Existence and ill-posedness for fluid PDEs with rough data

by

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Existence and ill-posedness for fluid PDEs with rough data

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Abstract

It has been of great interest in recent decades to know whether the incompressible Euler equations are well-posed in the borderline spaces. In order to understand the behavior of solutions in these spaces, the logarithmically regularized 2D Euler equations were introduced. In the borderline Sobolev space, the local well-posedness was proved by Chae-Wu when the regularization is sufficiently strong, while strong ill-posedness of the unregularized case was established by Bourgain-Li. The first part of the dissertation closes the gap between the two results, by establishing the strong ill-posedness in the remaining intermediate regime of the regularization.

The second part of the thesis considers the Cauchy problem of incompressible 3D Navier-Stokes equations with uniformly locally square integrable initial data. If the square integral of the initial datum on a ball vanishes as the ball goes to infinity, the existence of a time-global weak solution has been known. However, such data do not include constants, and the only known global solutions for non-decaying data are either for perturbations of constants or when the velocity gradients are in L^p with finite p. This work presents how to construct global weak solutions for non-decaying initial data whose local oscillations decay, no matter how slowly.

Lay Summary

For any given evolutionary partial differential equations, one of the fundamental questions is the existence and uniqueness of a solution to an equation. Here, the answer depends on the solution space. Once the existence of a solution is guaranteed, the follow-up questions are its lifespan and stability under a small change in initial data. The second one has a significance, especially in the physical application.

In this dissertation, two fluid models are considered to have a deeper understanding of the motion of fluid flows: the logarithmically regularized 2D Euler equations and Naiver-Stokes equations. For the former equations, we explain how to find initial data in some borderline space such that a corresponding unique solution exists in some other space but leaves the borderline space instantaneously. For the latter ones, a construction scheme is introduced for a globally existing solution to the Naiver-Stokes equations with non-decaying initial data.

Preface

This dissertation is based on two different previous works. One is submitted for publication in an academic journal, and the other will be submitted soon.

The material in Chapter 2 is original, independent work by the author, H. Kwon. I was responsible for developing the methodology, the solution, as well as the manuscript composition.

The material in Chapter 3 is based on the paper "Global Navier-Stokes flows for non-decaying initial data with slowly decaying oscillation". This work is done in collaboration with T. Tsai, and its preprint can be found in [30]. I proved the local existence of a local energy solution, defined as in [26], to the Navier-Stokes equations, under a revised scheme of the one in [31]. For the main theorem (see Theorem 3.1.1), based on the scheme suggested by T. Tsai, I laid the groundwork for the global existence under stronger assumptions on initial data, $\lim_{|x_0|\to\infty}\int_{B(x_0,1)}|\nabla v_0|dx \to 0$ and $\sup_{x_0\in\mathbb{R}^3}\int_{B(x_0,1)}|v_0|^3dx \le C$ for some positive constant C. Then, T. Tsai weakened these assumptions to the ones in the theorem. I wrote most part of the manuscript, but subchapter 3.2.4 is written by T. Tsai.

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Acknowledgments

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Chapter 1

Introduction

In the study of partial differential equations (PDEs), one of the first questions that one would ask is the existence of a solution to the equation. In mathematics, the existence is proved by indicating the space where the solution lies, so the answer for the existence depends on the solution space. Once we have the existence, the question on the uniqueness of the solution follows. If the existence and uniqueness of a solution of a time-evolutionary PDE are guaranteed in some space, an initial data in the solution space identifies the solution associated with the data. It is the usual way of indicating an object of discussion in PDEs.

On the other hand, it is also imperative to ask whether a small change in the data leads to a small change in the corresponding solutions. Here, the smallness is measured by the norm defined in the solution space. In the derivation of the equations describing a certain phenomenon, we often assume an ideal situation in which all the minor factors are ignored. In this way, we can simplify the complexity of the setting. However, it could lead a small error, for example in experiments, between the data measured in the real world and the one in the ideal setting. Also, data plugged into a computer program, which usually discretize the continuum, can carry small error with respect to the true one. In order to see that the solution obtained either from experiments or by the assistance of a computer program is close to the mathematical solution, the continuity of the solution in the initial data must be guaranteed.

When we have positive answers to all of the three fundamental questions, the existence, uniqueness, and continuous dependence of solutions in the initial data for an evolutionary PDE, we say that the PDE is well-posed. Then, we analyze the behavior of the solution further by examining the lifespan of the solution, its rate of growth, or its asymptotic behavior.

In this dissertation, we narrow our focus down to the incompressible fluid models, the Navier-Stokes equations, Euler, and equations derived from them.

The Navier-Stokes equation describes the flow of incompressible, homogeneous, viscous fluids, which is given as follows

$$\begin{cases} \partial_t u + (u \cdot \nabla) u + \nabla p = \nu \Delta u \\ \operatorname{div} u = 0 \\ u|_{t=0} = u_0. \end{cases}$$

Here, the unknown functions u and p represent the flow velocity and a pressure, respectively, and a positive number ν is the kinematic viscosity. The second term $(u \cdot \nabla)u$ in the first equation, called the non-linear term, explains convective phenomena of the fluids, while the last term $\nu \Delta u$ describes diffusive phenomena. Also, the divergence-free condition corresponds to the incompressibility of the fluids. Even though the equations were introduced in the 19th Century, we only have a limited understanding of the solutions. One of the difficulties arises from turbulence, which remains as one of the unsolved problems in physics. Mathematicians have made many efforts to understand the motion of the fluids, and the global-in-time existence of a unique smooth solution is one of the seven Millennium Prize Problems stated by the Clay Mathematics Institute.

The Euler equation, on the other hand, describes the motion of incompressible, homogeneous, ideal fluids as follows,

$$\begin{cases} \partial_t u + (u \cdot \nabla) u + \nabla p = 0 \\ \operatorname{div} u = 0 \\ u|_{t=0} = u_0. \end{cases}$$

In other words, we examine the fluid under a zero viscosity hypothesis. However, the Euler equation is often considered to have the same level of the difficulty as the

Navier-Stokes.

In the past decades, the local-in-time well-posedness theories of both Navier-Stokes and Euler equations have been well established for solutions with suitable regularity. However, for solutions starting from rough initial data, the question of the well-posedness has great mathematical challenges. Even in a solution space having some threshold regularity, the solution behavior is unpredictable. Therefore, analysis of such solutions has attracted massive interest among mathematicians. In this direction, two different problems are studied in this dissertation. One is about strong ill-posedness of Euler-type equations in the borderline Sobolev space, and the other is the existence of time-global Navier-Stokes solutions with non-decaying initial data.

The well-posedness of the Euler equations in the borderline spaces is a long-standing open problem. Many efforts have been made in order to solve this problem. One direction is finding a borderline solution space on which the standard energy method works. In particular, in such space, the $L^\infty(\mathbb{R}^n)$-norm of the gradient of the velocity is under control. For example, the local well-posedness of the Euler equations in $\mathbb{R}^n$, $n \geq 2$, is known on the critical Besov spaces $B^{\frac{n}{p}+1}_{p,1}(\mathbb{R}^n)$ for $1 < p \leq \infty$ (See [9, 39–41]). However, the borderline Sobolev space $H^{\frac{n}{2}+1}(\mathbb{R}^n)$ is not included in these critical Besov spaces; because the Sobolev embedding barely fails there and the gradient of velocity is out of control in L^∞ space. To tackle the well-posedness in the critical Sobolev space, regularized Euler equations are introduced [10, 11]. For example, in [11], the logarithmically regularized 2D Euler equations

$$\begin{cases} \partial_t \omega + (u \cdot \nabla)\omega = 0 \\ u = \nabla^\perp \psi, \quad \Delta\psi = T_\gamma \omega \\ \omega|_{t=0} = \omega_0 \end{cases}$$

are studied, where T_γ, $\gamma > 0$, is a Fourier multiplier defined by $T_\gamma = \log^{-\gamma}(10 - \Delta)$. In the absence of the operator T_γ, these equations correspond to the 2D Euler equations for the vorticity $\omega = \partial_1 u_2 - \partial_2 u_1$. Hence, the operator T_γ regularizes the velocity in the Euler vorticity equations at the level of logarithm of the Laplacian. In particular, a large positive number γ, in the index of the operator T_γ, indicates more regular velocity. As a result of the regularization, for $\gamma > \frac{1}{2}$, the local well-posedness in the borderline Sobolev space has been proved, [11]. In the end, Bourgain and Li recently have established that the Euler equation in $\mathbb{R}^n$, $n = 2,3$, is strongly ill-posed in the borderline Sobolev space $H^{\frac{n}{2}+1}(\mathbb{R}^n)$ in [4]. Furthermore, they showed the strong ill-posedness in $C^m(\mathbb{R}^n)$ and $C^{m-1,1}(\mathbb{R}^n)$, for $m \geq 1$ in [5]. Here, the strong ill-posedness holds in the sense that for any given smooth initial data, we can always find a perturbation such that it is arbitrarily small in the borderline space, but the perturbed solution leaves the borderline space instantaneously. Indeed, it breaks the existence of the solution in the borderline space and its continuous dependence at any initial data in a dense subset of the borderline spaces. Even though this long-standing problem is solved in two and three-dimensional spaces, it is unknown whether the strong ill-posedness of the logarithmically regularized 2D Euler equations still holds in the intermediate region $0 < \gamma \leq \frac{1}{2}$. This problem is discussed in Chapter 2 of this thesis. Indeed, the following theorem is established.

Theorem 1.0.1. *The logarithmically regularized 2D Euler equation for* $0 < \gamma \leq \frac{1}{2}$ *is strongly ill-posed in the borderline Sobolev space* $H^1(\mathbb{R}^2) \cap \dot{H}^{-1}(\mathbb{R}^2)$.

More precisely, we construct two types of perturbations; one is compactly supported, and the other is not. The proof is based on the scheme for the strong ill-posedness invented by Bourgain and Li in [4]: the creation of the large Lagrangian deformation, critical norm inflation, and the gluing procedure. However, in order to deal with the regularization, new technical ingredients and a new approach in the gluing procedure in the compact case are required. Combined with the local well-posedness result for $\gamma > \frac{1}{2}$, Theorem 1.0.1 gives a complete solution to the well-posedness problem of the logarithmically regularized 2D Euler equations in the borderline Sobolev space.

The work on the Navier-Stokes equation is about classifying non-decaying initial data for which the existence of a time-global weak solution is guaranteed. Non-decaying Navier-Stokes flows at spatial infinity are widely known in practice such as constant flows or periodic flows. However, most research has been toward decaying flows, because of difficulties arising from the pressure. For initial data with finite kinetic energy (i.e., square integrable), the global existence of a weak solution, called Leray-Hopf solution, is well-known. Unfortunately, such initial data decays at spatial infinity, in the sense that the square integral of the data in a unit ball vanishes as the center of the ball goes to infinity. In order to work on non-decaying data, we consider a local version of the Leray-Hopf solution, called *a local energy solution*, whose square integrals on unit balls are uniformly bounded. Such a solution is physically reasonable because the kinetic energy is rarely concentrated in a small region all of a sudden. The local existence is the only available result for a local energy solution with non-decaying initial data. In Chapter 3 of the dissertation, we consider non-decaying initial data with arbitrarily slow oscillation decay at spatial infinity and find a time-global weak solution with uniformly local kinetic energy. The more precise statement is as follows.

Theorem 1.0.2. *For any divergence-free vector field* $v_0 \in E^2_\sigma(\mathbb{R}^3) + L^3_{\mathrm{uloc},\sigma}(\mathbb{R}^3)$ *satisfying*

$$\lim_{|x_0|\to\infty} \int_{B(x_0,1)} |v_0 - (v_0)_{B(x_0,1)}|\,dx = 0, \tag{1.0.1}$$

we can find a time-global local energy solution (v,p) *to the Navier-Stokes equations* (NS) *in* $\mathbb{R}^3 \times (0,\infty)$*, in the sense of Definition 3.3.1.*

Here, $L^q_{\mathrm{uloc},\sigma}(\mathbb{R}^3)$, $1 \le q < \infty$, is the space of divergence-free vector fields whose $L^q(\mathbb{R}^3)$-norms on unit balls are uniformly bounded, and E^2_σ is the subspace of $L^2_{\mathrm{uloc},\sigma}(\mathbb{R}^3)$ with additional spatial decay assumption. $(v_0)_B$ denotes the average of v_0 on a set B. The oscillation decay assumption is motivated by a recent result [38] for global Navier-Stokes flows with initial data in $L^\infty(\mathbb{R}^3)$ whose gradients are in $L^q(\mathbb{R}^3)$ for some $q > 3$. However, applying the idea of Calderón in [7] to the local energy solution, the result in Theorem 1.0.2 made a great improvement from the most recent result [38].

We close the chapter by giving a brief presentation of subsequent chapters. In Chapter 2, we discuss the strong ill-posedness of the logarithmically regularized 2D Euler equations for $0 < \gamma \le \frac{1}{2}$ in the borderline Sobolev space. The construction of a non-compactly supported perturbation is considered in Section 2.3-2.6, while the compactly supported one is studied in Section 2.7. In Chapter 3, we consider the global existence of local energy solutions to the Navier-Stokes equation in $\mathbb{R}^3$ for non-decaying initial data with slowly decaying oscillations. In Section 3.3, we reprove the local existence of local energy solutions for initial data in $L^2_{\mathrm{uloc},\sigma}(\mathbb{R}^3)$. Then, based on the key estimate in Section 3.4, the global existence is established in Section 3.5.

Chapter 2

Strong ill-posedness of logarithmically regularized 2D Euler equations in the borderline Sobolev spaces

2.1 Introduction

The incompressible Euler equation describes the behaviour of inviscid and volume-preserving fluids. It has two unknown functions u and p which present the fluid velocity and pressure, respectively. The Euler equation for the vorticity $\omega = \nabla^{\perp} \cdot u$ in the domain $\mathbb{R}^2$ is

$$\begin{cases} \partial_t \omega + (u \cdot \nabla)\omega = 0, \qquad (x,t) \in \mathbb{R}^2 \times \mathbb{R} \\ u = \nabla^{\perp} \psi, \quad \Delta \psi = \omega, \\ \omega|_{t=0} = \omega_0, \end{cases} \tag{E}$$

where $\nabla^{\perp} = (-\partial_2, \partial_1)$. It is well-known that the 2D Euler vorticity equation (E) is well-posed globally in time in the Sobolev spaces $W^{s,p}(\mathbb{R}^2) \cap \dot{H}^{-1}(\mathbb{R}^2)$, $p > \frac{2}{s}$, $s \geq 1$ in the literature. (For example, see [12, 37]). In the effort of understanding the behaviour of the solutions in the borderline Sobolev space $H^1(\mathbb{R}^2)\cap\dot{H}^{-1}(\mathbb{R}^2)$, Chae and Wu [11] introduce logarithmically regularized 2D Euler equations

$$\begin{cases}\partial_t\omega+(u\cdot\nabla)\omega=0, & (x,t)\in\mathbb{R}^2\times\mathbb{R}\\ u=\nabla^{\perp}\psi, \quad \Delta\psi=T(|\nabla|)\omega, & \\ \omega|_{t=0}=\omega_0, & \end{cases}$$

where the Fourier multiplier $T(|\nabla|)$ satisfies

$$\int_1^\infty \frac{T^2(r)}{r}dr<+\infty. \tag{2.1.1}$$

Such operator T regularizes the velocity in the Euler vorticity equation (E) at the level of logarithm of the Laplacian. The particular integrability assumption (2.1.1) on T is imposed to guarantee the local well-posedness of the regularized model in the critical Sobolev space. As typical examples of T satisfying (2.1.1), we have

$$\widehat{T_\gamma\omega}(k)=\ln^{-\gamma}(e+|k|^2)\hat{\omega}(k), \quad \widehat{T_\gamma\omega}(k)=\ln^{-\gamma}(e+|k|)\widehat{\omega}(k), \quad \forall k\in\mathbb{R}^2. \tag{2.1.2}$$

for $\gamma>\frac{1}{2}$. In this chapter, we narrow our focus on these examples.

The logarithmically regularized 2D Euler equations

$$\begin{cases}\partial_t\omega+(u\cdot\nabla)\omega=0, & (x,t)\in\mathbb{R}^2\times\mathbb{R}\\ u=\nabla^{\perp}\psi, \quad \Delta\psi=T_\gamma\omega, & \\ \omega|_{t=0}=\omega_0. & \end{cases} \tag{LE}$$

can be seen as a family of transport equations for a scalar function $\omega:\mathbb{R}^2\times\mathbb{R}\to\mathbb{R}$ with the non-local velocity $u:\mathbb{R}^2\times\mathbb{R}\to\mathbb{R}^2$. The Fourier multiplier $T_\gamma=T_\gamma(|\nabla|)$ is defined as in (2.1.2) but we extend the range of γ to $\gamma>0$. By its definition, the operator T_γ in the extended range still plays a role of a logarithmic regularization. In the case of $\gamma=0$, this operator is considered as the identity, so that (LE) with $\gamma=0$ corresponds to the 2D Euler vorticity equation.

Under suitable regularity assumptions on the solution, the transport phenomena of the equation (LE) can be better described by the equivalent form

$$\omega(\phi(x,t),t) = \omega_0(x),$$

with the help of the characteristic $\phi : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ which solves

$$\begin{cases} \partial_t \phi(x,t) = u(\phi(x,t),t) \\ \phi(x,0) = x. \end{cases}$$

The global well-posedness result of the 2D Euler vorticity equation (the case $\gamma = 0$ in (LE)) in the subcritical spaces $W^{s,p}(\mathbb{R}^2) \cap \dot{H}^{-1}(\mathbb{R}^2)$, $p > \frac{2}{s}$ can be extended to that of (LE) for $\gamma \geq 0$. The local well-posedness follows from the usual energy method which requires two key estimates: commutator estimates and Sobolev inequalities. The critical space is determined by the Sobolev inequality

$$\|\nabla u\|_\infty = \left\| D\nabla^\perp \Delta^{-1} T_\gamma \omega \right\|_\infty \lesssim \|\omega\|_{W^{s,p}(\mathbb{R}^2)}, \quad p > \frac{2}{s}.$$

Then, we can extend the local solution to the global one by the Beale-Kato-Majda criterion in [3].

In [11], the regularized velocity $u = \nabla^\perp \Delta^{-1} T_\gamma \omega$ leads to the local well-posedness of (LE) even in the critical space $H^1(\mathbb{R}^2) \cap \dot{H}^{-1}(\mathbb{R}^2)$ for $\gamma > \frac{1}{2}$. Then, for $\gamma > \frac{3}{2}$, the global lifespan of the local-in-time solutions is obtained by Dong-Li [13]. On the other hand, the *strong ill-posedness* of 2D Euler equation ($\gamma = 0$) in the borderline space $H^1(\mathbb{R}^2) \cap \dot{H}^{-1}(\mathbb{R}^2)$ is established by Bourgain-Li [4]. More precisely, they prove that for any given compactly supported smooth initial data, a small perturbation in the borderline space can be always found such that the perturbed solution uniquely exists in some other solution space but leaves the borderline space instantaneously. Later, Elgindi-Jeong [14] gives a very delicate proof (based on Kiselev-Šverák [27]) and show the ill-posedness for some special initial data on the torus $\mathbb{T}^2$.

In this chapter, we prove that the logarithmically regularized 2D Euler equations (LE) for $0 < \gamma < \frac{1}{2}$ are strongly ill-posed in the critical Sobolev space $H^1(\mathbb{R}^2) \cap \dot{H}^{-1}(\mathbb{R}^2)$. The ill-posedness in the strong sense is defined as in [4]. This closes the gap between $\gamma = 0$ (ill-posed) and $\gamma > \frac{1}{2}$ (well-posed), and give complete answers to well/ill-posedness questions of logarithmically regularized 2D Euler equations.

We consider two types of perturbations: one has non-compact support and the other is compactly supported.

Theorem 2.1.1 (Non-compact case)**.** *Let $0 < \gamma \leq \frac{1}{2}$ and $a \in C_c^\infty(\mathbb{R}^2)$. Then, for any $\varepsilon > 0$, we can find a small perturbation $\zeta \in C^\infty(\mathbb{R}^2)$ in the sense of*

$$\|\zeta\|_{\dot{H}^1(\mathbb{R}^2)} + \|\zeta\|_{L^1(\mathbb{R}^2)} + \|\zeta\|_{L^\infty(\mathbb{R}^2)} < \varepsilon$$

such that for the perturbed initial data from a, we have a unique classical solution ω to (LE)

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0, & (x,t) \in \mathbb{R}^2 \times (0,1] \\ u = \nabla^\perp \psi, \quad \Delta \psi = T_\gamma \omega, & \\ \omega|_{t=0} = a + \zeta, & \end{cases}$$

satisfying $\omega(\cdot,t) \in C^\infty(\mathbb{R}^2)$ for $0 \leq t \leq 1$ and $\omega \in C([0,1];L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2))$, but the solution ω leaves the critical Sobolev space instantaneously. i.e., for each $0 < T \leq 1$,

$$\|\omega\|_{L^\infty([0,T];\dot{H}^1(\mathbb{R}^2))} = +\infty.$$

Theorem 2.1.2 (Compact case)**.** *Let $0 < \gamma \leq \frac{1}{2}$ and $a \in C_c^\infty(\mathbb{R}^2)$ which is odd in x_2. Then, for any $\varepsilon > 0$, we can find a small perturbation $\zeta \in C_c(\mathbb{R}^2)$ in the sense of*

$$\|\zeta\|_{\dot{H}^1(\mathbb{R}^2)} + \|\zeta\|_{L^\infty(\mathbb{R}^2)} + \|\zeta\|_{L^1(\mathbb{R}^2)} + \|\zeta\|_{\dot{H}^{-1}(\mathbb{R}^2)} < \varepsilon$$

such that for the perturbed initial data from a, we have a unique solution $\omega : \mathbb{R}^2 \times [0,1] \to \mathbb{R}$ in $C([0,1];C_c(\mathbb{R}^2))$ to (LE)

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0, & (x,t) \in \mathbb{R}^2 \times (0,1] \\ u = \nabla^\perp \psi, \quad \Delta \psi = T_\gamma \omega, & \\ \omega|_{t=0} = a + \zeta, & \end{cases}$$

satisfying L^∞-norm preservation, but the solution ω leaves the critical Sobolev space instantaneously.

The proof follows the outline of the strong ill-posedness scheme for the 2D Euler equations, developed in [4]. It consists of 3 steps: creation of the large Lagrangian deformation, local inflation of the critical norm, and patching argument. More precisely, we first do local construction of the perturbation ζ by finding a family of initial data with the large Lagrangian deformations $D\phi$ —large in the sense of L^∞-norm—in a shorter time. Then, the critical Sobolev norm inflation is induced by the large Lagrangian deformation, so that we get a family of local solutions whose critical norm becomes larger in a shorter time. Finally, we sequentially patch the local solutions in a way of minimizing the interaction between them. This makes the patched solution locally behave like local solutions and hence have the critical norm inflation property.

Difficulties first arise in the local construction of the perturbation. The velocity $u = \nabla^\perp \Delta^{-1} T_\gamma \omega$ in (LE) is more regular than the one in the Euler but the critical space remains same. This makes it more difficult for local solutions to be inflated in the critical norm. Furthermore, one of the main ingredients of getting the larger Lagrangian deformation is missing— a pointwise estimate of the kernels of $D\nabla^\perp \Delta^{-1} T_\gamma$. To solve these issues, we find essentially sharp pointwise lower bounds of the kernel. What's more, we construct local initial data having increasingly higher frequencies. Along these lines, the desired local construction can be achieved. Then, the successful construction of *non-compactly supported* perturbation follows as in [4], placing local solutions far from each other. However, for a *compactly supported* perturbation, the genuine difficulty moves to the patching process of local solutions. The increasingly higher frequencies of local initial data are likely to intensify interaction between local solutions. Moreover, in order to have a compact support, the local solutions must be placed at an infinitesimal distance from each other eventually. This enhances the interaction further. In a worse case, the active interaction can make high frequencies of local solutions canceled out, so the norm inflation of local solutions becomes meaningless for the global one. On the other hand, increasingly higher frequencies of local solutions most likely help to create the norm inflation. In order to see what really happens, a sharp control of the propagation of the current local initial data is required under the presence of the previously chosen ones. This can be done based on a keen analysis of the non-local operators. As a result, it can be shown that the existing local solution does not destroy the norm inflation of the current local solution in a very short time. This approach is different from the one in [4] based on the perturbation argument, and makes the behaviour of the solution more clear.

The chapter consists of the following sections. Based on the creation of the large Lagrangian deformation (Section 2.3), local critical norm inflation (Section 2.4), and patching argument (Section 2.5), we get the proof of Theorem 2.1.1 in Section 2.6. Then, the compact case (Theorem 2.1.2) follows in Section 2.7.

2.2 Notations

- For a point $x \in \mathbb{R}^2$ and a positive real number R, $B(x,R)$ is the Euclidean ball defined by
$$B(x,R) = \{y \in \mathbb{R}^2 : |x-y| < R\}.$$
For a set $A \subset \mathbb{R}^2$ and a positive real number R, a generalized ball $B(A,R)$ means
$$B(A,R) = \{y \in \mathbb{R}^2 : |x-y| < R \text{ for some } x \in A\}.$$
Obviously, when A is a single point set, $A = \{x\}$, we have $B(A,R) = B(x,R)$.

- Given two sets A and B in $\mathbb{R}^2$, the distance between two sets is denoted by
$$\operatorname{dist}(A,B) := \inf\{|x-y| : x \in A \text{ and } y \in B\}.$$

- For any function f on $\mathbb{R}^2$, we denote the Fourier transform of f by
$$\hat{f}(k) = \int_{\mathbb{R}^2} f(x) e^{-ik\cdot x} dx, \quad k \in \mathbb{R}^2,$$
and its inverse Fourier transform by
$$\check{f}(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \hat{f}(k) e^{ik\cdot x} dk.$$

- For any $1 \le p \le \infty$, $\|\cdot\|_{L^p(\mathbb{R}^2)}$ is the usual Lebesgue norm in $\mathbb{R}^2$ with its abbreviation $\|\cdot\|_p$. For any $m \in \mathbb{N}$ and $1 \le p \le \infty$, $\|\cdot\|_{W^{m,p}(\mathbb{R}^2)}$ denotes the usual Sobolev norm in $\mathbb{R}^2$. In the case of $p = 2$, we use $H^m(\mathbb{R}^2) = W^{m,2}(\mathbb{R}^2)$.

The homogeneous Sobolev norm is defined by

$$\|f\|_{\dot{H}^s(\mathbb{R}^2)} = \left(\int_{\mathbb{R}^2} |k|^{2s}|\hat{f}(k)|^2 dk\right)^{\frac{1}{2}}, \quad \forall s \in \mathbb{R},$$

which includes the definition of $\dot{H}^{-1}(\mathbb{R}^2)$-norm. We omit $(\mathbb{R}^2)$ in the expression of Sobolev norms, when the domain of a function is obvious.

- Given two comparable quantities X and Y, the inequality $X \lesssim Y$ stands for $X \le CY$ for some positive constant C. In a similar way, $X \gtrsim Y$ denotes $X \ge CY$ for some $C > 0$. We write $X \sim Y$ when both $X \lesssim Y$ and $Y \lesssim X$ hold. When the constants C in the inequalities depend on some quantities $Z_1, \cdots, Z_n$, we use $\lesssim_{Z_1,\cdots,Z_n}$, $\gtrsim_{Z_1,\cdots,Z_n}$, and $\sim_{Z_1,\cdots,Z_n}$. On the other hand, we say $X \ll Y$ if $X \le \varepsilon Y$ for some sufficiently small $\varepsilon > 0$. Similarly, $X \gg Y$ is defined.

Since we prove the strong ill-posedness of (LE) for each $0 < \gamma \le \frac{1}{2}$, we omit the dependence of γ below if it is not needed. Also, without mentioning, we assume $0 < \gamma \le \frac{1}{2}$.

2.3 Large Lagrangian deformation

In this section, we find a family of initial data which has the large Lagrangian deformation property. As we mentioned, one of the main ingredients is finding a sharp pointwise estimate of the kernel of the operator $-\partial_{12}\Delta^{-1}T_\gamma$ from below. We consider the case $T_\gamma(|\nabla|) = \ln^{-\gamma}(e-\Delta)$ first.

Lemma 2.3.1. *Let $\gamma > 0$ and K_{12} be the kernel of the Fourier multiplier $-\partial_{12}\Delta^{-1}\ln^{-\gamma}(e-\Delta)$. Then, for any $x = (x_1, x_2) \in \mathbb{R}^2$, $x_1 > 0$, $x_2 > 0$, we have*

$$K_{12}(x_1,x_2) \ge \frac{Cx_1x_2}{|x|^4} \ln^{-\gamma}\left(e + \frac{1}{|x|}\right) e^{-|x|^2} \tag{2.3.1}$$

for some positive constant C depending only on γ.

Proof. Using the equalities

$$\int_0^\infty e^{-|k|^2 s}|k|^2 ds = 1, \quad \text{for } k \neq 0,$$
$$\frac{1}{\Gamma(\gamma)}\int_0^\infty e^{-at}t^\gamma \frac{dt}{t} = a^{-\gamma}, \quad \text{for } a > 0,$$

the Fourier transform of K_{12} can be written as

$$\begin{aligned}\widehat{K_{12}}(k) &= -\frac{k_1k_2}{|k|^2}\ln^{-\gamma}(e+|k|^2) = \int_0^\infty e^{-|k|^2 s}(-k_1k_2)\ln^{-\gamma}(e+|k|^2)ds \\ &= \int_0^\infty \frac{1}{\Gamma(\gamma)}\int_0^\infty (e+|k|^2)^{-t}e^{-|k|^2 s}(-k_1k_2)t^\gamma \frac{dt}{t}ds \\ &= \frac{1}{\Gamma(\gamma)}\int_0^\infty \frac{1}{\Gamma(t)}\int_0^\infty e^{-e\beta}\int_0^\infty (-k_1k_2)e^{-|k|^2(\beta+s)}ds\beta^t\frac{d\beta}{\beta}t^\gamma\frac{dt}{t}, \qquad \forall k \neq 0.\end{aligned} \tag{2.3.2}$$

Taking the inverse Fourier transform, the kernel $K_{12}(x)$, for any $x \neq 0$, can be expressed as an integral form:

$$\begin{aligned}K_{12}(x) &= \frac{1}{\Gamma(\gamma)}\int_0^\infty \frac{1}{\Gamma(t)}\int_0^\infty e^{-e\beta}\left(\int_0^\infty \partial_{12}(e^{(s+\beta)\Delta}\delta_0)(x)ds\right)\beta^t\frac{d\beta}{\beta}t^\gamma\frac{dt}{t} \\ &\sim_\gamma x_1x_2\int_0^\infty \frac{1}{\Gamma(t)}\int_0^\infty e^{-e\beta}\left(\int_0^\infty \frac{1}{(s+\beta)^3}e^{-\frac{|x|^2}{4(s+\beta)}}ds\right)\beta^t\frac{d\beta}{\beta}t^\gamma\frac{dt}{t} \\ &= \frac{x_1x_2}{|x|^4}\int_0^\infty \frac{|x|^{2t}}{\Gamma(t)}\int_0^\infty e^{-e|x|^2\tilde{\beta}}\left(\int_0^\infty \frac{1}{(\tilde{s}+\tilde{\beta})^3}e^{-\frac{1}{4(\tilde{s}+\tilde{\beta})}}d\tilde{s}\right)\tilde{\beta}^t\frac{d\tilde{\beta}}{\tilde{\beta}}t^\gamma\frac{dt}{t},\end{aligned}$$

where $e^{t\Delta}\delta_0$ is the usual heat kernel. The last equality easily follows from the change of variables $\beta = |x|^2\tilde{\beta}$ and $s = |x|^2\tilde{s}$.

Then, the integral in $\tilde{s}$ can be computed as

$$\begin{aligned}\int_0^\infty \frac{1}{(\tilde{s}+\tilde{\beta})^3}e^{-\frac{1}{4(\tilde{s}+\tilde{\beta})}}d\tilde{s} &= \int_{\tilde{\beta}}^\infty \frac{1}{\tau^3}e^{-\frac{1}{4\tau}}d\tau = \int_{\tilde{\beta}}^\infty \frac{1}{\tau}(4e^{-\frac{1}{4\tau}})'d\tau \\ &= \frac{4}{\tau}e^{-\frac{1}{4\tau}}\Big|_{\tilde{\beta}}^\infty + \int_{\tilde{\beta}}^\infty \frac{4}{\tau^2}e^{-\frac{1}{4\tau}}d\tau \\ &= 16\left(1-e^{-\frac{1}{4\tilde{\beta}}}-\frac{1}{4\tilde{\beta}}e^{-\frac{1}{4\tilde{\beta}}}\right),\end{aligned} \tag{2.3.3}$$

so that we simplify the integral form as

$$K_{12}(x) \sim_\gamma \frac{x_1 x_2}{|x|^4} \int_0^\infty \frac{|x|^{2t}}{\Gamma(t)} \int_0^\infty e^{-e|x|^2\tilde{\beta}} \left(1 - e^{-\frac{1}{4\tilde{\beta}}} - \frac{1}{4\tilde{\beta}} e^{-\frac{1}{4\tilde{\beta}}}\right) \tilde{\beta}^t \frac{d\tilde{\beta}}{\tilde{\beta}} t^\gamma \frac{dt}{t}, \quad \forall x \neq 0.$$

Now, for each $x = (x_1, x_2)$ with $x_1 > 0$ and $x_2 > 0$, we find the lower bound of the kernel. Indeed, the desired lower bound (2.3.1) follows from

$$\begin{aligned} &\int_0^\infty \frac{|x|^{2t}}{\Gamma(t)} \int_0^\infty e^{-e|x|^2\tilde{\beta}} \left(1 - e^{-\frac{1}{4\tilde{\beta}}} - \frac{1}{4\tilde{\beta}} e^{-\frac{1}{4\tilde{\beta}}}\right) \tilde{\beta}^t \frac{d\tilde{\beta}}{\tilde{\beta}} t^\gamma \frac{dt}{t} \\ &\quad \gtrsim e^{-|x|^2} \int_0^1 \frac{|x|^{2t}}{\Gamma(t)} \int_0^{\frac{1}{e}} \tilde{\beta}^t \frac{d\tilde{\beta}}{\tilde{\beta}} t^\gamma \frac{dt}{t} \\ &\quad \gtrsim e^{-|x|^2} \int_0^1 \frac{|x|^{2t}}{t\Gamma(t)} t^\gamma \frac{dt}{t} \gtrsim e^{-|x|^2} \int_0^1 |x|^{2t} t^\gamma \frac{dt}{t} \\ &\quad \gtrsim_\gamma \ln^{-\gamma}\left(e + \frac{1}{|x|}\right) e^{-|x|^2}. \end{aligned}$$

□

Now, we consider the case of $T_\gamma(|\nabla|) = \ln^{-\gamma}(e + |\nabla|)$. To express the corresponding kernel as an integral form, we need the following identity.

Lemma 2.3.2. *(Subordination identity) For any $r \geq 0$, we have*

$$e^{-r} = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-\tau} e^{-\frac{r^2}{4\tau}} \tau^{-\frac{1}{2}} d\tau.$$

Proof. By using Fourier transform, it is easy to see

$$e^{-r} = \frac{1}{\pi} \int_{-\infty}^\infty \frac{1}{1+\theta^2} e^{i\theta r} d\theta, \quad \forall r \geq 0.$$

Since we can write

$$\frac{1}{1+\theta^2} = \int_0^\infty e^{-\tau} e^{-\tau\theta^2} d\tau,$$

the result follows from interchanging the $d\theta - d\tau$ integral. □

Lemma 2.3.3. *Let $\gamma > 0$ and $\tilde{K}_{12}$ be the kernel of the multiplier $-\partial_{12}\Delta^{-1}\ln^{-\gamma}(e+$*

$|\nabla|$). *Then, for any* $x = (x_1, x_2) \in \mathbb{R}^2$, $x_1 > 0$, $x_2 > 0$, *we have*

$$\tilde{K}_{12}(x_1, x_2) \geq \frac{Cx_1x_2}{|x|^4} \ln^{-\gamma}\left(e + \frac{1}{|x|}\right) e^{-|x|^2} \tag{2.3.4}$$

for some positive constant C depending only on $\gamma > 0$.

Proof. As we did in Lemma 2.3.1, the Fourier transform of $\tilde{K}_{12}$ can be expressed as follows:

$$\begin{aligned}\widehat{\tilde{K}_{12}}(k) &= -\frac{k_1k_2}{|k|^2} \ln^{-\gamma}(e + |k|) \\ &= \frac{1}{\Gamma(\gamma)} \int_0^\infty \frac{1}{\Gamma(t)} \int_0^\infty e^{-e\beta} \int_0^\infty (-k_1k_2) e^{-k|\beta|} e^{-|k|^2 s} ds \beta^t \frac{d\beta}{\beta} t^\gamma \frac{dt}{t}, \quad \forall k \neq 0.\end{aligned} \tag{2.3.5}$$

Using the identity in Lemma 2.3.2, for $\beta \geq 0$ we have

$$e^{-|k|\beta} = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-\tau} e^{-\frac{|k|^2\beta^2}{4\tau}} \tau^{-\frac{1}{2}} d\tau, \tag{2.3.6}$$

so that the kernel can be written as an integral form: for any $x \neq 0$,

$$\begin{aligned}\tilde{K}_{12}(x) &= \frac{1}{\sqrt{\pi}\Gamma(\gamma)} \int_0^\infty \frac{1}{\Gamma(t)} \int_0^\infty e^{-e\beta} \int_0^\infty \int_0^\infty e^{-\tau} (\partial_{12} e^{\left(\frac{\beta^2}{4\tau}+s\right)\Delta} \delta_0)(x) \tau^{-\frac{1}{2}} d\tau ds \beta^t \frac{d\beta}{\beta} t^\gamma \frac{dt}{t} \\ &\sim_\gamma x_1x_2 \int_0^\infty \frac{1}{\Gamma(t)} \int_0^\infty e^{-e\beta} \int_0^\infty e^{-\tau} \int_0^\infty \frac{1}{\left(\frac{\beta^2}{4\tau}+s\right)^3} e^{-\frac{|x|^2}{4\left(\frac{\beta^2}{4\tau}+s\right)}} ds \tau^{-\frac{1}{2}} d\tau \beta^t \frac{d\beta}{\beta} t^\gamma \frac{dt}{t} \\ &= \frac{x_1x_2}{|x|^4} \int_0^\infty \frac{|x|^t}{\Gamma(t)} \int_0^\infty e^{-e|x|\tilde{\beta}} \int_0^\infty e^{-\tau} \int_0^\infty \frac{1}{\left(\frac{\tilde{\beta}^2}{4\tau}+\tilde{s}\right)^3} e^{-\frac{1}{4\left(\frac{\tilde{\beta}^2}{4\tau}+\tilde{s}\right)}} d\tilde{s} \tau^{-\frac{1}{2}} d\tau \tilde{\beta}^t \frac{d\tilde{\beta}}{\tilde{\beta}} t^\gamma \frac{dt}{t}.\end{aligned}$$

In the last equality, we do the change of variables $\beta = |x|\tilde{\beta}$ and $s = |x|^2\tilde{s}$.

The integral in $\tilde{s}$ can be simplified as

$$\int_0^\infty \frac{1}{\left(\frac{\tilde{\beta}^2}{4\tau}+\tilde{s}\right)^3} e^{-\frac{1}{4\left(\frac{\tilde{\beta}^2}{4\tau}+\tilde{s}\right)}} d\tilde{s} = 16\left(1 - e^{-\frac{\tau}{\tilde{\beta}^2}} - \frac{\tau}{\tilde{\beta}^2} e^{-\frac{\tau}{\tilde{\beta}^2}}\right), \tag{2.3.7}$$

and the integral form also becomes simple,

$$\tilde{K}_{12}(x) \sim_\gamma \frac{x_1 x_2}{|x|^4} \int_0^\infty \frac{|x|^t}{\Gamma(t)} \int_0^\infty e^{-e|x|\tilde{\beta}} \left(\int_0^\infty e^{-\tau} (1 - e^{-\frac{\tau}{\tilde{\beta}^2}} - \frac{\tau}{\tilde{\beta}^2} e^{-\frac{\tau}{\tilde{\beta}^2}}) \tau^{-\frac{1}{2}} d\tau \right) \tilde{\beta}^t \frac{d\tilde{\beta}}{\tilde{\beta}} t^\gamma \frac{dt}{t}.$$

To get a lower bound, we first consider the integral in τ and $\tilde{\beta}$:

$$\begin{aligned} &\int_0^\infty e^{-e|x|\tilde{\beta}} \left(\int_0^\infty e^{-\tau} (1 - e^{-\frac{\tau}{\tilde{\beta}^2}} - \frac{\tau}{\tilde{\beta}^2} e^{-\frac{\tau}{\tilde{\beta}^2}}) \tau^{-\frac{1}{2}} d\tau \right) \tilde{\beta}^t \frac{d\tilde{\beta}}{\tilde{\beta}} \\ &\qquad \gtrsim \int_0^\infty e^{-\tau} e^{-\sqrt{e\tau}|x|} \left(\int_0^{\sqrt{\frac{\tau}{e}}} \tilde{\beta}^t \frac{d\tilde{\beta}}{\tilde{\beta}} \right) \tau^{-\frac{1}{2}} d\tau \\ &\qquad \geq \frac{1}{t\sqrt{e}^t} e^{-\frac{|x|}{e}} \int_0^{\frac{1}{e^3}} e^{-\tau} \tau^{\frac{t-1}{2}} d\tau \gtrsim \frac{1}{t(t+1)e^{2t}} e^{-\frac{|x|}{e}}, \quad \forall x \neq 0, t > 0. \end{aligned}$$

Then, for each $x = (x_1, x_2) \in \mathbb{R}^2$ with $x_1 > 0$ and $x_2 > 0$, the desired lower bound (2.3.4) of the kernel follows from

$$\begin{aligned} &\int_0^\infty \frac{|x|^t}{\Gamma(t)} \int_0^\infty e^{-e|x|\tilde{\beta}} \left(\int_0^\infty e^{-\tau} (1 - e^{-\frac{\tau}{\tilde{\beta}^2}} - \frac{\tau}{\tilde{\beta}^2} e^{-\frac{\tau}{\tilde{\beta}^2}}) \tau^{-\frac{1}{2}} d\tau \right) \tilde{\beta}^t \frac{d\tilde{\beta}}{\tilde{\beta}} t^\gamma \frac{dt}{t} \\ &\qquad \gtrsim e^{-\frac{|x|}{e}} \int_0^1 \frac{|x|^t}{t\Gamma(t)} \frac{1}{(t+1)e^{2t}} t^\gamma \frac{dt}{t} \gtrsim_\gamma \ln^{-\gamma}\left(e + \frac{1}{|x|}\right) e^{-|x|^2}. \end{aligned}$$

□

Remark 2.3.1. By Lemma 2.3.1 and Lemma 2.3.3, we can see that the kernels of $-\partial_{12}\Delta^{-1}T_\gamma$ for both $T_\gamma = \ln^{-\gamma}(e - \Delta)$ and $T_\gamma = \ln^{-\gamma}(e + |\nabla|)$ have the same lower bound. Therefore, we use the combined notations T_γ and its kernel K for both cases from now on.

Now, we are ready to estimate the Lagrangian deformation.

Proposition 2.3.4. *Let $\gamma > 0$. Suppose that a function $g \in C_c^\infty(\mathbb{R}^2)$ satisfies the following conditions.*

(i) g is odd in x_1 and x_2.

(ii) $g(x_1, x_2) \geq 0$ on $\{x_1 \geq 0, x_2 \geq 0\}$.

