Low-dimensional Lie Algebras And Control Theory

by

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Abstract

Lie groups and Lie algebras are important mathematical constructs first developed by Sophus Lie in the late nineteenth century to unify and extend known methods used to solve differential equations. The problem considered in this thesis emphasizes one way Lie groups and Lie algebras can be used in control theory.

Suppose an apparatus has mechanisms for moving in a limited number of ways with other movements generated by compositions of allowed motions. The question is then how to get a targeted motion using a minimal number of the allowed motions. Motions can often be represented by Lie groups which have associated Lie algebras as their building blocks. This research shows explicitly how one can obtain elements of Lie groups as compositions of products of other elements based on the structure of the associated Lie algebras. Here, the structure of a Lie algebra refers to its commutators which are the results that one gets by applying an operation known as the "commutator" to each pair of elements of a Lie algebra.

Two concrete examples of this problem, in control theory, are: (1) the restricted parallel parking problem where the commutator of the Lie algebra element representing translations in y and that representing rotations in the xy-plane yields translations in x. Here the control problem involves a vehicle that can only perform a series involving translations in y and rotations with the aim of efficiently obtaining a pure translation in x; (2) involves an apparatus that can only perform rotations about two axes and the aim is to perform a pure rotation about a third axis. Both examples involve three-dimensional Lie algebras.

In this thesis, the composition problem is solved for the nine three-

and four-dimensional Lie algebras with non-trivial solutions. Three different solution methods are presented. Two of these methods depend on operator and matrix representations of a Lie algebra. The other method is a differential equation method that depends solely on the commutator properties of a Lie algebra. Remarkably, for these distinguished Lie algebras the solutions involve arbitrary functions and can be expressed in terms of elementary functions.

Lay Summary

Lie groups can represent motion and have building blocks known as Lie algebras. The objective is to explicitly show how one can combine certain ”allowed” motions (Lie group elements) to obtain a different and desired type of motion (another Lie group element) in a minimal number of steps. An example is parallel parking where one aims to perform a pure translation in x using a minimal combination of translations in y and rotations. This problem was solved for the nine relevant three- and four-dimensional Lie algebras.

Preface

The problem considered in this research is related to a problem that arose in the 1967 thesis by George Bluman [4]. It consists of explicitly showing how one can combine ”certain” allowed motions (Lie group elements) to obtain a different and desired type of motion (another Lie group element) in a minimal number of steps. Bluman developed three methods to approach the problem which are the *operator method*, *matrix method*, and *differential equation method*. Using the differential equation method, Bluman worked with Deshin Finlay to solve the problem for all three-dimensional Lie algebras other than the Euler angles Lie algebra $so(3,\mathbb{R})$. Additionally, they laid out some of the basic work needed to solve the problem using the DE method for the four-dimensional Lie algebra $S_{4,7}$. My work consists of solving the composition problem for all relevant three- and four-dimensional Lie algebras using the DE method as well as two other methods which depend on the representation of Lie algebras using operators or matrices. I also did research that enabled us to find all matrix representations and some operator representations.

Once my research was completed, George Bluman, Deshin Finlay, and I wrote a paper to publish the results: Composition of Lie group elements from basis Lie algebra elements. JNMP.25, 528-557 (2018) [2]. During the revision process we were made aware of reference [12] which provided operator representations for low-dimensional Lie algebras. This was particularly useful as we were unaware of any operator representations for certain Lie algebras and hence one of the referees helped us make our research more complete. Chapters 3-6 of this thesis are significantly based on the paper.

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Chapter 1

Introduction

Lie groups and their representations play an important role in various applications. Lie groups of transformations describe rigid body motions (rotations and translations), scalings, as well as other transformations. This thesis is concerned with showing explicitly how one can obtain further elements of a low-dimensional Lie group as compositions of other elements of a chosen basis. Problems of this kind can arise naturally in control theory [5,10,11]. Here an apparatus has mechanisms for moving in a limited number of ways and the aim is to generate efficiently other movements from compositions of possible motions. Two concrete examples are:

(1) The restricted parallel parking problem where the commutator of translations in y and rotations in the xy-plane yields translations in x. Here the control problem involves a vehicle that only performs translation in y and rotations in the xy-plane with the aim of efficiently obtaining a pure translation in x;

(2) An apparatus that only performs rotations about two axes with the aim of efficiently generating rotations about a third axis. Here the commutator of rotations about two axes yields rotations about

the third axis.

Both examples involve three-dimensional Lie algebras with the property that the commutator of two of its elements generates a third element. In terms of the notation used in [13], examples (1) and (2) respectively include the Lie algebras $S_{3,3}$ with its parameter set to zero and $so(3,\mathbb{R})$.

Three distinct methods are presented to solve the composition problem. The first method (*operator method*) depends on realizing a Lie group as a Lie group of transformations. Such realizations can be found in [8] for some and in [12] for all three- and four-dimensional Lie algebras.

The second method (*matrix representation method*) involves matrix representations of finite-dimensional Lie algebras which are known to exist from Ado's theorem [1]. This theorem states that there exists a faithful square matrix representation for every finite-dimensional Lie algebra. There are many existing algorithms that generate such matrices including one developed by Willem de Graaf [9]. While the minimal dimension of a matrix representation is not known in general, it is known for all three- and four-dimensional Lie algebras (See [6] and [7]). This method is applied to a control theory problem in [10].

The third method (*DE method*) was initially presented in [4] for other purposes. This method only uses the commutator properties of a Lie algebra. In particular, it does not require the use of a representation of a Lie algebra. The DE method involves setting up and solving an initial value problem for a nonlinear system of first order ordinary differential equations. The DE method yields a necessary condition for solutions—it turns out that for all three- and four-dimensional cases, the DE method yields all solutions.

Remarkably, for all relevant n-dimensional Lie algebras, $n = 3$ or

4, the considered composition has $n + 1$ Lie group elements and the solution involves one arbitrary function and can be expressed in terms of elementary functions.

In Chapter 2, the necessary background is presented. We start by giving historical motivations for the study of Lie groups and Lie algebras. First, we define one-parameter Lie groups of transformations and present Lie's fundamental theorems needed for solving the research problem with illustrative examples. Then we define multiparameter Lie groups of transformations and associated Lie algebras. Here we present relevant fundamental theorems and illustrate them through examples.

In Chapter 3, we give a precise mathematical statement of the research problem. Then we describe fully the three different methods used to solve it. As an illustrative example, we focus on the Lie algebra $sl(2,\mathbb{R})$. In Chapter 4, in two tables we summarize our results for all relevant three- and four-dimensional Lie algebras. Following this, we show the essential details that yield these solutions for three- and four-dimensional Lie algebras in Chapters 5 and 6 respectively. In Chapter 7 we consider the case where the composition problems are stated with the order of the Lie group elements reversed. Finally in Chapter 8, we make further remarks and discuss the advantages and disadvantages of the three presented methods.

Chapter 2

Background

The problem considered in this thesis consists of explicitly determining how one can obtain further elements of Lie groups as compositions of other Lie group elements for all relevant three- and four-dimensional Lie algebras. However, before introducing the specifics of the research problem and the methods used to solve it, we begin by providing the reader with the necessary background on Lie groups and Lie algebras. We provide definitions and examples of Lie groups and Lie algebras (taken from [3]) as well as fundamental theorems.

In the late 19^{th} century, Sophus Lie initially developed the concept of Lie groups to unify the known techniques for solving ordinary differential equations. Lie's work resulted in a systematic approach to solving ordinary differential equations and, in particular, unified the topics of integrating factors, separable equation, homogeneous equation, variation of parameter, reduction of order, Fourier and Laplace transforms. Because of its systematic nature, Lie's work and its advancements led to the development of symbolic manipulation software that can be used to solve differential equations [3].

The key element in Lie's framework for solving differential equations is the use of *symmetries*, i.e. transformations that map solutions of a differential equation into other solutions. These symmetries are expressed in terms of continuous groups of transformations known as *Lie groups of transformations* which are defined in the next subsection. It is worth noting that an essential part of Lie's work is an algorithm to find *all* Lie groups of transformations admitted by a differential equation.

2.1 Lie Groups of Transformations

Here we introduce *Lie groups of transformations* acting on $\mathbb{R}^n$ after defining groups and groups of transformations.

2.1.1 Group

Definition.

A *group* G is a set of elements with a law of composition ϕ between elements satisfying the following properties.

(i) *Closure property*. For any elements a and b in G, $\phi(a,b)$ is an element of G.

(ii) *Associative property*. For any elements a, b and c in G, $\phi(a,\phi(b,c)) = \phi(\phi(a,b),c)$.

(iii) *Identity element*. There exist an identity element e in G such that for every element a in G, $\phi(a,e) = \phi(e,a) = a$.

(iv) *Inverse element*. For every element a in G there is an inverse element a^{-1} such that $\phi(a, a^{-1}) = (a^{-1}, a) = e$.

Definition.

A subset H of a group G that is a group under the same law of composition ϕ is a *subgroup of* G.

Examples of groups

(1) The set of all positive real numbers $\mathbb{R}^+$, with the law of composition ϕ given by multiplication $\phi(a, b) = ab$ and the identity element $e = 1$. Here the inverse of each element a is $a^{-1} = 1/a$.

(2) The set of all complex numbers $\mathbb{C}$, with the law of composition ϕ given by addition, $\phi(a, b) = a + b$. Here the identity element is $e = 0$ and the inverse of each element a is $a^{-1} = -a$.[The set of all real numbers $\mathbb{R}$ is a subgroup of $\mathbb{C}$ whereas the set of all integers $\mathbb{Z}$ is a subgroup of both $\mathbb{R}$ and $\mathbb{C}$.]

(3) The set of all invertible $n \times n$ matrices, where the law of composition ϕ is given by matrix multiplication, the identity element is the identity matrix I_n, and the inverse of each element is the inverse of a matrix under multiplication.

2.1.2 One-parameter Lie group of transformations

Definition.

Let $\mathbf{x} = (x_1, x_2, ..., x_n)$ lie in a region $\mathbf{D} \subset \mathbb{R}^n$ and let a parameter $\varepsilon \in S \subset \mathbb{R}$. The set of transformations

$$\mathbf{x}^* = \mathbf{X}(\mathbf{x}; \varepsilon) \tag{2.1.1}$$

defined for each $\mathbf{x} \in \mathbf{D}$ and $\varepsilon \in S$ with law of composition $\phi(\varepsilon, \delta)$ defined for all ε and δ in S, is a *one-parameter group of transformations* iff the following properties hold.

(i) For each ε in S the transformations are one-to-one onto $\mathbf{D}$.

(ii) S with the law of composition ϕ is a group.

(iii) For the identity element $e = \varepsilon_0$ and each $\mathbf{x} \in \mathbf{D}$: $\mathbf{x}^* = \mathbf{x}$, i.e.

$$\mathbf{x}^* = \mathbf{X}(\mathbf{x}; \varepsilon_0) = \mathbf{x}$$

(iv) If $\mathbf{x}^* = \mathbf{X}(\mathbf{x}, \varepsilon)$ and $\mathbf{x}^{**} = \mathbf{X}(\mathbf{x}^*, \delta)$, then

$$\mathbf{x}^{**} = \mathbf{X}(\mathbf{x}; \phi(\varepsilon, \delta)).$$

Definition.

A group of transformations is a *one-parameter Lie group of transformations* if in addition to properties (i) to (iv) it also satisfies:

(v) ε is a continuous parameter, i.e. S is an interval in $\mathbb{R}$.

(vi) $\mathbf{X}$ is infinitely differentiable with respect to $\mathbf{x}$ in $\mathbf{D}$ and an analytic function of ε in S.

(vii) $\phi(\varepsilon, \delta)$ is an analytic function of ε and δ, for all ε and δ in S.

Note that one can think of ε as a time parameter and $\mathbf{x}$ as a spatial variable. Here $\mathbf{x}^* = \mathbf{X}(\mathbf{x}, \varepsilon)$ represents the evolution over time of a point in the region $\mathbf{D}$.

Examples of one-parameter Lie groups of transformations include:

(1) *Group of x-translations in the plane*

Here $\mathbf{D} = \mathbb{R}^2$, $S = \mathbb{R}$, and $\phi(\varepsilon, \delta) = \varepsilon + \delta$ and for any $\mathbf{x} = (x, y) \in \mathbf{D}$ and $\varepsilon \in S$,

$$\mathbf{x}^* = \mathbf{X}(\mathbf{x}; \varepsilon) = (x + \varepsilon, y).$$

This group corresponds to translations parallel to the x-axis.

(2) *A group of scalings in the plane*

In this example $\mathbf{D} = \mathbb{R}^2$, $S = (0, \infty)$, $\phi(\varepsilon, \delta) = \varepsilon\delta$, the identity element is 1, and for any $\mathbf{x} = (x, y) \in \mathbf{D}$ and $\alpha \in S$,

$$\mathbf{x}^* = \tilde{\mathbf{X}}(\mathbf{x}, \alpha) = (\alpha x, \alpha^2 y).$$

In order to set the identity element to 0, one can reparameterize this group by setting $\varepsilon = \alpha - 1$. Hence S becomes $(-1, \infty)$ and the transformation becomes

$$\mathbf{x}^* = \mathbf{X}(\mathbf{x};\varepsilon) = \tilde{\mathbf{X}}(\mathbf{x};1+\varepsilon) = \Big((1+\varepsilon)x, (1+\varepsilon)^2 y\Big)$$

with law of composition $\phi(\varepsilon, \delta) = \varepsilon + \delta + \varepsilon\delta$.

First Fundamental Theorem of Lie; infinitesimal generators

Definition. Consider a one-parameter Lie group of transformations

$$\mathbf{x}^* = \mathbf{X}(\mathbf{x};\varepsilon) \tag{2.1.2}$$

with the identity $\varepsilon = 0$ and law of composition ϕ. From the Taylor expansion of (2.1.2) about $\varepsilon = 0$, one gets

$$\mathbf{x}^* = \mathbf{x} + \varepsilon\left(\frac{\partial \mathbf{X}(\mathbf{x};\varepsilon)}{\partial \varepsilon}\right)\bigg|_{\varepsilon=0} + O(\varepsilon^2) \tag{2.1.3}$$

Let

$$\xi(\mathbf{x}) = \frac{\partial \mathbf{X}(\mathbf{x};\varepsilon)}{\partial \varepsilon}\Big|_{\varepsilon=\mathbf{0}}. \tag{2.1.4}$$

The transformation $\mathbf{x} + \varepsilon\xi(\mathbf{x})$ is the *infinitesimal transformation* of the Lie group of transformations (2.1.2). The components of $\xi(\mathbf{x})$ are the *infinitesimals of* (2.1.2).

Theorem 2.1. First Fundamental Theorem of Lie. *There exists a parameterization $\tau(\varepsilon)$ such that the Lie group of transformations (2.1.2)* *is equivalent to the solution of an initial value problem for a system of first-order ODEs given by*

$$\frac{d\mathbf{x}^*}{d\tau} = \xi(\mathbf{x}^*) \tag{2.1.5}$$

with

$$\mathbf{x}^* = \mathbf{x} \;\; when \;\; \tau = 0. \tag{2.1.6}$$

In particular,

$$\tau(\varepsilon) = \int_0^{\varepsilon} \Gamma(\varepsilon')d\varepsilon', \tag{2.1.7}$$

where

$$\Gamma(\varepsilon) = \frac{\partial\phi(a,b)}{\partial b}\Big|_{(\varepsilon^{-1},\varepsilon)} \tag{2.1.8}$$

and

$$\Gamma(0) = 1. \tag{2.1.9}$$

Proof.

See Ref [3].

The First Fundamental Theorem of Lie shows that a one-parameter Lie group of transformations can be determined from its infinitesimals. Next, we look at some examples.

(1) *Group of x-translations in the plane*

Here $\mathbf{D} = \mathbb{R}^2$, $S = \mathbb{R}$, $\phi(\varepsilon,\delta) = \varepsilon + \delta$ and for any $\mathbf{x} = (x,y) \in \mathbf{D}$ and $\varepsilon \in S$,

$$\mathbf{x}^* = \mathbf{X}(\mathbf{x};\varepsilon) = (x+\varepsilon, y). \tag{2.1.10}$$

Clearly, the infinitesimals are

$$\xi(\mathbf{x}) = (1, 0).$$

Using equation (2.1.8), one can easily check that

$$\Gamma(\varepsilon) = 1.$$

Hence equation (2.1.7) leads to

$$\tau(\varepsilon) = \varepsilon.$$

Thus the system of equations (2.1.5), (2.1.6) becomes

$$\left(\frac{dx^*}{d\varepsilon}, \frac{dy^*}{d\varepsilon}\right) = (1, 0); \tag{2.1.11}$$

and when $\varepsilon = 0$

$$(x^*, y^*) = (x, y). \tag{2.1.12}$$

One can easily check that the solution to the initial value problem (2.1.11), (2.1.12) generates the group of transformations (2.1.10).

(2) *A group of scalings in the plane*

In this example $\mathbf{D} = \mathbb{R}^2$, $S = (-1, \infty)$, $\phi(\varepsilon, \delta) = \varepsilon + \delta + \varepsilon\delta$, and for any $\mathbf{x} = (x, y) \in \mathbf{D}$ and $\varepsilon \in S$ the transformation is given by

$$\mathbf{x}^* = \mathbf{X}(\mathbf{x}; \varepsilon) = \Big((1+\varepsilon)x, (1+\varepsilon)^2 y\Big). \tag{2.1.13}$$

It is easy to check that for each $\varepsilon \in S$, its inverse is given by

$$\varepsilon^{-1} = \frac{1}{1+\varepsilon}. \tag{2.1.14}$$

Clearly, the infinitesimals of this Lie group of transformations are

$$\xi(\mathbf{x}) = (x, y).$$

Using equations (2.1.8) and (2.1.14), one can show that

$$\Gamma(\varepsilon) = \frac{1}{1+\varepsilon}.$$

Hence equation (2.1.7) leads to

$$\tau(\varepsilon) = \ln(1+\varepsilon). \tag{2.1.15}$$

Thus the system of equations (2.1.5), (2.1.6) becomes

$$\left(\frac{dx^*}{d\tau}, \frac{dy^*}{d\tau}\right) = (x^*, 2y^*) \tag{2.1.16}$$

and when $\varepsilon = 0$,

$$(x^*, y^*) = (x, y). \tag{2.1.17}$$

One can easily check, using equation (2.1.15), that the solution to the IVP (2.1.16), (2.1.17) generates the group of transformations (2.1.13).

2.1.3 Infinitesimal generator

As another consequence of the First Fundamental Theorem of Lie, one sees that it is always possible to reparameterize a one-parameter Lie group of transformations so that, without loss of generality, one can assume that its law of composition is given by $\phi(a,b) = a + b$ with identity element 0, and the inverse for each $\varepsilon \in S$ is given by $\varepsilon^{-1} = -\varepsilon$. Hence the group of transformations in (2.1.2) becomes ($\tau = \varepsilon$)

$$\frac{d\mathbf{x}^*}{d\varepsilon} = \xi(\mathbf{x}^*) \tag{2.1.18}$$

with

$$\mathbf{x}^* = \mathbf{x} \ \text{ when } \ \varepsilon = 0. \tag{2.1.19}$$

The *infinitesimal generator* of a one-parameter Lie group of transformations is the operator given by

$$X = X(\mathbf{x}) = \xi(\mathbf{x}).\nabla = \sum_{i=1}^{n} \xi_i(\mathbf{x})\frac{\partial}{\partial x_i}, \tag{2.1.20}$$

where ∇ is the gradient operator

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right). \tag{2.1.21}$$

For any differentiable function $F(\mathbf{x}) = F(x_1, x_2, \ldots, x_n)$ one has

$$XF(\mathbf{x}) = \xi(\mathbf{x}).\nabla F(\mathbf{x}) = \sum_{i=1}^{n} \xi_i(\mathbf{x}) \frac{\partial F(\mathbf{x})}{\partial x_i}.$$

Note that $Xx_i = \xi_i(\mathbf{x})$ for $i = 1, \ldots, n$. It follows that the infinitesimal generator determines the infinitesimal transformation of a Lie group and, from the First Fundamental Theorem of Lie, the infinitesimal generator can be used to determine the Lie group of transformations. Next, we show another important and explicit method for determining a one-parameter Lie group of transformations from its infinitesimal generator.

