ENUMERATIVE PROBLEMS
in Algebraic Geometry Motivated from Physics

by

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Abstract

This thesis contains two chapters which reflect the two main viewpoints of modern enumerative geometry.

In chapter I we develop a theory for stable maps to curves with divisible ramification. For a fixed integer $r > 0$, we show that the condition of every ramification locus being divisible by r is equivalent to the existence of an rth root of a canonical section. We consider this condition in regards to both absolute and relative stable maps and construct natural moduli spaces in these situations. We construct an analogue of the Fantechi-Pandharipande branch morphism and when the domain curves are genus zero we construct a virtual fundamental class. This theory is anticipated to have applications to r-spin Hurwitz theory. In particular it is expected to provide a proof of the r-spin ELSV formula [SSZ'15, Conj. 1.4] when used with virtual localisation.

In chapter II we further the study of the Donaldson-Thomas theory of the banana threefolds which were recently discovered and studied in [Bryan'19]. These are smooth proper Calabi-Yau threefolds which are fibred by Abelian surfaces such that the singular locus of a singular fibre is a non-normal toric curve known as a "banana configuration". In [Bryan'19] the Donaldson-Thomas partition function for the rank 3 sub-lattice generated by the banana configurations is calculated. In this chapter we provide calculations with a view towards the rank 4 sub-lattice generated by a section and the banana configurations. We relate the findings to the Pandharipande-Thomas theory for a rational elliptic surface and present new Gopakumar-Vafa invariants for the banana threefold.

Lay Summary

In this thesis we use modern algebraic techniques to work on enumerative problems that are motivated by mathematical physics. The objects being counted are *complex curves* which are surfaces that don't have edges (e.g. spheres, donuts, etc.) with some extra structure. In string theory, these objects roughly translate to the path a vibrating string would sweep out as it travels forward in time. We are interested in counting the possible *complex curves* which can live within a given even-dimensional space. In Chapter One, we develop the theory for a method of counting special sub-classes of these *complex curves*. We use this theory to provide a generalisation of the classical concept of *Hurwitz numbers*. In Chapter Two we provide an explicit computation of the number are *complex curves* that can live within a space called the *banana threefold*. We show that formulas obtained from these numbers have interesting properties related to previous work.

Preface

This is to certify that this thesis comprises only my original work towards the Doctor of Philosophy in Mathematics and due acknowledgement has been made in the text to all other material used.

This dissertation was originally formatted in accordance with the regulations of the University of Melbourne and submitted in partial fulfillment of the requirements for a PhD degree awarded jointly by the University of Melbourne (lead university) and the University of British Columbia. Different versions of this dissertation will exist in the institutional repositories of both institutions.

Publication status of all chapters presented:

- *Chapter 1*: Submitted for publication on the 20th of February 2019.
- *Chapter 2*: Unpublished material not submitted for publication.

Declaration (The University of Melbourne)

This is to certify that

1. the thesis comprises only my original work towards the Doctor of Philosophy in Mathematics except where indicated in the preface;
2. due acknowledgement has been made in the text to all other material used; and
3. the thesis is less than 100,000 words in length, exclusive of tables, maps, bibliographies and appendices.

Oliver Leigh

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Note that these chapters are separate self-contained works. They do not refer to each other and different notation is used in each.

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Introduction

There have been links between geometry and physics for millennia. Indeed, many great discoveries from historical figures such as Newton, Maxwell and Einstein have both arisen from and strengthened these links. However, some areas of geometry have not always played significant roles in these links. Enumerative geometry was one of these areas until relatively recently. The last thirty years have seen a rapid development and expansion of the links between enumeration of geometric structures and theoretical physics, particularly in the area of string theory.

This link can be intuitively described as follows: As a string moves around in spacetime, it sweeps out a Riemann surface called a "worldsheet". In complex geometry, this is one-dimensional, so we call it a curve. Counting curves, that live within a spacetime, gives information about interactions and probabilities of changing states.

In studying curve enumeration, two main viewpoints have arisen:

> *Gromov-Witten Theory:* Curves are external with a map to the space, and are parameterised by the moduli space of stable maps, $\overline{\mathcal{M}}_g(X, \beta)$.
>
> *Donaldson-Thomas Theory:* Curves are internal with structure coming from an embedding in the ambient space, and are parameterised by the Hilbert scheme $\mathrm{Hilb}^{\beta,1-g}(X)$.

We should note that these viewpoints are not completely separate and they have been proven equivalent in many cases. However, the techniques employed can vary greatly between the two approaches. One key similarity between the two theories is that the moduli spaces involved are not equidimensional, and a "virtual fundamental class" is required for their definitions. This is an inherent technical issue which one must deal with to use the theories.

This thesis reflects the separation of these two viewpoints by having each of the two chapters devoted to one side. They are separate self-contained works, do not refer to each other and have distinct notation.

Spin Structures and Map Enumeration

One link between theoretical physics and enumerative geometry was proposed by Witten in 1991 during an investigation into two-dimensional quantum gravity. He conjectured that certain curve counts would satisfy the KdV integrable hierarchy. This is a well known set of differential equations which possess soliton solutions. Witten's conjecture was subsequently proven by Kontsevich with an ingenious use of combinatorial methods.

There is another set of differential equations arising in soliton theory called the 2-Toda Hierarchy. In some ways it can be thought of as a more fundamental object than the KdV hierarchy. Okounkov and Pandharipande show in [OP] that this hierarchy has solutions arising from a generalised form of Hurwitz numbers. Classically, Hurwitz numbers count maps from smooth curves to the complex projective line where ramification is specified to be simple. Okounkov and Pandharipande generalise this definition using the representation theory of the symmetric group.

Moreover, it has since been conjectured in [SSZ] that these generalised Hurwitz numbers are actually natural intersections on the moduli space of curves with r-spin structure. The form of this conjecture generalises the celebrated ELSV formula for classical Hurwitz numbers. However a proof of the formula and its underlying geometric mechanism, has proved elusive.

In *Chapter I*, a moduli space is introduced that gives a geometric interpretation of the objects being counted by these generalised Hurwitz numbers. These are called stable maps with divisible ramification. They are maps where the ramification number at every point is divisible by r.

The definition of such maps is clear for a smooth curve C. We simply specify that the ramification divisor be divisible by r. However, when the curve is nodal this doesn't work because the ramification divisor cannot be defined. For a morphism $f : C \to \mathbb{P}^1$ the ramification divisors is determined by the differential map $df : f^*\Omega_{\mathbb{P}^1} \to \Omega_C$. The divisor construction relies on the fact that Ω_C in invertible when C is smooth and when C is nodal this is no longer the case.

In *Chapter I* we overcome this using the observation that the ramification divisor is defined by a canonical section $\delta : \mathcal{O}_C \to \omega_C \otimes f^*\omega_{\mathbb{P}^1}^\vee$ and this section is still well defined when C is nodal. We then use the theory of r-spin structures to take an rth root of this section. We show that this condition gives exactly the curves with ramification order divisible by r.

There are three main results of *Chapter I*. The first is Theorem A, which shows that the space described above is an appropriate space for enumerative study. Namely that it is a proper Deligne-Mumford stack. The theorem also gives a comparison between this space and the moduli space of stable maps which is the main moduli space studied in Gromov-Witten theory.

The second and third main results of *Chapter I* develop theory to allow enumerative study of this space. Theorem B gives an extension of the branching morphism of [FP] and an interpretation of the ramification properties of maps with nodal domains. A perfect obstruction theory for genus 0 is constructed in Theorem C which allows intersection theory to be used on this space. When combined, the three main theorems allow the definition and future study of the generalised Hurwitz problem in a geometric setting.

Enumeration of Subschemes in Calabi-Yau Threefolds

Many conventional string theory models require ten real dimensions. These consist of the four usual dimensions, and six extra hidden “curled-up" ones coming from *Calabi-Yau threefolds* (three complex dimensions). This makes Calabi-Yau threefolds a natural choice for enumerative study.

Even better, there are certain properties of Calabi-Yau threefolds that make the expected dimension of the Hilbert scheme zero. This suggests that counting subschemes may be related to the Euler characteristic. In fact, Behrend showed in [B1] that the virtual curve counting theory known as Donaldson-Thomas theory is a weighted Euler characteristic.

However, computing Donaldson-Thomas invariants is *very* hard. Even when we use the Euler characteristic approach. In fact computing them for compact threefolds is so hard that the full Donaldson-Thomas theory is only known in computationally trivial cases. An example of this is the product of a $K3$ surface with an elliptic curve. The group action of the elliptic curve extends to the Hilbert scheme making all the invariants trivial to compute. In non-trivial cases, there is not even a conjectural solution for the full Donaldson-Thomas theory of a compact threefold. However, there are many beautiful results that appear when we restrict our attention to subsets of the the full theory.

These results will often not manifest themselves until one assembles the invariants into a *partition function*. These are formal generating functions that store the enumerative invariants as coefficients of power series expansions. Partition functions will often have properties that are related to physical theories and modular forms. It is these properties and connections that make a full Donaldson-Thomas partition function highly desirable.

Recent advances in techniques and the discoveries of new Calabi-Yau threefolds have opened up new avenues for calculations in Donaldson-Thomas theory. One such technique was recently introduced by Bryan and Kool in [BK] for studying local elliptic surfaces. This method is extended in *Chapter II* to allow its use in a more general setting. The full generality in which this method can be used is currently unknown. However, it can certainly be used to study the Donaldson-Thomas theory of any Calabi-Yau threefold when the curve classes can be understood and when the subschemes are locally determined by monomial ideals.

In *Chapter II* these methods are then used to provide new calculations for the "banana" Calabi-Yau threefold recently introduced in [Br]. This is a smooth proper Calabi-Yau threefold which is fibred by Abelian surfaces such that the singular locus of a singular fibre is a non-normal toric curve known as a "banana configuration". In [Br] the Donaldson-Thomas partition function for the rank 3 sub-lattice generated by the banana configurations is calculated. In *Chapter II* we provide calculations with a view towards the rank 4 sub-lattice generated by a section and the banana configurations.

Chapter I

The Moduli Space of Stable Maps with Divisible Ramification

Introduction

Consider a smooth curve X and the moduli space parameterising degree d maps $f : C \to X$ where C is a smooth curve of genus g. This space is denoted by $\mathcal{M}_g(X,d)$ and point a $[f] \in \mathcal{M}_g(X,d)$ has an associated exact sequence

$$0 \longrightarrow f^*\Omega_X \otimes \Omega_C^\vee \xrightarrow{\delta^\vee} \mathcal{O}_C \longrightarrow \mathcal{O}_{R_f} \longrightarrow 0 \tag{I.1}$$

where R_f is the ramification divisor. If $[f]$ is a generic point then R_f is the union of disjoint points on C. In other words, f has simple ramification everywhere.

As an alternative, we consider a space $\mathcal{M}_g^{1/r}(X,d)$ where a generic point $[f]$ gives a ramification divisor of the form $R_f = r \cdot p_1 + \cdots + r \cdot p_m$ for disjoint points $p_1, \ldots, p_m \in C$. Specifically, we define $\mathcal{M}_g^{1/r}(X,d)$ as the following sub-moduli space of $\mathcal{M}_g(X,d)$:

$$\mathcal{M}_g^{1/r}(X,d) = \left\{ \left[f : C \to X\right] \in \mathcal{M}_g(X,d) \,\middle|\, R_f = r \cdot D \text{ for some } D \in \mathrm{Div}(C) \right\} / \sim .$$

In this chapter we construct a natural compactification of $\mathcal{M}_g^{1/r}(X,d)$. We develop the enumerative geometry of this space by constructing a virtual fundamental class in the case $g = 0$ and by constructing a branch morphism.

The above construction of the ramification divisor relies on the domain curve C being smooth. This means that Ω_C is locally free and that $df : f^*\Omega_X \to \Omega_C$ is injective. If C is allowed to be singular either of these may be false and we no longer have a straightforward definition of ramification. This leads us to rephrase the moduli problem using rth roots of δ which is defined in (I.1). One can show that $\mathcal{M}_g^{1/r}(X,d)$ is naturally isomorphic to:

$$\left\{ \left[f : C \to X\right] \in \mathcal{M}_g(X,d) \,\middle|\, \begin{array}{l} \text{There is a line bundle } L \text{ on } C,\ \sigma \in H^0(L) \text{ and} \\ \text{an isom. } L^{\otimes r} \xrightarrow{e} \omega_C \otimes f^*\omega_X^\vee \text{ with } e(\sigma^r) = \delta. \end{array} \right\} / \sim .$$

We now have the moduli problem in a form which can be naturally compactified. First we note that for nodal domain curves there is a natural morphism $\Omega_C \to \omega_C$. Here we have used standard notation for the sheaf of differentials and the dualising sheaf noting that the latter is locally free. This is combined with the differential map $df : f^*\Omega_X \to \Omega_C$ to obtain a morphism which we denote by

$$\delta : \mathcal{O}_C \longrightarrow \omega_C \otimes f^*\omega_X^\vee. \tag{I.2}$$

Definition 1 Denote by $\overline{\mathcal{M}}_g^{1/r}(X,d)$ the moduli stack parameterising morphisms $f : C \to X$ where

1. C is a genus g r-prestable curve (a stack such that the coarse space $\overline{C}$ is a prestable curve, where points mapping nodes of $\overline{C}$ are balanced r-orbifold points, and $C^{\mathrm{sm}} \cong \overline{C}^{\mathrm{sm}}$);
2. f is a morphism such that the induced morphism $\overline{f} : \overline{C} \to X$ on the course space is a stable map;
3. there exists a line bundle L on C, an isomorphism $e : L^{\otimes r} \xrightarrow{\sim} \omega_C \otimes f^*\omega_X^\vee$, and a morphism $\sigma : \mathcal{O}_C \to L$ such that $e(\sigma^r) = \delta$, where δ is defined in (I.2).

Remark 1 Throughout the chapter we will also be considering the same moduli problem in the context of stable maps relative to a point $x \in X$ and a partition μ of $d > 0$. The moduli space of relative stable maps $\overline{\mathcal{M}}_g(X,\mu)$ generically parameterises maps where the pre-image of x is smooth and locally has monodromy given by μ. We will leave the specifics of this moduli problem until section 1.1, however all of the following results will hold when $\overline{\mathcal{M}}_g^{1/r}(X,d)$ is replaced by $\overline{\mathcal{M}}_g^{1/r}(X,\mu)$, and $2g-2-d(2g_X-2)$ is replaced by $2g-2+l(\mu)+|\mu|(1-2g_X)$.

Remark 2 The r-prestable curves in definition 1 arise naturally when taking rth roots of line bundles on nodal curves [AJ, Ch1]. We review this in section 1.3.

Theorem A *$\overline{\mathcal{M}}_g^{1/r}(X,d)$ is a proper DM stack. It is non-empty only when r divides $2g-2-d(2g_X-2)$. The natural forgetful map*

$$\chi : \overline{\mathcal{M}}_g^{1/r}(X,d) \longrightarrow \overline{\mathcal{M}}_g(X,d)$$

is both flat and of relative dimension 0 onto its image. It is an immersion when restricted to $\mathcal{M}_g^{1/r}(X,d)$.

The image of χ has an explicit point-theoretic description. Let $f : C \to X$ be a stable map and consider the locus in C where f is not étale. Following [V, GV] a connected component of this locus is called a *special locus*. A special locus will be one of the following:

(a) A smooth point of C where f is locally of the form $z \mapsto z^{a+1}$ with $a \in \mathbb{N}$.

(b) A node of C such that on each branch f is locally like $z \mapsto z^{a_i}$ with $a_i \in \mathbb{N}$.

(c) A genus g' component B of C where $f|_B$ is constant and on the branches of C meeting B the map f is locally of the form $z \mapsto z^{a_i}$ with $a_i \in \mathbb{N}$.

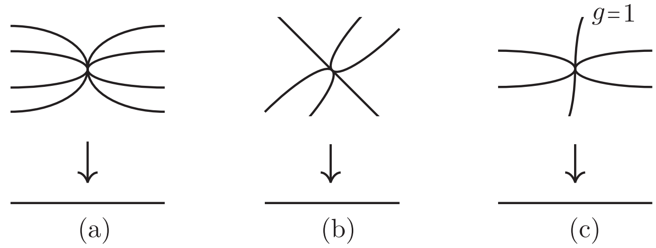


Figure I.1: Loci with ramification order 3. (a) A smooth point where the map is locally like $z \mapsto z^{3+1}$. (b) A node where the map is locally like $z \mapsto z^2$ on one branch and $z \mapsto z$ on the other. (c) A genus one component meeting its complement at a node, where the map is constant on the sub-curve and locally like $z \mapsto z^2$ on the complement.

Note that a slightly different definition is used for the relative case (see remark 3.3.1). Now, following [V, GV] again, we define a *ramification order* (or sometimes simply *order*) for each type of special locus by:

(a) a. (b) $a_1 + a_2$. (c) $2g' - 2 + \sum(a_i + 1)$.

This gives us an extended concept of ramification. There is also an extended concept of branching constructed in [FP] which agrees with the ramification order assigned to special loci. Specifically there is a well defined morphism of stacks which agrees with the classical definition of branching on the smooth locus:

$$br: \quad \mathcal{M}_g(X,d) \quad \longrightarrow \quad \mathrm{Sym}^{2g-2-d(2g_X-2)}X.$$

Theorem B *The objects in $\overline{\mathcal{M}}^{1/r}_g(X,d)$ have the following ramification and branching properties:*

1. *The closed points in the image of $\tau : \overline{\mathcal{M}}^{1/r}_g(X,d) \longrightarrow \overline{\mathcal{M}}_g(X,d)$ are the closed points of $\overline{\mathcal{M}}_g(X,d)$ with the property:*

   "Every special locus of the associated map has order divisible by r".

2. *There is a morphism of stacks*

$$\mathtt{br} : \overline{\mathcal{M}}^{1/r}_g(X,d) \longrightarrow \mathrm{Sym}^{\frac{1}{r}(2g-2-d(2g_X-2))}X$$

   that commutes with the branch morphism of [FP] via the diagram

$$\begin{tikzcd} \overline{\mathcal{M}}^{1/r}_g(X,d) \arrow[r, "\mathtt{br}"] \arrow[d, "\chi"'] & \mathrm{Sym}^{\frac{1}{r}(2g-2-d(2g_X-2))}X \arrow[d, "\Delta"] \\ \overline{\mathcal{M}}_g(X,d) \arrow[r, "br"] & \mathrm{Sym}^{2g-2-d(2g_X-2)}X \end{tikzcd}$$

   where Δ is defined by $\sum_i x_i \mapsto \sum_i r x_i$.

Just like for regular stable maps $\overline{\mathcal{M}}_g(X,d)$, the smooth-domain locus $\mathcal{M}^{1/r}_g(X,d)$ can be empty while $\overline{\mathcal{M}}^{1/r}_g(X,d)$ is non-empty. For explicit examples consider degree one maps to $\mathbb{P}^1$ with $g > 0$.

The properties of $\overline{\mathcal{M}}_g^{1/r}(X,d)$ can be quite different to those of $\overline{\mathcal{M}}_g(X,d)$. For example, if we consider genus zero domains we have that $\overline{\mathcal{M}}_0(X,d)$ is smooth, but in general $\overline{\mathcal{M}}_0^{1/r}(X,d)$ is not. An explicit example of this is $\overline{\mathcal{M}}_0^{1/3}(\mathbb{P}^1,4)$, which is not smooth as it contains components of dimensions 2 and 3. However, we do have the existence of a virtual fundamental class for $g=0$.

Theorem C $\overline{\mathcal{M}}_0^{1/r}(\mathbb{P}^1,d)$ *has a natural perfect obstruction theory giving a virtual fundamental class of dimension* $\frac{1}{r}(2d-2)=\frac{1}{r}\mathrm{virdim}(\overline{\mathcal{M}}_0(\mathbb{P}^1,d))$.

The moduli space $\overline{\mathcal{M}}_g^{1/r}(\mathbb{P}^1,\mu)$ has expected applications to r-spin Hurwitz theory. For example, in genus 0 using both theorems B and C we have the following natural intersection

$$\int_{[\overline{\mathcal{M}}_0^{1/r}(\mathbb{P}^1,\mu)]^{\mathrm{vir}}} \mathtt{br}^* H^{\frac{1}{r}(l(\mu)+|\mu|-2)} \tag{I.3}$$

where H is the hyperplane class in $\mathrm{Sym}^{\frac{1}{r}(l(\mu)+|\mu|-2)}\mathbb{P}^1 \cong \mathbb{P}^{\frac{1}{r}(l(\mu)+|\mu|-2)}$. This is a direct analogue of the characterisation of simple Hurwitz numbers given in [FP, Prop. 2]. This was the first step towards a proof via virtual localisation of the ELSV formula. After applying the virtual localisation techniques of [GP], (I.3) is expected to be related to the r-ELSV formula of [SSZ, BKLPS].

In the case where $r=l(\mu)+|\mu|-2$ the space $\overline{\mathcal{M}}_0^{1/r}(\mathbb{P}^1,\mu)$ has virtual dimension 1. These spaces are characterised by having exactly one free special locus of order r. In this situation the intersections given in (I.3) are expected to have a direct relation to the completion coefficients and one-point invariants of [OP]:

$$\int_{[\overline{\mathcal{M}}_{0,1}(\mathbb{P}^1,\mu)]^{\mathrm{vir}}} \psi_1^r ev_1^*[pt].$$

This chapter is structured as follows:

Section 1: Review the necessary theory of stable maps, r-prestable curves and line bundles on twisted curves required for the construction of $\overline{\mathcal{M}}_g^{1/r}(X,d)$ and its relative version.

Section 2: Extend the theory of roots of line bundles to the space of stable maps, construct the moduli space $\overline{\mathcal{M}}_g^{1/r}(X,d)$ and then prove theorem A.

Section 3: Consider properties of $\overline{\mathcal{M}}_g^{1/r}(X,d)$ related to branching and ramification while proving theorem B.

Section 4: Consider the cotangent complex of $\overline{\mathcal{M}}_g^{1/r}(X,d)$ and related properties while proving theorem C.

Conventions All stacks and schemes are over $\mathbb{C}$. By *local picture* we will mean the following. Let $f:X\to Y$ and $g:U\to V$ be morphisms of stacks. The *local picture* of f at $x\in X$ is the same as the local picture of g at $u\in U$ if:

- There is an isomorphism between the strict henselization $f^{\mathrm{sh}}:X^{\mathrm{sh}}\to Y^{\mathrm{sh}}$ of f at x and the strict henselization $g^{\mathrm{sh}}:U^{\mathrm{sh}}\to V^{\mathrm{sh}}$ of g at u.

Throughout the chapter we will consider both absolute and relative stable maps. The theory will be similar so we introduce the following simplifying notation.

Notation:

- $\mathcal{M}$ is either $\overline{\mathcal{M}}_g(X,d)$ or $\overline{\mathcal{M}}_g(X,\mu)$ for $g \geq 0$, $d > 0$ and μ a partition of d.
- $\mathcal{C} \to \mathcal{M}$ is the associated universal curve.
- If $\mathcal{M}$ is $\overline{\mathcal{M}}_g(X,d)$ (resp. $\overline{\mathcal{M}}_g(X,\mu)$) then $\mathfrak{M}$ is $\mathfrak{M}_g$ (resp. $\mathfrak{M}_{g,l(\mu)}$).
- The expected number of order r special loci in the generic case is denoted by m. When $\mathcal{M} = \overline{\mathcal{M}}_g(X,d)$ we have $m = \frac{1}{r}(2g-2-d(2g_X-2))$ and when $\mathcal{M} = \overline{\mathcal{M}}_g(X,\mu)$ we have $m = \frac{1}{r}(2g-2+l(\mu)+|\mu|(1-2g_X))$.
- Throughout, the notation used for a space without r or $\frac{1}{r}$ will carry through to analogous spaces involving r or $\frac{1}{r}$. For $\overline{\mathcal{M}}_g(X,\mu)$ the two key spaces are:
  - $\mathcal{M}^{[r]} = \overline{\mathcal{M}}_g^{[r]}(X,\mu)$ and $\mathcal{C}^{[r]} = \overline{\mathcal{C}}_g^{[r]}(X,\mu)$ defined in 2.1.3.
  - $\mathcal{M}^{[\frac{1}{r}]} = \overline{\mathcal{M}}_g^{[1/r]}(X,\mu)$ and $\mathcal{C}^{[\frac{1}{r}]} = \overline{\mathcal{C}}_g^{[1/r]}(X,\mu)$ are defined in 2.3.1.3.

  And the associated spaces are:
  - $\mathcal{M}^r = \overline{\mathcal{M}}_g^r(X,\mu)$ and $\mathcal{C}^r = \overline{\mathcal{C}}_g^r(X,\mu)$ defined in 2.1.
  - $\mathcal{M}^{\frac{1}{r},\mathcal{E}} = \overline{\mathcal{M}}_g^{\frac{1}{r},\mathcal{E}}(X,\mu)$ and $\mathcal{C}^{\frac{1}{r},\mathcal{E}} = \overline{\mathcal{C}}_g^{\frac{1}{r},\mathcal{E}}(X,\mu)$ are defined in 2.1.1.
  - $\mathcal{M}^{\frac{1}{r}} = \overline{\mathcal{M}}_g^{1/r}(X,\mu)$ and $\mathcal{C}^{1/r} = \overline{\mathcal{C}}_g^{1/r}(X,\mu)$ are defined in 2.3.1.1.

1 Review of Stable Maps and r-Stable Curves

1.1 Stable Maps and Relative Stable Maps

For the rest of this chapter we will set X to be a non-singular projective curve. Recall that a stable map $f : C \to X$ is a degree d morphism from a genus g prestable curve to X which has no infinitesimal automorphisms. We denote by $\overline{\mathcal{M}}_g(X,d)$ the moduli stack of these objects. Specifically this is the groupoid containing the objects:

$$\xi = \big(\ \pi : C \to S, \quad f : C \to X\ \big)$$

where π is a proper flat morphism and for each geometric point $p \in S$ we have $f_p : C_p \to X$ is a degree d genus g stable map to X. A morphism $\xi_1 \to \xi_2$ in $\overline{\mathcal{M}}_g(X,d)$ between objects $\xi_i = (\pi_i : C_i \to S_i, f_i : C_i \to X)$ is a commutative diagram where the left square is cartesian:

$$\begin{tikzcd} S_1 \arrow[d] & C_1 \arrow[l, "\pi_1"'] \arrow[r, "f_1"] \arrow[d] & X \arrow[d, equal] \\ S_2 & C_2 \arrow[l, "\pi_2"'] \arrow[r, "f_2"] & X \end{tikzcd}$$

Let x be a geometric point of X and μ a partition of $d > 0$. As we mentioned in remark 1, we will also be considering the moduli problem in the case of stable maps relative to (x,μ). We use the algebro-geometric definition of this moduli space and its obstruction theory provided in [L1, L2].

The goal of relative stable maps is to parameterise maps where the pre-image of x lies in the smooth locus of C and where the map has monodromy given by μ locally above x. However, this condition will not give a compact space. The solution provided in [L1] is to allow the target to degenerate in a controlled manner by allowing X to sprout a chain of $\mathbb{P}^1$'s.

Specifically we can define the nth degeneration $X[i]$ inductively from $X[0] := X$ by:

- $X[i+1]$ is given by the union $X[i] \cup \mathbb{P}^1$ meeting at a node n_{i+1}.
- The node n_1 is at $x \in X$. For $i > 0$ the node n_{i+1} is in the ith component of $X[i+1]$, i.e. the node is not in $X[i-1] \subset X[i+1]$.

Then a ***degenerated target*** is a pair (T,t) where $T = X[i]$ for some $i \geq 0$ and t is a geometric point in the smooth locus of ith component of T.

A genus g stable map to X relative to (μ, x) is given by

$$\Big(\ h : C \longrightarrow T,\ p : T \longrightarrow X,\ q_1, \ldots, q_{l(\mu)}\ \Big)$$

where (C, q_i) is a $l(\mu)$-marked prestable curve, h is a genus g stable map sending q_i to t and p is a morphism sending t to x such that:

1. There is an equality of divisors on C given by $h^{-1}(t) = \sum \mu_i q_i$.
2. We have $p|_X$ is an isomorphism and $p|_{T\setminus X} : T \setminus X \to \{x\}$ is constant.
3. The pre-image of each node n of T is a union of nodes of C. At any such node n' of C, the two branches of n' map to the two branches of n, and their orders of branching are the same.
4. The data has finitely many automorphisms (recall, an automorphism is a a pair of isomorphisms $a : C \to C$ and $b : T \to T$ taking q_i to q_i and t to t such that $h \circ a = b \circ h$ and $p = p \circ b$).

We denote by $\overline{\mathcal{M}}_g(X,\mu)$ the moduli stack of genus g stable maps relative to (μ, x). This is the groupoid containing the objects:

$$\xi = \left(\begin{tikzcd} C \arrow[d, "\pi"'] \\ S \arrow[u, bend right, "q_i"'] \end{tikzcd} ,\quad \begin{tikzcd} T \arrow[d, "\pi'"'] \\ S \arrow[u, bend right, "t"'] \end{tikzcd} ,\quad h : C \to T,\quad p : T \to X \right)$$

where π and π' are flat proper morphisms, h is a morphism over S and for each geometric point $z \in S$ we have ξ_z is a genus g stable map relative to (μ, x). Furthermore, we require that in a neighbourhood of a node of C_z mapping to a singularity of T_z we can choose étale-local coordinates on S, C and T with charts of the form $\operatorname{Spec} R$, $\operatorname{Spec} R[u,v]/(uv-a)$ and $\operatorname{Spec} R[x,y]/(xy-b)$ respectively such that the map is of the form $x \mapsto \alpha u^k$ and $y \mapsto \alpha v^k$ with α and β units. A morphism $\xi_1 \to \xi_2$ in $\overline{\mathcal{M}}_g(X,\mu)$ between two appropriately label objects is a pair of cartesian diagrams

$$\begin{tikzcd} C_1 \arrow[r, "a'"] \arrow[d, "\pi_1"'] & C_2 \arrow[d, "\pi_2"] \\ S_1 \arrow[r, "a"] & S_2 \end{tikzcd} \qquad\qquad \begin{tikzcd} T_1 \arrow[r, "b'"] \arrow[d, "\pi_1'"'] & T_2 \arrow[d, "\pi_2'"] \\ S_1 \arrow[r, "b"] & S_2 \end{tikzcd}$$

that are compatible with the other data (i.e. we have $a' \circ q_{1,i} = q_{2,i} \circ a$, $b' \circ t_1 = t_2 \circ b$, $b' \circ h_1 = h_2 \circ a'$ and $p_1 = p_2 \circ b'$).

1.2 The Canonical Ramification Section

As we saw in the introduction, for a moduli point $[f] \in \overline{\mathcal{M}}_g(X,d)$ we have two natural morphisms

$$f^*\omega_X \longrightarrow \Omega_C \qquad \text{and} \qquad \Omega_C \longrightarrow \omega c \tag{I.4}$$

which we can combine into a single morphism $\delta : \mathcal{O}_C \to \omega_C \otimes f^*\omega_X^\vee$. This morphism reflects the ramification properties of f which we will see in section 3. Hence, we will call the bundle $\omega_C \otimes f^*\omega_X^\vee$ the ramification bundle of f.

Considering the universal curve $\boldsymbol{\pi} : \overline{\mathcal{C}}_g(X,d) \to \overline{\mathcal{M}}_g(X,d)$. The above construction still holds for the universal stable map $\boldsymbol{f} : \overline{\mathcal{C}}_g(X,d) \to X$. We then have a universal section

$$\boldsymbol{\delta} : \mathcal{O}_{\overline{\mathcal{C}}_g(X,d)} \longrightarrow \overline{\mathcal{R}} \tag{I.5}$$

where we have denoted the universal ramification bundle $\overline{\mathcal{R}} := \omega_{\boldsymbol{\pi}} \otimes \boldsymbol{f}^*\omega_X^\vee$.

For the above case of stable maps we are interested in a subspace where a generic point $[f]$ corresponds to a map f with a ramification divisor of the form $R_f = r \cdot z_1 + \cdots + r \cdot z_m$ for disjoint points $z_1, \ldots, z_m$. However, the key concept of relative stable maps is that the ramification above a fixed point is determined by a given divisor. The ramification is allowed to be free elsewhere.

Hence for the relative case we will be interested in a subspace of $\overline{\mathcal{M}}_g(X,\mu)$ where a generic $[f]$ corresponds to a map f with a ramification divisor of the form:

$$R_f = D_\mu + r \cdot z_1 + \cdots + r \cdot z_m$$

where $D_\mu = \sum(\mu_i - 1)q_i$ is the ramification divisor supported at the points q_i mapping to $x \in X$. So, we are interested in taking rth roots of a section of the bundle $\omega_C \otimes f^*\omega_X^\vee \otimes \mathcal{O}_C(-D_\mu) \cong \omega_C^{\log} \otimes f^*(\omega_X^{\log})^\vee$. The situation is slightly more complicated because of the possibility of a degenerated target. So we consider a general genus g stable map to X relative to (μ, x) over S:

$$\xi = \left(\begin{array}{c} C \\ {\scriptstyle\pi}\big\downarrow \big\uparrow {\scriptstyle q_i} \\ S \end{array} , \quad \begin{array}{c} T \\ {\scriptstyle\pi'}\big\downarrow \big\uparrow {\scriptstyle t} \\ S \end{array} , \quad h : C \to T, \quad p : T \to X \right).$$

Now, letting $q = q_1 + \cdots + q_{l(\mu)}$, we have three line bundles which we are interested in:

$$\omega_{C/S}^{\log} = \omega_{C/S}(q), \qquad \omega_{T/S}^{\log} = \omega_{T/S}(t) \quad \text{and} \quad \omega_X^{\log} = \omega_X(x)$$

and we make choices of morphisms defining the divisors q, t and x respectively:

$$D_q : \mathcal{O}_C(-q) \to \mathcal{O}_C, \qquad D_t : \mathcal{O}_T(-t) \to \mathcal{O}_T \quad \text{and} \quad D_x : \mathcal{O}_X(-x) \to \mathcal{O}_X. \tag{I.6}$$

Now there is a unique choice of isomorphism $p^*\omega_X^{\log} \xrightarrow{\sim} \omega_{T/S}^{\log}$ such that the following diagram commutes

$$\begin{tikzcd} p^*\omega_X \arrow[r, "p^*D_x"] \arrow[d] & p^*\omega_X^{\log} \arrow[d, "\cong"] \\ \omega_{T/S} \arrow[r, "D_t"] & \omega_{T/S}^{\log} \end{tikzcd}$$

where the left vertical morphism is the natural morphism coming from (I.4) applied to $p : T \to X$.

After using the isomorphism $p^*\omega_X^{\log} \xrightarrow{\sim} \omega_{T/S}^{\log}$ we are interested in a canonical morphism $h^*\omega_{T/S}^{\log} \longrightarrow \omega_{C/S}^{\log}$. The construction used in (I.5) breaks down here because there we used the fact that $\Omega_X \cong \omega_X$ is locally free. In general, $\Omega_{T/S} \ncong \omega_{T/S}$ and $\Omega_{T/S}$ is not locally free, because of the nodes on the degenerated target. However, the admissibility condition allows us to define a morphism $h^*\omega_{T/S} \longrightarrow \omega_{C/S}$ directly.

Away from the nodes of T we can simply define the morphism in the usual way. Locally at the nodes we have that $S = \operatorname{Spec} R$, $T = \operatorname{Spec} R[x,y]/(xy-\xi)$ and $C = \operatorname{Spec} R[u,v]/(uv-\zeta)$ with the map h defined by

$$\begin{array}{rccc} H: & R[x,y]/(xy-\xi) & \longrightarrow & R[u,v]/(uv-\zeta) \\ & x & \longmapsto & \alpha u^a \\ & y & \longmapsto & \beta v^a \end{array}$$

for α and β units and with $H(\xi) = \alpha\beta\zeta^a$. Also, locally we have that $\omega_{T/X}$ and $\omega_{C/X}$ are generated by

$$\frac{dx \wedge dy}{(xy-\xi)} \qquad \text{and} \qquad \frac{du \wedge dv}{(uv-\zeta)}$$

respectively. Hence, we have a natural isomorphism locally defined by:

$$\frac{d(H(x)) \wedge d(H(y))}{\big(H(x)H(y) - \Phi(\xi)\big)} = \frac{d(\alpha u^a) \wedge d(\beta v^b)}{\big(\alpha\beta u^a v^a - \alpha\beta\xi^a\big)} = \frac{du \wedge dv}{(uv-\zeta)}.$$

Hence we have the following lemma.

Lemma 1.2.1. *Let T^{sm} be the smooth locus of T relative to S and $B = h^{-1}(T^{\mathrm{sm}})$. There is a canonical morphism $\widetilde{\delta} : h^*\omega_{T/S} \longrightarrow \omega_{C/S}$ such that:*

1. *The restriction $\widetilde{\delta}|_B$ to is the usual morphism $(h|_B)^*\omega_{T^{\mathrm{sm}}/S} \to \omega_{C/S}$,*
2. *$\widetilde{\delta}$ is locally an isomorphism at the nodes of T.*

Now, we restrict the morphism h to the smooth locus of C over S and denote these by h^{sm} and C^{sm} respectively. The morphism from lemma 1.2.1 restricted to C^{sm} is injective and is the divisor sequence for the ramification divisor. Both the divisors $q_1 + \cdots + q_{l(\mu)}$ and $h^{-1}(t)$ are in C^{sm}. Now using the choices from (I.6) it is straightforward to show that there is now a unique map $\widetilde{\delta}^{\log}$ making the following diagram commute

$$\begin{tikzcd} h^*\omega_{T/S} \arrow[r, "h^*D_t"] \arrow[d, "\widetilde{\delta}"] & h^*\omega_{T/S}^{\log} \arrow[d, "\widetilde{\delta}^{\log}"] \\ \omega_{C/S} \arrow[r, "D_q"] & \omega_{C/S}^{\log} \end{tikzcd}$$

Now, using the isomorphism $p^*\omega_X^{\log} \xrightarrow{\sim} \omega_{T/S}^{\log}$ we have the canonical morphism which we desire:

$$\delta^{\log} : \mathcal{O}_{C/S} \longrightarrow \omega_{C/S}^{\log} \otimes f^*(\omega_{T/S}^{\log})^\vee. \tag{I.7}$$

The above construction immediately lends itself to a universal construction. Consider the universal curve $\boldsymbol{\pi} : \overline{\mathcal{C}}_g(X,\mu) \to \overline{\mathcal{M}}_g(X,\mu)$, the universal degenerated target

$\boldsymbol{\pi'} : \mathcal{T} \to \overline{\mathcal{M}}_g(X,\mu)$ with universal maps $\boldsymbol{h} : \overline{\mathcal{C}}_g(X,\mu) \to \mathcal{T}$ and $\boldsymbol{p} : \mathcal{T} \to X$, and universal sections $\boldsymbol{q_i} : \overline{\mathcal{M}}_g(X,\mu) \to \overline{\mathcal{C}}_g(X,\mu)$ and $\boldsymbol{t} : \overline{\mathcal{M}}_g(X,\mu) \to \mathcal{T}$. Then we make, once and for all, choices

$$\boldsymbol{D_q} : \mathcal{O}_{\overline{\mathcal{C}}_g(X,\mu)}(-\boldsymbol{q}) \to \mathcal{O}_{\overline{\mathcal{C}}_g(X,\mu)}, \quad \boldsymbol{D_t} : \mathcal{O}_{\mathcal{T}}(-\boldsymbol{t}) \to \mathcal{O}_{\mathcal{T}} \text{ and } D_x : \mathcal{O}_X(-x) \to \mathcal{O}_X \tag{I.8}$$

which allows us to define the universal section

$$\boldsymbol{\delta}^{\mathbf{log}} : \mathcal{O}_{\overline{\mathcal{C}}_g(X,\mu)} \longrightarrow \overline{\mathcal{R}}^{\log} \tag{I.9}$$

where we have denoted the universal ramification bundle $\overline{\mathcal{R}}^{\log} := \omega_{\boldsymbol{\pi}}^{\log} \otimes \boldsymbol{f}^*(\omega_{\boldsymbol{\pi'}}^{\log})^\vee$.

1.3 r-Prestable curves

Our moduli problem requires the use of nodal curves where the nodes have a balanced r-orbifold structure. These curves are also called twisted curves and were introduced in [AV] to study stable maps where the target is a DM stack. They have since been extensively studied in [ACV, O, AGV, FJR1, FJR2]. In this chapter we are interested in using them in relation to taking rth roots of line bundles, which have been studied in [AJ, Ch1, Ch2].

Definition 1.3.1. Let S be a scheme. An r*-prestable curve* over S of genus g with n markings is:

$$\left(\begin{array}{c} C \\ \downarrow \pi \\ S \end{array} , \left(\begin{array}{c} C \\ x_i \uparrow \\ S \end{array} \right)_{i \in \{1,\ldots,n\}} \right)$$

where

1. π is a proper flat morphism from a tame stack to a scheme;
2. each x_i is a section of π that maps to the smooth locus of C,
3. the fibres of π are purely one dimensional with at worst nodal singularities,
4. the smooth locus C^{sm} is an algebraic space,
5. the coarse space $\overline{\pi} : \overline{C} \to S$ with sections $\overline{x}_i$ is a genus g, n-pointed prestable curve
$$\left(\overline{C}, \;\; \overline{\pi} : \overline{C} \to S, \;\; (\overline{x}_i : S \to \overline{C})_{i \in \{1,\ldots,n\}} \right)$$
6. the local picture at the nodes is given by $[U/\mu_r] \to T$, where
   - $T = \mathrm{Spec} A$, $U = \mathrm{Spec} A[z,w]/(zw - t)$ for some $t \in A$, and the action of μ_r is given by $(z,w) \mapsto (\xi_r z, \xi_r^{-1} w)$.

We denote the space parameterising r-prestable curves by $\mathfrak{M}^r_{g,n}$. This space is shown to be a smooth proper stack in [Ch1]. There is a natural forgetful map

$$\mathfrak{M}^r_{g,n} \longrightarrow \mathfrak{M}_{g,n}$$

which maps an r-prestable curve to its coarse space. This map is flat and surjective of degree 1, but it is not an isomorphism. Its restriction to the boundary is degree $\frac{1}{r}$.

One can also consider r-orbifold structure at smooth marked points as well. Specifically we can include in the definition étale gerbes $\mathcal{X}_i \to X$ which are closed sub-stacks of the smooth locus of the curve $\mathcal{X}_i \hookrightarrow C^{\mathrm{sm}}$. The local picture at an r-orbifold marked point is given by $[V/\mu_r] \to T$, where

$T = \mathrm{Spec} A$, $V = \mathrm{Spec} A[z]$, and the action of μ_r is given by $z \mapsto \xi_r z$.

1.4 Line Bundles and their rth Roots on r-Twisted Curves

The theory of rth roots on line bundles on prestable curves has origins related to theta characteristics [Co]. This led naturally to the study of r-spin structures studied in [J1, J2] using torsion free sheaves for rth roots. r-spin structures were also studied using twisted curves in [AJ, Ch1, Ch2] and other methods in [CCC]. The results of [Ch1] will be of particular interest to us.

Consider a family of r-prestable curves $C \xrightarrow{\gamma} \overline{C} \xrightarrow{\overline{\pi}} S$, and let E be a line bundle on C pulled back from $\overline{C}$ with relative degree divisible by r. Define the following groupoid $\mathrm{Root}^r_C(E)$ containing as objects:

$$\Big(\; h : Z \to S, \quad L, \quad e : L^r \xrightarrow{\sim} E_Z \; \Big)$$

where h is a morphism of schemes, L is a line bundle on C_Z and e is an isomorphism.

Theorem 1.4.1. *[Ch1, Prop 3.7, Thm 3.9] In the situation above we have:*

1. *For each geometric point $p \in S$, we have r^{2g} roots of E_p.*
2. *$\mathrm{Root}^r_C(\mathcal{O}_C)$ is a finite group stack.*
3. *$\mathrm{Root}^r_C(E)$ is a finite torsor under $\mathrm{Root}^r_C(\mathcal{O}_C)$.*
4. *$\mathrm{Root}^r_C(E) \to S$ is étale of degree r^{2g-1}.*

1.4.2. Consider an r-prestable curve over $\mathbb{C}$ with an r-orbifold marked point p. The local picture at the marking p is given by $[(\mathrm{Spec}\,\mathbb{C}[z])/\mu_r]$ where the action of μ_r is given by $z \mapsto \xi_r z$. Consider a line bundle L supported at p. Then (the sheaf) L is locally generated at p by $\phi = z^n$ for $n \in \mathbb{Z}$ and we have $\phi(\xi_r z) = \xi_r^k \phi(z)$ for some $k \in \mathbb{Z}/r$. We call k the *multiplicity* of L at p.

Similarly, at a node q the the local picture is given by $[(\mathrm{Spec}\,\mathbb{C}[u,v]/uv)/\mu_r]$, where the action of μ_r is given by $(u,v) \mapsto (\xi_r u, \xi_r^{-1} v)$. So a line bundle L on C supported at q is locally generated by $\psi = u^{n_1} - v^{n_2}$ for $n_1, n_2 \in \mathbb{Z} \setminus \{0\}$ such that $\psi(\xi_r u, \xi_r v) = \xi_r^a \psi(u,v)$ for some $a \in \mathbb{Z}/r$. In fact a is determined only up to a choice of branch. Hence we obtain a pair numbers $a, b \in \mathbb{Z}/r$ with either $a = b = 0$ or $a + b = r$. We call this pair the *multiplicity* of L at the node q.

We can also consider an associated sheaf on the coarse space $\overline{C}$. Locally the coarse space is given by the invariant sections of the structure sheaf. Then the sheaf $\overline{L} := \gamma_* L$ is similarly given by the locally invariant sections of L. When C is a smooth curve $\overline{L}$ is a line bundle. However when C is singular, then $\overline{L}$ is only torsion free in general. Using these ideas one can easily show the following lemma is true.

Lemma 1.4.3. *Let C be a r-prestable curve over $\mathbb{C}$, with n smooth orbifold points $x_1, \dots, x_n$ and let $\beta : C \to \mathsf{C}$ be the map forgetting the orbifold structure at the smooth points. Also let L be a line bundle on C with multiplicities $a_1, \dots, a_n$ at the the orbifold points and let D be the divisor $\sum a_i x_i$. Then for a section $\sigma : \mathcal{O}_C \to L$ there is a commuting diagram where the bottom row is the divisor sequence:*

$$\begin{tikzcd}
 & \mathcal{O}_{\mathsf{C}} \arrow[r, equal] \arrow[d, "(\beta_*\sigma)^r"] & \mathcal{O}_{\mathsf{C}} \arrow[d, "\beta_*(\sigma^r)"] & & \\
0 \arrow[r] & (\beta_* L)^r \arrow[r] & \beta_*(L^r) \arrow[r] & \mathcal{O}_D \arrow[r] & 0
\end{tikzcd}$$

2 Stable Maps with Roots of Ramification

Section 2 Notation: Recall the notation convention:

- $\mathcal{M}$ is either $\overline{\mathcal{M}}_g(X,d)$ or $\overline{\mathcal{M}}_g(X,\mu)$ for $g \geq 0$, $d > 0$ and μ a partition of d.
- $\mathcal{C} \to \mathcal{M}$ is the associated universal curve.
- If $\mathcal{M}$ is $\overline{\mathcal{M}}_g(X,d)$ (resp. $\overline{\mathcal{M}}_g(X,\mu)$) then $\mathfrak{M}$ is $\mathfrak{M}_g$ (resp. $\mathfrak{M}_{g,l(\mu)}$).
- The expected number of order r special loci in the generic case is denoted by m. When $\mathcal{M} = \overline{\mathcal{M}}_g(X,d)$ we have $m = \frac{1}{r}(2g-2-d(2g_X-2))$ and when $\mathcal{M} = \overline{\mathcal{M}}_g(X,\mu)$ we have $m = \frac{1}{r}(2g-2+l(\mu)+|\mu|(1-2g_X))$.
- Throughout, the notation used for a space without r or $\frac{1}{r}$ will carry through to analogous spaces involving r or $\frac{1}{r}$. For $\overline{\mathcal{M}}_g(X,\mu)$ the two key spaces are:
  - $\mathcal{M}^{[r]} = \overline{\mathcal{M}}_g^{[r]}(X,\mu)$ and $\mathcal{C}^{[r]} = \overline{\mathcal{C}}_g^{[r]}(X,\mu)$ defined in 2.1.3.
  - $\mathcal{M}^{[\frac{1}{r}]} = \overline{\mathcal{M}}_g^{[1/r]}(X,\mu)$ and $\mathcal{C}^{[\frac{1}{r}]} = \overline{\mathcal{C}}_g^{[1/r]}(X,\mu)$ are defined in 2.3.1.3.

  And the associated spaces are:
  - $\mathcal{M}^r = \overline{\mathcal{M}}_g^r(X,\mu)$ and $\mathcal{C}^r = \overline{\mathcal{C}}_g^r(X,\mu)$ defined in 2.1.
  - $\mathcal{M}^{\frac{1}{r},\mathcal{E}} = \overline{\mathcal{M}}_g^{\frac{1}{r},\mathcal{E}}(X,\mu)$ and $\mathcal{C}^{\frac{1}{r},\mathcal{E}} = \overline{\mathcal{C}}_g^{\frac{1}{r},\mathcal{E}}(X,\mu)$ are defined in 2.1.1.
  - $\mathcal{M}^{\frac{1}{r}} = \overline{\mathcal{M}}_g^{1/r}(X,\mu)$ and $\mathcal{C}^{1/r} = \overline{\mathcal{C}}_g^{1/r}(X,\mu)$ are defined in 2.3.1.1.

2.1 r-Stable Maps with Roots of the Ramification Bundle

In this subsection we will be considering the results of [Ch1] in the context of stable maps. We will begin by considering stable maps where the domain curve is r-prestable. We call these r-stable maps. The moduli stack of these and its universal curve fit into the two cartesian squares:

$$\begin{tikzcd}
\mathcal{C}^r \arrow[r] \arrow[d, "\gamma"'] & \mathcal{M}^r \arrow[r] \arrow[d] & \mathfrak{M}^r \arrow[d] \\
\mathcal{C} \arrow[r] & \mathcal{M} \arrow[r] & \mathfrak{M}
\end{tikzcd}$$

Now we will considering stable maps with an rth root of a line bundle. Let $\overline{\mathcal{E}}$ be a line bundle on $\mathcal{C}$ of degree divisible by r and define the line bundle $\mathcal{E}$ on $\mathcal{C}^r$ by $\mathcal{E} := \gamma^*\overline{\mathcal{E}}$.

Definition 2.1.1. Denote by $\mathcal{M}^{\frac{1}{r},\mathcal{E}}$ the moduli stack of *r-stable maps with roots of $\mathcal{E}$* which contains families:

$$\left(\ \xi,\quad \mathcal{L},\quad e : \mathcal{L}^r \xrightarrow{\sim} \mathcal{E}_\xi\ \right)$$

where

1. ξ is a family of r-stable maps in $\mathcal{M}$;
2. $\mathcal{L}$ is a line bundle on $\mathcal{C}_\xi$;
3. e is an isomorphism of line bundles on $\mathcal{C}_\xi$.

