_+\mathcal{L}_\xi g_{ab}^{(0)} = 2g_{(0)}^{ab}\partial_+\nabla_a\xi_b, \tag{9.61}$$

and the fact that since g^{ab} is independent of $r^\pm$ all Christoffel symbols of the form $\Gamma^\pm_{ab}$ vanish at leading order, the problem reduces to a problem of only the angular coordinates. We obtain

$$g_{(0)}^{ab}\partial_+\mathcal{L}_\xi g_{ab}^{(0)} = 2\nabla_a\partial_+\xi^a = \frac{2}{\sqrt{\gamma}}\partial_a\left(\sqrt{\gamma}\partial_+\xi^a\right). \tag{9.62}$$

Substituting the general form of ξ^a from equation (9.54) and using that $g_{ab}^{(0)} = \gamma_{ab}^{(0)}$ we end up with

$$\gamma_{(0)}^{ab}\partial_+\mathcal{L}_\xi\gamma_{ab}^{(0)} = -8\pi\frac{1}{\sqrt{\gamma}}\partial_a\left(\frac{\sqrt{\gamma}\gamma_{(0)}^{ab}}{\sin^2\theta}\partial_b\frac{1}{\rho_0^+ - \Lambda}\right). \tag{9.63}$$

$g_{\mu\nu}^{(0)}$ and $g_{\mu\nu}^{(1)}$ are the bulk metric and its perturbation and $\gamma_{ab}^{(0)}, \gamma_{ab}^{(1)}$ are the induced metric and the induced metric perturbation, respectively. This expression is proportional to the equation for an extremal surface, equation (9.42), and therefore vanishes.

If we drop the assumption that the Einstein equations are satisfied, one can show that the first law of entanglement entropy implies that the Einstein equations hold at first order around pure AdS. This was done in [119] where only ball-shaped CFT subregions were considered. Utilizing more general subregions bounded by a lightcone does not yield new (in-)equalities at first order.

Relative entropy at second order

We will now provide a more explicit expression for the leading perturbative contribution to relative entropy, which appears at second order in the perturbations. Starting from (9.59) and using our explicit expression for ω, we obtain four potentially contributing terms,

$$\begin{aligned}\frac{d^2}{d\lambda^2}S(g(\lambda)||g_0)|_{\lambda=0} =&\frac{1}{16\pi G_N}\int_\Sigma \epsilon_+^{(1)}P_{(0)}^{+\nu\alpha\beta\sigma\rho}\left(\mathcal{L}_\xi g_{\nu\alpha}^{(0)}\nabla_\beta g_{\sigma\rho}^{(1)} - g_{\nu\alpha}^{(1)}\nabla_\beta \mathcal{L}_\xi g_{\sigma\rho}^{(0)}\right)\\ &+\frac{1}{16\pi G_N}\int_\Sigma \epsilon^{(0)}P_{(1)}^{+\nu\alpha\beta\sigma\rho}\left(\mathcal{L}_\xi g_{\nu\alpha}^{(0)}\nabla_\beta g_{\sigma\rho}^{(1)} - g_{\nu\alpha}^{(1)}\nabla_\beta \mathcal{L}_\xi g_{\sigma\rho}^{(0)}\right)\\ &+\frac{1}{16\pi G_N}\int_\Sigma \epsilon^{(0)}P_{(0)}^{+\nu\alpha\beta\sigma\rho}\left(\mathcal{L}_\xi g_{\nu\alpha}^{(0)}\nabla_\beta g_{\sigma\rho}^{(2)} - g_{\nu\alpha}^{(2)}\nabla_\beta \mathcal{L}_\xi g_{\sigma\rho}^{(0)}\right)\\ &+\frac{1}{16\pi G_N}\int_\Sigma \epsilon^{(0)}P_{(0)}^{+\nu\alpha\beta\sigma\rho}\left(\mathcal{L}_\xi g_{\nu\alpha}^{(1)}\nabla_\beta g_{\sigma\rho}^{(1)} - g_{\nu\alpha}^{(1)}\nabla_\beta \mathcal{L}_\xi g_{\sigma\rho}^{(1)}\right).\end{aligned} \tag{9.64}$$

The first and third terms vanish because of our first order results of section 9.4.4. The last term is reminiscent of the standard canonical energy associated with the interior of the entanglement wedge, except that ξ is no longer a Killing vector. The non-zero contributions take the form

$$\begin{aligned}\delta^{(2)}H_\xi =& \int_\Sigma \omega\left(g(\lambda), \frac{d}{d\lambda}g, \mathcal{L}_\xi\frac{d}{d\lambda}g\right)\Big|_{\lambda=0}\\ &-\frac{1}{16\pi G_N}\int_\Sigma \epsilon_+^{(0)}\left(g_{(0)}^{+-}\right)^2\left[g_{-c}^{(1)}g_{(0)}^{ca}g_{(0)}^{db}g_{-d}^{(1)} - g_{--}^{(1)}g_{(1)}^{ab}\right]\partial_+\mathcal{L}_\xi g_{ab}^{(0)}.\end{aligned} \tag{9.65}$$

Here, a, b, c, d run over angular coordinates, μ, ν run over all coordinates. Note that although we are calculating relative entropy at second order, the expression only depends on first order metric perturbations. Due to the fact that ξ is no longer a Killing vector field, we appear to have a contribution in addition to the first term which appears for the case of ball-shaped regions.

However, we have not yet imposed the Hollands-Wald gauge condition on the first order metric perturbations, for which the coordinate location of the extremal surface is the same as in the case of pure AdS. We have additional gauge freedom on top of this, and it may be that for a suitable gauge choice, the final term in the expression above can be eliminated. We have checked that this is the case for a planar black hole in AdS_4. We discuss this, as well as the procedure of choosing the Hollands-Wald gauge condition in more detail in appendix D.4.

9.5 Holographic proof of the Markov property of the vacuum state

In [118] it was pointed out that the vacuum states of subregions of a CFT bounded by curves $\rho^- = \Lambda_A$ and $\rho^- = \Lambda_B$ on the lightcone $\rho^+ = \rho_0^+$ saturate strong subadditivity, i.e.,

$$S_A + S_B - S_{A\cap B} - S_{A\cup B} = 0. \tag{9.66}$$

This is also known as the Markov property. Moreover, even for CFTs deformed by relevant perturbations, the reduced density matrices for regions A and B describe Markov states if A and B have their boundary on a null-plane. In its most general form the proof used that the modular Hamiltonians for such regions obey

$$H_A + H_B - H_{A\cap B} - H_{A\cup B} = 0, \tag{9.67}$$

which can be proved using methods of algebraic QFT. In this section we will give a holographic proof of the Markov property which uses the Ryu-Takayanagi proposal for entanglement entropy. We will start with the proof for a subregion of a deformed CFT with boundary on a null-plane and after that also show the property for subregions of CFTs with boundary on a lightcone.