Theorem 2.1.1. *The one-parameter Lie group of transformations (2.1.2) is equivalent to*

$$\mathbf{x}^* = e^{\varepsilon X}\mathbf{x} = \mathbf{x} + \varepsilon X\mathbf{x} + \frac{1}{2}\varepsilon^2 X^2 \mathbf{x} + \ldots = \sum_{0}^{\infty} \frac{\varepsilon^k}{k!} X^k \mathbf{x}, \tag{2.1.22}$$

where the operator $X = X(\mathbf{x})$ is defined by (2.1.20) and $X^k F(\mathbf{x})$ is the function obtained by applying the operator X k times with $k = 1, 2, \ldots$. Moreover, $X^0 F(\mathbf{x}) \equiv F(\mathbf{x})$.

Proof

. See Ref [3].

This theorem presents a computationally useful alternative to the First Fundamental Theorem of Lie that can be used to generate a one-parameter Lie group of transformations from its infinitesimal transformation. The series shown in equation (2.1.22) is the *Lie series* of a one-parameter Lie group of transformations.

Theorem (2.1.1) leads to the following important corollary.

Corollary 2.1.1.1. *If $F(x)$ is infinitely differentiable, then for a one-parameter Lie group of transformations (2.1.2) with infinitesimal generator (2.1.20), one has*

$$F(\mathbf{x}^*) = F(e^{\varepsilon X}\mathbf{x}) = e^{\varepsilon X} F(\mathbf{x}) \tag{2.1.23}$$

Proof.

See Ref [3].

As an example consider the rotation group

$$x^* = \cos\varepsilon x + \sin\varepsilon y, \tag{2.1.24}$$

$$y^* = -\sin\varepsilon x + \cos\varepsilon y. \tag{2.1.25}$$

The infinitesimal for this group of transformations is

$$\xi(\mathbf{x}) = \left(\xi_1(\mathbf{x}), \xi_2(\mathbf{x})\right) = (y, -x)$$

Hence the infinitesimal generator is given by

$$X = X(\mathbf{x}) = \xi(\mathbf{x}).\nabla = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \tag{2.1.26}$$

One can use equation (2.1.26) to obtain the Lie series of this group of transformations as follows: $Xx = y$, $X^2x = -x$. Thus

$$X^k x = \begin{cases} (-1)^{\frac{k-1}{2}} x \text{ if } k \text{ is odd} \\ (-1)^{\frac{k}{2}} y \text{ if } k \text{ is even} \end{cases}$$

Hence

$$\mathbf{x}^* = e^{\varepsilon X} x = x\left(\sum_{k=0}^{\infty} \frac{(-1)^k \varepsilon^{2k}}{(2k)!}\right) + y\left(\sum_{k=0}^{\infty} \frac{(-1)^k \varepsilon^{2k+1}}{(2k+1)!}\right) = \cos\varepsilon x + \sin\varepsilon y.$$

Similarly, one can show that

$$y^* = -\sin\varepsilon x + \cos\varepsilon y.$$

2.2 Multiparameter Lie group of transformations

In this section we present some key definitions and results related to multiparameter Lie groups of transformations. We only consider Lie groups of transformations with a finite number of parameters. Each parameter of an r-parameter Lie group of transformations leads to an infinitesimal generator which in turn leads to an r-dimensional vector space that has an additional ”commutator” operation. The vector space generated by the infinitesimals is a *Lie algebra* (r-dimensional Lie algebra).

One can obtain a multiparameter Lie group of transformations through the exponentiation of its associated Lie algebra elements, as in the case of one-parameter Lie groups of transformations.

2.2.1 r-parameter Lie group of transformations

An *r-parameter Lie group of transformation* is defined in a way that is analogous to the definition of a one-parameter Lie group of transformations.

Definition

Let $\mathbf{x} = (x_1, x_2, ..., x_n) \in \mathbf{D} \subset \mathbb{R}^n$ and $\varepsilon = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_r)$, $\delta = (\delta_1, \delta_2, ..., \delta_r) \in \mathbf{S} \subseteq \mathbb{R}^r$. The transformation

$$\mathbf{x}^* = \mathbf{X}(\mathbf{x}; \varepsilon) \tag{2.2.1}$$

with law composition $\phi(\varepsilon, \delta)$ is an *r-parameter Lie group of transformation* if it is a one-to-one transformation onto $\mathbf{D}$ such that

(i) $\mathbf{S}$ with law of composition $\phi(\varepsilon, \delta)$ is a group with identity element $\varepsilon = 0$ which corresponds to $\varepsilon_1 = \varepsilon_2 = ... = \varepsilon_r = 0$.

(ii) $\phi(\varepsilon, \delta)$ is analytic in ε and δ.

(iii) For each $\mathbf{x} \in \mathbf{D}$,

$$\mathbf{x}^* = \mathbf{X}(\mathbf{x}; 0) = \mathbf{x}.$$

(iv) If $\mathbf{x}^* = \mathbf{X}(\mathbf{x};\varepsilon)$ and $\mathbf{x}^{**} = \mathbf{X}(\mathbf{x}^*;\delta)$, then

$$\mathbf{x}^{**} = \mathbf{X}(\mathbf{x};\phi(\varepsilon,\delta)).$$

Next, we present the *First Fundamental Theorem of Lie* for r-parameter Lie groups of transformations after defining the *infinitesimal matrix* of the r-parameter Lie group of transformations (2.2.1).

Definition.

The *infinitesimal matrix* is the $r \times n$ matrix $\Xi(\mathbf{x})$ with entries

$$\xi_{ij}(\mathbf{x}) = \left.\frac{\partial x_j^*}{\partial \varepsilon_i}\right|_{\varepsilon=0} = \left.\frac{\partial X_j(\mathbf{x},\varepsilon)}{\partial \varepsilon_i}\right|_{\varepsilon=0}, \quad i = 1,2,\ldots,r, \quad j = 1,2,\ldots,n. \tag{2.2.2}$$

Theorem 2.3. The First Fundamental Theorem of Lie. *Let* $\Theta(\varepsilon)$ *be the* $r \times r$ *matrix with elements*

$$\Theta_{kl}(\varepsilon) = \left.\frac{\partial \phi_l}{\partial \delta_k}\right|_{\delta=0}, \quad k,l = 1,2,\ldots r. \tag{2.2.3}$$

Let its inverse matrix be given by

$$\Psi(\varepsilon) = \Theta^{-1}(\varepsilon). \tag{2.2.4}$$

In some neighborhood of $\varepsilon = 0$, *the Lie group of transformations* (2.2.1) *is equivalent to the solution of the initial value problem for the system of* *nr first-order PDEs given by*

$$\begin{pmatrix} \frac{\partial x_1^*}{\partial \varepsilon_1} & \frac{\partial x_2^*}{\partial \varepsilon_1} & \cdots & \frac{\partial x_n^*}{\partial \varepsilon_1} \\ \frac{\partial x_1^*}{\partial \varepsilon_2} & \frac{\partial x_2^*}{\partial \varepsilon_2} & \cdots & \frac{\partial x_n^*}{\partial \varepsilon_2} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial x_1^*}{\partial \varepsilon_r} & \frac{\partial x_2^*}{\partial \varepsilon_r} & \cdots & \frac{\partial x_n^*}{\partial \varepsilon_r} \end{pmatrix} = \Psi(\varepsilon)\Xi(x^*), \tag{2.2.5}$$

with

$$\mathbf{x}^* = \mathbf{x} \;\; at \;\; \varepsilon = \mathbf{0}. \tag{2.2.6}$$

Proof.

See Ref [3].

Definition.

The *infinitesimal generator* X_α corresponding to the parameter ε_α of the r-parameter Lie group of transformations (2.2.1), is given by

$$X_\alpha = \sum_{j=1}^{n} \xi_{\alpha j} \frac{\partial}{\partial x_j}, \;\; \alpha = 1, ..., r. \tag{2.2.7}$$

One can show that the Lie group of transformations (2.2.1) is equivalent to

$$\mathbf{x}^* = \prod_{\alpha=1}^{r} e^{\mu_\alpha X_\alpha}\mathbf{x} = e^{\mu_1 X_1} e^{\mu_2 X_2} \ldots e^{\mu_r X_r}\mathbf{x}, \tag{2.2.8}$$

where, $\mu_1, \mu_2, \ldots, \mu_r$ are arbitrary real constants [3]. Note that the order of the operations in (2.2.8) can be rearranged by renumbering the infinitesimal generators. This corresponds to a different parameterization, i.e., $\Psi(\varepsilon)$ changes.

As an example, consider the two-parameter Lie group of transformations

$$\begin{aligned} x^* &= e^{\varepsilon_1} x + \varepsilon_2, \\ y^* &= e^{2\varepsilon_1} y. \end{aligned} \tag{2.2.9}$$

Here $\mathbf{D} = \mathbf{S} = \mathbb{R}^2$ and the law of composition is

$$\phi(\varepsilon, \delta) = \phi\Big((\varepsilon_1, \varepsilon_2), (\delta_1, \delta_2)\Big) = \Big(\varepsilon_1 + \varepsilon_2, \varepsilon_2 e^{\delta_1} + \delta_2\Big); \quad \varepsilon, \delta \in \mathbf{S}. \tag{2.2.10}$$

It is easy to check that the infinitesimal matrix is

$$\Xi(x, y) = \begin{pmatrix} x & y \\ 1 & 0 \end{pmatrix}.$$

Additionally,

$$\Theta(\varepsilon_1, \varepsilon_2) = \begin{pmatrix} 1 & \varepsilon_2 \\ 0 & 1 \end{pmatrix},$$

with its inverse given by

$$\Psi(\varepsilon_1, \varepsilon_2) = \begin{pmatrix} 1 & -\varepsilon_2 \\ 0 & 1 \end{pmatrix}.$$

Thus the system of PDEs (2.2.5) with initial conditions (2.2.6) becomes

$$\begin{pmatrix} \frac{\partial x^*}{\partial \varepsilon_1} & \frac{\partial y^*}{\partial \varepsilon_1} \\ \frac{\partial x^*}{\partial \varepsilon_2} & \frac{\partial y^*}{\partial \varepsilon_2} \end{pmatrix} = \Psi(\varepsilon_1, \varepsilon_2)\Xi(x^*, y^*) = \begin{pmatrix} x^* - \varepsilon_2 & 2y^* \\ 1 & 0 \end{pmatrix}, \tag{2.2.11}$$

with

$$(x^*, y^*) = (x, y) \text{ when } (\varepsilon_1, \varepsilon_2) = (0, 0). \tag{2.2.12}$$

One can easily check that the solution to the system (2.2.11), (2.2.12) is given by (2.2.9). Next, we check that (2.2.9) can also be generated using equation (2.2.8).

The infinitesimal generators for the two-parameter Lie group of transformations (2.2.9) are given by

$$\begin{aligned} X_1 &= x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}, \\ X_2 &= \frac{\partial}{\partial x}. \end{aligned}$$

It is useful to note that by analogy with equation (2.1.23), one has for every differentiable function $F(x, y)$

$$\begin{aligned} e^{\varepsilon X_1} F(x,y) &= F\Big(e^{\varepsilon X_1} x, e^{\varepsilon X_1} y\Big) = F\Big(e^{\varepsilon} x, e^{2\varepsilon} y\Big), \\ e^{\varepsilon X_2} F(x,y) &= F\Big(e^{\varepsilon X_2} x, e^{\varepsilon X_2} y\Big) = F\Big(x + \varepsilon, y\Big). \end{aligned}$$

Hence equation (2.2.8) becomes

$$e^{\mu_1 X_1} e^{\mu_2 X_2}(x,y) = e^{\mu_1 X_1}(x+\mu_2, y) = (e_1^{\mu} x + \mu_2, e^{2\mu_1} y) = (x^*, y^*). \quad (2.2.13)$$

2.3 Lie Algebras

Definition.

Consider an r-parameter Lie group of transformations (2.2.1) with infinitesimal generators X_α, $\alpha = 1,...,r$ defined by (2.2.7). The *commutator (Lie bracket)* of X_α and X_β is a first-order operator given by

$$\begin{aligned}
[X_\alpha, X_\beta] &= X_\alpha X_\beta - X_\beta X_\alpha \\
&= \sum_{i,j=1}^{n} \left[\left(\xi_{\alpha i}(\mathbf{x}) \frac{\partial}{\partial x_i} \right) \left(\xi_{\beta j}(\mathbf{x}) \frac{\partial}{\partial x_j} \right) - \left(\xi_{\beta i}(\mathbf{x}) \frac{\partial}{\partial x_i} \right) \left(\xi_{\alpha j}(\mathbf{x}) \frac{\partial}{\partial x_j} \right) \right] \\
&= \sum_{j=1}^{n} \eta_j(\mathbf{x}) \frac{\partial}{\partial x_j},
\end{aligned} \quad (2.3.1)$$

where

$$\eta_j = \sum_{i=1}^{n} \left(\xi_{\alpha i}(\mathbf{x}) \frac{\partial \xi_{\beta j}}{\partial x_i} - \xi_{\beta i}(\mathbf{x}) \frac{\partial \xi_{\alpha j}}{\partial x_i} \right). \quad (2.3.2)$$

From equation (2.3.2), it is clear that the commutator operation is anti-symmetric, i.e,

$$[X_\alpha, X_\beta] = -[X_\beta, X_\alpha]. \quad (2.3.3)$$

By direct computation, one can show that for any three infinitesimal generators X_α, X_β, X_γ, one has the *Jacobi identity*

$$[X_\alpha[X_\beta, X_\gamma]] + [X_\beta[X_\gamma, X_\alpha]] + [X_\gamma[X_\alpha, X_\beta]] = 0. \tag{2.3.4}$$

Theorem 2.4. The Second Fundamental Theorem of Lie. *The commutator of any two infinitesimal generators of an r-parameter Lie group of transformations is also an infinitesimal generator. In particular,*

$$[X_\alpha, X_\beta] = \sum_{\gamma=1}^{r} C^\gamma_{\alpha\beta} X_\gamma, \tag{2.3.5}$$

where the coefficients $C^\gamma_{\alpha\beta}$ *are real constants called the* structure constants, α, β, $\gamma = 1, ..., r$.

Proof.

See ref [3].

Equations (2.3.5) are the *commutation relations* of the r-parameter Lie group of transformations (2.2.1).

Theorem 2.5. *The anti-symmetric property* (2.3.3) *of the infinitesimal generators in* (2.2.7) *and Jacobi's identity* (2.3.4) *lead to the following relations satisfied by the structure constants.*

$$C^\gamma_{\alpha\beta} = C^\gamma_{\beta\alpha}, \tag{2.3.6}$$

$$\sum_{\rho=1}^{r}\left[C^{\rho}_{\alpha\beta}C^{\delta}_{\rho\gamma}+C^{\rho}_{\beta\gamma}C^{\delta}_{\rho\alpha}+C^{\rho}_{\gamma\alpha}C^{\delta}_{\rho\beta}\right]=0. \tag{2.3.7}$$

Definition.

A *Lie algebra L* is a vector space with a bilinear bracket operation (*the commutator*) satisfying the properties (2.3.3), (2.3.4) and (2.3.5). In particular, the set of infinitesimal generators $\{X_\alpha\}$, $\alpha = 1, 2, \ldots, r$, of an r-parameter Lie group of transformations (2.2.1) forms an r-dimensional Lie algebra over $\mathbb{R}$.

It is important to note that in this thesis, we are interested in a more abstract definition of a Lie algebra. In particular, we define a Lie algebra solely based on its properties (2.3.3)-(2.3.5). Starting from this abstract definition, we then seek natural representations of a Lie algebra, e.g. an operator and matrix representations, in order to solve the research problem. Most importantly, one can also obtain a necessary condition to solving the research problem, without any representation i.e., just using the algebraic properties (2.3.3)-(2.3.5) (differential equation method).

Chapter 3

Research problem

Consider a three-dimensional Lie algebra L with basis elements B_1, B_2, and B_3 such that the commutator of B_1 and B_2 is a linear combination of the basis elements of L, i.e.,

$$[B_1, B_2] = B_1 B_2 - B_2 B_1 = \sum_{k=1}^{3} C_{12}^k B_k, \tag{3.0.1}$$

in terms of real structure constants C_{12}^1, C_{12}^2, and C_{12}^3, with $C_{12}^3 \neq 0$. In other words, the Lie algebra element B_3 is generated by the other two elements. All six three-dimensional Lie algebras presented in [13] have at least one commutator that satisfies this non-zero property. In particular, for the problem under consideration, it does not matter what the other commutators of L are. The question of interest is whether the Lie group element generated by B_3 can be obtained from the Lie group elements generated by B_1 and B_2 as illustrated by the two examples mentioned in the Introduction. Motivated by the commutator property (3.0.1) with $C_{12}^3 \neq 0$, the aim is to find continuous functions $a(\varepsilon), b(\varepsilon), c(\varepsilon)$, and $d(\varepsilon)$ so that the equation

$$e^{a(\varepsilon)B_1} e^{b(\varepsilon)B_2} e^{c(\varepsilon)B_1} e^{d(\varepsilon)B_2} = e^{\varepsilon B_3}$$

with

$$a(0) = b(0) = c(0) = d(0) = 0 \tag{3.0.2}$$

holds for an arbitrary value of ε. Next, to clarify how the research problem given by (3.0.2) follows from (3.0.1) with $C_{12}^3 \neq 0$, a concrete example is presented.

The parallel parking problem has commutators given by $[R, Y] = X,\ [R, X] = -Y,\ [X, Y] = 0$, where

$$X = -\frac{\partial}{\partial x},\ \ Y = \frac{\partial}{\partial y},\ \ R = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

The first commutator indicates that X can be generated from R and Y. This leads one to consider either equation

$$e^{a(\varepsilon)R}e^{b(\varepsilon)Y}e^{c(\varepsilon)R}e^{d(\varepsilon)Y} = e^{\varepsilon X}; \tag{3.0.3}$$

or

$$e^{a(\varepsilon)Y}e^{b(\varepsilon)R}e^{c(\varepsilon)Y}e^{d(\varepsilon)R} = e^{\varepsilon X}. \tag{3.0.4}$$

In equation (3.0.3) R, Y, and X correspond to B_1, B_2, and B_3, respectively. The solution to equation (3.3) will be presented in Chapter 4. Note that when the commutator of a Lie algebra used in equation (3.0.2) does not satisfy the assumptions stated above, one would expect only trivial solutions. For example, since in the parking problem the commutators of X and Y do not generate R, the equation

$$e^{a(\varepsilon)X}e^{b(\varepsilon)Y}e^{c(\varepsilon)X}e^{d(\varepsilon)Y} = e^{\varepsilon R}$$

only has the trivial solution.

When the DE method is used, an additional assumption about these functions is needed. In particular, here $a(\varepsilon), b(\varepsilon), c(\varepsilon)$, and $d(\varepsilon)$ are differentiable everywhere except at $\varepsilon = 0$. This assumption is needed since the DE method relies on finding a system of differential equations that the four functions must satisfy.

