REALIZING HIGH-ENERGY PHYSICS IN TOPOLOGICAL SEMIMETALS

by

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Abstract

The discovery of topological phases of matter has brought high-energy and condensed matter communities together by giving us shared interests and challenges. One fruitful outcome is the broadened range of possibilities to study high-energy physics in cost-effective table-top experiments. I have investigated scenarios in which influential high-energy ideas emerge in solid-state systems built from topological semimetals – gapless topological phases which have drawn intense research efforts in recent years. My Thesis details three proposals for realizing Majorana fermions, Adler-Bell-Jackiw anomaly, and holographic black holes in superconductor-Weyl-semimetal heterostructures, mechanically strained Weyl semimetal nanowires/films, and graphene flakes subject to strong magnetic fields, respectively. By analyzing the effects of realistic experimental conditions, I wish to demonstrate that these proposals are experimentally tangible with existing technologies.
Lay Summary

Topological phases of matter are exotic phases that can only be described by an intricate mathematical language – topology. This exciting field of research has brought high-energy and condensed matter physicists together by giving us shared interests and challenges. One fruitful outcome is the broadened range of possibilities to study high-energy physics in cost-effective table-top experiments. This Thesis details three proposals to manipulate topological semimetals such that elusive fundamental particles, quantum anomalies, or black hole holograms can emerge inside a mesoscopic piece of material. By analyzing the effects of realistic experimental conditions, I wish to demonstrate that these proposals are experimentally tangible with existing technologies.
Preface

The majority of this Thesis has been published in peer-reviewed journals.

Chapter 3 is based on [A. Chen and M. Franz, Phys. Rev. B 93, 201105 (2016)] and [A. Chen, D. I. Pikulin, and M. Franz, Phys. Rev. B 95, 174505 (2017)]. In the former, I conducted the numerical analysis while my supervisor M. Franz penned the manuscript; in the latter, my collaborator D. I. Pikulin contributed the Random Matrix Theory analysis (Sec. 3.3.5) while the rest was done by me.

Chapter 4 is based on [D. I. Pikulin, A. Chen, and M. Franz, Phys. Rev. X 6, 041021 (2016)], in which I contributed the numerical analysis in Sec. 4.5.2. M. Franz wrote most of the paper and worked out the basic theory in Sec. 4.4, 4.5 and 4.6. D. I. Pikulin contributed to developing of the ideas behind the basic theory and performed calculations leading to Figs. A.1, A.2, and C.1.

Chapter 5 is based on [A. Chen, R. Ilan, F. de Juan, D. I. Pikulin, and M. Franz, Phys. Rev. Lett. 121, 036403 (2018)]. The idea was conceived jointly by R. Ilan, F. de Juan, D. I. Pikulin, and M. Franz in a conversation held at the 2017 APS March Meeting. The paper was written mostly by M. Franz with contributions from R. Ilan, F. de Juan, and D. I. Pikulin. M. Franz performed most of the analytical calculations with help from D. I. Pikulin. I conducted the numerical simulations which underlie all of the figures in the paper (except for the schematic illustration in Fig. 5.1).
Contents

Abstract ................................................................. iii

Lay Summary ............................................................ iv

Preface ................................................................. v

Contents ................................................................. vi

List of Tables ......................................................... ix

List of Figures ........................................................ x

Acknowledgments ...................................................... xxi

Dedication .............................................................. xxii

1 Introduction ......................................................... 1

2 Brief Review of Topological Semimetals ......................... 5
   2.1 Weyl Semimetals .............................................. 7
   2.2 Graphene ..................................................... 11

3 Majorana Fermions at Weyl-Semimetal-Superconductor Interface 13
   3.1 Overview ...................................................... 13
   3.2 Superconducting Proximity Effect and Majorana Flat Bands .... 16
       3.2.1 2D Continuum Model of Surface States ................. 17
3.2.2 3D Tight-Binding Model ........................................ 20
3.2.3 Stability Against Non-Magnetic Disorder ..................... 23
3.2.4 Summary and Outlook ........................................... 24
3.3 Experimental Detection by Josephson Current Measurement .... 25
  3.3.1 Heuristic Argument ............................................. 26
  3.3.2 Experimental Set-Up with Dirac Semimetal Cd₃As₂ .......... 27
  3.3.3 Majorana Flat Bands at Cd₃As₂-SC Interface ................. 29
  3.3.4 Current-Phase Relation at Finite Temperatures ............... 33
  3.3.5 Random Matrix Theory Analysis of Non-Magnetic Disorder  36
  3.3.6 Application to Other 7-Invariant Weyl Semimetals .......... 39
4 Chiral Anomalies in Strained Weyl Semimetals ..................... 41
  4.1 Overview .......................................................... 41
  4.2 Gauge-Fields Induced by Mechanical Strains ................... 44
  4.3 Intuitive Picture of Chiral Anomalies in Weyl Semimetals ..... 46
  4.4 Analytical Calculations with $\frac{1}{2}$-Cd₃As₂ Model .......... 48
  4.5 Numerical Simulations ............................................ 55
    4.5.1 Pseudomagnetic Field $b$ from Torsion ..................... 55
    4.5.2 Pseudo electric Field $e$ from Unidirectional Strain ....... 58
  4.6 Experimental Manifestations .................................... 65
    4.6.1 Topological Coaxial Cable .................................. 65
    4.6.2 Chiral Torsional Effect ..................................... 67
    4.6.3 Ultrasonic Attenuation and EM Field Emission ............... 70
    4.6.4 Chiral Anomaly in the Absence of EM Fields ............... 76
  4.7 Summary and Outlook ............................................ 76
5 Holographic Black Hole on a Graphene Flake ........................ 81
  5.1 Overview .......................................................... 81
List of Tables

5.1 Gaussian ensembles for the SY model. The relevant probability distributions are given by Eq. (5.6) with $Z = \frac{8}{27}, \frac{4\pi}{81\sqrt{3}}, \frac{4\pi}{729\sqrt{3}}$ and $\beta = 1, 2, 4$ for GOE, GUE, GSE, respectively. ........................................... 90

A.1 Material parameters taken from Refs. [121] and [139]. The last row represents the effective lattice constant used for our numerical simulations. ...................................................... 112
List of Figures

2.1 This figure is from Ref. [10]. Panel (a) illustrates schematically a Weyl semimetal system (in $k$-space) with two nodes. At every internodal momentum, the embedded Chern insulator (red plane) exhibits a chiral edge state. If the system is given an open boundary parallel to the displacement of the nodes, as shown by the top surface, all the chiral edge states collectively give rise to the Fermi-arc surface states. Panel (b) shows the band structure (at some fixed $k_x$). At the Fermi energy, the chiral-propagating surface states appear as a Fermi arc ending at the Weyl nodes. .......................................................... 10

2.2 This figure is from Ref. [24]. Panel (a) shows the graphene lattice with sublattices A and B. $a_1$ and $a_2$ are the lattice unit vectors and $\delta_i$'s are the nearest-neighbour vectors. Panel (b) is the corresponding Brillouin zone. The Dirac nodes are located at valleys $K$ and $K'$. .............. 11

3.1 Surface Fermi arcs in a minimal model of a Weyl semimetal with (a) and without (b) time reversal symmetry $\mathcal{T}$. Weyl nodes are represented by circles with positive (negative) chirality; green arrows indicate the possible direction of electron spin along the arcs. Panel (c) shows the $\pi$ junction setup on the surface of a Weyl semimetal. .............. 17
3.2 Tight binding simulations of the SC/Weyl proximity effect. Panels (a) and (e) show the surface spectral function \( A(k, \omega) \) for the tight binding Hamiltonian \( h_{\text{latt}} + \delta h_{\text{latt}} \) and \( \omega = 0.15 \). In all panels we use \( m = 0.5 \), \( \lambda = 1 \), \( L_x = 50 \) and \( L_z = 40 \); while \( (\mu, \epsilon) = (0, 0) \) and \( (0.1, 1.0) \) for top and bottom row respectively. Panel (b) shows the normal state spectrum with flat bands representing the surface states. The effect of the SC proximity effect with \( \Delta_0 = 0.5 \) is indicated in panel (c) while panel (d) shows the effect of two parallel equidistant \( \pi \) junctions (protected zero modes indicated in red). The bottom row displays our results for rotated arcs; panel (f) the uniform SC surface state, panels (g) and (h) two parallel \( \pi \) junctions along \( y \) and \( x \) axes, respectively.

The small gap at Weyl points in panels (c) and (f) is a finite size effect – the gap closes as \( L_z \to \infty \).

3.3 By placing a superconducting ring with a narrow opening on top of a \( \text{Cd}_3\text{As}_2 \) slab, one obtains a Josephson junction mediated by the surface states of \( \text{Cd}_3\text{As}_2 \). We label the length of the junction by \( d \) and the width by \( W \). An external flux going through the ring generates a phase difference \( \varphi = 2\pi(\Phi/\Phi_0) \) across the junction, which leads to supercurrent measurable by a capacitively-coupled SQUID sensor. To study the lowest ABS, we assume \( d \ll \xi \), the proximity-induced coherence length.
3.4 Spectrum of the $L_x a \times L_y a \times L_z a = 40a \times 52a \times 60a$ ($a = 20\text{Å}$) slab of Cd$_3$As$_2$ with periodic boundary condition in the $y$- and $z$-directions. The material parameters are listed in Table A.1. Energy states are color-coded according to the expectation value of distance from the surface. Panel (a) shows the normal-state energy spectrum with the Dirac cones separated along $k_z$ and the surface Fermi arcs spanned between them. In Panel (b) a uniform pairing potential of magnitude $\Delta_0 = 10\text{meV}$ is introduced on the surfaces; the Fermi arcs are gapped out. In Panel (c), two junctions of $\pi$ phase difference are implemented on the top surface and the bottom surface remains uniformly superconducting; the resulting zero-energy surface flat bands linking the Dirac nodes are four-fold degenerate and decoupled. In Panel (d), the phase difference deviates from $\pi$ to $1.1\pi$; the flat bands get split from zero energy and obtain small dispersion, while the rest of the bands stay the same. In Panels (b)-(d), the small gaps near the Dirac nodes are due to the finite-size effect.
3.5 The Josephson current (per junction) computed from Eq. (3.18) is shown at various $\varphi$. At $k_B T = 0$ and 0.1meV, there is a clear current jump of magnitude $60\text{meV} e/\hbar$ or $15\mu\text{A}$. At $k_B T = 1\text{meV}$, the CPR retains little skewness. For comparison, the solid blue curve shows the sinusoidal CPR of a conventional junction with one ABS. The magnitude is small (bounded by $\frac{\Delta_0^2}{2\hbar}$) compared to our numerical results, mainly because our junction hosts a large number of ABSs. Note that $\Delta_0 = 10\text{meV}$ in the numerics, which is unrealistically large so that the superconducting gap is larger than the finite-size gaps that are always present in finite systems. While $k_B T = 0.1$ and 1meV appear to be fairly high temperatures, it is the ratio $k_B T/\Delta_0$ (1% and 10% respectively) that dictates the thermalization of quasiparticles. In a more realistic system with $\Delta_0 = 0.1\text{meV}$, the predicted blue and green curves correspond to $T \sim 10\text{mK}$ and $100\text{mK}$ respectively.

3.6 The first few quasiparticle excitation energies $E_i$ at $k_z = 0$ are shown at various $\varphi$. The subgap ABS (in red) has a kink at $\pi$ while the other states show little phase dependence. Same behaviour holds for any $k_z \in (-Q, Q)$; outside of this range there is no ABS.

3.7 Panel (a) plots $\nu(E)$ at $\pi$ and shows that non-magnetic disorder broadens the Majorana flat bands by $c/\Delta_0$. Panel (b) shows $\nu(E)$ at $c/\Delta_0 = 0.1$ and different $\varphi$’s. The broadening diminishes as $\varphi$ deviates from $\pi$.

3.8 Panel (a) shows the disordered CPR near $\pi$ at $k_B T/\Delta_0 = 0.01$ and various $c/\Delta_0$. The discontinuity is rounded off but the jump size is unaffected. Panel (b) shows the CPR slope at $\pi$ as $c/\Delta_0$ increases (again at $k_B T/\Delta_0 = 0.01$). For comparison, the slope at $\pi$ of a conventional sinusoidal CPR is 1.
4.1 With external magnetic field threaded through the Weyl semimetal nanowire, counter-propagating zeroth Landau levels are formed at the opposite Weyl nodes, shown in Panel a. Twisting the nanowire induces a pseudo-magnetic field $\mathbf{b}$, whose chiral nature is seen in the parallel zeroth Landau levels of the bulk, as illustrated in Panel b. When a parallel electric field is applied, the electron states begin to evolve semiclassically towards higher momenta. 

4.2 The effect of strain on the hopping amplitudes in the tight binding model. a) Unidirectional strain along the $z$ axis simply changes the distance between the neighboring orbitals leading to the modification of the hopping amplitude $t_1$ that is linear in $u_{33}$ to leading order in small displacement. b) Torsional strain changes the relative orientation of the orbitals and brings about hopping amplitudes that are disallowed by symmetry in the unstrained crystal, such as $t_{sp}$. The corresponding mathematical expression encodes the expectation that $t_{sp}$ would become equal to $\Lambda$ if the $p$ orbital were displaced all the way to the horizontal position. In the real material one of course expects Eq. (4.12) to be valid only for displacements small compared to the lattice parameter $a$. 

4.3 The displacement field $\mathbf{u}$ in the presence of torsion. Consecutive layers of the crystal are rotated by relative angle $\varphi_0 = \Omega(L/a)$. 

xiv
4.4 Tight-binding model simulations of a Weyl semimetal wire under torsional strain and applied magnetic field $\mathbf{B} = \hat{z}B$. Top row of figures shows the band structure of the lattice Hamiltonian defined by Eqs. (4.8) and (4.13) computed for $\frac{1}{2}$-Cd$_3$As$_2$ model parameters, for a wire with a rectangular cross section of $30 \times 30$ sites and a lattice constant $a = 40\text{Å}$. (We use larger lattice constant here and in subsequent simulations than in real Cd$_3$As$_2$ in order to be able to model nanowires and films of realistic cross sections with available computational resources. Note that this does not affect the physics at low energies because the lattice Hamiltonian is designed to reproduce the relevant $\mathbf{k} \cdot \mathbf{p}$ theory independent of $a$.) Open boundary conditions are imposed along $x$ and $y$, periodic along $z$. Parameters appropriate for Cd$_3$As$_2$ are used. Middle and bottom rows show spectral functions $A^{\text{bulk}}(\mathbf{k},\omega)$ and $A^{\text{surf}}(\mathbf{k},\omega)$. The former is obtained by averaging the full spectral function $A_j(\mathbf{k},\omega)$ over sites $j$ in the central $10 \times 10$ portion of the wire while the latter averages over the sites located at the perimeter of the wire. The torsion applied in columns c and d corresponds to the maximum displacement at the perimeter of $0.5a$, or $\varphi_0 \simeq 2^\circ$ between consecutive layers.
4.5 Tight-binding model simulations of a Weyl semimetal under applied magnetic field $B = \hat{z}B$ and unidirectional strain. Parameters for Cd$_3$As$_2$ listed in Appendix A are used in all panels. Only spin up sector of the model is considered with $B = 10$T. a) Band structure of the system with periodic boundary conditions in all directions (no surfaces) projected onto the $z$ axis ($k$ denotes the crystal momentum along the $z$ direction). Solid (dashed) lines show occupied (empty) states. Occupation of the strained system is determined by adiabatically evolving the single-electron states of the unstrained system. b) Band structure of a slab with thickness $d = 1000\,\text{Å}$ (50 lattice sites). Only positive values of $k$ are displayed but the band structure is symmetric about $k = 0$. Red (black) lines show occupied (empty) states. The central panel indicates the nonequilibrium occupancy of the strained system obtained by adiabatically evolving the single-electron states of the unstrained system. The right panel shows the occupancy of the strained system once the electrons relaxed back to equilibrium. All three panels correspond to the same total number of electrons $N$. c) Change in the electron density in response to the applied strain as a function coordinate $y$ perpendicular to the slab surfaces. $\delta \rho$ refers to the nonequilibrium distribution while $\delta \rho_{\text{eq}}$ refers to the relaxed state. Note that density oscillations near the edges apparent in $\delta \rho$ average to zero: there is no net charge transfer between the bulk and the surface in the nonequilibrium state, as can also be deduced from the vanishing $\delta \rho$ in the bulk.
4.6 Numerically calculated change in the bulk charge density $\delta \rho^{\text{bulk}}$ in response to unidirectional strain $\alpha = 0.03$ as a function of the applied field $B$. Parameters for Cd$_3$As$_2$ are used with $\mu = 0$ and $d = 1000\text{Å}$ (50 lattice sites). Solid black symbols give result for the p-h symmetric version of the $\frac{1}{2}$-Cd$_3$As$_2$ model obtained by setting all $C_j$ parameters to zero. ................................. 63

4.7 Band structure of the spin down sector of Cd$_3$As$_2$ in magnetic field $B = 10\text{T}$. Two chiral branches are visible at low energy but they are now strongly distorted by p-h symmetry breaking terms and they no longer traverse the gap between the valence and the conduction band. 65

4.8 Equilibrium current density in the Weyl semimetal wire under torsion.  
   a) Schematic depiction of the bulk/surface current flow.  
   b) Ground state current density computed from the lattice model Eqs. (4.8) and (4.13) at chemical potential $\mu = 5\text{meV}$. Warm (cold) colors represent positive (negative) current density $j$. The ring-shaped inhomogeneity in $j$ apparent in the bulk of the wire reflects Friedel-like oscillations in electron wavefunctions caused by the presence of the surface. .......... 66

4.9 Proposed geometry for the EM field emission measurement in the limit when all the dimensions of the crystal are much larger than the sound wavelength $\lambda_s$.  
   a) A slab of thickness $d = 2d'$ is subjected to magnetic field $B$ and a longitudinal acoustic sound wave propagating along the $z$ direction.  
   b) A snapshot of the electric field distribution near the surface calculated from Eq. (4.43). As a function of time the entire pattern moves in the $z$ direction at the speed of sound $c_s$. . . . . . . 74
5.1 An irregular shaped graphene flake in an applied magnetic field gives rise to the (0+1)-dimensional SY model, holographically dual to a black hole in (1+1)-dimensional anti-de Sitter space. Inset: lattice structure of graphene with A and B sublattices marked and nearest neighbor vectors denoted by $\delta_j$. ................................. 83

5.2 Electronic properties of an irregular graphene flake in the absence of interactions. a) Single-particle energy levels $\epsilon_j$ of the Hamiltonian $H_0$ as a function of the magnetic flux $\Phi = SB$ through the flake. The flake used for this calculation, depicted in the inset, consists of 1952 carbon atoms with equal number of A and B sites. The energy spectrum, calculated here in the Landau gauge $A = Bx\hat{y}$ and with open boundary conditions, shows the same generic features irrespective of the detailed flake geometry. b) Typical wavefunction amplitudes of the eigenstates $\Phi_j(r)$ belonging to LL$_0$ at $\Phi = 40\Phi_0$ and the edge modes. The numerals above each panel denote the energy $\epsilon_j$ of the state in eV, scale bar shows the magnetic length $l_B = \sqrt{\hbar c/eB}$. ................................. 86

5.3 Statistical properties of the coupling constants and the thermal entropy. a) Histogram of $|J_{ij;kl}|$ as calculated from Eq. (5.4) with $V_1 = 1$ for the graphene flake depicted in Fig. 5.2 and $N = 16$, compared to the Gaussian distribution (orange line) with the same variance 0.000805$V_1$. Inset shows the histogram of real and imaginary components of $J_{ij;kl}$. The mirror symmetry about the horizontal follows from the hermiticity property $J_{ij;kl} = J_{kl;ij}^*$. b) Entropy $S(T)$ of the SY Hamiltonian (5.1) calculated with $J$s shown in panel (a). ................................. 88
5.4 Many-body level statistics for the interacting electrons in LL$_0$ of the graphene flake. Blue bars show the calculated distributions for the graphene flake. Orange, green and red curves indicate the expected distributions given by Eq. (5.6) for GOE, GUE and GSE, respectively. To obtain smooth distributions, results for $N = 14, 15, 16$ have been averaged over 8 (4) distinct flake geometry realizations while $N = 17, 18$ reflect a single realization.

A.1 Dispersion relations for the spin-up sector of the lattice Hamiltonian (4.7) describing Cd$_3$As$_2$ (top row) and Na$_3$Bi (bottom row). The parameters used in the simulations include the particle-hole symmetry breaking terms and are summarized in Table A.1. We used a lattice with $40 \times 40$ sites and the magnetic fields shown in the green boxes for each of the material. Notice the different magnitude of effective magnetic fields for different compounds – this is due to the different lattice constants, and different sign of the physical magnetic field between the two rows. Different sign of magnetic fields shows that the physical magnetic field compensates the torsional one in opposite Weyl points for opposite directions of magnetic field in accordance with the interpretation in the main text.

A.2 Persistent currents in a Cd$_3$As$_2$ nanowire under torsion and magnetic field. a) Band structure detail for spin up (blue) and spin down (green) sectors in a $30 \times 30$ lattice with $B = b = 3.2$T and other parameters as in Fig. A.1. b) Calculated current density $j_z$ for $\mu = 0$ including contributions from both spin sectors.
C.1 Ratio of conductance of a disordered $W \times W \times 20$ system to the conductance of the clean system averaged over 100 disorder realizations. Green line is the best fit to the data – parabolic, grey curves show the failure of the linear (with non-negative $G(0)$) and cubic fits.

E.1 Effects of the second neighbor hopping $t'$. a) Single-particle energy spectrum of a flake (the same geometry as Fig. 5.2) with second neighbor hopping $t' = 0.037t$. b) Average shift $\delta \epsilon = \bar{K}_{ij}$ and standard deviation $K$ of 40 energy levels that comprise LL$_0$ as a function of $t'$.

E.2 Effects of random on-site potential. a) Low-energy part of the numerically calculated energy spectrum for the flake with $n_I = 1\%$ of defected sites as a function of the disorder potential strength $w$ and $N = 40$. b) Average shift $\delta \epsilon = \bar{K}_{ij}$ and standard deviation $K$ of 40 energy levels that evolve from the zero modes which comprise LL$_0$ in the pure sample. These levels are marked in red in panel (a).
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For my parents
Chapter 1

Introduction

*To see a world in a grain of sand,*

*And a heaven in a wild flower,*

*Hold infinity in the palm of your hand,*

*And eternity in an hour.*

- *from “Auguries of Innocence” by William Blake*

Throughout the history of science and philosophy, the most prominent view of the Universe is that it is made up of indivisible fundamental particles. Nevertheless, modern physicists consider the possibility that all things and the interactions among them are merely collective excitations from an underlying quantum vacuum. This emergent picture is familiar from solid-state physics, where quasiparticles seemingly moving in free space emerge at long distances from complex many-body systems. It remains an open question whether we can construct, at least theoretically, a crystal lattice from which everything the high-energy physicists know – fermions, bosons, gauge fields, chiral interactions, and so on – emerge. If we dare to include gravity, then we would be searching for the Theory of Everything – inside a rock. The great poet probably never thought that one day physicists would take his imagery literally – perhaps the whole Universe can be held in the palm of your hand.
It seems that the research community is naturally moving towards that end, as there has been productive interaction between high-energy and condensed matter physicists. These two seemingly disparate fields – high-energy physics mainly concerns with single-particle processes at extremely high temperatures while condensed matter physics deals with many-body problems near absolute zero – both found tremendous benefits in using quantum field theoretical approaches and had many back-and-forth exchanges of ideas. For example, the Green’s function approach and Feynman diagrams were first used in high-energy physics and later adapted in many-body problems, and spontaneous symmetry-breaking during phase transitions in condensed matter systems is implemented by high-energy physicists to understand the very early Universe.

These interactions escalated in recent years partly due to the discovery of topological phases of matter. As their names suggest, these phases can only be distinguished by their underlying topological properties, beyond the traditional Landau classification by local symmetries. Several epitomes of this growing class of phases are described in the low-energy regime by the Chern-Simons theory, a topological field theory, originally considered by high-energy physicists [2, 3]. These phases also exhibit exotic, topologically-protected boundary states, some of which are physical realizations of high-energy theories. For instance, the chiral edge states of Fractional Quantum Hall states are described by chiral conformal gauge theories previously studied by string theorists [4], and the surface states of 3D topological insulators are massless relativistic fermions [5]. Entanglement, the mechanism which gives rise to topological orders [6], is another shared interest with the high-energy community. It seems to be closely related to the geometry of space-time, as suggested by quantum-gravity research [7].

Having high-energy physics hidden inside these materials is an incredible experimental advantage. Typically experimentalists must go to great lengths to observe
high-energy physics directly because of the immense energy-scale involved. It is therefore exciting to find table-top testing grounds in topological materials, which have become increasingly accessible in laboratories due to the intense interest and competition among experimental groups worldwide. The technologies for growing, manipulating, and measuring these materials are mature and innovative. So far important topological signatures such as quantized conductance, exotic surface states, and anomalous response functions have been confirmed by transport experiments, angle-resolved photoemission spectroscopy (ARPES), and scanning tunneling microscopy (STM) [8, 9].

Exploring this context, my collaborators and I devised ways to realize high-energy physics in topological semimetals [10]. These topological phases can be studied using band theory and by definition exhibit band-touching points or lines at the Fermi energy. Gapless topological phases have recently gathered great interests after it was realized that topological invariants are well-defined in the gapped subspace of the Brillouin zone. Notable examples include graphene and Weyl semimetals, which are the focus of this work. Both systems have gapless points around which the energy bands disperse linearly. Graphene, an atomic layer of graphite, is well-understood in theory and accessible experimentally thanks to the Nobel-prize-winning "peeling" technique [11]. Only until recently do people recognize its topological attribute. Weyl semimetals are 3D analogies of graphene. Although they entered the scene fairly recently, intense research efforts have made them available in labs and led to observations of their topological signatures [12–16].

Our research developed into three distinct projects:

- **Majorana Fermions at Weyl-Semimetal-Superconductor Interface (Chapter 3)**

- **Chiral Anomalies in Strained Weyl Semimetals (Chapter 4)**

- **Holographic Black Hole on a Graphene Flake (Chapter 5)**
Each was completed by extensive analysis done with experimental realization in mind. This Thesis is the culmination of our investigation and results. I will begin by giving a brief review of the topological semimetals in Chapter 2.
Chapter 2

Brief Review of Topological Semimetals

Up until recently, physicists upheld the traditional Landau theory of phase transition, which states that distinct phases of matter are characterized by the symmetries describing the organizations of the microscopic components. For instance, water turns into ice when water molecules lose their freedom to rotate and move continuously; we say that the translational and rotational symmetries of the system are broken. Without breaking symmetries, the topological phases of matter undergo phase transitions when the so-called topological order changes [6]. Therefore its discovery was revolutionary and justifiably recognized by the 2016 Nobel prize. We now know that there is a realm of quantum (i.e. zero-temperature) materials living beyond symmetry classification. A topological order is robust against local perturbations of the Hamiltonian (caused by impurities inside the material, for example), in much the same way that, in mathematics, the topology of a space is invariant under smooth deformation. It is manifested through exotic properties such as non-abelian excitations [17] and quantized edge conductance [5].
There are two types of topological phases, depending on the range of quantum entanglement among the microscopic constituents [6]. Most topological phases with long-range entanglement – such as quantum spin liquids [18] and fractional quantum hall effect [4] – occur in strongly correlated systems, which calls for advanced theoretical techniques. On the other hand, the topological phases with short-range entanglement, also known as symmetry-protected topological phases (SPT) [19], can be described effectively by topological band theory, stemming from conventional band theory of non-interacting quasiparticles. The SPT phases are less robust in the sense that certain symmetries must be present to maintain the topological order.

