Multilinear Restriction Estimates on Fractal Sets

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Abstract

In harmonic analysis there is a rich history of restriction theory for measures supported on smooth manifolds, and recently much focus has turned to restriction for measures supported on fractal sets. On the other hand, the use of multilinear restriction estimates has propelled most current progress on classical restriction theory. In this thesis we discuss the existing literature on both of these main interests in restriction theory, and then consider their combination. We analyze the existence of multilinear restriction estimates for a collection of singular measures, particularly measures supported on Cantor sets. We generalize a linear restriction estimate of Chen to a multilinear setting and provide a class of Cantor sets to which this result applies. Furthermore, we give necessary conditions for the existence of multilinear restriction for singular measures. We are hopeful that the success of multilinear restriction estimates in furthering classical restriction theory may be reproduced in our context.
Lay Summary

We provide an introduction to restriction theory which is a classical area in pure mathematics. Classically, restriction theory studies mathematical properties of smooth surfaces like the sphere. Recently, there has been interest in studying restriction theory in a more general context. We provide an exposition of recent progress made to this generalized form of restriction theory, and discuss techniques that have recently made progress in classical restriction theory. Our research seeks to understand how these powerful new techniques may be adapted to the more general problem.
Preface

This thesis includes an exposition by the author of recent progress in restriction theory for singular measures. In chapter 5, the author also presents the details of original, unpublished results to this field, which is joint work with her supervisors Drs. Izabella Laba and Malabika Pramanik. The research questions and main arguments of these results were generated in research meetings between the author and her supervisors.
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Chapter 1

Introduction

Restriction theory is a classical subject in harmonic analysis that studies the Fourier analytic properties of measures in \( \mathbb{R}^d \). Restriction estimates for the natural measures of smooth manifolds have been well studied, and their existence has been linked to geometric properties of the manifold such as curvature and dimension. Only recently, restriction estimates for more general measures have been explored. In particular, interest has garnered in restriction estimates for measures supported on fractal sets, as the existence of such an estimate is connected to the arithmetic structure of the fractal.

In this thesis we provide an overview of the literature on restriction estimates for singular measures supported on fractal sets. We are interested in extending this research to multilinear restriction estimates for measures supported on fractal sets, motivated by the effectiveness of multilinear restriction estimates in furthering classical restriction theory. Indeed, most recent progress on restriction estimates on manifolds has involved a decomposition of the manifold and the use of multilinear restriction estimates to control the Fourier interaction between pieces [17].

In this chapter we discuss some important background; in sections 1.1 and 1.2 we define the restriction problem and present some known and pertinent restriction estimates. These results indicate that for a given measure \( \mu \), the Hausdorff dimension of its support and its rate of Fourier decay are key factors in the existence of restriction estimates for \( \mu \). Consequently, in section 1.3 we briefly define Hausdorff dimension and discuss its relation to the Fourier decay of a measure. Finally, in section 1.4 we introduce the type of fractal sets for which we are especially interested in proving restriction estimates, namely, Cantor sets. We use the construction of Cantor sets given in [11], and define the associated Cantor measure.

In chapter 2 we continue to discuss Cantor measures and specifically analyze the existence of Fourier decay. As a general rule, high levels of arithmetic structure in the set
1.1. The restriction problem

indicate an absence of Fourier decay, and hence an absence of restriction estimates. Cantor sets with Fourier decay have been constructed via a random procedure. We provide two random constructions of Cantor sets given by Chen in [4], which generalize constructions due to Laba and Pramanik in [10], and Hambrook and Laba in [7], from which we can obtain Cantor sets whose associated measure achieves a given rate of Fourier decay.

Equipped with this background, in chapter 3 we expound on the restriction estimates of Mockenhaupt [15] and Chen [3] introduced in section 1.2, which hold for classes of singular measures supported on fractal sets, such as the construction of Cantor measures from [4] presented in chapter 2. Finally, to motivate our interest in multilinear restriction on Cantor sets, we introduce the multilinear restriction problem in chapter 4 and provide an example from [6] to illustrate such an estimate’s effectiveness in proving new linear restriction estimates.

In chapter 5 we present our work on multilinear restriction estimates for singular measures, which is joint work by the author and her supervisors, Izabella Laba and Malabika Pramanik. In Theorem 5.1 we generalize Chen’s restriction estimate to a multilinear form, and provide a class of Cantor measures for which the estimate applies. Then we discuss necessary conditions for multilinear restriction estimates, by illustrating how the dimension of the measures can restrict the exponents for which multilinear restriction may hold.

1.1 The restriction problem

For a function \( f : \mathbb{R}^d \rightarrow \mathbb{C} \), define its Fourier transform \( \hat{f} : \mathbb{R}^d \rightarrow \mathbb{C} \) by

\[
\hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \xi} f(x) dx.
\]

Restriction theory is rooted in the following question:

**Question 1.1 (Restriction problem).** Given a finite, compactly supported measure \( \mu \), for what values of \( p, q \) does there exist a constant \( C > 0 \) such that

\[
\| \hat{f} \|_{L^q(d\mu)} \leq C \| f \|_{L^p(\mathbb{R}^d)}
\]

for all functions \( f \in L^p(\mathbb{R}^d) \)?
1.1. The restriction problem

Classically this question was posed in the case where $\mu$ is the natural measure of a compact subset of a hypersurface in $\mathbb{R}^d$, such as the sphere or the paraboloid. The restriction problem asks to which surfaces may we meaningfully restrict the Fourier transform of functions on $\mathbb{R}^d$.

We are interested in this question for values of $p \in (1, 2)$. When $p = 1$, $\hat{f}$ is a continuous and bounded function, and it may be meaningfully restricted to any subset of $\mathbb{R}^d$. On the other hand, when $p = 2$, $\hat{f}$ may merely be an $L^2$ function and hence not meaningfully defined on sets of Lebesgue measure zero.

The restriction problem can be written in its dual formulation (called the extension problem), which is given by the following:

**Question 1.2** (Extension problem). Given a finite, compactly supported measure $\mu$, for what values of $p'$, $q$ does there exist a constant $C > 0$ such that

$$
\| \hat{f} d\mu \|_{L^{p'}(\mathbb{R}^d)} \leq C \| f \|_{L^p(d\mu)}
$$

for all functions $f \in L^q(d\mu)$?

Here, $p'$ and $q'$ are the dual exponents of $p$ and $q$, meaning that $1/p + 1/p' = 1$ and likewise for $q$. Moreover, we define $\hat{f} d\mu : \mathbb{R}^d \to \mathbb{C}$ via

$$
\hat{f} d\mu(\xi) = \int f(x) e^{-2\pi i x \cdot \xi} d\mu(x).
$$

**Proposition 1.3.** The restriction problem and extension problem are equivalent. That is, there is some $C > 0$ so that

$$
\| \hat{f} \|_{L^{q}(d\mu)} \leq C \| f \|_{L^p(\mathbb{R}^d)} \tag{1.1}
$$

for all $f \in L^p(\mathbb{R}^d)$ if and only if there is some $C' > 0$ so that

$$
\| \hat{g} d\mu \|_{L^{p'}(\mathbb{R}^d)} \leq C' \| g \|_{L^{q'}(d\mu)} \tag{1.2}
$$

for all $g \in L^{q'}(d\mu)$. 

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1.1. The restriction problem

Proof. Assume (1.2) holds and let \( B = \{ g : \| g \|_{L^q(d\mu)} \leq 1 \} \). Then for \( f \in L^p(\mathbb{R}^d) \), we have

\[
\| \hat{f} \|_{L^q(d\mu)} = \sup_{g \in B} \int \hat{f} g d\mu = \sup_{g \in B} \int \hat{g} \mu f dx
\]

by the duality relation of the Fourier transform. Applying Hölder’s inequality and (1.2), we have

\[
\| \hat{f} \|_{L^q(d\mu)} \leq \sup_{g \in B} \| \hat{g} \mu \|_{L^p} \| f \|_p \leq C \sup_{g \in B} \| g \|_{L^q(d\mu)} \| f \|_p \leq C \| f \|_p
\]

and so (1.1) holds. A similar argument gives the converse direction.

More abstractly, we can note the equivalence of the restriction problem and the extension problem by considering the operator \( T : L^p(\mathbb{R}^d) \rightarrow L^q(d\mu) \) defined by \( T : f \mapsto \hat{f}|_{\text{supp}(\mu)} \). Then its adjoint or extension operator \( T^* : L^q(d\mu) \rightarrow L^p(\mathbb{R}^d) \) satisfies

\[
\int T f \cdot g = \int f \cdot T^* g,
\]

and \( T \) is a bounded map if and only if \( T^* \) is bounded. By the duality relation of the Fourier transform, \( T^* : g \mapsto \hat{g} d\mu \).

1.1.1 A remark on notation

In the inequalities given in questions [1.1] and [1.2], the constant \( C \) is independent of the function \( f \), but may depend on quantities such as the dimension \( d \), the exponents \( p, q \), and properties of the measure \( \mu \). In such a case we absorb the constant by writing \( \lesssim \) or \( \gtrsim \). We may add subscripts to these inequality symbols to emphasize the dependence of the implicit constant on the indicated quantity. Furthermore, we sometimes abbreviate the norm of a function in \( L^p(\mathbb{R}^d) \) as \( \| \cdot \|_p \). If \( N \in \mathbb{N} \), we use \([N]\) to denote the set \( \{0, 1, \ldots, N - 1\} \). Finally, we use \( | \cdot | \) to denote the \( d \)-dimensional Lebesgue measure of a set in \( \mathbb{R}^d \), and \( B(x;r) \) for the ball in \( \mathbb{R}^d \) centered at \( x \) with radius \( r \).
1.2 Literature review: key restriction estimates

The classical cornerstone of restriction theory is the Tomas-Stein restriction theorem for the sphere. It gives a sharp range of permissible values of $p$ when $\mu$ is the natural measure of the sphere and $q = 2$.

**Theorem 1.4** (Tomas-Stein). Let $\sigma$ be the surface measure on $S^{d-1}$ induced by the Lebesgue measure on $\mathbb{R}^d$. If $1 \leq p \leq \frac{2(d+1)}{d+3}$, we have

$$\|\hat{f}\|_{L^2(\sigma)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}$$

for all $f \in L^p(\mathbb{R}^d)$. Equivalently, if $p' \geq \frac{2(d+1)}{d-1}$, then

$$\|\hat{f}d\sigma\|_{L^{p'}(\mathbb{R}^d)} \lesssim \|f\|_{L^2(\sigma)}$$

for all $f \in L^2(\sigma)$.

The proof of the Tomas-Stein theorem uses the dimensionality and curvature of $S^{d-1}$, and may be generalized to $(d-1)$-dimensional hypersurfaces with non-vanishing Gaussian curvature. The classic Knapp example establishes that the range of exponents given by Tomas-Stein is sharp:

**Example 1.5** (Knapp example). Let $x_0 \in S^{d-1}$ and $\delta > 0$ sufficiently small. Consider a disk centered at $x_0$ of thickness $\delta^2$ and radius $\delta$, oriented in the normal direction $n$ to $S^{d-1}$ at $x_0$. Intersecting this disk with $S^{d-1}$ gives a spherical cap $\kappa \subset S^{d-1}$ of surface measure $\sim \delta^{d-1}$. We take $f = \mathbb{1}_{\kappa}$.

Consider a tube $T$ centered at the origin and oriented in direction $n$, with length $c\delta^{-2}$ and radius $c\delta^{-1}$. Then for $\xi \in T$ and $x \in \kappa$, $|\xi \cdot x| \leq cd$. By taking $c$ sufficiently small, we may ensure $e^{-2\pi i x \cdot \xi}$ is approximately 1, giving $|\hat{f}d\sigma| \gtrsim |\kappa| \sim \delta^{d-1}$ on $T$. Thus

$$\|\hat{f}d\sigma\|_{L^{p'}(\mathbb{R}^d)} \gtrsim |T|^{1/p'} \delta^{d-1} \sim \delta^{-(d+1)/p'} \delta^{d-1}.$$  

But we also have

$$\|f\|_{L^{p'}(\sigma)} = |\kappa|^{1/q'} \sim \delta^{(d-1)/q'}.$$
Thus in order for a restriction estimate to hold,
\[ \delta^{-(d+1)/p'} \delta^{d-1} \lesssim \delta^{(d-1)/q'}. \]
Taking \( \delta \to 0 \) yields
\[ p' \geq \frac{q(d+1)}{d-1} \]
which, for \( q = 2 \), corresponds to the range of exponents in Tomas-Stein.

Heuristically, the Knapp example exploits the idea that locally the sphere is well approximated by a tangent plane having zero curvature.

Classical restriction theory focuses on estimates for measures supported on manifolds in Euclidean space, and has become a diligently explored topic. More recently, interest has developed in restriction estimates for more general measures, such as measures supported on fractal sets. Consider the following result, which generalizes the restriction theorem of Tomas-Stein:

**Theorem 1.6** (Mockenhaupt; Bak and Seeger (endpoint)). Let \( \mu \) be a compactly supported positive measure on \( \mathbb{R}^d \), with \( \alpha, \beta \in (0,d) \) such that
\[ \mu(B(x,r)) \lesssim r^\alpha \quad \forall x \in \mathbb{R}^d, r > 0, \]  
and
\[ |\hat{\mu}(\xi)| \lesssim (1 + |\xi|)^{-\beta/2} \quad \forall \xi \in \mathbb{R}^d. \]  
Then for \( 1 \leq p \leq p_{d,\alpha,\beta} = (4(d-\alpha) + 2\beta)/(4(d-\alpha) + \beta) \), we have the estimate
\[ \| \hat{f} \|_{L^2(d\mu)} \lesssim \| f \|_{L^p(\mathbb{R}^d)} \]
for all \( f \in L^p(\mathbb{R}^d) \). Equivalently, for \( p' \geq p'_{d,\alpha,\beta} = 2 + 4(d-\alpha)/\beta \), we have the estimate
\[ \| \hat{f} d\mu \|_{L^{p'}(\mathbb{R}^d)} \lesssim \| f \|_{L^2(d\mu)} \]
for all \( f \in L^2(d\mu) \).

We refer to conditions (1.3) and (1.4) as the \( \alpha \)-dimensional ball condition and the Fourier decay condition, respectively. Note that the implicit constants in these conditions
are independent of $x$, $r$, and $\xi$. We will see that the ball condition connects to the Hausdorff dimension of $\text{supp}(\mu)$. Moreover, note that by taking $\alpha = \beta = d - 1$ we recover the range of exponents given in the Tomas-Stein restriction theorem. This selection reflects the $d - 1$ dimensionality of the sphere in $\mathbb{R}^d$.

While we have seen that the range of exponents in the Tomas-Stein restriction estimate is sharp, in general the same is not true of Theorem 1.6. For some classes of measures the following result improves the range of exponents:

**Theorem 1.7 (Chen).** Let $\mu$ be a compactly supported positive measure on $\mathbb{R}^d$ such that $\mu^{*n} \in L^r(\mathbb{R}^d)$, for some $1 \leq r \leq \infty$. Let $1 \leq p \leq 2n/(2n - 1)$ if $r \geq 2$ and $1 \leq p \leq nr'/(nr' - 1)$ if $1 \leq r \leq 2$. Let $1 \leq q \leq p'/nr'$. Then

$$\|\hat{f}\|_{L^q(d\mu)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}$$

for all $f \in L^p(\mathbb{R}^d)$.

In particular, taking $r = \infty$ gives a restriction estimate with $q = 2$ and the range of exponents $1 \leq p \leq 2n/(2n - 1)$. For a class of measures we study later, this range is optimal.