In general, it turns out that the problem as stated always has a degree of freedom in its solution. Moreover, a minimum number of four terms are needed on the left hand side of equation (3.0.2). This follows from the origin of the commutator equation (3.0.1). In particular, from the form of equation (3.0.2), one would expect, as will be seen later in this thesis, that there are solutions for which $a(\varepsilon)$, $b(\varepsilon), c(\varepsilon)$, and $d(\varepsilon)$ are of order $\sqrt{\varepsilon}$ *as* $\varepsilon \to 0$ to generate a commutator element of order ε on the right-hand side of equation (3.0.2).

Now consider a four-dimensional Lie algebra L with basis elements B_1, B_2, B_3, and B_4 such that the elements B_1, B_2, and B_3 do not form a subalgebra. For the research problem we require that the commutator of B_1 and B_2 satisfies

$$[B_1, B_2] = B_1 B_2 - B_2 B_1 = \sum_{k=1}^{4} C_{12}^{k} B_k, \tag{3.0.5}$$

with real structure constants $C_{12}^1, ..., C_{12}^4$ where $C_{12}^3 \neq 0$ and $C_{12}^4 = 0$. Here the Lie algebra element B_3 can be generated from the elements B_1 and B_2. The problem of interest is whether the Lie group element generated by B_3 can be obtained from the Lie group elements generated by B_1 and B_2. In particular, in view of the commutator property (3.0.5), we are interested in finding continuous functions $a(\varepsilon), b(\varepsilon), c(\varepsilon), d(\varepsilon), f(\varepsilon)$, and $g(\varepsilon)$ so that the equation

$$e^{a(\varepsilon)B_1} e^{b(\varepsilon)B_2} e^{c(\varepsilon)B_1} e^{d(\varepsilon)B_2} e^{f(\varepsilon)B_1} e^{g(\varepsilon)B_2} = e^{\varepsilon B_3}, \tag{3.0.6}$$

with $a(0) = b(0) = c(0) = d(0) = f(0)) = g(0) = 0$ holds for an arbitrary value of ε.

It is important to note that it is essential that the left hand side of (3.0.6) is composed of the product of six Lie group elements: in all considered cases there exist solutions where one of $a(\varepsilon)$ or $g(\varepsilon)$ is zero, but this is not obvious a priori.

One should also note that one can also state the problems in (3.0.2) and (3.0.6) with the roles of B_1 and B_2 interchanged when the number of terms to the left of these equations is even. This does not change the nature of the problem and, in fact, it leads to isomorphic solutions as will be shown in Chapter 7.

In this thesis, for all relevant three- and four-dimensional Lie algebras, we present three different methods that can yield the general solution for their respective equations (3.0.2) and (3.0.6). In what follows, we will describe the different methods used to solve (3.0.2) and (3.0.6). As a simple example, in this Chapter we solve the composition problem for the three-dimensional Lie algebra $sl(2,\mathbb{R})$ to illustrate how these different methods work. The solution of the composition problem for the other relevant three- and four-dimensional Lie algebras will be presented in Chapter 4.

We first note that $sl(2,\mathbb{R})$ has the commutators

$$[X,Y]=Z,\ [Z,X]=2X,\ [Y,Z]=2Y. \tag{3.0.7}$$

3.1 Operator method

The *operator method* requires a representation of a Lie algebra in terms of differential operators. The operator representation is not necessarily unique. This lack of uniqueness is illustrated by the example of $sl(2,\mathbb{R})$.

3.1.1 Description of the operator method

Let $\{\Delta_1,\ldots,\Delta_r\}$ be a differential operator representation of a Lie algebra L which respectively has basis elements $\{B_1,\ldots,B_r\}$, where

$\Delta_i = \gamma_j^i(\boldsymbol{x})\frac{\partial}{\partial x_j}, \quad i = 1,\ldots,r,$ and $\boldsymbol{x} = (x_1,\ldots,x_m)$.

Then equations (3.0.2) and (3.0.6) become respectively the equations

$$e^{a(\varepsilon)\Delta_1}e^{b(\varepsilon)\Delta_2}e^{c(\varepsilon)\Delta_1}e^{d(\varepsilon)\Delta_2}\boldsymbol{x} = e^{\varepsilon\Delta_3}\boldsymbol{x}, \tag{3.1.1}$$

and

$$e^{a(\varepsilon)\Delta_1}e^{b(\varepsilon)\Delta_2}e^{c(\varepsilon)\Delta_1}e^{d(\varepsilon)\Delta_2}e^{f(\varepsilon)\Delta_1}e^{g(\varepsilon)\Delta_2}\boldsymbol{x} = e^{\varepsilon\Delta_3}\boldsymbol{x}, \tag{3.1.2}$$

where $e^{\varepsilon\Delta_i}\boldsymbol{x} = \sum\limits_{n=0}^{\infty}\frac{\varepsilon^n}{n!}\Delta_i^n\boldsymbol{x} \;\; = (\boldsymbol{x}^{\boldsymbol{i}})^*$ for $i = 1,\ldots,r$; This corresponds to generating the Lie algebra elements using Theorem 2.1.2.
$(\boldsymbol{x}^{\boldsymbol{i}})^* = \left(\ ({x_1}^i)^*,\ldots,\ ({x_m}^i)^*\right)$ is the image of $\boldsymbol{x}$ with respect to the ith basis element of the Lie group of transformations connected with the differential operator Δ_i, with $i = 1,\ldots,r$.

From the First Fundamental Theorem of Lie 2.1, one can also obtain $(\boldsymbol{x}^{\boldsymbol{i}})^*$ by solving the system of differential equations

$\frac{d({x_j}^i)^*}{d\varepsilon} = {\gamma_j}^i(\boldsymbol{x}),\ j = 1,\ldots,m,$ with initial condition $({x_j}^i)^*(0) = x_j$.

3.1.2 Example $sl(2,\mathbb{R})$

Operator representations for $sl(2,\mathbb{R})$ include

$$\Delta_1 = X = -x^2\frac{\partial}{\partial x},\ \Delta_2 = Y = \frac{\partial}{\partial x},\ \Delta_3 = Z = 2x\frac{\partial}{\partial x},\ \text{in } \mathbb{R}; \tag{3.1.3}$$

$$\Delta_1 = X = -y\frac{\partial}{\partial x},\ \Delta_2 = Y = -x\frac{\partial}{\partial y},\ \ \Delta_3 = Z = y\frac{\partial}{\partial y} - x\frac{\partial}{\partial x},\ \text{in } \mathbb{R}^2. \tag{3.1.4}$$

For the operator representation (3.1.3) for $sl(2,\mathbb{R})$, one obtains

$$e^{\varepsilon X}(x) = \frac{x}{1+\varepsilon x},\ \ e^{\varepsilon Y}(x) = x + \varepsilon,\ \ e^{\varepsilon Z}(x) = e^{2\varepsilon}x. \tag{3.1.5}$$

There are two well-known methods to obtain (3.1.5).

Method I. From the First Fundamental Theorem of Lie (Theorem 2.1), the operator representation (3.1.3) leads to solving separately the three IVPs

$$\frac{dx^*}{d\varepsilon} = -x^{*2}, x^*(0) = x; \tag{3.1.6}$$
$$\frac{dx^*}{d\varepsilon} = 1,\ x^*(0) = x; \tag{3.1.7}$$
$$\frac{dx^*}{d\varepsilon} = 2x^*, x^*(0) = x. \tag{3.1.8}$$

It is easy to show that the three one-parameter Lie groups of transformations (3.1.5) respectively solve the IVPs (3.1.6)-(3.1.8).

Method II. Using induction, it is easy to show that

$X^n x = (-1)^n n! x^{n+1},\ n \geq 0.$
Hence $e^{\varepsilon X} x = \sum\limits_{n=0}^{\infty} \frac{\varepsilon^n}{n!} X^n x = \sum\limits_{n=0}^{\infty} \varepsilon^n (-1)^n x^{n+1} = \frac{x}{1+\varepsilon x}$. Since $Yx = 1, Y^n x = 0,\ n \geq 2,$ it follows that $e^{\varepsilon Y} x = x + \varepsilon$. It is easy to show that $Z^n x = 2^n x,\ n \geq 0.$ Hence $e^{\varepsilon Z} x = e^{2\varepsilon} x.$

To proceed, it is convenient to rewrite expression (3.1.1) in the form

$$e^{b(\varepsilon)Y} e^{c(\varepsilon)X} e^{d(\varepsilon)Y} x = e^{-a(\varepsilon)X} e^{\varepsilon Z} x. \tag{3.1.9}$$

Then from (3.1.5), one obtains

$$e^{b(\varepsilon)Y} e^{c(\varepsilon)X} e^{d(\varepsilon)Y} x = e^{b(\varepsilon)Y} e^{c(\varepsilon)X} (x + d) = e^{b(\varepsilon)Y} \left(d + \frac{x}{1 + cx} \right) =$$

$$d+\frac{x+b}{1+c(x+b)}, e^{-a(\varepsilon)X}e^{\varepsilon Z}(x)=e^{-a(\varepsilon)X}\left(e^{2\varepsilon}x\right)=e^{2\varepsilon}\frac{x}{1-ax}.$$

Then expression (3.1.9) becomes $d+\frac{x+b}{1+c(x+b)}=e^{2\varepsilon}\frac{x}{1-ax}$. Hence for all x, one has

$$-\left(a(dc+1)+ce^{2\varepsilon}\right)x^2+\left(-a(d+b+bdc)+dc+1-e^{2\varepsilon}(1+bc)\right)x+b+d$$

$$+bdc=0.$$

This yields the set of equations

$$b+d+bdc=0,\ \ dc+1=e^{2\varepsilon}(1+bc),\ \ -a(dc+1)=ce^{2\varepsilon}. \tag{3.1.10}$$

The solution to the system of equations (3.1.10) is given by

$$a(\varepsilon)=\frac{e^{\varepsilon}-e^{2\varepsilon}}{d(\varepsilon)},\ \ b(\varepsilon)=-e^{-\varepsilon}d(\varepsilon),\ \ c(\varepsilon)=\frac{e^{\varepsilon}-1}{d(\varepsilon)}, \tag{3.1.11}$$

where $d(\varepsilon)$ is any continuous function chosen so that $a(\varepsilon)$ and $c(\varepsilon)$ are continuous functions, and satisfying $d(\varepsilon)\neq 0$ for any $\varepsilon\neq 0$ with $a(0)=b(0)=c(0)=d(0)=0$.

3.2 Matrix representation method

The *matrix representation method* involves a matrix representation of a Lie algebra L. In particular, in this thesis we seek an appropriate matrix for each basis element of L using the Lie algebra package of the computer software GAP (Group, Algorithms, Programming) [9]. A difficulty arose in the case of the four-dimensional Lie algebra $S_{4,7}$ (in terms of the nomenclature used in the classification of Lie algebras in [11]). Here the matrix representation obtained from the software package [9] could not be used since the obtained representation is not isomorphic to $S_{4,7}$. For this Lie algebra, we used a matrix representation given in [6].

3.2.1 Description of the procedure used for the matrix representation method

Let L be a k-dimensional Lie algebra with basis elements B_i with M_i denoting a matrix representation of B_i, $i = 1,\ldots,k$.

Step 1. Find matrices $\{M_i\}$ that represent L using computer software or relevant literature ([6], [9]).

Step 2. Attempt to find a closed form representation for each element $e^{\varepsilon B_i}$ of the Lie group associated with L from the Taylor expansion $e^{\varepsilon M_i} = \sum\limits_{n=0}^{\infty} \frac{\varepsilon^n}{n!} M_i^n$ where ${M_i}^0 = I$ for $i = 1,\ldots,k$. (In all three- and four-dimensional cases, this step was successful in leading to such a closed form representation.)

Step 3. Compute $\prod_{i=1}^{2k-2} e^{a_i(\varepsilon)B_j}$ where $\begin{cases} j = 1 \text{ if } i \text{ is odd} \\ j = 2 \text{ if } i \text{ is even} \end{cases}$.

Step 4. Solve equations (3.0.2) and (3.0.6).

3.2.2 Example $sl(2,\mathbb{R})$

Although $sl(2,\mathbb{R})$ is a three-dimensional Lie algebra, a matrix representation is given by the 2×2 matrices $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$,

$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then $X^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Hence $X^n = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $n \geq 2$. Thus $e^{\varepsilon X} = I + \varepsilon X = \begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix}$. Similarly, one can show that $e^{\varepsilon Y} = \begin{pmatrix} 1 & 0 \\ \varepsilon & 1 \end{pmatrix}$. One can easily show that $Z^n = \begin{pmatrix} 1 & 0 \\ 0 & (-1)^n \end{pmatrix}$. Hence $e^{\varepsilon Z} = \begin{pmatrix} e^{\varepsilon} & 0 \\ 0 & e^{-\varepsilon} \end{pmatrix}$. Consequently,

$$M = e^{a(\epsilon)X} e^{b(\epsilon)Y} e^{c(\epsilon)X} e^{d(\epsilon)Y} = \begin{pmatrix} abcd + ab + ad + cd + 1 & a + c + abc \\ b + d + bcd & bc + 1 \end{pmatrix}.$$

After setting $M = e^{\varepsilon Z}$, one obtains the equations

$$bc + 1 = e^{-\varepsilon},\ b + d + bcd = 0,\ a + c + abc = 0,\ abcd + ab + ad + cd + 1 = e^{\varepsilon},$$

whose solution is given by (3.1.11).

3.3 DE method

The *DE method* requires differentiation of the unknown functions in equations (3.0.2) and (3.0.6). It involves setting up a nonlinear system of first order ordinary differential equations that must be satisfied by *all* differentiable solutions of equations (3.0.2) and (3.0.6). Here the solutions respectively satisfy initial conditions

$$a(0) = b(0) = c(0) = d(0) = 0, \tag{3.3.1}$$

$$a(0) = b(0) = c(0) = d(0) = f(0) = g(0) = 0\ , \tag{3.3.2}$$

in the three- and four-dimensional cases. Note that when $\varepsilon = 0$, the unknown functions will be continuous but not differentiable. The DE method yields a necessary condition for solutions. Sufficiency is shown from the solutions obtained by the other two methods.

3.3.1 Description of the DE method

After differentiating equations (3.0.2) and (3.0.6) with respect to ε, one obtains respectively,

$$a_1'(\varepsilon)B_1\prod_{i=1}^{4}e^{a_iB_i}+a_2'(\varepsilon)e^{a_1B_1}B_2\prod_{i=2}^{4}e^{a_iB_i}+a_3'(\varepsilon)\prod_{i=1}^{2}e^{a_iB_i}B_1\prod_{i=3}^{4}e^{a_iB_i}$$
$$+a_4'(\varepsilon)\prod_{i=1}^{3}e^{a_iB_i}B_2e^{a_4B_2}=B_3e^{\varepsilon B_3}, \tag{3.3.3}$$

$$a_1'(\varepsilon)B_1\prod_{i=1}^{6}e^{a_iB_i}+a_2'(\varepsilon)e^{a_1B_1}B_2\prod_{i=2}^{6}e^{a_iB_i}+\cdots+a_6'(\varepsilon)\prod_{i=1}^{5}e^{a_iB_i}B_2e^{a_6B_2}$$
$$=B_3e^{\varepsilon B_3}, \tag{3.3.4}$$

where $B_i=\begin{cases}B_1 \text{ if } i \text{ is odd}\\ B_2 \text{ if } i \text{ is even}\end{cases}$ and $a_1=a,\ a_2=b,\ a_3=c,\ a_4=d,\ a_5=f,\ a_6=g$.

From equations (3.3.3) and (3.3.4), one sees that a formula is needed for pulling the products of the exponentials appropriately to the right of each B_p in order to get back $\prod_{i=1}^{n}e^{a_i(\varepsilon)B_i}=e^{\varepsilon B_3},\ \ n=4,\ 6,$ respectively.

In general, one proceeds as follows.

Step 1. Find $\{f_j\}$ so that

$$e^{\varepsilon B_i}B_p=\sum_{j=1}^{k}f_jB_je^{\varepsilon B_i} \tag{3.3.5}$$

where $i = 1, 2, \; p = 1, \ldots, k$, and k is the dimension of the Lie algebra L. Since Ado's theorem [**?**] guarantees the existence of a matrix representation for every finite-dimensional Lie algebra, one can treat the operations in L as matrix elements so that, without loss of generality, L is associative.

Step 2. Differentiate equations (3.0.2) and (3.0.6) with respect to ε to obtain equations (3.3.3) and (3.3.4), respectively. Then appropriately and recursively substitute equation (3.3.5) into equations (3.3.3) and (3.3.4). Thus in each case one obtains an equation of the form

$$\sum_{j=1}^{k} \alpha_j(\varepsilon) B_j \prod_{i=1}^{k} e^{a_i B_i} = B_3 e^{\varepsilon B_3}, \tag{3.3.6}$$

for specific functions $\alpha_j(\varepsilon)$.

Step 3. Assume that expressions (3.0.2) and (3.0.6) hold. Consequently, this yields necessary conditions that $\{\alpha_j(\varepsilon)\}$ must satisfy, namely the nonlinear system of first order ODEs

$$\begin{aligned} &\alpha_3(\varepsilon) = 1, \\ &\alpha_i(\varepsilon) = 0, \; i \neq 3, \end{aligned} \tag{3.3.7}$$

with initial conditions (3.3.1) and (3.3.2), respectively.

Step 4. Check that the solution of the ODE system (3.3.7) solves respectively expressions (3.0.2) or (3.0.6) using either the matrix or operator method.

3.3.2 *Example* $sl(2,\mathbb{R})$

Theorem 3.3.1. *For $sl(2,\mathbb{R})$ the following identities hold for any ε.*

$$\begin{aligned} &e^{\varepsilon X}Y \equiv (Y+\varepsilon Z-\varepsilon^2 X)e^{\varepsilon X}, \;\; e^{\varepsilon X}Z \equiv (Z-2\varepsilon X)\,e^{\varepsilon X}, \\ &e^{\varepsilon Y}X \equiv (X-\varepsilon Z-\varepsilon^2 Y)e^{\varepsilon Y}, e^{\varepsilon Y}Z = (Z+2\varepsilon Y)e^{\varepsilon Y}. \end{aligned} \tag{3.3.8}$$

Proof. From the commutator relations (3.0.7), one directly obtains

$$\begin{aligned} XY &= YX + Z, \\ XZ &= ZX - 2X, \\ YZ &= ZY + 2Y. \end{aligned}$$

Hence $X^2Y = YX^2 + 2ZX - 2X$. Then it is easy to show that $X^nY = YX^n + nZX^{n-1} - n(n-1)X^{n-1}$, $n \in \mathbb{N}$.