The pioneering examples of the SPT phases are topological insulators (TI) and topological superconductors (TSC). They have been fully categorized by a ten-fold-way table according to symmetry protection [20] and realized experimentally in a variety of materials [5, 21]. Their topological orders are defined by topological invariants computed using their ground state wavefunctions [3, 22]. Topological invariants are usually integer-valued and robust against any perturbations that respect the required symmetries. They dictate the quantization of certain physical observables such as the quantized Hall conductance in integer quantum Hall systems.

By construction it seems that topological order exists only in a fully gapped system, in which the ground state is well-separated from excitations. More recently, physicists pondered the question of whether there exists gapless topological states. It turns out that one can compute topological invariants in semimetals. A semimetal is a solid-state system in which the conduction and valence bands touch at isolated points or along a continuous line, so it is possible to construct a compact subspace of the Brillouin zone that avoids the gapless region and compute topological invariants on it. If a topological invariant is found to be nontrivial, the system is a topological semimetal. Weyl semimetals are the leading examples of topological semimetals [10, 23]. They have topologically protected band-touching points. Graphene can
be seen as their 2D analogue [24]. Nodal-loop semimetals are another example with a continuous line of band-touching points [23, 25, 26]. Nodal superconductors are generalizations of topological semimetals [27, 28]. In the following, we focus on introducing Weyl semimetals and graphene as they are the main materials of interest in this Thesis.

2.1 Weyl Semimetals

A Weyl semimetal is a 3D solid-state system in which the valence and conduction bands are nondegenerate and touch at points in the Brillouin zone. The nondegenerate band-touching points are called the Weyl nodes. Ideally, the Fermi level lies right at the Weyl nodes and there are no other Fermi surfaces. The electronic band structure near a Weyl node can be effectively described by the Weyl fermion Hamiltonian

\[ H = \sum_q c_q^\dagger v q \cdot \sigma c_q, \tag{2.1} \]

where momentum \( q \) is measured from the node, \( \sigma = (\sigma_x, \sigma_y, \sigma_z) \) are the Pauli matrices in the pseudospin space, \( v \) is the velocity of the massless Weyl fermion. The corresponding energy spectrum is a linearly dispersing Weyl cone. The fact that the low-energy theory describes a Weyl fermion is important for our discussion on chiral anomaly in Chapter 4. Note that all three Pauli matrices are used up by the Hamiltonian, so any additional mass term only shifts the position or energy of the node. In other words, the Weyl node is stable against any perturbations to the Hamiltonian.

To describe the nontrivial topology of a Weyl semimetal, consider a 2D subspace of the Brillouin zone defined by an arbitrary closed surface \( S \) enclosing a Weyl node. The electronic band structure is fully gapped in this subspace, on which it is therefore possible to compute topological invariants. In particular, one can compute the Chern
number, which is a measure of the total Berry flux through the surface. One begins by computing the Berry vector potential

$$A(k) = -i \sum_{n \text{ occupied}} \langle u_{n,k} | \nabla_k | u_{n,k} \rangle,$$  \hspace{1cm} (2.2)

where $u_{n,k}$ is the Bloch wavefunction and $n$ runs through the filled bands. The Berry curvature is the curl of the Berry vector potential

$$B(k) = \nabla_k \times A,$$  \hspace{1cm} (2.3)

analogous to the fact that magnetic field is the curl of magnetic vector potential. Intuitively, the Berry curvature can be seen as magnetic field in the momentum space. Chern number is the integral of Berry curvature through the surface

$$C = \frac{1}{2\pi} \int_S B \cdot ds = n \in \mathbb{Z}.$$  \hspace{1cm} (2.4)

It must be an integer, the proof of which can be found in Sec. 3.6 of Ref. [3]. A system with nonzero Chern number is called a Chern insulator. To gain physical intuition of the Chern number, one can invoke the Stokes’ theorem and rewrite Eq. (2.4) as the line integral of the Berry vector potential $A$ over the boundary of $S$. Since $S$ is a closed surface without a boundary, the integral vanishes as long as $A$ is well-defined over the momentum space enclosed by $S$. Nevertheless, a nonzero Chern number can result from any nontrivial structure of $A$, which obstructs the application of Stokes’ theorem. Weyl nodes are singularities of the Berry vector potential $A$, so the Chern number for the surface enclosing each node is nontrivial. In most Weyl semimetals, each Weyl node is a source or a sink of $2\pi$ Berry flux, corresponding to Chern number of 1 or -1 – the chirality of the node. As magnetic monopoles in the momentum space, two Weyl nodes of opposite chiralities can annihilate and result in
a band gap; whereas a single Weyl node cannot simply vanish as explained below Eq. (2.1).

The nonzero Chern numbers embedded in the band structure of Weyl semimetals give rise to oddly-shaped surface states – the Fermi arcs. To illustrate this, we consider the simplest Weyl semimetal with two nodes of opposite chiralities.\(^1\) Due to Gauss’ law, any closed surface enclosing a single Weyl node experiences \(2\pi\) Berry flux, so we can choose a closed surface such that it touches the boundary of the Brillouin zone, as shown by a combination of the red and blue planes in Fig. 2.1a. Since it contains a Weyl node, it has Chern number 1. Now let us consider the red and blue planes separately. Each of them is a closed surface on its own due to the periodicity of the Brillouin zone, so we can compute their Chern numbers. We know that one of them must be 0 and the other 1 because together they give one. Without loss of generality, let us suppose that the red plane has Chern number 1. The red plane can be anywhere between the two nodes and still has the same amount of Berry flux through it, i.e. the same Chern number. In other words, there is an embedded Chern insulator in the band structure at every crystal momentum between the nodes. Chern insulators are known to exhibit chiral edge states.\(^3\) If the Weyl semimetal system has a surface parallel to the displacement between the Weyl nodes (so the momentum in this direction remains a good quantum number), then the chiral edge states – one at each internodal momentum – accumulatively form the surface states of the Weyl semimetal. At the Fermi energy, they form a Fermi arc ending at the Weyl nodes, which is strikingly different from the Fermi surface in a conventional 2D metal. This is one of the most pronounced signatures of the nontrivial topology of Weyl semimetals.

Fig. 2.1b shows the band structure of a semi-infinite Weyl semimetal with one open boundary. Between the bulk Weyl nodes span the chiral-propagating surface states which appear as a Fermi arc at the Fermi energy.

\(^1\)Weyl nodes must occur in positive-negative pairs so that the total chirality in the Brillouin zone vanishes according to the fermion-doubling theorem [29].
Figure 2.1: This figure is from Ref. [10]. Panel (a) illustrates schematically a Weyl semimetal system (in $k$-space) with two nodes. At every internodal momentum, the embedded Chern insulator (red plane) exhibits a chiral edge state. If the system is given an open boundary parallel to the displacement of the nodes, as shown by the top surface, all the chiral edge states collectively give rise to the Fermi-arc surface states. Panel (b) shows the band structure (at some fixed $k_z$). At the Fermi energy, the chiral-propagating surface states appear as a Fermi arc ending at the Weyl nodes.

The above argument for the existence of Fermi-arc surface states assumes translational symmetry, which is broken in any realistic materials with impurities. Nevertheless, Fermi arcs have been experimentally confirmed by angle-resolved photoemission spectroscopy (ARPES) measurements in the first Weyl semimetal candidates—a family of compounds consisting of TaAs, TaP, NbAs and NbP [15, 16, 30, 31]. Their topological response—negative magnetoresistance due to chiral anomaly (more in Chapter 4)—has also been observed [13, 14]. Dirac semimetal $\text{Cd}_2\text{As}_3$ [32–37] has the topology of a Weyl semimetal, but its Weyl cones coincide pairwise due to crystal symmetry (hence realizing Dirac fermions in (3+1)-dimension). It is great for theoretical studies because it has only four Weyl nodes. Its Fermi arcs and negative magnetoresistance have been observed [12, 38–40].
2.2 Graphene

Graphene is a 2D honeycomb lattice of carbon atoms, as shown in Fig. 2.2a. It is a semimetal with two band-touching points at $K$ and $K'$ in the Brillouin zone (see Fig. 2.2b). Near each node, the electronic band structure can be effectively described by the Hamiltonian of a massless Dirac fermion in 2D:

$$ h(K + q) = q_x \sigma_x + q_y \sigma_y $$

and

$$ h(K' + q) = -q_x \sigma_x + q_y \sigma_y. $$

It can be shown that the gaplessness of the Dirac cones is protected by inversion and time-reversal symmetries. Any perturbation that breaks either of them makes graphene a gapped insulator. In fact, it is possible to construct a mass term that renders graphene a Chern insulator.\footnote{Since our work in Chapter 5 employs the semimetallic nature of graphene, I will leave the interested readers to learn more about the insulating graphene in Ref. [3].}

The Dirac nodes in graphene, like the Weyl nodes in Weyl semimetals, are topologically nontrivial. They are vortices in the Berry vector potential $A$ and can be detected by a 1D topological invariant. One computes the Berry phase for a
constant-energy loop $C$ surrounding the node

$$\gamma = \int_{C} \mathbf{d} \mathbf{k} \cdot \mathbf{A}(\mathbf{k})$$

(2.7)

which gives $\pi$ or $-\pi$, the vorticity of the node. Two Dirac nodes of opposite vorticities would annihilate when brought close.

Similar to the argument for the surface states of Weyl semimetals, the edge states in graphene can be understood in terms of the embedded 1D topological insulators. At each fixed $k_y$ between $K$ and $K'$ is a polyacetylene-type 1D topological insulator [41], which is known to exhibit isolated zero modes at the boundary. As a result, the band structure of graphene with boundary (e.g. a ribbon) displays zero-energy flat bands of edge states spanned between the bulk Dirac nodes. Detailed derivation of the edge states can be found in Ref. [3].
Chapter 3

Majorana Fermions at Weyl-Semimetal-Supercollider Interface

3.1 Overview

All fermions in the Standard Model are described by the Dirac equation. Mathematically, one can tweak the equation and find self-conjugating solutions representing the Majorana fermions. These hypothetical particles, indistinguishable from their anti-particles, have not been observed in nature, but they play important roles in neutrino physics and theories beyond the Standard Model [42]. Following the discovery of neutrino mass in the neutrino-oscillation experiments [43–45], Majorana fermions replaced massless Weyl fermions as the most suitable candidate for neutrinos. Notably this would allow the process of neutrinoless double beta decay, which violates the conservation of lepton number and thus provides a primordial mechanism that could originate the matter, anti-matter asymmetry. Furthermore, Majorana fermions
fill the role of several promising dark-matter candidates and superpartners required by Supersymmetry.

It is therefore exciting that Majorana fermions have been found as emergent quasi-particles in solid-state systems. In fact, they appear ubiquitously in superconductors as Bogoliubov quasiparticles, which are superpositions of electrons and holes in the Bogoliubov-de-Gennes theory of superconductivity [46]. A more interesting case arises when Majorana quasiparticles sit at zero energy, separated from the other excited states by the superconducting gap. These so-called Majorana zero modes (MZMs) exhibit non-Abelian exchange statistics, instead of the usual fermionic statistics, due to phase contribution from the superconducting condensate. In principle one can store quantum information in a group of these localized modes by spatially “braiding” them. This method for storing quantum information is more fault-tolerant than many other proposals and thus has drawn intense theoretical and experimental research interests [47].

The canonical example for realizing MZMs is the Kitaev chain, a 1D superconductor with a topological invariant defined in the bulk and MZMs localized at the end points [48]. More proposals have ensued and demonstrated that generally MZMs can be found at the boundary or the defects of topological superconductors [22, 49–66]. One particular example is a 2D topological insulator (TI) coupled to a conventional s-wave superconductor [49, 50]. The metallic edge of the 2D TI becomes superconducting due to proximity effect and resembles the Kitaev chain. One can artificially create an “end point” at the edge of the 2D TI by implementing a Josephson junction in the superconductor. Two decoupled MZMs show up localized at the junction when the phase difference across the junction is $\pi$, which restores the time-reversal-invariant symmetry needed by TI.

In Sec. 3.2, we show that Weyl semimetals with time-reversal symmetry $T$ can also be suitable platforms for hosting MZMs, and the underlying mechanism is similar
to the aforementioned case of 2D TI. The nontrivial topology of a $\mathcal{T}$-invariant Weyl semimetal can be understood from the hidden 2D TIs embedded in the band structure. The Fermi arcs can be seen as stacks of helical edge states of the embedded 2D TIs. Proximity-coupling the surface of the Weyl semimetal by a superconductor thus amounts to proximitizing the edges of the 2D TIs. It follows from the previous paragraph that a junction mediated by the Weyl semimetal surface states would also localize MZMs at $\pi$ phase difference. An important difference is that the system here is one dimension higher, so the geometric width of the junction and the extra momentum degree of freedom allow the localization of a large population of MZMs. In particular, while the junction mediated by the edge states of a 2D TI localizes a pair of MZMs, one mediated by the surface states of a $\mathcal{T}$-invariant Weyl semimetal contains $N$ pairs of MZMs where $N$ scales linearly as the momentum space distance between the Weyl nodes and the junction width.

While this result may not be applicable to quantum computation as the zero modes are not spatially separated, the conglomeration of MZMs is ideal for studying interacting Majorana fermions. Moreover, the set-up is ready to be realized with existing experimental technologies. To determine the presence of MZMs, one can measure Josephson current across the junction as phase difference varies. In Sec. 3.3, we predict a characteristic step discontinuity in the current-phase relation, which is a type of fractional Josephson effect previously studied in other unconventional junctions [50, 67–71]. We also demonstrate that this experimental signature is robust against finite temperature and weak non-magnetic disorder.
3.2 Superconducting Proximity Effect and Majorana Flat Bands

Interfacing topological materials with conventionally ordered states of matter, such as magnets and superconductors, has led to important conceptual advances over the past decade. Notable examples of this approach include the Fu-Kane superconductor [49] that occurs in the interface of a 3D strong topological insulator (STI) and a conventional s-wave SC, the “fractional” quantum Hall effect that arises when STI is interfaced with a magnetic insulator [72–74], as well as many interesting phenomena that occur when both SC and magnetic domains are present [75, 76]. Rich physics, including Majorana zero modes, also results when the edge of a 2D topological insulator is interfaced with magnets and superconductors [50, 77, 78]. More recently various exotic phases of quantum matter have been predicted to occur based on these same ingredients in strongly interacting systems [79–89].

In this section, we explore the physics of the interface between a Weyl semimetal and an s-wave superconductor. Surface states of a Weyl semimetal exhibit characteristic Fermi arcs that terminate at surface projections of the bulk Weyl nodes [90, 91]. Such “open” Fermi surfaces are fundamentally impossible in a purely 2D system and we thus expect the resulting SC state to also be anomalous. In this respect the situation is similar to the Fu-Kane superconductor [49] whose existence hinges on the odd number of Dirac fermions occurring on the surface of an STI. As we shall see there are several notable differences between STI/SC and Weyl/SC interfaces which make the latter a distinct and potentially more versatile platform for explorations of new phenomena.
3.2.1 2D Continuum Model of Surface States

Nondegenerate Weyl points can occur in crystals with broken time reversal symmetry $\mathcal{T}$ or broken bulk inversion symmetry $P$. Recent experimental work reported convincing evidence for Weyl nodes and surface Fermi arcs in $\mathcal{T}$-preserving noncentrosymmetric crystals in the TaAs family of semimetals [15, 16, 92–95]. We therefore focus our discussion on the SC proximity effect in this class of materials. When a crystal respects $\mathcal{T}$ the minimum number of Weyl points $n$ is 4. This is because under $\mathcal{T}$ a Weyl point at crystal momentum $Q$ maps onto a Weyl point at $-Q$ with the same chirality. Since the total chiral charge in the Brillouin zone must vanish there has to be another pair of $\mathcal{T}$-conjugate Weyl nodes with an opposite chirality. We begin by discussing the SC proximity effect in this minimal case with $n = 4$. We note that although the currently known Weyl semimetals exhibit larger number of nodes ($n = 24$ in the TaAs family) recent theoretical work identified materials that could realize instances with smaller $n$ [96, 97], including the minimal case with $n = 4$ predicted in MoTe$_2$ [98]. The generic surface state of such a minimal $\mathcal{T}$-preserving Weyl semimetal is depicted in Fig. 3.1a.

![Figure 3.1: Surface Fermi arcs in a minimal model of a Weyl semimetal with (a) and without (b) time reversal symmetry $\mathcal{T}$. Weyl nodes are represented by circles with positive (negative) chirality; green arrows indicate the possible direction of electron spin along the arcs. Panel (c) shows the $\pi$ junction setup on the surface of a Weyl semimetal.](image)

Interfacing such a surface with a spin singlet $s$-wave SC, one expects formation of a paired state from time-reversed Bloch electrons at crystal momenta $k$ and $-k$.
along the arcs. A minimal model describing this situation is defined by the second quantized Bogoliubov-de Gennes (BdG) Hamiltonian \( \mathcal{H} = \sum_k \hat{\Psi}_k^\dagger h_{\text{BdG}}(k) \hat{\Psi}_k \) where 

\[
\hat{\Psi}_k = (c_{k \uparrow}, c_{k \downarrow}, c_{-k \downarrow}^\dagger, -c_{-k \uparrow}^\dagger)^T
\]
is the Nambu spinor and

\[
h_{\text{BdG}} = \begin{pmatrix} h_0(k) & \Delta \\ \Delta^\dagger & -s^y h_0^*(-k)s^y \end{pmatrix}.
\] (3.1)

The normal state surface Hamiltonian for two parallel Fermi arcs with electron spin locked perpendicular to its momentum as in Fig. 3.1a can be written as

\[
h_0(k) = vs^zk_x - \mu, \quad \text{for } |k_y| < K.
\] (3.2)

Here \( s^\alpha \) are Pauli matrices acting in electron spin space and \( K \) controls the extent of the arc in the \( y \) direction. It is possible to make this model more generic by allowing the velocity \( v \) and chemical potential \( \mu \) to depend on \( k_y \) and the cutoff \( K \) on \( k_x \) (which would curve and shift the arcs similar to Fig. 3.1a) but the minimal model defined above already captures the essential physics we wish to describe. We note that the model based on Eq. (3.1) is valid only away from the surface projections of the Weyl points, specifically when \( v||k_y| - K| > \Delta \), as it does not capture the bulk bands that become gapless near the Weyl points.

If we define another set of Pauli matrices \( \tau^\alpha \) in the Nambu space we can write Eq. (3.1) for each \( |k_y| < K \) as

\[
h_{\text{BdG}} = (vs^zk_x - \mu)\tau^x + \Delta_1 \tau^x - \Delta_2 \tau^y, \quad \text{(3.3)}
\]

where \( \Delta = \Delta_1 + i\Delta_2 \). For spatially uniform \( \Delta \) the spectrum of \( h_{\text{BdG}} \) is fully gapped. Its topological character can be exposed by noting that for each allowed value of \( k_y \) Eq. (3.3) coincides with the Hamiltonian describing the edge of a 2D TI in contact
with a SC. It is well known that unpaired MZMs exist in such an edge at domain walls between SC and magnetic regions [50]. This is because $\mathcal{T}$-breaking also opens up a gap in 2D TI edge and MZMs occur at boundaries between two differently gapped regions. In our present case of the Weyl semimetal breaking $\mathcal{T}$ generically does not open a gap. However, we can achieve a similar result by creating a domain wall in the complex order parameter $\Delta(x)$. Specifically, we show below that a $\pi$ junction (i.e. a boundary between two regions whose SC phase $\varphi$ differs by $\pi$, see Fig. 3.1c) hosts a pair of protected MZMs (one for each arc and for each allowed value of $k_y$), separated by a gap from the rest of the spectrum. As a function of $k_y$, then, the band structure of such a $\pi$ junction exhibits a flat Majorana band pinned to zero energy. We demonstrate below that such flat Majorana bands are experimentally observable and represent a generic robust property of any $\mathcal{T}$-preserving Weyl/SC interface.

To exemplify this property within the effective surface theory (3.3) we replace $k_x \to -i\partial_x$ and assume $\Delta(x)$ to be purely real with a soliton profile such that $\Delta(x) \to \pm \Delta_0$ as $x \to \pm \infty$. We next note that Hamiltonian (3.3) can be brought to a block diagonal form $h_{\text{BdG}} = \text{diag}(h_+, h_-)$ by a unitary transformation that exchanges its second and third rows and columns. Here $h_\pm$ are $2 \times 2$ matrices

$$h_s = \begin{pmatrix} -ivs\partial_x - \mu & \Delta(x) \\ \Delta(x) & ivs\partial_x + \mu \end{pmatrix},$$

and $s = \pm$. It is easy to show that for each allowed value of $k_y$ and each $s$ the above Hamiltonian (3.4) supports an exact Jackiw-Rossi zero mode [99] with a wave function

$$\psi_s(x) = A \begin{pmatrix} i \\ s \end{pmatrix} \exp \left\{ -\frac{1}{v} \int_0^x dx' [\Delta(x') - is\mu] \right\}$$

(3.5)

exponentially localized near the junction.
We shall demonstrate below by an exact numerical diagonalization of a 3D lattice model that the Majorana flat bands remain robust beyond the minimal low energy surface theory. Their stability however can be deduced from the surface theory alone and relies on the time reversal symmetry $T$ and the bulk-boundary correspondence present in the Weyl semimetal. The most general surface Hamiltonian consistent with these requirements is of the form $h(k) = (v_{k_y} \cdot s)k_x - \mu_{k_y}$ for $|k_y| < K$ where both the velocity vector and the chemical potential are even functions of $k_y$. Time reversal symmetry further permits a term $(u_{k_y} \cdot s)k_y$ but this is not consistent with $h(k)$ describing the surface state of a Weyl semimetal. For each $k_y$ one can now perform an SU(2) rotation in spin space to bring $h(k)$ to the form indicated in Eq. (3.2). Because the pairing term in the BdG Hamiltonian (3.3) is not affected by this transformation the calculation demonstrating the zero modes proceeds as before, with the result (3.5) modified in two minor ways: $v$ and $\mu$ may now depend on $k_y$ and the spinor structure reflects the SU(2) rotation. The exact zero modes persist for all allowed values of $k_y$ in this more general case.

It is easy to see that breaking $T$ lifts the zero modes: for instance interpolating between $-\Delta_0$ and $+\Delta_0$ through complex values of $\Delta(x)$ moves MZMs to finite energies. Flat Majorana bands are thus protected by $T$ and by translation symmetry that is necessary to establish the underlying momentum space Weyl structure.

### 3.2.2 3D Tight-Binding Model

To ascertain the validity and robustness of the above analytic results, we now study the problem using a lattice model for the Weyl semimetal. We consider electrons in a simple cubic lattice with two orbitals per site and the following minimal Bloch

---

1For $u$ nonzero the surface state exhibits a massless Dirac dispersion which would imply that the Fermi arc can shrink to a point when $\mu$ coincides with the Dirac point.
Hamiltonian,
\[ h_{\text{latt}}(\mathbf{k}) = \lambda \sum_{\alpha=x,y,z} s^\alpha \sin k_{\alpha} - \mu + \sigma^y s^y M_k. \]  
(3.6)

Here \( M_k = (m + 2 - \cos k_x - \cos k_z) \) and \( \sigma^\alpha \) are Pauli matrices acting in the orbital space. The Hamiltonian (3.6) is inspired by Ref. [100] and adapted to respect \( \mathcal{T} \) (generated here by \( is^y \mathcal{K} \) with \( \mathcal{K} \) the complex conjugation). It has a simple phase diagram. For \( m > \lambda \) (taking \( \lambda \) positive) it describes a trivial insulator. At \( m = \lambda \), two Dirac points appear at \( \mathbf{k} = (0, \pm \pi/2, 0) \) which then split into two pairs of Weyl nodes positioned along the \( k_y \) axis. These persist as long as \(|m| < \lambda\). In this phase, surfaces parallel to the \( y \) crystal axis exhibit two Fermi arcs terminating at the surface projections of the bulk Weyl nodes illustrated in Fig. 3.2.

![Figure 3.2: Tight binding simulations of the SC/Weyl proximity effect. Panels (a) and (e) show the surface spectral function \( A(\mathbf{k}, \omega) \) for the tight binding Hamiltonian \( h_{\text{latt}} + \delta h_{\text{latt}} \) and \( \omega = 0.15 \). In all panels we use \( m = 0.5, \lambda = 1, L_x = 50 \) and \( L_z = 40 \); while \( (\mu, \epsilon) = (0, 0) \) and \( (0.1, 1.0) \) for top and bottom row respectively. Panel (b) shows the normal state spectrum with flat bands representing the surface states. The effect of the SC proximity effect with \( \Delta_0 = 0.5 \) is indicated in panel (c) while panel (d) shows the effect of two parallel equidistant \( \pi \) junctions (protected zero modes indicated in red). The bottom row displays our results for rotated arcs; panel (f) the uniform SC surface state, panels (g) and (h) two parallel \( \pi \) junctions along \( y \) and \( x \) axes, respectively. The small gap at Weyl points in panels (c) and (f) is a finite size effect – the gap closes as \( L_z \to \infty \).](image-url)
We have performed extensive numerical computations based on $h_{\text{latt}}(k)$ defined above focusing on the slab geometry with surfaces parallel to the $x$-$y$ plane and thickness of $L_z$ sites. The proximity effect is studied using the BdG Hamiltonian (3.1) with $h_0$ replaced by $h_{\text{latt}}$ and $\Delta$ taken to be non-zero in the surface layers only $^2$. Top row in Fig. 3.2 summarizes our results. The normal state shows gapless Fermi arcs (Fig. 3.2a,b) while uniform SC order is seen to gap out the Fermi arc states except in the vicinity of the Weyl points where they merge into the gapless bulk continuum (Fig. 3.2c). Presence of the $\pi$ junctions (which we define parallel to the $y$ crystal axis) generates perfectly flat bands at zero energy between the projected Weyl nodes, Fig. 3.2d. We note that due to the periodic boundary conditions adopted in both $x$ and $y$ directions, our numerics by necessity implement two parallel $\pi$ junctions which results in twice the number of flat bands compared to a single junction. We also find that, remarkably, the bands remain completely flat even when the system size $L_x$ is small along the direction perpendicular to the junctions (in which case one would naively expect a large overlap between the bound state wavefunctions resulting in significant energy splitting). This property can be understood by noting that wavefunctions (3.5) remain exact zero modes of the Hamiltonian (3.1) for periodic boundary conditions along $x$ as long as $\Delta(x)$ averages to zero over all $x$ and $\mu = (2\pi v/L_x)p$ with $p$ integer. These conditions are satisfied for two equally spaced junctions and $\mu = 0$ used in Fig. 3.2. We checked that violating these conditions indeed leads to zero mode splitting that depends exponentially on the junction distance. Importantly, the fact that this detailed property of the simple model (3.2) is borne out in a more realistic lattice model gives us confidence that the low energy theory provides a correct description of the physical surface state of a Weyl semimetal.