Lemma 2.1.2. *$\mathcal{M}^{\frac{1}{r},\mathcal{E}}$ has the following properties:*

1. *$\mathcal{M}^{\frac{1}{r},\mathcal{E}}$ is a proper DM stack.*
2. *When $\mathcal{E}$ is the trivial bundle $\mathcal{M}^{\frac{1}{r},\mathcal{O}_\mathcal{C}} \to \mathcal{M}^r$ is a finite group stack.*
3. *The forgetful map $\mathcal{M}^{\frac{1}{r},\mathcal{E}} \to \mathcal{M}^r$ is a finite torsor under $\mathcal{M}^{\frac{1}{r},\mathcal{O}_\mathcal{C}}$ and is étale of degree r^{2g-1}.*

Proof. Let $a : S \to \mathcal{M}^r$ be the morphism of stacks defined by the family $\xi \in \mathcal{M}^r$. Also let $C = (\mathcal{C}^r)_\xi$ and $E = (\mathcal{E})_\xi$. Then we have the following cartesian diagrams:

$$\begin{tikzcd} \mathrm{Root}^r_C(\mathcal{O}_C) \arrow[r] \arrow[d] & \mathcal{M}^{\frac{1}{r},\mathcal{O}_\mathcal{C}} \arrow[d] \\ S \arrow[r] & \mathcal{M}^r \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Root}^r_C(E) \arrow[r] \arrow[d] & \mathcal{M}^{\frac{1}{r},\mathcal{E}} \arrow[d] \\ S \arrow[r] & \mathcal{M}^r \end{tikzcd}$$

The lemma now follows from theorem 1.4.1. □

Definition 2.1.3. In the special case where $\mathcal{E} = \mathcal{R}$ we call $\mathcal{M}^{\frac{1}{r},\mathcal{R}}$ the moduli space of *r-stable maps with roots of the ramification bundle* and denote it with the simplifying notation:

$$\mathcal{M}^{[r]} := \mathcal{M}^{\frac{1}{r},\mathcal{R}}.$$

2.2 Power Map of Abelian Cone Stacks

Let $\overline{\mathcal{E}}$ be a line bundle on $\mathcal{C}$ of degree divisible by r and define the line bundle $\mathcal{E}$ on $\mathcal{C}^{[r]}$ by $\mathcal{E} := \boldsymbol{\gamma}^*\overline{\mathcal{E}}$ where $\boldsymbol{\gamma} : \mathcal{C}^r \to \mathcal{C}$ is the map forgetting the r-orbifold structure of the curves. Consider $\mathcal{M}^{\frac{1}{r},\mathcal{E}}$, the space of r-stable maps with roots of $\mathcal{E}$ defined in 2.1.1 with universal curve $\boldsymbol{\pi} : \mathcal{C}^{\frac{1}{r},\mathcal{E}} \to \mathcal{M}^{\frac{1}{r},\mathcal{E}}$, universal section $\boldsymbol{s} : \mathcal{M}^{\frac{1}{r},\mathcal{E}} \to \mathcal{C}^{\frac{1}{r},\mathcal{E}}$ and universal rth root bundle $\mathcal{L}$ and isomorphism $\boldsymbol{e} : \mathcal{L}^r \xrightarrow{\sim} \mathcal{E}$.

Definition 2.2.1. For a line bundle $\mathcal{F}$ on $\mathcal{C}^{\frac{1}{r},\mathcal{E}}$, we define the the following notation:

1. $\mathsf{Tot}\,\boldsymbol{\pi}_*\mathcal{F} := \mathrm{Spec}_{\mathcal{M}^{\frac{1}{r},\mathcal{E}}}\left(\mathrm{Sym}^\bullet R^1\boldsymbol{\pi}_*(\mathcal{F}^\vee \otimes \omega_{\boldsymbol{\pi}})\right)$ which contains objects:
$$\Big(\ \xi,\quad \sigma : \mathcal{O}_C \longrightarrow \mathcal{F}_\xi\ \Big)$$
where ξ is an object of $\mathcal{M}^{\frac{1}{r},\mathcal{E}}$ and $C := (\mathcal{C}^{\frac{1}{r},\mathcal{E}})_\xi$ (discussed in [CL, Prop 2.2] and [CLL, Thm 2.11]). Also, let $\boldsymbol{\alpha} : \mathsf{Tot}\,\boldsymbol{\pi}_*\mathcal{F} \to \mathcal{M}^{\frac{1}{r},\mathcal{E}}$ denote the natural forgetful map.
2. $\psi : \mathcal{C}_{\mathsf{Tot}\,\boldsymbol{\pi}_*\mathcal{F}} \to \mathsf{Tot}\,\boldsymbol{\pi}_*\mathcal{F}$ is the universal curve and $\widehat{\boldsymbol{\alpha}} : \mathcal{C}_{\mathsf{Tot}\,\boldsymbol{\pi}_*\mathcal{F}} \to \mathcal{C}^{\frac{1}{r},\mathcal{E}}$ is the natural forgetful map.
3. $\mathsf{Tot}\,\mathcal{F} := \mathrm{Spec}_{\mathcal{C}^{\frac{1}{r},\mathcal{E}}}\left(\mathrm{Sym}^\bullet \mathcal{F}^\vee\right)$ which contains objects:
$$\Big(\ \zeta,\quad \lambda : \mathcal{O}_S \longrightarrow s^*\mathcal{F}_\zeta\ \Big)$$
where ζ is an object of $\mathcal{C}^{\frac{1}{r},\mathcal{E}}$ over S and $s := \boldsymbol{s}_\zeta$. Also, let $\check{\boldsymbol{\alpha}} : \mathsf{Tot}\,\mathcal{F} \to \mathcal{C}^{\frac{1}{r},\mathcal{E}}$ denote the natural forgetful map.

Remark 2.2.2. Note that while we use the notation $\mathsf{Tot}\,\boldsymbol{\pi}_*\mathcal{F}$, it is often the case that $\boldsymbol{\pi}_*\mathcal{F}$ is not locally free. This space is called an *abelian cone* in [BF].

Let ζ be a family in $\mathcal{C}^{\frac{1}{r},\mathcal{E}}$ over S with $\boldsymbol{\pi}_\zeta = \pi : C \to S$ and $s := \boldsymbol{s}_\zeta$. There is a natural evaluation morphism $\mathfrak{e} : \mathcal{C}_{\mathsf{Tot}\,\boldsymbol{\pi}_*\mathcal{F}} \to \mathsf{Tot}\,\mathcal{F}$ defined by

$$\mathfrak{e} \;:\; \Big(\ \zeta,\ \ \sigma : \mathcal{O}_C \longrightarrow \mathcal{F}_\xi\ \Big) \longmapsto \Big(\ \zeta,\ \ s^*\sigma : \mathcal{O}_S \longrightarrow s^*\mathcal{F}_\zeta\ \Big). \tag{I.10}$$

This gives the following commutative diagram where the left-most square is Cartesian:

$$\begin{tikzcd} \mathsf{Tot}\,\boldsymbol{\pi}_*\mathcal{F} \arrow[d, "\boldsymbol{\alpha}"] & \mathcal{C}_{\mathsf{Tot}\,\boldsymbol{\pi}_*\mathcal{F}} \arrow[l, "\boldsymbol{\psi}"'] \arrow[r, "\mathfrak{e}"] \arrow[d, "\widehat{\boldsymbol{\alpha}}"] & \mathsf{Tot}\,\mathcal{F} \arrow[d, "\check{\boldsymbol{\alpha}}"] \\ \mathcal{M}^{\frac{1}{r},\mathcal{E}} & \mathcal{C}^{\frac{1}{r},\mathcal{E}} \arrow[l, "\boldsymbol{\pi}"] \arrow[r, equal] & \mathcal{C}^{\frac{1}{r},\mathcal{E}} \end{tikzcd} \tag{I.11}$$

There are special cases when $\mathcal{F} = \mathcal{L}$ and $\mathcal{F} = \mathcal{L}^r$ and we have we have canonical maps:

Definition 2.2.3. The *rth power map over $\mathcal{M}^{\frac{1}{r},\mathcal{E}}$* is the map $\tau : \mathsf{Tot}\,\boldsymbol{\pi}_*\mathcal{L} \to \mathsf{Tot}\,\boldsymbol{\pi}_*\mathcal{L}^r$ defined by:

$$\Big(\ \xi,\ \ \sigma\ \Big) \longmapsto \Big(\ \xi,\ \ \sigma^r\ \Big)$$

and the *rth power map over $\mathcal{C}^{\frac{1}{r},\mathcal{E}}$* is the similarly defined map $\check{\boldsymbol{\tau}} : \mathsf{Tot}\,\mathcal{L} \to \mathsf{Tot}\,\mathcal{L}^r$.

Out of the two maps defined here, $\check{\boldsymbol{\tau}}$ is nicer. It is a fibre-wise r-fold cover of the total space of $\mathsf{Tot}\,\mathcal{L}^r$ ramified at the zero section. However, τ is the map more directly related to out moduli problem.

Lemma 2.2.4. *The rth power map over $\mathcal{M}^{\frac{1}{r},\mathcal{E}}$ is factors via*

$$\begin{tikzcd} \mathsf{Tot}\,\boldsymbol{\pi}_*\mathcal{L} \arrow[rr, "\tau"] \arrow[dr, "\varphi"'] & & \mathsf{Tot}\,\boldsymbol{\pi}_*\mathcal{L}^r \\ & \mathcal{X} \arrow[ur, "j"'] & \end{tikzcd}$$

where j is a closed immersion and φ is the quotient by the following action of $\mathbb{Z}_r$ on $\mathsf{Tot}\,\boldsymbol{\pi}_*\mathcal{L}$*:*

$$\zeta_r \cdot \Big(\ \xi,\ \ \sigma\ \Big) = \Big(\ \xi,\ \ \zeta_r \cdot \sigma\ \Big)$$

Proof. We first show that the image of τ is a closed substack of $\mathsf{Tot}\,\boldsymbol{\pi}_*\mathcal{L}^r$. Denote the closed immersion defined by taking the graph of τ by $i : \mathsf{Tot}\,\boldsymbol{\pi}_*\mathcal{L} \to \mathsf{Tot}\,\boldsymbol{\pi}_*\mathcal{L}^r \times_{\mathcal{M}^{\frac{1}{r},\mathcal{E}}} \mathsf{Tot}\,\boldsymbol{\pi}_*\mathcal{L}$. Then τ factors via:

$$\begin{tikzcd} \mathsf{Tot}\,\boldsymbol{\pi}_*\mathcal{L} \arrow[r, hook, "i"] \arrow[dr, "\tau"'] & \mathsf{Tot}\,\boldsymbol{\pi}_*\mathcal{L}^r \times_{\mathcal{M}^{\frac{1}{r},\mathcal{E}}} \mathsf{Tot}\,\boldsymbol{\pi}_*\mathcal{L} \arrow[d, "\mathrm{pr}_1"] \\ & \mathsf{Tot}\,\boldsymbol{\pi}_*\mathcal{L}^r \end{tikzcd}$$

We claim that pr_1 is a closed map. To see this we let $\boldsymbol{\psi} : \mathcal{C}_{\mathsf{Tot}\,\boldsymbol{\pi}_*\mathcal{L}^r} \to \mathsf{Tot}\,\boldsymbol{\pi}_*\mathcal{L}^r$ be the universal family. Then we have the following abelian cone stack over $\mathsf{Tot}\,\boldsymbol{\pi}_*\mathcal{L}^r$

$$p : \mathrm{Spec}_{\mathsf{Tot}\,\boldsymbol{\pi}_*\mathcal{L}^r}\Big(\mathrm{Sym}^\bullet R^1\psi_*(\psi^*\mathcal{L}^\vee \otimes \omega_\psi)\Big) \longrightarrow \mathsf{Tot}\,\boldsymbol{\pi}_*\mathcal{L}^r$$

which is isomorphic over $\mathsf{Tot}\,\boldsymbol{\pi}_*\mathcal{L}^r$ to the pullback:

$$\begin{tikzcd} \mathsf{Tot}\,\boldsymbol{\pi}_*\mathcal{L}^r \times_{\mathcal{M}^{\frac{1}{r},\mathcal{E}}} \mathsf{Tot}\,\boldsymbol{\pi}_*\mathcal{L} \arrow[rr, "\sim"] \arrow[dr, "\mathrm{pr}_1"'] & & \mathrm{Spec}_{\mathsf{Tot}\,\boldsymbol{\pi}_*\mathcal{L}^r}\Big(\mathrm{Sym}^\bullet R^1\psi_*(\psi^*\mathcal{L}^\vee \otimes \omega_\psi)\Big) \arrow[dl, "p"] \\ & \mathsf{Tot}\,\boldsymbol{\pi}_*\mathcal{L}^r & \end{tikzcd}$$

p is a closed map, so pr_1 is also closed and $\mathrm{im}(\tau)$ is a well defined closed substack. It is clear that $\mathrm{im}(\tau)$ is isomorphic to the quotient of $\mathsf{Tot}\,\boldsymbol{\pi}_*\mathcal{L}$ by the action of $\mathbb{Z}_r$. $\square$

2.3 Proof of Theorem A

In this section we will prove theorem A about the properties of $\mathcal{M}^{1/r}$. We will also consider related spaces that contain extra information which we will denote by $\mathcal{M}^{[1/r]}$ and $\mathcal{M}^{(1/r)}$. In particular $\mathcal{M}^{[1/r]}$ will be the key space for study in the later sections of this chapter.

Let $\boldsymbol{\pi} : \mathcal{C}^{[r]} \to \mathcal{M}^{[r]}$ be the universal curve of $\mathcal{M}^{[r]}$ and $\boldsymbol{f} : \mathcal{C}^{[r]} \to X$ be the universal r-stable map and $\boldsymbol{\delta} : \mathcal{O}_{\mathcal{C}^r} \to \mathcal{R}$ be the pullback by $\boldsymbol{\gamma} : \mathcal{C}^r \to \mathcal{C}$ of canonical ramification section defined in (I.5) and (I.9). Where $\boldsymbol{\gamma}$ is the map which forgets the r-orbifold structure. Also let $\mathcal{L}$ be the universal rth root on $\mathcal{M}^{[r]}$.

Definition 2.3.1. The moduli spaces of stable maps with divisible ramification are:

1. $\mathcal{M}^{1/r}$ is the substack of $\mathcal{M}^r$ containing families ξ where there exists:

   (a) a line bundle L on $C := (\mathcal{C}^r)_\xi$;

   (b) an isomorphism $e : L^r \xrightarrow{\sim} \mathcal{R}_\xi$;

   (c) a morphism $\sigma : \mathcal{O}_C \to L$;

   such that $e(\sigma^r) = \boldsymbol{\delta}_\xi$.

2. $\mathcal{M}^{(1/r)}$ is the substack of $\mathcal{M}^{[r]}$ containing families $\zeta = (\xi, L, e)$ where ξ, L and e are as above and there exists a morphism $\sigma : \mathcal{O}_C \to L$ as above.

3. $\mathcal{M}^{[1/r]}$ is the substack of $\mathsf{Tot}\,\boldsymbol{\pi}_*\mathcal{L}$ containing families $\chi = (\xi, L, e, \sigma)$ where ξ, L, e and σ are as above.

These three stacks are related by the following diagram where the horizontal arrows are forgetful maps and the vertical arrows are inclusions.

$$\begin{tikzcd}
\mathcal{M}^{[\frac{1}{r}]} \arrow[r] \arrow[d] & \mathcal{M}^{(\frac{1}{r})} \arrow[r] \arrow[d] & \mathcal{M}^{\frac{1}{r}} \arrow[d] \\
\mathsf{Tot}\,\boldsymbol{\pi}_*\mathcal{L} \arrow[r] & \mathcal{M}^{[r]} \arrow[r] & \mathcal{M}^r
\end{tikzcd}$$

After pulling back to $\mathcal{M}^{[r]}$, the canonical ramification section $\boldsymbol{\delta} : \mathcal{O}_{\mathcal{C}^r} \to \mathcal{R}$ and the universal rth root $\boldsymbol{e} : \mathcal{L}^r \xrightarrow{\sim} \mathcal{R}$ define a natural inclusion:

$$\begin{array}{rccc} \boldsymbol{i}' : & \mathcal{M}^{[r]} & \longrightarrow & \mathsf{Tot}\,\boldsymbol{\pi}_*\mathcal{L}^r \\ & \xi & \longmapsto & \big(\,\xi,\, \boldsymbol{e}_\xi^{-1}(\boldsymbol{\delta}_\xi)\,\big). \end{array} \tag{I.12}$$

$\mathcal{M}^{[1/r]}$ now fits into the following cartesian diagram defining $\boldsymbol{\nu}$:

$$\begin{tikzcd}
\mathcal{M}^{[\frac{1}{r}]} \arrow[r, "\boldsymbol{i}"] \arrow[d, "\boldsymbol{\nu}"'] & \mathsf{Tot}\,\boldsymbol{\pi}_*\mathcal{L} \arrow[d, "\boldsymbol{\tau}"] \\
\mathcal{M}^{[r]} \arrow[r, "\boldsymbol{i}'"'] & \mathsf{Tot}\,\boldsymbol{\pi}_*\mathcal{L}^r
\end{tikzcd}$$

Lemma 2.2.4 shows that $\mathcal{M}^{[1/r]}$ is a proper DM stack. We have that $\mathcal{M}^{(1/r)}$ is the quotient of $\mathcal{M}^{[1/r]}$ by the action of $\mathbb{Z}/r$, showing $\mathcal{M}^{(1/r)}$ is a closed substack of $\mathcal{M}^{[r]}$. Also, since $\mathcal{M}^{[r]} \to \mathcal{M}^r$ is proper we can define $\mathcal{M}^{1/r}$ to be the closed substack of $\mathcal{M}^r$ coming from the image of $\mathcal{M}^{(1/r)}$. Hence we have proved theorem A after composing with $\mathcal{M}^r \to \mathcal{M}$, which is flat and proper.

Note that the forgetful map $\mathcal{M}^{[1/r]} \to \mathcal{M}^{(1/r)}$ is étale of degree r. However, the map $\mathcal{M}^{(1/r)} \to \mathcal{M}^{1/r}$ is more complicated and in general not étale. There are cases where the map is étale such as when the genus is zero, then the map is degree $1/r$. For another example, consider the space $\mathcal{M}_g^{(1/r)}(\mathbb{P}^1, 1)$ where the map is étale of degree r^{2g-1}.

3 Branching and Ramification of r-Stable Maps

Section 3 Notation: Consider the universal objects of $\mathcal{M}^{[1/r]}$:

1. The universal curve, $\boldsymbol{\rho} : \mathcal{C}^{[1/r]} \to \mathcal{M}^{[1/r]}$
2. The universal stable map, $\boldsymbol{f} : \mathcal{C}^{[1/r]} \to X$ and $\boldsymbol{F} : \mathcal{C}^{[1/r]} \to X \times \mathcal{M}^{[1/r]}$
3. The universal canonical section, $\boldsymbol{\delta} : \mathcal{O}_{\mathcal{C}^{[1/r]}} \to \mathcal{R}$
4. The universal rth root of $\boldsymbol{\delta}$, $(\mathcal{L}, \boldsymbol{e} : \mathcal{L}^r \xrightarrow{\sim} \mathcal{R}, \boldsymbol{\sigma} : \mathcal{O}_{\mathcal{C}^{[1/r]}} \to \mathcal{L})$

For a family ξ over S in $\mathcal{M}^{[1/r]}$ we will use the following notation

$$C := \mathcal{C}_\xi^{[1/r]},\ \rho := \boldsymbol{\rho}_\xi,\ f := \boldsymbol{f}_\xi,\ F := \boldsymbol{F}_\xi,\ \delta := \boldsymbol{\delta}_\xi,\ L := \mathcal{L}_\xi,\ e := \boldsymbol{e}_\xi \text{ and } \sigma := \boldsymbol{\sigma}_\xi.$$

We denote the expected number of order r ramification loci in the generic case by $m = \frac{1}{r}(2g-2-d(2g_X-2))$ in the case $\mathcal{M} = \overline{\mathcal{M}}_g(X,d)$ and $m = \frac{1}{r}(2g-2+l(\mu)+|\mu|(1-2g_X))$ in the case $\mathcal{M} = \overline{\mathcal{M}}_g(X,\mu)$. For a morphism of sheaves $a : A \to B$, we will denote the associated complex in degree $[-1,0]$ by $[a : A \to B]$.

3.1 Divisor Construction and the Branch Morphism for Stable Maps

As we saw in the introduction and discussed in 1.2 the ramification divisor is not well defined for stable maps. However it is possible to define a branch divisor using the canonical ramification section defined in 1.2. To do this, we must first review a construction of Mumford [MFK, §5.3] which allows us to assign a Cartier divisor to certain complexes of sheaves.

Let Z be a scheme and recall that a complex of sheaves $E^\bullet$ is torsion if the support of each $\mathcal{H}^i(E^\bullet)$ does not contain any of the associated points of Z. Let $E^\bullet$ be a finite torsion complex of free sheaves on Z and let $U \subset Z$ be the complement of $\bigcup_i \operatorname{Supp} \mathcal{H}^i(E^\bullet)$. Then $E^\bullet|_U$ is exact and U contains all the associated points of Z. There are two ways to construct isomorphisms

$$\det E^\bullet|_U \xrightarrow{\sim} \mathcal{O}_U.$$

1. κ: This is a canonical isomorphism which arises from the exactness of $E^\bullet|_U$.
2. Ψ: Which is from an explicit choice of isomorphism $E_i \xrightarrow{\sim} \mathcal{O}_U$ for each i.

So a choice of Ψ defines a section $\Psi \circ \kappa^{-1} \in H^0(U, \mathcal{O}_U^*)$. Also, it is shown in [FP, Lemma 1] that if U contains all the associated points of Z then a section of $H^0(U, \mathcal{O}_U^*)$ defines a canonical section λ of $H^0(Z, \mathcal{K}^*)$. A different choice of Ψ amounts to multiplication of λ by an element of $H^0(Z, \mathcal{O}_Z^*)$. In this way $E^\bullet$ defines an element of $H^0(Z, \mathcal{K}^*/\mathcal{O}_Z^*)$. This construction also holds when $E^\bullet$ is a perfect complex (i.e. locally isomorphic to a finite complex of locally free sheaves).

Definition 3.1.1. Let $E^\bullet$ be a perfect torsion complex. The divisor associated to $E^\bullet$ is the divisor constructed above and is denoted by $\operatorname{div}(E^\bullet)$.

The divisor construction has the following important properties.

Lemma 3.1.2. *[FP, Prop 1] Let $E^\bullet$ be a perfect torsion complex.*

1. $\operatorname{div}(E^\bullet)$ *depends only on the isomorphism class of $E^\bullet$ in the derived category of Z.*
2. *If F is a coherent sheaf admitting a finite free resolution $F^\bullet$, then we have* $\operatorname{div}(F) := \operatorname{div}(F^\bullet)$ *is an effective divisor.*
3. *If D is an effective divisor then* $\operatorname{div}(\mathcal{O}_D) = D$.
4. *The divisor construction is additive on distinguished triangles.*
5. *If $h : Z' \to Z$ is a base change such that $h^*E^\bullet$ is torsion, then we have* $\operatorname{div}(h^*E^\bullet) = h^*\operatorname{div}(E^\bullet)$.
6. *If L is a line bundle then* $\operatorname{div}(E^\bullet \otimes L) = \operatorname{div}(E^\bullet)$.

The divisor construction is used in [FP] to construct a morphism

$$br: \quad \mathcal{M} \quad \longrightarrow \quad \operatorname{Sym}^{m'} X.$$

where m' is the virtual dimension of $\mathcal{M}$. In particular, if $\zeta \in \mathcal{M}$ is a family of stable maps over S and $\mathrm{pr}_2 : X \times S \to S$ is the projection, then they show that the canonical ramification section (see 1.2) defines a pr_2-relative effective Cartier divisor of degree m':

$$B_\zeta := \operatorname{div}\Big(R(\boldsymbol{F}_\zeta)_*[\mathcal{O}_{\mathcal{C}_\zeta} \xrightarrow{\boldsymbol{\delta}_\zeta} \overline{\mathcal{R}}_\zeta]\Big).$$

Hence the map br is defined by $\zeta \mapsto B_\zeta$.

Remark 3.1.3. In [FP] the relative case $\mathcal{M} = \overline{\mathcal{M}}_g(X, \mu)$ is not considered. However, the results and proofs required to define br work in this case when we use the section $\boldsymbol{\delta}$ constructed in section 1.2 and $m' = 2g - 2 - d(2g_X - 2)$ is replaced by $m' = 2g - 2 + l(\mu) + |\mu|(1 - 2g_X)$.

3.2 A Branch Morphism for Maps with Divisible Ramification

We will show in this section that a branch morphism can be constructed for stable maps with divisible ramification. The role of the canonical ramification section will be replaced by its universal rth root.

Lemma 3.2.1. *The direct image $RF_*[\mathcal{O}_C \xrightarrow{\sigma} L]$ is a perfect torsion complex.*

Proof. Recall that F factors via the forgetful map to the coarse space $F = \overline{F} \circ \gamma$ where $\overline{F} := (\overline{f}, \overline{\rho})$. Also recall that γ_* is an exact functor, so we have the quasi-isomorphism $RF_*[\mathcal{O}_C \xrightarrow{\sigma} L] \cong R\overline{F}_*[\mathcal{O}_{\overline{C}} \xrightarrow{\gamma_*\sigma} \gamma_*L]$. $\overline{F}$ has finite tor-dimension, so $R\overline{F}_*[\gamma_*\sigma]$ is quasi-isomorphic to a finite complex of quasi-coherent sheaves on $X \times S$ flat over S. Denote this complex by $E^\bullet$.

Perfect is a local property so we can assume that $S = \mathrm{Spec}A$. Also, let $pr_1 : X \times S \to X$ and $pr_2 : X \times S \to S$ be the natural projections. Thus we have that $M := pr_1^*\mathcal{O}_X(1)$ is an ample line bundle on the fibres of pr_2. Then for sufficiently large n we have for each E_i the following properties:

1. $A_i := \rho^*\rho_*(E_i \otimes M^n) \otimes M^{-n}$ is locally free.
2. The natural map $a_i : A_i \longrightarrow E_i$ is surjective.

Let $K_i = \ker a_i$ and note that these sheaves are all flat over S. Hence, restricting to the fibres of $s \in S$ we have an exact sequence

$$0 \longrightarrow (K_i)_s \longrightarrow (A_i)_s \xrightarrow{a_i} (E_i)_s \longrightarrow 0.$$

We have pr_2 is smooth of relative dimension 1 so any module on the fibres has homological dimension at most 1. Thus showing that $(K_i)_s$ is locally free and hence K_i is locally free. A finite complex of locally free sheaves quasi-isomorphic to $E^\bullet$ can be constructed from the total complex associated to the double complex of these resolutions.

By [FP, Lemma 5] we can show $RF_*[\sigma]$ is torsion on $X \times S$ by showing is when S is a point. Define $Y \subset C$ to be the locus where f is not étale and $Z = f(Y) \subset C$. Note that Z is a finite collection of points in X. Define $\widetilde{C} = C \setminus Y$ with inclusion $j : \widetilde{C} \to C$ and $\widetilde{X} = X \setminus Z$ with inclusion $i : \widetilde{X} \to X$. We also have that $Y = \mathrm{Supp}(\ker\sigma) \cup \mathrm{Supp}(\mathrm{coker}\,\sigma)$ so $[j^*\sigma]$ is exact. Letting $\widetilde{f} = f|_{\widetilde{C}}$ be the restriction we have $i^*Rf_*[\sigma] = R\widetilde{f}_*[j^*\sigma]$. Hence, $i^*Rf_*[\sigma]$ is exact also, showing that the cohomology of $Rf_*[\sigma]$ is supported on points. □

Lemma 3.2.2. *Let $S = \mathrm{Spec}\,A$ be Noetherian and let E be a line bundle on C. There exists a ρ-relative line bundle M on C such that $H^0(C, M)$ and $H^0(C, M \otimes E)$ contain sections which define injective morphisms.*

Proof. Let G be the bundle $\omega_{\overline{\rho}} \otimes \overline{f}^*\mathcal{O}_X(3)$ which is an ample $\overline{\rho}$-relative line bundle on $\overline{C}$. Let $n \in \mathbb{N}$ be large enough that both G^N and $\overline{E} \otimes G^N$ are generated by global sections. We claim that $M := \gamma^*G^N$ has the desired properties. To see this note that C is quasi-compact and so has a finite number of associated points. A standard argument then shows that the subspaces of $H^0(\overline{C}, G^N)$ and $H^0(\overline{C}, \overline{E} \otimes G^N)$ which are not injective morphisms will then have strictly lower dimension. Then consider the isomorphism

$$\gamma_* : H^0(C, E \otimes M) \xrightarrow{\sim} H^0(\overline{C}, \overline{E} \otimes G^N)$$

and consider the pre-image s of a regular section $\overline{s} \in H^0(\overline{C}, \overline{E} \otimes G^N)$. γ_* is an exact functor and $\gamma_*K = 0$ if and only if $K = 0$. Hence we have that s is injective if and only if $\overline{s}$ is. □

Lemma 3.2.3. *Let $\widetilde{M}$ be a relative line bundle on C and an injective morphism $s : \mathcal{O}_C \to \widetilde{M}$. Let D be the divisor on C defined by $s^\vee$. Then*

$$\mathtt{D} := \mathrm{div}\Big(RF_*\big[\mathcal{O}_D \otimes \mathcal{O}_C \xrightarrow{\mathrm{id}\otimes\sigma^k} \mathcal{O}_D \otimes L^k\big]\Big) = 0.$$

Proof. We first show that $\mathtt{D}$ is an effective divisor on $X \times S$ by considering the case where $S = \mathrm{Spec}A$. Note that the map forgetting the stack structure $\gamma : C \to \overline{C}$ has the property that γ_* is left exact. Also, $\mathcal{O}_{\overline{D}}$ is supported in relative dimension 0, so $\mathtt{D}$ is given by

$$\mathtt{D} = \mathrm{div}\Big(\overline{F}_*\big[\mathcal{O}_{\overline{D}} \longrightarrow \gamma_*(\mathcal{O}_D \otimes L^k)\big]\Big).$$

We have that $i : D \to C$ is a relative effective divisor on C with coarse space $j : \overline{D} \to \overline{C}$. The natural map $\phi : \overline{D} \to S$ is quasi-finite and proper so it is also finite. Thus $\overline{D}$ is affine which shows that $(\gamma|_D)_*(\mathcal{O}_D \otimes L^k)$ is generated by global sections and so has sections which give injective morphism.

Let Φ be a section of $(\gamma|_D)_*(\mathcal{O}_D \otimes L^k)$ giving rise to an injective morphism. Then since the divisor construction is additive on distinguished triangles we have:

$$\mathtt{D} = \operatorname{div}\Big(\overline{F}_* j_*[\Phi]\Big)$$

Also, Φ is regular so it is injective and we have $\mathtt{D} = \operatorname{div}(\overline{F}_* j_* \operatorname{coker} \Phi)$. Hence showing that it is a relative effective Cartier divisor.

The degree of a relative effective Cartier divisor for a smooth morphism is locally constant. Hence we can compute the degree at geometric points. We see that the degree of $(\mathtt{D})_z$ is zero for geometric points $z \in S$. $\square$

Corollary 3.2.4. *There is an equality of divisors*

$$\operatorname{div}\big(RF_*[\mathcal{O}_C \xrightarrow{\sigma^k} L^k]\big) = \operatorname{div}\big(RF_*(L \otimes [\mathcal{O}_C \xrightarrow{\sigma^k} L^k])\big)$$

Proof. We have that div is additive on exact sequences. So, to show that two sequences give the same divisor, it will suffice to show that the cone of a morphism between the two complexes is the zero divisor.

We have two distinguished triangles coming from injective sections of M and $M \otimes L^{-1}$, where M is the line bundle from lemma 3.2.2:

$$[\sigma^k] \xrightarrow{s_1} [\sigma^k] \otimes M \longrightarrow \operatorname{Cone}(s_1) \longrightarrow [\sigma^k][1]$$

$$[\sigma^k] \otimes L \xrightarrow{s_2} [\sigma^k] \otimes M \longrightarrow \operatorname{Cone}(s_2) \longrightarrow [\sigma^k][1]$$

We saw in the lemma 3.2.3 that $\operatorname{div}(\operatorname{Cone}(s_1)) = \operatorname{div}(\operatorname{Cone}(s_2)) = 0$ which shows that $[\sigma^k]$ and $[\sigma^k] \otimes L$ have the same divisor.

$\square$

Lemma 3.2.5. *Let $E^\bullet \xrightarrow{a} G^\bullet \xrightarrow{b} H^\bullet$ be morphisms in the derived category. Then there is a distinguished triangle:*

$$\operatorname{cone}(a) \longrightarrow \operatorname{cone}(b \circ a) \longrightarrow \operatorname{cone}(b) \longrightarrow \operatorname{cone}(a)[1]$$

Proof. The result follows immediately from the following commuting diagram with distinguished triangles for rows and columns:

$$\begin{tikzcd}
E^\bullet \arrow[r, "\mathrm{id}"] \arrow[d, "a"] & E^\bullet \arrow[r] \arrow[d, "b\circ a"] & 0 \arrow[r] \arrow[d] & E^\bullet[1] \arrow[d, "a[1]"] \\
G^\bullet \arrow[r, "b"] \arrow[d] & H^\bullet \arrow[r] \arrow[d] & \operatorname{cone}(b) \arrow[r] \arrow[d] & G^\bullet[1] \arrow[d] \\
\operatorname{cone}(a) \arrow[r] \arrow[d] & \operatorname{cone}(b \circ a) \arrow[r] \arrow[d] & \operatorname{cone}(b) \arrow[r] \arrow[d] & \operatorname{cone}(a)[1] \arrow[d] \\
E^\bullet[1] \arrow[r, "\mathrm{id}[1]"] & E^\bullet[1] \arrow[r] & 0 \arrow[r] & E^\bullet[2]
\end{tikzcd}$$

$\square$

Corollary 3.2.6. *Let $a : E \to G$ and $b : G \to H$ be morphisms of coherent sheaves. Then there is a distinguished triangle:*

$$\big[\, E \xrightarrow{a} G \,\big] \longrightarrow \big[\, E \xrightarrow{b\circ a} H \,\big] \longrightarrow \big[\, G \xrightarrow{b} H \,\big] \longrightarrow \big[\, E \xrightarrow{a} G \,\big][1]$$

where $[a]$, $[b]$ and $[b \circ a]$ are considered to be in degree $[-1, 0]$.

Proof. The proof is immediate from lemma 3.2.5. □

Corollary 3.2.7. *We have the equality of divisors on $X \times S$:*

$$\operatorname{div}\Big(RF_*[\mathcal{O}_C \xrightarrow{\sigma^r} L^r]\Big) = r \cdot \operatorname{div}\Big(RF_*[\mathcal{O}_C \xrightarrow{\sigma} L]\Big)$$

Proof. From corollary 3.2.6 we have the distinguished triangle:

$$\big[\, \mathcal{O}_C \xrightarrow{\sigma} L \,\big] \longrightarrow \big[\, \mathcal{O}_C \xrightarrow{\sigma^{n+1}} L^{n+1} \,\big] \longrightarrow L \otimes \big[\, \mathcal{O}_C \xrightarrow{\sigma^n} L^n \,\big] \longrightarrow \big[\, \mathcal{O}_C \xrightarrow{\sigma} L \,\big][1].$$

After applying corollary 3.2.4 this shows that $\operatorname{div}\big(RF_*[\sigma^{n+1}]\big) = \operatorname{div}\big(RF_*[\sigma^n]\big) + \operatorname{div}\big(RF_*[\sigma]\big)$. The result follows from the induction hypothesis. □

Corollary 3.2.8. *(Theorem B) The divisor $\operatorname{div}(RF_*[\sigma])$ is relative effective and the associated morphism $b_\xi : S \to \operatorname{Sym}^m(X)$ defines a morphism of stacks:*

$$\begin{array}{rccc} \mathtt{br}: & \mathcal{M}^{[1/r]} & \longrightarrow & \operatorname{Sym}^m(X) \\ & \xi & \longmapsto & b_\xi. \end{array}$$

which satisfies the following commutative diagram:

$$\begin{tikzcd} \mathcal{M}^{[1/r]} \arrow[r, "\mathtt{br}"] \arrow[d, "\chi"'] & \operatorname{Sym}^m X \arrow[d, "\Delta"] \\ \mathcal{M} \arrow[r, "br"] & \operatorname{Sym}^{rm} X \end{tikzcd}$$

Proof. We have a natural quasi-isomorphism $[\sigma^r] \xrightarrow{\sim} [\delta]$. It is shown in [FP, 3.2] that $\operatorname{div}(RF_*[\delta])$ is a relative effective divisor of degree rm. Hence, corollary 3.2.7 shows that $\operatorname{div}(RF_*[\sigma])$ is relative effective as well and is of degree m. Corollary 3.2.7 also shows that the given diagram is commutative. □

3.3 Special Loci of the Moduli Points

In this subsection we will prove theorem B part 1 by considering the case when $S = \operatorname{Spec}\mathbb{C}$ and examining the ramification properties induced by the rth root condition.

Following [V, GV] we will call a *special loci* a connected component where the map $f : C \to X$ is not étale. Then each special locus is one of:

1. A smooth point of C where f is locally of the form $z \mapsto z^{a+1}$ with $a \in \mathbb{N}$.
2. A node of C such that on each branch f is locally of the form $z \mapsto z^{a_i}$ with $a_i \in \mathbb{N}$.
3. A genus g component B of C where $f|_B$ is constant and on the branches of C meeting B the map f is locally of the form $z \mapsto z^{a_i}$ with $a_i \in \mathbb{N}$.

We can also define a *ramification order* to each type of locus by:

1. a. (2) $a_1 + a_2$. (3) $2g_B - 2 + \sum(a_i + 1)$.

Remark 3.3.1. We use a slightly different definition for stable maps relative to a point $x \in X$. Let $(h : C \to T, p \to X)$ be over $S = \operatorname{Spec}\mathbb{C}$ in $\overline{\mathcal{M}}_g^{[1/r]}(X, \mu)$ with $f = p \circ h$. Then a special locus of f will be a connected component where the map $h : C \to T$ is not étale and not in the pre-image of a node of x. Everything else is the same. This agrees with lemma 1.2.1 which shows that δ will be an isomorphism at pre-images of nodes of T.

We will show that the existence of an rth root of δ is equivalent to each of these special loci having ramification order divisible by r.

Suppose we have $\xi \in \mathcal{M}^{1/r}$ over $S = \operatorname{Spec}\mathbb{C}$. Then locally on the coarse space $\overline{C}$ for each of the types of special loci $\overline{\delta} : \mathcal{O}_{\overline{C}} \to \overline{\mathcal{R}}_\xi$ is of the form:

1. $\mathbb{C}[x] \to \frac{1}{x^a}\mathbb{C}[x]$ given by $a \mapsto a\frac{x^a}{x^a}$.
2. $\mathbb{C}[x,y]/(xy) \to \frac{1}{x^{a_1} - x^{a_2}}\mathbb{C}[x,y]/(xy)$ given by $a \mapsto a\frac{x^{a_1} - x^{a_2}}{x^{a_1} - x^{a_2}}$.
3. At each node $\mathbb{C}[x,y]/(xy) \to \frac{1}{x^{a_i} - x^{b_i}}\mathbb{C}[x,y]/(xy)$ given by $a \mapsto a\frac{x^{a_i}}{x^{a_i} - x^{b_i}}$.

3.3.2. The rth root condition $\sigma^r = e(\delta)$ forces there to be local roots for special loci of types 1 and 2. This forces the divisibility of the ramification order:

> *For type* 1*:* Locally we must have σ being of the form $\mathbb{C}[x] \to \frac{1}{x^{a/r}}\mathbb{C}[x]$ and thus r divides a.
>
> *For type* 2*:* Pulling back from the coarse space via γ we see that δ is of the form $\mathbb{C}[u,v]/(uv) \to \frac{1}{u^{a_1 r} - v^{a_2 r}}\mathbb{C}[u,v]/(uv)$. Then taking the rth root we see that σ is of the form $\mathbb{C}[u,v]/(uv) \to \frac{\zeta_r^k}{u^{a_1} - \zeta_r v^{a_2}}\mathbb{C}[u,v]/(uv)$ for some $k \in \mathbb{Z}/r$. However, there are multiplicities e_1 and e_2 of L at the node with $e_1 + e_2 = r$ or $e_1 = e_2 = 0$. Also we have $a_i = e_i + n_i r$. Hence, r divides $a_1 + a_2$.

3.3.3. We now consider special loci of type 3. Suppose there is a genus g sub-curve B of C where $f|_B$ is constant and on the branches of C meeting B, the map f is locally of the form $z \mapsto z^{a_i}$ with $a_i \in \mathbb{N}$.

Let $A = C \setminus B$ and $\alpha : A \sqcup B \to C$ be the partial normalisation of C separating the contracted component B from A. Also, let p_i be the pre-images of the nodes on A and q_i the pre-images on B. Finally, let a_i and b_i be the multiplicities of L corresponding to the branches on the nodes on A and B respectively.

Now e restricts to an isomorphism $e_B : (L_B)^r \xrightarrow{\sim} (\mathcal{R}_\xi)_B \cong \omega_B(\sum q_i)$. We have a map g : $B \to$ B which forgets the orbifold structure at the points q_i. Pushing forward via g we have the following isomorphism coming from lemma 1.4.3:

$$e_{\mathsf{B}} : (L_{\mathsf{B}})^r \xrightarrow{\sim} \omega_{\mathsf{B}}\left(\sum q_i - \sum b_i q_i\right)$$

Hence, $\omega_{\mathsf{B}}(\sum q_i - \sum b_i q_i)$ must have degree divisible by r. Then r divides $2g - 2 + \sum(1 - bi)$ and also divides $2g - 2 + \sum(a_i + 1)$.

Remark 3.3.4. To consider the relative case in 3.3.3 we must replace $f : C \to X$ by $h : C \to T$. Everything else remains the same.

3.3.5. To finish the proof of the theorem we to show that such an rth root can be constructed if the ramification loci are of the desired form. First we observe that there is a tensor product decomposition of δ:

$$\delta = \delta_{\mathrm{sm}} \otimes \delta_{\mathrm{n}} \otimes \delta_{\mathrm{cn}}$$

where $\delta_{\mathrm{sm}} : \mathcal{O}_C \to R_{\mathrm{sm}}$ and $\delta_{\mathrm{n}} : \mathcal{O}_C \to R_{\mathrm{n}}$ define the divisors of δ supported on the smooth locus of C and nodes of C not meeting contracted components. Then $\delta_{\mathrm{cn}} : \mathcal{O}_C \to R_{\mathrm{cn}}$ is the unique section such that the above decomposition holds.

After reversing the reasoning of 3.3.2 we have rth roots $\sigma_{\mathrm{sm}} : \mathcal{O}_C \to L_{\mathrm{sm}}$ and $\sigma_{\mathrm{n}} : \mathcal{O}_C \to L_{\mathrm{n}}$ of δ_{sm} and δ_{n} respectively. For the contracted components δ_{cn} the line bundle R_{cn} will locally be of the form $\frac{1}{u^{a_i r} - v^{b_i r}}\mathbb{C}[u, v]/(uv)$ at the connecting nodes and δ_{cn} will be of the form $1 \mapsto \frac{u^{a_i r}}{u^{a_i r} - v^{b_i r}}$. Then let L_{cn} be an rth root of R_{cn} which is locally of the form $\frac{1}{u^{a_i} - \zeta_r v^{b_i}}\mathbb{C}[u, v]/(uv)$ at the connecting nodes and σ_{cn} will be of the form $1 \mapsto \frac{u^{a_i}}{u^{a_i} - v^{b_i}}$ and identical to δ_{cn} elsewhere.

Hence we have proved theorem B part 1.

4 Cotangent Complex of $\overline{\mathcal{M}}_g^{1/r}(X, d)$

Section 4 Notation: Recall the notation convention. The following diagram shows the relationships between the relevant spaces. It is commutative and many of the squares are cartesian.

$$\begin{tikzcd}
 & \mathcal{M}^{[\frac{1}{r}]} \arrow[dl, "\boldsymbol{\nu}"'] \arrow[dd, "\boldsymbol{i}"] & & \mathcal{C}^{[\frac{1}{r}]} \arrow[ll, "\boldsymbol{\rho}"'] \arrow[dl, "\widehat{\boldsymbol{\nu}}"'] \arrow[dd, "\boldsymbol{j}"] \arrow[ddrr, "\mathfrak{f}"] & & \\
\mathcal{M}^{[r]} \arrow[dd, "\boldsymbol{i}'"'] & & \mathcal{C}^{[r]} \arrow[ll, "\boldsymbol{\pi}"'] \arrow[dd, "\boldsymbol{j}'"'] \arrow[ddrr, "\mathfrak{f}'"] & & & \\
 & \mathsf{Tot}\,\boldsymbol{\pi}_*\mathcal{L} \arrow[dl, "\boldsymbol{\tau}"'] \arrow[dd, "\boldsymbol{\beta}"] & & \mathcal{C}_{\mathsf{Tot}\,\boldsymbol{\pi}_*\mathcal{L}} \arrow[ll, "\boldsymbol{\psi}"'] \arrow[rr, "\mathfrak{e}"] \arrow[dl, "\widehat{\boldsymbol{\tau}}"'] \arrow[dd, "\widehat{\boldsymbol{\beta}}"] & & \mathsf{Tot}\,\mathcal{L} \arrow[dl, "\check{\boldsymbol{\tau}}"'] \arrow[dd, "\check{\boldsymbol{\beta}}"] \\
\mathsf{Tot}\,\boldsymbol{\pi}_*\mathcal{L}^r \arrow[dd, "\boldsymbol{\alpha}"'] & & \mathcal{C}_{\mathsf{Tot}\,\boldsymbol{\pi}_*\mathcal{L}^r} \arrow[ll, "\boldsymbol{\varphi}"'] \arrow[rr, "\mathfrak{e}'"] \arrow[dd, "\widehat{\boldsymbol{\alpha}}"] & & \mathsf{Tot}\,\mathcal{L}^r \arrow[dd, "\check{\boldsymbol{\alpha}}"] & \\
 & \mathcal{M}^{[r]} \arrow[dl, equal] & & \mathcal{C}^{[r]} \arrow[ll, "\boldsymbol{\pi}"'] \arrow[rr, equal] \arrow[dl, equal] & & \mathcal{C}^{[r]} \arrow[dl, equal] \\
\mathcal{M}^{[r]} & & \mathcal{C}^{[r]} \arrow[ll, "\boldsymbol{\pi}"'] \arrow[rr, equal] & & \mathcal{C}^{[r]} &
\end{tikzcd} \tag{I.13}$$

Here $\boldsymbol{\pi}$ and $\boldsymbol{\rho}$ are the universal curves of their respective spaces. The maps $\boldsymbol{i}$ and $\boldsymbol{i}'$ are the natural inclusions defined by definition 2.3.1.3 and equation (I.12) respectively. The maps $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ are the universal curves defined in 2.2.1 with $\mathfrak{e}$ and $\mathfrak{e}'$ being the natural evaluation maps defined by equation (I.10). The power maps $\boldsymbol{\tau}$ and $\check{\boldsymbol{\tau}}$ are defined in 2.2.3 and $\widehat{\boldsymbol{\tau}}$ is the pullback by $\boldsymbol{\varphi}$ of $\boldsymbol{\tau}$. The maps $\boldsymbol{\alpha}$, $\widehat{\boldsymbol{\alpha}}$, $\check{\boldsymbol{\alpha}}$, $\boldsymbol{\beta}$, $\widehat{\boldsymbol{\beta}}$ and $\check{\boldsymbol{\beta}}$ are the natural projection maps. The maps $\boldsymbol{j}$ and $\boldsymbol{j}'$ are pullbacks of $\boldsymbol{i}$ and $\boldsymbol{i}'$ by $\boldsymbol{\psi}$ and $\boldsymbol{\varphi}$ respectively. Lastly, we also define the maps $\mathfrak{f} := \boldsymbol{i} \circ \mathfrak{e}$ and $\mathfrak{f}' := \boldsymbol{i}' \circ \mathfrak{e}'$.

We denote the expected number of special loci of order r in the generic case by $m = \frac{1}{r}(2g - 2 - d(2g_X - 2))$ in the case $\mathcal{M} = \overline{\mathcal{M}}_g(X, d)$ and $m = \frac{1}{r}(2g - 2 + l(\mu) + |\mu|(1 - 2g_X))$ in the case $\mathcal{M} = \overline{\mathcal{M}}_g(X, \mu)$.

4.1 Perfect Relative Obstruction Theory

Recall, that for a proper representable Gorenstein morphism $a : \mathcal{X} \to \mathcal{Y}$ of relative dimension n with relative dualising sheaf ω_a and any complexes $\mathcal{F}^\bullet \in \mathrm{D}(\mathcal{X})$ and $\mathcal{G}^\bullet \in \mathrm{D}(\mathcal{Y})$ one has the following functorial isomorphism coming from Serre duality (see for example [BBH, eq. C.12]):

$$\mathrm{Hom}_{\mathrm{D}(\mathcal{X})}\big(Ra_*\mathcal{F}^\bullet, \mathcal{G}^\bullet\big) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{D}(\mathcal{Y})}\big(\mathcal{F}^\bullet, a^*\mathcal{G}^\bullet \otimes \omega_a[n]\big). \tag{I.14}$$

Hence one obtains the following natural morphism by looking at the pre-image of the identity:

$$Ra_*(a^*\mathcal{G}^\bullet \otimes \omega_a)[n] \longrightarrow \mathcal{G}^\bullet. \tag{I.15}$$

Now, consider the following sub-diagram of (I.13) coming from the topmost horizontal square and the diagonal square:

$$\begin{tikzcd}
\mathcal{M}^{[1/r]} \arrow[d, "\boldsymbol{\nu}"'] & \mathcal{C}^{[1/r]} \arrow[l, "\boldsymbol{\rho}"'] \arrow[r, "\mathfrak{f}"] \arrow[d, "\widehat{\boldsymbol{\nu}}"] & \mathsf{Tot}\,\mathcal{L} \arrow[d, "\tilde{\boldsymbol{\tau}}"] \\
\mathcal{M}^{[r]} & \mathcal{C}^{[r]} \arrow[l, "\boldsymbol{\pi}"'] \arrow[r, "\mathfrak{f}'"] & \mathsf{Tot}\,\mathcal{L}^r
\end{tikzcd}$$

There are two natural maps arising from this diagram:

$$R\boldsymbol{\rho}_*(\boldsymbol{\rho}^*\mathbb{L}_{\boldsymbol{\nu}} \otimes \omega_{\boldsymbol{\rho}})[1] \longrightarrow \mathbb{L}_{\boldsymbol{\nu}} \qquad \text{and} \qquad L\mathfrak{f}^*\mathbb{L}_{\tilde{\boldsymbol{\tau}}} \longrightarrow \mathbb{L}_{\widehat{\boldsymbol{\nu}}} \cong \boldsymbol{\rho}^*\mathbb{L}_{\boldsymbol{\nu}}.$$

Combining these two we can we define the following morphism:

$$\phi_{\boldsymbol{\nu}} : R\boldsymbol{\rho}_*(L\mathfrak{f}^*\mathbb{L}_{\tilde{\boldsymbol{\tau}}} \otimes \omega_{\boldsymbol{\rho}})[1] \longrightarrow \mathbb{L}_{\boldsymbol{\nu}}.$$

We will show in this subsection that this morphism is a perfect relative obstruction theory.

We will begin by examining a related morphism constructed in the same way. Specifically, we consider the the following sub-diagram of (I.13) coming from the middle horizontal squares:

$$\begin{tikzcd}
\mathsf{Tot}\,\boldsymbol{\pi}_*\mathcal{L} \arrow[d, "\boldsymbol{\tau}"'] & \mathcal{C}_{\mathsf{Tot}\,\boldsymbol{\pi}_*\mathcal{L}} \arrow[l, "\boldsymbol{\psi}"'] \arrow[r, "\mathfrak{e}"] \arrow[d, "\widehat{\boldsymbol{\tau}}"] & \mathsf{Tot}\,\mathcal{L} \arrow[d, "\tilde{\boldsymbol{\tau}}"] \\
\mathsf{Tot}\,\boldsymbol{\pi}_*\mathcal{L}^r & \mathcal{C}_{\mathsf{Tot}\,\boldsymbol{\pi}_*\mathcal{L}^r} \arrow[l, "\boldsymbol{\varphi}"'] \arrow[r, "\mathfrak{e}'"] & \mathsf{Tot}\,\mathcal{L}^r
\end{tikzcd} \tag{I.16}$$

As before we have two natural maps

$$R\boldsymbol{\psi}_*(\boldsymbol{\psi}^*\mathbb{L}_{\boldsymbol{\tau}} \otimes \omega_{\boldsymbol{\psi}})[1] \longrightarrow \mathbb{L}_{\boldsymbol{\tau}} \qquad \text{and} \qquad L\mathfrak{e}^*\mathbb{L}_{\tilde{\boldsymbol{\tau}}} \longrightarrow \mathbb{L}_{\widehat{\boldsymbol{\tau}}} \cong \boldsymbol{\psi}^*\mathbb{L}_{\boldsymbol{\tau}}.$$

which combine to obtain the morphism:

$$\phi_{\boldsymbol{\tau}} : R\boldsymbol{\psi}_*(L\mathfrak{e}^*\mathbb{L}_{\tilde{\boldsymbol{\tau}}} \otimes \omega_{\boldsymbol{\psi}})[1] \longrightarrow \mathbb{L}_{\boldsymbol{\tau}}.$$

The following lemma shows that $\phi_{\boldsymbol{\tau}}$ is a relative obstruction theory.