Similarly, one can show that the following relations hold.

$$\begin{aligned} X^nZ &= ZX^n - 2nX^n, \\ Y^nX &= XY^n - nZY^{n-1} - n(n-1)Y^{n-1}, \\ Y^nZ &= ZY^n + 2nY^n. \end{aligned}$$

Consequently,

$$e^{\varepsilon X}Y = Ye^{\varepsilon X} + \varepsilon Z \sum_{n=1}^{\infty} \frac{\varepsilon^{n-1}}{(n-1)!}X^{n-1} - \varepsilon^2 X \sum_{n=2}^{\infty} \frac{\varepsilon^{n-2}}{(n-2)!}X^{n-2} =$$

$$(Y + \varepsilon Z - \varepsilon^2 X)e^{\varepsilon X}.$$

Similarly, one obtains the remaining relations in (3.3.8), completing the proof. □

Now to proceed further, we differentiate equation (3.0.6) with respect to ε. This yields

$$a'(\varepsilon)Xe^{\varepsilon Z} + b'(\varepsilon)e^{a(\varepsilon)X}Ye^{b(\varepsilon)Y}e^{c(\varepsilon)X}e^{d(\varepsilon)Y} + c'(\varepsilon)e^{a(\varepsilon)X}e^{b(\varepsilon)Y}Xe^{c(\varepsilon)X}e^{d(\varepsilon)Y} \\ + d'(\varepsilon)e^{a(\varepsilon)X}e^{b(\varepsilon)Y}e^{c(\varepsilon)X}Ye^{d(\varepsilon)Y} = Ze^{\varepsilon Z}. \tag{3.3.9}$$

Using the relations in Theorem (3.3.1), one finds that equation (3.3.9) becomes

$[\alpha_1(\varepsilon)X + \alpha_2(\varepsilon)Y + \alpha_3(\varepsilon)Z]e^{\varepsilon Z} = Ze^{\varepsilon Z}$ with

$$\alpha_1(\varepsilon) = a' - a^2\Big(b' + d' + b\Big(-bc' + bc^2d' + 2cd'\Big)\Big) + c' \\ + 2abc' - 2acd' - c^2d' - 2abc^2d' = 0, \tag{3.3.10}$$

$$\alpha_2(\varepsilon) = b' + d' + b\Big(-bc' + bc^2d' + 2cd'\Big) = 0, \tag{3.3.11}$$

$$\alpha_3(\varepsilon) = a\Big(b' + d' + b\Big(-bc' + bc^2d' + 2cd'\Big)\Big) - bc' + cd' + bc^2d' = 1. \tag{3.3.12}$$

After substituting equation (3.3.11) into equations (3.3.10) and (3.3.12), one gets respectively

$$a' + c' - c^2d' - 2a(-bc' + cd' + bc^2d') = 0, \tag{3.3.13}$$

$$-bc' + cd' + bc^2d' = 1. \tag{3.3.14}$$

After substituting equation (3.3.14) into each of equations (3.3.13) and (3.3.11), one obtains

$$a' + c' - c^2d' - 2a = 0, \tag{3.3.15}$$

$$b' + d' + bcd' + b = 0. \tag{3.3.16}$$

After subtracting c times equation (3.3.16) from equation (3.3.14), one finds that

$(bc)' + bc = -1$. Hence

$$b(\varepsilon) = \frac{e^{-\varepsilon} - 1}{c(\varepsilon)}. \tag{3.3.17}$$

Substitution of equation (3.3.17) into equation (3.3.16) leads to $d' = \frac{ce^{\varepsilon} - c' \, (e^{\varepsilon} - 1)}{c^2}$.

Consequently,

$$d(\varepsilon) = \frac{e^{\varepsilon} - 1}{c(\varepsilon)}. \tag{3.3.18}$$

After substituting equation (3.3.18) into equation (3.3.15), one gets

$2(a + e^{\varepsilon} c) = (a + e^{\varepsilon} c)'$. Hence $a(\varepsilon) = -c(\varepsilon)e^{\varepsilon}$.

Thus the solution to the system of differential equations (3.28)-(3.30) with initial condition (3.19) is given by (3.1.11).

Chapter 4

Results

Using the procedures described in chapter three, the results for all relevant three- and four-dimensional Lie algebras are presented in Tables 1 and 2, respectively.

Lie algebra; commutators	Composition equation	Solution
$sl(2,\mathbb{R})$ $[X,Y]=Z$ $[X,Z]=-2X$ $[Y,Z]=2Y$	$e^{a(\varepsilon)X}e^{b(\varepsilon)Y}e^{c(\varepsilon)X}e^{d(\varepsilon)Y}=e^{\varepsilon Z}$	$d(\varepsilon)$ is an arbitrary function satisfying $d(\varepsilon)\neq 0$ when $\varepsilon\neq 0$ $a(\varepsilon)=\frac{e^{\varepsilon}-e^{2\varepsilon}}{d(\varepsilon)}$ $b(\varepsilon)=\frac{e^{-\varepsilon}-1}{e^{\varepsilon}-1}d(\varepsilon)$ $c(\varepsilon)=\frac{e^{\varepsilon}-1}{d(\varepsilon)}$
Parallel parking problem, $S_{3,3}$ with constant $r=0$ $[R,Y]=X$ $[R,X]=-Y$ $[X,Y]=0$	$e^{a(\varepsilon)R}e^{b(\varepsilon)Y}e^{c(\varepsilon)R}e^{d(\varepsilon)Y}=e^{\varepsilon X}$	$d(\varepsilon)$ is an arbitrary function satisfying $d(\varepsilon)\neq k\pi$ for every $k\in\mathbb{Z}$ when $\varepsilon\neq 0$ $a(\varepsilon)=-\arctan\left(\frac{\varepsilon}{d(\varepsilon)}\right)$ $\lvert b(\varepsilon)\rvert=\sqrt{d^2+\varepsilon^2}$ $c(\varepsilon)=\arctan\left(\frac{\varepsilon}{d(\varepsilon)}\right)$
Euler angles problem, $so(3,\mathbb{R})$ $[X,Y]=Z$ $[X,Z]=-Y$ $[Y,Z]=X$	$e^{a(\varepsilon)X}e^{b(\varepsilon)Y}e^{c(\varepsilon)X}e^{d(\varepsilon)Y}=e^{\varepsilon Z}$	Any $c(\varepsilon)$ satisfying $c(\varepsilon)\neq k\pi$ for every $k\in\mathbb{Z}$ when $\varepsilon\neq 0$ and $\left\lvert\frac{\sin\varepsilon}{\sin c(\varepsilon)}\right\rvert\leq 1$ with $a(\varepsilon)=-\arccos\left(\frac{\cos c(\varepsilon)}{\cos\varepsilon}\right)$ $b(\varepsilon)=-\arcsin\left(\frac{\sin\varepsilon}{\sin c(\varepsilon)}\right)$ $d(\varepsilon)=\arccos\left(-\frac{\sin a(\varepsilon)}{\sin c(\varepsilon)}\right)$; $a(\varepsilon)=\arccos\left(\frac{\cos c(\varepsilon)}{\cos\varepsilon}\right)$ $b(\varepsilon)=\pi+\arcsin\left(\frac{\sin\varepsilon}{\sin c(\varepsilon)}\right)$ $d(\varepsilon)=\arccos\left(-\frac{\sin a(\varepsilon)}{\sin c(\varepsilon)}\right)$
$n_{3,1}$ $[X,Y]=Z$ $[X,Z]=0$ $[Z,Y]=0$	$e^{a(\varepsilon)X}e^{b(\varepsilon)Y}e^{c(\varepsilon)X}e^{d(\varepsilon)Y}=e^{\varepsilon Z}$	$d(\varepsilon)$ is an arbitrary function satisfying $d(\varepsilon)\neq 0$ when $\varepsilon\neq 0$ $a(\varepsilon)=-\frac{\varepsilon}{d(\varepsilon)}$ $b(\varepsilon)=-d(\varepsilon)$ $c(\varepsilon)=\frac{\varepsilon}{d(\varepsilon)}$
$S_{3,1}$ $[Y,Z]=-Y$ $[Y,X]=0$ $[Z,X]=rX+Y$ where r is a constant satisfying $\lvert r\rvert\leq 1$	$e^{a(\varepsilon)X}e^{b(\varepsilon)Z}e^{c(\varepsilon)X}e^{d(\varepsilon)Z}=e^{\varepsilon Y}$	$d(\varepsilon)$ is an arbitrary function satisfying $d(\varepsilon)\neq 0$ when $\varepsilon\neq 0$ $b(\varepsilon)=-d(\varepsilon)$ $a(\varepsilon)=\frac{\varepsilon(1-r)}{1-e^{(r-1)d(\varepsilon)}}$ $c(\varepsilon)=-a\,e^{rd(\varepsilon)}$
$S_{3,2}$ $[Z,X]=X$ $[Z,Y]=X+Y$ $[X,Y]=0$	$e^{a(\varepsilon)Y}e^{b(\varepsilon)Z}e^{c(\varepsilon)Y}e^{d(\varepsilon)Z}=e^{\varepsilon X}$	$d(\varepsilon)$ is an arbitrary function satisfying $d(\varepsilon)\neq 0$ when $\varepsilon\neq 0$ $b(\varepsilon)=-d(\varepsilon)$ $c(\varepsilon)=-\frac{\varepsilon e^{d(\varepsilon)}}{d(\varepsilon)}$ $a(\varepsilon)=\frac{\varepsilon}{d(\varepsilon)}$
$S_{3,3}$ general case $[R,X]=rX-Y$ $[R,Y]=X+rY$ $[X,Y]=0$ where r is a non-negative constant.	$e^{a(\varepsilon)R}e^{b(\varepsilon)Y}e^{c(\varepsilon)R}e^{d(\varepsilon)Y}=e^{\varepsilon X}$	$d(\varepsilon)$ is an arbitrary function satisfying $d(\varepsilon)\neq k\pi$ for every $k\in\mathbb{Z}$ when $\varepsilon\neq 0$ $a(\varepsilon)=-\arctan\left(\frac{\varepsilon}{d(\varepsilon)}\right)$ $\lvert b(\varepsilon)\rvert=\sqrt{d^2+\varepsilon^2}\,e^{r\arctan\left(\frac{\varepsilon}{d(\varepsilon)}\right)}$ $c(\varepsilon)=\arctan\left(\frac{\varepsilon}{d(\varepsilon)}\right)$

Table 4.1: Results for three-dimensional Lie algebras

Lie algebra; commutators	Composition equation	Solution
$S_{4,2}$ $[W,X]=X$ $[W,Y]=X+Y$ $[W,Z]=Y+Z$ $[X,Y]=0$ $[X,Z]=0$ $[Z,Y]=0$	$e^{a(\varepsilon)W}e^{b(\varepsilon)Z}e^{c(\varepsilon)W}e^{d(\varepsilon)Z}e^{f(\varepsilon)W}=e^{\varepsilon Y}$	$f(\varepsilon)$ is an arbitrary function satisfying $f(\varepsilon)\neq 0$ when $\varepsilon\neq 0$ $a(\varepsilon)=f(\varepsilon)$ $b(\varepsilon)=\frac{\varepsilon}{2f(\varepsilon)}e^{-f(\varepsilon)}$ $c(\varepsilon)=-2f(\varepsilon)$ $d(\varepsilon)=-\frac{\varepsilon}{2f(\varepsilon)}e^{f(\varepsilon)}$
$S_{4,7}$ $[Y,Z]=X$ $[W,Y]=-Z$ $[W,Z]=Y$ $[W,X]=0$ $[X,Y]=0$ $[X,Z]=0$	$e^{a(\varepsilon)Z}e^{b(\varepsilon)W}e^{c(\varepsilon)Z}e^{d(\varepsilon)W}e^{f(\varepsilon)Z}=e^{\varepsilon Y}$	$f(\varepsilon)$ is an arbitrary function satisfying $f(\varepsilon)\neq k\pi$ for every k $\in\mathbb{Z}$ when $\varepsilon\neq 0$ $a(\varepsilon)=f(\varepsilon)$ $b(\varepsilon)=-\arctan(\frac{\varepsilon}{2f(\varepsilon)})$ $c(\varepsilon)=\varepsilon\frac{\sqrt{\varepsilon^2+4f(\varepsilon)^2}}{2f(\varepsilon)}$ $d(\varepsilon)=\arctan(\frac{\varepsilon}{2f(\varepsilon)})$
$S_{4,9}$ $[Y,Z]=X$ $[W,Y]=rY-Z$ $[W,Z]=Y+rZ$ $[W,X]=2rX$ $[X,Y]=0$ $[X,Z]=0$	$e^{a(\varepsilon)W}e^{b(\varepsilon)Z}e^{c(\varepsilon)W}e^{d(\varepsilon)Z}e^{f(\varepsilon)W}=e^{\varepsilon Y}$	$f(\varepsilon)$ is an arbitrary function satisfying $f(\varepsilon)\neq k\pi$ for every k $\in\mathbb{Z}$ when $\varepsilon\neq 0$ $a(\varepsilon)=f(\varepsilon)$ $b(\varepsilon)=-\arctan(\frac{\varepsilon}{2f(\varepsilon)})$ $c(\varepsilon)=\varepsilon\frac{\sqrt{\varepsilon^2+4f(\varepsilon)^2}}{2f(\varepsilon)}e^{-rb}$ $d(\varepsilon)=\arctan(\frac{\varepsilon}{2f(\varepsilon)})$
$S_{4,10}$ $[Y,Z]=X$ $[W,Y]=Y$ $[W,Z]=Y+Z$ $[W,X]=2X$ $[X,Y]=0$ $[X,Z]=0$	$e^{a(\varepsilon)W}e^{b(\varepsilon)Z}e^{c(\varepsilon)W}e^{d(\varepsilon)Z}e^{f(\varepsilon)W}e^{g(\varepsilon)Z}=e^{\varepsilon Y}$	$a(\varepsilon)$ and $c(\varepsilon)$ are arbitrary functions satisfying $a(\varepsilon)c(\varepsilon)\neq 0$, $a(\varepsilon)+c(\varepsilon)\neq 0$, and $c(\varepsilon)^2+a(\varepsilon)c(\varepsilon)\geq 0$ $f(\varepsilon)=-(a(\varepsilon)+c(\varepsilon))$ $b(\varepsilon)=$ $\varepsilon\left(\frac{c(\varepsilon)\pm\sqrt{c(\varepsilon)^2+a(\varepsilon)c(\varepsilon)}}{a(\varepsilon)c(\varepsilon)}\right)e^{-a(\varepsilon)}$ $g(\varepsilon)=-\frac{\varepsilon+b(\varepsilon)c(\varepsilon)e^{a(\varepsilon)}}{a(\varepsilon)+c(\varepsilon)}$ $d(\varepsilon)=-g(\varepsilon)e^{-(a(\varepsilon)+c(\varepsilon))}-b(\varepsilon)e^{-c(\varepsilon)}$ and in the limiting case when $a(\varepsilon)=0$, $f(\varepsilon)=-c(\varepsilon)$ $d(\varepsilon)=\frac{\varepsilon e^{-c(\varepsilon)}}{c(\varepsilon)}$ $b(\varepsilon)=g(\varepsilon)=-\frac{\varepsilon}{2c(\varepsilon)}$

Table 4.2: Results for four-dimensional Lie algebras

The sketch of the proofs of the results, exhibited in Tables 1 and 2, follow in Chapters 5 and 6, respectively.

Chapter 5

Three-dimensional Lie algebras

In this chapter, we present the proofs of the results presented in Table 1.

5.1 Parallel parking problem (Lie algebra $S_{3,3}$ with constant $r = 0$)

5.1.1 Model example

To illustrate parallel parking, as an example, consider a unicycle that performs forward and backward translations as well as rotations. The unicycle is represented by a straight line with centre located at (x, y) and initial orientation parallel to the y-axis with its centre located at $(0, 0)$ in Figure 5.1. The aim is to move the unicycle so that its centre finishes at $(\varepsilon, 0)$ with the vehicle parallel to its initial orientation by a succession of rotations and translations in the same direction as the straight line. As will be illustrated in Figure 5.1, the minimum number of steps that start with a non-zero translation is four. Let $d(\varepsilon)$ and $b(\varepsilon)$ be the translations in the first and third steps, respectively and let $c(\varepsilon)$ and $a(\varepsilon)$ be the angles of rotation in the second and last steps, respectively. Since the direction the vehicle is facing at the end must be the same as at the start, one must have $a(\varepsilon) = -c(\varepsilon)$. From Figure 5.1, one sees that $d(\varepsilon)+b(\varepsilon)\cos c(\varepsilon) = 0$ and $b(\varepsilon)\sin c(\varepsilon) = \varepsilon$. Hence the solution to this problem is given by

$$a(\varepsilon) = -c(\varepsilon), \;\; b(\varepsilon) = \frac{\varepsilon}{\sin\ c(\varepsilon)}, \;\; d(\varepsilon) = -\varepsilon\cot c(\varepsilon), \tag{5.1.1}$$

where $c(\varepsilon)$ is any continuous function chosen so that $b(\varepsilon)$ and $d(\varepsilon)$ are continuous functions, and satisfying $c(\varepsilon) \neq k\pi$ for every $k \in \mathbb{Z}$ and $\varepsilon \neq 0$ with $a(0) = b(0) = c(0) = d(0) = 0$.

Alternatively, one could treat $d(\varepsilon)$ as an arbitrary function. Here the solution as reflected by Figure 5.1 is given by

$$a(\varepsilon) = -c(\varepsilon), \;\; c(\varepsilon) = -\arctan(\frac{\varepsilon}{d}), \;\; |b(\varepsilon)| = \sqrt{d^2 + \varepsilon^2},$$

where $d(\varepsilon)$ is any continuous function chosen so that $c(\varepsilon)$ is a continuous function, and satisfying $d(\varepsilon) \neq 0$ and $-\frac{2}{\pi\varepsilon} < d(\varepsilon) < \frac{2}{\pi\varepsilon}$ for every $\varepsilon \neq 0$ with $a(0) = b(0) = c(0) = d(0) = 0$.
Note that, as a special case, one obtains the trivial solution $d = 0, \;\; c = \frac{\pi}{2}, \;\; b = \varepsilon, \;\; a = -\frac{\pi}{2}$.

Next, the solution to the parallel parking problem is presented using the methods described in Chapter 3 which can be applied to any Lie algebra.

5.1.2 Solution using the operator method

An operator representation for this Lie algebra is given by

$$X = -\frac{\partial}{\partial x}, \;\; Y = \frac{\partial}{\partial y}, \;\; R = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

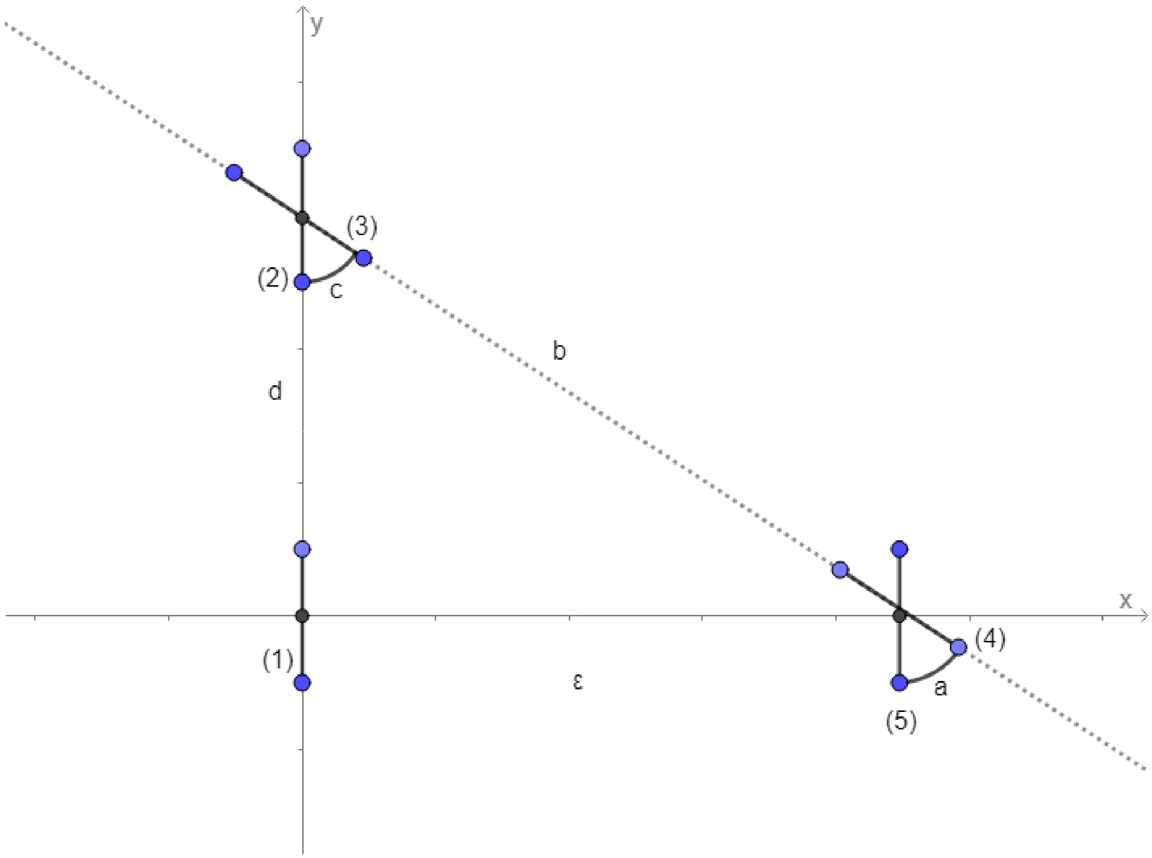

Figure 5.1: Illustration of solution of parallel parking problem. (1) represents initial configuration of vehicle with successive configurations represented by (2) to (5).