The lattice Hamiltonian (3.6) has high symmetry with Weyl nodes confined to lie on the $k_y$ axis. It is important to verify that the phenomena discussed above are not $^2$Other order parameter configurations (e.g. $\Delta$ decaying exponentially into the bulk) were also considered with substantially similar results.
dependent on such a fine tuned lattice symmetry. To this end we perturb \( h_{\text{latt}} \) by adding to it

\[
\delta h_{\text{latt}}(k) = \epsilon \sigma^y s^x (1 - \cos k_y - \cos k_z), \tag{3.7}
\]

which respects \( \mathcal{T} \) but breaks the \( C_4 \) rotation symmetry around the \( y \) axis, thus allowing Weyl nodes to detach from \( k_y \). For \( \epsilon \neq 0 \) the arcs rotate and curve as illustrated in Fig. 3.2e. Majorana flat bands however remain robustly present in this low symmetry case (Fig. 3.2g). They now also appear for a junction parallel to the \( x \) crystal direction (Fig. 3.2h) because the Weyl points project onto distinct \( k_x \) momenta in the boundary BZ.

### 3.2.3 Stability Against Non-Magnetic Disorder

We now address the stability of Majorana bands against nonmagnetic disorder that will inevitably be present in any real material. Scaling arguments \([23, 101-103]\) and numerical simulations \([104-106]\) show that disorder is a strongly irrelevant perturbation in a Weyl semimetal: electron density of states (DOS) at low energies \( D(\omega) \sim \omega^2 \) remains unchanged up to a critical disorder strength \( U_c \) at which point a transition occurs into a diffusive regime with finite DOS at \( \omega = 0 \). The Fermi arcs likewise remain stable in the weak disorder regime. These theoretical results are confirmed by experimental studies which show clear evidence for Weyl points and Fermi arcs in real materials using momentum resolved probes such as ARPES \([15, 16, 92-95]\) and FT-STS \([107, 108]\), in agreement with the predictions of momentum space band theory. In \( \mathcal{T} \)-preserving Weyl semimetals one furthermore expects the surface superconducting order to be stable against non-magnetic impurities due to Anderson’s theorem \([109]\). Finally, stability of Majorana flat bands can be argued as follows. In the clean system our calculations show MZMs localized in the vicinity of the junction at each momentum \( k \) between the projected Weyl points. Except in the vicinity of
the latter, these are separated by a gap $\Delta$ from the excited states in the system. Turning on weak random potential of strength $U$ will cause mixing between MZMs at different momenta $k$ as well as with the bulk modes. We expect the former to be a more important effect (except for MZMs in the close vicinity of the Weyl points) because of the low bulk DOS and the assumption of predominantly small momentum scattering. Importantly, the disorder averaged spectral function of the system must evolve continuously with the disorder strength $U$. For weak disorder, therefore, the Majorana band cannot abruptly disappear; instead disorder will produce a lifetime broadening, giving the $\delta$-function peak present in the clean spectral function a finite width proportional to $U$. This behavior is indeed observed in numerical simulations of disorder in Majorana flat bands at the edges of 2D topological superconductors [110]. We expect the broadened Majorana band to remain observable until its width becomes comparable to the gap $\Delta$ implying a significant range of stability in a weakly disordered sample.

### 3.2.4 Summary and Outlook

SC proximity effect in the surface of a Weyl semimetal produces an interesting interfacial superconductor with unique properties. In $\mathcal{T}$-preserving models (relevant to the recently identified Weyl semimetals [15, 16, 92–95]) the observable signature consists of protected flat Majorana bands that occur at zero energy in a linear Josephson $\pi$-junction defined on the surface. The flat bands span projections of the bulk Weyl points with opposite chirality onto the one dimensional edge BZ along the junction. We have demonstrated the existence and robustness of such flat bands in minimal models with $n = 4$ Weyl nodes but expect the signature to survive in materials with larger $n$. In this case each Kramers pair of Fermi arcs will produce a Majorana band at zero energy. In the presence of weak disorder there may be some additional broadening due to interband scattering, but given that Fermi arcs
are quite easy to resolve experimentally \[15, 16, 92-95\] even when \( n = 24 \), we expect no significant difficulties to arise when \( n > 4 \).

Majorana flat bands provide a unique opportunity to study interaction effects in Majorana systems because even weak interactions can have profound consequences in a flat-band setting \[111, 112\]. They have been theoretically predicted to occur in various 2D and 3D topological and nodal superconductors \[55-59, 61, 66\]. With the exception of high-\( T_c \) cuprates, where dispersionless edge states are known to exist (and were reinterpreted recently as Majorana flat bands \[111\]), these have not been observed experimentally because of the general paucity of topological superconductors. Majorana flat bands discussed here only require ingredients that are currently known to exist – a \( \mathcal{T} \)-preserving Weyl semimetal interfaced with an ordinary superconductor – and are thus promising candidates for experimental detection.

### 3.3 Experimental Detection by Josephson Current Measurement

The Majorana flat bands discussed above should be directly observable in tunneling spectroscopy of the junction region by a technique developed in the context of other materials \[113, 114\]. At phase difference \( \varphi = \pi \), flat bands will manifest through a distinctive zero-bias peak in the tunneling conductance (with the spectral weight proportional to the length of the flat portion of the band). The peak will split symmetrically about zero energy as \( \varphi \) is tuned away from \( \pi \) and will merge into the continuum when \( \varphi \) approaches zero. We note that bulk DOS in a Weyl semimetal \( D(\omega) \) is vanishingly small near the \( \omega = 0 \) neutrality point, even in the presence of weak disorder. The prominent zero-bias peak should thus be well visible in a tunneling experiment.
In this section, we consider a more conventional probe for the detection of the Majorana flat bands. We predict that, as the local parity of the junction switches due to the zero-mode degeneracy, the current-phase relation $I(\varphi)$ of the junction would exhibit a step discontinuity at $\varphi = \pi$. We substantiate this prediction by numerically simulating a Josephson junction mediated by the surface states of Cd$_3$As$_2$, a $T$-invariant Dirac semimetal. Our analysis further shows that the distinctive current jump can withstand the effect of finite temperatures and non-magnetic disorder.

### 3.3.1 Heuristic Argument

A well-known set-up for hosting MZMs is a Josephson junction mediated by the edge of a 2D topological insulator (TI) [49, 50]. When the phase difference across the junction is $\pi$, the lowest Andreev bound state (ABS) at the junction crosses zero energy and can be represented by two unpaired MZMs. For arbitrary phase difference the energy of the ABS is proportional to $\cos(\varphi/2)$. Since the ABS can have positive or negative energy depending on $\varphi$, naively thinking, the ABS would alternate between being occupied and unoccupied as $\varphi$ varies in order to minimize the energy. Nonetheless, because the ABS is nondegenerate, whether it is occupied or not is determined by the fermion number parity (an odd parity means it is occupied and vice versa), as long as $k_B T \ll \Delta_0$ where $\Delta_0$ is the superconducting gap. The many-body ground state of the system is thus $4\pi$-periodic because it has contributions from this special $4\pi$-periodic ABS, on top of the usual $2\pi$-periodic ABSs at higher energies and $\varphi$-independent quasiparticles above the gap $\Delta_0$. The $4\pi$-periodicity is even more obvious in the limit of short junction, where only the lowest ABS remains with energy given by

$$E(\varphi) = \Delta_0 \cos(\varphi/2).$$ (3.8)
where $\Delta_0$ is the superconducting gap. As a consequence, the Josephson current exhibits a characteristic $4\pi$-periodicity, in contrast to the $2\pi$-periodicity for conventional Josephson junctions. This so-called fractional Josephson effect would be a substantial evidence for MZMs [50, 67–71]. So far the experiments have shown first signs of possible $4\pi$-periodicity [115–119].

In Sec. 3.2, we have demonstrated that a Josephson junction with $\pi$ phase difference on the surface of a $\mathcal{T}$-invariant Weyl semimetal would localize many pairs of decoupled MZMs labeled by different momenta $k$. In fact, each pair can be attributed to a 2D TI embedded in the Weyl semimetal band structure [10]. Naturally it seems that the fractional Josephson effect should also carry over, but in this case it does not violate the fermion parity to change the occupancies of ABSs. Indeed, a Cooper pair can split and occupy ABSs at different $k$’s. The $4\pi$-periodicity is no longer protected by parity; instead we have a distinct Josephson current jump at odd-$\pi$ phase differences where many ABSs simultaneously change their occupancies in order to minimize the ground state energy.

### 3.3.2 Experimental Set-Up with Dirac Semimetal Cd$_3$As$_2$

The above heuristic argument applies to any $\mathcal{T}$-invariant Weyl or Dirac semimetals. Among all candidates, the Dirac semimetal Cd$_3$As$_2$ has the advantage of being experimentally accessible and having the minimal number of four Weyl points [120, 121]. For concreteness and experimental implication, we focus our analysis on this material.

Consider a linear Josephson junction obtained by proximitizing the top surface of a Cd$_3$As$_2$ slab with an open superconducting ring, as shown in Fig. 3.3. To study the lowest ABS, we work in the short-junction limit where the junction length $d \ll \xi$, the proximity-induced coherence length. An applied flux $\Phi$ through the ring creates phase

\footnote{For our purposes, a Dirac semimetal can be considered as a special case of Weyl semimetal with coinciding Weyl nodes due to point group symmetry.}
difference $\varphi = 2\pi(\Phi/\Phi_0)$ across the junction, where $\Phi_0 = h/2e$ is the superconducting flux quantum. The phase difference drives a supercurrent around the ring, which can be detected by a SQUID sensor as in [122, 123], or by including the junction into a SQUID with a stronger junction in parallel and measuring the modulation of the total supercurrent, as done for example in [124].

Figure 3.3: By placing a superconducting ring with a narrow opening on top of a Cd$_3$As$_2$ slab, one obtains a Josephson junction mediated by the surface states of Cd$_3$As$_2$. We label the length of the junction by $d$ and the width by $W$. An external flux going through the ring generates a phase difference $\varphi = 2\pi(\Phi/\Phi_0)$ across the junction, which leads to supercurrent measurable by a capacitively-coupled SQUID sensor. To study the lowest ABS, we assume $d \ll \xi$, the proximity-induced coherence length.

We predict that the current-phase relation (CPR) exhibits a discontinuous current jump at $\varphi = \pi$. The jump size is proportional to the number of MZMs, which is maximized when the junction width is oriented along the direction in which the Dirac nodes are separated, and decreases as $\cos(\theta)$ when the junction is rotated by an angle $\theta$. Low non-zero temperature and weak non-magnetic disorder smoothen the discontinuity to some extent, but the CPR profile remains highly skewed and the jump size is unaffected. We estimate that a current jump on the order of $1\mu$A can be observed in typical experiments. Our prediction is substantiated by the following analysis.
### 3.3.3 Majorana Flat Bands at Cd₃As₂-SC Interface

The low-energy effective Hamiltonian of tetragonal Cd₃As₂ has been extracted from symmetry considerations [120, 121]. In the basis \{ |P_{J=\frac{3}{2}}, J_z = \frac{3}{2} \rangle, |S_{\frac{1}{2}}, \frac{1}{2} \rangle, |S_{\frac{1}{2}}, -\frac{1}{2} \rangle, |P_{\frac{3}{2}}, -\frac{3}{2} \rangle \}`, where \( S \) and \( P \) refer to spin-orbit-coupled Cd-5s and As-4p states, the \( 4 \times 4 \) Hamiltonian takes the form

\[
H_0(k) = \epsilon_0(k) + \begin{pmatrix}
M(k) & Ak_- & 0 & 0 \\
Ak_+ & -M(k) & 0 & 0 \\
0 & 0 & -M(k) & -Ak_- \\
0 & 0 & -Ak_+ & M(k)
\end{pmatrix}
\]  

(3.9)

up to second order in \( k \) near the \( \Gamma \) point. \( \epsilon_0(k) = C_0 + C_1 k_2^2 + C_2 (k_x^2 + k_y^2) \), \( M(k) = M_0 + M_1 k_z^2 + M_2 (k_x^2 + k_y^2) \), \( k_\pm = k_x \pm ik_y \). The parameters in the Hamiltonian are best-fit values to the ab initio calculation [121], and are listed in Table A.1 of Appendix A.

Each diagonal block describes a Weyl semimetal. Diagonalizing this Hamiltonian gives the energy spectrum

\[
E(k) = \epsilon_0(k) \pm \sqrt{M(k)^2 + A^2 (k_x^2 + k_y^2)}
\]

(3.10)

The Weyl nodes are located at \((0,0,\pm Q)\) where \( Q = \sqrt{-M_0/M_1} \) such that the square-root term vanishes. Each location has two Weyl nodes of opposite chiralities, one from each diagonal block in Eq. (3.9), and the point group symmetry prevents them from mixing. In our analysis, the chemical potential \( \mu \) is always set to the energy of Weyl nodes, \( E_0 = C_0 - C_1 M_0/M_1 \) to minimize the influence of bulk states.

To model a lattice, we turn \( H_0(k) \) into a tight-binding Hamiltonian \( H_{tb}(k) \) via the substitutions \( k_i^2 \sim \frac{2}{a^2} (1 - \cos(k_i a)) \) and \( k_i \sim \frac{1}{a} \sin(k_i a) \), where \( a \) is the lattice constant for the low-energy effective model. \( Q \) now satisfies \( \cos(aQ) = 1 + (a^2/2)(M_0/M_1) \).
Figure 3.4: Spectrum of the $L_x a \times L_y a \times L_z a = 40a \times 52a \times 60a$ ($a = 20\text{Å}$) slab of Cd$_3$As$_2$ with periodic boundary condition in the $y$- and $z$-directions. The material parameters are listed in Table A.1. Energy states are color-coded according to the expectation value of distance from the surface. Panel (a) shows the normal-state energy spectrum with the Dirac cones separated along $k_z$ and the surface Fermi arcs spanned between them. In Panel (b) a uniform pairing potential of magnitude $\Delta_0 = 10\text{meV}$ is introduced on the surfaces; the Fermi arcs are gapped out. In Panel (c), two junctions of $\pi$ phase difference are implemented on the top surface and the bottom surface remains uniformly superconducting; the resulting zero-energy surface flat bands linking the Dirac nodes are four-fold degenerate and decoupled. In Panel (d), the phase difference deviates from $\pi$ to $1.1\pi$; the flat bands get split from zero energy and obtain small dispersion, while the rest of the bands stay the same. In Panels (b)-(d), the small gaps near the Dirac nodes are due to the finite-size effect.

We let $a = 20\text{Å}$ even though $a_x = a_y = 3\text{Å}$ and $a_z = 5\text{Å}$ are the best-fit values to the ab initio calculation [120]. The larger lattice constants make the relevant low-energy band structure more numerical resolvable. This approach is not an issue because our numerics mainly serve illustrative purposes; later on the quantitative predictions will be made with realistic parameters. We consider a slab of Cd$_3$As$_2$ with $L_x \times L_y \times L_z$ sites. It is periodic in the $y$- and $z$-directions and finite in the $x$-direction. We Fourier transform $H_{tb}(k)$ along the $x$-direction in order to define individual layers. Then numerical diagonalization gives the normal-state energy spectrum, showing two Dirac cones in the bulk and two Fermi arcs on each surface. Fig. 3.4(a) projects the spectrum onto $k_z$-axis, so the Fermi arcs overlap and appear as curved bands (each corresponding to a different $k_y$) connecting the Dirac cones.

Next we verify that the surface states can be gapped out by $s$-wave pairing of time-reversed states. We introduce a pairing potential to both top and bottom surfaces.
The matrix of Bogoliubov-de Gennes (BdG) Hamiltonian reads

\[
H_{\text{BdG}} = \begin{pmatrix}
H_{tb} - \mu & \Delta \\
\Delta^\dagger & -TH_{tb}T^{-1} + \mu
\end{pmatrix}
\]  

(3.11)

where every entry is a \(4L_x \times 4L_x\) matrix whose basis are tensor products of spin-coupled orbitals and \(x\)-layers. \(H_{tb}\) is as discussed above, and its time-reversal conjugate \(TH_{tb}T^{-1} = H_{tb}\) because it is time-reversal invariant. The pairing potential \(\Delta\) is a diagonal matrix whose nonzero elements are \(\Delta_0\) when the \(x\)-layer index is 1 or \(L_x\). Diagonalizing \(H_{\text{BdG}}\) gives a particle-hole symmetric band structure with gapped-out surface Fermi arcs and intact bulk Dirac cones, as shown by Fig. 3.4(b).

Lastly we Fourier transform \(H_{tb}\) again along \(y\); now each block in \(H_{\text{BdG}}\) is a matrix whose basis are tensor products of spin-coupled orbitals, \(x\)-layers, and \(y\)-sites. Then we let the nontrivial diagonal elements of \(\Delta\) be \(\Delta_0\) when the \(x\)-layer index is \(L_x\) and

\[
\begin{cases}
\Delta_0 e^{i\varphi} & \text{for } L_y/4 < y/a \leq 3L_y/4 \\
\Delta_0 & \text{for } y/a \leq L_y/4 \text{ or } y/a > 3L_y/4
\end{cases}
\]  

(3.12)

when the \(x\)-layer index is 1. This defines two evenly spaced short junctions each of phase difference \(\varphi\) on the top surface. Two evenly spaced junctions are needed to satisfy the periodic boundary condition along \(y\). We keep the bottom surface uniformly superconducting so that the bottom surface states do not obstruct the zero-energy flat bands in the figures. Note that the junctions are periodic widthwise, but a realistic junction would have a finite width. In principle, sides of the sample in Fig. 3.3 could also carry supercurrent, which is not accounted for in our model. However, in the geometry where the junction width is much larger than the thickness of Cd₃As₂ slab, the contribution of the side modes is negligible. This is additionally substantiated by Ref. [123], a S/TI/S junction experiment set up like Fig. 3.3: the authors did not consider this geometric effect, yet found good agreement between their
Theoretical model and experimental data. The energy spectrum of $H_{\text{BDG}}(\varphi = \pi)$ in Fig. 3.4(c) clearly shows flat bands of zero-energy surface states, which are Majorana or self-conjugate for reasons explained in [46, 125]. There are in total four Majorana flat bands. Each junction locally hosts two such bands, which hybridize and gain a non-zero energy when $\varphi$ moves away from $\pi$, as shown by Fig. 3.4(d). As long as the time-reversal symmetry $\mathcal{T}$ is respected, $\varphi$ must be either 0 or $\pi$, so we can say that the Majorana flat bands are protected by $\mathcal{T}$. Moreover, translational symmetry prohibits MZMs in the same band from mixing because $k_z$ is a good quantum number. Lastly, the flat bands at different junctions are decoupled due to the artificial symmetry of evenly spaced junctions.

The number of MZM pairs, $N$, in each junction is determined by the junction width, $W = aL_z$, and the distance between the Dirac nodes, $2Q$:

$$N = \frac{WQ}{\pi}$$  \hspace{1cm} (3.13)

If the junction is rotated by an angle $\theta$,

$$N = \frac{WQ\cos(\theta)}{\pi}$$  \hspace{1cm} (3.14)

because the distance between the Dirac nodes, as projected onto the 1D Brillouin zone parallel to the junction width, becomes $2Q \cos \theta$. In our numerical model, $aQ \sim 0.21 \pi$ and $L_z = 60$, so $N \sim 13$ at each junction. In a realistic model of Cd$_3$As$_2$, $Q = \sqrt{-M_0/M_1} = 0.033\text{Å}^{-1}$. Assuming a typical junction width of $W = 1\mu\text{m}$, one finds that $N \sim 100$. Hence a significant MZM population can be easily obtained in experiments, especially with a wide junction.
3.3.4 Current-Phase Relation at Finite Temperatures

One can detect the Majorana flat bands by measuring the current-phase relation (CPR). The MZMs cause a current jump at $\pi$ phase difference. The jump size is proportional to $N$. To demonstrate this in Cd$_3$As$_2$, we compute the Josephson current given by

$$ J(\varphi) = \frac{2e}{\hbar} \frac{d\varepsilon(\varphi)}{d\varphi} $$

(3.15)

where $\varepsilon(\varphi)$ is the many-body ground state energy of the system [126]. We diagonalize $H_{BdG}(\varphi)$ for every fixed $\varphi$. The positive energies, labelled by $E_i(\varphi)$, are the excitation energies of Bogoliubov quasiparticles. Due to particle-hole symmetry, the negative energies of the filled states below $E_F = 0$ are simply $-E_i(\varphi)$, so

$$ \varepsilon(\varphi) = -\frac{1}{2} \sum_i E_i(\varphi) $$

(3.16)

where the factor of $\frac{1}{2}$ compensates the particle-hole doubling in the BdG construction. As shown in Fig. 3.5, the resulting $J(\varphi)$ displays a $2\pi$-periodic CPR with a current jump of magnitude $\sim 60\text{meV}e/\hbar$ or $15\mu\text{A}$.

At non-zero temperature, quasiparticles below $E_F = 0$ can be thermally excited to occupy positive energy states. With the free energy $F(\varphi)$ in place of $\varepsilon(\varphi)$, the Josephson current becomes [127, 128]

$$ F = -k_B T \ln Z = -k_B T \sum_i \ln \left( 2 \cosh \left( \frac{E_i(\varphi)}{2k_B T} \right) \right) $$

(3.17)

$$ J(\varphi) = \frac{2e}{\hbar} \frac{dF(\varphi)}{d\varphi} = -\frac{e}{\hbar} \sum_i \frac{dE_i(\varphi)}{d\varphi} \tanh \left( \frac{E_i(\varphi)}{2k_B T} \right) $$

(3.18)
Figure 3.5: The Josephson current (per junction) computed from Eq. (3.18) is shown at various $\varphi$. At $k_BT = 0$ and 0.1meV, there is a clear current jump of magnitude 60meV$e/h$ or 15$\mu$A. At $k_BT = 1$meV, the CPR retains little skewness. For comparison, the solid blue curve shows the sinusoidal CPR of a conventional junction with one ABS. The magnitude is small (bounded by $\Delta_0^2e/\hbar$) compared to our numerical results, mainly because our junction hosts a large number of ABSs. Note that $\Delta_0 = 10$meV in the numerics, which is unrealistically large so that the superconducting gap is larger than the finite-size gaps that are always present in finite systems. While $k_BT = 0.1$ and 1meV appear to be fairly high temperatures, it is the ratio $k_BT/\Delta_0$ (1% and 10% respectively) that dictates the thermalization of quasiparticles. In a more realistic system with $\Delta_0 = 0.1$meV, the predicted blue and green curves correspond to $T \sim 10$mK and 100mK respectively, which agrees with Eq. (3.15) when $T \to 0$.

Fig. 3.5 shows that the current jump is slightly rounded at $k_BT/\Delta_0 = 0.01$ but significantly loses skewness when $k_BT/\Delta_0 = 0.1$. In experiments, one should aim at lowering $k_BT/\Delta_0$ to 1%. For a typical induced gap of 0.1meV, ideally the temperature should be 10mK or lower. Note that electron-phonon interaction would also knock the system off the ground state, which further smoothens the current discontinuity. The extent of this effect is left for future studies.

How do we understand this current jump? A slice of spin-up sector of the Cd$_3$As$_2$ Hamiltonian at any fixed $k_z \in (-Q, Q)$ combined with a slice of spin-down sector at $-k_z$ can be regarded as a 2D TI. Although a short junction on the edge of 2D TI hosts

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Footnote: Our expression for Josephson current differs from that in Refs. 127, 128 by a factor of 2, which was there to account for spin degeneracy. In our case, any degeneracy is already included in the summation because we assign a unique label to every eigenstate.
a $4\pi$-periodic ABS with energy given by Eq. (3.8), the 2D TI embedded in Cd$_3$As$_2$ does not locally conserve the fermion parity, so its ground state energy evolves as

$$\tilde{\varepsilon}(\varphi) = -\frac{1}{2}\Delta_0 \left| \cos\left(\frac{\varphi}{2}\right) \right|$$

(3.19)

following the lower of the two particle-hole-conjugate ABS bands in order to minimize the energy. Above-gap states are not included in $\tilde{\varepsilon}(\varphi)$ because they are phase-independent and do not contribute to the supercurrent. $\tilde{\varepsilon}(\varphi)$ has a kink at $\pi$, resulting in a characteristic supercurrent jump of magnitude

$$\delta J = \frac{e}{\hbar}\Delta_0$$

(3.20)

The current jump in Fig. 3.5 can thus be understood as the cumulative effect of many Josephson junctions mediated by the edge states of 2D TI. In Fig. 3.6, we confirm that a slice of our model at $k_z = 0$ indeed shows the same phase dependence as described.

Figure 3.6: The first few quasiparticle excitation energies $E_i$ at $k_z = 0$ are shown at various $\varphi$. The subgap ABS (in red) has a kink at $\pi$ while the other states show little phase dependence. Same behaviour holds for any $k_z \in (-Q, Q)$; outside of this range there is no ABS.
With $N$ ABSs, the total current jump is

\[ \delta J_{\text{total}} = N \delta J \]  

(3.21)

For our numerics, $N \sim 13$ and $\Delta_0 = 10 \text{meV}$, so $\delta J_{\text{total}} \sim 31 \mu\text{A}$, on the same order of magnitude as what Fig. 3.5 suggests. It is an overestimation because the actual superconducting gap as seen in Fig. 3.4(b) is smaller than $10 \text{meV}$ and shrinking as $k_z$ approaches the nodes. This happens because the opening of surface superconducting gap is interfered by the Dirac cones in the bulk, which is inevitable when we use an unrealistically large $\Delta_0$ (in order to overcome the finite-size effect). In a realistic setting where $\Delta_0$ is much smaller than the normal-state bulk gap, we expect the superconducting gap to be uniform except in the vicinity of the nodes. In that case Eq. (3.21) provides an accurate estimate of the jump size. With realistic parameters, $N \sim 100$, so $\delta J_{\text{total}} \sim 2.4 \mu\text{A}$ assuming a typical induced gap of $\Delta_0 = 0.1 \text{meV}$. Current jump of this magnitude is readily observable in experiments.

Due to Eq. (3.14), the jump size varies as $\cos(\theta)$ where $\theta$ is the angle between the junction width and the direction in which the Dirac nodes are separated ($z$-axis). To observe this, one can prepare multiple samples with different junction orientations and compare their CPR measurements.

### 3.3.5 Random Matrix Theory Analysis of Non-Magnetic Disorder

Disorder breaks the translational symmetry, allowing MZMs at different $k_z$’s to couple. We show that non-magnetic disorder broadens the Majorana flat bands to cover a range of near-zero energies. Consequently the discontinuity in Josephson current is rounded off but remains a robust signature for weak disorder.
Having in mind an experiment in which the chemical potential is tuned to the neutrality point, we consider only the subgap states, consisting of ABSs at the junction and bulk states on the Dirac cones. There are many ABSs; whereas the bulk density of states is vanishingly small at low energy: \( D(\omega) \sim \omega^2 \). Moreover, the bulk wavefunctions spread out over the entire bulk, so their probability densities near the surface are small. Based on the above arguments, we assume that any scattering between ABSs and bulk states is negligible in comparison to scattering among the ABSs, and neglect the presence of bulk states altogether.