9.5.1 The Markov property for states on the null-plane

The vacuum state of a deformed CFT is dual to a geometry of the form

$$ds^2 = f(z)dz^2 + g(z)(-2dx^+dx^- + dx_\perp^\mu dx_{\perp\mu}). \tag{9.68}$$

An undeformed CFT corresponds to the special case $f(z) = g(z) = \frac{1}{z^2}$. The entanglement entropy of a subregion A can then be calculated using the RT prescription, following the same steps as in section D.2. We assume that the boundary ∂A is described by $x^- = \text{const}$ and $x^+ = x^+(\vec{x}_\perp)$. To describe the corresponding extremal surface we go to static gauge, where z and $x_\perp$ are our coordinates and $x^\pm(z, x_\perp)$ is the embedding. The ansatz $x^- = \text{const}$ and $x^+ = x^+(\vec{x}_\perp, z)$ simplifies the equation to

$$0 = \partial_a(\sqrt{\gamma}\gamma^{ab}\partial_b x^+ g_{+-}). \tag{9.69}$$

The relevant solution to this equation in the case of pure AdS is discussed in appendix D.2 and is given by

$$x^+(z, x_\perp^i) = \frac{2^{\frac{2-d}{2}} k^{d/2}}{\Gamma(d/2)} \int d^{d-2}k a_{k^i} z^{d/2} K_{d/2}(zk) e^{ik^i x^i}. \tag{9.70}$$

Here, $K_{d/2}$ is the modified Bessel function of the second kind and the coefficients a_{k^i} are given in terms of the entangling surface $x^+(0, x_\perp^i)$ as

$$a_k = \int \frac{d^{d-2}x_\perp}{(2\pi)^{d-2}} e^{-ik\cdot x_\perp} x^+(0, x_\perp^i). \tag{9.71}$$

More generally, the induced metric on the extremal surface in the bulk is

$$ds^2 = f(z)dz^2 + g(z)(dx_\perp^\mu dx_{\perp\mu}) \tag{9.72}$$

and independent of the embedding $x^+(\vec{x}_\perp, z)$. Thus, it is clear that the areas of all extremal surfaces ending on $x^- = \text{const}$ are the same, potentially up to terms which depend on how the area of the extremal surface is regularized as we approach the boundary. The standard prescription given by cutting off z at some distance ϵ away from the boundary gives a universal cutoff term for all such extremal surfaces and therefore the entanglement entropies for all regions with boundary on x^- are identical and strong subadditivity is saturated. Our argument is an explicit version of very similar arguments which have been used to show the saturation of the Quantum Null Energy condition [123].[25]

[25] We thank Adam Levine for pointing this out to us.

9.5.2 The Markov property for states on the lightcone

If we consider an arbitrary region on the lightcone we expect the Markov property to hold for undeformed CFTs, since the lightcone is conformally equivalent to the null-plane. The solution for an extremal surface in pure AdS ending on a lightcone at the boundary was already discussed in section 9.4.2. Consider the case where we have two different entangling surfaces given by $\rho^- = \Lambda_A(\phi_i)$ and $\rho^- = \Lambda_B(\phi_i)$. We have seen before that the metric on the extremal surface is in fact r^- independent. However, again the dependence on the entangling surface can enter through regularization of the integral and would show up in the cutoff-dependent term.

In the coordinates of our choice θ, ϕ^i the divergent term in the area comes from the integral over θ. Following the standard way of regulating the surface integral we introduce a cutoff $z = \epsilon$, which translates into cutting off the integral at $\theta = \frac{\epsilon}{r} \approx \frac{\epsilon}{\rho}$. From this is follows that if we choose the canonical way of regulating the entropy, the θ integral runs from $\frac{2\epsilon}{(\rho_0^+ - \Lambda)} \equiv \theta_-$ to $\pi/2$.

The entropy which is proportional to the area term can now be calculated using the explicit form of the induced metric, equation (9.41), and is given by

$$\int \sqrt{\gamma} = \int d\Omega \int_{\theta_-}^{\pi/2} d\theta \frac{\cos^{d-2}\theta}{\sin^{d-1}\theta}. \tag{9.73}$$

The only way the shape of the entangling surface appears is through the cutoff, i.e., the surface area can be expanded as

$$A = \sum_{\alpha=d-2}^{0} c_n \left(\frac{\rho_0^+ - \Lambda(\phi^i)}{2\epsilon} \right)^{\alpha}, \tag{9.74}$$

where the coefficients c_n are the same for all entangling surfaces. In the light of equation (9.73) saturation of strong subadditivity for two regions on a lightcone defined by Λ_A and Λ_B is guaranteed if

$$\int d\Omega \Big((\rho_0^+ - \Lambda_A(\phi^i))^\alpha + (\rho_0^+ - \Lambda_B(\phi^i))^\alpha \\ - \max(\rho_0^+ - \Lambda_A(\phi^i), \rho_0^+ - \Lambda_B(\phi^i))^\alpha - \min(\rho_0^+ - \Lambda_A(\phi^i), \rho_0^+ - \Lambda_B(\phi^i))^\alpha \Big) = 0, \tag{9.75}$$

which is trivially pointwise true. This again shows that strong subadditivity is saturated, or in other words, reduced density matrices for regions on the lightcone describe Markovian states. For more details on the form of the coefficients c_n in the expansion, see [124].

The authors of [118] also speculated about the possibility of introducing a cutoff to regulate the area of extremal surfaces such that the area of the extremal surfaces of subregions on the lightcone are all exactly equal. The previous discussion explicitly shows that choosing to introduce a cutoff $\theta = \epsilon$ instead of $z = \epsilon$ realizes such a regularization procedure in which all entanglement entropies for regions on the lightcone are in fact the same.

9.6 Discussion

The results of this chapter imply that for any classical asymptotically AdS spacetime arising in a consistent theory of quantum gravity, the energy ΔH_ξ must be positive and must not decrease as we increase the size of region A. It would be interesting to understand if it is possible to prove this result directly in general relativity, by requiring that the matter stress-energy tensor satisfy some standard energy condition.

It may be useful to point out that there is a differential quantity whose positivity implies all the other positivity and monotonicity results considered here. If we consider a deformation of the region A by an infinitesimal amount $\epsilon v(\Omega)$, where v is some vector field on ∂A pointing along the lightcone away from p, the change in relative entropy to first order must take the form

$$\delta S(\rho_A || \rho_A^{\text{vac}}) = \epsilon \int \delta\Omega v(\Omega) S_A(\Omega) \tag{9.76}$$

The monotonicity property implies that the quantity $S_A(\Omega)$ must be positive for all A and all Ω.[26] It would be interesting to make use of our results to come up with a more explicit expression for the gravitational analogue of the quantity $S_A(\Omega)$. One approach to providing a GR proof of the subsystem energy theorems would be to prove positivity of this.

The Markov property discussed in section 9.5 suggests that it should be interesting to consider (for general states) the gravitational dual of the combination $S(A) + S(B) - S(A \cup B) - S(A \cap B)$ of entanglement entropies for regions A and B on a lightcone. Since strong subadditivity is saturated for the vacuum state, this gravitational quantity will vanish for pure AdS, but must be positive for any nearby physical asymptotically AdS spacetime according to strong subadditivity. Thus, strong subadditivity for these regions on a light cone will lead to a constraint on gravitational physics that appears even when considering small perturbations away from AdS. For two-dimensional CFTs, this quantity was already considered previously in [34, 37]; the analysis there suggests that this gravitational constraint takes the form of a spatially integrated null-energy condition. See [114] for some additional discussion of gravitational constraints from strong subadditivity.

[26] A special case of this positivity result was utilized in the proof of the averaged null energy condition in [125].

Chapter 10

Conclusions

10.1 Infrared quantum information

Part I of this thesis is concerned with the definition of information theoretic quantities for scattering in four dimensions in the presence of long range forces mediated by photons and gravitons. The presence of long range forces results in infrared divergences in the calculation of scattering amplitudes which need to be dealt with by choosing one of two approaches. In the first, we only ask questions which can also be operationally answered, and restrict our attention to inclusive observables. The construction of inclusive quantities involves summing over all possible states which yield outcomes compatible with our measurements. We have seen in chapter 4 that this treatment results in an essentially complete decoherence of the outgoing density matrix. The condition under which an off-diagonal density matrix element in the momentum basis does not decohere can be phrased in terms of a condition between an infinite number of current operators. The decoherence makes it particularly easy to calculate the entanglement entropy between the hard and soft modes. However, this procedure makes quantum electrodynamics and perturbative quantum gravity inherently non-unitary.