### 1.3 Hausdorff dimension and Fourier decay

We briefly recall Hausdorff measures and define the Hausdorff dimension of a set.

Fix $E \subset \mathbb{R}^d$, $\alpha > 0$, and $\varepsilon > 0$. Let $H_\varepsilon^\alpha(E)$ be the infimum of $\sum_{j=1}^{\infty} r_j^\alpha$ over all countable coverings $\{B(x_j; r_j)\}$ of $E$ with $r_j < \varepsilon$. Note that $H_\varepsilon^\alpha(E)$ is a decreasing function of $\varepsilon$, and so $H_\alpha(E) := \lim_{\varepsilon \to 0} H_\varepsilon^\alpha(E)$ exists, although is possibly infinite. Moreover, $H_\alpha(E)$ defines a Borel measure on $\mathbb{R}^d$.

One can show that $H_\alpha(E)$ is a decreasing function of $\alpha$, and most importantly, there exists some exponent $\alpha_0$ such that $H_\alpha(E) = \infty$ for all $\alpha < \alpha_0$, and $H_\alpha(E) = 0$ for all $\alpha > \alpha_0$. This fact motivates the following definition:

**Definition 1.8.** The *Hausdorff dimension* of $E \subset \mathbb{R}^d$ is defined as

$$\dim_H(E) = \sup\{\alpha \in [0, d] : H_\alpha(E) = \infty\}.$$
1.3. Hausdorff dimension and Fourier decay

As previously alluded, the Hausdorff dimension of a set is related to the ball condition via the following lemma:

**Lemma 1.9** (Frostman). Let $E \subset \mathbb{R}^d$ be compact, and let $\mathcal{P}(E)$ denote the set of probability measures supported on $E$. Then

$$\dim_{\mathbb{H}}(E) = \sup \left\{ \alpha \in [0, d] : \exists \mu \in \mathcal{P}(E), \sup_{x \in \mathbb{R}^d, r > 0} \mu(B(x; r)) r^{-\alpha} < \infty \right\}.$$ 

Moreover, there is also a connection between the Fourier decay of a measure and the Hausdorff dimension of its support (ergo, between the ball condition and the Fourier decay condition). To establish this relationship, we introduce energy integrals and their relation to Hausdorff dimension.

**Definition 1.10.** For a compactly supported positive measure $\mu$,

$$I_s(\mu) = \int \int |x - y|^{-s} d\mu(x) d\mu(y)$$

is the $s$-dimensional energy of $\mu$.

We will be able to relate Fourier decay and Hausdorff dimension via the following two lemmas:

**Lemma 1.11.** Let $E$ be compact. Then

$$\dim_{\mathbb{H}}(E) = \sup \{ s : \exists \mu \in \mathcal{P}(E) \text{ such that } I_s(\mu) < \infty \}.$$ 

**Lemma 1.12.** For a compactly supported positive measure $\mu$ on $\mathbb{R}^d$ and $0 < s < d$,

$$I_s(\mu) = \gamma(s, d) \int |\hat{\mu}(\xi)|^2 |\xi|^{s-d} d\xi,$$

where $\gamma(s, d)$ is a universal positive constant.

Proofs of these lemmas may be found in [21, Chapter 8].

**Proposition 1.13.** Let $\mu$ be a compactly supported positive measure on $\mathbb{R}^d$, and suppose that $|\hat{\mu}(\xi)| \lesssim (1 + |\xi|)^{-\beta/2}$ for all $\xi \in \mathbb{R}^d$. Then

$$\beta \leq \dim_{\mathbb{H}}(\text{supp}(\mu)).$$
1.4 Cantor sets

Proof. For all $s < \beta$,

$$I_s(\mu) = \gamma(s, d) \int |\hat{\mu}(\xi)|^2 |\xi|^{s-d} d\xi \leq \int (1 + |\xi|)^{-\beta} |\xi|^{s-d} d\xi,$$

which is finite as $\beta - s + d > d$. Then $\beta \leq \dim_H(\text{supp}(\mu))$.

\[ \square \]

1.4 Cantor sets

We have already alluded to more recent interest in restriction estimates on fractal sets. A classical example of a fractal set is the middle third Cantor set; however, we will explore Cantor sets of a more general construction that will allow for varying degrees of arithmetic structure. This variance is imperative as there is a connection between arithmetic structure and the existence of restriction estimates.

We will follow the Cantor set construction given in [11].

Construction 1.14 (General Cantor set). We construct a Cantor set in which the size of the subintervals and number of intervals chosen may vary at each iteration in its construction. Let $E_0 = [0, 1]$ and $E_k$ the $k^{th}$ iteration of the Cantor set.

Let $\{N_k\}_{k=1}^\infty \subset \mathbb{N}$ be non-decreasing, and take $M_k = \delta_k^{-1} = N_1 \cdots N_k$. Moreover, fix binary sequences

$$X_k = \{x_k(i) : i \in J_k\}, \quad Y_k = \{y_k(i) : i \in J_k\}$$

where $J_k = \{i = (i_1, \ldots, i_k) \in \mathbb{Z}^k : 0 \leq i_j \leq N_j - 1\}$ and we have the relation

$$x_{k+1}(i_1, \ldots, i_{k+1}) = x_k(i_1, \ldots, i_k)y_{k+1}(i_1, \ldots, i_{k+1}).$$

Take

$$T_k = \#\{i \in J_k : x_k(i) = 1\}.$$

Given $E_k$, we subdivide it into $N_{k+1}$ intervals of equal length $\delta_{k+1}$. Let

$$a_{k+1}(i) = \frac{i_1}{N_1} + \frac{i_2}{N_1N_2} + \cdots + \frac{i_k}{N_1 \cdots N_{k+1}},$$
1.4. Cantor sets

and

\[ A_{k+1} = \{ a_{k+1}(i) : i \in J_{k+1}, x_{k+1}(i) = 1 \}, \]

which will govern the endpoints of the \( T_{k+1} \) subintervals chosen to form \( E_{k+1} \). Let \( I_{k+1}(a) = [a, a + \delta_{k+1}] \) for each \( a \in A_{k+1} \). Then

\[ E_{k+1} = \bigcup_{a \in A_{k+1}} I_{k+1}(a), \]

and the resulting Cantor set is \( E = \bigcap_{k=0}^{\infty} E_k \). We frequently use \( I_k \) to denote the intervals \( I_k(a) \) with \( a \in A_k \).

**Construction 1.15** (Self-similar Cantor set). We can specialize the general Cantor set construction to one with a guaranteed high level of arithmetic structure and self-similarity. Fix \( N, t \in \mathbb{N} \) with \( 0 < t < N \). Then for all \( k \in \mathbb{N} \) take \( N_k = N \) and \( T_k = t^k \). Moreover, we impose the restriction that

\[ y_k(i_1, \ldots, i_k) = 1 \]

if and only if

\[ i_k \in A := \{ i \in [N] : x_1(i) = 1 \}. \]

Hence the set of endpoints is given by

\[ A_k = N^{-1}A + N^{-2}A + \cdots + N^{-k}A = \left\{ \frac{a_1}{N} + \frac{a_2}{N^2} + \cdots + \frac{a_k}{N^k} : a_i \in A \right\}. \]

In particular, to form the middle third Cantor set, take \( N = 3, t = 2, \) and \( A = \{0, 2\} \).

1.4.1 Cantor measures

Consider the following construction of a measure \( \nu \) on \( E \). Take \( \mathcal{B} = \bigcup_k \mathcal{B}_k \) where

\[ \mathcal{B}_0 = [0, 1], \quad \mathcal{B}_k = \bigcup_{a \in A_k} \{ I_k(a) \}. \]
1.4. Cantor sets

Let the weight function \( w \) be defined on \( B \) via \( w([0, 1]) = 1 \) and \( w(I) = T_k^{-1} \) for \( I \in B_k \). For each \( k \in \mathbb{N} \) define an outer measure \( \nu_k \) by assigning

\[
\nu_k(A) = \inf \left\{ \sum_{j=1}^{\infty} w(I_j) : A \subset \cup I_j, I_j \in B, |I_j| \leq \delta_k \right\}
\]

for \( A \subset E \). Since \( \nu_k(A) \) is a monotonic function of \( k \), we may take

\[
\nu(A) = \lim_{k \to \infty} \nu_k(A).
\]

Then \( \nu \) is a Borel measure.

We use \( \nu \) to bound the Hausdorff dimension of the set \( E \) in the following lemma, which is given as Lemma 2.1 in [11].

Lemma 1.16. Let \( E \) be a Cantor set given by construction I.14. Then

\[
\dim_H(E) \leq \lim inf \log(T_k)/\log(M_k)
\]

and

\[
\dim_H(E) \geq \lim \inf \log(T_k/N_k)/\log(M_{k-1}).
\]

In particular, in the case of construction I.15, these limits agree and

\[
\dim_H(E) = \log(t)/\log(N).
\]

Proof. For the first inequality, note that for each \( k \) there is a covering of \( E_k \) with \( T_k \) intervals of length \( \delta_k \), so that \( H^0_{\alpha_k}(E) \leq T_k \delta_k^\alpha \). Take \( \alpha = \lim inf \log(T_k)/\log(M_k) \). For any \( \varepsilon > 0 \), there is some subsequence \( \{k_j\} \) so that \( \log(T_{k_j})/\log(M_{k_j}) < \alpha + \varepsilon \). Then \( H^0_{\alpha+k}(E) \) will be bounded, and so by monotonicity \( H_{\alpha+k}(E) < \infty \). As this holds for all \( \varepsilon \), \( \dim_H(E) \leq \alpha \).

Next take \( \alpha' = \lim inf \log(T_k/N_k)/\log(M_{k-1}) \). Let \( J \) be an interval with \( |J| < \delta_1 \), and let \( k \) be such that \( \delta_{k+1} \leq |J| < \delta_k \). The number of intervals \( I_{k+1} \) intersecting \( J \) is at most \( 2N_{k+1} \), as \( J \) intersects at most 2 of the intervals \( I_k \). Furthermore, this number is at most \( |J|/\delta_{k+1} \) as the intervals \( I_{k+1} \) have trivial intersection. So

\[
\nu_{k+1}(J) \leq T_{k+1}^{-1} \min \left\{ 2N_{k+1}, \frac{|J|}{\delta_{k+1}} \right\} \leq T_{k+1}^{-1} (2N_{k+1})^{1-s} \left( \frac{|J|}{\delta_{k+1}} \right)^s
\]
1.4. Cantor sets

for any $0 < s < 1$. Thus

$$\frac{\nu_{k+1}(J)}{|J|^s} \leq 2^{1-s}T_{k+1}^{-1}N_{k+1}^{1-s}M_{k+1}^s = 2^{1-s}T_{k+1}^{-1}N_{k+1}^{1-s}M_{k}^s$$

As this quantity is bounded for all $s < \alpha'$, $\dim_\mathbb{H}(E) \geq \alpha'$ by Frostman’s Lemma.

\[ \square \]

1.4.2 Existence of weak-* limiting measure

Let $\varphi_k = 1_{E_k}/|E_k|$, and let $\mu_k$ be the measure on $E_k$ given by $d\mu_k = \varphi_k dx$. If the $\mu_k$ converge in the weak-* sense to a measure $\mu$, that is,

$$\int f d\mu_k \rightarrow \int f d\mu$$

for all continuous bounded functions $f$, then we say the limiting measure $\mu$ is a Cantor measure or the natural measure on $E$. The following lemma (Lemma 2.2 in [11]) gives a sufficient condition for the existence of such a measure. The condition may be understood in quantifying an approximate uniformity in the distribution of the subintervals of $E_k$ within the intervals of $E_k$.

**Lemma 1.17.** Suppose

$$\sup_{l \geq k} \sum_{a \in A_k} \left| \int_{I_k(a)} (\varphi_l - \varphi_k)(x) dx \right| \xrightarrow{k \to \infty} 0.$$

Then there is a probability measure $\mu$ on $[0,1]$ that is the weak-* limit of the $\mu_k$.

In particular, if each interval of $E_k$ contains the same number of subintervals of $E_{k+1}$ (so, exactly $T_{k+1}T_k^{-1}$), then the weak-* limit $\mu$ exists. Moreover, as for any $l \geq k$ we have $\mu_l(I_k) = T_k^{-1}$, we have $\mu(I_k) = T_k^{-1}$, a property shared with $\nu$. This property determines the integrals of continuous functions, giving $\mu = \nu$. 

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Chapter 2

Fourier decay of Cantor measures

Let $\mu$ be a Cantor measure. We define the Fourier transform of $\mu$ by

$$\hat{\mu}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \xi} d\mu(x).$$

We are interested in the values of $\beta$, if any, for which $\hat{\mu}$ satisfies the Fourier decay condition

$$|\hat{\mu}(\xi)| \leq C(1 + |\xi|)^{-\beta/2} \quad \forall \xi \in \mathbb{R}^d,$$  \hspace{1cm} (2.1)

where the constant $C > 0$ may depend on $\beta$. By Proposition 1.13, if such an estimate exists, we must have $\beta \geq \dim_H(\text{supp}(\mu))$.

In this chapter, we explore the connection between the levels of arithmetic structure of the Cantor set on which $\mu$ is supported, and whether or not $\mu$ exhibits Fourier decay. First we consider an example of a self-similar Cantor set, and show that its natural Cantor measure does not have Fourier decay. This example illustrates the approximate rule that highly structured Cantor sets do not achieve Fourier decay.

We then follow by an example of a “random” construction of a Cantor set. Often, “random” Cantor sets will obey (2.1) for some $\beta > 0$. In section 2.2 we begin with a construction of Salem examples of a Cantor measure from [4], which generalize a construction in [10]. A Salem measure $\mu$ is a measure that obeys the Fourier decay condition (2.1) for all $\beta < \dim_H(\text{supp}(\mu))$. In fact, for the measure $\mu$ constructed in this section, $\mu$ obeys (2.1) with $\beta = \dim_H(\text{supp}(\mu))$.

Finally, we will modify the construction in section 2.2 to produce a “Knapp” example of a Cantor set, again from [4], which is based on a construction from [7]. Once this theorem is presented, we justify the name of the construction. What we emphasize now, is that this construction will produce Cantor measures $\mu$ supported on sets of Hausdorff dimension $\alpha$ and satisfying (2.1) with $\beta$, where $\beta \leq \alpha$ are given parameters.
2.1 Self-similarity and lack of Fourier decay

In this section we compute the Fourier transform of the natural measure of the middle third Cantor set, and observe that it does not decay to zero at infinity, following the procedure given in [13].

Let $E$ be the middle third Cantor set, and $E_k$ the $k$th iteration in its construction. As in construction [1.15], we let $A$ be the set

$$A_k = \left\{ a_13^{-1} + \cdots + a_k3^{-k} : a_1, \ldots, a_k \in A \right\}.$$

Let $\mu$ be the natural Cantor measure on $E$, so that $\mu$ is the weak-* limit of $\mu_k$, where

$$d\mu_k = \varphi_k dx = 1_k/|E_k| dx.$$ Recall that $\mu$ may be defined via the property $\mu(I_k) = 2^{-k}$, where $I_k$ is one of the subintervals forming $E_k$.

Define

$$\nu_k = 2^{-k} \sum_{a \in A_k} \delta_a.$$ Then $\mu$ is also the weak-* limit of $\nu_k$. We find $\hat{\mu}$ by computing $\hat{\nu}_k$ and taking the limit.