At each \( k_z \in (-Q, Q) \), there is a pair of particle-hole symmetric Andreev levels dispersing according to \( E = \pm \Delta_0 \cos \varphi / 2 \). Non-magnetic disorder randomly couples \( N \) pairs of ABSs such that the effective Hamiltonian reads:

\[
H = \begin{pmatrix}
I \Delta_0 \cos \varphi / 2 & M \\
M^\dagger & -I \Delta_0 \cos \varphi / 2
\end{pmatrix}.
\]  

(3.22)

Here \( M \) is a \( N \times N \) random phase-independent matrix coupling the ABSs at different \( k_z \)'s, and \( I \) is the \( N \times N \) identity matrix. Note that we have chosen the shape of \( H \) such that the disorder matrix resides in the off-diagonal blocks. One can show that the Hamiltonian matrix can be chosen this way by diagonalizing whichever part of the disorder matrix is in the diagonal part of the Hamiltonian and noting that the disorder potential is time-reversal invariant and cannot shift the phase difference at which the states cross zero from \( \pi \). Let us also use time-reversal and particle-hole symmetries to further constrain \( M \), one of the choices being \( M = M^* \) and \( M = -M^T \). Thus \( M \) is real antisymmetric matrix and its eigenvalues are purely imaginary. The statistics of energy levels of such a random matrix are well known [129],

\[
P(m_i) = \frac{1}{\pi c^2} \text{Re} \sqrt{c^2 - m_i^2},
\]

(3.23)
where $c$ is a constant describing the disorder strength. We can now diagonalize $M$ and $M^\dagger$ to obtain the eigenvalues of the whole matrix $H$

$$e_i = \sqrt{\Delta_0^2 \cos^2(\varphi/2) + m_i^2}. \quad (3.24)$$

By change of variables in the probability distribution (3.23), we obtain disorder-averaged density of states as a function of phase difference and disorder strength:

$$\nu(E) \propto |E| \Re \sqrt{\frac{c^2}{E^2 - \Delta_0^2 \cos^2(\varphi/2)} - 1}, \quad (3.25)$$

Figure 3.7: Panel (a) plots $\nu(E)$ at $\pi$ and shows that non-magnetic disorder broadens the Majorana flat bands by $c/\Delta_0$. Panel (b) shows $\nu(E)$ at $c/\Delta_0 = 0.1$ and different $\varphi$'s. The broadening diminishes as $\varphi$ deviates from $\pi$.

As shown by Fig. 3.7, $\nu(E)$ describes the broadening of Andreev levels, which diminishes as $\varphi$ deviates from $\pi$, so we expect the CPR to be perturbed mostly near the jump. To confirm this, we compute the Josephson current using the free energy integrated with $\nu(E)$.

Fig. 3.8(a) shows that the discontinuity is rounded off but the overall jump size is not affected. The rounding is akin to that due to nonzero temperature as seen in Fig. 3.5, but to a lesser degree because $c/\Delta_0 = 0.1$ smoothen the discontinuity while $k_B T/\Delta_0 = 0.1$ almost completely destroys the skewness. We also measure the degree of rounding by the slope of CPR at $\pi$. It decreases with increasing $c/\Delta_0$, as shown in
Figure 3.8: Panel (a) shows the disordered CPR near \( \pi \) at \( k_B T/\Delta_0 = 0.01 \) and various \( c/\Delta_0 \). The discontinuity is rounded off but the jump size is unaffected. Panel (b) shows the CPR slope at \( \pi \) as \( c/\Delta_0 \) increases (again at \( k_B T/\Delta_0 = 0.01 \)). For comparison, the slope at \( \pi \) of a conventional sinusoidal CPR is 1.

Fig. 3.8(b), but remains significantly larger than 1, the slope at \( \pi \) of a conventional sinusoidal CPR.

The above RMT analysis does not consider non-averaged fluctuations, which can in principle obscure the jump. This is however not the case here. Supercurrent in the form of Eq. (3.18) is essentially a sum of \( N \) random variables when the energy eigenvalues are randomized as in Eq. (3.24). By the Central Limit Theorem, the sum grows as \( N \), but the standard deviation of the sum grows as \( \sqrt{N} \).

### 3.3.6 Application to Other \( T \)-Invariant Weyl Semimetals

We have shown that flat bands of MZMs occur in a linear Josephson junction mediated by the surface states of \( \text{Cd}_3\text{As}_2 \). Numerical simulation of CPR shows a significant supercurrent jump at \( \pi \), the size of which is determined by Eqs. (3.14), (3.20) and (3.21). Low temperature (\( \lesssim 10\text{mK} \)) and weak non-magnetic disorder mildly round off the jump.

This result applies to every \( T \)-invariant Weyl/Dirac semimetal, which in general has an even number of Weyl nodes. With a junction of \( \pi \) phase difference on the surface, each pair of Weyl nodes that used to terminate the same Fermi arc become the end points of a Majorana flat band, living in the 1D Brillouin zone parallel to the junction width. As before the CPR would have a jump, but the dependence of
jump size on junction orientation varies. For example in TaAs, a $\mathcal{T}$-invariant Weyl semimetal with 24 nodes [15, 16], the jump size is the largest when the junction width is parallel to the [110]-direction onto which every pair of Weyl nodes projects to two distinct points.
Chapter 4

Chiral Anomalies in Strained Weyl Semimetals

4.1 Overview

The Dirac equation for spin-$\frac{1}{2}$ fermions can be decoupled into left- and right-handed Weyl equations if the fermions are massless. In this case the current densities of left- and right-handed fermions are separately conserved: $\partial_t \rho_{L/R} + \nabla \cdot j_{L/R} = 0$ or equivalently $\partial_t \rho_5 + \nabla \cdot j_5 = 0$ where $(\rho_5, j_5) = (\rho_R, j_R) - (\rho_L, j_L)$ is the chiral current density. Further incorporating electric and magnetic fields by coupling the equations to a gauge potential does not affect the chiral symmetry. However, once the theory is quantized as in massless quantum electrodynamics (QED), the conservation of chiral current is violated in order for the theory to remain gauge invariant. There is an extra anomalous contribution that is nonzero in the presence of parallel electric and magnetic fields:

$$\partial_t \rho_5 + \nabla \cdot j_5 = \frac{e^2}{2\pi^2 \hbar^2 c} E \cdot B$$

This quantum phenomenon is called the Adler-Bell-Jackiw anomaly or the chiral anomaly [130, 131]. It can be applied to any gauge theories with massless fermions.
and has significant implications in the Standard Model where certain light fermions can be treated as massless. Remarkably, it explains the rapid neutral pion decay $\pi^0 \rightarrow 2\gamma$ and predicts successive generations of quarks and leptons by enforcing anomaly cancellation in the electroweak theory [132].

In their seminal paper, Nielsen and Ninomiya demonstrated that the chiral anomaly occurs in semiconductors where the conduction and valence bands meet at point-like degeneracies [29]. Today we call these materials Weyl semimetals because the low-energy excitations near the gapless nodes are described by the Weyl equations. They showed that in the presence of parallel electric and magnetic fields, electrons semiclassically evolve along the lowest Landau levels from one Weyl node to another with opposite chirality, resulting in excess of chiral charge. They also argued that the magneto-conduction is strong because any backscattering has to be internodal. This has been observed in experiments as a negative magnetoresistance [12–14].

Recent theoretical work shows that small mechanical strain acts like pseudo-electromagnetic fields $e, b$ in Weyl semimetals [133]. They differ from the ordinary fields $E, B$ by coupling with an extra minus sign to Weyl cones of different chirality. It is straightforward to derive the chiral anomaly equation with consideration of the chiral gauge fields:

$$\partial_t \rho_5 + \nabla \cdot \mathbf{j}_5 = \frac{e^2}{2\pi^2 \hbar^2 c}(E \cdot B + e \cdot b)$$ (4.1)

$$\partial_t \rho + \nabla \cdot \mathbf{j} = \frac{e^2}{2\pi^2 \hbar^2 c}(E \cdot b + e \cdot B)$$ (4.2)

where $(\rho, \mathbf{j}) = (\rho_R, \mathbf{j}_R) + (\rho_L, \mathbf{j}_L)$ is the electric current density [134]. It seems odd that the conservation of electric charge can be violated by parallel ordinary and chiral gauge fields. Indeed this is unphysical in the context of high-energy physics, but in a solid state system the nonconservation only describes what happens inside the bulk near the energy level of Weyl nodes – the electric charge of the entire system, including both surface and bulk, is always conserved.
In this work, we investigate the strain-induced chiral anomaly in Weyl semimetals, especially scenarios where the second chiral anomaly equation is nontrivial. By closely examining the band structure of a two-node tight-binding model in finite geometries, we shed new light to the anomalous nonconservation of electric charge: there is a dynamical charge re-distribution between bulk and surface that produces the chiral anomaly effect near the Weyl nodes while conserving the total charge of the system. First, we show that twisting a nanowire of Weyl semimetal induces a pseudo-magnetic field $b$, whose chiral nature is seen in the parallel zeroth Landau levels of the bulk, as shown in Fig. 4.1b (normally the zeroth Landau levels at opposite Weyl nodes are counter-propagating, as in Fig. 4.1a). Numerical simulation of the band structure unveils surface modes travelling in the opposite direction. Hence when a parallel $E$ is turned on, electrons evolve to occupy more bulk modes, in agreement with the second chiral anomaly equation, and less surface modes. Next, we demonstrate that the repetitive motion of stretching and compressing a film of Weyl semimetal induces an oscillating pseudo-electric field $e$. In the presence of a parallel magnetic field $B$, the oscillating $e$ moves the chemical potential up and down in the zeroth Landau levels according to the second chiral anomaly equation. Once again the band structure reveals the presence of surface states, which would equilibrate any change in the bulk chemical potential. Therefore the oscillating bulk chemical potential leads to periodic charge transfers between bulk and surface.
Figure 4.1: With external magnetic field threaded through the Weyl semimetal nanowire, counter-propagating zeroth Landau levels are formed at the opposite Weyl nodes, shown in Panel a. Twisting the nanowire induces a pseudo-magnetic field $\mathbf{b}$, whose chiral nature is seen in the parallel zeroth Landau levels of the bulk, as illustrated in Panel b. When a parallel electric field is applied, the electron states begin to evolve semiclassically towards higher momenta.

The interesting interplay between surface and bulk charges has several experimentally observable consequences. Since the surface and bulk modes in a twisted Weyl semimetal travel in the opposite directions, any backscattering in such a “topological coaxial cable” necessarily involves transverse hydrodynamic charge flow. Thus the current driven by longitudinal $\mathbf{E}$ has a large relaxation time and gives a conductivity proportional to the square of the torsion strength, which can be measured in transport experiments. The pseudo-electric field $\mathbf{e}$ in practice can be generated by sound waves. In the presence of nontrivial $\mathbf{e} \cdot \mathbf{B}$, the periodic bulk-surface charge transfers would lead to anomalous sound attenuation and electromagnetic field emission, both detectable by conventional probes. Lastly, we demonstrate that our results are applicable to realistic Weyl semimetals by conducting additional numerical analysis with ab-initio tight-binding models for Cd$_3$As$_2$ and Na$_3$Bi.

4.2 Gauge-Fields Induced by Mechanical Strains

Mechanical strain that varies smoothly on the interatomic scale is known to affect the low-energy Dirac fermions in graphene in a way that is similar to the externally
applied magnetic field. More precisely, strain acts in graphene as a “chiral” vector potential that couples to Dirac fermions oppositely in the two valleys $K$ and $K'$ [135]. The pseudomagnetic field that arises from this effect in a curved graphene sheet can be larger than 300T, and has been observed through the spectroscopic measurement of the Landau levels in the seminal experiment on graphene nanobubbles [136]. In terms of their low-energy physics Weyl and Dirac semimetals [23, 137, 138] can be thought of as three dimensional generalization of graphene. The question thus immediately arises whether strain in these materials gives rise to similar effects. Recent theoretical work [133] showed that this is indeed the case at least in a simple toy model of a Weyl semimetal with broken time reversal symmetry $T$. The authors predicted that the electron-phonon coupling in such a system will lead to non-zero phonon Hall viscosity, an interesting but notoriously difficult quantity to measure. We consider here the effect of strain in more realistic models relevant to Dirac semimetals Cd$_3$As$_2$ [32–37] and Na$_3$Bi [139–141] and the related Weyl semimetals [30, 142–145].

One reason why strain can generate pseudomagnetic fields as large as 300T in graphene [136] lies in its mechanical flexibility: substantial curvature can be achieved without breaking the graphene sheet. This suggests that to probe strain-induced effects in Dirac and Weyl semimetals one should focus on films or wires as these will be much more flexible than bulk crystals. In this work we thus concentrate on these geometries and show that strain leads to phenomena that are both striking and experimentally measurable. We note that high-quality nanowires of Dirac semimetal Cd$_3$As$_2$ have been grown and shown to exhibit giant negative magnetoresistance due to the chiral anomaly [39] as well as Aharonov-Bohm oscillations indicative of the protected surface states [40]. These wires bend easily and show mechanical flexibility that is required to study strain related phenomena. We also discuss consequences of lattice distortions caused by sound waves (phonons). These can be used to study the
above phenomena in crystalline flakes and films which are readily available for nearly all known Dirac and Weyl materials.

### 4.3 Intuitive Picture of Chiral Anomalies in Weyl Semimetals

Our results can be most easily understood by thinking about the simplest Weyl semimetal with a single pair of Weyl points [146] although many aspects translate to more complicated Weyl and Dirac semimetals. The low-energy effective theory is then defined by the Hamiltonian $H = \int d^3r \Psi^\dagger h(r) \Psi$ where $\Psi^\dagger = (\psi^\dagger_{r,R}, \psi^\dagger_{r,L})$ and

$$h = v \chi^z \sigma \cdot (p - eA - \chi^z e a) - \mu.$$  \hfill (4.3)

Here $\psi^\dagger_{r,R/L}$ represent two-component right and left-handed Weyl fermion creation operators, $\chi^z = \pm 1$ labels the chirality of the two Weyl nodes, $\sigma$ is a vector of Pauli matrices in the pseudospin space and $p = -i\hbar \nabla$. $A$ and $a$ denote gauge potentials of the ordinary EM and the chiral field, respectively.

To develop some intuition for the chiral anomaly let us consider the Hamiltonian (4.3) in the presence of a static uniform (pseudo)magnetic field. We begin with the ordinary magnetic field $B = B \hat{z}$. The solution of the corresponding Schrödinger equation $h \Phi = e \Phi$ is well known and consists of the set of Dirac Landau levels with energies \[29\]

$$\epsilon_n(k) = \pm \hbar v \sqrt{k^2 + 2ne\frac{|B|}{\hbar c}}, \quad n = 1, 2, \ldots, \quad (4.4)$$

for each Weyl fermion. There is also one chiral $n = 0$ level per valley with $\epsilon_0(k) = \chi^z \text{sgn}(B) \hbar v k$. If a parallel electric field $E = E \hat{z}$ is now applied to the system then the electron momenta begin to evolve according to the semiclassical equation of motion $k(t) = k(0) - eEt/\hbar$. Because of the existence of the two chiral branches in the
spectrum this leads to charge pumping between the two Weyl points, as illustrated in Fig. 4.1a, at a rate consistent with the chiral anomaly equation (4.1). The key point here is that in a real solid where the Hamiltonian is defined on the lattice the two chiral branches are connected away from the Weyl points and the chiral anomaly equation simply describes the semiclassical evolution of the electron states through the Brillouin zone [29]. In the presence of relaxation processes a steady state non-equilibrium distribution of electrons with nonzero chiral density $\rho_5$ is obtained which is responsible for the anomalous $\sim B^2$ contribution to the magnetoresistance.

Now consider the effect of the chiral magnetic field $b = b\hat{z}$. The solution consists of the same Dirac Landau levels Eq. (4.4) but the $n = 0$ levels now disperse in the same direction for the two Weyl points, $\epsilon_0(k) = \text{sgn}(b)\hbar v k$, as illustrated in Fig. 4.1b. Now if a parallel electric field $E = E\hat{z}$ is applied to the system we see that the charge density seemingly begins to change. Since the total charge is conserved this extra charge density must come from somewhere. We will demonstrate below that it comes from the edge of the system. Indeed this is plausible if we note that the energy spectrum sketched in Fig. 4.1b does not represent a legitimate dispersion of a lattice system which, due to the periodicity of the energy bands in the momentum space, must exhibit the same number of left and right moving modes. Since the Landau levels are the correct eigenstates in the bulk we conclude that the missing left moving modes must exist at the boundary. Our numerical simulations of a lattice model below indeed confirm this conclusion. Thus, in the presence of $b$ and $E$ the chiral anomaly can be understood as pumping of charge between the bulk and the edge of the system. The effects of nonzero $e \cdot B$ and $e \cdot b$ terms are more subtle, as they involve relaxational dynamics, but can be understood from similar arguments. Indeed, the difference between the effects lies in the directions of magnetic and electric fields as applied to the two Weyl cones. These effects and their experimental consequences constitute the main result of this work.
Several interesting observations follow from the above discussion. First, we conclude that electric transport in a twisted Weyl semimetal wire will be highly unusual because the right-moving modes occur in the bulk whereas the left-moving modes are localized near the boundary. (More precisely we may say that there is a net imbalance between the number of left and right moving modes in the bulk and at the boundary.) Since the left and right moving modes are spatially segregated one expects backscattering to be suppressed in such wires giving rise to anomalously long mean free paths. In addition, transport will sensitively depend on the applied torsion, giving rise to the new chiral torsional effect (CTE) that we describe in detail below. Second, we will see that charge transfer between the bulk and the boundary leads to interesting effects when time-dependent e field is generated e.g. by driving a longitudinal sound wave through the crystal when B field is also present. Such a sound wave will experience an anomalous attenuation that can be attributed to the chiral anomaly. It will also produce charge density oscillations in the crystal that can be observed through electric field measurement outside the sample. Third, the chiral anomaly can be observed even in the complete absence of real EM fields when the crystal is put simultaneously under torsion and time-periodic uniaxial strain. Then nonzero e \cdot b term is generated and according to Eq. (4.1) the chiral charge fails to be conserved. We argue that this has observable consequences for sound attenuation in the crystal.

4.4 Analytical Calculations with $\frac{1}{2}$-Cd$_3$As$_2$ Model

We now proceed to justify the above claims by detailed model calculations. For simplicity and concreteness, we once again adopt the low-energy effective model of Dirac semimetal Cd$_3$As$_2$ introduced in Chapter 2. The model captures the band inversion of the atomic Cd-5s and As-4p levels near the Γ point. In the basis of the
relevant spin-orbit coupled states \(|P_{\frac{3}{2}}\), \(|S_{\frac{1}{2}}\rangle\), \(|S_{\frac{1}{2}}, -\frac{1}{2}\rangle\) and \(|P_{\frac{3}{2}}\) it is defined by a 4 × 4 matrix Hamiltonian [32]

\[
H(k) = \epsilon_0(k) + \begin{pmatrix}
M(k) & Ak_- & 0 & 0 \\
Ak_+ & -M(k) & 0 & 0 \\
0 & 0 & -M(k) & -Ak_- \\
0 & 0 & -Ak_+ & M(k)
\end{pmatrix}.
\] (4.5)

Here \(\epsilon_0(k) = C_0 + C_1 k_z^2 + C_2 (k_x^2 + k_y^2), k_\pm = k_x \pm i k_y\), and \(M(k) = M_0 + M_1 k_z^2 + M_2 (k_x^2 + k_y^2)\). Parameters \(C_j, A\) and \(M_j\) follow from the \(k \cdot p\) expansion of the first principles calculation [32] and are summarized in Appendix A. We note that \(H(k)\) (with different parameters) also describes Dirac semimetal Na\(\text{\textsubscript{3}}\)Bi [139].

The low-energy spectrum of the model (4.5) consists of a pair of Dirac points located at

\[
\mathbf{K}_\eta = (0, 0, \eta Q), \quad Q = \sqrt{-M_0/M_1},
\] (4.6)

where \(\eta = \pm\) is the valley index. The model respects time reversal symmetry \(\mathcal{T} = i\sigma^y \tau^z K\), where \(K\) denotes complex conjugation and \(\sigma, \tau\) are Pauli matrices in spin and orbital space, respectively. \(\mathcal{T}\) maps the upper diagonal (spin up) block \(h(k)\) of \(H(k)\) onto the lower diagonal (spin down) block \(-h(k)\) and vice versa.

Since spin up and spin down blocks are effectively decoupled in the model Hamiltonian (4.5), we can analyze them separately. It is easy to see that each diagonal block taken in isolation can be regarded as describing a minimal \(\mathcal{T}\)-breaking Weyl semimetal with one pair of Weyl nodes located at \(\mathbf{K}_\pm\). In the following we will often focus our discussion on the spin up block of Hamiltonian (4.5) and refer to it as "\(\frac{1}{2}\)-Cd\(\text{\textsubscript{3}}\)As\(\text{\textsubscript{2}}\)" model. Once we have understood the physics of this \(\frac{1}{2}\)-Cd\(\text{\textsubscript{3}}\)As\(\text{\textsubscript{2}}\) model, it will be straightforward to deduce the behavior of the actual Cd\(\text{\textsubscript{3}}\)As\(\text{\textsubscript{2}}\) by simply adding a time-reversal conjugate set of states to the results obtained for \(\frac{1}{2}\)-Cd\(\text{\textsubscript{3}}\)As\(\text{\textsubscript{2}}\).
We emphasize that although $\frac{1}{2}$-Cd$_3$As$_2$ model taken on its own does not describe any specific real material, the results we report for this model are relevant to a broad class of Weyl semimetals with broken $T$ such as the Burkov-Balents layered heterostructure [146] and more recently proposed magnetic Weyl materials [147, 148]. We will explain in detail how these results apply to $T$-preserving Weyl and Dirac semimetals.

For many considerations and for numerical calculations, it will be useful to regularize the model defined by Eq. (4.5) on a lattice. Although real Cd$_3$As$_2$ crystal has a complex structure with 40 atoms per unit cell, Ref. [32] showed that its low-energy physics can be well described by an effective tight binding model with $s$ and $p$ orbitals on vertices of the tetragonal lattice and lattice constants $a_x, a_y = 3.0\text{Å}$ and $a_z = 5.0\text{Å}$. Here we simplify the model one step further and assume a simple cubic lattice with a lattice constant $a$. We checked that this leads to only minor deviations from the tetragonal model of Ref. [32]. We construct the tight-binding model for Cd$_3$As$_2$, as further explained in Appendix A, such that in the vicinity of the $\Gamma$ point it matches the $k \cdot p$ Hamiltonian (4.5) to the leading order in the expansion in small $ak$. For quantitative estimates we use $a = 4\text{Å}$ while in the numerics we use larger values of $a$ as this will allow us to simulate systems of sufficient size with the available computational resources. This does not affect the qualitative features of the physics we wish to describe. The Cd$_3$As$_2$ Hamiltonian regularized on the lattice thus becomes

$$H^{\text{latt}} = \epsilon_k + \begin{pmatrix} h^{\text{latt}} & 0 \\ 0 & -h^{\text{latt}} \end{pmatrix},$$

(4.7)

where $\epsilon_k$ is the lattice version of $\epsilon_0(k)$ given in Appendix A while

$$h^{\text{latt}}(k) = m_k \tau^z + \Lambda(\tau^x \sin ak_x + \tau^y \sin ak_y).$$

(4.8)
Here \( m_k = t_0 + t_1 \cos ak_x + t_2(\cos ak_x + \cos ak_y) \) and \( t_0 = M_0 + 2(M_1 + 2M_2)/a^2 \), \( t_{1/2} = -2M_{1/2}/a^2 \), \( \Lambda = A/a \).

The Hamiltonian (4.8) exhibits a single pair of Weyl nodes at \( K_\eta = (0,0,\eta Q) \), with \( Q \) given by \( \cos(aQ) = -(t_0 + 2t_2)/t_1 \) which coincides with Eq. (4.6) in the limit \( aQ \ll 1 \). In the vicinity of the nodes, we can expand \( h^{\text{latt}}(K_\pm + q) \) in \( q \) to obtain the Weyl Hamiltonian

\[
h_\eta(q) = \hbar v_\eta^j \tau^j q_j, \tag{4.9}\]

with the velocity vector

\[
v_\eta = \hbar^{-1} a(\Lambda, \Lambda, -\eta t_1 \sin aQ). \tag{4.10}\]

For \( \text{Cd}_2\text{As}_3 \) parameters and a physical lattice constant \( a = 4 \)Å, this gives \( \hbar v_\eta = (0.89, 0.89, -1.24\eta)\)eVÅ. From Eq. (4.10), we can read off the chiral charge of the Weyl node located at valley \( \eta \),

\[
\chi_\eta = \text{sgn}(v_\eta^x v_\eta^y v_\eta^z) = -\eta. \tag{4.11}\]

The effect of strain on the lattice Hamiltonian (4.8) is implemented using the method developed in Refs. [133, 149]. The key observation is that certain tunneling amplitudes that are prohibited by symmetry in the unstrained crystal become allowed when the strain is applied because of the displacement and rotation of the relevant orbitals in the neighboring atoms. For our purposes the most important modification of the Hamiltonian (4.8) comes from the replacement of the hopping amplitude along the \( \hat{z} \)-direction [133, 149],

\[
t_1 \tau^z \rightarrow t_1(1 - u_{33})\tau^z + i\Lambda \sum_{j \neq 3} u_{3j} \tau^j, \tag{4.12}\]
where \( u_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i) \) is the symmetrized strain tensor and \( \mathbf{u} = (u_1, u_2, u_3) \) represents the displacement vector. The physics of Eq. (4.12) has been discussed at length in Ref. [149] and is easy to understand intuitively by inspecting the two examples of strain configurations given in Fig. 4.2. The first term in Eq. (4.12) reflects the change in the hopping amplitude \( t_1 \) between two like orbitals (Fig. 4.2a) when the distance \( d \) between the neighboring atoms changes due to strain. The amplitude depends exponentially on \( d \), but for small strain it can be expanded to leading order in the atomic displacements which leads to a correction proportional to \( u_{33} \). The second term describes generation of hopping processes along the \( \hat{z} \)-direction between different orbitals (Fig. 4.2b) which are prohibited in the unstrained crystal due to their \( s \) and \( p \) symmetry. The underlying mechanism is outlined in the caption of Fig. 4.2.

![Figure 4.2](image)

Figure 4.2: The effect of strain on the hopping amplitudes in the tight binding model.

a) Unidirectional strain along the \( z \) axis simply changes the distance between the neighboring orbitals leading to the modification of the hopping amplitude \( t_1 \) that is linear in \( u_{33} \) to leading order in small displacement. b) Torsional strain changes the relative orientation of the orbitals and brings about hopping amplitudes that are disallowed by symmetry in the unstrained crystal, such as \( t_{sp} \). The corresponding mathematical expression encodes the expectation that \( t_{sp} \) would become equal to \( \Lambda \) if the \( p \) orbital were displaced all the way to the horizontal position. In the real material one of course expects Eq. (4.12) to be valid only for displacements small compared to the lattice parameter \( a \).