(iii)

$$G \equiv \int_{x_1>0,x_2>0} g(x)\frac{x_1x_2}{|x|^4}\ln^{-\gamma}\left(e+\frac{1}{|x|}\right)e^{-|x|^4}dx > 0.$$

Let ϕ be the characteristic line defined by

$$\begin{cases}\partial_t\phi(x,t) = \nabla^{\perp}\Delta^{-1}T_\gamma\omega(\phi(x,t),t)\\ \phi(x,0) = x,\end{cases}$$

where ω is a smooth solution to (LE) *for the initial data $\omega_0 = g$. Then, the Lagrangian deformation $D\phi$ satisfies*

$$\int_0^t e^{-\|D\phi(\cdot,\tau)\|_\infty^4}d\tau \leq \frac{1}{CG}\ln(1+CGt), \quad \forall t\geq 0 \tag{2.3.8}$$

for some positive constant $C = C(\gamma)$. In particular, we have

$$\max_{0\leq\tau\leq t}\|D\phi(\cdot,\tau)\|_\infty \geq \ln^{\frac{1}{4}}\left(\frac{CGt}{\ln(1+CGt)}\right), \quad \forall t>0. \tag{2.3.9}$$

Proof. Using the parity of g, it can be easily checked that ω is odd in x_1 and x_2, and hence $\phi(x,t) = (\phi_1(x_1,x_2,t),\phi_2(x_1,x_2,t))$ satisfies

$$\begin{aligned}\phi_1(0,x_2,t)\equiv 0, \quad \phi_2(x_1,0,t)\equiv 0 \qquad &\forall x_1\in\mathbb{R}, x_2\in\mathbb{R},\\ \phi(0,t)\equiv 0.&\end{aligned} \tag{2.3.10}$$

Also, the Frechet derivative $[Du(0,t)]_{ij} = \partial_j u_i(0,t)$ of $u = \nabla^{\perp}\Delta^{-1}T_\gamma\omega$ at $x=0$ takes the form

$$Du(0,t) = \begin{pmatrix}\lambda(t) & 0\\ 0 & -\lambda(t)\end{pmatrix},$$

where $\lambda(t) = -\partial_{12}\Delta^{-1}T_\gamma w(0,t)$. Then, this implies

$$(D\phi)(0,t) = \begin{pmatrix}\exp\left(\int_0^t\lambda(\tau)d\tau\right) & 0\\ 0 & \exp\left(-\int_0^t\lambda(\tau)d\tau\right)\end{pmatrix}.$$

On the other hand, by (2.3.10) and the sign preservation property of ϕ_1 and ϕ_2, we obtain for any $x_1 \geq 0$, $x_2 \geq 0$, and $t \geq 0$,

$$\begin{aligned} &\frac{1}{\|D\phi(\cdot,t)\|_\infty}\phi_1(x_1,x_2,t) \leq x_1 \leq \phi_1(x_1,x_2,t)\,\|D\phi(\cdot,t)\|_\infty, \\ &\frac{1}{\|D\phi(\cdot,t)\|_\infty}\phi_2(x_1,x_2,t) \leq x_2 \leq \phi_2(x_1,x_2,t)\,\|D\phi(\cdot,t)\|_\infty. \end{aligned} \tag{2.3.11}$$

Thus, for any $x_1 > 0$, $x_2 > 0$, and $t \geq 0$,

$$\frac{\phi_1\phi_2}{\phi_1^2+\phi_2^2} = \frac{1}{\frac{\phi_1}{\phi_2}+\frac{\phi_2}{\phi_1}} \geq \frac{1}{\|D\phi\|_\infty^2}\frac{x_1x_2}{|x|^2}.$$

Recall that we denote the kernel of the operator $-\partial_{12}\Delta^{-1}T_\gamma$ by K. By Lemma 2.3.1 and Lemma 2.3.3, for any $x = (x_1,x_2)$ with $x_1 > 0$ and $x_2 > 0$, and $t \geq 0$,

$$\begin{aligned} K(\phi(x,t)) &\gtrsim_\gamma \left(\frac{\phi_1\phi_2}{|\phi|^2}\right)\frac{1}{|\phi|^2}\ln^{-\gamma}\left(e+\frac{1}{|\phi|}\right)e^{-|\phi|^2} \\ &\gtrsim \frac{1}{\|D\phi\|_\infty^4}\frac{x_1x_2}{|x|^4}\ln^{-\gamma}\left(e+\frac{\|D\phi\|_\infty}{|x|}\right)e^{-\|D\phi\|_\infty^2|x|^2} \\ &\gtrsim \frac{1}{\|D\phi\|_\infty^4}\frac{x_1x_2}{|x|^4}\ln^{-\gamma}\left(e+\frac{1}{|x|}\right)\left(1+\ln\left(1+\|D\phi\|_\infty\right)\right)^{-\gamma}e^{-\frac{1}{4}\|D\phi\|_\infty^4}e^{-|x|^4} \\ &\gtrsim_\gamma e^{-\|D\phi(\cdot,t)\|_\infty^4}\frac{x_1x_2}{|x|^4}\ln^{-\gamma}\left(e+\frac{1}{|x|}\right)e^{-|x|^4}. \end{aligned}$$

Now, we estimate $\lambda(t)$ from below

$$\begin{aligned} \lambda(t) &= \int_{\mathbb{R}^2} K(y)\omega(y,t)dy = 4\int_{y_1>0,y_2>0} K(y)\omega(y,t)dy \\ &= 4\int_{x_1>0,x_2>0} K(\phi(x,t))g(x)dx \\ &\gtrsim_\gamma e^{-\|D\phi(\cdot,t)\|_\infty^4}\int_{x_1>0,x_2>0} g(x)\frac{x_1x_2}{|x|^4}\ln^{-\gamma}\left(e+\frac{1}{|x|}\right)e^{-|x|^4}dx \\ &= e^{-\|D\phi(\cdot,t)\|_\infty^4}G. \end{aligned}$$

Then, since

$$\|D\phi(\cdot,t)\|_\infty \geq |D\phi(0,t)| \geq \exp\left(\int_0^t \lambda(\tau)d\tau\right), \quad \forall t \geq 0$$

where $|\cdot|$ is the usual matrix norm, we have a positive constant $C > 0$ depending only on γ such that

$$\|D\phi(\cdot,t)\|_\infty \geq \exp\left(\frac{1}{4}CG\int_0^t e^{-\|D\phi(\tau)\|_\infty^4}d\tau\right), \quad \forall t \geq 0.$$

This implies that

$$\begin{aligned}\frac{d}{dt}\exp\left(CG\int_0^t e^{-\|D\phi(\tau)\|_\infty^4}d\tau\right) &= \exp\left(CG\int_0^t e^{-\|D\phi(\tau)\|_\infty^4}d\tau\right)CGe^{-\|D\phi(t)\|_\infty^4}\\ &\leq CG\|D\phi(\tau)\|_\infty^4 e^{-\|D\phi(\tau)\|_\infty^4} \leq CG.\end{aligned}$$

Therefore, we obtain

$$\exp\left(CG\int_0^t e^{-\|D\phi(\tau)\|_\infty^4}d\tau\right) \leq 1 + CGt.$$

The inequalities (2.3.8) and (2.3.9) then follow easily. □

Remark 2.3.2. By a slight modification of the proof, we can restrict the region where the large Lagrangian deformation occurs;

$$\max_{0\leq\tau\leq t}\|D\phi(\cdot,\tau)\|_{L^\infty(B(0,R))} \geq \ln^{\frac{1}{4}}\left(\frac{CGt}{\ln(1+CGt)}\right), \quad \forall 0 < t \leq 1, \qquad (2.3.12)$$

if $R > 0$ satisfies

$$\text{supp}(g) \subset B(0,R) \quad \text{and} \quad \phi^{-1}(B_g,t) \subset B(0,R)$$

for all $0 \leq t \leq 1$, where $B_g = B(0,R_g)$ is the smallest ball containing $\bigcup_{0\leq t\leq 1}\text{supp}(\omega(\cdot,t))$. Indeed, if x is in $\text{supp}(g)$, then $\phi(x,t) \subset \text{supp}(\omega(\cdot,t))$ and $|\phi(x,t)| \leq R_g$ when $0 \leq t \leq 1$. This implies that for $0 \leq t \leq 1$

$$\left\|D(\phi^{-1})(\cdot,t)\right\|_{L^\infty(B_g)} = \left\|(D\phi)^{-1}(\phi^{-1}(\cdot,t),t)\right\|_{L^\infty(B_g)} \leq \|D\phi(\cdot,t)\|_{L^\infty(B(0,R))}.$$

In the inequality, we use $|\det(D\phi(\cdot,t))| = 1$ for any $t \geq 0$. Then, a modification of

(2.3.11) holds; for $x=(x_1,x_2)\in \text{supp}(g)$, $x_1\geq 0$, $x_2\geq 0$, and $0\leq t\leq 1$, we have

$$\frac{1}{\|D\phi(\cdot,t)\|_{L^\infty(B(0,R))}}\phi_1(x_1,x_2,t)\leq x_1\leq \phi_1(x_1,x_2,t)\,\|D\phi(\cdot,t)\|_{L^\infty(B(0,R))},$$
$$\frac{1}{\|D\phi(\cdot,t)\|_{L^\infty(B(0,R))}}\phi_2(x_1,x_2,t)\leq x_2\leq \phi_2(x_1,x_2,t)\,\|D\phi(\cdot,t)\|_{L^\infty(B(0,R))}.$$

The rest of the proof is almost identical.

2.4 Local critical Sobolev norm inflation

In this section, we show that the inflation of the critical Sobolev norm can be induced from the largeness of the Lagrangian deformation. Then, based on this, we construct a family of local solutions whose critical norm gets larger in a shorter time, while the critical norm of initial data gets smaller.

We first recall Lemma 4.1 in [4].

Lemma 2.4.1. *Suppose $u=u(x,t)$ and $v=v(x,t)$ are smooth vector fields on $\mathbb{R}^2\times\mathbb{R}$. Let $\phi:\mathbb{R}^2\times\mathbb{R}\to\mathbb{R}^2$ and $\tilde{\phi}:\mathbb{R}^2\times\mathbb{R}\to\mathbb{R}^2$ be the solutions to*

$$\begin{cases}\partial_t\phi(x,t)=u(\phi(x,t),t)\\ \phi(x,0)=x\end{cases}$$

and

$$\begin{cases}\partial_t\tilde{\phi}(x,t)=u(\tilde{\phi}(x,t),t)+v(\tilde{\phi}(x,t),t)\\ \tilde{\phi}(x,0)=x.\end{cases}$$

Then, we have positive constants C and C_1 satisfying

$$\max_{0\leq t\leq 1}\left(\left\|(\tilde{\phi}-\phi)(\cdot,t)\right\|_\infty+\left\|(D\tilde{\phi}-D\phi)(\cdot,t)\right\|_\infty\right)$$
$$\leq C\max_{0\leq t\leq 1}\|v(\cdot,t)\|_{W^{1,\infty}}\cdot\exp\left(C_1\max_{0\leq t\leq 1}\|Dv(\cdot,t)\|_\infty\right),$$

where C depends on $\left\|D^2u(\cdot,t)\right\|_{L^\infty([0,1]\times\mathbb{R}^2)}$ and $\|Du(\cdot,t)\|_{L^\infty([0,1]\times\mathbb{R}^2)}$, and C_1 is an absolute constant.

The following is the main proposition in this section.

Proposition 2.4.2. *Suppose that* ω *is a smooth solution to* (LE) *with the initial data* ω_0 *and its velocity* $u = -\nabla^{\perp}\Delta^{-1}T_\gamma\omega$, $\gamma > 0$, *and satisfies the following properties.*

(i) $\|\omega_0\|_\infty + \|\omega_0\|_1 + \|\omega_0\|_{\dot{H}^{-1}} < \infty$.

(ii) There exists $R_0 > 0$ *such that*

$$\mathrm{supp}(\omega_0) \subset B(0,R_0)$$

and the characteristic line ϕ, *i.e., the solution to*

$$\begin{cases} \partial_t \phi(x,t) = u(\phi(x,t),t) & \mathbb{R}^2 \times (0,\infty) \\ \phi(x,0) = x & \mathbb{R}^2, \end{cases}$$

satisfies

$$\|(D\phi)(\cdot,t_0)\|_{L^\infty(B(0,R_0))} > L \tag{2.4.1}$$

for some $0 < t_0 \leq 1$ *and* $L > 8^9 \cdot 10^6$.

Then, we can construct a new smooth solution $\tilde{\omega}$ *to* (LE) *for a new initial data* $\tilde{\omega}_0$ *which satisfies the following conditions.*

(i) The size of the new initial data is controlled by that of the original one,

$$\|\tilde{\omega}_0\|_{\dot{H}^{-1}} \leq 2\|\omega_0\|_{\dot{H}^{-1}} \tag{2.4.2}$$

$$\|\tilde{\omega}_0\|_1 \leq 2\|\omega_0\|_1, \quad \|\tilde{\omega}_0\|_\infty \leq 2\|\omega_0\|_\infty, \tag{2.4.3}$$

$$\|\tilde{\omega}_0\|_{\dot{H}^1} \leq \|\omega_0\|_{\dot{H}^1} + L^{-\frac{1}{2}}. \tag{2.4.4}$$

(ii) The new initial data is compactly supported,

$$\mathrm{supp}(\tilde{\omega}_0) \subset B(0,R_0). \tag{2.4.5}$$

(iii) The large Lagrangian deformation at t_0 *induces* $\dot{H}^1$*-norm inflation:*

$$\|\tilde{\omega}(\cdot,t_0)\|_{\dot{H}^1(\mathbb{R}^2)} > L^{\frac{1}{3}}. \tag{2.4.6}$$

Proof of the Proposition.
`Sketch of the idea.` Let $\tilde{\phi}$ be the characteristic line corresponding to the new smooth solution $\tilde{\omega}$. Then, it solves

$$\begin{cases} \partial_t \tilde{\phi}(x,t) = \tilde{u}(\tilde{\phi}(x,t),t) & \mathbb{R}^2 \times (0,\infty) \\ \tilde{\phi}(x,0) = x & \mathbb{R}^2, \end{cases}$$

where $\tilde{u} = \nabla^{\perp}\Delta^{-1}T_{\gamma}\tilde{\omega}$. Since $\tilde{\omega}(\tilde{\phi}(x,t),t) = \tilde{\omega}_0(x)$, we can write the $\dot{H}^1$-norm of $\tilde{\omega}$ as

$$\|\nabla\tilde{\omega}(\cdot,t)\|_2^2 = \int_{\mathbb{R}^2} |\nabla\tilde{\omega}_0(x)\cdot(\nabla^{\perp}\tilde{\phi}_2)(x,t)|^2 dx + \int_{\mathbb{R}^2} |\nabla\tilde{\omega}_0(x)\cdot(\nabla^{\perp}\tilde{\phi}_1)(x,t)|^2 dx. \tag{2.4.7}$$

By Lemma 2.4.1, if we choose a new initial data $\tilde{\omega}_0$ to make $\|u-\tilde{u}\|_{W^{1,\infty}}$ small, $\left\|D\phi - D\tilde{\phi}\right\|_{\infty}$ also gets small. It follows that the main part in the right hand side of (2.4.7) is the one in which $\tilde{\phi}$ is replaced by ϕ. Then, we can produce the $\dot{H}^1$-norm inflation of $\tilde{\omega}$ at t_0 from largeness of the Lagrangian deformation $D\phi$ in (2.4.1) sense. Indeed, we construct the desired new initial data by adding a perturbation, localized at the point where the large Lagrangian deformation occurs, to the original initial data.

`Step 1.` Construction of the new initial data $\tilde{\omega}_0$.

Assume

$$\|\nabla\omega(\cdot,t_0)\|_2 \le L^{\frac{1}{3}}.$$

Otherwise, $\tilde{\omega}_0 = \omega_0$ completes the proof.

By the assumption (2.4.1) and the smoothness of ϕ, we can find $x_L = (x_L^1, x_L^2)$, $x_L^1 x_L^2 \neq 0$, in $B(0,R_0)$ such that one of the entries of $D\phi(x_L,t_0)$, say $\partial_2\phi_2(x_L,t_0)$, satisfies

$$|\partial_2\phi_2(x_L,t_0)| > L.$$

If we further use the continuity of $D\phi$, we can choose sufficiently small $\delta > 0$ satisfying $\delta \ll \min(x_L^1, x_L^2)$, $B(x_L,\delta) \subset B(0,R_0)$, and

$$|\partial_2\phi_2(x,t_0)| > L, \qquad \forall |x - x_L| < \delta.$$

Choose Ψ to be a smooth radial bump function which is compactly supported on the unit ball $B(0,1)$ and satisfies $\Psi \equiv 1$ on $B(0,\frac{1}{2})$ and $0 \le \Psi \le 1$. Set $\Psi_\delta = \frac{1}{\delta}\Psi(\frac{x-x_L}{\delta})$. By the choice of x_L and δ, we note that the support of Ψ_δ lies on one of the four quadrants. Now, let b be the odd extension of Ψ_δ in both variables. Then, we define the new initial data $\tilde{\omega}_0$, adding a perturbation

$$\eta_0(x) = \tilde{\omega}_0(x) - \omega_0(x) = \frac{1}{20k\sqrt{L}}\cos(kx_1)b(x),$$

to the original one ω_0 where k will be chosen later sufficiently large. We can easily see that the perturbation η_0 is odd in both variables.

`Step 2.` Check the required conditions on $\tilde{\omega}$.

By its construction, the support of η_0 is contained in $B(0,R_0)$, so that (2.4.5) holds.

To get (2.4.2) and (2.4.3), we estimate the corresponding Sobolev norms of η_0,

$$\|\eta_0\|_1 \le \frac{1}{20k\sqrt{L}}\|b\|_1 \qquad \|\eta_0\|_\infty \le \frac{1}{20k\sqrt{L}}\|b\|_\infty$$
$$\|\eta_0\|_{\dot{H}^{-1}} \lesssim \left\|\widehat{x\eta_0}\right\|_\infty + \|\eta_0\|_2 \lesssim \frac{1}{k},$$

where the estimate for the negative Sobolev norm follows from the parity of η_0. For sufficiently large k, both (2.4.2) and (2.4.3) hold true.

Finally, (2.4.4) follows from

$$\|b\|_2 \le 4\|\Psi_\delta\|_2 = 4\|\Psi\|_2 < 4\sqrt{\pi},$$

and

$$\|\nabla\eta_0\|_2 \le \frac{1}{20k\sqrt{L}}(k\|b\|_2 + \|\nabla b\|_2) \le \frac{1}{\sqrt{L}},$$

provided that k is sufficiently large.

Now, consider the $\dot{H}^1$-norm inflation of the new solution $\tilde{\omega}$. As we mentioned, we first show that the perturbation in the Lagrangian deformation is small. For this purpose, we consider the perturbation of velocity in $W^{1,\infty}(\mathbb{R}^2)$.

Since we have

$$\|\nabla(\tilde{u}-u)\|_\infty \lesssim_\gamma (\|\nabla\tilde{\omega}\|_4 + \|\nabla\omega\|_4)^{\frac{2}{3}} \|\tilde{\omega}-\omega\|_2^{\frac{1}{3}}, \tag{2.4.8}$$

it is enough to consider the terms on the right hand side. The terms $\|\nabla\tilde{\omega}\|_4$ and $\|\nabla\omega\|_4$ are estimated by the usual energy method. From the equation for $\tilde{\omega}$, we have

$$\frac{d}{dt}\|\nabla\tilde{\omega}\|_4^4 \leq 4\|\nabla\tilde{u}\|_\infty \|\nabla\tilde{\omega}\|_4^4. \tag{2.4.9}$$

By the log-type interpolation inequality,

$$\|\nabla\tilde{u}(\cdot,t)\|_\infty \lesssim 1 + \|\tilde{\omega}_0\|_\infty \log(10 + \|\tilde{\omega}_0\|_2 + \|\nabla\tilde{\omega}(\cdot,t)\|_4^4),$$

we obtain

$$\max_{0\leq t\leq 1} \|\nabla\tilde{\omega}(\cdot,t)\|_4 \leq C, \tag{2.4.10}$$

for some constant $C = C(\|\nabla\tilde{\omega}_0\|_4, \|\tilde{\omega}_0\|_2)$. Note that we can choose an upper bound C which is independent of k. Similarly, we have

$$\max_{0\leq t\leq 1} \|\nabla\omega(\cdot,t)\|_4 \leq C \tag{2.4.11}$$

for some positive constant C independent of k.

On the other hand, from the equations for $\tilde{\omega}$ and ω, we get the equation for $\eta = \omega - \tilde{\omega}$,

$$\partial_t\eta + \nabla^\perp\Delta^{-1}T_\gamma\eta\cdot\nabla\omega + \nabla^\perp\Delta^{-1}T_\gamma\tilde{\omega}\cdot\nabla\eta = 0.$$

Taking $\int\cdot\eta\,dx$ on both side, η satisfies

$$\frac{1}{2}\frac{d}{dt}\|\eta(\cdot,t)\|_2^2 \leq \left\|\nabla^\perp\Delta^{-1}T_\gamma\eta\right\|_4 \|\nabla\omega\|_4 \|\eta\|_2 \lesssim \|\nabla\omega\|_4 \|\eta\|_2^2.$$

Here, the last inequality follows from the Hardy-Littlewood Sobolev inequality and the compactness of the support of η. By the Grönwall inequality, we obtain

$$\max_{0\le t\le 1}\|\eta(\cdot,t)\|_2 \lesssim \|\eta_0\|_2 \lesssim \frac{1}{k}. \tag{2.4.12}$$

Combining with (2.4.8), (2.4.10), and (2.4.11), the perturbation of u can be estimated by

$$\|\nabla(\tilde{u}-u)\|_\infty \lesssim k^{-\frac{1}{3}}.$$

Finally, by the Gagliardo-Nirenberg interpolation inequality, for any $0 \le t \le 1$, we have

$$\|(\tilde{u}-u)(\cdot,t)\|_\infty \lesssim \|\nabla(\tilde{u}-u)\|_\infty^{\frac{1}{3}} \|\tilde{u}-u\|_4^{\frac{2}{3}} \lesssim k^{-\frac{1}{9}} \|\eta\|_2^{\frac{2}{3}} \lesssim k^{-\frac{7}{9}}.$$

Therefore, Lemma 2.4.1 gives the desired estimate for the perturbation of the Lagrangian deformation,

$$\max_{0\le t\le 1}\left(\left\|(\tilde{\phi}-\phi)(\cdot,t)\right\|_\infty + \left\|(D\tilde{\phi}-D\phi)(\cdot,t)\right\|_\infty\right) \lesssim k^{-\frac{1}{3}}.$$

Now, we are ready to get $\dot{H}^1$-norm inflation. Recall (2.4.7) and we further estimate its right hand side as follows.

$$\begin{aligned}
\|\nabla\tilde{\omega}(\cdot,t_0)\|_2^2 &\ge \int_{\mathbb{R}^2} |\nabla\tilde{\omega}_0(x)\cdot(\nabla^\perp\tilde{\phi}_2)(x,t_0)|^2 dx \\
&\ge \frac{1}{2}\int_{\mathbb{R}^2} |\nabla\tilde{\omega}_0(x)\cdot(\nabla^\perp\phi_2)(x,t_0)|^2 dx - O(k^{-\frac{2}{3}}) \\
&\ge \frac{1}{4}\int_{\mathbb{R}^2} |\nabla\eta_0(x)\cdot(\nabla^\perp\phi_2)(x,t_0)|^2 dx \\
&\quad - \frac{1}{2}\int_{\mathbb{R}^2} |\nabla\omega_0(x)\cdot(\nabla^\perp\phi_2)(x,t_0)|^2 dx - O(k^{-\frac{2}{3}}).
\end{aligned} \tag{2.4.13}$$

By the assumption on ω, we have

$$\int_{\mathbb{R}^2} |\nabla\omega_0(x)\cdot(\nabla^\perp\phi_2)(x,t_0)|^2 dx \le \|\nabla\omega(\cdot,t_0)\|_2^2 \le L^{\frac{2}{3}}.$$

On the other hand, by the construction of the perturbation η_0, we obtain

$$\begin{aligned}\int_{\mathbb{R}^2} |\nabla\eta_0(x)\cdot(\nabla^{\perp}\phi_2)(x,t_0)|^2 dx &\geq \frac{1}{800L}\int_{\mathbb{R}^2} |\sin(kx_1)b(x)\partial_2\phi_2(x,t_0)|^2 dx - O(k^{-2}) \\ &\geq \frac{L}{800}\frac{1}{\delta^2}\int_{|x-x_L|<\frac{1}{2}\delta} |\sin(kx_1)|^2 dx - O(k^{-2}) \\ &\geq \frac{1}{2^6\cdot 10^2}L - O(k^{-1}).\end{aligned}$$

Therefore, we get the desired norm inflation

$$\|\nabla\tilde{\omega}(\cdot,t_0)\|_2^2 \geq \frac{1}{2^8\cdot 10^2}L - \frac{1}{2}L^{\frac{2}{3}} - O(k^{-\frac{2}{3}}) > L^{\frac{2}{3}}$$

provided that $L > 8^9\cdot 10^6$ and k is sufficiently large. In other words, (2.4.6) is obtained.

□

Remark 2.4.1. Based on Proposition 2.3.4 and Proposition 2.4.2, we can construct a family of initial data having $\dot{H}^1$-norm inflation.

Choose a nonzero radial bump function $\varphi \in C_c^\infty(\mathbb{R}^2)$ satisfying $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ on $B(0,\frac{1}{2})$, and $\mathrm{supp}(\varphi) \subset B(0,1)$. Then, we define $\rho \in C_c^\infty(\mathbb{R}^2)$ by

$$\rho(x) = \rho(x_1,x_2) = \sum_{a_1,a_2=\pm 1} a_1 a_2 \varphi\left(\frac{x_1-a_1, x_2-a_2}{2^{-100}}\right). \tag{2.4.14}$$

Clearly, the function ρ is odd in both variables, and

$$\int_{x_1>0,x_2>0} \rho(x)\frac{x_1x_2}{|x|^4}e^{-|x|^4}dx > 0.$$

Now, for each $0 < \gamma \leq \frac{1}{2}$, define $g_A \in C_c^\infty(\mathbb{R}^2)$ by

$$g_A(x) = \begin{cases} C_A \sum_{a_A\leq j<b_A} \frac{1}{j^\gamma}\rho(2^j x), & 0<\gamma<\frac{1}{2} \\ C_A \sum_{\ln A\leq j<A+\ln A} \frac{1}{\sqrt{j}}\rho(2^j x), & \gamma = \frac{1}{2} \end{cases} \tag{2.4.15}$$

where $C_A = \frac{1}{\sqrt{\ln A}}\frac{1}{\ln\ln A}$, $a_A = A^{\frac{1}{1-2\gamma}}$, and $b_A = (A+\ln A)^{\frac{1}{1-2\gamma}}$. Note that the summations in (2.4.15) are over integer j in the range.

First, g_A satisfies all assumptions in Proposition 2.3.4. Obviously, g_A is an odd function in x_1 and x_2, and $g_A(x_1,x_2) \geq 0$ for $x_1 \geq 0$ and $x_2 \geq 0$. Using disjoint supports of $\rho(2^j\cdot)$, $j \in \mathbb{N}$, we have for $A \geq e^2$,

$$\begin{aligned} G_A &= \int_{x_1>0,x_2>0} g_A(x)\frac{x_1x_2}{|x|^4}\ln^{-\gamma}\left(e+\frac{1}{|x|}\right)e^{-|x|^4}dx \\ &= C_A\sum_j \frac{1}{j^\gamma}\int_{x_1>0,x_2>0}\rho(2^jx)\frac{x_1x_2}{|x|^4}\ln^{-\gamma}\left(e+\frac{1}{|x|}\right)e^{-|x|^4}dx \\ &= C_A\sum_j \frac{1}{j^\gamma}\int_{\substack{x_1>0,x_2>0\\ x\in\text{supp}(\rho)}}\rho(x)\frac{x_1x_2}{|x|^4}\ln^{-\gamma}\left(e+\frac{2^j}{|x|}\right)e^{-\frac{|x|^4}{2^{4j}}}dx \\ &\geq C_A\sum_j \frac{1}{j^{2\gamma}}\left(\int_{x_1>0,x_2>0}\rho(x)\frac{x_1x_2}{|x|^4}e^{-|x|^4}dx\right) > 0. \end{aligned} \tag{2.4.16}$$

Here, the range of summation over j depends on γ, which follows to the one in (2.4.15).

Since for $A \gg 1$, we have

$$\begin{aligned} \sum_j \frac{1}{j^{2\gamma}} &\sim \begin{cases} \int_{a_A}^{b_A}\frac{1}{x^{2\gamma}}dx = \frac{1}{1-2\gamma}(b_A^{1-2\gamma}-a_A^{1-2\gamma}) = \frac{1}{1-2\gamma}\ln A, & 0<\gamma<\frac{1}{2} \\ \int_{\ln A}^{A+\ln A}\frac{1}{x}dx = \ln(A+\ln A)-\ln\ln A, & \gamma=\frac{1}{2} \end{cases} \\ &\sim_\gamma \ln A, \end{aligned}$$

G_A has a lower bound

$$G_A \gtrsim_\gamma \frac{\sqrt{\ln A}}{\ln\ln A}.$$

Then, by Proposition 2.3.4, for any A with $A \geq A_0$ for some $A_0 = A_0(\gamma)$, we can find $t_A \in \left(0, \frac{1}{\ln\ln A}\right]$ such that the characteristic line ϕ_A corresponding to each initial data g_A has the large Lagrangian deformation

$$\|D\phi_A(\cdot,t_A)\|_{L^\infty(B(0,\frac{1}{2}))} > \ln^{\frac{1}{4}}\ln\ln\ln A. \tag{2.4.17}$$

Now, we induce the critical norm inflation from the large Lagrangian deformation.

Observe that all assumptions in Proposition 2.4.2 hold for $\omega_0 = g_A$, $t_0 = t_A$, $L = \ln^{\frac{1}{4}} \ln\ln\ln A$, and $R_0 = 1$, provided that A is sufficiently large. Indeed, using

$$|\phi_A(x,t) - x| \le \int_0^t |\partial_s \phi_A(x,s)| ds \le \left\| \nabla^\perp \Delta^{-1} T_\gamma (g_A \circ \phi_A^{-1}) \right\|_{L^\infty_{x,t}} t \lesssim \|g_A\|_1^{\frac{1}{2}} \|g_A\|_\infty^{\frac{1}{2}} t$$

for all $x \in \mathbb{R}^2$ and $t \ge 0$, we have $\phi^{-1}(B_{g_A}, t) \subset B(0,1)$ for sufficiently large A, where B_{g_A} is defined as in Remark 2.3.2. In what follows, we have a desired family $\{\tilde{g}_A\}$ of a new initial data which has the following properties:

(i) $\tilde{g}_A$ gets small as A goes to infinity in the following sense:

$$\begin{aligned} \|\tilde{g}_A\|_1 &\le 2\|g_A\|_1 \lesssim \frac{1}{A^{\ln 4}}, \\ \|\tilde{g}_A\|_\infty &\le 2\|g_A\|_\infty \le \frac{2}{\sqrt{\ln A}} \\ \|\nabla \tilde{g}_A\|_2 &\le \|\nabla g_A\|_2 + \ln^{-\frac{1}{8}} \ln\ln\ln A \le \frac{C_\gamma}{\ln\ln A} + \ln^{-\frac{1}{8}} \ln\ln\ln A \end{aligned} \tag{2.4.18}$$

where C_γ is independent of A.

(ii) $\operatorname{supp}(\tilde{g}_A) \subset B(0,1)$.

(iii) The smooth solution $\tilde{\omega}_A$ to (LE) for the initial data $\tilde{g}_A$ has local critical norm inflation:

$$\|\nabla \tilde{\omega}_A(\cdot, t_A)\|_2 > \ln^{\frac{1}{12}} \ln\ln\ln A.$$

2.5 Patching argument

In this section, we introduce useful lemmas and a proposition for the construction of the desired global solution from local ones. For the non-compactly supported case, our strategy is using a huge distance between local solutions so that they barely interact with each other. This leads the global solution to locally behave like local solutions. The following proposition describes this in detail.

Proposition 2.5.1. *Let $\{\omega_{j0}\} \subset C_c^\infty(B(0,1))$ be a sequence of functions satisfying*

$$\sum_{j=1}^\infty (\|\omega_{j0}\|_{H^1}^2 + \|\omega_{j0}\|_1) + \sup_j \|\omega_{j0}\|_\infty \le M \tag{2.5.1}$$

for some $M > 1$. For each $\gamma > 0$, let C_0 be an absolute constant such that

$$\left\|\nabla^{\perp}\Delta^{-1}T_{\gamma}f\right\|_{\infty} \le C_0(\|f\|_1 + \|f\|_{\infty}).$$

Then, we can find a sequence $\{x_j\}$ of centers with $|x_j - x_k| \gg 1$ for $j \ne k$ such that there exists a unique classical solution ω to (LE) *for the initial data*

$$\omega_0(x) = \sum_{j=1}^{\infty} \omega_{j0}(x - x_j) \in L^1 \cap L^{\infty} \cap H^1 \cap C^{\infty}$$

such that the following hold.

(i) For any $0 \le t \le 1$, $\omega(\cdot,t)$ is supported in the union of disjoint balls:

$$\operatorname{supp}(\omega(\cdot,t)) \subset \bigcup_{j=1}^{\infty} B(x_j, 3C_0M). \tag{2.5.2}$$

(ii) For each $0 \le t \le 1$, $\omega(\cdot,t) \in C^{\infty}(\mathbb{R}^2)$, and $\omega \in C([0,1]; L^1(\mathbb{R}^2) \cap L^{\infty}(\mathbb{R}^2))$.

(iii) For any $\varepsilon > 0$, we can find a sufficiently large integer $j_0 = j_0(\varepsilon)$ so that for $j \ge j_0$, we have

$$\max_{0 \le t \le 1} \left\|(\omega - \omega_j)(\cdot,t)\right\|_{H^2(B(x_j, 3C_0M))} < \varepsilon, \tag{2.5.3}$$

where a local solution ω_j solves (LE) *for the initial data*

$$\omega_j|_{t=0} = \omega_{j0}(\cdot - x_j).$$

Before we prove this proposition, we consider some preliminary lemmas.

Lemma 2.5.2. *Suppose that $f \in H^k \cap L^1$ for some $k \ge 2$ and $g \in H^2 \cap L^1$ satisfy*

$$\begin{aligned} \|f\|_1 + \|g\|_1 + \sup(\|f\|_{\infty}, \|g\|_{\infty}) \le M < \infty, \\ \operatorname{dist}(\operatorname{supp}(f), \operatorname{supp}(g)) \ge 100C_0M > 0 \end{aligned} \tag{2.5.4}$$

for some constant $M > 1$, and the Lebesgue measure of the support of f is bounded by some positive constant M_1.

Then, the solution ω to

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0 \quad \mathbb{R}^2 \times (0,1] \\ u = \nabla^{\perp} \Delta^{-1} T_{\gamma} \omega \\ \omega|_{t=0} = f + g \end{cases}$$

has the following properties.

(i) The solution ω can be decomposed as $\omega = \omega_f + \omega_g$ such that

$$\omega_f|_{t=0} = f, \quad \omega_g|_{t=0} = g$$

$$\mathrm{supp}(\omega_f(\cdot,t)) \subset B(\mathrm{supp}(f), 2C_0M), \tag{2.5.5}$$

$$\mathrm{supp}(\omega_g(\cdot,t)) \subset B(\mathrm{supp}(g), 2C_0M), \tag{2.5.6}$$

$$\mathrm{dist}(\mathrm{supp}(\omega_f(\cdot,t)), \mathrm{supp}(\omega_g(\cdot,t))) \geq 90C_0M, \quad \forall 0 \leq t \leq 1, \tag{2.5.7}$$

where C_0 is defined as in Proposition 2.5.1.

(ii) The Sobolev norms of ω_f can be estimated by

$$\max_{0 \leq t \leq 1} \left\| \omega_f(\cdot,t) \right\|_{H^k} \leq C \tag{2.5.8}$$

for some constant $C = C(\|f\|_{H^k}, k, M, M_1)$ independent of $\|g\|_{H^k}$.

Proof. Define ω_f and ω_g by the solutions to

$$\begin{cases} \partial_t \omega_f + u \cdot \nabla \omega_f = 0 \\ \omega_f|_{t=0} = f \end{cases} \tag{2.5.9}$$

and

$$\begin{cases} \partial_t \omega_g + u \cdot \nabla \omega_g = 0 \\ \omega_g|_{t=0} = g. \end{cases} \tag{2.5.10}$$

Let ϕ be the characteristic line which solves

$$\begin{cases} \partial_t \phi(x,t) = u(\phi(x,t),t) \\ \phi(x,0) = x. \end{cases}$$

Then, the equations (2.5.9) and (2.5.10) can be written as

$$\omega_f(\phi(x,t),t) = f(x), \quad \text{and} \quad \omega_g(\phi(x,t),t) = g(x).$$

From these forms, it follows that for $1 \leq p \leq \infty$

$$\left\|\omega_f(\cdot,t)\right\|_p = \|f\|_p, \quad \text{and} \quad \|\omega_g(\cdot,t)\|_p = \|g\|_p, \quad \forall 0 \leq t \leq 1,$$

and

$$\max_{0\leq t\leq 1} \|u(\cdot,t)\|_\infty \leq C_0 M.$$

Since we have

$$|\phi(x,t) - x| \leq \int_0^t |\partial_s \phi(x,s)| ds \leq \max_{0\leq s\leq 1} \|u(\cdot,s)\|_\infty t \leq C_0 M t,$$

(2.5.5) and (2.5.6) easily follows from

$$\begin{aligned} &\operatorname{supp}(\omega_f(\cdot,t)) \subset \phi(\operatorname{supp}(f),t) \subset B(\operatorname{supp}(f), 2C_0 M), \\ &\operatorname{supp}(\omega_g(\cdot,t)) \subset \phi(\operatorname{supp}(g),t) \subset B(\operatorname{supp}(g), 2C_0 M), \quad \forall 0 \leq t \leq 1. \end{aligned}$$

Using the assumption (2.5.4) additionally, the triangle inequality implies

$$\operatorname{dist}(\operatorname{supp}(\omega_f(\cdot,t)), \operatorname{supp}(\omega_g(\cdot,t))) \geq 90 C_0 M, \quad \forall 0 \leq t \leq 1. \tag{2.5.11}$$

In other words, (2.5.7) is obtained.

To control the Sobolev norm of ω_f, we first estimate $\nabla^\perp \Delta^{-1} T_\gamma \omega_g$ when $0 \leq t \leq 1$ and $x \in \operatorname{supp}(\omega_f(\cdot,t))$. Since the supports of $\omega_f(\cdot,t)$ and $\omega_g(\cdot,t)$ are apart from each other for $0 \le t \le 1$ (see (2.5.11)), we have for $0 \le t \le 1$ and $x \in \mathrm{supp}(\omega_f(\cdot,t))$,

$$\begin{aligned}\left|\partial^\alpha \nabla^\perp \Delta^{-1} T_\gamma \omega_g(x,t)\right| &= \left|\int_{|y-x|\ge 90C_0M} \partial^\alpha H(x-y)\omega_g(y)dy\right| \\ &\le \|\partial^\alpha H\|_{L^\infty(|z|\ge 90C_0M)} \|g\|_1 ,\end{aligned} \tag{2.5.12}$$

where H is the kernel of the Fourier multiplier $\nabla^\perp \Delta^{-1} T_\gamma$. By Lemma 2.8.1, for any multi-index α with $|\alpha| \ge 0$, H satisfies

$$|\partial^\alpha H(z)| \lesssim_{\alpha,\gamma} \frac{1}{|z|^{|\alpha|+1}}, \quad \forall z \ne 0$$

and therefore

$$\max_{0\le t\le 1} \max_{x\in \mathrm{supp}(\omega_f(\cdot,t))} |\partial^\alpha \nabla^\perp \Delta^{-1} T_\gamma \omega_g(x,t)| \lesssim_{\alpha,\gamma} 1. \tag{2.5.13}$$

To get (2.5.8), we use the energy method. We consider the Sobolev norm $W^{1,p}(\mathbb{R}^2)$ for $2 < p \le +\infty$ first. From the equation (2.5.9) for ω_f, we have

$$\frac{1}{p}\frac{d}{dt}\|\nabla \omega_f\|_p^p \le (\left\|D\nabla^\perp \Delta^{-1} T_\gamma \omega_f\right\|_\infty + \left\|D\nabla^\perp \Delta^{-1} T_\gamma \omega_g\right\|_{L^\infty(\mathrm{supp}(\omega_f(\cdot,t)))}) \|\nabla \omega_f\|_p^p \tag{2.5.14}$$

By a log-type interpolation inequality together with the L^p-norm preservation of ω_f,

$$\left\|D\nabla^\perp \Delta^{-1} T_\gamma \omega_f(\cdot,t)\right\|_\infty \lesssim_p 1 + \|f\|_\infty \log(10 + \|f\|_2 + \|\nabla \omega_f(\cdot,t)\|_p^p), \quad \forall 0 \le t \le 1.$$

Combining with (2.5.13) and (2.5.14), this implies

$$\max_{0\le t\le 1} \|\omega_f(\cdot,t)\|_{W^{1,p}(\mathbb{R}^2)} \le C(\|f\|_{W^{1,p}(\mathbb{R}^2)}, p, M).$$

We now estimate in $H^k(\mathbb{R}^2)$, $k \ge 2$. By the commutator estimate in [34, Theorem 1.9], for $J=(1-\Delta)^{\frac{1}{2}}$, we get

$$\begin{aligned}\frac{d}{dt}\left\|J^k\omega_f\right\|_2 &\leq \left\|[J^k,\nabla^\perp\Delta^{-1}T_\gamma\omega_f\cdot\nabla]\omega_f\right\|_2+\left\|[J^k,\nabla^\perp\Delta^{-1}T_\gamma\omega_g\cdot\nabla]\omega_f\right\|_2\\ &\lesssim \left\|J^{k-1}D\nabla^\perp\Delta^{-1}T_\gamma\omega_f\right\|_3\left\|\nabla\omega_f\right\|_6+\left\|D\nabla^\perp\Delta^{-1}T_\gamma\omega_f\right\|_\infty\left\|J^k\omega_f\right\|_2\\ &\quad+\max_{|\alpha|\leq k}\max_{0\leq t\leq 1}\left\|D^\alpha\nabla^\perp\Delta^{-1}T_\gamma\omega_g(\cdot,t)\right\|_{L^\infty(\mathrm{supp}(\omega_f(\cdot,t)))}\left\|J^k\omega_f\right\|_2\\ &\leq C\left\|J^k\omega_f\right\|_2,\end{aligned}$$

where the constant in the last inequality depends on $\|f\|_{H^2}$, M, M_1, and k.

Therefore, by the Grönwall inequality, we obtain (2.5.8).

□

Lemma 2.5.3. *Suppose that f is in $H^3(\mathbb{R}^2)\cap L^1(\mathbb{R}^2)$ with* $\mathrm{Leb}(\mathrm{supp}(f))\leq M_1$ *for some M_1, g is in $H^2(\mathbb{R}^2)\cap L^1(\mathbb{R}^2)$, and they satisfy*

$$\|f\|_1+\|g\|_1+\sup(\|f\|_\infty,\|g\|_\infty)\leq M$$

for some $M>1$. Let ω and $\tilde{\omega}$ be solutions to (LE) *for the initial data $f+g$ and f, respectively.*

Then, for each $\varepsilon>0$, we can find sufficiently large $R=R(\varepsilon,\|f\|_{H^3},M,M_1)>0$ such that if

$$\mathrm{dist}(\mathrm{supp}(f),\mathrm{supp}(g))\geq R, \tag{2.5.15}$$

then ω can be decomposed as $\omega=\omega_f+\omega_g$ such that ω_f and ω_g satisfy (2.5.5)-(2.5.7) *and*

$$\max_{0\leq t\leq 1}\left\|(\omega_f-\tilde{\omega})(\cdot,t)\right\|_{H^2}<\varepsilon. \tag{2.5.16}$$

Remark 2.5.1. Similar to (2.5.5) and (2.5.6), we have

$$\mathrm{supp}(\tilde{\omega}(\cdot,t))\subset B(\mathrm{supp}(f),2C_0M),\quad \forall 0\leq t\leq 1, \tag{2.5.17}$$

where C_0 is defined as in Proposition 2.5.1. It follows from $\max_{0\leq t\leq 1}\|\tilde{u}(\cdot,t)\|_\infty\leq$

C_0M for $\tilde{u} = \nabla^{\perp}\Delta^{-1}T_\gamma\tilde{\omega}$.

Proof. We use the same decomposition $\omega = \omega_f + \omega_g$ in Lemma 2.5.2. Then, we have (2.5.5) and (2.5.6). Furthermore, (2.5.7) is also obtained, provided that $R \geq 100C_0M$. In fact, using (2.5.15), we have

$$\mathrm{dist}(\mathrm{supp}(\omega_f(\cdot,t),\mathrm{supp}(\omega_g(\cdot,t)))) \geq R - 10C_0M \geq \frac{1}{2}R, \quad \forall 0 \leq t \leq 1 \tag{2.5.18}$$

for sufficiently large R.

To get (2.5.16), we recall the equation for ω_f,

$$\begin{cases} \partial_t\omega_f + u\cdot\nabla\omega_f = 0 \\ \omega_f|_{t=0} = f. \end{cases}$$

By the Gagliardo-Nirenberg inequality,

$$\left\|(\omega_f - \tilde{\omega})(\cdot,t)\right\|_{H^2} \lesssim (\left\|\omega_f(\cdot,t)\right\|_{H^3} + \|\tilde{\omega}(\cdot,t)\|_{H^3})^{\frac{2}{3}} \left\|(\omega_f - \tilde{\omega})(\cdot,t)\right\|_2^{\frac{1}{3}}. \tag{2.5.19}$$

By Lemma 2.5.2, we obtain

$$\max_{0\leq t\leq 1}\left\|\omega_f(\cdot,t)\right\|_{H^3} \leq C(\|f\|_{H^3},M,M_1). \tag{2.5.20}$$

Also, by the usual energy method, we also have a similar inequality for $\tilde{\omega}$

$$\max_{0\leq t\leq 1}\|\tilde{\omega}(\cdot,t)\|_{H^3} \leq C(\|f\|_{H^3},M,M_1). \tag{2.5.21}$$

Therefore, it is enough to consider $\|\eta(\cdot,t)\|_2$ for $\eta = \omega_f - \tilde{\omega}$.

The equation for η is

$$\begin{cases} \partial_t\eta + \nabla^{\perp}\Delta^{-1}T_\gamma\tilde{\omega}\cdot\nabla\eta + \nabla^{\perp}\Delta^{-1}T_\gamma\eta\cdot\nabla\omega_f + \nabla^{\perp}\Delta^{-1}T_\gamma\omega_g\cdot\nabla\omega_f = 0 \\ \eta|_{t=0} = 0. \end{cases}$$

Taking $\int \cdot \eta dx$ on both side of the first equation and using (2.5.20), we get

$$\begin{aligned}\frac{d}{dt}\|\eta(\cdot,t)\|_2 &\le \left\|\nabla^\perp\Delta^{-1}T_\gamma\eta\cdot\nabla\omega_f\right\|_2 + \left\|\nabla^\perp\Delta^{-1}T_\gamma\omega_g\cdot\nabla\omega_f\right\|_2 \\ &\lesssim_{M_1} \|\eta\|_2\left\|\nabla\omega_f\right\|_6 + \left\|\nabla^\perp\Delta^{-1}T_\gamma\omega_g\right\|_{L^\infty(\mathrm{supp}(\omega_f(\cdot,t)))}\left\|\nabla\omega_f\right\|_2 \\ &\le C(\|\eta\|_2 + \left\|\nabla^\perp\Delta^{-1}T_\gamma\omega_g\right\|_{L^\infty(\mathrm{supp}(\omega_f(\cdot,t)))}),\end{aligned}$$

for some positive constant C depending on $\|f\|_{H^3}$, M, and M_1. Then by the Grönwall inequality, we have

$$\max_{0\le t\le 1}\|\eta(\cdot,t)\|_2 \le C(\|f\|_{H^3},M,M_1)\max_{0\le t\le 1}\left\|\nabla^\perp\Delta^{-1}T_\gamma\omega_g\right\|_{L^\infty(\mathrm{supp}(\omega_f(\cdot,t)))}. \quad (2.5.22)$$

Using Lemma 2.8.1 and (2.5.18), we have for any $0\le t\le 1$ and $x\in\mathrm{supp}(\omega_f(\cdot,t))$,

$$\begin{aligned}|\nabla^\perp\Delta^{-1}T_\gamma\omega_g(x,t)| &= |H*\omega_g(x,t)| \\ &\lesssim \int_{|x-y|\ge\frac{1}{2}R}\frac{1}{|x-y|}|\omega_g(y,t)|dy \lesssim R^{-1}\|g\|_1 \le MR^{-1}.\end{aligned} \quad (2.5.23)$$

Finally, combining (2.5.19)-(2.5.23), we can find $R = R(\varepsilon,\|f\|_{H^3},M,M_1) > 100C_0M$ sufficiently large such that

$$\max_{0\le t\le 1}\left\|(\omega_f-\tilde{\omega})(\cdot,t)\right\|_{H^2} \le C(\|f\|_{H^3},M,M_1)R^{-\frac{1}{3}} < \varepsilon.$$

□

Now we are ready to prove the proposition.