Note first that

$$\hat{\nu}_k(\xi) = 2^{-k} \sum_{a_i \in \{0,2\}} e^{-2\pi i (a_1/3 + \cdots + a_k/3^k)} \xi = \prod_{j=1}^{k} \frac{1}{2} \left( 1 + e^{-2\pi i(2/3^j)\xi} \right).$$

Using the identity $(1 + e^{ix})/2 = e^{ix/2} \cos(x/2)$, we may write

$$\hat{\nu}_k(\xi) = \prod_{j=1}^{k} e^{-2\pi i 3^{-j} \xi} \cos(2\pi 3^{-j} \xi) = e^{-2\pi i \xi} \sum_{j=1}^{k} 3^{-j} \prod_{j=1}^{k} \cos(2\pi 3^{-j} \xi)$$

$$= e^{-\pi i (1-3^{-k})} \prod_{j=1}^{k} \cos(2\pi 3^{-j} \xi).$$

For fixed $\xi$, note that this infinite product converges. There are some constants $c, C > 0$ so that for sufficiently large $j$, $c(1 - \frac{q_j^2}{2}) \leq \cos(q_j) \leq C(1 - \frac{q_j^2}{2})$, where $q_j = 2\pi 3^{-j} \xi$. But the convergence of $\sum_{j=1}^{\infty} q_j$ implies the convergence of the infinite product.
2.2 Random construction: Salem measures

By the weak-* convergence of \( \hat{\nu}_k \) to \( \mu \), we have \( \hat{\mu}(\xi) = \lim_{k \to \infty} \hat{\nu}_k(\xi) \). Therefore,

\[
\hat{\mu}(\xi) = e^{-\pi \xi} \prod_{j=1}^\infty \cos(2\pi 3^{-j} \xi).
\]

By considering \( \xi = 3^k, k \in \mathbb{N} \), we see that \( \hat{\mu}(\xi) \to 0 \) as \( |\xi| \to \infty \). Indeed, \( \hat{\mu}(3^k) \) is a constant sequence as

\[
\hat{\mu}(3^k) = e^{-3k\pi i} \prod_{j=1}^k \cos(2\pi 3^{k-j}) \prod_{j=k+1}^\infty \cos(2\pi 3^{-(j-k)}) = -\prod_{j=1}^\infty \cos(2\pi 3^{-j}) = \hat{\mu}(1).
\]

This process extends to the natural Cantor measure \( \mu \) on a general self-similar Cantor set given by construction 1.15. The Fourier transform of \( \mu \) may be written as

\[
\hat{\mu}(\xi) = \prod_{j=1}^\infty \left( \frac{1}{\#A} \sum_{a \in A} e^{-2\pi i a \xi / N^j} \right).
\]

Then \( \hat{\mu}(N^k) = \hat{\mu}(1) \) for all \( k \in \mathbb{N} \), so \( \hat{\mu} \) does not decay to zero at infinity.

2.2 Random construction: Salem measures

We present the construction of Chen in [4, section 2], which generalizes a construction due to Laba and Pramanik in [10] for dimensions of the form \( \alpha = \log t / \log N \), where \( t, N \in \mathbb{N} \) and \( t < N \).

**Theorem 2.1** (Chen). Given \( 0 < \alpha < 1 \), there exists a probability measure \( \mu \) supported on a compact set \( E \) of Hausdorff dimension \( \alpha \) such that

\[
|\hat{\mu}(\xi)| \lesssim (1 + |\xi|)^{-\alpha/2}, \quad \forall \xi \in \mathbb{R}
\]

and

\[
\mu(I) \sim \frac{|I|^\alpha}{\log(|I|^{-1})}
\]

for any interval \( I \) centered in \( E \) and with \( |I| < 1/2 \).

Note that in particular (2.3) implies that for any \( \delta > 0 \), \( |I|^\alpha + \delta \lesssim \delta \mu(I) \lesssim |I|^\alpha \), so \( \mu \)
2.2. Random construction: Salem measures

satisfies the \( \alpha \)-dimensional ball condition, and \( E \) has Hausdorff dimension \( \alpha \) [13, Theorem 5.7].

2.2.1 Outline of construction and selection of parameters

Construction 2.2. Let \( \{t_k\}_{k=1}^\infty \) and \( \{N_k\}_{k=1}^\infty \) be given sequences, with \( 1 \leq t_k \leq N_k \), \( N_k \geq 2 \), and \( N_k \) non-decreasing. Let \( M_k = \delta_k^{-1} = N_1 \cdots N_k \), and assume also that 
\[
\frac{\log N_k}{\log M_k} \to 0 \quad \text{as} \quad k \to \infty.
\]

Set \( A_0 = \{0\} \) and suppose \( A_k \) is defined. For each \( a \in A_k \), let \( A_{k+1,a} \subset [N_{k+1}]/M_k \) so that \( \#A_{k+1,a} = t_{k+1} \). Then we define \( A_{k+1} \) by
\[
A_{k+1} = \bigcup_{a \in A_k} (a + A_{k+1,a}).
\]

Then setting \( I_{k+1}(a) = [a, a + \delta_k] \), we take
\[
E_{k+1} = \bigcup_{a \in A_{k+1}} I_{k+1}(a).
\]

The resulting Cantor set is \( E = \bigcap_{k=0}^\infty E_k \).

We first note that each interval composing \( E_k \) contains exactly \( t_{k+1} \) subintervals of \( E_{k+1} \), and so by Lemma 1.17 the weak-* limit \( \mu \) of the measures \( d\mu_k = 1_{E_k}/|E_k|dx \) exists.

Selection of the sequence \( t_k \). In order to guarantee the estimates (2.2) and (2.3), we select the sequences \( t_k \) and \( N_k \) so that the sequences \( T_k = t_1 \cdots t_k \) and \( M_k \) obey
\[
T_{k-1}^{-1/2} \log^{1/2} (2^3 M_k) \sim M_k^{-\alpha/2}. \tag{2.4}
\]

Then, there is some \( N_0 \in \mathbb{N} \) so that for \( k \geq N_0 \),
\[
4 \log (2^3 M_k) \leq T_{k-1}. \tag{2.5}
\]

Selection of the sets \( A_k \). Choose \( A_0, A_1, \ldots, A_{N_0} \) arbitrarily provided that \( E_{N_0} \) is compactly supported in \((0,1)\). For \( k > N_0 \), we select \( A_k \) inductively.
2.2. Random construction: Salem measures

First, we introduce the notation

\[ S_A(l) = \sum_{a \in A} e^{-2\pi i al}, \quad l \in \mathbb{Z} \]

for any finite set \( A \subset \mathbb{R} \). To establish the Fourier decay condition (2.2), we bound the differences \(|\hat{\mu}_k(l) - \hat{\mu}_{k-1}(l)|\) for \( l \in \mathbb{Z} \), which may be derived from bounds on the sum

\[
\sum_{a \in A_{k-1}} \left( \frac{S_{A_{k,a}}(l)}{t_k} - \frac{S_{[N_k]/M_k}(l)}{N_k} \right) e^{-2\pi i al}.
\]

If \( t_k = N_k \), then the sum is zero. In order to ensure a favourable bound in all cases, we randomly select a \( t_k \)-element subset of \([N_k]/M_k\) via the following method:

Let

\[ B_k = \{0, 1, \ldots, t_k - 1\}. \]

Consider translates of \( B_k \) within \([N_k]\); that is, for \( x \in [N_k] \), consider

\[ B'_{k,x} = \{y + x \pmod{N_k} : y \in B_k\} \]

and let \( B_{k,x} = B'_{k,x}/M_k \). Let \( \{x(a)\}_{a \in A_{k-1}} \) be independent random variables uniformly distributed on \([N_k]\). Then for fixed \( l \in \mathbb{Z} \), we have for \( a \in A_{k-1} \) the random variable

\[ \chi_a(l) = \left( \frac{S_{B_{k,x(a)}}(l)}{t_k} - \frac{S_{[N_k]/M_k}(l)}{N_k} \right) e^{-2\pi i al}. \]

We therefore seek bounds on \( \sum_{a \in A_{k-1}} \chi_a(l) \). One can show by Bernstein’s inequality that, given (2.5) it is possible to choose the \( x(a) \) so that

\[
\left| \frac{1}{T_{k-1}} \sum_{a \in A_{k-1}} \chi_a(l) \right| \leq 4 T_{k-1}^{-1/2} \log^{1/2}(2^3 M_k)
\]

for all \( l = 0, 1, \ldots, M_k - 1 \). Thus by periodicity and (2.4),

\[
\left| \frac{1}{T_{k-1}} \sum_{a \in A_{k-1}} \chi_a(l) \right| \approx M_k^{-\alpha/2}
\]
for all \( l \in \mathbb{Z} \).

For this choice of \( x(a) \), take \( A_{k,a} = B_{k,x(a)} \), which in turn sets \( A_k \).

### 2.2.2 Existence of Fourier decay

Let \( l \in \mathbb{Z} \), \( l \neq 0 \), and \( k > N_0 \). Calculating \( \tilde{\mu}_k(l) - \tilde{\mu}_{k-1}(l) \) and bounding \( \hat{1}_{[0,\delta_k]} \) yields

\[
|\tilde{\mu}_k(l) - \tilde{\mu}_{k-1}(l)| = \left| M_{k} \hat{1}_{[0,\delta_k]}(l) \right| \left| \frac{1}{T_{k-1}} \sum_{a \in A_{k-1}} \chi_a(l) \right| \\
\lesssim \min \left( 1, \frac{M_k}{|l|} \right) M_k^{-\alpha/2}.
\]

Let \( N_1 \) be the largest integer such that \( M_{N_1} \leq |l| \) and \( N_2 = N_1 + 1 \). Note that

\[
\sum_{k \leq N_1} M_k^{1-\alpha/2} \lesssim M_N^{1-\alpha/2} \quad \text{and} \quad \sum_{k \geq N_2} M_k^{-\alpha/2} \lesssim M_N^{-\alpha/2}.
\]

Then

\[
\sum_{k=1}^{\infty} \min \left( 1, \frac{M_k}{|l|} \right) M_k^{-\alpha/2} = \sum_{k \leq N_1} M_k M_k^{-\alpha/2} + \sum_{k \geq N_2} M_k^{-\alpha/2} \\
\lesssim \frac{1}{|l|} M_N^{1-\alpha/2} + M_N^{-\alpha/2} \\
= |l|^{-\alpha/2} \left( \frac{M_{N_1}}{|l|} \right)^{1-\alpha/2} + M_N^{-\alpha/2} \\
\lesssim |l|^{-\alpha/2}
\]

as \( M_{N_1}/|l| \leq 1 \) and \( M_N^{-1} \leq |l|^{-1} \) by choice of \( N_1 \) and \( N_2 \). Therefore for \( k > N_0 \) we have

\[
|\tilde{\mu}_k(l) - \tilde{\mu}_{k-1}(l)| \lesssim |l|^{-\alpha/2}.
\]

For \( k \leq N_0 \) we have the estimate

\[
|\tilde{\mu}_k(l)| \lesssim |l|^{-1}.
\]
2.2. Random construction: Salem measures

By the weak convergence of $\mu_k$ to $\mu$, $\hat{\mu}_k(l) \to \hat{\mu}(l)$, so

$$|\hat{\mu}(l)| \leq |\hat{\mu}_1(l)| + \sum_{k=2}^{\infty} |\hat{\mu}_k(l) - \hat{\mu}_{k-1}(l)| \lesssim |l|^{-\alpha/2}$$

for all $l \in \mathbb{Z}$, $l \neq 0$. The estimate extends to any $\xi \in \mathbb{R}$, $\xi \neq 0$, because $\mu$ is compactly supported in $(0,1)$; for this claim we follow the argument given in [21, Chapter 9]. Let $\varphi \in S$ so that $\text{supp}(\varphi) \subset (0,1)$ and $\varphi \equiv 1$ on $\text{supp}(\mu)$. Then $d\mu(x) = \varphi(x)d\mu\{x\}$, where $\{x\}$ is the fractional part of $x$. Then we may write

$$\hat{\mu}(\xi) = \sum_{l \in \mathbb{Z}} \hat{\mu}(l)\hat{\varphi}(\xi - l);$$

see [21, p. 68] for the details. Split this sum into the domains $|\xi - l| \leq |\xi|/2$ and $|\xi - l| > |\xi|/2$. For the former region, in which we note $|l| \geq |\xi|/2$, we use the Fourier decay bound on $\mu$ and the Schwartz decay bound on $\hat{\varphi}$ to obtain

$$\left| \sum_{l:|\xi - l| \leq |\xi|/2} \hat{\mu}(l)\hat{\varphi}(\xi - l) \right| \lesssim_n |\xi|^{-\alpha/2} \sum_{l:|\xi - l| \leq |\xi|/2} (1 + |\xi - l|)^{-n} \lesssim_n |\xi|^{-\alpha/2} \sum_{l \in \mathbb{Z}} (1 + |l|)^{-n} \lesssim_n |\xi|^{-\alpha/2}$$

as we may choose $n$ to be arbitrarily large.

For the latter region, we may use the Schwartz decay bound on $\hat{\varphi}$ and the $L^\infty$ bound on $\hat{\mu}$ to obtain

$$\left| \sum_{l:|\xi - l| > |\xi|/2} \hat{\mu}(l)\hat{\varphi}(\xi - l) \right| \lesssim_n \sum_{l:|\xi - l| > |\xi|/2} (1 + |\xi - l|)^{-n} \lesssim_n (1 + |\xi|)^{-\alpha/2}$$

again for $n$ sufficiently large. Thus $|\hat{\mu}(\xi)| \lesssim |\xi|^{-\alpha/2}$, as desired.

2.2.3 Proof of ball condition and Hausdorff dimension

We establish the $\alpha$-dimensional ball condition for $\mu$ using a similar argument as given for the lower bound in Lemma 1.16. If $I$ is an interval centered in $E$ with $\delta_{k+1} \leq |I| < \delta_k$, there are at most two subintervals of $E_k$ intersecting $I$. Hence $\mu(I) \leq 2 \cdot T_k^{-1} \sim M_k^{-\alpha} / \log(2^3 M_{k+1})$
by (2.4). But \( M_{k+1}^{-1} \leq |I| \) so \( \mu(I) \lesssim |I|^\alpha / \log(|I|^{-1}) \). Note this gives \( \dim_{\mathbb{H}}(E) \geq \alpha \) by Lemma 1.9.

To see that \( \dim_{\mathbb{H}}(E) = \alpha \), we note that by (2.4), \( T_k \leq cM_k^\alpha \log(2^3M_{k+1}) \), so

\[
\dim_{\mathbb{H}}(E) \leq \liminf_{k \to \infty} \frac{\log T_k}{\log M_k} \leq \alpha
\]

by Lemma 1.16 and the assumption \( \frac{\log N_{k+1}}{\log M_k} \to 0 \).

## 2.3 Random construction: “Knapp” example

We present the construction of Chen in [4, Section 4].

**Theorem 2.3** (Chen). Let \( 0 < \beta \leq \alpha < 1 \) and let \( \varphi : [2, \infty) \to (0, \infty) \) be a nondecreasing function so that \( \lim_{t \to \infty} \varphi(t) = \infty \) and \( \varphi(2t) \leq \varphi(t) + C \). Then there exists a Borel probability measure \( \mu \) on \( \mathbb{R} \) supported on a compact set \( E \) of Hausdorff dimension \( \alpha \) such that

\[
|\hat{\mu}(\xi)| \lesssim (1 + |\xi|)^{-\beta/2}, \quad \forall \xi \in \mathbb{R}
\]

and

\[
\frac{|I|^{\alpha}}{\varphi(|I|^{-1})\log(|I|^{-1})} \lesssim \mu(I) \lesssim \frac{|I|^{\alpha}}{\log(|I|^{-1})}
\]

for any interval \( I \) centered in \( E \) and with \( |I| < 1/2 \).