Lemma 4.1.1. *There is a commuting diagram where the rows are distinguished triangles:*

$$\begin{tikzcd}
L\boldsymbol{\tau}^*R\boldsymbol{\varphi}_*\big(\widehat{\boldsymbol{\alpha}}^*\mathcal{L}^{-r} \otimes \omega_{\boldsymbol{\varphi}}\big)[1] \arrow[r] \arrow[d, "L\boldsymbol{\tau}^*\phi_{\boldsymbol{\alpha}}"] & R\boldsymbol{\psi}_*\big(\widehat{\boldsymbol{\beta}}^*\mathcal{L}^{-1} \otimes \omega_{\boldsymbol{\psi}}\big)[1] \arrow[r] \arrow[d, "\phi_{\boldsymbol{\beta}}"] & R\boldsymbol{\psi}_*\big(L\mathfrak{e}^*\mathbb{L}_{\tilde{\boldsymbol{\tau}}} \otimes \omega_{\boldsymbol{\psi}}\big)[1] \arrow[r] \arrow[d, "\phi_{\boldsymbol{\tau}}"] & L\boldsymbol{\tau}^*R\boldsymbol{\varphi}_*\big(\widehat{\boldsymbol{\alpha}}^*\mathcal{L}^{-r} \otimes \omega_{\boldsymbol{\varphi}}\big)[2] \arrow[d, "\phi_{\boldsymbol{\tau}}[1]"] \\
L\boldsymbol{\tau}^*\mathbb{L}_{\boldsymbol{\alpha}} \arrow[r] & \mathbb{L}_{\boldsymbol{\beta}} \arrow[r] & \mathbb{L}_{\boldsymbol{\tau}} \arrow[r] & L\boldsymbol{\tau}^*\mathbb{L}_{\boldsymbol{\alpha}}[1]
\end{tikzcd}$$

such that $\phi_{\boldsymbol{\beta}}$ and $\phi_{\boldsymbol{\alpha}}$ are relative obstruction theories. Moreover, $\phi_{\boldsymbol{\tau}}$ is also a relative obstruction theory.

Proof. Consider the leftmost square of (I.16) and note that it is cartesian. The distinguished triangle arising from the cotangent complex gives the following diagram where the rows are distinguished triangles:

$$\begin{tikzcd} R\boldsymbol{\psi}_*(\boldsymbol{\psi}^* L\boldsymbol{\tau}^*\mathbb{L}_{\boldsymbol{\alpha}} \otimes \omega_{\boldsymbol{\psi}})[1] \arrow[r] \arrow[d] & R\boldsymbol{\psi}_*(\boldsymbol{\psi}^*\mathbb{L}_{\boldsymbol{\beta}} \otimes \omega_{\boldsymbol{\psi}})[1] \arrow[r] \arrow[d] & R\boldsymbol{\psi}_*(\boldsymbol{\psi}^*\mathbb{L}_{\boldsymbol{\tau}} \otimes \omega_{\boldsymbol{\psi}})[1] \arrow[r] \arrow[d] & R\boldsymbol{\psi}_*(\boldsymbol{\psi}^* L\boldsymbol{\tau}^*\mathbb{L}_{\boldsymbol{\alpha}} \otimes \omega_{\boldsymbol{\psi}})[2] \arrow[d] \\ L\boldsymbol{\tau}^*\mathbb{L}_{\boldsymbol{\alpha}} \arrow[r] & \mathbb{L}_{\boldsymbol{\beta}} \arrow[r] & \mathbb{L}_{\boldsymbol{\tau}} \arrow[r] & L\boldsymbol{\tau}^*\mathbb{L}_{\boldsymbol{\alpha}}[1] \end{tikzcd} \tag{I.17}$$

We also have isomorphisms:

$$R\boldsymbol{\psi}_*(\boldsymbol{\psi}^* L\boldsymbol{\tau}^*\mathbb{L}_{\boldsymbol{\alpha}} \otimes \omega_{\boldsymbol{\psi}}) \cong R\boldsymbol{\psi}_* L\widehat{\boldsymbol{\tau}}^*(\boldsymbol{\varphi}^*\mathbb{L}_{\boldsymbol{\alpha}} \otimes \omega_{\boldsymbol{\varphi}}) \cong L\boldsymbol{\tau}^* R\boldsymbol{\varphi}_*(\boldsymbol{\varphi}^*\mathbb{L}_{\boldsymbol{\alpha}} \otimes \omega_{\boldsymbol{\varphi}})$$

making the first column into the derived pullback of the canonical morphism from equation (I.15):

$$R\boldsymbol{\varphi}_*(\boldsymbol{\varphi}^*\mathbb{L}_{\boldsymbol{\alpha}} \otimes \omega_{\boldsymbol{\varphi}})[1] \longrightarrow \mathbb{L}_{\boldsymbol{\alpha}}.$$

Now consider the rightmost square of (I.16) and note that it has all morphisms over $\mathcal{C}$. This gives the following commutative diagram with distinguished triangles as rows, noting that $L\widehat{\boldsymbol{\tau}}^* L\mathfrak{e}'^*\mathbb{L}_{\check{\boldsymbol{\alpha}}} \cong L\mathfrak{e}^* L\check{\boldsymbol{\tau}}^*\mathbb{L}_{\check{\boldsymbol{\alpha}}}$:

$$\begin{tikzcd} L\widehat{\boldsymbol{\tau}}^* L\mathfrak{e}'^*\mathbb{L}_{\check{\boldsymbol{\alpha}}} \arrow[r] \arrow[d] & L\mathfrak{e}^*\mathbb{L}_{\check{\boldsymbol{\beta}}} \arrow[r] \arrow[d] & L\mathfrak{e}^*\mathbb{L}_{\check{\boldsymbol{\tau}}} \arrow[r] \arrow[d] & L\widehat{\boldsymbol{\tau}}^* L\mathfrak{e}'^*\mathbb{L}_{\check{\boldsymbol{\alpha}}}[1] \arrow[d] \\ L\widehat{\boldsymbol{\tau}}^*\mathbb{L}_{\widehat{\boldsymbol{\alpha}}} \arrow[r] & \mathbb{L}_{\widehat{\boldsymbol{\beta}}} \arrow[r] & \mathbb{L}_{\widehat{\boldsymbol{\tau}}} \arrow[r] & L\widehat{\boldsymbol{\tau}}^*\mathbb{L}_{\widehat{\boldsymbol{\alpha}}}[1] \\ L\widehat{\boldsymbol{\tau}}^*\boldsymbol{\varphi}^*\mathbb{L}_{\boldsymbol{\alpha}} \arrow[r] \arrow[u, "\cong"'] & \boldsymbol{\psi}^*\mathbb{L}_{\boldsymbol{\beta}} \arrow[r] \arrow[u, "\cong"'] & \boldsymbol{\psi}^*\mathbb{L}_{\boldsymbol{\tau}} \arrow[r] \arrow[u, "\cong"'] & L\widehat{\boldsymbol{\tau}}^*\boldsymbol{\varphi}^*\mathbb{L}_{\boldsymbol{\alpha}}[1] \arrow[u, "\cong"'] \end{tikzcd} \tag{I.18}$$

Also, note that $L\mathfrak{e}^*\mathbb{L}_{\check{\boldsymbol{\beta}}} \cong L\mathfrak{e}^*(\check{\boldsymbol{\beta}}^*\mathcal{L}^\vee) \cong L(\check{\boldsymbol{\beta}}\circ\mathfrak{e})^*\mathcal{L}^\vee \cong \widehat{\boldsymbol{\beta}}^*\mathcal{L}^\vee$ and similarly, $L\mathfrak{e}'^*\mathbb{L}_{\check{\boldsymbol{\alpha}}} \cong \widehat{\boldsymbol{\alpha}}^*(\mathcal{L}^r)^\vee$. We now obtain the desired diagram by combining (I.18) with (I.17). We denote the appropriate morphisms by $\phi_{\boldsymbol{\tau}}$, $\phi_{\boldsymbol{\beta}}$ and $\phi_{\boldsymbol{\alpha}}$. It is shown in [CL, Prop. 2.5] that $\phi_{\boldsymbol{\beta}}$ and $\phi_{\boldsymbol{\alpha}}$ are perfect relative obstruction theories.

To show that $\phi_{\boldsymbol{\tau}}$ is an obstruction theory it will suffice to show that $\mathcal{H}^{-1}(\mathrm{cone}(\phi_{\boldsymbol{\tau}})) = \mathcal{H}^0(\mathrm{cone}(\phi_{\boldsymbol{\tau}})) = 0$. We have that $\boldsymbol{\beta} : \mathsf{Tot}\,\boldsymbol{\pi}_*\mathcal{L}^r \to \mathcal{M}$ is representable, so $\mathcal{H}^1(\mathbb{L}_{\boldsymbol{\beta}}) = 0$ and $\mathcal{H}^i(\phi_{\boldsymbol{\beta}}) = 0$ for all $i \geq -1$. Also, $\mathrm{cone}(\phi_{\boldsymbol{\alpha}})$ is quasi-isomorphic to a flat complex $\mathcal{F}^\bullet$ which is zero in all degrees greater than -2. Now by definition $L\boldsymbol{\tau}^*\mathrm{cone}(\phi_{\boldsymbol{\alpha}}) = \boldsymbol{\tau}^*\mathcal{F}^\bullet$ also vanished in degrees greater than -2, making $\mathcal{H}^i(L\boldsymbol{\tau}^*\phi_{\boldsymbol{\alpha}}) = 0$ for all $i \geq -1$. The result now follows from taking the cohomology exact sequence of the distinguished triangle of the cones:

$$\mathcal{H}^{-1}(\mathrm{cone}(\phi_{\boldsymbol{\beta}})) \longrightarrow \mathcal{H}^{-1}(\mathrm{cone}(\phi_{\boldsymbol{\tau}})) \longrightarrow \mathcal{H}^0(\mathrm{cone}(L\boldsymbol{\tau}^*\phi_{\boldsymbol{\alpha}}))$$

$$\longrightarrow \mathcal{H}^0(\mathrm{cone}(\phi_{\boldsymbol{\beta}})) \longrightarrow \mathcal{H}^0(\mathrm{cone}(\phi_{\boldsymbol{\tau}})) \longrightarrow \mathcal{H}^1(\mathrm{cone}(L\boldsymbol{\tau}^*\phi_{\boldsymbol{\alpha}})).$$

□

Lemma 4.1.2. *$\phi_{\boldsymbol{\nu}}$ is the composition of $Li^*\phi_{\boldsymbol{\tau}}$ and the natural differential morphism $Li^*\mathbb{L}_{\boldsymbol{\tau}} \to \mathbb{L}_{\boldsymbol{\nu}}$. In particular, $\phi_{\boldsymbol{\nu}}$ is a relative obstruction theory.*

Proof. There is a commuting diagram, noting that there is an isomorphism $L\boldsymbol{j}^*\boldsymbol{\psi}^*\mathbb{L}_{\boldsymbol{\tau}} \cong \boldsymbol{\rho}^*Lj^*\mathbb{L}_{\boldsymbol{\tau}}$:

$$\begin{tikzcd}
L\boldsymbol{j}^*L\mathfrak{e}^*\mathbb{L}_{\check{\boldsymbol{\tau}}} \arrow[r] \arrow[d, equal] & L\boldsymbol{j}^*\mathbb{L}_{\widehat{\boldsymbol{\tau}}} \arrow[d] & L\boldsymbol{j}^*\boldsymbol{\psi}^*\mathbb{L}_{\boldsymbol{\tau}} \arrow[l, "\cong"'] \arrow[d] \\
L\mathfrak{f}^*\mathbb{L}_{\check{\boldsymbol{\tau}}} \arrow[r] & \mathbb{L}_{\widehat{\boldsymbol{\nu}}} & \boldsymbol{\rho}^*\mathbb{L}_{\boldsymbol{\nu}} \arrow[l, "\cong"']
\end{tikzcd}$$

which gives the left square of the following diagram after applying the functor $R\boldsymbol{\rho}_*(\_\otimes \omega_{\boldsymbol{\rho}})[1]$ and using the isomorphism of functors $R\boldsymbol{\rho}_*(L\boldsymbol{j}^*\_\otimes\omega_{\boldsymbol{\rho}})[1] \cong L\boldsymbol{i}^*R\boldsymbol{\psi}_*(\_\otimes \omega_{\boldsymbol{\psi}})[1]$:

$$\begin{tikzcd}
L\boldsymbol{i}^*R\boldsymbol{\psi}_*(L\mathfrak{e}^*\mathbb{L}_{\check{\boldsymbol{\tau}}}\otimes\omega_{\boldsymbol{\rho}})[1] \arrow[r] \arrow[d, "\cong"'] & L\boldsymbol{i}^*R\boldsymbol{\psi}_*(\boldsymbol{\psi}^*\mathbb{L}_{\boldsymbol{\tau}}\otimes\omega_{\boldsymbol{\psi}})[1] \arrow[r] \arrow[d] & L\boldsymbol{i}^*\mathbb{L}_{\boldsymbol{\tau}} \arrow[d] \\
R\boldsymbol{\rho}_*(L\mathfrak{f}^*\mathbb{L}_{\check{\boldsymbol{\tau}}}\otimes\omega_{\boldsymbol{\rho}})[1] \arrow[r] & R\boldsymbol{\rho}_*(\boldsymbol{\rho}^*\mathbb{L}_{\boldsymbol{\nu}}\otimes\omega_{\boldsymbol{\rho}})[1] \arrow[r] & \mathbb{L}_{\check{\boldsymbol{\tau}}}
\end{tikzcd}$$

Now, $L\boldsymbol{i}^*\phi_{\boldsymbol{\tau}}$ is the composition of the top row and $\phi_{\boldsymbol{\nu}}$ is the composition of the bottom row. Hence, $\phi_{\boldsymbol{\nu}}$ is the composition of the desired morphisms.

The maps $\boldsymbol{i}$ and $\boldsymbol{i'}$ are immersions and $\boldsymbol{\tau}$ and $\boldsymbol{\nu}$ are representable, so $L\boldsymbol{i}^*\mathbb{L}_{\boldsymbol{\tau}} \to \mathbb{L}_{\boldsymbol{\nu}}$ is a relative obstruction theory (see for example [BF, §7]). We now consider the distinguished triangle of cones coming from composition of lemma 3.2.5. The reasoning that $\phi_{\boldsymbol{\nu}}$ is a relative obstruction theory is now the same as for $\phi_{\boldsymbol{\tau}}$ in the previous lemma, lemma 4.1.1. □

Lemma 4.1.3. *The left derived pullback by* $\mathfrak{f}$ *of the map* $\check{\boldsymbol{\tau}}^*\mathbb{L}_{\check{\boldsymbol{\alpha}}} \to \mathbb{L}_{\check{\boldsymbol{\beta}}}$ *is the map:*

$$r\,\boldsymbol{j}^*\boldsymbol{\sigma}^{r-1} : \boldsymbol{j}^*\widehat{\boldsymbol{\beta}}^*\mathcal{L}^{\text{-}r} \longrightarrow \boldsymbol{j}^*\widehat{\boldsymbol{\beta}}^*\mathcal{L}^{\text{-}1}$$

where σ *is the universal* r*th root.*

Proof. It will suffice to show this locally. The local situation is described by the diagram

$$\begin{tikzcd}
\mathbb{A}^1\times U \arrow[r] \arrow[d, "\mathtt{t}"] & {[\mathbb{A}^1\times U/G]} \arrow[r] \arrow[d] & \mathsf{Tot}\,\mathcal{L} \arrow[d, "\check{\boldsymbol{\tau}}"] \\
\mathbb{A}^1\times U \arrow[r] \arrow[d, "\mathtt{a}"] & {[\mathbb{A}^1\times U/G]} \arrow[r] \arrow[d] & \mathsf{Tot}\,\mathcal{L}^r \arrow[d, "\check{\boldsymbol{\alpha}}"] \\
U \arrow[r] & {[U/G]} \arrow[r] & \mathcal{C}^{[r]}
\end{tikzcd}$$

where $U = \mathrm{Spec} B$, G is a finite group and $\mathtt{t}$ is defined by the morphism of B-algebras $B[z] \to B[w]$ with $z \mapsto w^r$. On U we have that $\mathcal{L}$ is given by an equivariant line bundle E defined by a local generator ϕ. Then $\mathcal{L}^r$ corresponds to E^r with ϕ^r.

We have that $\mathbb{L}_{\check{\boldsymbol{\beta}}} \cong \Omega_{\check{\boldsymbol{\beta}}} \cong \check{\boldsymbol{\beta}}^*\mathcal{L}^{\text{-}1}$ and the last morphism is defined locally by $dw \mapsto \frac{1}{\phi}$. Similarly, $\mathbb{L}_{\check{\boldsymbol{\alpha}}} \cong \Omega_{\check{\boldsymbol{\alpha}}} \cong \check{\boldsymbol{\alpha}}^*\mathcal{L}^{\text{-}r}$ is defined by $dz \mapsto \frac{1}{\phi^r}$ and the map $d\check{\boldsymbol{\tau}}$ is defined by:

$$\begin{array}{rccc}
d\mathtt{t}: & B[w]\otimes A[z]dz & \longrightarrow & A[w]dw \\
 & 1\otimes dz & \longmapsto & rw^{r-1}dw.
\end{array}$$

Let $\sigma_B \in B$ be the pullback of σ to U. Locally on U the map $\boldsymbol{j}$ is defined by the section σ_B via the following map $B[w] \to B : w \mapsto \sigma_B$. Hence, pulling back via $\boldsymbol{j}$ we have that the map $\boldsymbol{j}^*d\check{\boldsymbol{\tau}}$ is locally defined by:

$$\begin{array}{rccc}
\boldsymbol{j}^*d\mathtt{t}: & B\frac{1}{\phi^r} & \longrightarrow & B\frac{1}{\phi} \\
 & \frac{1}{\phi^r} & \longmapsto & r(\sigma_B)^{r-1}\frac{1}{\phi}.
\end{array}$$

□

Theorem 4.1.4. *The map $\phi_{\boldsymbol{\nu}} : R\boldsymbol{\rho}_*(L\mathfrak{f}^*\mathbb{L}_{\check{\boldsymbol{\tau}}} \otimes \omega_{\boldsymbol{\rho}})[1] \longrightarrow \mathbb{L}_{\boldsymbol{\nu}}$ is a perfect relative obstruction theory with relative virtual dimension $(1-r)m$. (Note that rm is the virtual dimension of $\mathcal{M}^{[r]}$. This follows from the discussion after this theorem.)*

Proof. Let $\rho : C \to S$ be a family in $\mathcal{M}^{[\frac{1}{r}]}$. In the derived category we have the following isomorphisms

$$L\mathfrak{f}^*\mathbb{L}_{\check{\boldsymbol{\tau}}} \otimes \omega_{\boldsymbol{\rho}} \cong L\mathfrak{f}^*\big([\check{\boldsymbol{\tau}}^*\mathbb{L}_{\check{\boldsymbol{\alpha}}} \longrightarrow \mathbb{L}_{\check{\boldsymbol{\beta}}}] \otimes \omega_{\boldsymbol{\rho}}\big) \cong \big[\,\widehat{\boldsymbol{\beta}}^*\mathcal{L}^{-r} \otimes \omega_{\boldsymbol{\rho}} \longrightarrow \widehat{\boldsymbol{\beta}}^*\mathcal{L}^{-1} \otimes \omega_{\boldsymbol{\rho}}\,\big].$$

Denote the restriction to S of this complex by

$$E^{\bullet} = [E_{-1} \xrightarrow{\theta} E_0] = [f^*\omega_X \otimes \mathcal{O}_C \overset{\mathrm{id}\otimes\sigma^{r-1}}{\longrightarrow} f^*\omega_X \otimes L^{r-1}] \tag{I.19}$$

with the last equality following from lemma 4.1.3. Let M be a line bundle on C which is ample on the fibres of ρ. Then for sufficiently large n we have for each E_i the following properties:

1. $\rho^*\rho_*(E_i \otimes M^n) \otimes M^{-n} \longrightarrow E_i$ is surjective.
2. $R^1\rho_*E_i \otimes M^n = 0$.
3. For all $z \in S$ we have $H^0(C_z, \rho^*\rho_*(E_i \otimes M^n) \otimes M^{-n}) = 0$.

Denote the locally free sheaf $\rho^*\rho_*(E_0 \otimes M^n) \otimes M^{-n}$ by A_{E_0} and the associated map from property 1 above by a. Then using the fibre product for modules we have the following commuting diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \ker(a) \arrow[r] \arrow[d, equal] & G \arrow[r] \arrow[d, "\widetilde{\theta}"] & E_{-1} \arrow[r] \arrow[d, "\theta"] & 0 \\
0 \arrow[r] & \ker(a) \arrow[r] & A_{E_0} \arrow[r, "a"] & E_0 \arrow[r] & 0
\end{tikzcd} \tag{I.20}$$

where G also fits into the exact sequence:

$$0 \longrightarrow G \longrightarrow E_{-1} \oplus A_{E_0} \xrightarrow{\binom{-\theta}{a}} E_0 \longrightarrow 0. \tag{I.21}$$

The diagram in (I.20) shows that there is an isomorphism $[G \xrightarrow{\widetilde{\theta}} A_{E_0}] \cong [E_{-1} \xrightarrow{\theta} E_0]$ in the derived category.

The exact sequence in (I.21) shows that G is locally free and hence the diagram in (I.20) contains only flat modules. Hence for $z \in S$ we may restrict to the fibre C_z and maintain exactness. Then using the snake lemma we have an isomorphism

$$\ker\theta_z \cong \ker\widetilde{\theta}_z.$$

We claim that $H^0(C_z, \ker\theta_z) = 0$. To see this take $s \in H^0(C_z, \ker\theta_z)$ and note that $s \in H^0(C_z, E_{-1})$ and s is in the kernel of θ_z. From (I.19) we know that $(E_{-1})_z = f_z^*\omega_X$ and $\theta_z = \sigma_z^{r-1}$, so θ_z only vanishes where f_z is constant. Hence, we let $B \subset C_z$ be the union of components contracted by f_z. Then we have $\operatorname{Supp} s \subset B$ and $(E_{-1})_z|_B = (f_z^*\omega_X)|_B \cong \mathcal{O}_B$ so we must have $s = 0$. Hence, $H^0(C_z, \ker\theta_z) = H^0(C_z, \ker\widetilde{\theta}_z) = 0$.

From property 3 of the definition of A_{E_0} we have $H^0(C_z, (A_{E_0})_z) \cong 0$. So the following exact sequence shows that $H^0(C_z, G_z) = 0$:

$$0 \longrightarrow H^0(C_z, \ker \widetilde{\theta}_z) \longrightarrow H^0(C_z, G_z) \longrightarrow H^0(C_z, (A_{\widetilde{L}})_z)$$

Hence, we have that $R^1\rho_* G$ is locally free and $R\rho_* G \cong [R^1\rho_* G][-1]$. Moreover $R\rho_*[G \xrightarrow{\widetilde{\theta}} A_{\widetilde{L}}] \cong [R^1\rho_* G \xrightarrow{R^1\rho_*\widetilde{\theta}} R^1\rho_* A_{\widetilde{L}}][-1]$ is a complex of locally free sheaves concentrated in degree $[0,1]$.

The virtual dimension follows immediately from Riemann-Roch for twisted curves (see [AGV, §7.2]) applied to $E^\bullet$. □

The space $\mathcal{M}^{[r]}$ has a natural perfect obstruction theory following from the construction of [B2, L2]. It will suffice to show that the space of r-stable maps $\mathcal{M}^r$ has a perfect obstruction theory since by lemma 2.1.2 the map $\mathcal{M}^{[r]} \to \mathcal{M}^r$ is étale. Recall the construction of the perfect obstruction theory for the moduli space of stable maps $\mathcal{M}$ is defined via the relative to the forgetful morphism

$$\overline{\mathcal{M}}_g(X,d) \longrightarrow \mathfrak{M}_g.$$

It is pointed out in [GV, §2.8] that in the case of relative stable maps the perfect obstruction theory can be constructed relative to the morphism

$$\overline{\mathcal{M}}_g(X,\mu) \longrightarrow \mathfrak{M}_{g,l(\mu)} \times \mathcal{T}_X$$

where $\mathcal{T}_X$ is the moduli space parameterising the degenerated targets. We have the following two cartesian squares where the bottom arrows are flat:

$$\begin{tikzcd} \overline{\mathcal{M}}^r_g(X,d) \arrow[r] \arrow[d, "\boldsymbol{p}^{\mathrm{abs}}"'] & \overline{\mathcal{M}}_g(X,d) \arrow[d] \\ \mathfrak{M}^r_g \arrow[r] & \mathfrak{M}_g \end{tikzcd} \qquad \begin{tikzcd} \overline{\mathcal{M}}^r_g(X,\mu) \arrow[r] \arrow[d, "\boldsymbol{p}^{\mathrm{rel}}"'] & \overline{\mathcal{M}}_g(X,\mu) \arrow[d] \\ \mathfrak{M}^r_{g,l(\mu)} \times \mathcal{T}_X \arrow[r] & \mathfrak{M}_{g,l(\mu)} \times \mathcal{T}_X \end{tikzcd}$$

We let $\boldsymbol{p} : \mathcal{M}^r \to \mathcal{X}$ be one of $\boldsymbol{p}^{\mathrm{abs}}$ or $\boldsymbol{p}^{\mathrm{rel}}$ maps depending on the choice of $\mathcal{M}^r$. Then we have a natural perfect relative obstruction for $\boldsymbol{p}$ by pulling back via $\mathcal{M}^r \to \mathcal{M}$:

$$\phi_{\boldsymbol{p}} : E^\bullet_{\boldsymbol{p}} \longrightarrow \mathbb{L}_{\boldsymbol{p}}.$$

Then a perfect obstruction theory for $\mathcal{M}^r$ is given by the following cone construction:

$$\begin{tikzcd} E^\bullet_{\boldsymbol{p}}[-1] \arrow[r] \arrow[d, "\phi_{\boldsymbol{p}}[-1]"] & \boldsymbol{p}^*\mathbb{L}_{\mathcal{X}} \arrow[r] \arrow[d, equal] & F^\bullet_{\mathcal{M}^m} \arrow[r] \arrow[d, "\phi"] & E^\bullet_{\boldsymbol{p}} \arrow[d, "\phi_{\boldsymbol{p}}"] \\ \mathbb{L}_{\boldsymbol{p}}[-1] \arrow[r] & \boldsymbol{p}^*\mathbb{L}_{\mathcal{X}} \arrow[r] & \mathbb{L}_{\mathcal{M}^r} \arrow[r] & \mathbb{L}_{\boldsymbol{\nu}} \end{tikzcd}$$

Corollary 4.1.5. *(Theorem C) If $g = 0$ there is a perfect obstruction theory for $\mathcal{M}^{[1/r]}$ giving virtual dimension m. Moreover, since $\mathcal{M}^{[1/r]} \to \mathcal{M}^{1/r}$ is étale in genus 0, there is a perfect obstruction theory for $\mathcal{M}^{1/r}$.*

Proof. Let $E^\bullet = R\boldsymbol{\rho}_*(L\mathfrak{f}^*\mathbb{L}_{\boldsymbol{\tau}} \otimes \omega_{\boldsymbol{\rho}})$. Then there the following is a commutative diagram with distinguished triangles for rows:

$$\begin{tikzcd} E^\bullet[-1] \arrow[r] \arrow[d, "\phi_{\boldsymbol{\nu}}[-1]"] & \boldsymbol{\nu}^*\mathbb{L}_{\mathcal{M}^{[r]}} \arrow[r] \arrow[d, equal] & F^\bullet \arrow[r] \arrow[d, "\phi"] & E^\bullet \arrow[d, "\phi_{\boldsymbol{\nu}}"] \\ \mathbb{L}_{\boldsymbol{\nu}}[-1] \arrow[r] & \boldsymbol{\nu}^*\mathbb{L}_{\mathcal{M}^{[r]}} \arrow[r] & \mathbb{L}_{\mathcal{M}^{[1/r]}} \arrow[r] & \mathbb{L}_{\boldsymbol{\nu}} \end{tikzcd}$$

Here $F^\bullet$ is defined via the cone construction. As before we have an exact sequence of cohomology of the cones:

$$\begin{aligned}&\mathcal{H}^{-1}(\mathrm{cone}(\mathrm{id})) \longrightarrow \mathcal{H}^{-1}(\mathrm{cone}(\phi)) \longrightarrow \mathcal{H}^{-1}(\mathrm{cone}(\phi_{\boldsymbol{\nu}}))\\ \longrightarrow\ &\mathcal{H}^{0}(\mathrm{cone}(\mathrm{id})) \longrightarrow \mathcal{H}^{0}(\mathrm{cone}(\phi)) \longrightarrow \mathcal{H}^{0}(\mathrm{cone}(\phi_{\boldsymbol{\nu}})).\end{aligned}$$

Which shows that $\mathcal{H}^{-1}(\mathrm{cone}(\phi)) = \mathcal{H}^{0}(\mathrm{cone}(\phi)) = 0$. □

Chapter II

The Donaldson-Thomas Theory of the Banana Threefold with Section Classes

1 Introduction

1.1 Donaldson-Thomas Partition Functions

Donaldson-Thomas theory provides a virtual count of curves on a threefold. It gives us valuable information about the structure of the threefold and has strong links to high-energy physics.

For a non-singular Calabi-Yau threefold Y over $\mathbb{C}$ we let

$$\mathrm{Hilb}^{\beta,n}(Y) = \left\{ Z \subset Y \;\middle|\; [Z] = \beta \in H_2(Y), n = \chi(\mathcal{O}_Z) \right\}$$

be the Hilbert scheme of one dimensional proper subschemes with fixed homology class and holomorphic Euler characteristic. We can define the (β, n) ***Donaldson-Thomas invariant*** of Y by:

$$\mathsf{DT}_{\beta,n}(Y) = 1 \cap [\mathrm{Hilb}^{\beta,n}(Y)]^{\mathrm{vir}}.$$

Behrend proved the surprising result in [B1] that the Donaldson-Thomas invariants invariants are actually weighted Euler characteristics of the Hilbert scheme:

$$\mathsf{DT}_{\beta,n}(Y) = e(\mathrm{Hilb}^{\beta,n}(Y), \nu) := \sum_{k \in \mathbb{Z}} k \cdot e(\nu^{-1}(k)).$$

Here $\nu : \mathrm{Hilb}^{\beta,n}(Y) \to \mathbb{Z}$ is a constructible function called the ***Behrend function*** and its values depend formally locally on the scheme structure of $\mathrm{Hilb}^{\beta,n}(Y)$. We also define the ***unweighted Donaldson-Thomas*** invariants to be:

$$\widehat{\mathsf{DT}}_{\beta,n}(Y) = e(\mathrm{Hilb}^{\beta,n}(Y)).$$

These are often closely related to Donaldson-Thomas invariants and their calculation provides insight to the structure of the threefold. Moreover, many important properties Donaldson-Thomas invariants such as the PT/DT correspondence and the flop formula also hold for the unweighted case [T1, T2].

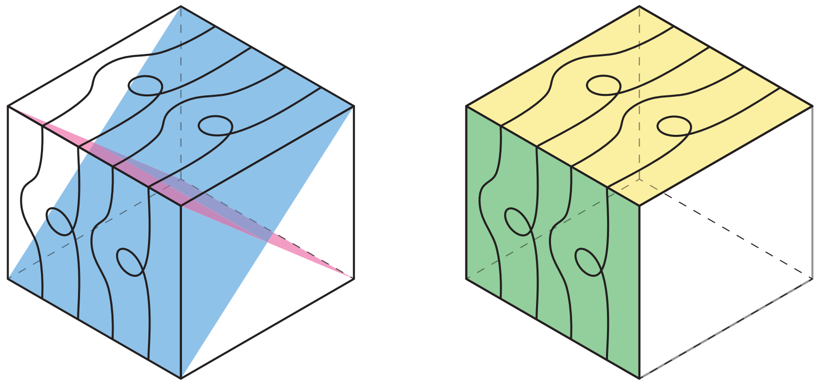


Figure II.1: A visual representation of the banana threefold. On the left the diagonal S_Δ and the anti-diagonal S_{op} are highlighted. On the right the two rational elliptic surfaces S_1 and S_2 are highlighted.

The depth of Donaldson-Thomas theory is often not clear until one assembles the invariants into a partition function. Let $\{C_1, \ldots, C_N\}$ be a basis for $H_2(Y, \mathbb{Z})$, chosen so that if $\beta \in H_2(Y, \mathbb{Z})$ is effective then $\beta = d_1C_1 + \cdots + d_NC_N$ with each $d_i \geq 0$. The Donaldson-Thomas partition function of Y is:

$$Z(Y) := \sum_{\beta \in H_2(Y,\mathbb{Z})} \sum_{n \in \mathbb{Z}} \mathsf{DT}_{\beta,n}(Y) Q^\beta p^n$$
$$= \sum_{d_1,\ldots,d_N \geq 0} \sum_{n \in \mathbb{Z}} \mathsf{DT}_{(\sum_i d_iC_i),n}(Y) Q_i^{d_i} p^n.$$

We also define the analogous partition function $\widehat{Z}$ for the unweighted Donaldson-Thomas invariants.

Remark 1.1.1. This choice of variable is not necessarily the most canonical as shown in [Br] where the variable p is substituted for $-p$. However, in this chapter we will be focusing on the unweighted Donaldson-Thomas invariants where this choice makes the most sense.

This partition function is *very* hard to compute and for proper Calabi-Yau threefolds, the only known complete examples are in computationally trivial cases. However, when we restrict our attention to subsets of $H_2(Y, \mathbb{Z})$ there are many remarkable results. Two case which we will be related to computations are the Schoen (Calabi-Yau) threefold of [S] and the banana (Calabi-Yau) threefold of [Br].

We will employ computational techniques developed in [BK] for studying Donaldson-Thomas theory of local elliptic surfaces.

1.2 Donaldson-Thomas Theory of Banana Threefolds

The banana threefold is of primary interest to us and is defined as follows. Let $\pi : S \to \mathbb{P}^1$ be a generic rational elliptic surface with a section $\zeta : \mathbb{P}^1 \to S$. We will take S to be $\mathbb{P}^2$ blown-up at 9 points which gives rise to 9 natural choices for ζ. The associated *banana threefold* is the blow-up

$$X := \mathrm{Bl}_\Delta(S \times_{\mathbb{P}^1} S) \tag{1}$$

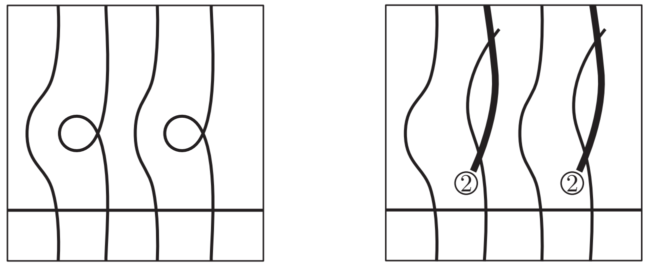


Figure II.2: On the left is a visual representation of the rational elliptic surfaces S_1, S_2 and S_{op}. On the right is the diagonal surface S_Δ. Note that the exceptional curves in the fibres of the pencil have order 2.

where Δ is the diagonal divisor in $S \times_{\mathbb{P}^1} S$. The surface S is smooth but the morphism $\pi : S \to \mathbb{P}^1$ is not. It is singular at 12 points of S and this gives rise to 12 conifold singularities of $S \times_{\mathbb{P}^1} S$ that all lie on the divisor Δ. This makes X a conifold resolution of $S \times_{\mathbb{P}^1} S$. It is a non-singular simply connected proper Calabi-Yau threefold as shown in [Br, Prop. 28].

The section $\zeta : \mathbb{P}^1 \to S$ gives a section $\sigma : \mathbb{P}^1 \to X$ of the natural map $\mathrm{pr} : X \to \mathbb{P}^1$. It also gives natural sections of the projections $\mathrm{pr}_i : X \to S$ which we denote by S_1 and S_2. These are both divisors of X that are copies of the rational elliptic surface. The diagonal Δ and anti-diagonal Δ^{op} of $S \times_{\mathbb{P}^1} S$ are also divisors which are copies of S. The anti-diagonal intersects the diagonal in a curve on Δ^{op}, so it is unaffected by the blow-up. We denote the anti-diagonal divisor in X by S_{op} and the proper transform of the diagonal by S_Δ. The latter is a rational elliptic surface blown-up at the 12 nodal points of the fibres.

The generic fibres of the map $\mathrm{pr} : X \to \mathbb{P}^1$ are Abelian surfaces of the form $E \times E$ where $E = \pi^{-1}(x)$ is an elliptic curve that is the fibre of a point $x \in \mathbb{P}^1$. The projection map pr also has 12 singular fibres which are non-normal toric surfaces. They are each compactifications of $\mathbb{C}^* \times \mathbb{C}^*$ by a *banana configuration* and their normalisations are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ blown up at 2 points [Br, Prop. 24].

Definition 1.2.1. A *banana configuration* is a union of three curves $C_1 \cup C_2 \cup C_3$ where $C_i \cong \mathbb{P}^1$ with $N_{C_i/X} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1)$ and $C_1 \cap C_2 = C_1 \cap C_3 = C_2 \cap C_3 = \{z_1, z_2\}$ where $z_1, z_2 \in X$ are distinct points. Also, there exist formal neighbourhoods of z_1 and z_2 such that the curves C_i become the coordinate axes in those coordinates. We label these curves by their intersection with the natural surfaces in X. That is C_1 is the unique banana curve that intersects S_1 at one point. Similarly, C_2 intersects S_2 and C_3 intersects S_{op}.

The banana threefold contains 12 copies of the banana configuration. We label the individual banana curves by $C_i^{(j)}$ (and simply C_i when there is no confusion). The banana curves C_1, C_2, C_3 generate a sub-lattice $\Gamma_0 \subset H_2(X, \mathbb{Z})$ and we can consider the partition function restricted to these classes:

$$Z_{\Gamma_0} := \sum_{\beta \in \Gamma_0} \sum_{n \in \mathbb{Z}} \mathsf{DT}_{\beta,n}(X) Q^\beta p^n.$$

In [Br, Thm. 4], this rank three partition function is computed to be:

$$Z_{\Gamma_0} = \prod_{d_1, d_2, d_3 \geq 0} \prod_k (1 - Q_1^{d_1} Q_2^{d_2} Q_3^{d_3} (-p)^k)^{-12c(\|\boldsymbol{d}\|, k)} \tag{2}$$

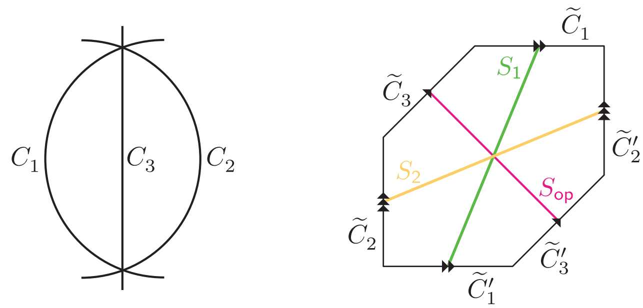


Figure II.3: On the left is a depiction of the banana configuration. On the right is the normalisation of the singular fibre $F_{\mathsf{ban}} = \mathrm{pr}^{-1}(x)$ with the restrictions of the surfaces S_1, S_2, S_{op}.

where $\boldsymbol{d} = (d_1, d_2, d_3)$ and the second product is over $k \in \mathbb{Z}$ unless $\boldsymbol{d} = (0,0,0)$ in which case $k > 0$. (Note the change in variables from [Br].) The powers $c(\|\boldsymbol{d}\|, k)$ are defined by

$$\sum_{a=-1}^{\infty} \sum_{k \in \mathbb{Z}} c(a,k) Q^a y^k := \frac{\sum_{k \in \mathbb{Z}} Q^{k^2} (-y)^k}{\left(\sum_{k \in \mathbb{Z} + \frac{1}{2}} Q^{2k^2} (-y)^k \right)^2} = \frac{\vartheta_4(2\tau, z)}{\vartheta_1(4\tau, z)^2}$$

and $\|\boldsymbol{d}\| := 2d_1 d_2 + 2d_2 d_3 + 2d_3 d_1 - d_1^2 - d_2^2 - d_3^2$.

Remark 1.2.2. We can pass to the unweighted $\widehat{Z}_{\Gamma_0}$ from the weighted partition function Z_{Γ_0} by the change of variables $Q_i \mapsto -Q_i$ and $p \mapsto -p$.

We can include the class of the section σ to generate a larger sub-lattice $\Gamma \subset H_2(X, \mathbb{Z})$. The partition function of this sub-lattice is currently unknown. The purpose of this chapter is to make progress towards understanding this partition function. We will be calculating the unweighted Donaldson-Thomas theory in the classes:

$$\beta = \sigma + (0, d_2, d_3) := \sigma + 0\, C_1 + d_2\, C_2 + d_3\, C_3,$$

by computing the following the partition function

$$\widehat{Z}_{\sigma+(0,\bullet,\bullet)} := \sum_{d_2, d_3 \geq 0} \sum_{k \in \mathbb{Z}} \widehat{\mathsf{DT}}_{\beta,n}(Y) Q_2^{d_2} Q_3^{d_3} p^n,$$

which we give in terms of the MacMahon functions $M(p,Q) = \prod_{m>0} (1 - p^m Q)^{-m}$ and their simpler version $M(p) = M(p,1)$.

Theorem A *The above unweighted Donaldson-Thomas functions are:*

$\widehat{Z}_{\sigma+(0,\bullet,\bullet)}$ *is:*

$$\frac{\widehat{Z}_{(0,\bullet,\bullet)}}{(1-p)^2} \prod_{m>0} \frac{1}{(1 - Q_2^m Q_3^m)^8 (1 - p Q_2^m Q_3^m)^2 (1 - p^{-1} Q_2^m Q_3^m)^2}$$

where $\widehat{Z}_{(0,\bullet,\bullet)}$ *is the* Q_1^0 *part of the unweighted version of the* Γ_0 *partition function (2) and is given by:*

$$M(p)^{24} \prod_{d>0} \frac{M(Q_2^d Q_3^d, p)^{24}}{(1 - Q_2^d Q_3^d)^{12} M(-Q_2^{d-1} Q_3^d, p)^{12} M(-Q_2^d Q_3^{d-1}, p)^{12}}.$$

The connected unweighted *Pandharipande-Thomas* version of the above formula is identified as the connected version of the Pandharipande-Thomas theory for a rational elliptic surface [BK, Cor. 2] in the following corollary.

Corollary B *The connected unweighted Pandharipande-Thomas partition function is:*

$$\begin{aligned}\widehat{Z}^{\mathsf{PT},\mathsf{Con}}_{\sigma+(0,\bullet,\bullet)} &:= \log\left(\frac{\widehat{Z}_{\sigma+(0,\bullet,\bullet)}}{\widehat{Z}_{(0,\bullet,\bullet)}|_{Q_i=0}}\right)\\ &= \frac{-1}{(1-p)^2}\prod_{m>0}\frac{-1}{(1-Q_2^mQ_3^m)^8(1-pQ_2^mQ_3^m)^2(1-p^{-1}Q_2^mQ_3^m)^2}.\end{aligned}$$

We will also be computing the unweighted Donaldson-Thomas theory in the classes:

$$\beta = b\,\sigma + (0,0,d_3), \qquad \beta = b\,\sigma + (0,1,d_3) \qquad \text{and} \qquad \beta = b\,\sigma + (1,1,d_3)$$

and the permutations involving C_1, C_2. So for $i, j \in \{0,1\}$ we define

$$\widehat{Z}_{\bullet\sigma+(i,j,\bullet)} := \sum_{b,d_3\geq 0}\sum_{k\in\mathbb{Z}}\widehat{\mathsf{DT}}_{\beta,n}(Y)Q_\sigma^bQ_3^{d_3}p^n.$$

The formulas will be given in terms of the functions which are defined for $g \in \mathbb{Z}$:

$$\psi_g = \psi_g(p) := \left(p^{\frac{1}{2}} - p^{-\frac{1}{2}}\right)^{2g-2} = \left(\frac{p}{(1-p)^2}\right)^{1-g}.$$

Theorem C *The above unweighted Donaldson Thomas functions are:*

1. $\widehat{Z}_{\bullet\sigma+(0,0,\bullet)}$ *is:*

$$M(p)^{24}\prod_{m>0}(1+p^mQ_\sigma)^m(1+p^mQ_3)^{12m}.$$

2. $\widehat{Z}_{\bullet\sigma+(0,1,\bullet)} = \widehat{Z}_{\bullet\sigma+(1,0,\bullet)}$ *is:*

$$\widehat{Z}_{\bullet\sigma+(0,0,\bullet)}\cdot\Big(\big(12\psi_0 + Q_3(24\psi_0+12\psi_1) + Q_3^2(12\psi_0)\big) + Q_\sigma Q_3\big(12\psi_0+2\psi_1\big)\Big)$$

3. $\widehat{Z}_{\bullet\sigma+(1,1,\bullet)}$ *is:*

$$\begin{aligned}&\widehat{Z}_{\bullet\sigma+(0,0,\bullet)}\\ &\cdot\Bigg(\Big((144\psi_{\text{-}1}+24\psi_0+12\psi_1) + Q_3(576\psi_{\text{-}1}+384\psi_0+72\psi_1+12\psi_2)\\ &\qquad + Q_3^2(864\psi_{\text{-}1}+720\psi_0+264\psi_1+24\psi_2)\\ &\qquad + Q_3^3(576\psi_{\text{-}1}+384\psi_0+72\psi_1+12\psi_2) + Q_3^4(144\psi_{\text{-}1}+24\psi_0+12\psi_1)\Big)\\ &\quad + Q_\sigma\Big(\big(12\psi_0+2\psi_1\big) + Q_3\big(288\psi_{\text{-}1}+96\psi_0+44\psi_1\big)\\ &\qquad\qquad + Q_3^2\big(576\psi_{\text{-}1}+600\psi_0+156\psi_1+24\psi_2\big)\\ &\qquad\qquad + Q_3^3\big(288\psi_{\text{-}1}+96\psi_0+44\psi_1\big) + Q_3^4\big(12\psi_0+2\psi_1\big)\Big)\\ &\quad + Q_\sigma^2Q_3^2\Big(144\psi_{\text{-}1}+48\psi_0+4\Big)\Bigg).\end{aligned}$$

The connected unweighted *Pandharipande-Thomas* versions of the above formula contain the same information but are given in the much more compact form. In fact we can present the same information in an even more compact form using the unweighted *Gopakumar-Vafa* invariants $\widehat{n}^g_\beta$ via the expansion

$$\begin{aligned}
&\widehat{Z}^{\mathsf{PT,Con}}_{\Gamma}(X)\\
&= \sum_{\beta\in\Gamma\setminus\{\mathbf{0}\}} \sum_{g\geq 0} \sum_{m>0} \widehat{n}^g_\beta \, \psi_g(p^m) \, (-Q)^{m\beta}\\
&= \sum_{\substack{b,\, d_1,\, d_2,\, d_3 \,\geq\, 0\\ (b,\, d_1,\, d_2,\, d_3) \,\neq\, \mathbf{0}}} \sum_{g\geq 0} \sum_{m>0} \widehat{n}^g_{(b,d_1,d_2,d_2)} \, \psi_g(p^m) \, (\text{-}Q_\sigma)^{mb} (\text{-}Q_1)^{md_1} (\text{-}Q_2)^{md_2} (\text{-}Q_3)^{md_3}.
\end{aligned}$$

As noted before, these express the same information as the above generating functions. For $\beta = (d_1, d_2, d_3)$, these invariants are given in [Br, §A.5]. We present the new invariants for $\beta = b\sigma + (i, j, d_3)$ where $b > 0$.

Corollary D *Let $i, j \in \{0, 1\}$, $b > 0$ and $\beta = b\sigma + (i, j, d_3)$. The unweighted Gopakumar-Vafa invariants $\widehat{n}^g_\beta$ are given by:*

1. *If $b > 1$ we have $\widehat{n}^g_\beta = 0$.*
2. *If $b = 1$ then the non-zero invariants are given in the following table:*

Table 1: The non-zero $\widehat{n}^g_\beta$ for $\beta = \sigma + (i, j, d_3)$ where $i, j \in \{0, 1\}$ and $d_3 \geq 0$.

(d_1, d_2, d_3)	$(0,0,0)$	$(0,1,1)$	$(1,0,1)$	$(1,1,0)$	$(1,1,1)$	$(1,1,2)$	$(1,1,3)$	$(1,1,4)$
$g = 0$	1	12	12	12	48	216	48	12
$g = 1$	0	2	2	2	44	108	44	2
$g = 2$	0	0	0	0	0	24	0	0

Remark 1.2.3. We note that the values given only depend on the quadratic form $\|\boldsymbol{d}\| := 2d_1d_2 + 2d_1d_3 + 2d_2d_3 - d_1^2 - d_2^2 - d_3^3$ appearing in the rank 3 Donaldson-Thomas partition function of [Br, Thm. 4]. However, there is no immediate geometric explanation for this fact.

Corollaries B and D will be proved in section 6.1.

1.3 Notation

The main notations for this chapter have been defined above in section 1.2. In particular X will always denote the banana threefold as defined in equation (1).

1.4 Future

The calculation here is for the unweighted Donaldson-Thomas partition function. However, the method of [BK] also provides a route (up to a conjecture) of computing the Donaldson-Thomas partition function. The following are needed in order to convert the given calculation:

1. A proof showing the invariance of the Behrend function under the $(\mathbb{C}^*)^2$-action used on the strata.
2. A computation of the dimensions of the Zariski tangent spaces for the various strata.

A comparison of the unweighted and weighted partition functions of the rank 3 lattice of [Br] reveals the likely differences:

> In the variables chosen in this chapter one can pass from the unweighted to the weighted partition functions by the change of variables $Q_i \mapsto -Q_i$ and $p \mapsto -p$.

Moreover, the conifold transition formula reveals further insight by comparing to the Donaldson-Thomas partition function of the Schoen variety with a single section and all fibre classes, which was shown in [ObPi] (via the reduced theory of the product of a $K3$ surface with an elliptic curve) to be given by the weight 10 Igusa cusp form.

As we mentioned previously the Donaldson-Thomas partition function is very hard to compute. So much so that for proper Calabi-Yau threefolds, the only known complete examples are in computationally trivial cases. This is even true conjecturally and even a conjecture for the rank 4 partition function is highly desirable. The work here shows underlying structures that a conjectured partition function must have.

2 Overview of the Computation

2.1 Overview of the Method of Calculation

We will closely follow the method of [BK] developed for studying the Donaldson-Thomas theory of local elliptic surfaces. However, due to some differences in geometry a more subtle approach is required in some areas. In particular, the local elliptic surfaces have a global action which reduces the calculation to considering only the so-called *partition thickened curves*.

Our method is based around the following *continuous map*:

$$\mathsf{Cyc} : \mathrm{Hilb}^1(X) \to \mathrm{Chow}^1(X).$$

which takes a one dimensional subscheme to its 1-cycle. The fibres of this map are of particular importance and we denote them by $\mathrm{Hilb}^\bullet_{\mathsf{Cyc}}(X, \mathfrak{q})$ where $\mathfrak{q} \in \mathrm{Chow}^1(X)$. The bullet notation will be elaborated on further in this section.