Proof of Proposition 2.5.1. Let $\omega_{\le n}$, $n\in\mathbb{N}$, be a smooth solution to

$$\begin{cases}\partial_t\omega_{\le n} + \nabla^\perp\Delta^{-1}\omega_{\le n}\cdot\nabla\omega_{\le n} = 0, \\ \omega_{\le n}|_{t=0} = \sum_{k=1}^n\omega_{k0}(x-x_k).\end{cases} \quad (2.5.24)$$

Our strategy is to construct a sequence $\{x_k\}_{k\in\mathbb{N}}$ of centers such that the following hold.

(i) For each $j\in\mathbb{N}$, $\{\omega_{\le n}\}$ is Cauchy in $C([0,1];H^2(B(x_j,3C_0M)))$.

(ii) For any $n \in \mathbb{N}$,

$$\operatorname{supp}(\omega_{\le n}(\cdot,t)) \subset \bigcup_{j=1}^{\infty} B(x_j, 3C_0M).$$

(iii) For any $n \in \mathbb{N}$ and $1 \le j \le n$,

$$\max_{0\le t\le 1} \left\|(\omega_{\le n} - \omega_j)(\cdot,t)\right\|_{H^2(B(x_j,3C_0M))} < \frac{1}{2^{j+1}}.$$

Then, the limit solution of $\{\omega_{\le n}\}$ becomes the desired one ω.

`Step 1` Construction of the sequence $\{x_k\}_{k\in\mathbb{N}}$.

For each $j \in \mathbb{N}$, apply Lemma 2.5.3 for $f = \omega_{j0}$ and $\varepsilon = \frac{1}{2^{j+1}}$. Then, we can find $R_j > 0$ such that for any $h \in H^2 \cap L^1$ with

$$\begin{gathered} \left\|\omega_{j0}\right\|_1 + \|h\|_1 + \sup(\left\|\omega_{j0}\right\|_\infty, \|h\|_\infty) \le M, \\ \operatorname{dist}(\operatorname{supp}(\omega_{j0}), \operatorname{supp}(h)) \ge R_j, \end{gathered} \tag{2.5.25}$$

where M is given in (2.5.1), the solutions ω and $\tilde{\omega}_j$ to (LE) for the initial data $\omega_{j0} + h$ and ω_{j0}, respectively, satisfy

$$\max_{0\le t\le 1} \left\|(\omega - \tilde{\omega}_j)(\cdot,t)\right\|_{H^2(B(0,3C_0M))} < \frac{1}{2^{j+1}}, \tag{2.5.26}$$

and

$$\operatorname{supp}(\omega(\cdot,t)) \subset B(0,3C_0M) \cup B(\operatorname{supp}(h), 2C_0M). \tag{2.5.27}$$

Here, (2.5.27) is an easy consequence of (2.5.5) and (2.5.6).

We find $\{x_n\}$ inductively. Indeed, we can relax the conditions on $\{x_n\}$ as follows; for any $n \ge 2$ in $\mathbb{N}$ with $x_1 = 0$,

(a) x_n is located at a far distance from previously chosen points

$$|x_n - x_l| > \sum_{i=1}^{n} R_i + 10C_0M + 2^n, \quad \forall 1 \le j < n,$$

(b) A smooth solution $\omega_{\leq n}$ to (2.5.24) satisfies

$$\mathrm{supp}(\omega_{\leq n}(\cdot,t)) \subset \bigcup_{j=1}^{n} B(x_j, 3C_0M), \quad \forall 0 \leq t \leq 1.$$

(c) Denoting $B(x_j, 3C_0M)$ by B_j,

$$\max_{0\leq t\leq 1} \|(\omega_{\leq n} - \omega_{\leq n-1})(\cdot,t)\|_{H^2(\bigcup_{j=1}^{n-1} B_j)} < \frac{1}{2^n}.$$

Then, the requirements (i) and (ii) easily follow from (c) and (b), respectively. We can also check that (a) implies (iii). For each $n \in \mathbb{N}$ and $1 \leq j \leq n$, plug

$$h(x) = \sum_{\substack{k=1 \\ k\neq j}}^{n} \omega_{k0}(x - x_k + x_j) \tag{2.5.28}$$

into (2.5.25). We can easily see that (2.5.25) holds true

$$\left\|\omega_{j0}\right\|_1 + \|h\|_1 + \sup(\left\|\omega_{j0}\right\|_\infty, \|h\|_\infty) \leq \sum_{k=1}^{n} \|\omega_{k0}\|_1 + \sup_{1\leq k\leq n} \|\omega_{k0}\|_\infty \leq M,$$

and

$$\begin{aligned}
\mathrm{dist}(\mathrm{supp}(\omega_{j0}), \mathrm{supp}(h)) &= \mathrm{dist}(\mathrm{supp}(\omega_{j0}(\cdot - x_j)), \mathrm{supp}(h(\cdot - x_j))) \\
&\geq \inf_{\substack{1\leq k\leq n \\ k\neq j}} \mathrm{dist}(B(x_j,1), B(x_k,1)) \\
&\geq \inf_{\substack{1\leq k\leq n \\ k\neq j}} |x_j - x_k| - 2 \geq R_j.
\end{aligned}$$

Therefore, using the translation invariant property of (LE), we have

$$\max_{0\leq t\leq 1} \left\|\omega_{\leq n}(\cdot + x_j, t) - \tilde{\omega}_j(\cdot,t)\right\|_{H^2(B(0,3C_0M))} < \frac{1}{2^{j+1}},$$

which follows (iii).

Now, we choose $\{x_j\}$ satisfying (a)-(c) by induction. At the end of each inductive step, we also find $\tilde{R}_n \geq \tilde{R}_{n-1}$ satisfying the following condition

(d) For any $g \in H^2 \cap L^1$ with

$$\left\| \sum_{j=1}^{n} \omega_{j0}(\cdot - x_j) \right\|_1 + \|g\|_1 + \sup\left(\left\| \sum_{j=1}^{n} \omega_{j0}(\cdot - x_j) \right\|_\infty, \|g\|_\infty \right) \le M,$$
$$\operatorname{dist}\left(\operatorname{supp}\left(\sum_{j=1}^{n} \omega_{j0}(\cdot - x_j) \right), \operatorname{supp}(g) \right) \ge \tilde{R}_n, \tag{2.5.29}$$

the solution ω to (LE) for the initial data $\sum_{j=1}^{n} \omega_{j0}(x - x_j) + g$ satisfies

$$\operatorname{supp}(\omega(\cdot,t)) \subset \left(\bigcup_{j=1}^{n} B_j \right) \bigcup B(\operatorname{supp}(g), 2C_0M), \quad \forall 0 \le t \le 1$$

and

$$\max_{0 \le t \le 1} \|(\omega - \omega_{\le n})(\cdot,t)\|_{H^2(\bigcup_{k=1}^{n} B_k)} < \frac{1}{2^{n+1}}. \tag{2.5.30}$$

Set $x_1 = 0$ and $\tilde{R}_1 = R_1$. We first choose x_2 satisfying

$$|x_2 - x_1| > \sum_{i=1}^{2} R_i + 10C_0M + 2^2 + \tilde{R}_1.$$

Clearly, (a) for $n = 2$ is obtained. Also, $j = 1$ and $h = w_{20}(x - x_2)$ satisfies (2.5.25), which implies (b)-(c) for $n = 2$. Here, we use $\omega_{\le 1} = \omega_1 = \tilde{\omega}_1$.

The choice of $\tilde{R}_2 \ge \tilde{R}_1 = R_1$ satisfying (d) for $n = 2$ follows from Lemma 2.5.3; apply it to $f = \omega_{\le 2}|_{t=0}$ and $\varepsilon = \frac{1}{2^3}$.

Assume that $\{x_j\}_{j=1}^{n}$ and $\tilde{R}_n$ are given and satisfy (a)-(d). Then, we pick x_{n+1} such that

$$|x_{n+1} - x_j| > \sum_{i=1}^{n+1} R_i + 10C_0M + 2^{n+1} + \tilde{R}_n, \quad \forall j = 1, \cdots, n.$$

which follows (a). To achieve (b) and (c) for $n+1$, we observe that $g = \omega_{(n+1)0}(x-$

x_{n+1}) satisfies (2.5.29),

$$\left\|\sum_{j=1}^{n}\omega_{j0}(\cdot-x_j)\right\|_1+\|g\|_1+\sup\left(\left\|\sum_{j=1}^{n}\omega_{j0}(\cdot-x_j)\right\|_\infty,\|g\|_\infty\right)$$
$$\leq\sum_{j=1}^{\infty}\left\|\omega_{j0}\right\|_1+\sup_j\left\|\omega_{j0}\right\|_\infty\leq M$$

and

$$\operatorname{dist}\left(\operatorname{supp}\left(\sum_{j=1}^{n}\omega_{j0}(x-x_j)\right),\operatorname{supp}(g)\right)\geq\inf_{1\leq j\leq n}\operatorname{dist}(B(x_j,1),B(x_{n+1},1))$$
$$\geq\inf_{1\leq j\leq n}|x_{n+1}-x_j|-2\geq\tilde{R}_n.$$

Then by (d) for n, the conditions (b) and (c) for $n+1$ hold; we have

$$\operatorname{supp}(\omega_{\leq n+1}(\cdot,t))\subset\left(\bigcup_{j=1}^{n}B_j\right)\cup B(x_{n+1},2C_0M+1)\subset\bigcup_{j=1}^{n+1}B_j$$

and

$$\max_{0\leq t\leq 1}\|(\omega_{\leq n+1}-\omega_{\leq n})(\cdot,t)\|_{H^2(\bigcup_{k=1}^{n}B_k)}<\frac{1}{2^{n+1}}.$$

Applying again Lemma 2.5.3 for $f=\omega_{\leq n+1}|_{t=0}=\sum_{j=1}^{n+1}\omega_{j0}(x-x_j)$ and $\varepsilon=\frac{1}{2^{n+2}}$, we can find $\tilde{R}_{n+1}\geq\tilde{R}_n$ satisfying (d). Therefore, we have (a)-(d) at $(n+1)$th step, so that they hold true for any $n\geq 2$.

`Step 2.` Check the required conditions.

By condition (i), $\{\omega_{\leq n}\}$ is Cauchy in $C([0,1];H^2(B(x_j,3C_0M)))$ for each $j\in\mathbb{N}$. On the other hand, by Lemma 2.5.2, for each $j\in\mathbb{N}$ and $k\geq 2$, $\{\omega_{\leq n}\}$ is uniformly bounded in $C([0,1];H^k(B(x_j,3C_0M)))$, so that $\{\omega_{\leq n}\}$ is Cauchy even in $C([0,1];H^k(B(x_j,3C_0M)))$. This implies that for each $0\leq t\leq 1$, we have a pointwise limit solution

$$\omega(x,t)=\begin{cases}\lim_{n\to\infty}\omega_{\leq n}(x,t) & x\in\bigcup_{j=1}^{\infty}B(x_j,3C_0M)\\ 0 & \text{otherwise.}\end{cases}$$

Obviously, $\omega(\cdot,t) \in C^\infty$ and ω satisfies (2.5.2) and (2.5.3) by the conditions (ii) and (iii). Furthermore, $\omega \in C([0,1];L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2))$. This is because for any $0 \le t \le 1$, we have

$$\|\omega(\cdot,t)\|_1 = \sum_{j=1}^{\infty} \|\omega(\cdot,t)\|_{L^1(B_j)} = \sum_{j=1}^{\infty} \lim_{n\to\infty} \|\omega_{\le n}(\cdot,t)\|_{L^1(B_j)} = \sum_{j=1}^{\infty} \left\|\omega_{j0}\right\|_1 = \|\omega_0\|_1$$

and

$$\begin{aligned}\|\omega(\cdot,t)\|_\infty &= \sup_j \|\omega(\cdot,t)\|_{L^\infty(B_j)} = \sup_j \lim_{n\to\infty} \|\omega_{\le n}(\cdot,t)\|_{L^\infty(B_j)} \\ &= \sup_j \left\|\omega_{j0}\right\|_\infty = \|\omega_0\|_\infty .\end{aligned}$$

Finally, we prove that the limit solution ω is the unique classical solution to (LE) for the initial data

$$\omega|_{t=0}(x) = \sum_{j=1}^{\infty} \omega_{j0}(x-x_j).$$

We first show that the limit solution ω solves (LE) in the sense of

$$\omega(x,t) = \omega_0(x) - \int_0^t (\nabla^\perp \Delta^{-1} T_\gamma \omega \cdot \nabla\omega)(x,s)ds, \quad \forall (x,t) \in \mathbb{R}^2 \times (0,1). \tag{2.5.31}$$

At $t=0$, it is apparent that the limit solution is the same with ω_0. Since $\omega_{\le n}$ solves (2.5.31) with $\omega_0 = \sum_{j=1}^n \omega_{j0}(\cdot - x_j)$ for any $n \in \mathbb{N}$, it is enough to prove the uniform convergence $\nabla^\perp\Delta^{-1}T_\gamma\omega_{\le n} \to \nabla^\perp\Delta^{-1}T_\gamma\omega$ on each $B(x_j,3C_0M)\times[0,1]$, $j \in \mathbb{N}$. For notational simplicity, we suppress the dependence on the variable t, if it's not needed. Fix $j \in \mathbb{N}$. For $n > j$ and $x \in B(x_j,3C_0M) = B_j$, we have

$$\begin{aligned}|(\Delta^{-1}\nabla^\perp T_\gamma(\omega_{\le n}-\omega)(x)| &\le \int |H(x-y)||(\omega_{\le n}-\omega)(y)|dy \\ &= \left(\sum_{\substack{m=1\\ m\ne j}}^{n} \int_{B_m} + \int_{B_j} + \sum_{l=n+1}^{\infty} \int_{B_l}\right) |H(x-y)||(\omega_{\le n}-\omega)(y)|dy \\ &= I_1^n + I_2^n + I_3^n.\end{aligned}$$

By the choice of the centers, we have for any $x \in B_j$ and $y \in B_m$, $m \neq j$,

$$|x-y| \geq |x_j - x_m| - 6C_0 M \geq 2^{\max(j,m)}.$$

This implies that I_1^n converges to 0, as n goes to infinity; for $x \in B_j$,

$$\begin{aligned} I_1^n &\lesssim \sum_{\substack{m=1 \\ m\neq j}}^{n} \int_{B_m} \frac{1}{|x-y|} |(\omega_{\leq n} - \omega)(y,t)| dy \leq \sum_{m=1}^{n} 2^{-m} \|(\omega_{\leq n} - \omega)(\cdot,t)\|_{L^1(B_m)} \\ &\lesssim \sum_{m=1}^{n} 2^{-m} \|\omega_{\leq n} - \omega\|_{C([0,1];L^\infty(B_m))} \to 0, \quad \text{as } n \to \infty. \end{aligned}$$

In a similar way, I_3^n approaches to 0, as n goes to infinity;

$$I_3^n \lesssim \sum_{l=n+1}^{\infty} \int_{B_l} \frac{1}{|x-y|} |\omega(y)| dy \leq \sum_{l=n+1}^{\infty} 2^{-l} \|\omega_0\|_1 \to 0, \quad \text{as } n \to \infty.$$

Finally, since $|x-y| \leq |x-x_j| + |y-x_j| \leq 6C_0 M$, we obtain

$$I_2^n \lesssim_M \max_{0\leq t\leq 1} \|(\omega_{\leq n} - \omega)(\cdot,t)\|_{L^\infty(B_j)} \to 0, \quad \text{as } n \to \infty.$$

Therefore, we get the uniform convergence of $\nabla^\perp \Delta^{-1} T_\gamma \omega_{\leq n}$ and hence ω solves (LE) in the sense of (2.5.31). Using the equation, we can improve the regularity of the solution in time, so that ω is a classical solution to (LE).

For the uniqueness of the classical solution, let $\overline{\omega}$ be another classical solution to (LE) for the same initial data. Note that the statement in Lemma 2.5.3 holds also for a classical solution ω for initial data $f+g$ where $g \in C^\infty(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$. Then, in the same way of obtaining (2.5.30), we have

$$\begin{aligned} \max_{0\leq t\leq 1} \|(\omega - \omega_{\leq n})(\cdot,t)\|_{H^2(\cup_{j=1}^n B_j)} &< \frac{1}{2^n} \\ \max_{0\leq t\leq 1} \|(\overline{\omega} - \omega_{\leq n})(\cdot,t)\|_{H^2(\cup_{j=1}^n B_j)} &< \frac{1}{2^n}. \end{aligned}$$

This follows from that $g = \sum_{j=n+1}^{\infty} \omega_{j0}(\cdot - x_j)$ satisfies (2.5.29) for the same M, f, and ε in the construction of $\tilde{R}_{n+1}$. Therefore, we have $\omega = \overline{\omega}$. In other words, the uniqueness of the classical solution holds. This completes the proof. □

2.6 Proof of Theorem 2.1.1

Proof of Theorem 2.1.1. Recall the family of initial data $\tilde{g}_A$ in Remark 2.4.1. By its construction, for fixed $0 < \gamma \le \frac{1}{2}$ and $0 < \varepsilon < 1$, we can find a sequence $\{A_j\}$ such that for any $j \in \mathbb{N}$, $\zeta_j = \tilde{g}_{A_j}$ satisfies $\mathrm{supp}(\zeta_j) \subset B(0,1)$ and

$$\|\zeta_j\|_1 + \|\zeta_j\|_\infty + \|\nabla \zeta_j\|_2 < \frac{\varepsilon}{2^j}, \tag{2.6.1}$$

and the smooth solution $\tilde{\omega}_j$ to (LE) with initial data ζ_j achieves

$$\|\nabla \tilde{\omega}_j(\cdot, t_j)\|_2 > j \tag{2.6.2}$$

for some t_j which converges to 0 as $j \to \infty$.

Since the solution to (LE) is translation-invariant, in the case of $\mathrm{supp}(a) \subset B(0,1)$ up to translation, we can apply Proposition 2.5.1 to $\omega_{10} = a$ and $\omega_{j0} = \zeta_j$ for $j \ge 2$. Then, we have a sequence $\{x_j\}_{j\in\mathbb{N}}$ of centers with $x_1 = 0$ such that for the initial data

$$\omega_0(x) = a(x - x_1) + \sum_{j=2}^{\infty} \zeta_j(x - x_j) =: a(x) + \zeta(x)$$

we have a unique classical solution ω to (LE) and the solution satisfies $\omega(\cdot,t) \in C^\infty(\mathbb{R}^2)$ for any $0 \le t \le 1$, $\omega \in C([0,1]; L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2))$, and

$$\max_{0 \le t \le 1} \|(\omega - \omega_j)(\cdot,t)\|_{H^2(B(x_j, 3C_0M))} < 1 \tag{2.6.3}$$

for sufficiently large j. Here, ω_j is a smooth solution to (LE) for the initial data $\zeta_j(x - x_j)$, C_0 is the constant defined in Proposition 2.5.1, and $M > 1$ is a bound of the initial data in the sense of

$$1 + \|a\|_{H^1}^2 + \|a\|_1 + \|a\|_\infty + \|\zeta\|_{H^1}^2 + \|\zeta\|_1 + \|\zeta\|_\infty \le M.$$

Note that ω_j for any $j \in \mathbb{N}$ satisfies

$$\mathrm{supp}(\omega_j) \subset B(x_j, 3C_0M), \quad \omega_j(x,t) = \tilde{\omega}_j(x - x_j, t).$$

It is easy to see that $\zeta \in C^\infty(\mathbb{R}^2)$ because of $\zeta_j \in C_c^\infty(B(0,1))$ and $|x_j - x_k| \gg 1$ for $j \neq k$. By (2.6.1), we also get

$$\|\zeta\|_{\dot{H}^1(\mathbb{R}^2)} + \|\zeta\|_1 + \|\zeta\|_\infty \leq \sum_{j=2}^{\infty} \left\|\nabla\zeta_j\right\|_2 + \left\|\zeta_j\right\|_1 + \left\|\zeta_j\right\|_\infty < \varepsilon.$$

By direct computation, we can also see that $\zeta \notin W^{1,p}(\mathbb{R}^2)$ for $p > 2$.

On the other hand, (2.6.2), (2.6.3), and $\mathrm{supp}(\omega_j(\cdot,t)) \subset B(x_j, 3C_0M)$, $0 \leq t \leq 1$, implies that

$$\begin{aligned}
\left\|\omega(\cdot,t_j)\right\|_{\dot{H}^1} &\geq \left\|\omega_j(\cdot,t_j)\right\|_{\dot{H}^1(B(x_j,3C_0M))} - \left\|(\omega-\omega_j)(\cdot,t_j)\right\|_{\dot{H}^1(B(x_j,3C_0M))} \\
&\geq \left\|\tilde{\omega}_j(\cdot,t_j)\right\|_{\dot{H}^1(\mathbb{R}^2)} - \max_{0\leq t\leq 1} \left\|(\omega-\omega_j)(\cdot,t)\right\|_{\dot{H}^1(B(x_j,3C_0M))} \\
&> j-1.
\end{aligned}$$

Therefore, the constructed perturbation ζ satisfies all requirements in Theorem 2.1.1. If $\mathrm{supp}(a) \not\subset B(0,1)$ up to translation, we slightly modify the proof of the Proposition and obtain the same conclusion.

□

2.7 The compact case

In this section, we prove Theorem 2.1.2, the compact case. Unlike the non-compact case, a large distance between local solutions cannot be used in order to minimize their interactions and make a global solution locally behave like local ones. For this reason, we adopt a different scheme; use the smallness in the L^1-norm of the tail part of a global solution.

The following proposition describes a simple scenario of patching.

Proposition 2.7.1. *Suppose that $f \in C_c^\infty(\mathbb{R}^2)$ satisfies*

$$\begin{aligned}
&\mathrm{supp}(f) \subset \{(x_1,x_2) \in \mathbb{R}^2 : x_1 \leq -2R_0\} \quad \textit{for some } R_0 > 0, \\
&f(x_1,x_2) = -f(x_1,-x_2) \qquad \forall (x_1,x_2) \in \mathbb{R}^2.
\end{aligned} \tag{2.7.1}$$

Then, for any $0 < \varepsilon_0 < \frac{R_0}{100}$, we can find $\delta = \delta(f,\varepsilon_0,R_0) > 0$, $t_0 = t_0(f,\varepsilon_0,R_0) \in (0,\varepsilon_0)$, and $g = g(f,\varepsilon_0,R_0) \in C_c^\infty(B(0,\varepsilon_0))$ such that the following holds.

(i) g satisfies

$$\|g\|_{\dot{H}^1} + \|g\|_\infty + \|g\|_1 + \|g\|_{\dot{H}^{-1}} < \varepsilon_0$$
$$g(x_1,x_2) = -g(x_1,-x_2), \qquad \forall (x_1,x_2) \in \mathbb{R}^2.$$

(ii) For any given $h \in C_c^\infty(\mathbb{R}^2)$ with

$$\mathrm{supp}(h) \subset \{(x_1,x_2) : x_1 \geq R_0\}, \quad \|h\|_1 + \|h\|_\infty \leq \delta, \tag{2.7.2}$$

the smooth solution ω to (LE) *for the initial data $\omega|_{t=0} = f+g+h$ has a decomposition*

$$\omega = \omega_f + \omega_g + \omega_h, \quad \text{on } \mathbb{R}^2 \times [0,t_0]$$

such that

$$\begin{aligned}
&\mathrm{supp}(\omega_f(\cdot,t)) \subset B(\mathrm{supp}(f), \frac{1}{8}R_0),\\
&\mathrm{supp}(\omega_g(\cdot,t)) \subset B(0, \varepsilon_0 + \frac{1}{8}R_0),\\
&\mathrm{supp}(\omega_h(\cdot,t)) \subset B(\mathrm{supp}(h), \frac{1}{8}R_0), \quad \forall 0 \leq t \leq t_0
\end{aligned} \tag{2.7.3}$$

and

$$\|\omega_g(\cdot,t_0)\|_{\dot{H}^1} > \frac{1}{\varepsilon_0}. \tag{2.7.4}$$

To prove this proposition, we need some preliminary lemmas. The first lemma is about the finite time propagation.

Lemma 2.7.2. *Let Ω be a smooth solution to*

$$\begin{cases}
\partial_t\Omega + \nabla^\perp\Delta^{-1}T_\gamma\Omega \cdot \nabla\Omega + (B+E-C)\cdot\nabla\Omega = 0\\
C(t) = (-\partial_2\Delta^{-1}T_\gamma\Omega(0,0,t),\ 0)^\intercal\\
\Omega|_{t=0} = \Omega_0
\end{cases} \tag{2.7.5}$$

where B, E, and Ω_0 are smooth functions satisfying

- $$\|\Omega_0\|_\infty \leq B_0, \quad \textit{for some } B_0 > 0,$$
$$\mathrm{supp}(\Omega_0) \subset B(0,R), \quad \textit{for some } R > 0, \tag{2.7.6}$$

- *B and E are divergence-free*
$$\nabla \cdot B = \nabla \cdot E = 0.$$

- *For some positive numbers B_1 and B_2,*
$$|B(y,t)| \leq B_1|y|, \quad |E(y,t)| \leq B_2|y|^2, \quad \forall (y,t) \in \mathbb{R}^2 \times [0,1].$$

Then, we can find $R_0 > 0$ and $0 < t_0 < 1$ both depending only on B_0, B_1 and B_2 such that if $0 < R \leq R_0$, a characteristic line Φ which solves

$$\begin{cases} \partial_t \Phi(y,t) = (\nabla^\perp \Delta^{-1} T_\gamma \Omega + B + E - C)(\Phi(y,t),t) \\ \Phi(y,0) = y \end{cases} \tag{2.7.7}$$

satisfies
$$|\Phi(y,t)| \leq 2R, \quad \forall |y| \leq R,\ t \in [0,t_0].$$

In particular, the solution Ω satisfies

$$\mathrm{supp}(\Omega(\cdot,t)) \subset B(0,2R), \quad \forall 0 \leq t \leq t_0.$$

Proof. From (2.7.7), we obtain

$$\partial_t |\Phi(y,t)| \leq 2 \left\| \nabla^\perp \Delta^{-1} T_\gamma \Omega \right\|_\infty + B_1 |\Phi(y,t)| + B_2 |\Phi(y,t)|^2. \tag{2.7.8}$$

By using L^p-norm preservation and (2.7.6), we have

$$\left\| \nabla^\perp \Delta^{-1} T_\gamma \Omega \right\|_\infty \lesssim \|\Omega\|_1^{\frac{1}{2}} \|\Omega\|_\infty^{\frac{1}{2}} \lesssim R \|\Omega_0\|_\infty \leq RB_0.$$

Combining with (2.7.8), we can find $t_0 > 0$ and $R_0 > 0$ such that if $0 < R \le R_0$,

$$|\Phi(y,t)| \le 2R, \quad \forall |y| \le R,\ t \in [0,t_0].$$

Furthermore, using the characteristic, (2.7.5) can be written as $\Omega(\Phi(y,t),t) = \Omega_0(y)$, so that

$$\mathrm{supp}(\Omega(\cdot,t)) \subset \Phi(\mathrm{supp}(\Omega_0),t).$$

Then, it easily follows that $\mathrm{supp}(\Omega(\cdot,t)) \subset B(0,2R)$ for any $0 \le t \le t_0$. $\square$

Recall the definition of g_A in (2.4.15). This family of initial data was used in order to create the large Lagrangian deformation. Now, we redefine g_A when $\gamma = \frac{1}{2}$ by

$$g_A(x) = \frac{1}{\ln\ln\ln A} \frac{1}{\sqrt{\ln\ln A}} \sum_{A \le j < A\ln A} \frac{1}{\sqrt{j}} \rho(2^j x), \tag{2.7.9}$$

where ρ is given as in (2.4.14). In the case of $0 < \gamma < \frac{1}{2}$, we use the same g_A in (2.4.15).

Then, g_A satisfies

- $\mathrm{supp}(g_A) \subset B(0, 2 \cdot 2^{-A})$.
-
$$\|g_A\|_1 \lesssim 2^{-2A}, \quad \|g_A\|_\infty \lesssim \frac{1}{A^\gamma}, \quad \|g_A\|_{\dot{H}^{-1}} \lesssim 2^{-A}, \quad \|\nabla g_A\|_2 \lesssim \frac{1}{\ln\ln\ln A}.$$
-
$$\int_{z_1>0, z_2>0} \frac{1}{|z|^2} \ln^{-\gamma}\left(e + \frac{1}{|z|}\right) e^{-|z|^4} g_A(z) dz \gtrsim \frac{\sqrt{\ln\ln A}}{\ln\ln\ln A}.$$

In the next Lemma, we prove that for this newly redefined family, we can create the large Lagrangian deformation even in the presence of a compactly supported perturbation f whose support is away from the origin. In other words, the support of f is away from that of g_A.

Lemma 2.7.3. *Suppose that f satisfies* (2.7.1)*. Let ω be a smooth solution to*

$$\begin{cases} \partial_t \omega + \nabla^\perp \Delta^{-1} T_\gamma \omega \cdot \nabla \omega = 0 \\ \omega|_{t=0} = f + g_A. \end{cases}$$

Then, a characteristic line ϕ which solves

$$\begin{cases} \partial_t \phi(x,t) = \nabla^\perp \Delta^{-1} T_\gamma \omega(\phi(x,t),t) \\ \phi(x,0) = x \end{cases} \tag{2.7.10}$$

satisfies

$$\max_{0 \le t \le \frac{1}{\ln\ln\ln A}} \|D\phi(\cdot,t)\|_{L^\infty(B(0,10\cdot 2^{-A}))} > \ln^{\frac{1}{4}} \ln\ln\ln A \tag{2.7.11}$$

for sufficiently large A.

Proof. Suppose that (2.7.11) doesn't hold true. i.e.,

$$\max_{0 \le t \le \frac{1}{\ln\ln\ln A}} \|D\phi(\cdot,t)\|_{L^\infty(|x| \le 10\cdot 2^{-A})} \le \ln^{\frac{1}{4}} \ln\ln\ln A.$$

First, we decompose the solution ω into ω_f and ω_g, where ω_g solves

$$\begin{cases} \partial_t \omega_g + \nabla^\perp \Delta^{-1} T_\gamma \omega_g \cdot \nabla \omega_g + \nabla^\perp \Delta^{-1} T_\gamma \omega_f \cdot \nabla \omega_g = 0 \\ \omega_g|_{t=0} = g_A. \end{cases}$$

Since both f and g_A are odd in x_2, so are ω and ω_g. Also, we have

$$\phi_1(x_1,-x_2,t) = \phi_1(x_1,x_2,t), \quad \phi_2(x_1,-x_2,t) = -\phi_2(x_1,x_2,t)$$

and therefore $\phi_2(x_1,0,t) = 0$ for any $x_1 \in \mathbb{R}$ and $t \ge 0$. Let $a(t) = \phi_1(0,0,t)$. Then, it satisfies

$$\begin{cases} a'(t) = -\partial_2 \Delta^{-1} T_\gamma \omega(a(t),0,t) \\ a(0) = 0. \end{cases}$$

Similar to (2.5.5)-(2.5.7), we can easily see that the supports of ω_f and ω_g are apart from each other for a short time. Indeed, on $[0, t_A]$, $t_A = \frac{1}{\ln\ln\ln A}$,

$$
\begin{aligned}
\operatorname{supp}(\omega_f(\cdot,t)) &\subset B\left(\operatorname{supp}(f), \frac{1}{8}R_0\right) \subset \left\{x_1 \leq -\frac{15}{8}R_0\right\},\\
\operatorname{supp}(\omega_g(\cdot,t)) &\subset B\left(\operatorname{supp}(g_A), \frac{1}{8}R_0\right) \subset B\left(0, \frac{1}{4}R_0\right),
\end{aligned}
$$

provided that A is sufficiently large. It follows that $\nabla^\perp \Delta^{-1} T_\gamma \omega_f$ is smooth and has Sobolev norm bounds on $B(0, \frac{1}{4}R_0) \times [0, t_A]$, where the bounds depend only on f and R_0. Therefore, we can expand it at the point $(a(t), 0)$, which is in $B(0, \frac{1}{8}R_0)$ for $0 \leq t \leq t_A$, to get

$$
\nabla^\perp \Delta^{-1} T_\gamma \omega_f(a(t)+y_1, y_2, t) = \begin{pmatrix} a'(t) + \partial_2 \Delta^{-1} T_\gamma \omega_g(a(t), 0, t) \\ 0 \end{pmatrix} + b(t) \begin{pmatrix} -y_1 \\ y_2 \end{pmatrix} + E(y,t)
$$

for any $(a(t)+y_1, y_2, t) \in B(0, \frac{1}{4}R_0) \times [0, t_A]$. Here, $b(t) = \partial_{12} \Delta^{-1} T_\gamma \omega_f(a(t), 0, t)$ has a bound $|b(t)| \leq B_1$ for some $B_1 = B_1(R_0, f)$ and a divergence-free vector E can be chosen satisfying

$$
|E(y,t)| \leq B_2 |y|^2, \quad |DE(y,t)| \leq B_2 |y|, \quad |D^2 E(y,t)| \leq B_2, \quad \forall y \in \mathbb{R}^2,
$$

for some $B_2 = B_2(R_0, f)$. In the expansion, we use the oddness of ω_f in x_2 and

$$
\partial_1 \Delta^{-1} T_\gamma \omega_f(a(t), 0, t) = \partial_{11} \Delta^{-1} T_\gamma \omega_f(a(t), 0, t) = \partial_{22} \Delta^{-1} T_\gamma \omega_f(a(t), 0, t) = 0.
$$

We do the change of variables $(x_1, x_2, t) = (a(t)+y_1, y_2, t)$ and denote the solution in a new coordinate system (y,t) by $\Omega(y,t) = \omega_g(a(t)+y_1, y_2, t) = \omega_g(x_1, x_2, t)$. Then, the equation for Ω on $\mathbb{R}^2 \times [0, t_A]$ can be written as

$$
\begin{cases}
\partial_t \Omega + \left(\nabla^\perp \Delta^{-1} T_\gamma \Omega + B + E - C\right) \cdot \nabla \Omega = 0 \\
\Omega|_{t=0} = g_A,
\end{cases}
$$

where B and C are

$$
B = b \begin{pmatrix} -y_1 \\ y_2 \end{pmatrix}, \quad C = \begin{pmatrix} -\partial_2 \Delta^{-1} T_\gamma \Omega(0,0,t) \\ 0 \end{pmatrix}.
$$

Also, we let Φ be a characteristic in a new coordinate, which solves

$$\begin{cases} \partial_t \Phi(y,t) = \left(\nabla^\perp \Delta^{-1} T_\gamma \Omega + B + E - C\right)(\Phi(y,t),t) \\ \Phi(y,0) = y. \end{cases}$$

From now on, without mentioning, we only consider $t \in [0,t_A]$.

We can easily check that $\phi^{-1}(x,t) = \Phi^{-1}(y,t)$ and $\Phi^{-1}(0,t) = \phi^{-1}(a(t),0,t) = 0$. Furthermore, Φ^{-1} satisfies

$$\Phi_1^{-1}(y_1,y_2,t) = \Phi_1^{-1}(y_1,-y_2,t), \quad \Phi_2^{-1}(y_1,y_2,t) = -\Phi_2^{-1}(y_1,-y_2,t).$$

By Lemma 2.7.2, on the other hand, we have

$$|\Phi(y,t)| \le 4 \cdot 2^{-A}, \quad \forall |y| \le 2 \cdot 2^{-A},$$

for sufficiently large A. Also, if $|y| \le 4 \cdot 2^{-A}$ and t_A is sufficiently small, by finite speed propagation, $|\phi^{-1}(a(t)+y_1,y_2,t)| \le 10 \cdot 2^{-A}$. It follows that

$$\begin{aligned} \max_{0\le t\le t_A} \|D\Phi(\cdot,t)\|_{L^\infty(|y|\le 2\cdot 2^{-A})} &\le \max_{0\le t\le t_A} \left\|D(\Phi^{-1})(\cdot,t)\right\|_{L^\infty(|y|\le 4\cdot 2^{-A})} \\ &\le \max_{0\le t\le t_A} \|D\phi(\cdot,t)\|_{L^\infty(|x|\le 10\cdot 2^{-A})} \\ &\le \ln^{\frac{1}{4}} \ln\ln\ln A = M_A. \end{aligned} \tag{2.7.12}$$

Indeed, $(D\Phi(x,t))^{-1} = D(\Phi^{-1})(\Phi(x,t))$ and $(D\phi(x,t))^{-1} = D(\phi^{-1})(\phi(x,t))$ are used in the first and second inequalities, respectively.

Then, by Lemma 2.8.3 with $\phi = \Phi^{-1}$, we have

$$\sup_{0\le t\le t_A} \left\|\mathscr{R}_{11} T_\gamma \Omega(\cdot,t)\right\|_\infty + \left\|\mathscr{R}_{22} T_\gamma \Omega(\cdot,t)\right\|_\infty \le \frac{C_\gamma M_A}{\sqrt{\ln\ln A}} \tag{2.7.13}$$

for $\mathscr{R}_{ij}\omega = \Delta^{-1}\partial_{ij}\omega$ and for some constant $C_\gamma > 0$ depending only on γ.

Now, we find a lower bound of $D\Phi$ which makes a contradiction to (2.7.12).

From the equation for Φ, we get

$$\begin{cases} \partial_t D\Phi(y,t) = \left(D\nabla^\perp \Delta^{-1} T_\gamma \Omega + b \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + DE \right) (\Phi(y,t),t) D\Phi(y,t) \\ D\Phi(y,0) = I, \end{cases} \tag{2.7.14}$$

and the derivative of the velocity can be rewritten as

$$\begin{aligned} & D\nabla^\perp \Delta^{-1} T_\gamma \Omega + b \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + DE \\ & \qquad = \begin{pmatrix} -\mathscr{R}_{12} T_\gamma \Omega - b & 0 \\ 0 & \mathscr{R}_{12} T_\gamma \Omega + b \end{pmatrix} + \begin{pmatrix} 0 & -\mathscr{R}_{22} T_\gamma \Omega \\ \mathscr{R}_{11} T_\gamma \Omega & 0 \end{pmatrix} + DE \\ & \qquad = \begin{pmatrix} -\mathscr{R}_{12} T_\gamma \Omega - b & 0 \\ 0 & \mathscr{R}_{12} T_\gamma \Omega + b \end{pmatrix} + P. \end{aligned}$$

By Grönwall's inequality, we have

$$\begin{aligned} D\Phi(y,t) = & \exp \begin{pmatrix} \int_0^t \lambda(y,s) - b(s) ds & 0 \\ 0 & \int_0^t -\lambda(y,s) + b(s) ds \end{pmatrix} \\ & + \int_0^t \exp \begin{pmatrix} \int_\tau^t \lambda(y,s) - b(s) ds & 0 \\ 0 & \int_\tau^t -\lambda(y,s) + b(s) ds \end{pmatrix} P(\Phi(y,\tau),\tau) D\Phi(y,\tau) d\tau \end{aligned} \tag{2.7.15}$$

where $\lambda(y,t) = -\mathscr{R}_{12} T_\gamma \Omega(\Phi(y,t),t)$.

Since $|\Phi(y,t)| \leq 4 \cdot 2^{-A}$ for $|y| \leq 2 \cdot 2^{-A}$, $A \gg 1$, we have $|DE(\Phi(y,t),t)| \lesssim B_2 2^{-A}$. Combining (2.7.15) with (2.7.12) and (2.7.13), we obtain for $|y| \leq 2 \cdot 2^{-A}$,

$$\exp \left| \int_0^t \lambda(y,s) - b(s) ds \right| \leq M_A + \frac{C_\gamma M_A^2}{\sqrt{\ln \ln A}} \max_{0 \leq \tau \leq t} \exp \left(2 \left| \int_0^\tau \lambda(y,s) - b(s) ds \right| \right).$$

Then, by the continuation argument, we get

$$\exp \left| \int_0^t \lambda(y,s) - b(s) ds \right| \leq 2M_A$$

for sufficiently large A, so that we can consider the second term in (2.7.15) as an error term.

The remaining analysis is similar to the proof of Proposition 2.3.4. Using $\Phi(0,t)=0$, it follows that for $|y|\le 2\cdot 2^{-A}$

$$\Phi(y,t)=\Phi(y,t)-\Phi(0,t)=\int_0^1 \frac{\partial}{\partial\theta}[\Phi(\theta y,t)]d\theta=\left(\int_0^1 D\Phi(\theta y,t)d\theta\right)y$$
$$=\left(y_1\int_0^1 \exp\left(\int_0^t \lambda(\theta y,s)-b(s)ds\right)d\theta, y_2\int_0^1 \exp\left(-\int_0^t \lambda(\theta y,s)-b(s)ds\right)d\theta\right)+e \tag{2.7.16}$$

where

$$|e(y,t)|\lesssim_\gamma \frac{M_A^4}{\sqrt{\ln\ln A}}|y|.$$

Since $\frac{1}{M_A}\gg\frac{M_A^4}{\sqrt{\ln\ln A}}$ if $A\gg 1$ and $y_1\sim y_2$ for $y=(y_1,y_2)\in\operatorname{supp}(g_A)$, it follows that for sufficiently large A, Φ has a sign preserving property;

$$\Phi_1(y,t)>0,\quad \Phi_2(y,t)>0,\qquad y\in\operatorname{supp}(g_A)\cap\{y_1>0,y_2>0\}$$

$$\Phi_1(y,t)<0,\quad \Phi_2(y,t)>0,\qquad y\in\operatorname{supp}(g_A)\cap\{y_1<0,y_2>0\}$$

Based on this, we get

$$\begin{aligned}\lambda(0,t)&=-\mathscr{R}_{12}T_\gamma\Omega(\Phi(0,t),t)=-\mathscr{R}_{12}T_\gamma\Omega(0,t)\\&=\int_{\mathbb{R}^2}K(-z,t)\Omega(z,t)dz=2\int_{z_2>0}K(z,t)\Omega(z,t)dz=2\int_{z_2>0}K(\Phi(z,t),t)g_A(z)dz\\&\ge\int_{z_1>0,z_2>0}K(\Phi(z,t),t)g_A(z)dz\end{aligned}$$

where K is the kernel of the operator $-\partial_{12}\Delta^{-1}T_\gamma$. The fourth equality follows from the parity of K and Ω in z_2. The last inequality follows from the positiveness of the integrand on $\{z_1<0,z_2>0\}$.

Note that if $z\in\operatorname{supp}(g_A)\cap\{z_1>0,z_2>0\}$, we have

$$\frac{1}{2}<\frac{z_1}{z_2}<2$$

and hence by (2.7.16)

$$\frac{1}{10M_A^2} < \frac{\Phi_1(z,t)}{\Phi_2(z,t)} < 10M_A^2.$$

Also, we have $\frac{|z|}{M_A} \le |\Phi(z,t)| \le M_A|z|$ for $z \in \mathrm{supp}(g_A)$. Then, by Lemma 2.3.1 and Lemma 2.3.3, we get

$$\begin{aligned}
&\int_{z_1>0,z_2>0} K(\Phi(z,t),t)g_A(z)dz \\
&\qquad \gtrsim_\gamma \int_{z_1>0,z_2>0} \frac{\Phi_1(z,t)\Phi_2(z,t)}{|\Phi(z,t)|^4} \ln^{-\gamma}\left(e+\frac{1}{|\Phi(z,t)|}\right) e^{-|\Phi(z,t)|^2} g_A(z)dz \\
&\qquad \ge \frac{1}{M_A^2}\int_{z_1>0,z_2>0} \frac{1}{\frac{\Phi_1(z,t)}{\Phi_2(z,t)}+\frac{\Phi_2(z,t)}{\Phi_1(z,t)}} \cdot \frac{1}{|z|^2}\ln^{-\gamma}\left(e+\frac{M_A}{|z|}\right) e^{-M_A^2|z|^2} g_A(z)dz \\
&\qquad \gtrsim \frac{e^{\frac{3}{4}M_A^4}}{M_A^4(1+\ln(1+M_A))^\gamma} e^{-M_A^4} \int_{z_1>0,z_2>0} \frac{1}{|z|^2}\ln^{-\gamma}\left(e+\frac{1}{|z|}\right) e^{-|z|^4} g_A(z)dz \\
&\qquad \gtrsim e^{-M_A^4}\int_{z_1>0,z_2>0} \frac{1}{|z|^2}\ln^{-\gamma}\left(e+\frac{1}{|z|}\right) e^{-|z|^4} g_A(z)dz \\
&\qquad \gtrsim_\gamma \frac{\sqrt{\ln\ln A}}{\ln\ln\ln A} e^{-M_A^4} = \frac{\sqrt{\ln\ln A}}{(\ln\ln\ln A)^2},
\end{aligned}$$

provided that $A \gg 1$. Therefore, we get

$$\begin{aligned}
\max_{0\le t\le t_A} \|D\Phi(\cdot,t)\|_{L^\infty(|y|\le 2\cdot 2^{-A})} &\ge \max_{0\le t\le t_A} |D\Phi(0,t)| \\
&\ge \exp\left(\left(\frac{C_\gamma\sqrt{\ln\ln A}}{(\ln\ln\ln A)^2} - B_1\right)\frac{1}{\ln\ln\ln A}\right) - 1,
\end{aligned}$$

which makes a contradiction to (2.7.12)

$$\max_{0\le t\le t_A} \|D\Phi(\cdot,t)\|_{L^\infty(|y|\le 2\cdot 2^{-A})} \le \ln^{\frac{1}{4}}\ln\ln\ln A$$

for sufficiently large A. □

Now, we give a proof of the main proposition.

Proof of Proposition 2.7.1.

`Step 1.` Critical norm inflation of a local solution.