Moreover, there exists a sequence of functions \( \{f_l\}_{l \geq 1} \) so that \( \|f_l\|_{L^2(d\mu)} > 0 \), and

\[
\sup_{l \geq 1} \frac{\|\hat{f_l}d\mu\|_{L^q(\mathbb{R})}}{\|f_l\|_{L^2(d\mu)}} = \infty
\]

for all \( 2 \leq q < 2 + \frac{4(1-\alpha)}{\beta} = p'_{1,\alpha,\beta} =: q_{\alpha,\beta} \).

This theorem generalizes a result of Hambrook and Laba in [7]. Hambrook and Laba show that given \( \alpha = \log t / \log N \) for \( t, N \in \mathbb{N} \) and \( t < N \), there exists a measure \( \mu \) so that \( \mu \) satisfies the \( \alpha \)-dimensional ball condition, and the Fourier decay condition (2.6) for all \( \beta < \alpha \). Moreover, they provide a sequence of functions \( \{f_l\}_{l \geq 1} \) so that (2.8) holds for all \( 2 \leq q \leq 4/\alpha - 2 = q_{\alpha,\alpha} \).
2.3. Random construction: “Knapp” example

The existence of the sequence of functions satisfying (2.8) imply that for the measure $\mu$ of Theorem 2.3, the range of exponents in the restriction estimate of Theorem 1.6 is sharp. Recall that Theorem 1.6 generalizes the Tomas-Stein restriction theorem for the sphere. In Example 1.5, we presented the Knapp example, which proves sharpness of the Tomas-Stein restriction theorem. The Knapp example exploits the idea that the sphere contains caps of low curvature. Comparably, the construction giving Theorem 2.3 exploits the idea that a random Cantor set may contain a small but highly structured subset. Due to this analogy, we refer to this construction as a “Knapp” example.

2.3.1 Outline of construction and selection of parameters

Construction 2.4. First, set

$$N_k = \lceil \varphi(2^k)^{1/2} \rceil + 2$$

and take $M_k = N_1 \cdots N_k$. Each subinterval in the $k^{th}$ stage of the Cantor set will be subdivided into $N_{k+1}$ intervals of length $\delta_k = M_{k+1}^{-1}$. Moreover, the number of intervals selected will be governed by a given sequence of integers $\{t_k\}_{k=1}^{\infty}$ satisfying $1 \leq t_k < N_k$. The sets $A_k$ are defined as in construction 2.2.

However, we also require that $A_k$ contains a subset $P_k$, where the subsets $P_k$ are highly structured. The size of $P_k$ is determined by a given sequence of integers $\{\tau_k\}_{k=1}^{\infty}$, where $\tau_k$ satisfies $1 \leq \tau_k \leq t_k$. We define $P_k$ inductively via $P_0 = \{0\}$ and

$$P_{k+1} = \bigcup_{a \in P_k} \left( a + \{1, \ldots, \tau_{k+1}\}/M_{k+1} \right).$$

Then $E_k$ is the union of the intervals $I_k(a) = [a, a + \delta_k]$ for $a \in A_k$, and contains the highly structured subset

$$F_k = \bigcup_{a \in P_k} I_k(a).$$

We will show that the functions $f_k = 1_{F_k}$ satisfy (2.8).

Note again that each interval composing $E_k$ contains exactly $t_{k+1}$ subintervals of $E_{k+1}$, and so by Lemma 1.17 the weak-* limit $\mu$ of the measures $d\mu_k = 1_{E_k}/|E_k|dx$ exists.
Moreover, if we take $\varphi(t) = \log^\varepsilon(t)$, then (2.7) implies that

$$|I|^\alpha + \delta \lesssim \mu(I) \lesssim |I|^\alpha$$

for all $\delta > 0$, and so $\mu$ satisfies the $\alpha$-dimensional ball condition, and $E = \bigcap_{k=1}^\infty E_k$ has Hausdorff dimension $\alpha$.

**Selection of the sequences $t_k$ and $\tau_k$.** As before, for sufficiently large $k$, we may select the sequence $t_k$ so that the sequence $T_k = t_1 \cdots t_k$ obeys

$$T_{k-1}^{-1/2} \log^{1/2}(2^3M_k) \sim M_k^{-\alpha/2}. \tag{2.9}$$

Moreover, in order to preserve the Fourier decay bound, the size of the structured subset $F_k$ must obey

$$\frac{\tau_1 \cdots \tau_k}{t_1 \cdots t_k} \sim M_k^{-\beta/2}. \tag{2.10}$$

Finally, let $N_0$ be sufficiently large so that for $k \geq N_0$ we have

$$4\log(2^3M_k) \leq T_{k-1}. \tag{2.11}$$

**Selection of the sets $A_k$ and existence of Fourier decay.** Let $k > N_0$. In a similar process as for construction 2.2, we may choose $A_k$ randomly by first defining $B_k'$ as before, and then considering translations of $B_k$ within $[N_k]$: for $x \in [N_k]$ we define $B_{k,x}' = x + B_k$ (mod $N_k$), and then set $B_{k,x} = B_{k,x}'/M_k$.

As before, there exists a collection of random variables $x(a)$ so that for

$$\chi_a(l) = \left( \frac{S_{B_{k,x}(a)}(l)}{t_k} - \frac{S_{[N_k]/M_k}(l)}{N_k} \right) e^{-2\pi i al}$$

we have the bound

$$\left| \frac{1}{T_{k-1}} \sum_{a \in A_{k-1}} \chi_a(l) \right| \leq 4T_{k-1}^{-1/2} \log^{1/2}(2^3M_k) \sim M_k^{-\alpha/2}.$$
for all \( l \in \mathbb{Z} \). Since \( \beta \leq \alpha \), we then have

\[
\left| \frac{1}{T_{k-1}} \sum_{a \in A_{k-1}} \chi_a(l) \right| \lesssim M_k^{-\beta/2}.
\]  

(2.12)

For \( a \notin P_{k-1} \), set \( A_{k,a} = B_{k,x(a)} \).

If \( a \in P_{k-1} \), we need to modify \( B_{k,x(a)} \) so that it contains \( P_k \) while preserving the bound (2.12). If \( a \in P_{k-1} \), adjoin \( \{1, \ldots, \tau_k\}/M_k \) to \( B_{k,x(a)} \) and remove the points \( 0 \) and \( (\tau_k + 1)/M_k \) if they are members of the set. Finally, remove as many other points as necessary so that \( B_{k,x(a)} \) still contains \( \{1, \ldots, \tau_k\}/M_k \) and the cardinality of the set remains \( t_k \). Let \( A_{k,a} \) be the resulting set.

Let

\[
\tilde{\chi}_a(l) = \left( \frac{S_{A_{k,a}}(l)}{t_k} - \frac{S_{[N_k/M_k]}(l)}{N_k} \right) e^{-2\pi il}.
\]

Then for \( a \in P_{k-1} \),

\[
|\tilde{\chi}_a(l) - \chi_a(l)| \leq \frac{2\tau_k + 4}{t_k} \leq \frac{6\tau_k}{t_k}
\]
as the symmetric difference of the sets \( A_{k,a} \) and \( B_{k,x(a)} \) has at most \( 2\tau_k + 4 \) elements. Then

\[
\left| \frac{1}{T_{k-1}} \sum_{a \in A_{k-1}} \tilde{\chi}_a(l) - \frac{1}{T_{k-1}} \sum_{a \in A_{k-1}} \chi_a(l) \right| \lesssim \frac{\#P_{k-1} - 6\tau_k}{T_{k-1}} \cdot \frac{6\tau_1 \cdots \tau_k}{T_k}.
\]

But by our choice of \( \tau_k \) and \( t_k \), \( \tau_1 \cdots \tau_k/T_k \sim M_k^{-\beta/2} \), and so the bound (2.12) is indeed preserved for \( \tilde{\chi}_a \). The Fourier decay bound (2.6) follows in a similar manner as for Theorem 2.1. We may also establish the ball condition (2.7) and the Hausdorff dimension of \( E \) using Lemma 1.16 as for Theorem 2.1.

### 2.3.2 The structured subset: lack of restriction for \( q < p'_{1,\alpha,\beta} \)

**Arithmetic progressions.** Let \( k,l \) be integers so that \( N_0 < l < k \). Writing

\[
f_l \mu_k = \frac{1}{T_k} \left( \sum_{a \in F_l \cap A_k} \delta_a \right) * M_k \mathbb{1}_{[0,\delta_k]}.
\]
we see that
\[ \hat{f_l\mu_k}(\xi) = \frac{e^{-i\pi\xi\delta_k}}{T_k} \left( \sum_{a \in F_l \cap A_k} e^{-2\pi i a \xi} \right) \hat{\chi}(\xi/M_k) \]
where \( \chi = 1_{[-1/2,1/2]} \). Then for \( r \in \mathbb{N} \) the \( L^{2r} \) norm of \( \hat{f_l\mu_k} \) is given by
\[ \|\hat{f_l\mu_k}\|_{2r} = \frac{M_k}{T_k^{2r}} \sum_{a_j \in F_l \cap A_k} e^{-2\pi i (\sum_{j=1}^r a_j - \sum_{j=r+1}^{2r} a_j) M_k \xi} \chi^{2r}(\xi) d\xi. \]

Using the Fourier inversion theorem,
\[ \|\hat{f_l\mu_k}\|_{2r}^2 = \frac{M_k}{T_k^{2r}} \sum_{a_j \in F_l \cap A_k} \chi^{2r} \left( \left( \sum_{j=1}^r a_j - \sum_{j=r+1}^{2r} a_j \right) M_k \right). \quad (2.13) \]

Since \( \chi^{2r} \) is non-negative, we can show that \( \|\hat{f_l\mu_k}\|_{2r} \) is large by finding a lower bound for
\[ M_{l,k,r} = \# \left\{ (a_1, \ldots, a_{2r}) \in (F_l \cap A_k)^{2r} : \sum_{j=1}^r a_j = \sum_{j=r+1}^{2r} a_j \right\}, \]
for which we exploit the arithmetic progressions \( P_l \) embedded in \( E \). To this end, define
\[ g(z) = \# \{(a_1, \ldots, a_r) \in F_l \cap A_k : a_1 + \cdots + a_r = z\}. \]

As each \( z \in \mathbb{Z}/M_k \) contributes \( g(z)^2 \) \( r \)-tuples to \( M_{l,k,r} \), \( \|g\|_{l^2}^2 = M_{l,k,r} \). By the Cauchy-Schwarz inequality, a lower bound for \( M_{l,k,r} \) may be found via an upper bound on \( Z \), the support of \( g \). Any element of \( Z \) may be written in the form \( \sum_{j=1}^l \frac{z_j}{M_l} + \frac{z_k}{M_k} \) where \( z_j \in \{r, r + 1, \ldots, r\tau_l\} \) and \( z_k \in \{0, 1, \ldots, rN_{l+1} \cdots N_k\} \), giving
\[ \#Z \leq (r\tau_l) \cdots (r\tau_l)(rN_{l+1} \cdots N_k) = r^{l+1} \tau_l \cdots \tau_l M_k M_t. \]

Thus, as \( \|g\|_{l^1} = (\#F_l \cap A_k)^r = (\tau_1 \cdots \tau_t l_{t+1} \cdots t_k)^r \)
\[ M_{l,k,r} = \|g\|_{l^2}^2 \geq \frac{\|g\|_{l^1}^2}{\#Z} \geq \frac{(\tau_1 \cdots \tau_t l_{t+1} \cdots t_k)^{2r} M_l}{r^{l+1} \tau_l \cdots \tau_l M_k}. \]
Returning to (2.13), we have
\[
\|\hat{f}_l \hat{\mu}_k\|_{2r} \geq \frac{C_r M_k M_{l,k,r}}{T_k^{2r}} \geq C_r \frac{M_l (\tau_1 \cdots \tau_l)^{2r-1}}{T_l^{2r} r^{l+1}}
\]  
(2.14)

for \(C_r = \chi^{2r}(0) > 0\).

**Sharpness of Mockenhaupt’s theorem.** Let \(2 \leq q < q_{\alpha, \beta} = 2 + 4(1 - \alpha)/\beta\), and fix \(r \in \mathbb{Z}\) so that \(r > 1/\beta\) and \(2r > q_{\alpha, \beta}\). Since \(F_l\) is isolated in \(E_l\), there is a Schwartz function \(\chi_l\) so that \(f_l d\mu_k = \chi_l d\mu_k\), and so for \(\xi \neq 0\)
\[
|f_l d\mu_k(\xi)| = |\chi_l d\mu_k(\xi)| \lesssim |\xi|^{-\beta/2}.
\]
As \(r > 1/\beta\), \(|f_l d\mu_k(\xi)|^{2r}\) is dominated by an integrable function. Then taking \(k \to \infty\) and appealing to the dominated convergence theorem the lower bound (2.14) holds for \(\hat{f}_l d\mu\).

This bound can be extended to one for \(\|\hat{f}_l d\mu\|_q\) by noting
\[
\|\hat{f}_l d\mu\|_{2r}^{2r} \leq \|\hat{f}_l d\mu\|_q^q \cdot \|\hat{f}_l d\mu\|_{\infty}^{2r-q} \\
\leq \|\hat{f}_l d\mu\|_q^q \cdot \|f_l\|_{L^1(d\mu)}^{2r-q}.
\]
As \(\|f_l\|_{L^1(d\mu)} = (\tau_1 \cdots \tau_l)/T_l\),
\[
\|\hat{f}_l d\mu\|_{L^q(\mathbb{R})}^q \gtrsim \frac{M_l (\tau_1 \cdots \tau_l)^{q-1}}{r^{l+1} T_l^q}.
\]
Next, assume toward contradiction that
\[
\|\hat{f}_l d\mu\|_{L^q(\mathbb{R})}^q \lesssim \|f_l\|_{L^2(d\mu)}^q,
\]
which implies
\[
M_l (\tau_1 \cdots \tau_l)^{q-1} \gtrsim (\tau_1 \cdots \tau_l)^{q/2} 
\]
\[
\frac{r^{l+1} T_l^q}{(\tau_1 \cdots \tau_l)^{q/2}}.
\]
Rearranging gives
\[
M_l \lesssim r^{l+1} \frac{T_l^{q/2}}{(\tau_1 \cdots \tau_l)^{q/2}} \tau_1 \cdots \tau_l.
\]  
(2.15)
2.3. Random construction: “Knapp” example

From (2.9) and (2.10) we have

\[ \tau_1 \cdots \tau_l \sim M_l^{-\beta/2} T_l \sim M_l^{-\beta/2} \log(2^3 M_{l+1}) M_l^\alpha \]

so (2.15) becomes

\[ M_l^{1-\frac{3}{2} \frac{\beta}{2} + \frac{\alpha}{2}} \lesssim t^{l+1} \log(2^3 M_{l+1}) N_{l+1}^\alpha. \]

For \( q < q_{\alpha,\beta} \) the exponent on the left-hand side of the inequality is positive. Since \( N_l \to \infty \), \( M_l^{1-\frac{3}{2} \frac{\beta}{2} + \frac{\alpha}{2}} \) grows faster than any geometric rate. But, as \( N_l \lesssim l \), the right-hand side of the inequality is bounded by geometric growth, giving a contradiction. Thus the initial assumption is false, proving (2.8).
Chapter 3

Restriction estimates for singular measures

In section 1.2 we introduced two restriction estimates that may be applied to singular measures satisfying particular conditions, namely, Mockenhaupt’s restriction theorem and Chen’s restriction theorem. In this chapter we provide a detailed exploration of these results in order to analyze the range of exponents for which there holds a restriction estimate of the form

\[ \| \hat{f} \|_{L^2(d\mu)} \lesssim \| f \|_{L^p(\mathbb{R}^d)}, \]  

(3.1)

3.1 A comparison of Mockenhaupt’s and Chen’s restriction estimates

First we present a proof of Mockenhaupt’s argument, which is based on Stein’s original argument for the restriction estimate on the sphere. It covers the range of \( p \) excluding the endpoint \( p_{d,\alpha,\beta} \). The proof employs what is known as a “TT* argument,” detailed for this case in the following lemma:

**Lemma 3.1.** Let \( \mu \) be a finite positive measure on \( \mathbb{R}^d \). Then (3.1) holds for all \( f \in S \) if and only if

\[ \| \hat{\mu} \ast g \|_{L^{p'}(\mathbb{R}^d)} \lesssim \| g \|_{L^p(\mathbb{R}^d)} \]  

(3.2)

for all \( g \in S \).