Alternatively we can use so-called dressed formalisms which add a finely tuned set of soft bosons to scattering states. These formalisms do not require a sum over outgoing soft bosons and the S-matrix is formally unitary. Furthermore, they allow one to ask questions about amplitudes and other "unphysical" quantities. Also, in this case we can calculate the entanglement entropy between soft and hard modes. In chapter 5 we found agreement with the calculation in the inclusive formalism.

Chapter 6 discussed an important difference between the two formalisms. In the previous chapters, the calculations were done using incoming and outgoing momentum eigenstates. If we replace momentum eigenstates by wavepackets, the predictions of the inclusive and dressed formalism disagree; the reduced outgoing density matrix in the inclusive formalism becomes trivial. This behavior can be traced back to the fact that all components of the wavepackets after scattering are orthogonal. In the dressed formalism, however, everything works as expected. This suggests that the use of dressed states is not simply an alternative to the inclusive formalism, but – at least in four dimensions – in fact required if one wants to treat questions beyond scattering of momentum eigenstates.

In chapter 7 we tackled two issues. First, the total decoherence found by tracing out soft modes only depends on the fact that S-matrix scattering assumes a limit in which incoming and outgoing states have had an infinite amount of time to interact, so that bosons of infinitely long wavelength can be produced. We thus tried to understand late-but-finite time behavior of decoherence. Second, all dressed state proposals have the issue that they either do not come with a well-defined Hilbert space, their Hilbert space is non-separable, or that their Hilbert space is not a representation of the canonical commutation relations of the soft-boson canonical commutation relations, but instead a set of vectors coming from different, unitarily inequivalent representations. We solve both problems for quantum electrodynamics by showing that, if charged asymptotic states are equipped with the correct electric field and additional radiative dressing, they form states in a single representation of the CCR. Repeating the decoherence calculation with states in such a representation, it was possible to extract the time-dependence of decoherence at late time.

The above results can be used to calculate time dependence of quantum information theoretic quantities such as relative entropy between different photon states. A logical next step would be the extension of the Hilbert space construction in chapter 7 to the case of perturbative quantum gravity. As we have seen in the course of this thesis, many of our results are related to or can naturally be interpreted in the context of asymptotic symmetries related to Weinberg's soft theorems. It would be interesting to investigate the relation between our results and symmetries related to subleading soft theorems. A better understanding of dressed states along those lines will be a crucial contribution to understanding the Hilbert space structure of flat space holography.

10.2 Quantum information and holography

In part II of this thesis we turned to more established applications of quantum information theory in the context of the AdS/CFT correspondence. It was previously shown that relative entropy between the density matrices of the vacuum and some other holographic CFT state, reduced on a ball-shaped region, is dual to a measure of energy of the associated entanglement wedges. In chapter 9 we showed that this statement can be generalized to deformed ball-shaped regions which can be expressed as a cone cut. This measure of energy inherits properties from relative entropy, like monotonicity under inclusion of subregions and positivity. Moreover, we gave an explicit form for the bulk extremal surface as a function of the CFT entangling surface which bounds the deformed ball-shaped region. This was then used to give a holographic proof of the Markov property of the vacuum state on deformed ball-shaped regions.

It would be interesting to better understand how these class of new found positive energy theorems relate to existing energy theorems in gravity, such as the various energy conditions.

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Appendix A

Infrared quantum information

Here, we show that the exponents $\Delta A, \Delta B$ controlling the infrared divergences are always positive or zero, and give necessary and sufficient conditions for these exponents to vanish.

The first step is to notice that the expressions for the differential exponents (4.8) between the processes $\alpha \to \beta$ and $\alpha \to \beta'$ are the same as the exponents (4.6) for the divergences in the process $\beta \to \beta'$, that is

$$\begin{aligned} \Delta A_{\beta\beta',\alpha} &= A_{\beta',\beta}/2, \\ \Delta B_{\beta\beta',\alpha} &= B_{\beta',\beta}/2. \end{aligned} \tag{A.1}$$

To see this, note from the definitions (4.4), (4.6), and (4.8) that there are terms in each of $A_{\beta,\alpha}, A_{\beta',\alpha}$, and $\tilde{A}_{\beta\beta',\alpha}$ coming from contractions between pairs of incoming legs, pairs of an incoming and outgoing leg, and pairs of outgoing legs. One can easily check that the in/in and in/out terms cancel pairwise between the A and $\tilde{A}$ terms in ΔA. The remainder is the terms involving contractions between pairs of outgoing legs:

$$\Delta A_{\beta\beta',\alpha} = \frac{1}{2} \sum_{\mathbf{p},\mathbf{p}' \in \beta} \gamma_{\mathbf{pp}'} + \frac{1}{2} \sum_{\mathbf{p},\mathbf{p}' \in \beta'} \gamma_{\mathbf{pp}'} - \sum_{\mathbf{p} \in \beta, \mathbf{p}' \in \beta} \gamma_{\mathbf{pp}'} \tag{A.2}$$

where we defined $\gamma_{\mathbf{pp}'} = e_{\mathbf{p}} e_{\mathbf{p}'} \beta_{\mathbf{pp}'}^{-1} \log[(1+\beta_{\mathbf{pp}'})/(1-\beta_{\mathbf{pp}'})]$. We have used the fact that every $\eta_{\mathbf{p}}$ that would have been in (A.2) is a -1 since every line being summed is an outgoing particle. But then we have a relative minus sign and factor of 2 between the first two terms and the third; this is precisely the same factor that would have come from the relative $\eta_{\text{in}} = -1$ and $\eta_{\text{out}} = +1$ terms in exponent for the process $\beta \longrightarrow \beta'$, namely

$$A_{\beta',\beta} = \sum_{\mathbf{p},\mathbf{p}' \in \beta} \gamma_{\mathbf{pp}'} + \sum_{\mathbf{p},\mathbf{p}' \in \beta'} \gamma_{\mathbf{pp}'} - 2 \sum_{\mathbf{p} \in \beta, \mathbf{p}' \in \beta'} \gamma_{\mathbf{pp}'}. \tag{A.3}$$

This proves (A.1) for ΔA; an identical combinatorial argument shows that the gravitational exponent obeys the analogous relation, $\Delta B_{\beta\beta',\alpha} = B_{\beta',\beta}/2$.

Now we prove that for the process $\alpha \to \beta +$ (soft) the exponent $A_{\beta,\alpha}$ is always greater or equal to zero with equality if and only if the in and outgoing currents agree; we can then take $\alpha = \beta'$ to get the results quoted in the text. Referring to Weinberg's derivation [56], we can write $A_{\beta,\alpha}$ as

$$A_{\beta,\alpha} = \frac{1}{2(2\pi)^3} \int_{S^2} d\hat{\mathbf{q}}\; t^{\mu}(\hat{\mathbf{q}}) t_{\mu}(\hat{\mathbf{q}}). \tag{A.4}$$

Here,

$$t^{\mu}(\hat{\mathbf{q}}) \equiv \sum_{n} \frac{e_n \eta_n p_n^{\mu}}{p_n \cdot q} = c(q) q^{\mu} + c_i(q)(q_{\perp}^{i})^{\mu}. \tag{A.5}$$

In this equation, we have defined a lightlike vector $q^{\mu} = (1, \hat{\mathbf{q}})$ and $q_{\perp}^{i}$, $i = 1,2$ are two unit normalized, mutually orthogonal, purely spatial vectors perpendicular to q^{μ}. The sum on $n \in \alpha, \beta$ runs over in- and out-going particles. By charge conservation, $t \cdot q = 0$, which justifies the decomposition in the second equality in (A.5). With this decomposition we may write

$$A_{\beta,\alpha} = \frac{1}{2(2\pi)^3} \int_{S^2} d\hat{\mathbf{q}} (c_1^2(q) + c_2^2(q)) \geq 0, \tag{A.6}$$

which immediately proves the statement that $A_{\beta,\alpha} \geq 0$.