By Lemma 2.7.3, we can create the large Lagrangian deformation (2.7.11) at the presence of f satisfying (2.7.1). Then, similar to Proposition 2.4.2, we can find a perturbed initial data $\tilde{g}_A \in C_c^\infty(|x| \lesssim 2^{-A})$ from g_A such that it satisfies

$$\tilde{g}_A(x_1,x_2) = -\tilde{g}_A(x_1,-x_2),$$
$$\|\tilde{g}_A\|_{\dot{H}^1} + \|\tilde{g}_A\|_\infty + \|\tilde{g}_A\|_1 + \|\tilde{g}_A\|_{\dot{H}^{-1}} \lesssim \ln^{-\frac{1}{8}} \ln\ln\ln A,$$

and the smooth solution $\tilde{\omega}^{(A)}$ to

$$\begin{cases} \partial \tilde{\omega}^{(A)} + \nabla^\perp \Delta^{-1} T_\gamma \tilde{\omega}^{(A)} \cdot \nabla \tilde{\omega}^{(A)} = 0 \\ \tilde{\omega}^{(A)}|_{t=0} = f + \tilde{g}_A \end{cases}$$

has a decomposition $\tilde{\omega}^{(A)} = \tilde{\omega}_f^{(A)} + \tilde{\omega}_{\tilde{g}}^{(A)}$ where $\tilde{\omega}_g^{(A)}$ satisfies

$$\max_{0 \le t \le \frac{1}{\ln\ln\ln A}} \left\| \nabla \tilde{\omega}_g^{(A)}(\cdot,t) \right\|_2 \ge \ln^{\frac{1}{12}} \ln\ln\ln A,$$

for sufficiently large A. Indeed, $\tilde{\omega}_g^{(A)}$ solves

$$\begin{cases} \partial \tilde{\omega}_g^{(A)} + \nabla^\perp \Delta^{-1} T_\gamma \tilde{\omega}^{(A)} \cdot \nabla \tilde{\omega}_g^{(A)} = 0 \\ \tilde{\omega}_g^{(A)}|_{t=0} = \tilde{g}_A. \end{cases}$$

Then, we construct g by choosing $A_0 = A_0(\varepsilon_0) \gg 1$ such that $g = \tilde{g}_{A_0} \in C_c^\infty(B(0,\varepsilon_0))$ and $\tilde{\omega}_g = \tilde{\omega}_g^{(A_0)}$ satisfies

$$\|g\|_{\dot{H}^1} + \|g\|_\infty + \|g\|_1 + \|g\|_{\dot{H}^{-1}} < \varepsilon_0,$$

and

$$\max_{0 \le t \le \frac{1}{\ln\ln\ln A_0}} \|\nabla \tilde{\omega}_g(\cdot,t)\|_2 > \frac{2}{\varepsilon_0}.$$

In particular, we can find $0 < t_0 \le \frac{1}{\ln\ln\ln A_0} < \varepsilon_0$ such that

$$\|\nabla \tilde{\omega}_g(\cdot,t_0)\|_2 > \frac{2}{\varepsilon_0}. \tag{2.7.17}$$

`Step 2.` Patch a function h.

Suppose that h satisfies (2.7.2) and $\delta < 1$. Let ω be a solution to

$$\begin{cases} \partial_t \omega + \nabla^\perp \Delta^{-1} T_\gamma \omega \cdot \nabla \omega = 0 \\ \omega|_{t=0} = f + g + h. \end{cases}$$

We decompose $\omega = \omega_f + \omega_g + \omega_h$ where ω_f and ω_g are defined as solutions to

$$\begin{cases} \partial_t \omega_f + \nabla^\perp \Delta^{-1} T_\gamma \omega \cdot \nabla \omega_f = 0 \\ \omega|_{t=0} = f \end{cases}$$

and

$$\begin{cases} \partial_t \omega_g + \nabla^\perp \Delta^{-1} T_\gamma \omega \cdot \nabla \omega_g = 0 \\ \omega|_{t=0} = g, \end{cases}$$

respectively. Since

$$\left\| \nabla^\perp \Delta^{-1} T_\gamma \omega \right\|_\infty \lesssim \|\omega\|_1 + \|\omega\|_\infty = \|\omega|_{t=0}\|_1 + \|\omega|_{t=0}\|_\infty \lesssim 1 + \|f\|_1 + \|f\|_\infty,$$

similar to (2.5.5) and (2.5.6), we can easily check ω_f, ω_g and ω_h satisfies (2.7.3), provided that t_0 is sufficiently small. If necessary, we can adjust the choice of A_0 to make t_0 small enough.

Now, recall (2.5.23). By the assumption (2.7.2) on h and (2.7.3), we have

$$\left\| \nabla^\perp \Delta^{-1} T_\gamma \omega_h(\cdot,t) \right\|_{L^\infty(B(\mathrm{supp}(f), \frac{1}{8}R_0) \cup B(0, \varepsilon_0 + \frac{1}{8}R_0))} \lesssim_{R_0} \|h\|_1 \le \delta$$

for any $0 \le t \le t_0$. Then, by the same arguments in Lemma 2.5.3, we get

$$\begin{aligned} \|(\omega_g - \tilde{\omega}_g)(\cdot,t_0)\|_{H^2} &\le \max_{0 \le t \le t_0} \left\| ((\omega_f + \omega_g) - \tilde{\omega})(\cdot,t) \right\|_{H^2} \\ &\le C(\|f\|_{H^3}, R_0, \mathrm{supp}(f))\delta \le \frac{1}{\varepsilon_0}, \end{aligned}$$

provided that $\delta \in (0,1)$ is sufficiently small. Combining with (2.7.17), we obtain the desired inflation (2.7.4).

□

Before we prove Theorem 2.1.2, we need the following lemma for the uniqueness.

Lemma 2.7.4. *Suppose that $f \in C_c^\infty(\mathbb{R}^2)$ with the compact support in $B(0,R)$ for some $R > 0$ and $g \in L^\infty(\mathbb{R}^2) \cap \dot{H}^{-1}(\mathbb{R}^2)$ with $\|g\|_\infty \leq M$ for some $M > 0$. Let $\tilde{\omega}$ be a smooth solution to*

$$\begin{cases} \partial_t \tilde{\omega} + \nabla^\perp \Delta^{-1} T_\gamma \tilde{\omega} \cdot \nabla \tilde{\omega} = 0 \\ \tilde{\omega}|_{t=0} = f \end{cases}$$

and ω be a weak solution in $C([0,1]; L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2))$ to

$$\begin{cases} \partial_t \omega + \nabla^\perp \Delta^{-1} T_\gamma \omega \cdot \nabla \omega = 0 \\ \omega|_{t=0} = f + g \end{cases}$$

satisfying L^∞-norm preservation

$$\|\omega(\cdot,t)\|_\infty = \|f+g\|_\infty, \quad \forall 0 \leq t \leq 1.$$

Then, for any $\varepsilon > 0$, we can find a constant $\delta = \delta(\varepsilon, f, M) > 0$ such that if $\|g\|_{\dot{H}^{-1}} < \delta$,

$$\max_{0 \leq t \leq 1} \|(\omega - \tilde{\omega})(\cdot,t)\|_{\dot{H}^{-1}(\mathbb{R}^2)} < \varepsilon.$$

Furthermore, under the additional assumption $g \in C_c^\infty(B(0,R))$, we have $\tilde{\delta} = \tilde{\delta}(\varepsilon, R, f) > 0$ such that if $\|g\|_\infty < \tilde{\delta}$,

$$\max_{0 \leq t \leq 1} \|(\omega - \tilde{\omega})(\cdot,t)\|_{L^\infty(\mathbb{R}^2)} < \varepsilon.$$

Proof. The equation for $\eta = \omega - \tilde{\omega}$ is

$$\begin{cases} \partial_t \eta + \nabla^\perp \Delta^{-1} T_\gamma \eta \cdot \nabla \omega + \nabla^\perp \Delta^{-1} T_\gamma \tilde{\omega} \cdot \nabla \eta = 0 \\ \eta|_{t=0} = g. \end{cases}$$

Taking $\int_{\mathbb{R}^2} \cdot \Lambda^{-2}\eta\, dx$, $\Lambda = (-\Delta)^{\frac{1}{2}}$, on both side of the equation, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\eta\|_{\dot{H}^{-1}(\mathbb{R}^2)}^2 &\leq \left|\int \omega(\nabla^\perp\Delta^{-1}T_\gamma\eta\cdot\nabla)\Lambda^{-2}\eta\, dx\right| + \left|\int \eta(\nabla^\perp\Delta^{-1}T_\gamma\tilde{\omega}\cdot\nabla)\Lambda^{-2}\eta\, dx\right| \\
&\leq \|\omega\|_\infty \|\eta\|_{\dot{H}^{-1}(\mathbb{R}^2)}^2 + \left\|\Lambda^{-1}\eta\right\|_2 \left\|[\Lambda, \nabla^\perp\Delta^{-1}T_\gamma\tilde{\omega}\cdot\nabla]\Lambda^{-2}\eta\right\|_2 \\
&\lesssim (\|f\|_\infty + \|g\|_\infty + \left\|D\nabla^\perp\Delta^{-1}T_\gamma\tilde{\omega}\right\|_\infty)\|\eta\|_{\dot{H}^{-1}}^2.
\end{aligned}$$

Here, the second inequality follows from

$$\int \Lambda^{-1}\eta(\nabla^\perp\Delta^{-1}T_\gamma\tilde{\omega}\cdot\nabla)\Lambda^{-1}\eta\, dx = \frac{1}{2}\int(\nabla^\perp\Delta^{-1}T_\gamma\tilde{\omega}\cdot\nabla)|\Lambda^{-1}\eta|^2 dx = 0$$

and the third one from the commutator estimate

$$\|\Lambda(lm) - l(\Lambda m)\|_2 \lesssim \|Dl\|_\infty \|m\|_2.$$

By the Grönwall inequality, we have

$$\max_{0\leq t\leq 1}\|\eta(\cdot,t)\|_{\dot{H}^{-1}(\mathbb{R}^2)} \leq \|g\|_{\dot{H}^{-1}(\mathbb{R}^2)} \exp(C(\|f\|_\infty + \|f\|_{W^{1,4}(\mathbb{R}^2)} + M)).$$

Therefore, for given $\varepsilon > 0$, we can find the desired $\delta = \delta(\varepsilon, f, M)$.

Now, we further assume that g is in $C_c^\infty(B(0,R))$. Then, the weak solution ω becomes a smooth solution. The equation for η can be rewritten as

$$\partial_t\eta + \nabla^\perp\Delta^{-1}T_\gamma\omega\cdot\nabla\eta + \nabla^\perp\Delta^{-1}T_\gamma\eta\cdot\nabla\tilde{\omega} = 0,$$

so that we have

$$\|\eta(\cdot,t)\|_\infty \leq \|g\|_\infty + \int_0^t \left\|(\nabla^\perp\Delta^{-1}T_\gamma\eta)(\cdot,s)\right\|_\infty \|\nabla\tilde{\omega}(\cdot,s)\|_\infty\, ds. \qquad (2.7.18)$$

By the usual energy estimate, we have

$$\max_{0\leq t\leq 1}\|\nabla\tilde{\omega}(\cdot,t)\|_\infty \lesssim_f 1.$$

Using $f, g \in C_c^\infty(B(0,R))$ and Lebesgue measure preservation of the supports ω

and $\tilde{\omega}$,

$$\begin{aligned}\left\|(\nabla^{\perp}\Delta^{-1}T_{\gamma}\eta)(\cdot,s)\right\|_{\infty} &\lesssim \|\eta(\cdot,t)\|_{1}^{\frac{1}{2}}\|\eta(\cdot,t)\|_{\infty}^{\frac{1}{2}} \\ &\leq (|\operatorname{supp}(\omega(\cdot,0))|+|\operatorname{supp}(\tilde{\omega}(\cdot,0))|)^{\frac{1}{2}}\|\eta(\cdot,t)\|_{\infty} \\ &\lesssim R\|\eta(\cdot,t)\|_{\infty}.\end{aligned}$$

Then, combining with (2.7.18) and using Grönwall inequality, we have

$$\max_{0\leq t\leq 1}\|\eta(\cdot,t)\|_{\infty}\leq C(R,f)\|g\|_{\infty}.$$

This completes the proof. □

Finally, we find the compactly supported perturbation in our main theorem.

Proof of Theorem 2.1.2.

Fix $0<\varepsilon<\frac{1}{200}$. Without loss of generality, we may assume the support of the given initial data lies on $\{x=(x_1,x_2): x_1\leq -1\}\cap B(0,\overline{R})$ for some $\overline{R}\geq 10$. (Otherwise, using the translation invariant property of the solution, we apply the proof for a suitably translated initial data in x_1 direction. Note that the translated one is still odd in x_2.) Let $\{x_n=(x_n^1,0)\}$ be a sequence of centres with

$$x_1^1=0,\qquad x_n^1=\sum_{j=1}^{n-1}\frac{1}{2^j}\quad\text{for } n\geq 2.$$

Now, we construct sequences $\{\zeta_n\}_{n\in\mathbb{N}}\subset C_c^{\infty}(B(0,2^{-(n+1)}))$, $\{(\delta_n,\tilde{\delta}_n,t_n)\}_{n\in\mathbb{N}}\subset\mathbb{R}_+^3$ such that for any $n\in\mathbb{N}$,

- ζ_n is odd in x_2 and satisfies

$$\|\zeta_n\|\equiv\|\zeta_n\|_{\dot{H}^1}+\|\zeta_n\|_{\infty}+\|\zeta_n\|_1+\|\zeta_n\|_{\dot{H}^{-1}}<\min\left(\frac{\varepsilon}{2^n},\frac{\delta_{n-1}}{2^{n-1}},\frac{\tilde{\delta}_{n-1}}{2^{n-1}}\right),$$

  where $\delta_0=\tilde{\delta}_0=1$.

- for any $h \in C_c^\infty(\mathbb{R}^2)$ with

$$\begin{aligned} \operatorname{supp}(h) \subset \{x = (x_1, x_2) : x_1 \geq \frac{1}{2^{n+1}}\} \\ \|h\|_1 + \|h\|_\infty \leq \delta_n, \end{aligned} \tag{2.7.19}$$

  a smooth solution ω to (LE) for the initial data

$$\omega|_{t=0}(x) = a(x + x_n) + \sum_{j=1}^{n-1} \zeta_j(x - x_j + x_n) + \zeta_n(x) + h(x)$$

  has a decomposition

$$\omega = \omega_{\leq n-1} + \omega_n + \omega_h$$

  such that the supports of $\omega_{\leq n-1}$, ω_n, and ω_h are disjoint for $t \in [0, t_n]$, and

$$\|\omega_n(\cdot, t_n)\|_{\dot{H}^1} > 2^n.$$

- $\{\delta_n\}$ and $\{\tilde{\delta}_n\}$ are decreasing sequences. Also, t_n converges to 0.

- for any g satisfying $\|g\|_\infty \leq 1$ and $\|g\|_{\dot{H}^{-1}} \leq \tilde{\delta}_n$,

$$\max_{0 \leq t \leq 1} \|(\tilde{\omega} - \tilde{\omega}_{\leq n})(\cdot, t)\|_{\dot{H}^{-1}} < \frac{1}{2^n}. \tag{2.7.20}$$

  where $\tilde{\omega} \in C([0,1]; L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2))$ is a weak solution having L^∞-norm preservation and $\tilde{\omega}_{\leq n}$ is a smooth solution to (LE) for initial data

$$\tilde{\omega}|_{t=0}(x) = a(x) + \sum_{j=1}^{n} \zeta_j(x - x_j) + g, \quad \tilde{\omega}_{\leq n}|_{t=0} = a(x) + \sum_{j=1}^{n} \zeta_j(x - x_j). \tag{2.7.21}$$

  Furthermore, if $g \in C_c^\infty(B(0, \overline{R}))$ with $\|g\|_\infty \leq \tilde{\delta}_n$, we have

$$\max_{0 \leq t \leq 1} \|(\tilde{\omega} - \tilde{\omega}_{\leq n})(\cdot, t)\|_\infty < \frac{1}{2^n}.$$

The construction is based on induction. First, we choose ζ_1, and $(\delta_1, \tilde{\delta}_1, t_1)$. By

Proposition 2.7.1 with

$$f = a(x) = a(x + x_1), \quad R_0 = \frac{1}{4}, \quad \varepsilon_0 = \frac{\varepsilon}{2},$$

there exist an smooth function ζ_1 odd in x_2 and compactly supported in $B(0, \frac{1}{4})$, and positive constants $0 < \delta_1 < \delta_0$ and $0 < t_1 < \frac{1}{2}$ which satisfy the following:

- $$\|\zeta_1\| < \frac{\varepsilon}{2}$$
- If $h \in C_c^\infty(\mathbb{R}^2)$ satisfies

  $$\text{supp}(h) \subset \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 \geq \frac{1}{4}\},$$
  $$\|h\|_1 + \|h\|_\infty \leq \delta_1,$$

  a smooth solution ω to (LE) for the initial data

  $$\omega|_{t=0} = a(x) + \zeta_1(x) + h(x)$$

  has a decomposition

  $$\omega = \omega_a + \omega_1 + \omega_h, \quad \text{on } \mathbb{R}^2 \times [0, t_1]$$

  such that the supports of ω_a, ω_1, and ω_h are disjoint for $t \in [0, t_1]$ and

  $$\|\omega_1(\cdot, t_1)\|_{\dot{H}^1} > 2.$$

Then, we apply Lemma 2.7.4 for $f = a + \zeta_1$, $R = \overline{R}$, $M = 1$, and $\varepsilon = \frac{1}{2}$, so that we obtain $0 < \tilde{\delta}_1 \leq \tilde{\delta}_0$ such that if $\|g\|_\infty \leq 1$, and $\|g\|_{\dot{H}^{-1}} \leq \tilde{\delta}_1$, then we have

$$\max_{0 \leq t \leq 1} \|(\tilde{\omega} - \tilde{\omega}_{\leq 1})(\cdot, t)\|_{\dot{H}^{-1}(\mathbb{R}^2)} < \frac{1}{2},$$

where $\tilde{\omega}$ and $\tilde{\omega}_{\leq 1}$ are solutions to (LE) for the initial data

$$\tilde{\omega}|_{t=0} = a + \zeta_1 + g, \quad \tilde{\omega}_{\leq 1}|_{t=0} = a + \zeta_1.$$

Furthermore, if $g \in C_c^\infty(B(0,\overline{R}))$ satisfies $\|g\|_\infty \le \tilde{\delta}_1$, then we have

$$\max_{0\le t\le 1} \|(\tilde{\omega} - \tilde{\omega}_{\le 1})(\cdot,t)\|_\infty < \frac{1}{2},$$

Therefore, we obtain the desired ζ_1 and $(\delta_1, \tilde{\delta}_1, t_1)$.

Assume that we have $\{\zeta_j\}_{j=1}^n$ and $\{(\delta_j, \tilde{\delta}_j, t_j)\}_{j=1}^n$ satisfying all conditions above. Then, applying Proposition 2.7.1 for

$$f = a(x+x_{n+1}) + \sum_{j=1}^n \zeta_j(x - x_j + x_{n+1}), \quad R_0 = \frac{1}{2^{n+2}}, \quad \varepsilon_0 = \min\left(\frac{\varepsilon}{2^{n+1}}, \frac{\delta_n}{2^n}, \frac{\tilde{\delta}_n}{2^n}\right),$$

we can find $\zeta_{n+1} \in C_c^\infty(B(0, 2^{-(n+2)}))$ odd in x_2, and $0 < \delta_{n+1} \le \delta_n$ and $0 < t_{n+1} < \frac{1}{2^{n+1}}$ such that

- $$\|\zeta_{n+1}\| < \min\left(\frac{\varepsilon}{2^{n+1}}, \frac{\delta_n}{2^n}, \frac{\tilde{\delta}_n}{2^n}\right).$$

- for any $h \in C_c^\infty(\mathbb{R}^2)$ with

  $$\operatorname{supp}(h) \subset \{x = (x_1, x_2) : x_1 \ge \frac{1}{2^{n+2}}\}$$
  $$\|h\|_1 + \|h\|_\infty \le \delta_{n+1},$$

  the smooth solution ω to (LE) for the initial data

  $$\omega|_{t=0}(x) = a(x+x_{n+1}) + \sum_{j=1}^n \zeta_j(x - x_j + x_{n+1}) + \zeta_{n+1}(x) + h(x)$$

  has a decomposition

  $$\omega = \omega_{\le n} + \omega_{n+1} + \omega_h, \quad \text{on } \mathbb{R}^2 \times [0, t_{n+1}]$$

  such that the supports of $\omega_{\le n}$, ω_{n+1}, and ω_h are disjoint for $t \in [0, t_{n+1}]$, and

  $$\|\omega_{n+1}(\cdot, t_{n+1})\|_{\dot{H}^1} > 2^{n+1}.$$

Once we obtain ζ_{n+1}, applying Lemma 2.7.4 for $f(x) = a(x) + \sum_{j=1}^{n+1} \zeta_j(x - x_j)$, $R = \overline{R}$, $M = 1$, and $\varepsilon = 2^{-(n+1)}$, we can find $0 < \tilde{\delta}_{n+1} \le \tilde{\delta}_n$ such that for any g with $\|g\|_\infty \le 1$ and $\|g\|_{\dot{H}^{-1}} \le \tilde{\delta}_{n+1}$, we have

$$\max_{0 \le t \le 1} \|(\tilde{\omega} - \tilde{\omega}_{\le n+1})(\cdot, t)\|_\infty < \frac{1}{2^{n+1}},$$

where $\tilde{\omega}$ and $\tilde{\omega}_{\le n+1}$ solves (LE) for the initial data

$$\tilde{\omega}|_{t=0} = a + \sum_{j=1}^{n+1} \zeta_j(\cdot - x_j) + g, \quad \tilde{\omega}_{\le n+1}|_{t=0} = a + \sum_{j=1}^{n+1} \zeta_j(\cdot - x_j).$$

If g further satisfies $g \in C_c^\infty(B(0, \overline{R}))$ and $\|g\|_\infty \le \tilde{\delta}_{n+1}$, we get

$$\max_{0 \le t \le 1} \|(\tilde{\omega} - \tilde{\omega}_{\le n+1})(\cdot, t)\|_\infty < \frac{1}{2^{n+1}}.$$

Therefore, by the induction argument, we obtain the desired sequences $\{\zeta_n\}$, $\{(\delta_n, \tilde{\delta}_n, t_n)\}$.

Now, we set the perturbation as

$$\zeta(x) = \sum_{j=1}^{\infty} \zeta_j(x - x_j).$$

Obviously, the perturbation satisfies

$$\|\zeta\|_{\dot{H}^1} + \|\zeta\|_\infty + \|\zeta\|_1 + \|\zeta\|_{\dot{H}^{-1}} = \|\zeta\| \le \sum_{j=1}^{\infty} \|\zeta_j\| < \varepsilon.$$

Since $\zeta_{n+1}(\cdot - x_{n+1}) \in C_c^\infty(B(0, \overline{R}))$ and $\|\zeta_{n+1}\|_\infty \le \|\zeta_{n+1}\| \le \tilde{\delta}_n$, we plug $g = \zeta_{n+1}(\cdot - x_{n+1})$ into (2.7.21) to get

$$\max_{0 \le t \le 1} \|(\tilde{\omega}_{\le n+1} - \tilde{\omega}_{\le n})(\cdot, t)\|_\infty < \frac{1}{2^n}. \tag{2.7.22}$$

Indeed, for any $n \in \mathbb{N}$, $\zeta_n(\cdot - x_n) \in C_c^\infty(B(0, \overline{R}))$, and by finite speed propagation we have $\tilde{\omega}_{\le n} \in C([0,1] \times \overline{B(0, R_*)})$ for some finite number R_*. Then, (2.7.22) implies that $\{\tilde{\omega}_{\le n}\}$ is Cauchy in $C([0,1] \times \overline{B(0, R_*)})$, and hence we have its limit $\omega \in C([0,1]; C_c(\mathbb{R}^2))$. On the other hand, since the L^∞-norm of $\tilde{\omega}_{\le n}$ is preserved for any $n \in \mathbb{N}$, so is that of ω.

Now, we check that ω is the unique weak solution in $C([0,1];L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2))$ to the equation (LE) for the initial data

$$\omega|_{t=0} = a+\zeta, \tag{2.7.23}$$

having L^∞-norm preservation. Since $\tilde{\omega}_{\leq n}$ is smooth solution to (LE), it satisfies for any $\varphi \in C^1([0,1];C_c^1(\mathbb{R}^2))$ and $n \in \mathbb{N}$,

$$\int_{\mathbb{R}^2} \tilde{\omega}_{\leq n}(x,1)\varphi(x,1)dx = \int_{\mathbb{R}^2} \tilde{\omega}_{\leq n}(x,0)\varphi(x,0)dx + \int_0^1 \int_{\mathbb{R}^2} (\partial_s \varphi + \nabla^\perp \Delta^{-1} T_\gamma \tilde{\omega}_{\leq n} \cdot \nabla \varphi)\tilde{\omega}_{\leq n} dxds.$$

Sending n to infinity, ω solves (LE) in a weak sense. Then the uniqueness follows from (2.7.20). Indeed, for any weak solution $\overline{\omega} \in C^1([0,1];L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2))$ to (LE) for the same initial data with ω having L^∞-norm preservation, we have

$$\max_{0\leq t\leq 1} \|\overline{\omega} - \tilde{\omega}_{\leq n}(\cdot,t)\|_{\dot{H}^{-1}(\mathbb{R}^2)} < \frac{1}{2^n},$$

for sufficiently large n. Here, we use $\sup_j \left\|\zeta_j\right\|_\infty \leq 1$ and

$$\sum_{j=n+1}^{\infty} \left\|\zeta_j(\cdot - x_j)\right\|_{\dot{H}^{-1}(\mathbb{R}^2)} < \sum_{j=n+1}^{\infty} \frac{\tilde{\delta}_{j-1}}{2^{j-1}} \leq \tilde{\delta}_n \sum_{j=n+1}^{\infty} \frac{1}{2^{j-1}} \leq \tilde{\delta}_n.$$

Therefore, if the weak solution is not unique, i.e., $\overline{\omega} \neq \omega$, then it makes a contradiction to

$$\max_{0\leq t\leq 1} \|(\overline{\omega} - \omega)(\cdot,t)\|_{\dot{H}^{-1}(\mathbb{R}^2)} < \frac{1}{2^{n-1}}, \quad \forall n \in \mathbb{N}.$$

Therefore, we obtain uniqueness.

Finally, since $h = \sum_{j=n+1}^{\infty} \zeta_j(x - x_j + x_n)$ satisfies the conditions (2.7.19), we have

$$\|\omega(\cdot,t_n)\|_{\dot{H}^1} \geq \|\omega_n(\cdot,t_n)\|_{\dot{H}^1} > 2^n. \tag{2.7.24}$$

Indeed, in Proposition 2.7.1, the assumption $h \in C_c^\infty(\mathbb{R}^2)$ can be dropped if we have a unique weak solution $\omega \in C([0,1];L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2))$ to (LE) with initial data $\omega|_{t=0} = f + g + h$. This leads to (2.7.24).

Using the continuity of $\|\omega_n(\cdot,t)\|_{H^1}$, we have a short time interval $[t_n^l, t_n^r]$, $t_n^l \le t_n \le t_n^r$ such that t_n^r converges to 0 and

$$\|\omega(\cdot,t)\|_{\dot{H}^1} > n, \quad \forall t_n^l \le t \le t_n^r.$$

This implies the desired critical Sobolev norm inflation.

□

2.8 Analysis of the velocity

In this section, we provide proofs of some inequalities for self-containedness.

2.8.1 Kernel for the velocity

In this section, we estimate the kernel H in the velocity $u = \nabla^{\perp}\Delta^{-1}T_\gamma\omega = H * \omega$.

Lemma 2.8.1. *Let $\gamma > 0$ and H is the kernel of the multiplier $\nabla^{\perp}\Delta^{-1}T_\gamma$, where T_γ is either*

$$T_\gamma = \ln^{-\gamma}(e - \Delta), \quad \textit{or} \quad T_\gamma = \ln^{-\gamma}(e + |\nabla|).$$

Then, for each α with $|\alpha| \ge 0$, we have

$$|\partial^\alpha H(x)| \lesssim_\alpha \frac{1}{|x|^{|\alpha|+1}}, \qquad \forall x \neq 0. \tag{2.8.1}$$

Proof. By a similar argument in Lemma 2.3.1 and Lemma 2.3.3, we have an explicit expression of the kernel H_Δ of the multiplier $\nabla^{\perp}\Delta^{-1}\ln^{-\gamma}(e - \Delta)$,

$$H_\Delta(x) = \frac{C}{\Gamma(\gamma)}\frac{x^{\perp}}{|x|^2}\int_0^\infty \frac{1}{\Gamma(t)}\int_0^\infty e^{-e\beta}(1 - e^{-\frac{|x|^2}{4\beta}})\beta^t\frac{d\beta}{\beta}t^\gamma\frac{dt}{t} =: \frac{x^{\perp}}{|x|^2}H_r(x)$$

where $x^{\perp} = (-x_2, x_1)$ for some absolute constant $C > 0$.

Also, the kernel $\tilde{H}_{|\nabla|}$ of the multiplier $\nabla^{\perp}\Delta^{-1}\ln^{-\gamma}(e+|\nabla|)$ is

$$\begin{aligned}\tilde{H}_{|\nabla|} &= \frac{\tilde{C}}{\Gamma(\gamma)}\frac{x^{\perp}}{|x|^2}\int_0^{\infty}\frac{1}{\Gamma(t)}\int_0^{\infty}e^{-\tau}\int_0^{\infty}e^{-e\beta}\left(1-e^{-\frac{\tau|x|^2}{\beta^2}}\right)\beta^t\frac{d\beta}{\beta}\tau^{-\frac{1}{2}}d\tau t^{\gamma}\frac{dt}{t},\\ &=: \frac{x^{\perp}}{|x|^2}\tilde{H}_r(x)\end{aligned}$$

for some constant $\tilde{C}>0$.

Using $|t^n e^{-t}|\leq C(n)$ for any $t\geq 0$, we have for each $|\alpha|\geq 0$,

$$|\partial^{\alpha}(1-e^{-\frac{|x|^2}{4\beta}})|\lesssim_{\alpha}\frac{1}{|x|^{|\alpha|}}\qquad \forall x\neq 0,\ \beta>0,$$

where the constant in the inequality is independent of β. Since

$$\frac{1}{\Gamma(\gamma)}\int_0^{\infty}\frac{1}{\Gamma(t)}\int_0^{\infty}e^{-e\beta}\beta^t\frac{d\beta}{\beta}t^{\gamma}\frac{dt}{t}=\frac{1}{\Gamma(\gamma)}\int_0^{\infty}e^{-t}t^{\gamma}\frac{dt}{t}=1,$$

we can easily get

$$|\partial^{\alpha}H_r(x)|\lesssim_{\alpha}\frac{1}{|x|^{|\alpha|}}.$$

On the other hand, we have

$$\frac{1}{\Gamma(\gamma)}\int_0^{\infty}\frac{1}{\Gamma(t)}\int_0^{\infty}e^{-\tau}\int_0^{\infty}e^{-e\beta}\beta^t\frac{d\beta}{\beta}\tau^{-\frac{1}{2}}d\tau t^{\gamma}\frac{dt}{t}=\int_0^{\infty}e^{-\tau}\tau^{-\frac{1}{2}}d\tau\lesssim 1.$$

Therefore, we also obtain

$$|\partial^{\alpha}\tilde{H}_r(x)|\lesssim_{\alpha}\frac{1}{|x|^{|\alpha|}}.$$

Finally, since for each $|\alpha|\geq 0$, we have

$$\left|\partial^{\alpha}\left(\frac{x_i}{|x|^2}\right)\right|\lesssim_{\alpha}\frac{1}{|x|^{|\alpha|+1}},\qquad \forall x\neq 0,$$

the desired estimate (2.8.1) follows easily. □

2.8.2 Operator norm of T_γ on L^p

Lemma 2.8.2. *Let $\gamma > 0$ and $f \in C_c^\infty(\mathbb{R}^2)$. For any $1 \le p \le \infty$, we have*

$$\left\|T_\gamma f\right\|_p \le \|f\|_p .$$

Proof. Let K_γ be the kernel for T_γ. In other words, $T_\gamma f = K_\gamma * f$. Then, by Young's inequality, it is enough to show that $\left\|K_\gamma\right\|_1 = 1$. First, consider $T_\gamma = \ln^{-\gamma}(e-\Delta)$. Since we have

$$\ln^{-\gamma}(e+|\xi|^2) = \frac{1}{\Gamma(\gamma)}\int_0^\infty \frac{1}{\Gamma(t)}\int_0^\infty e^{-e\beta}e^{-|\xi|^2\beta}\beta^t\frac{d\beta}{\beta}t^\gamma\frac{dt}{t},$$

we take the inverse Fourier transform to get the corresponding kernel

$$K_\gamma(x) = \frac{1}{\Gamma(\gamma)}\int_0^\infty \frac{1}{\Gamma(t)}\int_0^\infty e^{-e\beta}e^{\beta\Delta}\delta_0(x)\beta^t\frac{d\beta}{\beta}t^\gamma\frac{dt}{t}.$$

Therefore using $\left\|e^{\beta\Delta}\delta_0\right\|_1 = 1$, we can easily get $\left\|K_\gamma\right\|_1 = 1$. Here, $e^{t\Delta}\delta_0$ is the usual heat kernel.

Similarly, when $T_\gamma = \ln^{-\gamma}(e+|\nabla|)$, the integral expression of the kernel is

$$K_\gamma(x) = \frac{1}{\Gamma(\gamma)}\int_0^\infty \frac{1}{\Gamma(t)}\int_0^\infty e^{-e\beta}\frac{1}{\sqrt{\pi}}\int_0^\infty e^{-\tau}e^{\frac{\beta}{4\tau}\Delta}\delta_0(x)\tau^{-\frac{1}{2}}d\tau\beta^t\frac{d\beta}{\beta}t^\gamma\frac{dt}{t},$$

and hence again $\left\|e^{\frac{\beta}{4\tau}\Delta}\delta_0\right\|_1 = 1$ implies $\left\|K_\gamma\right\|_1 = 1$. □

2.8.3 Estimate for $\Delta^{-1}\partial_{ii}T_\gamma W$

In this section, we estimate $\mathscr{R}_{ii}T_\gamma(g_A\circ\phi_A) = \Delta^{-1}\partial_{ii}T_\gamma(g_A\circ\phi_A)$ in the L^∞-norm. Here, g_A is defined as in (2.4.15) for $0<\gamma<\frac{1}{2}$ and (2.7.9) for $\gamma=\frac{1}{2}$. The function ϕ_A is a bi-Lipschitz function having certain properties. More precisely, we obtain the following lemma by a slight modification of the proof of Lemma 3.2 in [4].

Lemma 2.8.3. *Let $\{g_A\}$ be a family of functions defined as in* (2.4.15) *for* $0<\gamma<\frac{1}{2}$ *and* (2.7.9) *for* $\gamma=\frac{1}{2}$. *Suppose that $\phi_A = (\phi_A^1,\phi_A^2) : \mathbb{R}^2\to\mathbb{R}^2$ is a bi-Lipschitz function such that*

- $\phi_A(0) = 0$.
- $\phi_A^1(y_1, -y_2) = \phi_A^1(y_1, y_2)$ *and* $\phi_A^2(y_1, -y_2) = -\phi_A^2(y_1, y_2)$.
- *For some integer* $m_A \geq 1$,

$$\|D\phi_A\|_{L^\infty(|y|\leq 4\cdot 2^{-A})} \leq 2^{m_A}, \quad \left\|D(\phi_A^{-1})\right\|_{L^\infty(|y|\leq 2\cdot 2^{-A})} \leq 2^{m_A}. \tag{2.8.2}$$

- $|\det(D\phi_A)| = |\det(D(\phi_A^{-1}))| = 1$.
- *If* $|\phi_A(y)| \leq 2\cdot 2^{-A}$, *then* $|y| \leq 4\cdot 2^{-A}$.

Then, we have

$$\left\|\mathscr{R}_{11}T_\gamma(g_A\circ\phi_A)\right\|_\infty + \left\|\mathscr{R}_{22}T_\gamma(g_A\circ\phi_A)\right\|_\infty \lesssim_\gamma \frac{2^{m_A}}{\sqrt{\ln\ln A}}. \tag{2.8.3}$$

Proof. Recall the definition of g_A,

$$g_A(y) = \begin{cases} C_A \sum_{a_A\leq j<b_A} \frac{1}{j^\gamma}\rho(2^j y), & 0<\gamma<\frac{1}{2} \\ \frac{1}{\ln\ln\ln A}\frac{1}{\sqrt{\ln\ln A}}\sum_{A\leq j<A\ln A}\frac{1}{\sqrt{j}}\rho(2^j y), & \gamma = \frac{1}{2} \end{cases}$$

where $C_A = \frac{1}{\sqrt{\ln A}}\frac{1}{\ln\ln A}$, $a_A = A^{\frac{1}{1-2\gamma}}$, and $b_A = (A+\ln A)^{\frac{1}{1-2\gamma}}$. Here, ρ is an odd function in both variables and satisfies $\frac{1}{2} \leq |x| \leq 2$ for $x \in \operatorname{supp}(\rho)$. (See (2.4.14))

We first consider $\mathscr{R}_{ii}T_\gamma(\rho_j\circ\phi_A)$ for $j \geq A$, where $\rho_j = \rho(2^j\cdot)$. For convenience, we drop the index A in g_A, ϕ_A and m_A below. Denote the kernel for the operator $\mathscr{R}_{ii}T_\gamma$ by K_{ii} for $i = 1,2$ and fix $y \in \mathbb{R}^2\setminus\{0\}$ with $|y| \sim 2^{-l}$ for some l. Note that the kernel K_{ii}, $i = 1,2$, satisfies $|K_{ii}(y)| \lesssim \frac{1}{|y|^2}$ for $y \neq 0$.

Case 1. $2^j \ll 2^{l-m}$.

By the assumption on ϕ, for x with $|\phi(x)| \leq 2\cdot 2^{-A}$, we have $|x| \leq 4\cdot 2^{-A}$. Then, using $\phi(0) = 0$ and (2.8.2), x with $2^{-j-1} \leq |\phi(x)| \leq 2^{-j+1}$ satisfies

$$2^{-j+m} \gtrsim |x| \gtrsim 2^{-j-m}. \tag{2.8.4}$$

Therefore, if y and z satisfy $\phi(y-z) \in \operatorname{supp}(\rho_j)$, we have $2^{-j-1} \leq |\phi(y-z)| \leq 2^{-j+1}$ and hence $2^{-l} \ll 2^{-j-m} \lesssim |y-z| \lesssim 2^{-j+m}$. Combining with $|y| \sim 2^{-l}$, for such y and z, we get

$$2^{-j-m} \lesssim |z| \lesssim 2^{-j+m}.$$

Now, we estimate $\mathscr{R}_{ii}T_\gamma(\rho_j \circ \phi)$ for $i = 1, 2$.

$$\begin{aligned}
|\mathscr{R}_{ii}T_\gamma(\rho_j \circ \phi)(y)| &= \left| \int (\rho_j \circ \phi)(y-z) K_{ii}(z) dy \right| \\
&\leq \int_{2^{-j-m} \lesssim |z| \lesssim 2^{-j+m}} |(\rho_j \circ \phi)(y-z) - (\rho_j \circ \phi)(-z)| |K_{ii}(z)| dy \\
&\lesssim |y| \left\| \nabla(\rho_j \circ \phi) \right\|_\infty \int_{2^{-j-m} \lesssim |z| \lesssim 2^{-j+m}} \frac{1}{|z|^2} dz \\
&\lesssim 2^{-l+m+j} m.
\end{aligned}$$

In the first inequality, we use $\phi(y-z) \in \operatorname{supp}(\rho_j)$ and

$$\mathscr{R}_{ii}T_\gamma(\rho_j \circ \phi)(0) = \int_{c \leq |z| \leq C} (\rho_j \circ \phi)(-z) K_{ii}(z) dz = 0$$

for any arbitrary constants $0 < c < C < +\infty$. This is because ϕ^1 and K_{ii} for $i = 1, 2$ are even in z_2, while ϕ^2, and ρ are odd in z_2.

Case 2. $2^j \gg 2^{l+m}$

By (2.8.4) with $2^{-l} \gg 2^{-j+m}$, we have $|z| \sim 2^{-l}$ when $\phi(y-z) \in \operatorname{supp}(\rho_j)$ and $|y| \sim 2^{-l}$. This implies that for $i = 1, 2$

$$|\mathscr{R}_{ii}T_\gamma(\rho_j \circ \phi)(y)| \leq \|K_{ii}\|_{L^\infty(|y| \sim 2^{-l})} \left\| \rho_j \circ \phi \right\|_1 \lesssim 4^{l-j}.$$

Case 3. $2^{l-m} \lesssim 2^j \lesssim 2^{l+m}$

$$\left\| \mathscr{R}_{ii}T_\gamma(\rho_j \circ \phi) \right\|_\infty \lesssim \left\| \rho_j \circ \phi \right\|_2^{\frac{1}{2}} \left\| \nabla(\rho_j \circ \phi) \right\|_\infty^{\frac{1}{2}} \lesssim \left\| \rho_j \right\|_2^{\frac{1}{2}} \left\| \nabla \rho_j \right\|_\infty^{\frac{1}{2}} 2^{\frac{m}{2}} \lesssim 2^{\frac{m}{2}}.$$

Combining all the cases, we have

$$\sum_j \left\| \mathscr{R}_{ii} T_\gamma(\rho_j \circ \phi) \right\|_\infty \lesssim 2^{\frac{m}{2}} m + m \lesssim 2^{\frac{m}{2}} m, \quad i = 1, 2.$$

Then, (2.8.3) easily follows. □

Chapter 3

Global Navier-Stokes flows for non-decaying initial data with slowly decaying oscillation

3.1 Introduction

In this chapter, we consider the incompressible Navier-Stokes equations

$$\begin{cases} \partial_t v - \Delta v + (v\cdot\nabla)v + \nabla p = 0 \\ \operatorname{div} v = 0 \\ v|_{t=0} = v_0 \end{cases} \tag{NS}$$

in $\mathbb{R}^3 \times (0,T)$ for $0 < T \le \infty$. These equations describe the flow of incompressible viscous fluids, so the solution $v : \mathbb{R}^3 \times (0,T) \to \mathbb{R}^3$ and $p : \mathbb{R}^3 \times (0,T) \to \mathbb{R}$ represent the flow velocity and the pressure, respectively.

For an initial datum with finite kinetic energy, $v_0 \in L^2(\mathbb{R}^3)$, the existence of a time-global weak solution dates back to Leray [33]. This solution has a finite global energy, i.e, it satisfies the energy inequality:

$$\|v(\cdot,t)\|_{L^2(\mathbb{R}^3)}^2 + 2\,\|\nabla v\|_{L^2(0,t;L^2(\mathbb{R}^3))}^2 \le \|v_0\|_{L^2(\mathbb{R}^3)}^2, \quad \forall t > 0. \tag{3.1.1}$$

In Hopf [20], this result is extended to smooth bounded domains with the Dirichlet boundary condition. We say v is *a Leray-Hopf weak solution* to (NS) in $\Omega \times (0,T)$ for a domain $\Omega \subset \mathbb{R}^3$, if

$$v \in L^\infty(0,T;L^2_\sigma(\Omega)) \cap L^2(0,T;H^1_{0,\sigma}(\Omega)) \cap C_{wk}([0,T);L^2_\sigma(\Omega))$$

satisfies the weak form of (NS) and the energy inequality (3.1.1).

However, when a fluid fills an unbounded domain, it is possible to have finite local energy but infinite global energy. One such example is a fluid with constant velocity. There are also many interesting non-decaying infinite energy flows like time-dependent spatially periodic flows (flows on torus) and *two-and-a-half dimensional flows*; see [37, Section 2.3.1] and [16]. Can we get global existence for such data? To analyze the motion of such fluids, one may consider the class L^2_{uloc} for the velocity field v_0 in $\mathbb{R}^3$ whose kinetic energy is uniformly locally bounded. Here, for $1 \le q \le \infty$, we denote by L^q_{uloc} the space of functions in $\mathbb{R}^3$ with

$$\|v_0\|_{L^q_{\text{uloc}}} := \sup_{x_0 \in \mathbb{R}^3} \|v_0\|_{L^q(B(x_0,1))} < \infty.$$

In [31], Lemarié-Rieusset introduced the class of *local energy solutions* for initial data $v_0 \in L^2_{\text{uloc}}$ (see Section 3.3 for details). He proved the short time existence for initial data in L^2_{uloc}, and the global in time existence for $v_0 \in E^2$, those initial data in L^2_{uloc} which further satisfy the spatial decay condition

$$\lim_{|x_0| \to \infty} \int_{B(x_0,1)} |v_0|^2 dx = 0. \tag{3.1.2}$$

Then, Kikuchi-Seregin [26] added more details to the results in [31], especially the careful treatment of the pressure. They also allowed a force term g in (NS) which satisfies $\operatorname{div} g = 0$ and

$$\lim_{|x_0| \to \infty} \int_0^T \int_{B(x_0,1)} |g(x,t)|^2 dx dt = 0, \quad \forall T > 0.$$

Recently, Maekawa-Miura-Prange [35] generalized this result to the half-space $\mathbb{R}^3_+$. The treatment of the pressure in [35] is even more complicated.

One key difficulty in the study of infinite energy solutions is the estimates of the pressure. While finite energy solutions have enough decay at spatial infinity and one may often get the pressure from the equation $p = (-\Delta)^{-1}\partial_i\partial_j(v_i v_j)$, this is not applicable to infinite energy solutions because of their slow (or no) spatial decay.

To estimate the pressure, the definition of a local energy solution in [26] includes a locally-defined pressure decomposition near each point in $\mathbb{R}^3$, see condition (v) in Definition 3.3.1. (It is already in [31] but not part of the definition.) In [21]-[22], on the other hand, Jia and Šverák use a slightly different definition by replacing the decomposition condition by the spatial decay of the velocity

$$\lim_{|x_0|\to\infty}\int_0^{R^2}\int_{B(x_0,R)}|v(x,t)|^2 dxdt = 0, \quad \forall R > 0. \tag{3.1.3}$$

Under the decay assumption (3.1.2) on initial data, these two definitions can be shown to be equivalent; see [23, 35]. However, for general non-decaying initial data, the decay condition (3.1.3) is not expected, while the decomposition condition still works. For this reason, we follow the definition of Kikuchi-Seregin [26] in this chapter.