The lemma is an easy consequence of the rule

\[ \int \hat{f} g d\mu = \int f \cdot (\overline{g} \ast \hat{\mu}) dx \]  

(3.3)
3.1. A comparison of Mockenhaupt’s and Chen’s restriction estimates

which follows from the duality relation of the Fourier transform and the Fourier inversion theorem.

**Theorem 3.2** (Mockenhaupt; Bak and Seeger (endpoint)). Let \( \mu \) be a compactly supported positive measure on \( \mathbb{R}^d \), with \( \alpha, \beta \in (0, d) \) such that

\[
\mu(B(x, r)) \lesssim r^\alpha \quad \forall x \in \mathbb{R}^d, \quad r > 0, \tag{3.4}
\]

and

\[
|\hat{\mu}(\xi)| \lesssim (1 + |\xi|)^{-\beta/2} \quad \forall \xi \in \mathbb{R}^d. \tag{3.5}
\]

Then for \( 1 \leq p \leq p_{d,\alpha,\beta} = (4(d - \alpha) + 2\beta)/(4(d - \alpha) + \beta) \), we have the estimate

\[
\|\hat{f}\|_{L^2(\hat{d}\mu)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}. \tag{3.6}
\]

Equivalently, in its dual form, for \( p' \geq p'_{d,\alpha,\beta} = 2 + 4(d - \alpha)/\beta \), we have the estimate

\[
\|\hat{f}d\hat{\mu}\|_{L^{p'}(\mathbb{R}^d)} \lesssim \|f\|_{L^2(\hat{d}\mu)}. \tag{3.7}
\]

**Proof.** Let \( T \) be the operator defined by \( Tf = f * \hat{\mu} \). By Lemma 3.1 it suffices to show that \( T \) is a bounded operator from \( L^p \) to \( L^{p'} \).

Consider a partition of unity \( \{\varphi_k\}_{k=0}^{\infty} \) where for \( k \geq 1 \), \( \varphi_k = \varphi(\cdot/2^k) \) for some \( \varphi \in C^\infty(\mathbb{R}^d) \) so that \( \text{supp}(\varphi_k) \subset \{2^{k-1} \leq |x| \leq 2^{k+1}\} \), and \( \text{supp}(\varphi_0) \subset \{|x| \leq 2\} \). Decompose \( T \) into the operators \( T_k \) defined by \( T_k f = (\varphi_k \hat{\mu}) * f \). We will bound the operator norms of \( T_k : L^1 \to L^\infty \) and \( T_k : L^2 \to L^2 \) and interpolate to recover a bound for the full range of \( p \) (excluding the endpoint \( p_{d,\alpha,\beta} \)).

Using the Fourier decay condition (3.5) and the support properties of \( \varphi_k \),

\[
\|T_k\|_{L^1 \to L^\infty} \leq \|\varphi_k \hat{d}\mu\|_{\infty} \lesssim 2^{-k\beta/2}. \tag{3.6}
\]

Note that by Plancherel’s theorem

\[
\|T_k f\|_2 = \| (\hat{\varphi_k} * \mu) \hat{f} \|_2 \leq \|\varphi_k \hat{\mu}\|_\infty \|f\|_2,
\]

so

\[
\|T_k\|_{L^2 \to L^2} \lesssim \|\varphi_k \hat{\mu}\|_{\infty}.
\]
But $\hat{\varphi} \in S$, so for $N > 1$
\[
\|T_k\|_{L^2 \rightarrow L^2} \lesssim 2^{kd} \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{(1 + 2^k|x-y|)^N} d\mu(y) \\
\lesssim 2^{kd} \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^\infty 1_{B(x,r/2^k)}(y)(1 + r)^{-(N+1)} dr d\mu(y) \\
= 2^{kd} \sup_{x \in \mathbb{R}^d} \int_0^\infty \mu(B(x, r/2^k))(1 + r)^{-(N+1)} dr.
\]
Invoking condition (3.4) and taking $N$ sufficiently large then gives
\[
\|T_k\|_{L^2 \rightarrow L^2} \lesssim 2^{k(d-\alpha)}.
\] (3.7)
Interpolating the bounds (3.6) and (3.7) gives
\[
\|T_k\|_{L^p \rightarrow L^{p'}} \lesssim 2^k \left( \frac{2d-2\alpha+\beta}{p'} - \frac{\beta}{p} \right),
\]
which is summable for $p' > \frac{4(d-\alpha)+2\beta}{\beta}$. Thus $T = \sum_{k=0}^\infty T_k$ is a bounded operator from $L^p$ to $L^{p'}$ for the corresponding range of $p$.

While Mockenhaupt’s theorem applies to measures satisfying the ball condition and the Fourier decay condition with some exponents $0 < \alpha, \beta < d$, Chen’s theorem applies to measures for which there is some $n$ such that the $n$-fold convolution belongs to $L^r$, for some $1 \leq r \leq \infty$. The convolution between two Borel measures $\mu$ and $\nu$ on $\mathbb{R}^d$ is defined via
\[
\mu * \nu(A) = \int \int 1_A(x+y) d\mu(x) d\nu(y)
\]
for any Borel set $A$. The $n$-fold convolution of a measure $\mu$,
\[
\mu^{*n} = \mu * \cdots * \mu,
\]
involves the convolution of $n$ copies of $\mu$.

**Theorem 3.3** (Chen). Let $\mu$ be a compactly supported positive measure on $\mathbb{R}^d$ such that $\mu^{*n} \in L^r(\mathbb{R}^d)$, for some $1 \leq r \leq \infty$. Let $1 \leq p \leq 2n/(2n-1)$ if $r \geq 2$ and $1 \leq p \leq
3.1. A comparison of Mockenhaupt’s and Chen’s restriction estimates

Let \( nr'/(nr' - 1) \) if \( 1 \leq r \leq 2 \). Let \( 1 \leq q \leq p'/(nr') \). Then

\[
\| \hat{f} \|_{L^2(d\mu)} \lesssim \| f \|_{L^p(\mathbb{R}^d)}
\]

for all \( f \in L^p(\mathbb{R}^d) \).

A construction of Körner in [8] shows the existence of measures \( \mu \) satisfying the conditions of the theorem in the case \( d = 1, n = 2 \). More precisely, Körner constructed measures \( \mu \) supported on compact sets of Hausdorff dimension 1/2, and such that \( \mu \ast \mu \) is a continuous function with compact support. In [5] Chen and Seeger extended this construction to generate measures satisfying the conditions with general \( d \) and \( n \).

We are particularly interested in Chen’s restriction estimate when \( r = \infty \), for then \( q = 2 \), matching Mockenhaupt’s estimate. For convenience we rewrite Chen’s restriction estimate with \( r = \infty \), and present an adaption of Chen’s proof to this case.

**Theorem 3.4** (Chen; \( r = \infty \)). Let \( \mu \) be a compactly supported positive measure on \( \mathbb{R}^d \) such that \( \mu \ast^n \in L^\infty(\mathbb{R}^d) \). Then for \( 1 \leq p \leq p_{d,n} = 2n/(2n - 1) \), the restriction estimate

\[
\| \hat{f} \|_{L^2(d\mu)} \lesssim \| f \|_{L^p(\mathbb{R}^d)}
\]

holds for all \( f \in L^p(\mathbb{R}^d) \).

**Proof.** As we have the trivial estimate \( \| \hat{f} \|_{L^2(d\mu)} \lesssim \| f \|_{L^1(\mathbb{R}^d)} \) it suffices to prove the restriction estimate for \( p = p_{d,n} \) as we may interpolate to recover the full range of exponents.

Let \( \varphi \in C_c^\infty(\mathbb{R}^d) \) be a non-negative function satisfying \( \| \varphi \|_1 = 1 \), and let \( \varphi_\varepsilon(\xi) = \varepsilon^{-d} \varphi(\xi/\varepsilon) \). Define \( \mu_\varepsilon = \varphi_\varepsilon \ast \mu \) so that \( \mu_\varepsilon \) converges weakly to \( \mu \) as \( \varepsilon \to 0 \). We show that the dual restriction estimate holds for \( \mu_\varepsilon \); that is, for all bounded continuous functions \( g \),

\[
\| g \hat{\mu}_\varepsilon \|_{L^{p'}(\mathbb{R}^d)} \lesssim \| g \|_{L^2(d\mu_\varepsilon)}
\]
3.1. A comparison of Mockenhaupt’s and Chen’s restriction estimates

when \( p' = 2n \). By the Plancherel theorem,

\[
\left( \int_{\mathbb{R}^d} \left| g_{\mu_{\varepsilon}}(x) \right|^{p'} \, dx \right)^{\frac{n}{p'}} = \left( \int_{\mathbb{R}^d} \left| g_{\mu_{\varepsilon}}^{*n}(x) \right|^2 \, dx \right)^{1/2} = \left( \int_{\mathbb{R}^d} \left( |g_{\mu_{\varepsilon}}|^n(\xi) \right)^2 \, d\xi \right)^{1/2} = \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^{(n-1)d}} G(\xi, \eta) M_{\varepsilon}(\xi, \eta) \, d\eta \right)^2 \, d\xi \right)^{1/2}
\]

where \( \eta = (\eta_1, \ldots, \eta_{n-1}) \),

\[
G(\xi, \eta) = g(\eta_{n-1}) \prod_{j=1}^{n-1} g(\eta_{j-1} - \eta_j), \quad M_{\varepsilon}(\xi, \eta) = \mu_{\varepsilon}(\eta_{n-1}) \prod_{j=1}^{n-1} \mu_{\varepsilon}(\eta_{j-1} - \eta_j)
\]

and we take \( \eta_0 = \xi \). Split the inner integral into the factors \( M_{\varepsilon}^{1/2} \) and \( G \cdot M_{\varepsilon}^{1/2} \) and apply Cauchy-Schwarz to obtain

\[
\left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^{(n-1)d}} G(\xi, \eta) M_{\varepsilon}(\xi, \eta) \, d\eta \right)^2 \, d\xi \right)^{1/2} \leq \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^{(n-1)d}} |G(\xi, \eta)|^2 M_{\varepsilon}(\xi, \eta) \, d\eta \right) \, d\xi \right)^{1/2} \leq ||\mu_{\varepsilon}||_{L^n}^{1/2} \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^{(n-1)d}} |G(\xi, \eta)|^2 M_{\varepsilon}(\xi, \eta) \, d\eta \, d\xi \right)^{1/2} = ||\mu_{\varepsilon}||_{L^n}^{1/2} \left( \int_{\mathbb{R}^d} |g(\xi)|^2 \mu_{\varepsilon}(\xi) \, d\xi \right)^{n/2} \leq ||\mu_{\varepsilon}||_{L^n}^{1/2} \left( \int_{\mathbb{R}^d} |g(\xi)|^2 \mu_{\varepsilon}(\xi) \, d\xi \right)^{n/2},
\]

where the last inequality is obtained by an application of Young’s convolution inequality.

\[\square\]

In Proposition 3.5 and Corollary 3.6 we examine the relationship between the conditions of Mockenhaupt’s and Chen’s theorems, which will allow us to compare the range of exponents for which their results provide a restriction estimate.
Proposition 3.5 (Chen, Proposition 1). Let $\mu$ be a compactly supported positive measure on $\mathbb{R}^d$. If $\mu^{|n}$ satisfies the $\alpha$-dimensional ball condition with $0 \leq \alpha \leq d$, then $\mu$ satisfies the $\alpha/n$-dimensional ball condition.

Proof. Assume toward contradiction that for all $k \in \mathbb{N}$ there is some ball $B_{r_k}$ of radius $r_k > 0$ so that $\mu(B_{r_k}) \geq kr_k^{\alpha/n}$. Let $B_k = B_{r_k} + \cdots + B_{r_k}$, the $n$-fold Minkowski sum of $B_{r_k}$, which is a ball of radius $nr_k$. As $\mu^{|n}$ satisfies the $\alpha$-dimensional ball condition,

$$\mu^{|n}(B_k) \lesssim (nr_k)^{\alpha} \lesssim r_k^{\alpha}.$$

Moreover,

$$\mu^{|n}(B_k) = \int \cdots \int 1_{B_k}(x_1 + \cdots + x_n)d\mu(x_1)\cdots d\mu(x_n)$$

$$\geq \int \cdots \int 1_{B_{r_k}}(x_1)\cdots 1_{B_{r_k}}(x_n)d\mu(x_1)\cdots d\mu(x_n)$$

$$= \mu(B_{r_k})^n \geq k^n r_k^{\alpha}.$$

Taking $k \to \infty$ gives the desired contradiction.

In particular, if $\mu^{|n} \in L^\infty(\mathbb{R}^d)$, then $\mu^{|n}$ satisfies the $d$-dimensional ball condition. Thus we have the following corollary:

Corollary 3.6. Let $\mu$ be a compactly supported positive measure on $\mathbb{R}^d$. If $\mu^{|n} \in L^\infty$, then $\mu$ satisfies the $\alpha$-dimensional ball condition with $\alpha = d/n$.

The endpoint $p_{d,n} = 2n/(2n - 1)$ given in Theorem 3.4 may be rewritten in terms of $\alpha = d/n$, obtaining

$$p_{d,n} = \frac{2d}{2d - \alpha}.
(3.8)$$

Assume that a finite compactly supported measure $\mu$ satisfies $\mu^{|n} \in L^\infty(\mathbb{R}^d)$ and satisfies the Fourier decay condition with exponent $\beta$. Then the restriction estimate (3.1) holds for $1 \leq p \leq \max\{p_{d,n}, p_{d,\alpha,\beta}\}$. The exponent giving the endpoint depends on the relationship between $\alpha$ and $\beta$.

First, observe that when $d$ and $\alpha$ are fixed, $p_{d,\alpha,\beta}$ is an increasing function of $\beta$, and
3.2 Sharpness of Chen’s and Mockenhaupt’s restriction estimates

that \( p_{d,\alpha,\beta} = p_{d,n} \) when \( \beta = 2\alpha \). Thus, if \( \beta < 2\alpha \), Theorem 3.4 gives the larger range of exponents, \( 1 \leq p \leq p_{d,n} \). If \( \beta > 2\alpha \), then Theorem 3.2 gives the larger range of exponents, \( 1 \leq p \leq p_{d,\alpha,\beta} \). By Proposition 1.13 we see that this case is only possible if \( \alpha < \dim_H(\text{supp}(\mu)) \).