Consequently,

$$
\begin{aligned}
M &= e^{a(\varepsilon)R}e^{b(\varepsilon)Y}e^{c(\varepsilon)R}e^{d(\varepsilon)Y}(x,y) = e^{a(\varepsilon)R}e^{b(\varepsilon)Y}e^{c(\varepsilon)R}(x,y+d) \\
&= e^{a(\varepsilon)R}e^{b(\varepsilon)Y}(x\cos\, c\, + y\sin\, c\,, d+y\cos\, c\, - x\sin\, c\,) \\
&= e^{a(\varepsilon)R}(b\sin\, c\, + x\cos\, c\, + y\sin\, c\,, d+b\cos c + y\cos\, c\, - x\sin\, c\,) \\
&= (b\sin\, c\, + x\cos\,(a+c) + y\sin\,(a+c), d+b\cos\, c\, - x\sin\,(a+c) \\
&\quad + y\cos\,(a+c)).
\end{aligned}
$$

Then $M = e^{\varepsilon X}(x,y) = (x-\varepsilon, y)$ iff

$$\sin\,(a+c) = 0,\ \cos\,(a+c) = 1, d+b\cos\, c\, = 0, -b\sin\, c\, = \varepsilon. \quad (5.1.2)$$

The solution to the system of equations (5.1.2) is given by (5.1.1).

5.1.3 Solution using the matrix representation method

A matrix representation of $S_{3,3}$ with constant $r = 0$ is given by

$$X = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence $X^n = Y^n = 0\,, \quad n \geq 2.$

Thus $e^{\varepsilon Y} = I + \varepsilon Y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\varepsilon \\ 0 & 0 & 1 \end{pmatrix}$, and $e^{\varepsilon X} = \begin{pmatrix} 1 & 0 & \varepsilon \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

For all $k > 0$, one has

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^k = \begin{cases} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{if } k = 1 \pmod{4} \\ \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{if } k = 2 \pmod{4} \\ \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{if } k = 3 \pmod{4} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{if } k = 4 \pmod{4} \end{cases}.$$

Consequently,

$$e^{\varepsilon R} = \begin{pmatrix} \cos\,\varepsilon & -\sin\,\varepsilon & 0 \\ \sin\,\varepsilon & \cos\,\varepsilon & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence

$$M = e^{a(\varepsilon)R} e^{b(\varepsilon)Y} e^{c(\varepsilon)R} e^{d(\varepsilon)Y} =$$

$$\begin{pmatrix} \cos\,(a+c) & -\sin\,(a+c) & d\,\sin\,(a+c)\,+b\,\sin\,a \\ \sin\,(a+c) & \cos\,(a+c) & -d\,\cos\,(a+c)\,-b\,\cos\,a \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $M = e^{\varepsilon X}$ is satisfied iff

$$\cos\,(a+c)\,=1,\quad \sin\,(a+c)\,=0,\; b\,\sin\,a\,=\varepsilon,\; d+b\,\cos\,a\,=0. \quad (5.1.3)$$

The solution to the system of equations (5.1.3) is given by (5.1.1).

5.1.4 *Solution using the DE method*

For all $n \geq 0$, one can show that

$$\begin{aligned} &X^n Y = Y X^n, \quad X^n R = R X^n - nY X^{n-1}, \\ &R^n X = X \sum_{\substack{i=0 \\ i \text{ even}}}^{n} (-1)^{\frac{i}{2}} \binom{n}{i} R^{n-i} - Y \sum_{\substack{i=1 \\ i \text{ odd}}}^{n} (-1)^{\frac{i+1}{2}} \binom{n}{i} R^{n-i}, \\ &R^n Y = Y \sum_{\substack{i=0 \\ i \text{ even}}}^{n} (-1)^{\frac{i}{2}} \binom{n}{i} R^{n-i} + X \sum_{\substack{i=1 \\ i \text{ odd}}}^{n} (-1)^{\frac{i+1}{2}} \binom{n}{i} R^{n-i}. \end{aligned} \quad (5.1.4)$$

Using these results, one can easily obtain the identities

$$e^{\varepsilon Y}X \equiv Xe^{\varepsilon Y},\ \ e^{\varepsilon Y}R \equiv (R-\varepsilon X)e^{\varepsilon Y},\ \ e^{\varepsilon R}X \equiv (\cos\varepsilon\, X - \sin\varepsilon\, Y)e^{\varepsilon R},$$
$$e^{\varepsilon R}Y \equiv (\cos\varepsilon\, Y + \sin\varepsilon\, X)e^{\varepsilon R}. \tag{5.1.5}$$

Now to proceed, we differentiate with respect to ε the equation

$$e^{a(\varepsilon)R}e^{b(\varepsilon)Y}e^{c(\varepsilon)R}e^{d(\varepsilon)Y} = e^{\varepsilon X}. \tag{5.1.6}$$

Thus

$$\begin{aligned} &a'Re^{a(\varepsilon)R}e^{b(\varepsilon)Y}e^{c(\varepsilon)R}e^{d(\varepsilon)Y} + b'e^{a(\varepsilon)R}Ye^{b(\varepsilon)Y}e^{c(\varepsilon)R}e^{d(\varepsilon)Y} + \\ &\ c'e^{a(\varepsilon)R}e^{b(\varepsilon)Y}Re^{c(\varepsilon)R}e^{d(\varepsilon)Y} + d'e^{a(\varepsilon)R}e^{b(\varepsilon)Y}e^{c(\varepsilon)R}Ye^{d(\varepsilon)Y} = Xe^{\varepsilon X}. \end{aligned} \tag{5.1.7}$$

Using the identities in (5.1.5), one can show that equation (5.1.7) leads to the ODE system

$$\begin{aligned} &a' + c' = 0, \quad \cos a\, b' + b\sin a\, c' + \cos(a+c)\, d' = 0, \\ &\sin a\, b' - b\cos a\, c' + \sin(a+c)\, d' = 1. \end{aligned} \tag{5.1.8}$$

It is easy to show that the solution to the ODE system (5.1.8) is given by (5.1.1).

5.2 Euler angles problem (Lie algebra so $(3,\mathbb{R})$)

5.2.1 Solution using the operator method

An operator representation for this Lie algebra is given by

$$X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x},\ \ Y = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y},\ \ Z = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}.$$

Here setting $e^{a(\varepsilon)X}e^{b(\varepsilon)Y}e^{c(\varepsilon)X}e^{d(\varepsilon)Y}=e^{\varepsilon Z}$ leads *to* the system of nine equations

$$\begin{gathered}
\sin c\ \sin b=-\sin\varepsilon,\ \cos c-\cos a\ \cos\varepsilon=0,\ \sin c\ \cos b+\\
\sin a\ \cos\varepsilon=0,\\
\cos d\ \sin c+\sin a=0,\ \cos b\ \cos c\ \cos d-\sin b\ \sin d=\cos a,\\
\sin b\ \cos c\ \cos d+\cos b\ \sin d=0,\ \sin b\ \cos d+\\
\cos b\ \cos c\ \sin d=\sin a\ \sin\varepsilon,\\
\sin c\ \sin d-\cos a\ \sin\varepsilon=0,\ \cos b\ \cos d\\
-\sin b\ \cos c\ \sin d=\cos\varepsilon,
\end{gathered}\tag{5.2.1}$$

whose solutions are given by

$$\begin{gathered}
a(\varepsilon)=-\arccos\left(\frac{\cos c(\varepsilon)}{\cos\varepsilon}\right),\ b(\varepsilon)=-\arcsin\left(\frac{\sin\varepsilon}{\sin c(\varepsilon)}\right),\\
d(\varepsilon)=\arccos\left(-\frac{\sin a(\varepsilon)}{\sin c(\varepsilon)}\right);
\end{gathered}\tag{5.2.2}$$

$$\begin{gathered}
a(\varepsilon)=\arccos\left(\frac{\cos c(\varepsilon)}{\cos\varepsilon}\right),\ b(\varepsilon)=\pi+\arcsin\left(\frac{\sin\varepsilon}{\sin c(\varepsilon)}\right),\\
d(\varepsilon)=\arccos\left(-\frac{\sin a(\varepsilon)}{\sin c(\varepsilon)}\right).
\end{gathered}\tag{5.2.3}$$

In both solutions, for any $\varepsilon\neq0$, $c(\varepsilon)$ is any continuous function chosen so that $a(\varepsilon)$ and $d(\varepsilon)$ are continuous, and satisfying $\left|\frac{\sin\varepsilon}{\sin c(\varepsilon)}\right|\leq 1$ with $c(\varepsilon)\neq k\pi$ for every k in$\mathbb{Z}$ and such that $a(0)=b(0)=c(0)=d(0)=0$.

5.2.2 Solution using the matrix representation method

A matrix representation of $so(3,\mathbb{R})$ is given by

$$X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Analogous to the way of obtaining the rotation matrix e^{cR} in the parking problem, one finds that

$$e^{\varepsilon X} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varepsilon & -\sin\varepsilon \\ 0 & \sin\varepsilon & \cos\varepsilon \end{pmatrix}, \quad e^{\varepsilon Y} = \begin{pmatrix} \cos\varepsilon & 0 & \sin\varepsilon \\ 0 & 1 & 0 \\ -\sin\varepsilon & 0 & \cos\varepsilon \end{pmatrix},$$

$$e^{\varepsilon Z} = \begin{pmatrix} \cos\varepsilon & -\sin\varepsilon & 0 \\ \sin\varepsilon & \cos\varepsilon & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Consequently, one can show that the entries $\{a_{ij}\}$ of the matrix $M = e^{a(\varepsilon)X}e^{b(\varepsilon)Y}e^{c(\varepsilon)X}e^{d(\varepsilon)Y}$ are given by

$$a_{11} = \cos b \cos d - \sin b \cos c \sin d, \quad a_{12} = \sin b \sin c,$$
$$a_{13} = \cos b \sin d + \sin b \cos c \cos d,$$
$$a_{21} = \sin a \sin b \cos d + \cos a \sin c \sin d + \sin a \cos b \cos c \sin d,$$
$$a_{22} = \cos a \cos c - \sin a \cos b \sin c,$$
$$a_{23} = \sin a \sin b \sin d - \cos a \sin c \cos d - \sin a \cos b \cos c \cos d,$$
$$a_{31} = -\cos a \sin b \cos d + \sin a \sin c \sin d - \cos a \cos b \cos c \sin a,$$
$$a_{32} = \sin a \cos c + \sin c \cos a \cos b,$$
$$a_{33} = -\sin d \cos a \sin b - \sin a \sin c \cos d + \cos a \cos b \cos c \cos d.$$

From the matrix equation $M = e^{\varepsilon Z}$, one obtains a system of equations that can be simplified to (5.2.1). Hence the solutions are given by (5.2.2) and (5.2.3).

5.2.3 Solution using the DE method

One can show the following identities hold for all ε.

$$\begin{aligned} &e^{\varepsilon X} Y \equiv (\cos\ \varepsilon\ Y + \sin\ \varepsilon\ Z) e^{\varepsilon X}, \quad e^{\varepsilon X} Z \equiv (\cos\ \varepsilon\ Z - \sin\ \varepsilon\ Y) e^{\varepsilon X}, \\ &e^{\varepsilon Y} X \equiv (\cos\ \varepsilon\ X - \sin\ \varepsilon\ Z) e^{\varepsilon Y}, \quad e^{\varepsilon Y} Z \equiv (\cos\ \varepsilon\ Z + \sin\ \varepsilon\ X) e^{\varepsilon Y}. \end{aligned} \tag{5.2.4}$$

After differentiating with respect to ε the equation $e^{a(\varepsilon)X} e^{b(\varepsilon)Y} e^{c(\varepsilon)X} e^{d(\varepsilon)Y} = e^{\varepsilon Z}$ and using the identities in (5.2.4), one obtains the simplified ODE system

$$\begin{aligned} \cos\ b\ a' + c' = -\sin\ b\ \cos\ a\ ,\ \sin\ b\ a' + \sin\ c\ d' \\ = \cos\ a\ \cos\ b,\ b' + \cos\ c\ d' \\ = \sin\ a\ . \end{aligned} \tag{5.2.5}$$

The ODE system (5.2.5) admits (5.2.2) and (5.2.3) as solutions.

5.3 Lie algebra $n_{3,1}$

5.3.1 Solution using the operator method

From [12], an operator representation for $n_{3,1}$is given by
$X = \frac{\partial}{\partial x}, \ \ Y = x\frac{\partial}{\partial z}, \ \ Z = \frac{\partial}{\partial z}$.

Consequently, equation $e^{a(\varepsilon)X} e^{b(\varepsilon)Y} e^{c(\varepsilon)X} e^{d(\varepsilon)Y}(x,z) = e^{\varepsilon Z}(x,z)$ leads to the equation

$$(x + a + c, (b + d)(x + a) + dc + z) = (x, z + \varepsilon). \tag{5.3.1}$$

It is easy to see that the solution to equation (5.3.1) is given by

$$a(\varepsilon) = -\frac{\varepsilon}{d(\varepsilon)}, \quad b(\varepsilon) = -d(\varepsilon), \quad c(\varepsilon) = \frac{\varepsilon}{d(\varepsilon)}, \tag{5.3.2}$$

where $d(\varepsilon)$ is any continuous function chosen so that $a(\varepsilon)$ and $c(\varepsilon)$ are continuous functions, and satisfying $d(\varepsilon) \neq 0$ for any $\varepsilon \neq 0$ with $a(0) = b(0) = c(0) = d(0) = 0$.

5.3.2 Solution using the matrix representation method

From [9], a matrix representation of $n_{3,1}$is given by

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence $X^2 = Y^2 = Z^2 = 0$. Then one can show that

$$e^{\varepsilon X} = \begin{pmatrix} 1 & \varepsilon & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad e^{\varepsilon Y} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \varepsilon \\ 0 & 0 & 1 \end{pmatrix}, \quad e^{\varepsilon Z} = \begin{pmatrix} 1 & 0 & \varepsilon \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Accordingly, one can show that

$$M = e^{a(\varepsilon)X} e^{b(\varepsilon)Y} e^{c(\varepsilon)X} e^{d(\varepsilon)Y} = \begin{pmatrix} 1 & a+c & cd+ad+ab \\ 0 & 1 & b+d \\ 0 & 0 & 1 \end{pmatrix}.$$

Consequently $M = e^{\varepsilon Z}$ yields the system of equations

$$a + c = 0, \quad b + d = 0, \quad cd + ad + ab = \varepsilon,$$

whose solution is given by (5.3.2).

5.3.3 *Solution using the DE method*

One can readily obtain the following identities which hold for all ε.

$$e^{\varepsilon X} Y \equiv (X+\varepsilon Z) e^{\varepsilon X}, \;\; e^{\varepsilon X} Z \equiv Z e^{\varepsilon X}, \;\; e^{\varepsilon Y} X \equiv (Y-\varepsilon Z)\, e^{\varepsilon Y}, \;\; e^{\varepsilon Y} Z \equiv Z e^{\varepsilon Y}. \tag{5.3.3}$$

After differentiating with respect to ε the equation $e^{a(\varepsilon)X} e^{b(\varepsilon)Y} e^{c(\varepsilon)X} e^{d(\varepsilon)Z} = e^{\varepsilon Z}$ and using the identities (5.3.3), one obtains the ODE system $a' + c' = 0, \;\; b' + d' = 0, \;\; ab' - bc' + (a+c)d' = 1$, whose solution is given by (5.3.2).

5.4 Lie algebra $S_{3,1}$

5.4.1 *Solution using the operator method*

An operator representation [12] for $S_{3,1}$ is given by

$$X = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \;\; Y = (1-r)\,\frac{\partial}{\partial x}, \;\; Z = -x\frac{\partial}{\partial x} - ry\frac{\partial}{\partial y}.$$

Consequently, the equation $e^{a(\varepsilon)X} e^{b(\varepsilon)Z} e^{c(\varepsilon)X} e^{d(\varepsilon)Z}\,(x,y) = e^{\varepsilon Y}\,(x,y)$ leads to the equation

$$\left(e^{-(b+d)}(x+a+ce^{b}), e^{-r(b+d)}(y+a+ce^{rb})\right) = ((1-r)\,\varepsilon + x, y). \tag{5.4.1}$$

It is easy to see that equation (5.4.1) is satisfied iff

$$b = -d, \;\; a + ce^{-d} = (1-r)\,\varepsilon, \;\; a + ce^{-rd} = 0. \tag{5.4.2}$$

The solution to the system of equations (5.4.2) is given by

$$a\,(\varepsilon) = \frac{\varepsilon(1-r)}{1-e^{(r-1)d(\varepsilon)}}, \;\; b\,(\varepsilon) = -d\,(\varepsilon), \;\; c\,(\varepsilon) = -a\; e^{rd(\varepsilon)}, \tag{5.4.3}$$

where $d\,(\varepsilon)$ is any continuous function chosen so that $a(\varepsilon)$ is a continuous function, and satisfying $d\,(\varepsilon) \neq 0$ for any $\varepsilon \neq 0$ with $a(0) = b(0) = c(0) = d(0) = 0$.

Note that in the limiting case where $r \to 1$, equation (5.4.3) becomes $a(\varepsilon) = -\frac{\varepsilon}{d(\varepsilon)}$, $b(\varepsilon) = -d(\varepsilon)$, and $c(\varepsilon) = -a\, e^{d(\varepsilon)}$.

5.4.2 Solution using the matrix representation method

A matrix representation of $S_{3,1}$ is given by

$$X = \begin{pmatrix} 0 & -r & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} r & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Hence for all $n \geq 2$ one has $X^n = Y^n = 0$ and $Z^n = \begin{pmatrix} r^n & 0 & 0 \\ 0 & 0 & 0 \\ r^{n-1} + r^{n-2} + \ldots + r + 1 & 0 & 1 \end{pmatrix}$.

Then one can easily show that

$$e^{\varepsilon X} = \begin{pmatrix} 1 & -r\varepsilon & 0 \\ 0 & 1 & 0 \\ 0 & -\varepsilon & 1 \end{pmatrix}, \quad e^{\varepsilon Y} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\varepsilon & 1 \end{pmatrix}, \quad e^{\varepsilon Z} = \begin{pmatrix} e^{\varepsilon r} & 0 & 0 \\ 0 & 1 & 0 \\ \frac{e^{\varepsilon} - e^{\varepsilon r}}{1-r} & 0 & e^{\varepsilon} \end{pmatrix}.$$

Hence

$$M = e^{a(\varepsilon)X} e^{b(\varepsilon)Z} e^{c(\varepsilon)X} e^{d(\varepsilon)Z} = \begin{pmatrix} e^{r(b+d)} & -ra - cre^{br} & 0 \\ 0 & 1 & 0 \\ \frac{e^{b+d} - e^{r(b+d)}}{1-r} & \frac{cr(e^{b} - e^{br})}{r-1} - a - ce^{b} & e^{b+d} \end{pmatrix}.$$

Then equation $M = e^{\varepsilon Y}$ leads to the system of equations

$$a + ce^{br} = 0, \quad b + d = 0, \quad \frac{cr(e^{b} - e^{br})}{1-r} + a + ce^{b} = \varepsilon, \tag{5.4.4}$$

with solution given by (5.4.3).