A new feature in the study of infinite energy solutions with non-decaying initial data is the abundance of *parasitic solutions*,

$$v(x,t) = f(t), \quad p(x,t) = -f'(t)\cdot x$$

for a smooth vector function $f(t)$. They solve the Navier-Stokes equations with initial data $f(0)$. If we choose $f_1(t) \neq f_2(t)$ with $f_1(0) = f_2(0)$, the corresponding parasitic solutions give two different local energy solutions with the same initial data. Such solutions have non-decaying initial data, and can be shown to fail the pressure decomposition condition. More generally, if (v, p) is a solution to (NS), then the following *parasitic transform*

$$u(x,t) = v(y,t) + q'(t), \quad \pi(x,t) = p(y,t) - q''(t)\cdot y, \quad y = x - q(t) \tag{3.1.4}$$

gives another solution (u, π) to (NS) with the same initial data v_0 for any vector function $q(t)$ satisfying $q(0) = q'(0) = 0$.

We now summarize the known existence results in $\mathbb{R}^3$. In addition to the weak solution approach based on the a priori bound (3.1.1) following Leray and Hopf, another fruitful approach is the theory of *mild solutions*, treating the nonlinear term as a source term of the nonhomogeneous Stokes system. In the framework of $L^q(\mathbb{R}^3)$, there exist short time mild solutions in $L^q(\mathbb{R}^3)$ when $3 \le q \le \infty$ ([15, 18, 24]). When $q = 3$, these solutions exist for all time for sufficiently small initial data in $L^3(\mathbb{R}^3)$; see [24]. Similar small data global existence results hold for many other spaces of similar scaling property, such as L^3_{weak}, Morrey spaces $M_{p,3-p}$, negative Besov spaces $\dot{B}^{3/q-1}_{q,\infty}$, $3 < q < \infty$, and the Koch-Tataru space BMO^{-1}; See e.g. [1, 2, 8, 19, 25, 28, 29].

For any data $v_0 \in L^q(\mathbb{R}^3)$, $2 < q < 3$, Calderón [7] constructed a global solution. His strategy is to first decompose $v_0 = a_0 + b_0$ with small $a_0 \in L^3(\mathbb{R}^3)$ and large $b_0 \in L^2(\mathbb{R}^3)$. A solution is then obtained as $v = a + b$, where a is a global small mild solution of (NS) in $L^3(\mathbb{R}^3)$ with $a(0) = a_0$, and b is a global weak solution of the a-perturbed Navier-Stokes equations in the energy class with $b(0) = b_0$.

This idea is then used by Lemarié-Rieusset [31] to construct global local energy solutions for $v_0 \in E^2$; also see Kikuchi-Seregin [26].

We now summarize the known existence results for non-decaying initial data. For the local existence, many mild solution existence theorems mentioned earlier allow non-decaying data. The most relevant to us are Giga-Inui-Matsui [18] for initial data in $L^\infty(\mathbb{R}^3)$ and $BUC(\mathbb{R}^3)$, and Maekawa-Terasawa [36] for initial data in the closure of $\bigcup_{p>3} L^p_{\text{uloc}}$ in L^3_{uloc}-norm, and any small initial data in L^3_{uloc}. Smallness is needed for L^3_{uloc} data even for short time existence.

When it comes to the global existence for non-decaying data, a solution theory for perturbations of constant vectors seems straightforward. The only other result we are aware of is the recent paper Maremonti-Shimizu [38], which proved the global existence of weak solutions for initial data v_0 in $L^\infty(\mathbb{R}^3) \cap \overline{C_0(\mathbb{R}^3)}^{\dot{W}^{1,q}}$, $3 < q < \infty$. In particular, they assume $\nabla v_0 \in L^q(\mathbb{R}^3)$. Their strategy is to decompose the solution $v = U + w$, $U = \sum_{k=1}^n v^k$, where v^1 solves the Stokes equations with the given initial data, and v^{k+1}, $k \ge 1$, solves the linearized Navier-Stokes equations with the force $f^k = -v^k \cdot \nabla v^k$ and homogeneous initial data. The force $f^1 \in L^q(0,T; L^q(\mathbb{R}^3))$ thanks to the assumption on v_0. In each iteration, we get an additional decay of the force f^k. The perturbation w is then solved in the framework of weak solutions. The paper [38] motivated this chapter.

We now state our main theorem. Denote the average of a function v in a set $E \subset \mathbb{R}^3$ by $(v)_E = \frac{1}{|E|}\int_E v(x)\,dx$. We denote $w \in E^2_\sigma$ if $w \in E^2$ and $\operatorname{div} w = 0$.

Theorem 3.1.1. *For any vector field* $v_0 \in E^2_\sigma + L^3_{\mathrm{uloc}}$ *satisfying* $\operatorname{div} v_0 = 0$ *and*

$$\lim_{|x_0|\to\infty} \int_{B(x_0,1)} |v_0 - (v_0)_{B(x_0,1)}|\,dx = 0, \tag{3.1.5}$$

we can find a time-global local energy solution (v,p) *to the Navier-Stokes equations* (NS) *in* $\mathbb{R}^3 \times (0,\infty)$*, in the sense of Definition 3.3.1.*

Our main assumption is the "*oscillation decay*" condition (3.1.5). Note that all $v_0 \in L^2_{\mathrm{uloc}}$ satisfying (3.1.2) also satisfy (3.1.5). Furthermore, for $v_0 \in L^2_{\mathrm{uloc}}$, either $v_0 \in E^1$ or $\nabla v_0 \in E^1$ implies the condition (3.1.5). Here E^q for $1 \le q \le \infty$ is the space of functions in L^q_{uloc} whose L^q-norm in a ball $B_1(x_0)$ goes to zero as $|x_0|$ goes to infinity. In particular, our result generalizes the global existence for decaying initial data $v_0 \in E^2$ in [31] and [26]. It also extends [38] for $v_0 \in L^\infty$ and $\nabla v_0 \in L^q$.

The condition $v_0 \in E^2_\sigma + L^3_{\mathrm{uloc}}$ gives us more regularity on the nondecaying part of v_0. We do not know if it is necessary for the global existence, but it is essential for our proof, and enables us to prove that for small $t > 0$,

$$\|w(t)\chi_R\|_{L^2_{\mathrm{uloc}}} \lesssim (t^{\frac{1}{20}} + \|w_0\chi_R\|_{L^2_{\mathrm{uloc}}}), \tag{3.1.6}$$

where $\chi_R(x)$ is a cut-off function supported in $|x| > R$, we decompose $v_0 = w_0 + u_0$ with $w_0 \in E^2_\sigma$ and $u_0 \in L^3_{\mathrm{uloc}}$, and $w(t) = v(t) - e^{t\Delta}u_0$ with $w(0) = w_0$. This estimate shows that $\|w(t)\chi_R\|_{L^2_{\mathrm{uloc}}}$ vanishes as $t \to 0_+$ and $R \to \infty$.

The idea of our proof is as follows. First, we construct a local energy solution in a short time. For $v_0 \in L^2_{\mathrm{uloc}}$, this is done in [31] but not in [26]. However, we use a slightly revised approximation scheme to make all statements about the pressure easy to verify. In our scheme, we not only mollify the non-linear term as in [33] and [31], but also insert a cut-off function, so that the non-linear term $(v\cdot\nabla)v$ is replaced by $(\mathcal{J}_\varepsilon(v)\cdot\nabla)(v\Phi_\varepsilon)$, where $\mathcal{J}_\varepsilon$ is a mollification of scale ε and Φ_ε is a radial bump function supported in the ball $B(0,2\varepsilon^{-1})$.

Once we have a local-in-time local energy solution, we need some smallness to extend the solution globally in time. To this end, we decompose the solution as $v = V + w$ where $V(t) = e^{t\Delta}u_0$ solves the heat equation. The main effort is to show that $w(t) \in E^2$ for all t and $w(t) \in E^6$ for almost all t. The proof is similar to the decay estimates in [26, 31] and we try to do local energy estimate for $w\chi_R$. The background V has no spatial decay, but we can show the decay of $\nabla V(x,t)$ in $L^\infty(B_R^c \times (t_0,\infty))$ as $R \to \infty$ for any $t_0 > 0$. This decay is not uniform up to $t_0 = 0$ as u_0 is rather rough. We need a new decomposition formula of the pressure, so that in the intermediate regions we can show the decay of the pressure using the decay of ∇V. Because the decay of ∇V is not up to $t_0 = 0$, we need to do the local energy estimate in the time interval $[t_0, T)$, $0 < t_0 \ll 1$. This forces us to prove the estimate (3.1.6), and the *strong local energy inequality* for w away from $t = 0$.

Once we have shown $w(t) \in E^6$ for almost all $t < T$, we can extend the solution as in [31] and [26]. However, we avoid using the strong-weak uniqueness as in [26, 31], and choose to verify the definition of local energy solutions directly as in [35].

The rest of the chapter consists of the following sections. In Section 3.2, we discuss the properties of the heat flow $e^{t\Delta}u_0$, especially the decay of its gradient at spatial infinity assuming (3.1.5). In Section 3.3, we recall the definition of local energy solutions as in [26] and use our revised approximation scheme to find a local energy solution local-in-time. In Section 3.4, we find a new pressure decomposition formula suitable of using the decay of ∇V, prove the estimate (3.1.6) and the strong local energy inequality, and then do the local energy estimate of $w\chi_R$, which implies $w(t) \in E^6$ for almost all t. In Section 3.5, we construct the desired time-global local energy solution.

3.2 Notations and preliminaries

3.2.1 Notation

Given two comparable quantities X and Y, the inequality $X \lesssim Y$ stands for $X \le CY$ for some positive constant C. In a similar way, $\gtrsim$ denotes $\ge C$ for some $C > 0$. We write $X \sim Y$ if $X \lesssim Y$ and $Y \lesssim X$. Furthermore, in the case that a constant C

in $X \leq CY$ depends on some quantities $Z_1, \cdots, Z_n$, we write $X \lesssim_{Z_1,\cdots,Z_n} Y$. The notations $\gtrsim_{Z_1,\cdots,Z_n}$ and $\sim_{Z_1,\cdots,Z_n}$ are similarly defined.

For a point $x \in \mathbb{R}^3$ and a positive real number r, $B(x,r)$ is the Euclidean ball in $\mathbb{R}^3$ centered at x with a radius r,

$$B(x,r) = B_r(x) = \{y \in \mathbb{R}^3 : |y-x| < r\}.$$

When $x = 0$, we denote $B_r = B(0,r)$. For a point $x \in \mathbb{R}^3$ and $r > 0$, we denote the open cube centered at x with a side length $2r$ as

$$Q(x,r) = Q_r(x) = \left\{ y \in \mathbb{R}^3 : \max_{i=1,2,3} |y_i - x_i| < r \right\}.$$

We denote the mollification $\mathscr{J}_\varepsilon(v) = v * \eta_\varepsilon$, $\varepsilon > 0$, where the mollifier is $\eta_\varepsilon(x) = \varepsilon^{-3}\eta\left(\frac{x}{\varepsilon}\right)$ and η is a fixed nonnegative radial bump function in $C_c^\infty(\mathbb{R}^3)$ supported in $B(0,1)$ satisfying $\int \eta \, dx = 1$.

Various test functions in this chapter are defined by rescaling and translating a non-negative radially decreasing bump function Φ satisfying $\Phi = 1$ on $B(0,1)$ and $\operatorname{supp}(\Phi) \subset B(0, \frac{3}{2})$.

For $k \in \mathbb{N} \cup \{0, \infty\}$, let $C_c^k(\mathbb{R}^3)$ be the subset of functions in $C^k(\mathbb{R}^3)$ with compact supports, and

$$C_{c,\sigma}^k(\mathbb{R}^3) = \left\{ u \in C_c^k(\mathbb{R}^3;\mathbb{R}^3) : \operatorname{div} u = 0 \right\}.$$

3.2.2 Uniformly locally integrable spaces

To consider infinite energy flows, we work in the spaces L^q_{uloc}, $1 \leq q \leq \infty$, and $U^{s,p}(t_0,t)$ for $1 \leq s, p \leq \infty$ and $0 \leq t_0 < t \leq \infty$, defined by

$$L^q_{\mathrm{uloc}} = \left\{ u \in L^1_{\mathrm{loc}}(\mathbb{R}^3) : \|u\|_{L^q_{\mathrm{uloc}}} = \sup_{x_0 \in \mathbb{R}^3} \|u\|_{L^q(B_1(x_0))} < +\infty \right\}$$

and

$$U^{s,p}(t_0,t) = \left\{ u \in L^1_{\text{loc}}(\mathbb{R}^3 \times (t_0,t)) : \|u\|_{U^{s,p}(t_0,t)} = \sup_{x_0 \in \mathbb{R}^3} \|u\|_{L^s(t_0,t;L^p(B_1(x_0)))} < +\infty \right\}.$$

When $t_0 = 0$, we simply use $U^{s,p}_T = U^{s,p}(0,T)$. Note that $U^{\infty,p}(t_0,t) = L^\infty(t_0,t;L^p_{\text{uloc}})$, $1 \le p \le \infty$, but for general $1 \le s < \infty$ and $1 \le p \le \infty$, $U^{s,p}(t_0,t)$ and $L^s(t_0,t;L^p_{\text{uloc}})$ are not equivalent norms. Indeed, we can only guarantee that

$$\|u\|_{U^{s,p}(t_0,t)} \le \|u\|_{L^s(t_0,t;L^p_{\text{uloc}})}, \tag{3.2.1}$$

but not the inequality of the other direction.

Example 3.2.1. Fix $1 \le s < \infty$ and $p \in [1,\infty]$. Let x_k be a sequence in $\mathbb{R}^3$ with disjoint $B_1(x_k)$, $k \in \mathbb{N}$, and let $t_k = t_0 + 2^{-k}$. Define a function u by $u(x,\tau) = 2^{k/s}$ on $B_1(x_k) \times (t_0,t_k)$, $k \in \mathbb{N}$, and $u(x,\tau) = 0$ otherwise. It is defined independently of p. We have $u \in U^{s,p}(t_0,t)$, but

$$\int_{t_0}^{t_1} \|u(\cdot,\tau)\|^s_{L^p_{\text{uloc}}} d\tau = \sum_{k=1}^\infty \int_{t_{k+1}}^{t_k} c_p 2^k d\tau = \sum_{k=1}^\infty \frac{1}{2} c_p = \infty,$$

and hence $u \notin L^s(t_0,t;L^p_{\text{uloc}})$. □

We define a local energy space $\mathscr{E}(t_0,t)$ by

$$\mathscr{E}(t_0,t) = \left\{ u \in L^2_{\text{loc}}([t_0,t] \times \mathbb{R}^3;\mathbb{R}^3) \ : \ \operatorname{div} u = 0, \ \|u\|_{\mathscr{E}(t_0,t)} < +\infty \right\}, \tag{3.2.2}$$

where

$$\|u\|_{\mathscr{E}(t_0,t)} := \|u\|_{U^{\infty,2}(t_0,t)} + \|\nabla u\|_{U^{2,2}(t_0,t)}.$$

When $t_0 = 0$, we use the abbreviation $\mathscr{E}_T = \mathscr{E}(0,T)$.

The spaces E^p and $G^p(t_0,t)$, $1 \le p \le \infty$, are defined by an additional decay condition at infinity,

$$E^p := \{ f \in L^p_{\text{uloc}} : \|f\|_{L^p(B(x_0,1))} \to 0, \quad \text{as } |x_0| \to \infty \},$$

and

$$G^p(t_0,t) := \{u \in U^{p,p}(t_0,t) : \ \|u\|_{L^p([t_0,t]\times B(x_0,1))} \to 0, \quad \text{as } |x_0| \to \infty\}.$$

We let $L^p_{\mathrm{uloc},\sigma}$, E^p_σ and $G^p_\sigma(t_0,t)$ denote divergence-free vector fields with components in L^p_{uloc}, E^p and $G^p(t_0,t)$, respectively.

The space E^p, $1 \le p < \infty$, can be characterized as $\overline{C_c^\infty(\mathbb{R}^3)}^{L^p_{\mathrm{uloc}}}$. The analogous statement for E^p_σ is true.

Lemma 3.2.2. ([26, Appendix]) *Suppose that $f \in E^p_\sigma$ for some $1 \le p < \infty$. Then, for any $\varepsilon > 0$, we can find $f^\varepsilon \in C^\infty_{c,\sigma}(\mathbb{R}^3)$ such that*

$$\|f - f^\varepsilon\|_{L^p_{\mathrm{uloc}}} < \varepsilon.$$

3.2.3 Heat and Oseen kernels on L^q_{uloc}

Now, we study the operators $e^{t\Delta}$ and $e^{t\Delta}\mathbb{P}\nabla\cdot$ on L^q_{uloc}. Here $\mathbb{P}$ denotes the Helmholtz projection in $\mathbb{R}^3$. Both are defined as convolution operators

$$e^{t\Delta}f = H_t * f, \quad \text{and} \quad e^{t\Delta}\mathbb{P}_{ij}\partial_k F_{jk} = \partial_k S_{ij} * F_{jk},$$

where H_t and S_{ij} are the heat kernel and the Oseen tensor, respectively,

$$H_t(x) = \frac{1}{\sqrt{4\pi t}^3}\exp\left(-\frac{|x|^2}{4t}\right),$$

and

$$S_{ij}(x,t) = H_t(x)\delta_{ij} + \frac{1}{4\pi}\frac{\partial^2}{\partial_{x_i}\partial_{x_j}}\int_{\mathbb{R}^3}\frac{H_t(y)}{|x-y|}dy.$$

In this paper, we use $(\mathrm{div}\, F)_i = (\nabla\cdot F)_i = \partial_j F_{ji}$. Note that the Oseen tensor satisfies the following pointwise estimates

$$|\nabla_x^l \partial_t^k S(x,t)| \le C_{k,l}(|x| + \sqrt{t})^{-3-l-2k}. \tag{3.2.3}$$

We have the following estimates.

Lemma 3.2.3 (Remark 3.2 in [36]). *For $1 \le q \le p \le \infty$, the following holds. For any vector field f and any 2-tensor F in $\mathbb{R}^3$,*

$$\left\|\partial_t^\alpha \partial_x^\beta e^{t\Delta} f\right\|_{L^p_{\mathrm{uloc}}} \lesssim \frac{1}{t^{|\alpha|+\frac{|\beta|}{2}}} \left(1+\frac{1}{t^{\frac{3}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}}\right) \|f\|_{L^q_{\mathrm{uloc}}},$$

$$\left\|\partial_t^\alpha \partial_x^\beta e^{t\Delta} \mathbb{P}\nabla\cdot F\right\|_{L^p_{\mathrm{uloc}}} \lesssim \frac{1}{t^{|\alpha|+\frac{|\beta|}{2}+\frac{1}{2}}} \left(1+\frac{1}{t^{\frac{3}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}}\right) \|F\|_{L^q_{\mathrm{uloc}}}.$$

Note $p=\infty$ is allowed, with $L^\infty_{\mathrm{uloc}} = L^\infty$.

Lemma 3.2.4. *For any $T>0$, if $f \in L^2_{\mathrm{uloc}}$ and $F \in U_T^{2,2}$, then we have*

$$\left\|e^{t\Delta} f\right\|_{\mathscr{E}_T} \lesssim (1+T^{\frac{1}{2}})\, \|f\|_{L^2_{\mathrm{uloc}}},$$
$$\left\|\int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot F(s)\,ds\right\|_{\mathscr{E}_T} \lesssim (1+T)\, \|F\|_{U_T^{2,2}}.$$

Recall $\|u\|_{\mathscr{E}_T} = \|u\|_{U_T^{\infty,2}} + \|\nabla u\|_{U_T^{2,2}}$. Similar estimates can be found in the proof of [32, Theorem 14.1]. We give a slightly revised proof here for completeness.

Proof. Fix $x_0 \in \mathbb{R}^3$ and let $\phi_{x_0}(x) = \Phi\left(\frac{x-x_0}{2}\right)$. We decompose f and F as

$$f = f\phi_{x_0} + f(1-\phi_{x_0}) = f_1 + f_2$$

and

$$F = F\phi_{x_0} + F(1-\phi_{x_0}) = F_1 + F_2.$$

Since $f_1 \in L^2(\mathbb{R}^3)$ and $F_1 \in L^2(0,T;L^2(\mathbb{R}^3))$, by the usual energy estimates for the heat equation and the Stokes system, we get

$$\left\|e^{t\Delta} f_1\right\|_{\mathscr{E}_T} \lesssim \|f_1\|_2 \lesssim \|f\|_{L^2_{\mathrm{uloc}}} \tag{3.2.4}$$

and

$$\left\|\int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot F_1(s)ds\right\|_{\mathscr{E}_T} \lesssim \|F_1\|_{L^2(0,T;L^2(\mathbb{R}^3))} \lesssim \|F\|_{U_T^{2,2}}. \tag{3.2.5}$$

On the other hand, by Lemma 3.2.3,

$$\left\|e^{t\Delta} f_2\right\|_{U_T^{\infty,2}} = \left\|e^{t\Delta} f_2\right\|_{L^\infty(0,T;L^2_{\mathrm{uloc}})} \lesssim \|f_2\|_{L^2_{\mathrm{uloc}}} \lesssim \|f\|_{L^2_{\mathrm{uloc}}}.$$

Together with (3.2.4), we get

$$\left\|e^{t\Delta} f\right\|_{U_T^{\infty,2}} \lesssim \|f\|_{L^2_{\mathrm{uloc}}}. \tag{3.2.6}$$

(This also follows from Lemma 3.2.3.) By the heat kernel estimates,

$$\begin{aligned}
\left\|\nabla e^{t\Delta} f_2\right\|_{L^2((0,T)\times B(x_0,1))} &\lesssim T^{\frac{1}{2}} \left\|\nabla e^{t\Delta} f_2\right\|_{L^\infty((0,T)\times B(x_0,1))} \\
&\lesssim T^{\frac{1}{2}} \int_{B(x_0,2)^c} \frac{1}{|x_0-y|^4} |f_2(y)|dy \\
&\le T^{\frac{1}{2}} \sum_{k=1}^{\infty} \int_{B(x_0,2^{k+1})\setminus B(x_0,2^k)} \frac{1}{|x_0-y|^4} |f(y)|dy \\
&\lesssim T^{\frac{1}{2}} \sum_{k=1}^{\infty} \frac{1}{2^{4k}} \int_{B(x_0,2^{k+1})} |f(y)|dy.
\end{aligned}$$

We may cover $B(x_0,2^{k+1})$ by $\bigcup_{j=1}^{J_k} B(x_j^k,1)$ with J_k bounded by $C_0 2^{3k}$ for some constant $C_0>0$. Then

$$\left\|\nabla e^{t\Delta} f_2\right\|_{L^2((0,T)\times B(x_0,1))} \lesssim T^{\frac{1}{2}} \sum_{k=1}^{\infty} \frac{1}{2^{4k}} \sum_{j=1}^{J_k} \int_{B(x_j^k,1)} |f(y)|dy \lesssim T^{\frac{1}{2}} \|f\|_{L^2_{\mathrm{uloc}}}.$$

Together with (3.2.4), we get

$$\left\|\nabla e^{t\Delta} f\right\|_{L^2((0,T)\times B(x_0,1))} \lesssim (1+T^{\frac{1}{2}}) \|f\|_{L^2_{\mathrm{uloc}}}.$$

Taking supremum in x_0, we obtain

$$\left\|\nabla e^{t\Delta} f\right\|_{U_T^{2,2}} \lesssim (1+T^{\frac{1}{2}}) \|f\|_{L^2_{\mathrm{uloc}}}.$$

This and (3.2.6) show the first bound of the lemma, $\left\|e^{t\Delta} f\right\|_{\mathscr{E}_T} \lesssim (1+T^{\frac{1}{2}}) \|f\|_{L^2_{\mathrm{uloc}}}$.

Denote $\Psi F(t) = \int_0^t e^{(t-s)\Delta} \mathbb{P}\nabla\cdot F(s)ds$. By the pointwise estimates (3.2.3) for the Oseen tensor, we have

$$\begin{aligned}
\|\Psi F_2\|_{L^\infty(0,T;L^2(B(x_0,1)))} &\lesssim \int_0^t \int_{B(x_0,2)^c} \frac{1}{|x_0-y|^4} |F_2(y,s)| dyds \\
&\le \sum_{k=1}^{\infty} \frac{1}{2^{4k}} \int_0^t \int_{B(x_0,2^{k+1})} |F(y,s)| dyds \\
&\le \sum_{k=1}^{\infty} \frac{1}{2^{4k}} \sum_{j=1}^{J_k} \int_0^t \int_{B(x_j^k,1)} |F(y,s)| dyds \\
&\lesssim \|F\|_{U_T^{1,1}} \lesssim T^{\frac{1}{2}} \|F\|_{U_T^{2,2}}
\end{aligned}$$

and

$$\begin{aligned}
\|\nabla\Psi F_2\|_{L^2((0,T)\times B(x_0,1))} &\lesssim T^{\frac{1}{2}} \|\nabla\Psi F_2\|_{L^\infty((0,T)\times B(x_0,1))} \\
&\lesssim T^{\frac{1}{2}} \int_0^t \int_{B(x_0,2)^c} \frac{1}{|x_0-y|^5} |F_2(y,s)| dyds \\
&\le T^{\frac{1}{2}} \sum_{k=1}^{\infty} \frac{1}{2^{5k}} \int_0^t \int_{B(x_0,2^{k+1})} |F(y,s)| dyds \\
&\lesssim T \|F\|_{U_T^{2,2}} .
\end{aligned}$$

Combined with (3.2.5), we have

$$\|\Psi F\|_{L^\infty(0,T;L^2(B(x_0,1)))} \lesssim (1+T^{\frac{1}{2}}) \|F\|_{U_T^{2,2}}$$

and

$$\|\nabla\Psi F\|_{L^2((0,T)\times B(x_0,1))} \lesssim (1+T) \|F\|_{U_T^{2,2}} .$$

Finally, we take suprema in x_0 to get

$$\left\| \int_0^t e^{(t-s)\Delta} \mathbb{P}\nabla \cdot F(s) ds \right\|_{\mathscr{E}_T} \lesssim (1+T) \|F\|_{U_T^{2,2}} .$$

This is the second bound of the lemma. □

3.2.4 Heat kernel on L^1_{uloc} with decaying oscillation

In this subsection, we investigate how the decaying oscillation assumption (3.1.5) on initial data affects the heat flow. Recall

$$(u)_{Q_r(x)} = \fint_{Q_r(x)} u(y)\,dy = \frac{1}{|Q_r(x)|}\int_{Q_r(x)} u(y)\,dy.$$

Lemma 3.2.5. *Suppose that $u \in L^1_{\text{uloc}}(\mathbb{R}^3)$ satisfies*

$$\lim_{|x_0|\to\infty} \int_{Q_1(x_0)} |u-(u)_{Q_1(x_0)}|dx = 0. \tag{3.2.7}$$

Then, for any $r>0$, we have

$$\lim_{|x_0|\to\infty} \int_{Q_r(x_0)} |u-(u)_{Q_r(x_0)}|dx = 0, \tag{3.2.8}$$

and

$$\lim_{|x_0|\to\infty} \sup_{y\in Q_{2r}(x_0)} |(u)_{Q_r(y)}-(u)_{Q_r(x_0)}| = 0. \tag{3.2.9}$$

Proof. First note that $(u)_{Q_r(x)}$ is finite for any $x\in\mathbb{R}^3$ and $r>0$. Indeed,

$$|(u)_{Q_r(x)}| \le C_r \|u\|_{L^1_{\text{uloc}}}$$

for a constant C_r independent of x, $C_r < C$ for $r>1$, and $C_r \sim r^{-3}$ for $r \ll 1$.

Fix $x_0 \in \mathbb{R}^3$ and $r>0$. For any constant $c\in\mathbb{R}$, we get

$$\begin{aligned}
\fint_{Q_r(x_0)} |u-(u)_{Q_r(x_0)}|dx &\le \fint_{Q_r(x_0)} |u-c| + |(u)_{Q_r(x_0)}-c|dx \\
&= \fint_{Q_r(x_0)} |u-c|dx + \left|\fint_{Q_r(x_0)} (u-c)\,dx\right| \\
&\le 2\fint_{Q_r(x_0)} |u-c|dx.
\end{aligned}$$

Then, for $Q_r = Q_r(x_1) \subset Q_R(x_0)$, $R > r$, we get

$$\fint_{Q_r} |u-(u)_{Q_r}|dx \le 2\fint_{Q_r} |u-(u)_{Q_R(x_0)}|dx \le \frac{2R^3}{r^3}\fint_{Q_R(x_0)} |u-(u)_{Q_R(x_0)}|dx. \tag{3.2.10}$$

With $x_0 = x_1$ and $R = 1$ in (3.2.10), (3.2.7) implies (3.2.8) for all $r \in (0,1)$.

If $y \in \overline{Q_{2r}(x_0)}$, then

$$Q_r(x_0) \cup Q_r(y) \subset Q_R(x_1), \quad x_1 = \frac{1}{2}(x_0+y), \quad R \ge 2r.$$

Thus,

$$\begin{aligned} \left|(u)_{Q_r(x_0)} - (u)_{Q_r(y)}\right| &\le \left|\fint_{Q_r(x_0)} u-(u)_{Q_R(x_1)}dx\right| + \left|\fint_{Q_r(y)} u-(u)_{Q_R(x_1)}dx\right| \\ &\le \fint_{Q_r(x_0)} |u-(u)_{Q_R(x_1)}|dx + \fint_{Q_r(y)} |u-(u)_{Q_R(x_1)}|dx \\ &\le \frac{2R^3}{r^3}\fint_{Q_R(x_1)} |u-(u)_{Q_R(x_1)}|dx. \end{aligned} \tag{3.2.11}$$

With $R = 1$, this and (3.2.7) imply (3.2.9) for all $r \in (0, \frac{1}{2}]$.

Now, for any $Q_r(x_0)$ with $r > 1$, choose the smallest integer $N > 2r$ and let $\rho = r/N < \frac{1}{2}$. We can find a set $S = S_{x_0,r}$ of N^3 points such that $\{Q_\rho(z) : z \in S\}$ are disjoint and

$$\overline{Q_r(x_0)} = \bigcup_{z\in S} \overline{Q_\rho(z)}.$$

For any $z, z' \in S$, we can connect them by points z_j in S, $j = 0,1,\dots,N$, such that $z_0 = z$, $z_N = z'$, and $z_j \in \overline{Q_{2\rho}(z_{j-1})}$, $j = 1,\dots,N$. We allow $z_{j+1} = z_j$ for some j. Thus

$$|(u)_{Q_\rho(z)} - (u)_{Q_\rho(z')}| \le \sum_{j=1}^{N} |(u)_{Q_\rho(z_j)} - (u)_{Q_\rho(z_{j-1})}|,$$

and hence

$$\max_{z,z'\in S_{x_0,r}} |(u)_{Q_\rho(z)} - (u)_{Q_\rho(z')}| = o(1) \quad \text{as } |x_0| \to \infty \tag{3.2.12}$$

by (3.2.9) as $\rho \in (0, \frac{1}{2})$. We have

$$\begin{aligned}
&⨍_{Q_r(x_0)} |u - (u)_{Q_r(x_0)}| dx \\
&\qquad = \sum_{z \in S} N^{-3} ⨍_{Q_\rho(z)} |u - (u)_{Q_r(x_0)}| dx \\
&\qquad \le \sum_{z \in S} N^{-3} \left(⨍_{Q_\rho(z)} |u - (u)_{Q_\rho(z)}| + |(u)_{Q_r(x_0)} - (u)_{Q_\rho(z)}| dx \right) \\
&\qquad \le \left(\sum_{z \in S} N^{-3} ⨍_{Q_\rho(z)} |u - (u)_{Q_\rho(z)}| dx \right) + \max_{z, z' \in S} |(u)_{Q_\rho(z)} - (u)_{Q_\rho(z')}| \\
&\qquad = o(1) \qquad \text{as } |x_0| \to \infty
\end{aligned}$$

by (3.2.8) and (3.2.12) for $\rho \in (0, \frac{1}{2})$. This shows (3.2.8) for all $r > 1$.

Finally, (3.2.9) for $r > 1/2$ follows from (3.2.8) and (3.2.11). □

The following lemma says that decaying oscillation over *cubes* is equivalent to decaying oscillation over *balls*.

Lemma 3.2.6. *Suppose $u \in L^1_{\mathrm{uloc}}$. Then u satisfies* (3.2.7) *if and only if*

$$\lim_{|x_0| \to \infty} \int_{B_1(x_0)} |u - (u)_{B_1(x_0)}| dx = 0. \tag{3.2.13}$$

Proof. Let $\rho = 3^{-1/2}$. We have $Q_\rho(x_0) \subset B_1(x_0) \subset Q_1(x_0)$. Similar to the proof of (3.2.10), we have

$$\int_{B_1(x_0)} |u - (u)_{B_1(x_0)}| dx \le C \int_{Q_1(x_0)} |u - (u)_{Q_1(x_0)}| dx$$

and hence (3.2.13) follows from (3.2.7). Similarly, we also have

$$\int_{Q_\rho(x_0)} |u - (u)_{Q_\rho(x_0)}| dx \le C \int_{B_1(x_0)} |u - (u)_{B_1(x_0)}| dx$$

and hence (3.2.8) for $r = \rho$ follows from (3.2.13). Then $v(x) = u(\rho x)$ satisfies (3.2.7). By Lemma 3.2.5, v satisfies (3.2.8) for any $r > 0$, and we get (3.2.7) for u. □

Lemma 3.2.7. *Suppose $v_0 \in L^1_{\text{uloc}}$ and*

$$\int_{Q(x_0,1)} |v_0 - (v_0)_{Q(x_0,1)}| \to 0, \quad \text{as } |x_0| \to \infty.$$

Let $V = e^{t\Delta} v_0$. Then $(\nabla V)(t_0) \in C_0(\mathbb{R}^3)$ for every $t_0 > 0$. Furthermore, for any $t_0 > 0$, we have

$$\sup_{t > t_0} \|\nabla V(\cdot, t)\|_{L^\infty(B(x_0,1))} \to 0, \quad \text{as } |x_0| \to \infty. \tag{3.2.14}$$

Proof. For $k \in \mathbb{Z}^3$, let Σ_k denote the set of its neighbor integer points,

$$\Sigma_k = \mathbb{Z}^3 \cap Q(k, 1.01) \setminus \{k\}.$$

Let

$$a_k = (v_0)_{Q_1(k)}, \quad b_k = \max_{k' \in \Sigma_k} |a_{k'} - a_k|, \quad c_k = \int_{Q_1(k)} |v_0(x) - a_k| dx.$$

By the assumption, $c_k \to 0$ as $|k| \to \infty$ and by Lemma 3.2.5, $b_k \to 0$ as $|k| \to \infty$.

Choose a nonnegative $\phi \in C_c^\infty(\mathbb{R}^3)$ with $\operatorname{supp} \phi \subset Q_1(0)$ and

$$\sum_{k \in \mathbb{Z}^3} \phi_k(x) = 1 \quad \forall x \in \mathbb{R}^3, \quad \phi_k(x) = \phi(x - k).$$

Define

$$v_1(x) = \sum_{k \in \mathbb{Z}^3} a_k \phi_k(x).$$

Since $|a_k| \lesssim \|v_0\|_{L^1_{\text{uloc}}}$, v_1 is in $L^\infty(\mathbb{R}^3)$. For $x \in Q_1(k)$, it can be written as

$$v_1(x) = a_k + \sum_{k' \in \Sigma_k} (a_{k'} - a_k) \phi_{k'}(x).$$

Thus

$$\begin{aligned} \int_{Q_1(k)} |v_0(x) - v_1(x)| dx &\le \int_{Q_1(k)} |v_0(x) - a_k| dx + \sum_{k' \in \Sigma_k} \int_{Q_1(k)} |a_k - a_{k'}| \phi_{k'}(x) dx \\ &\le c_k + C b_k, \end{aligned} \tag{3.2.15}$$

and

$$\sup_{x\in Q_1(k)} |\nabla v_1(x)| \leq \sup_{x\in Q_1(k)} \sum_{k'\in\Sigma_k} |a_{k'} - a_k| \cdot |\nabla \phi_{k'}(x)| \leq Cb_k. \tag{3.2.16}$$

Let $\psi_R(x) = \Phi\left(\frac{x}{R}\right)$. We decompose

$$\begin{aligned}
\nabla V(x,t) = & \int \nabla H_t(x-y) v_0(y)(1-\psi_R(x-y))dy \\
& + \int \nabla H_t(x-y)[v_0(y) - v_1(y)]\,\psi_R(x-y)dy \\
& + \int \nabla H_t(x-y) v_1(y)\,\psi_R(x-y)dy = I_1 + I_2 + I_3.
\end{aligned}$$

By integration by parts, we can rewrite I_3,

$$I_3 = \int H_t(x-y)\nabla v_1(y)\,\psi_R(x-y)dy - \int H_t(x-y) v_1(y)(\nabla\psi_R)(x-y)dy = I_{31} + I_{32}.$$

Fix $\varepsilon > 0$ and consider $t > t_0 > 0$. Since for any $t > 0$ and $x \in \mathbb{R}^3$, we have

$$\begin{aligned}
|I_1| & \lesssim \int_{B(x,R)^c} \frac{|x-y|^5}{t^{\frac{5}{2}}} e^{-\frac{|x-y|^2}{4t}} \frac{1}{|x-y|^4} |v_0(y)| dy \\
& \lesssim \int_{B(x,R)^c} \frac{1}{|x-y|^4} |v_0(y)| dy \lesssim \frac{1}{R} \|v_0\|_{L^1_{\text{uloc}}},
\end{aligned}$$

and

$$|I_{32}| \lesssim \|H_t\|_1 \|v_1\|_\infty \|\nabla\psi_R\|_\infty \lesssim \frac{1}{R} \|v_0\|_{L^1_{\text{uloc}}},$$

we can choose sufficiently large $R > 0$ such that

$$|I_1, I_{32}| < \varepsilon.$$

The integrands of both I_2 and I_{31} are supported in $|y-x| \leq 2R$. If $|x| > 2\rho$ with $\rho > 2R$ and $|y-x| \leq 2R$, then $|y| \geq |x| - |x-y| > \rho$. Let

$$1_{>\rho}(y) = 1 \quad \text{for} \quad |y| > \rho, \quad \text{and} \quad 1_{>\rho}(y) = 0 \quad \text{for} \quad |y| \leq \rho.$$

We have

$$|I_2| \le \left\| |\nabla H_t| * |v_0 - v_1| 1_{>\rho} \right\|_{L^\infty(\mathbb{R}^3)} \lesssim t_0^{-\frac{1}{2}} \left(1 + t_0^{-\frac{3}{2}} \right) \left\| |v_0 - v_1| 1_{>\rho} \right\|_{L^1_{\mathrm{uloc}}}$$

by Lemma 3.2.3, and

$$|I_{31}| \le \left\| e^{t\Delta}(|\nabla v_1| 1_{>\rho}) \right\|_{L^\infty(\mathbb{R}^3)} \lesssim \left\| |\nabla v_1| 1_{>\rho} \right\|_{L^\infty(\mathbb{R}^3)}.$$

If we take ρ sufficiently large, by (3.2.15) and (3.2.16), we have $|I_2| + |I_{31}| \le 2\varepsilon$.

Since for any $t > t_0$ and $\varepsilon > 0$, we can choose $\rho > 0$ such that

$$\sup_{t > t_0} \|\nabla V(\cdot, t)\|_{L^\infty(B(0,2\rho)^c)} < 4\varepsilon,$$

we get (3.2.14). □

3.3 Local existence

In this section, we recall the definition of local energy solutions and prove their *time-local* existence using a revised approximation scheme. Note that we do not assume spatial decay of initial data for the time-local existence.

As mentioned in the introduction, we follow the definition in Kikuchi-Seregin [26].

Definition 3.3.1 (local energy solution)**.** *Let $v_0 \in L^2_{\mathrm{uloc}}$ with* $\operatorname{div} v_0 = 0$*. A pair (v, p) of functions is a local energy solution to the Navier-Stokes equations* (NS) *with initial data v_0 in $\mathbb{R}^3 \times (0,T)$, $0 < T < \infty$, if it satisfies the following.*

(i) *$v \in \mathcal{E}_T$, defined in* (3.2.2)*, and $p \in L^{\frac{3}{2}}_{\mathrm{loc}}([0,T) \times \mathbb{R}^3)$.*

(ii) *(v, p) solves the Navier-Stokes equations* (NS) *in the distributional sense.*

(iii) *For any compactly supported function $\varphi \in L^2(\mathbb{R}^3)$, the function $\int_{\mathbb{R}^3} v(x,t) \cdot \varphi(x)\, dx$ of time is continuous on $[0,T]$. Furthermore, for any compact set $K \subset \mathbb{R}^3$,*

$$\|v(\cdot, t) - v_0\|_{L^2(K)} \to 0, \quad \text{as } t \to 0^+.$$

(iv) (v,p) *satisfies the local energy inequality (LEI) for any* $t \in (0,T)$*:*

$$\begin{aligned} &\int_{\mathbb{R}^3} |v|^2 \xi(x,t) dx + 2\int_0^t \int_{\mathbb{R}^3} |\nabla v|^2 \xi \, dxds \\ &\qquad \le \int_0^t \int_{\mathbb{R}^3} |v|^2(\partial_s \xi + \Delta \xi) + (|v|^2 + 2p)(v\cdot\nabla)\xi \, dxds, \end{aligned} \tag{3.3.1}$$

for all non-negative smooth functions $\xi \in C_c^\infty((0,T)\times\mathbb{R}^3)$.

(v) For each $x_0 \in \mathbb{R}^3$*, we can find* $c_{x_0} \in L^{\frac{3}{2}}(0,T)$ *such that*

$$p(x,t) = \widehat{p}_{x_0}(x,t) + c_{x_0}(t), \qquad \text{in } L^{\frac{3}{2}}(B(x_0,\tfrac{3}{2})\times(0,T)), \tag{3.3.2}$$

where

$$\begin{aligned} \widehat{p}_{x_0}(x,t) = &-\frac{1}{3}|v(x,t)|^2 + \text{p.v.}\int_{B(x_0,2)} K_{ij}(x-y)v_iv_j(y,t)dy \\ &+ \int_{B(x_0,2)^c} (K_{ij}(x-y) - K_{ij}(x_0-y))v_iv_j(y,t)dy \end{aligned} \tag{3.3.3}$$

for $K(x) = \frac{1}{4\pi|x|}$ *and* $K_{ij} = \partial_{ij}K$.

We say the pair (v,p) *is a local energy solution to* (NS) *in* $\mathbb{R}^3\times(0,\infty)$ *if it is a local energy solution to* (NS) *in* $\mathbb{R}^3\times(0,T)$ *for all* $0 < T < \infty$. □

For an initial data $v_0 \in L^2_{\text{uloc}}$ whose local kinetic energy is uniformly bounded, we reprove the local existence of a local energy solution of [31, Chapt 32].

Theorem 3.3.2 (Local existence)**.** *Let* $v_0 \in L^2_{\text{uloc}}$ *with* $\operatorname{div} v_0 = 0$*. If*

$$T \le \frac{\varepsilon_1}{1 + \|v_0\|^4_{L^2_{\text{uloc}}}}$$

for some small constant $\varepsilon_1 > 0$*, we can find a local energy solution* (v,p) *on* $\mathbb{R}^3 \times (0,T)$ *to the Navier-Stokes equations* (NS) *for the initial data* v_0*, satisfying* $\|v\|_{\mathscr{E}_T} \le C\|v_0\|_{L^2_{\text{uloc}}}$.

Note that we do not assume $v_0 \in E^2$, i.e., we do not assume spatial decay of v_0. Although the local existence theorem is proved in [31, Chapt 32], a few details are missing there, in particular those related to the pressure. These details are given in [26] for the case $v_0 \in E^2$. Here we treat the general case $v_0 \in L^2_{\text{uloc}}$.

Recall the definitions of $\mathscr{J}_\varepsilon(\cdot)$ and Φ in Section 3.2 and let $\Phi_\varepsilon(x) = \Phi(\varepsilon x)$, $\varepsilon > 0$. To prove Theorem 3.3.2, we consider approximate solutions $(v^\varepsilon, p^\varepsilon)$ to the localized-mollified Navier-Stokes equations

$$\begin{cases} \partial_t v^\varepsilon - \Delta v^\varepsilon + (\mathscr{J}_\varepsilon(v^\varepsilon)\cdot\nabla)(v^\varepsilon\Phi_\varepsilon) + \nabla p^\varepsilon = 0 \\ \operatorname{div} v^\varepsilon = 0 \\ v^\varepsilon|_{t=0} = v_0 \end{cases} \tag{3.3.4}$$

in $\mathbb{R}^3 \times (0,T)$.

Since $v_0 \in L^2_{\text{uloc}}$ has no decay, it cannot be approximated by L^2-functions, as was done in [26] when $v_0 \in E^2$. Hence the approximation solution v^ε cannot be constructed in the energy class $L^\infty(0,T;L^2(\mathbb{R}^3)) \cap L^2(0,T;\dot{H}^1(\mathbb{R}^3))$, and has to be constructed in $\mathscr{E}_T$ directly.

Compared to [26, 31], our mollified nonlinearity has an additional localization factor Φ_ε. It makes the decay of the Duhamel term apparent when the approximation solutions have no decay.

We first construct a mild solution v^ε of (3.3.4) in $\mathscr{E}_T$.

Lemma 3.3.3. *For each $0<\varepsilon<1$ and v_0 with $\|v_0\|_{L^2_{\text{uloc}}} \le B$, if $0<T<\min(1, c\varepsilon^3B^{-2})$, we can find a unique solution $v = v^\varepsilon$ to the integral form of* (3.3.4)

$$v(t) = e^{t\Delta}v_0 - \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(\mathscr{J}_\varepsilon(v)\otimes v\Phi_\varepsilon)(s)ds \tag{3.3.5}$$

satisfying

$$\|v\|_{\mathscr{E}_T} \le 2C_0B,$$

where $c>0$ and $C_0>1$ are absolute constants and $(a\otimes b)_{jk} = a_jb_k$.