As a simple example consider stretching the crystal along the \( \hat{z} \)-direction. This is represented by a displacement field \( \mathbf{u} = (0, 0, \alpha z) \) where \( \alpha = \Delta L/L \) measures the elongation of the crystal. The only nonzero component of the strain tensor is \( u_{33} = \alpha \).
and Eq. (4.12) thus gives \( t_1 \rightarrow t_1(1 - \alpha) \). It is easy to deduce that for small \( \alpha \) this changes the value of \( Q \rightarrow Q - \alpha Q/(aQ)^2 \) thus moving the Weyl nodes closer together or farther apart depending on the sign of \( \alpha \). We see that stretching the crystal has the same effect on the Weyl fermions as the \( z \)-component of the chiral gauge field \( a \).

More generally, elastic distortion expressed through Eq. (4.12) generates additional terms in the lattice Hamiltonian (4.8) of the form

\[
\delta h^{\text{latt}}(k) = -t_1 u_{33} \tau^z \cos ak_z + \Lambda (u_{13} \tau^x - u_{23} \tau^y) \sin ak_z.
\] (4.13)

Expanding again in the vicinity of \( K_{\pm} \), we obtain the linearized Hamiltonian of the distorted crystal

\[
h_\eta(q) = v^{ij}_\eta \tau^j (\hbar q_j - \eta^e c a_j),
\] (4.14)

where the gauge potential is given by

\[
a = -\frac{\hbar c}{e a} (u_{13} \sin aQ, u_{23} \sin aQ, u_{33} \cot aQ).
\] (4.15)

For \( aQ \ll 1 \), we may approximate \( \sin aQ \approx aQ \approx a\sqrt{-M_0/M_1} \) and \( \cot aQ \approx 1/aQ \).

We thus conclude that in a Weyl semimetal with nodes located on the \( k_z \) axis, components \( u_{j3} \) of the strain field act on the low-energy fermions as a gauge potential. \( a \) represents a chiral gauge field because it couples with the opposite sign to the Weyl fermions with different chirality \( \chi \).

We saw above that \( a_3 \sim u_{33} \) can be generated by stretching or compressing the crystal along its \( \hat{z} \) axis. Time-dependent distortion of this type will thus produce a pseudoelectric field \( e = -\frac{1}{e} \partial_t a \) directed along \( \hat{z} \). In combination with an applied magnetic field \( B \parallel \hat{z} \), this will generate nonzero \( e \cdot B \) term and, as we discuss below, allow to test the second chiral anomaly equation (4.2). It is also possible to generate the pseudomagnetic field by applying torsion to the crystal prepared in
a wire geometry. To see this, consider the displacement field $\mathbf{u}$ that results from twisting a wire-shaped crystal of length $L$ by angle $\Omega$. As illustrated in Fig. 4.3, we have

$$\mathbf{u} = \frac{\Omega}{L} \mathbf{r} \times \hat{z}, \quad (4.16)$$

where $\mathbf{r}$ denotes the position relative to the origin located on the axis of the wire. Nonzero components of the strain field are $u_{13} = (\Omega/2L)y$ and $u_{23} = -(\Omega/2L)x$. Via Eq. (4.15) we then get the pseudomagnetic field

$$\mathbf{b} = \nabla \times \mathbf{a} = b_0 \hat{z}, \quad b_0 = \frac{\hbar c}{2La_e} \sin aQ. \quad (4.17)$$

Figure 4.3: The displacement field $\mathbf{u}$ in the presence of torsion. Consecutive layers of the crystal are rotated by relative angle $\varphi_0 = \Omega(L/a)$.

To close this section, we estimate the magnitude of the strain-induced field $\mathbf{b}$ that can be achieved in a typical Cd$_3$As$_2$ nanowire described in Ref. [39]. We consider a cylindrical wire with a diameter $d = 100$nm, length $L = 1\mu$m and lattice parameter $a = 4\text{Å}$. Eq. (4.6) gives $Q = 0.033\text{Å}^{-1}$, so the condition $aQ \ll 1$ is satisfied and we may expand the sine in Eq. (4.17). Recalling further that $\Phi_0 = \hbar c/e \simeq 4.12 \times 10^5$TÅ$^2$, we find $b_0 \approx 1.8 \times 10^{-3}$T per angular degree of twist. The maximum attainable field strength in a given wire will depend on how much torsion can the wire sustain before breaking. While we were unable to find any data on the mechanical properties of Cd$_3$As$_2$, we note that Ref. [39] characterized the nanowires as “greatly flexible”. We
take this to imply that they can withstand substantial torsion. Based on this, a twist angle \( \Omega \simeq 180^\circ \) would appear sustainable and will produce \( b_0 \approx 0.3 \text{T} \). For the wire under consideration, such a twist translates to a maximum displacement at the outer radius of the wire of about 0.3Å between the neighboring atoms, or about 8\% of the unit cell. Because the maximum twist angle is limited by the maximum distortion, higher effective fields can be achieved in thinner wires.

### 4.5 Numerical Simulations

To further confirm the validity of the analytical results presented in the previous section, we carried out extensive numerical simulations of the lattice Hamiltonian (4.8) in the presence of magnetic field \( B \) as well as torsional and unidirectional strain implemented via Eq. (4.13). Magnetic field was implemented through the usual Peierls substitution. Our results below indeed validate the general concepts discussed above and illustrate them in a concrete setting of a lattice model relevant to \( \text{Cd}_3\text{As}_2 \) and \( \text{Na}_3\text{Bi} \).

#### 4.5.1 Pseudomagnetic Field \( b \) from Torsion

We start by studying a wire grown along the crystallographic \( z \) axis in the presence of magnetic field \( B = \hat{z}B \) and torsion. Representative results are displayed in Fig. 4.4. For simplicity and ease of interpretation, we used here parameters appropriate for \( \text{Cd}_3\text{As}_2 \) (summarized in Appendix A), neglecting terms in \( \epsilon_k \). We have verified that substantially similar results are obtained when \( \epsilon_k \) is retained as well as for parameters appropriate for \( \text{Na}_3\text{Bi} \). These results are given in Appendix A.
Figure 4.4: Tight-binding model simulations of a Weyl semimetal wire under torsional strain and applied magnetic field $B = \hat{z}B$. Top row of figures shows the band structure of the lattice Hamiltonian defined by Eqs. (4.8) and (4.13) computed for $\frac{1}{2}$-Cd$_3$As$_2$ model parameters, for a wire with a rectangular cross section of 30 × 30 sites and a lattice constant $a = 40\AA$. (We use larger lattice constant here and in subsequent simulations than in real Cd$_3$As$_2$ in order to be able to model nanowires and films of realistic cross sections with available computational resources. Note that this does not affect the physics at low energies because the lattice Hamiltonian is designed to reproduce the relevant $k \cdot p$ theory independent of $a$.) Open boundary conditions are imposed along $x$ and $y$, periodic along $z$. Parameters appropriate for Cd$_3$As$_2$ are used. Middle and bottom rows show spectral functions $A^\text{bulk}(k, \omega)$ and $A^\text{surf}(k, \omega)$. The former is obtained by averaging the full spectral function $A_j(k, \omega)$ over sites $j$ in the central 10 × 10 portion of the wire while the latter averages over the sites located at the perimeter of the wire. The torsion applied in columns c and d corresponds to the maximum displacement at the perimeter of 0.5$a$, or $\varphi_0 \simeq 2^\circ$ between consecutive layers.

Column (a) in Fig. 4.4 shows the spectrum of an unstrained wire in zero field. Gapless Weyl points are apparent at $k = \pm Q$ and are connected by surface states that originate from the Fermi arcs, expected to occur in the surface of a Weyl semimetal.
Spectral functions computed in the bulk, $A^{\text{bulk}}(k, \omega)$, and at the surface, $A^{\text{surf}}(k, \omega)$, confirm this identification of bulk and surface electron states. Column (b) exhibits our results for an unstrained wire in magnetic field $B = 3.2\, \text{T}$ along the axis of the wire. As expected on the basis of arguments that led to Fig. 4.1a, we observe at low energies a pair of left and right moving chiral modes. These originate from the $n = 0$ Landau level and occur in the bulk of the sample. We also observe that the surface states remain largely unaffected by the field.

Our main finding is illustrated in column (c). Torsional strain applied to the wire produces two right-moving chiral modes that are localized in the bulk of the sample as evidenced by $A^{\text{bulk}}(k, \omega)$. The bulk spectrum has the structure depicted in Fig. 4.1b expected to occur in the presence of the chiral magnetic field $b$. We are thus led to identify the torsional strain with the chiral vector potential $a$. Surface states discernible in the corresponding $A^{\text{surf}}(k, \omega)$ are seen to compensate for the bulk band structure by providing the required left moving chiral modes.

Column (d) shows the spectrum for the case when the strength of $B$ is chosen to exactly equal $b$. As a result, vector potentials $A$ and $a$ add in one Weyl point but cancel in the other. The resulting spectrum exhibits a set of right moving bulk chiral modes present in only one of the two Weyl points. This establishes the complete equivalence of the real magnetic field $B$ and the strain-induced pseudomagnetic field $b$ insofar as their action on the low-energy Weyl fermions is concerned.

We note that pseudomagnetic field $b \simeq 3.2\, \text{T}$ indicated in Fig. 4.4 is larger than the maximum achievable field in the realistic Cd$_3$As$_2$ wire estimated in the previous section. This is because for clarity we employed here larger torsion (resulting in the maximum displacement of about half the lattice spacing) than can likely be sustained in a real wire. For weaker torsion strengths, the effect remains qualitatively unchanged but becomes less clearly visible in the numerical data for system sizes that are accessible to our simulations.
Results presented in Fig. 4.4 pertain to a Weyl semimetal described by Hamiltonian (4.8) but are easily extended to Cd$_3$As$_2$ as long as we continue neglecting the particle-hole symmetry breaking term $\epsilon_k$. In this limit, spectra for Cd$_3$As$_2$ are obtained by simply superimposing bands $E_k$ and $-E_k$ shown in Fig. 4.4 or by forming spectral functions $A(k, \omega) + A(k, -\omega)$. Full spectra, including the p-h breaking terms, are more complicated but show the same qualitative features. Some relevant examples are given in Appendix A.

4.5.2 Pseudoelectric Field $e$ from Unidirectional Strain

According to our previous discussion, pseudoelectric field $e$ should emerge when the $u_{33}$ component of the strain tensor becomes time dependent. This can be achieved through dynamically stretching and compressing the crystal along its $z$ axis, e.g. by driving longitudinal sound waves through the crystal. To see how the lattice model realizes the chiral anomaly under these conditions, we first consider an infinite bulk crystal in the presence of a uniform magnetic field $B = \hat{z}B$ and investigate the effect of the static $u_{33}$ strain. The spectrum of an unstrained crystal in the field $B = 10\, \text{T}$ is displayed in Fig. 4.5a (we use once again Cd$_3$As$_2$ parameters and include this time also $\epsilon_k$). At low energies the spectrum exhibits the expected chiral branches that result from the $n = 0$ Dirac Landau level. We assume the system is initially in its ground state with all energy levels below the chemical potential $\mu_0$ occupied and all levels above $\mu_0$ empty. We now implement unidirectional strain through Eq. (4.13) which amounts to rescaling the hopping amplitudes $t_1 \rightarrow t_1(1-\alpha)$ and $c_1 \rightarrow c_1(1-\alpha)$. Here $c_1$ is the hopping amplitude along the $z$ direction in $\epsilon_k$ defined below Eq. (A.1). We imagine doing this sufficiently slowly so that the ground state evolves adiabatically in response to the increasing strain. The new ground state for $\alpha = 0.03$ is depicted in Fig. 4.5a. It exhibits a slightly modified band structure with the chemical potential shifted to a new value $\mu'$. The shift in $\mu$ occurs because under adiabatic evolution an
electron initially in the quantum state with momentum $k$ in the $n$th band remains in that state as the band energy $E_n(k)$ evolves in response to strain.

Figure 4.5: Tight-binding model simulations of a Weyl semimetal under applied magnetic field $\mathbf{B} = \hat{z}B$ and unidirectional strain. Parameters for Cd$_3$As$_2$ listed in Appendix A are used in all panels. Only spin up sector of the model is considered with $B = 10$T. a) Band structure of the system with periodic boundary conditions in all directions (no surfaces) projected onto the $z$ axis ($k$ denotes the crystal momentum along the $z$ direction). Solid (dashed) lines show occupied (empty) states. Occupation of the strained system is determined by adiabatically evolving the single-electron states of the unstrained system. b) Band structure of a slab with thickness $d = 1000$ Å (50 lattice sites). Only positive values of $k$ are displayed but the band structure is symmetric about $k = 0$. Red (black) lines show occupied (empty) states. The central panel indicates the nonequilibrium occupancy of the strained system obtained by adiabatically evolving the single-electron states of the unstrained system. The right panel shows the occupancy of the strained system once the electrons relaxed back to equilibrium. All three panels correspond to the same total number of electrons $N$. c) Change in the electron density in response to the applied strain as a function coordinate $y$ perpendicular to the slab surfaces. $\delta \rho$ refers to the nonequilibrium distribution while $\delta \rho_{eq}$ refers to the relaxed state. Note that density oscillations near the edges apparent in $\delta \rho$ average to zero: there is no net charge transfer between the bulk and the surface in the nonequilibrium state, as can also be deduced from the vanishing $\delta \rho$ in the bulk.

From the point of view of the low-energy theory, the lateral shift of the chiral branches is consistent with the effect of the uniform chiral gauge potential $a_z$ which according to our discussion below Eq. (4.12) moves the Weyl points closer together for $\alpha > 0$. From Eqs. (4.14) and (4.15) we can estimate the amount of this shift $\delta Q \simeq (e/\hbar c)a_z = -u_{33} \cot aQ/a$. This in turn gives an estimate for the required
change in the chemical potential \( \delta \mu = \mu' - \mu_0 = -\hbar v \delta Q \), or

\[
\delta \mu = -\frac{e}{c} a z = \alpha \frac{\hbar v}{a} \cot aQ.
\]  

(4.18)

For Cd\(_3\)As\(_2\) parameters including the particle-hole symmetry breaking terms in \( \epsilon_k \), we have \( \hbar v \approx 1.94 \text{eVÅ} \), which implies \( \delta \mu = 3.75 \text{meV} \) for \( \alpha = 0.03 \). This estimate compares favorably with the value \( \delta \mu_{\text{num}} = 3.46 \text{meV} \) obtained from our lattice model simulation presented in Fig. 4.5a.

If we continue focusing solely on the low energy degrees of freedom, we would conclude that a change \( \delta \mu \) in the chemical potential in a linearly dispersing band with degeneracy \( (B/\Phi_0) \) brings about a change in the electron density

\[
\delta \rho = 2 \frac{\delta \mu}{2\pi \hbar v} \left( \frac{B}{\Phi_0} \right),
\]  

(4.19)

where the factor of 2 accounts for two chiral branches. Using Eq. (4.18) it is easy to verify that Eq. (4.19) coincides exactly with the prediction of the second chiral anomaly equation (4.2) for uniform static magnetic field and a time dependent pseudo-electric field \( e = -\frac{1}{c} \partial_t a \).

If on the other hand we espouse a band theory point of view, then we see that in reality the charge density remains unchanged. This is because precisely the same number of single electron states are filled before and after the deformation. The chemical potential changes in order to accommodate the fixed number of electrons in the modified band structure. We may thus conclude that in an infinite crystal pseudoelectric field induced by strain does not bring about any change in charge density. The chiral anomaly equation (4.2) however correctly predicts the strain induced change in the chemical potential \( \delta \mu \).

A change in the chemical potential, even if time dependent (as would be the case when strain is induced by a sound wave), is not easily measurable when not accompa-
ned by a density change. So it would seem that this effect does not have observable consequences. Consider however a finite system with boundaries. The key point is that topologically protected surface states that are present in a Weyl semimetal will generally not respond to strain in the same way as the bulk states. To a good approximation one may consider the surface state to remain basically unaffected by a small unidirectional strain. This is verified by our numerical simulations summarized in Fig. 4.5b. In that case application of strain will bring about a nonequilibrium distribution of electrons ($\mu$ changes in the bulk but remains unchanged at the surface). This is illustrated in Fig. 4.5b where we simulate the effect of a 3% strain in a slab of thickness $d$ with surfaces perpendicular to the $y$ direction and magnetic field along $z$.

We observe that strain shifts the chemical potential for the bulk states by the same amount as in the infinite system but leaves it essentially unchanged for the surface states.

Several interesting consequences follow from the above observation. First, we may expect the charge density to remain essentially unchanged in the strained crystal with nonequilibrium distribution of electrons. This is because the bulk density remains unchanged (as per our discussion above) and since the total charge is conserved there can be no charge transfer to the surface. Second, in a real material the nonequilibrium electron distribution brought about by strain will relax towards equilibrium, causing dissipation in the system which is in principle observable. When the strain is induced by a sound wave this dissipation will provide a new mechanism for sound attenuation related to the chiral anomaly. Third, the relaxed charge density $\rho'(y)$ in the strained crystal will differ from the the original charge density $\rho_0(y)$ of the unstrained crystal because relaxation necessarily involves transfer of charge between the bulk and the surface of the sample. This is illustrated in Fig. 4.5c which shows the numerically calculated change in the charge density $\delta\rho(y) = \rho'(y) - \rho_0(y)$ in both nonequilibrium and equilibrium state following the application of a 3% strain. We note that modulo
some local fluctuations near the edge the charge density indeed behaves as expected on the basis of the above arguments.

We conclude by elaborating on this last effect. If the sound frequency \( \omega \) is small compared to the electron relaxation rate \( \tau^{-1} \), as it will be the case in the typical experimental situation, the electron distribution will always remain close to an equilibrium characterized by a global chemical potential \( \mu'_{\text{eq}} \). The corresponding charge density should then exhibit significant variations as the chemical potential oscillates. Such a time dependent variation in the charge density will produce EM fields outside the sample which are measurable and can provide direct experimental evidence for the strain-induced chiral anomaly. We shall estimate the distribution and the amplitude of these fields in the next section.

To this end it will be useful to estimate the chemical potential \( \mu'_{\text{eq}} \) of the equilibrated strained system (see also Fig. 4.5b). A straightforward calculation for a slab of thickness \( d \) (summarized in Appendix B) gives

\[
\mu'_{\text{eq}} = \mu_0 + \frac{\delta \mu}{1 + \xi_B/d},
\]

(4.20)

where \( \xi_B = 2Q\ell_B^2 \) is the characteristic lengthscale and \( \delta \mu \) is the chemical potential change in the system without surfaces given by Eq. (4.18). The physics of Eq. (4.20) is quite simple: it reflects the fact that a surface can accommodate only a limited amount of charge from the bulk. For a thick slab \( d \gg \xi_B \) we recover the bulk result \( \mu'_{\text{eq}} \approx \mu_0 + \delta \mu \) because the effect of the surface becomes negligible.

From Eq. (4.20) it is easy to obtain an estimate for the corresponding change in the bulk charge density

\[
\delta \rho^\text{bulk} = -\frac{\alpha}{\pi a} \left( \frac{B}{\Phi_0} \right) \frac{\cot aQ}{1 + d/\xi_B}.
\]

(4.21)
In the limit of a thin slab, $d \ll \xi_B$, this result approaches the charge density change (4.19) derived based on the naive application of the chiral anomaly equation, except for the opposite overall sign. In this limit, physically, almost all the non-equilibrium charge density produced in the bulk can be absorbed by the surface. The bulk charge density thus goes down by the amount close to that predicted by the chiral anomaly.

Fig. 4.6 shows the bulk charge density $\delta \rho_{\text{bulk}}$ in response to the unidirectional strain $\alpha = 0.03$ as a function of the applied field $B$ in a relaxed system, numerically calculated from the lattice model. A good agreement with Eq. (4.21) is seen both in the magnitude of the effect and its functional form. The lattice model shows a somewhat stronger response than expected which we attribute to the p-h anisotropy that was not included in the analytical calculation. That this is indeed the case is confirmed by the same calculation performed for the p-h symmetric version of the $\frac{1}{2}$-Cd$_3$As$_2$ model which shows closer agreement with Eq. (4.21), modulo finite size effect induced fluctuations (solid black symbols in Fig. 4.6). We however note that in this case the contribution from the spin-down sector exactly cancels that from spin up so p-h asymmetry is required to obtain a nonzero result.

<table>
<thead>
<tr>
<th>$B$ [Tesla]</th>
<th>$\delta \rho_{\text{bulk}}$ [e per site]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>-0.00002</td>
</tr>
<tr>
<td>4</td>
<td>-0.00004</td>
</tr>
<tr>
<td>6</td>
<td>-0.00006</td>
</tr>
<tr>
<td>8</td>
<td>-0.00008</td>
</tr>
</tbody>
</table>

Figure 4.6: Numerically calculated change in the bulk charge density $\delta \rho_{\text{bulk}}$ in response to unidirectional strain $\alpha = 0.03$ as a function of the applied field $B$. Parameters for Cd$_3$As$_2$ are used with $\mu = 0$ and $d = 1000\text{Å}$ (50 lattice sites). Solid black symbols give result for the p-h symmetric version of the $\frac{1}{2}$-Cd$_3$As$_2$ model obtained by setting all $C_j$ parameters to zero.
We note that Eqs. (4.20) and (4.21) were derived assuming quantum limit, i.e., chemical potential in the \( n = 0 \) Landau level. The corresponding results valid away from the quantum limit are given in Appendix B.

We close this subsection by considering Dirac semimetals. Naively one could think that the effects discussed above will cancel once we include both spin sectors. This would indeed be the case in a perfectly particle-hole symmetric system. However, the band structures of both Cd\(_3\)As\(_2\) and Na\(_3\)Bi exhibit strong particle-hole asymmetry which prevents such cancellations. To elucidate this we show in Fig. 4.7 the band structure of the spin-down sector of Cd\(_3\)As\(_2\) in the field of 10T. Compared to the spin-up sector (Fig. 4.5a) it indicates a spectral gap at low energies. It is clear that when \( \mu \) lies inside this gap then all the physics comes exclusively from the spin-up sector. Specifically, there is nothing here to cancel or even weaken the effects discussed above. We find that this remains true more generally. Even when \( \mu \) is outside the gap the contributions to various effects discussed above generically do not cancel but remain of a similar magnitude as they would be in a Weyl semimetal with a single pair of Weyl points. This is illustrated in Fig. 4.6 where the chemical potential is chosen to lie outside the bandgap; the effect is only slightly reduced when contributions from both sectors are added up. We thus expect the effects discussed above to generically remain present in Dirac semimetals such as Cd\(_3\)As\(_2\) and Na\(_3\)Bi.
Figure 4.7: Band structure of the spin down sector of Cd$_3$As$_2$ in magnetic field $B = 10$T. Two chiral branches are visible at low energy but they are now strongly distorted by p-h symmetry breaking terms and they no longer traverse the gap between the valence and the conduction band.

### 4.6 Experimental Manifestations

#### 4.6.1 Topological Coaxial Cable

The phenomena discussed above have several observable consequences which we now discuss. According to Fig. 4.4c Weyl semimetal wire under torsion exhibits spatial separation between left and right moving modes at low energies: the former are localized near the boundary while the latter occur in the bulk. At a generic chemical potential we thus expect persistent equilibrium currents to flow in such a wire as indicated in Fig. 4.8a. This can be argued as follows. Suppose the current density $j_z(r)$ is uniformly zero at some reference chemical potential $\mu_0$. If we now change the chemical potential to $\mu = \mu_0 + \delta \mu$ we are populating additional right moving modes in the bulk and left moving modes at the surface of the wire. Although the total current carried by the wire remains zero, as it must be in any normal metal in equilibrium [150], there is now a non-vanishing positive current density flowing in the bulk compensated by the negative current density flowing along the surface. We
have verified numerically that this is indeed the case in the lattice model (4.8) and (4.13): for any chemical potential $\mu \neq 0$ a ground-state current density develops as illustrated in Fig. 4.8b.

![Diagram of current density](image)

**Figure 4.8**: Equilibrium current density in the Weyl semimetal wire under torsion. a) Schematic depiction of the bulk/surface current flow. b) Ground state current density computed from the lattice model Eqs. (4.8) and (4.13) at chemical potential $\mu = 5\text{meV}$. Warm (cold) colors represent positive (negative) current density $j$. The ring-shaped inhomogeneity in $j$ apparent in the bulk of the wire reflects Friedel-like oscillations in electron wavefunctions caused by the presence of the surface.

Such a current flow generates magnetic fields outside the wire which are, at least in principle, measurable e.g by scanning SQUID microscopy. In practice, however, we expect this to be a challenging experiment. The currents occur only in a Weyl semimetal with broken $\mathcal{T}$ which is most likely to be realized in a magnetic material. It might be difficult to distinguish the fields produced by torsion-induced persistent currents from the sample magnetization. We note that in Dirac semimetals, like Cd$_3$As$_2$ or Na$_3$Bi, the total current density will vanish upon including the contribution from the lower diagonal block in the Hamiltonian (4.5). This has to be the case because non-zero $j$ would violate the $\mathcal{T}$ symmetry of the material, which should remain unbroken under strain. The current density can be nonzero, however, when both torsion and magnetic field are applied. This is demonstrated in Fig. A.2 of Appendix A.
4.6.2 Chiral Torsional Effect

The physics described above however has a simple manifestation observable in transport measurements in both Weyl and Dirac semimetals. Consider a measurement of longitudinal resistivity in a twisted wire. Once again we start by discussing a Weyl semimetal. When electric field $E$ is applied to the twisted wire it begins to produce charge density $\delta \rho = \rho - \rho_0$ in the bulk at the rate given by the anomaly equation (4.2). In view of our discussion above we interpret $\delta \rho$ as charge density imbalance between the bulk and the surface of the wire. Such an imbalance can relax back to equilibrium only through processes that induce backscattering between the bulk right moving modes and the surface left moving modes. If we denote the relevant scattering time by $\tau$, we get an equation

$$\frac{d}{dt} \delta \rho = \frac{e^2}{2\pi^2\hbar^2 c} E \cdot b - \frac{\delta \rho}{\tau}. \quad (4.22)$$

At long times $t \gg \tau$ the steady state solution reads

$$\delta \rho = \frac{e^2 \tau}{2\pi^2\hbar^2 c} E \cdot b. \quad (4.23)$$

The wire clearly carries non-zero electrical current. The expression for the current depends on the relative position of the chemical potential $\mu$ and the bottom of the first Landau level $\epsilon_1(0) = \hbar v \sqrt{2eB}/hc$. In the quantum limit, $|\mu| < \epsilon_1(0)$, only the chiral modes in the $n = 0$ Landau level are populated. These all move at the same velocity $v \text{sgn}(b)$ and the non-equilibrium charge density $\delta \rho$ thus gives electrical current

$$J_{CTE} = -e v \text{sgn}(b) \delta \rho = \frac{e^3 v \tau}{2\pi^2\hbar^2 c} E \cdot b \text{sgn}(b). \quad (4.24)$$

For a constant relaxation time $\tau$ we thus have chiral torsional contribution to the conductivity $\sigma_{CTE} \sim |b|$, similar to the ordinary chiral magnetic effect $\sigma_{CME} \sim |B|$
in the quantum limit [29]. However, if the wire radius $R$ significantly exceeds the magnetic length $\ell_b = \sqrt{\hbar c/eb} \simeq 256\AA \sqrt{1T/b}$, then we find that the appropriate relaxation time becomes field-dependent, namely,

$$\tau \simeq \tau_0 \frac{R^2}{\ell_b^2} \sim |b|,$$

(4.25)

where $\tau_0$ is the microscopic scattering time. This is because the bulk electron wavefunctions have spatial extent $\ell_b$ in the direction transverse to the axis of the wire. Deep in the bulk impurities cause scattering between the individual bulk modes but since these are all right moving such processes cannot relax the current. Only those electrons that have diffused all the way to the boundary through repeated scattering processes can backscatter into left moving surface modes. Electrons thus experience hydrodynamic flow whereby dissipation occurs only at the boundary. Eq. (4.25) is derived in Appendix C and expresses the fact that an electron that is produced near the center of the wire has to travel distance $R$ to the boundary and this takes on average $(R/\ell_b)^2$ scattering events. We conclude that $\sigma_{CTE} \sim b^2$ in the quantum limit when $\ell_b \ll R$.