We define new classes of measures \( \mathcal{C}_n(\mathbb{R}^d) \), which is a subset of the set of positive compactly supported measures on \( \mathbb{R}^d \). Precisely, \( \mathcal{C}_n(\mathbb{R}^d) \) restricts to \( \mu \) satisfying \( \mu^{*n} \in L^\infty(\mathbb{R}^d) \), the Fourier decay condition with some exponent \( \beta > 0 \), and \( \dim_H(\text{supp}(\mu)) = \alpha = \frac{d}{n} \). First we note that this class is non-empty: in [5] Chen and Seeger show that for any exponent of the form \( \alpha = d/n, n \geq 2 \), there exists measures \( \mu \) satisfying these conditions with \( \beta = \alpha \).

On \( \mathcal{C}_n(\mathbb{R}^d) \), Theorem 3.4 gives the larger range of exponents \( 1 \leq p \leq p_{d,n} = \frac{2d}{2d-\alpha} \), because \( \beta \leq \dim_H(\text{supp}(\mu)) = \alpha \). In the next section, we show that this range is optimal.

3.2 Sharpness of Chen’s and Mockenhaupt’s restriction estimates

First we note that by Theorem 2.3, which constructs “Knapp” Cantor sets, there are some measures \( \mu \) for which the range of exponents given in Theorem 3.2 is sharp. However, we have seen that for \( \mu \in \mathcal{C}_n(\mathbb{R}^d) \), the restriction estimate (3.1) holds for a larger range of \( p \), implying that in general, the range of exponents given in Theorem 3.2 is not sharp.

The following result implies that the range of exponents \( 1 \leq p \leq 2d/(2d-\alpha) \) is sharp for measures \( \mu \) satisfying the ball condition with \( \alpha = \dim_H(\text{supp}(\mu)) \):

**Theorem 3.7.** Let \( \mu \) be a finite positive measure on \( \mathbb{R}^d \). Suppose \( \mu \) is supported on a compact set of Hausdorff dimension \( \alpha \) and the restriction estimate \( \|f\|_{L^2(\mu)} \lesssim \|f\|_{L^p(\mathbb{R}^d)} \) holds for all \( f \in L^p(\mathbb{R}^d) \). Then \( p \leq 2d/(2d-\alpha) \).

**Proof.** We proceed by contradiction, considering the dual estimate. Suppose for \( p' < 2d/\alpha \) we have \( \|\hat{g}\mu\|_{L^{p'}(\mathbb{R}^d)} \lesssim \|g\|_{L^2(\mu)} \) for all \( g \in L^2(\mu) \). Taking \( g = 1 \), we therefore have

\[
\int |\hat{\mu}(\xi)|^{p'} d\xi < \infty.
\]

Let \( \alpha' > \alpha \) be such that \( p' < 2d/\alpha' < 2d/\alpha \). Recall the \( \alpha' \)-dimensional energy integral may
3.2. Sharpness of Chen’s and Mockenhaupt’s restriction estimates

be given by

\[ I_{\alpha'}(\mu) = \gamma(\alpha', d) \int_{\mathbb{R}^d} |\widehat{\mu}(\xi)|^2 |\xi|^{-(d-\alpha')} \, d\xi \]

via Lemma 1.12. Decompose the domain of the integral into the regions $|\xi| \leq 1$ and $|\xi| \geq 1$. The integral over the first region is finite as $\widehat{\mu}$ is uniformly bounded. Applying Hölder’s inequality to the integral over the latter region, we have

\[ \int_{|\xi| \geq 1} |\widehat{\mu}(\xi)|^2 |\xi|^{-(d-\alpha')} \, d\xi \leq \left( \int_{|\xi| \geq 1} |\widehat{\mu}(\xi)|^{p'} \, d\xi \right)^{2/p'} \left( \int_{|\xi| \geq 1} |\xi|^{-(d-\alpha')p'/(p'-2)} \, d\xi \right)^{(p'-2)/p'} \]

The first integral on the right-hand side of the inequality is finite by assumption; the second is finite as the inequality $p' < 2d/\alpha'$ implies that $(d-\alpha')p'/(p'-2) > d$.

So $I_{\alpha'}(\mu) < \infty$. But then by Lemma 1.11, $\dim_h(\text{supp}(\mu)) \geq \alpha' > \alpha$, a contradiction.

In particular, Theorem 3.7 implies that Theorem 3.4 is sharp for measures $\mu \in C_n(\mathbb{R}^d)$. The following table summarizes the theorems discussed in this chapter:

<table>
<thead>
<tr>
<th>1 ≤ p ≤ p_{d,\alpha,\beta}</th>
<th>1 ≤ p ≤ p_{d,n}</th>
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<td>$\mu(B(x, r)) \lesssim r^\alpha$,</td>
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<tr>
<td>Sharp for</td>
<td>“Knapp” measures [4, 7]</td>
<td>$C_n(\mathbb{R}^d)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>[5, 8]</td>
</tr>
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Table 3.1: Range of exponents and existence of restriction estimate (3.1)
Chapter 4

Multilinear restriction theory

Consider the following generalization of the restriction problem:

Question 4.1 (Multilinear restriction problem). Given finite, compactly supported measures $\mu_1, \ldots, \mu_n$ on $\mathbb{R}^d$, for what values of $p, q$ does there exist a constant $C > 0$ such that

$$\left\| \prod_{j=1}^{n} f_j \, d\mu_j \right\|_{L^q(\mathbb{R}^d)} \leq C \prod_{j=1}^{n} \| f_j \|_{L^p(d\mu_j)}$$

for all $f_1 \in L^p(d\mu_1), \ldots, f_n \in L^p(d\mu_n)$?

Multilinear restriction estimates have been studied in the case where $\mu_1, \ldots, \mu_n$ are measures supported on disjoint compact subsets $S_1, \ldots, S_n$ of a smooth hypersurface in $\mathbb{R}^d$, such as the sphere or paraboloid. Multilinear restriction theorems exploit the transversality of the subsets $S_1, \ldots, S_n$, meaning that their unit normals are separated by some fixed positive distance. The transversality assumption makes it possible to obtain multilinear restriction estimates in cases where linear ones cannot hold. In recent years, multilinear restriction theory has become a powerful tool in proving new linear restriction estimates. In this chapter we discuss these ideas with focus on bilinear restriction.

4.1 Relationship between linear and multilinear restriction

Some bilinear restriction estimates follow directly from linear restriction estimates. However, there is more interest in obtaining new linear restriction estimates, and bilinear restriction estimates have proved a powerful tool to do so. In section 4.1.1 we discuss how we may obtain bilinear restriction estimates from linear ones, but also give an example of a bilinear restriction estimate that cannot be obtained from a linear estimate. Then in section 4.1.2 we illustrate how bilinear restriction estimates can be used to prove linear...
ones through the example given in [6] of restriction on the circle. We refer to [17] for a more complete overview of bilinear restriction and its contribution to recent progress in restriction theory.

### 4.1.1 Linear restriction implies bilinear restriction

Suppose we have the linear restriction estimate

\[ \| \hat{f} \|_{L^q(\mathbb{R}^d)} \lesssim \| f \|_{L^p(d\mu)}. \]  

(4.1)

By the Cauchy-Schwarz inequality,

\[ \| \hat{f}_1 \hat{f}_2 \|_{L^{q/2}(\mathbb{R}^d)} \lesssim \| \hat{f}_1 \|_{L^q(\mathbb{R}^d)} \| \hat{f}_2 \|_{L^q(\mathbb{R}^d)}. \]

Applying (4.1) to both terms gives the bilinear restriction estimate

\[ \| \hat{f}_1 \hat{f}_2 \|_{L^{q/2}(\mathbb{R}^d)} \lesssim \| f_1 \|_{L^p(d\mu)} \| f_2 \|_{L^p(d\mu)}. \]

There are bilinear restriction estimates that cannot be obtained from linear ones. Consider, for example, \( S_1 = \{ (x_1, 0) : 0 \leq x_1 \leq 1 \} \subset \mathbb{R}^2 \) and \( S_2 = \{ (0, x_2) : 0 \leq x_2 \leq 1 \} \subset \mathbb{R}^2 \), equipped with the one-dimensional Lebesgue-measure \( \mu_1 \) and \( \mu_2 \) respectively. Then \( \hat{\mu}_1(\xi) \) and \( \hat{\mu}_2(\xi) \) depend only on the first coordinate and second coordinate of \( \xi \) respectively, so they do not decay to zero at infinity. Thus no linear restriction estimate for \( \mu_1 \) or \( \mu_2 \) can hold, unless \( q = \infty \).

However, we have a bilinear restriction estimate with \( p = q = 2 \). Since \( \hat{f}_i (\xi) = \hat{g}_i (\xi_i) \) for some \( g_i \) supported on \([0, 1] \), we may write

\[ \| \hat{f}_1 \hat{f}_2 \|_{L^2(\mathbb{R}^2)} = \| \hat{g}_1 \|_{L^2(\mathbb{R})} \| \hat{g}_2 \|_{L^2(\mathbb{R})}. \]

The restriction estimates follows from applying Plancheral and observing that

\[ \| g_i \|_{L^2(\mathbb{R})} = \| f_i \|_{L^2(d\mu_i)}. \]

This result can be generalized to pairs of transverse compact hypersurfaces in higher dimensions; see for example [14, Theorem 25.1].
4.1. Relationship between linear and multilinear restriction

4.1.2 Obtaining linear restriction estimates via bilinear ones

We present a proof of Theorem 3 in [6], which employs the bilinear restriction method developed by Wolff, and Tao, Vargas, and Vega in [20] and [18] respectively. Let \( \sigma \) be the natural measure on the circle. The main idea of the bilinear approach is to decompose the circle into pieces \( \theta_j \). Then if \( f \) is supported on the circle, we write
\[
fd\sigma = \sum_j f_j d\sigma,
\]
where \( f_j \) is supported on \( \theta_j \). Thus, for \( k \in \mathbb{N} \),
\[
\| \hat{f}d\sigma \|_{2k}^2 = \sum_{i,j} |\hat{f}_i d\sigma|^k |\hat{f}_j d\sigma|^k = \sum_j \| \hat{f}_j d\sigma \|_{2k}^2 + \sum_{i \neq j} |\hat{f}_i d\sigma|^k |\hat{f}_j d\sigma|^k.
\]

Consider the off-diagonal terms; if \( \theta_i \) and \( \theta_j \) are transversal, then we can estimate the integral \( |\hat{f}_i d\sigma|^k |\hat{f}_j d\sigma|^k \) using a bilinear restriction estimate. In particular, if the distance between \( \theta_i \) and \( \theta_j \) is comparable to their lengths, we have a lower bound on the angle between them, and may use bilinear restriction; this idea is captured in Lemma 4.4.

Moreover, a common strategy is to localize the estimate. If \( \sigma \) is the natural measure on the circle and \( R > 1 \), consider a local restriction estimate of the form
\[
\| \tilde{f} \|_{L^p(B(0,R))} \lesssim_R \|f\|_{L^p(d\sigma)},
\]
where \( \nu \) is non-negative. This estimate is equivalent to the estimate
\[
\| \tilde{f} \|_{L^p(B(0,R))} \lesssim_R R^{\nu-1/p} \|f\|_{L^p(N_{R^{-1}})} \tag{4.2}
\]
where \( N_{R^{-1}} \) is the \( R^{-1} \)-neighbourhood of the unit circle [17]; heuristically, the equivalence of these estimates is due to the uncertainty principle.

Via Tao’s “epsilon removal” argument (so-called as we frequently wish to prove inequality (4.2) with \( \nu = \varepsilon \) for all \( \varepsilon > 0 \)), if such an estimate holds for all \( R > 1 \), we may obtain a restriction estimate on the circle, independent of \( R \), but only for a more restrictive range of \( p \) [16].

The result of Erdogan that we use to illustrate the bilinear method is closely related to the restriction problem. First, by rescaling the estimate (4.2) we may consider the annulus \( A_R(c) \) given by
\[
A_R(c) = \{ \xi \in \mathbb{R}^2 : R - c < |\xi| < R + c \}.
\]
4.1. Relationship between linear and multilinear restriction

The estimate given in Theorem 4.2 is of the form
\[ \| \tilde{f} \|_{L^2(\mu)} \lesssim_R R^\alpha \| f \|_{L^2(A_R(1))}, \]
where \( \mu \) is a finite positive measure supported on \( B(0; 1) \). Note that this estimate generalizes (4.2), as we can specialize to this case by taking \( \mu \) to be the Lebesgue measure on \( B(0; 1) \). Erdogān’s work in [6] contributed to progress on Falconer’s distance problem, indicating the applicability of multilinear restriction to many areas of mathematics. We state the theorem now.

**Theorem 4.2 (Erdogān).** Let \( \alpha \in [1, 2] \) and let \( \mu \) be a finite positive measure supported in the unit ball in \( \mathbb{R}^2 \), so that \( \mu(B(x, r)) \lesssim r^\alpha \) for all \( x \in \mathbb{R}^2 \) and \( r > 0 \). Let \( f \in L^2(\mathbb{R}^2) \) be such that \( \text{supp}(f) \subset A_R(1) \). Then for all \( \varepsilon > 0 \) and \( R > 1 \), we have
\[ \| \tilde{f} \|_{L^2(\mu)} \lesssim_R R^{1/2-\alpha/4+\varepsilon} \| f \|_{L^2(A_R(1))}. \]

First we transcribe, without proof, Lemmas 3.1 and 3.2 of [6]. Let \( \varphi \in S \) be such that \( \varphi \equiv 1 \) on \( Q \), the unit cube, and \( \text{supp}(\varphi) \subset 2Q \). Moreover, choose \( \varphi \) so that for any \( M > 0 \) and \( \xi \in \mathbb{R}^2 \),
\[ |\tilde{\varphi}(\xi)| \lesssim \sum_{j=1}^\infty 2^{-Mj} \mathbb{1}_{2^jQ}(\xi). \]

For a rectangle \( D \) in the plane define \( \varphi_D = \varphi \circ a_D^{-1} \) where \( a_D \) is an affine map taking \( Q \) to \( D \). The following lemma provides some properties of \( \mu_D := |\tilde{\varphi}_D|(-\cdot) \ast \mu \):

**Lemma 4.3.** Let \( \alpha \in [1, 2] \) and \( \mu \) a positive measure on \( \mathbb{R}^2 \) with \( \mu(B(x, r)) \lesssim r^\alpha \) for all \( x \in \mathbb{R}^2 \) and \( r > 0 \). Let \( D \) be a rectangle of dimensions \( R_1 \times R_2 \) with \( R_1 \leq R_2 \), and \( D' \) the dual of \( D \) centered at the origin. Then \( \| \mu_D \|_\infty \lesssim R^\alpha_2 \) and for all \( K \geq 1 \) and \( x \in \mathbb{R}^2 \)
\[ \mu_D(x + KD') := \int_{KD'} \mu_D(x + y) dy \lesssim K^\alpha R^{1-\alpha}_2 R^{-1}_1. \]

As we will decompose \( A_R(c) \) into dyadic pieces, for an interval \( I \), let
\[ A_R(c, I) = \{(\rho \cos \theta, \rho \sin \theta) : R - c < \rho < R + c, \theta \in I \}. \]

The following lemma is key in obtaining a bilinear restriction estimate on transversal pieces
4.1. Relationship between linear and multilinear restriction

Lemma 4.4. Let $R > 1$. Let $I, J \subset [-1/2, 1/2]$ be such that $|I| = |J| = l \gtrsim R^{-1/2}$, and $d(I, J) \gtrsim l$. Then for any $x \in \mathbb{R}^2$,

$$|(x - A_R(1, I)) \cap A_R(1, J)| \lesssim l^{-1}.$$ 

Decomposition of the annulus: As it suffices to consider $f \in L^2(\mathbb{R}^2)$ such that $\text{supp}(f) \subset A_R(1, [-1/2, 1/2])$, we decompose $A_R(1, [-1/2, 1/2])$ into dyadic pieces.