Now it remains to be shown that equality holds if and only if all of the in- and out-going currents match. From the previous paragraph we know that $A_{\beta,\alpha}$ vanishes if and only if both $c_i(q) = 0$ for all q, that is if and only if $t \cdot q_{\perp}^{i} = 0$. Assume that $A_{\beta,\alpha} = 0$, so that $q_{\perp} \cdot t(q) = 0$. Now suppose also that $j_{\mathbf{v}_0}(\alpha) \neq j_{\mathbf{v}_0}(\beta)$

for some $\mathbf{v}_0$, where these are the eigenvalues of $j_{\mathbf{v}} |\alpha\rangle = j_{\mathbf{v}}(\alpha) |\alpha\rangle$ and similarly for β. We derive a contradiction. For any finite set of velocities, the functions $f_{\mathbf{v}}(\hat{\mathbf{q}}) = (\mathbf{v} \cdot \mathbf{q}_\perp)/(1 - \mathbf{v} \cdot \hat{\mathbf{q}})$ are linearly independent. Therefore the terms in

$$0 = t \cdot q_\perp = \sum_n \frac{e_n \eta_n v_n \cdot q_\perp}{v_n \cdot q} \tag{A.7}$$

must cancel separately for each velocity in the list of $\mathbf{v}_n$. Consider in particular the term for $\mathbf{v}_0$. For this to vanish, the sum of the coefficients must vanish, i.e.,

$$0 = \sum_{n | v_n = v_0} e_n \eta_n = \left[j_{v_0}(\alpha) - j_{v_0}(\beta) \right], \tag{A.8}$$

the relative minus coming from the η factors. But this contradicts our assumption that $j_{\mathbf{v}_0}(\alpha) \neq j_{\mathbf{v}_0}(\beta)$. This completes the proof for A.

The proof for gravitons goes similarly. Again referring to Weinberg we write B as

$$B_{\beta,\alpha} = \frac{G}{4\pi^2} \int_{S^2} d\hat{\mathbf{q}} t^{\mu\nu} D_{\mu\nu\rho\sigma} t^{\rho\sigma}. \tag{A.9}$$

Here, $D_{\mu\nu\rho\sigma} = \eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho}$ is the numerator of the graviton propagator, and

$$t^{\mu\nu} = \sum_n \frac{\eta_n p_n^\mu p_n^\nu}{p_n \cdot q} = c q^{(\mu} q^{\nu)} + c^i q^{(\mu} q_{\perp,i}^{\nu)} + c^{ij} q_{\perp,i}^{(\mu} q_{\perp,j}^{\nu)}. \tag{A.10}$$

This symmetric tensor obeys $t^{\mu\nu} q_\nu = 0$ by energy-momentum conservation, which justifies the decomposition in the second equality. Using this we have

$$t^{\mu\nu} D_{\mu\nu\rho\sigma} t^{\rho\sigma} = 2 c^i_j c^j_i - \left(c^i_i\right)^2 = (\lambda_1 - \lambda_2)^2 \tag{A.11}$$

with $\lambda_{1,2}$ the two eigenvalues of the matrix c^{ij}. Plugging this into (A.9) we immediately see that $B \geq 0$. The condition for vanishing of $B_{\beta,\beta'}$ is that the eigenvalues are equal $\lambda_1 = \lambda_2$, which means that c^{ij} is proportional to the identity matrix. Hence, if B vanishes we have that

$$0 = t^{\mu\nu} q_\mu^{\perp,1} q_\nu^{\perp,2} = \sum_n \eta_n E_n \frac{(v_n \cdot q_\perp^1)(v_n \cdot q_\perp^2)}{v_n \cdot q}. \tag{A.12}$$

As before, any finite set of functions $g_v(q) = (v \cdot q_\perp^1)(v \cdot q_\perp^2)/(v \cdot q)$ are linearly independent functions of q, and so by direct analogy with the previous proof, $B = 0$ if and only if $j_{\mathbf{v}}^{\mathrm{gr}}(\alpha) = j_{\mathbf{v}}^{\mathrm{gr}}(\beta)$ for every $\mathbf{v}$.

Appendix B

Dressed soft factorization

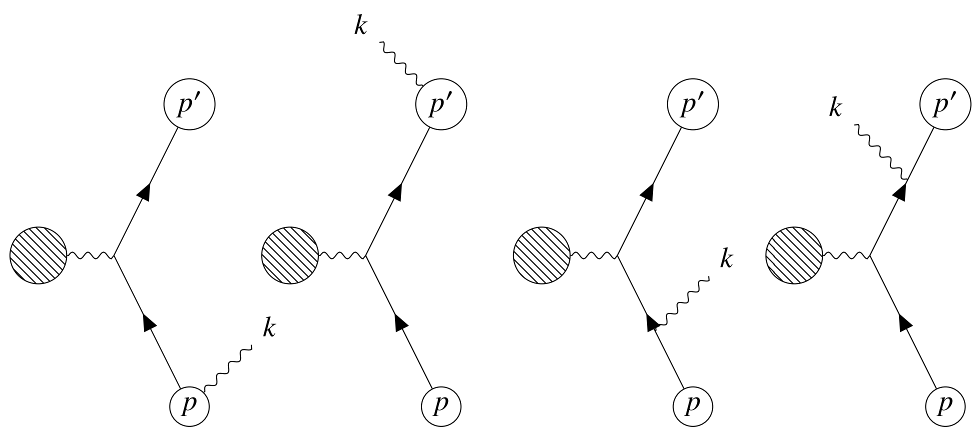

Figure B.1: Diagrams contributing to the dressed scattering with additional bremsstrahlung. The first two diagrams correspond to photons coming from the dressing, while the latter two diagrams correspond to the usual Feynman diagrams where the photon is emitted from the electron lines.

The soft photon theorem looks somewhat different in dressed QED. In standard, undressed QED, the theorem says that the amplitude for a process $\mathbf{p} \to \mathbf{q}$ accompanied by emission of an additional soft photon of momentum $\mathbf{k}$ and polarization ℓ has amplitude

$$S_{\mathbf{qk}\ell,\mathbf{p}} = e\left[\frac{q \cdot \epsilon^*_\ell(\mathbf{k})}{q \cdot k} - \frac{p \cdot \epsilon^*_\ell(\mathbf{k})}{p \cdot k}\right] S_{\mathbf{q},\mathbf{p}}. \tag{B.1}$$

This is singular in the $k \to 0$ limit. On the other hand, in the dressed formalism of QED, the statement is that

$$\tilde{S}_{\mathbf{qk}\ell,\mathbf{p}} = e f(\mathbf{k}) \tilde{S}_{\mathbf{q},\mathbf{p}}, \tag{B.2}$$

where $f(\mathbf{k}) \sim \mathcal{O}(|\mathbf{k}|^0)$, so that the right-hand side is finite as $k \to 0$. We can see this by straightforward computation. In computing equation (B.2), there will be four Feynman diagrams at lowest order in the charge (see figure B.1). We will get the usual pair of Feynman diagrams coming from contractions of the interaction Hamiltonian with the external photon state, leading to the poles, equation (B.1). Moreover we will get a pair of terms coming from contractions of the interaction Hamiltonian with dressing operators. These contribute a factor

$$\left[f_\ell^*(\mathbf{k},\mathbf{p}) - f_\ell^*(\mathbf{k},\mathbf{q})\right] \to \left[\frac{q \cdot \epsilon_\ell^*(\mathbf{k})}{q \cdot k} - \frac{p \cdot \epsilon_\ell^*(\mathbf{k})}{p \cdot k}\right] + \mathcal{O}(|\mathbf{k}|^0), \tag{B.3}$$

times $-e$, where the limit as $k \to 0$ follows from the definition (3.50). This extra contribution precisely cancels the poles in (B.1), leaving only the order $\mathcal{O}(|\mathbf{k}|^0)$ term.