In the semiclassical limit, $|\mu| \gg \epsilon_1(0)$, we must take into account the additional equilibration that occurs between the individual Landau levels within a given Weyl point. We assume that the relaxation time for this process is very short and essentially instantaneous compared to $\tau$. In this case, electron density produced through Eq. (4.23) gets distributed among all the bulk states and leads to a shift in the chemical potential $\mu \rightarrow \mu + \delta \mu$. In the semiclassical limit we can approximate the density of states by the expression valid in the zero field, $D(\epsilon) = e^2/\pi^2\hbar^3v^3$, where for simplicity we also assume isotropic velocities. In the limit of interest, $\delta \mu \ll k_B T \ll \mu$, it is easy to find from this the shift in the chemical potential caused by a small change in
density,
\[
\delta \mu \simeq \frac{2\pi^2 \hbar^3 v^3}{\mu^2 + \frac{2\pi^2}{3} k_B^2 T^2} \delta \rho.
\]  
(4.26)

We can now calculate the current by noting that, once again, only the chiral branches contribute. We thus obtain

\[
J_{\text{CTE}} = -ev \left( \frac{\delta \mu}{2\pi \hbar v} \right) \left( \frac{b}{\Phi_0} \right),
\]  
(4.27)

where the first bracket represents the number of extra modes \( \delta n \) that have been populated on the chiral branch and the second reflects their degeneracy. Combining this with Eqs. (4.26) and (4.23) we find

\[
J_{\text{CTE}} = \frac{e^4 v^3}{8\pi^3 \hbar c^2} \frac{\tau}{\mu^2 + \frac{2\pi^2}{3} k_B^2 T^2} (E \cdot b)b.
\]  
(4.28)

In view of Eq. (4.25) in a Weyl semimetal under torsion (parametrized here by \( b \propto \Omega \)) we thus predict a positive contribution to the conductivity

\[
\sigma_{\text{CTE}} \propto \begin{cases} 
|b|^2, & \mu < \hbar v \sqrt{\frac{2eb}{\hbar c}} \text{ quantum limit,} \\
|b|^3, & \mu \gg \hbar v \sqrt{\frac{2eb}{\hbar c}} \text{ semiclassical limit.}
\end{cases}
\]  
(4.29)

The predicted field dependence is different from the analogous effect encountered in the presence of the real magnetic field \( B \) (where \( \sigma_{\text{CME}} \) behaves as \( \sim B \) and \( \sim B^2 \) in the two limits). This reflects the hydrodynamic nature of the electron flow that occurs when \( R \gg \ell_b \). The right and left moving modes are then segregated to the bulk and the boundary respectively, which leads to an extra power of \( b \) due to \( b \)-dependent transport scattering rate (4.25). We also note that when \( R \gg \ell_b \), Eq. (4.25) implies significant enhancement of the transport lifetime and thus leads us to expect a strong effect. In the opposite limit, \( R \ll \ell_b \), the transport scattering rate becomes field
independent and the more conventional behavior with \( \sigma_{\text{CTE}} \propto b \left( b^2 \right) \) in quantum (semiclassical) limit is restored.

The effect will persist in a Dirac semimetal such as Cd\(_3\)As\(_2\) and Na\(_3\)Bi, which can be thought of as two \( \mathcal{T} \)-conjugate copies of the Weyl semimetal discussed above. In the presence of a twist the spectrum will consist of that indicated in Fig. 4.4c for the spin-up sector plus a time-reversed copy (obtained by reversing \( k \rightarrow -k \)) for the spin down sector. The same analysis we just performed applies unchanged for each spin sector if one can ignore spin-flip scattering events. In this case Eq. (4.29) continues to hold in a Dirac semimetal. Spin-flip processes, if present, open additional channel for relaxation by scattering between left and right moving bulk modes. In the limit when the spin-flip relaxation rate \( \tau_{\text{sf}}^{-1} \) exceeds \( \tau^{-1} \) the hydrodynamic flow will cease and the behavior will cross over to the regular chiral anomaly with \( \sigma_{\text{CTE}} \propto b \left( b^2 \right) \) in the quantum (semiclassical) limit. In clean samples of \( \mathcal{T} \)-preserving Cd\(_3\)As\(_2\) and Na\(_3\)Bi we expect the hydrodynamic behavior to prevail. This is because ordinary non-magnetic impurities cannot cause spin-flip scattering. Time reversal symmetry permits spin-orbit scattering terms of the form \( \hat{z} \cdot (\sigma \times k) \). These do contribute to \( \tau_{\text{sf}}^{-1} \) but we expect such contributions to be small.

## 4.6.3 Ultrasonic Attenuation and EM Field Emission

We now consider the experimental manifestations of the \( e \cdot B \) term in the second chiral anomaly equation (4.2). For concreteness we again start with a Weyl semimetal and consider a sample in the shape of a slab with thickness \( d \) and surfaces perpendicular to the \( y \) axis. Magnetic field \( B \) is applied along the \( z \)-direction. The requisite \( e \) field is generated by a longitudinal sound wave with frequency \( \omega \) that is driven along the \( z \) direction. This produces a time dependent displacement field

\[
u = u_0 \hat{z} \sin(qz - \omega t),
\]

(4.30)
where \( q = \omega / c_s \) is the wavenumber and \( c_s \) the sound velocity. The nonzero component of the strain tensor is \( u_{33} = u_0 q \cos (q z - \omega t) \) which through Eq. (4.18) yields an oscillating component of the bulk chemical potential,

\[
\delta \mu(t) = u_0 q \left( \frac{\hbar v}{a} \cot a Q \right) \cos(qz - \omega t).
\] (4.31)

As mentioned in Sec. 4.5.2, electron relaxation dissipates energy which will be manifested by the attenuation of the sound wave as it propagates through the medium. Specifically, as explained e.g. in Ref. [151] the energy flux \( I \) carried by the sound wave obeys \( I(z) = I_0 e^{-2\Gamma z} \) where \( \Gamma \) is the sound attenuation coefficient. We now proceed to estimate \( \Gamma \) which is given by \( \Gamma = Q/2I \), where \( Q \) denotes the amount of energy dissipated in a unit volume per unit time. To provide a crude estimate of \( Q \) we assume for a moment that the electron relaxation rate \( \tau^{-1} \) is comparable to the driving frequency, \( \omega \tau \approx 1 \). In this case relaxational dynamics is maximally out of phase with the sound wave and we can estimate \( Q \) simply by calculating the energy difference between the nonequilibrium distribution of electrons (see Fig. 4.5b) reached at the crest of the wave (assuming no dissipation has occurred until then) and the corresponding equilibrium distribution with the chemical potential \( \mu'_{\text{eq}} \). For this estimate consider a slice of the system perpendicular to \( z \) of length \( l \) such that \( l \ll \lambda_s \). Inside the slice the strain can be considered uniform, implying a uniform chemical potential \( \delta \mu(t) \propto \cos \omega t \). We may thus estimate the total dissipated electron energy per cycle as

\[
E_{\text{diss}} = lw \int_{\mu'_\text{eq}}^{\mu'} \epsilon D_b(\epsilon) d\epsilon - lw \int_{\mu_0}^{\mu'_\text{eq}} \epsilon D_s(\epsilon) d\epsilon,
\] (4.32)

where \( w \) is the width of the slab along the \( x \) direction and \( D_{b/s}(\epsilon) \) is the bulk/surface density of states given in Appendix B. It is easy to evaluate the requisite integrals.
After some algebra and with help of Eq. (4.20) one obtains, assuming quantum limit,

$$E_{\text{dis}} \approx \frac{lwd}{2\pi\hbar v} \left( \frac{B}{\Phi_0} \right) \frac{1}{1 + d/\xi_B} \delta\bar{\mu}^2,$$  \hspace{1cm} (4.33)

where $\delta\bar{\mu}$ is the amplitude of $\delta\mu(t)$ given in Eq. (4.31).

A more complete treatment of the relaxational dynamics, which we omit here for the sake of brevity, gives a result for the energy dissipated per cycle valid for any frequency,

$$E_{\text{dis}} = \frac{lwd}{2\pi\hbar v} \left( \frac{B}{\Phi_0} \right) \frac{\omega\tau}{(1 + d/\xi_B)^2 + (\omega\tau)^2} \delta\bar{\mu}^2.$$  \hspace{1cm} (4.34)

The energy density of the sound wave, averaged over one cycle, is $\rho E = \frac{1}{2} \rho c_s^2 u_0^2 q^2$, where $\rho$ denotes the mass density of the crystal. Noting that the corresponding energy flux is $I = c_s \rho E$ one obtains the sound attenuation coefficient

$$\Gamma = \frac{(\omega E_{\text{dis}}/lwd)}{2c_s\rho E}.$$  \hspace{1cm} (4.35)

To estimate its magnitude we assume the limit of a thin slab $d \ll \xi_B$ and fast relaxation $\omega\tau \ll 1$ in Eq. (4.34). In this limit $\Gamma$ becomes independent of $d$:

$$\Gamma \simeq \left( \frac{\omega}{2\pi c_s} \right) \frac{\hbar v}{a^2} \left( \frac{B}{\Phi_0} \right) \frac{2c_s^2 aQ}{\rho c_s^2} (\omega\tau).$$  \hspace{1cm} (4.36)

For our estimate we take $f = \omega/2\pi = 200\text{MHz}$, the mass density of Cd$_3$As$_2$ is $\rho = 7.0 \times 10^3 \text{ kg/m}^3$ while the speed of sound is $c_s = 2.3 \times 10^3 \text{ m/s}$ [152] which gives $\lambda_s \simeq 11\mu\text{m}$ at this frequency. For these parameters we obtain

$$\Gamma \simeq 3.6 \times 10^3 \text{m}^{-1} \left[ \frac{B}{1T} \right] (\omega\tau).$$  \hspace{1cm} (4.37)

We see that depending on the magnitude of the electron scattering rate the sound attenuation can be substantial. There are of course many conventional sources
of ultrasonic attenuation in metals [151]. Given the specific dependence of $\Gamma$ on frequency, magnetic field and the fact that it depends only on the component of $B$ parallel to $q$, it should be possible to separate the contribution of the chiral anomaly from the more conventional contributions.

At $B = 1T$ for material parameters relevant to Cd$_3$As$_2$ we have $\xi_B \simeq 430$nm so the above estimate applies to thin films or wires. For thicker films one must include the additional suppression factor $(1 + d/\xi_B)^{-2}$ from Eq. (4.34) that we neglected so far. This factor reflects the fact that the relaxation mechanism involves charge transfer from the bulk to the surface of the sample. For the same reason, however, we expect in this limit to obtain an enhanced relaxation time $\tau \simeq \tau_0(d/\ell_B)^2$, where $\tau_0$ is the microscopic relaxation time as in Eq. (4.25). This is because to relax the non-equilibrium distribution brought about by the chiral anomaly bulk electrons must diffuse to the surface and this takes on average $(d/\ell_B)^2$ scattering events. In the end we expect only a weak dependence of $\Gamma$ on the sample thickness $d$ although a detailed treatment of the combined spatial and temporal distribution of electrons during the relaxation process is an interesting topic for future research.

The oscillating charge density that occurs in the system in response to the sound wave will generate EM fields that can be detected outside the sample. We show below that in a typical situation the electric field close to the surface can be substantial and thus detectable. The field decays as $\sim e^{-r/\lambda_s}$ away from the surface but since $\lambda_s$ is tens or hundreds of microns at typical ultrasonic frequencies the detection of such fields should not be difficult [153].

To estimate the amplitude of the electric field we assume once again that electron relaxation is fast compared to the driving frequency, $\omega \tau \ll 1$. This means that electrons will locally always be close to equilibrium characterized by the charge density $\bar{\rho} + \delta\rho^{\text{bulk}}(z,t)$ where $\bar{\rho}$ is the bulk charge density of the unstrained crystal and $\delta\rho^{\text{bulk}}(z,t)$ is given by Eq. (4.21) with $\alpha$ now describing the local strain field at
In a slab of thickness $d = 2d'$ illustrated in Fig. 4.9 the oscillating component of the charge density therefore reads

$$\rho_{\text{bulk}} = \rho_0 \cos(qz - \omega t),$$

(4.38)

with

$$\rho_0 = -\frac{u_0 q}{\pi a} \left( \frac{B}{\Phi_0} \right) \cot aQ \frac{1}{(1 + d/\xi_B)}.$$

(4.39)

Figure 4.9: Proposed geometry for the EM field emission measurement in the limit when all the dimensions of the crystal are much larger than the sound wavelength $\lambda_s$. a) A slab of thickness $d = 2d'$ is subjected to magnetic field $B$ and a longitudinal acoustic sound wave propagating along the $z$ direction. b) A snapshot of the electric field distribution near the surface calculated from Eq. (4.43). As a function of time the entire pattern moves in the $z$ direction at the speed of sound $c_s$.

From our previous discussion, we know that the charge generated in the bulk comes from the boundary. The total charge density that reflects the overall charge conservation in each constant-$z$ slice of the sample can thus be written as

$$\rho = \rho_0 [\theta(d' - |y|) - d' \delta(y \pm d')] \cos(qz - \omega t),$$

(4.40)

where $\rho_0$ is given by Eq. (4.39). In the near field (static) region we may neglect the magnetic effects and determine the electric field $E = -\nabla \Phi$ by solving the Poisson equation $\nabla^2 \Phi = -4\pi \rho$. Adopting the ansatz $\Phi(r, t) = f(y) \cos(qz - \omega t)$ we are led
to a 1D equation for $f(y)$ of the form

$$(\partial_y^2 - q^2)f = -4\pi \rho_0 [\theta(d' - |y|) - d' \delta(y \pm d')].$$

(4.41)

This has a solution

$$f(y) = \begin{cases} 
\frac{4\pi \rho_0}{q^2} + B \cosh qy & |y| < d' \\
A e^{-q|y|} & |y| > d'.
\end{cases}$$

(4.42)

The function $f(y)$ must be continuous at $y = \pm d'$ and the discontinuity in its first derivative must match the surface charge in Eq. (4.41), $f'(d' + \epsilon) - f'(d' - \epsilon) = 8\pi \rho_0 d'$. This determines constants $A$ and $B$. The full solution for the potential outside the sample reads

$$\Phi(\mathbf{r}, t) = 4\pi \rho_0 \frac{A e^{-q|y|}}{q^2} \cos(qz - \omega t),$$

(4.43)

with $A = \sinh q d' - 2q d \cosh q d' \approx -q d' e^{qd'}$. The electric field that follows from this potential is depicted in Fig. 4.9b. For $d = 1$mm, $u_0 = 0.01a$ and all the other parameters as before the maximum electric field (that occurs right at the sample surface) can be estimated as $|\mathbf{E}| \approx 4\pi \rho_0 e d \approx 2.4 \times 10^4$V/m. This is a large field which should be easily detectable.

In a realistic semimetal we should include screening effects which can substantially reduce the electric field amplitude estimated above. A crude estimate of the screened field can be obtained by replacing $\Phi(q) \rightarrow \Phi(q)/\epsilon(q)$ where $\epsilon(q) = 1 + k_{TF}^2/q^2$ is the dielectric function in the Thomas-Fermi approximation and $k_{TF}^2 = 4\pi e^2 D(\mu)$. It is physically more transparent to write $(k_{TF}/q)^2 = (\lambda_s/\lambda_{TF})^2$ where $\lambda_{TF} = 1/k_{TF}$ is the Thomas-Fermi screening length. Using the experimentally determined electron velocity $v \approx 1.5 \times 10^6$m/s [34] to obtain density of states $D(\epsilon) = \epsilon^2/\pi^2 \hbar^3 v^3$ we find that $\lambda_{TF} \approx 32\mu$m[$1$meV/$\mu$]. Thus, depending on the chemical potential, the screening length can be quite long. For instance if $\mu = 10$meV the screening length $\lambda_{TF} \approx 3.2\mu$m is comparable to $\lambda_s = 11\mu$m and the electric potential will be suppressed
only modestly. Even for the experimentally observed $\mu \approx 200\text{meV}$ [34] the suppression is about a factor of $1.6 \times 10^4$ which still leaves a significant field strength of several V/m at the surface. We conclude that the effect remains measurable even in the presence of realistic levels of screening that can be expected in a Dirac semimetal with the chemical potential not too far from the neutrality point.

4.6.4 Chiral Anomaly in the Absence of EM Fields

We finally mention an attractive possibility of testing the chiral anomaly using purely strain-induced gauge fields and no real EM fields. According to our previous discussion, a simultaneous application of torsion and time-dependent unidirectional strain in a wire geometry generates both $b$ and $e$ pointed along the $z$ direction of the crystal. In this situation the right hand side of the first anomaly equation (4.1) becomes nonzero even in the complete absence of $E$ and $B$ and pumping of charge between the Weyl nodes occurs. The nonequilibrium electron distribution thus created will relax via internodal scattering and produce dissipation of energy. This dissipation is in principle measurable. For instance when the pseudoelectric field $e$ is generated by a longitudinal sound wave its amplitude will be attenuated by this effect and the attenuation coefficient will be proportional to the amount of torsion on the wire. This can be demonstrated by considerations that are similar to those that lead to Eq. (4.36). We shall not repeat this analysis here but simply note that a substantial contribution to the attenuation can be achieved by this effect.

4.7 Summary and Outlook

We studied the chiral anomaly in Weyl semimetals in a new context, when the sign of the anomaly is the same in the two Weyl cones. This takes place when a chiral gauge field is present in addition to the ordinary EM gauge field. Specifically, this
type of chiral anomaly occurs when pseudomagnetic field \( b \), produced by torsion in the material, is present together with the real electric field \( E \). Alternatively, pseudoelectric \( e \) field produced by unidirectional strain combined with a real magnetic field \( B \) gives rise to the anomaly. Contrary to the usually discussed chiral anomaly, density of electron grows in both Weyl cones when the fields are applied, thus making the bulk theory of the material truly anomalous. The apparent contradiction with charge conservation is resolved when one takes into account the surface of the material – fermions are taken from the surface into the bulk.

In the presence of the \( b \cdot E \) term the transfer of charge from the surface to the bulk occurs through the ordinary semiclassical evolution of the electron states in the Brillouin zone. This is facilitated by the phenomenon of spatial segregation of the right and left moving modes between the surface and the bulk of the wire under torsion as discussed in Sec. 4.5.1. In the presence of the \( e \cdot B \) term the situation is different: here the charge transfer occurs through relaxation of the nonequilibrium state that is generated by the chiral anomaly. This disparity in the action of the two types of terms has a very simple physical reason. A uniform \( b \) field can only exist in a finite system with boundaries because it requires an increasing strain field in some spatial directions (just like the uniform \( B \) field requires increasing vector potential \( A \)). However, in a realistic crystal strain can only increase to a certain point after which the crystal breaks. We thus see that a uniform \( b \) necessarily implies the existence of surfaces. A consequence of this is that the band structure of the relevant system will have an equal number of left and right moving modes which are however unbalanced between the surface and the bulk. Semiclassical evolution in the presence of \( E \parallel b \) will thus transfer charge from the bulk to the surface (or vice versa). By contrast the \( e \) field can be created by a time-dependent unidirectional strain which does not require spatial boundaries. We have seen that in a system without boundaries the \( e \cdot B \) term simply changes the chemical potential in accord with the chiral anomaly equation.
However surfaces exist, this creates a nonequilibrium distribution which can relax by transferring charge to the surface. Furthermore, if sufficient number of surface states are available, then the bulk charge density change can be close to that predicted by the chiral anomaly.

When both torsion and unidirectional strain are applied a new form of the chiral anomaly can be created via the $e \cdot b$ term in Eq. (4.1). In this case charge is transferred between the two Weyl points with opposite chirality but remarkably no physical EM fields are required.

Based on these general concepts we make several specific predictions for the experimentally observable signatures of the anomaly. In the case of the $b \cdot E$ term we predict a negative contribution to the resistance that has a square or cubic dependence on the torsion strength, depending on the regime. In the case of the $e \cdot B$ term we predict bulk-boundary charge transfer, resulting in EM field emission and ultrasonic attenuation. Similarly, we predict that the $e \cdot b$ term will contribute to ultrasonic attenuation. These predictions are most clear cut in a semimetal with a single pair of Weyl points. We showed, however, that substantial observable consequences occur also in Dirac semimetals $\text{Cd}_3\text{As}_2$ and $\text{Na}_3\text{Bi}$ whose electronic structure can be viewed as two time-reversed copies of such an elemental Weyl semimetal. On general grounds we also expect these phenomena to be manifested in more complex Weyl semimetals such as $\text{TaAs}$, $\text{ZrTe}_5$ or $\text{WTe}_2$ which host several pairs of Weyl points. Because their electronic structures are complex, detailed quantitative modeling of strain effects will require delving into the details of the band structures and we leave this for future study. We nevertheless anticipate that in these materials each pair of Weyl points will contribute to various chiral anomaly related effects discussed in this work. The contributions will have different magnitudes and signs depending on the relative positions and Fermi velocities of the Weyl cones. Partial cancellations can occur but it appears unlikely that the effect would vanish completely, except
perhaps for very specific strain patterns with high symmetry. Pronounced transport signatures of the ordinary chiral anomaly have already been detected in several materials [13, 154–157] including some with multiple Weyl points. This strongly suggests that the novel strain-induced effects predicted in this work should also be observable in these materials.

Our work draws upon several previous studies. Some of our results regarding the physical consequences of the second anomaly equation (4.2) have been foreshadowed in Ref. [158] which considered Weyl fermions in magnetically doped topological insulators. Here the chiral gauge field can arise from magnetic fluctuations in the system and was predicted to produce one-dimensional chiral modes in a ferromagnetic vortex line and a novel plasmon-magnon coupling. As far as we know Weyl fermions have not yet been observed in magnetically doped topological insulators. Also, it may be challenging to create and control the magnetic textures envisioned in Ref. [158]. By contrast the phenomena predicted in our work only require existing materials, such as Cd$_3$As$_2$ or Na$_3$Bi. Also, producing the chiral gauge field from strain is not expected to pose an exceptional experimental challenge. Our work also draws upon the results of Refs. [133, 135, 149] which established the equivalence between strain and chiral gauge field in various materials ranging from graphene and topological insulators to a simple model of a Weyl semimetal. Our study however goes far beyond the scope of Ref. [133] by considering the effect of strain in specific materials and geometries and by providing concrete quantitative predictions for experimentally measurable quantities related to the chiral anomaly.

When our work was substantially completed we became aware of a preprint [159] which discussed a fictitious magnetic field in a Weyl semimetal created by crystal dislocations. This effect is closely related to our $b$ field but is different in that unlike externally applied strain, dislocations in a crystal cannot be easily controlled. Therefore, experimental detection of this effect may prove challenging. Very recently
Schuster et al. [160] discussed the concept of a topological coaxial cable in a gapped Weyl semimetal with a vortex in the Higgs field that is responsible for the gap. In this situation the vortex line is predicted to carry protected fermionic modes and contribute exactly quantized conductance. This effect is very interesting but also very different from our concept of topological coaxial cable which occurs in an ungapped Dirac or Weyl semimetal and does not in general produce quantized conductance.

Given both the fundamental nature of the new anomaly discussed here and its obvious potential for future applications, we envision numerous possible extensions of this work. On the theory side there are multiple questions that one can ask: Which of the EM effects in solids translate to pseudo-EM fields discussed here? What are the best materials to study the effects? Do the chiral states predicted by our work have prospects for designing more exotic many-body states in the presence of interactions? We also expect experimental activity to be stimulated since our predictions made for real materials yield effects that should be both unusual and eminently observable by conventional experimental probes such as charge transport, ultrasonic attenuation and EM field emission.
Chapter 5

Holographic Black Hole on a Graphene Flake

5.1 Overview

Originally proposed to settle a debate over the black hole information paradox [161], the holographic principle has evolved into a universal conjecture that the information content of all the physics in a volume of spacetime is bounded by the surface area [162]. Relating spacetime geometry to number of quantum states, it opens up a new direction for the unification of quantum mechanics and gravity. If the conjecture continues to withstand the test of time, a revolution in physics may be in sight because the holographic principle challenges the principle of locality fundamental to existing theories.

The most successful example of the holographic principle is the AdS/CFT correspondence, which is the duality between string theories in \((n+1)\)-dimensional anti-de Sitter (AdS) spacetime and conformal field theories (CFT) defined on the \(n\)-dimensional spacetime boundary [163]. It is a concrete starting point from which the more general holographic principle is probed. Moreover, it provides a novel
approach for tackling mathematically intractable problems. For example, strongly coupled quantum field theories can be translated with one-to-one correspondence to weakly interacting string theories on the other side of the duality. It has also sparked interaction among the string theory, nuclear physics, and condensed matter communities.

The Sachdev-Ye-Kitaev (SYK) model, a fermionic system with all-to-all random interactions, is recently shown to be dual to a planar charged black hole [164–167]. More specifically, the SYK model in the strong coupling regime displays an emergent \((0+1)\)-dimensional CFT, dual to a charged black hole in the \((1+1)\)-dimensional AdS\(_2\) spacetime described by the Einstein-Maxwell-dilaton (EMD) theory, a quantum gravity theory. Remarkably the strongly interacting SYK model can be solved exactly when the number of fermionic modes \(N\) approaches infinity. It is shown to be maximally chaotic and exhibit finite entropy at zero temperature, which are characteristic of black holes. Outside of its intriguing connection with black holes, the SYK model is a fascinating quantum system in its own right. It is a non-Fermi liquid and potentially a microscopic model for the strange metal phase in hole-doped cuprates.

The SYK models can be written in terms of either Majorana fermions or ordinary complex fermions, with the latter having an additional \(U(1)\) charge conserving symmetry. Since Kitaev’s work mostly concerns the Majorana SYK model, we dub the complex SYK model the Sachdev-Ye (SY) model. In this project, we propose an experimental realization of the SY model using graphene. Consider a mesoscopic graphene flake with irregular boundary, as shown in Fig. 5.1. In the presence of a strong perpendicular magnetic field, Dirac Landau levels are formed in the bulk. In particular, the zeroth Landau level is highly degenerate with the number of zero-energy modes proportional to the total flux through the flake. The wavefunctions of the zero modes are randomly distributed over the flake in response to the irregular boundary. Since the highly disordered wavefunctions overlap significantly for a suf-
sufficiently small flake, one can imagine that the interaction arising from the Coulomb repulsion (as projected onto the zero modes) would be random and all-to-all. Indeed our numerical simulation shows that the four-fermion coupling constants are complex random variables with a distribution similar to the Gaussian distribution used in the theoretical SY models.