Remark 2.1.1. No such morphism exists in the algebraic category. In fact we note from [K, Thm. 6.3] that there is only a morphism from the semi-normalisation $\mathrm{Hilb}^1(X)^{SN} \to \mathrm{Chow}^1(X)$. However, $\mathrm{Hilb}^1(X)^{SN}$ is homeomorphic to $\mathrm{Hilb}^1(X)$, which gives rise to the above continuous map.

Broadly, we will be calculating the Euler characteristics $e\big(\mathrm{Hilb}^{\beta,n}(X)\big)$ using the following method:

1. Push forward the calculation to an Euler characteristic on $\mathrm{Chow}^1(X)$, weighted by the constructible function $(\mathsf{Cyc}_*1)(\mathfrak{q}) := e\big(\mathrm{Hilb}^\bullet_{\mathsf{Cyc}}(X, \mathfrak{q})\big)$. This is further described in sections 2.2 and 2.3.

2. Analyse the image of Cyc and decompose it into combinations of symmetric products where the strata are based on the types of subscheme in the fibres $\mathrm{Hilb}^\bullet_{\mathsf{Cyc}}(X, \mathfrak{q})$. This is done in section 3.

3. Compute the Euler characteristic of the fibres $e\big(\mathrm{Hilb}^\bullet_{\mathsf{Cyc}}(X, \mathfrak{q})\big)$ and show that they form a constructible function on the combinations of symmetric products. This is done in section 5.

4. Use the following lemma to give the Euler characteristic partition function.

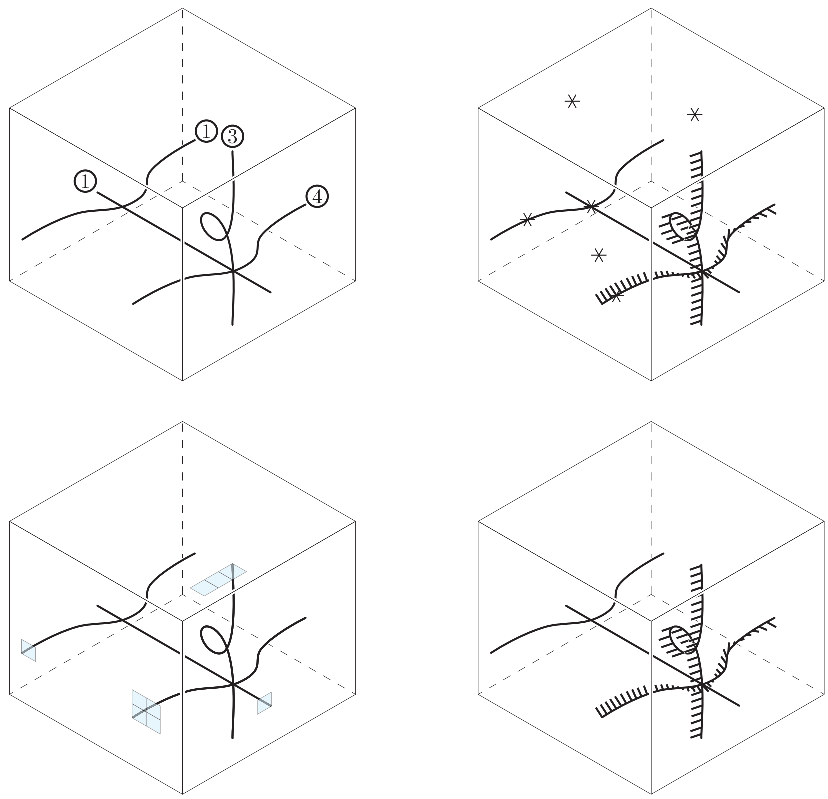


Figure II.4: A depiction of the process for reducing to partition thickened curves. Clockwise from the top-left we have: a) Consider a 1-cycle in the Chow scheme; b) Consider the fibre of the given 1-cycle; c) Reduce to a computation on the open subset of Cohen-Macaulay subschemes; d) Reduce to a computation on partition thickened schemes.

Lemma 2.1.2. *[BK, Lemma 32] Let Y be finite type over $\mathbb{C}$ and let $g : \mathbb{Z}_{\geq 0} \to \mathbb{Z}(\!(p)\!)$ be any function with $g(0) = 1$. Let $G : \mathrm{Sym}^d(Y) \to \mathbb{Z}(\!(p)\!)$ be the constructible function defined by*

$$G(\boldsymbol{ax}) = \prod_i g(a_i)$$

where $\boldsymbol{ax} = \sum_i a_i x_i \in \mathrm{Sym}^d(Y)$ and $x_i \in Y$ are distinct points. Then

$$\sum_{d=0}^{\infty} e(\mathrm{Sym}^d(Y), G) q^d = \left(\sum_{a=0}^{\infty} g(a) q^a \right)^{e(Y)}$$

To compute the Euler characteristics of the fibres $(\mathsf{Cyc}_* 1)(\mathfrak{q}) := e\big(\mathrm{Hilb}^{\bullet}_{\mathsf{Cyc}}(X, \mathfrak{q})\big)$ we use the following method made rigorous in section 4:

1. Consider the image of the fibre under the constructible morphism denoted κ : $\mathrm{Hilb}^1(X) \to \mathrm{Hilb}^1(X)$ which takes a subscheme Z to the maximal Cohen-Macaulay subscheme $Z_{\mathsf{CM}} \subset Z$.

2. Denote the open subset contain Cohen-Macaulay subschemes by $\mathrm{Hilb}^{\bullet}_{\mathsf{CM}}(X, \mathfrak{q}) \subset \mathrm{Hilb}^{\bullet}_{\mathsf{Cyc}}(X, \mathfrak{q})$.

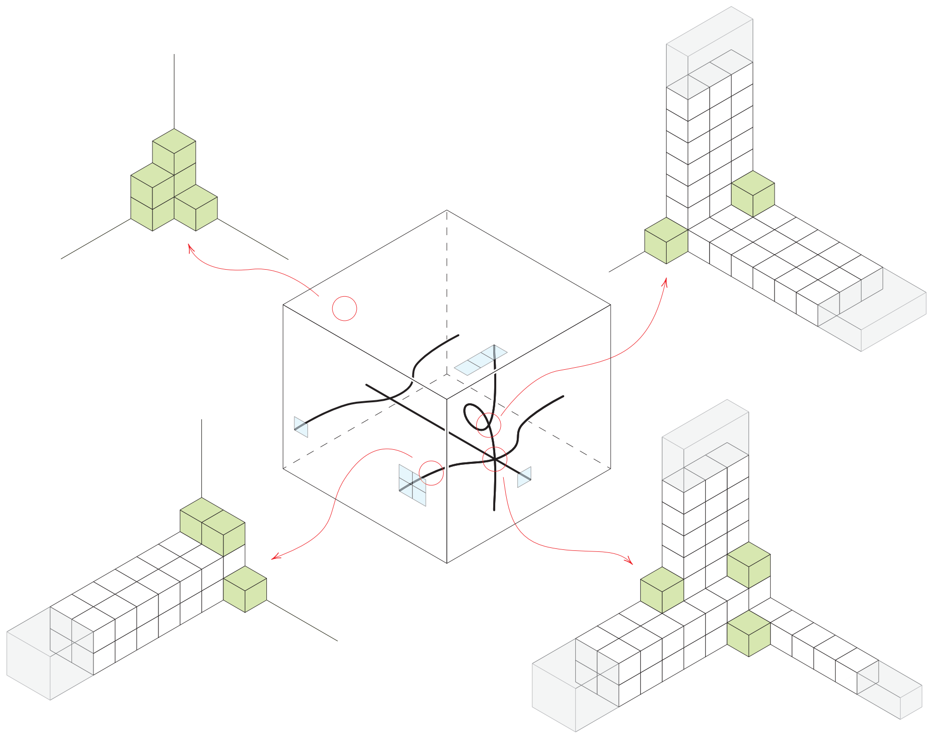


Figure II.5: A depiction of how the topological vertex is applied to calculate Euler characteristic of a given strata.

3. Note the equality of the Euler characteristic $e\big(\mathrm{Hilb}^{\bullet}_{\mathsf{Cyc}}(X,\mathfrak{q})\big)$ and that of the weighted Euler characteristic $e\big(\mathrm{Hilb}^{\bullet}_{\mathsf{CM}}(X,\mathfrak{q}),\kappa_*1\big)$ where κ_*1 is the constructible function $(\kappa_*1)(\mathsf{p}) = e(\kappa^{-1}(\mathsf{p}))$.

4. Define a $(\mathbb{C}^*)^2$-action on $\mathrm{Hilb}^{\bullet}_{\mathsf{CM}}(X,\mathfrak{q})$ and show that $\kappa_*1(\mathsf{p}) = \kappa_*1(\alpha\cdot\mathsf{p})$ meaning $e\big(\mathrm{Hilb}^{\bullet}_{\mathsf{Cyc}}(X,\mathfrak{q})\big) = e\big(\mathrm{Hilb}^{\bullet}_{\mathsf{CM}}(X,\mathfrak{q})^{(\mathbb{C}^*)^2},\kappa_*1\big)$. This technique is discussed in section 4.2.

5. Identify the $(\mathbb{C}^*)^2$-fixed points $\mathrm{Hilb}^{\bullet}_{\mathsf{CM}}(X,\mathfrak{q})^{(\mathbb{C}^*)^2}$ as a discrete subset containing partition thickened curves. These neighbourhoods and this action are given explicitly in section 4.4.

6. Calculate the Euler characteristics $e\big(\mathrm{Hilb}^{\bullet}_{\mathsf{CM}}(X,\mathfrak{q})^{(\mathbb{C}^*)^2},\kappa_*1\big)$ using the Quot scheme decomposition and topological vertex method of [BK]. The concept of this is depicted in figure 2.1 and described below. Further technical details are given in section 4.5.

The Euler characteristic calculation of $e\big(\mathrm{Hilb}^{\bullet}_{\mathsf{CM}}(X,\mathfrak{q})^{(\mathbb{C}^*)^2},\kappa_*1\big)$ for theorems A and C follow similar methods but have different decompositions. The calculations are completed by considering the different types of topological vertex that occur for each fixed point in $\mathrm{Hilb}^{\bullet}_{\mathsf{CM}}(X,\mathfrak{q})^{(\mathbb{C}^*)^2}$.

Since the fixed locus $\mathrm{Hilb}^{\bullet}_{\mathsf{CM}}(X,\mathfrak{q})^{(\mathbb{C}^*)^2}$ will be disjoint we can consider individual subschemes $C \in \mathrm{Hilb}^{\bullet}_{\mathsf{CM}}(X,\mathfrak{q})^{(\mathbb{C}^*)^2}$ and their contribution to the Euler characteristic

$e\big(\mathrm{Hilb}^{\bullet}_{\mathsf{CM}}(X,\mathfrak{q})^{(\mathbb{C}^*)^2}, \kappa_* 1\big)$. To compute the contribution from C we must decompose X as follows:

1. Take the complement $W = X \setminus C$.

2. Consider set of singularities of the underlying reduced curve. Denote this set $C^{\diamond}$.

3. Define $C^{\circ} = C^{\mathsf{red}} \setminus C^{\diamond}$ to be its complement.

The curve C will be partition thickened. So each formal neighbourhood of a point $x \in C^{\diamond}$ will give rise to a 3D partition. Similarly points on $U \in C^{\circ}$ will also give rise to 3D partitions and points on W will give rise to the empty partition. Using techniques from section 4.5 the Euler characteristics can then be determined.

This calculation for theorem A is finalised in section 5.1. Generalities for the proof of theorem C are given in section 5.2 and the individual calculations are given in sections 5.3, 5.4 and 5.5.

2.2 Review of Euler characteristic

We begin by recalling some facts about the (topological) Euler characteristic. For a scheme Y over $\mathbb{C}$ we denote by $e(Y)$ the topological Euler characteristic in the complex analytic topology on Y. This is independent of any non-reduced structure of Y, is additive under decompositions of Y into open sets and their complements, and is multiplicative on Cartesian products. In this way we see that the Euler characteristic defines a ring homomorphism from the Grothendieck ring of varieties to the integers:

$$e : K_0(\mathrm{Var}_{\mathbb{C}}) \longrightarrow \mathbb{Z}.$$

If Y has a $\mathbb{C}^*$-action with fixed locus $Y^{\mathbb{C}^*} \subset Y$ the Euler characteristic also has the property $e(Y^{\mathbb{C}^*}) = e(Y)$.

The interaction of Euler characteristic with constructible functions and morphisms also plays a key role in this chapter. Recall that a function $\mu : T \to \mathbb{Z}$ is constructible if $\mu(T)$ is finite and $\mu^{-1}(c)$ is the union of finitely many locally closed sets for all non-zero $c \in \mu(T)$. The μ-weight Euler characteristic is defined to be $e(Y,\mu) = \sum_{k\in\mathbb{Z}} k \cdot e(\mu^{-1}(k))$. Note that we have $e(Y) = e(Y,1)$ where 1 is the constant function.

For a scheme Z over $\mathbb{C}$, a constructible morphism $f : Y \to Z$ is a finite collection of morphisms $f_i : Y_i \to Z_i$ where $Y = \coprod_i Y_i$ and $Z = \coprod_i Z$ are decompositions into locally closed subschemes. We can defined a constructible function $f_*\mu : Z \to \mathbb{Z}$ by

$$(f_*\mu)(x) := e(f^{-1}(x), \mu).$$

This has the important property $e(Z, f_*\mu) = e(Y,\mu)$. If $\nu : Z \to \mathbb{Z}$ is a constructible function, then $\mu \cdot \nu$ is a constructible function on $Y \times Z$ and $e(Y \times Z, \mu \cdot \nu) = e(Y,\mu) \cdot e(Z,\nu)$.

It will also be helpful to extend these definitions to the rings of formal power series in Q_i and formal Laurent series in p. This will allow us to make use of lemma 2.1.2.

2.3 Pushing Forward to the Chow Variety

Recall that the Chow scheme $\mathrm{Chow}^1(X)$ is a space parametrising the one dimensional cycles of X. We will consider the subspace of this $\mathrm{Chow}^\beta(X)$ parametrising 1-cycles in the class $\beta \in H_2(X, \mathbb{Z})$. We will then define a constructible morphism

$$\rho_\beta : \sum_n p^n \mathrm{Hilb}^{\beta,n}(X) \to \mathrm{Chow}^\beta(X).$$

The strategy for calculating the partition functions is to analyse $\mathrm{Chow}^\beta(X)$ and the fibres of the map ρ_β. These will often involve the symmetric product, and where possible we will apply lemma 2.1.2.

It will be convenient to employ the following $\bullet$ notations for the Hilbert schemes:

$$\mathrm{Hilb}^{\sigma+(0,\bullet,\bullet),n}(X) := \sum_{d_2,d_3\geq 0} \sum_{n\in\mathbb{Z}} Q_\sigma Q_2^{d_2} Q_3^{d_3} p^n \, \mathrm{Hilb}^{\sigma+(0,d_2,d_3),n}(X)$$
$$\mathrm{Hilb}^{\bullet\sigma+(i,j,\bullet),n}(X) := \sum_{b,d_3\geq 0} \sum_{n\in\mathbb{Z}} Q_\sigma^b Q_1^i Q_2^j Q_3^{d_3} p^n \, \mathrm{Hilb}^{b\sigma+(i,j,d_3),n}(X)$$

and for the Chow schemes:

$$\mathrm{Chow}^{\sigma+(0,\bullet,\bullet)}(X) := \sum_{d_2,d_3\geq 0} Q_\sigma Q_2^{d_2} Q_3^{d_3} \mathrm{Chow}^{\sigma+(0,d_2,d_3)}(X)$$
$$\mathrm{Chow}^{\bullet\sigma+(i,j,\bullet)}(X) := \sum_{b,d_3\geq 0} Q_\sigma^b Q_1^i Q_2^j Q_3^{d_3} \mathrm{Chow}^{b\sigma+(i,j,d_3)}(X)$$

where $i, j \in \{0, 1\}$. Note, that here he have viewed the Hilbert and Chow schemes in the Grothendieck ring of varieties. We also extend the $\bullet$ notation to symmetric products in the following way:

$$\mathrm{Sym}^\bullet(Y) := \sum_{n\in\mathbb{Z}_{\geq 0}} Q^n \mathrm{Sym}^n(Y),$$

an we use the following notation for elements of the symmetric product

$$\boldsymbol{ay} := \sum_i a_i y_i \in \mathrm{Sym}^n(Y)$$

where y_i are distinct points on Y and $a_i \in \mathbb{Z}_{\geq 0}$. We also denote a tuple of partitions $\boldsymbol{\alpha}$ of a tuple of non-negative integers $\boldsymbol{a}$ by $\boldsymbol{\alpha} \vdash \boldsymbol{a}$.

Using the $\bullet$-notation for the maps ρ_β we create the following constructible morphisms:

$$\rho_\bullet : \mathrm{Hilb}^{\sigma+(0,\bullet,\bullet),n}(X) \longrightarrow \mathrm{Chow}^{\sigma+(0,\bullet,\bullet)}(X)$$
$$\eta_\bullet^{ij} : \mathrm{Hilb}^{\bullet\sigma+(i,j,\bullet),n}(X) \longrightarrow \mathrm{Chow}^{\bullet\sigma+(i,j,\bullet)}(X)$$

and we also use the notation $\eta_\bullet = \eta_\bullet^{00} + \eta_\bullet^{01} + \eta_\bullet^{11}$. The fibres of these morphisms will be subspaces of the Hilbert scheme parametrising one dimensional subschemes with a fixed 1-cycle. Specifically, let $C \subset X$ be a one dimensional subscheme in the class $\beta \in H_2(X)$ with 1-cycle $\mathsf{Cyc}(C)$. Define $\mathrm{Hilb}^n(X, \mathsf{Cyc}(C)) \subset \mathrm{Hilb}^{\beta,n}(X)$ to be the subscheme.

$$\mathrm{Hilb}^n(X, \mathsf{Cyc}(C)) = \left\{ Z \in \mathrm{Hilb}^{\beta,n}(X) \;\middle|\; \mathsf{Cyc}(Z) = \mathsf{Cyc}(C) \right\}.$$

The maps $\rho_\bullet$ and $\eta_\bullet$ are explicitly described in lemmas 3.5.3 and 3.5.1 respectively.

3 Parametrising Underlying 1-cycles

3.1 Related Linear Systems in Rational Elliptic Surfaces

In this section we consider some basic results about linear systems on a rational elliptic surface. Some of these result can be found in [BK, §A.1].

Recall our notation that $\pi : S \to \mathbb{P}^1$ is a generic rational elliptic surface with a canonical section $\zeta : \mathbb{P}^1 \to S$. Consider the following classical results for rational elliptic surfaces from [Mi, II.3]:

$$\pi_*\mathcal{O}_S \cong \pi_*\mathcal{O}_S(\zeta) \cong \mathcal{O}_{\mathbb{P}^1}, \qquad R^1\pi_*\mathcal{O}_S \cong \mathcal{O}_{\mathbb{P}^1}(-1) \qquad \text{and} \qquad R^1\pi_*\mathcal{O}_S(\zeta) \cong 0.$$

After applying the projection formula we have the following:

$$\pi_*\mathcal{O}_S(dF) \cong \pi_*\mathcal{O}_S(\zeta + dF) \cong \mathcal{O}_{\mathbb{P}^1}(d) \tag{II.5}$$

as well as

$$R^1\pi_*\mathcal{O}_S(dF) \cong \mathcal{O}_{\mathbb{P}^1}(d-1) \qquad \text{and} \qquad R^1\pi_*\mathcal{O}_S(\zeta + dF) \cong 0. \tag{II.6}$$

Lemma 3.1.1. *We have the following isomorphisms:*

$$H^1(S, \mathcal{O}_S(dF)) \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d-1)) \qquad \textit{and} \qquad H^1(S, \mathcal{O}_S(\zeta + dF)) \cong 0.$$

Proof. The second isomorphism is immediate from the vanishing of $R^i\pi_*\mathcal{O}_S(\zeta + dF)$ for $i > 0$ (see for example [H, III Ex. 8.1]) and $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) \cong 0$.

To show the first isomorphism we consider the following exact sequence arising from the Leray spectral sequence:

$$H^1(\mathbb{P}^1, \pi_*\mathcal{O}_S(dF)) \to H^1(S, \mathcal{O}_S(dF)) \to H^0(\mathbb{P}^1, R^1\pi_*\mathcal{O}_S(dF)) \to 0$$

We have from (II.5) that $H^1(\mathbb{P}^1, \pi_*\mathcal{O}_S(dF)) \cong 0$ and we have the desired isomorphism after considering (II.6). □

Lemma 3.1.2. *Consider a fibre F of a point $z \in \mathbb{P}^1$ by the map $S \to \mathbb{P}^1$ and the image of a section $\zeta : \mathbb{P}^1 \to S$. Then there are isomorphisms of the linear systems*

$$|d\,F|_S \cong |\zeta + d\,F|_S \cong |d\,z|_{\mathbb{P}^1} \qquad \textit{and} \qquad |b\,\zeta + F|_S \cong |z|_{\mathbb{P}^1}.$$

Proof. The isomorphism $|\zeta + d\,F|_S \cong |d\,z|_{\mathbb{P}^1}$ is immediate from the vanishing of $R^i\pi_*\mathcal{O}_S(\zeta + dF)$ for $i > 0$ and (II.5) (see for example [H, III Ex. 8.1]).

We continue by showing $|d\,F|_S \cong |\zeta + d\,F|_S$. Consider the long exact sequence arising from the divisor sequence for ζ twisted by $\mathcal{O}_S(\zeta + dF)$:

$$\begin{aligned} 0 \to H^0(S, \mathcal{O}_S(dF)) &\xrightarrow{f} H^0(S, \mathcal{O}_S(\zeta + dF)) \to H^0(S, \zeta_*\mathcal{O}_{\mathbb{P}^1}(\zeta + dF)) \\ &\xrightarrow{g} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d-1)) \to 0 \end{aligned}$$

where we have applied the results from lemma 3.1.1. from intersection theory we have that $\zeta_*\mathcal{O}_{\mathbb{P}^1}(\zeta + dF) \cong \zeta_*\mathcal{O}_{\mathbb{P}^1}(d-1)$. Hence, g is an isomorphism making f an isomorphism also.

The isomorphism $|b\,\zeta + F|_S \cong |z|_{\mathbb{P}^1}$ will follow inductively from the divisor sequence for ζ on S:

$$0 \longrightarrow \mathcal{O}_S(k\zeta + F) \longrightarrow \mathcal{O}_S\big((k+1)\zeta + F\big) \longrightarrow \mathcal{O}_\zeta\big((k+1)\zeta + F\big) \longrightarrow 0.$$

Intersection theory shows us that $\mathcal{O}_\zeta\big((k+1)\zeta + F\big)$ is a degree $-k$ line bundle on $\mathbb{P}^1$ which shows that its 0th cohomology vanishes. Hence, we have isomorphisms:

$$H^0\big(S, \mathcal{O}_S(F)\big) \cong \cdots \cong H^0\big(S, \mathcal{O}_S(b\zeta + F)\big).$$

□

3.2 Curve Classes and 1-cycles in the Threefold

Recall from definition 1.2.1 that the banana curves C_i are labelled by their unique intersections with the rational elliptic surfaces

$$S_1, \quad S_2 \quad \text{and} \quad S_{\mathsf{op}}.$$

These are smooth effective divisors on X. Hence a curve C in the class (d_1, d_2, d_3) will have the following intersections with these divisors:

$$C \cdot S_1 = d_2, \quad C \cdot S_2 = d_1 \quad \text{and} \quad C \cdot S_{\mathsf{op}} = d_3.$$

The full lattice $H_2(X, \mathbb{Z})$ is generated by

$$C_1,\ C_2,\ C_3,\ \sigma_{11},\ \sigma_{12},\ \ldots\ , \sigma_{19},\ \sigma_{21},\ \ldots\ ,\ \sigma_{99}$$

where the σ_{ij} are the 81 canonical sections of $\mathrm{pr} : X \to \mathbb{P}^1$ arising from the 9 canonical sections of $\pi : S \to \mathbb{P}^1$. However, there are 64 relations between the σ_{ij}'s giving the lattice rank of 20 (see [Br, Prop. 28 and Prop. 29]).

Lemma 3.2.1. *There are no relations in $H_2(X, \mathbb{Z})$ of the form:*

$$n \cdot \sigma_{i,j} + d_1C_1 + d_2C_2 + d_3C_3 = \sum_{(k,l)\neq(i,j)} a_{k,l} \cdot \sigma_{k,l} + d_1'C_1 + d_2'C_2 + d_3'C_3$$

where $n, a_{k,l}, d_t, d_t' \in \mathbb{Z}_{\geq 0}$ for all $k, l \in \{1, \ldots, 9\}$ and $t \in \{1, 2, 3\}$.

Proof. Any such relation must push forward to relations on S via the projections $\mathrm{pr}_i : X \to S_i$. However, S is isomorphic to $\mathbb{P}^1$ blown up at 9 points. The exceptional divisors of these blow-ups correspond to the sections $\zeta_i : \mathbb{P}^1 \to S$. Hence

$$\mathrm{Pic}\, S \cong \mathrm{Pic}\, \mathbb{P}^2 \times \zeta_1 \times \cdots \times \zeta_9 \cong \mathbb{Z}^{10}$$

and there are are no relations of this form. □

The next lemma allows us to consider the curves in our desired classes by decomposing them.

Lemma 3.2.2. *Let $d_1, d_2, d_3, b \in \mathbb{Z}_{\geq 0}$ and $i, j \in \{0, 1\}$.*

1. *Let C be a Cohen-Macaulay curve in the class (d_1, d_2, d_3). Then the support of C is contained in fibres of the projection map $\mathrm{pr} : X \to \mathbb{P}^1$.*
2. *A curve C in the class $\sigma + (d_1, d_2, d_3)$ is of the form*
$$C = \sigma \cup C_0$$
where C_0 is a curve in the class (d_1, d_2, d_3).
3. *A curve in the class $b\,\sigma + (i, j, d_3)$ is of the form*
$$C = C_\sigma \cup C_0$$
where C_σ is a curve in the class $b\,\sigma$ and C_0 is a curve in the class (i, j, d_3). The same result holds for permutations of $b\,\sigma + (i, j, d_3)$.

Proof. Consider a curve in one of the given classes and it's image under the two projections $\mathrm{pr}_i : X \to S_i$. For (1) these must be in the classes $|d_1 f_1|$ and $|d_1 f_1|$, for (2) the classes $|\zeta + d_1F_1|$ and $|\zeta + d_2F_2|$, and for (3) the classes $|i f_1|$ and $|j f_1|$. Lemma 3.1.2 now shows that the curve must have the given form. □

3.3 Analysis of 1-cycles in Smooth Fibres of pr

Consider a fibre $F_x = \mathrm{pr}^{-1}(x)$ which is smooth. Then there is an elliptic curve E such that $F_x \cong E \times E$. Consider a curve C with underlying 1-cycle contained in $E \times E$, then this gives rise to a divisor D in $E \times E$. Hence we must analyse divisors in $E \times E$ and their classes in X. The class of such a curve is determined uniquely by its intersection with the surfaces S_1, S_2 and S_{op}.

Lemma 3.3.1. *Let $C \subset X$ correspond to a divisor D in $E \times E$.*

1. *If C is in the class $(0, d_2, d_3)$ then $d_2 = d_3$ and D is the pullback of a degree d_2 divisor on E via the projection to the second factor.*
2. *The result in (2) is true for $(d_1, 0, d_3)$ and projection to the first factor.*

Proof. If C is in the class $(0, d_2, d_3)$ then it doesn't intersect with the surface S_2. When we restrict to $E \times E$ this is the same condition as not intersecting with a fibre of the projection to the second factor. The only divisors that this is true for are those pulled back from E via this projection. It is clear that that the intersection with S_2 is d_2, and that the intersection with S_{op} is d_2 as well. Hence we have that $d_2 = d_3$. The proof for part (2) is completely analogous. □

Lemma 3.3.2. *Let $C \subset X$ be in the class $(1, 1, d)$ and correspond to a divisor D in $E \times E$. Then $d \in \{0, \ldots, 4\}$ and occurs in the following situations:*

1. *If E has $j(E) \neq 0, 1728$ then:*
   (a) *$d = 0$ occurs when D is a translation of the graph $\{(x, -x)\}$.*
   (b) *$d = 4$ occurs when D is a translation of the graph $\{(x, x)\}$.*
   (c) *$d = 2$ occurs when D is the union of a fibre from the projection to the first factor and a fibre from the projection to the second factor.*
2. *If $j(E) = 1728$ and $E \cong \mathbb{C}/i$ we have the cases (a) to (c) as well as:*
   (d) *$d = 2$ occurs when D is a translation of the graph $\{(x, \pm ix)\}$.*
3. *If $j(E) = 0$ and $E \cong \mathbb{C}/\tau$ with $\tau = \frac{1}{2}(1 + i\sqrt{3})$ we have the cases (a) to (c) as well as:*
   (e) *$d = 1$ occurs when D is a translation of the graph $\{(x, -\tau x)\}$ or the graph $\{(x, (\tau - 1)x)\}$.*
   (f) *$d = 3$ occurs when D is a translation of the graph $\{(x, \tau x)\}$ or the graph $\{(x, (-\tau + 1)x)\}$.*

Proof. Denote the projection maps by $\mathsf{p}_i : E \times E \to E$ and let $C \subset X$ be in the class $(1, 1, d)$ and correspond to a divisor D in $E \times E$. Suppose D is reducible. Then from lemma 3.3.1 we see that D must be the union $\mathsf{p}_1^{-1}(x_1) \cup \mathsf{p}_2^{-1}(x_2)$ where $x_1, x_2 \in E$ are generic points. We also have that D is in the class $(1, 1, 2)$.

Suppose D is irreducible. The surfaces S_1 and S_2 intersect D exactly once and their restrictions correspond the fibres of the projection maps $\mathsf{p}_i : E \times E \to E$. So the projection maps must be isomorphisms when restricted to D. Hence D is the translation of the graph of an automorphism of E.

All elliptic curves have the automorphisms $x \mapsto \pm x$. Also

- if $E \cong \mathbb{C}/i$ (j-invariant $j(E) = 1728$) the E also has the automorphisms $x \mapsto \pm ix$, and

- if $E = \mathbb{C}/\tau$ with $\tau = \frac{1}{2}(1 + i\sqrt{3})$ (j-invariant $j(E) = 0$) then E also has the automorphisms $x \mapsto \pm\tau x$ and $x \mapsto \pm(\tau - 1)x$.

So to complete the proof we have to calculate the intersections $\#(\Gamma_\xi \cap S_{\mathsf{op}})$ where Γ_ξ is the graph of an automorphism ξ. Also, $S_{\mathsf{op}}|_{F_x} \cong \Gamma_{-1}$ hence we calculate $\#(\Gamma_\xi \cap \Gamma_{-1}) = \#\{(x, \xi(x)) = (x, -x)\}$ in the surface F_x. For all the elliptic curves we have:

(a) $\#(\Gamma_1 \cap \Gamma_{-1})$ is given by the four 2-torsion points $\{0, \frac{1}{2}, \frac{1}{2}\tau, \frac{1}{2}(1+\tau)\}$.

(b) $\#(\Gamma_{-1} \cap \Gamma_{-1}) = 0$ since one copy can be translated away from the other.

For $E \cong \mathbb{C}/i$ (j-invariant $j(E) = 1728$) we have:

(d) $\#(\Gamma_{\pm i} \cap \Gamma_{-1})$ is given by the two points $\{0, \frac{1}{2}(1+\tau)\}$.

For $E = \mathbb{C}/\tau$ with $\tau = \frac{1}{2}(1 + i\sqrt{3})$ (j-invariant $j(E) = 0$) we have:

(e) $\#(\Gamma_\tau \cap \Gamma_{-1})$ and $\#(\Gamma_{(1-\tau)i} \cap \Gamma_{-1})$ are both determined by the three points $\{0, \frac{1}{3}(1+\tau), \frac{2}{3}(1+\tau)\}$.

(f) $\#(\Gamma_{-\tau} \cap \Gamma_{-1})$ and $\#(\Gamma_{(\tau-1)i} \cap \Gamma_{-1})$ are both given by the single point $\{0\}$.

$\square$

3.4 Analysis of 1-cycles in Singular Fibres of pr

We denote the fibres of the projection pr by $F_x := \mathrm{pr}^{-1}(x)$. The singular fibres are all isomorphic so we denote a singular fibre by F_{ban} and its normalisation by $\nu : \widetilde{F}_{\mathsf{ban}} \to F_{\mathsf{ban}}$. From [Br, Prop. 24] we have that $\widetilde{F}_{\mathsf{ban}} \cong \mathrm{Bl}_{2\,\mathrm{pt}}(\mathbb{P}^1 \times \mathbb{P}^1)$ and if we choose the coordinates on the $\mathbb{P}^1$'s so that the 0 and ∞ map to a nodal singularity, then the two points blown-up are $z_1 = (0, \infty)$ and $z_2 = (\infty, 0)$.

$$\begin{tikzcd} \mathrm{Bl}_{z_1,z_2}(\mathbb{P}^1 \times \mathbb{P}^1) \arrow[r, "\nu"] \arrow[d, "\mathsf{bl}"'] & F_{\mathsf{ban}} \\ \mathbb{P}^1 \times \mathbb{P}^1 & \end{tikzcd}$$

Also, we recall the decomposition of S_{i} into

$$S_{\mathsf{i}}^\circ = \mathtt{Sm}_i \amalg \mathtt{N}_i$$

where $\mathtt{N}_i$ are the 12 nodal fibres with their nodes removed and $\mathtt{Sm}_i = S_{\mathsf{i}}^\circ \setminus \mathtt{N}_i$. Let $\overline{\mathtt{N}}_i = \{\overline{\mathtt{N}}_i^{(1)}, \ldots, \overline{\mathtt{N}}_i^{(12)}\}$ be the 12 nodal fibres with the nodes.

3.4.1. Denote the divisors in $\widetilde{F}_{\mathsf{ban}}$ corresponding to the banana curve C_i by $\widetilde{C}_i$ and $\widetilde{C}'_i$. They are identified in F_{ban} by

$$\nu(\widetilde{C}_i) = \nu(\widetilde{C}'_i) = C_i.$$

For $i = 1, 2$ we also denote $\widehat{C}_i = \mathsf{bl}(\widetilde{C}_i)$ and $\widehat{C}'_i = \mathsf{bl}(\widetilde{C}'_i)$ inside $\mathbb{P}^1 \times \mathbb{P}^1$. The curve classes in $\widetilde{F}_{\mathsf{ban}}$ are generated by the collection of $\widetilde{C}_i$ and $\widetilde{C}'_i$'s with the relations:

$$\widetilde{C}_1 + \widetilde{C}_3 \sim \widetilde{C}'_1 + \widetilde{C}'_3 \quad \text{and} \quad \widetilde{C}_2 + \widetilde{C}_3 \sim \widetilde{C}'_2 + \widetilde{C}'_3.$$

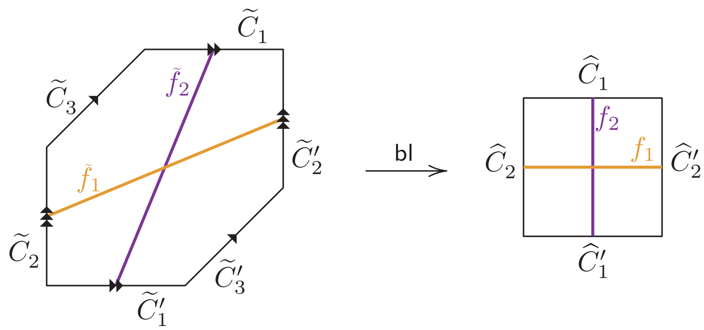


Figure II.6: On the left is a depiction of the normalisation $\widetilde{F}_{\mathsf{ban}}$ and on the right is a depiction of $\mathbb{P}^1 \times \mathbb{P}^1$. Here bl is the map blowing up $(0, \infty)$ and $(\infty, 0)$. On the right f_1 and f_2 are generic fibres of the projection maps $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ and on the left $\tilde{f}_1$ and $\tilde{f}_2$ are their proper transforms.

3.4.2. Let f_1 and f_2 be fibres of the projections $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ not equal to any $\widehat{C}_i$ or $\widehat{C}'_i$ and let $\tilde{f}_1$ and $\tilde{f}_2$ be their proper transforms. Then we also have the relations:

$$\tilde{f}_1 \sim \widetilde{C}_1 + \widetilde{C}_3 \quad \text{and} \quad \tilde{f}_2 \sim \widetilde{C}_2 + \widetilde{C}_3.$$

Moreover, if $\widetilde{D}$ is a divisor in $\widetilde{F}_{\mathsf{ban}}$ such that $\nu(\widetilde{D})$ is in the class (d_1, d_2, d_3) then D is in a class

$$a_1\widetilde{C}_1 + a'_1\widetilde{C}'_1 + a_2\widetilde{C}_2 + a'_2\widetilde{C}'_2 + a_3\widetilde{C}_3 + a'_3\widetilde{C}'_3$$

where $a_i + a'_i = d_i$.

Lemma 3.4.3. *Let $C \subset X$ correspond to a divisor D in F_{ban}.*

1. *C is in the class $(0, 0, d_3)$ if and only if D has 1-cycle $d_3 C_3$.*
2. *C is in the class $(0, d_2, d_3)$ if and only if D has 1-cycle $\widetilde{D} + a_2 C_2^{(j)} + a_3 C_3^{(j)}$ where $\widetilde{D}$ is the pullback of a degree a_f divisor from the smooth part of $\overline{\mathtt{N}}_2^{(j)}$ via the projection $F_{\mathsf{ban}} \to \overline{\mathtt{N}}_2^{(j)}$ such that $a_f + a_2 = D_2$ and $a_f + a_3 = D_3$. Moreover, $\widetilde{D}$ is in the class $(0, a_f, a_f)$.*

Proof. Let $C \subset X$ be a curve in the class $(0, d_2, d_3)$ and correspond to a divisor D in F_{ban}. There exists a divisor $\widetilde{D}$ in $\widetilde{F}_{\mathsf{ban}} \cong \mathrm{Bl}_{z_1,z_2}(\mathbb{P}^1 \times \mathbb{P}^1)$ with $\nu(\widetilde{D}) = D$.

From the discussion in 3.4.2 we have that $\mathsf{bl}(\widetilde{D})$ is in the class of $d_2 f_2$ and is hence in its corresponding linear system. So, $\widetilde{D}$ is the union of the the proper transform of $\mathsf{bl}(\widetilde{D})$ and curves supported at $\widetilde{C}_3$ and $\widetilde{C}'_3$. The result now follows. $\square$

Lemma 3.4.4. *Let $C \subset X$ be an irreducible curve in the class $(1, 1, d)$ and correspond to a divisor D in F_{ban}. Then D is the image under ν of the proper transform under bl of a smooth divisor in $|f_1 + f_2|$ on $|\mathbb{P}^1 \times \mathbb{P}^1|$. Moreover, the value of d is determined the intersection of D with points in $\mathsf{P} = \{(0,0), (0,\infty), (\infty,0), (\infty,\infty)\}$. That is, if D intersects*

1. *$(0,0)$ and (∞,∞) only, then $d = 2$.*
2. *$(0,\infty)$ and $(\infty,0)$ only, then $d = 0$.*
3. *$(0,0)$ only or (∞,∞) only, then $d = 2$.*
4. *$(0,\infty)$ only or $(\infty,0)$ only, then $d = 1$.*

5. *no points of* P*, then* $d = 2$.

Moreover, there are no smooth divisors in $|f_1 + f_2|$ *on* $|\mathbb{P}^1 \times \mathbb{P}^1|$ *that intersect other combinations of these points.*

Proof. Let $C \subset X$ be an irreducible curve in the class $(1, 1, d)$ and correspond to a divisor D in F_{ban}. There exists an irreducible divisor $\widetilde{D}$ in $\widetilde{F}_{\mathsf{ban}} \cong \mathrm{Bl}_{z_1,z_2}(\mathbb{P}^1 \times \mathbb{P}^1)$ with $\nu(\widetilde{D}) = D$. $\widetilde{D}$ does not contain either of the exceptional divisor $\widetilde{C}_3$ and $\widetilde{C}_3'$. Hence, it must be the proper transform of a curve in $\mathbb{P}^1 \times \mathbb{P}^1$.

From the discussion in 3.4.2 we have that $\mathsf{bl}(\widetilde{D})$ is in the class of $f_1 + f_2$ and is hence in its corresponding linear system. The only irreducible divisors in $|f_1 + f_2|$ are smooth and can only pass through the combinations of points in P that are given. We refer to the appendix 6.2.3 for the proof of this. The total transform in any divisor in $|f_1 + f_2|$ will correspond to a curve in the class $C_1 + C_2 + 2C_2$. Hence the classes of the proper transforms depend the number of intersections with the set $\{(0, \infty), (\infty, 0)\}$. The values are immediately calculated to be those given. □

3.5 Parametrising 1-cycles

We use the notation:

1. $B_{\mathsf{i}} = \{b_{\mathsf{i}}^1, \ldots, b_{\mathsf{i}}^{12}\}$ is the set of the 12 points in S_{i} that correspond to nodes in the fibres of the projection $\pi : S_{\mathsf{i}} \to \mathbb{P}^1$.
2. $S_{\mathsf{i}}^\circ = S_{\mathsf{i}} \setminus B_{\mathsf{i}}$ is the complement of B_{i} in S_{i}

Lemma 3.5.1. *In the case* $\beta = \sigma + (0, d_2, d_3)$ *there is a constructible morphism* $\rho_\bullet$ *where* $\mathrm{Chow}^{\sigma+(0,\bullet,\bullet)}(X)$ *has the decomposition:*

$$\mathrm{Chow}^{\sigma+(0,\bullet,\bullet)}(X) = \mathrm{Sym}^\bullet(S_2^\circ) \times \mathrm{Sym}^\bullet(B_2) \times \mathrm{Sym}^\bullet(B_{\mathsf{op}}).$$

Moreover, if $x = (\boldsymbol{ay}, \boldsymbol{mb_2}, \boldsymbol{nb_{\mathsf{op}}}) \in \mathrm{Chow}^{\sigma+(0,\bullet,\bullet)}(X)$ *then the fibre is given by* $\rho_\bullet^{-1}(x) = \mathrm{Hilb}^\bullet_{\mathsf{Cyc}}(X, \mathfrak{q})$ *where*

$$\mathfrak{q} = \sigma + \sum_i a_i \mathrm{pr}_2^{-1}(y_i) + \sum_i m_i C_2^{(i)} + \sum_i n_i C_3^{(i)}.$$

Proof. From lemma 3.2.2 part 2 it is enough to consider curves in the class $(0, d_2, d_3)$. Also from 3.2.2 part 1 we know that the curves are supported on fibres of the map $\mathrm{pr} : X \to \mathbb{P}^1$. From lemma 3.3.1 part 1 we know that the curves supported on smooth fibres of pr must be thicken fibres of the projection $\mathrm{pr}_2 : X \to S$. Similarly we know from lemma 3.4.3 part 2 that the curves supported on singular fibres of pr must be the union of thicken fibres of pr_2 and curves supported on the C_2 and C_3 banana curves. The result now follows. □

We also use the notation:

1. $\mathtt{N}_i \subset S_{\mathsf{i}}$ are the 12 nodal fibres of $\pi : S_{\mathsf{i}} \to \mathbb{P}^1$ with the nodes removed and:

   $\mathtt{N}_i = \mathtt{N}_i^\sigma \amalg \mathtt{N}_i^\emptyset$ where $\mathtt{N}_i^\sigma := \mathtt{N}_i \cap \sigma$ and $\mathtt{N}_i^\emptyset := \mathtt{N}_i \setminus \sigma$.
2. $\mathtt{Sm}_i = S_{\mathsf{i}}^\circ \setminus \mathtt{N}_i$ is the complement of $\mathtt{N}_i$ in S_{i}° and:

   $\mathtt{Sm}_i = \mathtt{Sm}_i^\sigma \amalg \mathtt{Sm}_i^\emptyset$ where $\mathtt{Sm}_i^\sigma := \mathtt{Sm}_i \cap \sigma$ and $\mathtt{Sm}_i^\emptyset := \mathtt{Sm}_i \setminus \sigma$.
3. $\mathtt{J}^0$ and $\mathtt{J}^{1728}$ to be the subsets of points $x \in \mathbb{P}^1$ such that $\pi^{-1}(x)$ has j-invariant 0 or 1728 respectively and $\mathtt{J} = \mathtt{J}^0 \amalg \mathtt{J}^{1728}$.

4. L to be the linear system $|f_1 + f_2|$ on $\mathbb{P}^1 \times \mathbb{P}^1$ with the singular divisors removed where f_1 and f_2 are fibres of the two projection maps.

5. $\widetilde{\mathrm{Aut}}(E) := \mathrm{Aut}(E) \setminus \{\pm 1\}$.

Remark 3.5.2. The following lemma should be parsed in the following way. For $i, j \in \{0, 1\}$ and $b, d_3 \in \mathbb{Z}_{\geq 0}$, a subscheme in the class $\beta = d\sigma + (i, j, d_3)$ will have 1-cycle of the following form:

$$\mathfrak{q} = b\sigma + D + \sum_i n_i C_3^{(i)}$$

where D is reduced and does not contain σ or and $C_3^{(i)}$. Then D is in the class (i, j, n) for some $n \in \mathbb{Z}_{\geq 0}$.

The Chow groups parameterise the different possible D and these possibilities depend on i and j:

- If $i = j = 0$ then D is the empty curve. If
- If $i = 0$ and $j = 1$ then D can be either a fibre of the projection pr_2 or $C_2^{(i)}$.
- If $i = j = i$ then and D then it can be combinations of fibres and banana curves. It can also be neither of these in the cases we call *diagonals.*

Lemma 3.5.3. *In the cases $\beta = d\sigma + (i, j, d_3)$ we have*

$$\mathrm{Chow}^{\bullet\sigma + (i,j,\bullet)}(X) \cong \mathbb{Z}_{\geq 0} \times \mathrm{Chow}^{(i,j,\bullet)}(X)$$

which agrees with constructible morphisms $\eta_\bullet^{ij}$ and the following decompositions of $\mathrm{Chow}^{(i,j,\bullet)}(X)$*:*

1. *For $i = j = 0$ we have the decomposition of* $\mathrm{Chow}^{(0,0,\bullet)}(X)$ *with parts:*

   (a) $Sym^\bullet(B_{\mathsf{op}})$.

   The corresponding fibres are then $(\eta_\bullet^{00})^{-1}(x) = \mathrm{Hilb}^\bullet_{\mathsf{Cyc}}(X, \mathfrak{q})$ *where:*

   (a) If $x = \boldsymbol{n b_{\mathsf{op}}}$ then $\mathfrak{q} = \sum_i n_i C_3^{(i)}$.

2. *For $i = 0$ and $j = 1$ we have a decomposition of* $\mathrm{Chow}^{(0,1,\bullet)}(X)$ *with parts:*

   (a) $S_2^\circ \times \mathrm{Sym}^\bullet(B_{\mathsf{op}})$

   (b) $\coprod_{k=1}^{12} \mathrm{Sym}^\bullet(\{b_{\mathsf{op}}^k\}) \times \mathrm{Sym}^\bullet(B_{\mathsf{op}} \setminus \{b_{\mathsf{op}}^k\})$.

   The corresponding fibres are then $(\eta_\bullet^{01})^{-1}(x) = \mathrm{Hilb}^\bullet_{\mathsf{Cyc}}(X, \mathfrak{q})$ *where:*

   (a) If $x = (y, \boldsymbol{n b_{\mathsf{op}}})$ then $\mathfrak{q} = \mathrm{pr}_2^{-1} + \sum_i n_i C_3^{(i)}$.

   (b) If $x = (a_k b_{\mathsf{op}}^k, \boldsymbol{n b_{\mathsf{op}}})$ then $\mathfrak{q} = a_k C_3^{(k)} + \sum_i n_i C_3^{(i)}$.

3. *For $i = j = 1$ we have a decomposition of* $\mathrm{Chow}^{(1,1,\bullet)}(X)$ *with parts:*

   (a) $S_1^\circ \times S_2^\circ \times \mathrm{Sym}^\bullet(B_{\mathsf{op}})$

(b) $\coprod_{k=1}^{12} S_1^\circ \times \mathrm{Sym}^\bullet(\{b_{\mathsf{op}}^k\}) \times \mathrm{Sym}^\bullet(B_{\mathsf{op}} \setminus \{b_{\mathsf{op}}^k\})$

(c) $\coprod_{k=1}^{12} S_2^\circ \times \mathrm{Sym}^\bullet(\{b_{\mathsf{op}}^k\}) \times \mathrm{Sym}^\bullet(B_{\mathsf{op}} \setminus \{b_{\mathsf{op}}^k\})$

(d) $\coprod_{\substack{k,\, l\, =\, 1 \\ k \neq l}}^{12} \mathrm{Sym}^\bullet(\{b_{\mathsf{op}}^k\}) \times \mathrm{Sym}^\bullet(\{b_{\mathsf{op}}^l\}) \times \mathrm{Sym}^\bullet(B_{\mathsf{op}} \setminus \{b_{\mathsf{op}}^k, b_{\mathsf{op}}^l\})$

(e) $\coprod_{k=1}^{12} \mathrm{Sym}^\bullet(\{b_{\mathsf{op}}^k\}) \times \mathrm{Sym}^\bullet(B_{\mathsf{op}} \setminus \{b_{\mathsf{op}}^k\})$

(f) $\coprod \mathtt{Diag}^\bullet$

where $\mathtt{Diag}^\bullet$ *will be defined by a further decomposition. The corresponding fibres of (a)-(e) are* $(\eta_\bullet^{11})^{-1}(x) = \mathrm{Hilb}_{\mathsf{Cyc}}^\bullet(X, \mathfrak{q})$ *where:*

(a) If $x = (y_1, y_2, \boldsymbol{nb}_{\mathsf{op}})$ *then* $\mathfrak{q} = \mathrm{pr}_1^{-1}(y_1) + \mathrm{pr}_1^{-1}(y_2) + \sum_i n_i C_3^{(i)}$.

(b) If $x = (y_1, a_k b_{\mathsf{op}}^k, \boldsymbol{nb}_{\mathsf{op}})$ *then* $\mathfrak{q} = \mathrm{pr}_1^{-1}(y_1) + C_2^{(k)} + a_k C_3^{(k)} + \sum_i n_i C_3^{(i)}$.

(c) If $x = (y_2, a_k b_{\mathsf{op}}^k, \boldsymbol{nb}_{\mathsf{op}})$ *then* $\mathfrak{q} = \mathrm{pr}_2^{-1}(y_2) + C_1^{(k)} + a_k C_3^{(k)} + \sum_i n_i C_3^{(i)}$.

(d) If $x = (a_k b_{\mathsf{op}}^k, a_l b_{\mathsf{op}}^l, \boldsymbol{nb}_{\mathsf{op}})$ *then* $\mathfrak{q} = C_1^{(k)} + C_2^{(l)} + a_k C_3^{(k)} + a_l C_3^{(l)} + \sum_i n_i C_3^{(i)}$.

(e) If $x = (a_k b_{\mathsf{op}}^k, \boldsymbol{nb}_{\mathsf{op}})$ *then* $\mathfrak{q} = C_1^{(k)} + C_2^{(k)} + a_k C_3^{(k)} + \sum_i n_i C_3^{(i)}$.

For part (f), $\mathtt{Diag}^\bullet$ *is defined by the further decomposition:*

(g) $\mathtt{Sm}_1 \times \mathrm{Sym}^\bullet(B_{\mathsf{op}})$

(h) $\coprod \mathtt{Sm}_2 \times \mathrm{Sym}^\bullet(B_{\mathsf{op}})$

(i) $\coprod_{y \in \mathtt{J}} E_{\pi(y)} \times \widetilde{\mathrm{Aut}}(E_{\pi(y)}) \times \mathrm{Sym}^\bullet(B_{\mathsf{op}})$

(j) $\coprod_{k=1}^{12} \mathtt{L} \times \mathrm{Sym}^\bullet(\{b_{\mathsf{op}}^k\}) \times \mathrm{Sym}^\bullet(B_{\mathsf{op}} \setminus \{b_{\mathsf{op}}^k\})$.