Appendix C

On the need for soft dressing

C.1 Proof of positivity of $\Delta A, \Delta B$

The exponent that is responsible for the decoherence of the system is defined as

$$\Delta A_{\beta\beta',\alpha\alpha'} = \frac{1}{2} A_{\beta,\alpha} + \frac{1}{2} A_{\beta',\alpha'} - \tilde{A}_{\beta\beta',\alpha\alpha'}. \tag{C.1}$$

The factor in the first two terms, $A_{\beta,\alpha}$, is defined as in [56]

$$A_{\beta,\alpha} = \frac{1}{2(2\pi)^3} \int_{S^2} d\hat{\mathbf{q}} \left(\sum_{n\in\beta} \frac{e_n \eta_n p_n^\mu}{p_n \cdot \hat{q}} \right) g_{\mu\nu} \left(\sum_{m\in\alpha} \frac{e_m \eta_m p_m^\mu}{p_m \cdot \hat{q}} \right). \tag{C.2}$$

Performing the integral over $\hat{\mathbf{q}}$ yields

$$A_{\beta,\alpha} = - \sum_{n,n'\in\alpha,\beta} \frac{e_n e_{n'} \eta_n \eta_{n'}}{8\pi^2} \beta_{nn'} \log \left[\frac{1+\beta_{nn'}}{1-\beta_{nn'}} \right]. \tag{C.3}$$

Similarly $\tilde{A}_{\beta\beta',\alpha\alpha'}$ can be written as

$$\tilde{A}_{\beta\beta',\alpha\alpha} = - \sum_{\substack{n\in\alpha,\beta \\ n'\in\alpha'\beta'}} \frac{e_n e_{n'} \eta_n \eta_{n'}}{8\pi^2} \beta_{nn'} \log \left[\frac{1+\beta_{nn'}}{1-\beta_{nn'}} \right]. \tag{C.4}$$

We rearrange the terms such that ΔA can be written as

$$\Delta A_{\beta\beta',\alpha\alpha'} = -\frac{1}{2}\sum_{n,n'\in\alpha,\bar{\alpha}',\beta,\bar{\beta}'}\frac{e_n e_{n'}\eta_n\eta_{n'}}{8\pi^2}\beta_{nn'}^{-1}\log\left[\frac{1+\beta_{nn'}}{1-\beta_{nn'}}\right], \tag{C.5}$$

where a bar means incoming particles are taken to be outgoing and vice versa (or equivalently, $\eta_{\bar{\alpha}'} = -\eta_{\alpha'}$). From equation (C.5), it is clear that incoming particles are found within the set $\{\alpha, \beta'\}$ while the outgoing particles are part of $\{\alpha', \beta\}$. Let us rename those sets σ and σ' respectively. ΔA now takes the form

$$\Delta A_{\beta\beta',\alpha\alpha'} = -\frac{1}{2}\sum_{n,n'\in\sigma,\sigma'}\frac{e_n e_{n'}\eta_n\eta_{n'}}{8\pi^2}\beta_{nn'}^{-1}\log\left[\frac{1+\beta_{nn'}}{1-\beta_{nn'}}\right] = \frac{1}{2}A_{\sigma\sigma'} \geq 0, \tag{C.6}$$

as was proven in [1]. This shows that $\Delta A_{\beta\beta',\alpha\alpha'} \geq 0$. The same proof goes through for $\Delta B_{\beta\beta',\alpha\alpha'}$.

C.2 The out-density matrix of wavepacket scattering

In this part of the appendix we flesh out the argument in section 6.3, namely that after tracing out soft radiation, the only contribution to the out-density matrix is coming from the identity term in the S-matrix. We will focus on the case of QED.

C.2.1 Contributions to the out-density matrix

First, let us decompose the IR regulated S-matrix into its trivial part and the $\mathcal{M}$-matrix element. For simplicity we ignore partially disconnected terms, where only a subset of particles interact. Then,

$$S^{\Lambda}_{\alpha,\beta} = \delta(\alpha-\beta) - 2\pi i\mathcal{M}^{\Lambda}_{\alpha\beta}\delta^{(4)}(p^{\mu}_{\alpha} - p^{\mu}_{\beta}), \tag{C.7}$$

where the first term is the trivial LSZ constribution to forward scattering. This trivial part does not involve any divergent loops and therefore exhibits no Λ-dependence. However, the factorization of the S-matrix into a cutoff dependent term times some power of λ/Λ remains valid since all exponents of the form $A_{\alpha,\beta}$ vanish identically for forward scattering. This decomposition of the S-matrix gives rise to three different terms for the outgoing density matrix, containing different powers of $\mathcal{M}$.

"No scattering"-term

The case where both S-matrices contribute the delta function term results – unsurprisingly – in the well-defined outgoing density matrix

$$\rho^{(\mathrm{I})}_{\beta\beta'} = \int d\alpha d\alpha' f(\alpha) f(\alpha')^* \delta(\alpha - \beta)\delta(\alpha' - \beta')\delta_{\alpha\alpha'} = f(\beta) f^*(\beta'). \tag{C.8}$$

Contribution from forward scattering

We would now expect to find an additional contribution to the density matrix reflecting the non-trivial scattering processes, coming from the cross-terms

$$-2\pi i \left(\delta(\alpha - \beta)\mathcal{M}^{\Lambda}_{\alpha'\beta}\delta^{(4)}(p^{\mu}_{\alpha'} - p^{\mu}_{\beta}) - \delta(\alpha' - \beta)\mathcal{M}^{\dagger\Lambda}_{\alpha\beta}\delta^{(4)}(p^{\mu}_{\alpha} - p^{\mu}_{\beta}) \right). \tag{C.9}$$

For simplicity, let us focus solely on the case in which S^* contributes the delta function and S contributes the connected part

$$\rho^{(\mathrm{II})}_{\beta\beta'} = -2\pi i f^*(\beta') \int d\alpha f(\alpha)\mathcal{M}^{\Lambda}_{\beta\alpha}\delta^{(4)}(p^{\mu}_{\alpha} - p^{\mu}_{\beta})\lambda^{\Delta A_{\alpha,\beta}}\mathcal{G}(E, E_T, \Lambda)_{\beta,\alpha} + \ldots, \tag{C.10}$$

where the ellipsis denotes the contribution coming from the omitted term of (C.9). The exponent of λ only vanishes if the currents in α and β agree. We will show in appendix C.2.2 that we can take the limit $\lambda \to 0$ before doing the integrals. Taking this limit, $\lambda^{\Delta A_{\alpha,\beta}}$ gets replaced by

$$\delta_{\alpha\beta} = \begin{cases} 1, & \text{if charged particles in } \alpha \text{ and } \beta \text{ have the same velocities} \\ 0, & \text{otherwise,} \end{cases} \tag{C.11}$$

which is zero almost everywhere. If the integrand was regular, we could conclude that the integrand is a zero measure subset and integrates to zero and thus

$$\rho^{(\mathrm{II})}_{\beta\beta'} = 0. \tag{C.12}$$

However, the integrand is not well-behaved. Singular behavior can come from the delta function or the matrix element, so let's consider the two possibilities.