Proof. Let $\Psi(v)$ be the map defined by the right side of (3.3.5) for $v \in \mathscr{E}_T$. By

Lemma 3.2.4 and $T \leq 1$,

$$\begin{aligned}\|\Psi(v)\|_{\mathscr{E}_T} &\lesssim \|v_0\|_{L^2_{\mathrm{uloc}}} + \|\mathscr{J}_\varepsilon(v)\otimes v\Phi_\varepsilon\|_{U^{2,2}_T} \\ &\lesssim \|v_0\|_{L^2_{\mathrm{uloc}}} + \|\mathscr{J}_\varepsilon(v)\|_{L^\infty(0,T;L^\infty(\mathbb{R}^3))}\|v\|_{U^{2,2}_T} \\ &\lesssim \|v_0\|_{L^2_{\mathrm{uloc}}} + \varepsilon^{-\frac{3}{2}}\sqrt{T}\,\|v\|^2_{U^{\infty,2}_T}.\end{aligned}$$

Thus

$$\|\Psi(v)\|_{\mathscr{E}_T} \leq C_0\|v_0\|_{L^2_{\mathrm{uloc}}} + C_1\varepsilon^{-\frac{3}{2}}\sqrt{T}\,\|v\|^2_{\mathscr{E}_T},$$

for some constants $C_0, C_1 > 0$. Similarly, for $v, u \in \mathscr{E}_T$,

$$\|\Psi(v)-\Psi(u)\|_{\mathscr{E}_T} \leq C_1\varepsilon^{-\frac{3}{2}}\sqrt{T}\left(\|v\|_{\mathscr{E}_T}+\|u\|_{\mathscr{E}_T}\right)\|v-u\|_{\mathscr{E}_T}.$$

By the Picard contraction theorem, if T satisfies

$$T < \frac{\varepsilon^3}{64(C_0C_1B)^2} = c\varepsilon^3B^{-2},$$

then we can always find a unique fixed point $v \in \mathscr{E}_T$ of $v = \Psi(v)$, i.e., (3.3.5), satisfying

$$\|v\|_{\mathscr{E}_T} \leq 2C_0B.$$

□

Lemma 3.3.4. *Let $v_0 \in L^2_{\mathrm{uloc}}$ with* $\operatorname{div} v_0 = 0$*. For each $\varepsilon \in (0,1)$, we can find v^ε in $\mathscr{E}_T$ and p^ε in $L^\infty(0,T;L^2(\mathbb{R}^3))$ for some positive $T = T(\varepsilon, \|v_0\|_{L^2_{\mathrm{uloc}}})$ which solves the localized-mollified Navier-Stokes equations* (3.3.4) *in the sense of distributions, and* $\lim_{t\to 0_+}\|v^\varepsilon(t)-v_0\|_{L^2(E)} = 0$ *for any compact subset E of* $\mathbb{R}^3$.

Proof. By Lemma 3.3.3, there is a mild solution $v^\varepsilon \in \mathscr{E}_T$ of (3.3.5) for some $T = T(\varepsilon, \|v_0\|_{L^2_{\mathrm{uloc}}})$. Apparently,

$$\begin{aligned}\left\|v^\varepsilon - e^{t\Delta}v_0\right\|_{U^{\infty,2}_t} &= \left\|\int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(\mathscr{J}_\varepsilon(v)\otimes v\Phi_\varepsilon)(s)ds\right\|_{U^{\infty,2}_t} \\ &\lesssim \|\mathscr{J}_\varepsilon(v)\otimes v\Phi_\varepsilon\|_{U^{2,2}_t} \lesssim \varepsilon^{-\frac{3}{2}}\sqrt{t}\,\|v\|^2_{U^{\infty,2}_T}.\end{aligned}$$

Also, for any compact subset E of $\mathbb{R}^3$, we have $\left\|e^{t\Delta}v_0 - v_0\right\|_{L^2(E)} \to 0$ as t goes to

0; by Lebesgue's convergence theorem

$$\left\|e^{t\Delta}v_0 - v_0\right\|_{L^2(E)} \le \frac{1}{(4\pi)^{\frac{3}{2}}} \int e^{-\frac{|z|^2}{4}} \left\|v_0(\cdot - \sqrt{t}z) - v_0\right\|_{L^2(E)} dz \to 0,$$

as $t \to 0+$. Then, it follows that $\lim_{t\to 0_+} \|v^\varepsilon(t) - v_0\|_{L^2(E)} = 0$ for any compact subset E of $\mathbb{R}^3$.

Note that $e^{t\Delta}v_0$ with $v_0 \in L^2_{\text{uloc}}$ solves the heat equation in the distributional sense. Also, using $\operatorname{div} v_0 = 0$, we can easily see that $\operatorname{div} e^{t\Delta}v_0 = 0$.

On the other hand, $\mathscr{J}_\varepsilon(v^\varepsilon) \in L^\infty(\mathbb{R}^3 \times [0,T])$ and $v^\varepsilon \in \mathscr{E}_T$ imply

$$\mathscr{J}_\varepsilon(v^\varepsilon) \otimes v^\varepsilon \Phi_\varepsilon \in L^\infty(0,T;L^2(\mathbb{R}^3))$$

and hence by the classical theory, $w^\varepsilon = v^\varepsilon - V$ and p^ε defined by

$$p^\varepsilon = (-\Delta)^{-1}\partial_i\partial_j(\mathscr{J}_\varepsilon(v_i^\varepsilon)v_j^\varepsilon\Phi_\varepsilon) \in L^\infty(0,T;L^2(\mathbb{R}^3)). \tag{3.3.6}$$

solves the Stokes system with the source term $\nabla \cdot (\mathscr{J}_\varepsilon(v^\varepsilon) \otimes v^\varepsilon \Phi_\varepsilon)$ in the distribution sense.

By adding the heat equation for V with $\operatorname{div} V = 0$ and the Stokes system for $(w^\varepsilon, p^\varepsilon)$, $v^\varepsilon = V + w^\varepsilon$ satisfies

$$\partial_t v^\varepsilon - \Delta v^\varepsilon + (\mathscr{J}_\varepsilon(v^\varepsilon)\cdot\nabla)(v^\varepsilon\Phi_\varepsilon) + \nabla p^\varepsilon = 0$$

in the sense of distribution. □

To extract a limit solution from the family $(v^\varepsilon, p^\varepsilon)$ of approximation solutions, we need a uniform bound of $(v^\varepsilon, p^\varepsilon)$ on a uniform time interval $[0,T]$, $T > 0$.

Lemma 3.3.5. *For each $\varepsilon \in (0,1)$, let $(v^\varepsilon, p^\varepsilon)$ be the solution on $\mathbb{R}^3 \times [0,T_\varepsilon]$, for some $T_\varepsilon > 0$, to the localized-mollified Navier-Stokes equations* (3.3.4) *constructed in Lemma 3.3.4. There is a small constant $\varepsilon_1 > 0$, independent of ε and $\|v_0\|^2_{L^2_{\text{uloc}}}$, such that, if $T_\varepsilon \le T_0 = \varepsilon_1(1 + \|v_0\|^4_{L^2_{\text{uloc}}})^{-1}$, then v^ε is uniformly bounded*

$$\|v^\varepsilon\|_{\mathscr{E}_{T_\varepsilon}} \le C\|v_0\|_{L^2_{\text{uloc}}}, \tag{3.3.7}$$

where the constant C on the right hand side is independent of ε and T_ε.

Proof. Let $\phi_{x_0} = \Phi(\cdot - x_0)$ be a smooth cut-off function supported around x_0. For the convenience, we drop the index x_0. Starting from $v^\varepsilon \in \mathscr{E}_{T_\varepsilon}$ and $p^\varepsilon \in L^\infty_{T_\varepsilon} L^2$, and using the interior regularity theory for perturbed Stokes system with smooth coefficients, we have

$$\|v^\varepsilon, \partial_t v^\varepsilon, \nabla v^\varepsilon, \Delta v^\varepsilon\|_{L^\infty((\delta, T_\varepsilon)\times\mathbb{R}^3)} < +\infty$$

for any $\delta \in (0, T_\varepsilon)$. Using $2v^\varepsilon \psi$ with $\psi \in C_c^\infty((0,T_\varepsilon)\times\mathbb{R}^3)$ as a test function in (3.3.4), we get

$$\begin{aligned}2\int_0^{T_\varepsilon}\int |\nabla v^\varepsilon|^2 \psi dxds &= \int_0^{T_\varepsilon}\int |v^\varepsilon|^2(\partial_s\psi + \Delta\psi)dxds + \int_0^{T_\varepsilon}\int |v^\varepsilon|^2 \Phi_\varepsilon(\mathscr{J}_\varepsilon(v^\varepsilon)\cdot\nabla)\psi dxds\\ &\quad + 2\int_0^{T_\varepsilon}\int p^\varepsilon v^\varepsilon \cdot \nabla\psi dxds - \int_0^{T_\varepsilon}\int |v^\varepsilon|^2\psi(\mathscr{J}_\varepsilon(v^\varepsilon)\cdot\nabla)\Phi_\varepsilon dxds.\end{aligned}$$

Using $\lim_{t\to 0_+}\|v^\varepsilon(t) - v_0\|_{L^2(B_n)} = 0$ for any $n \in \mathbb{N}$ (Lemma 3.3.4), we can show

$$\begin{aligned}\int |v^\varepsilon|^2\psi(x,t)dx + 2\int_0^t\int |\nabla v^\varepsilon|^2\psi dxds &= \int |v_0|^2\psi(\cdot,0)dx\\ &+ \int_0^t\int |v^\varepsilon|^2(\partial_s\psi + \Delta\psi)dxds + \int_0^t\int |v^\varepsilon|^2\Phi_\varepsilon(\mathscr{J}_\varepsilon(v^\varepsilon)\cdot\nabla)\psi dxds \quad (3.3.8)\\ &+ 2\int_0^t\int p^\varepsilon v^\varepsilon\cdot\nabla\psi dxds - \int_0^t\int |v^\varepsilon|^2\psi(\mathscr{J}_\varepsilon(v^\varepsilon)\cdot\nabla)\Phi_\varepsilon dxds\end{aligned}$$

for any $\psi \in C_c^\infty([0,T_\varepsilon)\times\mathbb{R}^3)$ and $0 < t < T_\varepsilon$.

We suppress the index ε in v^ε and p^ε, and take $\psi(x,s) = \phi(x)^2\theta(s)$ where

$\theta(s) \in C_c^\infty([0,T_\varepsilon))$ and $\theta(s)=1$ on $[0,t]$ to get

$$\begin{aligned}
&\|v(t)\phi\|_2^2 + 2\,\||\nabla v|\phi\|^2_{L^2([0,t]\times\mathbb{R}^3)} \\
&\quad \lesssim \|v_0\|^2_{L^2_{\rm uloc}} + \left|\int_0^t\int |v|^2|\Delta\phi^2|dxds\right| + \left|\int_0^t\int |v|^2\phi^2(\mathscr{J}_\varepsilon(v)\cdot\nabla)\Phi_\varepsilon dxds\right| \\
&\qquad + \left|\int_0^t\int |v|^2\Phi_\varepsilon(\mathscr{J}_\varepsilon(v)\cdot\nabla)\phi^2 dxds\right| + \left|\int_0^t\int 2\widehat{p}(v\cdot\nabla)\phi^2 dxds\right| \\
&\quad = \|v_0\|^2_{L^2_{\rm uloc}} + I_1 + I_2 + I_3 + I_4,
\end{aligned} \tag{3.3.9}$$

where $\widehat{p} = \widehat{p}^\varepsilon_{x_0}$ will be defined later in (3.3.11) as a function satisfying $\nabla(p-\widehat{p})=0$ on $B(x_0,\frac{3}{2})\times(0,T)$.

The bounds of I_1, I_2 and I_3 can be easily obtained by Hölder inequalities,

$$I_1 \lesssim \|v\|^2_{U_t^{2,2}}, \quad \text{and} \quad I_2, I_3 \lesssim \|v\|^3_{U_t^{3,3}}. \tag{3.3.10}$$

Here we have used $|\nabla\Phi_\varepsilon| \lesssim \varepsilon \le 1$.

On the other hand, I_4 can be estimated as

$$I_4 \lesssim \|\widehat{p}\|_{L^{\frac{3}{2}}([0,t]\times B(x_0,\frac{3}{2}))} \|v\|_{U_t^{3,3}}.$$

Now, we define $\widehat{p}^\varepsilon$ on $B(x_0,\frac{3}{2})\times[0,T]$ by

$$\begin{aligned}
\widehat{p}^\varepsilon(x,t) &= -\frac{1}{3}\mathscr{J}_\varepsilon(v^\varepsilon)\cdot v^\varepsilon\Phi_\varepsilon(x,t) + \text{p.v.}\int_{B(x_0,2)} K_{ij}(x-y)\mathscr{J}_\varepsilon(v_i^\varepsilon)v_j^\varepsilon(y,t)\Phi_\varepsilon(y)dy \\
&\quad + \int_{B(x_0,2)^c} (K_{ij}(x-y)-K_{ij}(x_0-y))\mathscr{J}_\varepsilon(v_i^\varepsilon)v_j^\varepsilon(y,t)\Phi_\varepsilon(y)dy \\
&= \widehat{p}^1 + \widehat{p}^2 + \widehat{p}^3.
\end{aligned} \tag{3.3.11}$$

Comparing the above with (3.3.6) for p^ε, which has the singular integral form

$$p^\varepsilon(x,t) = -\frac{1}{3}\mathscr{J}_\varepsilon(v^\varepsilon)\cdot v^\varepsilon(x,t)\Phi_\varepsilon(x) + \text{p.v.}\int K_{ij}(x-y)\mathscr{J}_\varepsilon(v_i^\varepsilon)v_j^\varepsilon(y,t)\Phi_\varepsilon(y)dy,$$

we see that $p-\widehat{p}$ depends only on t, and hence $\nabla\widehat{p}=\nabla p$ on $B(x_0,\frac{3}{2})\times[0,T]$.

Then, we take the $L^{\frac{3}{2}}([0,t]\times B(x_0,\frac{3}{2}))$-norm for each term to get

$$\left\|\widehat{p}^1\right\|_{L^{\frac{3}{2}}([0,t]\times B(x_0,\frac{3}{2}))}\lesssim\|v\|^2_{U_t^{3,3}},$$

and

$$\left\|\widehat{p}^2\right\|_{L^{\frac{3}{2}}([0,t]\times B(x_0,\frac{3}{2}))}\leq\left\|\widehat{p}^2\right\|_{L^{\frac{3}{2}}([0,t]\times\mathbb{R}^3)}\lesssim\left\|\mathscr{J}\varepsilon(v_i)v_j\Phi_\varepsilon\right\|_{L^{\frac{3}{2}}([0,t]\times B(x_0,2))}\lesssim\|v\|^2_{U_t^{3,3}}.$$

The second inequality for $\widehat{p}^2$ follows from the Calderon-Zygmund theorem. Finally, using

$$|K_{ij}(x-y)-K_{ij}(x_0-y)|\lesssim\frac{|x-x_0|}{|x_0-y|^4}$$

for $x\in B(x_0,\frac{3}{2})$ and $y\in B(x_0,2)^c$, we have

$$\begin{aligned}\left\|\widehat{p}^3\right\|_{L^{\frac{3}{2}}([0,t]\times B(x_0,\frac{3}{2}))}&\lesssim\left\|\int_{B(x_0,2)^c}\frac{1}{|x_0-y|^4}\mathscr{J}\varepsilon(v_i)v_j(y,s)\Phi_\varepsilon(y)dy\right\|_{L^{\frac{3}{2}}(0,t)}\\&\lesssim\left\|\sum_{k=1}^{\infty}\frac{1}{2^{4k}}\int_{B(x_0,2^{k+1})}|\mathscr{J}\varepsilon(v_i)v_j|(y,s)dy\right\|_{L^{\frac{3}{2}}(0,t)}\\&\leq\sum_{k=1}^{\infty}\frac{1}{2^{4k}}\left\|\sum_{j=1}^{J_k}\int_{B(x_j^k,1)}|\mathscr{J}\varepsilon(v_i)v_j|(y,s)dy\right\|_{L^{\frac{3}{2}}(0,t)}\\&\lesssim\sum_{k=1}^{\infty}\frac{J_k}{2^{4k}}\left\|\mathscr{J}\varepsilon(v_i)v_j\right\|_{U_t^{\frac{3}{2},\frac{3}{2}}}\lesssim\|v\|^2_{U_t^{3,3}}.\end{aligned}$$

Above we have taken $B(x_0,2^{k+1})\subset\cup_{j=1}^{J_k}B(x_j^k,1)$ with $J_k\lesssim 2^{3k}$.

Therefore, we get

$$\|\widehat{p}\|_{L^{\frac{3}{2}}([0,t]\times B(x_0,\frac{3}{2}))}\lesssim\|v\|^2_{U_t^{3,3}}\tag{3.3.12}$$

and

$$I_4\lesssim\|v\|^3_{U_t^{3,3}}.$$

Combining this with (3.3.10) and taking supremum on (3.3.9) over $\{x_0\in\mathbb{R}^3\}$, we have

$$\|v(t)\|_{L^2_{\text{uloc}}}^2 + 2\,\|\nabla v\|_{U_t^{2,2}}^2 \lesssim \|v_0\|_{L^2_{\text{uloc}}}^2 + \int_0^t \|v(s)\|_{L^2_{\text{uloc}}}^2\,ds + \|v\|_{U_t^{3,3}}^3\,.$$

Then, using the interpolation inequality and Young's inequality,

$$\begin{aligned}\|v\|_{U_t^{3,3}}^3 &\lesssim \|v\|_{U_t^{6,2}}^{3/2}\,\|v\|_{U_t^{2,6}}^{3/2} \\ &\lesssim \|v\|_{L^6(0,t;L^2_{\text{uloc}})}^6 + \|v\|_{L^2(0,t;L^2_{\text{uloc}})}^2 + \|\nabla v\|_{U_t^{2,2}}^2\,,\end{aligned}$$

we get

$$\|v(t)\|_{L^2_{\text{uloc}}}^2 + \|\nabla v\|_{U_t^{2,2}}^2 \lesssim \|v_0\|_{L^2_{\text{uloc}}}^2 + \int_0^t \|v(s)\|_{L^2_{\text{uloc}}}^2\,ds + \int_0^t \|v(s)\|_{L^2_{\text{uloc}}}^6\,ds. \quad (3.3.13)$$

Finally, we apply the Grönwall inequality, so that there is a small $\varepsilon_1 > 0$ such that, if v^ε exists on $[0,T]$ for $T \le T_0$, $T_0 = \varepsilon_1 \left(1 + \|v_0\|_{L^2_{\text{uloc}}}^4\right)^{-1}$, then we have

$$\sup_{0<t<T} \|v^\varepsilon(t)\|_{L^2_{\text{uloc}}} \lesssim \|v_0\|_{L^2_{\text{uloc}}} \left(1 - \frac{Ct\,\|v_0\|_{L^2_{\text{uloc}}}^4}{\min(1, \|v_0\|_{L^2_{\text{uloc}}})^4}\right)^{-\frac{1}{4}} \le \|v_0\|_{L^2_{\text{uloc}}} (1 - C\varepsilon_1)^{-\frac{1}{4}}.$$

Together with (3.3.13), this completes the proof. □

Lemma 3.3.6. *The distributional solutions* $\{(v^\varepsilon, p^\varepsilon)\}_{0<\varepsilon<1}$ *of* (3.3.4) *constructed in Lemma 3.3.4 can be extended to the uniform time interval* $[0,T_0]$*, where* T_0 *is as in Lemma 3.3.5.*

Proof. We will prove it by iteration. For the convenience, we fix $0 < \varepsilon < 1$ and drop the index ε in v^ε and p^ε. Denote the uniform bound in Lemma 3.3.5 by

$$B = C(\|v_0\|_{L^2_{\text{uloc}}}), \quad B \ge \|v_0\|_{L^2_{\text{uloc}}}\,.$$

If an initial data $v(t_0)$ satisfies $\|v(\cdot,t_0)\|_{L^2_{\text{uloc}}} \le B$, by Lemma 3.3.3, we get $S = S(\varepsilon, B) > 0$ and a unique solution $v(x,t+t_0)$ on $\mathbb{R}^3 \times [0,S]$ to (3.3.5) satisfying

$$\|v(t+t_0)\|_{\mathscr{E}_S} \le 2C_0 B.$$

Now, we start the iteration scheme. Since $\|v_0\|_{L^2_{\mathrm{uloc}}} \leq B$, a unique solution v exists in $\mathscr{E}_S$ to (3.3.5). By Lemma 3.3.4 and Lemma 3.3.5, v satisfies

$$\|v\|_{\mathscr{E}_S} \leq B.$$

Then, we choose $\tau \in (\frac{3}{4}S, S)$, so that $\|v(\tau)\|_{L^2_{\mathrm{uloc}}} \leq B$, and hence we obtain a solution $\tilde{v} \in \mathscr{E}(\tau, \tau+S)$ to

$$\tilde{v}(t) = e^{(t-\tau)\Delta} v|_{t=\tau} + \int_\tau^t e^{(t-s)\Delta} \mathbb{P} \nabla \cdot N^\varepsilon(\tilde{v})(s) ds,$$

where we denote $N^\varepsilon(v) = \mathscr{J}_\varepsilon(v) \otimes v \Phi_\varepsilon$.

Denote the glued solution by $u(x,t) = v(x,t) 1_{[0,\tau]}(t) + \tilde{v}(x,t) 1_{(\tau,\tau+S]}(t)$, where 1_E is a characteristic function of a set $E \subset [0,\infty)$. We claim that it solves (3.3.5) in $(0, \tau+S)$; it is obvious for $t \in (0, \tau]$, and for $t \in (\tau, \tau+S]$,

$$\begin{aligned}
u(t) &= \tilde{v}(t) \\
&= e^{(t-\tau)\Delta} \left(e^{\tau\Delta} v_0 + \int_0^\tau e^{(\tau-s)\Delta} \mathbb{P} \nabla \cdot N^\varepsilon(v)(s) ds \right) + \int_\tau^t e^{(t-s)\Delta} \mathbb{P} \nabla \cdot N^\varepsilon(\tilde{v})(s) ds \\
&= e^{t\Delta} v_0 + \int_0^\tau e^{(t-s)\Delta} \mathbb{P} \nabla \cdot N^\varepsilon(v)(s) ds + \int_\tau^t e^{(t-s)\Delta} \mathbb{P} \nabla \cdot N^\varepsilon(\tilde{v})(s) ds \\
&= e^{t\Delta} v_0 + \int_0^t e^{(t-s)\Delta} \mathbb{P} \nabla \cdot N^\varepsilon(u)(s) ds.
\end{aligned}$$

By Lemma 3.3.5 again, it satisfies

$$\|u\|_{\mathscr{E}(0,\tau+S)} \leq B.$$

By uniqueness, we get $u = v$ for $0 \leq t \leq S$. In other words, u is an extension of v.

Repeat this until the extended solution exists on $[0, T_0]$. Since at each iteration, we can extend the time interval by at least $\frac{3}{4}S$, in a finite number of iterations, we have a distributional solution $(v^\varepsilon, p^\varepsilon)$ of (3.3.4) on $\mathbb{R}^3 \times [0, T_0]$. □

Proof of Theorem 3.3.2. For $0 < \varepsilon \ll 1$, let $(v^\varepsilon, \overline{p}^\varepsilon)$ be the distributional solution to the localized-mollified Navier-Stokes equations (3.3.4) on $\mathbb{R}^3 \times [0, T]$ constructed in Lemmas 3.3.4 and 3.3.6, where $T = T(\|v_0\|_{L^2_{\mathrm{uloc}}})$ is independent of ε. By Lemma

3.3.5,

$$\|v^\varepsilon\|_{\mathscr{E}_T} \le C(\|v_0\|_{L^2_{\mathrm{uloc}}}).$$

We then define $p^\varepsilon \in L^{\frac{3}{2}}_{\mathrm{loc}}([0,T]\times\mathbb{R}^3)$ by

$$\begin{aligned} p^\varepsilon(x,t) &= -\frac{1}{3}\mathscr{J}_\varepsilon(v^\varepsilon)\cdot v^\varepsilon(x,t)\Phi_\varepsilon(x) + \text{p.v.}\int_{B_2} K_{ij}(x-y)N^\varepsilon_{ij}(y,t)dy \\ &\qquad + \text{p.v.}\int_{B_2^c}(K_{ij}(x-y)-K_{ij}(-y))N^\varepsilon_{ij}(y,t)dy, \\ N^\varepsilon_{ij}(y,t) &= \mathscr{J}_\varepsilon(v^\varepsilon_i)v^\varepsilon_j(y,t)\Phi_\varepsilon(y). \end{aligned} \tag{3.3.14}$$

Because $N^\varepsilon_{ij} \in L^\infty(0,T;L^2(\mathbb{R}^3))$, the right side of (3.3.14) is defined in $L^\infty(0,T;L^2(\mathbb{R}^3)) + L^\infty(0,T)$. Note that $\nabla(\overline{p}^\varepsilon - p^\varepsilon) = 0$ because

$$(\overline{p}^\varepsilon - p^\varepsilon)(t) = \int_{B_2^c} K_{ij}(-y)\mathscr{J}_\varepsilon(v^\varepsilon_i)v^\varepsilon_j(y,t)\Phi_\varepsilon(y)dy \in L^{\frac{3}{2}}(0,T).$$

Therefore, $(v^\varepsilon, p^\varepsilon)$ is another distributional solution to the localized-mollified equations (3.3.4). We will show that for each $n\in\mathbb{N}$, p^ε has a bound independent of ε in $L^{\frac{3}{2}}([0,T]\times B_{2^n})$. We drop the index ε in v^ε and p^ε for a moment.

For $n\in\mathbb{N}$, we rewrite (3.3.14) for $x\in B_{2^n}$ as follows.

$$\begin{aligned} p(x,t) &= -\frac{1}{3}\mathscr{J}_\varepsilon(v)\cdot v(x,t)\Phi_\varepsilon(x) + \text{p.v.}\int_{B_2} K_{ij}(x-y)N^\varepsilon_{ij}(y,t)dy \\ &\qquad + \left(\text{p.v.}\int_{B_{2^{n+1}}\setminus B_2} + \text{p.v.}\int_{B^c_{2^{n+1}}}\right)(K_{ij}(x-y)-K_{ij}(-y))N^\varepsilon_{ij}(y,t)dy \\ &= p_1+p_2+p_3+p_4. \end{aligned}$$

All p_i are defined in $L^\infty(0,T;L^2)+L^\infty(0,T)$.

By Lemma 3.3.5, we have

$$\left\|N^\varepsilon_{ij}\right\|_{U_T^{\frac{3}{2},\frac{3}{2}}} \lesssim \|\mathscr{J}_\varepsilon(v)\|_{U_T^{3,3}}\|v\|_{U_T^{3,3}} \le C(\|v_0\|_{L^2_{\mathrm{uloc}}}), \tag{3.3.15}$$

and

$$\left\|N_{ij}^{\varepsilon}\right\|_{L^{\frac{3}{2}}([0,T]\times B_{2^n})} \lesssim 2^{2n}\,\|\mathscr{J}_\varepsilon(v)\|_{U_T^{3,3}}\,\|v\|_{U_T^{3,3}} \le C(n,\|v_0\|_{L^2_{\mathrm{uloc}}}), \quad \forall n\in\mathbb{N}. \tag{3.3.16}$$

Then, the bound of p_1 can be obtained since

$$\|p_1\|_{L^{\frac{3}{2}}([0,T]\times B_{2^n})} \lesssim \sum_{i=1}^{3}\|N_{ii}^{\varepsilon}\|_{L^{\frac{3}{2}}([0,T]\times B_{2^n})}.$$

Using the Calderon-Zygmund theorem, we get

$$\|p_2\|_{L^{\frac{3}{2}}([0,T]\times B_{2^n})} \lesssim \left\|N_{ij}^{\varepsilon}\right\|_{L^{\frac{3}{2}}([0,T]\times B_2)},$$

and

$$\|p_{31}\|_{L^{\frac{3}{2}}([0,T]\times B_{2^n})} \lesssim \left\|N_{ij}^{\varepsilon}\right\|_{L^{\frac{3}{2}}([0,T]\times B_{2^{n+1}})},$$

where

$$p_{31}(x,t) = \text{p.v.}\int_{B_{2^{n+1}}\setminus B_2} K_{ij}(x-y)\mathscr{J}_\varepsilon(v_i)v_j(y,t)\Phi_\varepsilon(y)dy.$$

On the other hand, $p_{32} = p_3 - p_{31}$ satisfies

$$\begin{aligned}\|p_{32}\|_{L^{\frac{3}{2}}([0,T]\times B_{2^n})} &\lesssim 2^{2n}\left\|\frac{1}{|y|^3}\right\|_{L^3(B_{2^{n+1}}\setminus B_2)}\left\|N_{ij}^{\varepsilon}\right\|_{L^{\frac{3}{2}}([0,T]\times B_{2^{n+1}})}\\ &\lesssim 2^{2n}\left\|N_{ij}^{\varepsilon}\right\|_{L^{\frac{3}{2}}([0,T]\times B_{2^{n+1}})}.\end{aligned}$$

Since for $x\in B_{2^n}$ and $y\in B^c_{2^{n+1}}$, we have

$$|K_{ij}(x-y)-K_{ij}(-y)| \lesssim \frac{|x|}{|y|^4} \lesssim \frac{2^n}{|y|^4},$$

the bound of p_4 can be obtained as

$$\begin{aligned}\|p_4\|_{L^{\frac{3}{2}}([0,T]\times B_{2^n})} &\lesssim 2^{2n}\|p_4\|_{L^{\frac{3}{2}}(0,T;L^\infty(B_{2^n}))} \lesssim 2^{3n}\left\|\int_{B^c_{2^{n+1}}}\frac{1}{|y|^4}|N_{ij}^{\varepsilon}|(y,t)dy\right\|_{L^{\frac{3}{2}}(0,T)}\\ &\lesssim 2^{3n}\sum_{k=n+1}^{\infty}\frac{1}{2^{4k}}\left\|N_{ij}^{\varepsilon}\right\|_{L^{\frac{3}{2}}(0,T;L^1(B_{2^{k+1}}))} \lesssim_n \left\|N_{ij}^{\varepsilon}\right\|_{U_T^{\frac{3}{2},1}}.\end{aligned}$$

Adding the estimates and using (3.3.15)-(3.3.16), we get for each $n \in \mathbb{N}$,

$$\|p^{\varepsilon}\|_{L^{\frac{3}{2}}([0,T]\times B_{2^n})} \leq C(n, \|v_0\|_{L^2_{\mathrm{uloc}}}). \tag{3.3.17}$$

Now, we find a limit solution of $(v^{\varepsilon}, p^{\varepsilon})$ up to subsequence on each $[0,T]\times B_{2^n}$, $n \in \mathbb{N}$.

First, construct the solution v on the compact set $[0,T]\times B_2$. By uniform bounds on v^{ε} and the compactness argument, we can extract a sequence $v^{1,k}$ from $\{v^{\varepsilon}\}$ satisfying

$$\begin{aligned}
&v^{1,k} \overset{*}{\rightharpoonup} v^1 && \text{in } L^{\infty}(0,T;L^2(B_2)),\\
&v^{1,k} \rightharpoonup v^1 && \text{in } L^{2}(0,T;H^1(B_2)),\\
&v^{1,k} \to v^1 && \text{in } L^{3}(0,T;L^3(B_2)),\\
&\mathscr{J}_{1,k}(v^{1,k}) \to v^1 && \text{in } L^{3}(0,T;L^3(B_{2^-})),
\end{aligned}$$

as $k \to \infty$. Let $v = v^1$ on $[0,T]\times B_2$.

Then, we extend v to $[0,T]\times B_4$ as follows. In a similar way to getting v^1, we can find a subsequence $\{(v^{2,k}, p^{2,k})\}_{k\in\mathbb{N}}$ of $\{(v^{1,k}, p^{1,k})\}_{k\in\mathbb{N}}$ which satisfies the following convergence:

$$\begin{aligned}
&v^{2,k} \overset{*}{\rightharpoonup} v^2 && \text{in } L^{\infty}(0,T;L^2(B_4)),\\
&v^{2,k} \rightharpoonup v^2 && \text{in } L^{2}(0,T;H^1(B_4)),\\
&v^{2,k} \to v^2 && \text{in } L^{3}(0,T;L^3(B_4)),\\
&\mathscr{J}_{2,k}(v^{2,k}) \to v^2 && \text{in } L^{3}(0,T;L^3(B_{4^-})),
\end{aligned}$$

as $k \to \infty$. Here, we can easily check that $v^2 = v^1$ on $[0,T]\times B_2$, so that $v = v^2$ is the desired extension. By repeating this argument, we can construct a sequence $\{v^{n,k}\}$ and its limit v. Indeed, by the diagonal argument, v can be approximated by

$$v^{(k)} = \begin{cases} v^{k,k} & [0,T]\times B_{2^k}, \\ 0 & \text{otherwise} \end{cases}, \quad \forall k \in \mathbb{N}$$

More precisely, on each $[0,T]\times B_{2^n}$, $\{v^{(k)}\}_{k=n}^{\infty}$ enjoys the same convergence properties as above. This follows from that $\{v^{m,j}\}_{j\in\mathbb{N}}$, $m \geq n$ is a subsequence of $\{v^{n,j}\}_{j\in\mathbb{N}}$. Indeed, for each $v^{k,k}$, $k \geq n$, we can find $j_k \geq k$ such that

$$v^{k,k} = v^{n,j_k}.$$

Then, by its construction, for each $n \in \mathbb{N}$, $\{v^{(k)}\}_{k=n}^{\infty}$ satisfies

$$v^{(k)} \stackrel{*}{\rightharpoonup} v \qquad \text{in } L^\infty(0,T;L^2(B_{2^n})), \tag{3.3.18}$$

$$v^{(k)} \rightharpoonup v \qquad \text{in } L^2(0,T;H^1(B_{2^n})), \tag{3.3.19}$$

$$v^{(k)} \to v \qquad \text{in } L^3(0,T;L^3(B_{2^n})), \tag{3.3.20}$$

$$\mathscr{J}_{(k)}(v^{(k)}) \to v \qquad \text{in } L^3(0,T;L^3(B_{2^n-})) \tag{3.3.21}$$

as $k \to \infty$. Furthermore, since v^ε are uniformly bounded in $\mathscr{E}_T$, we can easily see that $v \in \mathscr{E}_T$ and $v \in U_T^{3,3}$,

$$\|v\|_{\mathscr{E}_T} + \|v\|_{U_T^{3,3}} \leq C(\|v_0\|_{L^2_{\text{uloc}}}).$$

Now, we construct a pressure p corresponding to v. Using (3.3.14), we define $p^{(k)}$ by

$$\begin{aligned} p^{(k)}(x,t) = &-\frac{1}{3}\mathscr{J}_{(k)}(v^{(k)})\cdot v^{(k)}(x,t)\Phi_{(k)}(x) \\ &+ \text{p.v.}\int_{B_2} K_{ij}(x-y)\mathscr{J}_{(k)}(v_i^{(k)})v_j^{(k)}(y,t)\Phi_{(k)}(y)dy \\ &+ \text{p.v.}\int_{B_2^c} (K_{ij}(x-y)-K_{ij}(-y))\mathscr{J}_{(k)}(v_i^{(k)})v_j^{(k)}(y,t)\Phi_{(k)}(y)dy. \end{aligned} \tag{3.3.22}$$

where $\Phi_{(k)} = \Phi_{\varepsilon_k}$ for ε_k satisfying $v^{k,k} = v^{\varepsilon_k}$. Also define

$$p(x,t) = \lim_{n\to\infty} \overline{p}^n(x,t) \tag{3.3.23}$$

where $\overline{p}^n(x,t)$ is defined for $|x| < 2^n$ by

$$\overline{p}^n(x,t) = -\frac{1}{3}|v(x,t)|^2 + \text{p.v.}\int_{B_2} K_{ij}(x-y)v_iv_j(y,t)dy + \overline{p}_3^n + \overline{p}_4^n, \tag{3.3.24}$$

with

$$\overline{p}_3^n(x,t) = \text{p.v.}\int_{B_{2^{n+1}}\setminus B_2} (K_{ij}(x-y) - K_{ij}(-y))v_iv_j(y,t)\,dy,$$
$$\overline{p}_4^n(x,t) = \int_{B^c_{2^{n+1}}} (K_{ij}(x-y) - K_{ij}(-y))v_iv_j(y,t)\,dy.$$

The first two terms in $\overline{p}^n$ are defined in $U_T^{\frac{3}{2},\frac{3}{2}}$. By estimates similar to those for p^ε, we get $\overline{p}_3^n, \overline{p}_4^n \in L^{3/2}((0,T)\times B_{2^n})$ and

$$\overline{p}_3^n + \overline{p}_4^n = \overline{p}_3^{n+1} + \overline{p}_4^{n+1}, \quad \text{in} \quad L^{3/2}((0,T)\times B_{2^n})$$

Thus $\overline{p}^n(x,t)$ is independent of n for $n > \log_2|x|$.

Our goal is to show that the strong convergences (3.3.20)-(3.3.21) of $\{v^{(k)}\}$ gives

$$p^{(k)} \to p \quad \text{in } L^{\frac{3}{2}}([0,T]\times B_{2^n}), \quad \text{for each } n\in\mathbb{N}, \tag{3.3.25}$$

Let $N_{ij}^{(k)} = \mathscr{J}_{(k)}(v_i^{(k)})v_j^{(k)}\Phi_{(k)}$ and $N_{ij} = v_iv_j$. For any fixed $R>0$, we have

$$\begin{aligned}
&\left\|N_{ij}^{(k)} - N_{ij}\right\|_{L^{\frac{3}{2}}([0,T]\times B_R)}\\
&\le \left\|\left(\mathscr{J}_{(k)}(v_i^{(k)}) - v_i\right)v_j^{(k)}\Phi_{(k)}\right\|_{L^{\frac{3}{2}}([0,T]\times B_R)} + \left\|v_i(v_j^{(k)} - v_j)\Phi_{(k)}\right\|_{L^{\frac{3}{2}}([0,T]\times B_R)}\\
&\quad + \left\|v_iv_j(1-\Phi_{(k)})\right\|_{L^{\frac{3}{2}}([0,T]\times B_R)}\\
&\lesssim \left\|\mathscr{J}_{(k)}(v^{(k)}) - v\right\|_{L^3([0,T]\times B_R)}\left\|v^{(k)}\right\|_{L^3([0,T]\times B_R)}\\
&\quad + \left\|v^{(k)} - v\right\|_{L^3([0,T]\times B_R)}\|v\|_{L^3([0,T]\times B_R)} + \left\||v|^2(1-\Phi_{(k)})\right\|_{L^{\frac{3}{2}}([0,T]\times B_R)} \longrightarrow 0
\end{aligned} \tag{3.3.26}$$

by (3.3.20), (3.3.21), and Lebesgue dominated convergence theorem. Then, it provides the convergence of $p^{(k)}$ to p: On $[0,T]\times B_{2^n}$, for $m>n$,

$$\begin{aligned} p^{(k)}-p &= -\frac{1}{3}\operatorname{tr}(N^{(k)}-N)+\text{p.v.}\int_{B_2} K_{ij}(\cdot-y)(N_{ij}^{(k)}-N_{ij})(y)dy \\ &\quad+\left[\text{p.v.}\int_{B_{2^{n+1}}\setminus B_2}+\int_{B_{2^m}\setminus B_{2^{n+1}}}+\int_{B_{2^m}^c}\right](K_{ij}(\cdot-y)-K_{ij}(-y))(N_{ij}^{(k)}-N_{ij})(y)dy \\ &= q_1+q_2+q_3+q_4+q_5. \end{aligned}$$

In a similar way to getting (3.3.17), we have

$$\|q_1,q_2,q_3\|_{L^{\frac{3}{2}}([0,T]\times B_{2^n})}\lesssim_n \left\|N^{(k)}-N\right\|_{L^{\frac{3}{2}}([0,T]\times B_{2^{n+1}})},$$

and

$$\|q_4\|_{L^{\frac{3}{2}}([0,T]\times B_{2^n})}\lesssim \left\|N^{(k)}-N\right\|_{L^{\frac{3}{2}}([0,T]\times B_{2^m})},$$

On the other hand, using

$$|K_{ij}(x-y)-K_{ij}(-y)|\lesssim \frac{|x|}{|y|^4}$$

we obtain

$$\|q_5\|_{L^{\frac{3}{2}}([0,T]\times B_{2^n})}\lesssim \frac{2^{3n}}{2^m}\left(\|v\|_{U_T^{3,3}}^2+\left\|\mathscr{J}_{(k)}(v^{(k)})\right\|_{U_T^{3,3}}\left\|v^{(k)}\right\|_{U_T^{3,3}}\right)\le C(n,\|v_0\|_{L^2_{\text{uloc}}},T)\frac{1}{2^m}.$$

Therefore, for fixed n, if we choose sufficiently large m, we can make q_5 very small in $L^{\frac{3}{2}}([0,T]\times B_{2^n})$ and then for sufficiently large k, q_1, q_2, q_3, and q_4 also become very small in $L^{\frac{3}{2}}([0,T]\times B_{2^n})$ because of (3.3.26). This gives the desired convergence (3.3.25) of $p^{(k)}$ to p.

Now, we check that (v,p) is a local energy solution. It is easy to prove that (v,p) solves the Navier-Stokes equation in the distributional sense by using the distributional form of (3.3.4) for $(v^{(k)},p^{(k)})$ and the convergence (3.3.18)-(3.3.21) and (3.3.25)-(3.3.26). For example, for any $\xi \in C_c^\infty((0,T)\times\mathbb{R}^3;\mathbb{R}^3)$,

$$\left.\begin{aligned} \int_0^T\int v^{(k)}\cdot\partial_t\xi\,dxdt &\to \int_0^T\int v\cdot\partial_t\xi\,dxdt \\ \int_0^T\int \mathscr{J}_{(k)}(v^{(k)})(v^{(k)}\Phi_{(k)}):\nabla\xi\,dxdt &\to \int_0^T\int v\otimes v:\nabla\xi\,dxdt \end{aligned}\right\} \quad \text{as} \quad k\to\infty.$$

Since we have

$$\begin{aligned} &\int_0^t\int(\Delta v-(v\cdot\nabla)v-\nabla p)\cdot\phi\,dxdt \\ &\quad\le\left|\int_0^t\int\nabla v\cdot\nabla\phi\,dxdt\right|+\left|\int_0^t\int v(v\cdot\nabla)\phi\,dxdt\right|+\left|\int_0^t\int p\,\mathrm{div}\,\phi\,dxdt\right| \\ &\quad\lesssim\|\nabla v\|_{L^2(0,T;L^2(B_{2^n}))}\|\nabla\phi\|_{L^2(0,T;L^2(\mathbb{R}^3))} \\ &\qquad+\left(\|v\|^2_{L^3(0,T;L^3(B_{2^n}))}+\|p\|_{L^{\frac32}(0,T;L^{\frac32}(B_{2^n}))}\right)\|\nabla\phi\|_{L^3(0,T;L^3(\mathbb{R}^3))} \\ &\quad\le C(n,T,\|v_0\|_{L^2_{\mathrm{uloc}}})\|\nabla\phi\|_{L^3(0,T;L^3(\mathbb{R}^3))}, \end{aligned}$$

for any $\phi\in C_c^\infty([0,T]\times B_{2^n})$, $n\in\mathbb{N}$, it follows that

$$\partial_t v=\Delta v-(v\cdot\nabla)v-\nabla p\in X_n$$

for any $n\in\mathbb{N}$, where X_n is the dual space of $L^3(0,T;W_0^{1,3}(B_{2^n}))$.

With this bound of $\partial_t v$, for each $n\in\mathbb{N}$, we may redefine $v(t)$ on a measure-zero subset Σ_n of $[0,T]$ such that the function

$$t\longmapsto\int_{\mathbb{R}^3}v(x,t)\cdot\zeta(x)\,dx \tag{3.3.27}$$

is continuous for any vector $\zeta\in C_c^\infty(B_{2^n})$. Redefine $v(t)$ recursively for all n so that (3.3.27) is true for any $\zeta\in C_c^\infty(\mathbb{R}^3)$. It is then true for any $\zeta\in L^2(\mathbb{R}^3)$ with a compact support using $v\in L^\infty(0,T;L^2_{\mathrm{uloc}})$.

Furthermore, consider the local energy equality (3.3.8) for $(v^{(k)},p^{(k)})$ on the time interval $(0,T)$ for a non-negative $\psi\in C_c^\infty([0,T)\times\mathbb{R}^3)$. The first term $\int|v^{(k)}|^2\psi(x,T)dx$ vanishes. Taking limit infimum as k goes to infinity, and using the weak convergence (3.3.19) and the strong convergence (3.3.20)-(3.3.21) and (3.3.25)-(3.3.26), we get

$$2\int_0^T\int|\nabla v|^2\psi\,dxds\le\int|v_0|^2\psi(\cdot,0)dx + \int_0^T\int|v|^2(\partial_s\psi+\Delta\psi)+(|v|^2+2\widehat{p})(v\cdot\nabla)\psi\,dxds, \tag{3.3.28}$$

for any non-negative $\psi\in C_c^\infty([0,T)\times\mathbb{R}^3)$.

Then, for any $t\in(0,T)$ and non-negative $\varphi\in C_c^\infty([0,T)\times\mathbb{R}^3)$, take $\psi(x,s)=\varphi(x,s)\theta_\varepsilon(s)$, $\varepsilon\ll 1$, where $\theta_\varepsilon(s)=\theta\left(\frac{s-t}{\varepsilon}\right)$ for some $\theta\in C^\infty(\mathbb{R})$ such that $\theta(s)=1$ for $s\le 0$ and $\theta(s)=0$ for $s\ge 1$, and $\theta'(s)\le 0$ for all s. Note that $\theta_\varepsilon(s)=1$ for $s\le t$ and $\theta_\varepsilon(s)=0$ for $s\ge t+\varepsilon$. Sending $\varepsilon\to 0$ and using

$$\int|v(t)|^2\varphi\,dx\le\liminf_{\varepsilon\to 0}\int_0^t\int|v|^2\varphi(-\theta_\varepsilon')\,dxds$$

due to the weak local L^2-continuity (3.3.27), we get

$$\begin{aligned}&\int|v(t)|^2\varphi\,dx+2\int_0^t\int|\nabla v|^2\varphi\,dxds\\&\quad\le\int|v_0|^2\varphi(\cdot,0)dx+\int_0^t\int\left\{|v|^2(\partial_s\varphi+\Delta\varphi)+(|v|^2+2\widehat{p})(v\cdot\nabla)\varphi\right\}dxds\end{aligned} \tag{3.3.29}$$

for any $t\in(0,T)$ and non-negative $\varphi\in C_c^\infty([0,T)\times\mathbb{R}^3)$. The local energy inequality (3.3.1) is a special case of (3.3.29) for test functions vanishing at $t=0$.