The corresponding fibres of (g)-(j) are $(\eta_\bullet^{11})^{-1}(x) = \mathrm{Hilb}^n(X, \mathfrak{q})$ *where:*

(g) If $x = (y, \boldsymbol{nb}_{\mathsf{op}})$ *then* $\mathfrak{q} = D_y + \sum_i n_i C_3^{(i)}$ *where* D_y *is the graph of the map* $f(z) = z + x|_{E_{\pi(y)}}$ *in the fibre* $F_{\pi(y)} = E_{\pi(y)} \times E_{\pi(y)}$.

(h) If $x = (y, \boldsymbol{nb}_{\mathsf{op}})$ *then* $\mathfrak{q} = D_y + \sum_i n_i C_3^{(i)}$ *where* D_y *is the graph of the map* $f(z) = -z + x|_{E_{\pi(y)}}$ *in the fibre* $F_{\pi(y)} = E_{\pi(y)} \times E_{\pi(y)}$.

(i) If $x = (y, \boldsymbol{nb}_{\mathsf{op}})$ *then* $\mathfrak{q} = D_y + \sum_i n_i C_3^{(i)}$ *where* D_y *is the graph of the map* $f(z) = A(z) + x$ *for some* $A \in \mathrm{Aut}(E_{\pi(y)}) \setminus \{\pm 1\}$.

(j) If $x = (z, a_k b_{\mathsf{op}}^k, \boldsymbol{nb}_{\mathsf{op}})$ *then* $\mathfrak{q} = \nu(\widetilde{L}_z) + a_k C_3^{(k)} + \sum_i n_i C_3^{(i)}$ *where* $\widetilde{L}_z$ *is the proper transform of the divisor* L_z *in* $\mathbb{P}^1 \times \mathbb{P}^1$ *and* ν *is the normalisation of the* k*th singular fibre.*

Proof. The decomposition $\mathrm{Chow}^{\bullet\sigma + (i,j,\bullet)}(X) \cong \mathbb{Z}_{\geq 0} \times \mathrm{Chow}^{(i,j,\bullet)}(X)$ is immediate from lemma 3.2.2 part 3. Hence it is enough to parametrise the curves in the class $\beta = (i, j, \bullet)$. Also from 3.2.2 part 1 we know that the curves are supported on fibres of the map $\mathrm{pr} : X \to \mathbb{P}^1$. We must have that

$$\mathsf{Cyc}(C) = a\sigma + D + \sum_{i=1}^{12} m_i C_3^{(i)}.$$

for some minimal reduces curve D in the class $(1, 1, n)$ for $n \geq 0$ minimal. The possible D curves are described in lemmas 3.3.1, 3.3.2, 3.4.3 and 3.4.4. The result now follows. □

4 Techniques for Calculating Euler Characteristic

4.1 Quot Schemes and their Decomposition

This section is a summary of required results from [BK]. First we consider the following subscheme of the Hilbert scheme.

Definition 4.1.1. Let $C \subset X$ be a Cohen-Macaulay subscheme of dimension 1. Consider the Hilbert scheme of subschemes $Z \subset X$ of class $[Z] = [C] \in H_2(X)$ and $\chi(\mathcal{O}_Z) = \chi(\mathcal{O}_C) + n$ for some $n \in \mathbb{Z}_{\geq 0}$. This contains the following closed subscheme:

$$\mathrm{Hilb}^n(X, C) := \big\{ Z \subset X \text{ such that } C \subset Z \text{ and } I_C/I_Z \text{ has finite length } n \big\}.$$

It is convenient to replace the Hilbert scheme here with a Quot scheme. Recall the Quot scheme $\mathrm{Quot}^n_X(\mathcal{F})$ parametrising quotients $\mathcal{F} \twoheadrightarrow Q$ on X, where Q is zero-dimensional of length n. It is related to the above Hilbert scheme in the following way.

Lemma 4.1.2. *[BK, Lemma 5]. The following equality holds in $K_0(\mathrm{Var}_{\mathbb{C}})((p))$:*

$$\mathrm{Hilb}^{\bullet}(X, C) = \mathrm{Quot}^{\bullet}_X(I_C).$$

We also consider the following subscheme of these Quot schemes.

Definition 4.1.3. [BK, Def. 12] Let $\mathcal{F}$ be a coherent sheaf on X, and $S \subset X$ a locally closed subset. We define the locally closed subset of $\mathrm{Quot}^n_X(\mathcal{F})$

$$\mathrm{Quot}^n_X(\mathcal{F}, S) := \big\{ [\mathcal{F} \twoheadrightarrow Q] \in \mathrm{Quot}^n_X(\mathcal{F}) \;\big|\; \mathrm{Supp}(Q^{\mathsf{red}}) \subset S \big\}.$$

This allows us to decompose the Quot schemes in the following way.

Lemma 4.1.4. *[BK, Prop. 13] Let $\mathcal{F}$ be a coherent sheaf on X, $S \subset X$ a locally closed subset and $Z \subset X$ a closed subset. Then if $Z \subset S$ and and $n \in \mathbb{Z}_{\geq 0}$ there is a geometrically bijective constructible morphism:*

$$\mathrm{Quot}^n_X(\mathcal{F}, S) \longrightarrow \coprod_{n_1+n_2=n} \mathrm{Quot}^{n_1}_X(\mathcal{F}, S \setminus Z) \times \mathrm{Quot}^{n_2}_X(\mathcal{F}, Z).$$

4.2 An Action on the Formal Neighbourhoods

Let $C \subset X$ be a one dimensional subscheme in the class $\beta \in H_2(X)$ with 1-cycle $\mathfrak{q} = \mathsf{Cyc}(C)$. We recall the our notation that $\mathrm{Hilb}^n_{\mathsf{Cyc}}(X, \mathfrak{q}) \subset \mathrm{Hilb}^{\beta,n}(X)$ is the following subscheme

$$\mathrm{Hilb}^n_{\mathsf{Cyc}}(X, \mathfrak{q}) := \Big\{ \; [Z] \in \mathrm{Hilb}^{\beta,n}(X) \mid \mathsf{Cyc}(C) = \mathfrak{q} \; \Big\}.$$

Furthermore, we define

$$\mathrm{Hilb}^n_{\mathsf{CM}}(X, \mathfrak{q}) \subset \mathrm{Hilb}^n_{\mathsf{Cyc}}(X, \mathfrak{q})$$

to be the open subscheme containing Cohen-Macaulay subschemes of Z.

Lemma 4.2.1. *Suppose $Z \subset X$ is a one dimensional Cohen-Macaulay subscheme such that:*

1. *Z has the decomposition $Z = C \cup_i Z_i$ where C is reduced and $Z_i \cap Z_j = \emptyset$ for $i \neq j$.*

2. *There are formal neighbourhoods* V_i *of* Z_i *in* X *such that* $(\mathbb{C}^*)^2$ *acts on each and fixes* Z_i^{red}.
3. *If* $\widetilde{C} := \overline{C \setminus (\cup V_i)}$ *then* $\widetilde{C} \cap (\cup V_i)$ *is invariant under the* $(\mathbb{C}^*)^2$*-action on* V_i.

Then there is a $(\mathbb{C}^*)^2$*-action on* $\mathrm{Hilb}^n_{\mathsf{CM}}(X, \mathsf{Cyc}(Z))$ *such that if* $\alpha \in (\mathbb{C}^*)^2$ *and* $Y \in \mathrm{Hilb}^n_{\mathsf{CM}}(X, \mathsf{Cyc}(Z))$ *then:*

$$\alpha \cdot Y \;=\; \widetilde{C} \;\cup\; \alpha \cdot (Y|_{\cup V_i}).$$

Proof. We show the action is well defined on a flat family in $\mathrm{Hilb}^n_{\mathsf{CM}}(X, \mathsf{Cyc}(Z))$. Let such a family be given by the diagram:

$$\begin{tikzcd} \mathcal{Z} \arrow[r, hook] \arrow[dr] & X \times S \arrow[d] \\ & S \end{tikzcd}$$

The reduced curves C, Z_i^{red} and the neighbourhoods V_i must all be constant on the family and we have a decomposition

$$\mathcal{Z} = (C \times S) \bigcup_i \mathcal{Z}_i$$

where $\mathcal{Z}_i \subset V_i \times S$. Hence, the action is given by

$$\alpha \cdot \mathcal{Z} \;=\; (\widetilde{C} \times S) \;\cup\; \alpha \cdot (\mathcal{Z}|_{\cup(V_i \times S)}).$$

$\square$

Consider the constructible map

$$\kappa : \mathrm{Hilb}^\bullet_{\mathsf{Cyc}}(X, \mathfrak{q}) \longrightarrow \mathrm{Hilb}^\bullet_{\mathsf{CM}}(X, \mathfrak{q})$$

where $Z \subset X$ is mapped to the maximal Cohen-Macaulay subscheme $Z_{\mathsf{CM}} \subset Z$. Then we have

$$\begin{aligned} e\Big(\mathrm{Hilb}^\bullet_{\mathsf{Cyc}}(X, \mathfrak{q})\Big) &= e\Big(\mathrm{Hilb}^\bullet_{\mathsf{CM}}(X, \mathfrak{q}), \kappa_* 1\Big) \\ &= e\Big(\mathrm{Hilb}^\bullet_{\mathsf{CM}}(X, \mathfrak{q})^{(\mathbb{C}^*)^2}, \kappa_* 1\Big) \end{aligned} \tag{7}$$

where $(\kappa_* 1)(z) := e(\kappa^{-1}(z))$ and the last line comes from the following lemma.

Lemma 4.2.2. *The constructible function* $\kappa_* 1$ *is invariant under the* $(\mathbb{C}^*)^2$*-action. That is if* $\alpha \in (\mathbb{C}^*)^2$ *and* $z \in \mathrm{Hilb}^n_{\mathsf{CM}}(X, \mathfrak{q})$ *then* $(\kappa_* 1)(z) = (\kappa_* 1)(\alpha \cdot x)$.

Proof. Let $\alpha \in (\mathbb{C}^*)^2$ and $z \in \mathrm{Hilb}^n_{\mathsf{CM}}(X, \mathfrak{q})$ correspond to $Z \subset X$. Also let $\widetilde{Z} = \cup Z_i$ and $\widetilde{V} = V_i$ be as in lemma 4.2.1. Then the fibre $\kappa^{-1}(x)$ is the

$$\kappa^{-1}(x) = \mathrm{Hilb}^\bullet(X, Z) = \mathrm{Quot}^\bullet_X(I_Z)$$

where the last equality is in $K_0(\mathrm{Var}_{\mathbb{C}})(\!(p)\!)$ from lemma 4.1.2. Also from lemma 4.1.4 we have a geometrically bijective constructible morphism:

$$\mathrm{Quot}^n_X(I_Z) \longrightarrow \coprod_{n_1+n_2=n} \mathrm{Quot}^{n_1}_X(I_Z, X \setminus \widetilde{V}) \times \mathrm{Quot}^{n_2}_X(I_Z, \widetilde{V}).$$

We have $I_{\alpha \cdot Z}|_{X \setminus \widetilde{V}} = I_Z|_{X \setminus \widetilde{V}}$ so $\mathrm{Quot}^{n_1}_X(I_Z, X \setminus \widetilde{V}) \cong \mathrm{Quot}^{n_1}_X(I_{\alpha \cdot Z}, X \setminus \widetilde{V})$. Moreover, we have isomorphisms

$$\mathrm{Quot}^{n_2}_X(I_Z, \widetilde{V}) \cong \mathrm{Quot}^{n_2}_{\widetilde{V}}(I_Z|_{\widetilde{V}})$$

and $Z_{\widetilde{V}} \cong \alpha \cdot Z_{\widetilde{V}}$ so we have an isomorphism

$$\mathrm{Quot}_X^{n_2}(I_Z, \widetilde{V}) \cong \mathrm{Quot}_X^{n_2}(I_{\alpha \cdot Z}, \widetilde{V})$$

Taking Euler characteristic now shows that $e\big(\kappa^{-1}(x)\big) = e\big(\kappa^{-1}(\alpha \cdot x)\big)$. □

4.2.3. We will now consider a useful tool in calculating Euler characteristics of the form given in (7). First let $z \in \mathrm{Hilb}^n_{\mathsf{CM}}(X, \mathfrak{q})$ correspond to $Z \subset X$ such that Z is locally monomial. Then the fibre $\kappa^{-1}(x)$ is

$$\kappa^{-1}(x) = \mathrm{Hilb}^{\bullet}(X, Z) = \mathrm{Quot}^{\bullet}_X(I_Z)$$

where the last equality is in $K_0(\mathrm{Var}_{\mathbb{C}})(\!(p)\!)$ from lemma 4.1.2. To compute this fibre we employ the following method:

1. Decompose X by $X = Z \ \amalg \ W$ where $W := X \setminus Z$
2. Let $Z^{\diamond}$ be set of singularities of Z^{red}.
3. Let $\coprod_i Z_i = Z \setminus Z^{\diamond}$ be a decomposition into irreducible components.

Then applying Euler characteristic to lemma 4.1.4 we have:

$$e\big(\mathrm{Quot}^{\bullet}_X(I_Z)\big) = e\big(\mathrm{Quot}^{\bullet}_X(I_Z, W)\big) \prod_{z \in Z^{\diamond}} e\big(\mathrm{Quot}^{\bullet}_X(I_Z, \{z\})\big) \prod_i e\big(\mathrm{Quot}^{\bullet}_X(I_Z, Z_i)\big).$$

4.3 Partitions and the topological vertex

We recall the terminology of 2D partitions, 3D partitions and the topological vertex from [ORV, BCY]. A *2D partition* λ is an infinite sequence of decreasing integers that is zero except for a finite number of terms. The size of a 2D partition $|\lambda|$ is the sum of the elements in the sequence and the length $l(\lambda)$ is the number of non-zero elements. We will also think of a 2D partition as a subset of $(\mathbb{Z}_{\geq 0})^2$ in the following way:

$$\lambda \ \leftrightsquigarrow \ \{(i,j) \in (\mathbb{Z}_{\geq 0})^2 \mid \lambda_i \geq j \geq 0 \text{ or } i = 0\}$$

A *3D partition* is a subset $\boldsymbol{\eta} \subset (\mathbb{Z}_{\geq 0})^3$ satisfying the following condition:

1. $(i,j,k) \in \boldsymbol{\eta}$ if and only if one of i, j or k is zero or one of $(i-1,j,k)$, $(i,j-1,k)$ or $(i,j,k-1)$ is also in $\boldsymbol{\eta}$.

Given a triple of 2D partitions (λ, μ, ν) we also define a *3D partition asymptotic to* (λ, μ, ν) is a 3D partition $\boldsymbol{\eta}$ that also satisfies the conditions:

1. $(j,k) \in \lambda$ if and only if $(i,j,k) \in \boldsymbol{\eta}$ for all $i \gg 0$.
2. $(k,i) \in \mu$ if and only if $(i,j,k) \in \boldsymbol{\eta}$ for all $j \gg 0$.
3. $(i,j) \in \nu$ if and only if $(i,j,k) \in \boldsymbol{\eta}$ for all $k \gg 0$.

The *leg of* $\boldsymbol{\eta}$ *in the* i*th direction* is the subset $\{(i,j,k) \in \boldsymbol{\eta} \mid (j,k) \in \lambda\}$. We analogously define the legs of $\boldsymbol{\eta}$ in the j and k directions. The *weight* of a point in $\boldsymbol{\eta}$ is defined to be

$$\xi_{\boldsymbol{\eta}}(i,j,k) := 1 - \#\{\text{legs of } \boldsymbol{\eta} \text{ containing } (i,j,k)\}.$$

Using this we define the *renormalised volume* of $\boldsymbol{\eta}$ by:

$$|\boldsymbol{\eta}| := \sum_{(i,j,k) \in \boldsymbol{\eta}} \xi_{\boldsymbol{\eta}}(i,j,k). \tag{II.9}$$

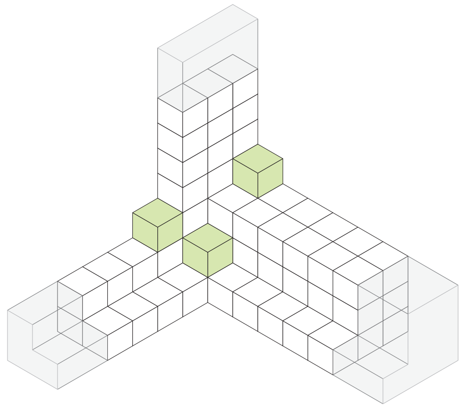


Figure II.7: A 3D partition asymptotic to $\big((2,1),(3,2,2),(1,1,1)\big)$. The partition containing only the white boxes has renormalised volume -14. The partition including the green boxes has renormalised volume -11.

The ***topological vertex*** is the formal Laurent series:

$$\mathsf{V}_{\lambda\mu\nu} := \sum_{\boldsymbol{\eta}} p^{|\boldsymbol{\eta}|}$$

where the sum is over all 3D partitions asymptotic to (λ, μ, ν). An explicit formula for $\mathsf{V}_{\lambda\mu\nu}$ is derived in [ORV, Eq. 3.18] to be:

$$\mathsf{V}_{\lambda\mu\nu} = M(p)p^{-\frac{1}{2}(\|\lambda\|^2+\|\mu^t\|^2+\|\nu\|^2)} S_{\nu^t}(p^{-\rho}) \sum_{\eta} S_{\lambda^t/\eta}(p^{-\nu-\rho}) S_{\mu/\eta}(p^{-\nu^t-\rho})$$

4.4 Partition Thickened Section, Fibre and Banana Curves

In this subsection we consider non-reduced structure for curves in our desired classes. The partition thickened structure will be the fixed points of a $(\mathbb{C}^*)^2$-action.

4.4.1. Recall that the section $\zeta \in S$ is the blow-up of a point in $z \in \mathbb{P}^2$. Choose once and for all a formal neighbourhood $\operatorname{Spec} \mathbb{C}[\![s,t]\!]$ of $z \in \mathbb{P}^2$. The blow-up gives the formal neighbourhood of $\zeta \in S$ with 2 coordinate charts:

$$\mathbb{C}[\![s,t]\!][u]/(t-su) \cong \mathbb{C}[\![s]\!][u] \quad \text{and} \quad \mathbb{C}[\![s,t]\!][v]/(s-tv) \cong \mathbb{C}[\![t]\!][v]$$

with change of coordinates $s \mapsto tv$ and $u \mapsto v^{-1}$. This gives the formal neighbourhood of $\sigma \in X$ with 2 coordinate charts:

$$\mathbb{C}[\![s_1,s_2]\!][u] \quad \text{and} \quad \mathbb{C}[\![t_1,t_2]\!][v]$$

with change of coordinates $s_i \mapsto t_i v$ and $u \mapsto v^{-1}$. We call these coordinates the **canonical formal coordinates** around $\sigma \in X$.

4.4.2. Now consider a reduced curve D in X that intersects σ transversely with length 1. When D is restricted to the formal neighbourhood of σ it is given by

$$\mathbb{C}[\![s_1,s_2]\!][u]/(a_0u-a_1, b_0s_1-b_1s_2) \quad \text{and} \quad \mathbb{C}[\![t_1,t_2]\!][v]/(a_0-a_1v, b_0t_1-b_1t_2)$$

for some $[a_0 : a_1], [b_0 : b_1] \in \mathbb{P}^1$. We use this to define the change of coordinates:

$$\tilde{s}_1 \mapsto b_0s_1 - b_1s_2 \quad \text{and} \quad \tilde{s}_2 \mapsto b_1s_1 + b_0s_2$$
$$\tilde{t}_1 \mapsto b_0t_1 - b_1t_2 \quad \text{and} \quad \tilde{t}_2 \mapsto b_1t_1 + b_0t_2$$

We call these coordinates the **canonical formal coordinates relative to** D.

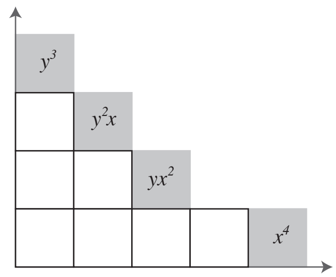


Figure II.8: Depiction of the subscheme in $\mathbb{C}^2$ given by the monomial ideal $(y^3, y^2x, y^1x^2, y^1x^3, x^4)$ associated to the partition $(3, 2, 1, 1, 0\ldots)$.

Definition 4.4.3. Let $\mathbb{C}[\![s_1, s_2]\!][u]$ and $\mathbb{C}[\![t_1, t_2]\!][v]$ be either of the above canonical coordinates. Then we define

1. The **canonical** $(\mathbb{C}^*)^2$**-action** on these coordinates by $(s_1, s_2) \mapsto (\lambda_1 s_1, \lambda_2 s_2)$ and $(t_1, t_2) \mapsto (\lambda_1 t_1, \lambda_2 t_2)$.
2. Let $\lambda = (\lambda_1, \ldots, \lambda_l, 0, \ldots)$ be a 2D partition. The λ-**thickened section** denoted by $\lambda\sigma$ is the subscheme of X defined by the ideal given in the coordinates by
$$(s_2^{\lambda_1}, \ldots, s_1^{l-1}s_2^{\lambda_l}, s^l) \quad \text{and} \quad (t_2^{\lambda_1}, \ldots, t_1^{l-1}t_2^{\lambda_l}, t^l).$$

We can now consider fibres of the projection map $\mathrm{pr}_2 : X \to S$.

Definition 4.4.4. Let $x \in S^\circ$ and $f_x = \mathrm{pr}_2^{-1}(x)$ the fibre. Then we define

1. **Canonical coordinates** on a formal neighbourhood V_x of f_x are given by formal coordinates $\mathbb{C}[\![s, t]\!]$ of x in S under where $V_x = f_x \times \operatorname{Spec} \mathbb{C}[\![s, t]\!]$.
2. The **canonical** $(\mathbb{C}^*)^2$**-action** on these coordinates by $(s, t) \mapsto (\lambda_1 s, \lambda_2 t)$.
3. Let $\lambda = (\lambda_1, \ldots, \lambda_l, 0, \ldots)$ be a 2D partition. The λ-**thickened fibre** at x denoted by λf_x is the subscheme of X given by the ideal:
$$(t^{\lambda_1}, \ldots, s^{l-1}t^{\lambda_l}, s^l)$$

4.4.5. We now consider a canonical formal neighbourhood of the banana curve C_3. We follow much of the reasoning from [Br, §5.2]. Let $x \in S$ correspond to a point where $\pi : S \to \mathbb{P}^1$ is singular. Let formal neighbourhoods in the two isomorphic copies of S be given by
$$\operatorname{Spec} \mathbb{C}[\![s_1, t_1]\!] \quad \text{and} \quad \operatorname{Spec} \mathbb{C}[\![s_2, t_2]\!]$$
and the map $S \to \mathbb{P}^1$ be given by $r \mapsto s_i t_i$. Then the formal neighbourhood of a conifold singularity in X is given by
$$\operatorname{Spec} \mathbb{C}[\![s_1, t_1, s_2, t_2]\!]/(s_1t_1 - s_2t_2),$$
and the restriction to a fibre of the projection $S \times_{\mathbb{P}^1} S \to \mathbb{P}^1$ is
$$\operatorname{Spec} \mathbb{C}[\![s_1, t_1, s_2, t_2]\!]/(s_1t_1, s_2t_2).$$

Now, blowing up along $\{s_1 = t_2 = 0\}$ (which is canonically equivalent to blowing up along $\{s_1 - t_1 = s_2 - t_2 = 0\}$), we have the two coordinate charts:
$$\mathbb{C}[\![s_1, t_2, s_2, t_2]\!][u]/(s_1 - ut_2, s_2 - ut_1) \cong \mathbb{C}[\![t_1, t_2]\!][u], \qquad \text{and}$$
$$\mathbb{C}[\![s_1, t_2, s_2, t_2]\!][v]/(t_1 - vs_2, t_2 - vs_1) \cong \mathbb{C}[\![s_1, s_2]\!][v],$$

where the change of coordinates is given by $t_1 \mapsto vs_2$, $t_2 \mapsto vs_1$ and $u \mapsto v^{-1}$. We call these coordinates the **canonical formal coordinates** around the banana curve C_3.

4.4.6. With these coordinates we have:

1. Then the restriction to the fibre of $\mathrm{pr} : X \to \mathbb{P}^1$ is

$$\mathbb{C}[\![t_1, t_2]\!][u]/(t_1t_2u) \quad \text{and} \quad \mathbb{C}[\![s_1, s_2]\!][v]/(s_1s_2v).$$

2. The banana curve C_3 is given by

$$\mathbb{C}[\![t_1, t_2]\!][u]/(t_1, t_2) \quad \text{and} \quad \mathbb{C}[\![s_1, s_2]\!][v]/(s_1, s_2).$$

4.4.7. Similar to 4.4.2 we also consider canonical relative coordinates for a C_3 banana curve. Recall 3.4.4 and let D is the image under $\nu : \widetilde{F}_{\mathsf{ban}} \to F_{\mathsf{ban}}$ of the proper transform under $\mathsf{bl} : \mathrm{Bl}_{(0,\infty),(\infty,0)}(\mathbb{P}^1 \times \mathbb{P}^1) \to \mathbb{P}^1 \times \mathbb{P}^1$ of a smooth divisor in $|f_1 + f_2|$ on $|\mathbb{P}^1 \times \mathbb{P}^1|$.

If D intersects $(0, 0)$ then the restriction of D to the formal neighbourhood of C_3 is given by:

$$\mathbb{C}[\![s_1, s_2]\!][v]/(s_1 - as_2, v)$$

for some $a \in \mathbb{C}^*$. In this case we define **canonical formal coordinates relative to** D around a C_3 banana by the following change of coordinates.

$$\tilde{s}_1 \mapsto s_1 - as_2 \quad \text{and} \quad \tilde{s}_2 \mapsto s_1 + as_2$$

$$\tilde{t}_1 \mapsto at_1 + t_2 \quad \text{and} \quad \tilde{t}_2 \mapsto -at_1 + t_2$$

We similarly define the same relative coordinates if for D intersects (∞, ∞) in the ideal $(-at_1 + t_2, u)$. Note that these coordinates are compatible if D intersects both $(0, 0$ and (∞, ∞).

Definition 4.4.8. Let $\mathbb{C}[\![s_1, s_2]\!][u]$ and $\mathbb{C}[\![t_1, t_2]\!][v]$ be either the canonical coordinates or relative coordinates.

1. The **canonical** $(\mathbb{C}^*)^2$**-action** on these coordinates is defined by

$$(s_1, s_2, v) \mapsto (\lambda_1 s_1, \lambda_2 s_2, v) \quad \text{and} \quad (t_1, t_2, u) \mapsto (\lambda_2 t_1, \lambda_1 t_2, u).$$

2. Let $\lambda = (\lambda_1, \ldots, \lambda_l, 0, \ldots)$ be a 2D partition. The λ**-thickened banana curve** C_3 denoted by λC_3 is the subscheme of X defined by the ideal given in the coordinates by

$$(s_2^{\lambda_1}, \ldots, s_1^{l-1}s_2^{\lambda_l}, s_1^l) \quad \text{and} \quad (t_1^{\lambda_1}, \ldots, t_2^{l-1}t_1^{\lambda_l}, t_2^l).$$

(Note the change in coordinates compared to definition 4.4.3.)

Remark 4.4.9. If D intersects both $(0, 0)$ and (∞, ∞) and λC_3 is partition thickened in the coordinates relative to D. Then ideals for $D \cup \lambda C_3$ at the points $(0, 0)$ and (∞, ∞) are

$$(s_2^{\lambda_1}, \ldots, s_1^{l-1}s_2^{\lambda_l}, s_1^l) \cap (s_1, v) \quad \text{and} \quad (t_1^{\lambda_1}, \ldots, t_2^{l-1}t_1^{\lambda_l}, t_2^l) \cap (t_2, u)$$

respectively. These both give 3D partitions asymptotic to $(\lambda, \emptyset, \square)$.

Lemma 4.4.10. *Let D be as described in the first paragraph of 4.4.7. If let V be the formal neighbourhood of C_3 in X. If D intersects $(0,0)$ and/or (∞,∞) then use the relative coordinates of 4.4.7, otherwise use the canonical coordinates of 4.4.5. Then $D \cap V$ is invariant under the $(\mathbb{C}^*)^2$-action.*

Proof. We have $D \cap V \neq 0$ if and only if it intersects at least one of $(0,0)$, $(0,\infty)$, $(\infty,0)$, (∞,∞). The possible combinations are:

1. $(0,0)$ *and/or* (∞,∞): This is by construction of the relative coordinates.
2. *Exactly one of* $(0,\infty)$ *or* $(\infty,0)$: Then D is given by the ideal $(v-a, s_1)$ or $(v-a, s_2)$ for some $a \in \mathbb{C}^*$, which are $(\mathbb{C}^*)^2$-invariant.
3. $(0,\infty)$ *and* $(\infty,0)$: Then D is given by the ideal $(v-a, s_1s_2)$ for some $a \in \mathbb{C}^*$ which is $(\mathbb{C}^*)^2$-invariant.

□

4.4.11. It is also shown in [Br, §5.2] that there are the following formal coordinates on C_2 compatible with the canonical formal coordinates around C_3:

$$\mathbb{C}[\![s_1, v]\!][s_2] \quad \text{and} \quad \mathbb{C}[\![t_1, u]\!][t_2]$$

where the change on coordinates is given by $s_2 \mapsto t_2$, $s_1 \mapsto t_1t_2$ and $v \mapsto t_2u$. We can define partition thickenings and a compatible $(\mathbb{C}^*)^2$-action in these coordinates.

Definition 4.4.12. Let $\mathbb{C}[\![s_1, v]\!][s_2]$ and $\mathbb{C}[\![t_1, u]\!][t_2]$ be the above canonical coordinates.

1. The **canonical** $(\mathbb{C}^*)^2$**-action** on these coordinates is defined by:
$$(s_1, v, s_2) \mapsto (\lambda_1 s_1, v, \lambda_2 s_2) \quad \text{and} \quad (t_1, u, t_2) \mapsto (\lambda_2 t_1, u, \lambda_1 t_2).$$
2. Let $\mu = (\mu_1, \dots, \mu_k, 0, \dots)$ be a 2D partition. The μ**-thickened banana curve** C_2 denoted by μC_2 is the subscheme of X defined by the ideal given in the coordinates by
$$(s_1^{\mu_1}, \dots, v^{k-1}s_1^{\mu_k}, v^k) \quad \text{and} \quad (t_1^{\mu_1}, \dots, u^{k-1}t_1^{\mu_k}, u^k).$$
(Note the change in coordinates compared to definition 4.4.8.)
3. Let $\lambda = (\lambda_1, \dots, \lambda_l, 0, \dots)$ be another 2D partition. The (μ, λ)**-thickened banana curve** denoted is the union $\mu C_2 + \lambda C_3$.

Remark 4.4.13. The C_2 and C_3 banana curves meet in exactly 2 points. At these two points a (μ, λ)-thickened banana curve will define define two 3D partitions. One will be asymptotic to $(\mu, \lambda, \emptyset)$ the other will be asymptotic to $(\mu^t, \lambda^t, \emptyset)$ (or equivalently $(\lambda, \mu, \emptyset)$).

Remark 4.4.14. The partition thickened curves described in this section are easily shown to be the only Cohen-Macaulay subschemes supported in these neighbourhoods that are invariant under the $(\mathbb{C}^*)^2$-action. This is because the invariant Cohen-Macaulay subschemes must be generated by monomial ideals.

Lemma 4.4.15. *Let $\lambda = (\lambda_1, \dots, \lambda_l, 0, \dots)$ and $\mu = (\mu_1, \dots, \mu_k, 0, \dots)$ be a 2D partitions. Then we have the holomorphic Euler characteristics :*

1. $\chi(\mathcal{O}_{\lambda\sigma}) = \frac{1}{2}\big(\|\lambda\|^2 + \|\lambda^t\|^2\big)$,
2. $\chi(\mathcal{O}_{\lambda f_x}) = 0$,
3. $\chi(\mathcal{O}_{\mu C_2 \,\cup\, \lambda C_3}) = |\boldsymbol{\eta}_1| + |\boldsymbol{\eta}_2| + \frac{1}{2}\big(\|\mu\|^2 + \|\mu^t\|^2 + \|\lambda\|^2 + \|\lambda^t\|^2\big)$ *where* $|\boldsymbol{\eta}_i|$ *are the renormalised volumes of the minimal 3D partitions associated to* $(\mu, \lambda, \emptyset)$ *and* $(\mu^t, \lambda^t, \emptyset)$.

Proof. (2) is straightforward and the rest are from [Br, Prop. 23]. □

4.5 Relation between Quot Schemes on $\mathbb{C}^3$ and the Topological Vertex

This section is predominately a summary of required results from [BK]. For 2D partitions λ, μ and ν we define the following subscheme of $\mathbb{C}^3$:

$$\mathsf{C}_{\lambda,\mu,\nu} = \mathsf{C}_{\lambda,\emptyset,\emptyset} \cup \mathsf{C}_{\emptyset,\mu,\emptyset} \cup \mathsf{C}_{\emptyset,\emptyset,\nu} \subset \mathrm{Spec}\mathbb{C}[r,s,t]$$

where $\mathsf{C}_{\lambda,\emptyset,\emptyset}$ is defined by the idea $I_{\lambda,\emptyset,\emptyset} := (t^{\lambda_1}, \dots, t^{l-1}s^{\lambda_l}, s^l)$, with $\mathsf{C}_{\emptyset,\mu,\emptyset}$ and $\mathsf{C}_{\emptyset,\emptyset,\nu}$ being cyclic permutations of this. Also define the ideal by $I_{\lambda\mu\nu} = I_{\lambda\emptyset\emptyset} \cap I_{\emptyset\mu\emptyset} \cap I_{\emptyset\emptyset\nu}$

Now we consider the Quot scheme of length n quotients that are supported at the origin and we employ the following simplifying notation:

$$\mathrm{Quot}^n(\lambda,\mu,\nu) := \mathrm{Quot}^n_{\mathbb{C}^3}(I_{\lambda\mu\nu}, \{0\})$$

The quotients parametrised here have kernels that are the ideal sheaf of a one-dimensional scheme Z with underlying Cohen-Macaulay curve $\mathsf{C}_{\lambda,\mu,\nu}$. The embedded points of this scheme are all supported at the origin, but Z doesn't have to be locally monomial. We use the following variation of the notation for the topological vertex:

$$\widetilde{\mathsf{V}}_{\lambda\mu\nu} := e\big(\mathrm{Quot}^\bullet(\lambda,\mu,\nu)\big) \in \mathbb{Z}[\![p]\!].$$

Lemma 4.5.1. *Let C be a partition thickened section, fibre or C_3-banana curve thickened by λ. Then*

1. *If $x \in C$ is a smooth point then $e\big(\mathrm{Quot}^n_X(I_C, \{x\})\big) = \widetilde{\mathsf{V}}_{\lambda\emptyset\emptyset}$.*
2. *If C is a thickened nodal fibre then $e\big(\mathrm{Quot}^n_X(I_C, \{x\})\big) = \widetilde{\mathsf{V}}_{\lambda\lambda^t\emptyset}$.*

Let C' be a reduced curve intersecting C at $y \in C$ such that $I_{C'} \cap I_C$ is locally monomial and there are formal local coordinates $\mathbb{C}[\![r,s,t]\!]$ at y such that:

1. *$I_{C'} \cap I_C = (t^{\lambda_1}, \dots s^{l-1}t^{\lambda_l}, s^l) \cap (r,s)$ then $e\big(\mathrm{Quot}^n_X(I_C, \{x\})\big) = \widetilde{\mathsf{V}}_{\lambda\emptyset\square}$.*
2. *$I_{C'} \cap I_C = (t^{\lambda_1}, \dots s^{l-1}t^{\lambda_l}, s^l) \cap (r,s) \cap (r,t)$ then $e\big(\mathrm{Quot}^n_X(I_C, \{x\})\big) = \widetilde{\mathsf{V}}_{\lambda\square\square}$.*

Proof. The proof is the same as [BK] Lemma 15. □

Lemma 4.5.2. *Let D be a one dimensional Cohen-Macaulay subscheme of X.*

1. *We have:*
$$e\big(\mathrm{Quot}^n_X(I_D, X \setminus D)\big) = \Big(\widetilde{\mathsf{V}}_{\emptyset\emptyset\emptyset}\Big)^{e(X)-e(C)}.$$
2. *Let λ be a 2D partition and $\lambda C \subset D$ be either a partition thickened section, fibre or C_3 banana and let T be finite set of points on C such that $C \setminus T$ is smooth. Then*
$$e\big(\mathrm{Quot}^n_X(I_D, C \setminus T)\big) = \Big(\widetilde{\mathsf{V}}_{\lambda\emptyset\emptyset}\Big)^{e(C)-e(T)}.$$

Proof. The argument is the same as that given for equation (9) in [BK]. □

The standard $(\mathbb{C}^*)^3$-action on $\mathbb{C}^3$ induces an action on the Quot schemes. The invariant ideals $I \subset \mathbb{C}[r,s,t]$ are precisely those generated by monomials. Also, since there is a bijection between locally monomial ideals and 3D partitions we see that

$$\begin{aligned}\widetilde{\mathsf{V}}_{\lambda\mu\nu} &= e\big(\mathrm{Quot}^\bullet(\lambda,\mu,\nu)^{(\mathbb{C}^*)^3}\big)\\ &= \sum_{\boldsymbol{\eta}} p^{n(\boldsymbol{\eta})}\end{aligned}$$

where we are summing over 3D partitions asymptotic to (λ, μ, ν) and $n(\boldsymbol{\eta})$ is the number of boxes not contained in any legs. Note that that the lowest order term in $\widetilde{\mathsf{V}}_{\lambda\mu\nu}$ is one, which is not true about $\mathsf{V}_{\lambda\mu\nu}$ in general. In fact we have the relationship:

$$\mathsf{V}_{\lambda\mu\nu} = p^{|\boldsymbol{\eta}_{min}|}\widetilde{\mathsf{V}}_{\lambda\mu\nu}$$

where $\boldsymbol{\eta}_{min}$ is the 3D partition associated to $\mathsf{C}_{\lambda\mu\nu}$, and $|\cdot|$ is the renormalised volume defined in eqn (II.9).

Lemma 4.5.3. *If λ is a 2D partition then we have the following equalities:*

1. $\mathsf{V}_{\lambda\emptyset\emptyset} = \widetilde{\mathsf{V}}_{\lambda\emptyset\emptyset}$
2. $\mathsf{V}_{\lambda\square\emptyset} = p^{-\lambda_1}\widetilde{\mathsf{V}}_{\lambda\square\emptyset}$
3. $\mathsf{V}_{\lambda\square\square} = p^{-\lambda_1-\lambda_1^t}\widetilde{\mathsf{V}}_{\lambda\square\square}$
4. $\mathsf{V}_{\lambda\lambda^t\emptyset} = p^{-\|\lambda\|^2}\widetilde{\mathsf{V}}_{\lambda\lambda^t\emptyset}$

Proof. Parts (1), (2) and (4) are directly from [BK] lemma 17. For part 3, there are λ_1 boxes that are in the λ-leg and one of the $\square$-legs. There are λ_1^t boxes that are in the λ-leg and the other $\square$-leg. There is one box that is contained in all three so the renormalised volume is calculated to be

$$(\lambda_i - 1)(1-2) + (\lambda_i^t - 1)(1-2) + (1)(1-3) = -\lambda_i - \lambda_i^t$$

$\square$

Let

$$\Psi^{\bullet,\bullet}(a, m) := \sum_{\alpha\vdash a}\sum_{\mu\vdash m} p^{\frac{1}{2}(\|\alpha\|^2+\|\alpha^t\|^2+\|\mu\|^2+\|\mu^t\|^2)}(\mathsf{V}_{\emptyset\mu\alpha}\mathsf{V}_{\emptyset\mu^t\alpha^t})$$

5 Calculating the Euler Characteristic from the Fibres of the Chow Map

5.1 Calculation for the class $\sigma + (0, \bullet, \bullet)$

We now recall some previously introduced notation:

1. $B_{\mathsf{i}} = \{b_{\mathsf{i}}^1, \ldots, b_{\mathsf{i}}^{12}\}$ is the set of the 12 points in S_{i} that correspond to nodes in the fibres of the projection $\pi : S_{\mathsf{i}} \to \mathbb{P}^1$.
2. $S_{\mathsf{i}}^\circ = S_{\mathsf{i}} \setminus B_{\mathsf{i}}$ is the complement of B_{i} in S_{i}
3. $\mathtt{N}_i \subset S_{\mathsf{i}}$ are the 12 nodal fibres of $\pi : S_{\mathsf{i}} \to \mathbb{P}^1$ with the nodes removed and:

$$\mathtt{N}_i = \mathtt{N}_i^\sigma \ \amalg \ \mathtt{N}_i^\emptyset \text{ where } \mathtt{N}_i^\sigma := \mathtt{N}_i \cap \sigma \text{ and } \mathtt{N}_i^\emptyset := \mathtt{N}_i \setminus \sigma.$$

4. $\mathtt{Sm}_i = S_{\mathsf{i}}^\circ \setminus \mathtt{N}_i$ is the complement of $\mathtt{N}_i$ in S_{i}° and:

$$\mathtt{Sm}_i = \mathtt{Sm}_i^\sigma \ \amalg \ \mathtt{Sm}_i^\emptyset \text{ where } \mathtt{Sm}_i^\sigma := \mathtt{Sm}_i \cap \sigma \text{ and } \mathtt{Sm}_i^\emptyset := \mathtt{Sm}_i \setminus \sigma.$$

Now from lemma 3.5.1 we can further decompose $\mathrm{Chow}^{\sigma+(0,\bullet,\bullet)}(X)$ as:

$$\mathrm{Sym}^\bullet(\mathtt{Sm}_2^\sigma) \times \mathrm{Sym}^\bullet(\mathtt{N}_2^\sigma) \times \mathrm{Sym}^\bullet(\mathtt{Sm}_2^\emptyset) \times \mathrm{Sym}^\bullet(\mathtt{N}_2^\emptyset) \times \mathrm{Sym}^\bullet(B_2) \times \mathrm{Sym}^\bullet(B_{\mathsf{op}}).$$

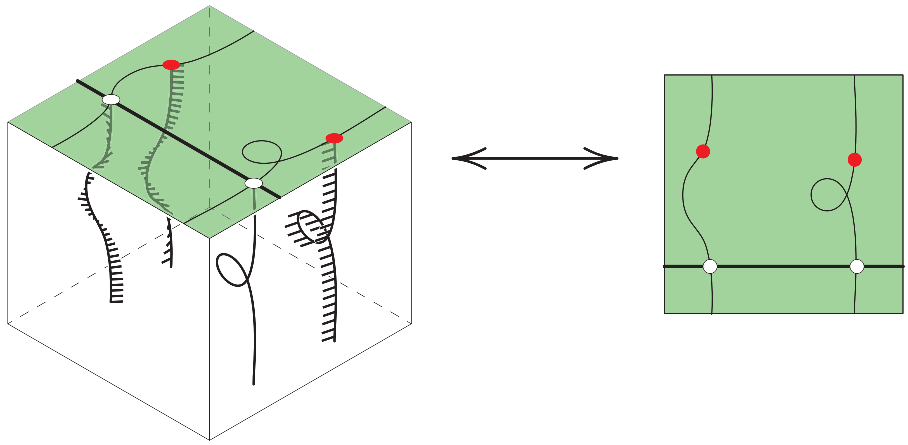


Figure II.9: Depiction of the decomposition of the Chow sub-scheme that parametrises the vertical fibres of pr_2. The red dots indicate when the fibres don't intersect the section i.e. $\mathtt{Sm}_2^{\emptyset}$ and $\mathtt{N}_2^{\emptyset}$. The white dots indicate when the fibres do intersect the section i.e. $\mathtt{Sm}_2^{\sigma}$ and $\mathtt{N}_2^{\sigma}$.

Moreover, if $\mathfrak{q} = (\boldsymbol{ax}, \boldsymbol{cy}, \boldsymbol{dz}, \boldsymbol{lw}, \boldsymbol{mb_2}, \boldsymbol{nb_{\mathsf{op}}}) \in \mathrm{Chow}^{\sigma+(0,\bullet,\bullet)}(X)$ then the fibre is given by $\rho_{\bullet}^{-1}(\mathfrak{q}) \cong \mathrm{Hilb}^{\bullet}_{\mathsf{Cyc}}(X, \mathfrak{q})$ where

$$\begin{aligned}\mathfrak{q} = \sigma &+ \sum_i a_i \mathrm{pr}_2^{-1}(x_i) + \sum_i c_i \mathrm{pr}_2^{-1}(y_i) \\ &+ \sum_i d_i \mathrm{pr}_2^{-1}(z_i) + \sum_i l_i \mathrm{pr}_2^{-1}(w_i) + \sum_i m_i C_2^{(i)} + \sum_i n_i C_3^{(i)}.\end{aligned}$$

5.1.1. Suppose C is Cohen-Macaulay with the cycle given above. Note that C can be decomposed into a part supported on C_2 and C_3 and a part supported away from the banana configuration. This gives the following formal neighbourhoods and $(\mathbb{C}^*)^2$-actions:

1. Let U_i be the formal neighbourhood of $C_2^{(i)} \cup C_3^{(i)}$ in X. These have a canonical $(\mathbb{C}^*)^2$-action described in 4.4.8 and 4.4.12.

2. Let V_i be the formal neighbourhood of $\mathrm{pr}_2^{-1}(y_i)$ in X. These have a canonical $(\mathbb{C}^*)^2$-action described in definition 4.4.4 and $\sigma \cap V_i$ is either empty of invariant under this action.

Hence the conditions of lemma 4.2.1 are satisfied and there is a $(\mathbb{C}^*)^2$-action defined on $\mathrm{Hilb}^n_{\mathsf{CM}}(X, \mathfrak{q})$. Using the partition thickened notation introduced in section 4.4 we introduce the subschemes:

$$C_{\boldsymbol{\alpha},\boldsymbol{\gamma},\boldsymbol{\delta},\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu}} := \sigma \bigcup_i \big(\alpha^{(i)} f_{x_i}\big) \bigcup_i \big(\gamma^{(i)} f_{y_i}\big) \bigcup_i \big(\delta^{(i)} f_{z_i}\big) \bigcup_i \big(\lambda^{(i)} f_{w_i}\big) \bigcup_i \big(\mu^{(i)} C_2^{(i)}\big) \bigcup_i \big(\nu^{(i)} C_3^{(i)}\big)$$

and their ideals $I_{\boldsymbol{\alpha},\boldsymbol{\gamma},\boldsymbol{\delta},\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu}}$ in X where $\boldsymbol{\alpha}$, $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$, $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ are tuples of partitions of $\boldsymbol{a}$, $\boldsymbol{c}$, $\boldsymbol{d}$, $\boldsymbol{l}$, $\boldsymbol{m}$ and $\boldsymbol{n}$ respectively. Then using this notation we can identify the fixed points of the action as the following discrete set:

$$\mathrm{Hilb}^{\bullet}_{\mathsf{CM}}(X, \mathfrak{q})^{(\mathbb{C}^*)^2} = \coprod_{\substack{\boldsymbol{\alpha} \vdash \boldsymbol{a},\ \boldsymbol{\gamma} \vdash \boldsymbol{c},\ \boldsymbol{\delta} \vdash \boldsymbol{d},\\ \boldsymbol{\lambda} \vdash \boldsymbol{l},\ \boldsymbol{\mu} \vdash \boldsymbol{m},\ \boldsymbol{\nu} \vdash \boldsymbol{n}}} \Big\{ C_{\boldsymbol{\alpha},\boldsymbol{\gamma},\boldsymbol{\delta},\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu}} \Big\}.$$

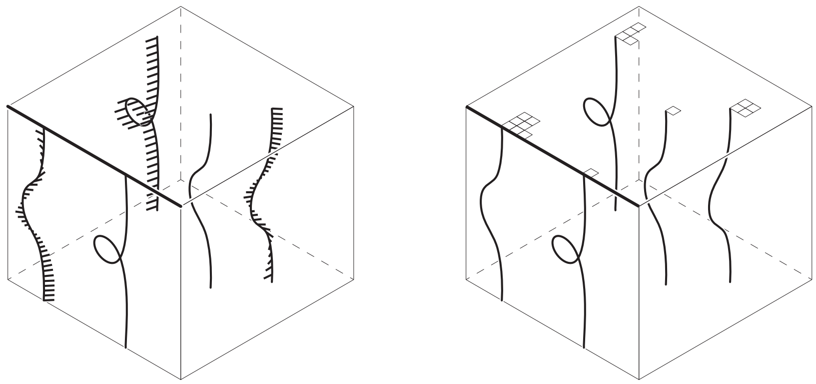


Figure II.10: Visual depictions of Cohen-Macaulay subschemes in the fibres of $\rho_\bullet$ that are supported away from the banana configurations. On the left is a general subscheme and on the right is a partition thickened curve which is a fixed point of the $(\mathbb{C}^*)^2$-action.

Using the result of 4.2.2 we have

$$\begin{aligned} e\Big(\mathrm{Hilb}^\bullet_{\mathsf{Cyc}}(X,\mathfrak{q})\Big) &= e\Big(\mathrm{Hilb}^\bullet_{\mathsf{CM}}(X,\mathfrak{q})^{(\mathbb{C}^*)^2}, \kappa_* 1\Big) \\ &= \sum_{\substack{\boldsymbol{\alpha}\vdash \boldsymbol{a},\ \boldsymbol{\gamma}\vdash \boldsymbol{c},\ \boldsymbol{\delta}\vdash \boldsymbol{d},\\ \boldsymbol{\lambda}\vdash \boldsymbol{l},\ \boldsymbol{\mu}\vdash \boldsymbol{m},\ \boldsymbol{\nu}\vdash \boldsymbol{n}}} e\Big((\mathrm{Hilb}^\bullet(X, C_{\boldsymbol{\alpha},\boldsymbol{\gamma},\boldsymbol{\delta},\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu}})\Big) \\ &= \sum_{\substack{\boldsymbol{\alpha}\vdash \boldsymbol{a},\ \boldsymbol{\gamma}\vdash \boldsymbol{c},\ \boldsymbol{\delta}\vdash \boldsymbol{d},\\ \boldsymbol{\lambda}\vdash \boldsymbol{l},\ \boldsymbol{\mu}\vdash \boldsymbol{m},\ \boldsymbol{\nu}\vdash \boldsymbol{n}}} e\Big(\mathrm{Quot}^\bullet_X(I_{\boldsymbol{\alpha},\boldsymbol{\gamma},\boldsymbol{\delta},\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu}})\Big). \end{aligned}$$

5.1.2. Using the decomposition method of 4.2.3 following method:

1. Decompose X by $X = W \ \amalg\ C_{\boldsymbol{\alpha},\boldsymbol{\gamma},\boldsymbol{\delta},\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu}}$ where $W := X \setminus C_{\boldsymbol{\alpha},\boldsymbol{\gamma},\boldsymbol{\delta},\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu}}$.

2. Let $C^\diamond_{\boldsymbol{\alpha},\boldsymbol{\gamma},\boldsymbol{\delta},\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu}}$ be set points given by the following disjoint sets:

   (a) $\sigma^\diamond_\alpha := \sigma \cap C^{\mathsf{red}}_{\boldsymbol{\alpha}}$

   (b) $\sigma^\diamond_\gamma := \sigma \cap C^{\mathsf{red}}_{\boldsymbol{\gamma}}$

   (c) $C^\diamond_{\boldsymbol{\gamma}}$ the set of nodes of $C_{\boldsymbol{\gamma}}$

   (d) $C^\diamond_{\boldsymbol{\lambda}}$ the set of nodes of $C_{\boldsymbol{\lambda}}$

   (e) $B^\diamond = \bigcup_i (C^{(i)}_2 \cap C^{(i)}_3)$.