The singular nature of the Dirac delta does not affect our conclusion: for n incoming particles, the measure $d\alpha$ runs over $3n$ momentum variables while the delta function constrains 4 of them, leaving us with $3n-4$ independent ones. If we managed to find a configuration for which $\Delta A_{\beta,\alpha} = 0$, any infinitesimal variation of the momenta in α along a direction that conserves energy and momentum would modify the eigenvalue of the current operator $j_{\mathbf{v}}(\alpha) - j_{\mathbf{v}}(\beta)$ and make $\Delta A_{\beta\alpha}$ non-zero. Therefore, the integrand would still be a zero-measure subset for the remaining integrals.

What could still happen is that $\mathcal{M}^{\Lambda}_{\beta\alpha}$ is so singular that it gives a contribution. For this to happen it would need to have contributions in the form of Dirac delta functions. However, also this does not happen, for example for Compton scattering which scatters into a continuum of states. Additional IR divergences also do not appear as guaranteed by the Kinoshita-Lee-Nauenberg theorem. We will not give a general proof since for our purposes it is problematic enough to know that no scattering is observed for *some* physical process.

The scattering term

It is evident that a similar argument goes through for the $\mathcal{M}^2$ term. One finds

$$\rho^{(\mathrm{III})}_{\beta\beta'} = -4\pi^2 \int d\alpha d\alpha' f(\alpha) f^*(\alpha') \mathcal{M}^{\Lambda}_{\beta\alpha} \mathcal{M}^{\Lambda *}_{\alpha'\beta'} \lambda^{\Delta A_{\alpha\alpha',\beta\beta'}} \tag{C.13}$$

$$\times \mathcal{F}(E, E_T, \Lambda)_{\beta\beta',\alpha\alpha'} \delta^{(4)}(p^{\mu}_{\alpha} - p^{\mu}_{\beta}) \delta^{(4)}(p^{\mu}_{\alpha'} - p^{\mu}_{\beta'}). \tag{C.14}$$

The analysis boils down the the question whether the term

$$\int d\alpha d\alpha' \lambda^{\Delta A_{\alpha\alpha',\beta\beta'}} \delta^{(4)}(p^{\mu}_{\alpha} - p^{\mu}_{\beta}) \delta^{(4)}(p^{\mu}_{\alpha'} - p^{\mu}_{\beta'}). \tag{C.15}$$

vanishes. As soon as there is at least one particle with charge, we need to obey the condition that the charged particles in α and β' agree with those in β and α' for the exponent of λ to vanish. Infinitesimal variations of α and α' that preserve the eigenvalue of the current operator $j_{\mathbf{v}}(\alpha) - j_{\mathbf{v}}(\alpha')$ form a zero-measure subset of the $6n-8$ directions that preserve momentum and energy, forcing us to conclude that the integration runs over a zero measure subset and the only contribution to the reduced density matrix comes from the trivial part of the scattering process. This means that

$$\rho^{\text{out, red.}}_{\beta\beta'} = f(\beta) f^*(\beta') = \rho^{\text{in}}_{\beta\beta'}, \tag{C.16}$$

or in other words it predicts that a measurement will not detect scattering for wavepackets. This is clearly in contradiction with reality and suggests that the standard formulation of QED and perturbative quantum gravity which relies on the existence of wavepackets is invalid.

C.2.2 Taking the cutoff $\lambda \to 0$ vs. integration

One might be concerned that the limit $\lambda \to 0$ and the integrals do not commute. In this part of the appendix, we will check the claim made in the preceding subsection, i.e., we will show that one can explicitly check that the integration and taking the IR regulator λ to zero commute. We assume in the following that we talk about QED with electrons and muons in the non-relativistic limit, which again is good enough as it is sufficient to show that we can find a limit in which no sign of scattering exists in the outgoing hard state. The wave packets are chosen to factorize for every particle and to be Gaussians in velocity centered around $v = 0$,

$$f(v) = \left(\frac{2}{\pi\kappa}\right)^{3/4} \exp\left(-\frac{v^2}{\kappa}\right). \tag{C.17}$$

In order to stay in the non-relativistic limit, κ must be sufficiently small. They are normalized such that

$$\int d^3v |f(v)|^2 = 1. \tag{C.18}$$

In the exponent of λ we set $\alpha' = \beta'$ for simplicity, i.e., we consider the case of forward scattering. In the non-relativistic limit, we can expand the exponent of λ

into

$$\Delta A_{\alpha,\beta} = \frac{e^2}{24\pi^2} \sum_{n,m\in\alpha,\beta} (v_\alpha - v_\beta)^2. \tag{C.19}$$

Thus, $\lambda^{\Delta A}$ has the form

$$\lambda^{\Delta A} \propto \exp\left(-\frac{1}{2}\gamma \sum_{n,m\in\alpha,\beta} (v_\alpha - v_\beta)^2\right), \tag{C.20}$$

where taking the cutoff λ to zero corresponds to $\gamma \propto -\log(\lambda) \to \infty$. The state α consists of a muon with well defined momentum and one electron with momentum mv, where v is centered around 0. The state β consists of the same muon (we assume it was not really deflected) and one electron with momentum mv'. To obtain the contribution to forward scattering, we have to perform the integral

$$\propto \int d^3v \left(\frac{2}{\pi\kappa}\right)^{3/4} \exp\left(-\frac{v^2}{\kappa}\right) \exp\left(-\gamma(v - v')^2\right) \cdot \text{(other terms)}. \tag{C.21}$$

Here, we assumed that the other terms which include the matrix element in the regime of interest is finite and approximately independent of v. The integral yields

$$\left(\frac{2\pi\kappa}{(1+\gamma\kappa)^2}\right)^{3/4} \exp\left(-\frac{\gamma v'^2}{1+\gamma\kappa}\right). \tag{C.22}$$

Taking the limit $\gamma \to \infty$, it is clear that this expression vanishes. If we want to consider an outgoing wave packet we have to integrate this over $f(v' - v_{\text{out}})$. The result is proportional to

$$\left(\frac{2\pi\kappa}{(1+2\gamma\kappa)^2}\right)^{3/4} \exp\left(-\frac{\gamma v_{\text{out}}^2}{1+2\gamma\kappa}\right) \tag{C.23}$$

and still vanishes if we remove the cutoff, $\gamma \to \infty$.

Appendix D

Cone Relative Entropies

D.1 Equivalence of H_ξ on the boundary and the modular Hamiltonian

In this appendix we will show that H_ξ reduces to the modular Hamiltonian on the boundary, even in the case of a deformed entangling surface. We take the infinitesimal difference between pure AdS and another spacetime that satisfies the linearized Einstein's equations around pure AdS, i.e., we want to calculate $\delta Q_\xi - \xi \cdot \theta$ on a constant z slice near the boundary. We can find in the appendix of [39] that

$$\delta Q_\xi - \xi\cdot\theta = \frac{1}{16\pi G_N}\hat{\epsilon}_{ab}\left[\delta g^{ac}\nabla_c\xi^b - \frac{1}{2}\delta g^c_c\nabla^a\xi^b + \xi^c\nabla^b\delta g^a_c - \xi^b\nabla_c\delta g^{ca} + \xi^b\nabla^a\delta g^c_c\right]. \tag{D.1}$$

The next step is to expand the sum over a and b. As we approach the boundary we consider volume elements on on constant z slices and thus the term involving the volume element $\hat{\epsilon}_{\mu\nu}$ vanishes. In Fefferman-Graham gauge ($\delta g_{zc} = 0$) we find

$$\begin{aligned}\delta Q_\xi - \xi\cdot\theta = &\frac{1}{16\pi G_N}\hat{\epsilon}_{\mu z}\left[\frac{1}{2}\delta g^\nu_\nu\nabla^z\xi^\mu - \xi^\mu\nabla^z\delta g^\nu_\nu - \xi^c\nabla^\mu\delta g^z_c + \xi^\mu\nabla_c\delta g^{cz}\right]\\ &+ \frac{1}{16\pi G_N}\hat{\epsilon}_{\mu z}\left[\delta g^{\mu\nu}\nabla_\nu\xi^z - \frac{1}{2}\delta g^\nu_\nu\nabla^\mu\xi^z + \xi^\nu\nabla^z\delta g^\mu_\nu - \xi^z\nabla_\nu\delta g^{\nu\mu} + \xi^z\nabla^\mu\delta g^\nu_\nu\right].\end{aligned} \tag{D.2}$$

Now all we need to do is find the leading order behaviour near $z = 0$. To this effect we assume that the vector fields have a asymptotic expansion near the conformal boundary given in equation (9.29).