Let $I_n$ be the set of all dyadic intervals in $[-1/2, 1/2]$ of length $2^{-n}$, and let $I = \cup_n I_n$. Consider the following relation on $I$: we say $I \sim J$ if $|I| = |J|$, and $I$ and $J$ are not adjacent, but their parents are adjacent. Note that for almost every $x \in [-1/2, 1/2]^2$, there is exactly one pair $I, J \in I$ such that $I \sim J$ and $x \in I \times J$. Then we can write

$$\left[ -\frac{1}{2}, \frac{1}{2} \right]^2 = \left( \bigcup_{n=1}^{\log R^{1/2}} \bigcup_{I, J \in I_n, I \sim J} I \times J \right) \cup D,$$

where $D$ is a subset of the $CR^{-1/2}$-neighbourhood of the diagonal of $[-1/2, 1/2]^2$, and the logarithm is base 2. Moreover, $D$ can be written as the union of boxes $I \times J$ where $I$ are taken from a collection $J$ of finitely overlapping intervals of length approximately $R^{-1/2}$.

Let $f_I = f 1_{A_R(1, I)}$. Using (4.3) we may write

$$(\tilde{f})^2 = \sum_{n=1}^{\log R^{1/2}} \sum_{I, J \in I_n, I \sim J} \tilde{f}_I \tilde{f}_J + \mathcal{E},$$

for which the error $\mathcal{E}$ satisfies

$$|\mathcal{E}| \lesssim \sum_{I \in J} |\tilde{f}_I|^2$$

as the intervals in $J$ are finitely overlapping. Therefore

$$\| \tilde{f} \|_{L^2(d\mu)}^2 \lesssim \sum_{I \in J} \| \tilde{f}_I \|_{L^2(d\mu)}^2 + \sum_{n=1}^{\log R^{1/2}} \sum_{I, J \in I_n} \| \tilde{f}_I \tilde{f}_J \|_{L^1(d\mu)}.$$ 

(4.4)

Let $S_1$ and $S_2$ be the first and second sums on the right-hand side of (4.4) respectively.
We will show that the diagonal terms satisfy

\[ \| \tilde{f}_I \|_{L^2(d\mu)}^2 \lesssim R^{1-\alpha/2} \| f_I \|_2^2 \]  \tag{4.5} \]

and that the off-diagonal terms satisfy

\[ \| \tilde{f}_I \tilde{f}_J \|_{L^1(d\mu)} \lesssim R^{1-\alpha/2} \| f_I \|_2 \| f_J \|_2 \]  \tag{4.6} \]

using bilinear restriction. Before proving these estimates we demonstrate how we may combine them to obtain the estimate given in Theorem 4.2.

**Combine estimates using \( L^2 \)-orthogonality:** First note that, for fixed \( n \), the supports of \( f_I \) are disjoint, where \( I \in \mathcal{I}_n \). As the intervals of \( \mathcal{J} \) are finitely overlapping, we have that \( \sum_{I \in \mathcal{J}} \| f_I \|_2^2 \lesssim \| f \|_2^2 \). Hence (4.5) gives \( S_1 \lesssim R^{1-\alpha/2} \| f \|_2^2 \).

Next we use (4.6) to bound \( S_2 \). Let \( n \in \{1, \ldots, \lfloor \log R^{1/2} \rfloor \} \). Note that for any fixed \( I \in \mathcal{I}_n \), \( I \sim J \) for \( O(1) \) many \( J \in \mathcal{I}_n \). Combining this observation and Cauchy-Schwarz, we see that

\[ \sum_{I_{I,J} \in \mathcal{I}_n, I \sim J} \| f_I \|_2 \| f_J \|_2 \lesssim \left( \sum_{I \in \mathcal{I}_n} \| f_I \|_2^2 \right)^{1/2} \left( \sum_{J \in \mathcal{I}_n} \| f_J \|_2^2 \right)^{1/2} = \sum_{I \in \mathcal{I}_n} \| f_I \|_2^2. \]

But the supports of the \( f_I \) are disjoint, and hence \( \sum_{I \in \mathcal{I}_n} \| f_I \|_2^2 = \| f \|_2^2 \). Thus using the estimate (4.6), we have

\[ S_2 \lesssim R^{1-\alpha/2} \sum_n \sum_{I_{I,J} \in \mathcal{I}_n, I \sim J} \| f_I \|_2 \| f_J \|_2 \lesssim R^{1-\alpha/2} \sum_n \sum_{I \in \mathcal{I}_n} \| f_I \|_2^2 \lesssim \epsilon R^{1-\alpha/2 + \epsilon} \| f \|_2^2 \]

as we sum over \( O(\log R^{1/2}) \) values of \( n \).

**Proof of (4.5):** Let \( I \in \mathcal{J} \). Note that \( \text{supp}(f_I) \) is contained in a rectangle \( D \) of dimensions \( C \times CR^{1/2} \). As \( \varphi_D = 1 \) on \( D \), we have \( \tilde{f}_I = \tilde{f}_I * \varphi_D \). Then via Cauchy-Schwarz,

\[ |\tilde{f}_I|^2 \leq (|\tilde{f}_I|^2 | \varphi_D|) || \varphi_D ||_1 \lesssim |\tilde{f}_I|^2 * | \varphi_D| \]
4.1. Relationship between linear and multilinear restriction

as \( \|\tilde{\varphi}_D\|_1 \lesssim 1 \). Combining this inequality with Fubini’s theorem, we have

\[
\|f_I\|_{L^2(d\nu)} \lesssim \int \int |\tilde{f}_I(x)|^2 |\tilde{\varphi}_D|(y-x)d\mu(y)dx = \int |\tilde{f}_I(x)|^2 \mu_D(x)dx.
\]

But Lemma 4.3 gives \( \|\mu_D\|_\infty \lesssim R^{1-\alpha/2} \). Therefore, via Parseval’s theorem we obtain the bound (4.5).

**Proof of (4.6) using bilinear restriction:** Fix \( I, J \in \mathcal{I}_n \) with \( I \sim J \). Note that \( f_I \) and \( f_J \) are supported in a rectangle of dimensions \( CR2^{-n} \times CR2^{-2n} \). Thus \( \text{supp}(f_I * f_J) \subset D \), where \( D \) is a rectangle of dimensions \( 2CR2^{-n} \times 2CR2^{-2n} \). Let \( e \) be the direction in which the longer side of \( D \) is oriented. Moreover, let \( D' \) be the dual of \( D \) centered at the origin. Next, tile \( \mathbb{R}^2 \) with rectangles \( P \) of dimension \( 100 \times 100 \cdot 2^{-n} \), oriented so that the short length is in direction \( e \). Note that each \( P \) is contained in a translate of \( CR2^{-2n}D' \).

Let \( \tilde{\varphi} \in \mathcal{S} \) be non-negative and greater than \( 1/2 \) on the unit cube \( Q \). Moreover, choose \( \tilde{\varphi} \) so that \( \text{supp}(\tilde{\varphi}) \subset Q \) and for all \( M > 0 \) and \( x \in \mathbb{R}^2 \), \( |\tilde{\varphi}(x)| \lesssim \sum_{j=1}^\infty 2^{-Mj}1_{2^jQ}(x) \).

For each rectangle \( P \) define \( \tilde{\varphi}_P \) via \( \tilde{\varphi}_P = \tilde{\varphi} \circ a_P^{-1} \), where \( a_P \) is an affine map bringing \( Q \) to \( P \). From the properties of \( \tilde{\varphi} \) we may derive \( 1 \lesssim \sum_P \tilde{\varphi}_P^3 \lesssim \sum_P \tilde{\varphi}_P^2 \lesssim 1 \).

Let \( f_I,P = (\tilde{f}_I \tilde{\varphi}_P) \). Mirroring an argument given when estimating \( S_1 \) we have

\[
\|\tilde{f}_I \tilde{f}_J\|_{L^1(d\mu)} \leq \int |\tilde{f}_I(x)\tilde{f}_J(x)|\mu_D(x)dx.
\]

Applying the property \( 1 \lesssim \sum_P \tilde{\varphi}_P^3 \) and Cauchy-Schwarz, we obtain

\[
\|\tilde{f}_I \tilde{f}_J\|_{L^1(d\mu)} \lesssim \sum_P \int |\tilde{f}_{I,P}(x)\tilde{f}_{J,P}(x)|\mu_D(x)\tilde{\varphi}_P(x)dx
\leq \sum_P \left( \int |\tilde{f}_{I,P}(x)\tilde{f}_{J,P}(x)|^2 dx \right)^{1/2} \left( \int \mu_D^2(x)\tilde{\varphi}_P^2(x)dx \right)^{1/2}.
\] (4.7)

We estimate the integral \( \int |\tilde{f}_{I,P}(x)\tilde{f}_{J,P}(x)|^2 dx \) first. Using Parseval’s Theorem we have

\[
\int |\tilde{f}_{I,P}(x)\tilde{f}_{J,P}(x)|^2 dx = \int |f_{I,P} * f_{J,P}(\xi)|^2 d\xi
= \int \left| \int 1_{\xi-A_{I,P} \cap A_{J,P}}(\eta)f_{I,P}(\xi-\eta)f_{J,P}(\eta)d\eta \right|^2 d\xi
\]
where $A_{I,P}$ and $A_{J,P}$ are the supports of $f_{I,P}$ and $f_{J,P}$ respectively. Applying Cauchy-Schwarz to the inner integral gives

$$
\int |\tilde{f}_{I,P}(x)\tilde{f}_{J,P}(x)|^2 dx \leq \left( \sup_{\xi} |(\xi - A_{I,P}) \cap A_{I,P}| \right) \int |f_{I,P}|^2 \ast |f_{J,P}|^2(\xi) d\xi \\
\leq \left( \sup_{\xi} |(\xi - A_{I,P}) \cap A_{I,P}| \right) \|f_{I,P}\|_2^2 \|f_{J,P}\|_2^2.
$$

As $f_{I,P} = f_I \ast \widehat{\phi}_P$, $A_{I,P} \subset \text{supp}(f_I) + \text{supp}(\widehat{\phi}_P) \subset A_R(1,1) + P'$. Recall that $\text{supp}(f_I)$ is contained in a rectangle of dimensions about $R2^{-n} \times R2^{-2n}$, with the longer side in direction $e$. Note that $P'$ is a rectangle of dimensions $1/100 \times 2^n/100$, with the longer side in direction $e$, and that $2^n \leq R2^{-n}$. The key geometric observation is

$$A_{R}(1,1) + P' \subset A_{R}(10,11/10I),$$

allowing us to invoke Lemma 4.4. Namely, we obtain $\sup_{\xi} |(\xi - A_{I,P}) \cap A_{I,P}| \lesssim 2^n$. Then

$$\int |\tilde{f}_{I,P}(x)\tilde{f}_{J,P}(x)|^2 dx \lesssim 2^n \|f_{I,P}\|_2^2 \|f_{J,P}\|_2^2. \tag{4.8}$$

Returning to (4.7), we use Lemma 4.3 to estimate $\int \mu_D(x)\tilde{\phi}_P(x) dx$. First, note that by using the property $|\tilde{\phi}_P(\xi)| \lesssim \sum_{j=1}^{\infty} 2^{-Mj}1_{2^jP}(\xi)$ and Lemma 4.3 with $K = CR2^{-2n}2^j$, we have

$$\int \mu_D(x)\tilde{\phi}_P(x) dx \lesssim \sum_{j=1}^{\infty} 2^{-Mj} \int \mu_D(x)1_{2^jP}(x) dx \lesssim \sum_{j=1}^{\infty} 2^{-Mj}2^{-n\alpha}2^j \lesssim 2^{n-n\alpha}.$$

Therefore

$$\int \mu_D^2(x)\tilde{\phi}_P^2(x) dx \lesssim \|\mu_D\|_\infty \int \mu_D(x)\tilde{\phi}_P(x) dx \lesssim R^{2-\alpha}2^{-n}$$

since $\|\mu_D\|_\infty \lesssim R^{2-\alpha}2^{n\alpha-2n}$ by Lemma 4.3. Combining this bound with (4.7) and (4.8), we have

$$\|\tilde{f}_{I,J}\|_{L^1(\mu)} \lesssim R^{1-\alpha/2} \sum_P \|f_{I,P}\|_2 \|f_{J,P}\|_2.$$
4.1. Relationship between linear and multilinear restriction

Finally, using Cauchy-Schwarz, the property that $\sum_P \tilde{\varphi}_P^2 \lesssim 1$, and Parseval’s Theorem (recall $f_{I,P} = (\tilde{f}_I \tilde{\varphi}_P)$), we obtain

$$\|\tilde{f}_I \tilde{f}_J\|_{L^1(d\mu)} \lesssim R^{1-\alpha/2} \left( \sum_P \|f_{I,P}\|_2^2 \right)^{1/2} \left( \sum_P \|f_{J,P}\|_2^2 \right)^{1/2} \lesssim R^{1-\alpha/2} \|f_I\|_2 \|f_J\|_2$$

as required.
Chapter 5

Multilinear restriction estimates for singular measures

We present our results on the existence of multilinear restriction estimates for a collection of singular measures. In section 5.1 we generalize Chen’s result Theorem 3.4 and show that a multilinear restriction estimate holds for measures \( \mu_1, \ldots, \mu_n \) satisfying \( \mu_1 \ast \cdots \ast \mu_n \in L^\infty(\mathbb{R}^d) \). Then we provide a class of Cantor measures for which this restriction estimate applies.

Then, in section 5.2 we give necessary conditions on the dimensionality of the measures \( \mu_1, \ldots, \mu_n \) so that we may have multilinear restriction for a given range of exponents.

5.1 Generalization of Chen’s restriction estimate

Consider the following generalization of Theorem 3.4:

**Theorem 5.1.** Let \( \mu_1, \ldots, \mu_n \) be finite compactly supported positive measures on \( \mathbb{R}^d \). Assume that \( \mu_1 \ast \cdots \ast \mu_n \in L^\infty(\mathbb{R}^d) \). Then, for all \( q \geq 2 \) we have the restriction estimate

\[
\left\| \prod_{i=1}^n \hat{f}_i d\mu_i \right\|_{L^q(\mathbb{R}^d)} \lesssim \prod_{i=1}^n \| f_i \|_{L^q(d\mu_i)}.
\]

**Proof.** Let \( \varphi \in C_c^\infty(\mathbb{R}^d) \) be a non-negative function satisfying \( \| \varphi \|_1 = 1 \), and let \( \varphi_\varepsilon(\xi) = \varepsilon^{-d} \varphi(\xi/\varepsilon) \). Define \( \mu_{i,\varepsilon} = \varphi_\varepsilon \ast \mu_i \) so that \( \mu_{i,\varepsilon} \) converges weakly to \( \mu_i \) as \( \varepsilon \to 0^+ \). We show that the restriction estimate holds for \( \mu_{1,\varepsilon}, \ldots, \mu_{n,\varepsilon} \).