Figure 5.1: An irregular shaped graphene flake in an applied magnetic field gives rise to the (0+1)-dimensional SY model, holographically dual to a black hole in (1+1)-dimensional anti-de Sitter space. Inset: lattice structure of graphene with A and B sublattices marked and nearest neighbor vectors denoted by $\delta_j$.

Furthermore, our proposal reproduces known signatures of the SY models. We show that the entropy dependence on temperature agrees well with the Gaussian-randomized SY model consisting of a finite $N$ number of degenerate modes. The limit $N \to \infty$ needed for achieving a nonvanishing zero-temperature entropy cannot be simulated numerically. However, we argue that with a reasonably strong field, a much larger $N$ can be obtained in experiments. The quantum chaotic nature of the SY model is revealed when one compares its many-body spectral statistics with that of the random matrices generated by the Gaussian orthogonal (GOE), unitary (GUE), and symplectic (GSE) ensembles. We show that the distribution of the ratios of consecutive level spacings follows the Wigner-Dyson level statistics in a $N(\text{mod } 4)$ cycle, as predicted by previous studies [168].

Lastly, we investigate various issues present in realistic experimental settings, such as impurities and second-nearest-neighbour hoppings. After examining their
effects through numerical studies, we conclude that the SY physics remains robust
given a strong field, low temperature, and a graphene flake with irregular boundary
and clean interior. Our proposal is relatively simple when compared to the others
which involve superconductivity, advanced fabrication techniques, or ultra-cold gases
[169–171]. Furthermore, follow-up research has indicated that the non-Fermi liquid
properties can been verified in transport experiments [172].

5.2 Definition of Sachdev-Ye Model

The Sachdev-Ye (SY) model [173–176], is defined by the second-quantized Hamiltonian

\[
\mathcal{H}_{\text{SY}} = \sum_{ij;kl} J_{ij;kl} c_j^\dagger c_k^\dagger c_l c_i - \mu \sum_j c_j^\dagger c_j, \tag{5.1}
\]

where \( c_j^\dagger \) creates a spinless fermion, \( J_{ij;kl} \) are zero-mean complex random variables
satisfying \( J_{ij;kl} = J_{kl;ij}^* \) and \( J_{ij;kl} = -J_{ji;kl} = -J_{ij;lk} \) and \( \mu \) denotes the chemical
potential. In what follows, we derive the effective low-energy Hamiltonian for electrons
in the zeroth Landau level (LL\(_0\)) of a graphene flake with an irregular boundary and
show that, under a broad range of conditions, it is given by Eq. (5.1). The system,
therefore, realizes the SY model.

5.3 Graphene Flake in a Magnetic Field

5.3.1 Tight-Binding Model

At the non-interacting level a flake of graphene is described by a simple tight-binding
Hamiltonian [24]

\[
H_0 = -t \sum_{r,\delta} (a_r^\dagger b_{r+\delta} + \text{h.c.}), \tag{5.2}
\]
where \( a_r^\dagger \) (\( b_{r+\delta}^\dagger \)) denotes the creation operator of the electron on the sublattice A (B) of the honeycomb lattice. These satisfy the canonical anticommutation relations
\[
\{a_r^\dagger, a_r\} = \{b_r^\dagger, b_r\} = \delta_{rr'},
\]
appropriate for fermion operators. \( r \) extends over the sites in sublattice A while \( \delta \) denotes the 3 nearest neighbor vectors (inset Fig. 5.1). \( t = 2.7 \) eV is the tunneling amplitude [177]. For simplicity we first ignore electron spin but reintroduce it later. The chiral symmetry \( \chi \) is generated by setting \((a_r, b_r) \rightarrow (-a_r, b_r)\) for all \( r \) which has the effect of flipping the sign of the Hamiltonian \( H_0 \rightarrow -H_0 \).

Magnetic field \( B \) is incorporated in the Hamiltonian (5.2) by means of the standard Peierls substitution which replaces \( t \rightarrow t_{r,r+\delta} = t \exp \left[ -i(e/\hbar c) \int_{r+\delta}^r A \cdot dl \right] \) where \( A \) is the vector potential \( B = \nabla \times A \). In the presence of \( \chi \) the Aharonov-Casher construction [178] implies \( N = N_\Phi \) exact zero modes in the spectrum of \( H_0 \) where \( N_\Phi = SB/\Phi_0 \) denotes the number of magnetic flux quanta \( \Phi_0 = \hbar c/e \) piercing the area \( S \) of the flake. It is clear that a flake with an arbitrary shape described by \( H_0 \) respects \( \chi \) which underlies the robustness of LL\(_0\) invoked above.

Hopping \( t' \) between second neighbor sites and random on-site potential are examples of perturbations that break \( \chi \) and are therefore expected to broaden LL\(_0\). These effects can be modeled by adding to \( \mathcal{H}_{SY} \) defined in Eq. (5.1) a term
\[
\mathcal{H}_2 = \sum_{ij} K_{ij} c_i^\dagger c_j
\]
which describes a small (undesirable) hybridization between the states in LL\(_0\) that will generically be present in any realistic experimental realization. We discuss the effect of these terms in Sec. 5.5.2.

### 5.3.2 Energy Spectrum and Zero-Mode Wavefunctions

In Fig. 5.2a we show the single-particle energy spectrum of \( H_0 \) for a graphene flake with a shape depicted in the inset. As a function of increasing magnetic field \( B \) we
observe new levels joining the zero-energy manifold \( \text{LL}_0 \) such that the number of zero modes follows \( N \approx N_\Phi \) in accordance with the Aharonov-Casher argument. Higher Landau levels and topologically protected edge modes are also visible. Despite the randomness introduced by the irregular boundary, \( \text{LL}_0 \) remains sharp as expected on the basis of the arguments presented above. This is the key feature in our construction of the SY Hamiltonian, which guarantees that the \( \mathcal{H}_2 \) term defined above vanishes as long as the chiral symmetry is respected. In the presence of e-e repulsion, the leading term in the effective description of \( \text{LL}_0 \) will therefore be a four-fermion interaction which we discuss in Sec. 5.3.3.

Electron wavefunctions \( \Phi_j(\mathbf{r}) \) belonging to \( \text{LL}_0 \) exhibit random spatial structure (Fig. 5.2b) owing to the irregular confining geometry imposed by the shape of the flake.

Figure 5.2: Electronic properties of an irregular graphene flake in the absence of interactions. a) Single-particle energy levels \( \epsilon_j \) of the Hamiltonian \( H_0 \) as a function of the magnetic flux \( \Phi = SB \) through the flake. The flake used for this calculation, depicted in the inset, consists of 1952 carbon atoms with equal number of A and B sites. The energy spectrum, calculated here in the Landau gauge \( \mathbf{A} = Bx\hat{y} \) and with open boundary conditions, shows the same generic features irrespective of the detailed flake geometry. b) Typical wavefunction amplitudes of the eigenstates \( \Phi_j(\mathbf{r}) \) belonging to \( \text{LL}_0 \) at \( \Phi = 40\Phi_0 \) and the edge modes. The numerals above each panel denote the energy \( \epsilon_j \) of the state in eV, scale bar shows the magnetic length \( l_B = \sqrt{\hbar c/eB} \).
5.3.3 Interaction Matrix Between Zero Modes

From the knowledge of the wavefunctions it is straightforward to evaluate the corresponding interaction matrix elements \(^1\) between the zero modes. The leading many-body Hamiltonian for electrons in LL\(_0\) will thus have the form of Eq. (5.1) with

\[ J_{ijkl} = \frac{1}{2} \sum_{r_1, r_2} \langle \Phi_i(r_1) \Phi_j(r_2) | V(r_1 - r_2) | \Phi_k(r_1) \Phi_l(r_2) \rangle, \tag{5.4} \]

where \( V(r) = (e^2/\epsilon r) e^{-r/\lambda_{TF}} \) is the screened Coulomb potential with Thomas-Fermi length \( \lambda_{TF} \) and dielectric constant \( \epsilon \). The summation extends over all sites of the honeycomb lattice. It is to be noted that only the part of \( J_{ijkl} \) antisymmetric in \((i,j)\) and \((k,l)\) contributes to the many-body Hamiltonian (5.1) so in the following we assume that \( J_{ijkl} \) has been properly antisymmetrized.

We numerically evaluated \( J_{ijkl} \) for various values of \( \lambda_{TF} \). The resulting \( J \)s are complex valued random variables satisfying

\[ \overline{J_{ijkl}} = 0, \quad |J_{ijkl}|^2 = \frac{1}{2N^3} J^2, \tag{5.5} \]

where \( J \) measures the interaction strength and the bar denotes averaging over randomness introduced by the irregular confining geometry. Fig. 5.3a shows the statistical distribution of \( J_{ijkl} \) calculated for the nearest-neighbor interactions \( V(r) = V_1 \sum_{\delta} \delta_{r,\delta} \) and the single-particle wavefunctions \( \Phi_j(r) \) depicted in Fig. 5.2b. The distribution of \( J_{ijkl} \) shows the expected randomness with some deviations from the ideal Gaussian.

\(^1\)See Appendix D for more details.
Figure 5.3: Statistical properties of the coupling constants and the thermal entropy. 
a) Histogram of $|J_{ijkl}|$ as calculated from Eq. (5.4) with $V_1 = 1$ for the graphene flake 
depicted in Fig. 5.2 and $N = 16$, compared to the Gaussian distribution (orange line) 
with the same variance $0.000805V_1$. Inset shows the histogram of real and imaginary 
components of $J_{ijkl}$. The mirror symmetry about the horizontal follows from the 
hermiticity property $J_{ijkl} = J_{kl}^{*}$. b) Entropy $S(T)$ of the SY Hamiltonian (5.1) 
calculated with $J$s shown in panel (a).

5.4 Exact Diagonalization of Many-Body Hamiltonian

To show that the low-energy fermions in the graphene flake are described by the SY 
model, we next perform numerical diagonalization of the many-body Hamiltonian 
(5.1) with coupling constants $J_{ijkl}$ obtained as described above. We then calculate
physical observables and compare them to the results obtained with random independent \( J_{ijkl} \).

### 5.4.1 Entropy

The SY model is known to exhibit non-zero ground state entropy per particle in the thermodynamic limit \( N \to \infty \), so we are motivated to compute the entropy of the flake. Given the many-body energy levels \( E_i \), we compute the partition function \( Z = \sum_i e^{-E_i/T} \), the free energy \( F = -k_B T \ln Z \), and the total energy \( \langle E \rangle = \sum_i E_i e^{-E_i/T} / Z \). Then entropy can be found using the thermodynamic law \( F = \langle E \rangle - TS \). Fig. 5.3b shows the thermal entropy per particle \( S(T)/N \) of the flake. At high temperature, it goes to \( \ln 2 \) as expected from the fact that each zero mode is equally likely to be occupied or unoccupied. At low temperature, our numerical simulation of 16 zero modes cannot demonstrate a nonzero entropy because \( S(T \to 0) \) vanishes for any finite \( N \) [179]. Nevertheless, the entropy calculated with random Gaussian \( J_{ijkl} \) indicates no significant difference, which is a good sign that our flake may indeed exhibit SY physics.

### 5.4.2 Level Statistics

Many-body energy level statistics provide another useful tool to validate our hypothesis that LL_0 electrons in the graphene flake behave according to the SY model. We thus arrange the energy eigenvalues \( E_n \) of the many-body Hamiltonian (5.1) in increasing order and form ratios of the subsequent levels \( r_n = (E_{n+1} - E_n)/(E_n - E_{n-1}) \). According to the random matrix theory applied to the SY model [168] probability distributions \( P(\{r_n\}) \) are given by different Gaussian ensembles, depending on \( N(\text{mod } 4) \) and the eigenvalue \( q \) of the total charge operator \( Q = \sum_j (c_j^\dagger c_j - 1/2) \) as summarized in Table 5.1. Here GOE, GUE and GSE stand for Gaussian orthogonal,
unitary and symplectic ensembles, respectively and

\[ P(r) = \frac{1}{Z} \frac{(r + r^2)^\beta}{(1 + r + r^2)^{1+3\beta/2}}, \tag{5.6} \]

with constants \( Z \) and \( \beta \) listed in Table 5.1. Since \( H_{\text{SY}} \) commutes with \( Q \) it can be block diagonalized in sectors with definite charge eigenvalue \( q \). As emphasized in Ref. [168], the level statistics analysis must be performed separately for each \( q \)-sector. Note that \( q \) has integer (half-integer) values for \( N \) even (odd) and this is why the neutrality condition \( q = 0 \) can be met only for even values of \( N \). Also note that \( q = 0 \) corresponds to \( N/2 \) particles.

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<tr>
<td>( q = 0 )</td>
<td>GOE</td>
<td>GSE</td>
<td></td>
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<tr>
<td>( q \neq 0 )</td>
<td>GUE</td>
<td>GUE</td>
<td>GUE</td>
<td>GUE</td>
</tr>
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</table>

Table 5.1: Gaussian ensembles for the SY model. The relevant probability distributions are given by Eq. (5.6) with \( Z = \frac{8}{27}, \frac{4\pi}{81\sqrt{3}}, \frac{4\pi}{729\sqrt{3}} \) and \( \beta = 1, 2, 4 \) for GOE, GUE, GSE, respectively.

Fig. 5.4 shows our results for the level statistics performed for a graphene flake with \( N = 14 \) through 18 and various values of \( q \). The obtained level spacing distributions are seen to unambiguously follow the prediction of the random matrix theory for the SY model summarized in Table 5.1. We are thus led to conclude that interacting electrons in \( \text{LL}_0 \) of a graphene flake with an irregular boundary indeed exhibit spectral properties characteristic of the SY model.
Figure 5.4: Many-body level statistics for the interacting electrons in LL$_0$ of the graphene flake. Blue bars show the calculated distributions for the graphene flake. Orange, green and red curves indicate the expected distributions given by Eq. (5.6) for GOE, GUE and GSE, respectively. To obtain smooth distributions, results for $N = 14, 15, 16$ have been averaged over 8 (4) distinct flake geometry realizations while $N = 17, 18$ reflect a single realization.

5.5 Further Considerations

We now discuss various aspects of the problem relevant to the laboratory realization.

5.5.1 Electron Spins

Electrons in graphene possess spin which we so far ignored. Given the weak spin-orbit coupling in graphene we may model the non-interacting system by two copies of the Hamiltonian Eq. (5.2) plus the Zeeman term, $H = H_0 + g^* \mu_B B \cdot S_{\text{tot}}$ where $S_{\text{tot}}$ is the total spin operator and $\mu_B = 5.78 \times 10^{-5}$ eV/T is the Bohr magneton. For graphene
on the SiO$_2$ or hBN substrate we may take $g^* \simeq 2$ which gives the bare Zeeman splitting $\Delta E_S(B) \simeq 0.12$ meV/T, or about 2.4 meV at $B = 20$ T. We expect this relatively small spin splitting to be significantly enhanced by the exchange effect of the Coulomb repulsion. The strength of the exchange splitting $\Delta E_C \simeq 8.8$ meV/T is estimated in Appendix D. For such a large spin splitting one may focus on a partially filled LL$_0$ for a single spin projection and disregard the other. The spinless model considered so far should therefore serve as an excellent approximation of the physical system in the strong field.

5.5.2 Chiral-Symmetry-Breaking Disorder

Disorder that breaks chiral symmetry will inevitably be present in real graphene samples. Such disorder tends to broaden LL$_0$ and compete with the interaction effects that underlie the SY physics. It is known that bilinear terms $\mathcal{H}_2$ that arise from such disorder constitute a relevant perturbation to $\mathcal{H}_{SY}$ and drive the system towards a disordered Fermi liquid (dFL) ground state. In Appendix E we analyze the symmetry-breaking effects and estimate their strength in realistic situations. We conclude that in carefully prepared samples a significant window should remain open at non-zero temperatures and frequencies in which the system exhibits behavior characteristic of the SY model.

5.6 Prospect of Experimental Realization

An ideal sample to observe the SY physics is a graphene flake with a highly irregular boundary and clean interior. These conditions promote random spatial structure of the electron wavefunctions and preserve degeneracy of LL$_0$. Disordered wavefunctions give rise to random interaction matrix elements $J_{ij;kl}$ while near-degeneracy of states in LL$_0$ guarantees that the two-fermion term $\mathcal{H}_2$ remains small. To observe signatures
of the emergent black hole the LL$_0$ degeneracy $N = SB/\Phi_0$ must be reasonably large—numerical simulations indicate that $N \gtrsim 10$ is required for the system to start showing the characteristic spectral features. Aiming at $N \simeq 100$, which is well beyond what one can conceivably simulate on a computer, implies the characteristic sample size $L \simeq \sqrt{S} = \sqrt{N\Phi_0/B} \simeq 150$ nm at $B = 20$ T. Signatures of the SY physics can be observed spectroscopically, e.g. by the differential tunneling conductance $g(V) = dI/dV$ which is predicted [169] to exhibit a characteristic square-root divergence $g(V) \sim |V|^{-1/2}$ in the SY regime at large $N$, easily distinguishable from the dFL behavior $g(V) \sim \text{const}$ at small $V$. We predict that a tunneling experiment will observe the SY behavior when the chemical potential of the flake is tuned to lie in LL$_0$ and dFL behavior for all LL$_n$ with $n \neq 0$. We also expect the two-terminal conductance across the flake to show interesting behavior in the SY regime [172].

In the limit of a large flake the irregular boundary will eventually become unimportant for the electrons in the bulk interior and the system should undergo a crossover to a more conventional “clean” phenomenology characteristic of graphene in applied magnetic field. The exact nature of this crossover poses an interesting theoretical as well as experimental problem which we leave to future study.
Chapter 6

Conclusion

This Thesis shows that topological materials can be testing grounds for high-energy physics ideas, especially those that may be keys to unsolved fundamental problems. This alternative approach for probing high-energy physics is an appealing opportunity because direct observation or detection is often difficult. Specifically, we illustrated three scenarios in which high-energy physics ideas are realized in topological semimetals. In Chapter 3, we proposed a superconductor-Weyl-semimetal-superconductor junction that can host Majorana flat bands at $\pi$-phase difference. Majorana flat bands are ideal for studying the interaction effects of Majorana fermions, which are favorable candidates for solving some crucial mysteries in the Standard Model. We further demonstrated that Josephson current measurement is a promising and experimentally feasible venue for detecting these flat bands. In Chapter 4, we investigated chiral anomaly present in Weyl semimetals subject to a combination of electromagnetic fields and mechanical strains. One key feature is the anomalous nonconservation of electron charges inside the material bulk, balanced out by the surface charges. Such phenomenon does not happen in the realm of high-energy physics since the Universe has no boundary. It is fascinating that solid-state systems can reveal more properties of a well-established high-energy concept. Furthermore,
in quantitative details, we pictured several novel experimental manifestations which could have potential technological applications. In Chapter 5, we provided evidences that a mesoscopic graphene flake subject to a perpendicular magnetic field gives rise to the SY model, a strongly correlated fermionic quantum dot holographically dual to a (1+1)-dimensional black hole. To our current knowledge, our proposal is the most experimentally feasible testing ground for the holographic conjecture.

Throughout the Thesis, we leaned heavily on numerical simulations to bring our theoretical proposals closer to experimental reality. The effects of realistic laboratory conditions - such as finite temperatures and impurities - were gauged by numerical analysis. Nevertheless, numerical approach is limited by computational power and simplifying assumptions in the theories. Real solid-state systems are extremely complex, and laboratory environments difficult to control. It is hard to predict what actually happens when one attempts the proposed experiments. Therefore to realize our proposals we must work closely with the experimentalists. Thus far our on-going discussions with experimental groups have already refined our interpretation of results. Once realized, these experiments could either confirm the physics we know or challenge our current understanding through unexpected findings.

Our work also begets more theoretical investigations. How can the interactions among Majorana zero modes be explicitly modeled? Would strain-induced chiral anomaly in the presence of interactions give rise to exotic many-body states? From Chapter 5, it seems that a lot more can be done along the line of “experimental holographic principle.” Can we find quantum matter systems that are holographically dual to black holes in higher dimensions, preferably the (3+1)-dimensional spacetime that we live in? What about other mysterious gravitational objects such as the wormholes [180, 181]? From the reverse perspective, how can we better understand strongly-correlated systems such as the high-Tc superconductors using the holographic
principle? These open questions may just lead us one step closer to unifying the disjointed worlds of high-energy and quantum matter physics.
Bibliography


Appendices

Appendix A: Tight-Binding Model, Dispersion Relations, and Parameters for Cd₃As₂ and Na₃Bi

From the low energy \( \mathbf{k} \cdot \mathbf{p} \) Hamiltonian (4.5) we construct the requisite lattice model by replacing \( A_{\pm} \rightarrow (A/a)(\sin ak_x \pm i \sin ak_y) \), \( C_1 k^2_z \rightarrow (2C_1/a^2)(1 - \cos ak_z) \), etc. For example \( \epsilon_0(k) \) defined below Eq. (4.5) becomes

\[
\epsilon_k = c_0 + c_1 \cos ak_z + c_2 (\cos ak_x + \cos ak_y), \tag{A.1}
\]

with \( c_0 = C_0 + 2(C_1 + 2C_2)/a^2 \), \( c_1 = -2C_1/a^2 \) and \( c_2 = -2C_2/a^2 \). The constants \( c_j \) are chosen such that \( \epsilon_k \) matches \( \epsilon_0(k) \) for small \( ak \) independent of the chosen value of the lattice constant \( a \). Treating all other terms in the Hamiltonian (4.5) in the similar fashion leads to the lattice Hamiltonian given by Eqs. (4.7) and (4.8).

In addition to the results presented in the main text, we performed detailed bandstructure simulations for Dirac semimetals Cd₃As₂ and Na₃Bi. Parameters for the model Hamiltonian (4.5) are taken from [32, 121] and [139] correspondingly and are summarized in Table A.1. In the main text we only presented results for the parameters of Cd₃As₂ with the asymmetry parameters \( C_i \) set to zero. In the Fig. A.1 we present the dispersion relation computation for the models of \( \frac{1}{2}\text{-Cd}_{3}\text{As}_{2} \) and \( \frac{1}{2}\text{-Na}_{3}\text{Bi} \) with all the asymmetry terms taken into account. The effects discussed in
the main text are present even in this more general case although to see them clearly now requires more effort due to the more complicated structure of the energy bands. For instance the equivalence of the torsional strain and magnetic field, pointed out in the main text, here can be only identified by a trained eye. One needs to notice that the right Weyl point is at $E = 0$ in the rightmost graph of the first column of Fig. A.1. Results between the two parameter sets are similar, but notice the larger gaps in Na$_3$Bi, which may make the experimental realization easier.

<table>
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<tr>
<th></th>
<th>Cd$_3$As$_2$</th>
<th>Na$_3$Bi</th>
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<tbody>
<tr>
<td>$C_0$ [eV]</td>
<td>-0.0145</td>
<td>-0.0638</td>
</tr>
<tr>
<td>$C_1$ [eVÅ$^2$]</td>
<td>10.59</td>
<td>8.75</td>
</tr>
<tr>
<td>$C_2$ [eVÅ$^2$]</td>
<td>11.5</td>
<td>-8.4</td>
</tr>
<tr>
<td>$M_0$ [eV]</td>
<td>0.0205</td>
<td>0.869</td>
</tr>
<tr>
<td>$M_1$ [eVÅ$^2$]</td>
<td>-18.77</td>
<td>-10.64</td>
</tr>
<tr>
<td>$M_2$ [eVÅ$^2$]</td>
<td>-13.5</td>
<td>-10.36</td>
</tr>
<tr>
<td>$A$ [eVÅ]</td>
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<td>2.46</td>
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<tr>
<td>$a$ [Å]</td>
<td>20</td>
<td>7.5</td>
</tr>
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</table>

Table A.1: Material parameters taken from Refs. [121] and [139]. The last row represents the effective lattice constant used for our numerical simulations.
Figure A.1: Dispersion relations for the spin-up sector of the lattice Hamiltonian (4.7) describing Cd$_3$As$_2$ (top row) and Na$_3$Bi (bottom row). The parameters used in the simulations include the particle-hole symmetry breaking terms and are summarized in Table A.1. We used a lattice with $40 \times 40$ sites and the magnetic fields shown in the green boxes for each of the material. Notice the different magnitude of effective magnetic fields for different compounds – this is due to the different lattice constants, and different sign of the physical magnetic field between the two rows. Different sign of magnetic fields shows that the physical magnetic field compensates the torsional one in opposite Weyl points for opposite directions of magnetic field in accordance with the interpretation in the main text.

To further confirm the validity of our ideas relating the torsional strain to the pseudomagnetic field we computed the equilibrium currents flowing along the wire using the full Hamiltonian including the p-h symmetry breaking terms. The results are as follows: (i) For both Cd$_3$As$_2$ and Na$_3$Bi we find equilibrium currents with the pattern similar to the one displayed in Fig. 4.8 in each spin sector when nonzero torsion is present. The total current density (summing up contributions from both spin up and down sectors) vanishes, as it must be in a $\mathcal{T}$-invariant system. (ii) When only magnetic field $B$ is present and no torsion, the current densities are zero in both sectors separately, in accord with the expectation. (In this case the band structure in each spin sector shows the same number of left and right moving modes in the
bulk of the system). (iii) When both torsion and magnetic field are present then we find non-zero persistent current density in both spin sectors. In this case $\mathcal{T}$ is absent and the currents from the two sectors generically do not cancel. This is illustrated in Fig. A.2. We observe an asymmetric band structure that supports different number of bulk left and right moving modes at various energies, leading to a net imbalance in the current flow between the bulk and the surface. The total current carried by the wire however still vanishes.

![Figure A.2](image)

Figure A.2: Persistent currents in a Cd$_3$As$_2$ nanowire under torsion and magnetic field. a) Band structure detail for spin up (blue) and spin down (green) sectors in a $30 \times 30$ lattice with $B = b = 3.2T$ and other parameters as in Fig. A.1. b) Calculated current density $j_z$ for $\mu = 0$ including contributions from both spin sectors.
Appendix B: Nonequilibrium Distribution in a Stretched Weyl Semimetal

In this appendix, we derive a quantitative estimate for the chemical potential denoted by $\mu_{eq}'$ in Fig. 4.5 as well as the corresponding bulk electron density. As discussed in Sec. 4.5.2, upon introducing strain the chemical potential in the bulk of the system rises from $\mu_0$ to $\mu'$ while that at the surfaces remains unchanged. In the process, the charge density also remains unchanged (except possibly for perturbations near the surface that average to zero and do not affect the bulk). This creates a nonequilibrium distribution of electrons illustrated in Fig. 4.5b which then relaxes to a new equilibrium characterized by a global chemical potential $\mu_{eq}'$. The latter can be calculated by demanding that the total electron number $N$ is conserved.