We also take $\delta g_{ab} = z^{d-2}\Gamma^{(d)}_{ab} + z^{d-1}\Gamma^{(d+1)}_{ab} + ...$ The leading order terms of equation (D.2) are

$$\frac{d}{16\pi G_N}\eta^{\mu\lambda}\hat{\epsilon}_{\mu z}\Gamma^{(d)}_{\lambda\nu}\xi^{\nu}z^{d+1} + ... = \mathcal{O}(1), \tag{D.3}$$

where we use the fact that for a CFT traceless stress-energy tensor implies that $\eta^{\nu\rho}g^{(d)}_{\nu\rho} = 0$ and $\hat{\epsilon}_{\mu z} = \mathcal{O}\left(z^{-(d+1)}\right)$. Finally, employing the relation between the metric perturbation in FG coordinates and the stress-energy tensor,

$$\Delta\langle T_{\mu\nu}\rangle = \frac{d}{16\pi G_N}\,\Gamma^{(d)}_{\mu\nu}\Big|_{z=0} \tag{D.4}$$

and the definition of ϵ given in section 9.2 we arrive at

$$\delta Q_{\xi} - \xi\cdot\theta = \epsilon^{\rho}\left\langle T_{\rho\sigma}\right\rangle\xi^{\sigma} + \mathcal{O}(z). \tag{D.5}$$

D.2 The HRRT surface ending on the null-plane

In order to derive the HRRT surface which ends on a curve located on a boundary null-plane, we split the coordinates into $x^{\pm} = t \pm x$ (here x is the spatial direction parallel to the null-plane), boundary directions $x^i_{\perp}$ orthogonal to the null-plane, and the bulk coordinate z. The metric on the Poincaré patch in these coordinates is

$$ds^2 = \frac{1}{z^2}(dz^2 - 2dx^+dx^- + dx^i_{\perp}dx_{\perp i}). \tag{D.6}$$

We choose static gauge for the coordinates on our extremal surface, such that $x^{\pm} = x^{\pm}(z, x^i_{\perp})$. The entangling surface on the boundary is then given by $x^{\pm} = x^{\pm}(0, x^i_{\perp})$. The equations which determine the embeddings $x^{\pm}(z, x^i_{\perp})$ are given by

$$\gamma^{ab}\frac{\partial\gamma_{ab}}{\partial x^{\pm}} = -\frac{1}{\sqrt{\gamma}}\partial_a\left(2\sqrt{\gamma}\gamma^{ab}g_{+-}\partial_b x^{\mp}\right), \tag{D.7}$$

where the induced metric is denoted by γ_{ab}. Having the extremal surface ending on a boundary null-plane means that either x^+ or x^- are constant. Without loss of generality, we choose $x^- = x_0^- =$ const. This reduces the two equations (D.7) to a single equation for $x^+(z, x_\perp^i)$. Making the ansatz $x^+(z, x_\perp^i) = h_k(z) g_k(x_\perp^i)$ we can separate the equation into

$$z^{d-1}\partial_z(z^{1-d}\partial_z h_k(z)) = -\Delta_\perp g_k(x_\perp^i). \tag{D.8}$$

The general solutions for the functions $h_k(z)$ and $g_k(x_\perp^i)$ are given by

$$g_k(x^\perp) = a_{k^i} e^{ik^i x_\perp^i}, \tag{D.9}$$

$$h_k(z) = c_k z^{d/2} I_{d/2}(zk) + d_k z^{d/2} K_{d/2}(zk), \tag{D.10}$$

where $k = |k^i|$ and $x_\perp^i k^i$ denotes the Euclidean inner product between the vectors k^i and x^i. I_ν and K_ν denote the modified Bessel functions of first and second kind, respectively. We also define $h_0 = \lim_{z\to 0} h_k(z)$. It turns out that we do not want the full solution for h_k. Intuitively, it is clear that the effect of deformations of the entangling surface on the boundary should die off as $z \to \infty$. At the same time we also require that the shape of the extremal surface is uniquely determined by boundary conditions. The asymptotic behavior of h_k as $z \to \infty$ and $z \to 0$ is

$$\lim_{z\to\infty} h_k(z) = c_k \sqrt{\frac{1}{2\pi k}} e^{kz} + d_k \sqrt{\frac{\pi}{2k}} e^{-kz}, \tag{D.11}$$

$$\lim_{z\to 0} h_k(z) = d_k 2^{\frac{d-2}{2}} \Gamma\left(\frac{d}{2}\right) k^{-d/2}. \tag{D.12}$$

We can only fulfill above requirements if we set $c_k = 0$. Hence any extremal surface ending on the null-plane $x^- = x_0^-$ is given by

$$x^+(z, x_\perp^i) = \frac{2^{\frac{2-d}{2}} k^{d/2}}{\Gamma(d/2)} \int d^{d-2}k a_{\vec{k}} z^{d/2} K_{d/2}(zk) e^{ik^i x^i}. \tag{D.13}$$

The normalization is chosen such that

$$\lim_{z\to 0} x^+(z, x_\perp^i) = \int d^{d-2}k a_{k^i} e^{ik^i x_\perp^i} \tag{D.14}$$

determines a_k in terms of the entangling surface $x^+(0, x^\perp)$.

D.3 Calculation of the binormal

The binormal $n^{\mu\nu}$ is defined as

$$n^{\mu\nu} = n_2^\mu n_1^\nu - n_2^\nu n_1^\mu, \tag{D.15}$$

where n_1 and n_2 are orthogonal ± 1 normalized normal vectors to the extremal surface. To calculate them start by calculating the $d-1$ tangent vectors to the surface which will be labeled by n as $t_n = t_n^\mu \partial_\mu$, $n \in \{1, 2, ..., d-1\}$. They satisfy $t_n^\mu \partial_\mu (r^+ - \rho_0^+) = 0$ and $t_n^\mu \partial_\mu (r^- - \Lambda(\theta, \phi^i)) = 0$. A convenient set of tangent vectors is given by

$$t_1 = \sqrt{g^{\theta\theta}}\left((\partial_\theta \Lambda)\partial_- + \partial_\theta\right), \tag{D.16}$$

$$t_2 = \sqrt{g^{\phi^1\phi^1}}\left((\partial_{\phi^1} \Lambda)\partial_- + \partial_{\phi^1}\right), \tag{D.17}$$

$$t_3 = \sqrt{g^{\phi^2\phi^2}}\left((\partial_{\phi^2} \Lambda)\partial_- + \partial_{\phi^2}\right), \tag{D.18}$$

$$t_4 = \dots \tag{D.19}$$

and so on for all ϕ^i. It is easy to see that these vectors form an orthonormal basis on the Ryu-Takayanagi surface. Requiring that n_1 and n_2 are orthogonal to all tangent vectors, $g_{\mu\nu} n_{1,2}^\mu t_a^\nu = 0$. This requirement is fulfilled by choosing

$$n_{1,2}^+ = g^{+-}, \qquad n_{1,2}^a = -\partial^a \Lambda, \tag{D.20}$$

where a stands again for all angular components. The condition that n_1 and n_2 be orthogonal and normalized to $+1$ and -1, respectively, is obeyed provided we choose

$$n_1^- = \frac{1}{2}\left(1 - \partial^a \Lambda \partial_a \Lambda\right), \qquad n_2^- = -\frac{1}{2}\left(1 + \partial^a \Lambda \partial_a \Lambda\right). \tag{D.21}$$

One can check that the only non-zero components of the binormal are given by:

$$n^{+-} = g^{+-}, \qquad\qquad n^{a-} = -\partial^a \Lambda. \tag{D.22}$$

D.4 Hollands-Wald gauge condition

In this appendix, we argue that for the example of a planar black hole in AdS$_4$, considered as a perturbation of pure AdS, we can choose a gauge where $g^{(1)}_{-a}|_\Sigma = 0 = g^{(1)}_{--}|_\Sigma$ which at the same time is compatible with Hollands-Wald gauge. In this case, the final term in our second order expression (9.65) for the relative entropy vanishes.