5.4.3 Solution using the DE method

One can show that the following identities hold for all ε.

$$\begin{aligned} &e^{\varepsilon X}Y \equiv Ye^{\varepsilon X},\ \ e^{\varepsilon X}Z \equiv (Z - r\varepsilon X - \varepsilon Y)\,e^{\varepsilon X}, \\ &e^{\varepsilon Z}X \equiv (e^{\varepsilon r}X + \frac{e^{\varepsilon} - e^{\varepsilon r}}{1-r}Y)e^{\varepsilon Z},\ \ e^{\varepsilon Z}Y \equiv (Y + \varepsilon Y)\,e^{\varepsilon Z}. \end{aligned} \tag{5.4.5}$$

After differentiating with respect to ε the equation $e^{a(\varepsilon)X}e^{b(\varepsilon)Z}e^{c(\varepsilon)X}e^{d(\varepsilon)Z} = e^{\varepsilon Y}$ and using the identities (5.4.5), one obtains the ODE system

$$a' - rab' + e^{rb}c' - r(a + ce^{rb})d' = 0,\ \ b' + d' = 0,\ \ \frac{e^{b} - e^{rb}}{1-r}(c' - rcd') - ce^{b}d' = 1, \tag{5.4.6}$$

whose solution is given by (5.4.3).

5.5 Lie algebra $S_{3,2}$

5.5.1 Solution using the operator method

From [12], an operator representation for $S_{3,2}$ is given by $X = \frac{\partial}{\partial x}$, $Y = y\frac{\partial}{\partial x}$, $Z = -x\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$.

Consequently, equation $e^{a(\varepsilon)Y}e^{b(\varepsilon)Z}e^{c(\varepsilon)Y}e^{d(\varepsilon)Z}(x,y) = e^{\varepsilon X}(x,y)$ leads to equation

$$\left(e^{-(b+d)}x + e^{-(b+d)}\left(a + ce^{b}\right)y + bce^{-d}, y + b + d\right) = (x + \varepsilon, y). \tag{5.5.1}$$

It is easy to see that equation (5.5.1) is satisfied iff

$$b = -d,\ a + ce^{-d} = 0,\ bce^{-d} = \varepsilon. \tag{5.5.2}$$

The solution to the system of equations (5.5.2) is given by

$$a(\varepsilon) = \frac{\varepsilon}{d(\varepsilon)},\ b(\varepsilon) = -d(\varepsilon)\ , c(\varepsilon) = -\frac{\varepsilon e^{d(\varepsilon)}}{d(\varepsilon)}, \tag{5.5.3}$$

where $d(\varepsilon)$ is any continuous function chosen so that $a(\varepsilon)$ and $c(\varepsilon)$ are continuous functions, and satisfying $d(\varepsilon) \neq 0$ for any $\varepsilon \neq 0$ with $a(0) = b(0) = c(0) = d(0) = 0$.

5.5.2 Solution using the matrix representation method

A matrix representation for $S_{3,2}$ is given by

$$X = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence for all $n \geq 2$ one has $X^n = Y^n = 0$ and $Z^n = \begin{pmatrix} 1 & n & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Then one can easily show that

$$e^{\varepsilon X} = \begin{pmatrix} 1 & 0 & -\varepsilon \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad e^{\varepsilon Y} = \begin{pmatrix} 1 & 0 & -\varepsilon \\ 0 & 1 & -\varepsilon \\ 0 & 0 & 1 \end{pmatrix}, \quad e^{\varepsilon Z} = \begin{pmatrix} e^{\varepsilon} & \varepsilon e^{\varepsilon} & 0 \\ 0 & e^{\varepsilon} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence

$$M = e^{a(\varepsilon)Y} e^{b(\varepsilon)Z} e^{c(\varepsilon)Y} e^{d(\varepsilon)Z} = \begin{pmatrix} e^{b+d} & e^{b+d}(b+d) & -ce^{b} - a - bce^{b} \\ 0 & e^{b+d} & -ce^{b} - a \\ 0 & 0 & 1 \end{pmatrix}.$$

Consequently, the equation $M = e^{\varepsilon X}$ leads to the system of equations

$$ce^{b} + a = 0, \quad b + d = 0, \quad ce^{b} + a + bce^{b} = \varepsilon,$$

whose solution is given by (5.5.3).

5.5.3 *Solution using the DE method*

One can show that the following identities hold for all ε.

$$e^{\varepsilon Y}X \equiv Xe^{\varepsilon Y},\ \ e^{\varepsilon Y}Z \equiv (Z-\varepsilon X-\varepsilon Y)\,e^{\varepsilon Y},\ \ e^{\varepsilon Z}X \equiv (e^{\varepsilon}X)\,e^{\varepsilon Z},$$
$$e^{\varepsilon Z}Y \equiv (e^{\varepsilon}Y+\varepsilon e^{\varepsilon}X)\,e^{\varepsilon Z}. \tag{5.5.4}$$

The differentiation with respect to ε of the equation $e^{a(\varepsilon)Y}e^{b(\varepsilon)Z}e^{c(\varepsilon)Y}e^{d(\varepsilon)Z} = e^{\varepsilon X}$ and the repeated use of the identities (5.5.4) leads to the ODE system

$$a'-ab'+e^{b}c'-(a+ce^{b})d'=0,\ \ b'+d'=0,\ \ -ab'+be^{b}c'-(a+cbe^{b})d'=1,$$

whose solution is given by (5.5.3).

5.6 Lie algebra $S_{3,3}$

5.6.1 *Solution using the operator method*

From [8] and [12], an operator representation for $S_{3,3}$ is given by

$$X=-\frac{\partial}{\partial x},\ \ Y=-\frac{\partial}{\partial y},\ \ R=(rx+y)\frac{\partial}{\partial x}+(ry-x)\frac{\partial}{\partial y}.$$

Consequently, equation $e^{a(\varepsilon)R}e^{b(\varepsilon)Y}e^{c(\varepsilon)R}e^{d(\varepsilon)Y}(x,y) = e^{\varepsilon X}(x,y)$ leads to equation

$$\begin{aligned}&(\cos(a+c)\,e^{-r(a+c)}x-\sin(a+c)\,e^{-r(a+c)}y+b\sin c\;e^{-rc},\\ &\cos(a+c)\,e^{-r(a+c)}\,y+\sin(a+c)\;e^{-r(a+c)}x-d-b\cos c\;e^{-rc})=\\ &(x-\varepsilon,y).\end{aligned} \tag{5.6.1}$$

It is easy to see that equation (5.6.1) is satisfied iff

$$\begin{aligned}&\sin(a+c)e^{-r(a+c)}=0,\ \cos(a+c)e^{-r(a+c)}=1,\ \ d+b\cos c\;\,e^{-rc}=0,\\ &b\sin c\;\,e^{-rc}=-\varepsilon.\end{aligned} \tag{5.6.2}$$

The solution to the system of equations (5.6.2) is given by

$$a = -c(\varepsilon), \;\; b = -\frac{\varepsilon e^{rc(\varepsilon)}}{\sin\, c(\varepsilon)}, \;\; d = \frac{\varepsilon}{\tan\, c(\varepsilon)}, \tag{5.6.3}$$

where $c(\varepsilon)$ is any continuous function chosen so that $b(\varepsilon)$ and $d(\varepsilon)$ are continuous functions, and satisfying $c(\varepsilon) \neq k\pi$ for every $k \in \mathbb{Z}$ and for any $\varepsilon \neq 0$ with $a(0) = b(0) = c(0) = d(0) = 0$.

5.6.2 Solution using the matrix representation method

A matrix representation of $S_{3,3}$ is given by

$$X = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -r \\ 0 & 0 & 0 \end{pmatrix}, \;\; Y = \begin{pmatrix} 0 & 0 & -r \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \;\; R = \begin{pmatrix} r & -1 & 0 \\ 1 & r & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence one can show that

$$e^{\varepsilon X} = \begin{pmatrix} 1 & 0 & \varepsilon \\ 0 & 1 & -r\varepsilon \\ 0 & 0 & 1 \end{pmatrix}, \;\; e^{\varepsilon Y} = \begin{pmatrix} 1 & 0 & -r\varepsilon \\ 0 & 1 & -\varepsilon \\ 0 & 0 & 1 \end{pmatrix},$$

$$e^{\varepsilon R} = \begin{pmatrix} \cos\,\varepsilon\, e^{r\varepsilon} & -\sin\,\varepsilon\, e^{r\varepsilon} & 0 \\ \sin\,\varepsilon\, e^{r\varepsilon} & \cos\,\varepsilon\, e^{r\varepsilon} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Consequently, the entries $\{a_{ij}\}$ of the matrix $M = e^{a(\varepsilon)R}e^{b(\varepsilon)Y}e^{c(\varepsilon)R}e^{d(\varepsilon)Y}$ are given by

$a_{11} = a_{22} = \cos\,(a+c)e^{r(a+c)}$, $\; a_{12} = -\sin\,(a+c)e^{r(a+c)}$,

$a_{21} = \sin\,(a+c)e^{r(a+c)}$,

$a_{13} = d(-r\cos\,(a+c) + \sin\,(a+c))e^{r(a+c)} + b(-r\cos\,a + \sin\,a)e^{ra}$,

$a_{23} = -d(\cos\,(a+c) + r\sin\,(a+c))e^{r(a+c)} - b(\cos\,a + r\sin\,a)e^{ra}$,

$a_{31} = a_{32} = 0, \;\; a_{33} = 1.$

Thus the relation $M = e^{\varepsilon X}$ yields the equations

$$
\begin{aligned}
&\cos(a+c)e^{r(a+c)} = 1, \quad \sin(a+c)\, e^{r(a+c)} = 0,\\
&d(-r\cos(a+c) + \sin(a+c))e^{r(a+c)} + b(-r\cos a + \sin a)e^{ra} = \varepsilon,\\
&d(\cos(a+c) + r\sin(a+c))e^{r(a+c)} + b(\cos a + r\sin a)e^{ra} = r\varepsilon,
\end{aligned}
$$

whose solution is given by (5.6.3).

5.6.3 Solution using the DE method

One can show that the following identities hold for all ε.

$$
\begin{aligned}
&e^{\varepsilon Y}X \equiv Xe^{\varepsilon Y}, \quad e^{\varepsilon Y}R \equiv (R - \varepsilon X - r\varepsilon Y)e^{\varepsilon Y},\\
&e^{\varepsilon R}X \equiv e^{r\varepsilon}(\cos\varepsilon\, X - \sin\varepsilon\, Y)e^{\varepsilon R},\\
&e^{\varepsilon R}Y \equiv e^{r\varepsilon}(\cos\varepsilon\, Y + \sin\varepsilon\, X)e^{\varepsilon R}.
\end{aligned}
\tag{5.6.4}
$$

To proceed, one differentiates with respect to ε the equation $e^{a(\varepsilon)R}e^{b(\varepsilon)Y}e^{c(\varepsilon)R}e^{d(\varepsilon)Y} = e^{\varepsilon X}$ and then uses the identities (5.6.4) recursively. This yields the ODE system

$$
\begin{aligned}
&a' + c' = 0,\\
&e^{ra}\Big(\cos a\, b' + (b\sin a - rb\cos a)c' + \cos(a+c)e^{rc}d'\Big) = 0,\\
&e^{ra}(\sin a\, b' - (rb\sin a + b\cos a)c' + sin\,(a+c)e^{rc}d') = 1.
\end{aligned}
\tag{5.6.5}
$$

It is easy to show that the solution to the ODE system (5.6.5) is given by (5.6.3).

Chapter 6

Four-dimensional Lie algebras

In this chapter, we present the proofs of the results presented in Table 2.

6.1 Lie algebra $S_{4,2}$

6.1.1 Solution using the operator method

From [12], an operator representation for $S_{4,2}$ is given by

$$W = -x\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + y\frac{\partial}{\partial z}, \quad X = \frac{\partial}{\partial x}, \quad Y = y\frac{\partial}{\partial x}, \quad Z = z\frac{\partial}{\partial x}.$$

Consequently, equation $e^{a(\varepsilon)W}e^{b(\varepsilon)Z}e^{c(\varepsilon)W}e^{d(\varepsilon)Z}e^{f(\varepsilon)W}(x,y,z) = e^{\varepsilon Y}(x,y,z)$ leads to

$$\begin{aligned}&((cd+ad+abe^{-c})e^{-f}y + \left((d+be^{-c})e^{-f}\right)z + e^{-(a+f+c)}x + (acd + \tfrac{1}{2}c^2d)e^{-f},\\ &y+a+c+f, (a+f+c)y + z + \tfrac{1}{2}(a+f+c)^2) = (x+\varepsilon y, y, z).\end{aligned} \tag{6.1.1}$$

It is easy to see that equation (6.1.1) is satisfied iff

$$a+f+c=0, \quad d+be^{-c}=0, \quad cde^{-f}=\varepsilon, \; a+c=0. \tag{6.1.2}$$

The solution to the system of equations (6.1.2) is given by

$$a(\varepsilon) = f(\varepsilon) = -\tfrac{1}{2}c(\varepsilon), \quad b(\varepsilon) = -\frac{\varepsilon}{c(\varepsilon)} e^{\frac{1}{2}c(\varepsilon)}, \quad d(\varepsilon) = \frac{\varepsilon}{c(\varepsilon)} e^{-\frac{1}{2}c(\varepsilon)}, \tag{6.1.3}$$

where $c(\varepsilon)$ is any continuous function chosen so that $b(\varepsilon)$ and $d(\varepsilon)$ are continuous functions, and satisfying $c(\varepsilon) \neq 0$ for any $\varepsilon \neq 0$ with $a(0) = b(0) = c(0) = d(0) = f(0) = 0$.

6.1.2 Solution using the matrix representation method

A matrix representation for $S_{4,2}$ is given by

$$X = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$W = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence for all $n \geq 2$ one has $Y^n = Z^n = 0$ and $W^n = \begin{pmatrix} 1 & n & \frac{n(n-1)}{2} & 0 \\ 0 & 1 & n & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Thus one can show that

$$e^{\varepsilon Y} = \begin{pmatrix} 1 & 0 & 0 & -\varepsilon \\ 0 & 1 & 0 & -\varepsilon \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad e^{\varepsilon Z} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\varepsilon \\ 0 & 0 & 1 & -\varepsilon \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$e^{\varepsilon W} = \begin{pmatrix} e^{\varepsilon} & \varepsilon e^{\varepsilon} & \frac{1}{2}\varepsilon^2 e^{\varepsilon} & 0 \\ 0 & e^{\varepsilon} & \varepsilon e^{\varepsilon} & 0 \\ 0 & 0 & e^{\varepsilon} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Consequently, the entries $\left\{a_{ij}\right\}$ of the matrix $M = e^{a(\epsilon)W}e^{b(\epsilon)Z}e^{c(\epsilon)W}e^{d(\epsilon)Z}e^{f(\varepsilon)W}$ are given by

$$\begin{aligned}
&a_{11} = a_{22} = a_{33} = e^{a+c+f}, \;\; a_{12} = a_{23} = (a+c+f)e^{a+c+f}, \\
&a_{13} = \left(af + fc + ac + \tfrac{1}{2}(a^2+f^2+c^2)\right)e^{a+c+f}, \\
&a_{14} = -d\left(a+c+ac+\tfrac{1}{2}(a^2+c^2)\right)e^{a+c} - ba\left(1+\tfrac{1}{2}a\right)e^{a}, \\
&a_{24} = -d\,(1+c+a)\,e^{a+c} - b\,(1+a)\,e^{a}, \\
&a_{34} = -de^{a+c} - be^{a}, \;\; a_{44} = 1, \;\; a_{ij} = 0 \text{ for } j < i.
\end{aligned}$$

After simplification, the relation $M = e^{\varepsilon Y}$ leads to the system of equations

$$a+c+f = 0, \;\; d\left(a+c+ac+\tfrac{1}{2}(a^2+c^2)\right)e^{a+c} + ba\left(1+\tfrac{1}{2}a\right)e^{a} = \varepsilon,$$

$d\,(1+a+c)\,e^{a+c} + b\,(1+a)\,e^{a} = \varepsilon$, $\quad de^{a+c} + be^{a} = 0$, whose solution is given by (6.1.3).

6.1.3 Solution using the DE method

One can show that the following identities hold for all ε.

$$\begin{aligned}
&e^{\varepsilon W}X \equiv e^{\varepsilon}Xe^{\varepsilon W}, \;\; e^{\varepsilon W}Y \equiv (e^{\varepsilon}Y + \varepsilon e^{\varepsilon}X)\,e^{\varepsilon W}, \\
&e^{\varepsilon W}Z \equiv \left(e^{\varepsilon}Z + \varepsilon e^{\varepsilon}Y + \tfrac{1}{2}\varepsilon^2 e^{\varepsilon}X\right)e^{\varepsilon W}, \\
&e^{\varepsilon Z}X \equiv Xe^{\varepsilon Z}, \;\; e^{\varepsilon Z}Y \equiv Ye^{\varepsilon Z}, \;\; e^{\varepsilon Z}W \equiv (W - \varepsilon Z - \varepsilon Y)\,e^{\varepsilon Z}.
\end{aligned} \tag{6.1.4}$$

After differentiating with respect to ε the equation $e^{a(\varepsilon)W}e^{b(\varepsilon)Z}e^{c(\varepsilon)W}e^{d(\varepsilon)Z}e^{f(\varepsilon)W} = e^{\varepsilon Y}$ and using the identities (6.1.4), one obtains the system of differential equations

$$
\begin{aligned}
& a' + c' + f' = 0, \\
& a^2 e^a b' - ba(2+a)e^a c' + (a+c)^2 e^{a+c} d' - (2ab + 2(a+c)\, d e^c + ba^2 \\
& + d(a+c)^2 e^c) e^a f' = 0, \\
& e^a b' - b e^a c' + e^{a+c} d' - (b e^a + d e^{a+c})\, f' = 0, \\
& a e^a b' - b(1+a)\, e^a c' + (a+c) e^{a+c} d' - (b(1+a) + d(1+a+c) e^c) e^a f' = 1.
\end{aligned}
\tag{6.1.5}
$$

The solution to the ODE system (6.1.5) is given by (6.1.3).