In the unstrained system we have $N = N_s + N_b$ where the subscripts refer to the surface and the bulk, respectively. In the nonequilibrium strained system $N_s$ and $N_b$ remain the same as per our discussion above. In the new equilibrium they change to $N_{s/b}' = N_{s/b} + \delta N_{s/b}$, where $\delta N_{s/b} \simeq \kappa_{s/b} \delta \mu_{s/b}$. Here $\kappa_{s/b} = dN_{s/b}/d\mu$ is the compressibility and $\delta \mu_{s/b}$ denotes the change in the chemical potential that is responsible for the change in $N_{s/b}$. Number conservation dictates that $\delta N_s = -\delta N_b$ which implies

$$\kappa_s (\mu_{eq}' - \mu_0) = -\kappa_b (\mu_{eq}' - \mu').$$  \hspace{1cm} (B.1)

We can solve for $\mu_{eq}'$ to obtain

$$\mu_{eq}' = \mu_0 + \frac{\delta \mu}{1 + \kappa_s/\kappa_b},$$  \hspace{1cm} (B.2)
where \( \delta \mu = \mu' - \mu_0 \). The corresponding change in the bulk number is \( \delta N_b = \kappa_b(\mu'_\text{eq} - \mu') \) which, together with Eq. (B.2) gives the change in the bulk density

\[
\delta \rho^\text{bulk} = -\frac{1}{Sd} \frac{\kappa_b}{1 + \kappa_b/\kappa_s} \delta \mu, \tag{B.3}
\]

where \( S \) is the area of the slab.

To complete the calculation we require the surface and bulk electron compressibilities. For the surface we assume that we have a single linearly dispersing band \( \epsilon^s_k = \hbar v k_x \) on each surface that extends between the surface projections of the two Weyl points, \( |k_z| < Q \). We furthermore assume that the surface state is essentially unaffected by the magnetic field, in accord with the results of our lattice simulations. This gives

\[
\kappa_s = SD_s(\mu) = \frac{SQ}{\pi^2 \hbar v}, \tag{B.4}
\]

where \( D_s(\epsilon) = Q/\pi^2 \hbar v \) is the surface density of states (counting both surfaces).

For the bulk we similarly have \( \kappa_b = SdD_b(\mu) \). We now must distinguish between the quantum and the semiclassical limits, as defined in Sec. 4.6.2. In the quantum limit we have a pair of linearly dispersing \( n = 0 \) Dirac Landau levels with degeneracy \( (B/\Phi_0) \) whereas in the semiclassical limit many Landau levels are populated so it is permissible to approximate the density of states by that of a zero-field system. We thus obtain

\[
\kappa_b = \begin{cases} 
\frac{Sd}{\pi \hbar v} \left( \frac{B}{\Phi_0} \right) & \text{quantum limit} \\
\frac{Sd \mu^2}{\pi^2 \hbar v^3} & \text{semiclassical limit}.
\end{cases} \tag{B.5}
\]

Substituting these results into Eqs. (B.2) and (B.3) we obtain results for \( \mu'_\text{eq} \) and \( \delta \rho^\text{bulk} \) quoted in the main text (Eqs. 4.20 and 4.21) for the quantum limit. In the semiclassical limit, we similarly obtain

\[
\mu'_\text{eq} = \mu_0 + \frac{\delta \mu}{1 + \lambda_Q/d}, \tag{B.6}
\]
and

\[ \delta \rho^{\text{bulk}} = -\frac{\alpha}{\pi a} \left( \frac{B}{\Phi_0} \right) \frac{\cot aQ}{1 + d/\lambda_Q}, \]  

(B.7)

where \( \lambda_Q = Q(\hbar v/\mu)^2/2 \) is the characteristic length.
Appendix C: Hydrodynamic Flow in a Twisted Weyl Nanowire

Consider a cylindrical nanowire of radius \( R \) made of a Weyl semimetal. Both torsion and electric field \( E \) are applied along the axis of the wire (taken here along the \( \hat{z} \) direction), giving a non-zero right hand side \( \propto b \cdot E \) in the second anomaly equation (4.2). Denoting the right hand side by \( g(r) \), we may write

\[
\partial_t \rho + \nabla \cdot j = g(r),
\]

where

\[
g(r) = g_0 \left[ \theta(R - r) - \frac{1}{2} R \delta(r - R) \right],
\]

with \( g_0 = \left( e^2 / 2\pi^2 \hbar^2 c \right) b E \). The first term in \( g(r) \) describes uniform production of electrons in the bulk of the wire at a rate given by the chiral anomaly. The second term reflects the fact that those electrons are removed from the surface, in accord with our discussion in Sec. 4.5. The total production in the wire is zero, \( \int_0^{R^+} r \, dr \, g(r) = 0 \), and the charge is conserved.

We now assume that the dominant relaxation mechanism for the nonequilibrium electrons produced in the bulk of the wire is diffusion towards the boundary. Electrons move ballistically along the \( \hat{z} \) direction and undergo occasional collisions that scatter them into neighboring Landau level states. Near the boundary bulk electrons can finally backscatter into the surface modes which are moving in the opposite direction. Under this assumption the diffusion current is

\[
j = -D \nabla \rho,
\]

with

\[
\rho = g(r) - \frac{1}{2} \left( b \cdot E \right) R \delta(r - R).
\]
where \( D = \ell_b^2/\tau_0 \) is the diffusion constant (\( \ell_b = \sqrt{\hbar c/e\phi} \) is the magnetic length and \( \tau_0^{-1} \) the microscopic scattering rate). The form of the diffusion constant reflects the fact that electron wavefunctions have Landau level character with the spatial extent \( \ell_b \) in the direction perpendicular to \( \hat{z} \) and scattering occurs predominantly between neighboring Landau level orbitals.

Substituting Eq. (C.3) into (C.1) and specializing to long time steady state with \( \partial_t \rho = 0 \), we obtain

\[
-D \nabla^2 \rho = g(r). \tag{C.4}
\]

Writing the Laplacian in the polar coordinates and assuming radially symmetric solution, we find

\[
\rho(r) = \frac{g_0}{D} \left( \frac{R^2 - r^2}{4} \right) \theta(R - r). \tag{C.5}
\]

The corresponding radial diffusion current density is \( j_r(r) = -D \partial_r \rho = \frac{1}{2}g_0r \). The total non-equilibrium charge in the bulk modes is

\[
\delta Q = -e \int_0^R dr 2\pi r \rho(r) = -e \frac{\pi}{8} \frac{g_0}{D} R^4. \tag{C.6}
\]

Since all these modes move in the same direction with velocity \( v \) this gives the total current along the \( \hat{z} \) direction in the wire,

\[
J_{\text{CTE}} = 2v\delta Q = -ev\frac{\pi}{4} \left( \frac{e^2}{2\pi^2 \hbar^2 c} \right) \frac{\tau}{\ell_b^2} R^4 bE. \tag{C.7}
\]

The factor 2 in the first equality reflects the fact that non-equilibrium charge \(-\delta Q\) must exist in the surface left moving modes to maintain overall charge neutrality. We assume for simplicity that these modes move at the same speed \( v \).

As mentioned, the transport current along the wire exhibits all the characteristics of the hydrodynamic flow: it is largest at the center and vanishes at the boundary. This is because momentum can only be relaxed by electrons that have reached the
boundary and can scatter into surface modes. The amount of current through the wire scales with $R^4$, just like fluid flowing through a pipe.

From Eq. (C.7) one can read off the chiral torsional conductivity $\sigma_{CTE}$ which can be written suggestively in the following way:

$$\sigma_{CTE} = \frac{e^2 v}{4\pi\hbar} \tau N,$$

where $N = \pi R^2/\ell_b^2$ is the number of chiral bulk modes in the wire and $\tau = \tau_0 R^2/\ell_b^2$ is the effective transport scattering time. The form of the latter reflects the fact that under diffusion the electron produced near the center of the wire must scatter on average $(R/\ell_b)^2$ times before it reaches the boundary.

To illustrate this point we have performed simulation of conductance in a disordered symmetric $\frac{1}{2}$-Cd$_3$As$_2$ model. The Hamiltonian parameters are the same as used in Fig. 4.4. We have performed the conductance simulations for $\mu = 5$meV and for the system of $W \times W \times 20$ sites. We have added on-site disorder $\delta \mu_i$ taken from normal distribution of width 10meV to simulate the hydrodynamic flow described above. The ratio of conductance of disordered system to the conductance of the clean system is plotted in Fig. C.1. Best fit to the data is in accordance with (C.8), where $\tau \propto R^2$. 

120
Figure C.1: Ratio of conductance of a disordered $W \times W \times 20$ system to the conductance of the clean system averaged over 100 disorder realizations. Green line is the best fit to the data – parabolic, grey curves show the failure of the linear (with non-negative $G(0)$) and cubic fits.
Appendix D: Exchange Splitting and Interaction Matrix Between Zero Modes

Here we discuss the enhancement of the Zeeman splitting due to the exchange interaction, derive the form of coupling constants $J_{ij,kl}$ quoted in Eq. (5.4) and estimate the characteristic interaction strength $J$.

General Considerations

We begin by writing the Hamiltonian for the electrons in graphene as $H = H_0 + H_{\text{int}}$ where

$$H_0 = - \sum_{(r,r'),\sigma} t_{rr'} f_{r\sigma}^\dagger f_{r'\sigma} + \frac{g^* \mu_B B}{2} \sum_r (\rho_{r\uparrow} - \rho_{r\downarrow}), \quad (D.1)$$

Here $f_{r\sigma}^\dagger$ creates an electron with spin $\sigma$ on the site $r$ of the honeycomb lattice, $t_{rr'} = t \exp \left[ -i(e/\hbar c) \int_{r'}^r A \cdot dl \right]$ is the hopping integral in the presence of the magnetic field $B = \nabla \times A$ and $\rho_{r\sigma} = f_{r\sigma}^\dagger f_{r\sigma}$. Interactions are described by

$$H_{\text{int}} = \frac{1}{2} \sum_{r,r'} \rho_r V(r - r') \rho_{r'}, \quad (D.2)$$

where $\rho_r = \rho_{r\uparrow} + \rho_{r\downarrow}$ represents the total charge on site $r$ and $V(r) = (e^2/\epsilon r) e^{-r/\lambda_{TF}}$ is the screened Coulomb potential.

Our strategy is to first solve the non-interacting problem defined by $H_0$ on a flake with an irregular boundary. This yields a set of single-particle energy levels $\epsilon_j$ and the corresponding eigenstates $\Phi_j(r)$. As discussed in Sec. 5.3, the energy levels consist of bulk Landau levels and edge modes. The Zeeman term simply offsets the spin-up bands by $\Delta E_S(B) = g^* \mu_B B$ with respect to spin-down bands.

Next we write the interaction term $H_{\text{int}}$ in the basis defined by the eigenstates $\Phi_j(r)$. If $c_{j\sigma}^\dagger$ creates a particle with spin $\sigma$ in eigenstate $\Phi_j(r)$ we have $\rho_r =$
\[ \sum_{i,j,\sigma} \Phi^*_i(r) \Phi_j(r) c_{i\sigma}^\dagger c_{j\sigma}. \]

Substituting into Eq. (D.2) and rearranging we find

\[ H_{\text{int}} = \sum_{i,j,k,l} \sum_{\sigma,\sigma'} J_{ijkl} c_{i\sigma}^\dagger c_{j\sigma'}^\dagger c_{k\sigma'} c_{l\sigma}, \]

(D.3)

where \( J_{ijkl} \) is given by Eq. (5.4).

Henceforth we focus on the states belonging to LL\(_0\), that is, we consider electron densities such that all Landau levels with negative energies are filled, while LL\(_0\) is partially filled. Given the LL degeneracy \( N = SB/\Phi_0 \) per spin we define the total number of LL\(_0\) electrons \( N_F \) such that \( N_F = 0 \) and \( N_F = 2N \) correspond to completely empty or filled LL\(_0\), respectively. Because higher LLs are separated by an energy gap, for sufficiently weak interactions we can disregard virtual transitions into these bands and project \( H_{\text{int}} \) onto LL\(_0\) by simply restricting all indices \((i, j, k, l)\) in Eq. (D.3) to those labeling eigenstates \( \Phi_j \) in LL\(_0\).

**Exchange Splitting**

We expect electrons to occupy LL\(_0\) in such a way as to maximize the total spin \( S_{\text{tot}} \) with \( S_{\text{tot}}^z \) aligned with the field. Such a state will minimize the Zeeman energy as well as the Coulomb repulsion due to the exchange effect. The latter arises because when the spin part of the many-body electron wavefunction is symmetric in spin degrees of freedom the spatial part must necessarily be antisymmetric. This forces \( \Psi(r_1, r_2, \ldots) \) to vanish whenever two electron positions coincide, which tends to minimize the short-range part of the Coulomb repulsion energy. While the Zeeman splitting is easy to determine (see Sec. 5.5), estimation of the exchange splitting magnitude for \( N_F \) fermions described by Eq. (D.3) is a non-trivial task. This is because couplings \( J_{ijkl} \) are all-to-all and essentially random. To get an idea about the expected magnitude of the exchange splitting we consider below a simple case of \( N = N_F = 2 \).
For two electrons the position space wavefunction can be either symmetric or antisymmetric under exchange depending on the spin state,

$$\Psi_{\pm}(r_1, r_2) = \frac{1}{\sqrt{2}} [\Phi_1(r_1)\Phi_2(r_2) \pm \Phi_1(r_2)\Phi_2(r_1)].$$  \hspace{1cm} (D.4)

The corresponding Coulomb energy is

$$E_{C}^{\pm} = \sum_{r_1, r_2} |\Psi_{\pm}(r_1, r_2)|^2 V(r_1 - r_2).$$  \hspace{1cm} (D.5)

The exchange splitting, then, becomes simply $\Delta E_C = E_{C}^{+} - E_{C}^{-}$ and reads

$$\Delta E_C = 2 \sum_{r_1, r_2} \text{Re} \left[ \Phi_1^*(r_1)\Phi_2(r_1)V(r_1 - r_2)\Phi_2^*(r_2)\Phi_1(r_2) \right].$$  \hspace{1cm} (D.6)

In order to estimate $\Delta E_C$ from Eq. (D.6) we make an assumption, motivated by our extensive numerical work, that on lengthscales larger than the magnetic length $l_B$ wavefunctions $\Phi_j(r)$ behave as random uncorrelated variables. We thus coarse grain the wavefunctions on a grid with sites denoted by $R$ and spacing $l_B$. The coarse-grained wavefunctions $\Phi_j(R)$ are then treated as complex-valued independent random variables with

$$\overline{\Phi_j(R)} = 0, \quad \Phi_i^*(R)\Phi_j(R') = \frac{1}{M_s} \delta_{ij} \delta_{RR'}.$$  \hspace{1cm} (D.7)

Overbar denotes averaging over independent realizations of the flake geometry. The second equality in Eq. (D.7) follows from the normalization of $\Phi_j$ and

$$M_s = S/l_B^2 = 2\pi N$$ \hspace{1cm} (D.8)

denotes the number of grid sites in the flake.
With this preparation we now recast Eq. (D.6) as a sum over the coarse grained grid, \( \sum_{r_1,r_2} \rightarrow \sum_{R_1,R_2} \) and \( \Phi_j(r) \rightarrow \Phi_j(R) \). Using Eq. (D.7) we then obtain an estimate for the typical exchange splitting

\[
\Delta E_C \simeq \frac{2}{M_s^2} \sum_{R_1,R_2} \delta_{R_1,R_2} V(R_1 - R_2) = \frac{2}{M_s} V(0). \tag{D.9}
\]

Here \( V(0) \) must be interpreted as the average Coulomb potential in a grid patch of the size \( l_B \), that is \( V(0) \approx \frac{1}{\pi l_B^2} \int_0^{l_B} V(r) 2\pi r dr = 2e^2/\epsilon l_B \), where we assumed \( \lambda_{TF} \gg l_B \). Taking the dielectric constant \( \epsilon = 2 \) and \( N = 2 \) we find the typical exchange splitting \( \Delta E_C \approx 8.8 \) meV/T. We expect this result to remain at least approximately valid for \( N > 2 \). Therefore, when \( N_F < N \), electrons will fill the spin-down states of \( LL_0 \) with empty spin-up states separated in energy by a significant exchange gap. The physics of such partially filled spin-down \( LL_0 \) can be described by the Hamiltonian (D.3) with \( \sigma = \sigma' = \downarrow \) which is precisely the SY Hamiltonian.

**Coupling Strength \( J \)**

To estimate the typical strength of couplings \( J_{ij;kl} \) that enter the SY Hamiltonian it is useful to first recast Eq. (5.4) such that it is explicitly antisymmetric in indices \((i,j)\) and \((k,l)\)

\[
J_{ij;kl} = \frac{1}{2} \sum_{R_1,R_2} \Omega^*_i(R_1,R_2) V(R_1 - R_2) \Omega_k(R_1,R_2), \tag{D.10}
\]

where \( \Omega_{ij}(R_1,R_2) = \frac{1}{2} [\Phi_i(R_1), \Phi_j(R_2)] \). We also passed to the coarse-grained variables, as described above. With help of Eq. (D.7) it is straightforward to show that \( J_{ij;kl} = 0 \) and

\[
|J_{ij;kl}|^2 = \frac{1}{M_s^3} \sum_{R\neq 0} V(R)^2. \tag{D.11}
\]
The sum can be approximated by an integral,

\[ \int_{l_B}^{\infty} \frac{2\pi R dR}{l_B^2} \left( \frac{e^2 e^{-R/\lambda_{TF}}}{\epsilon R} \right)^2 = \left( \frac{e^2}{\epsilon l_B} \right)^2 2\pi \Gamma(0, \frac{l_B}{\lambda_{TF}}), \]  

(D.12)

where \( \Gamma(0, x) = \int_x^{\infty} dy \frac{e^{-y}}{y} \) is the incomplete gamma function. Combining with Eq. (5.5) we thus obtain an estimate

\[ J \simeq 2 \left( \frac{e^2}{\epsilon l_B} \right) \left( \frac{N}{M_s} \right)^{3/2} \sqrt{\pi \Gamma(0, \frac{l_B}{\lambda_{TF}})}. \]  

(D.13)

For \( \epsilon = 2 \) this amounts to

\[ J \simeq 6.04 \text{ meV} \sqrt{B[T] \Gamma(0, \frac{l_B}{\lambda_{TF}})}. \]  

(D.14)

For \( x = l_B/\lambda_{TF} \ll 1 \), which is the limit of interest, \( \Gamma(0, x) \simeq \ln(1/x) \) so \( J \) is only very weakly dependent on the screening length. For \( B = 20 \text{ T} \) and \( \lambda_{TF}/l_B = 4 \) we obtain \( J \simeq 25 \text{ meV} \).

It is to be noted that our numerical calculations of \( J_{ij;kl} \) described in Sec. 5.3 [discussion below Eq. (5.5)] give larger values of \( J \) than the above estimate, in some cases by as much as an order of magnitude. The discrepancy is most likely attributable to the fact that LL\(_0\) wavefunctions are in fact disordered on a somewhat longer lengthscale than \( l_B \). This would modify the relation between \( M_s \) and \( N \) given by Eq. (D.8) and increase the ratio \( (N/M_s) \) that enters the estimate for \( J \) in Eq. (D.13). We may therefore regard Eq. (D.13) as a conservative lower bound on the expected magnitude of \( J \). This is already a large energy scale which should make the manifestations of the SY physics experimentally observable at low temperatures in clean graphene flakes.
Appendix E: Symmetry Breaking Perturbations in Real Graphene

To ascertain the experimental feasibility of our proposal, we discuss the effect of various chiral symmetry breaking perturbations that exist in real graphene. Such perturbations tend to broaden LL$_0$ and can be modeled by a bilinear term $H_2$ defined by Eq. (5.3). The matrix elements are

$$K_{ij} = \sum_r \Phi^*_i(r) H'(r) \Phi_j(r),$$  \hspace{1cm} (E.1)

where $H'$ denotes the Hamiltonian of the perturbation. The strength of these perturbations is measured by parameter $K$ defined as

$$K^2 = N \left( |K_{ij}|^2 - |\overline{K_{ij}}|^2 \right).$$  \hspace{1cm} (E.2)

It is known that since $H_2$ is a relevant perturbation to $H_{SY}$ (in the renormalization group sense) the ground state of the system becomes a (disordered) Fermi liquid for any nonzero $K$. Nevertheless, if $K \ll J$, a significant crossover region can exist at finite frequencies and temperatures in which the system behaves effectively as a maximally chaotic SY liquid. According to the analysis of Ref. [169] the zero-temperature propagator of the system with both $K$ and $J$ nonzero exhibits the SY conformal scaling $G(\omega) \sim |\omega|^{-1/2}$ for frequencies satisfying

$$16\sqrt{\pi} K^2 / J < \omega \ll J.$$  \hspace{1cm} (E.3)

In the following we consider two specific perturbations that are present in real graphene, the second neighbor hopping $t'$ and random on-site potential. Both break the chiral symmetry $\chi$ and produce non-zero parameter $K$. We derive limits on the admissible
strength of these perturbations based on the requirement that Eq. (E.3) yields a significant window in which SY behavior can be observed.

**Second-Neighbor Hopping**

We first consider second neighbor hopping with the Hamiltonian acting as \( H'(r)\Phi_j(r) = t' \sum_a \Phi_j(r + a) \). Here \( a \) denotes the 3 second neighbor vectors in the honeycomb lattice. Since \(|a| \ll l_B\) we find, upon coarse graining the sum in Eq. (E.1),

\[
K_{ij} \simeq 3t' \sum_R \Phi_i^*(R) \Phi_j(R).
\]  
(E.4)

With help of Eq. (D.7) it is straightforward to show that

\[
\overline{K_{ij}} \simeq 3t' \delta_{ij}, \quad |K_{ij}|^2 \simeq 9t'^2 \left( \delta_{ij} + M_s^{-1} \right).
\]  
(E.5)

From Eq. (E.2) we get \( K \simeq 3t' \sqrt{N/M_s} \simeq 3t'/\sqrt{2\pi} \) independent of the field.

Experimentally reported values of \( t' \) range between[24] 1-3\% of \( t \) which would produce a rather large broadening of LL\(_0\) in real graphene, \( K \simeq 30 - 90 \) meV. On the other hand existing experiments [177] indicate much smaller broadening of Landau levels in graphene of at most several meV which also includes broadening due to impurities and other defects. We therefore conclude that the above method must severely overestimate the contribution of second neighbor hopping to parameter \( K \).

This conclusion is supported by our numerical results presented below.

The numerically computed energy spectrum of the graphene flake with second neighbor hopping \( t' = 0.037t \) is displayed in Fig. E.1a. We observe that while LL\(_0\) is now significantly shifted away from zero energy it remains sharp and well defined. The overall upward shift of LL\(_0\) by about 0.25 eV is consistent with the estimate given in Eq. (E.5) which implies \( \overline{K_{ij}} \simeq 0.30 \) eV. The broadening induced by \( t' \) is quantified in Fig. E.1b and is well approximated by a linear dependence \( K \simeq 0.022t' \). This is
about a factor of 50 smaller than the estimate implied by Eq. (E.5). For $t' = 0.02t$ we obtain $K \simeq 1.2$ meV, a result that is much more in line with the experimental data.

Figure E.1: **Effects of the second neighbor hopping $t'$**. a) Single-particle energy spectrum of a flake (the same geometry as Fig. 5.2) with second neighbor hopping $t' = 0.037t$. b) Average shift $\delta \epsilon = \overline{K_{ij}}$ and standard deviation $K$ of 40 energy levels that comprise LL$_0$ as a function of $t'$.

The discrepancy between the analytical estimate and the numerical result can be understood as follows. In a large, disorder-free sample of graphene, inclusion of the second neighbor hopping produces changes in the band structure (and thus the position and spacing of LLs) but does not give rise to any LL broadening as long as $t'$ remains spatially uniform. The sharpness of LLs is protected by translational invariance, not the chiral symmetry. In our mesoscopic flake we see that the inclusion
of a spatially uniform $t'$ primarily shifts the position of LLs, as expected from the argument given above. Because randomness is present in the system due to its irregular geometry some broadening occurs. This broadening is, however, much weaker than what is predicted by the naive estimate.

**Random On-Site Potential**

Random on-site potential is implemented by taking $H'(r) = w \sum_{r' \in \mathcal{I}} \delta_{rr'}$, where $\mathcal{I}$ denotes a set of randomly chosen sites with number density $n_I$ in the graphene lattice and $w$ controls the disorder potential strength. Substituting $H'$ into Eq. (E.1) leads to the same result as indicated in Eq. (E.5) with $3t'$ replaced by $wn_I$. We therefore expect an overall energy shift of LL$_0$ by $wn_I$ accompanied by a broadening

$$K \simeq wn_I \sqrt{\frac{N}{M_s}}.$$  \hfill (E.6)

Fig. E.2a shows the numerically computed energy eigenvalues as a function of $w$ for a flake with $N = 40$ flux quanta and $n_I = 1\%$. We observe that LL$_0$ is shifted upward as well as broadened with increasing disorder strength. This shift $\delta \epsilon$ and broadening $K$ are quantified in Fig. E.2b. At small $w$ these satisfy $\delta \epsilon \simeq 0.8n_Iw$ and $K \simeq 1.3n_Iw$ while at larger values the dependence is no longer linear, presumably because the system enters a non-perturbative regime when $w$ becomes comparable to the bandwidth. We see that the numerically obtained shift in LL$_0$ is well aligned with the analytical estimate. The broadening $K$ also agrees if we take $\sqrt{N/M_s} \simeq 1.3$ (instead of $1/\sqrt{2\pi} \simeq 0.4$ implied by Eq. (D.8)). This result reinforces the conclusion, reached in Appendix D by comparing the interaction strength estimate to the numerical calculation, that the zero mode wavefunction disorder scale is somewhat longer than $l_B$. 

130
Figure E.2: **Effects of random on-site potential.** a) Low-energy part of the numerically calculated energy spectrum for the flake with \( n_I = 1\% \) of defected sites as a function of the disorder potential strength \( w \) and \( N = 40 \). b) Average shift \( \delta \epsilon = \mathbb{K}_{ij} \) and standard deviation \( K \) of 40 energy levels that evolve from the zero modes which comprise LL\(_0\) in the pure sample. These levels are marked in red in panel (a).

We finally remark that in the above example 40 flux quanta through a flake with 1952 carbon atoms correspond to an unrealistically high magnetic field of \( \sim 3200 \) T. Such high fields are needed for us to be able to numerically simulate meaningful number of zero modes \( N \) with available computational resources. To make a closer contact with experiment we may however reinterpret these results by viewing the honeycomb lattice not as the atomic carbon lattice but as a convenient regularization of the low energy theory of Dirac electrons in graphene. In such low energy theory the only important parameter is the Dirac velocity \( v_F = \frac{3}{2}ta \simeq 1.1 \times 10^6 \) m/s. The velocity is clearly unchanged if we rescale the lattice constant \( a \rightarrow \lambda a \) and the
tunneling amplitude $t \to t/\lambda$ with $\lambda$ an arbitrary positive parameter. Under the rescaling $B \to B/\lambda^2$ and all energy parameters defined through $t$ are changed as $E \to E/\lambda$. Thus, if we take $\lambda = 10$ in the above example we get a more reasonable field $B = 32$ T. According to Eq. (D.14) this corresponds to $J \simeq 34$ meV. Eqs. (E.6) and (E.3) then stipulate an upper bound on the disorder strength $n_I w \ll 9$ meV.

Clearly, like fractional quantum Hall effect and other exotic phases driven by interactions, observing the SY physics will require high fields, low temperatures and carefully prepared graphene flake with an irregular boundary and clean interior.