Hollands-Wald gauge is determined by requiring that the extremal surface in the deformed spacetime sits at the same coordinate location than the extremal surface in the undeformed spacetime. In particular this means that

$$r^- = \Lambda(\theta, \phi), \qquad\qquad r^+ = \rho_0^+. \tag{D.23}$$

The requirement that also after a perturbation of the metric the extremal surface $\tilde{A}$ sits at its old coordinate location translates into

$$0 = \partial_- \left(\gamma^{ab}_{(0)}\gamma^{(1)}_{ab}\right) - \partial_c\left(\sqrt{\gamma^{(0)}}\gamma^{ca}_{(0)}\partial_a x^\mu g^{(1)}_{-\mu}\right)\Big|_{\tilde{A}}, \tag{D.24}$$

$$\begin{aligned}0 = & -\frac{1}{2}\sqrt{\gamma^{(0)}}\gamma^{ab}_{(0)}\partial_a r^- g^{(0)}_{+-}\partial_b(\gamma^{cd}_{(0)}\gamma^{(1)}_{cd}) + \partial_c(\sqrt{\gamma^{(0)}}\partial_d r^- g^{(0)}_{+-}\gamma^{ca}_{(0)}\gamma^{(1)}_{ab}\gamma^{bd}_{(0)}) \\ & - \partial_b(\sqrt{\gamma^{(0)}}\gamma^{ab}_{(0)}\partial_a r^- g^{(1)}_{+-}) - \partial_b(\sqrt{\gamma^{(0)}}\gamma^{ab}_{(0)} g^{(1)}_{+a}) + \frac{1}{2}\sqrt{\gamma^{(0)}}\partial_+(\gamma^{ab}_{(0)}\gamma^{(1)}_{ab})\Big|_{\tilde{A}}.\end{aligned} \tag{D.25}$$

As a warm-up consider a ball-shaped entangling surface with a corresponding extremal surface at $r^+ = \rho_0^+, r^- = -\rho_0^+$ placed in a planar black hole background,

$$ds^2 = \frac{1}{z^2}\left(-(1-\mu z^d)dt^2 + \frac{dz^2}{(1-\mu z^d)} + dx^2\right), \tag{D.26}$$

at leading order in μ. The equations for the extremal surface now become at first order

$$0 = \frac{1}{2}\sqrt{\gamma^{(0)}}\partial_\pm\left(\gamma_{(0)}^{ab}\gamma_{ab}^{(1)}\right) - \partial_a\left(\sqrt{\gamma^{(0)}}\gamma_{(0)}^{ab}g_{\pm b}^{(1)}\right)\bigg|_{\tilde{A}}. \tag{D.27}$$

We can use the symmetry of the perturbation under time translations and regularity at the boundary to find a vector field v that generates a diffeomorphism $g \to \mathcal{L}_v g$ which locates the extremal surface in the perturbed geometry at the same coordinate location as the extremal surface in the unperturbed geometry.

$$v_+ = -\frac{\mu}{64}\sin\theta(1+\sin^2\theta)(r^+ - r^-)^2, \tag{D.28}$$

$$v_- = \frac{\mu}{64}\sin\theta(1+\sin^2\theta)(r^+ - r^-)^2, \tag{D.29}$$

$$v_\theta = \frac{\mu}{64}(r^+ - r^-)^3\cos^3\theta, \tag{D.30}$$

$$v_\phi = 0. \tag{D.31}$$

This diffeomorphism brings the metric perturbation into the form

$$\begin{aligned}\delta ds^2 =& \frac{\mu}{8}(r^+ - r^-)\frac{1+\sin^2\theta}{\sin\theta}dy^+dy^- + \frac{\mu}{32}(r^+ - r^-)^3\cos\theta\cot\theta d\theta^2 \\ &- \frac{\mu}{32}(r^+ - r^-)^3\cos^3\theta\cot\theta d\phi^2.\end{aligned} \tag{D.32}$$

The only non-vanishing components of the metric in the new coordinates are $g_{+-}, g_{\theta\theta}$ and $g_{\phi\phi}$. In particular, we have that $g_{-a}^{(1)} = 0 = g_{--}^{(1)}$. The main benefit of these coordinates is that equation (D.27) holds automatically. Hence at least for a ball-shaped entangling surface we are in Hollands-Wald gauge and the extremal surface is located at $r^\pm = \pm\rho_0^+$. It can be seen from the metric that lines of constant $r^\pm$ are lightlike and therefore we know that the new entangling surface still is on the bulk lightcone of a point p at the boundary.

From this we can conclude that the entanglement wedge associated to any region bounded by a lightcone does not contain any point outside the causal wedge. As we have seen this is true for ball-shaped regions. A deformation of the entangling surface cannot change this, since the boundary domain of dependence is smaller than that of some ball-shaped region. At the same time, the extremal surface cannot lie within the causal domain of dependence and therefore we must conclude that the extremal surface also lies on the lightcone.

This means that the transformations (D.28) – (D.31) bring the HRRT surface to its correction r^+ location. The only additional adjustment we need to make to the coordinate system is to reparameterize r^- around the extremal surface, e.g. by rescaling the r^- coordinate in an angle-dependent way.

To find a solution to the general Hollands-Wald gauge condition, equation (D.25), we alter the plus-component of the vector field, $v_+ \to v_+ + \tilde{v}_+(\theta, \phi)$, around the extremal surface such that it shifts the extremal surface into its new correct location on the lightcone. This vector field can be chosen such that at the extremal surface $\tilde{A}$ it remains constant along r^- and r^+ and thus depends only on θ and ϕ. It should be clear that such a solution exists, since at the boundary the correction $\tilde{v}_+(\theta, \phi)$ vanishes and is smooth everywhere else. More formally, in this case equation (D.25) reduces to

$$\begin{aligned} &\frac{\mu}{16}\partial_\theta(\cot\theta\partial_\theta(\rho_0^+ - \Lambda(\theta,\phi))^2) + \frac{\mu}{16}\tan\theta\partial_\phi^2(R - \Lambda(\theta,\phi))^2\Big|_{\tilde{A}} \\ &= \partial_\theta(\cos\theta(\partial_\theta\tilde{v}_+(\theta,\phi) + 2\cot\theta\tilde{v}_+(\theta,\phi))) + \frac{1}{\cos\theta}\partial_\phi^2\tilde{v}_+(\theta,\phi)\Big|_{\tilde{A}}. \end{aligned} \tag{D.33}$$

For small deformations of the ball shaped entangling surface we can write $\Lambda(\theta, \phi)$ as a series expansion in the deformations. At first order, this gives us a linear PDE which can be solved. Higher orders become inherently non-linear and thus this equation is in general very hard to solve. An interesting observation one can make for small $n = 1$ deformations of the entangling surface is that the linear order correction is zero.