6.2 Lie algebra $S_{4,7}$

6.2.1 *Solution using the operator method*

From [12], an operator representation for $S_{4,7}$ is given by

$$W = \frac{1}{2}\left(y^2 - z^2\right)\frac{\partial}{\partial x} - z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}, \;\; X = \frac{\partial}{\partial x}, \;\; Y = \frac{\partial}{\partial y}, \;\; Z = y\frac{\partial}{\partial x} + \frac{\partial}{\partial z}.$$

One can show that for all ε the following relations hold.

$$
\begin{aligned}
& e^{\varepsilon Y}(x,y,z) = (x, y+\varepsilon, z), \;\; e^{\varepsilon Z}(x,y,z) = (x + \varepsilon y, y, z + \varepsilon), \\
& e^{\varepsilon W}(x,y,z) = (\tfrac{1}{4}(\sin(2\varepsilon)(y^2 - z^2) + 2\cos(2\varepsilon)\, zy) + x - \tfrac{1}{2} zy, \\
& \cos\varepsilon\, y - \sin\varepsilon\, z, \cos\varepsilon\, z + \sin\varepsilon\, y).
\end{aligned}
$$

After much calculation and the use of the relations above, one can show that the equation

$e^{a(\varepsilon)Z}e^{b(\varepsilon)W}e^{c(\varepsilon)Z}e^{d(\varepsilon)W}e^{f(\varepsilon)Z}(x,y,z) = e^{\varepsilon Y}(x,y,z)$ leads to the simplified system of equations

$$f+c=0,\ \ d\ \sin\ f\ +\varepsilon=0,\ \ b+g+d\ \cos\ f\ \ =0,\ \ g\ \sin\ f\ -\tfrac{1}{2}\varepsilon\ \cos\ f\ \ =0. \tag{6.2.1}$$

The solution to the system of equations (6.2.1) is given by

$$a(\varepsilon)=f(\varepsilon)=-\frac{\varepsilon}{2\tan b(\varepsilon)},\ \ c(\varepsilon)=\frac{\varepsilon}{\sin b(\varepsilon)},\ \ d(\varepsilon)=-b(\varepsilon)\,, \tag{6.2.2}$$

where $b(\varepsilon)$ is any continuous function chosen so that $a(\varepsilon), f(\varepsilon)$ and $c(\varepsilon)$ are continuous functions, and satisfying $b(\varepsilon)\neq 0$ for every $\varepsilon\neq 0$ with $a(0)=b(0)=c(0)=d(0)=f(0)=0$.

6.2.2 Solution using the matrix representation method

The following matrix representation for $S_{4,7}$ was found after correction of the matrix representation of $S_{4,7}$ given in [8].

$$e^{\varepsilon Y}=\begin{pmatrix}1&0&-\varepsilon&0\\0&1&0&-\varepsilon\\0&0&1&0\\0&0&0&1\end{pmatrix},\quad e^{\varepsilon Z}=\begin{pmatrix}1&\varepsilon&0&0\\0&1&0&0\\0&0&1&-\varepsilon\\0&0&0&1\end{pmatrix},$$

$$e^{\varepsilon W}=\begin{pmatrix}1&0&0&0\\0&\cos\ \varepsilon&\sin\ \varepsilon&0\\0&-\sin\ \varepsilon&\cos\ \varepsilon&0\\0&0&0&1\end{pmatrix}.$$

Consequently the entries $\left\{a_{ij}\right\}$ of the matrix
$M=\ e^{a(\varepsilon)Z}e^{b(\varepsilon)W}e^{c(\varepsilon)Z}e^{d(\varepsilon)W}e^{f(\varepsilon)Z}$ are found to be

$$
\begin{aligned}
&a_{11} = a_{44} = 1,\ \ a_{12} = f + c \cos\ d\ + a \cos\,(b+d)\,,\\
&a_{13} = c \sin\ d\ + a \sin\,(b+d)\,,\\
&a_{14} = -fc \sin\ d\ - af\ \sin\,(b+d)\ - ac \sin\ b\ ,\ \ a_{22} = a_{33} = \cos\,(b+d),\\
&a_{23} = -a_{32} = \sin\,(b+d),\ \ a_{24} = -f\ \sin\ (b+d)\ \ - c \sin\ b\ ,\\
&a_{34} = -f\ \cos\ (b+d)\ - c \cos\ b\ - b,\ \ a_{21} = a_{31} = a_{41} = a_{42} = a_{43} = 0.
\end{aligned}
$$

Consequently, the relation $M = e^{\varepsilon Y}$ yields the system of equations

$$
b+d=0,\ \ fc \sin\ d\ + ac \sin\ b\ = 0,\ \ c \sin\ d\ = -\varepsilon, f + c \cos\ d\ + a = 0.
$$

whose solution is given by (6.2.1).

6.2.3 Solution using the DE method

The following identities hold for all ε.

$$
\begin{aligned}
&e^{\varepsilon W} Y \equiv (\ \cos\ \varepsilon\ Y - \ \sin\ \varepsilon\ Z) e^{\varepsilon W},\ \ e^{\varepsilon W} Z \equiv (\ \cos\ \varepsilon\ Z + \ \sin\ \varepsilon\ Y) e^{\varepsilon W},\\
&e^{\varepsilon Z} Y \equiv (Y - \varepsilon X) e^{\varepsilon Z},\ \ e^{\varepsilon Z} W \equiv (W - \varepsilon Y + \tfrac{1}{2}\varepsilon^2 X) e^{\varepsilon Z}.
\end{aligned}
\tag{6.2.3}
$$

After differentiating with respect to ε the equation $e^{a(\varepsilon)Z} e^{b(\varepsilon)W} e^{c(\varepsilon)Z} e^{d(\varepsilon)W} e^{f(\varepsilon)Z} = e^{\varepsilon Y}$ and repeatedly using the identities (6.2.3), one obtains the ODE system

$$
\begin{aligned}
&b' + d' = 0,\ \ a' + \ \cos\ b\ c' + c \sin\ b\ d'\ + \ \cos\ (b+d)\ f' = 0,\\
&-ab' + \ \sin\ b\ c' - (a + c \cos\ b)\ d' + \ \sin\ (b+d)\ f' = 1,\\
&\tfrac{1}{2}a^2 b' - a \sin\ b\ c' + \Big(ac \cos\ b + \tfrac{1}{2}(a^2 + c^2)\ \Big) d'\\
&-(c \sin\ d\ + a \sin\ (b+d))\ \ f' = 0.
\end{aligned}
\tag{6.2.4}
$$

The solution to the ODE system (6.2.4) is given by (6.2.1).

6.3 Lie algebra $S_{4,9}$

6.3.1 Solution using the operator method

From [12], an operator representation for $S_{4,9}$ is given by

$$W = \tfrac{1}{2}\left(y^2 - z^2 - 4rx\right)\frac{\partial}{\partial x} - (ry + z)\frac{\partial}{\partial y} + (y - rz)\frac{\partial}{\partial z}, \ \ X = \frac{\partial}{\partial x}, \ \ Y = \frac{\partial}{\partial y},$$

$$Z = y\frac{\partial}{\partial x} + \frac{\partial}{\partial z}.$$

One can show that for all ε the following relations hold.

$$\begin{aligned}
&e^{\varepsilon Y}(x,y,z) = (x, y+\varepsilon, z), \ \ e^{\varepsilon Z}(x,y,z) = (x+\varepsilon y, y, z+\varepsilon),\\
&e^{\varepsilon W}(x,y,z) = (\tfrac{1}{4}(\ \sin\ (2\varepsilon)(y^2 - z^2) + 2\ \cos\ (2\varepsilon)\ zy) + x - \tfrac{1}{2}zy)e^{-2\varepsilon r},\\
&e^{-\varepsilon r}(\ \cos\ \varepsilon\ y - \ \sin\ \varepsilon\ z), e^{-\varepsilon r}(\ \cos\ \varepsilon\ z + \ \sin\ \varepsilon\ y)).
\end{aligned}$$

These relations allow one to show that the equation $e^{a(\varepsilon)Z}e^{b(\varepsilon)W}e^{c(\varepsilon)Z}e^{d(\varepsilon)W}e^{f(\varepsilon)Z}(x,y,z) = e^{\varepsilon Y}(x,y,z)$ leads to the equations

$$\begin{aligned}
&f + c = 0, \ \ d\ \sin\ f\ + \ \varepsilon e^{fr} = 0, \ \ b + g + d\ \cos\ f\ \ e^{-fr} = 0,\\
&2g\ \sin\ f\ - \varepsilon\ \cos\ f\ = 0.
\end{aligned} \tag{6.3.1}$$

The solution to the system of equations (6.3.1) is given by

$$a(\varepsilon) = f(\varepsilon) = -\frac{\varepsilon}{2\tan b(\varepsilon)}, \ \ c(\varepsilon) = \frac{\varepsilon e^{-rb}}{\sin b(\varepsilon)}, \ \ d(\varepsilon) = -b(\varepsilon)\ , \tag{6.3.2}$$

where $b(\varepsilon)$ is any continuous function chosen so that $a(\varepsilon), f(\varepsilon)$ and $c(\varepsilon)$ are continuous functions, and satisfying $b(\varepsilon) \neq 0$ for any $\varepsilon \neq 0$ with $a(0) = b(0) = c(0) = d(0) = f(0) = 0$.

6.3.2 *Solution using the matrix representation method*

A matrix representation for $S_{4,9}$ is given by

$$X = \begin{pmatrix} 0 & 0 & 0 & -2r \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ Y = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -r \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ Z = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -r \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$W = \begin{pmatrix} 2r & 0 & 0 & 0 \\ 0 & r & 1 & 0 \\ 0 & -1 & r & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence for all $n \geq 3$ one has $Y^n = Z^n = 0$ and $Y^2 = Z^2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Thus the corresponding matrix representation of the associated Lie group is given by

$$e^{\varepsilon Y} = \begin{pmatrix} 1 & 0 & \varepsilon & \frac{1}{2}\varepsilon^2 \\ 0 & 1 & 0 & -r\varepsilon \\ 0 & 0 & 1 & \varepsilon \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad e^{\varepsilon Z} = \begin{pmatrix} 1 & -\varepsilon & 0 & \frac{1}{2}\varepsilon^2 \\ 0 & 1 & 0 & -\varepsilon \\ 0 & 0 & 1 & -r\varepsilon \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

Analogous to the way of obtaining the rotation matrix e^{cR} in the parking problem, one finds that

$$e^{\varepsilon W} = \begin{pmatrix} e^{2r\varepsilon} & 0 & 0 & 0 \\ 0 & \cos\, \varepsilon\, e^{r\varepsilon} & \sin\, \varepsilon\, e^{r\varepsilon} & 0 \\ 0 & -\sin\, \varepsilon\, e^{r\varepsilon} & \cos\, \varepsilon\, e^{r\varepsilon} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Consequently the entries $\left\{a_{ij}\right\}$ of the matrix $M = e^{a(\varepsilon)Z} e^{b(\varepsilon)W} e^{c(\varepsilon)Z} e^{d(\varepsilon)W} e^{f(\varepsilon)Z}$ are given by

$a_{11} = e^{2r(b+d)}$,
$a_{12} = -f e^{2r(b+d)} - c \cos\ d\ e^{r(2b+d)} - a \cos\ (b+d) e^{r(b+d)}$,
$a_{13} = -c \sin\ d e^{r(2b+d)}\ - a \sin\ (b+d)\ e^{r(b+d)}$,
$a_{14} = \frac{1}{2} f^2 e^{2r(b+d)} + cf\ \cos\ d\ e^{r(2b+d)} + rcf\ \sin\ d\ e^{r(2b+d)} +$
$af\ \cos\ (b+d) e^{r(b+d)}\ + raf\ \sin\ (b+d) e^{r(b+d)} + \frac{1}{2} c^2 e^{2rb}\ + ac \cos\ b\ e^{rb} +$
$rac \sin\ b\ e^{rb}\ + \frac{1}{2} a^2$,
$a_{22} = a_{33} = \cos\ (b+d) e^{r(b+d)}$, $a_{23} = -a_{32} = \sin\ (b+d) e^{r(b+d)}$,
$a_{24} = -f\ \cos\ (b+d) e^{r(b+d)}\ - rf\ \sin\ (b+d) e^{r(b+d)}\ - c \cos\ b\ e^{rb}$
$- rc \sin\ b\ e^{rb} - a$,
$a_{34} = f\ \sin\ (b+d) e^{r(b+d)}\ - rf\ \cos\ (b+d) e^{r(b+d)}\ + c \sin\ b\ e^{rb}$
$- rc \cos\ b\ e^{rb}\ - rb$,
$a_{44} = 1$, $a_{21} = a_{31} = a_{41} = a_{42} = a_{43} = 0$.

The relation $M = e^{\varepsilon Y}$ yields, after simplification, the system of equations

$$b + d = 0,\ \ c \sin\ b\ e^{rb} = \varepsilon,\ \ f + c \cos\ d\ e^{rb} + a = 0,$$
$$\frac{1}{2} f^2 + (a+f) c \cos\ b\ e^{rb} + r(a-f) c \sin\ b\ e^{rb} + af\ + \frac{1}{2} c^2 e^{2rb} + \frac{1}{2} a^2 = \frac{1}{2} \varepsilon^2. \tag{6.3.3}$$

The solution to the system of equations (6.3.3) is given by (6.3.1).

6.3.3 Solution using the DE method

The following identities hold for all ε.

$$e^{\varepsilon W}Y \equiv (\cos\varepsilon e^{r\varepsilon}\ Y - \sin\varepsilon\ e^{r\varepsilon}Z)e^{\varepsilon W}, e^{\varepsilon W}Z \equiv (\cos\varepsilon\ e^{r\varepsilon}Z + \sin\varepsilon e^{r\varepsilon}\ Y)e^{\varepsilon W},$$
$$e^{\varepsilon W}X \equiv e^{2r\varepsilon}Xe^{\varepsilon W}, e^{\varepsilon Z}Y \equiv (Y - \varepsilon X)e^{\varepsilon Z},$$
$$e^{\varepsilon Z}W \equiv (W - r\varepsilon Z - \varepsilon Y + \frac{1}{2}\varepsilon^2 X)e^{\varepsilon Z}, e^{\varepsilon Z}X \equiv Xe^{\varepsilon Z}. \tag{6.3.4}$$

After differentiating with respect to ε the equation $e^{a(\varepsilon)Z}e^{b(\varepsilon)W}e^{c(\varepsilon)Z}e^{d(\varepsilon)W}e^{f(\varepsilon)Z} = e^{\varepsilon Y}$ and using the identities (6.3.4) one obtains the ODE system

$$\begin{aligned}
&b' + d' = 0,\\
&a' + \cos\ b\ e^{rb}c' + (-rc\cos\ b\ e^{rb} + c\sin\ b\ e^{rb})\ d' + e^{r(b+d)}f' = 0,\\
&-ab' + \sin\ b\ e^{rb}c' + (-rc\sin\ b\ e^{rb} - c\cos\ b\ e^{rb})\ d' = 1,\\
&-a\sin\ b\ e^{rb}c' + \left(rac\sin\ b\ e^{rb} + ac\cos\ b\ e^{rb} + \tfrac{1}{2}c^2e^{2rb}\right)d'\\
&-c\sin\ d\ e^{rb}\ f' = 0.
\end{aligned} \tag{6.3.5}$$

The solution to the ODE system (6.3.4) is given by (6.3.1).

6.4 Lie algebra $S_{4,10}$

6.4.1 Solution using the operator method

From [12], an operator representation for $S_{4,10}$ is given by

$$W = -2x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + \frac{\partial}{\partial z},\ \ X = \frac{\partial}{\partial x},\ \ Y = \frac{\partial}{\partial y},\ \ Z = y\frac{\partial}{\partial x} + z\frac{\partial}{\partial y}.$$

Consequently, the equation $e^{a(\varepsilon)W}e^{b(\varepsilon)Z}e^{c(\varepsilon)W}e^{d(\varepsilon)Z}e^{f(\varepsilon)W}e^{g(\varepsilon)W}(x,y,z) = e^{\varepsilon Y}(x,y,z)$ leads to the simplified system of equations

$$\begin{aligned} &a+c+f=0, \quad g+de^{a+c}+be^{a}=0, \quad cde^{-f}+cg+fg=\varepsilon, \\ &d^2ce^{-2f}+2gdce^{-f}-ag^2=0, \end{aligned} \tag{6.4.1}$$

whose solutions are given by

$$\begin{aligned} &f(\varepsilon)=-(a(\varepsilon)+c(\varepsilon)), \quad b(\varepsilon)=\varepsilon e^{-a}\left(\frac{c(\varepsilon)\pm\sqrt{c(\varepsilon)^2+a(\varepsilon)c(\varepsilon)}}{a(\varepsilon)c(\varepsilon)}\right), \\ &g(\varepsilon)=-\frac{\varepsilon+b(\varepsilon)c(\varepsilon)e^{a(\varepsilon)}}{a(\varepsilon)+c(\varepsilon)}, \quad d(\varepsilon)=-g(\varepsilon)e^{-(a(\varepsilon)+c(\varepsilon))}-b(\varepsilon)e^{-c(\varepsilon)}, \end{aligned} \tag{6.4.2}$$

where $a(\varepsilon)$ and $c(\varepsilon)$ are any continuous function chosen so that $b(\varepsilon), d(\varepsilon)$ and $g(\varepsilon)$ are continuous functions, and satisfying $a(\varepsilon)c(\varepsilon)\neq 0$, $a(\varepsilon)+c(\varepsilon)\neq 0$, and $c(\varepsilon)^2+a(\varepsilon)c(\varepsilon)\geq 0$ for any $\varepsilon\neq 0$ with $a(0)=b(0)=c(0)=d(0)=f(0)=g(0)=0$.

In the limiting case when $a(\varepsilon)=0$, $f(\varepsilon)=-c(\varepsilon)$, $d(\varepsilon)=\frac{\varepsilon e^{-c(\varepsilon)}}{c(\varepsilon)}$, $b(\varepsilon)=g(\varepsilon)=-\frac{\varepsilon}{2c(\varepsilon)}$, with $c(\varepsilon)$ any continuous function chosen so that $b(\varepsilon), g(\varepsilon)$ and $d(\varepsilon)$ are continuous functions, and $c(\varepsilon)\neq 0$ when $\varepsilon\neq 0$.

6.4.2 *Solution using the matrix representation method*

A matrix representation of $S_{4,10}$ is given by

$$X = \begin{pmatrix} 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ Y = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ Z = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$W = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence for all $n \geq 2$ one has $X^n = Y^n = 0$ and $Z^2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

For all $n \geq 3$ $Z^3 = 0$ and $W^n = \begin{pmatrix} 2^n & 0 & 0 & 1 \\ 0 & 1 & n & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

One can show that

$$e^{\varepsilon X} = \begin{pmatrix} 1 & 0 & 0 & -2\varepsilon \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad e^{\varepsilon Y} = \begin{pmatrix} 1 & 0 & \varepsilon & 0 \\ 0 & 1 & 0 & -\varepsilon \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad e^{\varepsilon Z} = \begin{pmatrix} 1 & -\varepsilon & 0 & \frac{1}{2}\varepsilon^2 \\ 0 & 1 & 0 & -\varepsilon \\ 0 & 0 & 1 & -\varepsilon \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$e^{\varepsilon W} = \begin{pmatrix} e^{2\varepsilon} & 0 & 0 & 0 \\ 0 & e^{\varepsilon} & \varepsilon e^{\varepsilon} & 0 \\ 0 & 0 & e^{\varepsilon} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then the matrix $M = e^{a(\epsilon)W} e^{b(\epsilon)Z} e^{c(\epsilon)W} e^{d(\epsilon)Z} e^{f(\varepsilon)W} e^{g(\varepsilon)Z}$ has entries $\{a_{ij}\}$ given by

$$
\begin{aligned}
& a_{11} = e^{2(a+c+f)}, \;\; a_{12} = -(ge^{c+f} + de^{c} + b)e^{2a+c+f}, \\
& a_{13} = -(f\,de^{c} + f\,b + bc)\,e^{2a+c+f}, \\
& a_{14} = \left(\tfrac{1}{2}g^2 e^{c+f} + g\,(1+f)\,de^{c} + gb\,(1+f) + bcg\right)e^{2a+c+f} + \\
& (\tfrac{1}{2}d^2 e^{2c} + bd\,(1+c)\,e^{c} + \tfrac{1}{2}b^2)e^{2a}, \\
& a_{22} = a_{33} = e^{a+c+f}, \;\; a_{23} = (a+c+f)e^{a+c+f}, \\
& a_{24} = -g\,(1+a+c+f)\,e^{a+c+f} - (d\,(1+a+c)\,e^{c} + b\,(1+a))e^{a}, \\
& a_{34} = -(ge^{a+c+f} + de^{a+c} + be^{a}), \;\; a_{44} = 1, \;\; a_{ij} = 0 \text{ for } j < i.
\end{aligned}
$$

Consequently, after simplification, the relation $M = e^{\varepsilon Y}$ leads to the system of equations (6.4.1) whose solutions are given by (6.4.2).