By the Hausdorff-Young inequality,

\[
\left\| \prod_{i=1}^n \hat{f}_i d\mu_{i}\varepsilon \right\|_{L^q(\mathbb{R}^d)} \leq \left\| \prod_{i=1}^n f_i d\mu_{i,\varepsilon} \right\|_{L^q'(\mathbb{R}^d)}
\]
5.1. Generalization of Chen’s restriction estimate

Expanding the right-hand side out we obtain

\[
\left( \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^{d(n-1)}} F(\xi, \eta) M_\varepsilon(\xi, \eta) d\eta \right|^{q'} d\xi \right)^{1/q'}
\]

where

\[
F(\xi, \eta) = f_1(\eta_1) \prod_{j=2}^n f_j(\eta_j - \eta_{j-1}), \quad M_\varepsilon = \mu_1, \varepsilon(\eta_1) \prod_{j=2}^n \mu_j, \varepsilon(\eta_j - \eta_{j-1})
\]

with \( \eta = (\eta_1, \ldots, \eta_{n-1}) \) and \( \eta_n = \xi \). We can use Hölder’s Inequality to bound the inner integral by splitting up its integrand into the factors \( M_\varepsilon^{1/q} \) and \( FM_\varepsilon^{1/q'} \). This gives an upperbound of

\[
\left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^{d(n-1)}} M_\varepsilon(\xi, \eta) d\eta \right)^{q'/q} \left( \int_{\mathbb{R}^{d(n-1)}} |F(\xi, \eta)|^{q'} M_\varepsilon(\xi, \eta) d\eta \right)^{q'/q'} d\xi \right)^{1/q'}
\]

Since \( \int M_\varepsilon(\xi, \eta) d\eta = \mu_1, \varepsilon \cdots \mu_n, \varepsilon(\xi) \), this quantity is bounded by

\[
\|\mu_1, \varepsilon \cdots \mu_n, \varepsilon\|_{L^1} \left( \int \int |F(\xi, \eta)|^{q'} M_\varepsilon(\xi, \eta) d\eta d\xi \right)^{1/q'}
\]

\[
\lesssim \|\mu_1 \cdots \mu_n\|_{L^{1/q}} \|f_1\|_{L^{q'}(d\mu_1, \varepsilon)} \cdots \|f_n\|_{L^{q'}(d\mu_n, \varepsilon)}.
\]

\[\Box\]

Example 5.2. Consider the self-similar Cantor sets

\[
E = \left\{ \sum_{j=1}^\infty \frac{x_j}{4^j} : x_j \in \{0, 1\} \right\} \quad \text{and} \quad F = \left\{ \sum_{j=1}^\infty \frac{y_j}{4^j} : y_j \in \{0, 2\} \right\}
\]

and their natural Cantor measures, \( \mu \) and \( \nu \) respectively. Moreover, let \( \mu_k \) and \( \nu_k \) be defined as usual, so that \( \mu_k \to \mu \) and \( \nu_k \to \nu \) in the weak-* sense. Since \( E \) and \( F \) are self-similar, their natural measures do not have Fourier decay, and so there do not exist linear restriction estimates for \( \mu \) and \( \nu \). However, we show that \( \mu * \nu \in L^\infty(\mathbb{R}) \), and so the multilinear restriction estimate given in Theorem 5.1 applies.
5.1. Generalization of Chen’s restriction estimate

Note that

\[ \mu_k = 2^k \left( \sum_{a \in A_k} \delta_a \right) * 1_{[0, 4^{-k}]} , \quad \nu_k = 2^k \left( \sum_{a \in A_k'} \delta_a \right) * 1_{[0, 4^{-k}]} \]

where \( A_k = 4^{-1}\{0, 1\} + \cdots + 4^{-k}\{0, 1\} \) and \( A_k' = 4^{-1}\{0, 2\} + \cdots + 4^{-k}\{0, 2\} \). Since \( \{0, 1\} + \{0, 2\} = \{0, 1, 2, 3\} \), we have

\[ A_k + A_k' = 4^{-k}\{0, 1, \ldots, 4^k - 1\} . \]

Hence

\[ \mu_k * \nu_k = 4^k \left( \sum_{j=0}^{4^k-1} \delta_{j/4^k} \right) * \chi_k \]

where \( \chi_k = 1_{[0, 4^{-k}]} * 1_{[0, 4^{-k}]} \). Note that \( \text{supp}(\chi_k) \subset [0, 2 \cdot 4^{-k}] \) and for \( x \) in its support,

\[ \chi_k(x) = \min\{x, 4^{-k}\} - \max\{0, x - 4^{-k}\} \]

\[ = \begin{cases} x & 0 \leq x \leq 4^{-k} \\ 2 \cdot 4^{-k} - x & 4^{-k} \leq x \leq 2 \cdot 4^{-k} . \end{cases} \]

Thus, the supports of \( \delta_{j/4^k} \chi_k \) and \( \delta_{j/4^k} \chi_k \) intersect if and only if \(|i - j| \leq 1\). Consider

\[ \delta_{j/4^k} \chi_k + \delta_{(j+1)/4^k} \chi_k . \]

The supports of the individual functions intersect on \([ (j+1)4^{-k}, (j+2)4^{-k} ] \), where their sum is identically \( 4^{-k} \). Therefore

\[ \mu_k * \nu_k(x) = \begin{cases} 4^k x & 0 \leq x \leq 4^{-k} \\ 1 & 4^{-k} \leq x \leq 1 \\ 1 + 4^k - 4^k x & 1 \leq x \leq 1 + 4^{-k} \\ 0 & \text{otherwise} . \end{cases} \]

Then \( \mu * \nu = 1_{[0, 1]} \) and belongs to \( L^\infty(\mathbb{R}) \).

We note that the argument given in this example extends to a pair of Cantor sets with endpoints in the \( k^{th} \) iteration given by \( A_k = N^{-1}A + \cdots + N^{-K}A \) and \( A_k' = N^{-1}A' + \cdots + \)
$N^{-K} A'$ respectively, and $A \oplus A' = \mathbb{Z}/N\mathbb{Z}$.

### 5.2 Necessary conditions for multilinear restriction

Consider the following result, which uses the dimension of the measures $\mu_1, \ldots, \mu_n$ to restrict the range of exponents for which multilinear restriction may hold:

**Proposition 5.3.** Let $\mu_1, \ldots, \mu_n$ be compactly supported positive measures on $\mathbb{R}^d$. Suppose for each $1 \leq j \leq n$ there exists a sequence of points $\{x_{j,k}\}_{k=1}^\infty$ and a sequence $\{r_k\}_{k=1}^\infty$, $r_k \searrow 0$ so that

$$\mu_j(B(x_{j,k}, r_k)) \sim r_k^{\alpha_j}.$$  

If the restriction estimate

$$\left\| \prod_{j=1}^n f_j d\mu_j \right\|_{L^q(\mathbb{R}^d)} \lesssim \prod_{j=1}^n \|f_j\|_{L^p(d\mu_j)},$$

holds, then

$$q \geq \frac{dp'}{\alpha_1 + \cdots + \alpha_n}.$$  

**Proof.** Let $f_{j,k} = \mathbb{1}_{B(x_{j,k}, r_k)}$. Let $\varphi \in C_c^\infty(\mathbb{R}^d)$ be a non-negative function satisfying $\|\varphi\|_1 = 1$, and let $\varphi_\varepsilon(\xi) = \varepsilon^{-d} \varphi(\xi/\varepsilon)$. Define $\mu_{j,\varepsilon} = \varphi_\varepsilon * \mu_j$ so that $\mu_{j,\varepsilon}$ converges weakly to $\mu_j$ as $\varepsilon \to 0^+$. Note that

$$\int f_{1,k} d\mu_{1,\varepsilon} * \cdots * f_{n,k} d\mu_{n,\varepsilon}(x) = \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \xi} f_{1,k} d\mu_{1,\varepsilon} * \cdots * f_{n,k} d\mu_{n,\varepsilon}(x) dx = \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \xi} \sum_{j=1}^n f_{j,k}(y_j) \mu_{j,\varepsilon}(y_j) dy dx$$

where

$$\Sigma_x = \{y = (y_1, \ldots, y_n) \in (\mathbb{R}^d)^n : y_1 + \cdots + y_n = x\}.$$  

Making the substitution $u_j = y_j - x_{j,k}$ and $\bar{x} = x - \sum_{j=1}^n x_{j,k}$ we obtain

$$\int f_{1,k} d\mu_{1,\varepsilon} * \cdots * f_{n,k} d\mu_{n,\varepsilon}(x) = e^{-2\pi i \xi(\sum_{j=1}^n x_{j,k})} \int_{\mathbb{R}^d} e^{-2\pi i \bar{x} \cdot \xi} I(\bar{x}) d\bar{x}$$

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where

$$I(\overline{x}) = \int_{\Sigma} \prod_{j=1}^{n} \mathbb{1}_{|u_j| < r_k \mu_j,\epsilon}(u_j + x_{j,k}) du.$$

Next, write $f_{1,k}d\mu_{1,\epsilon} \ast \cdots \ast f_{n,k}d\mu_{n,\epsilon}(\xi)$ as

$$e^{-2\pi i \xi \cdot \sum_{j=1}^{n} x_{j,k}} \left( \int_{\mathbb{R}^d} I(\overline{x}) d\overline{x} + \int_{\mathbb{R}^d} (e^{-2\pi i \xi \cdot \overline{x}} - 1) I(\overline{x}) d\overline{x} \right).$$

Note that $I(\overline{x}) \neq 0$ only if $|\overline{x}| < nr_k$. Then for $|\xi| \leq \frac{1}{C r_k}$, where $C$ is a sufficiently large constant (not depending on $r_k$), $|e^{-2\pi i \xi \cdot \overline{x}} - 1| < 1/10$. Hence for $|\xi| \leq \frac{1}{C r_k}$, we have

$$|f_{1,k}d\mu_{1,\epsilon} \ast \cdots \ast f_{n,k}d\mu_{n,\epsilon}(\xi)| \gtrsim \int_{\mathbb{R}^d} I(\overline{x}) d\overline{x} = \prod_{j=1}^{n} \int_{\mathbb{R}^d} \mathbb{1}_{|x| < r_k \mu_j,\epsilon}(x + x_{j,k}) dx,$$

by the reverse triangle inequality. So taking $\epsilon \to 0^+$,

$$|f_{1,k}d\mu_{1,\epsilon} \ast \cdots \ast f_{n,k}d\mu_{n,\epsilon}(\xi)| \gtrsim \prod_{j=1}^{n} \mu_j(B(x_{j,k}, r_k)).$$

Thus

$$\left\| \prod_{i=1}^{n} \hat{f}_i d\mu_i \right\|_{L^q(\mathbb{R}^d)} \gtrsim r_k^{\alpha_1 + \cdots + \alpha_n} r_k^{-d/q}.$$

But we have the estimate

$$\prod_{i=1}^{n} \|f_i\|_{L^p(d\mu_i)} \lesssim r_k^{\frac{\alpha_1 + \cdots + \alpha_n}{p}},$$

implying that

$$r_k^{\alpha_1 + \cdots + \alpha_n - d/q} \lesssim r_k^{\frac{\alpha_1 + \cdots + \alpha_n}{p}}.$$

Taking $k \to \infty$ yields $q \geq \frac{dp'}{\alpha_1 + \cdots + \alpha_n}$. 

\[ \square \]
Chapter 6

Conclusion

We have seen that within restriction theory fractal sets exhibit a wide range of behaviour; the levels of arithmetic structure in the fractal sets are indicative of the existence of restriction estimates, and the range of exponents for which restriction is possible. While the Tomas-Stein restriction theorem provides a sharp range of exponents for smooth hypersurfaces of non-vanishing Gaussian curvature, the analogous restriction estimate given by Mockenhaupt is not sharp in general. However, Mockenhaupt’s restriction theorem is sharp for some measures, such as the “Knapp” Cantor measure, which exploits the idea that a Cantor measure may still achieve Fourier decay if its associated Cantor set contains a small subset of high arithmetic structure. Analogously, in illustrating the sharpness of the Tomas-Stein restriction theorem for the sphere, we exploit the idea that locally the sphere is well-approximated by its tangent plane. Through this comparison, we see that the role of arithmetic structure in restriction for Cantor measures is similar to the role of curvature in restriction for the natural measures of hypersurfaces.

This analogy carries over to multilinear restriction theory. In section 4.1.1 we showed that multilinear restriction may hold for measures supported on transverse segments of a hypersurface, where here transversality means that the norms of the segments are well separated. However, in the example given, linear restriction does not hold for the individual measures as the individual segments lack curvature. Then, in example 5.2 we considered a pair of self-similar Cantor sets with high levels of arithmetic structure, which implies a lack of linear restriction estimates. However, because the arithmetic structures of the Cantor sets are in some sense “transverse” (say, the sumset of their sets of endpoints is well spread out; in the example given, we could write \( \mathbb{Z}/4\mathbb{Z} \) as a direct sum of the sets defining the endpoints of the two Cantor sets) we are able to achieve multilinear restriction.

In section 4.1.2 we gave an overview of how multilinear restriction estimates are effective tools in improving linear restriction estimates, and we are hopeful that multilinear restric-
tion estimates for singular measures supported on fractal sets may similarly be effective in improving linear ones. However, multilinear restriction theory has proved to be extremely versatile in its application, contributing to many different problems within harmonic analysis. One such example was given in Theorem 4.2 which used bilinear restriction to prove a generalized form of the restriction problem on the circle, which in turn had applications to Falconer’s distance problem.

Another significant application of multilinear restriction is decoupling. Classically, decoupling considers functions $f$ which are Fourier supported on the $\delta$-neighbourhood of the unit paraboloid, $N_\delta$. Partitioning $N_\delta$ into caps $\theta$ of dimensions about $\delta^{1/2} \times \cdots \times \delta^{1/2} \times \delta$, and considering $f_\theta = (\hat{1}_\theta)^\circ$, we ask how much constructive interference may occur between the $f_\theta$. This idea may be captured in the $l^2$-decoupling inequality,

$$
\|f\|_{L^p(\mathbb{R}^d)} \lesssim_\varepsilon \delta^{-\varepsilon} \left( \sum_\theta \|f_\theta\|_{L^p(\mathbb{R}^d)}^2 \right)^{1/2},
$$

established by Bourgain and Demeter in [2], for which multilinear restriction was a key ingredient. Bourgain and Demeter’s result and techniques largely contributed to a renewed and reinvigorated interest in restriction theory. Recently, there has been study on decoupling for functions supported on Cantor sets (see [12]). We are interested in exploring the connection between multilinear restriction theory for measures supported on Cantor sets and decoupling on Cantor sets.

Furthermore, we anticipate the possibility of employing multilinear restriction theory to maximal functions defined via singular measures supported on fractal sets. If $\mu$ is such a measure, define the maximal operator

$$
\mathcal{M}f(x) = \sup_{t > 0} \int |f(x - ty)|d\mu(y).
$$

Of interest are estimates of the form

$$
\|\mathcal{M}f\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)} \tag{6.1}
$$

as they may be used to prove differentiation theorems. The classical analogue of this problem, which considers the natural measures of smooth surfaces such as the sphere, is well studied and well understood. The problem as we state it here has recently been studied
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in [11] and [9]. The latter result uses a decoupling inequality established in [12] to obtain a weaker form of (6.1), and applies to measures in \( \mathbb{R} \) supported on a particular class of Cantor sets. We are interested in extending these results, and generalizing the arguments to measures supported on Cantor sets in higher dimension.

Multilinear restriction has proved to be a powerful and versatile tool in contributing to many different classical problems in harmonic analysis. Moreover, there are many striking connections and interesting parallels between classical restriction theory and restriction theory for singular measures supported on fractal sets. Consequently, here we have focused on developing multilinear restriction theory for singular measures, and in the future wish to explore how multilinear restriction connects to the fractal set analogues of many classical results in harmonic analysis.
Bibliography