Sending $t\to 0_+$ in (3.3.29) we get $\limsup_{t\to 0_+}\int|v(t)|^2\varphi\,dx\le\int|v_0|^2\varphi(\cdot,0)dx$ for any non-negative $\varphi\in C_c^\infty$. Together with the weak continuity (3.3.27), we get $\lim_{t\to 0_+}\int_{B_n}|v(x,t)-v_0(x)|^2dx=0$ for any $n\in\mathbb{N}$.

Finally, we consider the decomposition of the pressure. Recall that the pressure p is defined recursively by (3.3.23)-(3.3.24). For any $x_0\in\mathbb{R}^3$ define $\widehat{p}_{x_0}\in$

$L^{\frac{3}{2}}([0,T]\times B(x_0,\frac{3}{2}))$ by (3.3.3), i.e.,

$$\widehat{p}_{x_0}(x,t) = -\frac{1}{3}|v|^2(x,t) + \text{p.v.}\int_{B(x_0,2)} K_{ij}(x-y)v_iv_j(y,t)dy + \int_{B(x_0,2)^c} (K_{ij}(x-y) - K_{ij}(x_0-y))v_iv_j(y,t)dy.$$

Let $c_{x_0} = p - \widehat{p}_{x_0}$. If $B(x_0,\frac{3}{2}) \subset B_{2^n}$, then

$$\begin{aligned} c_{x_0}(t) = &\int_{B_{2^{n+1}}\setminus B(x_0,2)} K_{ij}(x_0-y)v_iv_j(y,t)dy \\ &- \int_{B_{2^{n+1}}\setminus B_2} K_{ij}(-y)v_iv_j(y,t)dy \\ &+ \int_{B^c_{2^{n+1}}} (K_{ij}(x_0-y) - K_{ij}(-y))v_iv_j(y,t)dy. \end{aligned} \tag{3.3.30}$$

Note that $c_{x_0} \in L^{3/2}(0,T)$, and $c_{x_0}(t)$ is independent of $x \in B(x_0,\frac{3}{2})$, n, and T. Therefore, we get the desired decomposition (3.3.2) of the pressure. □

Remark 3.3.1. Our approach in this section is similar to that in Kikuchi-Seregin [26]. However, there are two significant differences:

1. Since we include initial data v_0 not in E^2, we add an additional localization factor $\Phi_{(k)}$ to the nonlinearity in the localized-mollified equations (3.3.4). Our approximation solutions v^ε live in L^2_{uloc} and are no longer in the usual energy class.

2. The pressure p and c_{x_0} are implicit in [26], but are explicit in this chapter. We first specify the formula (3.3.23) of the pressure and then justify the strong convergence and decomposition. In particular, our $c_{x_0}(t)$ is given by (3.3.30) and independent of T.

Remark 3.3.2. Estimate (3.3.12) and its proof for $\widehat{p}^{\varepsilon}_{x_0}$ are not limited to our approximation solutions. They are in fact also valid for any local energy solution (v,p) in $(0,T)$ with local pressure $\widehat{p}_{x_0}$ given by (3.3.3), that is,

$$\|\widehat{p}_{x_0}\|_{L^{\frac{3}{2}}([0,t]\times B(x_0,\frac{3}{2}))} \le C\,\|v\|^2_{U_t^{3,3}}, \quad \forall t < T, \tag{3.3.31}$$

with a constant C independent of t, T.

3.4 Spatial decay estimates

Recall that our initial data $v_0 \in E^2_\sigma + L^3_{\mathrm{uloc},\sigma}$. In Sections 3.4 and 3.5, we decompose

$$v_0 = w_0 + u_0, \quad w_0 \in E^2_\sigma, \quad u_0 \in L^3_{\mathrm{uloc},\sigma}. \tag{3.4.1}$$

Our goal in this section is to show that, although the solution v has no spatial decay, its difference from the linear flow, $w = v - V$, $V(t) = e^{t\Delta}u_0$, does decay due to the decay of the oscillation of u_0. Here, the oscillation decay of u_0 follows from that of v_0 and $w_0 \in E^2$. The main task is to show that the contribution from the nonlinear source term

$$(V \cdot \nabla)V = \nabla \cdot (V \otimes V)$$

has decay, although V itself does not. On the other hand, we also need the decay of the pressure. However, $\widehat{p}_{x_0}$ given by (3.3.3) does not decay. Thus we need a different decomposition of the pressure p near each point $x_0 \in \mathbb{R}^3$.

Lemma 3.4.1 (New pressure decomposition)**.** *Let $v_0 = w_0 + u_0$ with $w_0 \in E^2_\sigma$ and $u_0 \in L^3_{\mathrm{uloc},\sigma}$. Let (v, p) be any local energy solution of* (NS) *with initial data v_0 in $\mathbb{R}^3 \times (0,T)$, $0 < T < \infty$. Then, for each $x_0 \in \mathbb{R}^3$, we can find $q_{x_0} \in L^{\frac{3}{2}}(0,T)$ such that*

$$p(x,t) = \breve{p}_{x_0}(x,t) + q_{x_0}(t) \quad \text{in } L^{\frac{3}{2}}((0,T) \times B(x_0, \tfrac{3}{2}))$$

where

$$
\begin{aligned}
\breve{p}_{x_0} = & -\frac{1}{3}(|w|^2+2w\cdot V)+\text{p.v.}\int_{B(x_0,2)} K_{ij}(\cdot-y)(w_iw_j+V_iw_j+w_iV_j)(y)dy \\
& +\int_{B(x_0,2)^c}(K_{ij}(\cdot-y)-K_{ij}(x_0-y))(w_iw_j+V_iw_j+w_iV_j)(y)dy \\
& +\int K_i(\cdot-y)[(V\cdot\nabla)V_i]\rho_2(y)dy \\
& +\int (K_{ij}(\cdot-y)-K_{ij}(x_0-y))V_iV_j(1-\rho_2)(y)dy \\
& +\int (K_i(\cdot-y)-K_i(x_0-y))V_iV_j(\partial_j\rho_2)(y)dy.
\end{aligned} \tag{3.4.2}
$$

Here, $w=v-V$*,* $V(t)=e^{t\Delta}u_0$*,* $K_i=\partial_iK$*,* $K_{ij}=\partial_{ij}K$*,* $K(x)=\frac{1}{4\pi|x|}$*, and* $\rho_2=\Phi(\frac{\cdot-x_0}{2})$.

Proof. Consider $(x,t)\in B(x_0,\frac{3}{2})\times(0,T)$. Let $F_{ij}=w_iw_j+V_iw_j+w_iV_j$ and $G_{ij}=V_iV_j$. Substituting $v=V+w$ in (3.3.3), we get

$$
\begin{aligned}
\widehat{p}_{x_0} &= p^F_{x_0}+p^G_{x_0} \\
p^F_{x_0} &= -\frac{1}{3}\operatorname{tr}F+\text{p.v.}\int_{B(x_0,2)}K_{ij}(\cdot-y)F_{ij}(y)dy \\
&\quad +\int_{B(x_0,2)^c}(K_{ij}(\cdot-y)-K_{ij}(x_0-y))F_{ij}(y)dy
\end{aligned} \tag{3.4.3}
$$

and

$$
\begin{aligned}
p^G_{x_0} &= -\frac{1}{3}\operatorname{tr}G+\text{p.v.}\int_{B(x_0,2)}K_{ij}(\cdot-y)G_{ij}(y)dy \\
&\quad +\int_{B(x_0,2)^c}(K_{ij}(\cdot-y)-K_{ij}(x_0-y))G_{ij}[\rho_2+(1-\rho_2)](y)dy \\
&= -\frac{1}{3}\operatorname{tr}G+\text{p.v.}\int K_{ij}(\cdot-y)G_{ij}\rho_2(y)dy+p^G_{x_0,\text{far}}+\tilde{q}_{x_0}(t),
\end{aligned}
$$

where

$$
\begin{aligned}
p_{x_0,\text{far}}^G &= \int (K_{ij}(\cdot - y) - K_{ij}(x_0 - y))G_{ij}(1-\rho_2)(y)dy, \\
\tilde{q}_{x_0}(t) &= -\int_{B(x_0,2)^c} K_{ij}(x_0 - y)G_{ij}\rho_2(y)dy.
\end{aligned}
$$

Integrating by parts the principle value integral, we get

$$
p_{x_0}^G = \int K_i(\cdot - y)\partial_j[G_{ij}\rho_2(y)]dy + p_{x_0,\text{far}}^G + \tilde{q}_{x_0}(t).
$$

Note $\partial_j[G_{ij}\rho_2] = (V \cdot \nabla V_i)\rho_2 + G_{ij}\partial_j\rho_2$. Denote

$$
\widehat{q}_{x_0}(t) = \int K_i(x_0 - y)V_iV_j(\partial_j\rho_2)(y)dy.
$$

We get

$$
\begin{aligned}
\widehat{p}_{x_0}(x,t) = p_{x_0}^F &+ \int K_i(\cdot - y)(V\cdot\nabla)V_i\rho_2(y)dy + p_{x_0,\text{far}}^G + \tilde{q}_{x_0}(t) \\
&+ \int (K_i(\cdot - y) - K_i(x_0 - y))V_iV_j(\partial_j\rho_2)(y)dy + \widehat{q}_{x_0}(t) \\
&= \breve{p}_{x_0}(x,t) + \tilde{q}_{x_0}(t) + \widehat{q}_{x_0}(t).
\end{aligned}
\tag{3.4.4}
$$

Thus we have $p(x,t) = \breve{p}_{x_0}(x,t) + q_{x_0}(t)$ with

$$
q_{x_0}(t) = c_{x_0}(t) + \tilde{q}_{x_0}(t) + \widehat{q}_{x_0}(t).
$$

Note that using $\|G\|_{U_T^{\infty,1}} \le \|V\|_{U_T^{\infty,2}}^2$ and $|x_0 - y| > 2$ for $y \in \text{supp}(\partial_j\rho_2)$, we have

$$
\begin{aligned}
\|\tilde{q}_{x_0}(t)\|_{L^\infty(0,T)} + \|\widehat{q}_{x_0}(t)\|_{L^\infty(0,T)} &\lesssim \left\| \int_{B(x_0,3)\setminus B(x_0,2)} |G_{ij}|(y)dy \right\|_{L^\infty(0,T)} \\
&\lesssim \|G\|_{L^\infty(0,T;L^1(B(x_0,3)))} \lesssim \|V\|_{U_T^{\infty,2}}^2.
\end{aligned}
\tag{3.4.5}
$$

Since $\tilde{q}_{x_0}(t) + \widehat{q}_{x_0}(t)$ is in $L^{3/2}(0,T)$, so is $q_{x_0}(t)$. □

Although ∇V has spatial decay, it is not uniform in t. Thus, to show the spatial decay of w, we will first show (3.1.6), i.e., the smallness of w in L^2_{uloc} at far distance for a short time in Lemma 3.4.5. For that we need Lemmas 3.4.2, 3.4.3 and 3.4.4.

Lemma 3.4.2. *For $u_0 \in L^3(\mathbb{R}^3)$, if $\frac{2}{s}+\frac{3}{q}=1$ and $3 \le q < 9$, then*

$$\left\|e^{t\Delta}u_0\right\|_{L^s(0,\infty;L^q(\mathbb{R}^3))} \le C_q \left\|u_0\right\|_{L^3(\mathbb{R}^3)}.$$

This is proved in Giga [17].

Lemma 3.4.3. *Suppose $u_0 \in L^2_{\mathrm{uloc}}$ and $u_0 \in L^3(B(x_0,3))$. Then, $V = e^{t\Delta}u_0$ satisfies*

$$\|V\|_{L^8(0,T;L^4(B(x_0,\frac{3}{2})))} \lesssim \|u_0\|_{L^3(B(x_0,3))} + T^{\frac{1}{8}} \|u_0\|_{L^2_{\mathrm{uloc}}}. \tag{3.4.6}$$

Proof. Let $\phi(x)=\Phi(\frac{x-x_0}{2})$. Decompose

$$u_0 = u_0\phi + u_0(1-\phi) =: u_1 + u_2.$$

By Lemma 3.4.2,

$$\left\|e^{t\Delta}u_1\right\|_{L^8(0,T;L^4(B(x_0,\frac{3}{2})))} \le \left\|e^{t\Delta}u_1\right\|_{L^8(0,T;L^4(\mathbb{R}^3))} \lesssim \|u_1\|_{L^3(\mathbb{R}^3)} \le \|u_0\|_{L^3(B(x_0,3))}. \tag{3.4.7}$$

On the other hand, we have

$$\left\|e^{t\Delta}u_2\right\|_{L^8(0,T;L^4(B(x_0,\frac{3}{2})))} \lesssim \left\|\nabla e^{t\Delta}u_2\right\|_{L^8(0,T;L^2(B(x_0,\frac{3}{2})))} + \left\|e^{t\Delta}u_2\right\|_{L^8(0,T;L^2(B(x_0,\frac{3}{2})))}.$$

Obviously,

$$\left\|e^{t\Delta}u_2\right\|_{L^8(0,T;L^2(B(x_0,\frac{3}{2})))} \lesssim T^{\frac{1}{8}} \left\|e^{t\Delta}u_2\right\|_{L^\infty(0,T;L^2_{\mathrm{uloc}})} \lesssim T^{\frac{1}{8}} \|u_2\|_{L^2_{\mathrm{uloc}}}.$$

Using $\mathrm{supp}(u_2) \subset B(x_0,2)^c$ and heat kernel estimate, we get

$$\begin{aligned}
\left\|\nabla e^{t\Delta}u_2\right\|_{L^8(0,T;L^2(B(x_0,\frac{3}{2})))} &\lesssim T^{\frac{1}{8}} \left\|\nabla e^{t\Delta}u_2\right\|_{L^\infty((0,T)\times B(x_0,\frac{3}{2}))} \\
&\lesssim T^{\frac{1}{8}} \int_{B(x_0,2)^c} \frac{1}{|x_0-y|^4}|u_0(y)|dy \\
&\lesssim T^{\frac{1}{8}} \sum_{k=1}^{\infty} \int_{B(x_0,2^{k+1})\setminus B(x_0,2^k)} \frac{1}{2^{4k}}|u_0(y)|dy \\
&\lesssim T^{\frac{1}{8}} \|u_0\|_{L^2_{\mathrm{uloc}}}.
\end{aligned}$$

Therefore, we obtain

$$\left\|e^{t\Delta}u_2\right\|_{L^8(0,T;L^4(B(x_0,\frac{3}{2})))} \lesssim T^{\frac{1}{8}} \|u_0\|_{L^2_{\mathrm{uloc}}}.$$

Together with (3.4.7), we get (3.4.6). □

The perturbation $w = v - V$, $V(t) = e^{t\Delta}u_0$, satisfies the *perturbed Navier-Stokes equations* in the sense of distributions,

$$\begin{cases}
\partial_t w - \Delta w + (V+w)\cdot\nabla(V+w) + \nabla p = 0 \\
\operatorname{div} w = 0 \\
w|_{t=0} = w_0.
\end{cases} \tag{3.4.8}$$

It also satisfies the following local energy inequality for test functions supported away from $t = 0$.

Lemma 3.4.4 (Local energy inequality for w)**.** *Let $v_0, u_0 \in L^2_{\mathrm{uloc},\sigma}$. Let (v,p) be any local energy solution of* (NS) *with initial data v_0 in $\mathbb{R}^3\times(0,T)$, $0<T<\infty$. Then $w(t) = v(t) - e^{t\Delta}u_0$ satisfies*

$$\begin{aligned}
&\int |w|^2\varphi(x,t)\,dx + 2\int_0^t\int |\nabla w|^2\varphi\,dxds \\
&\quad\le \int_0^t\int |w|^2(\partial_s\varphi + \Delta\varphi + v\cdot\nabla\varphi)\,dxds \\
&\qquad + \int_0^t\int 2pw\cdot\nabla\varphi\,dxds + \int_0^t\int 2V\cdot(v\cdot\nabla)(w\varphi)\,dxds,
\end{aligned} \tag{3.4.9}$$

for any non-negative $\varphi \in C_c^\infty((0,T)\times\mathbb{R}^3)$ *and any* $t \in (0,T)$.

Note that φ vanishes near $t=0$. If φ does not vanish near $t=0$, the last integral in (3.4.9) may not be defined.

Proof. Recall that we have the local energy inequality (3.3.1) for (v,p). The equivalent form for (w,p) is exactly (3.4.9). Indeed, (3.3.1) and (3.4.9) are equivalent because they differ by an equality which is the sum of the weak form of the V-equation with $2v\varphi$ as the test function and the weak form of the w-equation (3.4.8) with $2V\varphi$ as the test function, after suitable integration by parts. This equality can be proved because V and ∇V are in $L^\infty_{\text{loc}}(0,T;L^\infty(\mathbb{R}^3))$, and φ has a compact support in space-time. □

For $r>0$, let

$$\chi_r(x) = 1 - \Phi\left(\frac{x}{r}\right),$$

so that $\chi_r(x)=1$ for $|x|\ge 2r$ and $\chi_r(x)=0$ for $|x|\le r$.

Lemma 3.4.5. *Let* $v_0 = w_0 + u_0$ *with* $w_0 \in E^2_\sigma$ *and* $u_0 \in L^3_{\text{uloc},\sigma}$. *Let* (v,p) *be any local energy solution of* (NS) *with initial data* v_0 *in* $\mathbb{R}^3\times(0,T)$, $0<T<\infty$. *Then, there exist* $T_0 = T_0(\|v_0\|_{L^2_{\text{uloc}}}) \in (0,1)$ *and* $C_0 = C_0(\|w_0\|_{L^2_{\text{uloc}}}, \|u_0\|_{L^3_{\text{uloc}}}) > 0$ *such that* $w(t) = v(t) - e^{t\Delta}u_0$ *satisfies*

$$\|w(t)\chi_R\|_{L^2_{\text{uloc}}} \le C_0(t^{\frac{1}{20}} + \|w_0\chi_R\|_{L^2_{\text{uloc}}}), \tag{3.4.10}$$

for any $R>0$ *and any* $t\in(0,T_1)$, $T_1 = \min(T_0,T)$.

In this lemma, we do not assume the oscillation decay.

Proof. By Lemma 3.2.4 and similar to (3.3.7), we can find $T_0 = T_0(\|v_0\|^2_{L^2_{\text{uloc}}}) \in (0,1)$ such that, for $T_1 = \min(T_0,T)$,

$$\|w\|_{\mathscr{E}_{T_1}} + \|V\|_{\mathscr{E}_{T_1}} \lesssim \|w_0\|_{L^2_{\text{uloc}}} + \|u_0\|_{L^2_{\text{uloc}}}.$$

By interpolation, it follows that for any $2\le s\le\infty$, and $2\le q\le 6$ satisfying $\frac{2}{s}+\frac{3}{q}=$

$\frac{3}{2}$, we have

$$\|w\|_{U^{s,q}_{T_1}} + \|V\|_{U^{s,q}_{T_1}} \lesssim \|w_0\|_{L^2_{\text{uloc}}} + \|u_0\|_{L^2_{\text{uloc}}}.$$

On the other hand, by Lemma 3.4.3, for any $t \in (0,1)$,

$$\|V\|_{U^{8,4}_t} \lesssim \|u_0\|_{L^3_{\text{uloc}}}.$$

Let $A = \|w_0\|_{L^2_{\text{uloc}}} + \|u_0\|_{L^3_{\text{uloc}}}$. Then, both inequalities can be combined for $t \le T_1$ as

$$\|w\|_{\mathscr{E}_t} + \|V\|_{\mathscr{E}_t} + \|w\|_{U_t^{\frac{10}{3},\frac{10}{3}}} + \|V\|_{U_t^{\frac{10}{3},\frac{10}{3}}} + \|V\|_{U_t^{8,4}} \lesssim A. \tag{3.4.11}$$

Fix $x_0 \in \mathbb{R}^3$ and $R > 0$, and let

$$\phi_{x_0} = \Phi(\cdot - x_0), \quad \xi = \phi_{x_0}^2 \chi_R^2. \tag{3.4.12}$$

Fix $\Theta \in C^\infty(\mathbb{R})$, $\Theta' \ge 0$, $\Theta(t) = 1$ for $t > 2$, and $\Theta(t) = 0$ for $t < 1$. Define $\theta_\varepsilon \in C_c^\infty(0,T)$ for sufficiently small $\varepsilon > 0$ by

$$\theta_\varepsilon(s) = \Theta\left(\frac{s}{\varepsilon}\right) - \Theta\left(\frac{s - T + 3\varepsilon}{\varepsilon}\right).$$

Thus $\theta_\varepsilon(s) = 1$ in $(2\varepsilon, T - 2\varepsilon)$ and $\theta_\varepsilon(s) = 0$ outside of $(\varepsilon, T - \varepsilon)$. We now consider the local energy inequality (3.4.9) for w with $\varphi(x,s) = \xi(x)\theta_\varepsilon(s)$. We may replace p by $\widehat{p}_{x_0}$ in (3.4.9) as $\operatorname{supp}\xi \subset B(x_0, \frac{3}{2})$ and $\iint c_{x_0}(t) w \cdot \nabla\xi \, dxdt = 0$. We now take $\varepsilon \to 0_+$. Since $\|v(t) - v_0\|_{L^2(B_2(x_0))} \to 0$ and $\|V(t) - u_0\|_{L^2(B_2(x_0))} \to 0$ as $t \to 0^+$, we get

$$\int_0^{2\varepsilon} \int |w|^2 \xi (\theta_\varepsilon)' \, dxds \to \int |w_0|^2 \xi \, dx.$$

The last term in (3.4.9) converges by Lebesgue dominated convergence theorem using

$$\int_0^t \int |V \cdot (v \cdot \nabla)(w\xi)| \, dxds \lesssim \|V\|_{L^8(0,T;L^4(B(x_0,\frac{3}{2})))} \|v\|_{U_T^{8/3,4}} (\|\nabla w\|_{U_T^{2,2}} + \|w\|_{U_T^{2,2}}),$$

where the right hand side of the inequality is bounded independently of ε.

In the limit $\varepsilon \to 0_+$, for any $t \in (0,T)$, we get

$$\begin{aligned}
&\int |w|^2(x,t)\xi(x)\,dx + 2\int_0^t \int |\nabla w|^2 \xi\,dxds \\
&\quad \le \int |w_0|^2 \xi\,dx + \int_0^t \int |w|^2(\Delta\xi + v\cdot\nabla\xi)\,dxds \\
&\qquad + \int_0^t \int 2\widehat{p}_{x_0} w\cdot\nabla\xi\,dxds + \int_0^t \int 2V\cdot(v\cdot\nabla)(w\xi)\,dxds,
\end{aligned} \tag{3.4.13}$$

for ξ given by (3.4.12). Now, we consider $t \le T_1$. Using (3.4.11), we have

$$\int_0^t \int |w|^2 \Delta\xi dxds \lesssim \|w\|^2_{U_t^{2,2}} \lesssim A^2 t,$$

$$\int_0^t \int |w|^2 (v\cdot\nabla)\xi dxds \lesssim \|v\|_{U_t^{3,3}} \|w\|^2_{U_t^{3,3}} \lesssim A^3 t^{\frac{1}{10}}.$$

For the convenience, we suppress the indexes x_0 and R in ϕ_{x_0}, $\widehat{p}_{x_0}$ and χ_R. By additionally using (3.3.31),

$$\begin{aligned}
\int_0^t \int \widehat{p} w\cdot\nabla\xi dxds &\lesssim \int_0^t \int_{B(x_0,\frac{3}{2})} |\widehat{p}||w| dxds \lesssim \|\widehat{p}\|_{L^{\frac{3}{2}}([0,t]\times B(x_0,\frac{3}{2}))} \|w\|_{U_t^{3,3}} \\
&\lesssim \|v\|^2_{U_t^{3,3}} \|w\|_{U_t^{3,3}} \lesssim A^3 t^{\frac{1}{10}}.
\end{aligned}$$

To estimate the last term in (3.4.13), we decompose it as

$$\begin{aligned}
&\int_0^t \int V\cdot(v\cdot\nabla)(w\xi)\,dxds = I_1 + I_2 + I_3 \\
&= \int_0^t \int \xi V\cdot(V\cdot\nabla)w\,dxds + \int_0^t \int \xi V\cdot(w\cdot\nabla)w\,dxds + \int_0^t \int V\cdot w(v\cdot\nabla)\xi\,dxds.
\end{aligned}$$

We have

$$|I_1| \lesssim \|V\|^2_{L^4(0,T;L^4(\mathrm{supp}(\xi)))} \|\nabla w\|_{U_t^{2,2}} \lesssim A^3 t^{\frac{1}{4}}.$$

On the other hand, by Poincaré inequality, we have

$$\int_0^t \|w\phi\chi\|_{L^6}^2\,ds \lesssim \int_0^t \|\nabla(w\phi\chi)\|_{L^2}^2\,ds + \int_0^t \|w\phi\chi\|_{L^2}^2\,ds$$
$$\lesssim \int_0^t \||\nabla w|\phi\chi\|_{L^2}^2\,ds + \|w\|_{U_t^{2,2}}^2\,,$$

which follows that (using Young's inequality $abc \le \varepsilon a^2 + \varepsilon b^{8/3} + C(\varepsilon)c^8$)

$$\begin{aligned}
|I_2| &\le \int_0^t \||\nabla w|\phi\chi\|_{L^2}\|w\phi\chi\|_{L^4}\|V\|_{L^4(\mathrm{supp}(\xi))}\,ds\\
&\le \int_0^t \||\nabla w|\phi\chi\|_{L^2}\|w\phi\chi\|_{L^6}^{\frac{3}{4}}\|w\phi\chi\|_{L^2}^{\frac{1}{4}}\|V\|_{L^4(\mathrm{supp}(\xi))}\,ds\\
&\le \varepsilon\int_0^t \left(\||\nabla w|\phi\chi\|_{L^2}^2 + \|w\phi\chi\|_{L^6}^2\right) ds\\
&\quad + C(\varepsilon)\int_0^t \|V\|_{L^4(\mathrm{supp}(\xi))}^8\|w\phi\chi\|_{L^2}^2\,ds\\
&\le \frac{1}{100}\int_0^t \||\nabla w|\phi\chi\|_{L^2}^2\,ds + A^2t + C\int_0^t \|V\|_{L^4(\mathrm{supp}(\xi))}^8\|w\phi\chi\|_{L^2}^2\,ds
\end{aligned}$$

by choosing suitable ε. It is easy to control I_3:

$$|I_3| \lesssim t^{\frac{1}{10}}\|V\|_{U_t^{\frac{10}{3},\frac{10}{3}}}\|v\|_{U_t^{\frac{10}{3},\frac{10}{3}}}\|w\|_{U_t^{\frac{10}{3},\frac{10}{3}}} \lesssim A^3t^{\frac{1}{10}}.$$

Therefore, we obtain

$$\left|\int_0^t\int V\cdot(v\cdot\nabla)(w\xi)\,dxds\right| \le \||\nabla w|\phi\chi\|_{L^2([0,t]\times\mathbb{R}^3)}^2$$
$$+ C(1+A^3)\left(t^{\frac{1}{10}} + \int_0^t \|V\|_{L^4(\mathrm{supp}(\xi))}^8\|w\phi\chi\|_{L^2}^2\,ds\right),$$

for some absolute constant C. Finally, we combine all the estimates to get from (3.4.13) that

$$\|w(t)\phi\chi\|_{L^2(\mathbb{R}^3)}^2 + \||\nabla w|\phi\chi\|_{L^2([0,t]\times\mathbb{R}^3)}^2$$
$$\lesssim \|w_0\chi_R\|_{L^2_{\mathrm{uloc}}}^2 + (1+A^3)\left(t^{\frac{1}{10}} + \int_0^t \|V\|_{L^4(\mathrm{supp}(\xi))}^8\|w\phi\chi\|_{L^2}^2\,ds\right)$$

Note that $\|w(t)\phi\chi\|^2_{L^2(\mathbb{R}^3)}$ is lower semicontinuous in t as $w\phi$ is weakly L^2-continuous in t. By Grönwall's inequality and (3.4.11), we have

$$\|w(t)\phi\chi\|^2_{L^2(\mathbb{R}^3)} \le C_0^2(\|w_0\chi_R\|^2_{L^2_{\mathrm{uloc}}} + t^{\frac{1}{10}}),$$

for some $C_0 = C_0(A) > 0$. Taking supremum in x_0, we get

$$\|w(t)\chi_R\|_{L^2_{\mathrm{uloc}}} \le C_0(t^{\frac{1}{20}} + \|w_0\chi_R\|_{L^2_{\mathrm{uloc}}}).$$

This finishes the proof of Lemma 3.4.5. □

Lemma 3.4.6 (Strong local energy inequality). *Let (v,p) be a local energy solution in $\mathbb{R}^3\times(0,T)$ to the Navier-Stokes equations* (NS) *for the initial data $v_0 \in L^2_{\mathrm{uloc}}$ constructed in Theorem 3.3.2, as the limit of approximation solutions $(v^{(k)},p^{(k)})$ of* (3.3.4)*. Then there is a subset $\Sigma \subset (0,T)$ of zero Lebesgue measure such that, for any $t_0 \in (0,T)\setminus\Sigma$ and any $t\in(t_0,T)$, we have*

$$\begin{aligned}\int |v|^2\varphi(x,t)\,dx + 2\int_{t_0}^t\int |\nabla v|^2\varphi\,dxds \\ \le \int |v|^2\varphi(x,t_0)\,dx + \int_{t_0}^t\int \left\{|v|^2(\partial_s\varphi+\Delta\varphi) + (|v|^2+2p)v\cdot\nabla\varphi\right\}dxds,\end{aligned} \tag{3.4.14}$$

for any $\varphi \in C_c^\infty(\mathbb{R}^3\times[t_0,T))$. If, furthermore, for some $u_0 \in L^2_{\mathrm{uloc},\sigma}$, $V(t) = e^{t\Delta}u_0$ and $w = v - V$, then for any $t_0\in(0,T)\setminus\Sigma$ and any $t\in(t_0,T)$, we have

$$\begin{aligned}\int |w|^2\varphi(x,t)\,dx + 2\int_{t_0}^t\int |\nabla w|^2\varphi\,dxds \\ \le \int |w|^2\varphi(x,t_0)\,dx + \int_{t_0}^t\int |w|^2(\partial_s\varphi+\Delta\varphi+v\cdot\nabla\varphi)dxds \\ + \int_{t_0}^t\int 2pw\cdot\nabla\varphi\,dxds - \int_{t_0}^t\int 2(v\cdot\nabla)V\cdot w\varphi\,dxds,\end{aligned} \tag{3.4.15}$$

for any $\varphi\in C_c^\infty(\mathbb{R}^3\times[t_0,T))$.

This lemma is not for general local energy solutions, but only for those constructed by the approximation (3.3.4). Also note that (3.4.14) is true for $t_0 = 0$ since it becomes (3.3.1), but (3.4.15) is unclear for $t_0 = 0$ since the last integral in (3.4.15) may not be defined without further assumptions; Compare it with (3.4.13).

Proof. For any $n \in \mathbb{N}$, the approximations $v^{(k)}$ satisfy

$$\lim_{k\to\infty} \left\| v^{(k)} - v \right\|_{L^2(0,T;L^2(B_n))} = 0.$$

Thus there is a set $\Sigma_n \subset (0,T)$ of zero Lebesgue measure such that

$$\lim_{k\to\infty} \left\| v^{(k)}(t) - v(t) \right\|_{L^2(B_n)} = 0, \quad \forall t \in [0,T) \setminus \Sigma_n.$$

Let

$$\Sigma = \cup_{n=1}^{\infty} \Sigma_n, \quad |\Sigma| = 0.$$

We get

$$\lim_{k\to\infty} \left\| v^{(k)}(t) - v(t) \right\|_{L^2(B_n)} = 0, \quad \forall t \in [0,T) \setminus \Sigma, \ \forall n \in \mathbb{N}. \tag{3.4.16}$$

The local energy equality of $(v^{(k)}, p^{(k)})$ in $[t_0, T]$ is derived similarly to (3.3.8)

$$\begin{aligned} 2\int_{t_0}^{T}\int |\nabla v^{(k)}|^2 \psi dxds = &\int |v^{(k)}|^2 \psi(x,t_0)\,dx + \int_{t_0}^{T}\int |v^{(k)}|^2(\partial_s \psi + \Delta\psi) dxds \\ &+ \int_{t_0}^{T}\int |v^{(k)}|^2 \Phi_{(k)}(\mathscr{J}_{(k)}(v^{(k)}) \cdot \nabla)\psi dxds \\ &+ \int_{t_0}^{T}\int 2p^{(k)} v^{(k)} \cdot \nabla\psi\, dxds \\ &- \int_{t_0}^{T}\int |v^{(k)}|^2 \psi(\mathscr{J}_{(k)}(v^{(k)}) \cdot \nabla)\Phi_{(k)} dxds, \end{aligned} \tag{3.4.17}$$

for any $\psi \in C_c^\infty(\mathbb{R}^3 \times [0,T))$. By (3.4.16), we have

$$\lim_{k\to\infty} \int |v^{(k)}|^2 \psi(x,t_0)\,dx = \int |v|^2 \psi(x,t_0)\,dx$$

for $t_0 \in [0,T) \setminus \Sigma$. Taking limit infimum $k \to \infty$ in (3.4.17), we get

$$\begin{aligned} &2\int_{t_0}^{T}\int |\nabla v|^2 \psi \, dxds \\ &\le \int |v|^2 \psi(x,t_0)\, dx + \int_{t_0}^{T}\int \left\{ |v|^2(\partial_s \psi + \Delta \psi) + (|v|^2 + 2p) v \cdot \nabla \psi \right\} dxds. \end{aligned}$$

By the same argument for (3.3.29), we get (3.4.14) from the above.

Finally, inequality (3.4.15) for $t_0 > 0$ is equivalent to (3.4.14) by the same argument of Lemma 3.4.4. We have integrated by parts the last term in (3.4.15), which is valid since $\nabla V \in L^\infty(\mathbb{R}^3 \times (t_0,t))$. $\square$

We now prove the main result of this section.

Proposition 3.4.7 (Decay of w and $\breve{p}$)**.** *Let $v_0 = w_0 + u_0$ with $w_0 \in E^2_\sigma$ and $u_0 \in L^3_{\mathrm{uloc},\sigma}$, and*

$$\lim_{|x_0|\to\infty} \int_{B(x_0,1)} |v_0 - (v_0)_{B(x_0,1)}| dx = 0.$$

Let (v,p) be a local energy solution in $\mathbb{R}^3 \times (0,T)$ to the Navier-Stokes equations (NS) *for the initial data $v_0 \in L^2_{\mathrm{uloc}}$ constructed in Theorem 3.3.2, as the limit of approximation solutions $(v^{(k)}, p^{(k)})$ of* (3.3.4)*. Let $w = v - V$ for $V(t) = e^{t\Delta}u_0$. Then, w and $\breve{p}_{x_0}$, defined in Lemma 3.4.1, decay at spatial infinity: For any $t_1 \in (0,T)$,*

$$\lim_{|x_0|\to\infty} \left(\|w\|_{L^\infty_t L^2_x \cap L^3(Q_{x_0})} + \|\nabla w\|_{L^2(Q_{x_0})} + \|\breve{p}_{x_0}\|_{L^{\frac{3}{2}}(Q_{x_0})} \right) = 0, \tag{3.4.18}$$

where $Q_{x_0} = B(x_0, \frac{3}{2}) \times (t_1, T)$.

Note that we do not assert uniform decay up to $t_1 = 0$. We assume the approximation (3.3.4) only to ensure the conclusion of Lemma 3.4.6, the strong local energy inequality.

Proof. Choose $A = A(\|w_0\|_{L^2_{\mathrm{uloc}}}, \|u_0\|_{L^2_{\mathrm{uloc}}}, T)$ such that

$$\|w\|_{\mathscr{E}_T} + \|V\|_{\mathscr{E}_T} + \|w\|_{U^{s,q}_T} + \|V\|_{U^{s,q}_T} \lesssim A,$$

for any $2 \le s \le \infty$, and $2 \le q \le 6$ satisfying $\frac{2}{s} + \frac{3}{q} = \frac{3}{2}$.

Fix $x_0 \in \mathbb{Z}^3$ and $R \in \mathbb{N}$. Let $\phi_{x_0} = \Phi(\cdot - x_0)$, $\chi_R(x) = 1 - \Phi\left(\frac{x}{R}\right)$, and

$$\xi = \phi_{x_0}^2 \chi_R^2.$$

For the convenience, we suppress the indexes x_0 and R in ϕ_{x_0}, $\breve{p}_{x_0}$ and χ_R.

Let Σ be the subset of $(0,T)$ defined in Lemma 3.4.6. For any $t_0 \in (0,t_1) \setminus \Sigma$ and $t \in (t_0, T)$, choose $\theta(t) \in C_c^\infty(0,T)$ with $\theta = 1$ on $[t_0, t]$. Let $\varphi(x,t) = \xi(x)\theta(t)$. By (3.4.15) of Lemma 3.4.6, using $t_0 \notin \Sigma$, we have

$$\begin{aligned}\int |w(x,t)|^2 \xi(x)\,dx + 2\int_{t_0}^t \int |\nabla w|^2 \xi\,dxds &\\ \le \int |w(x,t_0)|^2 \xi(x)\,dx + \int_{t_0}^t \int |w|^2 (\Delta \xi + (v\cdot\nabla)\xi)\,dxds &\\ + 2\int_{t_0}^t \int \breve{p}_{x_0} w \cdot \nabla \xi\,dxds - 2\int_{t_0}^t \int (v\cdot\nabla)V \cdot w\xi\,dxds.&\end{aligned} \tag{3.4.19}$$

Above we have replaced p by $\breve{p}_{x_0}$ using $\iint q_{x_0}(t) w \cdot \nabla \xi\,dxds = 0$.

By the choice of ξ, we can easily see that

$$\int |w(\cdot,t)|^2 \xi dx + 2\int_{t_0}^t \int |\nabla w|^2 \xi dxds \ge \|w(\cdot,t)\chi\|_{L^2(B(x_0,1))}^2 + 2\,\||\nabla w|\chi\|_{L^2([t_0,t]\times B(x_0,1))}^2,$$

$$\int |w(\cdot,t_0)|^2 \xi dx \lesssim \|w(\cdot,t_0)\chi\|_{L^2_{\text{uloc}}}^2,$$

$$\int_{t_0}^t \int |w|^2 \Delta\xi dxds \lesssim \|w\chi\|_{U^{2,2}(t_0,t)}^2 + \frac{1}{R}\|w\|_{U_T^{2,2}}^2,$$

and

$$\begin{aligned}\int_{t_0}^t \int |w|^2 (v\cdot\nabla)\xi dxds &\lesssim \|v\|_{U_T^{3,3}} \|w\chi\|_{U^{3,3}(t_0,t)}^2 + \frac{1}{R}\|v\|_{U_T^{3,3}} \|w\|_{U_T^{3,3}}^2 \\ &\lesssim_A \|w\chi\|_{U^{3,3}(t_0,t)}^2 + \frac{1}{R}.\end{aligned}$$

The last term can be also estimated by

$$\left|\int_{t_0}^{t}\int (v\cdot\nabla)V\cdot w\xi\,dxds\right| \lesssim \||\nabla V|\chi\|_{U^{\infty,3}(t_0,T)}\|v\|_{U_T^{2,6}}\|w\chi\|_{U^{2,2}(t_0,t)}$$
$$\lesssim_A \|w\chi\|^2_{U^{2,2}(t_0,t)} + \||\nabla V|\chi\|^2_{U^{\infty,3}(t_0,T)}.$$

The only remaining term is the one with pressure. Note

$$\int_{t_0}^{t}\int \breve{p}w\cdot\nabla\xi\,dxds \lesssim \int_{t_0}^{t}\int_{B(x_0,\frac{3}{2})}|\breve{p}||w|\chi^2 dxds + \frac{1}{R}\int_{t_0}^{t}\int_{B(x_0,\frac{3}{2})}|\breve{p}||w|\chi\,dxds$$
$$\lesssim \|\breve{p}\chi\|_{L^{\frac{3}{2}}([t_0,t]\times B(x_0,\frac{3}{2}))}\|w\chi\|_{U^{3,3}(t_0,t)} + \frac{1}{R}\|\breve{p}\|_{L^{\frac{3}{2}}([0,T]\times B(x_0,\frac{3}{2}))}\|w\chi\|_{U_T^{3,3}}.$$

For the second term, we can use a bound uniform in x_0

$$\|\breve{p}_{x_0}\|_{L^{\frac{3}{2}}([0,t]\times B(x_0,\frac{3}{2}))} \le C\|v\|^2_{U_t^{3,3}} + C(T)\|V\|^2_{U_T^{\infty,2}},$$

which follows from (3.3.31), (3.4.4) and (3.4.5). For the first term, although the other factor $\|w\chi\|_{U^{3,3}(t_0,t)}$ also has decay, it is larger than the left side of (3.4.19) by itself. Hence we need to estimate $\|\breve{p}\chi\|_{L^{\frac{3}{2}}([t_0,t]\times B(x_0,\frac{3}{2}))}$ and show its decay.