6.4.3 Solution using the DE method

One can show that the following identities hold for all ε.

$$
\begin{aligned}
& e^{\varepsilon W} X \equiv (e^{2\varepsilon} X)e^{\varepsilon W}, \;\; e^{\varepsilon W} Y \equiv (e^{\varepsilon} Y)e^{\varepsilon W}, \;\; e^{\varepsilon W} Z \equiv (e^{\varepsilon} Z + \varepsilon e^{\varepsilon} Y)e^{\varepsilon W}, \\
& e^{\varepsilon Z} X \equiv X e^{\varepsilon Z}, e^{\varepsilon Z} Y \equiv (Y - \varepsilon X)e^{\varepsilon Z}, \;\; e^{\varepsilon Z} W \equiv (W - \varepsilon Z - \varepsilon Y + \tfrac{1}{2}\varepsilon^2 X)e^{\varepsilon Z}.
\end{aligned}
\tag{6.4.3}
$$

After differentiating with respect to ε the equation $e^{a(\varepsilon)W} e^{b(\varepsilon)Z} e^{c(\varepsilon)W} e^{d(\varepsilon)Z} e^{f(\varepsilon)W} e^{g(\varepsilon)Z} = e^{\varepsilon Y}$ and using the identities in

(6.4.3), one obtains the ODE system

$$
\begin{aligned}
&a' + c' + f' = 0,\\
&e^{a}(ab' - b\,(1+a)\,c' + (a+c)\,e^{c}d' - (ab + ade^{c} + b + d\,(1+c)\,e^{c})\,f') = 1,\\
&e^{a}b' - be^{a}c' + e^{a+c}d' - (b + e^{c}d\ )\,e^{a}f' + g' = 0,\\
&\tfrac{1}{2}b^{2}e^{2a}c' - bce^{2a+c}d' + \left(\tfrac{1}{2}(b^{2} + d^{2}e^{2c}) + bd(1+c)e^{c}\right)e^{2a}f' -\\
&((bf + bc)\,e^{f+c+2a} + dfe^{f+2c+2a})g' = 0.
\end{aligned}
\tag{6.4.4}
$$

The solution to ODE system (6.4.3) is given by (6.4.2).

Chapter 7

Inverse Problem

Here we show that reversing the order of elements in the LHS of both equation (3.2) and equation (3.6) leads to an isomorphic solution if the number of elements that are not identically zero is even. Additionally, we present the isomorphism relating the inverse problem to the original problem. Then we present a table that shows the results for the inverse problems for all three-dimensional Lie algebras. Finally, we will examine the relation between the solution to the parking problem and the solution to its inverse problem.
The original problem is given by

$$e^{a(\varepsilon)B_1}e^{b(\varepsilon)B_2}e^{c(\varepsilon)B_1}e^{d(\varepsilon)B_2}e^{f(\varepsilon)B_1}e^{g(\varepsilon)B_2} = e^{\varepsilon B_3}. \tag{7.0.1}$$

To obtain the solution of the inverse problem from equation (7.0.1), we first multiply equation (7.0.1) by

$e^{-g(\varepsilon)B_2}e^{-f(\varepsilon)B_1}e^{-d(\varepsilon)B_2}e^{-c(\varepsilon)B_1}e^{-b(\varepsilon)B_2}e^{-a(\varepsilon)B_1}$. This yields

$1 = e^{-g(\varepsilon)B_2}e^{-f(\varepsilon)B_1}e^{-d(\varepsilon)B_2}e^{-c(\varepsilon)B_1}e^{-b(\varepsilon)B_2}e^{-a(\varepsilon)B_1}e^{\varepsilon B_3}$.

Next, we multiply both sides by $e^{-\varepsilon B_3}$. This leads to

$$e^{-g(\varepsilon)B_2}e^{-f(\varepsilon)B_1}e^{-d(\varepsilon)B_2}e^{-c(\varepsilon)B_1}e^{-b(\varepsilon)B_2}e^{-a(\varepsilon)B_1} = e^{-\varepsilon B_3}. \tag{7.0.2}$$

Thus the inverse problem given by

$$e^{\tilde{a}(\varepsilon)B_2}e^{\tilde{b}(\varepsilon)B_1}e^{\tilde{c}(\varepsilon)B_2}e^{\tilde{d}(\varepsilon)B_1}e^{\tilde{f}(\varepsilon)B_2}e^{\tilde{g}(\varepsilon)B_1} = e^{\varepsilon B_3}. \tag{7.0.3}$$

has a solution isomorphic to that of equation (7.0.1) and the isomorphism is given by

$$\begin{aligned}
\tilde{a}(\varepsilon) &= -g(-\varepsilon),\\
\tilde{b}(\varepsilon) &= -f(-\varepsilon),\\
\tilde{c}(\varepsilon) &= -d(-\varepsilon),\\
\tilde{d}(\varepsilon) &= -c(-\varepsilon),\\
\tilde{f}(\varepsilon) &= -b(-\varepsilon),\\
\tilde{g}(\varepsilon) &= -a(-\varepsilon).
\end{aligned} \tag{7.0.4}$$

The solution to the inverse problem is presented in the following table for all relevant Lie algebras

Lie algebra; commutators	Composition equation	Solution
$sl(2,\mathbb{R})$ $[X,Y]=Z$ $[X,Z]=-2X$ $[Y,Z]=2Y$	$e^{a(\varepsilon)Y}e^{b(\varepsilon)X}e^{c(\varepsilon)Y}e^{d(\varepsilon)X}=e^{\varepsilon Z}$	$d(\varepsilon)$ is an arbitrary function satisfying $d(\varepsilon)\neq 0$ when $\varepsilon\neq 0$ $a(\varepsilon)=-d(\varepsilon)$ $b(\varepsilon)=-\frac{e^{-\varepsilon}-1}{d(\varepsilon)}$ $c(\varepsilon)=-\frac{e^{\varepsilon}-1}{e^{-\varepsilon}-1}d(\varepsilon)$
Parallel parking problem, $S_{3,3}$ with constant $r=0$ $[R,Y]=X$ $[R,X]=-Y$ $[X,Y]=0$	$e^{a(\varepsilon)Y}e^{b(\varepsilon)R}e^{c(\varepsilon)Y}e^{d(\varepsilon)R}=e^{\varepsilon X}$	$d(\varepsilon)$ is an arbitrary function satisfying $d(\varepsilon)\neq k\pi$ for every $k\in\mathbb{Z}$ when $\varepsilon\neq 0$ $a(\varepsilon)=\varepsilon\cot d(\varepsilon)$ $b(\varepsilon)=-d(\varepsilon)$ $c(\varepsilon)=-\varepsilon\csc d(\varepsilon)$
Euler angles problem, $so(3,\mathbb{R})$ $[X,Y]=Z$ $[X,Z]=-Y$ $[Y,Z]=X$	$e^{a(\varepsilon)Y}e^{b(\varepsilon)X}e^{c(\varepsilon)Y}e^{d(\varepsilon)X}=e^{\varepsilon Z}$	Any $b(\varepsilon)$ satisfying $b(\varepsilon)\neq k\pi$ for every $k\in\mathbb{Z}$ when $\varepsilon\neq 0$ and $\left\lvert\frac{\sin\varepsilon}{\sin b(\varepsilon)}\right\rvert\leq 1$ with $a(\varepsilon)=-\arccos\left(\frac{\sin d(\varepsilon)}{\sin b(\varepsilon)}\right)$ $c(\varepsilon)=\arcsin\left(\frac{\sin\varepsilon}{\sin b(\varepsilon)}\right)$ $d(\varepsilon)=\arccos\left(\frac{\cos b(\varepsilon)}{\cos\varepsilon}\right)$; $a(\varepsilon)=-\arccos\left(\frac{\sin d(\varepsilon)}{\sin b(\varepsilon)}\right)$ $c(\varepsilon)=\pi-\arcsin\left(\frac{\sin\varepsilon}{\sin b(\varepsilon)}\right)$ $d(\varepsilon)=-\arccos\left(\frac{\cos b(\varepsilon)}{\cos\varepsilon}\right)$
$n_{3,1}$ $[X,Y]=Z$ $[X,Z]=0$ $[Z,Y]=0$	$e^{a(\varepsilon)Y}e^{b(\varepsilon)X}e^{c(\varepsilon)Y}e^{d(\varepsilon)X}=e^{\varepsilon Z}$	$d(\varepsilon)$ is an arbitrary function satisfying $d(\varepsilon)\neq 0$ when $\varepsilon\neq 0$ $a(\varepsilon)=\frac{\varepsilon}{d(\varepsilon)}$ $b(\varepsilon)=-d(\varepsilon)$ $c(\varepsilon)=-\frac{\varepsilon}{d(\varepsilon)}$
$S_{3,1}$ $[Y,Z]=-Y$ $[Y,X]=0$ $[Z,X]=rX+Y$ where r is a constant satisfying $\lvert r\rvert\leq 1$	$e^{a(\varepsilon)Z}e^{b(\varepsilon)X}e^{c(\varepsilon)Z}e^{d(\varepsilon)X}=e^{\varepsilon Y}$	$a(\varepsilon)$ is an arbitrary function satisfying $a(\varepsilon)\neq 0$ when $\varepsilon\neq 0$ $d(\varepsilon)=\frac{\varepsilon(1-r)}{1-e^{(1-r)a(\varepsilon)}}$ $b(\varepsilon)=d(\varepsilon)\,e^{-ra(\varepsilon)}$ $c(\varepsilon)=-a(\varepsilon)$
$S_{3,2}$ $[Z,X]=X$ $[Z,Y]=X+Y$ $[X,Y]=0$	$e^{a(\varepsilon)Z}e^{b(\varepsilon)Y}e^{c(\varepsilon)Z}e^{d(\varepsilon)Y}=e^{\varepsilon X}$	$d(\varepsilon)$ is an arbitrary function satisfying $d(\varepsilon)\neq 0$ when $\varepsilon\neq 0$ $a(\varepsilon)=-\frac{\varepsilon}{d(\varepsilon)}$ $b(\varepsilon)=-d(\varepsilon)e^{-a(\varepsilon)}$ $c(\varepsilon)=\frac{\varepsilon}{d(\varepsilon)}$
$S_{3,3}$ general case $[R,X]=rX-Y$ $[R,Y]=X+rY$ $[X,Y]=0$ where r is a non-negative constant.	$e^{a(\varepsilon)Y}e^{b(\varepsilon)R}e^{c(\varepsilon)Y}e^{d(\varepsilon)R}=e^{\varepsilon X}$	$d(\varepsilon)$ is an arbitrary function satisfying $d(\varepsilon)\neq k\pi$ for every $k\in\mathbb{Z}$ when $\varepsilon\neq 0$ $a(\varepsilon)=\varepsilon\cot d(\varepsilon)$ $b(\varepsilon)=-d(\varepsilon)$ $c(\varepsilon)=-\varepsilon\csc d(\varepsilon)e^{rd(\varepsilon)}$

Table 7.1: Results for the Inverse Problem

7.1 The inverse parking problem

As mentioned in Chapter 5 the parking problem models a vehicle that aims to perform a parallel translation in x by ε through translations in y and rotations starting with a translation. This was illustrated in Figure (5.1) and the solution was given by equation (5.1.1). By applying the isomorphism given in equation (7.1.1), one can see that the solution to the inverse problem, i.e, parking which starts with a rotation, is given by

$$a_1(\varepsilon) = -\varepsilon \cot b_1(\varepsilon), \quad c_1(\varepsilon) = \varepsilon \csc b_1(\varepsilon), \quad d_1(\varepsilon) = -b_1(\varepsilon), \tag{7.1.1}$$

where $b_1(\varepsilon)$ is an arbitrary function satisfying $b_1(\varepsilon) \neq k\pi$ for every $k \in \mathbb{Z}$ when $\varepsilon \neq 0$.

In the diagram below we represent both solutions to the parking problem. In red is the solution that starts with an arbitrary translation while in blue is the inverse solution that begins with an arbitrary rotation. As one can see, such solutions are reflections of each other with respect to the line $x = \frac{\varepsilon}{2}$. In practice, one solution might be more desirable than the other depending on other constraints that one might have. The solution in red would be most useful in a situation where one has to park between two vehicles with limited space between the vehicles. The inverse solution in blue, on the other hand, is practical when there is a lot of parking space available. The red solution requires a more skilled driver since it requires a precise rotation after the arbitrary translation.

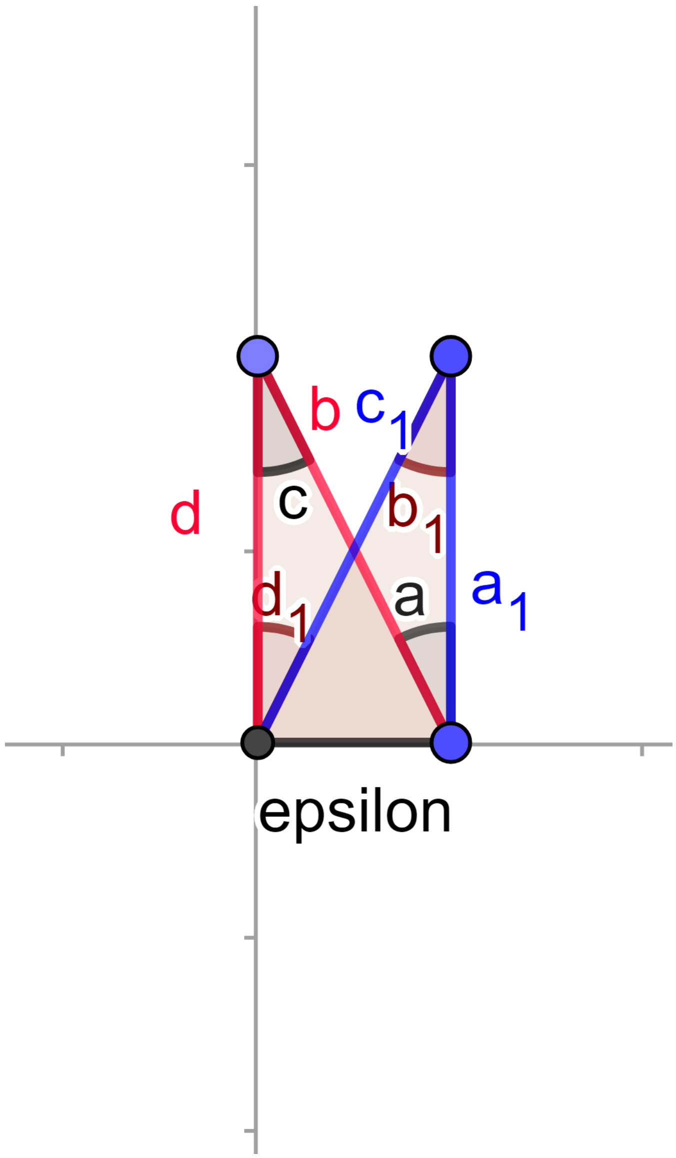

Figure 7.1: Illustration of the solutions of the parallel parking problem and its inverse.

Chapter 8

Discussion and conclusions

In this thesis, for all relevant three- and four-dimensional Lie algebras, we have shown explicitly how one can obtain elements of the associated Lie groups as compositions of products of other elements from the commutator properties of their Lie algebras. Three methods have been presented to accomplish this: an operator method, a matrix representation method, and a DE method. It turns out that in all cases solutions contain an arbitrary function of a parameter ε. In the parallel parking problem, the parameter ε is a translation in x arising from translations in y and rotations in the xy-plane and the arbitrary continuous function can be the angle of rotation or the initial translation. Interestingly, in all cases solutions can be expressed in terms of elementary functions involving an arbitrary continuous function. In practical applications, other constraints could be satisfied by appropriately restricting associated arbitrary functions.

There is an "initial condition" that constrains the arbitrary function. In particular as $\varepsilon \to 0$, if the arbitrary function is $O(\varepsilon^p)$ then it is easy to check that $0 < p < 1$ and that all other functions in the compositions are either $O(\varepsilon^p)$ or $O(\varepsilon^{1-p})$.

As noted earlier in the thesis, one can also state the problems in (3.0.2) and (3.0.6) with the roles of B_1 and B_2 interchanged. It was shown that doing so, when the number of terms to the left of equations (3.0.2) and (3.0.6) is even, leads to isomorphic solutions. However, if the number of terms to the left of (3.0.2) and (3.0.6) is odd, then interchanging the roles of B_1 and B_2 may lead to a problem with no solutions. For instance, considering the $S_{4,7}$ problem with the alternative order of Z and W leads to no solutions.

Each of the three methods, used to solve equations (3.0.2) and (3.0.6), have different strengths and challenges. When a useful operator representation exists, the operator method offers a computationally very simple and complete approach to solving (3.0.2) and (3.0.6). However, an appropriate operator representation of a Lie algebra is only known for three- and four-dimensional Lie algebras [12]. But one would expect an operator representation to exist for Lie algebras that arise in practical problems.

The matrix representation method requires a matrix representation of a Lie algebra. Such a representation may not always be readily available. In the case of $S_{4,7}$, for example, the matrix representation found using the software [9] was not isomorphic to $S_{4,7}$, and thus could not be used. Instead, our correction of the adjoint matrix representation found in [6] was used. Another issue with the matrix representation method is that the software GAP [9] cannot handle Lie algebras with algebraic values or non-integers in their structure constants. Accordingly, we had to make adjustments for the matrix representations for the Lie algebras $S_{3,1}$, $S_{3,3}$, and, $S_{4,9}$. Moreover, without carrying out all calculations, the number of independent equations one obtains from the matrix representation method and whether a solution exists cannot be determined a priori. The main strength of the matrix representation method is that in all cases it resulted in algebraic systems of equations that we were able to solve. Most importantly, the matrix representation method is complete since it leads to necessary and sufficient conditions for solutions.

Unlike the matrix and operator representation methods, the differential equation method (DE method) requires no Lie algebra representation. Moreover, it can handle all forms of structure constants. Furthermore, in the DE method, unlike the other two methods, one can see that the solution should depend on an arbitrary function before calculations are performed since the resulting system of ODEs has more unknowns than the number of ODEs in the system. However, the resulting first order system of nonlinear ODEs often presents a more significant challenge to solve than the system of equations obtained through the other two methods. For instance, we were unable to solve directly the ODE system associated with the Euler angles problem but obtained its solution using the operator and matrix representation methods. The most crucial issue with the DE method remains that it only yields a necessary condition. But for all cases considered, it turns out that the obtained solutions satisfied both necessary and sufficient conditions. Related to this, it is an open problem to prove the existence and uniqueness theorem for the nonlinear systems of first order ODEs that result from the DE method for any relevant n-dimensional Lie algebra without use of solutions arising from matrix or operator representations.

One should note that it is possible to extend the solutions presented in this thesis by not requiring the initial conditions (3.3.1) or (3.3.2) to be satisfied. For example, the parallel parking problem also has the solution

$$d(\varepsilon) = \varepsilon, \ \ c(\varepsilon) = -a(\varepsilon) = \frac{\pi}{4}, \ \ b(\varepsilon) = \sqrt{2}\varepsilon.$$

It is of interest to note that the operator and matrix representation methods are algebraic ways of solving nonlinear ODE systems arising from the DE method!

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