3. Denote the components supported on smooth reduced sub-curves by:

   (a) $\sigma^\circ := \sigma \setminus \sigma^\diamond$

   (b) $C^\circ_{\boldsymbol{\alpha}} := C_{\boldsymbol{\alpha}} \setminus \sigma^\diamond_\alpha$

   (c) $C^\circ_{\boldsymbol{\gamma}} := C_{\boldsymbol{\gamma}} \setminus (\sigma^\diamond_\gamma \cup C^\diamond_{\boldsymbol{\lambda}})$

   (d) $C^\circ_{\boldsymbol{\lambda}} := C_{\boldsymbol{\lambda}} \setminus C^\diamond_{\boldsymbol{\lambda}}$

   (e) $C^\circ_{\boldsymbol{\mu}} := C_{\boldsymbol{\mu}} \setminus B^\diamond$

   (f) $C^\circ_{\boldsymbol{\nu}} := C_{\boldsymbol{\nu}} \setminus B^\diamond$

5.1.3. Then applying Euler characteristic to lemma 4.1.4 we have:

$$\begin{aligned}
&e\big(\mathrm{Quot}^{\bullet}_X(I_{\boldsymbol{\alpha},\boldsymbol{\gamma},\boldsymbol{\delta},\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu}})\big)\\
&= e\big(\mathrm{Quot}^{\bullet}_X(I_{\boldsymbol{\alpha},\boldsymbol{\gamma},\boldsymbol{\delta},\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu}}, W)\big) e\big(\mathrm{Quot}^{\bullet}_X(I_{\boldsymbol{\alpha},\boldsymbol{\gamma},\boldsymbol{\delta},\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu}}, \sigma^{\circ})\big)\\
&\quad e\big(\mathrm{Quot}^{\bullet}_X(I_{\boldsymbol{\alpha},\boldsymbol{\gamma},\boldsymbol{\delta},\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu}}, \sigma^{\diamond}_{\boldsymbol{\alpha}})\big) e\big(\mathrm{Quot}^{\bullet}_X(I_{\boldsymbol{\alpha},\boldsymbol{\gamma},\boldsymbol{\delta},\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu}}, C^{\circ}_{\boldsymbol{\alpha}})\big)\\
&\quad e\big(\mathrm{Quot}^{\bullet}_X(I_{\boldsymbol{\alpha},\boldsymbol{\gamma},\boldsymbol{\delta},\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu}}, \sigma^{\diamond}_{\boldsymbol{\gamma}})\big) e\big(\mathrm{Quot}^{\bullet}_X(I_{\boldsymbol{\alpha},\boldsymbol{\gamma},\boldsymbol{\delta},\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu}}, C^{\diamond}_{\boldsymbol{\gamma}})\big) e\big(\mathrm{Quot}^{\bullet}_X(I_{\boldsymbol{\alpha},\boldsymbol{\gamma},\boldsymbol{\delta},\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu}}, C^{\circ}_{\boldsymbol{\gamma}})\big)\\
&\quad e\big(\mathrm{Quot}^{\bullet}_X(I_{\boldsymbol{\alpha},\boldsymbol{\gamma},\boldsymbol{\delta},\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu}}, C_{\boldsymbol{\delta}})\big)\\
&\quad e\big(\mathrm{Quot}^{\bullet}_X(I_{\boldsymbol{\alpha},\boldsymbol{\gamma},\boldsymbol{\delta},\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu}}, C^{\diamond}_{\boldsymbol{\lambda}})\big) e\big(\mathrm{Quot}^{\bullet}_X(I_{\boldsymbol{\alpha},\boldsymbol{\gamma},\boldsymbol{\delta},\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu}}, C^{\circ}_{\boldsymbol{\lambda}})\big)\\
&\quad e\big(\mathrm{Quot}^{\bullet}_X(I_{\boldsymbol{\alpha},\boldsymbol{\gamma},\boldsymbol{\delta},\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu}}, B^{\diamond})\big) e\big(\mathrm{Quot}^{\bullet}_X(I_{\boldsymbol{\alpha},\boldsymbol{\gamma},\boldsymbol{\delta},\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu}}, C^{\circ}_{\boldsymbol{\mu}})\big) e\big(\mathrm{Quot}^{\bullet}_X(I_{\boldsymbol{\alpha},\boldsymbol{\gamma},\boldsymbol{\delta},\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu}}, C^{\circ}_{\boldsymbol{\nu}})\big)
\end{aligned}$$

Applying lemmas 4.5.1 and 4.5.2 we have:

$$\begin{aligned}
&e\big(\mathrm{Quot}^{\bullet}_X(I_{\boldsymbol{\alpha},\boldsymbol{\gamma},\boldsymbol{\delta},\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu}})\big)\\
&\qquad = \Big(\widetilde{\mathsf{V}}_{\emptyset\emptyset\emptyset}\Big)^{e(W)} \Big(\widetilde{\mathsf{V}}_{\square\emptyset\emptyset}\Big)^{e(\sigma^{\circ})}\\
&\qquad\quad \prod_i \Big(p^{-\alpha_i}\widetilde{\mathsf{V}}_{\alpha^{(i)}\square\emptyset}\Big) \prod_i \Big(\widetilde{\mathsf{V}}_{\alpha^{(i)}\emptyset\emptyset}\Big)^{-1}\\
&\qquad\quad \prod_i \Big(p^{-\gamma_i}\widetilde{\mathsf{V}}_{\gamma^{(i)}\square\emptyset}\Big) \prod_i \Big(\widetilde{\mathsf{V}}_{\gamma^{(i)}(\gamma^{(i)})^t\emptyset}\Big) \prod_i \Big(\widetilde{\mathsf{V}}_{\gamma^{(i)}\emptyset\emptyset}\Big)^{-1}\\
&\qquad\quad \prod_i \Big(\widetilde{\mathsf{V}}_{\delta^{(i)}\emptyset\emptyset}\Big)^{0}\\
&\qquad\quad \prod_i \Big(\widetilde{\mathsf{V}}_{\lambda^{(i)}(\lambda^{(i)})^t\emptyset}\Big) \prod_i \Big(\widetilde{\mathsf{V}}_{\lambda^{(i)}\emptyset\emptyset}\Big)^{0}\\
&\qquad\quad \prod_i \Big(p^{\chi\left(\mathcal{O}_{\mu^{(i)}C_2^{(i)}\cup\nu^{(i)}C_3^{(i)}}\right)}\widetilde{\mathsf{V}}_{\mu^{(i)}\nu^{(i)}\emptyset}\widetilde{\mathsf{V}}_{(\mu^{(i)})^t(\nu^{(i)})^t\emptyset}\Big) \prod_i \Big(\widetilde{\mathsf{V}}_{\mu^{(i)}\emptyset\emptyset}\Big)^{0} \prod_i \Big(\widetilde{\mathsf{V}}_{\nu^{(i)}\emptyset\emptyset}\Big)^{0}.
\end{aligned}$$

We note that $e(X) = 24$ and $e(\sigma) = 2$ and:

$$\begin{aligned}
&p^{\chi\left(\mathcal{O}_{\mu^{(i)}C_2^{(i)}\cup\nu^{(i)}C_3^{(i)}}\right)}\widetilde{\mathsf{V}}_{\mu^{(i)}\nu^{(i)}\emptyset}\widetilde{\mathsf{V}}_{(\mu^{(i)})^t(\nu^{(i)})^t\emptyset}\\
&= p^{\frac{1}{2}(\|\mu^{(i)}\|^2+\|(\mu^{(i)})^t\|^2+\|\nu^{(i)}\|^2+\|(\nu^{(i)})^t\|^2)}\mathsf{V}_{\mu^{(i)}\nu^{(i)}\emptyset}\mathsf{V}_{(\mu^{(i)})^t(\nu^{(i)})^t\emptyset}.
\end{aligned}$$

So from lemma 4.5.3 we now have we have

$$\begin{aligned}
&e\big(\mathrm{Quot}^{\bullet}_X(I_{\boldsymbol{\alpha},\boldsymbol{\gamma},\boldsymbol{\delta},\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu}})\big)\\
&\quad = \Big(\mathsf{V}_{\emptyset\emptyset\emptyset}\Big)^{24}\Big(\frac{\mathsf{V}_{\square\emptyset\emptyset}}{\mathsf{V}_{\emptyset\emptyset\emptyset}}\Big)^{2} \prod_i \Big(\frac{\mathsf{V}_{\emptyset\emptyset\emptyset}}{\mathsf{V}_{\square\emptyset\emptyset}}\frac{\mathsf{V}_{\alpha^{(i)}\square\emptyset}}{\mathsf{V}_{\alpha^{(i)}\emptyset\emptyset}}\Big)\\
&\qquad \prod_i \Big(\frac{\mathsf{V}_{\emptyset\emptyset\emptyset}}{\mathsf{V}_{\square\emptyset\emptyset}} p^{\|\gamma^{(i)}\|^2}\frac{\mathsf{V}_{\gamma^{(i)}\square\emptyset}\mathsf{V}_{\gamma^{(i)}(\gamma^{(i)})^t\emptyset}}{\mathsf{V}_{\emptyset\emptyset\emptyset}\mathsf{V}_{\gamma^{(i)}\emptyset\emptyset}}\Big) \prod_i \Big(p^{\|\lambda^{(i)}\|^2}\frac{\mathsf{V}_{\lambda^{(i)}(\lambda^{(i)})^t\emptyset}}{\mathsf{V}_{\emptyset\emptyset\emptyset}}\Big)\\
&\qquad \prod_i \Big(p^{\frac{1}{2}(\|\mu^{(i)}\|^2+\|(\mu^{(i)})^t\|^2+\|\nu^{(i)}\|^2+\|(\nu^{(i)})^t\|^2)}\frac{\mathsf{V}_{\mu^{(i)}\nu^{(i)}\emptyset}\mathsf{V}_{(\mu^{(i)})^t(\nu^{(i)})^t\emptyset}}{\mathsf{V}_{\emptyset\emptyset\emptyset}\mathsf{V}_{\emptyset\emptyset\emptyset}}\Big)
\end{aligned}$$

We now define the functions:

1. $g_{\mathtt{Sm}^\sigma} : \mathrm{Sym}^\bullet(\mathtt{Sm}_2^\sigma) \longrightarrow Z((p))$ is defined by $g_{\mathtt{Sm}^\sigma}(a) = \frac{\mathsf{V}_{\emptyset\emptyset\emptyset}}{\mathsf{V}_{\Box\emptyset\emptyset}} \sum_{\alpha\vdash a} \frac{\mathsf{V}_{\alpha\Box\emptyset}}{\mathsf{V}_{\alpha^{(i)}\emptyset\emptyset}}$,

2. $g_{\mathtt{N}^\sigma} : \mathrm{Sym}^\bullet(\mathtt{N}_2^\sigma) \longrightarrow Z((p))$ is defined by $g_{\mathtt{N}^\sigma}(c) = \frac{\mathsf{V}_{\emptyset\emptyset\emptyset}}{\mathsf{V}_{\Box\emptyset\emptyset}} \sum_{\gamma\vdash c} p^{\|\gamma^{(i)}\|^2} \frac{\mathsf{V}_{\gamma\Box\emptyset}\mathsf{V}_{\gamma\gamma^t\emptyset}}{\mathsf{V}_{\emptyset\emptyset\emptyset}\mathsf{V}_{\gamma\emptyset\emptyset}}$,

3. $g_{\mathtt{Sm}^\emptyset} : \mathrm{Sym}^\bullet(\mathtt{Sm}_2^\emptyset) \longrightarrow Z((p))$ is defined by $g_{\mathtt{Sm}^\emptyset}(d) = \sum_{\delta\vdash d} 1$,

4. $g_{\mathtt{N}^\emptyset} : \mathrm{Sym}^\bullet(\mathtt{N}_2^\emptyset) \longrightarrow Z((p))$ is defined by $g_{\mathtt{N}^\emptyset}(l) = \sum_{\lambda\vdash l} p^{\|\lambda^{(i)}\|^2} \frac{\mathsf{V}_{\lambda\lambda^t\emptyset}}{\mathsf{V}_{\emptyset\emptyset\emptyset}}$,

5. $g_B : \mathrm{Sym}^\bullet(B_2) \times \mathrm{Sym}^\bullet(B_{\mathsf{op}}) \longrightarrow Z((p))$ is defined by the equation
$$g_B(m,n) = \sum_{\substack{\mu^{(i)} \vdash m_i \\ \nu^{(i)} \vdash n_i}} p^{\frac{1}{2}(\|\mu^{(i)}\|^2 + \|(\mu^{(i)})^t\|^2 + \|\nu^{(i)}\|^2 + \|(\nu^{(i)})^t\|^2)} \frac{\mathsf{V}_{\mu^{(i)}\nu^{(i)}\emptyset}\mathsf{V}_{(\mu^{(i)})^t(\nu^{(i)})^t\emptyset}}{\mathsf{V}_{\emptyset\emptyset\emptyset}\mathsf{V}_{\emptyset\emptyset\emptyset}}.$$

So the constructible function $(\rho_\bullet)_*1 : \mathrm{Chow}^{\sigma+(0,\bullet,\bullet)}(X) \to Z((p))$ is calculated for $\mathsf{q} = (\boldsymbol{ax}, \boldsymbol{cy}, \boldsymbol{dz}, \boldsymbol{lw}, \boldsymbol{mb_2}, \boldsymbol{nb_{\mathsf{op}}})$ by:

$$\begin{aligned}
&\big((\rho_\bullet)_*1\big)(\mathsf{q}) \\
&\quad = e\Big(\rho_\bullet^{-1}(\mathsf{q})\Big) \\
&\quad = \sum_{\substack{\boldsymbol{\alpha}\vdash \boldsymbol{a},\ \boldsymbol{\gamma}\vdash \boldsymbol{c},\ \boldsymbol{\delta}\vdash \boldsymbol{d}, \\ \boldsymbol{\lambda}\vdash \boldsymbol{l},\ \boldsymbol{\mu}\vdash \boldsymbol{m},\ \boldsymbol{\nu}\vdash \boldsymbol{n}}} e\Big(\mathrm{Quot}_X^\bullet(I_{\boldsymbol{\alpha},\boldsymbol{\gamma},\boldsymbol{\delta},\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu}})\Big) \\
&\quad = \Big(\mathsf{V}_{\emptyset\emptyset\emptyset}\Big)^{24} \Big(\frac{\mathsf{V}_{\Box\emptyset\emptyset}}{\mathsf{V}_{\emptyset\emptyset\emptyset}}\Big)^2 \prod_i g_{\mathtt{Sm}^\sigma}(a_i) \prod_i g_{\mathtt{N}^\sigma}(c_i) \prod_i g_{\mathtt{Sm}^\emptyset}(d_i) \prod_i g_{\mathtt{N}^\emptyset}(l_i) \prod_i g_B(m_i,n_i).
\end{aligned}$$

So we can now apply lemma 2.1.2 to obtain:

$$\begin{aligned}
&e\Big(\mathrm{Chow}^{\sigma+(0,\bullet,\bullet)}(X), (\rho_\bullet)_*1\Big) \\
&\quad = \Big(\mathsf{V}_{\emptyset\emptyset\emptyset}\Big)^{24} \Big(\frac{\mathsf{V}_{\Box\emptyset\emptyset}}{\mathsf{V}_{\emptyset\emptyset\emptyset}}\Big)^2 \Big(\frac{\mathsf{V}_{\emptyset\emptyset\emptyset}}{\mathsf{V}_{\Box\emptyset\emptyset}} \sum_\alpha (Q_2Q_3)^{|\alpha|} \frac{\mathsf{V}_{\alpha\Box\emptyset}}{\mathsf{V}_{\alpha\emptyset\emptyset}}\Big)^{e(\mathtt{Sm}^\sigma)} \\
&\quad \Big(\frac{\mathsf{V}_{\emptyset\emptyset\emptyset}}{\mathsf{V}_{\Box\emptyset\emptyset}} \sum_\gamma (Q_2Q_3)^{|\gamma|} p^{\|\gamma\|^2} \frac{\mathsf{V}_{\gamma\Box\emptyset}\mathsf{V}_{\gamma\gamma^t\emptyset}}{\mathsf{V}_{\emptyset\emptyset\emptyset}\mathsf{V}_{\gamma\emptyset\emptyset}}\Big)^{e(\mathtt{N}^\sigma)} \\
&\quad \Big(\sum_\delta (Q_2Q_3)^{|\delta|}\Big)^{e(\mathtt{Sm}^\emptyset)} \Big(\sum_\lambda (Q_2Q_3)^{|\lambda|} p^{\|\lambda\|^2} \frac{\mathsf{V}_{\lambda\lambda^t\emptyset}}{\mathsf{V}_{\emptyset\emptyset\emptyset}}\Big)^{e(\mathtt{N}^\emptyset)} \\
&\quad e\Big(\mathrm{Sym}^\bullet(B_2) \times \mathrm{Sym}^\bullet(B_{\mathsf{op}}), G_B\Big)
\end{aligned}$$

where G_B is the constructible function

$$G_B : \mathrm{Sym}^\bullet(B_2) \times \mathrm{Sym}^\bullet(B_{\mathsf{op}}) \to \mathbb{Z}((p))$$

defined by $G_B(\boldsymbol{mb_2}, \boldsymbol{b_{\mathsf{op}}}) := \prod_{i=1}^{12} g_B(m_i, n_i)$. However, since $B_2 = \{b_2^1, \ldots, b_2^{12}\}$ and $B_{\mathsf{op}} = \{b_{\mathsf{op}}^1, \ldots, b_{\mathsf{op}}^{12}\}$ we have:

$$\mathrm{Sym}^\bullet(B_2) \times \mathrm{Sym}^\bullet(B_{\mathsf{op}}) \cong \prod_{i=1}^{12} \mathrm{Sym}^\bullet\Big(\{b_2^{(i)}\}\Big) \times \mathrm{Sym}^\bullet\Big(\{b_{\mathsf{op}}^{(i)}\}\Big).$$

Which gives us:

$$
\begin{aligned}
&e\Big(\mathrm{Sym}^\bullet(B_2)\times\mathrm{Sym}^\bullet(B_{\mathsf{op}}),G_B\Big)\\
&\quad=\prod_{i=1}^{12} e\Big(\mathrm{Sym}^\bullet\Big(\{b_2^{(i)}\}\Big)\times\mathrm{Sym}^\bullet\Big(\{b_{\mathsf{op}}^{(i)}\}\Big),g_B\Big)\\
&\quad=\left(\sum_{\mu,\nu} Q_2^\mu Q_3^\nu p^{\frac{1}{2}(\|\mu\|^2+\|\mu^t\|^2+\|\nu\|^2+\|\nu^t\|^2)}\frac{\mathsf{V}_{\mu\nu\emptyset}\mathsf{V}_{\mu^t\nu^t\emptyset}}{\mathsf{V}_{\emptyset\emptyset\emptyset}\mathsf{V}_{\emptyset\emptyset\emptyset}}\right)^{12}
\end{aligned}
$$

5.1.4. Applying the vertex formulas of lemmas 6.3.6, 6.3.2 and corollary 6.3.4 we have

$$
\begin{aligned}
&e\Big(\mathrm{Chow}^{\sigma+(0,\bullet,\bullet)}(X),(\rho_\bullet)_*1\Big)\\
&\quad=M(p)^{24}\left(\frac{1}{1-p}\right)^2\left(\prod_{d>0}\frac{(1-Q_2^dQ_3^d)}{(1-pQ_2^dQ_3^d)(1-p^{-1}Q_2^dQ_3^d)}\right)^{-10}\\
&\qquad\left(\prod_{d>0}\frac{M(p,Q_2^dQ_3^d)}{(1-pQ_2^dQ_3^d)(1-p^{-1}Q_2^dQ_3^d)}\right)^{12}\\
&\qquad\left(\prod_{d>0}\frac{1}{(1-Q_2^dQ_3^d)}\right)^{10}\left(\prod_{d>0}\frac{M(p,Q_2^dQ_3^d)}{(1-Q_2^dQ_3^d)}\right)^{-12}\\
&\qquad\left(\prod_{d>0}\frac{M(Q_2^dQ_3^d,p)^2}{(1-Q_2^dQ_3^d)M(-Q_2^{d-1}Q_3^d,p)M(-Q_2^dQ_3^{d-1},p)}\right)^{12}\\
&\quad=\frac{M(p)^{24}}{(1-p)^2}\prod_{d>0}\frac{1}{(1-Q_2^dQ_3^d)^8(1-pQ_2^dQ_3^d)^2(1-p^{-1}Q_2^dQ_3^d)^2}\\
&\qquad\left(\prod_{d>0}\frac{M(Q_2^dQ_3^d,p)^2}{(1-Q_2^dQ_3^d)M(-Q_2^{d-1}Q_3^d,p)M(-Q_2^dQ_3^{d-1},p)}\right)^{12}
\end{aligned}
$$

Which completes the proof of theorem A.

5.2 Preliminaries for classes of the form $\bullet\sigma+(i,j,\bullet)$

We recall from lemma 3.5.3 that there is a decomposition of $\mathrm{Chow}^{\bullet\sigma+(i,j,\bullet)}(X)$ such that for any point $\mathfrak{q}\in\mathrm{Chow}^{\bullet\sigma+(i,j,\bullet)}(X)$ the fibre is

$$(\eta_\bullet)^{-1}(\mathfrak{q})\cong\mathrm{Hilb}^\bullet_{\mathsf{Cyc}}(X,\mathsf{Cyc}(C))$$

for some one dimensional subscheme C of X with

$$\mathsf{Cyc}(C)=\mathfrak{q}=a\sigma+D+\sum_{i=1}^{12}m_iC_3^{(i)}\tag{II.10}$$

where D is a one dimensional *reduced* subscheme of X. We see from lemma 3.5.3 that the intersection of D with σ has length 0, 1 or 2. We consider the following formal neighbourhoods around components of C:

1. Let U_i be the formal neighbourhood of $C_3^{(i)}$ in X. These have a canonical $(\mathbb{C}^*)^2$-action described in 4.4.8 and the $(\mathbb{C}^*)^2$-invariance of $D\cap U_i$ is shown in lemma 4.4.10.

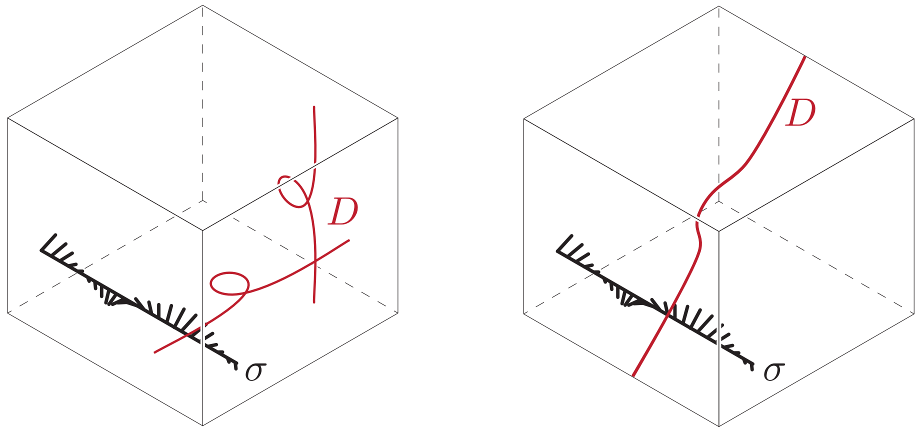


Figure II.11: Depiction of two typical curves (away from C_3) in the class $b\sigma + (1,1,d)$.

2. Let V be the formal neighbourhood of σ in X with the coordinates:
   (a) If $\#(D\cap\sigma) = 0,2$ the let V have the canonical coordinates of 4.4.1 of and $(\mathbb{C}^*)^2$-action described in 4.4.3.
   (b) If $\#(D\cap\sigma) = 1$ the let V have the canonical coordinates of 4.4.2 of and $(\mathbb{C}^*)^2$-action described in 4.4.3.

By construction the restrictions of D to these neighbourhoods are invariant under these actions. Hence the conditions of lemma 4.2.1 are satisfied and there is a $(\mathbb{C}^*)^2$-action defined on $\mathrm{Hilb}^n_{\mathsf{CM}}(X, \mathsf{Cyc}(C))$. We introduce the notation for subschemes of X:

$$C_{\alpha,\boldsymbol{\mu}} = C_{\alpha,\mu^{(1)},\dots,\mu^{(12)}} \;=\; \alpha\sigma \;\cup\; D \bigcup_{i=1}^{12} \mu_i C_3^{(i)}$$

and their ideals $I_{\alpha,\boldsymbol{\mu}}$. Then using this notation we can identify the fixed points of the action as the following discrete set:

$$\mathrm{Hilb}^{\bullet}_{\mathsf{CM}}\big(X, \mathfrak{q}\big)^{(\mathbb{C}^*)^2} = \coprod_{\alpha\vdash a,\ \boldsymbol{\mu}\vdash \boldsymbol{m}} \Big\{\ C_{\alpha,\boldsymbol{\mu}}\ \Big\}.$$

Using the result of 4.2.2 we have

$$\begin{aligned} e\Big(\mathrm{Hilb}^{\bullet}_{\mathsf{Cyc}}(X,\mathfrak{q})\Big) &= e\Big(\mathrm{Hilb}^{\bullet}_{\mathsf{CM}}(X,\mathfrak{q})^{(\mathbb{C}^*)^2}, \kappa_* 1\Big) \\ &= \sum_{\alpha\vdash a,\ \boldsymbol{\mu}\vdash\boldsymbol{m}} p^{\chi(\mathcal{O}_{C_{\alpha,\boldsymbol{\mu}}})} e\Big((\mathrm{Hilb}^{\bullet}(X, C_{\alpha,\boldsymbol{\mu}}\Big) \\ &= \sum_{\alpha\vdash a,\ \boldsymbol{\mu}\vdash\boldsymbol{m}} p^{\chi(\mathcal{O}_{C_{\alpha,\boldsymbol{\mu}}})} e\Big(\mathrm{Quot}^{\bullet}_X(I_{\alpha,\boldsymbol{\mu}}\Big). \end{aligned}$$

Where the holomorphic Euler characteristic $\chi(\mathcal{O}_{C_{\alpha,\boldsymbol{\mu}}})$ is given by the following lemma.

Lemma 5.2.1. *The holomorphic Euler characteristic of $C_{\alpha,\boldsymbol{\mu}}$ is:*

$$\begin{aligned} \chi(\mathcal{O}_{C_{\alpha,\boldsymbol{\mu}}}) =\ & \chi(\mathcal{O}_D) + \Big(\chi(\mathcal{O}_{\alpha\sigma}) - |D\cap\alpha\sigma|\Big) \\ & + \Big(\sum_{i=1}^{12}\chi(\mathcal{O}_{\mu^{(i)}C_3^{(i)}}) - \sum_{i=1}^{12}|D\cap\mu^{(i)}C_3^{(i)}|\Big). \end{aligned}$$

Proof. This is immediate from the exact sequence decomposing $C_{\alpha,\boldsymbol{\mu}}$ into irreducible components:

$$0 \to \mathcal{O}_{C_{\alpha,\boldsymbol{\mu}}} \to \mathcal{O}_D \oplus \chi(\mathcal{O}_{\alpha\sigma}) \bigoplus_i \mathcal{O}_{\mu^{(i)}C_3^{(i)}} \to \chi(\mathcal{O}_{D\cap\alpha\sigma}) \bigoplus_i \mathcal{O}_{D\cap\mu^{(i)}C_3^{(i)}} \to 0$$

□

5.2.2. Using the decomposition method of 4.2.3 we take the following steps:

1. Decompose X by $X = W \amalg C_{\alpha,\boldsymbol{\mu}}$ where $W := X \setminus C_{\alpha,\boldsymbol{\mu}}$.
2. Let $C^{\diamond}_{\alpha,\boldsymbol{\mu}}$ be set points given by the following disjoint sets:
   (a) $D^{\diamond}$ is the set of nodes of $D \setminus (\sigma \cup_i C_3^{(i)})$.
   (b) D^* is the set singularities of $D \setminus (\sigma \cup_i C_3^{(i)})$ that are locally isomorphic it the coordinate axes in $\mathbb{C}^3$.
   (c) $\sigma^{\diamond} := \sigma \cap D$,
   (d) $B_i^{\diamond} = (C_3^{(i)} \cap D)$ for $i \in \{1, \dots, 12\}$,

   Note that $D^{\diamond} \cup D^*$ is the set of singularities of $D \setminus (\sigma \cup_i C_3^{(i)})$.
3. Denote the components supported on smooth reduced sub-curves by:
   (a) $D^{\circ} = D \setminus (D^{\diamond} \cup D^*)$,
   (b) $\sigma^{\circ} := \sigma \setminus \sigma^{\diamond}$,
   (c) $B_i^{\circ} = \mathbb{C}_3^{(i)} \setminus B_i^{\diamond}$ for $i \in \{1, \dots, 12\}$.

5.2.3. Then applying Euler characteristic to lemma 4.1.4 we have:

$$\begin{aligned} &p^{\chi(\mathcal{O}_{C_{\alpha,\boldsymbol{\mu}}})} e\Big(\mathrm{Quot}^{\bullet}_X(I_{\alpha,\boldsymbol{\mu}})\Big) \\ &\quad= e\Big(\mathrm{Quot}^{\bullet}_X(I_{\alpha,\boldsymbol{\mu}}, W)\Big) \\ &\qquad p^{\chi(\mathcal{O}_D)} e\Big(\mathrm{Quot}^{\bullet}_X(I_{\alpha,\boldsymbol{\mu}}, D^{\circ})\Big) e\Big(\mathrm{Quot}^{\bullet}_X(I_{\alpha,\boldsymbol{\mu}}, D^{\diamond})\Big) e\Big(\mathrm{Quot}^{\bullet}_X(I_{\alpha,\boldsymbol{\mu}}, D^*)\Big) \\ &\qquad p^{\chi(\mathcal{O}_{\alpha\sigma}) - |D\cap\alpha\sigma|} e\Big(\mathrm{Quot}^{\bullet}_X(I_{\alpha,\boldsymbol{\mu}}, \sigma^{\circ})\Big) e\Big(\mathrm{Quot}^{\bullet}_X(I_{\alpha,\boldsymbol{\mu}}, \sigma^{\diamond})\Big) \\ &\qquad \prod_{i=1}^{12} p^{\chi(\mathcal{O}_{\mu^{(i)}C_3^{(i)}}) - |D\cap\mu^{(i)}C_3^{(i)}|} e\Big(\mathrm{Quot}^{\bullet}_X(I_{\alpha,\boldsymbol{\mu}}, B_i^{\circ})\Big) e\Big(\mathrm{Quot}^{\bullet}_X(I_{\alpha,\boldsymbol{\mu}}, B_i^{\diamond})\Big) \end{aligned}$$

5.2.4. We have that $e(X) = 24$ and $e(\sigma) = e(C_3^{(i)}) = 2$. So the Euler characteristic of W is:

$$\begin{aligned} e(W) &= e(X) - e(\sigma) - \sum_{i=1}^{12} e(C_3^{(i)}) - e(D^{\circ}) - e(D^{\diamond}) - e(D^*) \\ &= -2 - e(D^{\circ}) - e(D^{\diamond}) - e(D^*) \end{aligned}$$

Hence now have from lemma 4.5.2 that the first two lines from above will be:

$$\begin{aligned} \Psi(D) &:= p^{\chi(\mathcal{O}_D)} \left(\widetilde{\mathsf{V}}_{\emptyset\emptyset\emptyset}\right)^{e(W)} \left(\widetilde{\mathsf{V}}_{\square\emptyset\emptyset}\right)^{e(D^{\circ})} \left(\widetilde{\mathsf{V}}_{\square\square\emptyset}\right)^{e(D^{\diamond})} \left(\widetilde{\mathsf{V}}_{\square\square\square}\right)^{e(D^*)} \\ &= p^{\chi(\mathcal{O}_D)} \left(\mathsf{V}_{\emptyset\emptyset\emptyset}\right)^{-2} \left(\frac{\mathsf{V}_{\square\emptyset\emptyset}}{\mathsf{V}_{\emptyset\emptyset\emptyset}}\right)^{e(D^{\circ})} \left(p\frac{\mathsf{V}_{\square\square\emptyset}}{\mathsf{V}_{\emptyset\emptyset\emptyset}}\right)^{e(D^{\diamond})} \left(p^2\frac{\mathsf{V}_{\square\square\square}}{\mathsf{V}_{\emptyset\emptyset\emptyset}}\right)^{e(D^*)} \end{aligned}$$

The intersection of D and $\alpha\sigma$ will determine the third line. From lemma 4.5.2 and lemma 4.5.3 it will be one of:

1. $p^{\frac{1}{2}(\|\alpha\|^2+\|\alpha^t\|^2)}\left(\widetilde{\mathsf{V}}_{\emptyset\emptyset\alpha}\widetilde{\mathsf{V}}_{\emptyset\emptyset\alpha}\right) = p^{\frac{1}{2}(\|\alpha\|^2+\|\alpha^t\|^2)}\left(\mathsf{V}_{\emptyset\emptyset\alpha}\mathsf{V}_{\emptyset\emptyset\alpha^t}\right)$

2. $p^{\frac{1}{2}(\|\alpha\|^2+\|\alpha^t\|^2)-l(\alpha^t)}\left(\widetilde{\mathsf{V}}_{\square\emptyset\alpha}\widetilde{\mathsf{V}}_{\emptyset\emptyset\alpha}\right) = p^{\frac{1}{2}(\|\alpha\|^2+\|\alpha^t\|^2)}\left(\mathsf{V}_{\square\emptyset\alpha}\mathsf{V}_{\emptyset\emptyset\alpha^t}\right)$

3. $p^{\frac{1}{2}(\|\alpha\|^2+\|\alpha^t\|^2)-l(\alpha^t)-l(\alpha)}\left(\widetilde{\mathsf{V}}_{\square\emptyset\alpha}\widetilde{\mathsf{V}}_{\emptyset\square\alpha}\right) = p^{\frac{1}{2}(\|\alpha\|^2+\|\alpha^t\|^2)}\left(\mathsf{V}_{\square\emptyset\alpha}\mathsf{V}_{\square\emptyset\alpha^t}\right)$

4. $p^{\frac{1}{2}(\|\alpha\|^2+\|\alpha^t\|^2)-(l(\alpha)+l(\alpha^t)-1)}\left(\widetilde{\mathsf{V}}_{\square\square\alpha}\widetilde{\mathsf{V}}_{\emptyset\emptyset\alpha}\right) = p^{\frac{1}{2}(\|\alpha\|^2+\|\alpha^t\|^2)+1}\left(\mathsf{V}_{\square\square\alpha}\mathsf{V}_{\emptyset\emptyset\alpha^t}\right)$

Similarly the factors of the fourth line will determined by the intersections $D\cap C_3^{(i)}$ to be (the fourth comes from 4.4.9):

1. $p^{\frac{1}{2}(\|\alpha\|^2+\|\alpha^t\|^2)}\left(\widetilde{\mathsf{V}}_{\emptyset\emptyset\alpha}\widetilde{\mathsf{V}}_{\emptyset\emptyset\alpha^t}\right) = p^{\frac{1}{2}(\|\alpha\|^2+\|\alpha^t\|^2)}\left(\mathsf{V}_{\emptyset\emptyset\alpha}\mathsf{V}_{\emptyset\emptyset\alpha^t}\right)$

2. $p^{\frac{1}{2}(\|\alpha\|^2+\|\alpha^t\|^2)-(l(\alpha^t)+l(\alpha))}\left(\widetilde{\mathsf{V}}_{\square\emptyset\alpha}\widetilde{\mathsf{V}}_{\square\emptyset\alpha^t}\right) = p^{\frac{1}{2}(\|\alpha\|^2+\|\alpha^t\|^2)}\left(\mathsf{V}_{\square\emptyset\alpha}\mathsf{V}_{\square\emptyset\alpha^t}\right)$

3. $p^{\frac{1}{2}(\|\alpha\|^2+\|\alpha^t\|^2)-(l(\alpha)+l(\alpha^t))}\left(\widetilde{\mathsf{V}}_{\emptyset\square\alpha}\widetilde{\mathsf{V}}_{\emptyset\square\alpha^t}\right) = p^{\frac{1}{2}(\|\alpha\|^2+\|\alpha^t\|^2)}\left(\mathsf{V}_{\square\emptyset\alpha}\mathsf{V}_{\square\emptyset\alpha^t}\right)$

4. $p^{\frac{1}{2}(\|\alpha\|^2+\|\alpha^t\|^2)-2l(\alpha^t)}\left(\widetilde{\mathsf{V}}_{\emptyset\square\alpha}\right)^2 = p^{\frac{1}{2}(\|\alpha\|^2+\|\alpha^t\|^2)}\left(\mathsf{V}_{\emptyset\square\alpha}\right)^2$

5. $p^{\frac{1}{2}(\|\alpha\|^2+\|\alpha^t\|^2)-2(l(\alpha)+l(\alpha^t)-1)}\left(\widetilde{\mathsf{V}}_{\square\square\alpha}\widetilde{\mathsf{V}}_{\square\square\alpha^t}\right) = p^{\frac{1}{2}(\|\alpha\|^2+\|\alpha^t\|^2)+2}\left(\mathsf{V}_{\square\square\alpha}\mathsf{V}_{\square\square\alpha^t}\right)$

5.2.5. We can calculate $e\left(\mathrm{Hilb}^{\bullet}_{\mathsf{Cyc}}(X,\mathfrak{q})\right)$ using the above results and notation from 5.2.4:

$$\begin{aligned} e\left(\mathrm{Hilb}^{\bullet}_{\mathsf{Cyc}}(X,\mathfrak{q})\right) &= \sum_{\alpha\vdash a,\ \boldsymbol{\mu}\vdash\boldsymbol{m}} p^{\chi(\mathcal{O}_{C_{\alpha,\boldsymbol{\mu}}})} e\left(\mathrm{Quot}^{\bullet}_X(I_{\alpha,\boldsymbol{\mu}})\right) \\ &= \Psi(D)\Phi_\sigma(a)\prod_{i=1}^{12}\Phi_i(m_i). \end{aligned}$$

where Φ_σ and Φ_i are determined by the intersections of σ and $C_3^{(i)}$ respectively to be one of the following functions:

1. $\Phi^{\emptyset,\emptyset}(a) := \sum_{\alpha\vdash a} p^{\frac{1}{2}(\|\alpha\|^2+\|\alpha^t\|^2)}(\mathsf{V}_{\emptyset\emptyset\alpha}\mathsf{V}_{\emptyset\emptyset\alpha^t})$

2. $\Phi^{-,\emptyset}(a) := \sum_{\alpha\vdash a} p^{\frac{1}{2}(\|\alpha\|^2+\|\alpha^t\|^2)}(\mathsf{V}_{\square\emptyset\alpha}\mathsf{V}_{\emptyset\emptyset\alpha^t})$

3. $\Phi^{-,-}(a) := \sum_{\alpha\vdash a} p^{\frac{1}{2}(\|\alpha\|^2+\|\alpha^t\|^2)}(\mathsf{V}_{\square\emptyset\alpha}\mathsf{V}_{\square\emptyset\alpha^t})$

4. $\Phi^{-,|}(a) := \sum_{\alpha\vdash a} p^{\frac{1}{2}(\|\alpha\|^2+\|\alpha^t\|^2)}(\mathsf{V}_{\emptyset\square\alpha})^2$

5. $\Phi^{+,\emptyset}(a) := \sum_{\alpha\vdash a} p^{\frac{1}{2}(\|\alpha\|^2+\|\alpha^t\|^2)+1}(\mathsf{V}_{\square\square\alpha}\mathsf{V}_{\emptyset\emptyset\alpha^t})$

6. $\Phi^{+,+}(a) := \sum_{\alpha\vdash a} p^{\frac{1}{2}(\|\alpha\|^2+\|\alpha^t\|^2)+2}(\mathsf{V}_{\square\square\alpha}\mathsf{V}_{\square\square\alpha^t})$

5.3 Calculation for the class $\bullet\sigma + (0, 0, \bullet)$

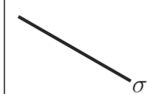
From lemma 3.5.3 have the decomposition of $\mathrm{Chow}^{\bullet\sigma+(0,0,\bullet)}(X)$ into:

$$\mathbb{Z}_{\geq 0} \times \mathrm{Sym}^{\bullet}(B_{\mathsf{op}})$$

Recall equation (II.10) from section 5.2 and the notation:

$$\mathsf{Cyc}(C) = a\sigma + D + \sum_{i=1}^{12} m_i C_3^{(i)}.$$

In this class we have $D = \emptyset$. Hence we have the following summary of results from 5.2.4 and 5.2.5.

$\chi(\mathcal{O}_D) = 0$	
$e(\eta_{\bullet}^{-1}(a, \boldsymbol{m})) = \frac{1}{(\mathsf{V}_{\emptyset\emptyset\emptyset})^2} \cdot Q_{\sigma}^{a} \Phi^{\emptyset,\emptyset}(a) \cdot \prod_{i=1}^{12} Q_3^{m_i} \Phi^{\emptyset,\emptyset}(m_i)$	

Now we have:

$$\begin{aligned} e\Big(\mathbb{Z}_{\geq 0} \times \mathrm{Sym}^{\bullet}(B_{\mathsf{op}}),\ (\eta_{\bullet})_* 1 \Big) &= \frac{1}{(\mathsf{V}_{\emptyset\emptyset\emptyset})^2} \left(\sum_a Q_{\sigma}^{a} \Phi^{\emptyset,\emptyset}(a) \right) \left(\sum_m Q_3^{m} \Phi^{\emptyset,\emptyset}(m) \right)^{12} \\ &= M(p)^{24} \prod_{m>0} (1 + p^m Q_{\sigma})^m (1 + p^m Q_3)^{12m}. \end{aligned}$$

Where the last equality is from 6.3.4 part 2 and 6.3.2 part 1.

5.4 Calculation for the class $\bullet\sigma + (0, 1, \bullet)$

Recall the previously introduced notation:

1. $B_{\mathsf{i}} = \{b_{\mathsf{i}}^1, \ldots, b_{\mathsf{i}}^{12}\}$ is the set of the 12 points in S_{i} that correspond to nodes in the fibres of the projection $\pi : S_{\mathsf{i}} \to \mathbb{P}^1$.
2. $S_{\mathsf{i}}^{\circ} = S_{\mathsf{i}} \setminus B_{\mathsf{i}}$ is the complement of B_{i} in S_{i}
3. $\mathtt{N}_i \subset S_{\mathsf{i}}$ are the 12 nodal fibres of $\pi : S_{\mathsf{i}} \to \mathbb{P}^1$ with the nodes removed and:
$$\mathtt{N}_i = \mathtt{N}_i^{\sigma} \ \amalg \ \mathtt{N}_i^{\emptyset} \text{ where } \mathtt{N}_i^{\sigma} := \mathtt{N}_i \cap \sigma \text{ and } \mathtt{N}_i^{\emptyset} := \mathtt{N}_i \setminus \sigma.$$
4. $\mathtt{Sm}_i = S_{\mathsf{i}}^{\circ} \setminus \mathtt{N}_i$ is the complement of $\mathtt{N}_i$ in S_{i}° and:
$$\mathtt{Sm}_i = \mathtt{Sm}_i^{\sigma} \ \amalg \ \mathtt{Sm}_i^{\emptyset} \text{ where } \mathtt{Sm}_i^{\sigma} := \mathtt{Sm}_i \cap \sigma \text{ and } \mathtt{Sm}_i^{\emptyset} := \mathtt{Sm}_i \setminus \sigma.$$

Now from lemma 3.5.3 we can further decompose $\mathrm{Chow}^{\bullet\sigma+(0,1,\bullet)}(X)$ into the four parts:

1. $\mathbb{Z}_{\geq 0} \times \mathtt{Sm}_2^{\sigma} \times \mathrm{Sym}^{\bullet}(B_{\mathsf{op}})$
2. $\mathbb{Z}_{\geq 0} \times \mathtt{Sm}_2^{\emptyset} \times \mathrm{Sym}^{\bullet}(B_{\mathsf{op}})$
3. $\mathbb{Z}_{\geq 0} \times \mathtt{N}_2^{\sigma} \times \mathrm{Sym}^{\bullet}(B_{\mathsf{op}})$
4. $\mathbb{Z}_{\geq 0} \times \mathtt{N}_2^{\emptyset} \times \mathrm{Sym}^{\bullet}(B_{\mathsf{op}})$
5. $\coprod_{k=1}^{12} \mathbb{Z}_{\geq 0} \times \mathrm{Sym}^{\bullet}(\{b_{\mathsf{op}}^k\}) \times \mathrm{Sym}^{\bullet}(B_{\mathsf{op}} \setminus \{b_{\mathsf{op}}^k\})$

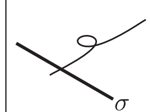
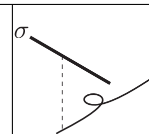
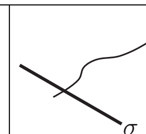
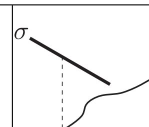
Recall equation (II.10) from section 5.2 and the notation:

$$\mathsf{Cyc}(C) = a\sigma + D + \sum_{i=1}^{12} m_i C_3^{(i)}.$$

Each part will be characterised by the type of D. We consider parts (1)-(4) separately to part (5).

5.4.1. Parts (1)-(4): In parts (1)-(4) the curve D is a fibre of the projection $\mathrm{pr}_2 : X \to S$. The following table is the summary of results from 5.2.4 and 5.2.5 when applied to the particular D's arising in each strata:

$$\mathbb{Z}_{\geq 0} \times U \times \mathrm{Sym}^{\bullet}(B_{\mathsf{op}}).$$

$U = \mathtt{N}_1^{\sigma}$	$e(U) = 12$	$\chi(\mathcal{O}_D) = 0$
$e(\eta_{\bullet}^{-1}(a, x, \boldsymbol{m})) = Q_1 Q_3 p \frac{(\mathsf{V}_{\square\square\emptyset})}{(\mathsf{V}_{\square\emptyset\emptyset})(\mathsf{V}_{\emptyset\emptyset\emptyset})^2} \cdot Q_{\sigma}^{a} \Phi^{-,\emptyset}(a) \cdot \prod_{i=1}^{12} Q_3^{m_i} \Phi^{\emptyset,\emptyset}(m_i)$		
$U = \mathtt{N}_1^{\emptyset}$	$e(U) = -12$	$\chi(\mathcal{O}_D) = 0$
$e(\eta_{\bullet}^{-1}(a, x, \boldsymbol{m})) = Q_1 Q_3 p \frac{(\mathsf{V}_{\square\square\emptyset})}{(\mathsf{V}_{\emptyset\emptyset\emptyset})^3} \cdot Q_{\sigma}^{a} \Phi^{\emptyset,\emptyset}(a) \cdot \prod_{i=1}^{12} Q_3^{m_i} \Phi^{\emptyset,\emptyset}(m_i)$		
$U = \mathtt{Sm}_1^{\sigma}$	$e(U) = -10$	$\chi(\mathcal{O}_D) = 0$
$e(\eta_{\bullet}^{-1}(a, x, \boldsymbol{m})) = Q_1 Q_3 \frac{1}{(\mathsf{V}_{\square\emptyset\emptyset})(\mathsf{V}_{\emptyset\emptyset\emptyset})} \cdot Q_{\sigma}^{a} \Phi^{-,\emptyset}(a) \cdot \prod_{i=1}^{12} Q_3^{m_i} \Phi^{\emptyset,\emptyset}(m_i)$		
$U = \mathtt{Sm}_1^{\emptyset}$	$e(U) = 10$	$\chi(\mathcal{O}_D) = 0$
$e(\eta_{\bullet}^{-1}(a, x, \boldsymbol{m})) = Q_1 Q_3 \frac{1}{(\mathsf{V}_{\emptyset\emptyset\emptyset})^2} \cdot Q_{\sigma}^{a} \Phi^{\emptyset,\emptyset}(a) \cdot \prod_{i=1}^{12} Q_3^{m_i} \Phi^{\emptyset,\emptyset}(m_i)$		

The union of parts (1)-(4) is $\mathbb{Z}_{\geq 0} \times S_2^{\circ} \times \mathrm{Sym}^{\bullet}(B_{\mathsf{op}})$ so we have:

$$\begin{aligned} e\Big(\mathbb{Z}_{\geq 0} \times S_2^{\circ} \times \mathrm{Sym}^{\bullet}(B_{\mathsf{op}}),\ (\eta_{\bullet})_* 1 \Big) \\ = e\Big(\mathbb{Z}_{\geq 0} \times \mathtt{Sm}_2^{\sigma} \times \mathrm{Sym}^{\bullet}(B_{\mathsf{op}}),\ (\eta_{\bullet})_* 1 \Big) \\ \amalg\ e\Big(\mathbb{Z}_{\geq 0} \times \mathtt{Sm}_2^{\emptyset} \times \mathrm{Sym}^{\bullet}(B_{\mathsf{op}}),\ (\eta_{\bullet})_* 1 \Big) \\ \amalg\ e\Big(\mathbb{Z}_{\geq 0} \times \mathtt{N}_2^{\sigma} \times \mathrm{Sym}^{\bullet}(B_{\mathsf{op}}),\ (\eta_{\bullet})_* 1 \Big) \\ \amalg\ e\Big(\mathbb{Z}_{\geq 0} \times \mathtt{N}_2^{\emptyset} \times \mathrm{Sym}^{\bullet}(B_{\mathsf{op}}),\ (\eta_{\bullet})_* 1 \Big) \end{aligned}$$

Which becomes:

$$
\begin{aligned}
e\Big(&\mathbb{Z}_{\geq 0} \times S_2^{\circ} \times \mathrm{Sym}^{\bullet}(B_{\mathsf{op}}),\ (\eta_{\bullet})_* 1 \Big) \\
&= e(\mathtt{Sm}_2^{\sigma}) Q_1 Q_3 p \frac{(\mathsf{V}_{\square\square\emptyset})}{(\mathsf{V}_{\square\emptyset\emptyset})(\mathsf{V}_{\emptyset\emptyset\emptyset})^2} \left(\sum_{a} Q_\sigma^a \Phi^{-,\emptyset}(a) \right) \left(\sum_{m} Q_3^m \Phi^{\emptyset,\emptyset}(m) \right)^{12} \\
&\quad \amalg\ e(\mathtt{Sm}_2^{\emptyset}) Q_1 Q_3 p \frac{(\mathsf{V}_{\square\square\emptyset})}{(\mathsf{V}_{\emptyset\emptyset\emptyset})^3} \left(\sum_{a \geq 0} Q_\sigma^a \Phi^{\emptyset,\emptyset}(a) \right) \left(\sum_{m \geq 0} Q_3^m \Phi^{\emptyset,\emptyset}(m) \right)^{12} \\
&\quad \amalg\ e(\mathtt{N}_2^{\sigma}) Q_1 Q_3 \frac{1}{(\mathsf{V}_{\square\emptyset\emptyset})(\mathsf{V}_{\emptyset\emptyset\emptyset})} \left(\sum_{a \geq 0} Q_\sigma^a \Phi^{-,\emptyset}(a) \right) \left(\sum_{m \geq 0} Q_3^m \Phi^{\emptyset,\emptyset}(m) \right)^{12} \\
&\quad \amalg\ e(\mathtt{N}_2^{\emptyset}) Q_1 Q_3 \frac{1}{(\mathsf{V}_{\emptyset\emptyset\emptyset})^2} \left(\sum_{a \geq 0} Q_\sigma^a \Phi^{\emptyset,\emptyset}(a) \right) \left(\sum_{m \geq 0} Q_3^m \Phi^{\emptyset,\emptyset}(m) \right)^{12}
\end{aligned}
$$

From lemmas 6.3.2, 6.3.4 and 6.3.5 we have:

1. $\mathsf{V}_{\emptyset\emptyset\emptyset} = M(p)$
2. $\mathsf{V}_{\square\emptyset\emptyset} = M(p)\frac{1}{1-p}$
3. $\mathsf{V}_{\square\square\emptyset} = M(p)\frac{p^2-p+1}{p(1-p)^2}$
4. $\sum_{m \geq 0} Q^m \Phi^{\emptyset,\emptyset}(m) = M(p)^2 \prod_{m>0} (1 + p^m Q)^m$
5. $\sum_{m \geq 0} Q^m \Phi^{-,\emptyset}(m) = M(p)^2 \dfrac{1+Q}{1-p} \prod_{m>0} (1 + p^m Q)^m$

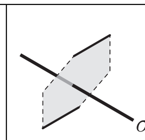
So we have:

$$
\begin{aligned}
e\Big(&\mathbb{Z}_{\geq 0} \times S_2^{\circ} \times \mathrm{Sym}^{\bullet}(B_{\mathsf{op}}),\ (\eta_{\bullet})_* 1 \Big) \\
&= Q_\sigma Q_1 Q_3 M(p)^{24} \left(\prod_{m>0} (1 + p^m Q_\sigma)^m (1 + p^m Q_3)^{12m} \right) \left(2 + 12 \frac{p}{(1-p)^2} \right) \\
&= Q_\sigma Q_1 Q_3 M(p)^{24} \left(\prod_{m>0} (1 + p^m Q_\sigma)^m (1 + p^m Q_3)^{12m} \right) (2\psi_1 + 12\psi_0)
\end{aligned}
$$

5.4.2. Part (5): We have 12 separate isomorphic strata:

$$\mathbb{Z}_{\geq 0} \times \mathrm{Sym}^{\bullet}(\{b_{\mathsf{op}}^k\}) \times \mathrm{Sym}^{\bullet}(B_{\mathsf{op}} \setminus \{b_{\mathsf{op}}^k\}).$$

These parameterise when $D = C_2^{(k)}$. The following is the summary of results from 5.2.4 and 5.2.5.