Let $F_{ij} = w_iw_j + w_iV_j + w_jV_i$ and $G_{ij} = V_iV_j$. The local pressure $\breve{p}$ defined in Lemma 3.4.1 can be further decomposed as

$$\breve{p}(x,t) = p^F + p^{G,1} + p^{G,2} + p^{G,3}$$

where $p^F = p^F_{x_0}$ is defined as in (3.4.3),

$$p^{G,1} = \int K_i(\cdot - y)[\partial_j G_{ij}]\rho_2(y)dy,$$

$$p^{G,2} = \int (K_{ij}(\cdot - y) - K_{ij}(x_0 - y))G_{ij}(\rho_\tau - \rho_2)(y)dy$$
$$- \int (K_i(\cdot - y) - K_i(x_0 - y))G_{ij}\partial_j(\rho_\tau - \rho_2)(y)dy,$$

for $\rho_\tau = \Phi\left(\frac{\cdot - x_0}{\tau}\right)$, $\tau > 4$, and

$$p^{G,3} = \int (K_{ij}(\cdot - y) - K_{ij}(x_0 - y))G_{ij}(1-\rho_\tau)(y)dy + \int (K_i(\cdot - y) - K_i(x_0 - y))G_{ij}\partial_j \rho_\tau(y)dy.$$

Recall $p^F = p^F_{x_0}$

$$\begin{aligned} p^F &= -\frac{1}{3}\operatorname{tr} F + \text{p.v.}\int_{B(x_0,2)} K_{ij}(\cdot - y)F_{ij}(y)dy \\ &\quad + \int_{B(x_0,2)^c} (K_{ij}(\cdot - y) - K_{ij}(x_0 - y))F_{ij}(y)dy \\ &= p^{F,1} + p^{F,2} + p^{F,3}. \end{aligned}$$

We estimate $p^{F,i}\chi$, $i = 1,2,3$. Obviously, we have

$$\left\|p^{F,1}\chi\right\|_{L^{\frac{3}{2}}([t_0,t]\times B(x_0,\frac{3}{2}))} \lesssim \|F\chi\|_{U^{\frac{3}{2},\frac{3}{2}}(t_0,t)}.$$

Using the L^p-norm preservation of Riesz transfroms and $\|\nabla\chi\|_\infty \lesssim \frac{1}{R}$,

$$\begin{aligned} \left\|p^{F,2}\chi\right\|_{L^{\frac{3}{2}}([t_0,t]\times B(x_0,\frac{3}{2}))} &\le \left\|\text{p.v.}\int_{B(x_0,2)} K_{ij}(\cdot - y)F_{ij}(y)\chi(y)dy\right\|_{L^{\frac{3}{2}}([t_0,t]\times B(x_0,\frac{3}{2}))} \\ &\quad + \left\|\text{p.v.}\int_{B(x_0,2)} K_{ij}(\cdot - y)F_{ij}(y)(\chi(\cdot) - \chi(y))dy\right\|_{L^{\frac{3}{2}}([t_0,t]\times B(x_0,\frac{3}{2}))} \\ &\lesssim \|F\chi\|_{U^{\frac{3}{2},\frac{3}{2}}(t_0,t)} + \frac{1}{R}\left\|\int_{B(x_0,2)} \frac{1}{|\cdot - y|^2}|F_{ij}(y)|dy\right\|_{L^{\frac{3}{2}}(t_0,t;L^3(\mathbb{R}^3)))} \\ &\lesssim \|F\chi\|_{U^{\frac{3}{2},\frac{3}{2}}(t_0,t)} + \frac{1}{R}\|F\|_{U^{\frac{3}{2},\frac{3}{2}}(t_0,t)}. \end{aligned}$$

The last inequality follows from the Riesz potential estimate. Since

$$|\chi(x) - \chi(y)| \le \|\nabla\chi\|_\infty |x-y| \lesssim \frac{1}{\sqrt{R}}$$

for $x \in B(x_0, \frac{3}{2})$ and $y \in B(x_0, \sqrt{R})$, and

$$|x-y| \geq |x_0 - y| - |x - x_0| \geq \frac{1}{4}|x_0 - y|$$

for $x \in B(x_0, \frac{3}{2})$ and $y \in B(x_0, 2)^c$, we get

$$\begin{aligned}
\left\| p^{F,3}\chi \right\|_{L^{\frac{3}{2}}([t_0,t]\times B(x_0,\frac{3}{2}))} \leq & \left\| \int_{B(x_0,\sqrt{R})\setminus B(x_0,2)} \frac{1}{|\cdot - y|^4} F_{ij}\chi(y)dy \right\|_{L^{\frac{3}{2}}([t_0,t]\times B(x_0,\frac{3}{2}))} \\
& + \left\| \int_{B(x_0,\sqrt{R})\setminus B(x_0,2)} \frac{1}{|\cdot - y|^4} F_{ij}(y)(\chi(\cdot) - \chi(y))dy \right\|_{L^{\frac{3}{2}}([t_0,t]\times B(x_0,\frac{3}{2}))} \\
& + \left\| \int_{B(x_0,\sqrt{R})^c} \frac{1}{|\cdot - y|^4} F_{ij}(y)dy\chi \right\|_{L^{\frac{3}{2}}([t_0,t]\times B(x_0,\frac{3}{2}))}.
\end{aligned}$$

Thus

$$\begin{aligned}
\left\| p^{F,3}\chi \right\|_{L^{\frac{3}{2}}([t_0,t]\times B(x_0,\frac{3}{2}))} \lesssim & \sum_{k=1}^{\infty} \left\| \int_{B(x_0,2^{k+1})\setminus B(x_0,2^k)} \frac{1}{|x_0 - y|^4} |F_{ij}\chi(y)|dy \right\|_{L^{\frac{3}{2}}(t_0,t;L^\infty(B(x_0,\frac{3}{2})))} \\
& + \frac{1}{\sqrt{R}} \sum_{k=1}^{\infty} \left\| \int_{B(x_0,2^{k+1})\setminus B(x_0,2^k)} \frac{1}{|x_0 - y|^4} |F_{ij}(y)|dy \right\|_{L^\infty([t_0,t]\times B(x_0,\frac{3}{2}))} \\
& + \sum_{\lfloor \log_2 \sqrt{R} \rfloor}^{\infty} \left\| \int_{B(x_0,2^{k+1})\setminus B(x_0,2^k)} \frac{1}{|x_0 - y|^4} |F_{ij}(y)|dy \right\|_{L^{\frac{3}{2}}([t_0,t]\times B(x_0,\frac{3}{2}))} \\
\lesssim & \left\| F\chi \right\|_{L^{\frac{3}{2}}([t_0,t]\times B(x_0,\frac{3}{2}))} + \frac{1}{\sqrt{R}} \left\| F \right\|_{U^{\infty,1}(t_0,t)}.
\end{aligned}$$

Combining the estimates for $p^{F,i}\chi$, $i = 1,2,3$, we obtain

$$\begin{aligned}
\left\| p^F\chi \right\|_{L^{\frac{3}{2}}([t_0,t]\times B(x_0,\frac{3}{2}))} \lesssim_T & \left\| F\chi \right\|_{U^{\frac{3}{2},\frac{3}{2}}(t_0,t)} + \frac{1}{R} \left\| F \right\|_{U^{\frac{3}{2},\frac{3}{2}}(t_0,t)} + \frac{1}{\sqrt{R}} \left\| F \right\|_{U^{\infty,1}(t_0,t)} \\
\lesssim_{A,T} & \left\| w\chi \right\|_{U^{3,3}(t_0,t)} + \frac{1}{\sqrt{R}}.
\end{aligned}$$

Now, we consider the $p^{G,i}$'s. Since for $x \in B(x_0, \frac{3}{2})$, $p^{G,1}$ satisfies

$$
\begin{aligned}
|p^{G,1}\chi(x,t)| &\leq \int_{|x_0-y|\leq 3} |(\nabla K)(x-y)||V||\nabla V|(y,t)(|\chi(y)| + |\chi(x)-\chi(y)|)dy \\
&\lesssim \int_{B_3(x_0)} \frac{1}{|x-y|^2}||V||\nabla V(y,t)|\chi(y)dy + \frac{1}{R}\int_{B_3(x_0)} \frac{1}{|x-y|}|V||\nabla V(y,t)|dy
\end{aligned}
$$

using $|\chi(x)-\chi(y)| \lesssim \|\nabla\chi\|_\infty |x-y|$, the estimate for $p^{G,1}\chi$ can be obtained from Young's convolution inequality;

$$
\begin{aligned}
\|p^{G,1}\chi\|_{L^{\frac{3}{2}}([t_0,t]\times B(x_0,\frac{3}{2}))} &\lesssim_T \left\| \int_{|x_0-y|\leq 3} \frac{1}{|\cdot - y|^2}||V||\nabla V|(y,t)\chi(y)dy \right\|_{L^2([t_0,t]\times\mathbb{R}^3)} \\
&\quad + \frac{1}{R}\left\| \int_{|x_0-y|\leq 3} \frac{1}{|\cdot - y|}|V||\nabla V|(y,t)dy \right\|_{L^{\frac{20}{13}}(t_0,t;L^{\frac{30}{7}}(\mathbb{R}^3))} \\
&\lesssim \left\| \frac{1}{|\cdot|^2} \right\|_{\frac{3}{2},\infty} \||\nabla V|\chi\|_{L^\infty_t(t_0,T;L^{\frac{3}{2}}(B(x_0,3)))} \|V\|_{L^2(0,T;L^6(B(x_0,3)))} \\
&\quad + \frac{1}{R}\left\| \frac{1}{|\cdot|} \right\|_{3,\infty} \|V\|_{L^{\frac{20}{3}}(0,T;L^{\frac{5}{2}}(B(x_0,3)))} \|\nabla V\|_{U_T^{2,2}} \\
&\lesssim_{A,T} \||\nabla V|\chi\|_{U^{\infty,\frac{3}{2}}(t_0,T)} + \frac{1}{R}.
\end{aligned}
$$

By integration by parts, for $x \in B(x_0, \frac{3}{2})$, $p^{G,2}$ can be rewritten as

$$
p^{G,2} = \int (K_i(\cdot - y) - K_i(x_0 - y))V_i\partial_j V_j(y,t)(\rho_\tau - \rho_2)(y)dy
$$

and then it satisfies

$$
\begin{aligned}
|p^{G,2}\chi(x,t)| &\lesssim \int_{2<|x_0-y|\leq 2\tau} \frac{1}{|x_0-y|^3}|V||\nabla V|(y,t)(|\chi(y)| + |\chi(x)-\chi(y)|)dy \\
&\lesssim \sum_{i=1}^{m_\tau} \int_{B_{i+1}\setminus B_i} \frac{1}{|x_0-y|^3}|V||\nabla V|(y,t)\left(|\chi(y)| + \frac{\tau}{R}\right)dy,
\end{aligned}
$$

where $m_\tau = \lceil \ln(2\tau)/\ln 2 \rceil$ and $B_i = B(x_0, 2^i)$. Taking $L^2(t_0,t)$ on it, we have

$$\begin{aligned}
\|p^{G,2}\chi\|_{L^2(t_0,t;L^\infty(B(x_0,\frac{3}{2})))} &\lesssim \left\| \sum_{i=1}^{m_\tau} \int_{B_{i+1}\setminus B_i} \frac{1}{|x_0-y|^3} |V||\nabla V|(y,t) \left(|\chi(y)| + \frac{\tau}{R}\right) dy \right\|_{L^2(t_0,t)} \\
&\lesssim \sum_{i=1}^{m_\tau} \frac{1}{2^{3i}} \left(\|V|\nabla V|\chi\|_{L^2(t_0,t;L^1(B_{i+1}))} + \frac{\tau}{R} \|V|\nabla V|\|_{L^2(t_0,t;L^1(B_{i+1}))} \right) \\
&\lesssim \sum_{i=1}^{m_\tau} \left((\| |V||\nabla V|\chi\|_{U^{2,1}(t_0,T)} + \frac{\tau}{R} \| |V||\nabla V|\|_{U^{2,1}_T} \right) \\
&\lesssim_T \ln\tau \|V\|_{U^{\infty,2}_T} \| |\nabla V|\chi\|_{U^{\infty,2}(t_0,T)} + \frac{\tau\ln\tau}{R} \|V\|_{U^{\infty,2}_T} \|\nabla V\|_{U^{2,2}_T}.
\end{aligned}$$

Lastly,

$$|p^{G,3}(x,t)| \le \int_{|x_0-y|\ge\tau} \frac{|V(y,t)|^2}{|x_0-y|^4} dy + \frac{1}{\tau} \int_{\tau\le|x_0-y|\le 2\tau} \frac{|V(y,t)|^2}{|x_0-y|^3} dy \le \frac{1}{\tau} \|V\|^2_{U^{\infty,2}_T}.$$

Hence

$$\left\|p^{G,3}\chi\right\|_{L^{\frac{3}{2}}([t_0,t]\times B(x_0,\frac{3}{2}))} \le \left\|p^{G,3}\right\|_{L^{\frac{3}{2}}([t_0,t]\times B(x_0,\frac{3}{2}))} \lesssim_{A,T} \frac{1}{\tau}$$

To summarize, we have shown

$$\sum_{i=1}^{3} \left\|p^{G,i}\chi\right\|_{L^{\frac{3}{2}}([t_0,t]\times B(x_0,\frac{3}{2}))} \lesssim_{A,T} \ln\tau \| |\nabla V|\chi\|_{U^{\infty,2}(t_0,T)} + \frac{\tau\ln\tau}{R} + \frac{1}{\tau},$$

and therefore

$$\|\breve{p}\chi\|_{L^{\frac{3}{2}}([t_0,t]\times B(x_0,\frac{3}{2}))} \lesssim_{A,T} \|w\chi\|_{U^{3,3}(t_0,t)} + \frac{1}{\sqrt{R}} + \ln\tau \| |\nabla V|\chi\|_{U^{\infty,2}(t_0,T)} + \frac{\tau\ln\tau}{R} + \frac{1}{\tau}. \tag{3.4.20}$$

Finally, combining all estimates and then taking supremum on (3.4.19) over $x_0 \in \mathbb{R}^3$, we obtain

$$\begin{aligned}
\|w(\cdot,t)\chi\|^2_{L^2_{\mathrm{uloc}}} &+ 2\| |\nabla w|\chi\|^2_{U^{2,2}(t_0,t)} \\
&\lesssim_{A,T} \|w(\cdot,t_0)\chi\|^2_{L^2_{\mathrm{uloc}}} + \|w\chi\|^2_{U^{2,2}(t_0,t)} + \|w\chi\|^2_{U^{3,3}(t_0,t)} \\
&\quad + (\ln\tau)^2 \| |\nabla V|\chi\|^2_{U^{\infty,3}(t_0,T)} + \frac{(\tau\ln\tau)^2}{R^2} + \frac{1}{\tau^2} + \frac{1}{R}.
\end{aligned} \tag{3.4.21}$$

Using the estimates

$$\|w\chi\|_{U^{3,3}(t_0,t)}^2 \lesssim \|w\chi\|_{U^{6,2}(t_0,t)} \left(\|w\chi\|_{U^{2,2}(t_0,t)} + \||\nabla w|\chi\|_{U^{2,2}(t_0,t)} + \frac{1}{R}\|w\|_{U_T^{2,2}} \right), \tag{3.4.22}$$

and Lemma 3.4.5, it becomes

$$\begin{aligned} &\|w(\cdot,t)\chi\|_{L^2_{\text{uloc}}}^2 + \||\nabla w|\chi\|_{U^{2,2}(t_0,t)}^2 \\ &\quad \lesssim_{A,T,C_0} t_0^{\frac{1}{10}} + \|w_0\chi\|_{L^2_{\text{uloc}}}^2 + \|w\chi\|_{L^6(t_0,t;L^2_{\text{uloc}})}^2 \\ &\qquad + (\ln\tau)^2 \||\nabla V|\chi\|_{U^{\infty,3}(t_0,T)}^2 + \frac{(\tau\ln\tau)^2}{R^2} + \frac{1}{\tau^2} + \frac{1}{R}, \end{aligned} \tag{3.4.23}$$

where C_0 is defined as in Lemma 3.4.5.

Note that $\|w(\cdot,t)\chi\|_{L^2_{\text{uloc}}}^2$ is lower semicontinuous in t as w is weakly $L^2(B_n)$-continuous in t for any n. By the Grönwall inequality, we have

$$\begin{aligned} \|w\chi\|_{L^6(t_0,T;L^2_{\text{uloc}})}^2 &\lesssim_{A,T,C_0} t_0^{\frac{1}{10}} + \|w_0\chi\|_{L^2_{\text{uloc}}}^2 \\ &\quad + (\ln\tau)^2 \||\nabla V|\chi\|_{U^{\infty,3}(t_0,T)}^2 + \frac{(\tau\ln\tau)^2}{R^2} + \frac{1}{\tau^2} + \frac{1}{R}. \end{aligned} \tag{3.4.24}$$

We now prove (3.4.18). Fix $t_1 \in (0,T)$. For every $n \in \mathbb{N}$ we can choose $t_0 = t_0(n) \in (0,t_1)\setminus\Sigma$ satisfying

$$t_0^{\frac{1}{10}} < \tfrac{1}{n}.$$

At the same time, we pick $\tau = \tau(n) > 4$ satisfying $\tau^{-2} \le 1/n$. After t_0 and τ are fixed, we can make all the remaining terms small by choosing $R = R(n, \|v_0\|_{L^2_{\text{uloc}}}, t_0, \tau)$ sufficiently large:

$$\|w_0\chi_R\|_{L^2_{\text{uloc}}}^2 + (\ln\tau)^2 \||\nabla V|\chi_R\|_{U^{\infty,3}(t_0,T)}^2 + \frac{(\tau\ln\tau)^2}{R^2} + \frac{1}{R} \le \frac{1}{n}.$$

Here, the smallness of the second term follows from ∇V decay (Lemma 3.2.7), using the oscillation decay of v_0. In conclusion, by (3.4.24), for each $n \in \mathbb{N}$, we can find t_0, τ and $R \gg 1$ so that

$$\|w\chi_R\|^2_{L^6(t_0,T;L^2_{\mathrm{uloc}})} \lesssim_{A,T,C_0} \frac{1}{n}.$$

By (3.4.23),

$$\|w\chi_R\|^2_{L^\infty(t_0,T;L^2_{\mathrm{uloc}})} + \||\nabla w|\chi_R\|_{U^{2,2}(t_0,T)} \lesssim_{A,T,C_0} \frac{1}{n}.$$

By (3.4.22),

$$\|w\chi_R\|^2_{U^{3,3}(t_0,T)} \lesssim_{A,T,C_0} \frac{1}{n}.$$

Restricted to the original time interval (t_1,T), the perturbation w satisfies

$$\lim_{R\to\infty} \|w\chi_R\|_{U^{3,3}(t_1,T)} = 0,$$

$$\lim_{R\to\infty} \|w\chi_R\|^2_{L^\infty(t_1,T;L^2_{\mathrm{uloc}})} + \||\nabla w|\chi_R\|_{U^{2,2}(t_1,T)} = 0.$$

Using (3.4.20), we also have

$$\lim_{R\to\infty} \sup_{x_0\in\mathbb{R}^3} \|\breve{p}_{x_0}\chi_R\|_{L^{\frac{3}{2}}(B(x_0,\frac{3}{2})\times(t_1,T))} = 0.$$

This completes the proof of Proposition 3.4.7. □

Corollary 3.4.8. *Under the same assumptions of Proposition 3.4.7, the perturbed Navier-Stokes flow* $w = v - e^{t\Delta}u_0$ *satisfies* $w(t) \in E^p(\mathbb{R}^3)$ *for almost all* $t \in (0,T]$ *for any* $3 \le p \le 6$.

Proof. By Proposition 3.4.7, for any fixed $x_0 \in \mathbb{R}^3$ and $t_1 \in (0,T)$, the perturbed local energy solution w to the Navier-Stokes equations satisfies

$$\|w\|_{L^3(B_{3/2}(x_0)\times(t_1,T))} + \|\breve{p}_{x_0}\|_{L^{3/2}(B_{3/2}(x_0)\times(t_1,T))} \to 0 \quad \text{as } |x_0| \to \infty.$$

Recall that $V \in C^1([\delta,\infty)\times\mathbb{R}^3)$ for any $\delta > 0$. Then, by the Caffarelli-Kohn-Nirenberg criteria [6], for any $t_2 \in (t_1,T]$, we can find $R_0 > 0$ such that if $|x_0| \ge R_0$,

$$\|w\|_{L^\infty([t_2,T]\times B_1(x_0))} \lesssim \|w\|_{L^3(B_{3/2}(x_0)\times(t_1,T))} + \|\breve{p}_{x_0}\|^{1/2}_{L^{3/2}(B_{3/2}(x_0)\times(t_1,T))},$$

and the constant in the inequality is independent of x_0. Moreover, $\|w\|_{L^\infty([t_2,T]\times B_1(x_0))} \to 0$ as $|x_0| \to \infty$. Although the system (3.4.8) satisfied by w is not the original (NS), similar proof works since $V \in C^1$. See [22, Theorem 2.1] for more singular $V \in L^m$, $m > 1$, but without the source term $V \cdot \nabla V$.

On the other hand, $w \in \mathscr{E}_T$ implies that

$$w \in L^s(0,T;L^p(B_{R_0}))$$

for any $s \in [2,\infty]$ and $p \in [2,6]$ with $\frac{2}{s} + \frac{3}{p} = \frac{3}{2}$, and therefore $w(t) \in E^p$ for a.e. $t \in (0,T]$. □

3.5 Global existence

In this section, we prove Theorem 3.1.1. We first give the following decay estimates.

Lemma 3.5.1. *Let (v,p) be a local energy solution in $\mathbb{R}^3 \times [t_0,T]$, $0 < t_0 < T < \infty$, to the Navier-Stokes equations* (NS) *for the initial data*

$$v|_{t=t_0} = w_* + e^{t_0\Delta}u_0$$

where $w_* \in E^2_\sigma$ and $u_0 \in L^3_{\mathrm{uloc},\sigma}$ satisfies the oscillation decay (3.1.5)*. Let $V(t) = e^{t\Delta}u_0$. Then, the perturbation $w = v - V$ also decays at infinity:*

$$\|w\|_{L^3([t_0,T]\times B(x_0,1))} + \|\breve{p}_{x_0}\|_{L^{\frac{3}{2}}([t_0,T]\times B(x_0,1))} \to 0,$$

and

$$\|w\|_{L^\infty(t_0,T;L^2(B(x_0,1)))} + \|\nabla w\|_{L^2(t_0,T;L^2(B(x_0,1)))} \to 0, \quad \text{as } |x_0| \to \infty.$$

Remark. This T is arbitrarily large, unlike the existence time given in the local existence theorem, Theorem 3.3.2. We assume $w_* \in E^2$, and we have $V \in C^1(\mathbb{R}^3 \times [t_0,T])$. We no longer need Lemma 3.4.5 nor the strong local energy inequality.

Proof. The proof is almost the same as that of Proposition 3.4.7 except for the way to estimate $\|w(\cdot,t_0)\chi_R\|_{L^2_{\mathrm{uloc}}}$ in (3.4.21). Indeed, $\lim_{R\to\infty}\|w(\cdot,t_0)\chi_R\|_{L^2_{\mathrm{uloc}}} = 0$ by the assumption $w(\cdot, t_0) = w_* \in E^2$. $\square$

Now, we prove the main theorem.

Proof of Theorem 3.1.1. Let (v, p) be a local energy solution to the Navier-Stokes equations in $\mathbb{R}^3 \times [0, T_0]$, $0 < T_0 < \infty$, for the initial data $v|_{t=0} = v_0$, constructed in Theorem 3.3.2. By Corollary 3.4.8, there exists $t_0 \in (0, T_0)$, arbitrarily close to T_0, with $w(t_0) = v(t_0) - e^{t_0 \Delta} u_0 \in E^4$. Then, by Lemma 3.2.2, for any small $\delta > 0$, we can decompose

$$w(t_0) = W_0 + h_0,$$

where $W_0 \in C^\infty_{c,\sigma}(\mathbb{R}^3)$ and $h_0 \in E^4(\mathbb{R}^3)$ with $\|h_0\|_{L^4_{\mathrm{uloc}}} < \delta$.

To construct a local energy solution $(\tilde{v}, \tilde{p})$ to (NS) for $t \geq t_0$ with initial data $\tilde{v}|_{t=t_0} = v(t_0)$, we decompose $(\tilde{v}, \tilde{p})$ as

$$\tilde{v} = V + h + W, \quad \tilde{p} = p_h + p_W$$

where $V(t) = e^{t\Delta} u_0$, (h, p_h) satisfies

$$\begin{cases} \partial_t h - \Delta h + \nabla p_h = -H \cdot \nabla H, \quad H = V + h, \\ \operatorname{div} h = 0, \quad h|_{t=t_0} = h_0, \end{cases} \tag{3.5.1}$$

so that H solves (NS) with $H(t_0) = e^{t_0 \Delta} u_0 + h_0$, and (W, p_W) satisfies

$$\begin{cases} \partial_t W - \Delta W + \nabla p_W = -[(H + W) \cdot \nabla] W - (W \cdot \nabla) H, \\ \operatorname{div} W = 0, \quad W|_{t=t_0} = W_0. \end{cases} \tag{3.5.2}$$

Our strategy is to first find, for each $\varepsilon > 0$, a distributional solution $(h^\varepsilon, p_h^\varepsilon)$ and a Leray-Hopf weak solution $(W^\varepsilon, p_W^\varepsilon)$ to ε-approximations of (3.5.1) and (3.5.2) for $t \in I$ for some $S = S(\delta, V) > 0$ uniform in ε. Then, we prove that they have a limit $(\tilde{v}, \tilde{p})$ which is a desired local energy solution to (NS) on I. By gluing two solutions v and $\tilde{v}$ at $t = t_0$, we can get the extended local energy solution on the time interval $[0, t_0 + S]$. Repeating this process, we get a time-global local energy solution. The detailed proof is given below.

Step 1. Construction of approximation solutions

Let $I = (t_0, t_0 + S)$ for some small $S \in (0,1)$ to be decided. For $0 < \varepsilon < 1$, we first consider the fixed point problem for

$$\Psi(h) = e^{(t-t_0)\Delta} h_0 - \int_{t_0}^{t} e^{(t-s)\Delta} \mathbb{P} \nabla \cdot (\mathscr{J}_\varepsilon H \otimes H \Phi_\varepsilon)(s) ds, \quad H = V + h, \tag{3.5.3}$$

where $\mathscr{J}_\varepsilon(H) = H * \eta_\varepsilon$ is the mollification of scale ε and $\Phi_\varepsilon(x) = \Phi(\varepsilon x)$ is a localization factor of scale ε^{-1}. We will solve for a fixed point $h = h^\varepsilon$ in the Banach space

$$\mathscr{F} = \mathscr{F}_{t_0,S} := \{h \in U^{\infty,4}(I) : (t-t_0)^{\frac{3}{8}} h(\cdot,t) \in L^\infty(I \times \mathbb{R}^3)\}$$

for some small $S > 0$ with

$$\|h\|_{\mathscr{F}} := \|h\|_{U^{\infty,4}(I)} + \left\| (t-t_0)^{\frac{3}{8}} h(t) \right\|_{L^\infty(I\times\mathbb{R}^3)}.$$

Denote $M = \|V\|_{L^\infty(I\times\mathbb{R}^3)} \lesssim (1 + t_0^{-4/3}) \|v_0\|_{L^2_{\text{uloc}}}$. By Lemma 3.2.3, we have

$$\begin{aligned}
\|\Psi h\|_{U^{\infty,4}(I)} &\lesssim \|h_0\|_{L^4_{\text{uloc}}} + S^{\frac{1}{8}} \|h\|_{U^{\infty,4}}^2 + S^{\frac{1}{2}} M \|h\|_{U^{\infty,4}} + S^{\frac{1}{2}} \|V\|_{L^\infty(I;L^8_{\text{uloc}})}^2 \\
&\lesssim \|h_0\|_{L^4_{\text{uloc}}} + S^{\frac{1}{8}} \|h\|_{\mathscr{F}}^2 + S^{\frac{1}{2}} M \|h\|_{\mathscr{F}} + S^{\frac{1}{2}} M^2,
\end{aligned}$$

and for $t \in I$,

$$\begin{aligned}
\|\Psi h(t)\|_{L^\infty(\mathbb{R}^3)} &\lesssim (t-t_0)^{-\frac{3}{8}} \|h_0\|_{L^4_{\text{uloc}}} + \int_{t_0}^{t} |t-s|^{-1/2} \left(\|h(s)\|_{L^\infty}^2 + M^2 \right) ds \\
&\lesssim (t-t_0)^{-\frac{3}{8}} \|h_0\|_{L^4_{\text{uloc}}} + (t-t_0)^{-1/4} \|h\|_{\mathscr{F}}^2 + (t-t_0)^{1/2} M^2.
\end{aligned}$$

Therefore, we get

$$\|\Psi h\|_{\mathscr{F}} \lesssim \|h_0\|_{L^4_{\text{uloc}}} + S^{\frac{1}{8}} \|h\|_{\mathscr{F}}^2 + S^{\frac{1}{2}} M \|h\|_{\mathscr{F}} + S^{\frac{1}{2}} M^2.$$

Similarly we can show

$$\|\Psi h_1 - \Psi h_2\|_{\mathscr{F}} \lesssim \left\{ S^{\frac{1}{8}} (\|h_1\|_{\mathscr{F}} + \|h_2\|_{\mathscr{F}}) + S^{\frac{1}{2}} M \right\} \|h_1 - h_2\|_{\mathscr{F}}.$$

By the Picard contraction theorem, we can find $S = S(\delta, \|V\|_{L^\infty(I\times\mathbb{R}^3)}) \in (0,1)$ such that a unique fixed point (mild solution) h^ε to (3.5.3) exists in $\mathscr{F}_{t_0,S}$ with

$$\|h^\varepsilon\|_{\mathscr{F}} \le C\delta, \qquad \forall 0<\varepsilon<1. \tag{3.5.4}$$

We also have the uniform bound

$$\|h^\varepsilon\|_{\mathscr{E}(I)} \lesssim \|\mathscr{J}_\varepsilon H^\varepsilon \otimes H^\varepsilon \Phi_\varepsilon\|_{U^{2,2}(I)} \lesssim \|h^\varepsilon\|_{\mathscr{F}}^2 + \|V\|_{U^{4,4}(I)}^2 \lesssim \delta^2 + M^2. \tag{3.5.5}$$

Now, we define $H^\varepsilon = V + h^\varepsilon$ and the pressure p_h^ε by

$$\begin{aligned} p_h^\varepsilon = &-\frac{1}{3}\mathscr{J}_\varepsilon H^\varepsilon \cdot H^\varepsilon \Phi_\varepsilon + \text{p.v.}\int_{B_2} K_{ij}(\cdot - y)(\mathscr{J}_\varepsilon H^\varepsilon)_i H_j^\varepsilon \Phi_\varepsilon(y,t)dy \\ &+ \text{p.v.}\int_{B_2^c} (K_{ij}(\cdot - y) - K_{ij}(-y))(\mathscr{J}_\varepsilon H^\varepsilon)_i H_j^\varepsilon \Phi_\varepsilon(y,t)dy. \end{aligned}$$

It is well defined thanks to the localization factor Φ_ε. For each $R>0$, we have a uniform bound

$$\|p_h^\varepsilon\|_{L^{\frac{3}{2}}(I\times B_R)} \le C(R) \tag{3.5.6}$$

in a similar way to getting (3.3.17). The pair $(h^\varepsilon, p_h^\varepsilon)$ solves, with $H^\varepsilon = V + h^\varepsilon$,

$$\begin{cases} \partial_t h^\varepsilon - \Delta h^\varepsilon + \nabla p_h^\varepsilon = -(\mathscr{J}_\varepsilon H^\varepsilon \cdot \nabla)(H^\varepsilon \Phi_\varepsilon), \\ \operatorname{div} h^\varepsilon = 0, \quad h^\varepsilon|_{t=t_0} = h_0 \in L^4_{\text{uloc}} \end{cases} \tag{3.5.7}$$

in $\mathbb{R}^3 \times I$ in the distributional sense.

We next consider the equation for $W = W^\varepsilon$,

$$\begin{cases} \partial_t W - \Delta W + \nabla p_W = f_W^\varepsilon \\ f_W^\varepsilon := -\mathscr{J}_\varepsilon(H^\varepsilon + W)\cdot\nabla W - \mathscr{J}_\varepsilon W \cdot \nabla H^\varepsilon, \\ \operatorname{div} W = 0, \quad W|_{t=t_0} = W_0 \in C^\infty_{c,\sigma}. \end{cases} \tag{3.5.8}$$

Note that (3.5.8) is a mollified and perturbed (NS), and has no localization factor Φ_ε.

Using W^ε itself as a test function, we can get an a priori estimate: for $t \in I$,

$$\|W(t)\|_{L^2(\mathbb{R}^3)}^2 + 2\|\nabla W\|_{L^2([t_0,t]\times\mathbb{R}^3)}^2 \le \|W_0\|_{L^2(\mathbb{R}^3)}^2 + \iint f_W^\varepsilon \cdot W.$$

Note that $\iint \mathscr{J}_\varepsilon(H+W)\cdot\nabla W \cdot W\,dxdt = 0$ and $-\iint(\mathscr{J}_\varepsilon W\cdot\nabla)h^\varepsilon\cdot W\,dxdt = \iint(\mathscr{J}_\varepsilon W\cdot\nabla)W\cdot h^\varepsilon\,dxdt$. Also recall that

$$\|h^\varepsilon W\|_{L^2(Q)} \lesssim \|h^\varepsilon\|_{L^\infty(I;L^3_{\mathrm{uloc}})}(\|\nabla W\|_{L^2(Q)} + \|W\|_{L^2(Q)})$$

for $Q = [t_0,t]\times\mathbb{R}^3$. Its proof can be found in [26, page 162]. Thus

$$\begin{aligned}\iint f_W^\varepsilon\cdot W &= \iint(\mathscr{J}_\varepsilon W\cdot\nabla)W\cdot h^\varepsilon - \iint(\mathscr{J}_\varepsilon W\cdot\nabla)V\cdot W\\ &\le C\|\nabla W\|_{L^2(Q)}\,\delta(\|\nabla W\|_{L^2(Q)} + \|W\|_{L^2(Q)}) + M_1\|W\|_{L^2(Q)}^2.\end{aligned}$$

where $M_1 = \|\nabla V\|_{L^\infty(I\times\mathbb{R}^3)}$. By choosing δ sufficiently small, we conclude

$$\|W(t)\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla W\|_{L^2([t_0,t]\times\mathbb{R}^3)}^2 \le \|W_0\|_{L^2(\mathbb{R}^3)}^2 + C(1+M_1)\|W\|_{L^2(Q)}^2.$$

By the Grönwall inequality (using that $\|W(t)\|_{L^2(\mathbb{R}^3)}^2$ is lower semicontinuous), we obtain

$$\|W^\varepsilon\|_{L^\infty(I;L^2(\mathbb{R}^3))}^2 + \|\nabla W^\varepsilon\|_{L^2(I\times\mathbb{R}^3)}^2 \le C(M_1)\|W_0\|_{L^2(\mathbb{R}^3)}^2. \tag{3.5.9}$$

With this uniform a priori bound, for each $0<\varepsilon<1$, we can use the Galerkin method to construct a Leray-Hopf weak solution W^ε on $I\times\mathbb{R}^3$ to (3.5.8).

Define $F_{ij}^\varepsilon = \mathscr{J}_\varepsilon(W^\varepsilon + H^\varepsilon)_i W_j^\varepsilon + (\mathscr{J}_\varepsilon W^\varepsilon)_i H_j^\varepsilon$. We have the uniform bound

$$\left\|F_{ij}^\varepsilon\right\|_{U^{3/2,3/2}(I)} \le C\||V| + |h^\varepsilon| + |W^\varepsilon|\|_{U^{3,3}(I)}^2 \le C(M, M_1, \|W_0\|_{L^2(\mathbb{R}^3)}).$$

Define $p_W^\varepsilon(x,t) = \lim_{n\to\infty} p_W^{\varepsilon,n}(x,t)$, and $p_W^{\varepsilon,n}(x,t)$ is defined for $|x| < 2^n$ by

$$\begin{aligned} p_W^{\varepsilon,n}(x,t) = &-\frac{1}{3}\operatorname{tr} F_{ij}^\varepsilon(x,t) + \text{p.v.}\int_{B_2(0)} K_{ij}(x-y)F_{ij}^\varepsilon(y,t)dy \\ &+ \left(\text{p.v.}\int_{B_{2^{n+1}}\setminus B_2} + \int_{B^c_{2^{n+1}}}\right)(K_{ij}(x-y)-K_{ij}(-y))F_{ij}^\varepsilon(y,t)dy. \end{aligned}$$

For each $R>0$, we have a uniform bound

$$\|p_W^\varepsilon\|_{L^{\frac{3}{2}}(I\times B_R)} \le C(R,M,M_1,\|W_0\|_{L^2(\mathbb{R}^3)}). \tag{3.5.10}$$

By the usual theory for the nonhomogeneous Stokes system in $\mathbb{R}^3$, the pair $(W^\varepsilon, p_W^\varepsilon)$ solves (3.5.8) in the distributional sense.

We now define

$$v^\varepsilon = H^\varepsilon + W^\varepsilon = V + h^\varepsilon + W^\varepsilon, \quad p^\varepsilon = p_h^\varepsilon + p_W^\varepsilon.$$

Summing (3.5.7) and (3.5.8), the pair $(v^\varepsilon, p^\varepsilon)$ solves in the distributional sense

$$\begin{cases} \partial_t v^\varepsilon - \Delta v^\varepsilon + \nabla p^\varepsilon = -\mathscr{J}_\varepsilon v^\varepsilon\cdot\nabla v^\varepsilon + E^\varepsilon, \\ \qquad\qquad\qquad\qquad E^\varepsilon = \mathscr{J}_\varepsilon H^\varepsilon\cdot\nabla(H^\varepsilon(1-\Phi_\varepsilon)), \\ \operatorname{div} v^\varepsilon = 0, \quad v^\varepsilon|_{t=t_0} = v(t_0). \end{cases} \tag{3.5.11}$$

Thanks to the mollification, h^ε and W^ε have higher local integrability by the usual regularity theory. Thus we can test (3.5.11) by $2v^\varepsilon\xi$, $\xi\in C_c^\infty([t_0,t_0+S)\times\mathbb{R}^3)$, and integrate by parts to get the identity

$$\begin{aligned} 2\iint_I |\nabla v^\varepsilon|^2\xi\,dxds = &\int |v|^2\xi(x,t_0)\,dx \\ &+\iint_I |v^\varepsilon|^2(\partial_s\xi+\Delta\xi) + (|v^\varepsilon|^2\mathscr{J}_\varepsilon v^\varepsilon + 2p^\varepsilon v^\varepsilon)\cdot\nabla\xi + E^\varepsilon\cdot 2v^\varepsilon\xi\,dxds. \end{aligned} \tag{3.5.12}$$

Note that v in $\int|v|^2\xi(x,t_0)\,dx$ is the original solution in $[0,T)$.

`Step 2.` A local energy solution on $I=(t_0,t_0+S)$

We now show that $(v^\varepsilon, p^\varepsilon)$ has a weak limit $(\tilde v, \tilde p)$ which is a local energy solution on I. Recall the uniform bounds (3.5.4), (3.5.5), (3.5.6), (3.5.9), and (3.5.10) for $h^\varepsilon, p_h^\varepsilon, W^\varepsilon$ and p_W^ε. As in the proof of Theorem 3.3.2, from the uniform estimates and the compactness argument, we can find a subsequence $(v^{(k)}, p^{(k)})$, $k \in \mathbb{N}$, from $(v^\varepsilon, p^\varepsilon)$ which converges to some $(\tilde v, \tilde p)$ in the following sense: for each $n \in \mathbb{N}$,

$$\begin{aligned}
v^{(k)} &\overset{*}{\rightharpoonup} \tilde v && \text{in } L^\infty(I; L^2(B_{2^n})),\\
v^{(k)} &\rightharpoonup \tilde v && \text{in } L^2(I; H^1(B_{2^n})),\\
v^{(k)}, \mathscr{J}_{(k)} v^{(k)} &\to \tilde v && \text{in } L^3(I \times B_{2^n}),\\
p^{(k)} &\to \tilde p && \text{in } L^{\frac{3}{2}}(I \times B_{2^n}),
\end{aligned}$$

where $\tilde p(x,t) = \lim_{n\to\infty} \tilde p^n(x,t)$, and $\tilde p^n(x,t)$ is defined for $|x| < 2^n$ by

$$\begin{aligned}
\tilde p^n(x,t) = &-\frac{1}{3}|\tilde v(x,t)|^2 + \text{p.v.}\int_{B_2} K_{ij}(x-y)\tilde v_i \tilde v_j(y,t)\,dy \\
&+ \left(\text{p.v.}\int_{B_{2^{n+1}}\setminus B_2} + \int_{B^c_{2^{n+1}}}\right)(K_{ij}(x-y) - K_{ij}(-y))\tilde v_i \tilde v_j(y,t)\,dy.
\end{aligned}$$

Taking the limit of the weak form of (3.5.11), we obtain that $(\tilde v, \tilde p)$ satisfies the weak form of (NS) for the initial data $\tilde v|_{t=t_0} = v(t_0)$. Furthermore, the limit of (3.5.12) gives us the local energy inequality: For any $\xi \in C_c^\infty([t_0, t_0+S) \times \mathbb{R}^3)$, $\xi \geq 0$, we have

$$\begin{aligned}
2\int_I\int |\nabla \tilde v|^2 \xi\, dxds \leq &\int |v|^2 \xi(x,t_0)\,dx \\
&+ \int_I\int |\tilde v|^2(\partial_s \xi + \Delta\xi) + (|\tilde v|^2 + 2\tilde p)\tilde v \cdot \nabla\xi\, dxds.
\end{aligned} \tag{3.5.13}$$

Here we have used that $\iint E^{(k)} \cdot v^{(k)}\xi = \iint \mathscr{J}_{(k)} H^{(k)} \cdot \nabla(H^{(k)}(1-\Phi_{(k)})) \cdot v^{(k)}\xi = 0$ for k sufficiently large. In a way similar to the proof of Theorem 3.3.2, we get the local pressure decomposition for $\tilde p$, weak local L^2-continuity of $\tilde v(t)$, and local L^2-convergence to the initial data. We also get (3.5.13) with the time interval I replaced by $[t_0, t]$ and an additional term $\int |\tilde v|^2 \xi(x,t)\,dx$ in the left side.

We have shown that $(\tilde v, \tilde p)$ is a local energy solution on $\mathbb{R}^3 \times I$ with initial data $\tilde v|_{t=t_0} = v(t_0)$.

`Step 3.` To extend to a time-global local energy solution.

We first prove that the combined solution

$$u = v1_{[0,t_0]} + \tilde{v}1_I, \quad q = p1_{[0,t_0]} + \tilde{p}1_I$$

is a local energy solution on the extended time interval $[0,T_1] = [0,t_0+S]$. It is obvious that u and q are bounded in $\mathscr{E}_{T_1}$ and $L^{\frac{3}{2}}_{\mathrm{loc}}([0,T_1]\times\mathbb{R}^3)$, respectively and q satisfies the decomposition at each point $x_0 \in \mathbb{R}^3$. Since we have for any $\zeta \in C_c^\infty([t_0,T_1)\times\mathbb{R}^3;\mathbb{R}^3)$

$$\int_{t_0}^{T_1} -(\tilde{v},\partial_t\zeta) + (\nabla\tilde{v},\nabla\zeta) + (\tilde{v},(\tilde{v}\cdot\nabla)\zeta) + (\tilde{p},\operatorname{div}\zeta)dt = (\tilde{v},\zeta)(t_0) = (v,\zeta)(t_0),$$

and for any $\zeta \in C_c^\infty((0,t_0]\times\mathbb{R}^3;\mathbb{R}^3)$

$$\int_0^{t_0} -(v,\partial_t\zeta) + (\nabla v,\nabla\zeta) + (v,(v\cdot\nabla)\zeta) + (p,\operatorname{div}\zeta)dt = -(v,\zeta)(t_0),$$

from the weak continuity of $\tilde{v}$ at t_0 from the right and that of v at t_0, we can prove that (u,p) satisfies (NS) in the distribution sense: For any $\zeta \in C_c^\infty((0,T_1)\times\mathbb{R}^3;\mathbb{R}^3)$

$$\int_0^{T_1} -(u,\partial_t\zeta) + (\nabla u,\nabla\zeta) + (u,(u\cdot\nabla)\zeta) + (q,\operatorname{div}\zeta)dt = 0.$$

Also, since we already have local L^2-weak continuity of u on $[0,T_1]\setminus\{t_0\}$, it is enough to check it at t_0; for any $\varphi \in L^2(\mathbb{R}^3)$ with a compact support,

$$\lim_{t\to t_0^-}(u,\varphi)(t) = \lim_{t\to t_0^-}(v,\varphi)(t) = (v,\varphi)(t_0) = \lim_{t\to t_0^+}(\tilde{v},\varphi)(t) = \lim_{t\to t_0^+}(u,\varphi)(t).$$

Finally, we prove the local energy inequality (3.3.1). Indeed, for any $t \in (0,t_0]$, the inequality follows from the one of v. For $t \in (t_0,T_1)$, we add the inequality of v

in $[0,t_0]$ to the one of $\tilde{v}$ in $[t_0,t]$ to get, for any non-negative $\xi \in C_c^\infty((0,T_1)\times\mathbb{R}^3)$,

$$\begin{aligned}
&\int |u|^2\xi(t)dx + 2\int_0^t\int |\nabla u|^2\xi dxds \\
&\quad = \int |\tilde{v}|^2\xi(t)dx + 2\int_0^{t_0}\int |\nabla v|^2\xi dxds + 2\int_{t_0}^t\int |\nabla\tilde{v}|^2\xi dxds \\
&\quad \le \int_0^{t_0}\int |v|^2(\partial_s\xi + \Delta\xi) + (|v|^2+2p)(v\cdot\nabla)\xi dxds \\
&\qquad + \int_{t_0}^t\int |\tilde{v}|^2(\partial_s\xi+\Delta\xi) + (|\tilde{v}|^2+2\tilde{p})(\tilde{v}\cdot\nabla)\xi dxds \\
&\quad = \int_0^t\int |u|^2(\partial_s\xi+\Delta\xi) + (|u|^2+2q)(u\cdot\nabla)\xi dxds.
\end{aligned}$$

Therefore, (u,q) is a local energy solution on $[0,T_1)$ and is an extension of (v,p).

Then, by Lemma 3.5.1 and the proof of Corollary 3.4.8, we can find $t_1 \in (t_0+\frac{7}{8}S, t_0+S)$ such that $u(t_1)-V(t_1)\in E^4$. Repeating the above argument with new initial time t_1, we can get a local energy solution in $[0,t_1+S)$. Iterating this process, we get a local energy solution global in time. Note that $\|V\|_{L^\infty([t_1,\infty)\times\mathbb{R}^3)} \le \|V\|_{L^\infty([t_0,\infty)\times\mathbb{R}^3)}$ whenever $t_1 > t_0$, so that on each step, we can extend the time interval for the existence by at least $\frac{7}{8}S$. $\square$

Chapter 4

Conclusion

In this Chapter, we summarize the works in previous chapters and provide open questions related to the materials discussed in the dissertation.

The work in Chapter 2 presents that the logarithmically regularized 2D Euler equation (LE) for $0 < \gamma \leq \frac{1}{2}$ is strongly ill-posed in the borderline Sobolev space $H^1(\mathbb{R}^2) \cap \dot{H}^{-1}(\mathbb{R}^2)$. More precisely, we first showed that for any given compactly supported smooth initial data, a non-compactly supported perturbation always exists such that it is arbitrary small in the borderline space but the unique solution for the perturbed initial data in some other space leaves the borderline space instantaneously. Then, we also construct a compactly supported perturbation with similar properties, but it requires an additional condition on the given smooth initial data: it is odd at least in one variable.

This result completely solves the well-posedness problem of (LE) in Sobolev spaces. Especially in the borderline space, it closes a gap between the local well-posedness for $\gamma > \frac{1}{2}$ and the strong ill-posedness for $\gamma = 0$. Namely, the local well-posedness result is sharp, while the strong ill-posedness still holds even under a slight regularization of the velocity in the Euler vorticity equation keeping the same borderline space. Furthermore, it provides a deeper understanding of the local well-posedness of the 2D Euler equation in the borderline Sobolev space which has attracted much attention in recent decades.

In a broader perspective, clarifying the well-posedness of an evolutionary partial differential equation has a great significance. For example, in the physical application, approximate solutions are mostly considered. For any smooth solution, a small error in initial data still makes the approximate solution stay close to the real one. However, in a borderline space, a solution can be very sensitive to a small change in initial data. Indeed, in our analysis on (LE), the difference between the real solution and the approximate one is infinity in the sense of the borderline space norm. In other words, it gives a warning sign in the usage of data in the borderline space in physical computing.

As an extension of this work, a possible direction is working on other spaces. For instance, one can consider Besov spaces of logarithmic smoothness, which is more sophisticated than Sobolev spaces. This will provide a sharper result on the well-posedness of (LE). On the other hand, the Lipschitz type spaces as in [5] can be another option. Along these lines, we can deepen our understanding of the 2D Euler dynamics in integer C^m spaces in [5].

The strong ill-posedness scheme can be applied to other problems. One of the most challenging but interesting problems is the well-posedness of the inviscid surface quasi-geostrophic equation

$$\begin{cases} \partial_t \theta + \nabla^{\perp}(-\Delta)^{\frac{1}{2}}\theta \cdot \nabla\theta = 0 \\ \theta|_{t=0} = \theta_0. \end{cases}$$

This equation in the two-dimensional whole domain has special importance among incompressible fluid equations because it has strong analogies with the 3D Euler equation. In this case, its borderline Sobolev space is also changed to $H^2(\mathbb{R}^2)$. The scheme perfectly works in the construction of local solutions having the critical Sobolev norm inflation. However, in order to apply the patching argument, we need to guarantee the uniform lifespan of the local solutions. It can be one of the defects in the scheme that we need to overcome.

In summary, our result in Chapter 2 improves the understanding of the behavior of the solution to the logarithmically regularized 2D Euler equations. Also, it strengthens the scheme and shows the potential of its applications to many other fluid equations.

In Chapter 3, we suggest a construction scheme for a global-in-time local energy solution of the Navier-Stokes with non-decaying initial data. More precisely, this scheme works for any initial data whose non-decaying part is in $L^3_{\mathrm{uloc}}(\mathbb{R}^3)$ and decaying part is in $E^2_\sigma(\mathbb{R}^3)$, having slow oscillation decay. Here, we do not restrict the rate of the decay. Furthermore, the analysis in the global existence explains how the solution behaves in any positive time. Indeed, the solution to the heat equation with the non-decaying part of given initial data governs the non-decaying part of the constructed solution.

In practice, non-decaying flows are widely known — for example, constant flows and periodic flows. Most research, however, has studied decaying flows because of difficulties arising from the pressure. In this context, our scheme not only has practical usages but also enrich the analysis of Navier-Stokes flows with non-decaying initial data. Indeed, it makes a vast improvement from most recent work on the global existence with non-decaying initial data in [38].

Despite the significant improvement, one of the defects of the scheme is the requirement on the local regularity of non-decaying part of the initial data. This assumption follows from a technical difficulty of the scheme. On the other hand, this suggests a possible direction of future work, weakening the assumption on this local regularity of non-decaying part.

In summary, the scheme introduced in Chapter 3 says that even for a rough initial datum with no decay at spatial infinity, a Navier-Stokes flow can exist globally in time. Our result has rich applications to the reality and also deepens the understanding regarding a solution to the Navier-Stokes equation with non-decaying initial data.

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