$U = \{k\}$	$e(U) = 1$	$\chi(\mathcal{O}_D) = 1$
$e(\eta_{\bullet}^{-1}(a, m_k, \boldsymbol{m})) = Q_1 p \frac{1}{(\mathsf{V}_{\emptyset\emptyset\emptyset})^2} \cdot Q_\sigma^a \Phi^{\emptyset,\emptyset}(a) \cdot Q_3^{m_k} \Phi^{-,-}(m_k) \cdot \prod_{i=1, i \neq k}^{12} Q_3^{m_i} \Phi^{\emptyset,\emptyset}(m_i)$		

From lemmas 6.3.2 and 6.3.4 we have:

1. $\mathsf{V}_{\emptyset\emptyset\emptyset} = M(p)$

2. $\sum_{m\geq 0} Q^m \Phi^{\emptyset,\emptyset}(m) = M(p)^2 \prod_{m>0} (1+p^m Q)^m$

3. $\sum_{m\geq 0} Q^m \Phi^{-,-}(m) = M(p)^2(\psi_0 + (\psi_1 + 2\psi_0)Q + \psi_0 Q^2) \prod_{m>0} (1+p^m Q)^m$

Since the strata are isomorphic we have:

$$
\begin{aligned}
&e\Big(\coprod_{k=1}^{12} \mathbb{Z}_{\geq 0} \times \mathrm{Sym}^\bullet(\{b^k_{\mathsf{op}}\}) \times \mathrm{Sym}^\bullet(B_{\mathsf{op}} \setminus \{b^k_{\mathsf{op}}\}),\ (\eta_\bullet)_* 1 \Big) \\
&\quad = 12\, e\Big(\mathbb{Z}_{\geq 0} \times \mathrm{Sym}^\bullet(\{b^k_{\mathsf{op}}\}) \times \mathrm{Sym}^\bullet(B_{\mathsf{op}} \setminus \{b^k_{\mathsf{op}}\}),\ (\eta_\bullet)_* 1 \Big) \\
&\quad = 12\, Q_1 \frac{1}{(\mathsf{V}_{\emptyset\emptyset\emptyset})^2} \left(\sum_{a\geq 0} Q_\sigma^a \Phi^{\emptyset,\emptyset}(a) \right) \left(\sum_{m\geq 0} Q_3^m \Phi^{-,-}(m) \right) \left(\sum_{m\geq 0} Q_3^m \Phi^{\emptyset,\emptyset}(m) \right)^{11} \\
&\quad = 12\, Q_1 M(p)^{24} \left(\prod_{m>0} (1+p^m Q_\sigma)^m (1+p^m Q_3)^{12m} \right) \Big(\psi_0 + (\psi_1 + 2\psi_0) Q_3 + \psi_0 Q_3^2 \Big)
\end{aligned}
$$

5.4.3. Thus combining parts (1)-(5) we have that the overall formula is:

$$
\begin{aligned}
&e\Big(\mathrm{Chow}^{\bullet\sigma+(0,1,\bullet)}(X), (\eta_\bullet)_* 1\Big) \\
&\quad = Q_1 M(p)^{24} \left(\prod_{m>0} (1+p^m Q_\sigma)^m (1+p^m Q_3)^{12m} \right) \\
&\qquad \cdot \Big(12\big(\psi_0 + (2\psi_0 + \psi_1) Q_3 + \psi_0 Q_3^2\big) + Q_\sigma Q_3 \big(12\psi_0 + 2\psi_1\big) \Big)
\end{aligned}
$$

5.5 Calculation for the class $\bullet\sigma + (1, 1, \bullet)$

We have a decomposition from lemma 3.5.3 of $\mathrm{Chow}^{(1,1,\bullet)}(X)$ into the parts:

(a) $S_1^\circ \times S_2^\circ \times \mathrm{Sym}^\bullet(B_{\mathsf{op}})$

(b) $\coprod_{k=1}^{12} S_1^\circ \times \mathrm{Sym}^\bullet(\{b^k_{\mathsf{op}}\}) \times \mathrm{Sym}^\bullet(B_{\mathsf{op}} \setminus \{b^k_{\mathsf{op}}\})$

(c) $\coprod_{k=1}^{12} S_2^\circ \times \mathrm{Sym}^\bullet(\{b^k_{\mathsf{op}}\}) \times \mathrm{Sym}^\bullet(B_{\mathsf{op}} \setminus \{b^k_{\mathsf{op}}\})$

(d) $\coprod_{k,\,l\,=\,1,\ k\,\neq\,l}^{12} \mathrm{Sym}^\bullet(\{b^k_{\mathsf{op}}\}) \times \mathrm{Sym}^\bullet(\{b^l_{\mathsf{op}}\}) \times \mathrm{Sym}^\bullet(B_{\mathsf{op}} \setminus \{b^k_{\mathsf{op}}, b^l_{\mathsf{op}}\})$

(e) $\coprod_{k=1}^{12} \mathrm{Sym}^\bullet(\{b^k_{\mathsf{op}}\}) \times \mathrm{Sym}^\bullet(B_{\mathsf{op}} \setminus \{b^k_{\mathsf{op}}\})$

(f) $\coprod \mathtt{Diag}^\bullet$

We also recall the notation from equation (II.10) from section 5.2 and the notation:

$$
\mathsf{Cyc}(C) = a\sigma + D + \sum_{i=1}^{12} m_i C_3^{(i)}.
$$

Each part will be characterised by the type of D. We will consider each case (a)-(f) separately and will use the following the previously introduced notation throughout:

1. $B_{\mathsf{i}} = \{b_{\mathsf{i}}^1, \ldots, b_{\mathsf{i}}^{12}\}$ is the set of the 12 points in S_{i} that correspond to nodes in the fibres of the projection $\pi : S_{\mathsf{i}} \to \mathbb{P}^1$.
2. $S_{\mathsf{i}}^{\circ} = S_{\mathsf{i}} \setminus B_{\mathsf{i}}$ is the complement of B_{i} in S_{i}
3. $\mathtt{N}_i \subset S_{\mathsf{i}}$ are the 12 nodal fibres of $\pi : S_{\mathsf{i}} \to \mathbb{P}^1$ with the nodes removed and:
$$\mathtt{N}_i = \mathtt{N}_i^{\sigma} \ \amalg \ \mathtt{N}_i^{\emptyset} \text{ where } \mathtt{N}_i^{\sigma} := \mathtt{N}_i \cap \sigma \text{ and } \mathtt{N}_i^{\emptyset} := \mathtt{N}_i \setminus \sigma.$$
4. $\mathtt{Sm}_i = S_{\mathsf{i}}^{\circ} \setminus \mathtt{N}_i$ is the complement of $\mathtt{N}_i$ in S_{i}° and:
$$\mathtt{Sm}_i = \mathtt{Sm}_i^{\sigma} \ \amalg \ \mathtt{Sm}_i^{\emptyset} \text{ where } \mathtt{Sm}_i^{\sigma} := \mathtt{Sm}_i \cap \sigma \text{ and } \mathtt{Sm}_i^{\emptyset} := \mathtt{Sm}_i \setminus \sigma.$$

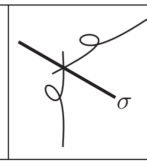
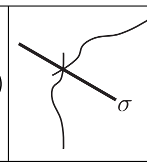
We will also use the new notation:
$$\mathtt{D} := \Big\{(x, x) \in S_1^{\circ} \times S\Big\}.$$

5.5.1. Part (a): We have the following stratification of $S_1^{\circ} \times S_1^{\circ}$:

1. $\Big((\mathtt{N}_1^{\sigma} \times \mathtt{N}_2^{\sigma}) \cap D \ \amalg \ (\mathtt{Sm}_1^{\sigma} \times \mathtt{Sm}_2^{\sigma}) \cap D \Big)$
2. $\amalg \ \Big(\mathtt{N}_1^{\sigma} \times \mathtt{N}_2^{\sigma} \setminus D \ \amalg \ \mathtt{N}_1^{\sigma} \times \mathtt{Sm}_2^{\sigma} \ \amalg \ \mathtt{Sm}_1^{\sigma} \times \mathtt{N}_2^{\sigma} \ \amalg \ \mathtt{Sm}_1^{\sigma} \times \mathtt{Sm}_2^{\sigma} \setminus D \Big)$
3. $\amalg \left(\begin{array}{c} \mathtt{N}_1^{\sigma} \times \mathtt{N}_2^{\emptyset} \setminus D \ \amalg \ (\mathtt{N}_1^{\sigma} \times \mathtt{N}_2^{\emptyset}) \cap D \ \amalg \ \mathtt{N}_1^{\sigma} \times \mathtt{Sm}_2^{\emptyset} \ \amalg \ \mathtt{Sm}_1^{\sigma} \times \mathtt{N}_2^{\emptyset} \\ \amalg \ \mathtt{Sm}_1^{\sigma} \times \mathtt{Sm}_2^{\emptyset} \setminus D \ \amalg \ (\mathtt{Sm}_1^{\emptyset} \times \mathtt{Sm}_2^{\sigma}) \cap D \end{array} \right)$
4. $\amalg \left(\begin{array}{c} \mathtt{N}_1^{\emptyset} \times \mathtt{N}_2^{\sigma} \setminus D \ \amalg \ (\mathtt{N}_1^{\emptyset} \times \mathtt{N}_2^{\sigma}) \cap D \ \amalg \ \mathtt{N}_1^{\emptyset} \times \mathtt{Sm}_2^{\sigma} \ \amalg \ \mathtt{Sm}_1^{\emptyset} \times \mathtt{N}_2^{\sigma} \\ \amalg \ \mathtt{Sm}_1^{\emptyset} \times \mathtt{Sm}_2^{\sigma} \setminus D \ \amalg \ (\mathtt{Sm}_1^{\sigma} \times \mathtt{Sm}_2^{\emptyset}) \cap D \end{array} \right)$
5. $\amalg \left(\begin{array}{c} \mathtt{N}_1^{\emptyset} \times \mathtt{N}_2^{\emptyset} \setminus D \ \amalg \ (\mathtt{N}_1^{\emptyset} \times \mathtt{N}_2^{\emptyset}) \cap D \ \amalg \ \mathtt{Sm}_1^{\emptyset} \times \mathtt{N}_2^{\emptyset} \ \amalg \ \mathtt{N}_1^{\emptyset} \times \mathtt{Sm}_2^{\emptyset} \\ \amalg \ \mathtt{Sm}_1^{\emptyset} \times \mathtt{Sm}_2^{\emptyset} \setminus D \ \amalg \ (\mathtt{Sm}_1^{\emptyset} \times \mathtt{Sm}_2^{\emptyset}) \cap D \end{array} \right)$

Here we have grouped by the number and type of intersection with σ.

Grouping (1): The following table is the summary of results from 5.2.4 and 5.2.5 for the strata in grouping (1):
$$\mathbb{Z}_{\geq 0} \times U \times \mathrm{Sym}^{\bullet}(B_{\mathsf{op}}).$$

$U = (\mathtt{N}_1^{\sigma} \times \mathtt{N}_2^{\sigma}) \cap \mathtt{D}$	$e(U) = 12$	$\chi(\mathcal{O}_D) = -1$	
$e(\eta_{\bullet}^{-1}(a, x, \boldsymbol{m})) = Q_1 Q_2 Q_3^2 p \frac{(\mathsf{V}_{\square\square\emptyset})^2}{(\mathsf{V}_{\square\emptyset\emptyset})^2 (\mathsf{V}_{\emptyset\emptyset\emptyset})^2} \cdot Q_\sigma^a \Phi^{+,\emptyset}(a) \cdot \prod_{i=1}^{12} Q_3^{m_i} \Phi^{\emptyset,\emptyset}(m_i)$			
$U = (\mathtt{Sm}_1^{\sigma} \times \mathtt{Sm}_2^{\sigma}) \cap \mathtt{D}$	$e(U) = -10$	$\chi(\mathcal{O}_D) = -1$	
$e(\eta_{\bullet}^{-1}(a, x, \boldsymbol{m})) = Q_1 Q_2 Q_3^2 p^{-1} \frac{1}{(\mathsf{V}_{\square\emptyset\emptyset})^2} \cdot Q_\sigma^a \Phi^{+,\emptyset}(a) \cdot \prod_{i=1}^{12} Q_3^{m_i} \Phi^{\emptyset,\emptyset}(m_i)$			

From lemmas 6.3.2 and 6.3.4 we have:

1. $\mathsf{V}_{\emptyset\emptyset\emptyset} = M(p)$
2. $\mathsf{V}_{\square\emptyset\emptyset} = M(p) \frac{1}{1-p}$
3. $\mathsf{V}_{\square\square\emptyset} = M(p) \frac{p^2 - p + 1}{p(1-p)^2}$

4. $\sum_{m\geq 0} Q^m \Phi^{\emptyset,\emptyset}(m) = M(p)^2 \prod_{m>0}(1+p^mQ)^m$

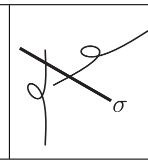
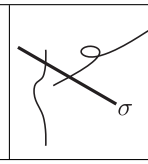
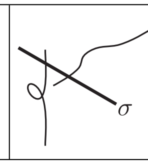
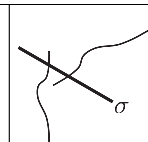
5. $\sum_{m\geq 0} Q^m \Phi^{+,\emptyset}(m) = M(p)^2(1+\psi_0+(\psi_1+2\psi_0)Q+\psi_0Q^2) \prod_{m>0}(1+p^mQ)^m.$

So the contribution is:

$$Q_1Q_2Q_3^2M(p)^{24}\left(\prod_{m>0}(1+p^mQ_\sigma)^m(1+p^mQ_3)^{12m}\right)(\psi_0+(\psi_1+2\psi_0)Q_\sigma+\psi_0Q_\sigma^2)$$
$$\cdot\left(\frac{2\left(p^4+8p^3-12p^2+8p+1\right)}{(p-1)^2p}\right)$$

Grouping (2): The following table is the summary of results from 5.2.4 and 5.2.5 for the strata in grouping (2):

$$\mathbb{Z}_{\geq 0}\times U\times \mathrm{Sym}^\bullet(B_{\mathsf{op}}).$$

<table>
<tr><td>$U=\mathtt{N}_1^\sigma\times\mathtt{N}_2^\sigma\setminus\mathtt{D}$</td><td>$e(U)=132$</td><td colspan="2">$\chi(\mathcal{O}_D)=0$</td></tr>
<tr><td colspan="3">$e(\eta_\bullet^{-1}(a,x,\boldsymbol{m}))=$
$Q_1Q_2Q_3^2p^2\frac{(\mathsf{V}_{\square\square\emptyset})^2}{(\mathsf{V}_{\square\emptyset\emptyset})^2(\mathsf{V}_{\emptyset\emptyset\emptyset})^2}\cdot Q_\sigma^a\Phi^{-,-}(a)\cdot\prod_{i=1}^{12}Q_3^{m_i}\Phi^{\emptyset,\emptyset}(m_i)$</td><td></td></tr>
<tr><td>$U=\mathtt{N}_1^\sigma\times\mathtt{Sm}_2^\sigma$</td><td>$e(U)=-120$</td><td colspan="2">$\chi(\mathcal{O}_D)=0$</td></tr>
<tr><td colspan="3">$e(\eta_\bullet^{-1}(a,x,\boldsymbol{m}))=$
$Q_1Q_2Q_3^2p\frac{(\mathsf{V}_{\square\square\emptyset})}{(\mathsf{V}_{\square\emptyset\emptyset})^2(\mathsf{V}_{\emptyset\emptyset\emptyset})}\cdot Q_\sigma^a\Phi^{-,-}(a)\cdot\prod_{i=1}^{12}Q_3^{m_i}\Phi^{\emptyset,\emptyset}(m_i)$</td><td></td></tr>
<tr><td>$U=\mathtt{Sm}_1^\sigma\times\mathtt{N}_2^\sigma$</td><td>$e(U)=-120$</td><td colspan="2">$\chi(\mathcal{O}_D)=0$</td></tr>
<tr><td colspan="3">$e(\eta_\bullet^{-1}(a,x,\boldsymbol{m}))=$
$Q_1Q_2Q_3^2p\frac{(\mathsf{V}_{\square\square\emptyset})}{(\mathsf{V}_{\square\emptyset\emptyset})^2(\mathsf{V}_{\emptyset\emptyset\emptyset})}\cdot Q_\sigma^a\Phi^{-,-}(a)\cdot\prod_{i=1}^{12}Q_3^{m_i}\Phi^{\emptyset,\emptyset}(m_i)$</td><td></td></tr>
<tr><td>$U=\mathtt{Sm}_1^\sigma\times\mathtt{Sm}_2^\sigma\setminus\mathtt{D}$</td><td>$e(U)=110$</td><td colspan="2">$\chi(\mathcal{O}_D)=0$</td></tr>
<tr><td colspan="3">$e(\eta_\bullet^{-1}(a,x,\boldsymbol{m}))=Q_1Q_2Q_3^2\frac{1}{(\mathsf{V}_{\square\emptyset\emptyset})^2}\cdot Q_\sigma^a\Phi^{-,-}(a)\cdot\prod_{i=1}^{12}Q_3^{m_i}\Phi^{\emptyset,\emptyset}(m_i)$</td><td></td></tr>
</table>

From lemmas 6.3.2 and 6.3.4 we have:

1. $\mathsf{V}_{\emptyset\emptyset\emptyset}=M(p)$

2. $\mathsf{V}_{\square\emptyset\emptyset}=M(p)\frac{1}{1-p}$

3. $\mathsf{V}_{\square\square\emptyset}=M(p)\frac{p^2-p+1}{p(1-p)^2}$

4. $\sum_{m\geq 0} Q^m \Phi^{\emptyset,\emptyset}(m) = M(p)^2 \prod_{m>0}(1+p^mQ)^m$

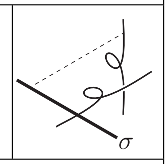
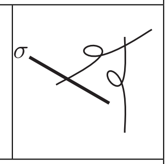
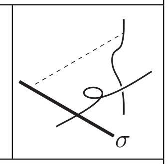
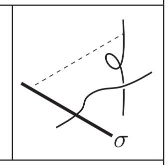
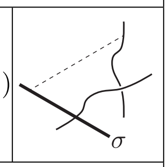
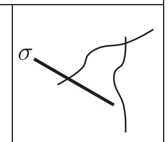
5. $\sum_{m\geq 0} Q^m \Phi^{-,-}(m) = M(p)^2\frac{1}{p}(\psi_0+(\psi_1+2\psi_0)Q+\psi_0Q^2) \prod_{m>0}(1+p^mQ)^m.$

So the contribution is:

$$Q_1Q_2Q_3^2M(p)^{24}\left(\prod_{m>0}(1+p^mQ_\sigma)^m(1+p^mQ_3)^{12m}\right)\frac{1}{p}(\psi_0+(\psi_1+2\psi_0)Q+\psi_0Q^2)$$
$$\cdot\left(\frac{2\left(p^4+8p^3+48p^2+8p+1\right)}{(p-1)^2}\right)$$

Grouping (3): The following table is the summary of results from 5.2.4 and 5.2.5 for the strata in grouping (3):

$$\mathbb{Z}_{\geq 0}\times U\times \mathrm{Sym}^\bullet(B_{\mathsf{op}}).$$

<table>
<tr><td>$U=\mathtt{N}_1^\sigma\times\mathtt{N}_2^\emptyset\setminus\mathtt{D}$</td><td>$e(U)=-132$</td><td colspan="2">$\chi(\mathcal{O}_D)=0$</td></tr>
<tr><td colspan="3">$e(\eta_\bullet^{-1}(a,x,\boldsymbol{m}))=$ $Q_1Q_2Q_3^2p^2\frac{(\mathsf{V}_{\square\square\emptyset})^2}{(\mathsf{V}_{\square\emptyset\emptyset})(\mathsf{V}_{\emptyset\emptyset\emptyset})^3}\cdot Q_\sigma^a\Phi^{-,\emptyset}(a)\cdot\prod_{i=1}^{12}Q_3^{m_i}\Phi^{\emptyset,\emptyset}(m_i)$</td><td></td></tr>
<tr><td>$U=(\mathtt{N}_1^\sigma\times\mathtt{N}_2^\emptyset)\cap\mathtt{D}$</td><td>$e(U)=-12$</td><td colspan="2">$\chi(\mathcal{O}_D)=-1$</td></tr>
<tr><td colspan="3">$e(\eta_\bullet^{-1}(a,x,\boldsymbol{m}))=$ $Q_1Q_2Q_3^2p^2\frac{(\mathsf{V}_{\square\square\emptyset})^3}{(\mathsf{V}_{\square\emptyset\emptyset})^3(\mathsf{V}_{\emptyset\emptyset\emptyset})^2}\cdot Q_\sigma^a\Phi^{-,\emptyset}(a)\cdot\prod_{i=1}^{12}Q_3^{m_i}\Phi^{\emptyset,\emptyset}(m_i)$</td><td></td></tr>
<tr><td>$U=\mathtt{N}_1^\sigma\times\mathtt{Sm}_2^\emptyset$</td><td>$e(U)=120$</td><td colspan="2">$\chi(\mathcal{O}_D)=0$</td></tr>
<tr><td colspan="3">$e(\eta_\bullet^{-1}(a,x,\boldsymbol{m}))=$ $Q_1Q_2Q_3^2p\frac{(\mathsf{V}_{\square\square\emptyset})}{(\mathsf{V}_{\square\emptyset\emptyset})(\mathsf{V}_{\emptyset\emptyset\emptyset})^2}\cdot Q_\sigma^a\Phi^{-,\emptyset}(a)\cdot\prod_{i=1}^{12}Q_3^{m_i}\Phi^{\emptyset,\emptyset}(m_i)$</td><td></td></tr>
<tr><td>$U=\mathtt{Sm}_1^\sigma\times\mathtt{N}_2^\emptyset$</td><td>$e(U)=120$</td><td colspan="2">$\chi(\mathcal{O}_D)=0$</td></tr>
<tr><td colspan="3">$e(\eta_\bullet^{-1}(a,x,\boldsymbol{m}))=$ $Q_1Q_2Q_3^2p\frac{(\mathsf{V}_{\square\square\emptyset})}{(\mathsf{V}_{\square\emptyset\emptyset})(\mathsf{V}_{\emptyset\emptyset\emptyset})^2}\cdot Q_\sigma^a\Phi^{-,\emptyset}(a)\cdot\prod_{i=1}^{12}Q_3^{m_i}\Phi^{\emptyset,\emptyset}(m_i)$</td><td></td></tr>
<tr><td>$U=\mathtt{Sm}_1^\sigma\times\mathtt{Sm}_2^\emptyset\setminus\mathtt{D}$</td><td>$e(U)=-110$</td><td colspan="2">$\chi(\mathcal{O}_D)=0$</td></tr>
<tr><td colspan="3">$e(\eta_\bullet^{-1}(a,x,\boldsymbol{m}))=Q_1Q_2Q_3^2\frac{1}{(\mathsf{V}_{\square\emptyset\emptyset})(\mathsf{V}_{\emptyset\emptyset\emptyset})}\cdot Q_\sigma^a\Phi^{-,\emptyset}(a)\cdot\prod_{i=1}^{12}Q_3^{m_i}\Phi^{\emptyset,\emptyset}(m_i)$</td><td></td></tr>
<tr><td>$U=(\mathtt{Sm}_1^\emptyset\times\mathtt{Sm}_2^\sigma)\cap\mathtt{D}$</td><td>$e(U)=10$</td><td colspan="2">$\chi(\mathcal{O}_D)=-1$</td></tr>
<tr><td colspan="3">$e(\eta_\bullet^{-1}(a,x,\boldsymbol{m}))=Q_1Q_2Q_3^2\frac{(\mathsf{V}_{\square\square\emptyset})}{(\mathsf{V}_{\square\emptyset\emptyset})^3}\cdot Q_\sigma^a\Phi^{-,\emptyset}(a)\cdot\prod_{i=1}^{12}Q_3^{m_i}\Phi^{\emptyset,\emptyset}(m_i)$</td><td></td></tr>
</table>

From lemmas 6.3.2, 6.3.4 and 6.3.5 we have:

1. $\mathsf{V}_{\emptyset\emptyset\emptyset}=M(p)$
2. $\mathsf{V}_{\square\emptyset\emptyset}=M(p)\frac{1}{1-p}$

3. $\mathsf{V}_{\square\square\emptyset} = M(p)\frac{p^2-p+1}{p(1-p)^2}$

4. $\sum_{m\geq 0} Q^m\Phi^{\emptyset,\emptyset}(m) = M(p)^2\prod_{m>0}(1+p^mQ)^m$

5. $\sum_{m\geq 0} Q^m\Phi^{-,\emptyset}(m) = M(p)^2\dfrac{1+Q}{1-p}\prod_{m>0}(1+p^mQ)^m$

The contribution from grouping (3) is:

$$Q_1Q_2Q_3^2M(p)^{24}\left(\prod_{m>0}(1+p^mQ_\sigma)^m(1+p^mQ_3)^{12m}\right)\left(\frac{1+Q_\sigma}{1-p}\right)$$
$$\cdot\left(\frac{2\left(p^2+10p+1\right)\left(p^4-2p^3+8p^2-2p+1\right)}{(p-1)^3p}\right)$$

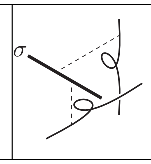
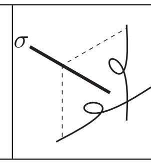
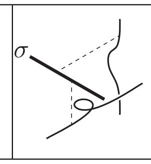
Grouping (4): The results for grouping (4) are identical to those of grouping (3) under the symmetry of the banana threefold.

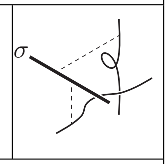
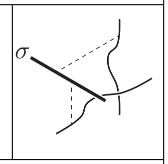
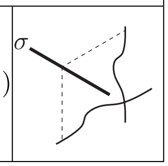
The contribution from grouping (4) is:

$$Q_1Q_2Q_3^2M(p)^{24}\left(\prod_{m>0}(1+p^mQ_\sigma)^m(1+p^mQ_3)^{12m}\right)\left(\frac{1+Q_\sigma}{1-p}\right)$$
$$\cdot\left(\frac{2\left(p^2+10p+1\right)\left(p^4-2p^3+8p^2-2p+1\right)}{(p-1)^3p}\right)$$

Grouping (5): The following table is the summary of results from 5.2.4 and 5.2.5 for the strata in grouping (5):

$$\mathbb{Z}_{\geq 0}\times U\times \mathrm{Sym}^\bullet(B_{\mathsf{op}}).$$

$U = \mathtt{N}_1^\emptyset\times\mathtt{N}_2^\emptyset\setminus\mathtt{D}$	$e(U) = 132$	$\chi(\mathcal{O}_D) = 0$
$e(\eta_\bullet^{-1}(a,x,\boldsymbol{m})) = Q_1Q_2Q_3^2p^2\frac{\mathsf{V}_{\square\square\emptyset})^2}{(\mathsf{V}_{\emptyset\emptyset\emptyset})^4}\cdot Q_\sigma^a\Psi^{\emptyset,\emptyset}(a)\cdot\prod_{i=1}^{12}Q_3^{m_i}\Phi^{\emptyset,\emptyset}(m_i)$		
$U = (\mathtt{N}_1^\emptyset\times\mathtt{N}_2^\emptyset)\cap\mathtt{D}$	$e(U) = 12$	$\chi(\mathcal{O}_D) = -1$
$e(\eta_\bullet^{-1}(a,x,\boldsymbol{m})) =$ $Q_1Q_2Q_3^2p^2\frac{(\mathsf{V}_{\square\square\emptyset})^3}{(\mathsf{V}_{\square\emptyset\emptyset})^2(\mathsf{V}_{\emptyset\emptyset\emptyset})^3}\cdot Q_\sigma^a\Psi^{\emptyset,\emptyset}(a)\cdot\prod_{i=1}^{12}Q_3^{m_i}\Phi^{\emptyset,\emptyset}(m_i)$		
$U = \mathtt{Sm}_1^\emptyset\times\mathtt{N}_2^\emptyset$	$e(U) = -120$	$\chi(\mathcal{O}_D) = 0$
$e(\eta_\bullet^{-1}(a,x,\boldsymbol{m})) = Q_1Q_2Q_3^2p\frac{(\mathsf{V}_{\square\square\emptyset})}{(\mathsf{V}_{\emptyset\emptyset\emptyset})^3}\cdot Q_\sigma^a\Psi^{\emptyset,\emptyset}(a)\cdot\prod_{i=1}^{12}Q_3^{m_i}\Phi^{\emptyset,\emptyset}(m_i)$		

$U = \mathtt{N}_1^{\emptyset} \times \mathtt{Sm}_2^{\emptyset}$	$e(U) = -120$	$\chi(\mathcal{O}_D) = 0$
$e(\eta_\bullet^{-1}(a, x, \boldsymbol{m})) = Q_1 Q_2 Q_3^2 p \frac{(\mathsf{V}_{\square\square\emptyset})}{(\mathsf{V}_{\emptyset\emptyset\emptyset})^3} \cdot Q_\sigma^a \Psi^{\emptyset,\emptyset}(a) \cdot \prod_{i=1}^{12} Q_3^{m_i} \Phi^{\emptyset,\emptyset}(m_i)$		
$U = \mathtt{Sm}_1^{\emptyset} \times \mathtt{Sm}_2^{\emptyset} \setminus \mathtt{D}$	$e(U) = 110$	$\chi(\mathcal{O}_D) = 0$
$e(\eta_\bullet^{-1}(a, x, \boldsymbol{m})) = Q_1 Q_2 Q_3^2 \frac{1}{(\mathsf{V}_{\emptyset\emptyset\emptyset})^2} \cdot Q_\sigma^a \Psi^{\emptyset,\emptyset}(a) \cdot \prod_{i=1}^{12} Q_3^{m_i} \Phi^{\emptyset,\emptyset}(m_i)$		
$U = (\mathtt{Sm}_1^{\emptyset} \times \mathtt{Sm}_2^{\emptyset}) \cap \mathtt{D}$	$e(U) = -10$	$\chi(\mathcal{O}_D) = -1$
$e(\eta_\bullet^{-1}(a, x, \boldsymbol{m})) = Q_1 Q_2 Q_3^2 \frac{(\mathsf{V}_{\square\square\emptyset})}{(\mathsf{V}_{\square\emptyset\emptyset})^2 (\mathsf{V}_{\emptyset\emptyset\emptyset})} \cdot Q_\sigma^a \Psi^{\emptyset,\emptyset}(a) \cdot \prod_{i=1}^{12} Q_3^{m_i} \Phi^{\emptyset,\emptyset}(m_i)$		

From lemmas 6.3.2 and 6.3.4 we have:

1. $\mathsf{V}_{\emptyset\emptyset\emptyset} = M(p)$
2. $\mathsf{V}_{\square\emptyset\emptyset} = M(p)\frac{1}{1-p}$
3. $\mathsf{V}_{\square\square\emptyset} = M(p)\frac{p^2-p+1}{p(1-p)^2}$
4. $\sum_{m\geq 0} Q^m \Phi^{\emptyset,\emptyset}(m) = M(p)^2 \prod_{m>0} (1 + p^m Q)^m$

$$
Q_1 Q_2 Q_3^2 M(p)^{24} \left(\prod_{m>0} (1 + p^m Q_\sigma)^m (1 + p^m Q_3)^{12m} \right)
\cdot \left(\frac{2\left(p^2 + 10p + 1\right)\left(p^4 - 2p^3 + 8p^2 - 2p + 1\right)}{(p-1)^4 p} \right)
$$

Summing the contributions from the above groupings we arrive at the overall contribution from part (a):

$$
\begin{aligned}
& e\Big(\mathbb{Z}_{\geq 0} \times S_1^\circ \times S_2^\circ \times \mathrm{Sym}^\bullet(B_{\mathsf{op}}),\ (\eta_\bullet)_* 1\Big) \\
&= Q_1 Q_2 Q_3^2 Q_\sigma M(p)^{24} \left(\prod_{m>0} (1 + p^m Q_\sigma)^m (1 + p^m Q_3)^{12m} \right) \\
&\cdot \left(120\psi_0 + Q_\sigma \left(144\psi_0^2 + 48\psi_0 + 4\right)\right)
\end{aligned}
$$

5.5.2. Part (b)-(c): By the symmetry of X we only need to consider part (b), with part (c) being completely analogous. For each $k \in \{1, \ldots, 12\}$ we begin by decomposing S_1° into the following six parts:

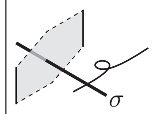
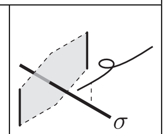
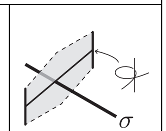
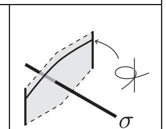
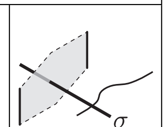
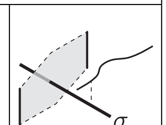
$$
\mathtt{Sm}_1^{\sigma} \amalg \mathtt{Sm}_1^{\emptyset} \amalg \mathtt{N}_1^{\sigma,(k)} \amalg \mathtt{N}_1^{\sigma,c} \amalg \mathtt{N}_1^{\emptyset,(k)} \amalg \mathtt{N}_1^{\emptyset,c}
$$

where $\mathtt{N}_1^{\sigma,(k)}$ is the connected component of $\mathtt{N}_1^{\sigma}$ corresponding the the kth banana configuration and $\mathtt{N}_1^{\sigma,c}$ is its complement in $\mathtt{N}_1^{\sigma}$. The same definition is true for $\mathtt{N}_1^{\emptyset}$.

We use the above size part decomposition for

$$\mathbb{Z}_{\geq 0} \times S_1^\circ \times \mathrm{Sym}^\bullet(\{b_{\mathsf{op}}^k\}) \times \mathrm{Sym}^\bullet(B_{\mathsf{op}} \setminus \{b_{\mathsf{op}}^k\}).$$

The following table is the summary of results from 5.2.4 and 5.2.5 for this stratification.

<table>
<tr><td>$U = \mathtt{N}_1^{\sigma,c}$</td><td>$e(U) = 11$</td><td>$\chi(\mathcal{O}_D) = 1$</td></tr>
<tr><td colspan="2">$e(\eta_\bullet^{-1}(a,x,n,\boldsymbol{m})) =$ $Q_1Q_2Q_3p^2\frac{(\mathsf{V}_{\square\square\emptyset})}{(\mathsf{V}_{\square\emptyset\emptyset})(\mathsf{V}_{\emptyset\emptyset\emptyset})^2} \cdot Q_\sigma^a\Phi^{-,\emptyset}(a) \cdot Q_3^n\Phi^{\cdot,\cdot}(n) \prod_{\substack{i=1\\ i\neq j}}^{12} Q_3^{m_i}\Phi^{\emptyset,\emptyset}(m_i)$</td><td>
</td></tr>
<tr><td>$U = \mathtt{N}_1^{\emptyset,c}$</td><td>$e(U) = -11$</td><td>$\chi(\mathcal{O}_D) = 1$</td></tr>
<tr><td colspan="2">$e(\eta_\bullet^{-1}(a,x,n,\boldsymbol{m})) =$ $Q_1Q_2Q_3p^2\frac{(\mathsf{V}_{\square\square\emptyset})}{(\mathsf{V}_{\emptyset\emptyset\emptyset})^3} \cdot Q_\sigma^a\Phi^{\emptyset,\emptyset}(a) \cdot Q_3^n\Phi^{\cdot,\cdot}(n) \prod_{\substack{i=1\\ i\neq j}}^{12} Q_3^{m_i}\Phi^{\emptyset,\emptyset}(m_i)$</td><td>
</td></tr>
<tr><td>$U = \mathtt{N}_1^{\sigma,(k)}$</td><td>$e(U) = 1$</td><td>$\chi(\mathcal{O}_D) = 0$</td></tr>
<tr><td colspan="2">$e(\eta_\bullet^{-1}(a,x,n,\boldsymbol{m})) =$ $Q_1Q_2Q_3p^2\frac{(\mathsf{V}_{\square\square\square})}{(\mathsf{V}_{\square\emptyset\emptyset})^2(\mathsf{V}_{\emptyset\emptyset\emptyset})} \cdot Q_\sigma^a\Phi^{-,\emptyset}(a) \cdot Q_3^n\Phi^{\cdot,\cdot}(n) \prod_{\substack{i=1\\ i\neq j}}^{12} Q_3^{m_i}\Phi^{\emptyset,\emptyset}(m_i)$</td><td>
</td></tr>
<tr><td>$U = \mathtt{N}_1^{\emptyset,(k)}$</td><td>$e(U) = -1$</td><td>$\chi(\mathcal{O}_D) = 0$</td></tr>
<tr><td colspan="2">$e(\eta_\bullet^{-1}(a,x,n,\boldsymbol{m})) =$ $Q_1Q_2Q_3p^2\frac{(\mathsf{V}_{\square\square\square})}{(\mathsf{V}_{\square\emptyset\emptyset})(\mathsf{V}_{\emptyset\emptyset\emptyset})^2} \cdot Q_\sigma^a\Phi^{\emptyset,\emptyset}(a) \cdot Q_3^n\Phi^{\cdot,\cdot}(n) \prod_{\substack{i=1\\ i\neq j}}^{12} Q_3^{m_i}\Phi^{\emptyset,\emptyset}(m_i)$</td><td>
</td></tr>
<tr><td>$U = \mathtt{Sm}_1^{\sigma}$</td><td>$e(U) = -10$</td><td>$\chi(\mathcal{O}_D) = 1$</td></tr>
<tr><td colspan="2">$e(\eta_\bullet^{-1}(a,x,n,\boldsymbol{m})) =$ $Q_1Q_2Q_3p\frac{1}{(\mathsf{V}_{\square\emptyset\emptyset})(\mathsf{V}_{\emptyset\emptyset\emptyset})} \cdot Q_\sigma^a\Phi^{-,\emptyset}(a) \cdot Q_3^n\Phi^{\cdot,\cdot}(n) \prod_{\substack{i=1\\ i\neq j}}^{12} Q_3^{m_i}\Phi^{\emptyset,\emptyset}(m_i)$</td><td>
</td></tr>
<tr><td>$U = \mathtt{Sm}_1^{\emptyset}$</td><td>$e(U) = 10$</td><td>$\chi(\mathcal{O}_D) = 1$</td></tr>
<tr><td colspan="2">$e(\eta_\bullet^{-1}(a,x,n,\boldsymbol{m})) =$ $Q_1Q_2Q_3p\frac{1}{(\mathsf{V}_{\emptyset\emptyset\emptyset})^2} \cdot Q_\sigma^a\Phi^{\emptyset,\emptyset}(a) \cdot Q_3^n\Phi^{\cdot,\cdot}(n) \prod_{\substack{i=1\\ i\neq j}}^{12} Q_3^{m_i}\Phi^{\emptyset,\emptyset}(m_i)$</td><td>
</td></tr>
</table>

From lemmas 6.3.2, 6.3.4 and 6.3.5 we have:

1. $\mathsf{V}_{\emptyset\emptyset\emptyset} = M(p)$
2. $\mathsf{V}_{\square\emptyset\emptyset} = M(p)\frac{1}{1-p}$
3. $\mathsf{V}_{\square\square\emptyset} = M(p)\frac{p^2-p+1}{p(1-p)^2}$
4. $\mathsf{V}_{\square\square\square} = M(p)\frac{p^4-p^3+p^2-p+1}{p^2(1-p)^3}$
5. $\sum_{m\geq 0} Q^m\Phi^{\emptyset,\emptyset}(m) = M(p)^2 \prod_{m>0}(1+p^mQ)^m$

6. $\sum_{m\geq 0} Q^m \Phi^{-,\emptyset}(m) = M(p)^2 \frac{1+Q}{1-p} \prod_{m>0} (1+p^mQ)^m$

7. $\sum_{m\geq 0} Q^m \Phi^{-,-}(m) = M(p)^2 \frac{1}{p}(\psi_0 + (\psi_1+2\psi_0)Q + \psi_0 Q^2) \prod_{m>0}(1+p^mQ)^m.$

There are 12 singular fibres of pr. So, we have that the combined contribution from parts (c) and (d) is:

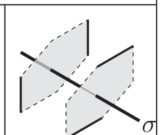
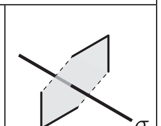
$$
\begin{aligned}
& e\Big(\coprod_{k=1}^{12} S_1^\circ \times \mathrm{Sym}^\bullet(\{b_{\mathsf{op}}^k\}) \times \mathrm{Sym}^\bullet(B_{\mathsf{op}} \setminus \{b_{\mathsf{op}}^k\}),\ (\eta_\bullet)_* 1\Big) \\
& \quad + e\Big(\coprod_{k=1}^{12} S_2^\circ \times \mathrm{Sym}^\bullet(\{b_{\mathsf{op}}^k\}) \times \mathrm{Sym}^\bullet(B_{\mathsf{op}} \setminus \{b_{\mathsf{op}}^k\}),\ (\eta_\bullet)_* 1\Big) \\
& = 24 Q_1Q_2Q_3Q_\sigma M(p)^{24} \left(\prod_{m>0} (1+p^mQ_\sigma)^m (1+p^mQ_3)^{12m} \right) \\
& \quad \cdot (\psi_0 + (\psi_1+2\psi_0)Q_3 + \psi_0 Q_3^2)\,(12\psi_0 + 4\psi_1 + \psi_2)
\end{aligned}
$$

5.5.3. Part (d)-(e): Parts (d) and (e) parametrise the cases when D is the union of $C_2^{(k)}$ and $C_2^{(l)}$. We have the spaces:

1. $\coprod_{\substack{k,\, l\, =\, 1 \\ k \neq l}}^{12} \mathrm{Sym}^\bullet(\{b_{\mathsf{op}}^k\}) \times \mathrm{Sym}^\bullet(\{b_{\mathsf{op}}^l\}) \times \mathrm{Sym}^\bullet(B_{\mathsf{op}} \setminus \{b_{\mathsf{op}}^k, b_{\mathsf{op}}^l\}),$

2. $\coprod_{k=1}^{12} \mathrm{Sym}^\bullet(\{b_{\mathsf{op}}^k\}) \times \mathrm{Sym}^\bullet(B_{\mathsf{op}} \setminus \{b_{\mathsf{op}}^k\}).$

The following table is the summary of results from 5.2.4 and 5.2.5 for this stratification.

<table>
<tr><td>$U = \{(k,l)\}$, $k \neq l$</td><td>$e(U) = 1$</td><td colspan="2">$\chi(\mathcal{O}_D) = 2$</td></tr>
<tr><td colspan="3">$e(\eta_\bullet^{-1}(a,c,d,\boldsymbol{m})) =$
$Q_1Q_2p^2 \frac{1}{(\mathsf{V}_{\emptyset\emptyset\emptyset})^2} \cdot Q_\sigma^a \Phi^{\emptyset,\emptyset}(a) \cdot Q_3^c \Phi^{-,-}(c) \cdot Q_3^d \Phi^{-,-}(d) \prod_{\substack{i\,=\,1 \\ i \neq j,\, k}}^{12} Q_3^{m_i} \Phi^{\emptyset,\emptyset}(m_i)$</td><td></td></tr>
<tr><td>$U = \{(k,k)\}$</td><td>$e(U) = 1$</td><td colspan="2">$\chi(\mathcal{O}_D) = 0$</td></tr>
<tr><td colspan="3">$e(\eta_\bullet^{-1}(a,m_k,\boldsymbol{m})) =$
$Q_1Q_2 \frac{1}{(\mathsf{V}_{\emptyset\emptyset\emptyset})^2} \cdot Q_\sigma^a \Psi^{\emptyset,\emptyset}(a) \cdot Q_3^{m_k} \Phi^{+,+}(m_k) \prod_{\substack{i\,=\,1 \\ i \neq j}}^{12} Q_3^{m_i} \Phi^{\emptyset,\emptyset}(m_i)$</td><td></td></tr>
</table>

From lemmas 6.3.2, 6.3.4 and 6.3.5 we have:

1. $\mathsf{V}_{\emptyset\emptyset\emptyset} = M(p)$

2. $\sum_{m\geq 0} Q^m \Phi^{\emptyset,\emptyset}(m) = M(p)^2 \prod_{m>0} (1+p^mQ)^m$

3. $\sum_{m\geq 0} Q^m \Phi^{-,-}(m) = M(p)^2 \frac{1}{p}(\psi_0 + (\psi_1+2\psi_0)Q + \psi_0 Q^2) \prod_{m>0}(1+p^mQ)^m.$

4. $\sum_{m\geq 0} Q^m \Phi^{+,+}(m)$
$= M(p)^2 \prod_{m>0}(1+p^mQ)^m \Big(Q^4(2\psi_0+\psi_1) + Q^3(8\psi_0+6\psi_1+\psi_2) + Q^2(12\psi_0$
$+10\psi_1+2\psi_2) + Q(8\psi_0+6\psi_1+\psi_2) + (2\psi_0+\psi_1)\Big)$

There are 136 choices for two distinct fibres. Hence the contribution from part (d) is:

$$e\Big(\coprod_{\substack{k,\,l\,=\,1\\ k\,\neq\,l}}^{12} \mathrm{Sym}^\bullet(\{b^k_{\mathsf{op}}\}) \times \mathrm{Sym}^\bullet(\{b^l_{\mathsf{op}}\}) \times \mathrm{Sym}^\bullet(B_{\mathsf{op}} \setminus \{b^k_{\mathsf{op}}, b^l_{\mathsf{op}}\}),\ (\eta_\bullet)_*1\Big)$$

$$= 132 Q_1 Q_2 M(p)^{24} \left(\prod_{m>0} (1+p^m Q_\sigma)^m (1+p^m Q_3)^{12m}\right) (\psi_0 + (\psi_1 + 2\psi_0)Q_3 + \psi_0 Q_3^2)^2.$$

The 12 singular fibres give the contribution of (e) as:

$$e\Big(\coprod_{k=1}^{12} \mathrm{Sym}^\bullet(\{b^k_{\mathsf{op}}\}) \times \mathrm{Sym}^\bullet(B_{\mathsf{op}} \setminus \{b^k_{\mathsf{op}}\}),\ (\eta_\bullet)_*1\Big)$$

$$= 12 Q_1 Q_2 M(p)^{24} \left(\prod_{m>0} (1+p^m Q_\sigma)^m (1+p^m Q_3)^{12m}\right)$$

$$\cdot \Big(\big(Q_3^2\psi_0 + Q_3(2\psi_0+\psi_1) + \psi_0\big)^2 + \big(Q_3^4(2\psi_0+\psi_1) + Q_3^3(8\psi_0 + 6\psi_1 + \psi_2) + Q_3^2(12\psi_0 + 10\psi_1 + 2\psi_2) + Q_3(8\psi_0 + 6\psi_1 + \psi_2) + (2\psi_0+\psi_1)\big)\Big).$$

Summing the contributions of (d) and (e) we have:

$$Q_1 Q_2 M(p)^{24} \left(\prod_{m>0} (1+p^m Q_\sigma)^m (1+p^m Q_3)^{12m}\right)$$

$$\cdot \Big(144\big(Q_3^2\psi_0 + Q_3(2\psi_0+\psi_1) + \psi_0\big)^2 + 12\big(Q_3^4(2\psi_0+\psi_1) + Q_3^3(8\psi_0 + 6\psi_1 + \psi_2) + Q_3^2(12\psi_0 + 10\psi_1 + 2\psi_2) + Q_3(8\psi_0 + 6\psi_1 + \psi_2) + (2\psi_0+\psi_1)\big)\Big).$$

5.5.4. Part (f): Recall from lemma 3.5.3 that part (f), $\mathtt{Diag}^\bullet$ has the further decomposition:

(g) $\mathtt{Sm}_1 \times \mathrm{Sym}^\bullet(B_{\mathsf{op}})$

(h) $\coprod \mathtt{Sm}_2 \times \mathrm{Sym}^\bullet(B_{\mathsf{op}})$

(i) $\coprod_{y\in \mathtt{J}} E_{\pi(y)} \times \widetilde{\mathrm{Aut}}(E_{\pi(y)}) \times \mathrm{Sym}^\bullet(B_{\mathsf{op}})$

(j) $\coprod_{k=1}^{12} \mathtt{L} \times \mathrm{Sym}^\bullet(\{b^k_{\mathsf{op}}\}) \times \mathrm{Sym}^\bullet(B_{\mathsf{op}} \setminus \{b^k_{\mathsf{op}}\})$.

Where we have used the notation:

1. $\mathtt{J}^0$ and $\mathtt{J}^{1728}$ to be the subsets of points $x \in \mathbb{P}^1$ such that $\pi^{-1}(x)$ has j-invariant 0 or 1728 respectively and $\mathtt{J} = \mathtt{J}^0 \ \coprod\ \mathtt{J}^{1728}$.
2. $\mathtt{L}$ to be the linear system $|f_1 + f_2|$ on $\mathbb{P}^1 \times \mathbb{P}^1$ with the singular divisors removed where f_1 and f_2 are fibres of the two projection maps.
3. $\widetilde{\mathrm{Aut}}(E) := \mathrm{Aut}(E) \setminus \{\pm 1\}$.

5.5.5. Parts (g)-(i):
The results for parts (g)-(i) will all be very similar. The key differences are:

1. The overall factor of Q_3 may be different.

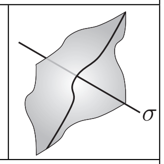
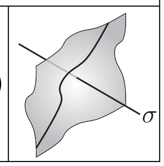
2. The Euler characteristics of the space parametrising the D's may be different.

We define U to be one of

(g) $\mathtt{Sm}_1$ noting that $e(\mathtt{Sm}_1 \cap \{\sigma\}) = -10$ and $e(\mathtt{Sm}_1 \setminus \{\sigma\}) = 10$.

(h) $\mathtt{Sm}_2$ noting that $e(\mathtt{Sm}_2 \cap \{\sigma\}) = -10$ and $e(\mathtt{Sm}_2 \setminus \{\sigma\}) = 10$.

(i) $E_{\pi(y)}$ for $y \in \mathtt{J}$ noting that $e(E_{\pi(y)} \cap \{\sigma\}) = 1$ and $e(E_{\pi(y)} \setminus \{\sigma\}) = -1$.

<table>
<tr><td>$U \cap \{\sigma\}$</td><td colspan="2">$\chi(\mathcal{O}_D) = 0$</td></tr>
<tr><td colspan="2">$e(\eta_\bullet^{-1}(a, x, m_k, \boldsymbol{m})) =$ $Q_1 Q_2 Q_3^n \frac{1}{(\mathsf{V}_{\square\emptyset\emptyset})(\mathsf{V}_{\emptyset\emptyset\emptyset})} \cdot Q_\sigma^a \Phi^{-,\emptyset}(a) \cdot \prod_{i=1}^{12} Q_3^{m_i} \Phi^{\emptyset,\emptyset}(m_i)$</td><td></td></tr>
<tr><td>$U \setminus \{\sigma\}$</td><td colspan="2">$\chi(\mathcal{O}_D) = 0$</td></tr>
<tr><td colspan="2">$e(\eta_\bullet^{-1}(a, x, m_k, \boldsymbol{m})) = Q_1 Q_2 Q_3^n \frac{1}{(\mathsf{V}_{\emptyset\emptyset\emptyset})^2} \cdot Q_\sigma^a \Phi^{\emptyset,\emptyset}(a) \cdot \prod_{i=1}^{12} Q_3^{m_i} \Phi^{\emptyset,\emptyset}(m_i)$</td><td></td></tr>
</table>

From lemmas 6.3.2, 6.3.4 and 6.3.5 we have:

1. $\mathsf{V}_{\emptyset\emptyset\emptyset} = M(p)$
2. $\mathsf{V}_{\square\emptyset\emptyset} = M(p)\frac{1}{1-p}$
3. $\sum_{m \geq 0} Q^m \Phi^{\emptyset,\emptyset}(m) = M(p)^2 \prod_{m>0} (1 + p^m Q)^m$
4. $\sum_{m \geq 0} Q^m \Phi^{-,\emptyset}(m) = M(p)^2 \dfrac{1+Q}{1-p} \prod_{m>0} (1 + p^m Q)^m$

The overall factors of Q_3^n are calculated in 3.3.2 to be:

1. $n = 4$ for (g) and $n = 0$ for (h).
2. If $j(E) = 1728$ and $E \cong \mathbb{C}/i$ then
   - $n = 2$ occurs when D is a translation of the graph $\{(x, \pm ix)\}$.
3. If $j(E) = 0$ and $E \cong \mathbb{C}/\tau$ with $\tau = \frac{1}{2}(1 + i\sqrt{3})$ then
   - $n = 1$ occurs when D is a translation of the graph $\{(x, -\tau x)\}$ or the graph $\{(x, (\tau - 1)x)\}$.
   - $n = 3$ occurs when D is a translation of the graph $\{(x, \tau x)\}$ or the graph $\{(x, (-\tau + 1)x)\}$.

Lastly, in a generic pencil we have $e(\mathtt{J}^0) = 4$ and $e(\mathtt{J}^{1728}) = 6$.

Hence the contribution for parts $(g) - (i)$ is:

$$Q_1 Q_2 Q_\sigma M(p)^{24} \left(\prod_{m>0} (1 + p^m Q_\sigma)^m (1 + p^m Q_3)^{12m} \right) \left(-10 + 8Q_3 + 12Q_3^2 + 8Q_3^3 - 10Q_3^4 \right)$$

5.5.6. Part (j):
In the appendix 6.2.2 we give the following decomposition for $\mathtt{L}$ into parts:

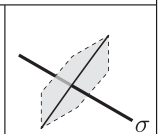
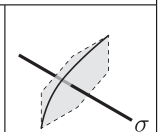
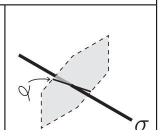
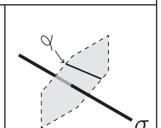
1. $\mathtt{L}^{\sigma}_{(0,0),(\infty,\infty)} \amalg \mathtt{L}^{\emptyset}_{(0,0),(\infty,\infty)} \amalg \mathtt{L}^{\sigma}_{(0,\infty),(\infty,0)} \amalg \mathtt{L}^{\emptyset}_{(0,\infty),(\infty,0)}$
2. $\amalg\, \mathtt{L}^{\sigma}_{(0,0)} \amalg \mathtt{L}^{\emptyset}_{(0,0)} \amalg \mathtt{L}^{\sigma}_{(\infty,\infty)} \amalg \mathtt{L}^{\emptyset}_{(\infty,\infty)}$
3. $\amalg\, \mathtt{L}^{\sigma}_{(0,\infty)} \amalg \mathtt{L}^{\emptyset}_{(0,\infty)} \amalg \mathtt{L}^{\sigma}_{(\infty,0)} \amalg \mathtt{L}^{\emptyset}_{(\infty,0)}$
4. $\amalg\, \mathtt{L}^{\sigma}_{\emptyset} \amalg \mathtt{L}^{\emptyset}_{\emptyset}$.

The Euler characteristics of the parts of this decomposition are computed in 6.2.3 and the overall factors of Q_1, Q_2 and Q_3 are calculated in lemma 3.4.4.

Grouping (1): The following table is the summary of results from 5.2.4 and 5.2.5 for the strata in grouping (1):

$$\mathbb{Z}_{\geq 0} \times U \times \mathrm{Sym}^{\bullet}(\{b^k_{\mathsf{op}}\}) \times \mathrm{Sym}^{\bullet}(B_{\mathsf{op}} \setminus \{b^k_{\mathsf{op}}\}).$$

Note that the vertex is different for $\mathtt{L}_{(0,0),(\infty,\infty)}$ as described in 4.4.9.

$U = \mathtt{L}^{\sigma}_{(0,0),(\infty,\infty)}$	$e(U) = 1$	$\chi(\mathcal{O}_D) = 1$
$e(\eta_{\bullet}^{-1}(a,x,m_k,\boldsymbol{m})) =$ $Q_1Q_2Q_3^2 p \frac{1}{(\mathsf{V}_{\square\emptyset\emptyset})(\mathsf{V}_{\emptyset\emptyset\emptyset})} Q_\sigma^a \Phi^{-,\emptyset}(a) Q_3^{m_k} \Phi^{-,\mid}(m_k) \prod_{\substack{i=1 \\ i \neq j}}^{12} Q_3^{m_i} \Phi^{\emptyset,\emptyset}(m_i)$		
$U = \mathtt{L}^{\emptyset}_{(0,0),(\infty,\infty)}$	$e(U) = -1$	$\chi(\mathcal{O}_D) = 1$
$e(\eta_{\bullet}^{-1}(a,x,m_k,\boldsymbol{m})) =$ $Q_1Q_2Q_3^2 p \frac{1}{(\mathsf{V}_{\emptyset\emptyset\emptyset})^2} Q_\sigma^a \Phi^{\emptyset,\emptyset}(a) Q_3^{m_k} \Phi^{-,\mid}(m_k) \prod_{\substack{i=1 \\ i \neq j}}^{12} Q_3^{m_i} \Phi^{\emptyset,\emptyset}(m_i)$		
$U = \mathtt{L}^{\sigma}_{(0,\infty),(\infty,0)}$	$e(U) = 1$	$\chi(\mathcal{O}_D) = 0$
$e(\eta_{\bullet}^{-1}(a,x,m_k,\boldsymbol{m})) =$ $Q_1Q_2 \frac{1}{(\mathsf{V}_{\square\emptyset\emptyset})(\mathsf{V}_{\emptyset\emptyset\emptyset})} \cdot Q_\sigma^a \Phi^{-,\emptyset}(a) \cdot Q_3^{m_k} \Phi^{+,\emptyset}(m_k) \prod_{\substack{i=1 \\ i \neq j}}^{12} Q_3^{m_i} \Phi^{\emptyset,\emptyset}(m_i)$		
$U = \mathtt{L}^{\emptyset}_{(0,\infty),(\infty,0)}$	$e(U) = -1$	$\chi(\mathcal{O}_D) = 0$
$e(\eta_{\bullet}^{-1}(a,x,m_k,\boldsymbol{m})) =$ $Q_1Q_2 \frac{1}{(\mathsf{V}_{\emptyset\emptyset\emptyset})^2} \cdot Q_\sigma^b \Phi^{\emptyset,\emptyset}(b) \cdot Q_3^{m} \Phi^{+,\emptyset}(m) \prod_{\substack{i=1 \\ i \neq j}}^{12} Q_3^{d_i} \Phi^{\emptyset,\emptyset}(d_i)$		

From lemmas 6.3.2, 6.3.4 and 6.3.5 we have:

1. $\mathsf{V}_{\emptyset\emptyset\emptyset} = M(p)$
2. $\mathsf{V}_{\square\emptyset\emptyset} = M(p)\frac{1}{1-p}$
3. $\sum_{m\geq 0} Q^m \Phi^{\emptyset,\emptyset}(m) = M(p)^2 \prod_{m>0} (1+p^mQ)^m$
4. $\sum_{m\geq 0} Q^m \Phi^{-,\emptyset}(m) = M(p)^2 \dfrac{1+Q}{1-p} \prod_{m>0} (1+p^mQ)^m$
5. $\sum_{m\geq 0} Q^m \Phi^{-,-}(m) = M(p)^2 \frac{1}{p}(\psi_0 + (\psi_1 + 2\psi_0)Q + \psi_0 Q^2) \prod_{m>0} (1+p^mQ)^m$.

6. $\sum_{m\geq 0} Q^m \Phi^{-,|}(m) = M(p)^2(\psi_0 + (2\psi_0+\psi_1)Q + (\psi_0+\psi_1)Q^2) \prod_{m>0}(1+p^mQ)^m.$

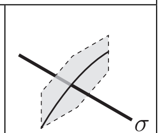
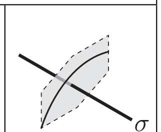
7. $\sum_{m\geq 0} Q^m \Phi^{+,\emptyset}(m) = M(p)^2(\psi_1+\psi_0+(\psi_1+2\psi_0)Q+\psi_0Q^2) \prod_{m>0}(1+p^mQ)^m$

So the after accounting for the 12 singular fibres we have the contribution from grouping (1) as:

$$Q_1Q_2M(p)^{24}\left(\prod_{m>0}(1-p^mQ_\sigma)^m(1-p^mQ_3)^{12m}\right)$$
$$\cdot\, 12Q_\sigma Q_3^2\left((\psi_0+\psi_1)+Q_3(2\psi_0+\psi_1)+2Q_3^2\psi_0+Q_3^3(2\psi_0+\psi_1)+Q_3^4(\psi_0+\psi_1)\right)$$

Grouping (2): We compute the results for $\mathsf{L}_{(0,0)}$ with $\mathsf{L}_{(\infty,\infty)}$ being completely analogous. The following table is the summary of results from 5.2.4 and 5.2.5 for the strata in grouping (2):

$$\mathbb{Z}_{\geq 0}\times U\times \mathrm{Sym}^\bullet(\{b^k_{\mathsf{op}}\})\times \mathrm{Sym}^\bullet(B_{\mathsf{op}}\setminus\{b^k_{\mathsf{op}}\}).$$

$U = \mathsf{L}^\sigma_{(0,0)}$	$e(U) = -1$	$\chi(\mathcal{O}_D) = 1$
$e(\eta_\bullet^{-1}(a,x,m_k,\boldsymbol{m})) =$ $Q_1Q_2Q_3^2p\frac{1}{(\mathsf{V}_{\emptyset\emptyset\emptyset})^2}Q_\sigma^a\Phi^{-,\emptyset}(a)Q_3^{m_k}\Phi^{-,\emptyset}(m_k)\prod_{i=1, i\neq j}^{12}Q_3^{m_i}\Phi^{\emptyset,\emptyset}(m_i)$		
$U = \mathsf{L}^\emptyset_{(0,0)}$	$e(U) = 1$	$\chi(\mathcal{O}_D) = 1$
$e(\eta_\bullet^{-1}(a,x,m_k,\boldsymbol{m})) =$ $Q_1Q_2Q_3^2p\frac{(\mathsf{V}_{\square\emptyset\emptyset})}{(\mathsf{V}_{\emptyset\emptyset\emptyset})^3}Q_\sigma^a\Phi^{\emptyset,\emptyset}(a)Q_3^{m_k}\Phi^{-,\emptyset}(m_k)\prod_{i=1, i\neq j}^{12}Q_3^{m_i}\Phi^{\emptyset,\emptyset}(m_i)$		

From lemmas 6.3.2, 6.3.4 and 6.3.5 we have:

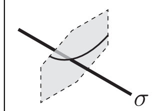
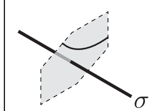
1. $\mathsf{V}_{\emptyset\emptyset\emptyset} = M(p)$
2. $\mathsf{V}_{\square\emptyset\emptyset} = M(p)\frac{1}{1-p}$
3. $\sum_{m\geq 0} Q^m\Phi^{\emptyset,\emptyset}(m) = M(p)^2\prod_{m>0}(1+p^mQ)^m$
4. $\sum_{m\geq 0} Q^m\Phi^{-,\emptyset}(m) = M(p)^2\dfrac{1+Q}{1-p}\prod_{m>0}(1+p^mQ)^m$

Accounting for both $\mathsf{L}_{(0,0)}$ and $\mathsf{L}_{(\infty,\infty)}$, the contribution for grouping (2) is:

$$Q_1Q_2M(p)^{24}\left(\prod_{m>0}(1+p^mQ_\sigma)^m(1+p^mQ_3)^{12m}\right)$$
$$\cdot\,(-24)Q_\sigma Q_3^2\left(\psi_0+Q_3\psi_0\right)$$

Grouping (3): We compute the results for $\mathsf{L}_{(0,\infty)}$ with $\mathsf{L}_{(\infty,0)}$ being completely analogous. The following table is the summary of results from 5.2.4 and 5.2.5 for the strata in grouping (3):

$$\mathbb{Z}_{\geq 0}\times U\times \mathrm{Sym}^\bullet(\{b^k_{\mathsf{op}}\})\times \mathrm{Sym}^\bullet(B_{\mathsf{op}}\setminus\{b^k_{\mathsf{op}}\}).$$

$U = \mathsf{L}^{\sigma}_{(0,\infty)}$	$e(U) = -1$	$\chi(\mathcal{O}_D) = 1$
$e(\eta_\bullet^{-1}(a,x,m_k,\boldsymbol{m})) = Q_1Q_2Q_3p\frac{1}{(\mathsf{V}_{\emptyset\emptyset\emptyset})^2}Q_\sigma^a\Phi^{-,\emptyset}(a)Q_3^{m_k}\Phi^{-,\emptyset}(m_k)\prod_{\substack{i=1\\ i\neq j}}^{12} Q_3^{m_i}\Phi^{\emptyset,\emptyset}(m_i)$		
$U = \mathsf{L}^{\emptyset}_{(0,\infty)}$	$e(U) = 1$	$\chi(\mathcal{O}_D) = 1$
$e(\eta_\bullet^{-1}(a,x,m_k,\boldsymbol{m})) = Q_1Q_2Q_3p\frac{(\mathsf{V}_{\square\emptyset\emptyset})}{(\mathsf{V}_{\emptyset\emptyset\emptyset})^3}Q_\sigma^a\Phi^{\emptyset,\emptyset}(a)Q_3^{m_k}\Phi^{-,\emptyset}(m_k)\prod_{\substack{i=1\\ i\neq j}}^{12} Q_3^{m_i}\Phi^{\emptyset,\emptyset}(m_i)$		

From lemmas 6.3.2, 6.3.4 and 6.3.5 we have:

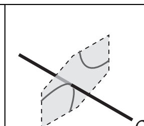
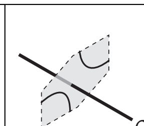
1. $\mathsf{V}_{\emptyset\emptyset\emptyset} = M(p)$
2. $\mathsf{V}_{\square\emptyset\emptyset} = M(p)\frac{1}{1-p}$
3. $\sum_{m\geq 0} Q^m\Phi^{\emptyset,\emptyset}(m) = M(p)^2\prod_{m>0}(1+p^mQ)^m$
4. $\sum_{m\geq 0} Q^m\Phi^{-,\emptyset}(m) = M(p)^2\dfrac{1+Q}{1-p}\prod_{m>0}(1+p^mQ)^m$

Accounting for both $\mathsf{L}_{(0,\infty)}$ and $\mathsf{L}_{(\infty,0)}$, the contribution for grouping (3) is:

$$Q_1Q_2M(p)^{24}\left(\prod_{m>0}(1+p^mQ_\sigma)^m(1+p^mQ_3)^{12m}\right)$$
$$\cdot(-24)Q_\sigma Q_3^2\left(\psi_0+Q_3\psi_0\right)$$

Grouping (4): The following table is the summary of results from 5.2.4 and 5.2.5 for the strata in grouping (4):

$$\mathbb{Z}_{\geq 0}\times U\times \mathrm{Sym}^\bullet(\{b^k_{\mathsf{op}}\})\times\mathrm{Sym}^\bullet(B_{\mathsf{op}}\setminus\{b^k_{\mathsf{op}}\}).$$

$U = \mathsf{L}^{\sigma}_{\emptyset}$	$e(U) = 2$	$\chi(\mathcal{O}_D) = 1$
$e(\eta_\bullet^{-1}(a,x,m_k,\boldsymbol{m})) = Q_1Q_2Q_3^2p\frac{(\mathsf{V}_{\square\emptyset\emptyset})}{(\mathsf{V}_{\emptyset\emptyset\emptyset})^3}Q_\sigma^a\Phi^{-,\emptyset}(a)\prod_{i=1}^{12} Q_3^{m_i}\Phi^{\emptyset,\emptyset}(m_i)$		
$U = \mathsf{L}^{\emptyset}_{\emptyset}$	$e(U) = -2$	$\chi(\mathcal{O}_D) = 1$
$e(\eta_\bullet^{-1}(a,x,m_k,\boldsymbol{m})) = Q_1Q_2Q_3^2p\frac{(\mathsf{V}_{\square\emptyset\emptyset})^2}{(\mathsf{V}_{\emptyset\emptyset\emptyset})^4}Q_\sigma^a\Phi^{\emptyset,\emptyset}(a)\prod_{i=1}^{12} Q_3^{m_i}\Phi^{\emptyset,\emptyset}(m_i)$		

From lemmas 6.3.2, 6.3.4 and 6.3.5 we have:

1. $\mathsf{V}_{\emptyset\emptyset\emptyset} = M(p)$
2. $\mathsf{V}_{\square\emptyset\emptyset} = M(p)\frac{1}{1-p}$
3. $\sum_{m\geq 0} Q^m\Phi^{\emptyset,\emptyset}(m) = M(p)^2\prod_{m>0}(1+p^mQ)^m$

4. $\sum_{m\geq 0} Q^m \Phi^{-,\emptyset}(m) = M(p)^2 \dfrac{1+Q}{1-p} \prod_{m>0} (1+p^m Q)^m$

So the contribution for grouping (4) is:

$$Q_1 Q_2 M(p)^{24} \left(\prod_{m>0} (1+p^m Q_\sigma)^m (1+p^m Q_3)^{12m} \right) \cdot 24 Q_\sigma Q_3^2 \psi_0$$

Combining groupings (1)-(4) we have the overall contribution for part (j) is:

$$Q_1 Q_2 M(p)^{24} \left(\prod_{m>0} (1+p^m Q_\sigma)^m (1+p^m Q_3)^{12m} \right)$$
$$\cdot 12 Q_\sigma \Big((\psi_0 + \psi_1) + Q_3 \psi_1 + Q_3^3 \psi_1 + Q_3^4 (\psi_0 + \psi_1) \Big)$$

6 Appendix

6.1 Connected Invariants and their Partition Functions

For the rank four sub-lattice $\Gamma \subset H_2(X, \mathbb{Z})$ generated by a section and banana curves, we can consider the *connected* unweighted Pandharipande-Thomas invariants. They are defined formally via the following partition function

$$\widehat{Z}_\Gamma^{\mathsf{PT,Con}}(X) := \log \left(\frac{\widehat{Z}_\Gamma(X)}{\widehat{Z}_{(0,\bullet,\bullet)}|_{Q_i=0}} \right).$$

For the partition function in theorem A we consider the first terms of the expansion in Q_σ and Q_1:

$$\begin{aligned} \frac{\widehat{Z}_\Gamma(X)}{\widehat{Z}_{(0,\bullet,\bullet)}|_{Q_i=0}} &= \frac{\widehat{Z}_{(0,\bullet,\bullet)}}{\widehat{Z}_{(0,\bullet,\bullet)}|_{Q_i=0}} + Q_\sigma \frac{\widehat{Z}_{\sigma+(0,\bullet,\bullet)}}{\widehat{Z}_{(0,\bullet,\bullet)}|_{Q_i=0}} + \cdots \\ &= \frac{\widehat{Z}_{(0,\bullet,\bullet)}}{\widehat{Z}_{(0,\bullet,\bullet)}|_{Q_i=0}} \Big(1 + Q_\sigma \frac{\widehat{Z}_{\sigma+(0,\bullet,\bullet)}}{\widehat{Z}_{(0,\bullet,\bullet)}} + \cdots \Big). \end{aligned}$$

So the first terms of the expansion in Q_σ and Q_1 of the connected partition function are:

$$\widehat{Z}_\Gamma^{\mathsf{PT,Con}}(X) = \frac{\widehat{Z}_{(0,\bullet,\bullet)}}{\widehat{Z}_{(0,\bullet,\bullet)}|_{Q_i=0}} - Q_\sigma \frac{\widehat{Z}_{\sigma+(0,\bullet,\bullet)}}{\widehat{Z}_{(0,\bullet,\bullet)}} + \cdots.$$

In particular we have the connected version of $\widehat{Z}_{\sigma+(0,\bullet,\bullet)}$ as:

$$\widehat{Z}^{\mathsf{PT,Con}}_{\sigma+(0,\bullet,\bullet)} = \frac{-1}{(1-p)^2} \prod_{m>0} \frac{1}{(1-Q_2^m Q_3^m)^8 (1-pQ_2^m Q_3^m)^2 (1-p^{-1} Q_2^m Q_3^m)^2},$$

proving corollary B. For the partition function in theorem C we consider the first terms of the expansion in Q_1 and Q_2:

$$\begin{aligned} &\frac{\widehat{Z}_\Gamma(X)}{\widehat{Z}_{(0,\bullet,\bullet)}|_{Q_i=0}} \\ &= \frac{\widehat{Z}_{(0,\bullet,\bullet)}}{\widehat{Z}_{(0,\bullet,\bullet)}|_{Q_i=0}} \Big(1 + Q_1 \frac{\widehat{Z}_{\bullet\sigma+(1,0,\bullet)}}{\widehat{Z}_{\bullet\sigma+(0,0,\bullet)}} + Q_2 \frac{\widehat{Z}_{\bullet\sigma+(0,1,\bullet)}}{\widehat{Z}_{\bullet\sigma+(0,0,\bullet)}} + Q_1 Q_2 \frac{\widehat{Z}_{\bullet\sigma+(1,1,\bullet)}}{\widehat{Z}_{\bullet\sigma+(0,0,\bullet)}} + \cdots \Big). \end{aligned}$$

So the first terms of the expansion in Q_1 and Q_2 of the connected partition function are:

$$\widehat{Z}_{\Gamma}^{\mathsf{PT},\mathsf{Con}}(X) = \log\left(\frac{\widehat{Z}_{(0,\bullet,\bullet)}}{\widehat{Z}_{(0,\bullet,\bullet)}|_{Q_i=0}}\right) - Q_1\frac{\widehat{Z}_{\bullet\sigma+(1,0,\bullet)}}{\widehat{Z}_{\bullet\sigma+(0,0,\bullet)}} - Q_2\frac{\widehat{Z}_{\bullet\sigma+(0,1,\bullet)}}{\widehat{Z}_{\bullet\sigma+(0,0,\bullet)}}$$
$$+ Q_1Q_2\left(\frac{\widehat{Z}_{\bullet\sigma+(1,0,\bullet)}\widehat{Z}_{\bullet\sigma+(0,1,\bullet)}}{\big(\widehat{Z}_{\bullet\sigma+(0,0,\bullet)}\big)^2} - \frac{\widehat{Z}_{\bullet\sigma+(1,1,\bullet)}}{\widehat{Z}_{\bullet\sigma+(0,0,\bullet)}}\right) + \cdots$$

In particular we have the connected version of $\widehat{Z}_{\bullet\sigma+(0,0,\bullet)}$ as

$$\begin{aligned}
\widehat{Z}_{\bullet\sigma+(0,0,\bullet)}^{\mathsf{PT},\mathsf{Con}} &= \log\left(\frac{\widehat{Z}_{\bullet\sigma+(0,0,\bullet)}}{\widehat{Z}_{(0,\bullet,\bullet)}|_{Q_i=0}}\right)\\
&= \log\left(\prod_{m>0}(1+p^mQ_\sigma)^m(1+p^mQ_3)^{12m}\right)\\
&= \sum_{n>0}\frac{p^n}{(1-p^n)^2}\frac{(-Q_\sigma)^n}{n} + \sum_{n>0}12\frac{p^n}{(1-p^n)^2}\frac{(-Q_3)^n}{n}\\
&= \sum_{n>0}\psi_0(p^n)\frac{(-Q_\sigma)^n}{n} + \sum_{n>0}12\psi_0(p^n)\frac{(-Q_3)^n}{n}
\end{aligned}$$

and the connected version of $\widehat{Z}_{\bullet\sigma+(0,1,\bullet)}$ (and also of $\widehat{Z}_{\bullet\sigma+(1,0,\bullet)}$) given by:

$$\widehat{Z}_{\bullet\sigma+(0,1,\bullet)}^{\mathsf{PT},\mathsf{Con}} = -\Big(\big(12\psi_0 + Q_3(24\psi_0+12\psi_1) + Q_3^2(12\psi_0)\big) + Q_\sigma Q_3\big(\psi_0+2\psi_1\big)\Big)$$

and the connected version of $\widehat{Z}_{\bullet\sigma+(1,1,\bullet)}$ given by:

$$\begin{aligned}
&\widehat{Z}_{\bullet\sigma+(1,1,\bullet)}^{\mathsf{PT},\mathsf{Con}}\\
&= \Big(12\big(Q_3^4(2\psi_0+\psi_1) + Q_3^3(8\psi_0+6\psi_1+\psi_2) + Q_3^2(12\psi_0+10\psi_1+2\psi_2)\\
&\qquad + Q_3(8\psi_0+6\psi_1+\psi_2) + (2\psi_0+\psi_1)\big)\Big)\\
&\quad + Q_\sigma\Big(\big(12\psi_0+2\psi_1\big) + Q_3\big(48\psi_0+44\psi_1\big) + Q_3^2\big(216\psi_0+108\psi_1+24\psi_2\big)\\
&\qquad + Q_3^3\big(48\psi_0+44\psi_1\big) + Q_3^4\big(12\psi_0+2\psi_1\big)\Big).
\end{aligned}$$

Corollary D now follows immediately.

6.2 Linear System in $\mathbb{P}^1 \times \mathbb{P}^1$

In this section we consider a stratification of the following linear system in $\mathbb{P}^1 \times \mathbb{P}^1$ with strata determined by the intersections of the associated divisors with a collection of points.

Consider the fibres of the projection maps $\mathrm{pr}_i : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ and a fibre from each f_i. The linear system in $\mathbb{P}^1 \times \mathbb{P}^1$ defined by the sum of a fibre from each map is $|f_1 + f_2| = \mathbb{P}^3$. This is the collection of bi-homogeneous polynomials of degree $(1,1)$:

$$\left\{\ ax_0y_0 + bx_0y_1 + cx_1y_0 + dx_1y_1 = 0\ \middle|\ [a:b:c:d] \in \mathbb{P}^3\ \right\}$$

6.2.1. There are five points in $\mathbb{P}^1 \times \mathbb{P}^1$ that are of interest to us:

$$\sigma = \big([1:1],[1:1]\big) \quad \text{and} \quad \mathsf{P} := \Big\{(0,0),(0,\infty),(\infty,0),(\infty,\infty)\Big\}.$$

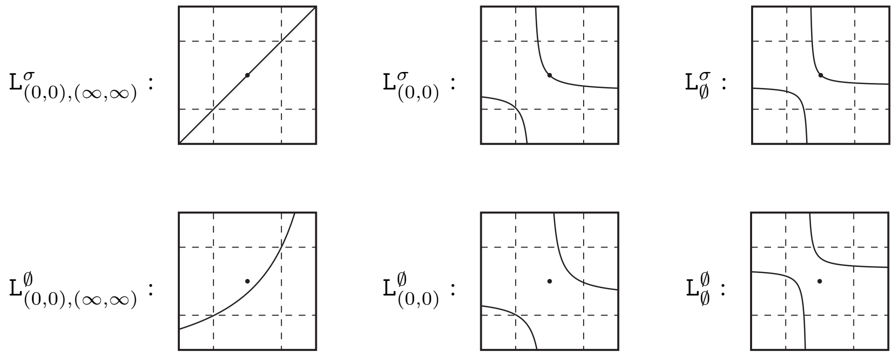


Figure II.12: Depictions of the curves in the decomposition of the linear system $|f_1+f_2|$ on $\mathbb{P}^1 \times \mathbb{P}^1$.

where we have used the standard notation $0 = [0 : 1]$ and $\infty = [1 : 0]$. We will decompose $|f_1 + f_2|$ into strata based on which points the divisor intersects. Consider a divisor $D \in |f_1 + f_2|$. Then D passes through:

1. $(0, 0)$ if and only if $d = 0$;
2. $(0, \infty)$ if and only if $c = 0$;
3. $(\infty, 0)$ if and only if $b = 0$;
4. (∞, ∞) if and only if $a = 0$.

6.2.2. Define the following convenient notation for $y, x \in \mathsf{P}$:

1. $\mathsf{Sing} \subset |f_1 + f_2|$ is the subset of singular divisors.
2. $\mathtt{L}_{\emptyset} \subset \big(|f_1 + f_2| \setminus \mathsf{Sing}\big)$ is the subset of smooth curves not passing through any points of P .
3. $\mathtt{L}_{x} \subset \big(|f_1 + f_2| \setminus \mathsf{Sing}\big)$ is the subset of smooth curve passing through x but no other points of P .
4. $\mathtt{L}_{x,y} \subset \big(|f_1 + f_2| \setminus \mathsf{Sing}\big)$ is the subset of smooth curve passing through x and y but no other points of P.
5. Also let $\mathtt{L}^{\sigma}_{\emptyset}$, $\mathtt{L}^{\sigma}_{x}$ and $\mathtt{L}^{\sigma}_{x,y}$ be subsets of $\mathtt{L}_{\emptyset}$, $\mathtt{L}_{x}$ and $\mathtt{L}_{x,y}$ respectively with the further condition that the curves pass through σ.
6. Let $\mathtt{L}^{\emptyset}_{\emptyset}$, $\mathtt{L}^{\emptyset}_{x}$ and $\mathtt{L}^{\emptyset}_{x,y}$ be the complements of $\mathtt{L}^{\sigma}_{\emptyset}$, $\mathtt{L}^{\sigma}_{x}$ and $\mathtt{L}^{\sigma}_{x,y}$ in $\mathtt{L}_{\emptyset}$, $\mathtt{L}_{x}$ and $\mathtt{L}_{x,y}$ respectively.

With this notation we have the following decomposition of $|F_1 + F_2|$:

$$\begin{aligned}|F_1 + F_2| = \mathsf{Sing} \ &\amalg \ \mathtt{L}_{(0,0),(\infty,\infty)} \ \amalg \ \mathtt{L}_{(0,\infty),(\infty,0)} \\ &\amalg \ \mathtt{L}_{(0,0)} \ \amalg \ \mathtt{L}_{(0,\infty)} \ \amalg \ \mathtt{L}_{(\infty,0)} \ \amalg \ \mathtt{L}_{(\infty,\infty)} \\ &\amalg \ \mathtt{L}_{\emptyset}.\end{aligned}$$

6.2.3. We now consider the strata of this collection and their Euler characteristics:

ban: A curve in $|f_1+f_2|$ is singular if and only if the equation for the curve factorises:

$$ax_0y_0 + bx_0y_1 + cx_1y_0 + dx_1y_1 = (\alpha x_0 + \beta x_1)(\gamma y_0 + \delta y_1) = 0$$

where $[\alpha:\beta], [\gamma:\delta] \in \mathbb{P}^1$. Hence $\mathsf{Sing} \cong \mathbb{P}^1 \times \mathbb{P}^1$ and the Euler characteristic is $e(\mathsf{ban}) = e(|f_1+f_2|) = 4$.

$\mathtt{L}_{x,y}$: We consider for $x = (0,0)$ and $y = (\infty,\infty)$ with the case $(0,\infty)$ and $(\infty,0)$ being completely analogous. The points $[a:b:d:c] \in |f_1+f_2|$ correspond to a curve passing through x and y if and only if $a = d = 0$. Moreover, this is singular when either $b = 0$ or $c = 0$. Hence $\mathtt{L}_{x,y} \cong \mathbb{P}^1 \setminus \{0,\infty\}$ and $e(\mathtt{L}_{x,y}) = 0$.

The set $\mathtt{L}^{\sigma}_{x,y}$ is when $b + c = 0$, which is a point in $\mathbb{P}^1$. So we have $e(\mathtt{L}^{\sigma}_{x,y}) = 1$ and $e(\mathtt{L}^{\emptyset}_{x,y}) = 1$.

$\mathtt{L}_x$: We consider the case $x = (0,0)$ with the other cases being completely analogous. So the subspace of all divisors passing through x is $[a:b:d:c] \in |f_1+f_2|$ where $d = 0$. This is a $\mathbb{P}^2 \subset \mathbb{P}^3$. The subspace where the curve doesn't pass through one of the other points is where $a, b, c \neq 0$ which is given by $\mathbb{C}^* \times \mathbb{C}^* \cong \mathbb{P}^2 \setminus \big(\{a=0\} \cup \{b=0\} \cup \{c=0\}\big)$. None of the equations for these curves factorise since such a factorisation would require either $b = 0$ or $c = 0$. Hence, $\mathtt{L}_x \cong \mathbb{C}^* \times \mathbb{C}^*$ and $e(\mathtt{L}_x) = 0$.

The subset $\mathtt{L}^{\sigma}_x$ is defined by the further condition $a + b + c = 0$ which gives

$$\mathtt{L}^{\sigma}_x = \left\{ [a:b:c] \in \mathbb{P}^2 \;\middle|\; a, b, c \neq 0 \text{ and } a + b = 1 \right\} \cong \mathbb{C}^* \setminus \mathrm{pt}$$

Hence we have the Euler characteristics $e(\mathtt{L}^{\sigma}_x) = -1$ and $e(\mathtt{L}^{\emptyset}_x) = 1$.

$\mathtt{L}_{\emptyset}$: The set of curves not passing through any points of P is given by

$$\left\{ [a:b:c:d] \in |f_1+f_2| \;\middle|\; a, b, c, d \neq 0 \right\} \cong \left\{ b, c, d \in (\mathbb{C}^*)^3 \right\}$$

The singular curves are given by the factorisation condition:

$$x_0y_0 + bx_0y_1 + cx_1y_0 + dx_1y_1 = (x_0 + \beta x_1)(y_0 + \delta y_1)$$

which is the condition that $d = bc$. So the subspace of curves which are singular is $(\mathbb{C}^*)^2 \subset (\mathbb{C}^*)^3$. Hence $\mathtt{L}_{\emptyset} \cong \{(b,c,d) \in (\mathbb{C}^*)^3 | b \neq dc\}$ and $e(\mathtt{L}_{\emptyset}) = 0$.

$\mathtt{L}^{\sigma}_{\emptyset}$ is given by the further condition that $b + c + d = 0$, so we have:

$$\begin{aligned}
\mathtt{L}^{\sigma}_{\emptyset} &\cong \left\{ (b,c,d) \in (\mathbb{C}^*)^3 \;\middle|\; d \neq bc \text{ and } 1 + b + c + d = 0 \right\} \\
&\cong \left\{ (b,c) \in \left(\mathbb{C}^*\right)^2 \;\middle|\; (b+1)(c+1) \neq 0 \text{ and } b + c \neq -1 \right\} \\
&\cong \left\{ (b,c) \in \left(\mathbb{C}^* \setminus \{-1\}\right)^2 \;\middle|\; b + c \neq -1 \right\} \\
&\cong \left(\mathbb{C}^* \setminus \{-1\}\right)^2 - \left(\mathbb{C} \setminus \{2\mathrm{pt}\}\right).
\end{aligned}$$

Hence we have the Euler characteristics $e(\mathtt{L}^{\sigma}_{\emptyset}) = 2$ and $e(\mathtt{L}^{\emptyset}_{\emptyset}) = -2$.

6.3 Topological Vertex Formulas

In this section of the appendix we collect some useful formulas for partition functions involving the topological vertex.
Define the "MacMahon" notation:

$$M(p,Q)=\prod_{m>0}(1-p^mQ)^{-m}$$

and the simpler version $M(p)=M(p,1)$.

Lemma 6.3.1. *We have the equality:*

$$\mathsf{V}_{\lambda\square\square}\mathsf{V}_{\lambda\emptyset\emptyset}=\frac{1}{p}\mathsf{V}_{\lambda\emptyset\emptyset}\mathsf{V}_{\lambda\emptyset\emptyset}+\mathsf{V}_{\lambda\square\emptyset}\mathsf{V}_{\lambda\emptyset\square}$$

Proof. We prove the equivalent equation:

$$\frac{\mathsf{V}_{\square\square\nu}}{\mathsf{V}_{\emptyset\emptyset\nu}}=\frac{1}{p}+\frac{\mathsf{V}_{\square\emptyset\nu}\mathsf{V}_{\emptyset\square\nu}}{(\mathsf{V}_{\emptyset\emptyset\nu})^2}$$

From the definition we have:

$$\begin{aligned}\frac{\mathsf{V}_{\square\square\nu}}{\mathsf{V}_{\emptyset\emptyset\nu}}=&\frac{1}{p}\sum_{\eta\subset\square}S_{\square/\eta}(p^{-\nu-\rho})S_{\square/\eta}(p^{-\nu^t-\rho})\\=&\frac{1}{p}\Big(S_{\square/\emptyset}(p^{-\nu-\rho})S_{\square/\emptyset}(p^{-\nu^t-\rho})+S_{\square/\square}(p^{-\nu-\rho})S_{\square/\square}(p^{-\nu^t-\rho})\Big)\\=&\frac{\mathsf{V}_{\square\emptyset\nu}\mathsf{V}_{\emptyset\square\nu}}{(\mathsf{V}_{\emptyset\emptyset\nu})^2}+\frac{1}{p}\end{aligned}$$

□

Lemma 6.3.2. *We have*

1. $\mathsf{V}_{\emptyset\emptyset\emptyset}=M(p)$
2. $\mathsf{V}_{\square\emptyset\emptyset}=M(p)\frac{1}{1-p}$
3. $\mathsf{V}_{\square\square\emptyset}=M(p)\frac{p^2-p+1}{p(1-p)^2}$
4. $\mathsf{V}_{\square\square\square}=M(p)\frac{p^4-p^3+p^2-p+1}{p^2(1-p)^3}$

Proof. Part (1) is immediate from the definition. For part (2) we have:

$$\begin{aligned}\mathsf{V}_{\square\emptyset\emptyset}&=M(p)p^{-\frac{1}{2}}S_\emptyset(p^{-\rho})\sum_\eta S_{\square/\eta}(p^{-\rho})S_{\emptyset/\eta}(p^{-\rho})\\&=M(p)\frac{1}{1-p}\end{aligned}$$

For part (3) we have:

$$\begin{aligned}\mathsf{V}_{\square\square\emptyset}&=M(p)p^{-1}S_\emptyset(p^{-\rho})\sum_\eta S_{\square/\eta}(p^{-\rho})S_{\square/\eta}(p^{-\rho})\\&=M(p)p^{-1}\Big(S_{\square/\emptyset}(p^{-\rho})S_{\square/\emptyset}(p^{-\rho})+S_{\square/\square}(p^{-\rho})S_{\square/\square}(p^{-\rho})\Big)\\&=M(p)p^{-1}\Big(\frac{p}{(1-p)^2}+1\Big)\\&=M(p)\frac{p^2-p+1}{p(1-p)^2}\end{aligned}$$

Part (4) follows from parts (2) and (3) and lemma 6.3.1:

$$\mathsf{V}_{\square\square\square} = \frac{1}{p}\mathsf{V}_{\square\emptyset\emptyset} + \frac{\mathsf{V}_{\square\square\emptyset}\mathsf{V}_{\square\emptyset\square}}{\mathsf{V}_{\square\emptyset\emptyset}}$$

□

6.3.3. It is shown in [Br, §4.3] the Donaldson-Thomas partition function of this is computed to be:

$$\begin{aligned}&\sum_{\nu,\alpha,\mu} Q_1^{|\nu|}Q_2^{|\alpha|}Q_3^{|\mu|}p^{\frac{1}{2}(\|\nu\|^2+\|\nu^t\|^2+\|\alpha\|^2+\|\alpha^t\|^2+\|\mu\|^2+\|\mu^t\|^2)}(\mathsf{V}_{\nu\mu\alpha}\mathsf{V}_{\nu^t\mu^t\alpha^t}).\\ &= \prod_{d_1,d_2,d_3\geq 0}\prod_k (1-(-Q_1)^{d_1}(-Q_2)^{d_2}(-Q_3)^{d_3}p^k)^{-c(\|\boldsymbol{d}\|,k)}\end{aligned}$$

where $\boldsymbol{d} = (d_1, d_2, d_3)$ and the second product is over $k \in \mathbb{Z}$ unless $\boldsymbol{d} = (0,0,0)$ in which case $k > 0$. The powers $c(\|\boldsymbol{d}\|, k)$ are defined by

$$\sum_{a=-1}^{\infty}\sum_{k\in\mathbb{Z}} c(a,k)Q^a y^k := \frac{\sum_{k\in\mathbb{Z}} Q^{k^2}(-y)^k}{\left(\sum_{k\in\mathbb{Z}+\frac{1}{2}} Q^{2k^2}(-y)^k\right)^2} = \frac{\vartheta_4(2\tau,z)}{\vartheta_1(4\tau,z)^2}$$

and $\|\boldsymbol{d}\| := 2d_1d_2 + 2d_2d_3 + 2d_3d_1 - d_1^2 - d_2^2 - d_3^2$. Also, if we recall the notation that

$$\psi_g := \left(\frac{p}{(1-p)^2}\right)^{1-g}$$

then we have the following corollary.

Corollary 6.3.4. *We have:*

1. $\sum_{\alpha,\mu} Q_2^{|\alpha|}Q_3^{|\mu|}p^{\frac{1}{2}(\|\alpha\|^2+\|\alpha^t\|^2+\|\mu\|^2+\|\mu^t\|^2)}(\mathsf{V}_{\emptyset\mu\alpha}\mathsf{V}_{\emptyset\mu^t\alpha^t})$
$= M(p)^2 \prod_{m>0} \frac{M(Q_2^mQ_3^m, p)^2}{(1-Q_2^mQ_3^m)M(-Q_2^{m-1}Q_3^m, p)M(-Q_2^mQ_3^{m-1}, p)}$
2. $\sum_{\alpha} Q^{|\alpha|}p^{\frac{1}{2}(\|\alpha\|^2+\|\alpha^t\|^2)}(\mathsf{V}_{\emptyset\emptyset\alpha}\mathsf{V}_{\emptyset\emptyset\alpha^t}) = M(p)^2\prod_{m>0}(1+p^mQ)^m$
3. $\sum_{\alpha} Q^{|\alpha|}p^{\frac{1}{2}(\|\alpha\|^2+\|\alpha^t\|^2)+1}(\mathsf{V}_{\square\emptyset\alpha}\mathsf{V}_{\square\emptyset\alpha^t})$
$= M(p)^2(\psi_0 + (\psi_1 + 2\psi_0)Q + \psi_0Q^2)\prod_{m>0}(1+p^mQ)^m$
4. $\sum_{\alpha} Q^{|\alpha|}p^{\frac{1}{2}(\|\alpha\|^2+\|\alpha^t\|^2)+2}(\mathsf{V}_{\square\square\alpha}\mathsf{V}_{\square\square\alpha^t})$
$= M(p)^2\prod_{m>0}(1+p^mQ)^m\Big(Q^4(2\psi_0+\psi_1)+Q^3(8\psi_0+6\psi_1+\psi_2)+Q^2(12\psi_0$
$+10\psi_1+2\psi_2)+Q(8\psi_0+6\psi_1+\psi_2)+(2\psi_0+\psi_1)\Big)$

Proof. These are all coefficients of the partition function in 6.3.3. For example part (3) is the coefficient of $Q_1^1Q_2^0$. □

Lemma 6.3.5. *We have the following equalities:*

1. $\sum_{\alpha} Q^{|\alpha|}p^{\frac{1}{2}(\|\alpha\|^2+\|\alpha^t\|^2)}(\mathsf{V}_{\square\emptyset\alpha}\mathsf{V}_{\emptyset\emptyset\alpha^t}) = M(p)^2\frac{1+Q}{1-p}\prod_{m>0}(1+p^mQ)^m$

2. $\sum_{\alpha} Q^{|\alpha|} p^{\frac{1}{2}(\|\alpha\|^2+\|\alpha^t\|^2)+1}(\mathsf{V}_{\square\square\alpha}\mathsf{V}_{\emptyset\emptyset\alpha^t})$
$= M(p)^2((\psi_0+\psi_1)+(2\psi_0+\psi_1)Q+\psi_0 Q^2)\prod_{m>0}(1+p^m Q)^m$

3. $\sum_{\alpha} Q^{|\alpha|} p^{\frac{1}{2}(\|\alpha\|^2+\|\alpha^t\|^2)+1}(\mathsf{V}_{\square\emptyset\alpha})^2$
$= M(p)^2(\psi_0+(2\psi_0+\psi_1)Q+(\psi_0+\psi_1)Q^2)\prod_{m>0}(1+p^m Q)^m$

Proof. Part (1) is given by:

$$\begin{aligned}
&\sum_{\alpha} Q^{|\alpha|} p^{\frac{1}{2}(\|\alpha\|^2+\|\alpha^t\|^2)}(\mathsf{V}_{\alpha\square\emptyset}\mathsf{V}_{\alpha^t\emptyset\emptyset})\\
&= p^{-\frac{1}{2}} M(p)^2 \sum_{\alpha} Q^{|\alpha|} \sum_{\eta} S_{\alpha^t/\eta}(p^{-\rho}) S_{\square/\eta}(p^{-\rho}) \sum_{\delta} S_{\alpha/\delta}(p^{-\rho}) S_{\emptyset/\delta}(p^{-\rho})\\
&= p^{-\frac{1}{2}} M(p)^2 \sum_{\alpha} Q^{|\alpha|} \Big(S_{\alpha^t}(p^{-\rho}) S_{\square}(p^{-\rho}) + S_{\alpha^t/\square}(p^{-\rho})\Big) S_{\alpha}(p^{-\rho})\\
&= p^{-\frac{1}{2}} M(p)^2 \Big(S_{\square}(p^{-\rho}) \sum_{\alpha\supset\emptyset} S_{\alpha^t/\emptyset}(p^{-\rho}) S_{\alpha/\emptyset}(Qp^{-\rho}) + \sum_{\alpha\supset\square} S_{\alpha^t/\square}(p^{-\rho}) S_{\alpha/\emptyset}(Qp^{-\rho})\Big)
\end{aligned}$$

After applying [Ma, Eqn. 2, pg. 96] the equation becomes

$$\begin{aligned}
&p^{-\frac{1}{2}} M(p)^2 \prod_{i,j>0}(1+p^{i+j}Q)\\
&\Big(S_{\square}(p^{-\rho}) \sum_{\tau\subset\emptyset} S_{\emptyset/\tau}(p^{-\rho}) S_{\emptyset/\tau}(Qp^{-\rho}) + \sum_{\tau\subset\emptyset} S_{\emptyset/\tau^t}(p^{-\rho}) S_{\square/\tau}(Qp^{-\rho})\Big)\\
&= p^{-\frac{1}{2}} M(p)^2 \prod_{m>0}(1+p^m Q)^m (1+Q)\frac{p^{\frac{1}{2}}}{1-p}.
\end{aligned}$$

Part (2) follows from lemma 6.3.1 and corollary 6.3.4:

$$\begin{aligned}
&\sum_{\alpha} Q^{|\alpha|} p^{\frac{1}{2}(\|\alpha\|^2+\|\alpha^t\|^2)+1}(\mathsf{V}_{\square\square\alpha}\mathsf{V}_{\emptyset\emptyset\alpha^t})\\
&= \sum_{\alpha} Q^{|\alpha|} p^{\frac{1}{2}(\|\alpha\|^2+\|\alpha^t\|^2)}(\mathsf{V}_{\emptyset\emptyset\alpha}\mathsf{V}_{\emptyset\emptyset\alpha^t}) + \sum_{\alpha} Q^{|\alpha|} p^{\frac{1}{2}(\|\alpha\|^2+\|\alpha^t\|^2)+1}(\mathsf{V}_{\square\emptyset\alpha}\mathsf{V}_{\square\emptyset\alpha^t}).
\end{aligned}$$

Part (3) is given by:

$$\begin{aligned}
&\sum_{\alpha} Q^{|\alpha|} p^{\frac{1}{2}(\|\alpha\|^2+\|\alpha^t\|^2)+1}(\mathsf{V}_{\square\emptyset\alpha})^2\\
&= \sum_{\alpha} Q^{|\alpha|} p^{\frac{1}{2}(\|\alpha\|^2+1)}(\mathsf{V}_{\alpha\square\emptyset})\, p^{\frac{1}{2}(\|\alpha^t\|^2+1)}(\mathsf{V}_{\emptyset\alpha\square})\\
&= M(p)^2 \sum_{\alpha} Q^{|\alpha|} S_{\square}(p^{-\rho}) \sum_{\delta} S_{\alpha/\delta}(p^{-\square-\rho}) S_{\emptyset/\delta}(p^{-\square-\rho})\\
&\qquad S_{\emptyset}(p^{-\rho}) \sum_{\eta} S_{\alpha^t/\eta}(p^{-\rho}) S_{\square/\eta}(p^{-\rho})\\
&= M(p)^2 S_{\square}(p^{-\rho}) \sum_{\alpha} S_{\alpha}(Qp^{-\square-\rho}) \Big(S_{\alpha^t}(p^{-\rho}) S_{\square}(p^{-\rho}) + S_{\alpha^t/\square}(p^{-\rho})\Big)
\end{aligned}$$

After applying [Ma, Eqn. 2, pg. 96] the equation becomes

$$M(p)^2 S_{\square}(p^{-\rho})(1+Q)\prod_{m>0}(1+Qp^m)^m \Big(S_{\square}(p^{-\rho}) + S_{\square}(p^{-\square-\rho})\Big).$$

The result follows from a quick computation involving $\mathsf{V}_{\emptyset\square\square} = \mathsf{V}_{\square\square\emptyset}$ showing that

$$S_\square(p^{-\rho})S_\square(p^{-\square-\rho}) = S_\square(p^{-\rho})^2 + 1.$$

□

Lemma 6.3.6. *The following are true*

1. $\sum_\alpha Q^{|\alpha|} = \prod_{d>0} \frac{1}{(1-Q^d)}$

2. $\sum_\alpha Q^{|\alpha|} \frac{(\mathsf{V}_{\alpha\square\emptyset})}{(\mathsf{V}_{\alpha\emptyset\emptyset})} = \frac{1}{1-p} \prod_{d>0} \frac{(1-Q^d)}{(1-pQ^d)(1-p^{-1}Q^d)}$

3. $\sum_\alpha p^{\|\alpha\|^2} Q^{|\alpha|} \frac{(\mathsf{V}_{\alpha\alpha^t\emptyset})}{(\mathsf{V}_{\emptyset\emptyset\emptyset})} = \prod_{d>0} \frac{M(p,Q^d)}{(1-Q^d)}$

4. $\sum_\alpha p^{\|\alpha\|^2} Q^{|\alpha|} \frac{(\mathsf{V}_{\alpha\alpha^t\emptyset})(\mathsf{V}_{\alpha\square\emptyset})}{(\mathsf{V}_{\alpha\emptyset\emptyset})(\mathsf{V}_{\emptyset\emptyset\emptyset})} = \frac{1}{1-p} \prod_{d>0} \frac{M(p,Q^d)}{(1-pQ^d)(1-p^{-1}Q^d)}$

Proof. The first is a classical result and the other three are the content of [BKY, Thm. 3]. □

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